A Multi-Regional View on Aging Societies Accounting for the Existence of Non-Tradable Goods

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Abstract

This thesis assesses the question, whether accounting for non-tradable goods sectors in a calibrated Auerbach-Kotlikoff multi-regional overlapping-generations-model significantly affects this model's results when simulating the economic impact of demographic change. Non-tradable goods constitute a major part of up to 80 percent of GDP of modern economies. At the same time, multi-regional overlapping-generations-models presented by literature on demographic change so far ignored their existence and counterfactually assumed perfect tradability between model regions. Moreover, this thesis introduces the assumption of an increasing preference share for non-tradable goods of old generations. This fact-based assumption is also not part of models in relevant literature.

These obvious simplifications of common models vis-à-vis reality notwithstanding, this thesis concludes that differences in results between a model featuring non-tradable goods and a common model with perfect tradability are very small. In other words, the common simplification of ignoring non-tradable goods is unlikely to lead to significant distortions in model results.

In order to ensure that differences in results between the 'new' model, featuring both non-tradable and tradable goods, and the common model solely reflect deviations due to the more realistic structure of the 'new' model, both models are calibrated to match exactly the same benchmark data and thus do not show deviations in their respective baseline steady states.

A variation analysis performed in this thesis suggests that differences between the common model and a model with non-tradable goods can theoretically be large, but only if the benchmark tradable goods sector is assumed to be unrealistically small.

Finally, this thesis analyzes potential real exchange rate effects of demographic change, which could occur due to regional price differences of non-tradable goods. However, results show that shifts in real exchange rate based on these price differences are negligible.

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II List of Symbols and Abbreviations

Symbol	Description
$a_{g,r}^{term}$	Terminal period assets per generation
$a_{i,j}$	Benchmark value share of input good *i*
$\overline{a}_{g,r}^{0}$	Initial assets of cohort *g*
$\hat{a}_{a,r}$	Value of assets held by age
$\hat{a}_{a,r}^{LC}$	Present value of assets over the lifecycle
$A_{g,r}^{0}$	Initial endowment
$A_{g,r}^{term}$	Endowment of terminal cohorts with post-terminal period utility
A_r^T , A_r^{NT}	Total factor productivity of (non-)tradable production
$B_{t,r}$	Aggregate pension benefits
$\overline{B}_{t,r}$	Benchmark aggregate pension benefits
$B_{g,r}^{term}$	Terminal capital stock
$c_{g,t,r}$, $c_{g,t,r}^{T}$, $c_{g,t,r}^{NT}$	Period goods consumption (total, tradable, non-tradable)
$\overline{c}_{g,t,r}$, $\overline{c}_{g,t,r}^{T}$, $\overline{c}_{g,t,r}^{NT}$	Benchmark period goods consumption (total, tradable, non-tradable)
C_j	Unit cost function
$\overline{C}_j$	Benchmark cost for activity level of *one*
$\overline{C}_r$, $\overline{C}_r^{T}$, $\overline{C}_r^{NT}$	Benchmark aggregate consumption from SAMs
$CA_{t,r}$	Current account
d	Duration of one model period in years
$d_{i,h}$	Demand of good *i* by household *h*
f	Function
F	Mapping
$Q_{t,r}$	Real exchange rate
g	Cohort index
h	Household index
$I_{t,r}$	Investment
j	Activity index
$\hat{j}_r^{avg}$	Average cohort income during working years
$K_{t,r}$, $K_{t,r}^{T}$, $K_{t,r}^{NT}$	Capital stock (total, tradable, non-tradable)
$\overline{K}_r$	Benchmark capital stock
K_r^{term}	Terminal capital stock
l , u	Lower/upper bounds

$L_{t,r}$, $L_{t,r}^{T}$, $L_{t,r}^{NT}$	Labor supply (total, tradable, non-tradable)
$\overline{L}_{t,r}^{T}$, $\overline{L}_{t,r}^{NT}$	Benchmark labor earnings (tradable, non-tradable)
$n_{g,r}$	Size of cohort
N^{eq}	Number of single, numerical model equations
$NX_{t,r}$	Net exports
$\overline{NX}_{r}$	Benchmark net exports from SAMs
$NY_{t,r}$	Net factor income
p_i	Price vector
$\overline{p}_t$	Reference price path
$p_{t,r}^{lab}$	Wage rate
p_t^{T} , $p_{t,r}^{NT}$	Price of (non-)tradable goods
$p_{g,a,r}^{z,term}$	Price of post-terminal period utility
$p_r^{K,term}$	Price of terminal capital stock
$\overline{p}_i^{I}$	Benchmark price of input good *i*
$\overline{p}_i^{O}$	Benchmark price of output good *i*
$\overline{p}_{g,t,r}^{z}$	Benchmark price of period utility
$\overline{p}_{g,a,r}^{z,term}$	Benchmark price of post-terminal period utility
r	Region index
r_t	Endogenous world interest rate
$\overline{r}$	Reference real interest rate
R_j	Unit revenue function
$\overline{R}_j$	Benchmark revenue for activity level of *one*
$\overline{R}_r$, $\overline{R}_r^{T}$, $\overline{R}_r^{NT}$	Benchmark capital earnings (total, tradable, non-tradable)
$S_{t,r}$	Aggregate savings
t	Time index
$t_{g,t,r}^{lab}$	Labor time per period and cohort
$t_{g,t,r}^{leis}$	Leisure time per period and cohort
T	Number of time periods
$u_{g,r}$	Lifetime utility
$\overline{u}_{g,r}$	Benchmark lifetime utility
$y_{i,j}$	Output quantity of good *i*
$Y_{t,r}^{T}$, $Y_{t,r}^{NT}$	Output (non-)tradable goods
$\overline{Y}_{t,r}^{T}$, $\overline{Y}_{t,r}^{NT}$	Benchmark output (non-)tradable goods
v	Vector
w	Vector

$x^{I}_{i,j}$	Input quantity of good *i*
$\bar{x}^{I}_{i,j}$	Benchmark input quantity of good *i*
$\bar{x}^{O}_{i,j}$	Benchmark output quantity of good *i*
z	Vector
$z_{g,t,r}$	Period utility
$\bar{z}_{g,t,r}$	Benchmark period utility
$z^{term}_{g,a,r}$	Post-terminal period utility
$\bar{z}^{term}_{g,a,r}$	Benchmark post-terminal period utility
~	Capital share
~ $_{i,j}$	Exogenous parameter
~ $^{retire}_{a,r}$	Share of retired cohort *a* in total retired population
˜	Goods share in period utility
$_{i,j}$	Exogenous parameter
$_{r}$	Preference share for goods consumption
¥	Preference share for tradable goods consumption
$\Gamma_{t,r}$	Aggregate pension contribution
¢ $_{j}$	Exogenous parameter
¢ $_{r}$	Depreciation rate
¢ ^{NT}	Parameter for increasing non-tradable goods preference
£ $_{r}$	Fixed pension contribution rate
£̄ $_{r}$	Benchmark contribution rate
¸ $_{g,r}$	Number of working periods
ˆ¸ $_{r}$	Correction factor
¦	Elasticity of transformation
¦ $_{a,r}$	Lagrange-multiplier
¯	Elasticity of intertemporal substitution
$\bar{\Theta}^{K}_{r}$	Benchmark share of capital in total initial assets
a $_{g,r}$	Cohort lifespan
â $_{r}$	Lagrange-multiplier
Λ	Lagrange function
«	Elasticity of substitution (general)
« $_{i,j}$	Benchmark value share of output good *i*
³ $_{r}$	Steady-state growth rate of youngest cohort
© $_{g,t,r}$	Cohort pension contribution
¶	Elasticity of substitution goods consumption vs. leisure time
Ξ_{i}	Net supply function (general)
$\Xi^{x}_{g,t,r}$	Net supply function for good *x*

®$_{g,t,r}$	Age-dependent labor productivity
Π_{j}	Unit profit function (general)
$\Pi^{x}_{g,r}$	Unit profit function of activity x
°	Time discount parameter
±	Elasticity of substitution tradable vs. non-tradable goods
²$_{a,r}$	Preference share for non-tradable goods consumption
¤$_{g,t,r}$	Cohort pension benefits
¤$_{g,t,r}$	Pension benefits for cohort size of *one*
i$_{j}$	Exogenous parameter
·$_{t,r}$	Benefit rate in period *t*
Ψ_{h}	Income of household *h*
µ	Time endowment per time period (cohort size of one)
µ$_{i,h}$	Exogenous endowment of good *i* to household *h*
Ω	Activity level (general)
$\Omega^{x}_{g,r}$	Activity level of activity *x*

Abbreviation	Description
CAP	Capital
CES	Constant elasticity of substitution
CET	Constant elasticity of transformation
CIES	Constant intertemporal elasticity of substitution
CON	Private consumer
c.p.	Ceteris paribus
GAMS	General Algebraic Modeling System (Software)
GDP	Gross domestic product
GOV	Government
GTAP	Global Trade Analysis Program (Purdue University)
LAB	Labor
MCP	Mixed Complementarity Problem
NCP	Non-linear complementarity problem
OECD	Organisation for Economic Cooperation and Development
OLG	Overlapping-generations(-model)
OUTTR	Tradable output
OUTNT	Non-tradable output
PAYGO	Pay-as-you-go (pension system)
REGHOUS	Regional household
ROW	Rest of world
SAM	Social Accounting Matrix
s.t.	Subject to
UN	United Nations
USD	US-Dollar

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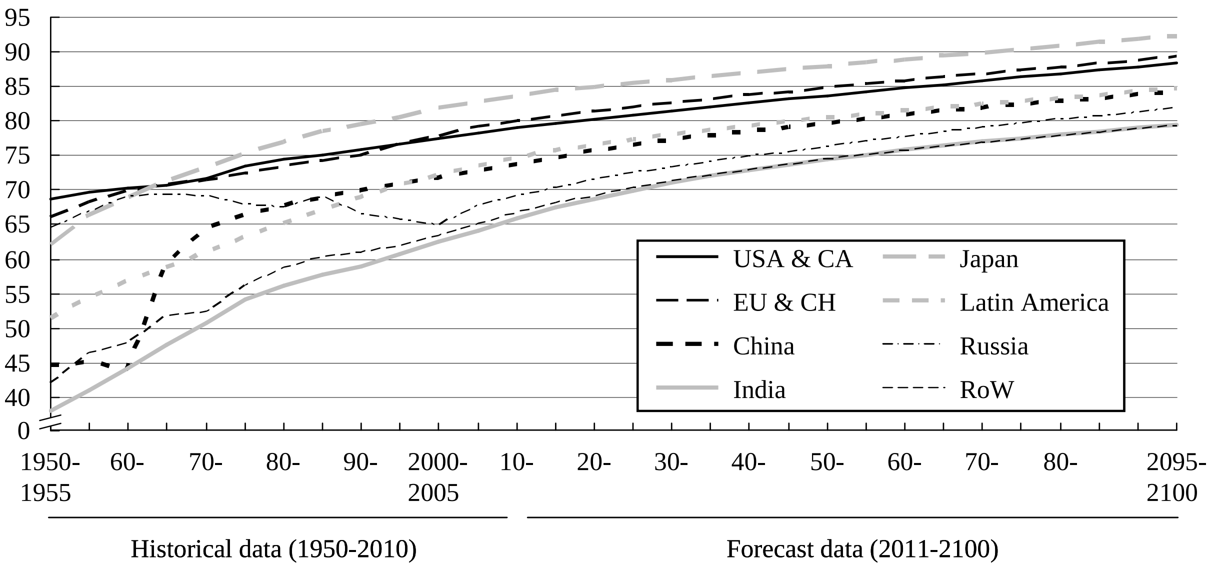
1 Introduction

Two distinct phenomena had and will continue to have an unprecedented and significant impact on the age structure of all world regions. First, life expectancy increased since 1950 by 10-20 years in developed nations and by 30 years in India and China and it is projected to further increase by additional 10-13 years until 2100 both in developed regions as well as in India and China. Figure 1-1 exhibits the weighted average life expectancy at birth for different world regions based on United Nations data.

Figure 1-1: Average life expectancy at birth in years in different world regions[1]

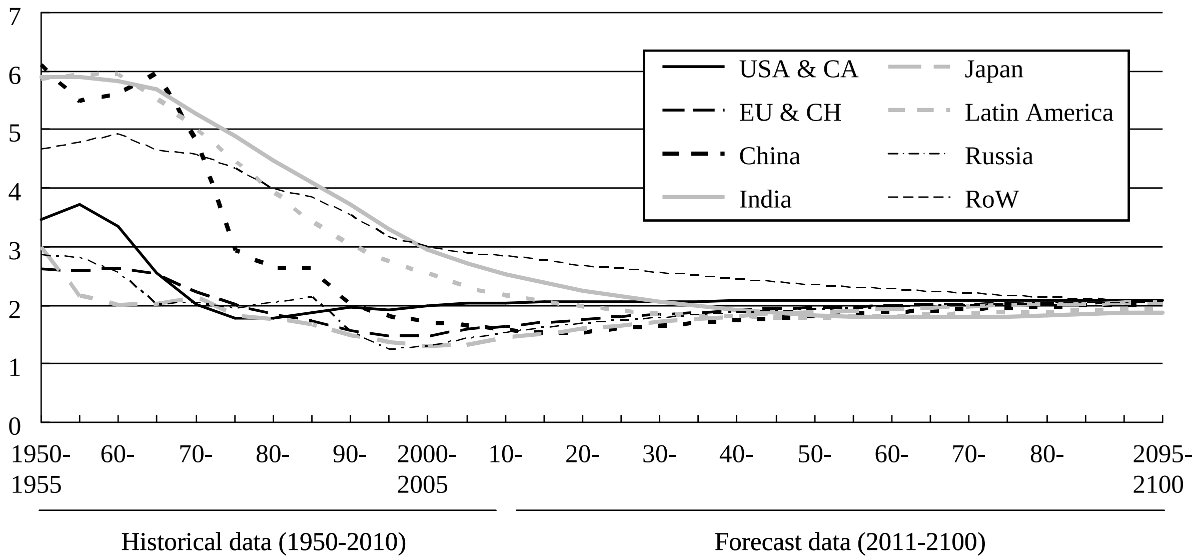


Second, many major economic regions experienced strong swings and a general downward trend in birth rate (Figure 1-2). Both may be due to sociological aspects or to political interventions, such as the one-child-policy of China. Furthermore, these fluctuations in birth rate are not synchronous between regions.

These two phenomena, the increasing life expectancy and the fluctuations in fertility, are together responsible for swings and a long-term downward trend in the share of working age population.

[1] Source: UN World Population Prospects, 2010 revision; own calculations (population-weighted aggregation of single country data to world regions data)

Figure 1-2: Fertility – average lifetime births per women[2]



This dramatic demographic transition will have a significant impact on the long-run global economic development. Longer-living agents would need to save more during their working years in order to finance their retirement, leading to an increase in capital supply. At the same time, a reduction in the share of working population leads to reduced labor supply, which in turn decreases the optimal level of capital stock, thereby reducing capital demand by firms. In a closed economy, these two effects lead to, among others, a long-term decrease in interest rates, increase in wages, and increase in capital per unit of labor[3]. Additionally, demographics impact the economy via further channels, such as imbalances in public pension systems[4]. In a setting with international trade and capital mobility, the differences in demographic development between regions induce cross-border effects such as trade flows[5] and potentially exchange rate effects.

In order to prepare and react appropriately, it is thus crucial for governments and private economic decision makers to be aware of the timing and extent of demographic impact on key economic parameters. This impact may play a role in political decisions such as pension reforms or level of public debt, but also in private decisions such as long-term capital investments.

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[2] Source: UN World Population Prospects, 2010 revision; own calculations (population-weighted aggregation of single country data to world regions data)
[3] See e.g. Krueger and Ludwig (2006)
[4] Börsch-Supan and Ludwig (2009): 1
[5] See e.g. Börsch-Supan et al. (2006)

Economists have been investigating the effects of demographics for roughly 25 years, applying both theoretical and empirical approaches. The most common model type for theoretical analyses of demographics is the overlapping-generations-model as pioneered by Auerbach and Kotlikoff (1987). This model type is characterized by a number of generations alive at each point in time, which optimize their economic behavior over their entire lifecycle. In terms of openness of trade, OLG-models may be arranged as a closed-economy, small-open-economy or multi-country model.

As Krueger and Ludwig (2002) point out, the model setup in modeling demographic change with regard to openness of the simulated economies is crucial to results. They find significant differences in model results between a closed-economy-model and a small-open-economy-model. In general, closed-economy-models tend to overestimate the effect of demographic change as they do not allow for the dampening effect of balancing trade flows, resulting in higher swings in capital stock and thereby in higher interest rate and wage effects. On the other hand, the key assumption of small-open-economy-models of a constant world interest rate is vulnerable in this context. Demographic change hits on all major world regions and is therefore likely to impact the world interest rate. Fueled by this finding, more recent papers[6] employed large-open-economy-models with endogenous interest rate and trade flows in order to even better grasp the effects of international openness for trade.

However, these models counterfactually assume perfect openness in the sense that all goods are perfectly tradable. In reality, a major share of e.g. roughly 80% of GDP for the United States is non-tradable in nature and stems from sectors such as services, infrastructure, or energy[7]. Accounting for this significant share of non-tradable goods in a large-open-economy-model could affect model results via several channels.

First, by definition, both individual and government foreign debt have to be repaid in tradable goods. This condition restricts the level of a country's[8] foreign debt to the net present value of future production of tradable goods by the respective country[9]. This link between foreign debt

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[6] E.g. Börsch-Supan et al. (2006), Börsch-Supan and Ludwig (2009), Saarenheimo (2005), Bloom et al. (2007), Fehr et al. (2008)

[7] Source: GTAP database version 8, Global Trade Analysis Project, Purdue University. See section 4.7.1 in this thesis for further details

[8] Some models, including the model developed in this thesis, treat a bundle of several real countries as one model-country (referred to as 'world-region' in this thesis).

[9] Harms (2008): 148-149

and future tradable production could propagate to other endogenous model variables. For instance, an increase in future tradable production could take effect on capital stock and wages, as the tradable production sector could have a different capital share as the non-tradable sector.

Second, large-open-economy-models in relevant literature[10] forecast an increase in wages and decrease in interest rate over time, since labor will become scarce relative to capital due to the demographic development. In a model with both non-tradable and tradable goods, this increase in wages and decrease in interest rate could lead to a price gap between tradable and non-tradable goods, if non-tradable goods are assumed to have a lower capital share[11] than tradable goods. A price gap between tradable and non-tradable goods would affect the effective real interest rate, as well as intra- and intertemporal optimization of cohorts' consumption.

Thus, this thesis sets out to investigate the question if accounting for the existence of non-tradable goods in analyzing demographic change with state-of-the-art multi-country OLG-models has a significant impact on model results vis-à-vis tradable-goods-only-models, as used in the vast majority of relevant literature[12].

Additionally, this thesis tests the impact of another assumption based on empirical data, which multi-country OLG-models in relevant literature on demographic effects do not cover. Empirical evidence shows that the consumption share of non-tradable goods tends to increase over the lifecycle[13]. That is, elderly persons have a higher preference for non-tradable goods, such as health services, than younger persons. By including this assumption, the model accounts for a potential shift of resources between sectors and potential price effects between tradable and non-tradable goods that may result from this fact-based assumption.

Most closely related to this research is a paper by Bettendorf and Heijdra (2005), who analyze population aging with an OLG-model featuring non-tradable goods. However, their model is

[10] E.g. Börsch-Supan et al. (2006), Börsch-Supan and Ludwig (2009), Saarenheimo (2005), Bloom et al. (2007), Fehr et al. (2008)

[11] According to the GTAP database (see section 4.7.1), a major part of non-tradable goods are services, which are assumed to be more labor-intensive than production goods.

[12] E.g. Börsch-Supan et al. (2006), Börsch-Supan and Ludwig (2009), Saarenheimo (2005), Domeiji and Floden (2003), INGENUE (2001)

[13] Lührmann (2005) provides data on the consumption of different goods categories over the lifecycle, indicating an increase in non-tradable categories such as health and energy with rising age.

highly stylized, non-quantitative and based on a small-open-economy setup rather than a multi-country simulation.

The approach in the present thesis is to set up two multi-country OLG-models, of which the first model acts as a baseline model. That is, the first model is a conventional tradable-goods-only-model with one production sector per region producing perfectly tradable goods. These goods are perfect substitutes between regions. The second model is identical, except that it features two production sectors in each region: one sector for tradable goods and one for non-tradable goods. These non-tradable goods must be fully consumed within the respective region. Allocation of capital and labor resources is endogenous and adjusts between these two sectors based on their respective marginal products of capital and labor. In both models, accumulation of capital stock, individual saving as well as individual labor vs. leisure choice are also modeled as endogenous variables. Both models are large-open-economy-models with an endogenous world interest rate and both feature a stylized pay-as-you-go[14] pension system for each region. One model period equals five years. Generations enter the model at age 20 and their lifespan is variable and exogenously defined based on data shown in Figure 1-1. Similarly, the size of a generation that enters the model in period *t*, corresponding to year *y*, is exogenously predetermined by the size of the cohort of age 20-24 in year *y* according to data from the UN World Population Prospects.

Even though the actual forecasting period is identical to that of the UN World Population Prospects and ranges from 2010 until 2100, the total model horizon spans a period of 315 years between 1950 and 2265. The model starts from an initial steady state in year 1950 and the period between 1950 and 2010 adjusts the model to the demographic development prior to 2010. During the period after 2100, the model converges towards a new steady state, holding population growth rates constant in all periods. This latter period is necessary due to the assumption of perfect foresight. Cutting off the model horizon too early after year 2100 would distort the results in the actual forecasting period prior to 2100.

A crucial element of this thesis is the calibration procedure, which ensures that both the 2-sector-model and the conventional 1-sector-model match exactly the same benchmark data in their benchmark steady-state-equilibrium. For instance, the single production sector in the

[14] This refers to a pension system that pays benefits in each period from contributions of the same period. It is abbreviated as ‘PAYGO pension system’ in the following.

tradable-goods-only-model has exactly the same parameters as the aggregate of the two production sectors of the 2-sector-model with non-tradable goods. Similarly, the consumption preference of consumers for tradable goods in the first model is equal to the aggregate of tradable and non-tradable goods in the second model. This step is important in order to ensure that differences between model results are solely due to the more comprehensive and more realistic setup of the model with non-tradable goods. If, e.g., the second model was set up by simply adding a non-tradable sector to an existing tradable sector from the first model without adjusting the tradable sector's parameters, the two models would be inconsistent in the sense that they would already show deviations in their steady states and thus overestimate the differences in final results. However, after running the calibration procedure detailed in chapter 4, any differences between the two models are solely due to the structure of the 2-sector-model and may thus be interpreted as a structural error of the conventional 1-sector-model vis-à-vis the 2-sector-model.

In order to achieve a realistic quantitative simulation, both models are calibrated using region-specific Social Accounting Matrices (SAMs) which were aggregated based on raw data from the database of the Global Trade Analysis Project (GTAP)[15]. Section 4 provides more details on the calibration methodology and the concept of Social Accounting Matrices.

As opposed to conventional tradable-goods-only multi-country OLG-models, the model as described above may show real exchange rate effects between regions due to potential regional price differences in non-tradable goods. This topic of real exchange rate effects induced by demographic change so far received relatively little attention in literature and no publications presenting quantitative multi-country OLG-analyses of demographic effects on real exchange rates could be identified. Rose and Supaat (2007) provide a theoretical explanation for the effects of fertility rate on real exchange rates and continue with an empirical analysis. Bryant et al. (2004) perform a theoretical analysis of cross-border effects (including exchange rates) of demographics with an emphasis on youth-dependency. However, their model is theoretical in nature, given the fact that Bryant et al. model two equally sized and parameterized economies rather than, e.g., a set of calibrated world regions. McMorrow and Röger (2003) developed an OLG-model capable of computing real exchange rate effects by modeling tradable goods as imperfect substitutes between regions. However, they do neither present any quanti-

[15] GTAP program at Purdue University, https://www.gtap.agecon.purdue.edu/

tative results regarding real exchange rate effects nor an overview of their underlying model equations.

The OLG-models in this thesis are constructed in MCP[16] equation format. In MCP format, each activity, e.g., goods production is associated with a zero-profit-condition and each good is associated with a market-clearing-condition. Zero-profit-conditions ensure that activities earn exactly zero profit, which is an equilibrium condition of all constant-return-to-scale-models[17]. Market-clearing-conditions match supply and demand of all goods in the model. Finally, each agent in the model who has a ‘final demand’[18] is associated with an income definition, matching his income with the costs of his consumption. All the above equations together constitute a Mixed Complementary Problem (MCP) and are numerically solved using the PATH-solver in the GAMS software environment.

The thesis is organized as follows. The next section provides an overview of the relevant literature. Section 3 exhibits the structure and equations of the OLG-models. Section 4 continues with the calibration procedure and input data and section 5 discusses the simulation results of the OLG-models. Section 6 performs a variation analysis of key parameters. Section 7 concludes with a summary and a proposal for possible future extensions of the analysis.

[16] Mixed Complementarity Problem
[17] See Mathiesen (1985) or Rasmussen and Rutherford (2004)
[18] A ‘final demand’ refers to a demand for consumption (e.g. private households) rather than transformation of goods (e.g. production sectors).

2 Literature Overview

Economic effects of demographic change have been in focus of literature for roughly 25 years. Since then, economists applied both empirical approaches as well as theoretical approaches based on structural economic models to grasp different facets of demographic impact on the aggregate economy.

Mankiw and Weil (1989) authored one of the first papers of this kind, focusing on the demographic impact of the baby boom/baby bust cycle in the USA on the domestic housing market. In subsequent years, a large number of papers examined several different aspects and potential effects of demographics. For instance, Cutler et al. (1990) explore aging effects for the United States based on dependency ratios and a two-country simulation model and provide normative inferences with regard to, e.g., the type of pension system. Following Cutler et al. (1990), a large number of papers focused on normative aspects, such as the 'optimal' type of pension system, intergenerational welfare effects and political means of ensuring intergenerational fairness, labor market effects and possible policy reactions, as well as fiscal policy in the context of demographic change[19].

In parallel to these papers of rather normative character, several aspects and effects of demographic change were assessed in a more positive analysis and often with the goal of forecasting certain economic developments. Most notably, a controversial discussion evolved between, among others, Poterba, Abel, and Brooks[20] on the existence of an impending 'asset-meltdown', i.e. a decrease in asset prices upon baby boomers' retirement. Optimists, such as Poterba (2001) and Brooks (2002), claim that there will be no significant asset market meltdown, while Abel (2002) stresses that theory predicts an asset meltdown even under the assumption of dampening factors such as the bequest motive. This controversy might be caused by differences in approaches: Abel focuses on a theoretical OLG-model while Poterba and Brooks base their conclusions mostly on empirical analyses.

[19] See e.g. Börsch-Supan and Chiappori (1991) or Roseveare et al. (1996) for early assessments of policy reactions on demographic change or Börsch-Supan et al. (2006) for a more recent analysis of pension reform scenarios

[20] See Poterba (2001), Abel (2001), Abel (2002), Brooks (2002)

Several other rather positive analyses consider the demographic impact on economic parameters such as interest rates, generations' welfare, capital per capita, and wages and typically rely on numerical OLG-models or statistical models. While earlier papers[21] of this kind mostly focused on single countries by applying closed or small-open-economy-models, more recent papers[22] employ multi-country models, typically with an endogenous world interest rate. These latter models allow for interregional trade flows and thus capture cross-border effects of demographic change.

Pioneers of the model type applied in these papers were Auerbach et al. (1989) who modified their previously developed[23] Auerbach-Kotlikoff overlapping-generations-model in order to simulate demographic changes and showed that based on their theoretical model, demographics indeed have a strong influence on economic parameters. In subsequent years, other economists enhanced the Auerbach-Kotlikoff model and analyzed different aspects of population aging. Theoretical approaches based on OLG-models can be further divided into more stylized models, which can be analytically solved, and more sophisticated models, which require a numerical, software-based solution algorithm. Both approaches have their respective strengths and downsides. An analytical solution offers more transparency on the mathematical relationships, as it provides closed-form formulas for the respective variables. On the other hand, numerically solved models may be more detailed and realistic.

This research builds on four strands of the literature. First, as introduced above, a number of papers[24] applied multi-country OLG-models to forecast demographic effects. The model developed in this thesis is conceptually closely related to the models described in these papers. Second, one paper by Bettendorf and Heijdra (2005) could be identified that assessed demographics with a model featuring non-tradable goods. In contrast to this thesis, however, their model is set up as a small-open-economy and solved analytically. The third strand of literature is related to the analysis of real exchange rates in context of demographic change. Finally, the fourth strand of literature is concerned with theory and modeling techniques of Auerbach-Kotlikoff OLG-models.

[21] See e.g. Abel (2001), Abel (2002), Bütler and Harms (2001)
[22] E.g. Börsch-Supan et al. (2006), Börsch-Supan and Ludwig (2009), Saarenheimo (2005)
[23] Auerbach and Kotlikoff (1987)
[24] E.g. Börsch-Supan et al. (2006), Börsch-Supan and Ludwig (2009), Saarenheimo (2005), Domeiji and Floden (2003), INGENUE (2001), Fehr et al. (2008), Bloom et al. (2007)

The models presented in the first strand of the literature are set up as large-open-economy-models with multiple countries or world regions and feature endogenous interest rates and realistic demographic input data based on historical demographic development and forecasts. Due to the high level of detail and complexity, these models generally require numerical solution methods. In contrast to the model presented in this thesis, the existing multi-country-models only feature tradable goods and ignore the existence of non-tradable goods, which may only be consumed domestically. Börsch-Supan et al. (2006) develop a dynamic multi-country Auerbach-Kotlikoff model comprising 7 world regions and a PAYGO pension system. Based on this detailed and numerically solved model, they conclude that initially, capital flows occur from fast-aging regions to the rest of the world. In the long-run, however, these flows will be reversed when households in the fast-aging regions decumulate their savings. At the same time, the interest rate would decrease by a bit less than one percentage point until 2070. They also assess a pension reform scenario, which freezes pension contribution rates at their 2004 levels and implements adjusting benefit rates, such that the PAYGO system is balanced. According to their analyses, the effect of this pension reform, replacing the existing fixed benefit rate system by a fixed contribution rate system, works in the opposite direction as the original, direct effects. Saarenheimo (2005) apply a model closely related to that of Börsch-Supan et al. (2006) and they conclude that the interest rate will fall by 70 base points if the current pension systems are maintained. This value is consistent with the projection by Börsch-Supan et al. (2006). Domeji and Floden (2003) also build a very similar multi-country OLG-model. However, their focus is not a projection into the future as in the papers mentioned above. Rather, they use their model in order to test how well the model results statistically describe historical capital flows between regions. They find evidence that their model explains a small but significant fraction of the capital flows between countries, especially in the period after 1985. According to the authors, rising capital mobility after 1985 could be an explanation for the increasing accurateness of their model after this point in time. They believe that their model may be ill-suited for earlier years, when capital mobility was still relatively low, due to its assumption of perfect capital mobility. INGENUE (2001) is one of the very first papers that analyzed demographics with a multi-country OLG-model. Their focus is on global capital flows. According to INGENUE (2001), the multi-country model approach yields significantly different results than the extreme cases of closed or small open economies. One of the more recent papers of this kind was authored by Vogel et al. (2013) and focuses on three factors, which could potentially dampen the effects of demographic change.

These three factors are the relatively young emerging countries, the endogenous adjustment of human capital formation as a response to demographic change, and a potential increase in retirement age. They conclude that especially endogenous human capital formation and a potential increase of retirement age would significantly dampen demographic effects on welfare.

The second strand of literature is related to the inclusion of a non-tradable goods sector in the OLG-model-analysis of demographic change. Only a very limited amount of literature exists with regard to this specific topic. More precisely, only one truly relevant paper by Bettendorf and Heijdra (2005) could be identified, which introduces non-tradable goods in a small-open-economy OLG-model. Bettendorf and Heijdra develop a highly stylized, analytically solved OLG-model featuring both a tradable and a non-tradable goods sector and analyze effects of demographic shocks and pension reforms. They point out specific effects that are related to the inclusion of a non-tradable goods sector. For instance, they find that under certain assumptions, pension reforms may lead to a crowding-out effect of investments due to increased non-traded goods consumption[25]. With regard to Bettendorf and Heijdra, this thesis enhances the analysis of non-traded goods in the context of demographic change in two respects. First, this thesis develops a numerically solved and calibrated model that allows for a higher level of model sophistication. For example, realistic demographic data input is used and the increasing preference for non-tradable goods over the lifecycle is covered. In addition, while Bettendorf and Heijdra develop a small-open-economy-model in order to eventually derive an analytical solution, the model presented in this thesis is set up as a multi-region OLG-model with 7+1 world regions. Finally, this thesis delivers a direct comparison between the model with non-tradable goods and a tradable-goods-only multi-region OLG-model that is identical in all other respects. Based on this comparison, the numeric impact on model results of including the non-tradable goods sector may be evaluated. Bettendorf and Heidjra, however, do not focus on differences in results between these two model types. Also relevant for the third strand of literature is a paper by Engel and Kletzer (1989), who assess a series of phenomena using a small-open-economy-model with non-tradable goods. However, they do not focus on demographic change and develop a single-consumer-model rather than an OLG-model.

[25] Bettendorf and Heijdra (2005): 3

For the third strand of literature regarding real exchange rates effects of population aging, similarly, only a few papers are available. Furthermore, there is no consensus among economists on the direction of real exchange rates effects induced by population aging. None of the identified papers assesses exchange rates using a quantitative multi-region Auerbach-Kotlikoff model as applied in this thesis. Rose and Supaat (2008) develop a highly stylized, single-country, small-open-economy OLG-model, which they assess analytically. Additionally, they backup their findings empirically using a broad data panel comprising 87 countries and years 1975 to 2005. Based on their theoretical and empirical work, they come to the conclusion that a decline in fertility in a country leads ceteris paribus to a real depreciation of that country's currency. In contrast, Bryant et al. (2004) infer from their also highly stylized and analytically solved two-country-model that an *increase* in fertility would lead to a depreciation of the respective country's currency. Cantor and Driskill (2000) suggest based on their also analytically solved small-open-economy-model that the sign of the effect depends on the indebtedness of the country. According to them, a country with a sufficiently high net debt position will face an appreciation as a short-run response to an increase in national savings; otherwise, it would face a depreciation. This increase in national savings may be induced by population aging. Finally, Andersson and Österholm (2005) estimate an autoregressive model to explain and forecast real exchange rate movements based on Swedish age structure data and subsequently apply their model to 25 OECD countries in Andersson and Österholm (2006). They find evidence for statistical coherence between age structure and real exchange rates and propose the use of age structure data upon forecasting real exchange rate movements.

Finally, regarding the fourth strand of the literature, this thesis draws on literature detailing the technical aspects of numerical overlapping-generation-models.

Auerbach and Kotlikoff (1987) and Auerbach et al. (1989) introduce the foundations of the Auerbach-Kotlikoff model type. The model developed in this thesis belongs to the class of Auerbach-Kotlikoff models, which generally feature more than two overlapping generations at each point in time and simulate a dynamic equilibrium path rather than pure steady states.

For the foundations of numerical economic modeling, Judd (1998) provides a wide, yet in-depth overview on modeling approaches and solution methods. Similarly, Heer et al. (2009) present a good introduction into this complex topic.

Moreover, the model developed in this thesis is constructed in the MCP-formulation, pioneered by Mathiesen (1985) and further developed by e.g. Rutherford (1995), Böhringer et al. (2003) and Rasmussen and Rutherford (2004). This format facilitates the numerical solution of the model and is detailed in section 3.5. Also with high relevance for this thesis are papers and lecture notes by Rutherford[26], who provides detailed explanatory notes on different issues in dynamic OLG modeling.

[26] Rutherford (1995), Rutherford (2002)

3 A Dynamic Multi-Region OLG-Model with Non-Tradable Goods

As introduced in chapter 1, this thesis develops two multi-region Auerbach-Kotlikoff OLG-models. The first features an additional non-tradable goods sector, while the second serves as a baseline model and is thus designed as a conventional model with only tradable goods. Besides this difference, both models are identical. In particular, this also holds for the calibration process, in the sense that both models respectively are calibrated to match baseline data as accurately as possible. As a result, the baseline size of the tradable goods sector in the second model is identical to the sum of the non-tradable and tradable sector in the first model. Similarly, the capital share $\sim$ in the Cobb-Douglas production function of the second model equals the weighted average of the capital shares of the non-tradable and tradable sectors in the first model. This calibration procedure ensures that differences in results between the two models are solely due to a different reaction to demographic change induced by the different model structure, rather than a mere result of differences in baseline calibration.

At first, the following derivation of model equations only focuses on the 2-sector model with non-tradable goods. Once the final equations in calibrated share form for the 2-sector model are developed, the 1-sector model with only tradable goods is derived on the basis of the 2-sector model by cutting out certain equations and modifying others in section 4.4.5.

3.1 Consumer Behavior

Several generational cohorts coexist in each region at each point in time. A cohort is uniquely identified by their time period of birth, g, and their regional origin r. The number of generations alive in a certain region depends on the lifespan of generations born prior to time t in that specific region and may fluctuate over the model horizon since region specific lifespans of cohorts are allowed to change over time.

Cohorts enter the model when they reach their working age and exit when they hit their respective cohort-specific, deterministic lifespan $a_{g,r}$. Each cohort is homogenous and chooses its economic activity so as to optimize its lifetime utility $u_{g,r}$ under the assumption of perfect foresight. Lifetime utility $u_{g,r}$ is aggregated via a constant elasticity of intertemporal substitution function (CIES) from the respective cohort's period utilities[27]:

$$u_{g,r} \rightarrow \max$$

$$u_{g,r} = \left[\sum_{t=g}^{g+ª_{g,r}-1} \left(\frac{1}{1+°} \right)^{t} \cdot z_{g,t,r}^{\frac{¯-1}{¯}} \right]^{\frac{¯}{¯-1}} \qquad \textbf{(1)}$$

$u_{g,r}$: Lifetime utility

$¯$: Elasticity of intertemporal substitution

$°$: Time discount parameter

$ª_{g,r}$: Lifespan of cohort in periods

$z_{g,t,r}$: Period utility

Due to the assumption of endogenous labor supply, the definition of period utility $z_{g,t,r}$ is a constant elasticity of substitution (CES) aggregate of compound period goods consumption $c_{g,t,r}$ and leisure time $t_{g,t,r}^{leis}$:

$$z_{g,t,r} = \left[\;_{r} \cdot c_{g,t,r}^{\frac{¶-1}{¶}} + \left(1- \;_{r}\right) \cdot t_{g,t,r}^{leis\,\frac{¶-1}{¶}} \right]^{\frac{¶}{¶-1}} \qquad \textbf{(2)}$$

$c_{g,t,r}$: Compound goods consumption

$¶$: Elasticity of substitution goods consumption vs. leisure time

$t_{g,t,r}^{leis}$: Leisure time

$\;_{r}$: Preference share parameter for goods consumption

Similarly, the compound goods consumption per period and per cohort $c_{g,t,r}$ is a CES aggregate of tradable and non-tradable goods consumption:

$$c_{g,t,r} = \left[¥ \cdot c_{g,t,r}^{T\,\frac{±-1}{±}} + \left(1-¥\right) \cdot c_{g,t,r}^{NT\,\frac{±-1}{±}} \right]^{\frac{±}{±-1}} \qquad \textbf{(3)}$$

$c_{g,t,r}^{T}$: Consumption of tradable goods

$±$: Elasticity of substitution tradable vs. non-tradable goods

$c_{g,t,r}^{NT}$: Consumption of non-tradable goods

$¥$: Preference share parameter for tradable goods consumption

Each cohort is endowed with a fixed amount of time $µ$ multiplied by the size of the cohort in each period, which may be either used as labor time or leisure. The size of the cohort is exogenously predetermined as part of the demographic data input.

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[27] The use of this specific notation of the CIES function supports the solution approach introduced in section 3.5.

$$n_{g,r} \cdot \mu = t^{lab}_{g,t,r} + t^{leis}_{g,t,r} \qquad \textbf{(4)}$$

$n_{g,r}$: Size of cohort

μ : Time endowment per cohort and time period

$t^{lab}_{g,t,r}$: Labor time per cohort and time period

Cohorts face a lifetime budget constraint, which limits their consumption expenditures to the sum of their labor and capital incomes. In the absence of a pension system, the lifetime budget constraint may be written as follows[28].

$$\sum_{t=g}^{g+a_{g,r}-1} \left[\frac{p^{T}_{t,r} \cdot c^{T}_{g,t,r} + p^{NT}_{t,r} \cdot c^{NT}_{g,t,r}}{\prod_{t'=g}^{t} \left(1+r_{t'}\right)} \right] \leq \sum_{t=g}^{g+a_{g,r}-1} \left[\frac{p^{lab}_{t,r} \cdot \text{®}_{g,t,r} \cdot t^{lab}_{g,t,r}}{\prod_{t'=g}^{t} \left(1+r_{t'}\right)} \right] \qquad \textbf{(5)}$$

$p^{T}_{t,r}$, $p^{NT}_{t,r}$: Price of (non-)tradable goods

r_t : Endogenous world interest rate

$\text{®}_{g,t,r}$: Age-dependent labor productivity[29]

$p^{lab}_{t,r}$: Wage rate

The left side of this budget constraint can be interpreted as the present value of lifetime expenditures and the right side as the present value of lifetime labor income. In this notation, capital income is implicitly covered by the present value term $\prod_{t'=g}^{t} \left(1+r_{t'}\right)$.

3.2 Production Sectors, Investment, and Capital Accumulation

Each region has two production sectors, one producing tradable goods and one producing non-tradable goods. Each sector is homogenous and may be interpreted as a single firm that follows a Cobb-Douglas production function.

[28] The pension system itself as well as the budget constraint including the pension system is introduced in section 3.3.

[29] This parameter determines the effective labor supply, which is computed as the product of labor productivity and labor time.

$$\begin{aligned} Y_{t,r}^{T} &= A_r^{T} \cdot \left(K_{t,r}^{T}\right)^{\sim_r^{T}} \cdot \left(L_{t,r}^{T}\right)^{1-\sim_r^{T}} \\ Y_{t,r}^{NT} &= A_r^{NT} \cdot \left(K_{t,r}^{NT}\right)^{\sim_r^{NT}} \cdot \left(L_{t,r}^{NT}\right)^{1-\sim_r^{NT}} \end{aligned} \qquad \textbf{(6)}$$

$Y_{t,r}^{T}, Y_{t,r}^{NT}$: Output of (non-) tradable goods $\quad A_r^{T}, A_r^{NT}$: Total factor productivities

$K_{t,r}^{T}, K_{t,r}^{NT}$: Capital stocks $\quad L_{t,r}^{T}, L_{t,r}^{NT}$: Labor supplies

The productivity factors A_r^{T} and A_r^{NT} are time-invariant, but region specific. The assumption of time invariant productivity assures that model results are solely due to demographic change and not mixed with effects stemming from exogenously determined changes in productivity. The sum of labor supplies $L_{t,r}^{T}$ and $L_{t,r}^{NT}$ are equal to the aggregated and productivity weighted labor time supplies of individual cohorts. The sum is defined over the generational index g for all generations that are alive[30] at time t. The sum index g starts with the oldest generation alive at time t, which entered the model at time $t - \text{ª}_{g,r} + 1$. The youngest generation alive was born at time t, constituting the upper limit for the sum index g:

$$L_{t,r}^{T} + L_{t,r}^{NT} = L_{t,r} = \sum_{t-\text{ª}_{g,r}+1 \le g \le t} \text{®}_{g,t,r} \cdot t_{g,t,r}^{lab} \qquad \textbf{(7)}$$

Due to the assumption of perfect labor mobility between tradable and non-tradable sectors, the allocation of total labor supply $L_{t,r}$ into sector labor supplies $L_{t,r}^{T}$ and $L_{t,r}^{NT}$ in equilibrium is chosen such that the wage rates and thus the marginal value product of labor in both sectors are identical.

The sum of capital stocks $K_{t,r}^{T} + K_{t,r}^{NT}$ in equation (6) is equal to the total capital stock $K_{t,r}$:

$$K_{t,r} = K_{t,r}^{T} + K_{t,r}^{NT} \qquad \textbf{(8)}$$

$K_{t,r}$: Total capital stock

Similar to labor, capital is perfectly mobile between sectors[31] and the allocation of total capital stock to sectors is chosen such that the marginal product of capital in each sector is equal

[30] The terms 'born' and 'alive' refer to the cohorts' active 'life' within the model, which does not cover childhood.

[31] Modeling capital as not mobile between sectors would not make any difference. This is due to the fact that the price of capital in period t is always equal to the price of tradable goods in period t-1 plus interest, since investment of one additional unit of tradable goods in t-1 results in one unit of additional capital in t. Since this holds true for both sectors, the prices of capital in both sectors are always equal. This would not hold true in case of capital adjustment costs; however, these are not part of this model.

to the world interest rate r_t. The level of capital stock is determined by the level of capital in the previous period net of depreciation plus last period's investment.

$$K_{t+1,r} = K_{t,r} \cdot (1 - \text{¢}_r) + I_{t,r} \qquad \textbf{(9)}$$

¢_r : Depreciation rate $\qquad$ $I_{t,r}$: Investment

Capital goods are assumed to be tradable and period investment is determined by the residual of aggregated production, consumption, and trade balance:

$$\begin{gathered} I_{t,r} = S_{t,r} - CA_{t,r} = Y_{t,r}^T - C_{t,r}^T - NX_{t,r} \\ with\ C_{t,r}^T = \sum_{t-\text{ª}_{g,r}+1 \le g \le t} c_{g,t,r}^T \\ CA_{t,r} = NX_{t,r} + NY_{t,r} \end{gathered} \qquad \textbf{(10)}$$

$C_{t,r}^T$: Aggregate consumption of tradable goods $\qquad$ $S_{t,r}$: Aggregate savings

$CA_{t,r}$: Current account $\qquad$ $NX_{t,r}$: Net exports

$NY_{t,r}$: Net factor income

Finally, the sum of net exports across all regions must be in balance in each period:

$$\sum_r NX_{t,r} = 0 \qquad \textbf{(11)}$$

3.3 Pay-As-You-Go Pension System

Pension systems significantly affect the saving behavior of agents and thereby the overall equilibrium results of OLG-models[32]. Accordingly, the model in this thesis features a stylized pay-as-you-go (PAYGO) pension system in each region.

A PAYGO pension system is characterized by a balanced budget in each period. That is, pension benefits for retirees in period t are paid by pension contributions from the working cohorts in that same period:

[32] See e.g. Börsch-Supan and Ludwig (2010)

$$\mathrm{B}_{t,r} = \Gamma_{t,r} \qquad \textbf{(12)}$$

$\mathrm{B}_{t,r}$: Aggregate pension benefits $\qquad \Gamma_{t,r}$: Aggregate pension contribution

PAYGO pension systems can be classified in two basic archetypes, depending on which side of the equal sign in equation (12) adjusts endogenously[32]. A constant-benefit pension system guarantees fixed benefit rates and contribution rates are allowed to float in order to satisfy equation (12). In the opposite case of a constant-contribution system, the contribution rate is fixed and the budget is balanced by endogenous adjustments in the benefit rate.

The present model features the latter approach of a constant contribution rate. This contribution rate $\pounds_r$ on labor income is defined as follows.

$$\copyright_{g,t,r} = \pounds_r \cdot p_{t,r}^{lab} \cdot \circledR_{g,t,r} \cdot t_{g,t,r}^{lab} \qquad \textbf{(13)}$$

$\copyright_{g,t,r}$: Cohort pension contribution $\qquad \pounds_r$: Fixed contribution rate

Using equation (7), aggregated pension contributions in period t can be written as:

$$\Gamma_{t,r} = \sum_{t-{}^{\text{a}}_{g,r}+1 \le g \le t} \copyright_{g,t,r} = \pounds_r \cdot p_{t,r}^{L} \cdot \sum_{t-{}^{\text{a}}_{g,r}+1 \le g \le t} \circledR_{g,t,r} \cdot t_{g,t,r}^{lab} = \pounds_r \cdot p_{t,r}^{L} \cdot L_{t,r} \qquad \textbf{(14)}$$

Similar to the contribution rate, the calculation of the benefit rate is also based on labor earnings. More precisely, the benefit rate in this model is defined as pension benefits per cohort in period t over average labor earnings per cohort in working age in period t.

$$¤_{g,t,r} = \cdot_{t,r} \cdot n_{g,r} \cdot \frac{p_{t,r}^{L} \cdot \sum_{t-¸_{g',r}+1 \le g' \le t} \circledR_{g',t,r} \cdot t_{g',t,r}^{lab}}{\sum_{t-¸_{g',r}+1 \le g' \le t} n_{g',r}} = \cdot_{t,r} \cdot n_{g,r} \cdot \frac{p_{t,r}^{L} \cdot L_{t,r}}{\sum_{t-¸_{g',r}+1 \le g' \le t} n_{g',r}} \qquad \textbf{(15)}$$

$¤_{g,t,r}$: Cohort pension benefits $\qquad \cdot_{t,r}$: Benefit rate in period t

$¸_{g,r}$: Number of working periods

The average labor earnings in period t are calculated as the sum of all labor earnings in that period divided by the size of the working population.

Aggregate benefits $\mathrm{B}_{t,r}$ in period t is given by the sum of individual cohort benefits $¤_{g,t,r}$ for all cohorts in retirement age.

$$B_{t,r} = \sum_{t-\text{ª}_{g,r}+1\le g\le t-\text{¸}_{g,r}} \text{¤}_{t,r} \tag{16}$$

Since the contribution rate is fixed, the benefit rate is endogenous in order to satisfy equation (12). The endogenous benefit rate can be derived by substituting equations (14), (15), and (16) in equation (12):

$$\begin{aligned} \sum_{t-\text{ª}_{g,r}+1\le g\le t-\text{¸}_{g,r}} \text{·}_{t,r}\cdot n_{g,r}\cdot \frac{p^L_{t,r}\cdot L_{t,r}}{\sum\limits_{t-\text{¸}_{g,r}+1\le g\le t} n_{g,r}} &= \text{£}_r\cdot p^L_{t,r}\cdot L_{t,r} \\ \Leftrightarrow \text{·}_{t,r} &= \text{£}_r\cdot \frac{\sum\limits_{t-\text{¸}_{g,r}+1\le g\le t} n_{g,r}}{\sum\limits_{t-\text{ª}_{g,r}+1\le g\le t-\text{¸}_{g,r}} n_{g,r}} \end{aligned} \tag{17}$$

Accordingly, the endogenous benefit rate $\text{·}_{t,r}$ that balances the pension budget in each period is the product of the pension contribution rate £_r and the ratio of the size of working population over the size of retired population in period t. This relationship is plausible, since a higher pension contribution rate £_r or a larger working population, which pays pension contributions, will increase the size of the pension budget. Holding the size of the retired population constant, this will increase the benefit rate. On the opposite, an increase in the size of the retired population will c.p. reduce the benefit rate, since the same pension budget needs to be distributed between more retirees.

The pension system impacts the budget constraint of individual cohorts. Due to the pension system, the constraint defined in equation (5) needs to be adjusted as follows:

$$\sum_{t=g}^{g+\text{ª}_{g,r}-1}\left[\frac{p^T_{t,r}\cdot c^T_{g,t,r}+p^{NT}_{t,r}\cdot c^{NT}_{g,t,r}}{\prod\limits_{t'=g}^{t}(1+r_{t'})}\right] \le \sum_{t=g}^{g+\text{¸}_{g,r}-2}\left[\frac{p^{lab}_{t,r}\cdot \text{®}_{g,t,r}\cdot t^{lab}_{g,t,r}}{\prod\limits_{t'=g}^{t}(1+r_{t'})}\right] + \sum_{t=g+\text{¸}_{g,r}-1}^{g+\text{ª}_{g,r}-1}\left[\frac{\text{¤}_{g,t,r}}{\prod\limits_{t'=g}^{t}(1+r_{t'})}\right] \tag{18}$$

As formulated in equation (18), the income side of the new budget constraint covers both labor earnings during working age ($g\le t\le g+\text{¸}_{g,r}-2$) and pension benefits during retirement age ($g+\text{¸}_{g,r}-1\le t\le g+\text{ª}_{g,r}-1$). The wage rate $p^{lab}_{t,r}$ is defined as the net wage rate, thus an explicit deduction of pension contribution in equation (18) is not necessary.

3.4 Closing the Model

Due to the fact that computation power and time are limited, dynamic numerical models in discrete time may only be solved for a finite number of periods[33]. The time span covered by these periods is referred to as the 'model horizon'. The model horizon in this model comprises $T+1$ periods, such that $0 \leq t \leq T$.

At the same time, several equations defined above refer to future or past time periods. For instance, the budget constraint in equation (18) refers to the future lifetime of a generation born in period t. However, for generations born close to the end of the model horizon ($g > T - a_{g,r} + 1$), parts of these generations' lifetime are not captured by the model horizon. More technically, the upper bound of the left sum term in equation (18) is greater than the terminal model period: $t_{\max} = g + a_{g,r} - 1 > T$, which is not feasible due to the definition of the time horizon $0 \leq t \leq T$.

The same holds true for cohort utility in equation (1). Similarly, the proliferation of the capital stock as defined in equation (9) links next period's capital stock with the current capital stock and investment.

Issues of variables running out of scope do not only arise at the end of the model horizon, but also at the beginning. Without closing equations, the capital stock $K_{0,r}$ at $t = 0$ is *zero* or not defined, since variables $K_{-1,r}$ and $I_{-1,r}$ in equation (9) do not exist. Additionally, for early time periods $0 \leq t \leq a_{0,r}$, parts of the cohorts are not defined. For instance, in period $t = 0$, only the cohort born in this period exists, but older cohorts are not defined yet.

The literature refers to this step of setting up a computational general equilibrium model with the term 'model closure' or 'closing the model' and provides different solutions to this problem. Auerbach et al. (1989) suggest to follow a three-step-procedure. First, they solve for an initial steady state, characterized by all exogenous parameters set to their initial values. When solving for a steady state, issues of variables running out of time index cannot occur, since a steady state is defined by time-invariant endogenous variables. Thus, the time index can be omitted even if it refers to future or past periods.

[33] See Rasmussen and Rutherford (2004)

Second, Auerbach et al. solve for the final steady state by setting all exogenous parameters to their final values. In the third step, they simulate the transition path between these two steady states by imposing fixed variable values from the respective steady-state solutions for variables with time index prior or past the model horizon.

This thesis, however, follows a different closing approach suggested by Rasmussen and Rutherford (2004). This approach differs from the one applied by Auerbach et al. (1989) mainly in the way of handling the end of the model horizon. Instead of pre-calculating a final steady state and forcing the simulation path to match these steady-state values in the last time period, Rasmussen and Rutherford (2004) turn out-of-scope-variables at the end of the model period into endogenous variables solved simultaneously with the simulation path. They do not impose a final steady state, but rather introduce additional equations as terminal conditions to solve for these additional endogenous variables.

More specifically, these additional terminal variables are terminal period assets per generation $a_{g,r}^{term}$, post-terminal utility $z_{g,a,r}^{term}$, and terminal capital stock K_R^{term}. Index a in post-terminal utility describes the age in model periods of the respective generation and thus implicitly refers to specific time periods beyond the model horizon, since $z_{g,a,r}^{term}$ is only defined for $T - g - 1 < a \leq \mathrm{a}_{g,r} - 1$.

Rasmussen and Rutherford (2004) introduce three terminal constraints in order to solve for these additional variables. They derive the value of terminal assets per generation $a_{g,r}^{term}$ by postulating that the lifetime utility of all generations, which are alive in post-terminal years, is equal.

$$u_{g,r} = u_{g-1,r} \text{ with } g > T - \mathrm{a}_{g,r} \qquad \textbf{(19)}$$

As the second terminal constraint, they impose that the post-terminal price of period utility $p_{g,a,r}^{z,term}$ must be consistent with the last period's interest rate. The post-terminal price of period utility describes the price of one unit of period utility in periods after the end of the model horizon *T*.

$$p^{z}_{g-a,T,r} = p^{z,term}_{g,a,r} \cdot \left(\frac{p^{T}_{T-1}}{p^{T}_{T}} \right)^{a} \text{ with } g > T - \text{a}_{g,r} \tag{20}$$

This terminal condition requires that the (shadow) price of consumption in terminal period T of generation of age a is equal to a generation of that same age a in a post-terminal period, multiplied by a factor $\left(p^{T}_{T-1} / p^{T}_{T}\right)^{a}$ to account for the definition of prices as present value prices. The constraint ensures that the present-value price of period utility declines with the final interest rate defined as follows.

$$r_T = \frac{p^{T}_{T-1}}{p^{T}_{T}} - 1 \tag{21}$$

The third terminal condition suggested by Rasmussen et al. fixes the terminal capital stock such that steady-state investment is achieved in the terminal period:

$$\frac{I_{T,r}}{I_{T-1,r}} = 1 + \neg_{r} \tag{22}$$

$\neg_r$: Steady-state growth rate of youngest cohort

With regard to the beginning of the model horizon, Rasmussen et al. close their model very similarly to Auerbach et al. by imposing steady-state values.

Both of the presented closing approaches are approximations. In the approach by Auerbach et al., the simulation path is forced to match certain steady-state values in the last period, even though for a finite amount of simulation periods, the ‘exact’ simulation path will actually not have reached the steady state by the end of the model horizon. The approximation error decreases with an increasing number of periods between the last exogenous ‘event’, e.g. an exogenous change in demographics, and the end of the model horizon. This time span is referred to as ‘phase-out’ in the literature.

Similarly, the closing approach by Rasmussen et al. is also an approximation of the exact simulation path. In this case, however, the simulation path may still deviate from the actual steady-state values by the end of the model horizon and the approximation error stems from the imposed terminal conditions, which artificially assume that the last periods of the simulation path are equivalent to the infinite-time steady state.

For both approaches, the phase-out should be chosen long enough in order to allow the exact simulation path to closely approach the infinite-time steady state. If the phase-out period is too short, the whole simulation path can be distorted and significant deviations may occur in the actual forecasting period. However, with both approaches, the model results do not contain any information on the quality of approximation, i.e. on whether the duration of phase-out is long enough. Therefore, the model is solved three times in this thesis: First with the longest phase-out that is numerically feasible[34], second with a phase-out, which is only $4/5$ of the first phase-out, and finally with a phase-out which is only $3/5$ of the first phase out duration. Since the second and third simulation paths do not differ significantly from each other, it is concluded that the phase-out of the first simulation path is long enough.

3.5 Introduction to the MCP-Format

The structural model equations as defined in sections 3.1, 3.2, and 3.3 together with the terminal conditions from section 3.4 are sufficient to fully determine the behavior of the model. The model may be interpreted as a set of g optimization problems, which are subject to the constraints defined in the sections above. These constraints interlink the optimization problems, such that maximizing utility for one specific cohort has impact on other cohorts' utility through their impact on prices. For instance, the choice of one cohort's consumption path has impact on savings and investment at macro level according to equation (10). This macro-impact has implications for other cohorts, for instance via the capital stock, which is dependent on aggregate investment.

At the same time, the model has forward-looking behavior, since cohorts optimize their entire lifetime utility 'at once' as defined in equation (1). While this assumption is preferred by several relevant papers[35] over the myopic approach of period-by-period optimization, it requires the model to be solved simultaneously for all time periods, rather than simulating period by period[36].

[34] Due to the model setup with present value prices, the numerical values for prices approach zero for large values of t. If T is chosen "too" large, price values become so small that they cannot be properly expressed with numerical double precision variables and the model becomes instable.

[35] E.g. Börsch-Supan et al. (2006), Saarenheimo (2005), Domeiji and Floden (2003), INGENUE (2001)

[36] Auerbach et al. (1989): 13

A straight-forward approach to solve the model would be to form the Lagrange function and solve the resulting system of non-linear equations. However, this approach, which is referred to as 'optimal planning approach' in the literature, has limitations for more complex models with multiple agents, production sectors, or regions. It requires relatively large computation efforts and common solution algorithms quickly become unstable when dimensionality of the model is high[37].

This thesis thus applies a different approach referred to as MCP[38]-formulation, which was pioneered by Mathiesen (1985) and further developed by Rutherford (1995), Böhringer et al. (2003), and Rasmussen and Rutherford (2004). This formulation approach is chosen for the present model for four reasons.

1) *Stability and robustness*. As Rausch (2009) points out, the MCP-formulation is more robust for complex models than the 'optimal planning approach'. This finding was confirmed within the work of this thesis by an early test model formulated using the optimal planning approach. This test model featured only two co-existing overlapping generations per time period and was first solved for a model-environment with only tradable goods. However, as soon as non-tradable goods were added to the model structure, the model failed to converge with available solver algorithms in GAMS.

2) *Convenient calibration.* The MCP-formulation features the 'calibrated share form'[39], which allows for a relatively convenient calibration to real world data. As already introduced, a careful calibration of both models to the exact same baseline data is especially important in the context of this thesis. Section 4.3 provides an introduction to the calibrated share form.

3) *Systematic modeling structure*. The MCP-formulation is based on a systematic representation of the model as a set of activities, goods, and final demand[40]. This very structured approach reduces the likelihood of modeling errors.

[37] E.g. Lau et al. (2002)
[38] Mixed Complementarity Problem
[39] See e.g. Böhringer et al. (2003), Rasmussen and Rutherford (2004)
[40] See e.g. Böhringer et al. (2003), Rasmussen and Rutherford (2004)

4) *Availability of off-the-shelf solution algorithms*. According to Rutherford (1995), the MCP-formulation is well-suited for model setup and solution within the GAMS software environment.

The Mixed Complementary Problem in the context of economic equilibrium models may be defined as follows[41]:

$$
\begin{aligned}
&Given: && f:\Re^n \to \Re^n,\ l,u \in \Re^n, \\
&Find: && z,w,v \in \Re^n \\
&s.t.: && f(z) - w + v = 0, \\
& && w \circ (z-l) = 0, \quad v \circ (u-z) = 0, \\
& && l \le z \le u,\ w \ge 0,\ v \ge 0
\end{aligned} \qquad \textbf{(23)}
$$

The lower and upper bound vectors l and u must be consistent in the sense that $-\infty \le l \le u \le \infty$. In order to form a numerically solvable MCP, the mapping f should be continuously differentiable.

The MCP as given in (23) may be interpreted as a generalization of a non-linear complementarity problem (NCP). While a non-linear complementarity problem comprises only inequalities and their complementarity conditions, an MCP as defined in equation (23) also allows equations without complementarity conditions. These latter equations will be introduced as auxiliary equations; however, the ‘main’ equations of an MCP-formulation of a general equilibrium model are complementary inequalitites.

This special case of a ‘pure’ NCP can be deducted from (23) by setting $l = 0$, $u = +\infty$. The resulting NCP can then be formulated in a more intuitive form as follows.

$$
\begin{aligned}
&Given: && f:\Re^n \to \Re^n, \\
&Find: && x \in \Re^n_+ \\
&s.t.: && f(z) \ge 0, \\
& && z \circ f(z) = 0
\end{aligned} \qquad \textbf{(24)}
$$

Thus, if an inequality is solved in an NCP as a strict inequality, its complementary variable must equal zero. This is due to the fact that if any inequality of the system $f(z) \ge 0$ is solved as a strict inequality (i.e. $f_i(z) > 0$), it follows that the complementary variable $z_i = 0$ due to the scalar product condition. As it turns out later, this is a natural formulation for equilibrium behavior of economic markets. If, for instance, the supply of a certain good is in balance with

[41] See Rutherford (1995)

its demand, then it must have a positive market clearing price. This price is defined as the complementary variable to the according quantity of any good. This case relates to a solution of the according inequality as a weak inequality. If, however, a certain good is in excess supply, such as salt water in the middle of the Atlantic Ocean for instance, the according inequality is solved as a strict inequality and the price of salt water is forced down to zero by the complementarity condition.

The MCP-formulation comprises four classes of equations: Zero-profit-conditions, market-clearing-conditions, income definitions, and auxiliary equations. The first two classes are formulated as an NCP, however, the income definitions and auxiliary equations are "normal" equations without inequality or complementarity features. Due to this fact, this formulation is a general MCP and cannot be constructed as a pure NCP.

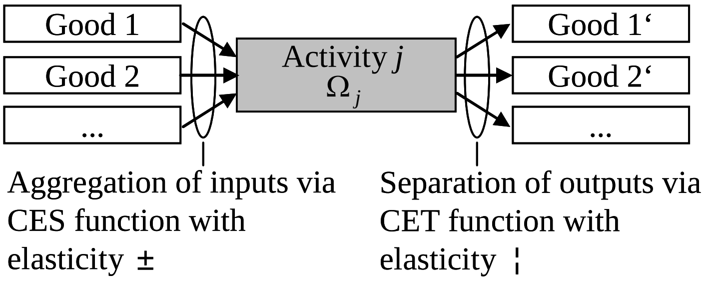
3.5.1 Zero-Profit-Conditions

Each 'activity' of the model is associated with a zero-profit-condition. The term 'activity' has an abstract meaning in this context, as it refers not only to 'physical' activities such as the production of goods, but also to 'virtual' activities that are introduced due to modeling reasons. One example for such a virtual activity is the formation of a compound consumption good from non-tradable goods and tradable goods. More generally, any transformation of one or more input goods into one or more output goods constitutes an 'activity'.

Literature suggests[42] to introduce an additional variable for each activity j referred to as the activity level Ω_j. This step simplifies the calibration procedure as described in section 4 significantly. Due to the constant-returns-to-scale properties of any given activity, input quantities and output quantities of an activity are proportional to its activity level. That is, doubling the quantities of input goods while leaving their relative shares constant will double the activity level. If the activity has only one output good, such as a common production activity, then the output quantity would be doubled as a result. Figure 3-1 visualizes the concept of activities.

[42] See Rasmussen and Rutherford (2004)

Figure 3-1: Visualization of an activity



The zero-profit-condition for activity *j* follows the basic scheme depicted below[43], implying that either the profit of a given activity is zero and its activity level (and thus output) may be positive, or the profit of a given activity is negative and the activity level is zero. The scheme constitutes a NCP as defined in equation (24).

$$-\Pi_j(p_i) \geq 0,\ \ \Omega_j \geq 0,\ \ \Omega_j \circ \left(-\Pi_j(p_i)\right) = 0 \qquad \textbf{(25)}$$

Π_j : Unit profit function of activity *j* $\qquad$ p_i : Price vector

Ω_j : Activity level of activity *j*

This is intuitive, since no agent in the model would pursue an activity with negative profit, as this particular agent would be better off by stopping this activity. On the other hand, positive profits are not allowed by definition in Mathiesen’s formulation. This assumption is consistent as long as only constant-returns-to-scale functions are allowed as activities, implying that positive profits cannot occur. In fact, this poses a restriction of Mathiesen’s formulation, which is not necessary in the common Lagrangian equation formulation. However, constant-returns-to-scale behavior of activities is a widely used assumption in economic modeling and is also applied in the present model.

In order to implement the zero-profit-conditions, the unit profit function for each activity needs to be specified. Unit profit is equal to the difference between unit revenue and unit costs[44]:

[43] See Mathiesen (1985), Paltsev (2004)
[44] See Rasmussen and Rutherford (2004), Paltsev (2004)

$$\Pi_j(p_i) = R_j(p_i) - C_j(p_i) \qquad \textbf{(26)}$$

R_j: Unit revenue function C_j: Unit cost function

The price vector p_i comprises all goods prices existing in the model. For any given activity *j*, one or more of these goods are input goods (factors) and one or more good are output goods.

The specification of all activities in this model follows a constant-elasticity-of-substitution-function (CES) for the input goods of activities, and a constant-elasticity-of-transformation-function (CET) for the output goods. The former is relevant for the cost function and the latter for the revenue function.

The unit cost function $C_j(p_i)$ calculates the minimal cost of achieving an activity level $\Omega_j = 1$ as a function of prices of input goods.

The specification of the underlying quantity-based CES function is given by:

$$\Omega_j = \left[¡_j \cdot \sum_i \sim_{i,j} \cdot x_{i,j}^{\frac{«-1}{«}} \right]^{\frac{«}{«-1}} \qquad \textbf{(27)}$$

$¡_j, \sim_{i,j}$: Exogenous parameters $«$: Elasticity of substitution

$x_{i,j}$: Input quantity of good *i*

The associated cost function $C_j(p_i)$ for the CES function defined above reads as:

$$C_j(p_i) = ¡_j^{-«} \cdot \left[\sum_i \sim_{i,j}^{«} \cdot p_i^{1-«} \right]^{\frac{1}{1-«}} \qquad \textbf{(28)}$$

The revenue function is similarly interlinked with the constant-elasticity-of-transformation function of output goods of activity *j*. The output quantities are associated with the activity level by the following CET function.

$$\Omega_j = \left[¢_j \cdot \sum_i \quad {}_{i,j} \cdot y_{i,j}^{\frac{¦+1}{¦}} \right]^{\frac{¦}{¦+1}} \tag{29}$$

$¢_j$, ${}_{i,j}$: Exogenous parameters ¦ : Elasticity of transformation

$y_{i,j}$: Ouput quantity of good *i*

Obviously, this function is algebraically very similar to the CES input function. Thus, the revenue of activity *j* for an activity level of *one* can be computed in an analogous way as the cost function above.

$$R_j\left(p_i\right) = ¢_j^{¦} \cdot \left[\sum_i \quad {}_{i,j} \cdot p_i^{1+¦} \right]^{\frac{1}{1+¦}} \tag{30}$$

It should be noted that in the present model, all activities feature only one output good. In this case, the revenue function is simply $R_j\left(p_i\right) = ¢_j^{¦} \cdot \quad {}_{i,j}^{\frac{1}{1+¦}} \cdot p_i$ (with p_i being the price of the activity's single output good). In other words, the revenue for an activity level of one is the price of the respective output good multiplied with a constant factor $¢_j^{¦} \cdot \quad {}_{i,j}^{\frac{1}{1+¦}}$. This constant factor is equivalent to the output quantity for an activity level of *one.*

On the other hand, the full revenue function given by equation (30) is required for activities with more than one output good. An example for such an activity would be the transformation of production goods into domestic goods and export goods in a model where these two classes of goods are assumed to be imperfect substitutes[45]. Again, this type of activity is not included in the present model and is only described in order to provide a more exhaustive description of Mathiesen's format.

Now that the cost and revenue functions are specified, the profit function for activity *j* can be assembled. The zero-profit-conditions as in Rasmussen and Rutherford (2004) follow directly from equation (25).

[45] See Rasmussen and Rutherford (2004) for an example of such a model.

$$\Pi\ \left(p_i\right)=\text{¢}_j^{\text{¦}}\cdot\left[\sum_i \quad _{i,j}\cdot p_i^{1+\text{¦}}\right]^{\frac{1}{1+\text{¦}}}-\text{¡}_j^{-\text{«}}\cdot\left[\sum_i \sim_{i,j}^{\text{«}}\cdot p_i^{1-\text{«}}\right]^{\frac{1}{1-\text{«}}} \qquad \textbf{(31)}$$

Most activities in actual models feature only one single output good or one single input good. In this case, the revenue or cost function degenerates to simply the price of the associated good.

Special Case of Cobb-Douglas Function

As noted above, a Cobb-Douglas type function is a special case of a CES function with $\text{«}\rightarrow 1$. The cost function for an activity associated with a Cobb-Douglas function can be written as follows.

$$C_j\left(p_i\right)=\text{¡}_j^{-1}\cdot\prod_i \sim_{i,j}^{-\sim_{i,j}}\cdot p_i^{\sim_{i,j}} \qquad \textbf{(32)}$$

3.5.2 Market-Clearing-Conditions

Each good in the model is associated with a market-clearing-condition that ensures consistency of supply and demand. More specifically, it allows either that supply and demand are equal in quantity and associated with a market-clearing price, or that supply exceeds demand and the price of this respective good is zero. Similar to the zero-profit-conditions, this logic again is formulated via a non-linear complementary problem[46]:

$$\Xi_i = S_i - D_i \geq 0,\ S_i \geq 0,\ D_i \geq 0,\ \Xi_i\, p_i = 0 \qquad \textbf{(33)}$$

S_i: Total supply of good *i* $\qquad$ D_i: Total demand of good *i*

The total supply of a good is the sum of this good's output quantities over all activities plus exogenous endowments[47].

[46] See Mathiesen (1985)
[47] See Rasmussen and Rutherford (2004)

$$S_i = \sum_j y_{ji} + \sum_h \mu_{i,h} \tag{34}$$

$\mu_{i,h}$: Exogenous endowment of good *i* to household *h*

Similarly on the demand side, good *i* may not only be demanded by activities, but also by households[47]:

$$D_i = \sum_j x_{ji} + \sum_h d_{i,h} \tag{35}$$

$d_{i,h}$: Demand of good *i* by household *h*

Following Shephard's Lemma, the net supplies of activities, i.e. output minus input quantities, are equal to the associated price derivatives of the activities' profit functions. Due to the fact that the profit functions Π_j are defined per unit of activity level Ω_j, the factor Ω_j needs to be added to the equation. Putting the above equations together, the general market-clearing-condition for good *i* is given by[47]:

$$\Xi_i = \sum_j \Omega_j \frac{\partial \Pi_j}{\partial p_i} + \sum_h \mu_{i,h} - \sum_h d_{i,h} \geq 0, \;\; p_i \geq 0, \;\; \Xi_i \cdot p_i \; = 0 \tag{36}$$

3.5.3 Income Definitions

Every agent in the model that has a 'final' demand for any good is associated with an income definition equation. As opposed to the demand for goods by activities, final demand does not transform an input good into output goods. Rather, a good entering final demand can be interpreted as 'consumed' and thus vanished from the model. Typically, household or governments are modeled as agents with final demand. Their demanded good is often utility, defined for example as an intertemporally aggregated compound of consumption goods and leisure time. Income definitions as defined below ensure that these final demands are consistent with the endowments of these agents at equilibrium prices[47].

$$\Psi_h = \sum_i p_i \cdot \mu_{i,h} = \sum_i p_i \cdot d_{i,h} \tag{37}$$

Ψ_h : Income of household h

3.5.4 Auxiliary Equations

The auxiliary equations in Mathiesen's format are simply all residual equations required to define the model. These are, for instance, closing conditions as introduced in section 3.4 or budget balancing equations for a pension system[48].

3.6 Transforming the Model into MCP-Format

While sections 3.1 through 3.4 have defined the model structure of the OLG-model in direct formulation, section 3.5 has introduced a general approach to construct complex general equilibrium models as suggested by Mathiesen (1985). The step of applying Mathiesen's format and to transform the present model accordingly is performed in this section.

Before setting up the equations, the model as defined in sections 3.1 through 3.4 is layed out in a flow chart as shown in Figure 3-2. This chart shows all regions of the model economy as a whole, thus current accounts of the respective regions are not explicitly shown. The chart shows the flow of goods through the various activities of the model. Private households receive *time* as an endowment in every period. On the other hand, they demand and optimize lifetime utility $U_{g,r}$. Each single activity in between optimizes its output in a way that the price of the output good is minimal. That is, the activity 'Intertemporal utility aggregation' allocates 'Total period utility' over all lifetime periods in a way that the price of the output good $U_{g,r}$ is minimal based on given prices of total period utility. These prices of total period utility are again endogenous and the activity 'Period utility aggregation' determines for each lifetime period the amount of leisure time and consumption in a way that the price of total period utility is minimized for given prices of leisure time and compound consumption good. This chain of price minimizing activities leads to a global optimal behavior of both cohorts and production sectors as shown in Mathiesen (1985) and thus leads to the same results as the solution of conventional equations introduced in sections 3.1 and 3.2.

[48] See Paltsev (2004), Rasmussen and Rutherford (2004)

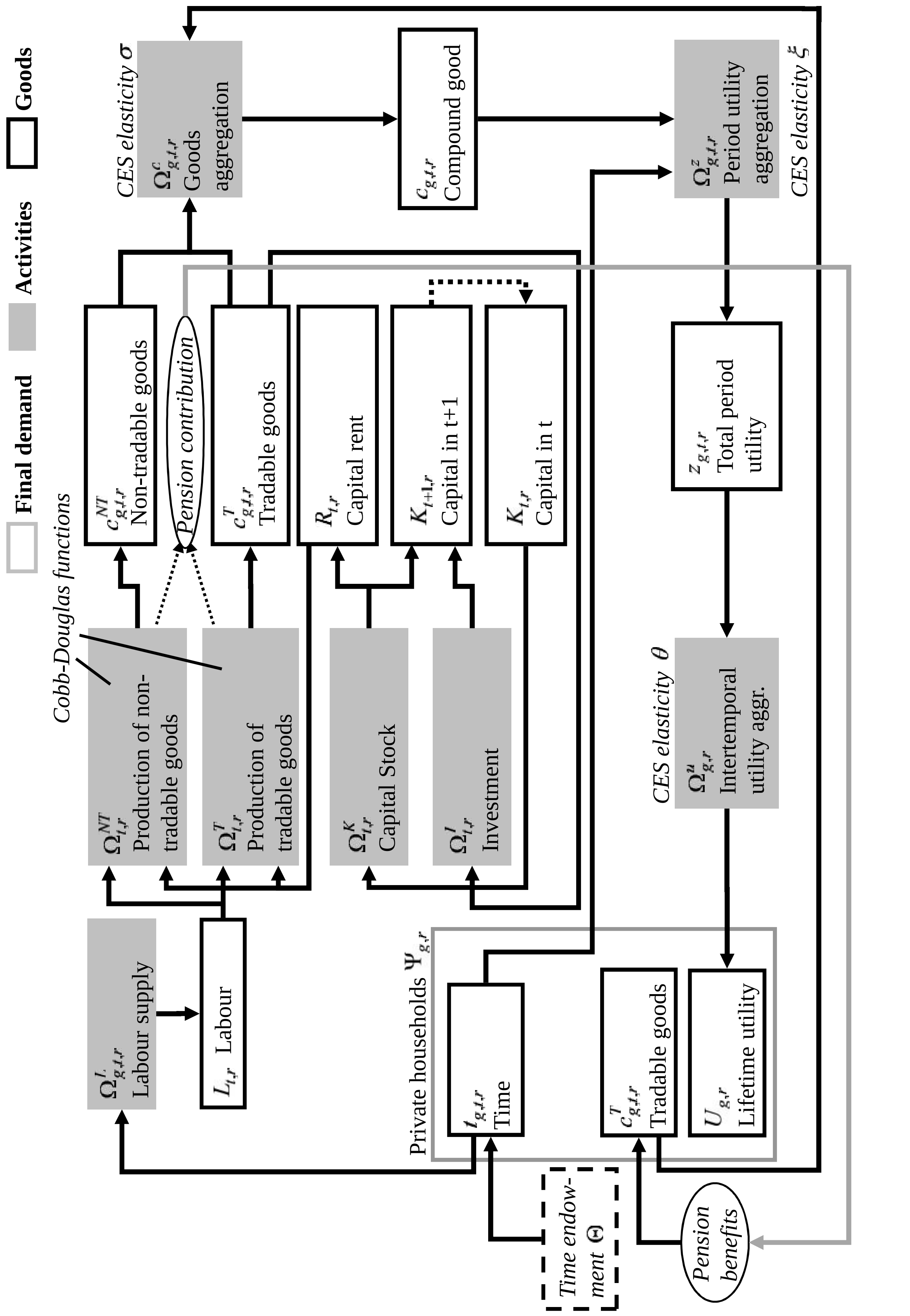
In a second step, the zero-profit-conditions for all activities are derived. Subsequently, the market-clearing-conditions for all goods and finally the income definition for consumers are composed. The auxiliary equations are identical to the budget balance of the pension system from section 3.3 and the closing constraints from section 3.4. These equations are not repeated in this chapter.

Following Rasmussen and Rutherford (2004) and in contrast to sections 3.1 through 3.4, all prices in the following are defined as present value prices in order to simplify algebra. Thus, they already incorporate the time discount by the respective endogenous interest rate. The price of tradable goods in time period 0 represents the numeraire. As a result, numerical values of prices will decline over the model horizon from a value close to 1 to values in the order of 10^{-5}.

3.6.1 Zero-Profit-Conditions of Model

Each activity array in Figure 3-2 adds to the model one array of variables and one array of equations of the same dimensionality. Each activity 'block' in Figure 3-2 constitutes an array of single activities. The dimensionality of these arrays is defined by indices *g*, *t*, and *r*. The variable arrays are the respective activity levels and the equation arrays are the zero-profit-conditions.

Figure 3-2: Model structure of 2-sector model



Intertemporal utility aggregation $\Omega^u_{g,r}$

This activity aggregates period utilities of each cohort $z_{g,t,r}$ to a lifetime utility value $u_{g,r}$ by a CES function with elasticity of substitution ¯ according to equation (1). This activity is per cohort, thus each cohort in each region is associated with one zero-profit-condition:

$$\Pi^u_{g,r} = p^u_{g,r} - \left[\sum_{t=g}^{g+\text{a}_{g,r}-1}\left(\frac{1}{1+\text{°}}\right)^t \cdot \left(p^z_{g,t,r}\right)^{1-\text{¯}}\right]^{\frac{1}{1-\text{¯}}} \leq 0 \qquad \textbf{(38)}$$
$$s.t. \quad \Pi^u_{g,r} \cdot \Omega^u_{g,r} = 0$$

As introduced above for the general case, the complementarity condition in the form $\Pi^u_{g,r} \cdot \Omega^u_{g,r} = 0$ ensures that the activity is only engaged if it has a non-negative 'profit' $\Pi^u_{g,r}$.

Period utility aggregation $\Omega^z_{g,t,r}$

Period utility aggregation takes a compound consumption good and leisure time as inputs in order to form a cohort-specific period utility value. The activity is defined by equation (2) and is performed once for each cohort in each period of lifetime. Its zero-profit-condition has the form:

$$\Pi^z_{g,t,r} = p^z_{g,t,r} - \left[\text{¶} \cdot \left(p^c_{g,t,r}\right)^{1-\text{¶}} + (1- \quad)^{\text{¶}} \cdot \left(p^{leis}_{g,t,r}\right)^{1-\text{¶}}\right]^{\frac{1}{1-\text{¶}}} \leq 0 \qquad \textbf{(39)}$$
$$s.t. \quad \Pi^z_{g,t,r} \cdot \Omega^z_{g,t,r} = 0$$

Consumption good aggregation $\Omega^c_{g,t,r}$

Consumers demand a compound consumption good, which consists of tradable and non-tradable goods as defined in equation (3). The constant elasticity of substitution between tradable and non-tradable goods is ± and the associated zero-profit-condition is structurally identical to period utility aggregation. Due to the assumption of perfect tradability of tradable goods between regions, the price of tradable goods p^T_t is equal in all regions and thus not associated with a regional index.

$$\Pi^c_{g,t,r} = p^c_{g,t,r} - \left[\text{¥}^{\pm} \cdot \left(p^T_t\right)^{1-\pm} + (1-\text{¥})^{\pm} \cdot \left(p^{NT}_{t,r}\right)^{1-\pm}\right]^{\frac{1}{1-\pm}} \leq 0 \qquad \textbf{(40)}$$
$$s.t. \quad \Pi^c_{g,t,r} \cdot \Omega^c_{g,t,r} = 0$$

Production sectors $\Omega^T_{t,r}$ *and* $\Omega^{NT}_{t,r}$

The two production sectors for each region are modeled with a Cobb-Douglas function according to equation (6). The wage rate $p^L_{t,r}$ is defined as the wage rate net of pension contribution. Accordingly, pension contribution needs to be accounted for by multiplying with $(1+\text{£})$ in order to form the actual labor costs from firms' perspective.

Applying the algebraic form of the associated zero-profit-condition derived in section 3.5.1 and taking the ad valorem pension contribution into account, the zero-profit-condition of the production sectors can be written as:

$$\Pi_{t,r}^{T} = p_{t}^{T} - \left(A_{r}^{T}\right)^{-1} \cdot \left(\sim_{r}^{T}\right)^{-\sim_{r}^{T}} \cdot \left(p_{t,r}^{R}\right)^{\sim_{r}^{T}} \cdot \left(1-\sim_{r}^{T}\right)^{-(1-\sim_{r}^{T})} \cdot \left(\left(1+\text{£}\right)\cdot p_{t,r}^{L}\right)^{(1-\sim_{r}^{T})} \leq 0$$

$$\begin{gathered} \Pi_{t,r}^{NT} = p_{t,r}^{NT} - \left(A_{r}^{NT}\right)^{-1} \cdot \left(\sim_{r}^{NT}\right)^{-\sim_{r}^{NT}} \cdot \left(p_{t,r}^{R}\right)^{\sim_{r}^{NT}} \cdot \left(1-\sim_{r}^{NT}\right)^{-(1-\sim_{r}^{NT})} \\ \cdot \left(\left(1+\text{£}\right)\cdot p_{t,r}^{L}\right)^{(1-\sim_{r}^{NT})} \leq 0 \\ s.t. \quad \Pi_{t,r}^{T} \cdot \Omega_{t,r}^{T} = 0, \qquad \Pi_{t,r}^{NT} \cdot \Omega_{t,r}^{NT} = 0 \end{gathered} \tag{41}$$

Capital stock and investment activities $\Omega_{t,r}^{K}$ *and* $\Omega_{t,r}^{I}$

As exhibited in Figure 3-2, the capital stock accumulation is modeled by an activity. However, an additional good 'rental capital' needs to be added in order implement the accumulation of capital and the usage of capital by firms in Mathiesen's format. Following Rasmussen and Rutherford (2004), the zero-profit-condition of the accumulation of capital stock takes the following form.

$$\begin{gathered} \Pi_{t,r}^{K} = p_{t+1,r}^{K} \cdot \left(1-\text{¢}\right) + p_{t,r}^{R} - p_{t,r}^{K} \leq 0 \\ s.t. \quad \Pi_{t,r}^{K} \cdot \Omega_{t,r}^{K} = 0 \end{gathered} \tag{42}$$

The output goods 'capital' and 'rental capital' of this activity have an elasticity of transformation of zero. In other words, their quantity ratio is fixed and the unit revenue is simply the sum of their prices. Depreciation affects the output quantity of this activity in the sense that for an activity level of *one*, only $\left(1-\text{¢}\right)$ units of capital in period *t+1* are generated. Therefore, the revenue per unit of $\Omega_{t,r}^{K}$ for capital in *t+1* needs to be adjusted. As an interpretation, equation (42) ensures that the period rental rate per unit of capital $p_{t,r}^{R}$ is just high enough to cover depreciation and interest. The latter is implicitly addressed via the present value price definition of $p_{t,r}^{K}$:

$$p^K_{t+1,r} \cdot = \frac{p^K_{t,r}}{1 + r_{t,r}} \qquad \textbf{(43)}$$

The second activity involved in capital formation is the investment activity. It transforms tradable goods into capital goods with a time lag of one period. Thus, this activity has only one input and one output:

$$\begin{aligned} &\Pi^I_{t,r} = p^K_{t+1,r} - p^T_t \leq 0 \\ &s.t. \quad \Pi^I_{t,r} \cdot \Omega^I_{t,r} = 0 \end{aligned} \qquad \textbf{(44)}$$

Its zero-profit-condition is straightforward and links the price of tradable goods in period *t* to the price of capital goods in period *t+1*.

Labor supply $\Omega^L_{g,t,r}$

As exhibited in Figure 3-2, labor supply is also modeled as an activity with only one input and one output good. The purpose of this activity is simply to multiply labor time per cohort and per period with its respective labor productivity $\circledR_{g,t,r}$ as defined in equation (7). This results in the following zero-profit-condition:

$$\begin{aligned} &\Pi^L_{g,t,r} = p^L_{t,r} - \frac{1}{\circledR_{g,t,r}} \cdot p^t_{g,t,r} \leq 0 \\ &s.t. \quad \Pi^L_{t,r} \cdot \Omega^L_{t,r} = 0 \end{aligned} \qquad \textbf{(45)}$$

3.6.2 Market-Clearing-Conditions of Model

As introduced in section 3.5.2, each good in the model is associated with a market-clearing-condition. Accordingly, each good adds one variable, its price, and one market-clearing-condition[49].

Lifetime utility is the final good demanded by consumers, thus its market-clearing-condition is linked to the income definition of private households. These income definitions are described in the next section.

Period utility $p^z_{g,t,r}$

Period utility has a price of $p^z_{g,t,r}$ and its market-clearing-condition ensures that supply by activity $\Omega^z_{g,t,r}$ equals the demand for period utility of activity $\Omega^u_{g,r}$:

[49] Mathiesen (1985)

$$\Xi^{z}_{g,t,r} = \Omega^{z}_{g,t,r} - \Omega^{u}_{g,r} \cdot \frac{\partial \Pi^{u}_{g,r}}{\partial p^{z}_{g,t,r}}$$

$$= \Omega^{z}_{g,t,r} - \Omega^{u}_{g,r} \cdot \left[\sum_{t=g}^{g+{}^{a}{}_{g,r}-1} \left(\frac{1}{1+{}^{\circ}} \right)^{t\cdot -} \cdot \left(p^{z}_{g,t,r} \right)^{1-} \right]^{\frac{-}{1-}} \cdot \left(\frac{1}{1+{}^{\circ}} \right)^{t\cdot -} \cdot \left(p^{z}_{g,t,r} \right)^{-} \geq 0 \tag{46}$$

$$s.t. \quad \Xi^{z}_{g,t,r} \cdot p^{z}_{g,t,r} = 0$$

Compound consumption good $p^{c}_{g,t,r}$

The market-clearing-condition of the compound consumption good balances the supply of compound consumption good by activity $\Omega^{c}_{g,t,r}$ with its demand by activity $\Omega^{z}_{g,t,r}$:

$$\Xi^{c}_{g,t,r} = \Omega^{c}_{g,t,r} - \Omega^{z}_{g,t,r} \cdot \frac{\partial \Pi^{z}_{g,t,r}}{\partial p^{c}_{g,t,r}}$$

$$= \Omega^{c}_{g,t,r} - \Omega^{z}_{g,t,r} \cdot \left[\P \cdot \left(p^{c}_{g,t,r} \right)^{1-\P} + \left(1- \quad \right)^{\P} \cdot \left(p^{leis}_{g,t,r} \right)^{1-\P} \right]^{\frac{\P}{1-\P}} \cdot \quad \P \cdot \left(p^{c}_{g,t,r} \right)^{-\P} \geq 0 \tag{47}$$

$$s.t. \quad \Xi^{c}_{g,t,r} \cdot p^{c}_{g,t,r} = 0$$

Tradable goods p^{T}_{t}

Tradable goods are produced in each region. However, due to the assumption of perfect tradability of tradable goods between regions, its price is equal in all regions in each period of time. Thus, tradable goods are modeled as one good globally and the regional index can be omitted. The market-clearing-condition ensures that the sum of tradable-goods-production over all regions plus the pension benefits equals global tradable-goods-consumption plus investment in each period:

$$\Xi^{T}_{t} = \sum_{r} \Omega^{T}_{t,r} + \sum_{r} \frac{B_{t,r}}{p^{T}_{t}} - \sum_{r} \Omega^{I}_{t,r} - \sum_{r} \sum_{g} \Omega^{c}_{g,t,r} \cdot \frac{\partial \Pi^{c}_{g,t,r}}{\partial p^{T}_{t}}$$

$$= \sum_{r} \Omega^{T}_{t,r} + \sum_{r} \frac{B_{t,r}}{p^{T}_{t}} - \sum_{r} \Omega^{I}_{t,r}$$

$$- \sum_{r} \sum_{g} \Omega^{c}_{g,t,r} \cdot \left[\yen^{\pm} \cdot \left(p^{T}_{t} \right)^{1-\pm} + \left(1-\yen \right)^{\pm} \cdot \left(p^{NT}_{t,r} \right)^{1-\pm} \right]^{\frac{\pm}{1-\pm}} \cdot \yen^{\pm} \cdot \left(p^{T}_{t} \right)^{-\pm} \geq 0 \tag{48}$$

$$s.t. \quad \Xi^{T}_{t} \cdot p^{T}_{t} = 0$$

Non-tradable goods p^{NT}_{t}

Non-tradable goods must be consumed in their region of production. As opposed to tradable goods, they are thus modeled as distinct goods in each region. As a result, prices for non-tradable goods may vary between regions and each region has its own market-clearing-condition for non-tradable goods. Due to the assumption that only tradable goods can be invested, the market-clearing-condition for non-tradable goods only balances production with consumption demand:

$$\Xi_{t,r}^{NT} = \Omega_{t,r}^{NT} - \sum_{t-\text{ª}_{g,r}+1\le g\le t} \Omega_{g,t,r}^{c} \cdot \frac{\partial \Pi_{g,t,r}^{c}}{\partial p_{t,r}^{NT}}$$

$$= \Omega_{t,r}^{NT} - \sum_{t-\text{ª}_{g,r}+1\le g\le t} \Omega_{g,t,r}^{c} \cdot \left[\text{¥}^{\pm} \cdot \left(p_{t}^{T}\right)^{1-\pm} + \left(1-\text{¥}\right)^{\pm} \cdot \left(p_{t,r}^{NT}\right)^{1-\pm}\right]^{\frac{\pm}{1-\pm}} \cdot \left(1-\text{¥}\right)^{\pm} \cdot \left(p_{t,r}^{NT}\right)^{-\pm} \qquad \textbf{(49)}$$

$$\geq 0, \quad s.t. \quad \Xi_{t,r}^{NT} \cdot p_{t,r}^{NT} = 0$$

Rented capital $p_{t,r}^{R}$

Rented capital is a good introduced due to modeling reasons. Its price is the rental rate of one unit of capital, i.e. the price that production sectors pay for the use of one unit of capital for one period. Rented capital is produced by the capital stock activity $\Omega_{t,r}^{K}$ and demanded by production sectors of tradable and non-tradable goods $\Omega_{t,r}^{T}$ and $\Omega_{t,r}^{NT}$.

$$\Xi_{t,r}^{R} = \Omega_{t,r}^{K} - \Omega_{t,r}^{T} \cdot \frac{\partial \Pi_{t,r}^{T}}{\partial p_{t,r}^{R}} - \Omega_{t,r}^{NT} \cdot \frac{\partial \Pi_{t,r}^{NT}}{\partial p_{t,r}^{R}}$$

$$= \Omega_{t,r}^{K} - \Omega_{t,r}^{T} \cdot \left(A_{r}^{T}\right)^{-1} \cdot \left(\sim_{r}^{T}\right)^{-\sim_{r}^{T}+1} \cdot \left(p_{t,r}^{R}\right)^{\sim_{r}^{T}-1} \cdot \left(1-\sim_{r}^{T}\right)^{-(1-\sim_{r}^{T})} \cdot \left(p_{t,r}^{L}\right)^{(1-\sim_{r}^{T})} \qquad \textbf{(50)}$$

$$- \Omega_{t,r}^{NT} \cdot \left(A_{r}^{NT}\right)^{-1} \cdot \left(\sim_{r}^{NT}\right)^{-\sim_{r}^{NT}+1} \cdot \left(p_{t,r}^{R}\right)^{\sim_{r}^{NT}-1} \cdot \left(1-\sim_{r}^{NT}\right)^{-(1-\sim_{r}^{NT})} \cdot \left(p_{t,r}^{L}\right)^{(1-\sim_{r}^{NT})} \geq 0$$

$$s.t. \quad \Xi_{t,r}^{R} \cdot p_{t,r}^{R} = 0$$

Capital goods $p_{t,r}^{K}$

The capital stock of each period consists of last period's capital stock net of depreciation plus last period's investment. Its market-clearing-condition thus has the following form:

$$\Xi_{t,r}^{K} = \Omega_{t,r}^{K} \cdot \left(1-\text{¢}\right) + \Omega_{t,r}^{I} - \Omega_{t+1,r}^{K} \qquad \textbf{(51)}$$

$$s.t. \quad \Xi_{t,r}^{K} \cdot p_{t,r}^{K} = 0$$

Labor $p_{t,r}^{L}$

The market-clearing-condition of labor balances aggregated labor supply by all cohorts alive in period *t* with labor demand by production sectors:

$$
\begin{aligned}
\Xi_{t,r}^{L} &= \sum_{t-\text{ª}_{g,r}+1 \le g \le t} \Omega_{g,t,r}^{L} - \Omega_{t,r}^{T} \cdot \frac{\partial \Pi_{t,r}^{T}}{\partial p_{t,r}^{L}} - \Omega_{t,r}^{NT} \cdot \frac{\partial \Pi_{t,r}^{NT}}{\partial p_{t,r}^{L}} \\
&= \Omega_{t,r}^{L} - \Omega_{t,r}^{T} \cdot \left(A_r^T\right)^{-1} \cdot \left(\sim_r^T\right)^{-\sim_r^T} \cdot \left(p_{t,r}^{R}\right)^{\sim_r^T} \cdot \left(1-\sim_r^T\right)^{\sim_r^T} \cdot \left(p_{t,r}^{L}\right)^{-\sim_r^T} \\
&\quad - \Omega_{t,r}^{NT} \cdot \left(A_r^{NT}\right)^{-1} \cdot \left(\sim_r^{NT}\right)^{-\sim_r^{NT}} \cdot \left(p_{t,r}^{R}\right)^{\sim_r^{NT}} \cdot \left(1-\sim_r^{NT}\right)^{\sim_r^{NT}} \cdot \left(p_{t,r}^{L}\right)^{-\sim_r^{NT}} \ge 0 \\
&\qquad s.t. \quad \Xi_{t,r}^{L} \cdot p_{t,r}^{L} = 0
\end{aligned}
\tag{52}
$$

Time $p_{g,t,r}^{t}$

Each cohort is endowed with a fixed amount of time proportional to its cohort size in each period, which can be consumed as leisure time or employed as labor time:

$$
\begin{aligned}
\Xi_{g,t,r}^{t} &= n_{g,r} \cdot \mu - \Omega_{g,t,r}^{z} \cdot \frac{\partial \Pi_{g,t,r}^{z}}{\partial p_{g,t,r}^{t}} - \Omega_{g,t,r}^{L} \cdot \frac{1}{\text{®}_{g,t,r}} \ge 0 \\
&\qquad s.t. \quad \Xi_{g,t,r}^{t} \cdot p_{g,t,r}^{t} = 0
\end{aligned}
\tag{53}
$$

3.6.3 Income Definitions of Model

The income definition links the value of total endowment of cohorts to the value of their lifetime utility. The total endowment comprises time endowment $n_{g,r} \cdot \mu$ as well as pension benefits $\text{¤}_{g,t,r}$. As described in section 3.3, pension benefits are *zero* before pension age.

$$
\Psi_{g,r} = p_{g,r}^{u} \cdot \Omega_{g,r}^{u} = \sum_{t=g}^{g+\text{ª}_{g,r}-1} \left(p_{g,t,r}^{t} \cdot n_{g,r} \cdot \mu + \text{¤}_{g,t,r} \right) \tag{54}
$$

4 Model Calibration and Data Input

The model structure laid out in chapter 3 does not contain numerical assignments for any exogenous parameters. In contrast to analytical solution methods that derive general, closed form solution formulae, a numerical solution requires specific value assignments for all exogenous parameters. The process of choosing these exogenous model parameters in a way that the model including its endogenous variables behaves consistently with observed, real economic data, is referred to as 'calibration'. Additionally, the sets of indices, such as the region index or the time index, need to be defined.

This chapter first introduces these index set definitions and then proceeds with the theoretical fundamentals of the applied calibration approach including the concept of Social Accounting Matrices (SAM), before shedding light on the actual parameter values employed in the model.

4.1 Definition of Index Sets

The equations introduced in chapter 3 contain three types of indices: The time index t, the cohort index g and the regional index r.

4.1.1 Time Index

Two crucial modeling decisions are associated with the time index t. The first decision is concerned with the actual duration of one time period in the model, while the second deals with the partitioning of the model horizon.

Real-Time Duration of Model Periods

In rather stylized overlapping-generation-models, only two generational cohorts live at each point in time. In this case, each cohort is 'young' for one period and 'old' for the following period. Accordingly, generations only live for two periods and one period thus has an actual duration of roughly 25 years[50]. To the other extreme, one period may correspond to one year or even less, implying that 50 or more cohorts are alive at each point in time.

[50] In such models, typically the assumption is made that cohorts enter the model with a real-life age of 20.

Reducing the actual duration of one period and thereby increasing time resolution rapidly drives up the number of single model equations and thus numerical complexity. The number of equations will approximately increase proportionally to the square of the reciprocal value of time period duration:

$$N^{eq} \sim \frac{1}{d^2} \tag{55}$$

(Approximation for relatively large values of T and small values of d)

N^{eq} : Number of single, numerical model equations

d : Duration of one model period in actual time (years)

This relationship may be inferred from the fact that for a relatively large number of time periods (T large) and for several cohorts alive in each period (d small), the largest share of single equations are generated by equation blocks over all three indices, such as equation (2) for instance. The number of these equations is proportional to the number of time periods t and the number of cohorts living at each point in time. As the resolution of cohort index g is linked to the time index, a reduction of d will anti-proportionally increase the number of time periods as well as the number of generations living at each point in time. Thus, the number of equations over all three indices will increase anti-proportionally to the square of d. Since the number of other equations is small under the conditions outlined above, the total number of equations is approximately anti-proportional to the square of single-period duration d.

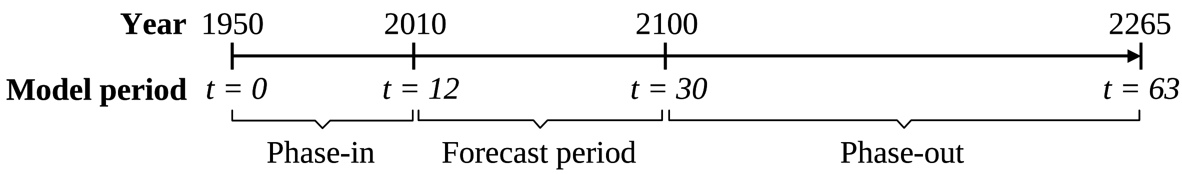
The increasing model complexity for relatively small values of d clearly is the main drawback of a high time resolution. On the other hand, the accuracy of model results suffers if d is chosen too large. That is, a stylized model with $d = 25$ not only has the drawback of poor resolution of results in the sense that dynamics taking place within each period of 25 years cannot be assessed, but also the drawback of worse approximation of the exact simulation path. The theoretical 'exact' path can be interpreted as the simulation result for $\lim d \to 0$. Thus, all discrete time general equilibrium models, which are characterized by finite values of d, are only approximations of an exact simulation path. This exact path may be interpreted as the result of a continuous time model, which is equivalent to $\lim d \to 0$.

Due to this relationship, d is chosen as small as possible such that the applied numerical solution algorithm is still able to find a solution in a reasonable amount of computation time. For the presented model with the already high complexity induced by the differentiation of eight regions and two production sectors, d is therefore defined as 5 years. Another reason to set d to 5 years is the frequency of available data. Most demographical time series data used in this model is available in steps of 5 years.

Segmentation of Model Horizon

As portrayed in section 3.4, a phase-out period is required at the end of the model horizon in order to transfer the simulation path close to the final steady state and to ensure that the simulation path in the actual forecast period is not significantly distorted by the end of the model horizon. For similar reasons, a phase-in period precedes the actual forecasting period as suggested by Börsch-Supan et al. (2006). Figure 4-1 exhibits the segmentation of the model horizon into the three phases introduced above.

Figure 4-1: Model horizon



Starting from an initial and artificial steady state in year 1950, a phase-in period from 1950 until 2010 allows the model to adjust to a realistic demographic structure until the start of the forecast period. Since the size of a cohort cannot be changed in this setup once the cohort entered the model and since the oldest cohorts alive in 2010 entered the model in the 1950s-1960s, this phase-in is necessary to achieve a realistic demographic structure in 2010.

Additionally, the phase-in allows the model variables to react to the exogenous demographic changes between 1950 and 2010. This demographic transition is likely to have repercussions on the forecasting period, thus making the phase-in period necessary in order to avoid distortions in results.

The actual forecasting period runs from 2010 until 2100 and matches the forecasted time span of United Nations demographic projections[51].

[51] Source: UN World Population Prospects, 2010 revision

4.1.2 Cohort Index

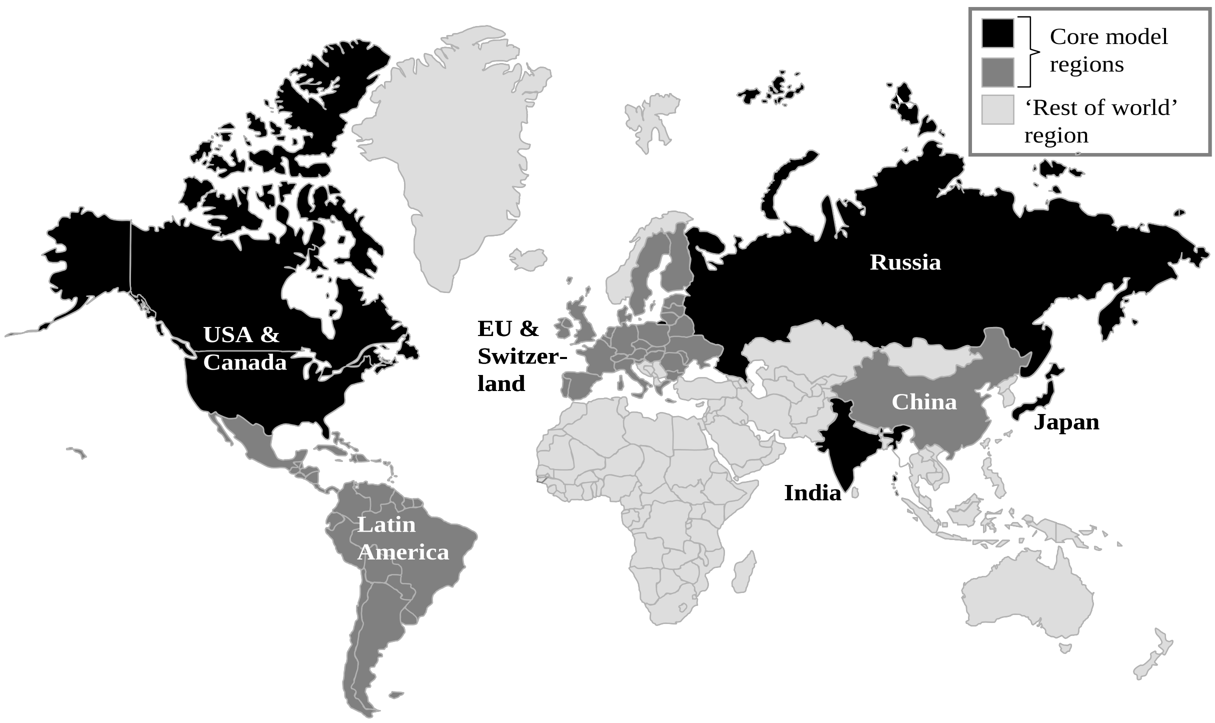
The cohort index g uniquely identifies each cohort in a given region by its period of birth. A cohort born in model period t is thus characterized by the index $g = t$. The index g runs from the period where the oldest generation alive in the first model period $t = 0$ was born and accordingly starts with a negative value. The upper limit of index g is equal to the end of the model horizon T. However, variables that depend on both indices *g* and *t*, such as period utility, are only defined for combinations of *g* and *t* for which cohort *g* is alive. This relationship can be written as follows.

$$t - g + 1 \leq a_{g,r} \quad \textbf{(56)}$$

4.1.3 Region Index

The set of the region index r comprises seven economically important world regions (‘core model regions’) and an additional ‘Rest of World’ region, covering all other countries that are included in the GTAP database. Figure 4-2 exhibits the seven core model regions in black and dark grey and the residual ‘Rest of World’ region in light grey.

Figure 4-2: Regional disaggregation



4.2 Calibration Approach

The calibration approach employed is based on the assumption that all model regions are in steady state in their reference year. In other words, the calibration procedure ensures that the model's steady-state equilibrium replicates benchmark year values observed in actual economic data. More specifically, the calibrated model replicates, for instance, the benchmark GDPs of the eight model regions, the benchmark shares of tradable vs. non-tradable production and consumption, the benchmark investment level, and benchmark labor and capital income shares. In addition to these aggregated benchmark values, the calibrated steady state also replicates cohort-specific values such as the benchmark consumption and saving over the lifecycle. While the aggregated values feed into the model from statistical data, the cohort-specific values are retro-fitted in order to be consistent with aggregate values. The following four-step approach is pursued in order to calibrate the model[52]:

1) Assemble consistent Social Accounting Matrices (SAMs) and consistent values for capital stocks for each region[53]

2) Define numerical values for exogenous parameters (elasticities, benchmark interest rate, and benchmark population growth rate)

3) Retro-fit cohort-specific values by solving an auxiliary steady-state equilibrium OLG-model only used for calibration purposes for each region

4) Transform model equations from section 3.6 into calibrated share form and assign benchmark values for quantities and prices from SAMs and from auxiliary steady-state models

The central concept of this calibration approach is the calibrated share form of model equations. For this reason, the next section first introduces the calibrated share form, before the subsequent sections shed light on how the parameters for the calibrated share form are derived by following steps 1-3.

[52] Based on Rasmussen and Rutherdord (2001)

[53] See section 4.5 for an introduction to the concept of Social Accounting Matrices

4.3 Introduction to the Calibrated Share Form

It is a crucial step with respect to the calibration procedure to translate the model equations into a form, which explicitly contains the benchmark values used for calibration. This form is referred to as the ‘calibrated share form’ by the literature[54]. This step is necessary, since the majority of exogenous parameters used in the model equations in section 3.6 cannot be directly pulled from available calibration data. In addition, the model equations from section 3.6 have not enough degrees of freedom, i.e. not enough exogenous parameters, to match real economic data as closely as the calibrated share form.

In calibrated share form, all price and activity level variables are normalized by benchmark prices and quantities. These benchmark prices and quantities are exogenous parameters, which remain constant over the model horizon, and which are based on real economic data. As a result of this normalization, all activity level variables equal unity in the benchmark equilibrium or are proportionate to the cohort size. This benchmark equilibrium is equivalent to the model’s initial steady state. In other words, the model replicates the ‘benchmark’ economic world, defined by the economic data used for calibration, in its initial steady state.

The CES function for activity levels takes the following form[55]:

$$\Omega_j = \left[\sum a_{i,j}^{-\pm} \cdot \left(\frac{x_{i,j}^I}{\overline{x}_{i,j}^I} \right)^{\frac{\pm -1}{\pm}} \right]^{\frac{\pm}{\pm -1}} \quad \textbf{(57)}$$

$a_{i,j} = \dfrac{\overline{p}_i^I \cdot \overline{x}_i^I}{\overline{C}_j}$: Benchmark value share of input good *i*

$\overline{C}_j = \sum_i \overline{p}_{i,j}^I \cdot \overline{x}_{i,j}^I$: Benchmark cost for activity level of *one*

$x_{i,j}^I$: Quantity of input good *i*

$\overline{x}_{i,j}^I$: Benchmark quantity of input good *i*

$\overline{p}_i^I$: Benchmark input price of good *i*

According to the algebraic derivations from section 3.5, the zero-profit-condition for activity *j* in calibrated share form can then be written as follows.

[54] See Rutherford (2002)
[55] See Rasmussen and Rutherford (2004)

$$\Pi_j(p_i) = \bar{R}_j \cdot \left[\sum_i \text{«}_{i,j} \cdot \left(\frac{p_i}{\bar{p}_i^O}\right)^{1+\text{¦}}\right]^{\frac{1}{1+\text{¦}}} - \bar{C}_j \cdot \left[\sum_i \sim_{i,j} \cdot \left(\frac{p_i}{\bar{p}_i^I}\right)^{1-\pm}\right]^{\frac{1}{1-\pm}} \quad \textbf{(58)}$$

$\bar{R}_j = \sum_i \bar{p}_i^O \cdot \bar{x}_{i,j}^O$: Benchmark revenue for activity level of *one*

$\text{«}_{i,j} = \dfrac{\bar{p}_i^O \cdot \bar{x}_{i,j}^O}{\bar{R}_j}$: Benchmark value share of output good *i*

$\bar{p}_i^O$: Benchmark output price of good *i*

Applying Shephard’s lemma, the net supply of good *i* by activity *j* can be derived as[56]:

$$\frac{\partial \Pi_j(p)}{\partial p_i} = \bar{x}_{i,j}^O \cdot \left(\frac{p_i / \bar{p}_i^O}{R_j(p)/\bar{R}_j}\right)^{\text{¦}} - \bar{x}_{i,j}^I \cdot \left(\frac{C_j(p)/\bar{C}_j}{p_i / \bar{p}_i^I}\right)^{\text{¦}} \quad \textbf{(59)}$$

$\bar{x}_{i,j}^O$: Benchmark output quantity of good *i*

In applied models, such as in the present one, the majority of sectors feature only one output good. In this case, the revenue function degenerates to $R_j(p) = p_j$. If it is clear upfront that such a sector must operate with a positive activity level in equilibrium, i.e. the zero-profit-condition is fulfilled as an equality, then the demand function for good *i* can be written in a more compact form. With $R_j(p) = p_j$ and the assumption of $\Pi_j(p_i) = 0$, it follows from equation (58)[55]:

$$\frac{C_j(p)}{\bar{C}_j} = \frac{p_j}{\bar{p}_j^O} \quad \textbf{(60)}$$

Substituting in equation (59) derives the net supply of input good *i* for an activity *j* with only one single output that is known to operate at positive activity level[57]:

[56] See Rasmussen and Rutherford (2004)
[57] See Rasmussen and Rutherford (2004)

$$\frac{\partial \Pi_j(p)}{\partial p_i} = -\overline{x}_{i,j}^{I} \cdot \left(\frac{p_j / \overline{p}_j^{O}}{p_i / \overline{p}_i^{I}} \right)^{¦} \qquad \textbf{(61)}$$

4.4 Model Equations in Calibrated Share Form

This section transforms the model equations assembled in section 3.6 into the calibrated share form. Finally, the form of equations presented in this section is the one, which is actually implemented in the OLG-model.

4.4.1 Zero-Profit-Conditions

The following equations define the zero-profit-conditions in calibrated share form. They are based on the zero-profit-conditions detailed in section 3.6.1 and the formulation given by equation (58). Most of the activities feature only a single output, thus the revenue term for these activities degenerates to simply a benchmark quantity times a benchmark price. For better numerical stability, these equations are raised to the power of $1-«$, with « representing the respective elasticity of substitution, and thus slightly deviate in its form from equation (58). Since both sides of each equation are raised to the power of $1-«$, the model solution is not affected.

Furthermore, to close the model, initial and terminal variables such as $\overline{z}_{g,a,r}^{term}$ are added to the equation system. These variables ensure that the model behaves smoothly at the beginning and end of the model horizon.

Intertemporal utility aggregation $\Omega^{u}_{g,r}$

$$\tilde{\Pi}^{u}_{g,r} = \bar{u}_{g,r} \cdot p^{u}_{g,r}{}^{1-\text{¯}} - \sum_{t} \bar{p}^{z}_{g,t,r} \cdot \bar{z}_{g,t,r} \cdot \left(\frac{p^{z}_{g,t,r}}{\bar{p}^{z}_{g,t,r}} \right)^{1-\text{¯}} - \sum_{a} \bar{p}^{z,term}_{g,a,r} \cdot \bar{z}^{term}_{g,a,r} \cdot \left(\frac{p^{z,term}_{g,a,r}}{\bar{p}^{z,term}_{g,a,r}} \right)^{1-\text{¯}} \leq 0 \qquad \textbf{(62)}$$

$$s.t. \quad \tilde{\Pi}^{u}_{g,r} \cdot \Omega^{u}_{g,r} = 0$$

$\bar{u}_{g,r}$: Benchmark cohort lifetime utility

$\bar{p}^{z}_{g,t,r}$: Benchmark price of period utility

$\bar{z}_{g,t,r}$: Benchmark period utility

$\bar{p}^{z,term}_{g,a,r}$: Benchmark price of post-terminal period utility

$\bar{z}^{term}_{g,a,r}$: Benchmark post-terminal period utility

Period utility aggregation $\Omega^{z}_{g,t,r}$

$$\tilde{\Pi}^{z}_{g,t,r} = \left(\frac{p^{z}_{g,t,r}}{\bar{p}^{z}_{g,t,r}} \right)^{1-\text{¶}} - \sim \cdot \left(\frac{p^{c}_{g,t,r}}{\bar{p}_{t}} \right)^{1-\text{¶}} - (1-\sim) \cdot \left(\frac{p^{t}_{g,t,r}}{\bar{p}^{t}_{g,t,r}} \right)^{1-\text{¶}} \leq 0 \qquad \textbf{(63)}$$

$$s.t. \quad \tilde{\Pi}^{z}_{g,t,r} \cdot \Omega^{z}_{g,t,r} = 0$$

$\sim_{r} = \dfrac{\bar{c}_{g,t,r} \cdot \bar{p}_{t}}{\bar{z}_{g,t,r} \cdot \bar{p}^{z}_{g,t,r}}$: Goods share in period utility

$\bar{p}_{t} = 1/(1+\bar{r})^{t}$: Reference price path

$\bar{r}$: Reference real interest rate

Consumption good aggregation $\Omega^c_{g,t,r}$

$$\tilde{\Pi}^c_{g,t,r} = \left(\frac{p^c_{g,t,r}}{\bar{p}_t}\right)^{1-\pm} - \frac{\bar{c}^T_{g,t,r}}{\bar{c}_{g,t,r}} \cdot \left(\frac{p^T_t}{\bar{p}_t}\right)^{1-\pm} - \frac{\bar{c}^{NT}_{g,t,r}}{\bar{c}_{g,t,r}} \cdot \left(\frac{p^{NT}_{t,r}}{\bar{p}_t}\right)^{1-\pm} \leq 0 \tag{64}$$

$$s.t. \quad \tilde{\Pi}^c_{g,t,r} \cdot \Omega^c_{g,t,r} = 0$$

$\bar{c}^T_{g,t,r}$: Benchmark tradable goods consumption

$\bar{c}^{NT}_{g,t,r}$: Benchmark non-tradable goods consumption

$\bar{c}_{g,t,r} = \bar{c}^T_{g,t,r} + \bar{c}^{NT}_{g,t,r}$: Benchmark goods consumption[58]

Production sectors $\Omega^T_{t,r}$ *and* $\Omega^{NT}_{t,r}$

$$\tilde{\Pi}^T_{t,r} = \frac{p^T_t}{\bar{p}_t} - \left(\frac{p^R_{t,r}}{\bar{p}_t}\right)^{\frac{\bar{R}^T_r}{\bar{Y}^T_r}} \cdot \left(\frac{p^L_{t,r} \cdot (1+\text{£})}{\bar{p}_t \cdot (1+\bar{\text{£}})}\right)^{\frac{\bar{L}^T_r \cdot (1+\bar{\text{£}})}{\bar{Y}^T_r}} \leq 0$$

$$\tilde{\Pi}^{NT}_{t,r} = \frac{p^{NT}_{t,r}}{\bar{p}_t} - \left(\frac{p^R_{t,r}}{\bar{p}_t}\right)^{\frac{\bar{R}^{NT}_r}{\bar{Y}^{NT}_r}} \cdot \left(\frac{p^L_{t,r} \cdot (1+\text{£})}{\bar{p}_t \cdot (1+\bar{\text{£}})}\right)^{\frac{\bar{L}^{NT}_r \cdot (1+\bar{\text{£}})}{\bar{Y}^{NT}_r}} \leq 0 \tag{65}$$

$$s.t. \quad \tilde{\Pi}^T_{t,r} \cdot \Omega^T_{t,r} = 0, \qquad \tilde{\Pi}^{NT}_{t,r} \cdot \Omega^{NT}_{t,r} = 0$$

$\bar{Y}^T_r, \bar{Y}^{NT}_r$: Benchmark (non-) tradable goods production

$\bar{L}^T_r, \bar{L}^{NT}_r$: Benchmark labor earnings from (non-) tradable sector

$\bar{R}^T_r, \bar{R}^{NT}_r$: Benchmark capital earnings from (non-) tradable sector

$\bar{\text{£}}$: Benchmark contribution rate

[58] Benchmark prices $\bar{p}^c_{g,t,r}$, $\bar{p}^T_{g,t,r}$ and $\bar{p}^{NT}_{g,t,r}$ are implicitly defined to be equal to a value of *one*. Thus, benchmark quantities in this equation are equal to benchmark values, making the direct summarization of these quantities feasible.

Capital stock and investment activities $\Omega_{t,r}^{K}$ *and* $\Omega_{t,r}^{I}$

$$\tilde{\Pi}_{t,r}^{K} = p_{t+1,r}^{K} \cdot (1 - \text{¢}) \cdot \overline{K}_r + p_{t,r}^{R} \cdot \overline{R}_r - p_{t,r}^{K} \cdot \overline{K}_r \leq 0 \quad \textbf{(66)}$$

$$\textit{with } p_{T+1,r}^{K} = p_r^{K,term}$$

$$s.t. \quad \tilde{\Pi}_{t,r}^{K} \cdot \Omega_{t,r}^{K} = 0$$

$$\tilde{\Pi}_{t,r}^{I} = p_{t+1,r}^{K} - p_t^{T} \leq 0 \quad \textbf{(67)}$$

$$s.t. \quad \tilde{\Pi}_{t,r}^{I} \cdot \Omega_{t,r}^{I} = 0$$

$\overline{K}_r$: Benchmark capital stock

$\overline{R}_r$: Benchmark capital earnings

$p_r^{K,term}$: Price of terminal capital stock

Labor supply $\Omega_{g,t,r}^{L}$

$$\tilde{\Pi}_{g,t,r}^{L} = p_{t,r}^{L} \cdot \text{®}_{g,t,r} - p_{g,t,r}^{t} \leq 0$$

$$s.t. \quad \tilde{\Pi}_{t,r}^{L} \cdot \Omega_{g,t,r}^{L} = 0 \quad \textbf{(68)}$$

4.4.2 Market-Clearing-Conditions

Market-clearing-conditions in calibrated share form are based on the market-clearing-conditions derived in section 3.6.2 and the general calibrated share formulations given by equations (59) and (61). Again, since most activities only have a single output good, the simplified formulation of equation (61) can be applied in the major part of market-clearing-equations. As stated above, the application of this simplified formula requires that the associated activity has a non-zero activity level in equilibrium. In this model, labor supply is the only activity that could actually have an activity level of zero in the case of endogenous retirement of individual cohorts. However, this activity only has a single input and a single output, such that the simplifying formulation of equation (61) is not required. All other activities must operate at a positive activity level in equilibrium. Thus, the simplifying formulation of equation (61) may be applied in this model without restrictions.

Period utility $p^z_{g,t,r}$

$$\tilde{\Xi}^z_{g,t,r} = \Omega^z_{g,t,r} - \Omega^u_{g,r} \cdot \left(p^u_{g,r} \frac{\bar{p}^z_{g,t,r}}{p^z_{g,t,r}} \right)^{-} \geq 0 \tag{69}$$

$$s.t. \quad \tilde{\Xi}^z_{g,t,r} \cdot p^z_{g,t,r} = 0$$

Compound consumption good $p^c_{g,t,r}$

$$\tilde{\Xi}^c_{g,t,r} = \Omega^c_{g,t,r} \cdot \bar{c}_{g,t,r} - \Omega^z_{g,t,r} \cdot \bar{c}_{g,t,r} \cdot \left(\frac{p^z_{g,t,r} \big/ \bar{p}^z_{g,t,r}}{p^c_{g,t,r} \big/ \bar{p}_t} \right)^{\P} \geq 0 \tag{70}$$

$$s.t. \quad \tilde{\Xi}^c_{g,t,r} \cdot p^c_{g,t,r} = 0$$

Tradable goods p^T_t

$$\tilde{\Xi}^T_t = \sum_r \Omega^T_{t,r} \cdot \bar{Y}^T_r + \sum_r \frac{B_{t,r}}{p^T_t} - \sum_r \Omega^I_{t,r} \cdot \bar{I}_r - \sum_r \sum_g \Omega^c_{g,t,r} \cdot \bar{c}^T_{g,t,r} \cdot \left(\frac{p^c_{g,t,r}}{p^T_t} \right)^{\pm} \geq 0 \tag{71}$$

$$s.t. \quad \tilde{\Xi}^T_t \cdot p^T_t = 0$$

Non-tradable goods p^{NT}_t

$$\tilde{\Xi}^{NT}_{t,r} = \Omega^{NT}_{t,r} \cdot \bar{Y}^{NT}_r - \sum_g \Omega^c_{g,t,r} \cdot \bar{c}^{NT}_{g,t,r} \cdot \left(\frac{p^c_{g,t,r}}{p^{NT}_t} \right)^{\pm} \geq 0 \tag{72}$$

$$s.t. \quad \tilde{\Xi}^{NT}_{t,r} \cdot p^{NT}_{t,r} = 0$$

Rented capital $p^R_{t,r}$

$$\tilde{\Xi}^R_{t,r} = \Omega^K_{t,r} \cdot \bar{R}_r - \Omega^T_{t,r} \cdot \bar{R}^T_r \cdot \frac{p^T_t}{p^R_{t,r}} - \Omega^{NT}_{t,r} \cdot \bar{R}^{NT}_r \cdot \frac{p^{NT}_{t,r}}{p^R_{t,r}} \tag{73}$$

$$s.t. \quad \tilde{\Xi}^R_{t,r} \cdot p^R_{t,r} = 0$$

Capital goods $p_{t,r}^{K}$

$$\tilde{\Xi}_{t,r}^{K} = \Omega_{t-1,r}^{K} \cdot \bar{K}_r \cdot \left(1-\text{¢}\right) + \Omega_{t-1,r}^{I} \cdot \bar{I}_r - \Omega_{t,r}^{K} \cdot \bar{K}_r$$
$$with\ \Omega_{-1,r}^{K} \cdot \bar{K}_r \cdot \left(1-\text{¢}\right) + \Omega_{-1,r}^{I} \cdot \bar{I}_r := \bar{\Theta}_r^{K} \cdot \sum_g \frac{\bar{a}_{g,r}^{0}}{1+\bar{r}} \qquad \textbf{(74)}$$
$$s.t. \quad \tilde{\Xi}_{t,r}^{K} \cdot p_{t,r}^{K} = 0$$

$\bar{\Theta}_r^{K}$: Benchmark share of capital stock in total initial assets

$\bar{a}_{g,r}^{0}$: Initial assets of cohort g

Labor $p_{t,r}^{L}$

$$\tilde{\Xi}_{t,r}^{L} = \sum_g \Omega_{g,t,r}^{L} \cdot \text{®}_{g,t,r} - \Omega_{t,r}^{T} \cdot \bar{L}_r^{T} \cdot \frac{p_t^{T}}{p_{t,r}^{L}} - \Omega_{t,r}^{NT} \cdot \bar{L}_r^{NT} \cdot \frac{p_{t,r}^{NT}}{p_{t,r}^{L}} \qquad \textbf{(75)}$$
$$s.t. \quad \tilde{\Xi}_{t,r}^{L} \cdot p_{t,r}^{L} = 0$$

Time $p_{g,t,r}^{t}$

$$\tilde{\Xi}_{g,t,r}^{t} = n_{g,r} \cdot \mu - \Omega_{g,t,r}^{L} - \bar{t}_{g,t,r}^{leis} \cdot \Omega_{g,t,r}^{z} \cdot \left(\frac{p_{g,t,r}^{z} / \bar{p}_{g,t,r}^{z}}{p_{g,t,r}^{t} / \bar{p}_{g,t,r}^{t}} \right)^{\text{¶}} \geq 0 \qquad \textbf{(76)}$$
$$s.t. \quad \tilde{\Xi}_{g,t,r}^{t} \cdot p_{g,t,r}^{t} = 0$$

4.4.3 Income Definitions

The income definition in calibrated share form is identical to the income definition described in section 3.6.3. However, following the logic of the calibrated share form, the activity level is redefined and multiplied with the associated benchmark value $\bar{u}_{g,r}$. Furthermore, this equation covers initial and terminal assets and utility provided to cohorts who live at the first time period or beyond the last time period, respectively. More precisely, cohorts born prior to the first model period are endowed with assets, such that the simulation starts with a positive capital stock. On the opposite, assets from cohorts living beyond the terminal period are exogenously taken away. However, cohorts living beyond the terminal period are endowed with post-terminal period utility. These latter steps are necessary to ensure that terminal cohorts do not consume their entire assets – and thus the capital stock of the economy – in the last model period.

$$\Psi_{g,r} = p^{u}_{g,r} \cdot \bar{u}_{g,r} \cdot \Omega^{u}_{g,r} = \sum_{t=g}^{g+ª_{g,r}-1} \left(p^{t}_{g,t,r} \cdot n_{g,r} \cdot \mu + ¤_{g,t,r} \right) + A^{0}_{g,r} + A^{term}_{g,r} - B^{term}_{g,r} \quad \textbf{(77)}$$

$$A^{0}_{g,r} = p^{K}_{0,r} \cdot \bar{\Theta}^{K}_{r} \cdot \frac{\bar{a}^{0}_{g,r}}{1+\bar{r}} + p^{T}_{0} \cdot \left(1 - \bar{\Theta}^{K}_{r}\right) \cdot \bar{a}^{0}_{g,r} \text{ for } t = 0$$

$$A^{0} = 0 \text{ for } t \neq 0$$

$$A^{term}_{g,r} = \sum_{a} p^{z,term}_{g,a,r} \cdot \bar{z}^{term}_{g,a,r} \cdot z^{term}_{g,a,r}$$

$$B^{term}_{g,r} = K^{term}_{r} \cdot p^{K,term}_{r} \cdot \bar{\Theta}^{K}_{r} \cdot \bar{a}^{term}_{g,r}$$

$A^{0}_{g,r}$: Initial endowment of capital stock and foreign tradable goods

$A^{term}_{g,r}$: Endowment of terminal cohorts with post-terminal period utilities

$B^{term}_{g,r}$: Closing of terminal capital stock

$z^{term}_{g,a,r}$: Auxiliary variable for post-terminal period utility

K^{term}_{r}: Auxiliary variable for terminal capital stock

4.4.4 Auxiliary Equations

Four auxiliary equations are implemented in the actual numerical model. These are the three closing conditions detailed in section 3.4 on the one hand and the budget balance of the PAYGO pension system on the other hand.

After the redefinitions due to the model transformation into Mathiesen's format and the subsequent transformation into the calibrated share form, the closing conditions from section 3.4 need to be updated accordingly. The first condition of constant welfare of terminal cohorts now reads as follows.

$$\Omega^{u}_{g,r} = \Omega^{u}_{g-1,r} \text{ with } g > T - ª_{g,r} \quad \textbf{(78)}$$

The second closing condition ensures that the post-terminal price of period utility declines with the steady-state interest rate. This equation is implemented directly as given by equation (20). The third closing condition imposes steady-state investment growth and is implemented follows.

$$\frac{\Omega^{I}_{t,r}}{\Omega^{I}_{t-1,r}} = 1 + \neg_{r} \tag{79}$$

$\neg_{r}$: Steady-state growth rate of size of youngest cohort

Finally, the budget balance of the pension system is implemented as an auxiliary equation. The PAYGO pension system is implemented in line with the concept detailed in section 3.3. For numerical simplicity, however, the explicit calculation of the benefit rate is omitted. However, the budget balance must ensure that pension benefits for cohorts are proportional to the respective cohort size, such that each individual receives the same benefits. For this reason, the variable $\tilde{¤}_{t,r}$ is introduced, which describes the pension benefits for a cohort size of one. By introducing this auxiliary variable, the number of additional single variables is reduced to match the number of single equations added by this budget balance. That is, one budget balance equation and one additional variable $\tilde{¤}_{t,r}$ is added per region and per time period.

$$\begin{gathered} \sum_{t-ª_{g,r}+1 \leq g \leq t-¸_{g,r}} \tilde{¤}_{t,r} \cdot n_{g,r} = £_{r} \cdot p^{L}_{t,r} \cdot \sum_{g} \Omega^{L}_{g,t,r} \cdot ®_{g,t,r} \\ \tilde{¤}_{t,r} = \frac{¤_{g,t,r}}{n_{g,r}} \end{gathered} \tag{80}$$

$\tilde{¤}_{t,r}$: Pension benefits for cohort size of *one*

4.4.5 Equations of 1-Sector-Model

As introduced in chapter 1, the impact of accounting for non-tradable goods on model results is derived by comparing the results of the 2-sector model detailed above with those of a conventional 1-sector model, which does not include a non-tradable goods sector. This section presents the modifications necessary to reduce the 2-sector model's equations from sections 4.4.1 through 4.4.4 to a consistent 1-sector model.

For full transparency, Figure 4-3 exhibits the structure of the 1-sector model in the layout already introduced for the 2-sector model in section 3.6. Several modifications become apparent when comparing Figure 4-3 with the 2-sector model structure in Figure 3-2. First, the activity of non-tradable goods production $\Omega^{NT}_{t,r}$ as well as the non-tradable good $c^{NT}_{g,t,r}$ are omitted. Now, the former activity $\Omega^{c}_{g,t,r}$, which aggregates non-tradable and tradable goods to a compound good consumption $c_{g,t,r}$, would only have tradable goods as a single input and the consumption good as single output. Thus, this aggregation activity is no longer required and may be removed as well. Accordingly, the compound consumption good $c_{g,t,r}$ is also no longer required and tradable goods directly enter the period utility aggregation activity $\Omega^{z}_{g,t,r}$.

In terms of model equations, one zero-profit-condition for each removed activity and one market-clearing-condition for each good are taken out from equations defined in section 4.4.1 through 4.4.4. Consequently, the following equations are not part of the 1-sector model:

- Zero-profit-condition for consumption good aggregation $\Omega^{c}_{g,t,r}$
- Market-clearing-condition for non-tradable goods p^{NT}_{t}
- Market-clearing-condition for compound consumption good $p^{c}_{g,t,r}$
- Zero-profit-condition for production sector $\Omega^{NT}_{t,r}$

Additionally, several of the remaining equations from sections 4.4.1 through 4.4.4 need to be revised for the 1-sector model. First, the zero-profit-condition of period utility aggregation now has tradable goods instead of the compound consumption good as input:

$$\tilde{\Pi}^{z}_{g,t,r} = \left(\frac{p^{z}_{g,t,r}}{\bar{p}^{z}_{g,t,r}}\right)^{1-\P} - \sim \cdot \left(\frac{p^{T}_{g,t,r}}{\bar{p}_{t}}\right)^{1-\P} - (1-\sim)\cdot\left(\frac{p^{t}_{g,t,r}}{\bar{p}^{t}_{g,t,r}}\right)^{1-\P} \leq 0 \qquad \textbf{(81)}$$

$$s.t. \quad \tilde{\Pi}^{z}_{g,t,r} \cdot \Omega^{z}_{g,t,r} = 0$$

Next, the market-clearing-condition of tradable goods also reflects the direct use of tradable goods in the period utility $\Omega^{z}_{g,t,r}$:

$$
\begin{aligned}
\tilde{\Xi}_t^T &= \sum_r \Omega_{t,r}^T \cdot \overline{Y}_r^T + \sum_r \frac{B_{t,r}}{p_t^T} - \sum_r \Omega_{t,r}^I \cdot \overline{I}_r \\
&- \sum_r \sum_g \Omega_{g,t,r}^z \cdot \overline{c}_{g,t,r} \cdot \left(\frac{p_{g,t,r}^z \big/ \overline{p}_{g,t,r}^z}{p_{g,t,r}^T \big/ \overline{p}_t} \right)^{\P} \geq 0 \\
&s.t. \quad \tilde{\Xi}_t^T \cdot p_t^T = 0
\end{aligned}
\tag{82}
$$

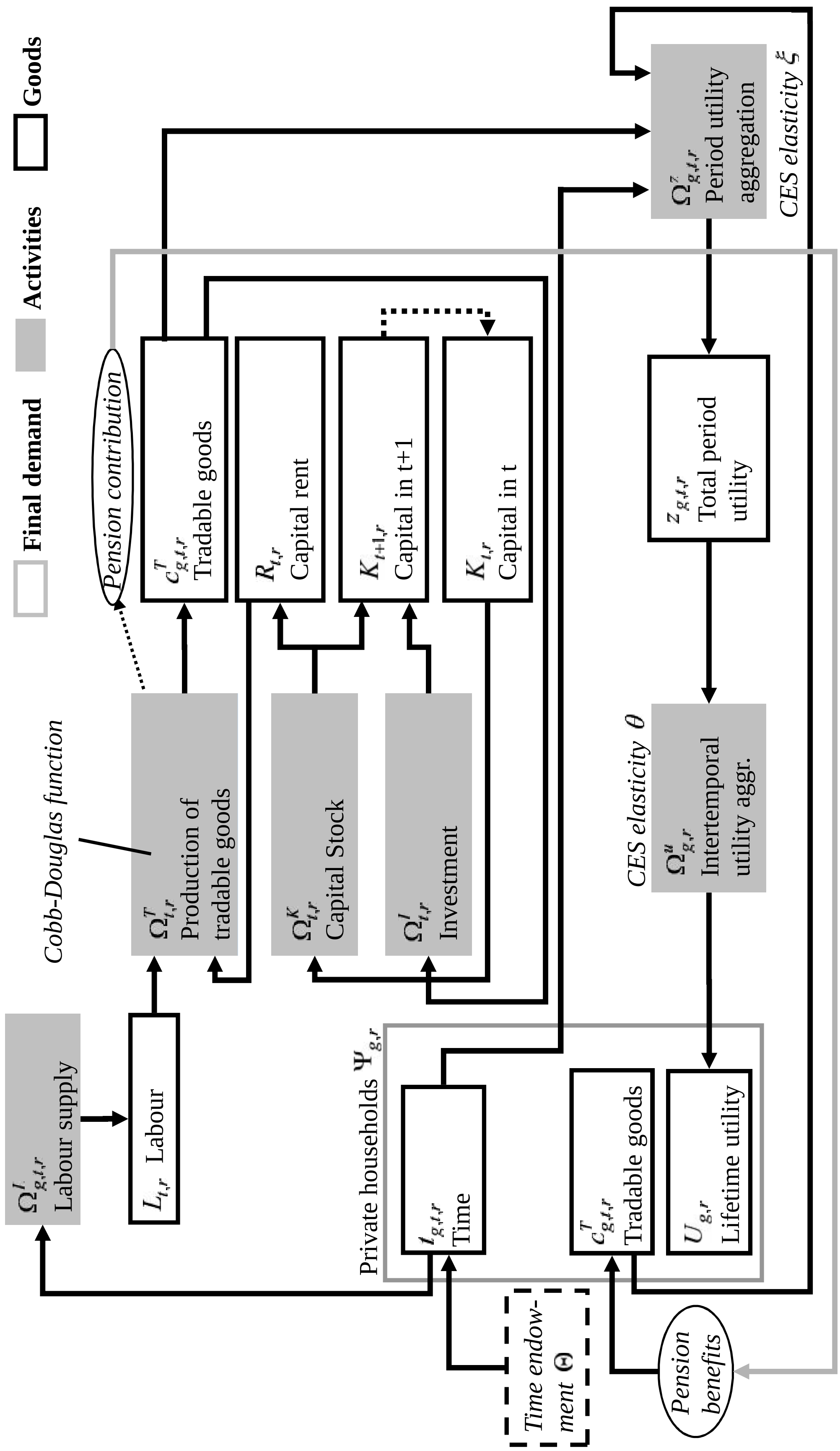
Finally, the market-clearing-conditions of rented capital and labor no longer incorporate the non-tradable production.

$$
\begin{aligned}
\tilde{\Xi}_{t,r}^R &= \Omega_{t,r}^K \cdot \overline{R}_r - \Omega_{t,r}^T \cdot \overline{R}_r \cdot \frac{p_t^T}{p_{t,r}^R} \\
&s.t. \quad \tilde{\Xi}_{t,r}^R \cdot p_{t,r}^R = 0
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
\tilde{\Xi}_{t,r}^L &= \sum_g \Omega_{g,t,r}^L \cdot \circledR_{g,t,r} - \Omega_{t,r}^T \cdot \overline{L}_r \cdot \frac{p_t^T}{p_{t,r}^L} \\
&s.t. \quad \tilde{\Xi}_{t,r}^L \cdot p_{t,r}^L = 0
\end{aligned}
\tag{84}
$$

After these modifications to the model equations from sections 4.4.1 through 4.4.4, the 1-sector model structure in calibrated share form is derived.

Figure 4-3: Model structure of 1-sector model



4.5 Concept of Social Accounting Matrices

The calibrated share form of model equations introduced in section 4.4 contains a large number of benchmark quantities and prices. These benchmark values are distinguished from their associated variables by an upper bar (e.g. $\overline{p}^{z}_{g,t,r}$). The choice of these benchmark values determines the calibration point and thus crucially impacts model behavior. Therefore, the goal is to find benchmark values that reflect reality as closely as possible. Accordingly, actual aggregated economic data based on National Accounts form the backbone of the derivation of benchmark values.

The most convenient data format for this purpose is the Social Accounting Matrix (SAM), which, in its core, is a matrix representation of National Accounts[59]. A SAM is a static representation of an economy's flows of value and is typically based on a period of one year. Even though SAMs are usually constructed based on National Account data for single countries, it is possible to aggregate SAMs across multiple countries in order to form SAMs for world regions as required for the model developed in this thesis. Thus, the model is calibrated by a total of eight SAMs, each covering one of the model's world regions.

Figure 4-4: Conceptual layout of a Social Accounting Matrix for an open economy[60]

	Activities	**Commodities**	**Factors**	**Households**	**Government**	**Savings and Investment**	**Rest of World**	**Total**
Activities		Domestic supply						Activity income
Commodities	Intermediate demand			Private consumption	Government spending	Investment	Exports	Total demand
Factors	Value added							Total factor income
Households			Factor payments to households		Social transfers		Net foreign remittances	Total household income
Government		Sales taxes and import tariffs	Direct taxes				Net foreign grants and loans	Government income
Savings and Investment				Private savings	Fiscal surplus		Current account balance	Total savings
Rest of World		Imports						Foreign exchange outflow
Total	Gross domestic product	Total supply	Total factor spending	Household usage of funds	Government usage of funds	Total investment	Foreign exchange inflow	

59 See Rasmussen and Rutherford (2004) for calibration using SAMs.
60 See Breisinger et al. (2010)

Figure 4-4 exhibits the conceptual layout of a SAM. Rows in this matrix represent income of the respective sectors and columns represent expenditures. A SAM is consistent if the column sums for each sector are equal to the respective row sum, or, in other words income equals expenditure for each sector. SAMs fully cover transactions in a given economy and provide transparency on both the source and usage of factors and commodities[60].

If required, each of the sectors in Figure 4-4 can be broken down into multiple subsectors to achieve higher disaggregation[60]. The present model, for instance, comprises tradable and non-tradable goods and therefore requires the commodity sector to be disaggregated into these two types of goods. Similarly, the model contains two factors, labor and capital, and thus requires this sector to be disaggregated accordingly. At the same time, the model allows several simplifications, since it does not cover intermediate supply and demand of production sectors and ignores small accounts such as 'net foreign remittances', for instance. The SAM used to calibrate each region in the model therefore has a conceptual layout as exhibited in Figure 4-5. Relevant for the calibration procedure are all values except imports, exports, private savings, and the current account balance. Due to the condition of equal row and column sums, capital income $\overline{p}_r^R \cdot \overline{R}_r$ must be equal to the sum of capital value added in the two production sectors. Similarly, net labor income $\overline{p}_r^L \cdot \overline{L}_r$ plus pension contribution must be equal to the labor value added in tradable and non-tradable production. Accordingly, only few parameters in this SAM layout are actually required in order to derive the residual figures.

All eight regional SAMs in this model are in units of US-Dollar billions. As a result, the benchmark prices shown in Figure 4-5 may be set to a value of *one* and the benchmark quantities in Figure 4-5 are equal to the associated SAM-values.

In addition to the benchmark parameters shown in Figure 4-5, the parameters $\overline{p}_r^T \cdot \overline{Y}_r^T$ and $\overline{p}_r^{NT} \cdot \overline{Y}_r^{NT}$ can also be directly inferred from the SAMs. Again setting the respective benchmark prices to *one*, the benchmark quantities of tradable and non-tradable outputs are equal to the sums of labor and capital value adds for tradable and non-tradable production, respectively:

$$\bar{p}^T \bar{Y}_r^T = \bar{p}_r^L \cdot \bar{L}_r^T + \bar{p}_r^R \cdot \bar{R}_r^T$$
$$\bar{p}^{NT} \bar{Y}_r^{NT} = \bar{p}_r^L \cdot \bar{L}_r^{NT} + \bar{p}_r^R \cdot \bar{R}_r^{NT} \tag{85}$$

Figure 4-5: Layout of SAM as used to calibrate each model region

	Tradable goods production	**Non-tradable goods production**	**Capital**	**Labor**	**Tax**	**Consumer**	**Savings and Investment**	**Pension system**	**Rest of World**
Trad. goods prod.						Tradable consumption $\bar{p}_r^T \cdot \bar{C}_r^T$	Investment $\bar{p}_r^T \cdot \bar{I}_r$		Exports
Non-trad goods prod.						Non-trad. consumption $\bar{p}_r^{NT} \cdot \bar{C}_r^{NT}$			
Capital	Capital return in trad. prod. $\bar{p}_r^R \cdot \bar{R}_r^T$	Cap. return in non-trad. prod. $\bar{p}_r^R \cdot \bar{R}_r^{NT}$							
Labor	Labor in trad. production $\bar{p}_r^L \cdot \bar{L}_r^T$	Labor in non-trad. production $\bar{p}_r^L \cdot \bar{L}_r^{NT}$							
Tax				Pension contribution $\bar{\Gamma}_r$					
Consumer			Capital income $\bar{p}_r^R \cdot \bar{R}_r$	Net labor income $\bar{p}_r^L \cdot \bar{L}_r$				Pension benefits $\bar{\mathrm{B}}_r$	
Savings and Investment						Private savings			Net Imports $-\overline{NX}_r$
Pension system					Pension contribution $\bar{\Gamma}_r$				
Rest of World	Imports								

However, a number of benchmark parameters appearing in the calibrated share form equations from section 4.4 cannot be directly inferred from the SAMs. First, the level of benchmark capital stock $\bar{K}_r$ is not directly defined in the SAMs, since the capital stock by its nature is not a value flow and is thus not explicitly included in the SAM concept. This issue may be resolved by exploiting the steady-state assumption introduced above. With this assumption, the consistent steady-state capital stock for given values from the SAM can be calculated as follows.

$$\bar{K}_r = \frac{\bar{R}_r}{\bar{r} + \text{¢}_r} \tag{86}$$

Second, several cohort-specific benchmark values, such as $\bar{c}_{g,t,r}$ for instance, are not fully determined by the SAMs, which only cover aggregated values. Additionally, the SAMs only cover the tangible flow of values, that is, they do not cover leisure time consumption or even welfare figures. The further assumptions and subsequent calculations that are required to calibrate these missing benchmark values are presented in the following section.

4.6 Auxiliary Steady-State Equilibrium Models

As laid out in the previous section, the SAMs do not contain all of the benchmark values required to fully parameterize the model equations in calibrated share form. In order to determine a complete set of benchmark values consistent with both the SAMs as well as further exogenous parameters, an auxiliary steady-state equilibrium model is constructed as suggested by Rasmussen and Rutherford (2004). As a second purpose, this model not only finds consistent benchmark values, but also consistent values for time discount rate $\circ_r$ and time endowment μ_r.

This first auxiliary model based on Rasmussen and Rutherford (2004) is referred to as ‘primary auxiliary model’ in the following. However, the benchmark values derived from this model are only suitable for one given value of cohort lifespan. Since cohort lifespan in the model of Rasmussen and Rutherford (2004) is constant, they can retrieve all required benchmark values from this single auxiliary model. In the model presented in this thesis, however, a series of secondary auxiliary models is introduced to calculate benchmark values for all cohort lifespan values appearing in the main OLG-model. In summary, the two auxiliary models serve the following purposes.

1) Primary Auxiliary Model

- Find unknown benchmark values for baseline lifespan in each region. Unknown benchmark values are those which cannot be directly extracted from the SAM, such as benchmark period consumption per cohort or benchmark labor supply per cohort.

- Find unknown values of utility discount rate $\circ_r$ and period time endowment μ_r. As proposed by Rasmussen and Rutherford (2004), these values are identified by the auxiliary model such that aggregate benchmark consumption equals of all cohorts in the auxiliary model equals aggregate consumption in the SAM and aggregate asset positions of all cohorts in the auxiliary model equal aggregate assets as implied by the SAM.

2) Secondary Auxiliary Model

- Find unknown benchmark values for other lifespans occurring in the main model using parameter values for $°_r$ and μ_r found by the primary auxiliary model as exogenous input. Thus, if a given region shows lifespan values of 8, 9, 10, 11, and 12 periods during the full horizon of the main model, the benchmark values have to be calculated for each of these lifespan values.

4.6.1 Primary Auxiliary Model

This model covers the cohorts alive in the reference period and their choices on consumption, endogenous labor supply, and savings. However, it is not necessary to cover the production side of the economy. In the following, the equations of the primary auxiliary model are introduced, which are an extension of the calibration model presented by Rasmussen and Rutherford (2004).

Generations in this model are indexed by their age *a* and the time index introduced in the dynamic equations in previous sections vanishes due to the steady-state assumption. All variables in this model are highlighted by carets to distinguish them from the benchmark values, marked with an upper bar, as well as the original variables in the main model.

The first model equation defines period utility $\hat{z}_{a,r}$ according to the CES function introduced in equation (2):

$$\hat{z}_{a,r} = \left({}_r \cdot \hat{c}_{a,r}^{\frac{¶-1}{¶}} + \left(1 - {}_r\right) \cdot \left(\hat{t}_{a,r}^{leis}\right)^{\frac{¶-1}{¶}} \right)^{\frac{¶}{¶-1}} \qquad \textbf{(87)}$$

In the next step, the first-order-condition of cohorts' choice on goods consumption over the lifecycle is derived by setting the derivative of lifetime utility with respect to consumption equal to a Lagrange-multiplier $\hat{ª}_r$ multiplied by the present value term. As introduced above, the model adjusts the time discount rate $°_r$ to be consistent with SAM-values and other exogenous inputs. Therefore the parameter $°_r$ becomes a variable $\hat{°}_r$ in this auxiliary model.

$$\hat{a}_r \cdot \frac{1}{(1+\bar{r})^a} = \frac{1}{(1+ {}^{\circ}{}_r)^a} \cdot \hat{z}_{a,r}^{\frac{1}{¶}-\frac{1}{-}} \cdot \quad {}_r \cdot \hat{c}_{a,r}^{\;-\frac{1}{¶}} \qquad \textbf{(88)}$$

The Lagrange-multiplier $\hat{a}_r$ can be interpreted as the shadow price of the lifetime budget constraint.

Similarly, the first-order-condition of cohorts' choice on leisure consumption can be written as follows.

$$\Gamma_{a,r} = \frac{1}{(1+ {}^{\circ}{}_r)^a} \cdot \hat{z}_{a,r}^{\frac{1}{¶}-\frac{1}{-}} \cdot (1- \quad {}_r) \cdot \hat{t}_{a,r}^{leis\;-\frac{1}{¶}} \qquad \textbf{(89)}$$

Here, the Lagrange-multiplier $\Gamma_{a,r}$ can be interpreted as the price of time endowment per period.

The third first-order-condition optimizes labor time per cohort. This condition is formulated as an inequality due to the possibility of a corner-solution of labor time. That is, if the price of leisure time $\Gamma_{a,r}$ is greater than the labor wage rate, the associated cohort will not work at all in the respective period(s) and consume its total period time endowment as leisure time. For this reason, labor time is modeled as a complementary variable to this inequality. This modeling approach thus combines endogenous labor with endogenous retirement. In other words, agents are forced to retire upon reaching pension age, however, they are free to retire earlier by offering zero labor.

$$\Gamma_{a,r} \geq \hat{a}_r \frac{1}{(1+\bar{r})^a} \cdot ®_{a,r} \qquad \textbf{(90)}$$

The consumer budget constraint of the auxiliary model relates the present value of lifetime consumption expenditure to the present value of lifetime labor and pension income.

$$\sum_a \frac{\hat{c}_{a,r}}{(1+\bar{r}_r)^a} = \sum_a \left[\frac{1}{(1+\bar{r}_r)^a} ®_{a,r} \cdot \hat{t}_{a,r}^{lab} + \sim_{a,r}^{retire} \cdot (1+ {}^{3}{}_r)^a \cdot \overline{\mathrm{B}}_r \right] \qquad \textbf{(91)}$$

$\sim_{a,r}^{retire}$: Share of retired cohort *a* in total retired population

Benchmark aggregate period pension income $\overline{\mathrm{B}}_r$ is allocated to the different retired cohorts such that each cohort receives pension benefits proportional to the respective cohort size. The share of a given retired cohort *a* in the total retired population can be calculated as:

$$\sim_{a,r}^{retire} = \frac{1/(1+{}^{3})^{a}}{\sum\limits_{a' \geq \bar{\text{¸}}_r+1} 1/(1+{}^{3})^{a'}} \text{ for } a \geq \bar{\text{¸}}_r+1 \text{ (pension age)}$$

(92)

$$\sim_{a,r}^{retire} = 0 \text{ for } a < \bar{\text{¸}}_r+1 \text{ (working age)}$$

The time endowment per period is equal to the sum of labor and leisure time. Similar to the time discount rate, the auxiliary model finds the consistent value for time endowment μ_r, which is thus modeled as an endogenous variable $\hat{\mu}_r$ in this primary auxiliary model.

$$\hat{\mu}_r = \hat{t}_{a,r}^{lab} + \hat{t}_{a,r}^{leis} \quad \textbf{(93)}$$

For consistency with the SAMs, the sum of consumption during each period across all cohorts must equal the SAM benchmark consumption. Since $\hat{c}_{a,r}$ describes the consumption over the lifecycle for a cohort of size *one*, it needs to be adjusted to reflect the respective cohort size of the cohort with age *a* in steady state.

$$\overline{C}_r = \sum_a \frac{\hat{c}_{a,r}}{(1+{}^{3}{}_r)^a} \quad \textbf{(94)}$$

$\overline{C}_r$: Aggregate benchmark consumption from SAMs

The age-dependent asset position in present value terms can be determined as the sum over previous labor and pension earnings minus previous consumption spending:

$$\hat{a}_{a,r}^{LC} = \sum_{a' \leq a}\left[\frac{1}{(1-\bar{r})^{a'}} \cdot \left(\text{®}_{a',r} \cdot \hat{t}_{a',r}^{lab} + \sim_{a',r}^{retire} \cdot (1+{}^{3}{}_r)^{a'} \cdot \overline{\mathrm{B}}_r - \hat{c}_{a',r}\right)\right] \quad \textbf{(95)}$$

$\hat{a}_{a,r}^{LC}$: Present value of assets over the lifecycle

These lifecycle asset positions can be transformed into cross-section asset positions for the different age cohorts:

$$\hat{a}_{a,r} = \hat{a}_{a,r}^{LC} \cdot \left(\frac{1+\bar{r}}{1+{}^{3}}\right)^a \quad \textbf{(96)}$$

$\hat{a}_{a,r}$: Value of assets held by age

Similar to the aggregate consumption, the cross-sectional sum of assets over all cohorts is equal to the aggregate assets held by the economy as defined by the respective SAM. Since government deficits are not in scope of the model, the aggregate asset position is determined by the regional capital stock and the net foreign asset position. Due to the steady-state assumption, a net foreign asset position that is consistent with a permanent current account balance as given by the net export value from the SAMs is implied. Thus, according to Rasmussen and Rutherford (2004), in case of negative benchmark net exports, the consistent net foreign asset position is positive and large enough to permanently finance the benchmark level of net exports.

$$(1+\bar{r})\cdot \bar{K}_r - \overline{NX}_r \cdot \frac{1+\bar{r}}{\bar{r} - {}^{3}{}_{r}} = \sum_a \hat{a}_{a,r} \qquad \textbf{(97)}$$

$\overline{NX}_r$: Net exports from SAMs

The auxiliary model so far does not differentiate between tradable and non-tradable goods. However, the main model in calibrated share form requires benchmark values for cohorts' tradable and non-tradable consumption.

The required preference share profile for non-tradable vs. tradable consumption $\text{¥}_{a,r}$ is determined exogenously. At this point, the assumption of increasing preference for non-tradable goods over the lifecycle is introduced. This assumption is based on Lührmann (2005), who assesses the impact of population ageing on demand of goods and services.

The according benchmark non-tradable consumption is calculated as follows.

$$\hat{c}^{NT}_{a,r} = \hat{c}_{a,r} \cdot \left(1-\text{¥}_{a,r}\right)\cdot \hat{s}_r \qquad \textbf{(98)}$$

$\hat{c}^{NT}_{a,r}$: Consumption of non-tradable goods per cohort

$\text{¥}_{a,r}$: Preference share for tradable goods consumption over lifecycle

$\hat{s}_r$: Correction factor

The preference share profile for non-tradable goods $\left(1-\text{¥}_{a,r}\right)$ is defined as a linear profile with an average equal to the aggregate share of non-tradable consumption and a spread between age *zero* and highest age of ¢^{NT}. Modeling the increase of non-tradable goods preference as a linear function is also suggested by Figure 1 in Lührmann (2005), which shows a rather linear change of different goods categories over lifecycle.

The exogenous parameter ¢^{NT} thus defines the difference in preference share for non-tradable consumption between youngest and oldest age in percentage points divided by 100.

$$\left(1-\text{¥}_{a,r}\right)=\max\left[0,\ \min\left[1,\ \frac{\overline{C}_r^{NT}}{\overline{C}_r}+\frac{\text{¢}^{NT}}{2}\cdot\left(\frac{2a}{\overline{\text{a}}_r-1}-1\right)\right]\right] \quad \textbf{(99)}$$

¢^{NT} : Exogenous parameter

$\overline{\text{a}}_r$: Number of lifetime periods of benchmark generation

The correction factor $\hat{\text{s}}_r$ in equation (98) adjusts small differences between the aggregate value of benchmark non-tradable production and the sum of consumption times the NT-preference profile $\hat{c}_{a,r}\cdot\left(1-\text{¥}_{a,r}\right)$. These deviations are due to second order errors resulting from a non-constant lifetime consumption profile, since the average of $\text{²}_{a,r}$ over all periods is fixed at the aggregate non-tradable consumption share. This correction factor is chosen such that the following equation holds.

$$\overline{C}_r^{NT}=\sum_a\frac{\hat{c}_{a,r}^{NT}}{\left(1+\text{³}\right)^a} \quad \textbf{(100)}$$

$\overline{C}_r^{NT}$: Aggregate benchmark consumption of non-tradable goods from SAMs

The tradable cohort consumption over the lifecycle $\hat{c}_{a,r}^{T}$ equals the difference between the already determined variables $\hat{c}_{a,r}$ and $\hat{c}_{a,r}^{NT}$.

$$\hat{c}_{a,r}^{T}=\hat{c}_{a,r}-\hat{c}_{a,r}^{NT} \quad \textbf{(101)}$$

Finally, the price of period utility $\hat{p}_{a,r}^{z}$ has not been calculated so far, but is required for the calibration of the main model:

$$\hat{p}_{a,r}^{z}=\left[\ {}_r^{\text{¶}}\cdot\left[\frac{1}{\left(1+\overline{r}\right)^a}\right]^{1-\text{¶}}+\left(1-\ {}_r\right)^{\text{¶}}\cdot\left(\frac{\text{r}_{a,r}}{\text{a}_r}\right)^{1-\text{¶}}\right]^{\frac{1}{1-\text{¶}}} \quad \textbf{(102)}$$

Now the auxiliary model comprises a total of 16 equations for an equal amount of variables and the model may be solved as a Mixed Complementary Problem in GAMS.

4.6.2 Secondary Auxiliary Models

The primary auxiliary calibration model yields consistent values for time discount rate and period time endowment as well as a number of benchmark values to parameterize the calibrated share form equations. However, due to the fact that the main OLG-model features exogenous, time-variant lifespans of cohorts, this first auxiliary model is not sufficient to fully calibrate the model. If the lifespan of a cohort changes, this will have impact on the reference consumption profile, making the benchmark values derived for the lifespan used in the first auxiliary model above inappropriate for cohorts with different lifespans.

The calibration approach presented by Rasmussen and Rutherford (2004) is thus extended by a series of secondary auxiliary models. An equation system similar to the first auxiliary model introduced above is solved for each value of lifespan $a_{g,t,r}$ that appears in the main model. However, the values for utility discount rate $^{\circ}{}_{r}$ and time endowment μ_r are fixed for these secondary auxiliary models, since these are assumed to be constant regardless of the lifespan. As a result, equations (94) and (97) from the first auxiliary model are omitted, reflecting the fact that the steady-state aggregated consumption and assets will be different if the lifespan of cohorts changes.

Another difference between the primary and secondary auxiliary systems is related with the pension system. Aggregate pension benefits in the first system are simply equal to the benchmark value from the SAMs. However, in the case of different lifespans, aggregate pension income as well as benefits may deviate from the SAM-values. For this reason, the pension system budget balance needs to be endogenized.

Three additional equations are added for the secondary auxiliary models in order to appropriately reflect the pension system. To simplify algebra, the first additional equation defines the average cohort income during working years, which forms the basis for the cohort pension benefit calculation:

$$\hat{j}_r^{avg} = \frac{\sum_{a \le \text{¸}_{g,r}-1} \text{®}_{a,r} \cdot \hat{p}_{a,r}^{lab}}{\text{¸}_{g,r}} \qquad \textbf{(103)}$$

$\hat{j}_r^{avg}$: Average cohort income during working years

Using this definition, the second additional equation determines the cohort pension benefits.

$$\hat{\text{¤}}_{a,r} = \hat{j}_r^{avg} \cdot \hat{\cdot}_r \qquad \textbf{(104)}$$

$\hat{\text{¤}}_{a,r}$: Cohort pension benefits $\hat{\cdot}_r$: Pension benefit rate

Finally the third additional equation ensures a balanced PAYGO pension system:

$$\sum_a \frac{\text{£}_r \cdot \text{®}_{a,r} \cdot \hat{t}_{a,r}^{lab}}{(1+\text{³}_r)^a} = \sum_{a \ge \text{¸}_{g,r}} \frac{\hat{\text{¤}}_{a,r}}{(1+\text{³}_r)^a} \qquad \textbf{(105)}$$

Also, the term $\sim_{a,r}^{retire} \cdot (1+\text{³}_r)^a \cdot \overline{\text{B}}$ in the first auxiliary system is replaced by the endogenized cohort benefits $\hat{\text{¤}}_{a,r}$. The secondary auxiliary model can now be solved once for each value of lifespan $\text{ª}_{g,t,r}$ appearing in the main model.

4.6.3 Transferring Benchmark Values from Auxiliary Models to Main Model Parameters

Upon transferring benchmark values from the auxiliary models developed in sections 4.6.1 and 4.6.2 to the actual OLG-main-model, some adjustments are necessary due to the fact that the auxiliary models are steady-state models and the actual OLG-main-model is dynamic. First, these values need to be multiplied by the respective cohort size $n_{g,r}$ of the main model. Second, a conditional loop in the simulation code ensures that for each index combination of *g*, *t*, and *r*, the correct benchmark values from the auxiliary systems are employed.

On top of that, the benchmark values for lifetime utility $\overline{u}_{g,r}$ need to be derived from the results of auxiliary models. The following formula is used to calculate these benchmark values.

$$\overline{u}_{g,r} = \sum_{t} \overline{p}^{z}_{g,t,r} \cdot \overline{z}_{g,t,r} \qquad \textbf{(106)}$$

Now, all benchmark parameters of the calibrated share form are determined, either directly from the SAMs or from the auxiliary equation systems.

4.7 Numerical Assembly of Social Accounting Matrices

The previous section laid out the approach to calibrate the model based on Social Accounting Matrices. However, so far the calibration was purely based on abstract parameters, which need to be specified in the next step. Accordingly, this section deals with the derivation of SAMs for the eight world regions in the format depicted in Figure 4-5.

4.7.1 Extraction of Social Accounting Matrices from GTAP Database

The Social Accounting Matrix in a layout as exhibited in Figure 4-5 is not readily available from general national statistic offices. However, the Global Trade Analysis Project (GTAP), which was initiated by Purdue University in 1993, collects SAMs for almost all countries in the GTAP database. The version used to calibrate the model is the last version available and provides SAM data for 129 countries for the year 2007. This database is flexible and allows to define sets of countries to aggregate SAMs on regional level. Additionally, the database comprises a total of 57 industry sectors and thus allows a high level of sectoral disaggregation. While this high level of disaggregation is not required in the present model, this feature allows to distinguish a tradable and a non-tradable sector. This disaggregation into these two sectors is achieved by defining upfront which of the 57 is assumed to be tradable or non-tradable. The disaggregation is based on the assumption that services, utilities, infrastructure including dwelling, public administration, and finally rapidly perishing goods (e.g. dairy products) and living animals are not tradable.

Subsequently, the ‘tradable’ and ‘non-tradable’ sectors are aggregated to form the two sectors required for the model.

Table 4-1: Allocation of GTAP sectors to tradable and non-tradable goods production

Tradable goods sectors

- pdr- Paddy rice
- wht - Wheat
- gro -Cereal grains nec
- v_f - Vegetables fruit nuts
- osd - Oil seeds
- c_b - Sugar cane sugar beet
- pfb - Plant-based fibers
- ocr - Crops nec
- oap - Mimal products nec
- wol - Wool silk-worm cocoons
- frs - Forestry
- fsh - Fishing
- coa - Coal
- oil - Oil
- gas - Gas
- omn - Minerals nec
- vol - Vegetable oils and fats
- pcr - Processed rice
- sgr - Sugar
- ofd - Food products nec

Tradable goods sectors (continued)

- b_t - Beverages and tobacco products
- tex - Textiles
- wap - Wearing apparel
- lea - Leather products
- lum - Wood products
- ppp - Paper products publishing
- p_c - Petroleum coal products
- crp - Chemical rubber plastic prods
- nmm - Mineral products nec
- i_s - Ferrous metals
- nfn - Metals nec
- fmp - Metal products
- mvh - Motor vehicles and parts
- otn - Transport equipment nec
- ele - Electronic equipment
- ome - Machinery and equipment nec
- omf - manufactures nec
- wtp - Sea transport
- atp - Air transport

Non-tradable goods sectors

- Ctl - Cattle sheep goats horses
- rmk - Raw milk
- cmt - Meat: cattle sheep goats horse
- omt - Meat products nec
- mil - Dairy products
- ely - Electricity
- gdt - Gas manufacture distribution
- wtr - Water
- cns - Construction
- trd -Trade
- otp - Transport nec
- cmn - Communication
- ofi - Financial services nec
- ins - Insurance
- obs - Business services nec
- ros - Recreation and other services
- osg - PubAdmin Defence Health Educat
- dwe - Dwellings

Using this allocation to aggregate the sectors, the GTAP database provides the following results for USA & Canada (Table 4-2). As a first plausibility check, the sum of all value adds to the production sectors for tradable and non-tradbale goods (OUTTR/OUTNT), except intermediate transfers and except imports should be equal to GDP. The sum of these positions in the SAM below is 15.20 trillion USD, which comes very close to the 2007 GDP of USA and Canada of 15.36 trillion USD[61]. Small deviations between SAMs and actual reported figures may occur due to the methodology applied by the GTAP project to assemble the SAMs. For each SAM, data from different sources are used as input and the balancing of the SAM is performed by an algorithm conceptually similar to a least-squares-approach[62]. Appendix A.1 exhibits raw SAMs for other model regions.

Table 4-2: Raw version of SAM from GTAP database for USA & Canada (2007)

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOVT	ROW	OTHER	REGHOUS	Total
OUTTR	11.207.429	2.176.973				1.946.522	1.155.210	152	1.396.925			17.883.211
OUTNT	1.664.060	25.566.280				8.506.217	1.827.250	2.536.858	384.657			40.485.321
CAP	815.342	3.163.014										3.978.356
LAB	1.530.459	7.332.194										8.862.652
TAX	404.377	1.825.522	328.094	1.923.611		304.806	32.632			6.324		4.825.366
CON											10.757.545	10.757.545
INV		70.718	1.403.483						753.084		787.807	3.015.092
GOVT											2.537.010	2.537.010
ROW	2.184.045	350.620										2.534.665
OTHER	77.499											77.499
REGHOUS			2.246.779	6.939.042	4.825.366					71.175		14.082.362
Total	17.883.211	40.485.321	3.978.356	8.862.652	4.825.366	10.757.545	3.015.092	2.537.010	2.534.665	77.499	14.082.361	

All values in millions USD

[61] Source: Worldbank
[62] See Narayanan et al. (2012)

OUTTR: Tradable output	**CON:** Private consumer
OUTNT: Non-tradable output	**GOVT:** Government
CAP: Capital	**ROW:** Rest of World
LAB: Labor	**OTHER:** Other items (mainly natural resources)
TAX: Taxes, tariffs and other contributions	**REGHOUS: '**Regional household'[63]

Furthermore, the data from this sample SAM supports the allocation of sectors in tradable and non-tradable goods production as defined in Table 4-1, because the relative size of imports and exports in the 'non-tradable' sector is very small compared to the 'tradable' sector. More specifically, in the 'non-tradable' sector as defined above, the import value according to the SAM is 350 billion USD, which constitutes a share of 2.8% of the non-tradable GDP of 12.4 trillion USD. On the other hand, the imports in the 'tradable' goods sector of 2.2 trillion USD represent a major share of 79% of the tradable GDP of 2.8 billion USD. The proportions for the export side are similar; here the non-tradable share of exports in non-tradable GDP is 3.1% and the tradable share of exports in tradable GDP is 51%.

Now that the data justifies the sector definition introduced above, it may be inferred from the SAM that in USA and Canada only ~19% of GDP are contributed by sectors that produce tradable goods and the vast majority of ~81% of GDP stems from non-tradable goods sectors. These shares follow from the division of sectoral GDPs (tradable: 2.9 trillion USD; non-tradable: 12.4 trillion USD) by the total GDP of 15.2 trillion USD.

The GDP of the tradable sector is given by the sum of the fields CAP/OUTTR[64], LAB/OUTTR, TAX/OUTTR, and OTHER/OUTTR. Similarly, the GDP of the non-tradable sector may be calculated as the sum of the fields CAP/OUTNT, LAB/OUTNT, TAX/OUTNT, and INV/OUTNT. The fields ROW/OUTTR and ROW/OUTNT reflect imports to the respective country and are thus not part of the GDP.

In contrast to the general SAM layout introduced in Figure 4-4, the fields OUTTR-OUTTR and OUTNT-OUTNT on the main diagonal are non-blank. This is due to the aggregation procedure from 57 industry sectors to a tradable and non-tradable sector. The interior value streams OUTTR to OUTTR and OUTNT to OUTNT stem from intermediate outputs between different tradable sectors or non-tradable sectors, respectively. If for instance intermediate products from sectors textiles (tex) and leather (lea) feed into the wearing apparel sector (wap), this transaction is summed up in the OUTTR-OUTTR field of the SAM.

[63] The GTAP SAM format features a 'Regional Household' that collects all income (private income and taxes) and pays all consumption (private and governmental) and investment.

[64] The first field refers to the row and the second field to the column of the SAM

4.7.2 Adjustments of SAMs for Consistency with Model Assumptions

Unfortunately, the raw SAM from the GTAP database still deviates in several aspects from the SAM layout required to calibrate the model (Figure 4-5). The adjustments necessary to transform the SAMs accordingly are detailed in the following.

Intermediate Output

As stated above, intermediate outputs between tradable and non-tradable sectors are not in scope of the model and are thus not required in the SAMs. The fields of intermediate output[65] are simply set to *zero* to be consistent with the assumption that all output by each of the two sectors is 'final' output for consumption or investment goods.

Direct and Indirect Taxes and Government Spending

The model does not include taxes and government, but only a stylized PAYGO pension system. For this reason, taxes and government spending needs to be netted out from the raw SAM. This step is done by setting both taxes and government spending to zero.

Assumption of Tradable Investment Goods

The model features the assumption that all investment goods are tradable. As a result, investment from non-tradable goods is not consistent with the model assumptions. However, in reality a certain share of investment goods stem from non-tradable sectors, leading to a non-zero value in the field OUTNT-INV in Table 4-2. To deal with this issue, the value in field OUTNT-INV is simply added to the tradable investments OUTTR-INV. In order to rebalance the SAM, the value added in tradable production is raised and the value added in non-tradable production is lowered accordingly.

Stylized Pension System

As described in detail in section 3.3, each model region features a simplified pension system with a fixed ad-valorem contribution rate on labor of 10%. This pension system is represented in the SAM by an additional sector PAYGO and an assignment of 10% of labor income to this sector. On the benefit side, the same amount is entered in the field CON-PAYGO, indicating that the consumer receives these pension benefits from the PAYGO system.

[65] Fields OUTTR-OUTTR, OUTTR-OUTNT, OUTNT-OUTNT, OUTNT-OUTTR in Table 4-2

The result of these 3 adjustments for the Social Accounting Matrix of USA and Canada is shown in Table 4-3. The values are converted to USD billions in order to avoid large numbers and thereby increase numerical stability of the solution algorithm.

Table 4-3: Adjusted Social Accounting Matrix for USA and Canada

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	PAYGO	ROW	Total
OUTTR	0	0	0	0	0	1.712	2.646	0	1.567	5.926
OUTNT	0	0	0	0	0	7.587	0	0	0	7.587
CAP	1.338	2.389	0	0	0	0	0	0	0	3.727
LAB	2.358	5.198	0	0	0	0	0	0	0	7.556
TAX	0	0	0	687	0	0	0	0	0	687
CON	0	0	3.727	6.869	0	0	0	687	0	11.283
INV	0	0	0	0	0	1.984	0	0	663	2.646
PAYGO	0	0	0	0	687	0	0	0	0	687
ROW	2.230	0	0	0	0	0	0	0	0	2.230
Total	5.926	7.587	3.727	7.556	687	11.283	2.646	687	2.230	

All values in USD billions

This format of the Social Accounting Matrix is now consistent with the one defined in Figure 4-5 and may serve as a basis for model calibration.

Correction of Capital Shares

Even though the format of the Social Accounting Matrix after these described adjustments is now consistent with model assumptions, one further adjustment step is performed. More specifically, the capital shares implied by the SAMs from GTAP are directly based on statistical data. However, as pointed out by Gollin (2002), the statistical measurement of capital shares is biased and inconsistent between countries. This is due to the fact that the capital value add is typically calculated as the residual from total output and labor income. However, according to Gollin, this methodology accounts self-employed labor to capital income, leading to a bias towards higher capital shares. Gollin adjusted for this error by introducing additional statistical data e.g. on the size of the workforce and proposed new values for the capital share for a number of countries. Wherever available, these capital shares are used in the model. For all regions, where Gollin did not provide estimates for the capital shares, a value of $\sim_r = 0.3$ is assumed, with the exception of China and Russia, for which $\sim_r$ is set at a slightly higher value. The latter assumption is based on Ortega and Rodriguez (2006), who show a negative correlation between GDP per capita and the capital share $\sim$. The following table provides an overview of the exogenously defined capital shares for the different regions.

Table 4-4: Values for capital shares used in the model

Region	$\sim_r$	**Region**	$\sim_r$
USA & Canada	0.257	Japan	0.308
EU and Switzerland	0.270	Latin America	0.300
China	0.350	Russia	0.350
India	0.172	Rest of World	0.300

The capital shares proposed by Gollin (2002) reflect the whole economy with both tradable and non-tradable goods production sectors. However, in the present model, the assumption is introduced that non-tradable goods production is generally more labor and less capital intensive than tradable goods production. This assumption is based on the fact that non-tradable goods production includes all sorts of services, which are typically more labor and less capital intensive than the production of physical goods. In order to reflect this assumption in the model, the ratio between capital shares for the respective sectors is chosen exogenously. More specifically, it is assumed that the capital share of tradable goods production as a percentage value is $x^{\sim} = 1.3$ times the capital share for non-tradable goods production[66]. Combining this condition with the constraint that the aggregated capital share for both sectors needs to be consistent with Table 4-4 yields the following formula to derive the respective capital share for the non-tradable goods sector.

$$\sim_r^{NT} = \frac{\sim_r \cdot \bar{Y}_r}{x^{\sim} \cdot \bar{Y}_r^{T} + \bar{Y}_r^{NT}} \qquad \textbf{(107)}$$

Once $\sim_r^{NT}$ is determined, the capital share for the tradable goods sector $\sim_r^{T}$ can be calculated as follows.

$$\sim_r^{T} = \frac{\sim_r \cdot \bar{Y}_r - \sim_r^{NT} \cdot \bar{Y}_r^{NT}}{\bar{Y}_r^{T}} \qquad \textbf{(108)}$$

With $x^{\sim} = 1.3$, this approach leads to the following values for $\sim_r^{T}$ and $\sim_r^{NT}$.

[66] According to Gollin (2002), agriculture and primary commodity production have low employee compensation shares, whereas manufacturing and services have relatively high employee compensation shares. Employee compensation share for mining is given by Gollin at 0.361, for manufacturing at 0.732 and for ‘Community, social and personal services’ at 0.751. Given these figures and the fact that primary goods are usually tradable in nature, the capital share of tradable goods in this thesis is assumed to be 30% higher than the capital share of non-tradable goods (factor 1.3).

Table 4-5: Capital shares in tradable and non-tradable goods sectors

Region	$\sim_r^T$	$\sim_r^{NT}$	Region	$\sim_r^T$	$\sim_r^{NT}$
USA & Canada	0.304	0.234	Japan	0.355	0.273
EU and Switzerland	0.308	0.237	Latin America	0.344	0.264
China	0.370	0.284	Russia	0.395	0.304
India	0.181	0.147	Rest of World	0.332	0.256

In a final step, the SAMs need to be updated to reflect the new capital shares. This is performed by keeping the outputs $\overline{Y}_r^T$ and $\overline{Y}_r^{NT}$ constant and adjusting the values for capital and labor value add. In order to rebalance the SAM, the capital and labor incomes as well as the pension contribution need to be updated accordingly. The SAMs after these adjustments are directly fed into the model. Again, the sample for USA and Canada after correction of the capital shares is exhibited in below. Appendix A.2 shows the SAMs for other model regions.

Table 4-6: Adjusted Social Accounting Matrix for USA and Canada after adjustment of capital shares

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	PAYGO	ROW	Total
OUTTR	0	0	0	0	0	1.712	2.646	0	1.567	5.926
OUTNT	0	0	0	0	0	7.587	0	0	0	7.587
CAP	1.338	2.389	0	0	0	0	0	0	0	3.727
LAB	2.358	5.198	0	0	0	0	0	0	0	7.556
TAX	0	0	0	687	0	0	0	0	0	687
CON	0	0	3.727	6.869	0	0	0	687	0	11.283
INV	0	0	0	0	0	1.984	0	0	663	2.646
PAYGO	0	0	0	0	687	0	0	0	0	687
ROW	2.230	0	0	0	0	0	0	0	0	2.230
Total	5.926	7.587	3.727	7.556	687	11.283	2.646	687	2.230	

All values in USD billions

4.8 Choice of Exogenous Parameters

In the calibration approach applied, the majority of parameters are defined by Social Accounting Matrices. However, some additional exogenous assumptions are required for certain parameters, such as elasticities or the benchmark interest rate. The choices for these values are shown in the table below.

Table 4-7: Choice of exogenous model parameters

¯	Elasticity of intertemporal substitution	0.5	$\bar{r}$	Benchmark global real interest rate	3.5%
¶	Elasticity of substitution goods vs. leisure consumption	0.8	¢	Depreciation rate	7%
±	Elasticity of substitution tradable vs. non-tradable goods consumption	1.2	$¢^{NT}$	Lifecycle preference change non-tradable vs. tradable goods	10%

Ogaki et al. (1996) assess the elasticity of intertemporal substitution in consumer utility for both poor and rich countries and come to the conclusion that this parameter tends to be smaller for poor countries. More specifically, they find a value of ¯ = 0.35 for India and a value of ¯ = 0.64 for the United States. Since the model in this thesis assumes an equal value of ¯ for all world regions, ¯ is thus chosen at an intermediate level of 0.5.

Following Altig et al. (2001), the elasticity of substitution of goods vs. leisure consumption is set to a value of 0.8.

According to Harms et al. (2010), estimates for the elasticity of intratemporal substitution of tradable vs. non-tradable goods consumption ± vary between 0.44 (Stockman and Tesar, 1995) and 1.2 (Ostry and Reinhart, 1992). The value is chosen at the upper bound of this range at a value of 1.2, to account for the fact that these estimates are based on single-country-data and the model in this thesis combines single countries to world regions. Large regions should tend to have a higher elasticity of substitution of non-tradable vs. tradable goods consumption than single countries, since the available domestic goods bundle is richer for a world region than for a single country.

The benchmark real interest rate is set to 3.5% p.a., which is equal to the average US real interest rate over the last 15 years (1998-2012) based on Worldbank data[67].

Regarding the annual depreciation rate of capital, typical assumptions range between 5% (Börsch-Supan et al., 2006) and 8% (Ludwig and Reiter, 2010)[68]. Thus, a midway value of 7% p.a. is chosen for ¢ .

[67] http://data.worldbank.org/indicator/FR.INR.RINR

[68] Ludwig and Reiter (2010) assume a depreciation of 34.5% over 5 years, equal to an annual depreciation of 8.3%.

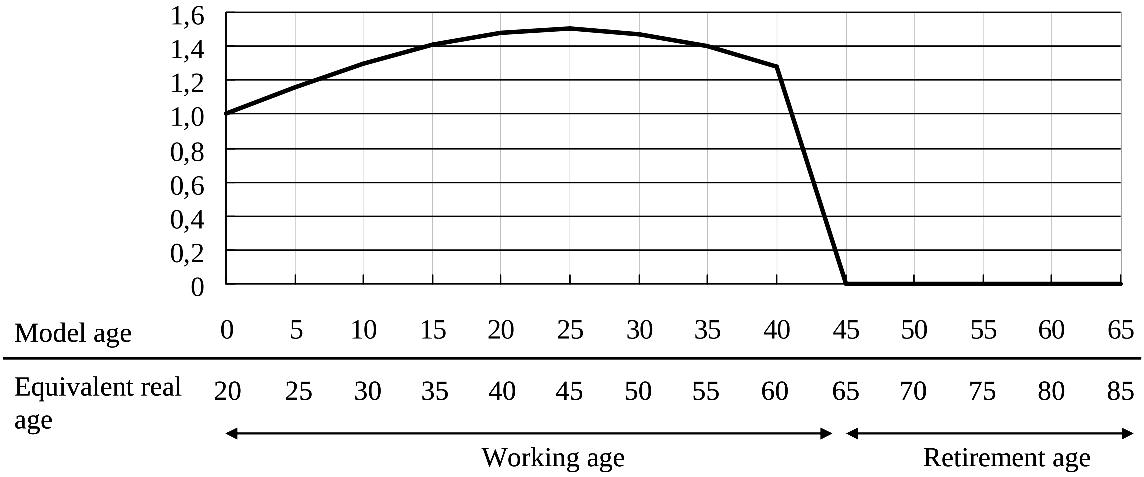
The choice of the lifecycle preference change for non-tradable goods ¢^{NT} of 10% over lifecycle is based on Figure 1 in Lührmann (2005)[69].

Furthermore, cohorts have age-specific labor productivity in order to account for changes in productivity over the lifecycle. These changes may be due to rising experience of workers in their early working years as well as their decreasing physical working capability when they approach retirement age. Auerbach and Kotlikoff (1987) provide a proposal for the parameterization of a hump-shaped lifecycle productivity profile defined by the following exponential function.

$$\text{®}_{a,r} = \frac{e^{4.47+0.033\cdot a-0.00067\cdot a^2}}{e^{4.47}} \tag{109}$$

This definition of $\text{®}_{a,r}$ results in a rising productivity in early years and a falling productivity in late years of the lifecycle as plotted in Figure 4-6. For period in retirement age, the productivity value $\text{®}_{a,r}$ is set to zero to implement exogenous retirement. It should further be noted that the absolute level of the productivity profile does not influence results. That is, multiplying the profile with a constant factor would be netted out by the first auxiliary model (section 4.6.1) by adjusting time endowment μ_r.

Figure 4-6: Labor productivity over lifecycle



[69] Lührmann (2005) shows the share of consumption of different goods categories over lifecycle. In order to transform into non-tradable vs. tradable goods consumption, the following categories are assumed to be non-tradable: Health, food, energy, transportation.

Another parameter that needs to be specified exogenously is the steady-state population growth rate. The compound population growth rate between years 1990 and 2030 according to the UN Population Prospects serves as a basis for the quantification of this parameter. Due to the fact that these population growth rates between 1990 and 2030 differ substantially between regions, this parameter is chosen on a regional rather than a global basis. The values assigned for the steady-state growth rate are shown in the table below.

Since negative steady-state population growth values lead to insolvable model behavior, those regions, which show negative growth between 1990 and 2030 according to the UN data, are assumed to have a steady-state growth rate of *zero*.

On the other hand, fast-growing regions such as India and Rest of World show very high population growth rates according to the UN data between 1990 and 2030. These values were clipped at a maximum of one percent per annum, since higher steady-state growth values lead to unrealistic population sizes over the course of the model horizon of 315 years.

Table 4-8: Assumed steady-state growth rates for world regions

Region	$^{3}{}_{r}$	**Region**	$^{3}{}_{r}$
USA & Canada	0.294%	Japan	0.000%
EU and Switzerland	0.000%	Latin America	0.860%
China	0.000%	Russia	0.000%
India	1.000%	Rest of World	1.000%

4.9 Benchmark Replication Test

As literature suggests[70], it is good practice in calibrated computational general equilibrium modeling to perform a benchmark replication test of the model. This step serves as a test for errors in the model structure or calibration procedure.

First, quantity and price variables are initialized with values equal to the associated benchmark values. All activity levels are set to a value of 1 for $t = 0$ and grow with the respective regional steady-state growth rate thereafter[71]. Second, the model (in)equations are evaluated with these values and a check is performed if all model (in)equations including their comple-

[70] See Rasmussen and Rutherford (2004), Paltsev (2004)

[71] Except $\Omega^{L}_{g,t,r}$ and $\Omega^{c}_{g,t,r}$, which are set to 1 for all periods, since benchmark values already reflect population growth for these activities

mentarity conditions are fulfilled. If one or more (in)equations are not satisfied, there must be errors in the model structure or in the calibration procedure.

This benchmark replication test is equivalent to a dynamic model, which is in perfect steady state in every period of the model horizon. However, in the present model, one additional adjustment is necessary in order to accomplish a benchmark replication test. Due to the fact that different steady-state growth rates for the eight world regions are assumed as exhibited in Table 4-8, a steady-state for the complete model with all eight regions does not exist. This is due to the fact that the different growth rates over time lead to shifts in relative size of the regions. Since a steady state generally requires balanced and proportionate growth, a steady state for the model with the original, regional growth rates does not exist.

Accordingly, all steady-state growth rates were temporarily set to a value of *0.294%* for the replication test procedure.

4.10 Calibration of 1-Sector Model

As introduced above, the 1-sector model serves as a reference model such that the differences between the 2-sector and 1-sector models may be interpreted as an error due to the lack of non-tradable goods in the 1-sector model. For this approach to be consistent, it is crucial that both models are calibrated to match exactly the same benchmark values in their respective steady states. Due to the convenient equation notation in the calibrated share form, this step does not require further complex calculations. Rather, all values from the 2-sector-model may be used. However, breakdowns of consumption and production into tradable and non-tradable goods are not required. Rather, aggregated values such as $\overline{c}_{g,t,r} = \overline{c}^{T}_{g,t,r} + \overline{c}^{NT}_{g,t,r}$, $\overline{R}_{t,r} = \overline{R}^{T}_{t,r} + \overline{R}^{NT}_{t,r}$, and $\overline{L}_{t,r} = \overline{L}^{T}_{t,r} + \overline{L}^{NT}_{t,r}$ are used. By applying these aggregate values, the 1-sector model is calibrated to match the same benchmark values as the 2-sector model in the sense that for instance the sum of the 2-sector model's benchmark tradable and non-tradable production are equal to the tradable production in the 1-sector model. The same logic applies to capital service $\overline{R}_{t,r}$ and labor supply $\overline{L}_{t,r}$.

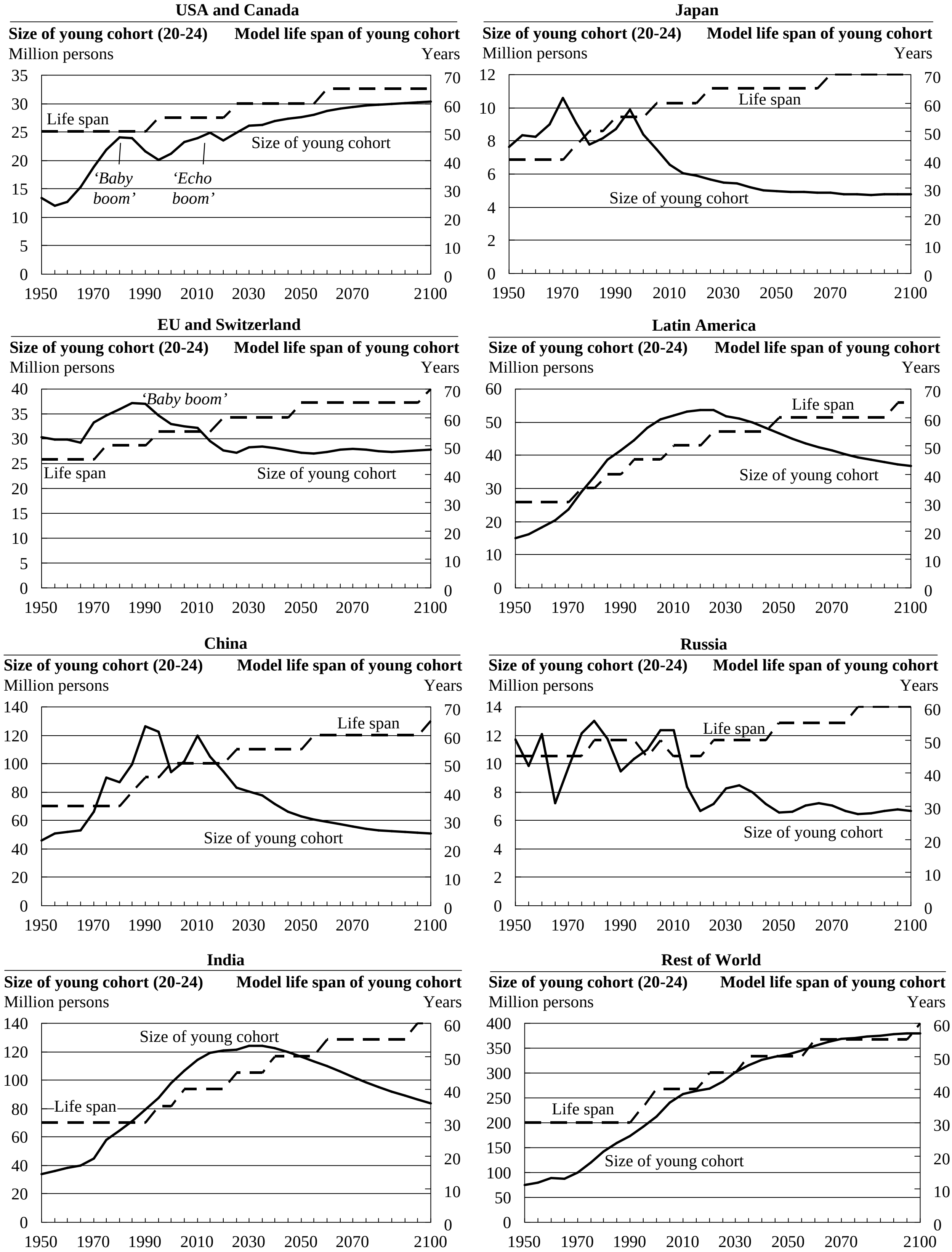
4.11 Overview on Demographical Data Input

The model in calibrated share form requires two distinct input series on demographics: the development of cohort size $n_{g,r}$ and lifespan $a_{g,r}$.

The raw data from UN World Population Prospects directly provides cohort sizes in steps of five years, starting from 1950 and then continuing 1955, 1960 and so on. This raw data provides the size of each age cohort, with a cohort defined as the population of a given 5-year age interval. Thus, the data for 1950 for instance provides the cohort size for the cohort of age 0-4 years, 5-9 years, 10-14 years and so on.

Since migration between regions is not in scope of the model and since every cohort in the model has its deterministic lifespan, it follows that the size of any given cohort in the model is constant from birth until death. As cohorts enter the model with an actual age of 20, the relevant cohort from the UN raw data is the age cohort 20-24. That is, in the model period that corresponds to year 2000 for instance, the size of the cohort with model age *zero* is equal to the size of the 20-24 year old cohort in year 2000 from UN data. The population sizes of this ‘young cohort’, corresponding to the newborn cohort in the model and the 20-24 year old cohort in the data, are depicted in Figure 4-7.

Figure 4-7: Demographical data input[72]



[72] Source: UN World Population Prospects, 2010 revision; own calculations (population-weighted aggregation of single country data to world regions data and rounding of life span to accommodate 5-year model periods)

Similarly, the second demographic series required, model lifespan, is based on data from UN World Population Prospects. In this case, for the lifespan of a cohort born in year *t*, the life expectancy at birth of the cohort born 20 years earlier (in period *t-20)* serves as the basis for this parameter. The shift of 20 years accounts for the fact that the 'newborn' model cohort in time period *t* was in fact born 20 years earlier in reality.

Due to the relatively poor availability of historical and forecast data for conditional life expectancies, the approach described above was applied. However, the theoretically correct data to calibrate this parameter would be the conditional life expectancy at age 20-24. This conditional life expectancy tends to be higher than life expectancy at birth, since probability of death in years 0-19 for an individual already of age 20-24 is zero. For this reason, the values used in the model somewhat underestimate actual life expectancies.

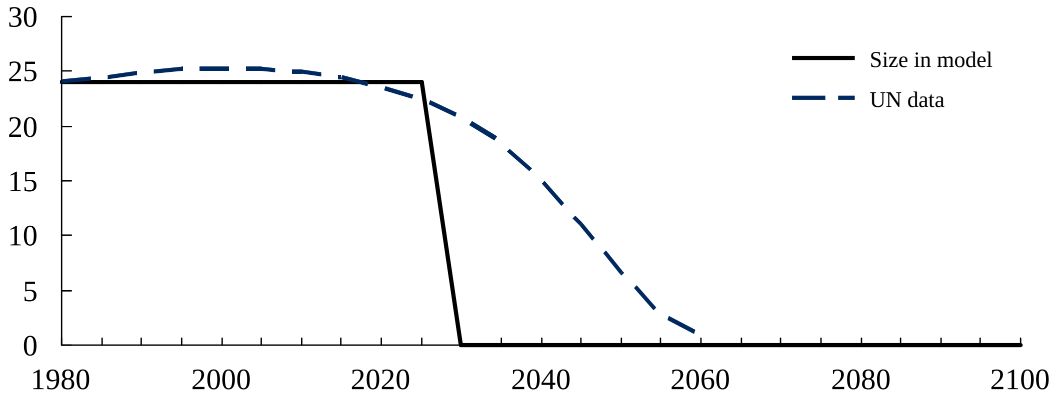
Due to the fact that the duration of one model period is 5 years, the life expectancies derived from the data with the approach detailed above are rounded to the next 5-year value. The resulting time series as used in the model is also exhibited in Figure 4-7.

4.12 Deviation of Simulated Demographics from Actual Data

As explained in section 4.11, the raw demographic historical and forecast data from the UN World Population Prospects were adjusted to be consistent with the model structure. These adjustments lead to deviations between simulated demographics and actual forecasts from UN raw data. In the following, the magnitude of these deviations is assessed in detail.

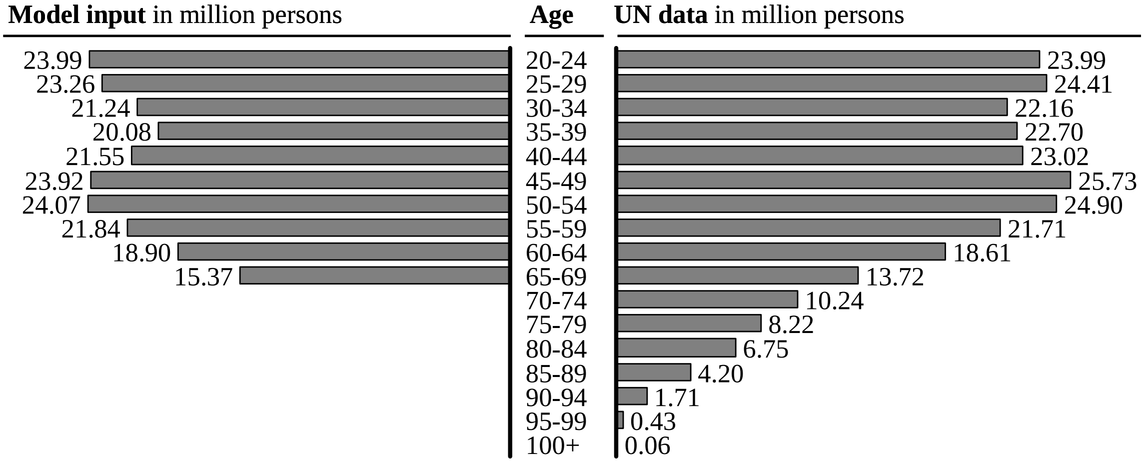
The first assessment compares the size of a sample cohort over time for actual UN data and model input. Figure 4-8 exhibits the size of the cohort with age 20-24 in year 1980. Due to the assumption of constant cohort size and deterministic lifespan, the size of this cohort is assumed to be constant in the model until the end of the lifespan. The UN raw data curve shows the cohort size of the cohort which was 20-24 in year 1980, that is, in year 1985 it shows the size of cohort 25-29, in year 1990 the size of cohort 30-34 according to the UN World Population Prospects and so on. Thus, the UN raw data curve incorporates both emigration to and from the respective region as well as real mortality rates. The slight increase of the cohort size between 1980 and 2000 is due to migration and the subsequent decrease is due to mortality.

Figure 4-8: Size of cohort that was 20-24 years old in 1980 in million persons



This adjustment leads to a demographic structure for the USA and Canada in year 2010 as exhibited in Figure 4-9. Again, small deviations in the structure for age groups between 20 and 60 are mainly due to migration. However, these devations are small compared to the fluctuations in cohort sizes. That is, the model input data reproduces the actual age structure quite well between ages 20 and 65.

Figure 4-9: Demographic structure in model and actual UN data for year 2010 in USA and Canada



5 Analysis and Interpretation of Model Results

This chapter presents and discusses the results of the OLG-model detailed in previous chapters. With regard to the objective of examining the impact of a non-tradable goods sector in the model, this chapter especially focuses on differences between a 2-sector-model with both non-tradable and tradable goods and a conventional 1-sector-model. This latter model has an equivalent structure and is identically calibrated as the 2-sector-model, except that it does not feature the non-tradable goods sector.

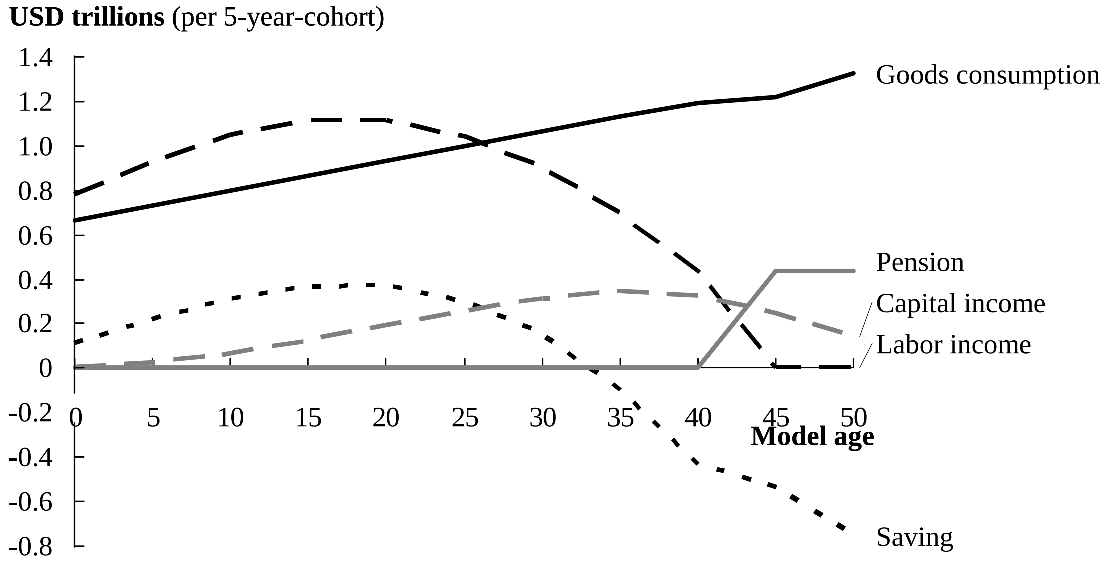
This chapter is organized in two sections. The first section discusses the results of the auxiliary calibration models outlined in section 4.6, while the second section presents the aggregated economic variables.

5.1 Results of Calibration Procedure

The calibration procedure determines consumer preferences and cohort-specific benchmark values that are consistent with the Social Accounting Matrices and other exogenous parameters. This chapter sheds light on this cohort benchmark behavior in order to test for plausibility, but also in order to better understand model behavior upon analysis of model results in the next section.

The first auxiliary model presented in section 4.6.1 yields consumer behavior in the benchmark case of steady-state cohort growth. As outlined above, this behavior leads to an aggregate economy for each region consistent with the respective Social Accounting Matrix. Figure 5-1 depicts the source and usage of funds over a cohort lifecycle in this benchmark scenario for region USA and Canada.

Figure 5-1: Benchmark source and usage of funds over lifecycle (USA and Canada)[73]



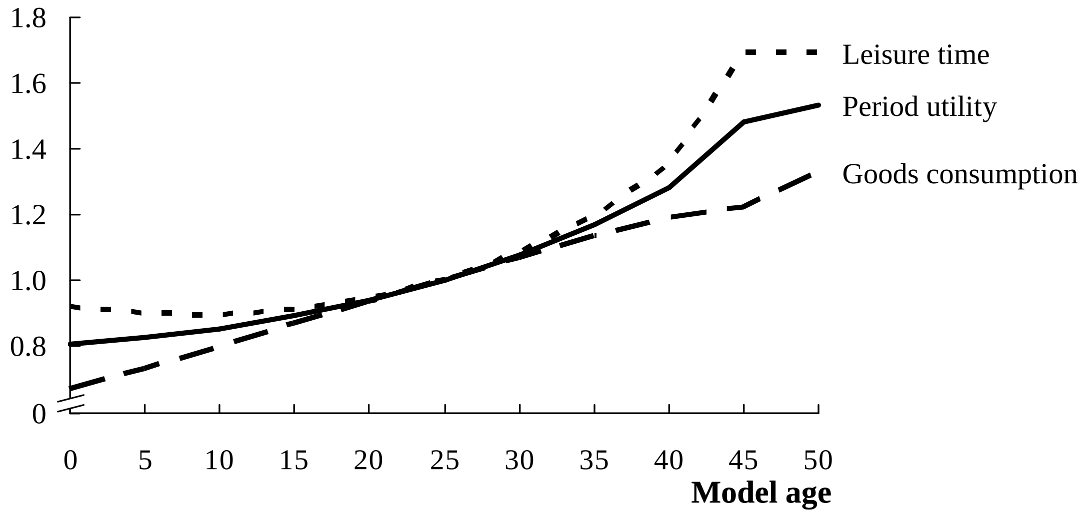
As the figure shows, cohorts in USA and Canada have a preference for rising goods consumption over the lifecycle. In order to finance this rising consumption, their savings balance is positive between model age 0 and 33[74]. Beyond this age, cohorts decumulate their savings. Due to the hump-shaped productivity profile introduced in Figure 4-6, cohorts earn highest labor incomes in their mid-years between model age 10 and 25. Due to the assumption of endogenous labor supply, cohorts work most when their productivity is highest. This effect amplifies the hump-shape of the productivity profile with regard to labor income. Pension benefits occur during the last two periods of lifespan, equivalent to an actual retirement age of 65. However, due to endogenous labor supply and declining productivity in old ages, cohorts already work significantly less with actual age 60 (model age 40) than at actual age 45 (model age 25).

This observation is supported by Figure 5-2, which shows period utility over the lifecycle and its two components leisure time and goods consumption. Leisure time is by definition equal to a fixed time endowment minus labor time. It thus follows from the graph that labor time performed by cohorts is highest between model ages 0 and 20, before it gradually decreases and reaches *zero* in retirement age.

[73] A linear path of steady state goods consumption is implied by the modeling approach in this thesis, using time discount rate $°_r$. This is a standard modeling approach in the literature (see e.g. Ludwig and Reiter (2005): 6, Fehr et al.: 9). However, this constitutes a simplification since goods consumption over lifecycle is in reality not a linear path.

[74] Actual age 20-53

Figure 5-2: Benchmark period utility, goods consumption, and leisure time



This figure also shows that period utility is increasing over the lifecycle. Thus, the intertemporal discount rate °$_r$ resulting from the auxiliary model for USA and Canada is smaller than the benchmark interest rate $\bar{r}$. Table 5-1 exhibits annualized utility discount rates for all world regions as determined by the primary auxiliary model. All annualized utility discount rates are smaller than the benchmark interest rate, such that all world regions are modeled with a preference for rising period utility over the lifecycle.

Table 5-1: Calibrated utility discount rates for world regions

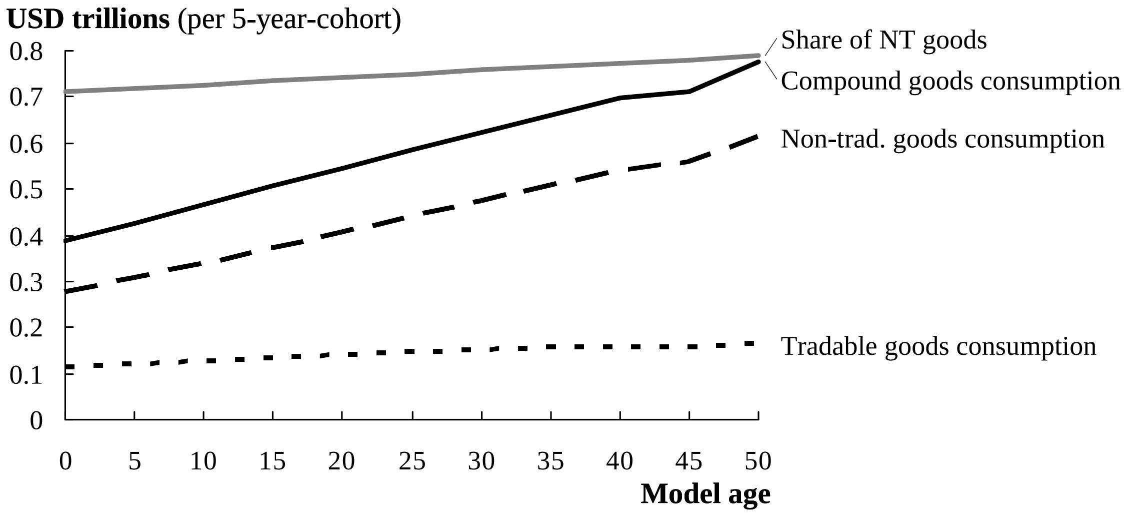
Region	°$_r$	**Region**	°$_r$
USA & Canada	0.74%	Japan	2.33%
EU and Switzerland	1.48%	Latin America	0.90%
China	2.50%	Russia	2.08%
India	-0.24%	Rest of World	3.08%

As detailed above, the main model allows for changing preferences over the lifecycle of consumers between tradable and non-tradable goods. This change of preferences is modeled according to equation (99), resulting in a linear increase in the preference share of non-tradable goods. This increase can be observed in Figure 5-3, where the share of NT goods in compound goods consumption rises linearly from 71.1% to 78.9% between model ages 0 and 50. The exogenously determined difference in share between age *zero* and highest model age ¢^{NT} is set to 10%. The full difference may only be observed for the highest lifespan occurring in the model, which is 65 years. However, extrapolating the values above prove the consistency of the calibration procedure with respect to the rising share of NT-preference:

$$\frac{78.9\% - 71.1\%}{50} \cdot 65 = 10.1\% \qquad \textbf{(110)}$$

This value is very close to the exogenously defined value of $¢^{NT} = 10\%$. The small deviation of 0.1 percentage points is due to the second order correction reflected by parameter $\hat{s}$ in equation (98).

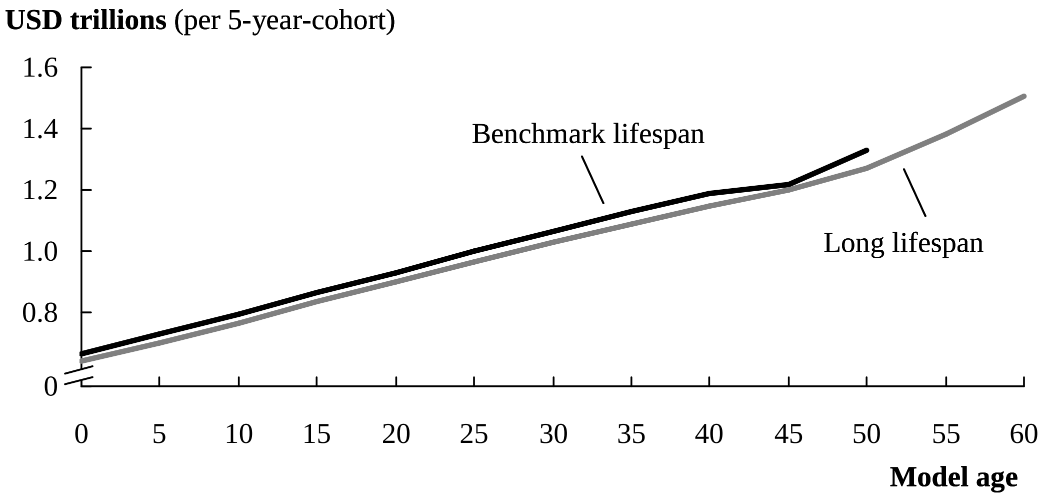
Figure 5-3: Benchmark tradable and non-tradable goods consumption (USA and Canada)



The absolute level of tradable and non-tradable consumption is chosen such that the aggregate tradable and non-tradable consumption values from the Social Accounting Matrices are fulfilled. Due to this reason, tradable consumption in Figure 5-3 is generally lower than non-tradable consumption for all cohort ages.

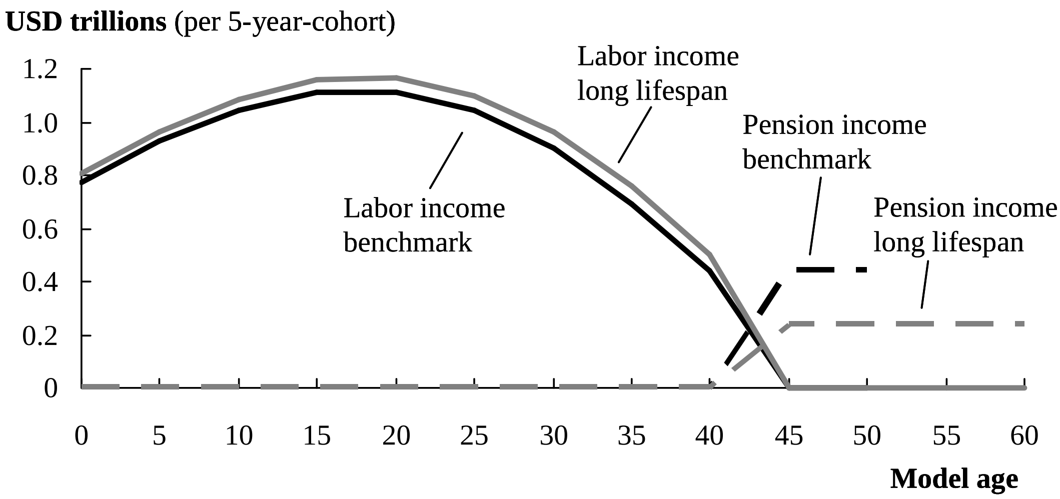
The secondary auxiliary models yield benchmark profiles for the parameters described above for cohort lifespans different from the baseline lifespan. For instance, Figure 5-4 compares benchmark consumption for the benchmark lifespan of 50 model years with a longer lifespan of 60 model years, which is achieved by cohorts born from 2060 onwards according to UN data. Consumption increases over the lifecycle with a very similar slope, however, the level of period goods consumption is smaller in the case of a long lifespan.

Figure 5-4: Comparison of goods consumption benchmark vs. long lifespan (USA and Canada)



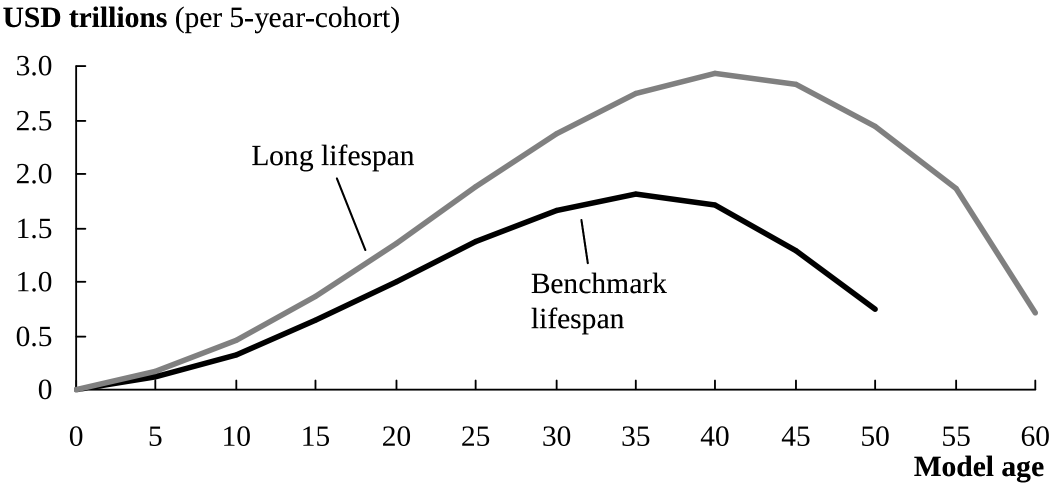
On the other hand, labor income and thus period labor supply is higher in the case of a long lifespan, as depicted in Figure 5-5. Due to the fact that the PAYGO pension system is modelled with a fixed contribution rate of 10% and the retirement age is fixed at an actual age of 65 years, pension benefits are significantly smaller for the longer lifespan. This is due to the fact that the number of working periods stays the same, while the number of pension periods doubles from 2 to 4.

Figure 5-5: Comparison of labor and pension income for benchmark and long lifespan (USA and Canada)



Both the lower level of consumption and the higher level of labor income for long lifespans lead to higher period savings. Cumulated savings, in turn, are equal to a cohort’s asset position. As Figure 5-6 shows, the differences in asset position are significant, even though the differences in consumption and labor income are not very large. These higher cohort assets over the lifecycle lead to significantly higher aggregate assets of the whole economy under the assumption of the described fixed contribution pension system.

Figure 5-6: Comparison of asset positions for benchmark and long lifespan



As will be discussed in more detail in the following sections, this increase in asset supply puts interest rates under pressure and evokes ceteris paribus higher capital stock per capita.

5.2 Model Results in Aggregate Variables

While the previous section presented the results of the auxiliary models used to calibrate the OLG-model, this section presents the actual model results in aggregated variables. As introduced above, the calculations are performed by both the 2-sector-model featuring non-tradable goods and by the conventional 1-sector-model, which only allows tradable-goods-production and assumes all goods to be perfectly tradable. Besides this assumption, both models are structurally equal and calibrated to both match exactly the same Social Accounting Matrices and exogenous parameters.

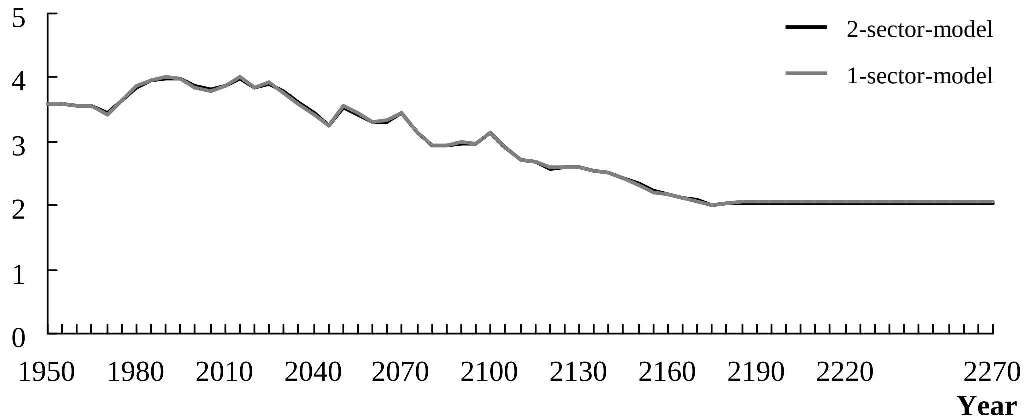
In the following, the discussion focuses on regions USA and Canada and EU and Switzerland. However, appendix A.3 provides simulation results for all eight model regions.

5.2.1 Interest Rate

Due to the fact that the model with all its eight regions represents a global closed economy, the global interest rate is endogenous. As Figure 5-7 exhibits, the simulated interest rate drops by 80 base points between 2010 and 2100. This result is in line with Krueger and Ludwig (2006), who predict an interest rate decrease of 86 base points between 2005 and 2080 in their scenario with a fixed-contribution PAYGO pension system.

With regard to the objective of this thesis, however, the most relevant observation is the very small difference between the model with non-tradable goods (2-sector-model) and the conventional 1-sector-model. The curves are very close to each other, with the highest difference of 2.4 base points occurring in year 1995.

Figure 5-7: Simulated interest rate development



The drop in interest rate development is mainly driven by the increased supply of capital due to increased lifetime savings of longer living cohorts (see Figure 5-6). This increased supply of capital in combination with a constant productivity-weighted time endowment per capita leads to a relative abundance of capital and scarcity of labor. The productivity-weighted time endowment per capita remains constant for increasing lifespans, since the retirement age, above which the productivity is set to *zero*, is fixed at 65 years.

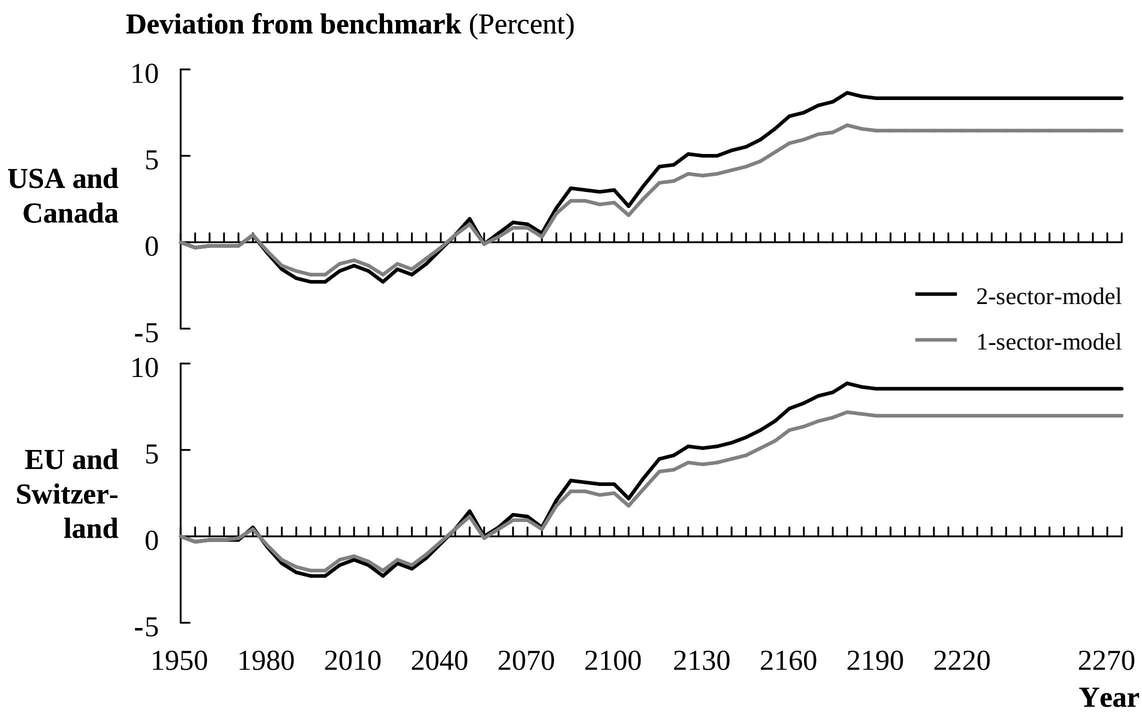
5.2.2 Wage Rate

This labor scarcity and capital abundance not only affects rates of returns, but also the wage rate. Within the forecast horizon between 2010 and 2100, wage rate is projected to rise by 4.4

percentage points (2-sector-model) or 3.3 percentage points (1-sector-model) in USA and Canada. In EU and Switzerland, the increase in wage rate is very similar with 4.4 percentage points for the 2-sector-model and 3.6 percentage points for the 1-sector-model.

In contrast to the interest rate, there is in fact a noticeable difference in wage rate development between the 2-sector-model with non-tradable goods and the conventional 1-sector-model. The numeraire in both models is the price of tradable goods. However, in the 1-sector-model, tradable goods have the calibrated production function equivalent to the aggregate of the tradable and non-tradable sectors in the 2-sector-model, resulting in the price difference between the respective numeraires. In other words, the difference in wages is driven by an increasing price difference between tradable and non-tradable goods over the model horizon. This increasing price difference is the result of the more labor-intensive non-tradable goods production in combination with increasing wages.

Figure 5-8: Simulated wage rate development

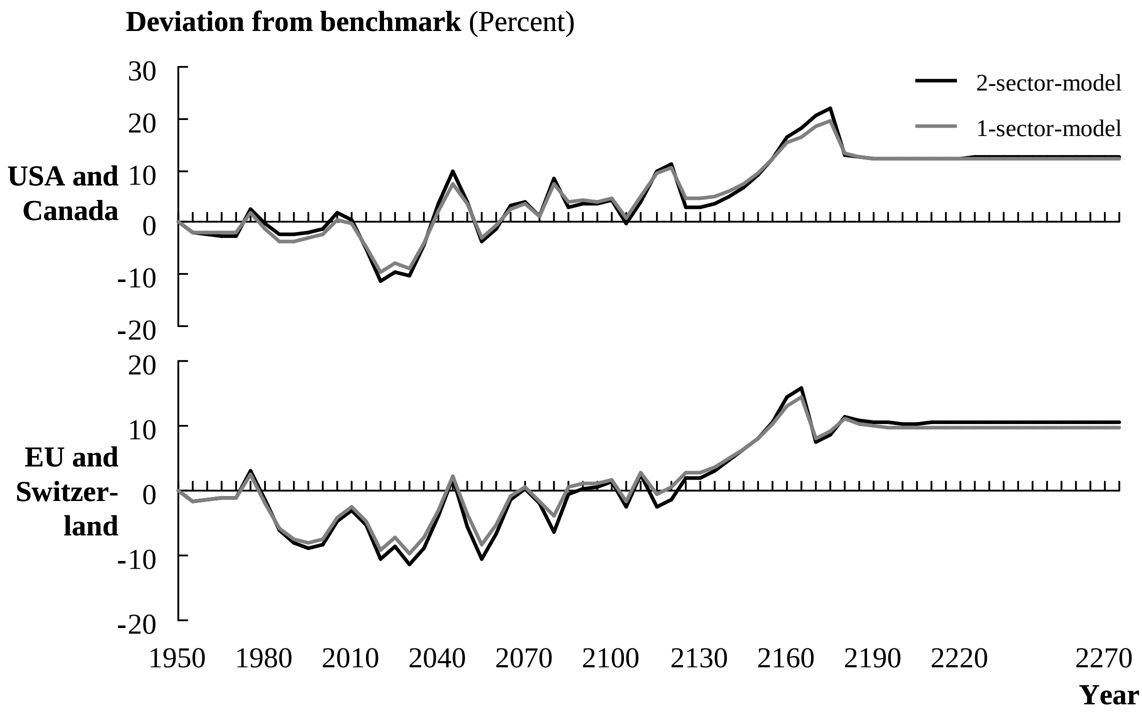


5.2.3 Capital Stock per Capita

Consistent with the reasoning of increasing per capita savings and capital supply, the per capita capital stock is also expected to show a long-term rising trend between 2010 and 2100. The simulated path shows some volatility, which is mainly due to the time resolution of 5 years per period. Whenever lifespan increases by one period in the model, an artificial peak in the simulated paths of capital stock and also other variables is the result.

Again, differences between the two model types are noticeable, but not highly relevant in their magnitude. Swings in capital stock development are slightly more pronounced in the 2-sector-model.

Figure 5-9: Simulated capital stock per capita development



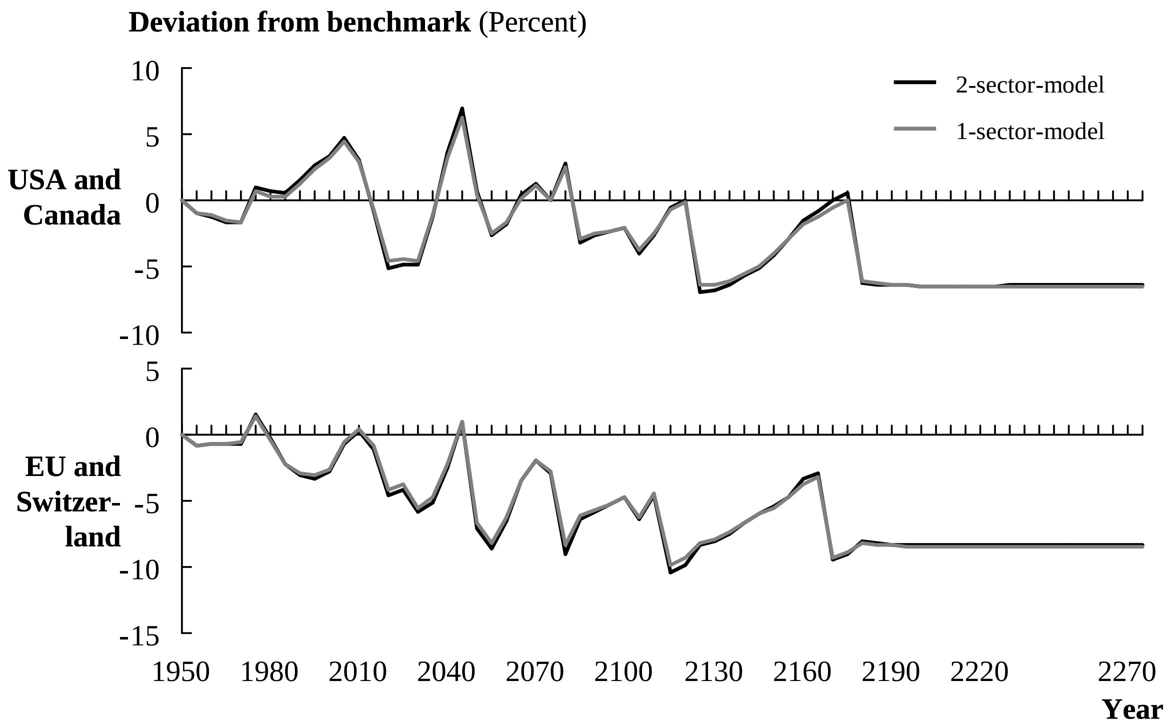
5.2.4 GDP per Capita

Due to the decrease in working population ratio, which results from both the increasing lifespan and the decreasing fertility, the long-term effect of demographics on GDP per capita in this model is negative. However, due to the increase in capital stock per capita and an increase endogenous labor supply, GDP per capita decreases by less than the working population ratio. More specifically, GDP per capita in USA and Canada is simulated to decrease by 5.2 percentage points for the 2-sector-model (5.0 in 1-sector-model) while the working population ratio of the input demographics drops by 17.8%[75] between 2010 and 2100. However, it should be noted that this decrease in GDP per capita is likely to be overcompensated by economic growth due to technological progress, which is not part of this model.

[75] From 92.8% to 76.3% (only for population of age 20 and above)

Similar to the variables presented above, the GDP per capita development also shows only small differences between the 2-sector-model and 1-sector-model.

Figure 5-10: Simulated GDP per capita development



5.2.5 Labor Supply per Capita

Due to the decrease in the share of working population, labor supply per capita decreases as well, even though labor supply during actual working years increases. That is, cohorts tend to work more in their working years in anticipation of an extended retirement period. This phenomenon is shown for USA and Canada and for EU and Switzerland in Figure 5-11 and Figure 5-12. Figure 5-11 shows a decline in labor supply per capita by 8.2 percentage points for USA and Canada, while at the same time, Figure 5-12 witnesses an increase in labor supply during working years by 13.2 percentage points for this region.

The increase of endogenous labor supply during working years is a result of two effects. First, the increased capital stock per capita leads to higher wages, which in turn fuels labor supply under the assumption that the substitution effect in the labor-leisure-choice dominates the income effect. Second, cohorts need to save more during their working years and react by increasing their labor supply.

Figure 5-11: Simulated labor supply per capita development

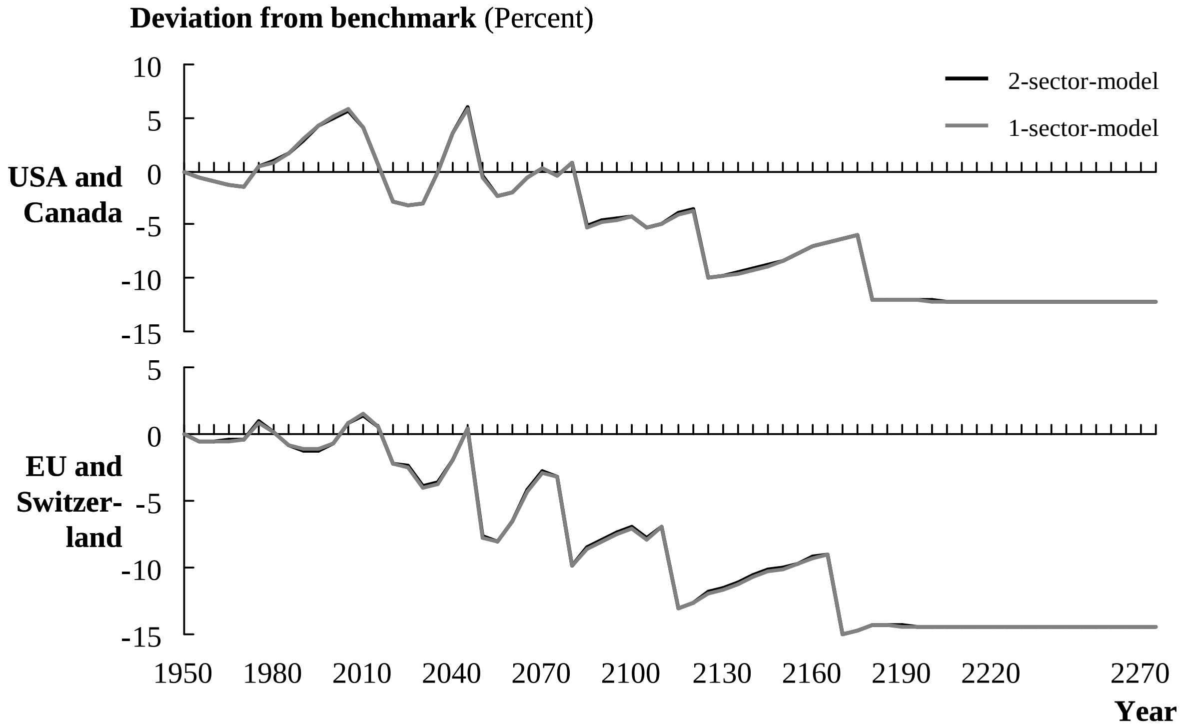
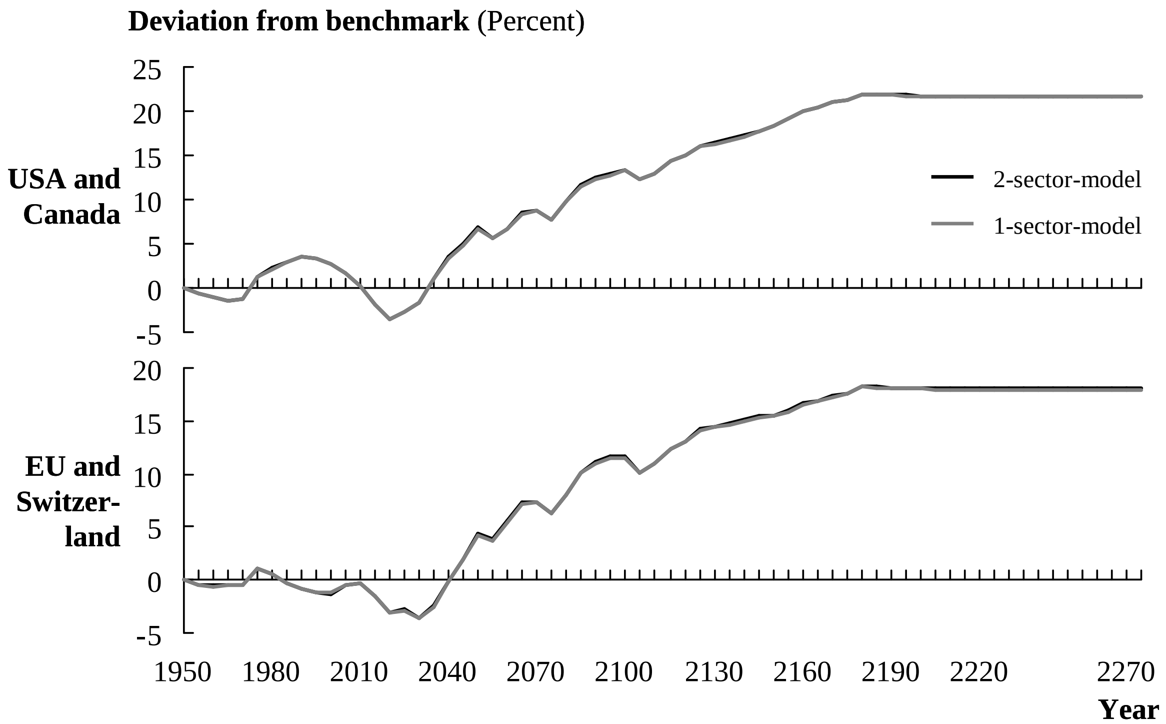


Figure 5-12: Simulated development of labor supply per unit of productivity-weighted time endowment



5.2.6 Welfare Effects

Welfare effects of demographics are generally hard to quantify. Any quantification of demographic welfare effects needs to answer the philosophical question of how to compare lifetime welfare of cohorts with different lifespans. How much is an additional period of lifetime worth in utility terms? Obviously, this question cannot be answered by calibration to actual data as, for instance, the question of a utility tradeoff between leisure and labor time. The latter tradeoff may easily be observed in the data. However, the first tradeoff never occurs, since no individual in reality makes a tradeoff between longer lifetime and more goods consumption, for instance.

For this reason, this thesis applies the following approach, which is based on the approach of Krueger and Ludwig (2006). In a first step, the baseline welfare for each cohort is calculated. This baseline welfare is defined as the lifetime utility that a cohort would achieve assuming benchmark interest rates and prices to be constant over its entire lifecycle. If the actual utility of any given cohort in the model is higher than its respective benchmark utility, the welfare effect will be positive and vice versa. Furthermore, the welfare effect is quantified as the percentage difference between this actual lifetime utility and benchmark lifetime utility.

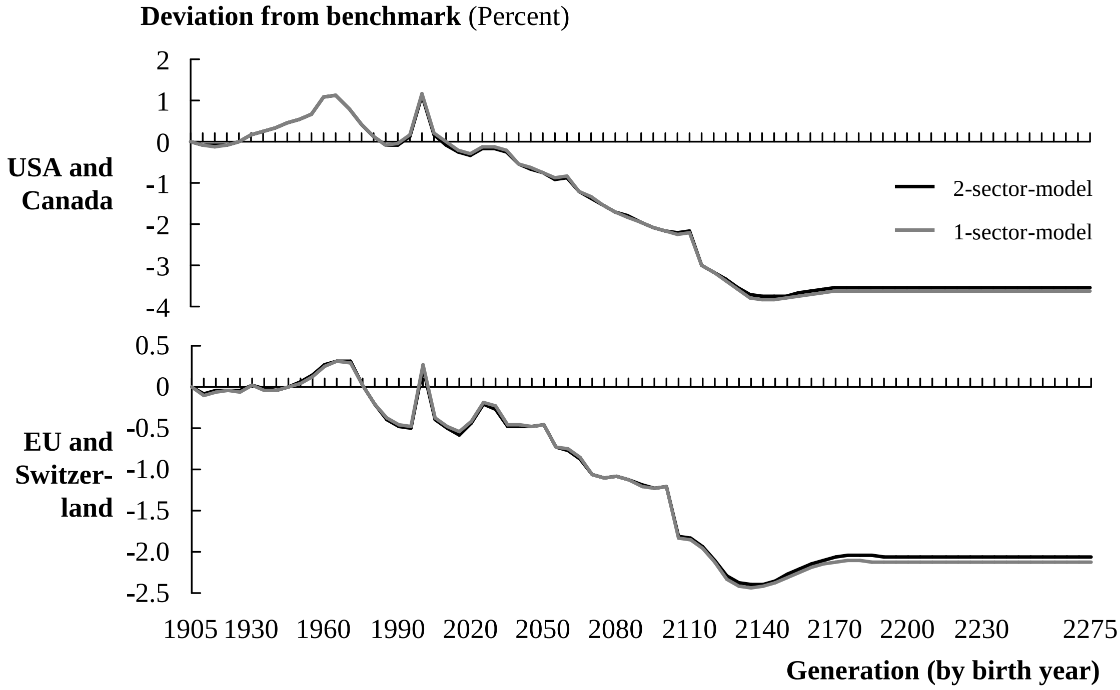
The baseline lifetime utilities $\bar{u}_{g,r}$ required to perform this calculation are already determined in the calibration process in equation (106). According to equation (77), total lifetime utility is defined as $\bar{u}_{g,r} \cdot \Omega^{u}_{g,r}$, with $\Omega^{u}_{g,r} = 1$ in the benchmark scenario. Thus, the welfare effect as defined above can be directly obtained by calculating $\Omega^{u}_{g,r} - 1$.

Figure 5-13 presents the result of this analysis again for two sample model regions. As supported by literature[76], the long-term welfare effects of demographic change, based on the welfare definition given above, are negative.

Moreover, as for most other model variables, the outcomes for the 2-sector-model and 1-sector-model are almost identical. Thus, accounting for non-tradable goods does not have a significant impact on welfare results for this model structure and calibration.

[76] See Krueger and Ludwig (2006)

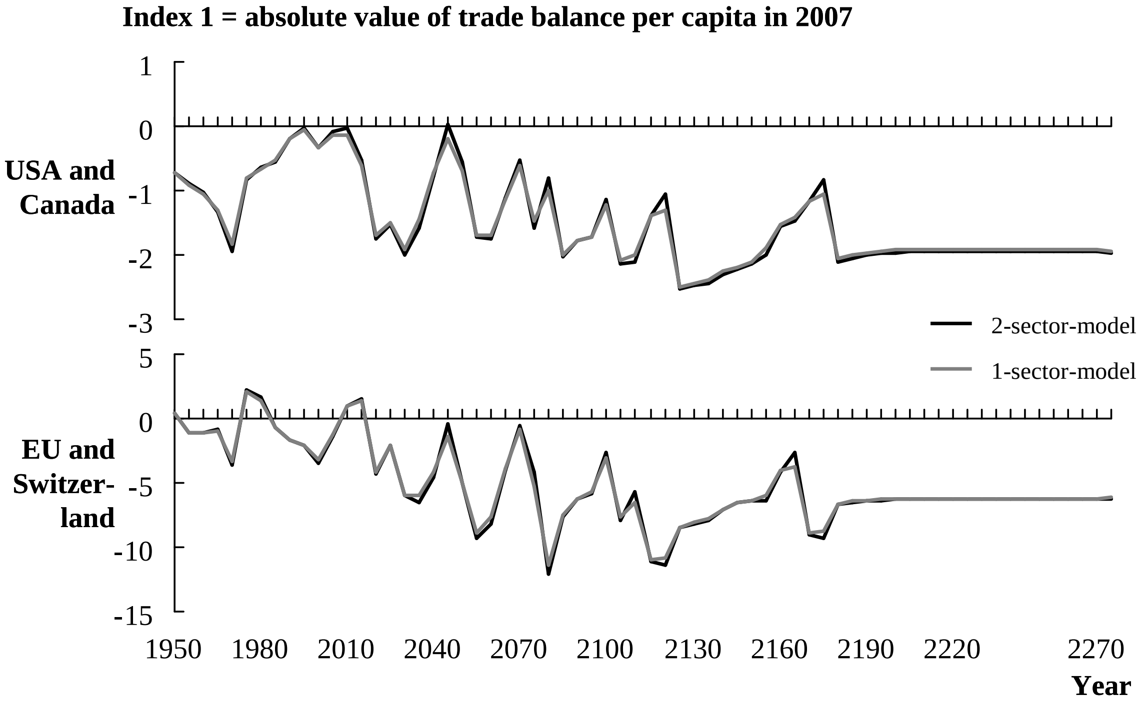
Figure 5-13: Simulated welfare effects



5.2.7 Trade Balances

In a long term trend, the demographic development exerts a downward pressure on trade balances for USA and Canada as well as for EU and Switzerland. Especially for EU and Switzerland, demographics have a strong negative effect on trade balance, as Figure 5-14 exhibits. The observation of negligible differences between the 1-sector-model and 2-sector model also holds true for trade balances.

Figure 5-14: Simulated trade balance per capita development



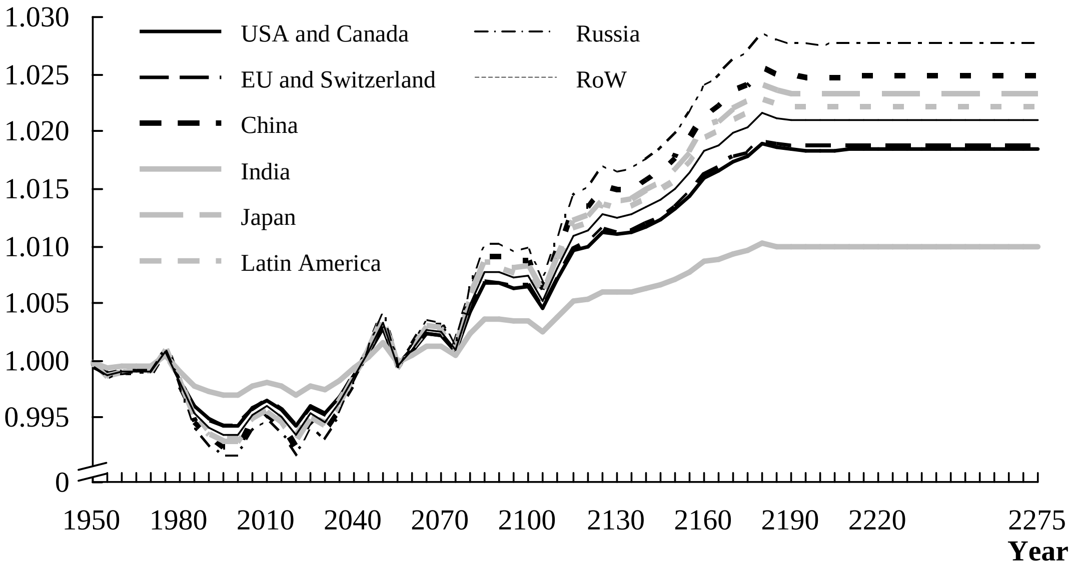
5.2.8 Price of Non-Tradable Goods

Both in the 1-sector and 2-sector-model, the price of tradable goods serves as a numeraire and is fixed across regions, due to the assumption of perfect tradability of tradable goods. However, in the 2-sector-model, the price of non-tradable goods may differ across regions.

The simulated effect of demographics on prices of non-tradable goods is exhibited in Figure 5-15. Clearly, the aging population exerts an upward pressure on non-tradable goods prices in all regions. However, the magnitude of this effect is relatively small with a shift of less than 1.5 percentage points between 2010 and 2100.

The likely reason for this increase in prices of non-tradable goods is the development of factor prices. Due to the fact that the non-tradable sector is calibrated to be more labor and less capital intensive than the tradable goods sector, the increase in wages and decrease in returns to capital (see sections 5.2.1 and 5.2.2) drives up the price of non-tradable goods relative to tradable goods.

Figure 5-15: Simulated price of non-tradable goods (Index 1 = Price of tradable goods)



5.2.9 Real Exchange Rate Effects

The shifts in prices of non-tradable goods presented in section 5.2.8 may induce real exchange rate effects between regions, depending on the mathematical definition of the real exchange rate. More specifically, if real exchange rates are defined according to the concept of terms of trade, for instance, the effects due to price changes in non-tradable goods would not be visible. Terms of trade are defined as the quotient of real export over import prices and since prices of tradable goods are identical across all regions due to the assumptions of perfect tradability and substitutability in this model, this quotient is equal to 1 in every time period.

Thus, a different definition of real exchange rates is used. The real exchange rate is calculated as the ratio of the compound price of domestic goods to the compound price of foreign goods of the respective period (Paasche index):

$$Q_{t,r} = \frac{p_{t,r}^D}{p_{t,r}^F} \tag{111}$$

$p_{t,r}^D$: Compound price of domestic goods

$p_{t,r}^F$: Compound price of foreign goods

The compound price of domestic goods p_t^D is calculated as the consumption-weighted average of the price of non-tradable goods in the respective region, $p_{t,r}^{NT}$ and the price of tradable goods p_t^T .

$$p_{t,r}^{D} = \frac{p_t^T \cdot c_{t,r}^T + p_{t,r}^{NT} \cdot c_{t,r}^{NT}}{c_{t,r}^T + c_{t,r}^{NT}} \tag{112}$$

$$c_{t,r}^{T} = \sum_g \Omega_{g,t,r}^{c} \cdot \bar{c}_{g,t,r}^{T} \cdot \left(\frac{p_{g,t,r}^{c}}{p_t^T} \right)^{\pm} \qquad c_{t,r}^{NT} = \sum_g \Omega_{g,t,r}^{c} \cdot \bar{c}_{g,t,r}^{NT} \cdot \left(\frac{p_{g,t,r}^{c}}{p_{t,r}^{NT}} \right)^{\pm}$$

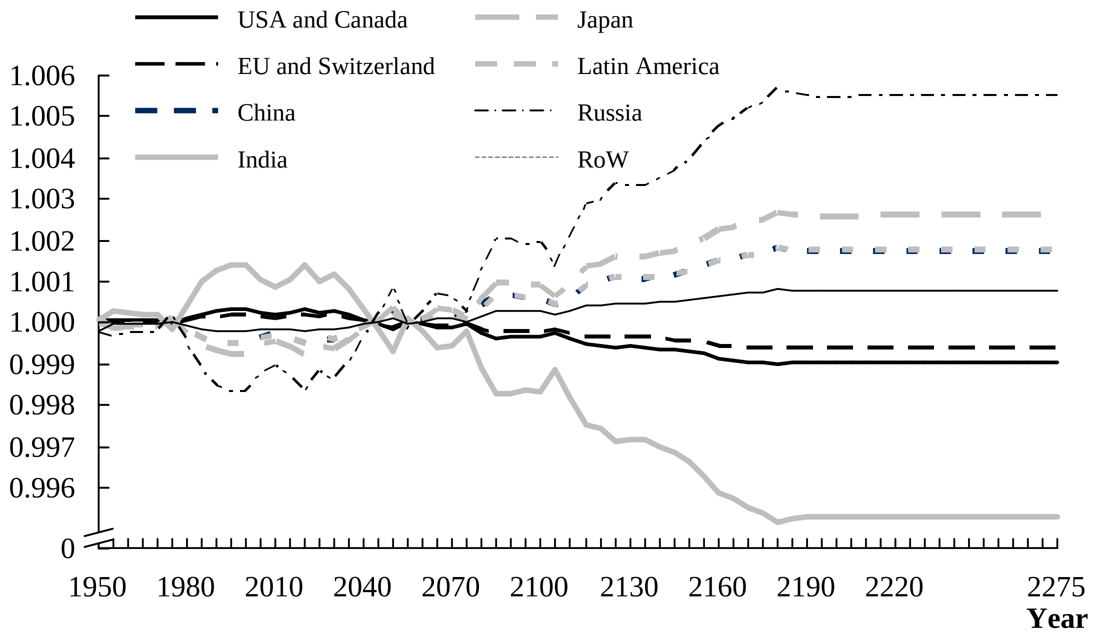
The compound price of foreign goods $p_{t,r}^{F}$ is calculated as a weighted average across tradable and non-tradable goods across all regions. It can be interpreted as the global average price of the goods bundle defined by $c_{t,r}^{T}$ and $c_{t,r}^{NT}$. Since this goods bundle is based on the respective domestic region, the foreign compound price includes index *r*.

$$p_{t,r}^{F} = \frac{\sum_{r'} \left[p_t^T \cdot c_{t,r}^T + p_{t,r'}^{NT} \cdot c_{t,r}^{NT} \right]}{\sum_{r'} \left[c_{t,r}^T + c_{t,r}^{NT} \right]} \tag{113}$$

Thus, the real exchange rate of a certain country is equal to the regional price of a goods bundle divided by the global average price of this same bundle.

The results of this analysis for all eight model regions are presented in Figure 5-16. Due to the relatively small effect on non-tradable prices discussed in section 5.2.8, the effects on exchange rates are also very small. In fact, the simulation suggests that exchange rates based on the demographic effect on non-tradable goods prices only move by less than 1 percent, even in the very long run.

Figure 5-16: Simulated real exchange rates in 2-sector-model



6 Variation Analysis of Key Model Parameters

The previous chapter leads to the conclusion that the inclusion of non-tradable goods in this thesis' global overlapping-generations-model only has a relatively small impact on model results. In order to further assess this finding, this chapter presents a variation analysis on several model parameters.

To better isolate the effects, the model is simplified down to two regions and now only includes EU and Switzerland as the primary region and 'Rest of World' as the secondary region. The 'Rest of World' region now covers all other regions except EU and Switzerland from the previous model setup with eight regions. Also for simplicity and transparency, the 'Rest of World' region is assumed to be on a balanced growth path with an annual growth rate of cohorts of 0.2% and lifespan of cohorts does not change for this region. That is, demographic phenomena only occur in the primary region, while the 'Rest of World' region is modeled without demographic effects and serves as a neutral trade partner for the primary region.

The objective of this chapter is no longer a realistic simulation, but rather the impact assessment of certain key parameters on model results. The strong simplification of the model as outlined above helps to separate and better understand the impact of changes in parameters. In other words, without these simplifications, the effect of parameter changes would be overlaid by complex demographic effects from other model regions, complicating the interpretation of parameter change effects.

Another modification compared to the original model setup involves the allocation of investment goods. While in the original setup, all investment goods were assumed to be tradable, they are now assumed to be fully non-tradable in the 2-sector-model. This modification is necessary in order to allow for very small sizes of the tradable sector without changing benchmark investment behavior of the model economies.

Using this modified setup, a series of parameter variations is performed in order to better understand the effects of a non-tradable goods sector on model results. The first analysis gradually increases the size of the tradable goods sector from zero to the status quo of the original model from the previous section. Next, the model is solved for different elasticities of substitution for tradable versus non-tradable goods consumption. Then, different values of parameter $¢^{NT}$ are assessed, which defines the rising demand for non-tradable goods over the lifecycle.

Finally, section 6.4 is again based on the full model setup with eight regions. This section analyzes the question, if the assumption of investment goods being fully tradable or non-tradable has any impact on model results.

6.1 Variation of the Size of the Non-Tradable Sector

The size of the non-tradable sector clearly should have an effect on results, since the extreme cases are equivalent to a closed economy on the one end and a perfectly open economy on the other end. That is, an economy with 100 percent non-tradable goods is not able to trade and is thus equivalent to a closed economy. Similarly, an economy with zero percent non-tradable goods is equivalent to the conventional 1-sector-model with perfect trade openness. Accordingly, this section addresses the question of how different shares between zero and 100 percent of the non-tradable goods sector impacts the model behavior.

Recalling the Social Accounting Matrix in Figure 4-5, the share of the tradable goods production sector is given by the sum of capital and labor input for the tradable goods sector divided by the sum of capital and labor input for both production sectors. In the original data from the SAM as used in the model (Table A.8), this share of tradable goods amounts to 16.5 percent for EU and Switzerland[77].

This share ¥ is exogenously set for the purpose of this analysis to 2 percent, 5 percent and 10 percent respectively. The following formulae are applied in order to redistribute the sector sizes without changing the aggregate size of both sectors.

[77] This is the case under the new assumption that all investment goods are non-tradable.

$$
\begin{aligned}
\overline{R}'^{T} &= \text{¥}\cdot\left(\overline{R}^{T}+\overline{R}^{NT}+\overline{L}^{T}+\overline{L}^{NT}\right)\cdot\frac{\overline{R}^{T}}{\overline{R}^{T}+\overline{L}^{T}}\\
\overline{L}'^{T} &= \text{¥}\cdot\left(\overline{R}^{T}+\overline{R}^{NT}+\overline{L}^{T}+\overline{L}^{NT}\right)\cdot\frac{\overline{L}^{T}}{\overline{R}^{T}+\overline{L}^{T}}\\
\overline{R}'^{NT} &= \overline{R}^{T}+\overline{R}^{NT}-\overline{R}'^{T}\\
\overline{L}'^{NT} &= \overline{L}^{T}+\overline{L}^{NT}-\overline{L}'^{T}
\end{aligned}
\qquad \textbf{(114)}
$$

$\overline{R}'^{T}, \overline{R}'^{NT}$, $\overline{L}'^{T}, \overline{L}'^{NT}$: Modified SAM-values

¥ : Exogenous share parameter tradable sector

$\overline{R}^{T}, \overline{R}^{NT}$: Original SAM-value for capital value added

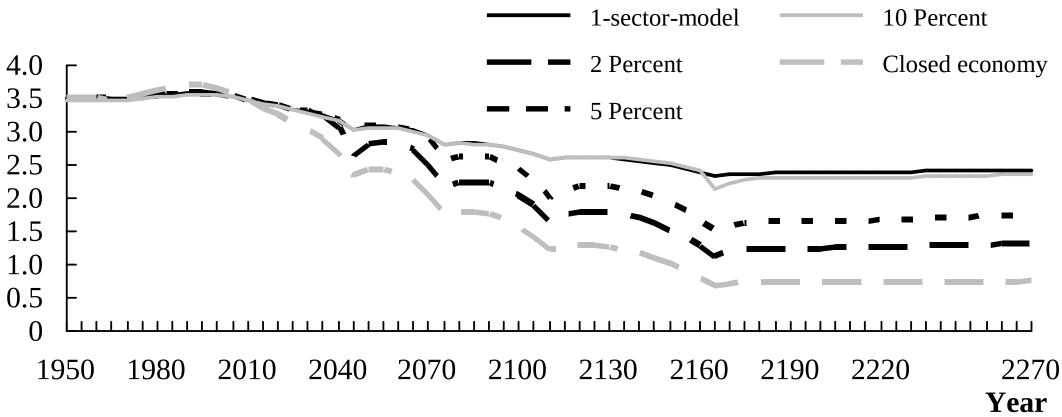
$\overline{L}^{T}, \overline{L}^{NT}$: Original SAM-value for labor value added

These modifications are performed for both the primary region (EU and Switzerland) as well as for the secondary 'Rest of World' region.

In addition to these modifications, the preference shares of tradable goods consumption versus non-tradable goods consumption in the SAMs are also modified to match the same respective shares ¥ .

The model was solved for five different parameter values ¥ . In addition to the three interior values of two, five, and ten percent, the model was also solved for the extreme cases of $\text{¥} = 1$, equivalent to the 1-sector-model with full trade openness, and $\text{¥} = 0$, equivalent to a closed economy. Figure 6-1 exhibits the resulting interest rate of these five model runs in the primary region.

Figure 6-1: Interest rate development in primary model region for different sizes of tradable goods sector

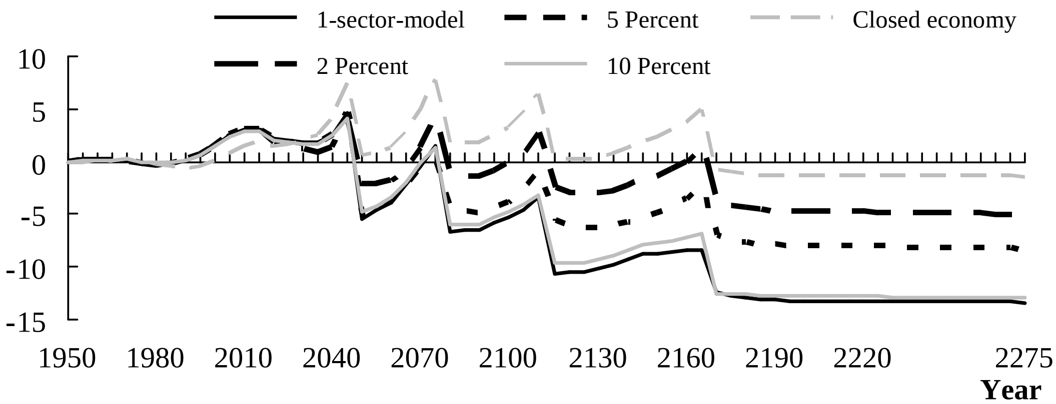


Clearly, the impact of demographics is larger for the closed economy than for the perfectly open economy (1-sector-model). This finding is in line with observations made in the literature[78]. Already for a benchmark share of tradable goods of 10 percent, the resulting interest rate is very close to the case of perfect openness. For values of ¥ below 10 percent, the non-tradable goods sector starts to play a significant role and the differences between the two models become more pronounced. For ¥ $= 0.05$, the maximum difference to the 1-sector-model is already as large as 70 base points. The next step from ¥ $= 0.05$ to ¥ $= 0.02$ significantly widens this difference to 110 base points. Finally the closed-economy model shows a maximum difference to the open-economy 1-sector-model of 170 base points.

This analysis suggests that the inclusion of non-tradable goods in the model indeed may have a very significant impact on model results. However, if the share of tradable goods is large enough, this difference becomes negligible. As can be inferred from the analysis above, tradable-goods-shares as low as 10 percent are sufficient to ensure negligible differences to the 1-sector-model in this sample simulation.

This finding is supported by other model variables as well. Figure 6-2 presents the simulated paths of GDP per capita. Again, the path of the model with ¥ $= 0.1$ is very close to the 1-sector-model, while the curves with smaller sizes of the tradable sector show significant deviations from the 1-sector-model.

Figure 6-2: Change in GDP per capita in percent in primary model region for different benchmark sizes of tradable sector[79]

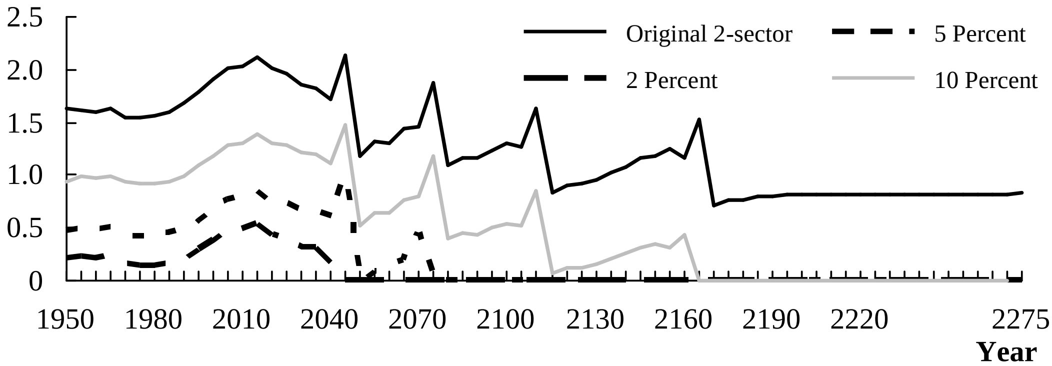


[78] E.g. Krueger and Ludwig (2006)
[79] The peaks in this graph are due to the assumption of 5-year-periods. In this simplified model with demographics occurring in only one region, every step-change in life-expectancy of one period (5 years) leads to a peak in GDP per capita. However, setting the period length to 5 years is in line with input data availability and has less impact on the main model developed in chapter 3.

To better understand the mechanisms that lead to this result, Figure 6-3 shows the absolute value of tradable production in the primary model region for different values of ¥ and for the 2-sector-model with the original size of the tradable goods sector. The curves with the original tradable sector size and the curve with ¥ = 0.1 are offset by roughly 0.6 trillions due to the different benchmark sector sizes. However, they run very much in parallel over the model horizon with equal amplitude in swings or peaks. The curves with ¥ = 0.05 and ¥ = 0.02 are clearly not able to follow with the same amplitude in swings, since they hit *zero* after year 2045. It may thus be inferred from this analysis, that the non-negativity constraint of the tradable production sector is the main reason for first-order differences occurring in Figure 6-1 and Figure 6-2.

Accordingly, significant differences between the conventional 1-sector-model and a 2-sector-model with non-tradable goods are expected to occur in case that the tradable production in the 2-sector-model falls down to zero at some point in the model horizon. This, in turn, is likely to happen for small benchmark sizes of the tradable sector and for strong exogenous demographic phenomena, leading to large swings in trade balance and thus tradable production.

Figure 6-3: Tradable production in trillion USD in primary model region for different benchmark sizes of tradable sector



The benchmark size of the tradable goods sector may also have a tremendous impact on prices of non-tradable goods. That is, the conclusion from section 5.2.8 and 5.2.9 that non-tradable goods prices and real exchange rates do not vary significantly over the model horizon is only true if the tradable goods sector is large enough. Figure 6-4 exhibits the price of non-tradable goods relative to prices of tradable goods in the primary model region for different benchmark sizes of the tradable goods sector.

Similar to the findings above, the price differences between tradable and non-tradable goods are small for the original size of the tradable goods sector and for the case of ¥ = 0.1. For ¥ = 0.05 and ¥ = 0.02, however, the price of non-tradable goods is significantly higher than the price of tradable goods.

The reason for the strong increase in relative prices of tradable goods is an abundance of tradable goods in the primary model region after year 2040. Due to their longer lifespan, cohorts try to accumulate more assets and thereby invoke a negative trade balance of the primary region. However, this negative trade balance is constrained by the very small consumption preference for tradable goods of consumers. That is, if domestic production of tradable goods is *zero*, the trade balance is equal to imports, and these imports must be fully consumed as tradable goods. In case this preference share is small (e.g. ¥ = 0.02) and the trade balance is strongly negative, then consumers are ‘forced’ to consume a higher share of tradable goods than they prefer, which reduces the price of tradable goods relative to non-tradable goods. Figure 6-3 serves as evidence for this abundance of tradable goods in the primary model region for small values of ¥, since domestic tradable production falls down to *zero* between 2050 and 2070.

Figure 6-4: Price of non-tradable goods in primary model region for different benchmark sizes of tradable sector (Index 1 = Price of tradable goods)

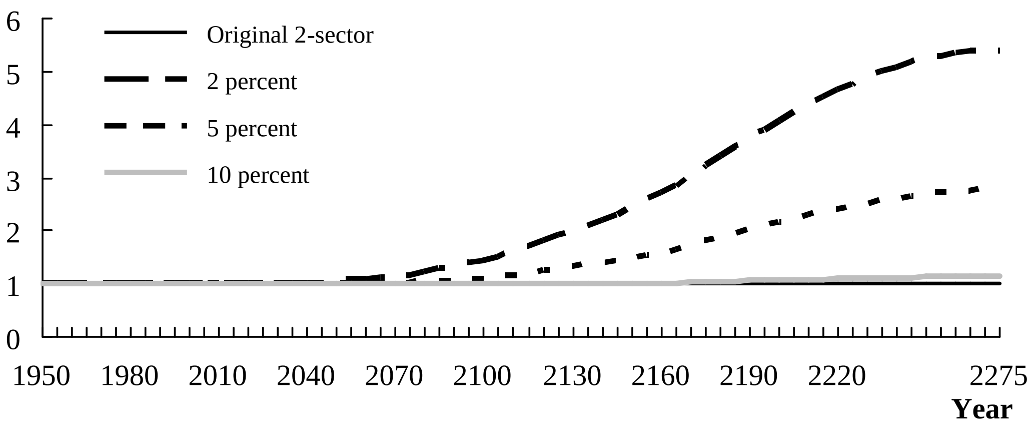
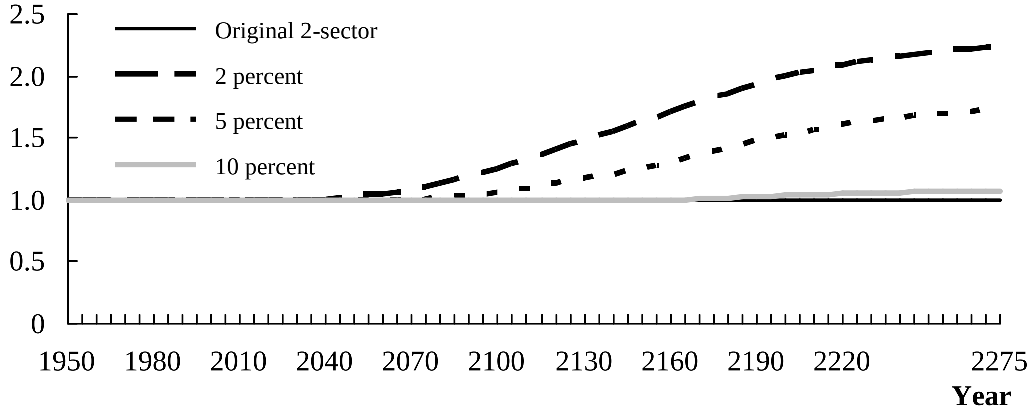


Figure 6-5 shows the resulting real exchange rates of the primary model region as defined in section 5.2.9. Due to the strong increase in prices of non-tradable goods in the primary model region for small benchmark sizes of the tradable goods sector, the real exchange rate also increases for these two scenarios. Thus, non-tradable goods may have a significant effect on real exchange rates as defined above, if the benchmark share of tradable goods is small.

Figure 6-5: Exchange rate of primary model region for different benchmark sizes of tradable production sector



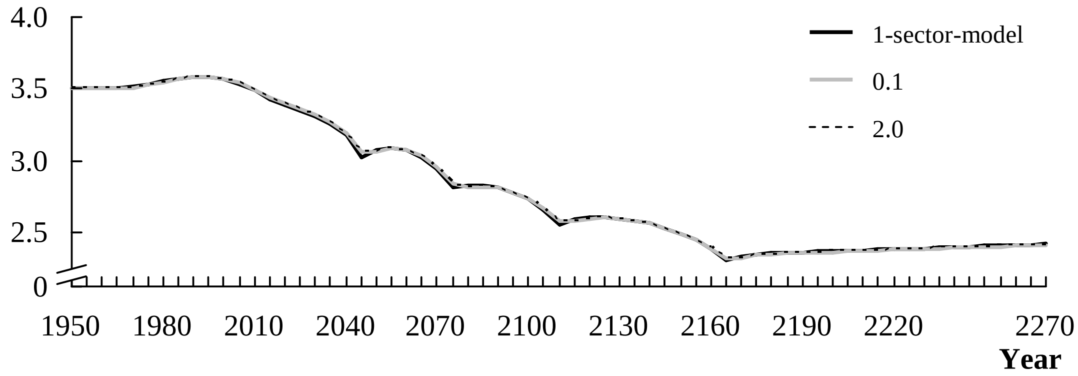
6.2 Variation of Elasticity of Substitution of Tradable versus Non-Tradable Goods Consumption

The elasticity of substitution of tradable vs. non-tradable goods in the preference function of consumers describes consumers' willingness to substitute consumption of non-tradable goods by tradable goods or vice versa. A very high value of this elasticity implies that tradable and non-tradable goods are almost perfect substitutes. This would lead to very small price differences between tradable and non-tradable goods in case of a shift in supply and demand. On the other hand, a very small value of this elasticity would c.p. lead to higher price differences, thereby 'forcing' the production side of the economy to more closely match the demanded relative shares of tradable and non-tradable goods. Thus, a very high value of this elasticity should lead to model results close to the 1-sector-model, while very small values should increase the difference of the 2-sector-model to the 1-sector-model.

In order to assess the impact of this parameter, the elasticity of substitution of tradable vs. non-tradable goods in the preference function of consumers is modified. The model is solved for $\pm = 0.1$ and the original value of $\pm = 2.0$, while the size of the tradable goods sector is at its original level.

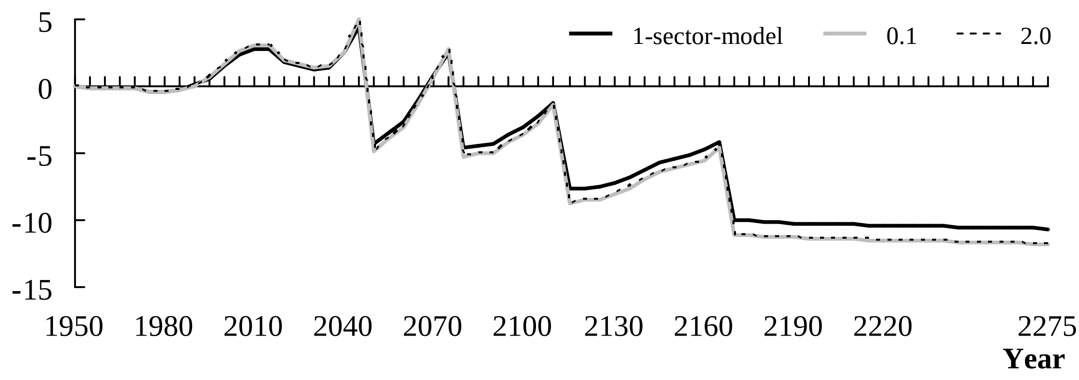
Figure 6-6 compares the resulting interest rates of the 2-sector-models with these two different elasticity values and the 1-sector-model. Clearly, for this setup, the influence of the modified elasticity value on the interest rate is negligible.

Figure 6-6: Interest rate in percent for different elasticities of substitution of tradable vs. non-tradable goods consumption



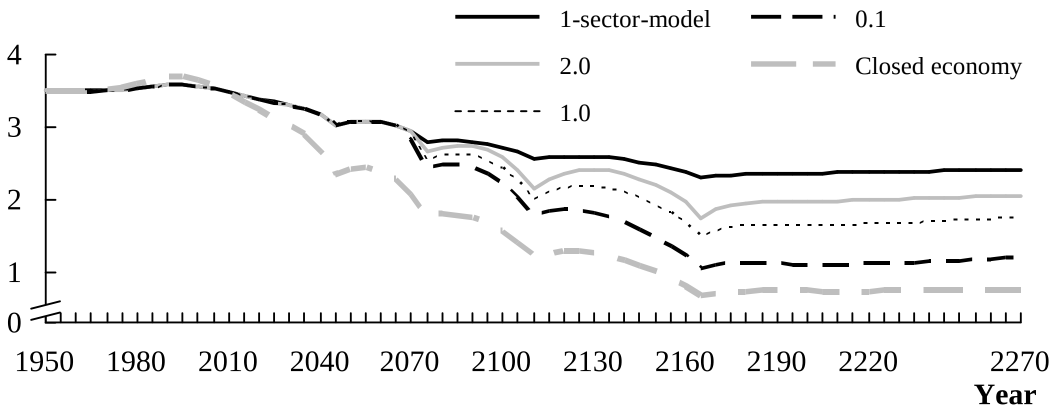
Also for other variables, this observation holds true. Figure 6-7 shows the GDP per capita for these different values of ± and for the 1-sector-model. While there is a small but visible difference between the 1-sector-model and the 2-sector-models, the difference between the models with ± = 2.0 and ± = 0.1 is again negligible.

Figure 6-7: Change in GDP per capita in percent in primary model region for different values of ±



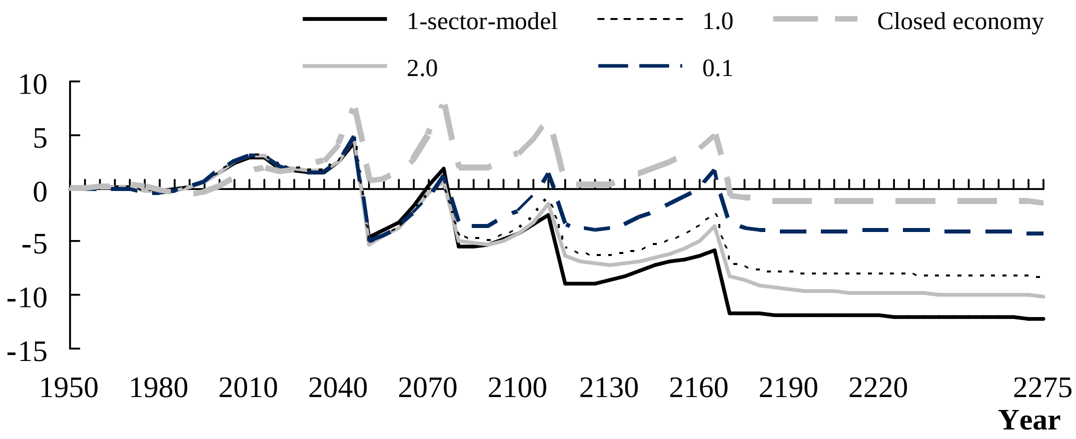
However, elasticity of substitution ± has a significant impact, if the size of the tradable goods sector is small enough. In the following experiment, the size of the tradable sector is set to 5 percent and the model is solved for different values of ± . Figure 6-8 compares the results of this experiment. Clearly, the curves for different elasticities of substitution ± deviate from each other, and again the spectrum is spanned by the extreme cases of the perfectly open 1-sector-model on the one side and the closed economy on the other side. The effect is highly non-linear, since the distance between the curves for ± = 1 and ± = 2 is roughly of the same magnitude as the distance between curves for ± = 0.1 and the closed economy.

Figure 6-8: Interest rate in percent in primary model region for different values of ± and with ¥ = 0.05



Again, this observation also holds true for other model variables, as Figure 6-9 shows for the example of GDP per capita.

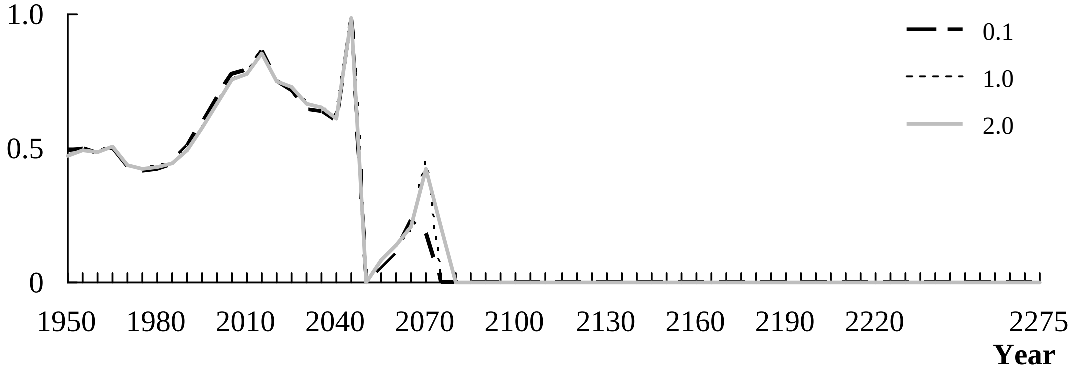
Figure 6-9: Change in GDP per capita in percent in primary model region for different values of ± and with ¥ = 0.05



However, as Figure 6-10 shows, the picture is different for the tradable production. This variable only shows significant deviations for different elasticity values in model years 2050 through 2080. This phenomenon is due to the fact that the model variables for different values of ± in general do not deviate significantly from each other until year 2040, as supported by

Figure 6-8 and Figure 6-9. Significant differences only occur after year 2040, when tradable production in Figure 6-10 also shows some differences, before it falls to zero thereafter.

Figure 6-10: Tradable production in trillion USD in primary model region for different values of ± and with ¥ = 0.05

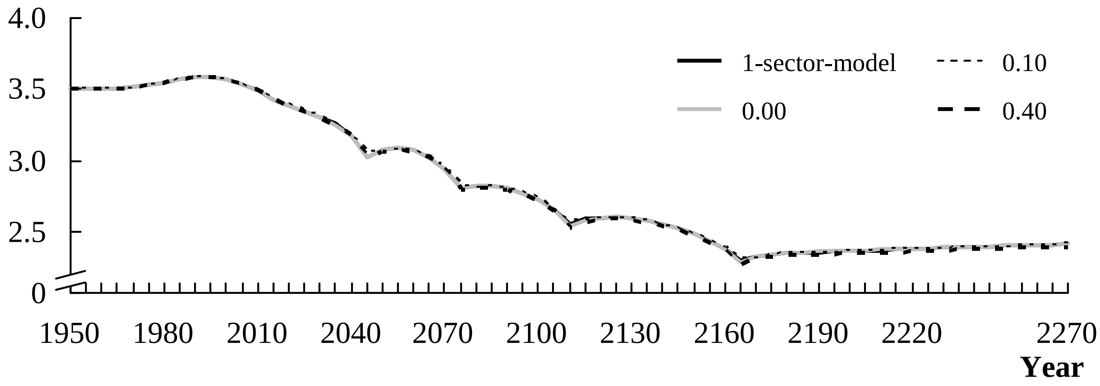


6.3 Variation of Preference for Non-Tradable Goods over Lifecycle

The third variation analysis solves the simplified model described above for different values of ¢^{NT}. This parameter defines the slope of the increasing preference for non-tradable goods over the lifecycle as specified in equation (99).

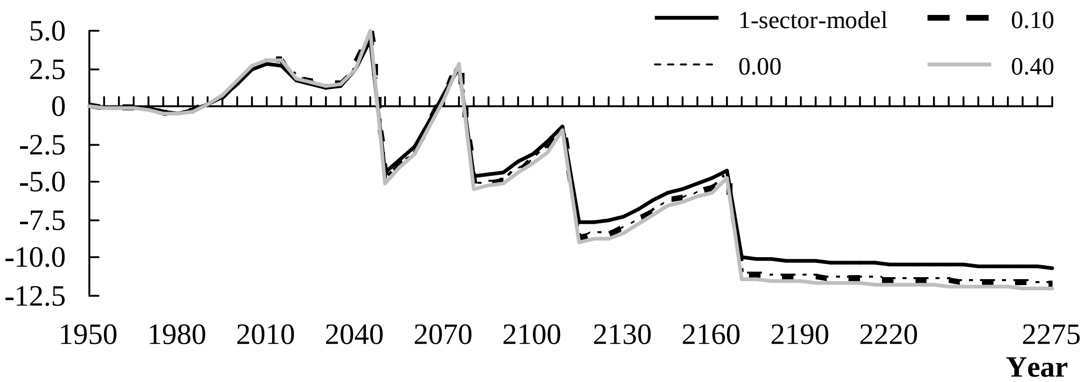
Intuitively, higher values of this parameter should lead to a shift from tradable towards non-tradable production over the course of the model horizon. This is due to the fact that the share of 'old' cohorts grows over the model horizon, and a high ¢^{NT} results in old cohorts having a relatively high preference for non-tradable goods. However, other effects of a change in this parameter are intuitively hard to predict. Figure 6-11 shows the results of this experiment with regards to the interest rate and it turns out that the influence on the interest rate of ¢^{NT} is in fact negligible. This analysis was performed for the original size of the tradable goods sector.

Figure 6-11: Interest rate in percent in primary model region for different values of $¢^{NT}$



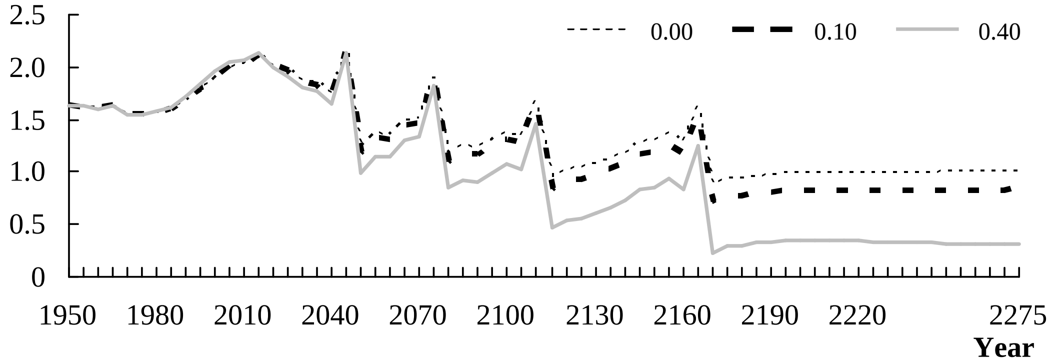
Also for the GDP per capita presented in Figure 6-12, deviations between different curves for different parameter values of $¢^{NT}$ are negligible. Again, upon interpreting this somewhat surprising result, one should keep in mind that the model is calibrated to match the exact same benchmark cases for each single value of $¢^{NT}$, such that the aggregate preference for all values is the same in benchmark year 2008.

Figure 6-12: Change in GDP per capita in percent in primary model region for different values of $¢^{NT}$



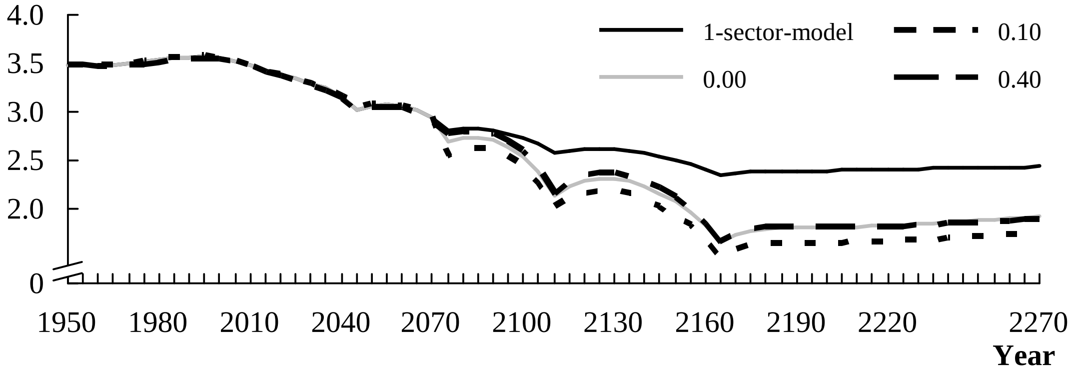
For tradable production, however, the picture is different. As Figure 6-13 shows, tradable production for higher values of $¢^{NT}$ tends to be lower, especially in the far future. This is due to the fact that the share of 'old' cohorts increases over the course of the model horizon due to longer lifespans and smaller birth rates. If $¢^{NT}$ is high, there is a high difference between the preference for non-tradable goods between old cohorts and young cohorts and old cohorts will relatively demand more non-tradable goods than the young. In combination, this leads to a decreasing aggregated preference share for tradable goods over the model horizon. Due to optimization behavior of the production sectors, the tradable production adjusts to the consumers' preferences. Accordingly, tradable production for $¢^{NT} > 0$ is smaller in future years than for $¢^{NT} = 0$.

Figure 6-13: : Tradable production in trillion USD in primary model region for different values of $¢^{NT}$



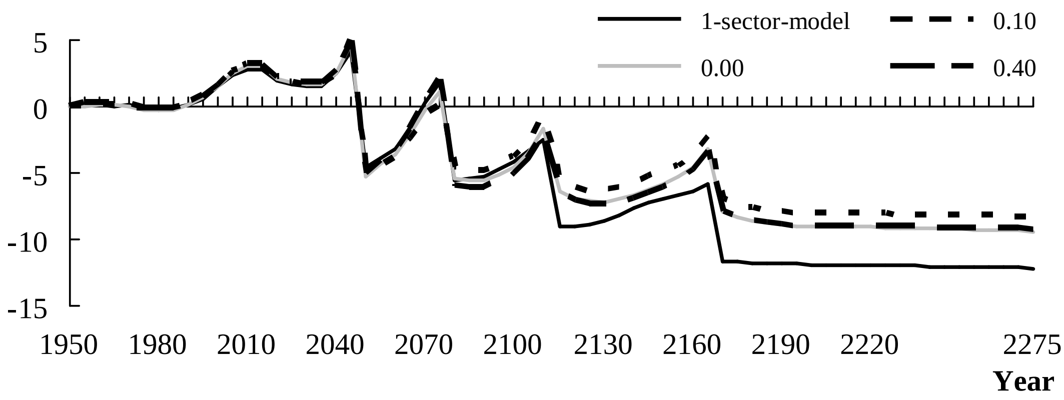
As for the previous parameter variation of elasticity of substitution, the variation of $¢^{NT}$ is also performed for a smaller share of tradable goods of $= 0.05$. Figure 6-14 compares the results of this analysis regarding interest rates and shows the interest rate development for three different values of $¢^{NT}$ and for the 1-sector-model. However, the outcome of section 6.2 that the elasticity parameter only makes a difference if the benchmark share of tradable goods is small, does not hold for $¢^{NT}$. In fact, within the range of $0 \leq ¢^{NT} \leq 0.40$, this parameter only has a weak influence on aggregate model results, regardless of the benchmark size of the tradable goods sector.

Figure 6-14: : Interest rate in percent in primary model region for different values of $¢^{NT}$ and with $= 0.05$



This finding is also supported by the GDP per capita development for $= 0.05$ and different values of $¢^{NT}$, which is presented in Figure 6-15. Again, the curves only show little deviations between the different values of ¢ ^{NT}. As already shown previously, the results for the 1-sector-model deviate from the other curves, because the tradable production share is set to $= 0.05$.

Figure 6-15: Change in GDP per capita in percent in primary model region for different values of ¢ ^{NT} and with $= 0.05$



6.4 Original Model with Non-Tradable Investment Goods

The original model presented in chapters 3, 4, and 5 incorporates the assumption that all investment goods are tradable in nature. However, this assumption artificially augments the tradable sector, since the calibration procedure as described in chapter 4 shifts any investment made from non-tradable goods according to the real data Social Accounting Matrices to the tradable sector. However, as pointed out in section 6.1, differences between the conventional 1-sector-model and the 2-sector-model with non-tradable goods are higher, if the benchmark size of the tradable goods sector is small. Thus, the assumption made in chapters 3, 4, and 5 of investment goods being fully tradable could lead to an underestimation of the impact of non-tradable goods on model results.

The raw versions of the SAMs reveal the actual values of tradable and non-tradable investment. For USA and Canada, investment of tradable goods amounts to 1.2 billion USD and investment of non-tradable goods amounts to 1.8 billion USD (Table 4-2). Thus, the larger portion of investment consists of non-tradable goods. However, as pointed out in section 4.7.1, the total GDP of the non-tradable goods sector is significantly higher than the GDP of the tradable goods sector. Therefore, in relative terms, the share of invested tradable output in total tradable output is higher than the invested share of non-tradable output. More specifically, 41% of tradable output are invested (1.2 billion USD tradable investment out of 2.9 billion USD tradable GDP), while only 15% of non-tradable output are invested (1.8 billion USD non-tradable investment out of 12.4 billion USD non-tradable output). Based on these findings, arguments for both modeling approaches exist. It can be argued that investment should be modeled as fully tradable, since the share of investment in tradable output is significantly higher than in non-tradable output. On the other hand, it can be argued that investment should be modeled as fully non-tradable, since the absolute amount of non-tradable investment is higher than the absolute amount of tradable investment.

Thus, to further assess this potentially crucial assumption, this section modifies the original model from chapters 3, 4, and 5 to the other extreme, such that all investment goods are now non-tradable.

For this purpose, the model equations presented in section 4.4 are revised as follows. First, the zero-profit-condition for the investment activity[80] is redefined, such that the price of capital goods in the following period is now linked to the price of non-tradable goods in the current period:

$$\begin{aligned} \tilde{\Pi}^{I}_{t,r} &= p^{K}_{t+1,r} - p^{NT}_{t} \leq 0 \\ s.t. \quad \tilde{\Pi}^{I}_{t,r} &\cdot \Omega^{I}_{t,r} = 0 \end{aligned} \tag{115}$$

Next, the market-clearing-conditions for tradable and non-tradable goods need to be updated accordingly. The investment term moves from the condition for tradable goods to the condition for non-tradable goods. The updated market-clearing-condition for tradable goods, which replaces equation (71), thus takes the following form:

$$\begin{aligned} \tilde{\Xi}^{T}_{t} &= \sum_{r} \Omega^{T}_{t,r} \cdot \overline{Y}^{T}_{r} + \sum_{r} \frac{B_{t,r}}{p^{T}_{t}} - \sum_{r}\sum_{g} \Omega^{c}_{g,t,r} \cdot \overline{c}^{T}_{g,t,r} \cdot \left(\frac{p^{c}_{g,t,r}}{p^{T}_{t}} \right)^{\pm} \geq 0 \\ s.t. \quad \tilde{\Xi}^{T}_{t} &\cdot p^{T}_{t} = 0 \end{aligned} \tag{116}$$

The revised market-clearing-condition for non-tradable goods now includes the investment term. However, the summation across regions in the investment term from equation (71)

[80] See equation (67)

needs to be omitted. Therefore, the original market-clearing-condition for non-tradable goods given by equation (72) is replaced by the following equation.

$$\tilde{\Xi}_{t,r}^{NT} = \Omega_{t,r}^{NT} \cdot \bar{Y}_r^{NT} - \Omega_{t,r}^{I} \cdot \bar{I}_r - \sum_g \Omega_{g,t,r}^{c} \cdot \bar{c}_{g,t,r}^{NT} \cdot \left(\frac{p_{g,t,r}^{c}}{p_t^{NT}} \right)^{\pm} \geq 0 \qquad \textbf{(117)}$$

$$s.t. \quad \tilde{\Xi}_{t,r}^{NT} \cdot p_{t,r}^{NT} = 0$$

After these modifications, the model structure now correctly reflects the assumption of investment goods being fully non-tradable.

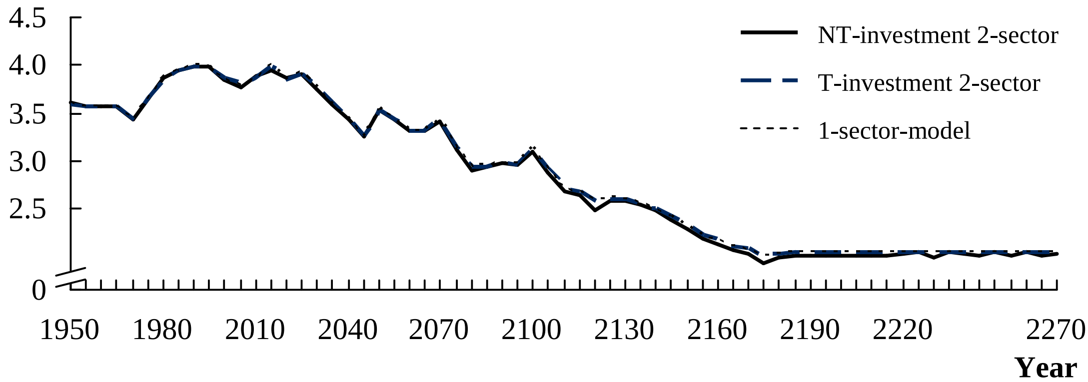
Modifications are also necessary in the Social Accounting Matrices, since the SAMs used to calibrate the model (see Table 4-6) allocate investment to the tradable goods sector. Thus, the investment is shifted from the tradable to the non-tradable sector and the sector GDPs are adjusted in proportion by changing the values CAP-OUTTR, CAP-OUTNT, LAB-OUTTR, and LAB-OUTNT. The result of these modifications is shown in Table 6-1 for the example of USA and Canada. As a side effect of this modification, the shift from tradable towards non-tradable production in combination with the higher labor value share of non-tradable production leads to less capital income and higher labor income. As the pension contribution rate is fixed at 10 percent, the pension contribution and pension benefits also increase slightly, such that the new Social Accounting Matrix is internally consistent.

Table 6-1: Social Accounting Matrix with non-tradable investment for USA and Canada

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	PAYGO	ROW	Total
OUTTR	0	0	0	0	0	1,712	0	0	1.567	3,280
OUTNT	0	0	0	0	0	7,587	2,646	0	0	10,233
CAP	341	2,559	0	0	0	0	0	0	0	2,900
LAB	709	7,675	0	0	0	0	0	0	0	8,383
TAX	0	0	0	762	0	0	0	0	0	762
CON	0	0	2,900	7,621	0	0	0	762	0	11,283
INV	0	0	0	0	0	1,984	0	0	663	2,646
PAYGO	0	0	0	0	762	0	0	0	0	762
ROW	2.230	0	0	0	0	0	0	0	0	2,230
Total	3,280	10,233	2,900	8,383	762	11,283	2,646	762	2,230	

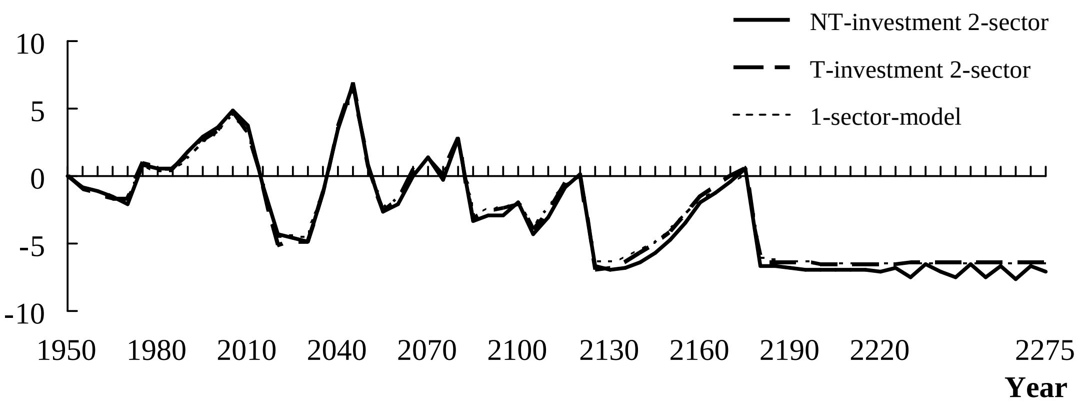
The result of this analysis with regard to interest rates is presented in Figure 6-16. The figure also compares the results of the modified model detailed above with the original 2-sector-model from chapters 3, 4, and 5 and the 1-sector-model. Clearly, all interest rate paths are very close to each other and the modeling assumption whether investment goods are tradable or non-tradable has no significant effect on resulting interest rates.

Figure 6-16: Interest rate in percent for models with tradable and non-tradable investment goods



Again, this observation is also true for other aggregate model variables. Figure 6-17 exhibits the GDP per capita paths for the three model types, which also show only minor deviations from each other.

Figure 6-17: Change in GDP per capita in percent in region USA and Canada for models with tradable and non-tradable investment goods



As a conclusion of this analysis, the assumption of investment goods being tradable or non-tradable does not have a significant impact on model results. That is, the size of the tradable goods sector in the actual model after this modification is larger than the threshold, below which significant deviations occur. The existence of such a threshold is shown in section 6.1.

7 Summary and Outlook

This thesis sets out to shed light on the question, whether incorporating non-tradable goods in a multi-country Auerbach-Kotlikoff overlapping-generations-model used to simulate demographic change has a significant impact on model results. This question is evoked by two facts. First, non-tradable goods, such as services for instance, constitute a major share of up to 80% of modern economies. Second, existing multi-country models from literature used to simulate demographic change ignore the existence of non-tradable goods and treat all goods as perfectly tradable. Thus, this thesis assesses the question, if this simplification of existing models might significantly distort simulation results.

In order to address this question, this thesis compares the simulation results of a 2-sector multi-region model, which features a tradable and a non-tradable goods sector in each region, with those of a 1-sector model with only tradable goods. Both models are calibrated to match exactly the same benchmark data in their respective steady states. That is, both yield exactly the same results in their steady states for all variables except those, which are broken down into tradable and non-tradable goods. For instance, the 1-sector model shows the same steady-state GDP, labor input, capital input, consumption, and trade balance as the 2-sector model. Even the cohort-specific calibration is equivalent, such that cohorts in both models show the same steady-state goods consumption, working time, and savings profile over their respective lifecycles. The only difference between models is the breakdown of production and consumption in the 2-sector model into a tradable and a non-tradable part, while the 1-sector model assumes all production and all consumption to be tradable. As a result, the sum of benchmark tradable and non-tradable production in the 2-sector model is equal to benchmark tradable production in the 1-sector model. Likewise, the sum of benchmark tradable and non-tradable consumption in the 2-sector model is equal to benchmark consumption in the 1-sector model.

This calibration effort is necessary to ensure that deviations between both models are only due to the more realistic model structure of the 2-sector model, and not just mere artifacts resulting from different baseline calibrations.

Both models are solved for a model horizon of a total of 325 years, corresponding to the period between 1950 and 2275 based on UN demographic historical data and projections.

However, the actual forecast data runs through 2100 and the period after 2100 serves as a phase-out to adjust the model to a final steady state. Both models cover a total of eight world regions, which represent the whole world.

A comparison of results of the 2-sector model with those of the 1-sector model clearly leads to the conclusion that the differences in results are small and negligible compared to other sources of error in such long-term projections.

For instance, differences in interest rate development are below 2.4 base points over the whole simulation horizon. Similarly, differences in GDP per capita, capital stock per capita, labor supply per capita, and welfare are also very small. Noticeable differences may only be observed for the development of wages, which show a deviation of 1.1 percentage points in the model region USA and Canada.

However, the differences between the 1-sector model with perfect tradability and the 2-sector model become very significant in magnitude, if the benchmark size of the tradable goods sector is set below a certain level. This result is derived from a variation analysis performed in this thesis, which solves a simplified version of the original 2-sector model with only two regions for different values of certain parameters. The analysis shows that significant differences occur for benchmark shares of the tradable goods sector of less than 10 percent. For a theoretical benchmark share of tradable goods as low as 2 percent, the model results approach the solution of an equivalent closed economy model. Clearly, benchmark shares of the tradable sector below 10 percent do not occur in the real world for significant economies. More specifically, USA and Canada have a tradable sector share of 20 percent, which is the smallest among the eight model regions. Thus, in practice, there is still a ‘safety margin’ in the sense that the tradable-goods sector would need to be less than half its actual size to lead to significant differences between 2-sector and 1-sector models.

Besides the tradable goods share, two further parameters are analyzed on the basis of a variation analyses. These two parameters are directly affiliated with the characteristics of the 2-sector model: the elasticity of substitution for tradable vs. non-tradable goods consumption in the consumers’ preference function and the increase of preference for non-tradable goods over the lifecycle. With regard to the first, the elasticity of substitution only has a significant impact on results, if at the same time the benchmark size of the tradable goods sector is very small. For the original sizes of tradable goods sectors as implied by the actual data, this parameter does not have a significant effect on model results.

The next parameter, describing the increase in preference for non-tradable goods for old cohorts, also does not have a significant impact on most model variables. This parameter describes the fact-based observation that old cohorts tend to consume a higher share of non-tradable goods than young cohorts. If this parameter is large, implying a high difference in preference for non-tradable goods between young and old cohorts, tradable production and consumption decreases over the model horizon, while non-tradable consumption increases at the same time. However, all other model variables, such as interest rates, total production, total consumption, prices and wages, are not significantly affected by this parameter.

One more analysis with a modified version of the original 2-sector model is motivated by the outcome of the variation analysis. It is shown that significant differences between the 2-sector and 1-sector model might occur for small benchmark sizes of the tradable goods sector. At the same time, for simplicity, the original model incorporates the assumption that all investment goods are tradable. This assumption, however, artificially inflates the size of the tradable sector and attenuates the size of the non-tradable sector compared to real data. Hence, this assumption might lead to an underestimation of the impact of non-tradable goods on results.

To assess this potentially critical assumption of tradable investment goods, the full model with all eight regions is again set up, but now with the assumption that all investment goods are non-tradable. However, as a comparison of results shows, this assumption also does not have a significant impact on model results.

As outlined in the introduction, a side effect of a non-tradable goods sector in the model structure could be price differences between non-tradable goods across regions, which in turn could lead to real exchange rate effects. However, as the results of the calibrated 2-sector model reveal, the price difference between non-tradable goods across regions is in the order of less than one percent. Accordingly, changes in real exchange rates calculated on the basis of these price differences are of a negligible order of magnitude.

Clearly, this thesis shows that ignoring the existence of non-tradable goods in a multi-region OLG-model to assess demographic change is a fair assumption in the sense that it is unlikely to significantly distort model results. As an outlook, further assumptions regarding trade openness in multi-region OLG-models used to simulate demographic change could be assessed.

First, imperfect trade openness could be modeled by introducing transport costs and/or tariffs on trade between world regions. Other than the concept of adding a non-tradable goods sector to a sector of perfectly tradable goods, this approach would lead to actual trade frictions already for small trade flows and could thus lead to different results.

Second, a combination of these concepts would be the most realistic, yet also the most complex model setup. This concept would include several sectors, each producing goods with different transport costs and thus different tradability. By introducing several sectors of this kind, a spectrum from almost perfectly tradable goods on the one end to non-tradable goods on the other end could be modeled.

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V Appendix

A.1. Raw SAMs from GTAP database for other model regions

Table A.1: Raw version of SAM from GTAP database for EU and Switzerland (2007)

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOVT	ROW	OTHER	REGHOUS	Total
OUTTR	17,039,751	2,936,598				2,891,919	1,134,511	43,105	4,650,704			28,696,588
OUTNT	2,453,086	28,534,552				6,151,440	2,624,706	3,512,228	1,235,983			44,511,994
CAP	1,313,924	5,015,139										6,329,063
LAB	1,671,173	4,671,975										6,343,148
TAX	1,324,171	2,233,141	446,087	1,619,090		1,148,500	17,594	1,450		3,606		6,793,640
CON											10,191,859	10,191,859
INV		-94,446	2,037,167						182,073		1,652,018	3,776,812
GOVT											3,556,782	3,556,782
ROW	4,853,726	1,215,034										6,068,760
OTHER	40,757											40,757
REGHOUS			3,845,809	4,724,058	6,793,640					37,151		15,400,659
Total	28,696,588	44,511,994	6,329,063	6,343,148	6,793,640	10,191,859	3,776,812	3,556,782	6,068,760	40,757	15,400,659	

All values in USD millions

Table A.2: Raw version of SAM from GTAP database for China (2007)

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOVT	ROW	OTHER	REGHOUS	Total
OUTTR	10,609,119	1,221,248				618,683	496,597	2,906	1,192,316			14,140,870
OUTNT	814,334	4,633,756				695,642	871,877	494,946	173,794			7,684,349
CAP	768,828	849,523										1,618,350
LAB	804,385	838,216										1,642,601
TAX	140,458	12,387	20,589	34,009		118,918	96,335	10,348		674		433,718
CON											1,433,243	1,433,243
INV		103	352,699						-294,978		1,406,987	1,464,811
GOVT											508,200	508,200
ROW	942,015	129,117										1,071,132
OTHER	61,731											61,731
REGHOUS			1,245,062	1,608,592	433,718					61,057		3,348,429
Total	14,140,869	7,684,349	1,618,350	1,642,601	433,718	1,433,243	1,464,810	508,200	1,071,132	61,731	3,348,431	

All values in USD millions

Table A.3: Raw version of SAM from GTAP database for India (2007)

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOVT	ROW	OTHER	REGHOUS	Total
OUTTR	1,639,927	237,068				315,560	168,196	11,308	157,118			2,529,177
OUTNT	234,007	1,504,408				390,981	230,842	121,493	64,702			2,546,432
CAP	203,283	348,619										551,902
LAB	153,955	395,185										549,139
TAX	56,086	12,008	19,983	16,391		26,116	25,232	1,295		400		157,512
CON											732,658	732,658
INV		5,086	105,036						53,122		261,026	424,270
GOVT											134,097	134,097
ROW	230,882	44,060										274,942
OTHER	11,037											11,037
REGHOUS			426,883	532,749	157,512					10,637		1,127,780
Total	2,529,177	2,546,432	551,902	549,139	157,512	732,658	424,270	134,097	274,942	11,037	1,127,780	

All values in USD millions

Table A.4: Raw version of SAM from GTAP database for Japan (2007)

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOVT	ROW	OTHER	REGHOUS	Total
OUTTR	4,638,715	715,267				557,638	316,389	402	681,919			6,910,330
OUTNT	680,895	7,003,208				1,831,069	673,694	786,917	60,204			11,035,987
CAP	317,802	1,265,273										1,583,075
LAB	437,121	1,540,286										1,977,407
TAX	265,929	418,934	150,203	230,280		108,911	19,443	50		398		1,194,149
CON											2,497,618	2,497,618
INV		-11,464	632,175						-71,967		460,781	1,009,526
GOVT											787,370	787,370
ROW	565,674	104,482										670,156
OTHER	4,194											4,194
REGHOUS			800,697	1,747,127	1,194,149					3,796		3,745,769
Total	6,910,330	11,035,987	1,583,075	1,977,407	1,194,149	2,497,618	1,009,526	787,370	670,156	4,194	3,745,769	

All values in USD millions

Table A.5: Raw version of SAM from GTAP database for Latin America (2007)

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOVT	ROW	OTHER	REGHOUS	Total
OUTTR	3,779,638	523,825				713,168	277,523	1,093	732,167			6,027,415
OUTNT	521,132	4,859,072				1,497,777	466,953	585,485	105,613			8,036,033
CAP	537,261	1,163,831										1,701,092
LAB	358,888	1,149,423										1,508,310
TAX	143,621	217,412	147,837	34,638		188,722	29,777	762		4,346		767,116
CON											2,399,668	2,399,668
INV		24,880	417,148						-106,511		438,737	774,254
GOVT											587,341	587,341
ROW	633,680	97,589										731,269
OTHER	53,196											53,196
REGHOUS			1,136,108	1,473,672	767,116					48,850		3,425,746
Total	6,027,415	8,036,033	1,701,092	1,508,310	767,116	2,399,668	774,253	587,341	731,269	53,196	3,425,746	

All values in USD millions

Table A.6: Raw version of SAM from GTAP database for Russia (2007)

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOVT	ROW	OTHER	REGHOUS	Total
OUTTR	1,357,220	270,203				192,731	87,952	1,996	335,716			2,245,818
OUTNT	217,795	1,741,307				424,325	186,161	235,050	41,628			2,846,266
CAP	165,458	411,419										576,877
LAB	131,783	324,653										456,436
TAX	101,767	38,600	30,648	103,957		51,519	13,658	40		3,231		343,422
CON											668,576	668,576
INV		6,039	60,826						-112,313		333,219	287,771
GOVT											237,087	237,087
ROW	210,986	54,045										265,031
OTHER	60,810											60,810
REGHOUS			485,402	352,478	343,422					57,579		1,238,882
Total	2,245,818	2,846,266	576,877	456,436	343,422	668,576	287,771	237,087	265,031	60,810	1,238,882	

All values in USD millions

Table A.7: Raw version of SAM from GTAP database for Rest of World (2007)

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOVT	ROW	OTHER	REGHOUS	Total
OUTTR	11,634,124	1,659,376				1,801,697	693,906	74,279	3,113,086			18,976,468
OUTNT	1,369,302	11,265,973				2,700,025	1,336,492	1,167,232	452,659			18,291,683
CAP	1,650,490	2,157,315										3,807,805
LAB	980,360	2,370,370										3,350,731
TAX	368,033	315,273	269,489	503,709		220,537	49,514	6,071		23,437		1,756,064
CON											4,722,259	4,722,259
INV		-916	921,031						-402,511		1,562,309	2,079,912
GOVT											1,247,583	1,247,583
ROW	2,638,942	524,292										3,163,234
OTHER	335,216											335,216
REGHOUS			2,617,285	2,847,022	1,756,064					311,778		7,532,149
Total	18,976,468	18,291,683	3,807,805	3,350,731	1,756,064	4,722,259	2,079,911	1,247,583	3,163,234	335,216	7,532,150	

All values in USD millions

A.2. Adjusted SAMs as used in the model for other model regions

Table A.8: SAM for EU and Switzerland as used in model

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOV	ROW	Total
OUTTR	0	0	0	0	0	1,949	3,303	0	3,967	9,218
OUTNT	0	0	0	0	0	5,950	0	0	0	5,950
CAP	1,581	1,411	0	0	0	0	0	0	0	2,991
LAB	3,548	4,540	0	0	0	0	0	0	0	8,088
TAX	0	0	0	735	0	0	0	0	0	735
CON	0	0	2,991	7,353	0	0	0	735	0	11,079
INV	0	0	0	0	0	3,181	0	0	123	3,303
GOV	0	0	0	0	735	0	0	0	0	735
ROW	4,089	0	0	0	0	0	0	0	0	4,089
Total	9,218	5,950	2,991	8,088	735	11,079	3,303	735	4,089	

All values in USD billions

Table A.9: SAM for China as used in model

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOV	ROW	Total
OUTTR	0	0	0	0	0	713	1,541	0	1,575	3,829
OUTNT	0	0	0	0	0	773	0	0	0	773
CAP	959	220	0	0	0	0	0	0	0	1,179
LAB	1,635	553	0	0	0	0	0	0	0	2,189
TAX	0	0	0	199	0	0	0	0	0	199
CON	0	0	1,179	1,990	0	0	0	199	0	3,367
INV	0	0	0	0	0	1,881	0	0	-340	1,541
GOV	0	0	0	0	199	0	0	0	0	199
ROW	1,235	0	0	0	0	0	0	0	0	1,235
Total	3,829	773	1,179	2,189	199	3,367	1,541	199	1,235	

All values in USD billions

Table A.10: SAM for India as used in model

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOV	ROW	Total
OUTTR	0	0	0	0	0	264	422	0	186	871
OUTNT	0	0	0	0	0	476	0	0	0	476
CAP	122	70	0	0	0	0	0	0	0	192
LAB	519	407	0	0	0	0	0	0	0	926
TAX	0	0	0	84	0	0	0	0	0	84
CON	0	0	192	842	0	0	0	84	0	1,118
INV	0	0	0	0	0	378	0	0	44	422
GOV	0	0	0	0	84	0	0	0	0	84
ROW	230	0	0	0	0	0	0	0	0	230
Total	871	476	192	926	84	1,118	422	84	230	

All values in USD billions

Table A.11: SAM for Japan as used in model

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOV	ROW	Total
OUTTR	0	0	0	0	0	420	949	0	559	1,928
OUTNT	0	0	0	0	0	1,932	0	0	0	1,932
CAP	505	528	0	0	0	0	0	0	0	1,033
LAB	918	1,404	0	0	0	0	0	0	0	2,321
TAX	0	0	0	211	0	0	0	0	0	211
CON	0	0	1,033	2,110	0	0	0	211	0	3,355
INV	0	0	0	0	0	1,003	0	0	-54	949
GOV	0	0	0	0	211	0	0	0	0	211
ROW	505	0	0	0	0	0	0	0	0	505
Total	1,928	1,932	1,033	2,321	211	3,355	949	211	505	

All values in USD billions

Table A.12: SAM for Latin America as used in model

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOV	ROW	Total
OUTTR	0	0	0	0	0	570	768	0	669	2,006
OUTNT	0	0	0	0	0	1,752	0	0	0	1,752
CAP	489	463	0	0	0	0	0	0	0	952
LAB	933	1,289	0	0	0	0	0	0	0	2,222
TAX	0	0	0	202	0	0	0	0	0	202
CON	0	0	952	2,020	0	0	0	202	0	3,174
INV	0	0	0	0	0	853	0	0	-85	768
GOV	0	0	0	0	202	0	0	0	0	202
ROW	584	0	0	0	0	0	0	0	0	584
Total	2,006	1,752	952	2,222	202	3,174	768	202	584	

All values in USD billions

Table A.13: SAM for Russia as used in model

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOV	ROW	Total
OUTTR	0	0	0	0	0	132	263	0	258	653
OUTNT	0	0	0	0	0	462	0	0	0	462
CAP	186	141	0	0	0	0	0	0	0	327
LAB	285	322	0	0	0	0	0	0	0	607
TAX	0	0	0	55	0	0	0	0	0	55
CON	0	0	327	552	0	0	0	55	0	934
INV	0	0	0	0	0	340	0	0	-77	263
GOV	0	0	0	0	55	0	0	0	0	55
ROW	181	0	0	0	0	0	0	0	0	181
Total	653	462	327	607	55	934	263	55	181	

All values in USD billions

Table A.14: SAM for Rest of World as used in model

	OUTTR	OUTNT	CAP	LAB	TAX	CON	INV	GOV	ROW	Total
OUTTR	0	0	0	0	0	1,523	1,984	0	3,014	6,521
OUTNT	0	0	0	0	0	2,823	0	0	0	2,823
CAP	1,279	722	0	0	0	0	0	0	0	2,001
LAB	2,568	2,101	0	0	0	0	0	0	0	4,669
TAX	0	0	0	424	0	0	0	0	0	424
CON	0	0	2,001	4,245	0	0	0	424	0	6,670
INV	0	0	0	0	0	2,324	0	0	-340	1,984
GOV	0	0	0	0	424	0	0	0	0	424
ROW	2,674	0	0	0	0	0	0	0	0	2,674
Total	6,521	2,823	2,001	4,669	424	6,670	1,984	424	2,674	

All values in USD billions

A.3. Model results for all regions

Figure A.1: Simulated wage rate development for all regions

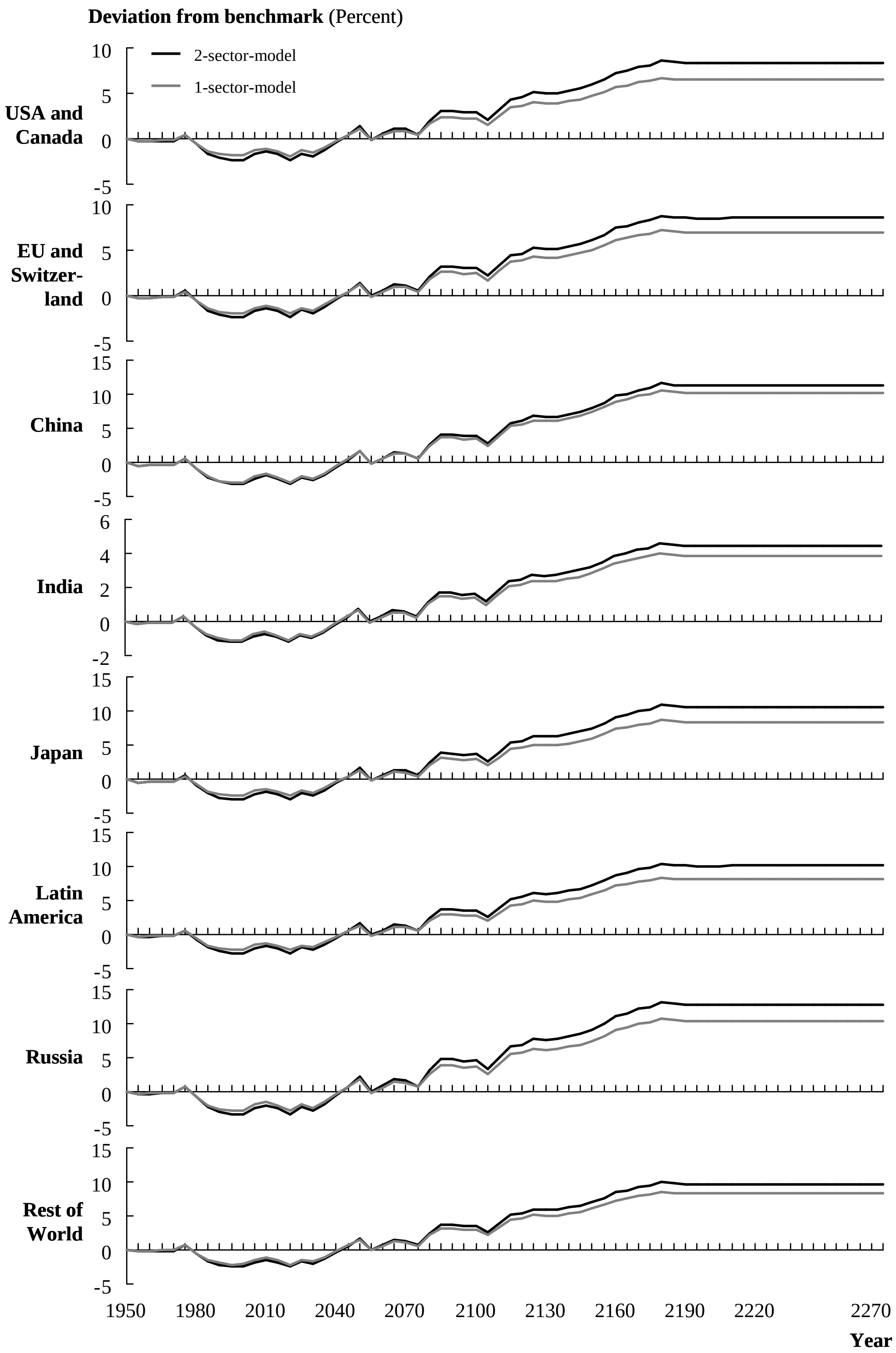

Figure A.2: Simulated capital stock per capita development for all regions

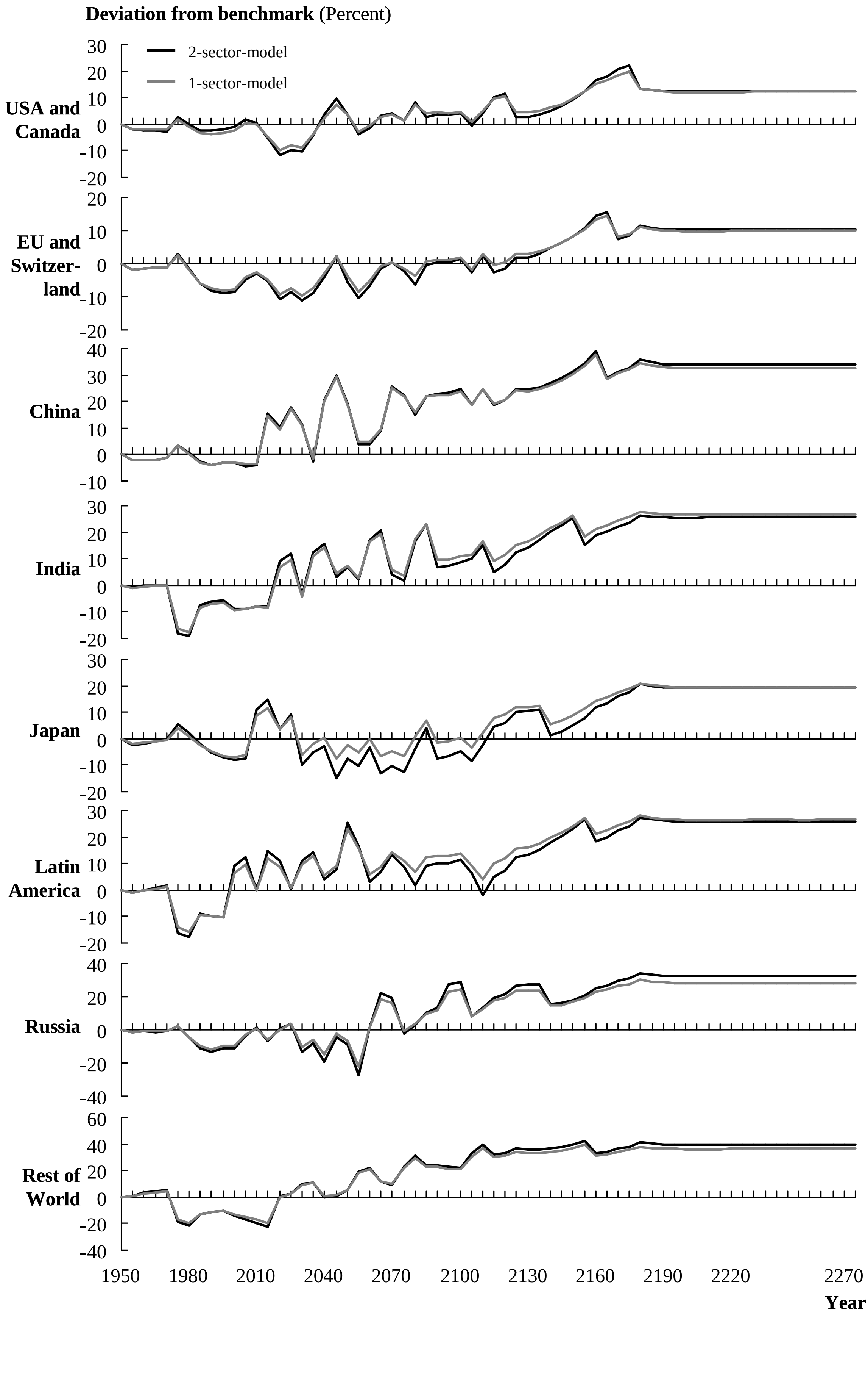

Figure A.3: Simulated GDP per capita development for all regions

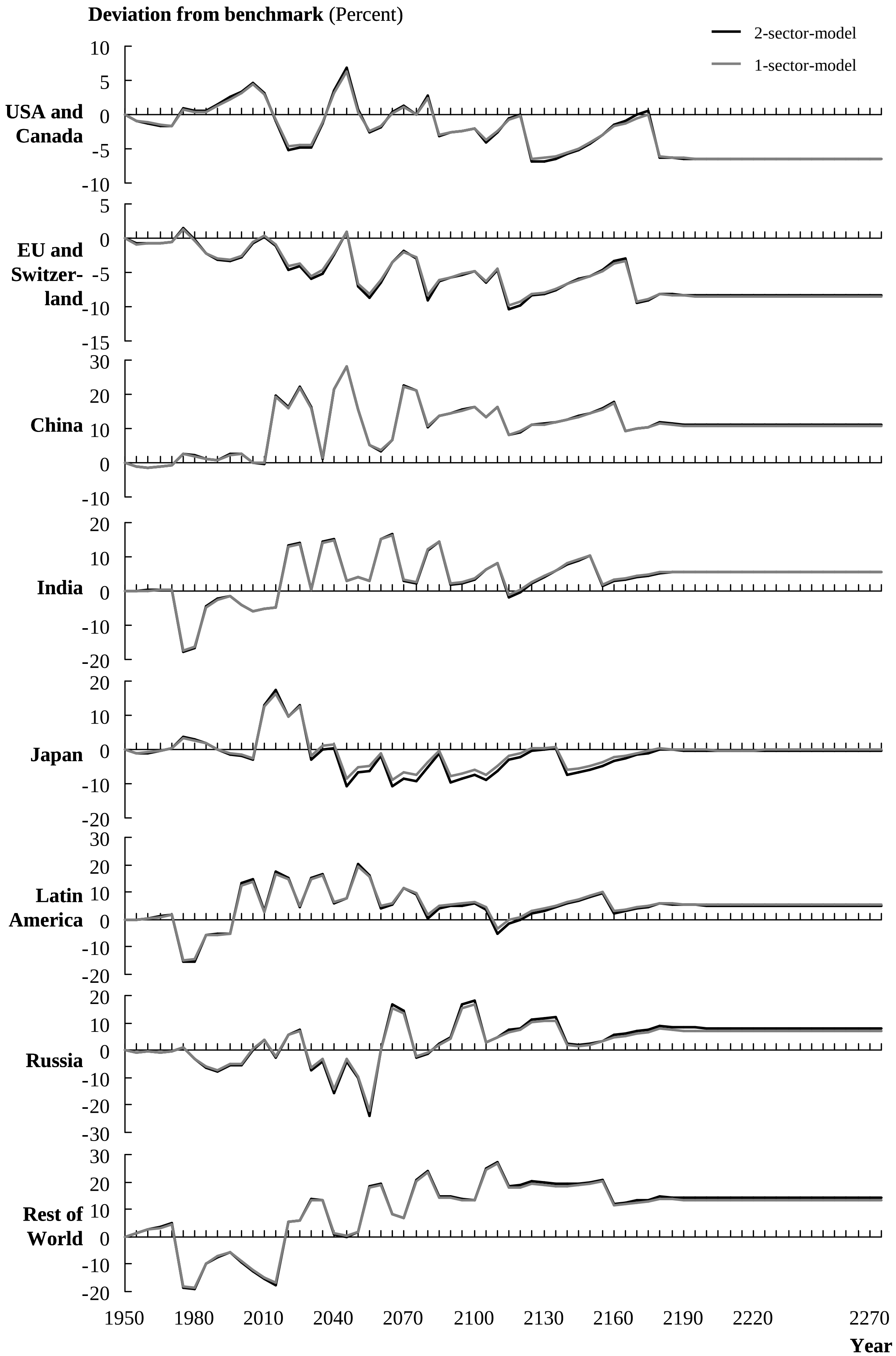

Figure A.4: Simulated labor supply per capita development for all regions

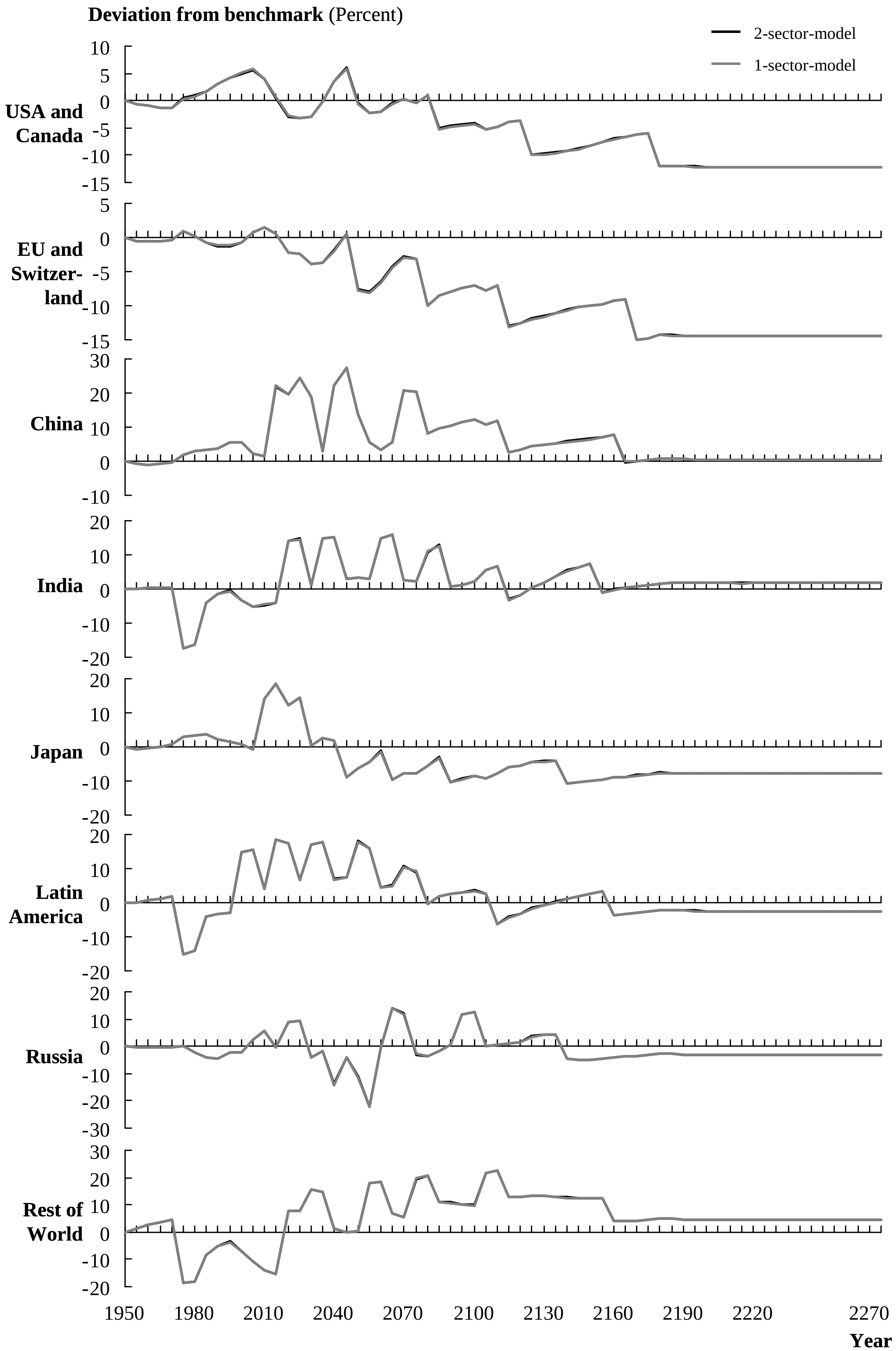

Figure A.5: Simulated development of labor supply per unit of productivity-weighted time endowment

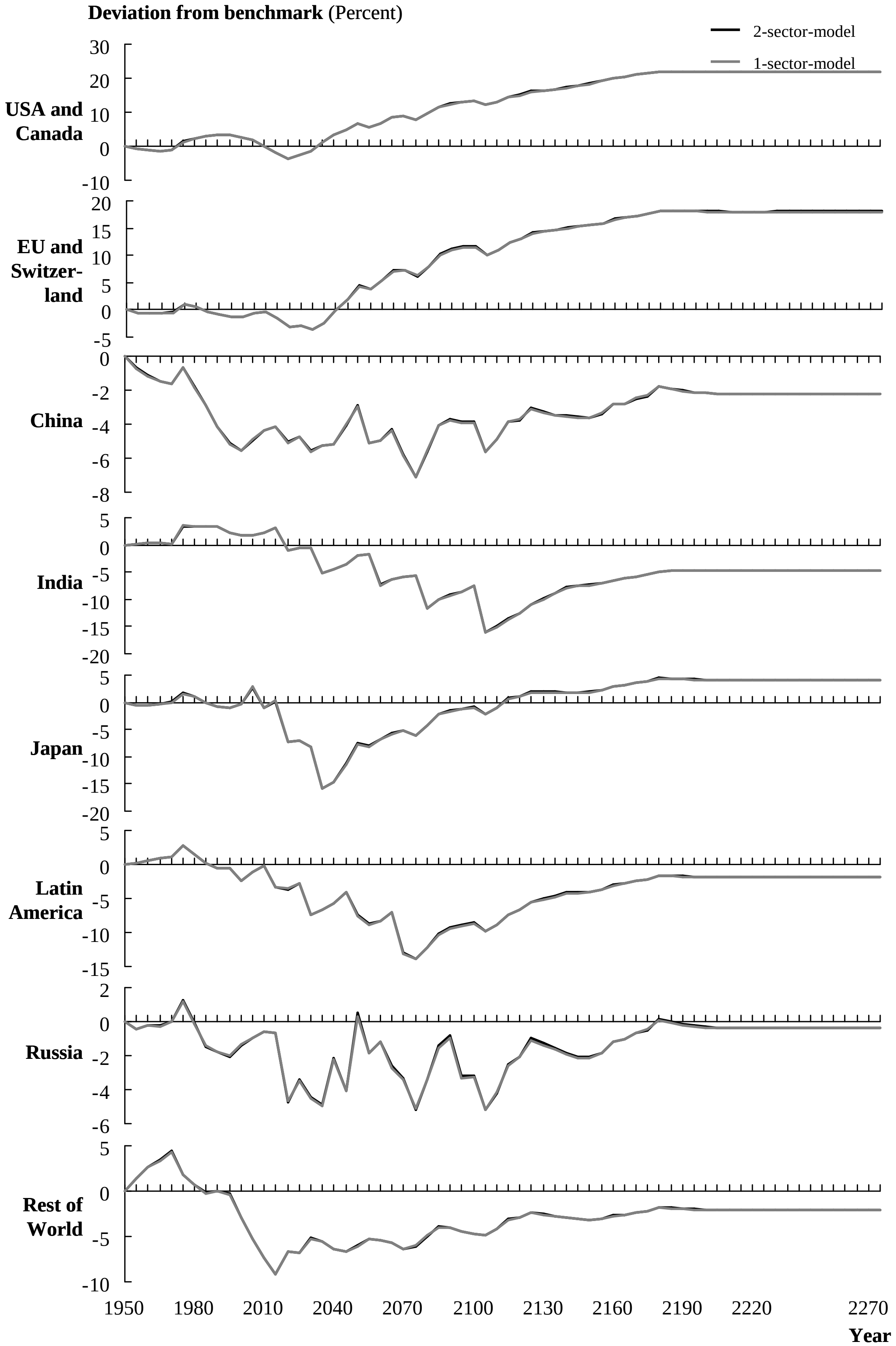

Figure A.6: Simulated welfare effects for all regions

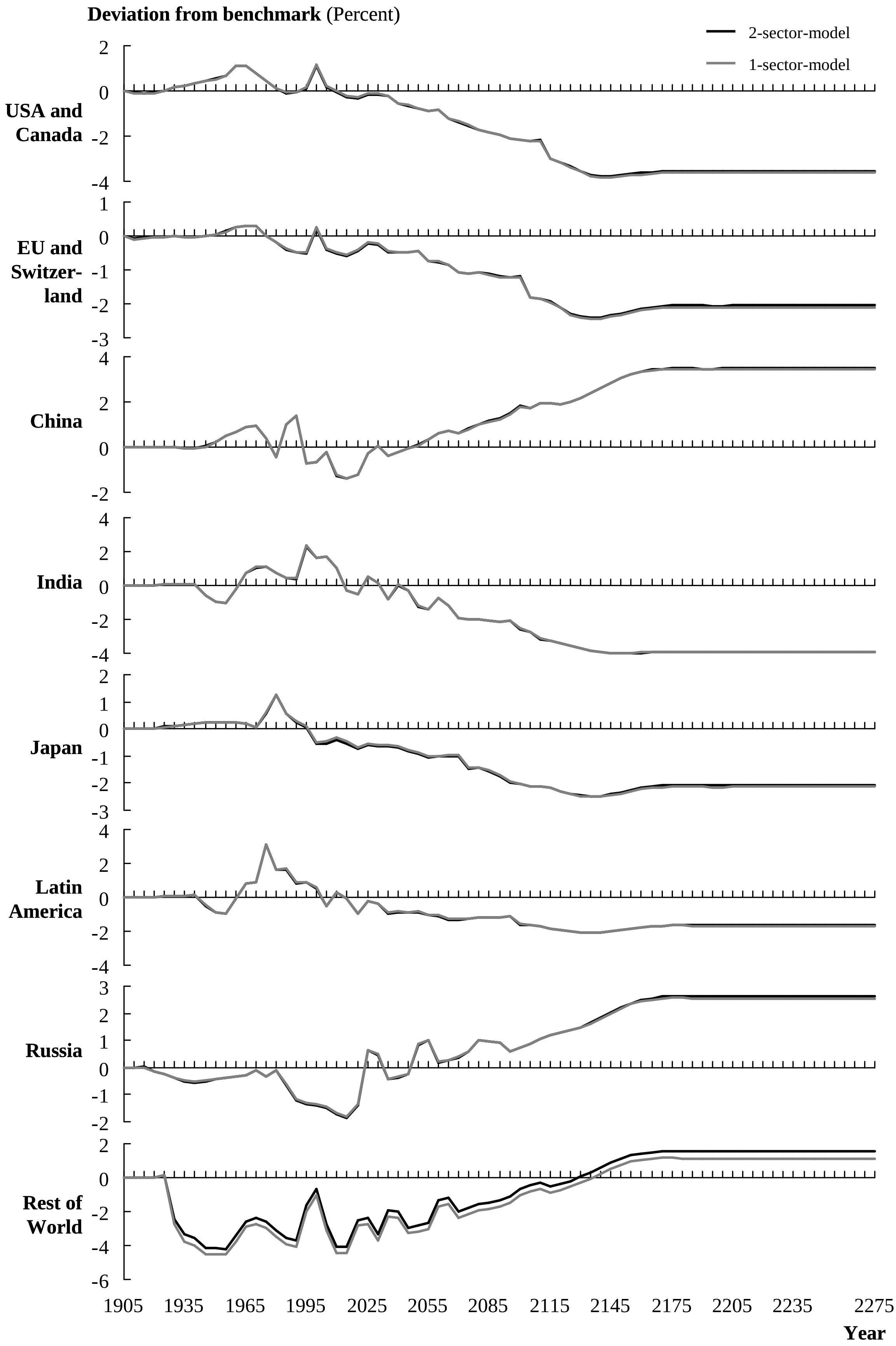

Figure A.7: Simulated trade balance per capita development