Brownian Motion in a Renormalized Inverse-Square Poisson Potential

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Abstract

We study a d-dimensional Brownian motion $(W_t)_{t\geq 0}$ moving in a Poisson potential $V(x) = \int_{\mathbb{R}^d} K(x-y)\omega(\mathrm{d}y)$, where ω is a standard Poisson point process on $\mathbb{R}^d$. It depends on the properties of the so called shape function K, whether the corresponding quenched and annealed Gibbs measures are well defined. In some cases, the lack of finiteness of the normalizing constant can be overcome by applying the renormalization

$$\overline{V}(x) = \int_{\mathbb{R}^d} K(x-y)[\omega(\mathrm{d}y) - \mathrm{d}y]$$

introduced in [CK12]. Taking $K(x) = \theta|x|^{-p}$, the finiteness of the positive quenched exponential moments $\mathbb{E}_0[\mathrm{e}^{\int_0^t \overline{V}(W_s)\,\mathrm{d}s}]$ depends on whether $p < 2$ or $p > 2$. In the case $p = 2, d = 3$ we show that a phase transition occurs at $\theta = \frac{1}{16}$ which is closely related to the optimal constant in the classical Hardy inequality. With the help of a *multipolar* Hardy inequality we determine the asymptotic behaviour of $\mathbb{E}_0[\mathrm{e}^{\int_0^t \overline{V}(W_s)\,\mathrm{d}s}]$ as $t \uparrow \infty$.

Zusammenfassung

Wir betrachten eine d-dimensionale Brownsche Bewegung $(W_t)_{t\geq 0}$ in einem zufälligen Potential der Form $V(x) = \int_{\mathbb{R}^d} K(x-y)\omega(\mathrm{d}y)$, wobei ω einen gewöhnlichen Poissonschen Punktprozess auf $\mathbb{R}^d$ bezeichnet. Es hängt von den Eigenschaften der Funktion K ab, ob die entsprechenden Gibbs-Maße wohldefiniert sind. Das Problem einer unendlichen Normalisierungskonstante kann in einigen Fällen mittels der in [CK12] vorgestellten Renormierung

$$\overline{V}(x) = \int_{\mathbb{R}^d} K(x-y)[\omega(\mathrm{d}y) - \mathrm{d}y]$$

gelöst werden; im Fall $K(x) = \theta|x|^{-p}$ hängt die Endlichkeit der quenched exponentiellen Momente $\mathbb{E}_0[\mathrm{e}^{\int_0^t \overline{V}(W_s)\,\mathrm{d}s}]$ davon ab, ob $p < 2$ oder $p > 2$. Im Fall $p = 2, d = 3$ zeigen wir, dass ein Phasenübergang im Wert $\theta = \frac{1}{16}$ auftritt. Dies steht in engem Zusammenhang zur optimalen Konstante in der klassischen Hardy-Ungleichung. Unter Zuhilfenahme einer *multipolaren* Hardy-Ungleichung bestimmen wir das asymptotische Verhalten von $\mathbb{E}_0[\mathrm{e}^{\int_0^t \overline{V}(W_s)\,\mathrm{d}s}]$ für $t \uparrow \infty$.

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For reasons of privacy protection, the acknowledgements are left blank in the electronic version of this thesis.

Contents

Introduction

In this thesis we study the model of Brownian motion in a renormalized Poisson potential. The model describes a particle following the path of a standard Brownian motion $(W_t)_{t\geq 0}$ in $\mathbb{R}^d$ under the influence of a random potential q which has the form

$$q(x) = \sum_{y\in\mathbb{R}^d\colon\, \omega(\{y\})=1} K(x-y) = \int_{\mathbb{R}^d} K(x-y)\omega(\mathrm{d}y). \tag{1}$$

Here, ω is a Poisson point process on $\mathbb{R}^d$ with the Lebesgue measure as its intensity measure and $K\colon \mathbb{R}^d \mapsto \mathbb{R}$ is a so called *shape function* modelling the impact that each Poisson point has on the particle. If K is negative, the Poisson points are viewed as *obstacles* or *traps* repelling the particle, whereas the particle is attracted by the Poisson points if K is positive. The general question in this kind of models is, how the presence of the potential q influences the motion of the Brownian particle. Amongst other things, the excitement in the subject comes from the fact, that in some cases the answers include some highly irregular behaviour compared to that of the free Brownian particle, and that intuitive strategies stand behind these results. The Brownian particle may search for a region where the potential takes extremely *high* values, i.e. where an extremely high concentration of Poisson points occurs, or, for a region with extremely *low* values, i.e. a large clearing in the forest of Poisson points. In the course of our investigation we want to determine (for a very specific choice of K) which regions are the best ones for the particle to reach, taking into account both the probabilistic costs of reaching some - possibly far away - region, and the energy $\int_0^t q(W_s)\,\mathrm{d}s$ absorbed this way.

Denoting the expectation with respect to the standard Wiener measure by $\mathbb{E}_0$, a lot of information is contained in the asymptotics of the quenched exponential moments

$$\mathbb{E}_0\left[\exp\left\{\theta\int_0^t q(W_s)\,\mathrm{d}s\right\}\right]. \tag{2}$$

It turns out that - for various choices of K - the main contribution to (2) comes from those Brownian paths which follow one of the two above indicated strategies. Many results have been obtained in the case where K is bounded and (or) compactly supported. However, a natural choice for K is

$$K(x) := \frac{1}{|x|^p}, \quad p\geq 1 \tag{3}$$

which is for example related to Newton's law of universal gravitation in the case $p = 2$. If we want to drop the restrictions of boundedness and a compact support, we have to be careful since $\int_{\mathbb{R}^d} K(x-y)\omega(\mathrm{d}y)$ might become infinite for all $x \in \mathbb{R}^d$ almost surely.

Recently, Chen and Kulik proposed a method called *renormalization* in order to solve this problem in some cases, cf. [CK12]. Here, the integration with respect to the Poisson point process $\omega(\mathrm{d}x)$ is replaced by an integration with respect to $\omega(\mathrm{d}x) - \mathrm{d}x$, where $\mathrm{d}x$ means integration with respect to the Lebesgue measure on $\mathbb{R}^d$. Chen et al. have studied the corresponding quenched and annealed exponential moments extensively in [CK11; CR11; Che12], focussing on the choice $K(x) = |x|^{-p}$. It turns out, that the renormalized potential

$$\overline{V}(x) = \int_{\mathbb{R}^d} K(x-y)[\omega(\mathrm{d}y) - \mathrm{d}y]$$

is well defined under the condition $d/2 < p < d$, and that for all $\theta > 0$, cf. [CK12, Theorem 1.5],

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right]\begin{cases} < \infty & \text{a.s., if } p < 2 \\ = \infty & \text{a.s., if } p > 2. \end{cases} \tag{4}$$

In the case $d = 3, p = 2$ we have according to [CR11, Theorem 2.1]

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right]\begin{cases} < \infty & \text{a.s., if } \theta < \frac{1}{16} \\ = \infty & \text{a.s., if } \theta > \frac{1}{16}. \end{cases} \tag{5}$$

Furthermore, the time asymptotics of $\mathbb{E}_0\left[\exp\left(\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right]$ reveal a strong and unusual shape dependence, i.e. a dependence on the parameters p and θ. In (5), the critical value $\theta = \frac{1}{16}$ is explained by the critical constant 4 in the famous Hardy inequality

$$\int_{\mathbb{R}^3} \frac{f(x)^2}{|x|^2}\,\mathrm{d}x \le 4\int_{\mathbb{R}^3} |\nabla f(x)|^2\,\mathrm{d}x, \quad f \in H^1(\mathbb{R}^3) \tag{6}$$

which can be used to bound the principal eigenvalues of the random Schrödinger operator $\frac{1}{2}\Delta + \overline{V}$ locally using the Rayleigh-Ritz variational formula. By the Feynman-Kac formula, spectral properties of this operator can be used to determine the asymptotics of the expectation (2).

In (5), the case $\theta = \frac{1}{16}$ is unanswered and this was the starting point for the present work. It is a plausible conjecture that $\mathbb{E}_0\left[\exp\left(\frac{1}{16}\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right] < \infty$, as the Hardy inequality still holds true at the critical point. However, the technique of proof used in [CR11] cannot be transferred to this case. In order to take into account the probabilistic costs of reaching regions far away from the origin, they make use of Hölder's inequality for some $q > 1$, i.e.

$$\begin{aligned} &\mathbb{E}_0\left[\exp\left(\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{W_s \text{ leaves a large ball until time } t\}}\right] \\ &\le \mathbb{P}[W_s \text{ leaves a large ball until time } t]^{\frac{1}{q}}\,\mathbb{E}_0\left[\exp\left(q'\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right]^{\frac{1}{q'}}. \end{aligned} \tag{7}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. When $\theta = \frac{1}{16}$, necessarily $q\theta > \frac{1}{16}$ and we are thrown into the 'bad' regime. To solve this problem, we have to implement a finer analysis of the interplay between the

Brownian particle and the random potential. This means, we are going to decompose the Brownian paths according to their entrance and exit times in and from some carefully chosen neighbourhoods of the set of Poisson points and control the contribution coming from each visit using semigroup and resolvent bounds. Furthermore, we have to make use of a more general variant of the classical Hardy inequality, called *multipolar Hardy inequality*.

This way, we are able to extend (5) to the case $\theta = \frac{1}{16}$. Moreover we determine the asymptotic scale $r(t)$ for which the family

$$r(t)^{-1} \log \mathbb{E}_0 \left[\exp \left(\theta \int_0^t \overline{V}(W_s) \, \mathrm{d}s \right) \right], \qquad t \geq 1 \tag{8}$$

is tight on the open interval $(0, \infty)$. We also characterize the limes superior and limes inferior in (8). With respect to the latter we get a result which conflicts with the corresponding Theorem 2.3 in [CR11]. However, it should be mentioned that we highly benefit from the work that has been done in [CK12; Che12; CR11] and that we adopt many of the results and methods appearing in these papers.

This work is structured as follows. In our first section, we give an overview of our model and a rigorous definition. We present results that have been obtained in the non-renormalized setup and finally introduce the concept of renormalization. In Section 2, we present our main results and the main ideas behind its proofs. Section 3 discusses techniques from semigroup theory and inequalities of multipolar Hardy type. We will explain the criticality of the multipolar inverse-square Schrödinger operator $\frac{1}{2}\Delta + \theta \sum_y |x-y|^{-2}$ from a functional analytic point of view and derive some upper and lower bounds on the Feynman-Kac functional. Section 4 contains the path decomposition theorem. In Section 5, we study asymptotics of small distances in Poisson clouds and translate the results into growth bounds for principal eigenvalues of Schrödinger operators. Finally, Section 6 is devoted to the proofs of the main theorems.

The proofs of the main theorems have been achieved in a joint work with Renato Soares dos Santos (WIAS Berlin), and will be published in a joint article. Because of this collaboration, I have to clarify my own proportion of this work. However, since many of the ideas came up during our joint work and during joint discussions, it is clearly not possible to trace back each contribution unequivocally.

I have worked out independently the overview section (Section 1), the sketches of the proofs (Section 2.2) and the supplementary information on Schrödinger semigroups, inverse-square potentials etc. in Sections 3.1 - 3.2. Among the results of Section 3, I credit Proposition 3.19 and the lower bounds in Section 3.3.2 to R.S. dos Santos, while the upper bounds in Section 3.3.1 could be attributed more to my work. The path expansion in Section 4 in its final version is due to R.S. dos Santos, even though there exists an earlier version, differing in its structure, that I have drawn up independently. The last two Sections 5 - 6 have been worked out by myself, but R.S. dos Santos supported me with many important remarks.

1 From “Soft” and “Hard” Obstacles to the Renormalized Poisson Potential: An Overview

In this section, we introduce the model of Brownian motion in a Poisson potential and present some of the main questions in the field. We explain, why it is useful to consider a renormalization to widen the class of admissible potentials, and give an overview about the results on the renormalized potential that have been obtained so far.

1.1 Brownian Motion in Random Potential

There exist many models describing random motion in random media both in a discrete and in a continuous setting. The most basic and well studied models in the discrete setting include the *random walk in random environment* on $\mathbb{Z}^d$ which is produced by endowing each vertex of the lattice with a $2d$ - dimensional random vector determining the transition probabilities to the neighbouring vertices, the *random walk among random conductances*, with transition probabilities proportional to random weights on each bond, and the *random walk in random potential*, which is the discrete analogue of the model we study in this work. Here, the potential is a family of random variables $(q(x))_{x\in\mathbb{Z}^d}$ and the law of a simple, time-continuous random walk X_t is tilted by the functional $\exp\big(-\int_0^t q(X_s)\,\mathrm{d}s\big)$. The resulting Gibbs measure puts weight on those paths that make $\int_0^t q(X_s)\,\mathrm{d}s$ small. Due to the relation to the Cauchy problem for the heat equation with random potential,

$$\begin{aligned} &\partial u(t,x) = \Delta u(t,x) + q(x)u(t,x), \qquad (t,x)\in(0,\infty)\times\mathbb{Z}^d,\\ &u(0,x)\equiv 1, \qquad x\in\mathbb{Z}^d, \end{aligned} \tag{1.1}$$

the model is also called the (discrete) *parabolic Anderson model*. Here, Δ is the discrete Laplacian on $\mathbb{Z}^d$, $\Delta f(x) = \sum_{y:|x-y|=1}[f(y)-f(x)]$. The relation comes from the fact that by the Feynman-Kac formula and under some regularity assumptions, the solution $u(t,x)$ to (1.1) is given by the functional $\mathbb{E}_x\left[\exp\left(-\int_0^t q(X_s)\,\mathrm{d}s\right)\right]$. For further introductory information about the discrete world of random media see e.g. [BS02; Zei04; DR14; Kön16].

Here we study the continuous version of the problem (1.1). A standard Brownian motion $(W_t)_{t\geq 0}$ is moving through a random field $(q(x))_{x\in\mathbb{R}^d}$. Mathematically, this is similarly described by tilting the standard Wiener measure by the functional $\exp\left(-\int_0^t q(W_s)\,\mathrm{d}s\right)$. Again, by the Feynman-Kac formula this model is closely related to the continuous parabolic Anderson model given by

$$\begin{aligned} \partial u(t,x) &= \Delta u(t,x) + q(x)u(t,x), \qquad (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) &= u_0(x), \qquad x \in \mathbb{R}^d, \end{aligned} \tag{1.2}$$

with the continuous Laplacian $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial^2 x_i}$.

For the potential q, various choices have been studied in the literature. Examples include:

- *Gaussian fields*, obeying sufficient regularity properties (e.g. the existence of a Hölder continuous modification), c.f. [GKM00] and [GK00]
- *Gaussian white noise*, where the high degree of irregularity yields serious technical problems to even get a rigorous definition. Recently this has led to the development of new methods (regularity structures, rough paths), c.f. [FH14] and [GIP15]
- *Poisson obstacles*, with $q(x)$ given by $-\sum_{y\in\mathbb{R}^d:\omega(\{y\}=1)} K(x-y)$ with a Poisson point process ω and a nonnegative shape function K,
- *Poisson shot noise potentials*, where q is given as in the Poisson obstacle case but K is taken nonpositive, and
- *Gibbsian point potentials*, allowing correlations between particles of the underlying point process, cf. [Szn93b].

In the following, we will only consider potentials based on a Poisson point process, beginning with a rigorous definition in the next section.

Literature on Poisson Potentials

For an overview on results obtained until 1998, the major reference is the book by Sznitman [Szn98] which mainly treats the case of Poisson obstacles. For the same topics cf. the review article [Kom00] by Komorowski. It is out of the scope of our work to list even the most important articles in the field. We refer to the book by König [Kön16] which is the most recent introduction to the field and contains a lot of results and further references concerning the continuous parabolic Anderson model, although its main topic is the discrete one. However, for the Poisson shot noise case we especially refer to [CM95], [GKM00] and [GK00].

Motivation from Physics

Besides the purely mathematical interest in the subject, a strong motivation for the treatment of random media comes from various fields like thermodynamics, hydrodynamics, geophysics (seismology), astrophysics, oceanography, biology and chemistry. In physics, think of a transport process: a tracer moving in an inhomogeneous hydrodynamic flow (caused by turbulence or porous media) or a charged particle (e.g. electron) moving in a medium with impurities. In these examples, the medium is too complex to be modelled deterministically or the composition is not known and thus it is described by a random process.

The physical interest has a long tradition. In 1892 Rayleigh [RS92] raises the problem of heat (or electricity) conductance in a medium with cylindrical obstacles arranged in rectangular order. Von Smoluchowski [Smo17] considers a diffusion among randomly distributed static traps in 1917 and Taylor [Tay22] studies a diffusion in a turbulent fluid in 1922. From questions of homogenization research turned to localization and intermittency in the 1960's. Grassberger and Procaccia [GP82] investigate Poisson distributed obstacles in 1982. A lot of results have been guessed here before being formulated rigorously by mathematicians. For more information and further references the interested reader is referred to [BG90] and [HB87] (especially Section 7 treating Poisson traps), [Mol91] and [Mol94], written from the probabilist's point of view, and [HK87] for the lattice case.

1.2 Results and Methods for Non-Renormalized Potentials

1.2.1 Model and Notation

Within the entire work we denote a standard d-dimensional Brownian motion by $(W_t)_{t\geq 0}$ and the corresponding Wiener measure by $\mathbb{P}_x$ when $W_0 = x$ for $x \in \mathbb{R}^d$. The expectation related to $\mathbb{P}_x$ will be denoted by $\mathbb{E}_x$. The random potentials we will consider in the following will be based on a standard Poisson point process ω on $\mathbb{R}^d$ with the Lebesgue measure $\mathrm{d}x$ as its intensity measure and with the corresponding probability measure and expectation denoted by $\mathbb{P}$ and $\mathbb{E}$ respectively. Write $\mathcal{P} = \{x \in \mathbb{R}^d : \omega(\{x\}) = 1\}$ for the random set of Poisson points, which is $\mathbb{P}$-a.s. a countable, closed set having no accumulation points. To describe the impact each Poisson point has on the Brownian motion, we take a so called 'shape function' $K\colon \mathbb{R}^d \to \mathbb{R}$ and define the corresponding Poisson potential by

$$q(x) = \sum_{y\in\mathcal{P}} K(x-y) = \int_{\mathbb{R}^d} K(x-y)\omega(\mathrm{d}y). \tag{1.3}$$

The amount of potential absorbed by the Brownian particle until time $t \geq 0$ is then described by the integral $\int_0^t q(W_s)\,\mathrm{d}s$ and we model the Brownian motion moving in the random potential generated by ω and q by transforming the Wiener measure $\mathbb{P}_x$ to the corresponding quenched and annealed Gibbs measures $\mathbb{Q}_{t,\omega}$ and $\mathbb{Q}_t$ defined as

$$\mathbb{Q}_{t,\omega} = \frac{1}{Z_{t,\omega}} \exp\left\{\pm\int_0^t q(W_s)\,\mathrm{d}s\right\}\mathbb{P}_0 \quad \text{and} \quad \mathbb{Q}_t = \frac{1}{Z_t} \exp\left\{\pm\int_0^t q(W_s)\,\mathrm{d}s\right\}\mathbb{P}\otimes\mathbb{P}_0 \tag{1.4}$$

with the normalizing constants

$$Z_{t,\omega} = \mathbb{E}_0\left[\exp\left\{\pm\int_0^t q(W_s)\,\mathrm{d}s\right\}\right] \quad \text{and} \quad Z_t = \mathbb{E}\otimes\mathbb{E}_0\left[\exp\left\{\pm\int_0^t q(W_s)\,\mathrm{d}s\right\}\right]. \tag{1.5}$$

In the quenched setting a specific configuration of the Poisson points is fixed. Thus, one is interested in statements that hold with high probability when this configuration is chosen randomly according to $\mathbb{P}$ or that hold $\mathbb{P}$-a.s. In the annealed setting instead, the average over all possible configurations is taken. Alternatively, the annealed measure can be viewed as an ergodic averaging over the starting point x of the Brownian motion.

It is one of the crucial questions, what regularity assumptions must be made on K to ensure that the Gibbs measures in (1.4) are well defined. Especially, we want to avoid the case $\mathbb{P} \otimes \mathbb{P}_0\big[\int_0^t q(W_s)\,\mathrm{d}s = \infty\big] > 0$. It turns out, that the so called *Kato class*

$$K_d = \left\{ K\colon \mathbb{R}^d \to \mathbb{R} \text{ Borel measurable, } \lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \mathbb{E}_x \left[\int_0^t |K(W_s)|\,\mathrm{d}s \right] = 0 \right\} \tag{1.6}$$

is a class of sufficiently nice shape functions, although later we will extend this class. For the moment we want to avoid all technicalities and assume that K is a bounded, measurable function with compact support. This obviously yields $\int_0^t q(W_s)\,\mathrm{d}s < \infty$ $\mathbb{P}_0 \otimes \mathbb{P}$-a.s. noting that $\omega(C) < \infty$ $\mathbb{P}$-a.s. for any compact set $C \subset \mathbb{R}^d$.

The two possible signs $\pm$ allowed in (1.4) and (1.5) are a tribute to the fact that in the literature, the equation (1.2) is sometimes formulated with a minus in front of the Laplacian, sometimes with a minus in front of the potential q and sometimes with the definition $q(x) := -\sum_{y\in\mathcal{P}} K(x-y)$, and so on. Here we make the following convention. When the potential does only take values in $[0,\infty)$ or $(-\infty, 0]$, which is the case most of the time, we always take q, i.e. K, nonnegative and distinguish the two cases by speaking of the *positive and negative exponential moments*, depending on which sign we choose in (1.4) and (1.5) or, equivalently in front of q in (1.2). However, from Section 2 on, we only consider the positive exponential moments.

With the negative sign chosen, the model is called *Brownian motion among Poisson obstacles* since the Gibbs measures in (1.4) put more mass on those paths that avoid to come too near to the Poisson points. The case where q is bounded and compactly supported has been studied extensively and is called the case of *soft* obstacles. A second important case are *hard obstacles*. Here one formally writes $K = \infty \mathbb{1}_C$ for some compact set $C \subset \mathbb{R}^d$, i.e. the Brownian path has to avoid the C - neighbourhoods of the Poisson points completely since the obstacles serve as a death trap. A third important example is given by shapes of type $K(x) = |x|^{-p}$, where one has integrability issues both at the singularity 0 but also at the tails. These problems can be circumvented either by substituting K by a truncated shape $|x|^{-p}| \wedge a$ or $|x|^{-p}|\mathbb{1}_{\{|x|\le a\}}$, or by a renormalization.

On the other hand, the Poisson points attract the Brownian particle when the positive sign is chosen in (1.4) and (1.5). In this case the potential is often called *Poisson shot noise potential* and has been studied under some regularity and decay assumptions on K, cf. (1.21) below.

1.2.2 Relation to Parabolic Anderson Model and Schrödinger Semigroups

The study of Brownian motion in random potential is closely related to that of the *parabolic Anderson model*, i.e. the initial-boundary value problem

$$\partial_t u(t,x) = \frac{\Delta}{2} u(t,x) \pm q(x)u(t,x), \quad (t,x) \in [0,\infty) \times D \tag{1.7}$$

$$u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial D \tag{1.8}$$

$$u(0,x) = u_0(x), \quad x \in D. \tag{1.9}$$

Here, $D \subset \mathbb{R}^d$ is either open and bounded or $D = \mathbb{R}^d$; $u_0 \in L^2(D)$ is the initial data. When $D = \mathbb{R}^d$ the Dirichlet boundary condition vanishes and (1.7)-(1.9) reduces to (1.2). We only consider standard Brownian motion here and neglect the case of a diffusion coefficient $\kappa \neq \frac{1}{2}$. This is no restriction since one can always get $\kappa = \frac{1}{2}$ by Brownian scaling and a simple time change.

The connection to (1.7)-(1.9) is explained by the well known Feynman-Kac formula stating that (e.g. when $q \in K_d$) the unique (mild) solution to (1.7)-(1.9) is given by

$$u(t,x) = \mathbb{E}_x \left[u_0(W_t) \exp\Big(\pm \int_0^t q(W_s) \Big) \mathbb{1}_{\{\tau_{D^c} > t\}} \right], \tag{1.10}$$

which mathematically goes back to Kac, cf. [Kac51]. Here, τ_{D^c} denotes the entrance time of the Brownian motion in D^c. In the case $q \equiv 0$, i.e. in the absence of a potential, (1.10) just reflects the classical observation that the fundamental solution to the heat equation is given by the Gaussian transition densities. We refer to Section 3 below for rigorous statements.

An important consequence of this connection is the possibility to use probabilistic tools to study (1.7)-(1.9) and to use functional analytic tools to study the expectation (1.10). The latter includes especially the spectral properties of the Schrödinger operator $\frac{1}{2}\Delta + q$. With the help of the principal eigenvalue $\lambda_{\max}(D,q)$ of $\frac{1}{2}\Delta + q$ acting on $L^2(D)$ we can derive upper and lower bounds on $u(t,x)$ using a Fourier expansion, see (3.29)-(3.30) below. Therefore computations of $\lambda_{\max}(D,q)$ are an essential ingredient in the field. We will use extensively the Rayleigh-Ritz variation formula

$$\lambda_{\max}(D,q) = \mathscr{V}(D,q) := \sup \left\{ \int_D \pm \frac{1}{2}|\nabla f|^2 + qf^2, f \in C_c^\infty(D), \int_D f^2 = 1 \right\}, \tag{1.11}$$

where $C_c^\infty(D)$ denotes the space of smooth and compactly supported functions $f\colon D \to \mathbb{R}$.

1.2.3 Problems, Results and Strategies

Various questions naturally arise after having introduced our model. These include:

- How fast do the quenched and annealed exponential moments grow or decay as t goes to infinity? Does a principle of large deviations hold true?

- Which Brownian paths and which configurations of the Poisson cloud yield the main contribution to the exponential moments? What is the typical behaviour of the paths under the quenched and annealed Gibbs measures? Can we observe homogenization or localization of the paths?
- What are the probabilistic costs for journeys from one point to another (possibly far away)?

In the present work, we focus on the study of the finiteness and the large-time asymptotics of the exponential moments, i.e. the first question. However, the second question is also concerned since for the lower bounds on the -say quenched- exponential moments, we have to figure out which behaviour of the Brownian paths provides the optimal strategy to absorb an extremely high or low amount of the potential. This amount should govern the sum of the contributions to the exponential moments of all other paths. Such a strategy can be a good candidate for the typical behaviour of W_t under the influence of the potential. With respect to the third question, let us only mention that this problem is treated in the study of the so called *Lyapounov-exponents*, cf. [Szn98, Section 5].

The results on the time asymptotics of the exponential moments heavily depend on whether we consider the obstacle or the Poisson shot noise case. Furthermore, the annealed and the quenched case differ fundamentally, and also the different classes of shapes functions K have an impact (soft, hard, bounded, $|x|^{-p}$-type). We now give a brief overview on some results on the time asymptotics. We do not aim to give a complete picture but restrict ourselves to the basic results. This shall serve, on the one hand, as a 'warm up' calculation showing some strategies and techniques used in the field and, on the other hand, as the basis for later comparison of the asymptotics in our main result.

Negative Annealed Exponential Moments We begin our brief review with the time asymptotics of the *negative annealed exponential moments*. In the case of hard obstacles, where the compact death-trap $C \subset \mathbb{R}^d$ is just an open ball $B_a = B_a(0) := \{x \in \mathbb{R}^d : |x| < a\}$ for some $a > 0$, the investigation of Brownian motion moving among these traps can be translated into that of the volume of the *Wiener sausage* $(W_t^{(a)})_{t\geq 0}$, with $W_t^{(a)} = \{x \in \mathbb{R}^d : \exists\, 0 \leq s \leq t \text{ with } |x - W_s| < a\}$, using the identity

$$Z_t = \mathbb{P} \otimes \mathbb{P}_0[\omega(W_t^{(a)}) = 0] = \mathbb{E}_0[\exp(-|W_t^{(a)}|)]. \tag{1.12}$$

Certainly, the main contribution to the expectation in (1.12) comes from paths that enter into a good compromise between the generation of a low volume and not too high probabilistic costs. A strategy to achieve a volume of order R_t^d is to stay inside (and fill) the ball of radius R_t until time t with costs of exponential rate tR_t^{-2}. Minimizing $R_t^d + tR_t^{-2}$ gives $R_t = t^{1/(d+2)}$. Indeed, this heuristic leads to the right asymptotic scale, which has been shown by Donsker and Varadhan in their celebrated article [DV75]. They show that

$$\lim_{t\to\infty} \frac{1}{t^{d/(d+2)}} \log Z_t = -\omega_d^{2/(d+2)} \left(\frac{d+2}{d}\right) \left(\frac{2\lambda_d}{d}\right)^{d/(d+2)}, \tag{1.13}$$

where ω_d denotes the volume of the d - dimensional unit ball and λ_d the principal Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ therein. In their proof, they use a large deviation principle for the normalized occupation times $\nu_t = \frac{1}{t}\int_0^t \delta_{W_s}\,\mathrm{d}s$ of Brownian motion. When W_t stays in a compact set $C \subset \mathbb{R}^d$, it is well known that for a Sobolev function $f \in H^1(C)$

$$\mathbb{P}_0[\nu_t(\mathrm{d}x) \approx f^2(x)\,\mathrm{d}x] \approx \exp\{-t\|\nabla f\|_2^2\} \quad \text{as } t \to \infty. \tag{1.14}$$

Minimizing the right-hand side over the set $\{f \in H^1(C)\colon \int_C f(x)^2\,\mathrm{d}x = 1\}$ explains the appearance of λ_d in the limit. The authors could also extend the asymptotic to the case

$$K(x) = \frac{\theta}{|x|^p} \wedge 1, \qquad d+2 < p < \infty, \theta \in (0,\infty). \tag{1.15}$$

In the regime $d < p < d+2$, Pastur showed in [Pas77] that

$$\lim_{t\to\infty} \frac{1}{t^{d/p}} \log Z_t = -\omega_d \theta^{d/p} \Gamma\left(\frac{p-d}{p}\right). \tag{1.16}$$

The critical case $p = d+2$ was established by Okûra, who showed in [Ôku81]

$$\lim_{t\to\infty} \frac{1}{t^{d/(d+2)}} \log Z_t = -\omega_d \theta^{d/p} \Gamma\left(\frac{p-d}{p}\right). \tag{1.17}$$

Note that only in the regime $d < p \leq d+2$ the asymptotic rate depends on p and the limit depends on θ and p.

Sznitman obtained (1.13) again and extended it to the soft obstacle case, using the so called technique of *enlargement of obstacles* to derive upper bounds on the principal eigenvalue, cf. [Szn98, Section 4]. To reduce the combinatorial complexity with which the Poisson points can be arranged inside the lattice $\alpha_t \mathbb{Z}^d$ of some given length α_t, the space $\mathbb{R}^d$ is divided into three random sets. The set of *good obstacles*, or density set, consists of those regions with a sufficiently high concentration of the Poisson cloud. Due to the already high concentration, these obstacles can be enlarged without increasing the eigenvalue too much. However, regions where points of $\mathcal{P}$ lie more isolated cannot be enlarged. One has to take into account the set of *bad obstacles*. One shows that this set is sufficiently small and that removing the bad Poisson points does therefore not change the eigenvalue too much. The third set is the 'empty' set, where no Poisson points lie. The analysis leads to the following, cf. [Szn98, Theorem 4.6]:

Theorem 1.1. *Let $d \geq 2$. There exists $\gamma > 0$ and $\chi \in (0,d)$ such that for $\mathbb{P}$- almost all ω for large l*

$$\frac{c(d)}{(\log(l)^{\frac{2}{d}})} - \frac{1}{(\log l)^{\frac{2+\chi}{d}}} \leq \lambda_{\max}((-l,l)^d, -q(\omega,\cdot)) \leq \frac{c(d)}{(\log(l)^{\frac{2}{d}})} + \frac{\gamma}{(\log l)^{\frac{3}{d}}}. \tag{1.18}$$

If $d = 1$, $\gamma(\log l)^{-3/d}$ is replaced by $\gamma(\log l)^{-3} \log\log l$.

The heuristic for the lower bound for (1.13) is based on a joint strategy of the Brownian motion, which stays in a rather small ball of radius $t^{1/(d+2)}$ until time t, and the Poisson point process, which leaves this ball empty. In dimension $d = 2$, Sznitman also proved the so called *confinement property*, which states that this strategy indeed becomes typical under the annealed law $\mathbb{Q}_t$. To be slightly more precise, this means that with probability tending to 1, inside of the ball with radius roughly $t^{\frac{1}{4}}$ around the origin one can find the centre of another ball of similar radius which contains Poisson points only near the boundary and where the Brownian particle is ‘confined’ in, cf. [Szn91, Theorem 4.5]:

Theorem 1.2 (Confinement Property, $d = 2$). *Let $d = 2$, $R_0 = \left(\frac{2\lambda_d}{d\omega_d}\right)^{\frac{d}{d+2}}$. There exists a constant $c > 0$ and a measurable map $D_t(\omega)$ with values in the ball of radius $t^{\frac{1}{4}}(R_0 + \mathrm{e}^{-c(\log t)^{\frac{1}{2}}})$ around 0, such that*

$$\sup_{0\le s\le t} |W_s - D_t| \le t^{\frac{1}{4}}(R_0 + \mathrm{e}^{-c(\log t)^{\frac{1}{2}}}) \tag{1.19}$$

and no Poisson point falls into the ball of radius $t^{\frac{1}{4}}(R_0 - \mathrm{e}^{-c(\log t)^{\frac{1}{2}}})$ around D_t with $\mathbb{Q}_t$ probability going to 1 as $t \to \infty$.

For higher dimensions a confinement property has been proven by Povel, [Pov99].

Theorem 1.3 (Confinement Property, $d \ge 3$). *For $d \ge 3$, there exists $\alpha > 0$ such that,*

$$\lim_{t\to\infty} \mathbb{Q}_t \left[\sup_{0\le u\le t} |W_u| \le 2t^{\frac{1}{d+2}}(R_0 + t^{-\alpha}) \right] = 1. \tag{1.20}$$

Table 1.1: Time asymptotics of logarithmic **annealed exponential moments**

Shape	Sign	Scale	Limit	Reference
Hard Obst.	$-$	$t^{d/(d+2)}$	$-\omega_d^{2/(d+2)}\left(\frac{d+2}{d}\right)\left(\frac{2\lambda_d}{d}\right)^{d/(d+2)}$	[DV75],[Szn93b]
Soft Obst.	$-$	$t^{d/(d+2)}$	$-\omega_d^{2/(d+2)}\left(\frac{d+2}{d}\right)\left(\frac{2\lambda_d}{d}\right)^{d/(d+2)}$	[DV75],[Szn93b]
$\lvert x\rvert^{-p}\wedge 1, p > d+2$	$-$	$t^{d/(d+2)}$	$-\omega_d^{2/(d+2)}\left(\frac{d+2}{d}\right)\left(\frac{2\lambda_d}{d}\right)^{d/(d+2)}$	[DV75]
$\lvert x\rvert^{-p}\wedge 1, p \in (d, d+2)$	$-$	$t^{d/p}$	$-\omega_d\theta^{d/p}\Gamma\left(\frac{p-d}{p}\right)$	[Pas77]
$\lvert x\rvert^{-p}\wedge 1, p = d+2$	$-$	$t^{d/(d+2)}$	$-\omega_d\theta^{d/p}\Gamma\left(\frac{p-d}{p}\right)$	[Ôku81]
Continuous, (1.21)	$+$	$\exp(t)$	$\sup_{x\in\mathbb{R}^d} K(x)$	[CM95],[GK00]
$\lvert x\rvert^{-p}, p \in (\frac{d}{2}, d)$,	$-$	$t^{d/p}$	$-\omega_d\theta^{d/p}\Gamma\left(\frac{p-d}{p}\right)$	[CK11]

Positive Annealed Exponential Moments We continue with a discussion of the *positive annealed exponential moments.* The first order asymptotics have been studied by Carmona and Molchanov in [CM95] under the following regularity assumptions on the shape function K. They assume K to be continuous on $\mathbb{R}^d$ and to have the decay property

$$|K(x)| \leq \frac{c}{1+|x|^{d+\beta}}, \quad x \in \mathbb{R}^d \tag{1.21}$$

with some constants $c, \beta > 0$. To avoid technical complications, we furthermore assume here that K is nonnegative, although it is sufficient to assume that $K(x) > 0$ for at least some $x \in \mathbb{R}^d$. Especially, it follows that K is bounded, and we assume without loss of generality that $K(0) = \max_{x \in \mathbb{R}^d} K(x)$. By the formula for the Laplace transform of a Poisson point process (adapted to nonnegative q, see Lemma 1.10 below) and the inequality $\mathrm{e}^x - 1 \leq x\,\mathrm{e}^x$ for $x > 0$ a simple upper bound on the annealed exponential moments is given by

$$\mathbb{E} \otimes \mathbb{E}_0 \left[\exp\left\{\int_0^t q(W_s)\,\mathrm{d}s\right\}\right] = \mathbb{E}_0 \left[\exp\left(\int_{\mathbb{R}^d} \exp\left(\int_0^t K(W_s - y)\,\mathrm{d}s\right) - 1\right)\mathrm{d}y\right] \tag{1.22}$$

$$\leq \exp\left[t\Big(\int_{\mathbb{R}^d} K(y)\,\mathrm{d}y\Big)\,\mathrm{e}^{tK(0)}\right]. \tag{1.23}$$

Similarly, a lower bound is derived by restricting the Brownian motion to stay inside an ε-ball around 0,

$$\mathbb{E} \otimes \mathbb{E}_0 \left[\exp\left\{\int_0^t q(W_s)\,\mathrm{d}s\right\}\right] \geq \mathbb{E}_0\left[\exp\left(\int_{\mathbb{R}^d} \mathrm{e}^{\int_0^t K(W_s - y)\,\mathrm{d}s} - 1\,\mathrm{d}y\right), \sup_{0\leq s\leq t} |W_s| \leq \varepsilon\right]$$

$$\geq \exp\left(\int_{\mathbb{R}^d} \exp\left(t\Big(\inf_{|x|\leq\varepsilon} K(y+x)\Big)\right) - 1\,\mathrm{d}y\right) \mathbb{P}_0\left[\sup_{0\leq s\leq t} |W_s| \leq \varepsilon\right].$$

As $t \to \infty$, it follows from Laplace's method that the integral in the second line is dominated by the global maximum of the function $y \mapsto \inf_{|x|\leq\varepsilon} K(y+x)$ in the sense that

$$\log\left\{\int_{\mathbb{R}^d} \exp\left(t\Big(\inf_{|x|\leq\varepsilon} K(y+x)\Big)\right) - 1\,\mathrm{d}y\right\} \sim t \sup_{y\in\mathbb{R}^d} \inf_{|x|\leq\varepsilon} K(y+x) = t(K(0) - f(\varepsilon)), \quad t \to \infty.$$

The equality holds, using the continuity of K, for some function f with $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. Noting the small ball estimate $\mathbb{P}\left[\sup_{0\leq s\leq t} |W_s| \leq \varepsilon\right] \approx \mathrm{e}^{-c\varepsilon^{-2}t}$, we get as a lower bound

$$\log \mathbb{E} \otimes \mathbb{E}_0 \left[\exp\left\{\int_0^t q(W_s)\,\mathrm{d}s\right\}\right] \geq \exp\left\{(1+o(1))t(K(0) - f(\varepsilon))\right\} - c\varepsilon^{-2}t, \quad t \to \infty$$

implying that

$$\lim_{t\to\infty} t^{-1} \log\log \mathbb{E} \otimes \mathbb{E}_0 \left[\exp\left\{\int_0^t q(W_s)\,\mathrm{d}s\right\}\right] \geq K(0) - f(\varepsilon). \tag{1.24}$$

Letting $\varepsilon \to 0$, we have identified the time asymptotics as

$$\lim_{t\to\infty} t^{-1} \log\log \mathbb{E} \otimes \mathbb{E}_0 \left[\exp\left\{\int_0^t q(W_s)\,\mathrm{d}s\right\}\right] = K(0) \tag{1.25}$$

which is Theorem 4.2 in [CM95]. Note that the costs for this small ball strategy are completely negligible on the double logarithmic scale.

Beyond this, the second order asymptotics have been derived by Gärtner and König in [GK00]. We will not comment on their proof further except for their technique of estimating the principal eigenvalues for later comparison. For $R \in (0, \infty)$, let $0 < r < \tilde{r} < R$ such that the macrobox $Q_R = [-R, R]^d$ is covered by the microboxes $2rz + Q_{\tilde{r}}$, $z \in \mathbb{Z}^d$, $|z| \leq R/r + 1$, and neighbouring microboxes are overlapping each other. Then it is shown that the principal eigenvalue in Q_R can be approximated by the maximum of the principal eigenvalues in the microboxes:

Proposition 1.4. *For every $r \geq 2$, there is a smooth function $\Phi_r \colon \mathbb{R}^d \to [0, \infty)$ with compact support inside the neighbourhood of radius 1 of $2r\mathbb{Z}^d + \partial Q_r$ such that for all $R > r$*

$$\lambda_{\max}(Q_R, q - \Phi_r) \leq \max_{z \in \mathbb{Z}^d, |z| < \frac{R}{r} + 1} \lambda_{\max}(2rz + Q_{r+1}, q). \tag{1.26}$$

Φ_r can be chosen periodic in each coordinate with period $2r$ and such that the estimate

$$\int_{Q_r} \Phi_r(x)\, \mathrm{d}x \leq \frac{C}{r} |Q_r| \tag{1.27}$$

holds for some constant $C \in (0, \infty)$ independent of r.

This is Proposition 1 in [GKM00]. The estimate is based on a partition of the one which allows us to somehow isolate the eigenvalue computation in each microbox. The function Φ_r describes the error of the approximation in terms of the gradient of the partition in the overlapping regions. We will use a similar construction, cf. Proposition 3.19 below. Note that this eigenvalue approximation reflects the fact, that the main contribution to the exponential moments comes from extremely high values of the potential in small boxes.

Negative Quenched Exponential Moments We continue with results on the time asymptotics of the *negative quenched exponential moments*. It is a fundamental difference to the annealed situation that now the random environment, as it is fixed, cannot contribute to a common strategy with the Brownian particle. Thus, the Brownian motion has to maximize $\exp\left\{-\int_0^t q(W_s)\, \mathrm{d}s\right\}$ 'on its own'. Let us first consider soft obstacles. Intuitively, the particle should search for a region which is not too small, not too far away and free of traps. It turns out, that with high probability there exists an empty ball $B_{r_t}(x_t)$ of radius roughly $r_t = (\log R(t))^{1/d}$ in a distance slightly less than $R(t)$ away from the origin. Then locally the potential can be described by the principal Dirichlet eigenvalue $\lambda_{\max}(B_{r_t}(x_t), 0)$ of the free Laplacian. Using the Fourier expansion bound from (3.30) below and the exponential decay of the principal eigenfunction e_1, $e_1(x) \leq \mathrm{e}^{-c|x|}$, the strategy of moving to $B_{r_t}(x_t)$ in time $R(t)$ and then staying there until time t produces a contribution to the Feynman-Kac functional of order

$$\exp\{-c|x_t| - (t - R(t))\lambda_{\max}(B_{r_t}(x_t), 0)\} \geq \exp\{-cR(t) - (t - R(t))c' r_t^2 \lambda_d\}.$$

Choosing $R(t)$ such that $t^{\alpha} << R(t) << \frac{t}{(\log t)^{2/d}}$ for any $0 < \alpha < 1$, the computation suggests the correct asymptotic

$$\lim_{t\to\infty} \frac{(\log t)^{\frac{2}{d}}}{t} \log \mathbb{E}_0 \left[\exp\left\{-\int_0^t q(W_s)\,\mathrm{d}s\right\}\right] = -\lambda_d \left(\frac{\omega_d}{d}\right)^{\frac{2}{d}} \quad \mathbb{P}\text{-a.s.}, \tag{1.28}$$

cf. [Szn98, Theorem 5.1]. While the rigorous proof of the lower bound indeed goes along the sketched lines, the upper bound is again derived by the eigenvalue asymptotics in (1.18) obtained by the enlargement of obstacles method.

In the case of hard obstacles, the above $\mathbb{P}$-a.s. asymptotic does not hold. First, with positive probability the Brownian motion may die immediately if its starting point lies too close to a Poisson point. On the other hand, the complement of the obstacles $\mathbb{P}$-a.s. will not have an infinite component if the radius of the hard obstacles is higher than the critical radius for continuum percolation. In this case the survival probability decays exponentially. Only if the radius of the obstacles is small enough, the asymptotic (1.28) holds with positive probability, i.e. with the probability that there is an obstacle free infinite cluster containing the origin, cf. [Szn93b].

Although the lower bound strategy above gives a substantial contribution to the exponential moments, this does not mean that it necessarily describes the typical behaviour of the Brownian path under $\mathbb{Q}_{t,\omega}$. It requires further work to show the so called *pinning effect*, cf. [Szn98, Chapter 6].

Theorem 1.5 (Pinning Effect)**.** *We have*

$$\lim_{t\to\infty} \mathbb{Q}_{t,\omega} \begin{bmatrix} W_s \text{ moves before time } t \text{ to the 1-neighboorhhod of some} \\ z_0 \in C_{t,\omega} \text{ and does not leave } B_{r_t}(z_0) \text{ afterwards} \end{bmatrix} = 1 \quad \mathbb{P}\text{-a.s.} \tag{1.29}$$

Here the skeleton

$$C_{t,\omega} = \left\{ z \in \frac{1}{\sqrt{d}}\mathbb{Z}^d : F_t(z,\omega) \le \mu_{t,\omega} + t(\log t)^{-(\chi+\frac{2}{d})} \right\} \tag{1.30}$$

with $\mu_{t,\omega} = \inf_{u>0}\{u - t\lambda_{\max}(\{x \in \mathbb{R}^d : \alpha_0(x) < u\}, q)\}$ contains the 'near minima' of the functional $F_t(z,\omega) = \alpha_0(z) - t\lambda_{\max}(B_{R_t}(z), 0)$ describing the contribution that enters the exponential moment when the Brownian particle moves to z. α_0 is the *Lyapounov norm* given by [Szn98, Theorem 5.2.5] and $r_t = \exp\left\{(\log t)^{1-\chi}\right\}$ for some sufficiently small $\chi > 0$.

Positive Quenched Exponential Moments The *positive quenched exponential moments* have also been studied by Carmona and Molchanov in [CM95] under the same continuity and decay properties as the annealed moments above. The boundedness assumption is crucial and results in a straight forward pinning-type strategy which allows the Brownian particle to make $\exp(\int_0^t q(W_s)\,\mathrm{d}s)$ large. It travels to a reachable region with a high concentration of Poisson points in a very short time and collects an amount of potential of order $tK(0)\#\{\text{Poisson points in this region}\}$. We want to quantify this. Fix $h > 0$ and write

$C_z := hz + [-h/2, h/2]^d$, $N_z = \omega(C_z)$ for $z \in \mathbb{Z}^d$. By the tails of the Poisson distribution we have for any z, $\varepsilon > 0$ and x large enough

$$\mathbb{P}[N_z > x] \le \mathrm{e}^{-(1-\varepsilon)x \log x}. \tag{1.31}$$

Now choose $x = (1+\alpha)d\frac{\log R}{\log\log R}$ with some $\alpha > 0$ and a large radius $R > 0$. Since $\# h\mathbb{Z}^d \cap B_R \approx R^d$, the Borel-Cantelli implies that

$$\limsup_{R\to\infty} \sup_{z \in h\mathbb{Z}^d \cap B_R} \frac{\log\log R}{d \log R} N_z \le 1 \quad \mathbb{P}\text{-a.s.} \tag{1.32}$$

A lower bound can be established similarly. Elementary bounds of this type will be proven rigorously in Section 5.1. By the continuity of K one can choose, for any $\delta > 0$, $h > 0$ small enough, such that $q(x) \ge (K(0) - \delta)N_z$ for $x \in C_z$. Now, roughly speaking, for any $\varepsilon > 0$ the costs for reaching a 'good' box C_z with $|z| \le t^{1-\varepsilon}$ in a short, on scale t negligible time $t^{1-\varepsilon}$ and staying there until time t are bounded from below by $\exp(-ct)$ for h fixed. Thus letting $\delta, \epsilon \downarrow 0$ this suggests that

$$\mathbb{E}_0\left[\exp\left\{\int_0^t q(W_s)\,\mathrm{d}s\right\}\right] \approx \exp\left(-ct + t\frac{K(0)d\log t}{\log\log t}\right). \tag{1.33}$$

Indeed, by Theorem 5.1 in [CM95],

$$\lim_{t\to\infty} \frac{\log\log t}{t \log t} \log \mathbb{E}_0\left[\exp\left\{\int_0^t q(W_s)\,\mathrm{d}s\right\}\right] = d \sup_{x\in\mathbb{R}^d} K(x) \quad \mathbb{P}\text{ -a.s.} \tag{1.34}$$

Obviously, this result is only satisfying if K is bounded. We will discuss in the following, what happens if we drop this restriction. For the asymptotic in (1.34) also the second order term has been derived, cf. [GKM00].

Table 1.2: Time asymptotics of logarithmic **quenched exponential moments**

Shape	Sign	Scale	Limit	Reference
Soft Obst.	$-$	$\frac{(\log t)^{2/d}}{t}$	$-\lambda_d\left(\frac{\omega_d}{d}\right)^{2/d}$	[Szn93a]
Hard Obst.	$-$	$\frac{(\log t)^{2/d}}{t}$	$-\lambda_d\left(\frac{\omega_d}{d}\right)^{2/d}$	[Szn93a]
Continuous, (1.21)	$+$	$\frac{\log\log t}{t\log t}$	$d\sup_{x\in\mathbb{R}^d} K(x)$	[CM95],[GKM00]
$\theta\lvert x\rvert^{-p}, \frac{d}{2} < p < d$	$-$	$t^{-1}(\log t)^{-\frac{d-p}{d}}$	$\frac{\theta d^2}{d-p}\left(\frac{\omega_d}{d}\Gamma\left(\frac{2p-d}{p}\right)\right)^{p/d}$	[Che12]
$\theta\lvert x\rvert^{-p}, \frac{d}{2} < p < 2\wedge d$	$+$	$\frac{1}{t}\left\{\frac{\log\log t}{\log t}\right\}^{\frac{2}{2-p}}$	$\frac{1}{2}p^{\frac{p}{2-p}}(2-p)^{\frac{4-p}{2-p}}\left(\frac{d\theta\sigma(d,p)}{2+d-p}\right)^{\frac{2}{2-p}}$	[Che12]
$\theta\lvert x\rvert^{-2}, d=3, \theta \le \frac{1}{16}$	$+$	$t^{\frac{k+1}{k-1}}$	?	[CR11]

1.3 The Renormalized Potential

1.3.1 Why Introducing a Renormalization?

The results we have presented so far treat Poisson potentials with bounded and / or compactly supported shape functions. We want to drop this restriction now and treat potentials which may have singularities and long range impacts. A natural motivation for this comes from physics since for example the choices $K(x) = |x|^{-(d-1)}$ and $K(x) = |x|^{-(d-2)}$ model the gravitational force and the gravitational potential. Thus, we consider shape functions of type

$$K(x) = |x|^{-p}, \quad p \geq 1, \tag{1.35}$$

with the corresponding Poisson potential given by

$$V(x) = \int_{\mathbb{R}^d} K(x-y)\omega(\mathrm{d}y). \tag{1.36}$$

When p is large compared to d, i.e. when $p > d$, V is well defined on $\mathbb{R}^d \setminus \mathcal{P}$. The strong singularity of K in 0 causes V to blow up at each $y \in \mathcal{P}$. In order to make the corresponding Gibbs measure well defined, K has to be truncated around the origin as in (1.15).

On the other hand, when p is small compared to d, i.e. when $p \leq d$, V is not well defined. The heavy tail of K at infinity and the summarized force of infinitely many Poisson obstacles cause $V(x)$ to become infinite at every $x \in \mathbb{R}^d$ with probability 1. The phase transition at $p = d$ corresponds to that of the Lebesgue integrability of $|x|^p$. Formally this follows from Theorem 1.11 below.

It seems possible that when $p < d$ the lack of integrability in (1.36) can be cured when the measure $\omega(\mathrm{d}x)$ is replaced by the *renormalized* measure $\omega(\mathrm{d}x) - \mathrm{d}x$, hoping that the mass collected at infinity by ω is neutralized by the Lebesgue part. Consider again a bounded and compactly supported shape K for a moment. We have

$$\mathbb{E}[V(x)] = \mathbb{E}\left[\int_{\mathbb{R}^d} K(y-x)\omega(\mathrm{d}y)\right] = \int_{\mathbb{R}^d} K(y-x)\,\mathrm{d}y = \int_{\mathbb{R}^d} K(y)\,\mathrm{d}y = \text{const.} \tag{1.37}$$

Thus, we may replace $V(x)$ by $\overline{V}(x) := V(x) - \mathbb{E}[V(x)]$ simultaneously in (1.4) and (1.5) without changing the quenched or annealed Gibbs measure. Now we can write formally

$$\overline{V}(x) = \int_{\mathbb{R}^d} K(x-y)[\omega(\mathrm{d}y) - \mathrm{d}y]. \tag{1.38}$$

Definition 1.6. *We call $\overline{V}$ the* ***renormalized Poisson potential****. The corresponding quenched and annealed Gibbs measures are defined as*

$$\overline{Q}_{t,\omega} = \frac{1}{\overline{Z}_{t,\omega}} \exp\left\{\pm\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right\}\mathbb{P}_0 \quad \text{and} \quad \overline{Q}_t = \frac{1}{\overline{Z}_t}\exp\left\{\pm\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right\}\mathbb{P}_0 \otimes \mathbb{P}.$$

The normalizing constants $\overline{Z}_{t,\omega}$ and $\overline{Z}_t$ are adapted in accordance.

The renormalization may also be applied to the parabolic Anderson model (1.7)-(1.9). Taking $\overline{V}$ as the potential, the effects of absorption and creation of mass are combined, the first one dependent on the random environment and the second one independent, or vice versa.

We now sketch how the construction of $\overline{V}$ has been done rigorously. It will follow that $\overline{V}$ provides a real generalization to V for some choices of K, i.e. $\overline{V}$ can we well defined for these shapes K but V explodes.

1.3.2 Construction and Basic Properties

The integration with respect to the signed measure $\omega(\mathrm{d}x) - \mathrm{d}x$ is defined canonically, i.e. by approximating the positive and negative part of a measurable function by simple functions and transporting the definition of the integral this way. Properties of the integral, as linearity, the monotone and dominated convergence theorems (in the sense of convergence in probability) etc. persist, but not monotonicity. The construction is executed thoroughly in [CK12, Section 2]; further results rely on methods from the general theory of random integration and the spectral representation of infinitely divisible random measures in the spirit of the Lévy-Kinchin formula, c.f. [RR89] for details. These general methods lead to the following important integrability result, cf. [CK12, Proposition 2.1].

Proposition 1.7. *Let $K\colon \mathbb{R}^d \to \mathbb{R}$ be Borel measurable. Then the integral $\int_{\mathbb{R}^d} K(x)\omega(\mathrm{d}x)$ exists if and only if*

$$\int_{\mathbb{R}^d} 1 - \exp(-|K(x)|)\,\mathrm{d}x < \infty. \tag{1.39}$$

The integral $\int_{\mathbb{R}^d} K(x)[\omega(\mathrm{d}x) - \mathrm{d}x]$ exists if and only if

$$\int_{\mathbb{R}^d} |K(x)| - 1 + \exp(-|K(x)|)\,\mathrm{d}x < \infty. \tag{1.40}$$

Corollary 1.8. *The integral $\int_{\mathbb{R}^d} |x|^{-p}\omega(\mathrm{d}x)$ is finite if and only if $p > d$. Furthermore, the integral $\int_{\mathbb{R}^d} |x|^{-p}[\omega(\mathrm{d}x) - \mathrm{d}x]$ is finite if and only if $d/2 < p < d$.*

If the definition of $\overline{V}$ has been done well, we should expect that

$$\int_{\mathbb{R}^d} K(x)[\omega(\mathrm{d}x) - \mathrm{d}x] = \int_{\mathbb{R}^d} K(x)\omega(\mathrm{d}x) - \int_{\mathbb{R}^d} K(x)\,\mathrm{d}x \tag{1.41}$$

whenever all of the integrals are finite. The following lemma (cf. [CK12, Proposition 2.5]) confirms this.

Lemma 1.9. *Any $K \in L^1(\mathbb{R}^d)$ is integrable w.r.t. both $\omega(\mathrm{d}x)$ and $\omega(\mathrm{d}x) - \mathrm{d}x$ and the identity* (1.41) *holds true.*

We collect further basic statements about the renormalized Poisson potential proven in [CK12].

Lemma 1.10. *Under* (1.40) *the following statements hold true.*

a) $\mathbb{E}\left|\int_{R^d} K(x)[\omega(\mathrm{d}x) - \mathrm{d}x]\right| < \infty$

b) $\mathbb{E}\left[\exp\left(\int_{\mathbb{R}^d} K(x)[\omega(\mathrm{d}x) - \mathrm{d}x]\right)\right] = \exp\left(\int_{\mathbb{R}^d} \mathrm{e}^{K(x)} - 1 - K(x)\,\mathrm{d}x\right) < \infty$

c) As a function of x, the random integral $\int_{\mathbb{R}^d} |K(x)(y-x)|)[\omega(\mathrm{d}y) - \mathrm{d}y]$ yields a measurable modification.

d) If $\lim_{x\to x_0} K(x) = \infty$ for some $x_0 \in \mathbb{R}^d$, then $\mathbb{P}[\sup_{x\in U} \overline{V}(x) = \infty] > 0$ for all $\emptyset \neq U \subset \mathbb{R}^d$ open.

Furthermore, a), c) and d) also hold true for the non-renormalized Poisson potential under (1.39). *Instead of b) we have* $\mathbb{E}\left[\exp\left(\int_{\mathbb{R}^d} K(x)\omega(\mathrm{d}x)\right)\right] = \exp\left(\int_{\mathbb{R}^d} \mathrm{e}^{K(x)} - 1\,\mathrm{d}x\right)$.

In view of Lemma 1.10 d), it is not obvious whether the path integral $\int_0^t \overline{V}(W_s)\,\mathrm{d}s$ or the normalizing constant $\overline{Z}_{t,\omega}$ is finite in the case of a shape function K having singularities. However, we obtain the $\mathbb{P}_0 \otimes \mathbb{P}$- a.s. finiteness of $\int_0^t |\overline{V}(W_s)|\,\mathrm{d}s$ from the identity

$$\mathbb{E}_0 \otimes \mathbb{E}\left(\int_0^t |\overline{V}(W_s)|\,\mathrm{d}s\right) = \mathbb{E}_0\left(\int_0^t \mathbb{E}|\overline{V}(W_s)|\,\mathrm{d}s\right) = \mathbb{E}_0\left(\int_0^t \mathbb{E}|\overline{V}(0)|\,\mathrm{d}s\right) = t\mathbb{E}[\overline{V}(0)] < \infty$$

using Fubini's Theorem, the spatial invariance of $\overline{V}$ and Lemma 1.10 a). For nonnegative K we have furthermore, using the Fubini type identity

$$\int_0^t \overline{V}(W_s)\,\mathrm{d}s = \int_{\mathbb{R}^d} \eta(t,x)[\omega(\mathrm{d}x) - \mathrm{d}x] \tag{1.42}$$

with $\eta(t,x) = \int_0^t K(x - W_s)\,\mathrm{d}s$,

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-\int_0^t \overline{V}(W_s)\,\mathrm{d}s}\right] = \mathbb{E}\left[\exp^{-\int_{\mathbb{R}^d} \eta(t,x)[\omega(\mathrm{d}x)-\mathrm{d}x]}\right] &= \mathrm{e}^{\int_{\mathbb{R}^d} \mathrm{e}^{-\eta(t,x)} - 1 + \eta(t,x)\,\mathrm{d}x} \leq \mathrm{e}^{\int_{\mathbb{R}^d} \mathrm{e}^{-tK(x)} - 1 + tK(x)\,\mathrm{d}x} \\ &\leq \mathrm{e}^{c_t \int_{\mathbb{R}^d} \mathrm{e}^{-K(x)} - 1 + K(x)\,\mathrm{d}x} < \infty.\end{aligned}$$

Here we have additionally used Lemma 1.10 b), the fact that by the convexity of the function $h\colon x \mapsto \mathrm{e}^{-x} - 1 + x$ and Jensen's inequality

$$\int_{\mathbb{R}^d} h(\eta(t,x))\,\mathrm{d}x \leq \frac{1}{t}\int_0^t \int_{\mathbb{R}^d} h(tK(x - W_s))\,\mathrm{d}x\,\mathrm{d}s = \int_{\mathbb{R}^d} h(tK(x))\,\mathrm{d}x \tag{1.43}$$

and that for all $t \geq 0$ there exists a constant $0 < c_t < \infty$ such that $h(tx) \leq c_t h(x)$. Our computation now suggests (cf. [CK12, Theorem 1])

Theorem 1.11. *Let $K \geq 0$. Under the integrability condition* (1.40), *the annealed (and so the quenched) exponential moment*

$$\mathbb{E}_0 \otimes \mathbb{E}\left[\exp\left\{-\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right\}\right] \tag{1.44}$$

is finite for all $t \geq 0$. Thus the Gibbs measures in Definition 1.6 (with the negative sign) are well defined.

1.3.3 Finiteness of Positive Exponential Moments

From now on we will consider the shape

$$K(x) = \theta|x|^{-p}, \qquad d/2 < p < d, \quad \theta \in (0,\infty).$$

While the existence of the *negative* quenched and annealed exponential moments has come without much effort, the story is different in the case of the positive moments. Let $\varepsilon > 0$ and note that on the event $A_\varepsilon = \{W \text{ stays in } B_\varepsilon \text{ until time } t\}$ we have for all x with $|x| \geq \varepsilon$

$$\eta(t,x) = \theta \int_0^t \frac{1}{|x - W_s|^p}\,\mathrm{d}s \geq \frac{t\theta}{2^p|x|^p}. \tag{1.45}$$

Since $\mathbb{P}[A_\varepsilon] \approx \mathrm{e}^{-ct\varepsilon^{-2}}$ for $\varepsilon \downarrow 0$, we have the lower bound

$$\begin{aligned}\mathbb{E}_0\left[\exp\Big(\int_{\mathbb{R}^d} \mathrm{e}^{\eta(t,x)} - 1 - \eta(t,x)\,\mathrm{d}x\Big)\right] &\geq \exp\Big(\int_{|x|\geq\varepsilon} \mathrm{e}^{\frac{t\theta}{2^p|x|^p}} - 1 - \frac{t\theta}{2^p|x|^p}\,\mathrm{d}x\Big)\mathbb{P}[A_\varepsilon]\\ &\geq \exp(\mathrm{e}^{c_t\varepsilon^{-p}})\mathbb{P}[A_\varepsilon] \xrightarrow{\varepsilon\downarrow 0} \infty.\end{aligned}$$

The rough second inequality can be derived easily using a Taylor expansion of the exponential inside of the integral. Together with (1.42) and Lemma 1.10 b) this shows that

$$\mathbb{E}_0 \otimes \mathbb{E}\left[\exp\left\{\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right\}\right] = \mathbb{E}_0\left[\exp\left\{\int_{\mathbb{R}^d} \mathrm{e}^{\eta(t,x)} - 1 - \eta(t,x)\,\mathrm{d}x\right\}\right] = \infty, \tag{1.46}$$

i.e., the annealed model is not well defined with the positive sign in front of $K(x) = |x|^{-p}$ as the singularity of K at 0 destroys the integrability.

In the case of the positive quenched exponential moments we cannot argue with an average over all Poisson point constellations. Rather, the Brownian particle has to search for a location of high values of the potential in order to, possibly, make the quenched exponential moment explode. Note that what makes the positive moments large is the singularity of K at 0 and not the long range dependencies. Thus, we truncate K at radius 1, drop the renormalization and study

$$V_1(x) = \int_{\mathbb{R}^d} K(x-y)\mathbb{1}_{\{|x-y|<1\}}\,\mathrm{d}y \tag{1.47}$$

instead of $\overline{V}$. On the event $A_R = \{\omega(B_R) \geq 1\}$, with $R \in (0,\infty)$, take some x inside of the ball B_R s.t. $\omega(\{x\}) = 1$. Then

$$\mathbb{P}[(W_s) \text{ reaches } B_{2^{-n}}(x) \text{ until time } t/2 \text{ and stays there until time } t] \tag{1.48}$$

$$\geq c_1|B_{2^{-n}}(x)|\exp\left(-\frac{2R^2}{t} - c_2 t 2^{2n}\right). \tag{1.49}$$

The energy absorbed by the particle using this strategy can be estimated as

$$\int_0^t V_1(W_s)\,\mathrm{d}s \geq \theta\int_{t/2}^t \frac{1}{|x-W_s|^p}\,\mathrm{d}s \geq \frac{t\theta}{2}2^{np}. \tag{1.50}$$

Thus we get on A_R

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t V_1(W_s)\,\mathrm{d}s\right)\right] \geq c_1|B_{2^{-n}}(x)|\exp\Big(-\frac{2R^2}{t} - c_2 t 2^{2n} + \frac{t\theta}{2}2^{np}\Big). \tag{1.51}$$

Now, if $p > 2$ or $p = 2$ and θ is sufficiently large, the r.h.s goes to infinity as $n \to \infty$. Thus, on A_R,

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t V_1(W_s)\,\mathrm{d}s\right)\right] = \infty. \tag{1.52}$$

Since R was arbitrarily large, (1.52) does even hold almost surely.

When $p < 2$, the above argument does not seem to lead anywhere, so we try to establish an upper bound. Decompose the exponential moment according to the exit times $\tau_n = \inf\{s\colon W_s \notin B_{2^n}\}$ from the balls of radius $2^n, n \in \mathbb{N}$ around the origin (with $\tau_0 = 0$):

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t V_1(W_s)\,\mathrm{d}s\right)\right] = \sum_{n=1}^{\infty}\mathbb{E}_0\left[\exp\left(\theta\int_0^t V_1(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{\tau_{n-1}\leq t<\tau_n\}}\right]. \tag{1.53}$$

Using the Cauchy-Schwarz inequality we bring into account the costs for long journeys,

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t V_1(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{\tau_{n-1}\leq t<\tau_n\}}\right] \leq \mathbb{P}[\tau_{n-1}\leq t]^{\frac{1}{2}}\mathbb{E}_0\left[\exp\left(2\theta\int_0^t V_1(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{t<\tau_n\}}\right]^{\frac{1}{2}}. \tag{1.54}$$

Now, separating the Brownian path until time 2^{-cn} with $c = \frac{2d}{2-p}$ from the path afterwards, we get, using again the Cauchy-Schwarz inequality and the Markov property,

$$\mathbb{E}_0\left[\exp\left(2\theta\int_0^t V_1(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{t<\tau_n\}}\right] \leq \mathbb{E}_0\left[\exp\left(4\theta\int_0^{2^{-cn}} V_1(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{t<\tau_n\}}\right]^{\frac{1}{2}} \tag{1.55}$$

$$\times\left\{\int_{B_{2^n}}\mathbb{E}_x\left[\exp\left(4\theta\int_0^{t-2^{-cn}} V_1(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{t-2^{-cn}<\tau_n\}}\right]\mathbb{P}[W_{2^{-cn}}\in\mathrm{d}x]\right\}^{\frac{1}{2}}. \tag{1.56}$$

The first expectation on the right-hand side is negligible. It can be controlled by using bounds on the exponential moments of $\sup_{x\in\mathbb{R}}\left(\int_0^1 \frac{1}{|W_s-x|^p}\,\mathrm{d}s\right)$, cf. [BCR09], Brownian scaling and a Cramér large deviation bound on $\omega(B_{2^n})$. We skip the details and rather go to the second expectation which is the one that produces large values. First, we bound the Gaussian transition density $\mathbb{P}[W_{2^{-cn}}\in\mathrm{d}x]$ uniformly on B_{2^n} by $\frac{2^{cdn/2}}{(2\pi)^{d/2}}$. This allows us to apply the Feynman-Kac eigenvalue bound

$$\int_{B_{2^n}}\mathbb{E}_x\left[\exp\left(4\theta\int_0^{t-2^{-cn}} V_1(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{t-2^{-cn}<\tau_n\}}\right]\mathrm{d}x \leq |B_{2^n}|\,\mathrm{e}^{t\lambda_{\max}(B_{2^n},V_1)}. \tag{1.57}$$

By Proposition 1.4, we can bound $\lambda_{\max}(B_{2^n}, V_1)$ by $\max_{z\in4\mathbb{Z}^d\cap B_{2^n}}\lambda_{\max}(B_3(z),V_1)$ modulo an error term. Note that we consider microballs $B_3(z)$ of a fixed (not n-dependent radius). Thus $\max_{z\in4\mathbb{Z}^d\cap B_{2^n}}\omega(B_3(z))$ will be large as n gets large. As the family $\{\lambda_{\max}(B_3(z),V_1),\ z\in 4\mathbb{Z}^d\cap B_{2^n}\}$ is identically distributed, we have for $\alpha > 0$ using a union bound

$$\mathbb{P}\left[\max_{z\in4\mathbb{Z}^d\cap B_{2^n}}\lambda_{\max}(B_3(z),V_1)\geq\alpha\right] \leq C2^{dn}\mathbb{P}\big[\lambda_{\max}(B_3,V_1)\geq\alpha\big]. \tag{1.58}$$

Using (1.11), $\lambda_{\max}(B_3, V_1)$ is given by the variation

$$\sup_{g\in H_0^1(Q_3),\|g\|_2=1} \left\{ 4\int_{B_3} V_1(x)g^2(x)\,\mathrm{d}x - \frac{1}{2}\int_{B_3} |\nabla g(x)|^2\,\mathrm{d}x \right\} \tag{1.59}$$

$$\leq (2\theta\omega(B_4))^{\frac{2}{2-p}} \sup_{g\in H_0^1(\mathbb{R}^d),\|g\|_2=1} \left\{ \int_{B_3} \frac{g^2(x)}{|x|^p}\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2\,\mathrm{d}x \right\}. \tag{1.60}$$

The inequality follows from a straightforward calculation using the substitution $g(x) = \theta^{d/(2(2-p))} f(\theta^{1/(2-p)}x)$. To determine the value of the above variational problem we can use the abstract result (1.19) in [BCR09]. We have

$$\sup_{g\in H_0^1(\mathbb{R}^d),\|g\|_2=1} \left\{ \int_{B_3} \frac{g^2(x)}{|x|^p}\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g(x)|^2\,\mathrm{d}x \right\} < \infty, \tag{1.61}$$

which is closely related to the inequality

$$\int_{\mathbb{R}^d} \frac{f^2(x)}{|x|^p}\,\mathrm{d}x \leq C\|f\|_2^{2-p}\|\nabla f\|_2^p, \qquad f\in H^1(\mathbb{R}^d). \tag{1.62}$$

Here the condition $p<2$ reveals its significance. Now put $\alpha = Cn^{2/(2-p)}$ in (1.58). As $\omega(B_4)$ has the Poisson distribution, one can choose C large enough to make the right-hand side in (1.58) summable. Then by the Borel-Cantelli lemma

$$\limsup_{n\to\infty} n^{-2/(2-p)} \max_{z\in 4\mathbb{Z}^d\cap B_{2^n}} \lambda_{\max}(B_3(z), V_1) \leq C \quad \mathbb{P}\text{-a.s.} \tag{1.63}$$

Thus, the costs $\mathbb{P}[\tau_{n-1}\leq t]^{\frac{1}{2}} \approx \exp(c_t 2^{-2n-1})$ for reaching far away balls dominate the growth of $\mathrm{e}^{t\lambda_{\max}(B_{2^n},V_1)}$ and therefore (1.53) is finite. Collecting the results about positive quenched integrability we have sketched so far, we get (cf. [CK12, Theorem 1.5])

Theorem 1.12. *Let $t,\theta>0$. Then*

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right] \begin{cases} <\infty & \mathbb{P}\text{-a.s. , if } p<2, \\ =\infty & \mathbb{P}\text{-a.s. , if } p>2. \end{cases}$$

Now the interesting question is, what happens in the case $p=2$. By the condition $d/2<p<d$ we then have to take $d=3$. It comes without surprise that $\frac{1}{|x|^2}$ is a critical potential, since it is well known in quantum mechanics that for $p>2$, $|x|^{-p}$ defines a singular potential pulling a particle to the origin with infinite speed. On the other hand, such a model is well defined and has finite energy when $p<2$.

From (1.51) we already know that for large coefficients θ,

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right] = \infty. \tag{1.64}$$

To determine what "large" exactly means, the estimates we have made are too rough. On the other hand, we can transfer the computations for the upper bound in the case $p < 2$ to $p = 2$. We arrive again at (1.57) and bound the principal eigenvalue by, instead of (1.60),

$$\sup_{f \in H^1(\mathbb{R}^3), \|f\|_2 = 1} \left\{ \theta \max_{z \in 2\delta_n \mathbb{Z}^3 \cap B_{2^n}} \omega(B_{b\delta_n}(z)) \int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} \, \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla f(x)|^2 \, \mathrm{d}x \right\} \tag{1.65}$$

where the n-dependent radius δ_n of the microboxes has to be chosen properly. This variational problem leads to the well known Hardy inequality

$$\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} \, \mathrm{d}x \leq 4 \|\nabla f\|_2^2, \qquad f \in H^1(\mathbb{R}^3). \tag{1.66}$$

The constant 4 is optimal, i.e. for any $\varepsilon > 0$ there exists a function $f \in H^1$ s.t.

$$\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} \, \mathrm{d}x > (4 - \varepsilon) \|\nabla f\|_2^2. \tag{1.67}$$

Now (1.66) and (1.67) imply that

$$\sup_{f \in H^1(\mathbb{R}^3), \|f\|_2 = 1} \left\{ a \int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} \, \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla f(x)|^2 \, \mathrm{d}x \right\} \begin{cases} = 0, & \text{if } a \leq \frac{1}{8}, \\ = \infty, & \text{if } a > \frac{1}{8}. \end{cases} \tag{1.68}$$

In the view of (1.65) and (1.68) the following problem arises. If $\theta \max_{z \in 2\delta_n \mathbb{Z}^3 \cap B_{2^n}} \omega(B_{b\delta_n}(z))$ finally becomes smaller than $\frac{1}{8}$ as n gets large, (1.65) becomes 0 and (1.53) will be summable. When $\theta \max_{z \in 2\delta_n \mathbb{Z}^3 \cap B_{2^n}} \omega(B_{b\delta_n}(z))$ finally becomes strictly larger than $\frac{1}{8}$, the ansatz does lead to a proper upper bound. To produce small values of $\max_{z \in 2\delta_n \mathbb{Z}^3 \cap B_{2^n}} \omega(B_{b\delta_n}(z))$ we have to choose the parameter δ_n small, but on the other hand, we cannot take δ_n too small, since the error term in the eigenvalue calculation in Proposition 1.4 grows as δ_n^{-2}. It turns out that the best we can get under this restriction is $\max_{z \in 2\delta_n \mathbb{Z}^3 \cap B_{2^n}} \omega(B_{b\delta_n}(z)) = 2$ for all large enough n. Thus we have to assume $\theta \leq \frac{1}{16}$ to get a finite upper bound. We will further investigate the problem using a *multipolar Hardy inequality*, cf. Section 3.2.2.

Note the difference in the size of the microcubes. In the case $p < 2$, the radius was fixed while here it converges to 0. Apparently only a finite number of very close Poisson points is needed in combination with the strong singularity of $\theta|x|^{-2}$ to produce sufficiently high values of the potential.

Due to this sensitivity to the coefficient θ another problem occurs when we adapt the proof of the upper bound in the case $p < 2$ to the case $p = 2$. The application of the Cauchy-Schwarz inequality in (1.54) enlarges the coefficient to 2θ, which, regarding (1.68), is too rough when $\theta \in (\frac{1}{8}, \frac{1}{16}]$. When $\theta \in (\frac{1}{8}, \frac{1}{16})$ We can instead apply Hölder's inequality with conjugated exponents q and q' with $\frac{1}{q} + \frac{1}{q'} = 1$ such that $q\theta < \frac{1}{16}$, but in the case $\theta = \frac{1}{16}$, our technique of proof fails. The application of Hölder's inequality in order to get the probabilistic costs for long journeys in (1.54) seems to be too rough; hence we will need a finer analysis. The solution to this problem is one of the main issues of this work. Let us collect what we have achieved until here in the following theorem due to Chen and Rosinski, [CR11, Theorem 2.1]:

Theorem 1.13. *For $d = 3$, $p = 2$, for all $t > 0$,*

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right]\begin{cases} < \infty & \mathbb{P}\text{-a.s., if } \theta < \frac{1}{16}, \\ = \infty & \mathbb{P}\text{-a.s., if } \theta > \frac{1}{16}. \end{cases}$$

See Theorem 2.1 below for the case $\theta = \frac{1}{16}$.

1.3.4 Time Asymptotics

We have found out that if $d/2 < p < d$, the negative annealed (and $\mathbb{P}$-a.s. the quenched exponential moments of $\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s$ are finite, as well as, $\mathbb{P}$-a.s., the positive quenched) exponential moments provided $p < 2$ or $p = 2$ and $\theta < \frac{1}{16}$. The natural follow-up task is the determination of the asymptotic behaviour as $t \to \infty$.

The asymptotics of the negative annealed exponential moments have been studied in [CK11]. From Theorem 2.1 therein we extract

Theorem 1.14. *For $p \in (d/2, d)$,*

$$\lim_{t\to\infty} t^{-d/p}\log\mathbb{E}\otimes\mathbb{E}_0\left[\exp\left(-\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right] = -\omega_d\theta^{d/p}\Gamma\left(\frac{p-d}{p}\right). \tag{1.69}$$

Apparently, (1.69) extends the asymptotics (1.16)-(1.17) in the regime $d < p < d+2$ to the regime $d/2 < p < d$ after implementing the renormalization. The heuristic picture of the Brownian particle staying inside a ball that is small relative to t and free of obstacles remains.

The negative quenched exponential moments have been treated in [Che12, Theorem 2.1].

Theorem 1.15. *For $d/2 < p < d$ and $\theta > 0$,*

$$\lim_{t\to\infty} t^{-1}(\log t)^{-(d-p)/d}\log\mathbb{E}_0\left[\mathrm{e}^{-\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s}\right] = \frac{\theta d^2}{d-p}\left(\frac{\omega_d}{d}\Gamma\left(\frac{2p-d}{p}\right)\right)^{p/d} \quad \mathbb{P}\text{-a.s.} \tag{1.70}$$

The result yields the following observations. First, as the constant in the limit is positive, the Lebesgue part of the renormalization must play a crucial role here since without it, the limit should be strictly negative. Secondly, the limit depends on the concrete shape of K, i.e. on p and θ in contrast to the asymptotics (1.28) in the soft obstacle case. Thirdly, the asymptotic rate depends on p and is much larger than the one in (1.28). This is indicated by the fact that, following the strategy used before (1.28), the Brownian particle moves to an obstacle free ball of radius $\approx \log t^{1/d}$ where the potential - in the absence of obstacles - locally equals the renormalization part

$$t\int_{\{|x|\le(\log t)^{1/d}\}} |x|^{-p}\,\mathrm{d}x = t(\log t)^{(d-p)/d}. \tag{1.71}$$

For the asymptotics of the positive quenched exponential moments we refer to [Che12, Theorem 2.2]. Write $\sigma(d,p)$ for the optimal constant in the inequality (1.62).

Theorem 1.16. *For $d/2 < p < \min(2, d)$ and $\theta > 0$,*

$$\lim_{t\to\infty} \frac{1}{t} \left\{ \frac{\log\log t}{\log t} \right\}^{\frac{2}{2-p}} \log \mathbb{E}_0 \left[e^{\theta \int_0^t \overline{V}(W_s)\, \mathrm{d}s} \right] = \frac{1}{2} p^{\frac{p}{2-p}} (2-p)^{\frac{4-p}{2-p}} \left(\frac{d\theta\sigma(d,p)}{2+d-p} \right)^{\frac{2}{2-p}} \quad \mathbb{P}\text{-}a.s. \tag{1.72}$$

The asymptotics (1.72) follows a similar strategy as discussed in the context of (1.34). This time we consider microballs of shrinking radii $r_t = \left\{ \frac{\log\log t}{\log t} \right\}^{\frac{1}{2-p}}$ not further than, roughly, distance t away from the origin. The Brownian motion visits the optimal one before time t; it turns out that there will be roughly $\frac{\log t}{\log\log t}$ points in this optimal microball. The potential is then bounded from below by $\frac{\log t}{\log\log t} r_t^{-p} = \left\{ \frac{\log t}{\log\log t} \right\}^{\frac{2}{2-p}}$. The probabilistic costs for this strategy have the same exponential scale which leads to the asymptotic (1.72). Note that we again have a p-dependent limit and a larger asymptotic rate compared to (1.34).

The asymptotics of the positive quenched exponential moments in the case $d = 3, p = 2$ are one of the main problems studied in the present work and will be treated in the following section.

2 Main Results and Ideas of the Proof

We are now ready to present our main theorems about the finiteness and the time asymptotics of the positive quenched exponential moments

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right]. \tag{2.1}$$

Recall that $(W_t)_{t\geq 0}$ is a standard three-dimensional Brownian motion and

$$\overline{V}(x) = \int_{\mathbb{R}^3} K(x-y)[\omega(\mathrm{d}y) - \mathrm{d}y]. \tag{2.2}$$

is the renormalized Poisson potential which is well defined under some integrability condition on K given by (1.39). We now choose an inverse-square shape, i.e.

$$K(x) = \frac{1}{|x|^2}.$$

After stating our main results and giving several remarks, we are going to discuss the main ideas behind the proofs. Finally, we will state some open problems.

2.1 Main Results

Recall that $\mathcal{P} = \mathrm{supp}(\omega)$. Our first theorem treats the finiteness of the quenched exponential moments (2.1), especially including the critical case $\theta = \frac{1}{16}$.

Theorem 2.1. *For each $\theta \in (0, \frac{1}{16}]$ and $\mathbb{P}$-almost surely for all $x \in \mathbb{R}^3 \setminus \mathcal{P}$ and all $t \geq 0$, we have*

$$\bar{u}_\theta(t,x) := \mathbb{E}_x\left[\exp\left(\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right] < \infty. \tag{2.3}$$

Remark 2.2. The statement of Theorem 2.1 cannot be extended to $x \in \mathcal{P}$, which is indicated by the fact that for all $t > 0$ we have $\mathbb{P}_0[\int_0^t |W_s|^{-2}\,\mathrm{d}s = \infty] = 1$, cf. [Szn98, Example 1.2.3].

With $\bar{u}_\theta$ we have found a *mild solution* to the parabolic Anderson model (1.2). We will recall the notion of a mild solution in Section 3.1 below.

Theorem 2.3. *For any $\theta \in (0, \frac{1}{16}]$, the function $\bar{u}_\theta(t,x)$ defined in* (2.3) *is a mild solution to* (1.2) *with $d = 3$ and $q = \theta\overline{V}$.*

The next result characterizes the time asymptotics of (2.1) via a tightness result. Define $k = k_\theta = \lfloor (8\theta)^{-1} \rfloor \in \{2, 3, ...\}$ for $\theta \in (0, \frac{1}{16}]$.

Theorem 2.4. *Let $\theta \in (0, \frac{1}{16}]$. For any function $\beta\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\beta(t) \overset{t\to\infty}{\longrightarrow} \infty$,*

$$\beta(t)t^{-\frac{k+1}{k-1}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right] \overset{t\to\infty}{\longrightarrow} \infty \quad \textit{in probability} \tag{2.4}$$

and

$$\beta(t)^{-1}t^{-\frac{k+1}{k-1}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right] \overset{t\to\infty}{\longrightarrow} 0 \quad \textit{in probability.} \tag{2.5}$$

In other words, the sequence $\left(t^{-\frac{k+1}{k-1}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right]\right)_{t\geq 0}$ is tight on the open interval $(0,\infty)$.

Remark 2.5. The asymptotic rate in (2.4) and (2.5) yields a rather unusual dependence on the parameter θ, since it depends on it only through $k = \lfloor (8\theta)^{-1} \rfloor$. Thus, moving θ inside of the intervals

$$\left(\frac{1}{8(l+1)}, \frac{1}{8l}\right], \quad l = 2, 3, \ldots \tag{2.6}$$

does not change the asymptotic rate, but shifting it from one of the intervals to another does change the rate significantly. However, the critical point $\theta = \frac{1}{16}$ yields no different behaviour since the rate is constant also on the interval $(\frac{1}{24}, \frac{1}{16}]$. We will explain this behaviour in the following sections by translating asymptotics of small distances in the Poisson cloud $\mathcal{P}$ into the asymptotics of the local eigenvalues of the Schrödinger operator $\frac{1}{2}\Delta + \overline{V}$ with the help of a *multipolar Hardy inequality*.

Remark 2.6. Note that the asymptotic rate $t^{\frac{k+1}{k-1}}$ in (2.4) and (2.5) is much larger than the one in the classical result (1.34) for bounded shapes K but also than the rate in (1.72) in the regime $K(x) = |x|^{-p}, p < 2$. Near below and at $\theta = \frac{1}{16}$, the quenched exponential moments grow like t^3. The large scale seems to correspond rather naturally to the strong singularity of our inverse-square shape.

Given the asymptotics in the non-critical setup, cf. (1.34), it would be no surprise if $\log \overline{u}_\theta(t,x)$ converged to a deterministic constant $\mathbb{P}$-almost surely on the scale $t^{\frac{k+1}{k-1}}$. However, the following results characterizing the lim inf and lim sup behaviour show that no almost sure convergence occurs at all. Recall that a function $\ell\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{x\to\infty} \ell(x) = \infty$ is called *slowly varying at infinity* if for any $a > 0$

$$\lim_{x\to\infty} \frac{\ell(ax)}{\ell(x)} = 1. \tag{2.7}$$

Theorem 2.7. *Let $\theta \in (0, \frac{1}{16}]$. For any $\ell\colon (0,\infty) \to (1,\infty)$ slowly varying at infinity, we have*

$$\limsup_{t\to\infty} t^{-\frac{k+1}{k-1}} \ell(t)^{-\frac{2}{3(k-1)}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right] = \begin{cases} 0 & \mathbb{P}\text{-a.s.}, \quad \text{if } \int_1^\infty \frac{\mathrm{d}r}{r\ell(r)} < \infty, \\ \infty & \mathbb{P}\text{-a.s.}, \quad \text{if } \int_1^\infty \frac{\mathrm{d}r}{r\ell(r)} = \infty. \end{cases} \tag{2.8}$$

Theorem 2.8. *For each* $\theta \in (0, \frac{1}{16}]$, *there exist* $0 < C_{\inf}(k) < C^{\inf}(k) < \infty$ *such that*

$$\liminf_{t\to\infty} t^{-\frac{k+1}{k-1}} \big(\log\log(t)\big)^{\frac{2}{3(k-1)}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right] \in [C_{\inf}(k), C^{\inf}(k)] \quad \mathbb{P}\text{-a.s.} \tag{2.9}$$

Remark 2.9. With respect to the choice of the function ℓ, there are two trivial regimes where both the *limes superior* and the *limes inferior* of the sequence

$$v(t) = t^{-\frac{k+1}{k-1}} \ell(t)^{-\frac{2}{3(k-1)}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right]$$

take the value 0 or ∞ and a nontrivial regime in between where the *limes inferior* is contained in $[0,\infty)$ and the *limes superior* becomes ∞. This nontrivial regime contains the cases $\ell \equiv$ *const.*, $\ell(t) = \log t$ and $\ell(t) = \frac{1}{\log\log t}$.

$\ell(t) =$	$\frac{1}{(\log\log t)^b}$	$\frac{1}{\log\log t}$	const	$\log t(\log\log t)$	$\log t(\log\log t)^b$
$\liminf_{t\to\infty} v(t,0)$	∞	$\in [C_{\inf}, C^{\inf}]$	0	0	0
$\limsup_{t\to\infty} v(t,0)$	∞	∞	∞	∞	0

Figure 2.1: Large time asymptotics of $v(t)$. The exponent b is considered as strictly larger than 1.

Remark 2.10. The difference between the asymptotics in (2.8) and (2.9) is explained as follows. The integral test in (2.8) distinguishes between the cases

$$\limsup_{t\to\infty} \sup_{x\in B_{R_k(t)}} \omega\Big(B_{r_k(t)}(x)\Big) \begin{cases} \le k & \mathbb{P}\text{-a.s.,} \\ \ge k+1 & \mathbb{P}\text{-a.s.,} \end{cases}$$

where $R_k(t) = t^{\frac{k}{k-1}} \ell(t)^{\frac{1}{3(k-1)}}$ and $r_k(t) = t^{-\frac{1}{k-1}} \ell(t)^{-\frac{1}{3(k-1)}}$, cf. Lemma 5.5 below. Note that the integrability condition in (2.8) is insensitive to the multiplication of ℓ by a fixed constant which indicates that the right-hand side of (2.8) can only take on the values 0 and ∞.

On the other hand, the appearance of the function $\ell(t) = \log\log(t)^{-1}$ in (2.9) is related to the test

$$\int_1^\infty \frac{1}{t} \exp(-\ell(t)^{-1})\,\mathrm{d}t \begin{cases} < \infty, \\ = \infty, \end{cases}$$

which distinguishes between

$$\liminf_{t\to\infty} \sup_{x\in B_{R_k(t)}} \omega\Big(B_{r_k(t)}(x)\Big) \begin{cases} \le k & \mathbb{P}\text{-a.s.,} \\ \ge k+1 & \mathbb{P}\text{-a.s.,} \end{cases}$$

where $R_k(t) = t^{\frac{k}{k-1}} \ell(t)^{-\frac{1}{3(k-1)}}$ and $r_k(t) = t^{-\frac{1}{k-1}} l(t)^{\frac{1}{3(k-1)}}$, cf. Lemma 5.9 below. The latter integrability condition is indeed sensitive when $\ell(t)$ is multiplied by a constant. Hence, the right-hand side of (2.9) is nontrivial.

Remark 2.11. In our proof of (2.9), we obtain different constants $C^{\inf} > C_{\inf}$. However, it seems possible that the limit in (2.9) is in fact constant almost surely. Our estimates used in the proof of (2.9) would have to be sharpened to prove this stronger result. We leave this problem to future study.

Remark 2.12. Given the lack of $\mathbb{P}$-a.s. convergence or convergence in probability we can hope to strengthen the tightness result from Theorem 2.4 to a convergence in distribution. It seems possible, that a refinement of our technique of proof where the estimates for lower and upper bound are sharpened could lead to such a result.

As already suggested, the asymptotic behaviour of $\mathbb{E}_0\left[\exp\left(\theta\int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right]$ is determined by the singularity of $|x|^{-2}$ and not by the long range interactions between the Brownian particle and the infinite number of far away Poisson points. Thus, we should be able to substitute the renormalized potential $\overline{V}$ by a truncated potential

$$V^{(a)}(x) = \int_{\mathbb{R}^d} |x-y|^{-2}\mathbb{1}_{\{|x-y|<a\}}\omega(\mathrm{d}y), \quad x \in \mathbb{R}^d \setminus \mathcal{P},\ d=3 \tag{2.10}$$

without changing our main results.

Moreover, the restriction of the dimension to $d=3$ was only necessary for the definition of the renormalized potential $\overline{V}$, whereas the integral defining $V^{(a)}(x)$ is finite in any dimension. In dimension $d=1$ we have, writing $\tau_{\mathcal{P}} = \inf\{t>0 : W_t \in \mathcal{P}\}$, $\int_0^{\tau_{\mathcal{P}}+\varepsilon} V^{(a)}(W_s)\,\mathrm{d}s = \infty$ $\mathbb{P}\otimes\mathbb{P}_0$-a.s. for all $\varepsilon>0$, which follows using the strong Markov property of Brownian motion from the facts that $\int_0^\varepsilon |W_s|^{-2}\,\mathrm{d}s = \infty$ $\mathbb{P}_0$-a.s. for all $\varepsilon>0$ and that $\mathbb{P}\otimes\mathbb{P}_0[\tau_{\mathcal{P}}<\infty]=1$. This implies that in dimension $d=1$ we have that for all $t>0$, $\theta\in(0,\infty)$ and $a>0$

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right] = \infty \quad \mathbb{P}\text{-a.s.} \tag{2.11}$$

The same seems to hold true in dimension $d=2$. By the Hardy inequality (cf. (3.36) below) the operator $\frac{1}{2}\Delta + \theta|x|^{-2}$ is not semibounded on $L^2(\mathbb{R}^2)$ for any $\theta>0$, which corresponds to the situation $d=3$ and $\theta > \frac{1}{16}$.

In the case $d \geq 4$ it is straightforward to extend Theorem 2.1. The critical constant in the d-dimensional Hardy inequality is $\left(\frac{d-2}{2}\right)^2$, thus setting $h_d = \frac{(d-2)^2}{8}$ we have the following.

Theorem 2.13. *For all $d\geq 3$, $a\in(0,\infty)$ and $\theta\in(0,\frac{h_d}{2}]$, it holds $\mathbb{P}$-almost surely that, for all $x\in\mathbb{R}^d\setminus\mathcal{P}$ and all $t\geq 0$,*

$$u_\theta^{(a)}(t,x) := \mathbb{E}_x\left[\exp\left(\theta\int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right] < \infty. \tag{2.12}$$

Theorem 2.14. *For all $d\geq 3$ and any $\theta\in(0,\frac{h_d}{2}]$, the function $u_\theta^{(a)}(t,x)$ defined in* (2.3) *is a mild solution to* (1.2) *with $q=\theta V^{(a)}$.*

To reformulate the asymptotics in (2.4), (2.5), (2.8) and (2.9), define for $d\geq 3$ and $\theta\in(0,\frac{h_d}{2}]$, $k=k_\theta=\lfloor\frac{h_d}{\theta}\rfloor$, which is consistent with the previous definition of k_θ in the case $d=3$.

Theorem 2.15. *Let $d \geq 3$, $a \in (0, \infty)$ and $\theta \in (0, \frac{h_d}{2}]$. For any function $\beta \colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\beta(t) \overset{t\to\infty}{\longrightarrow} \infty$,*

$$\beta(t) t^{-\frac{k+1}{k-1}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t V^{(a)}(W_s)\, \mathrm{d}s \right) \right] \overset{t\to\infty}{\longrightarrow} \infty \quad \textit{in probability} \tag{2.13}$$

and

$$\beta(t)^{-1} t^{-\frac{k+1}{k-1}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t V^{(a)}(W_s)\, \mathrm{d}s \right) \right] \overset{t\to\infty}{\longrightarrow} 0 \quad \textit{in probability.} \tag{2.14}$$

Theorem 2.16. *Let $d \geq 3$, $a \in (0, \infty)$ and $\theta \in (0, \frac{h_d}{2}]$. For any $\ell \colon (0, \infty) \to (1, \infty)$ slowly varying at infinity, we have*

$$\limsup_{t\to\infty} t^{-\frac{k+1}{k-1}} \ell(t)^{-\frac{2}{d(k-1)}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t V^{(a)}(W_s)\, \mathrm{d}s \right) \right] = \begin{cases} 0 & \mathbb{P}\text{-a.s.}, \quad \text{if } \int_1^\infty \frac{\mathrm{d}r}{r\ell(r)} < \infty, \\ \infty & \mathbb{P}\text{-a.s.}, \quad \text{if } \int_1^\infty \frac{\mathrm{d}r}{r\ell(r)} = \infty. \end{cases} \tag{2.15}$$

Theorem 2.17. *Let $d \geq 3$ and $\theta \in (0, \frac{h_d}{2}]$. There exist $0 < C_{\inf}(k,d) < C^{\inf}(k,d) < \infty$ such that, for all $a \in (0, \infty)$,*

$$\liminf_{t\to\infty} t^{-\frac{k+1}{k-1}} \log\log(t)^{\frac{2}{d(k-1)}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t V^{(a)}(W_s)\, \mathrm{d}s \right) \right] \in [C_{\inf}(k,d), C^{\inf}(k,d)] \quad \mathbb{P}\text{-a.s.} \tag{2.16}$$

We will focus on the proofs of Theorems 2.13-2.17 and derive Theorems 2.1-2.8 by estimating the error $|\overline{V} - V^{(a)}|$.

2.2 Main Ideas of the Proofs

We are going to present a heuristic derivation of our main results which is supposed to serve as both an illustration of the main ideas and a sketch of the subsequent rigorous proofs. For simplicity we only consider the case $d = 3$, the case $d \geq 4$ yields no significant differences other than the larger critical constants $h_d > \frac{1}{8}$.

Truncation of the Potential As a first remark we note that it is easier to prove our main theorems for the truncated potential $V^{(a)}$ instead of for the renormalized potential $\overline{V}$. To prove the results for the latter, we have to control the error $|V^{(a)} - \overline{V}|$. Fortunately, it is shown in [CR11, Section 3], that the contribution of $\overline{V} - V^{(a)}$ is negligible on the exponential scales we are interested in, since

$$\sup_{|x|<R} |\overline{V}(x) - V^{(a)}(x)| = \mathcal{O}(\log R), \quad \mathbb{P}\text{-a.s. }, \ R \to \infty. \tag{2.17}$$

In [CR11], (2.17) is derived by computing the exponential moments of $\overline{V} - V^{(a)}$ with the help of Lemma 1.10 b) and an application of the exponential Markov inequality. For the proof of the upper bounds in our main theorems, we need to derive a similar result in the situation where $V^{(a)}$ is further truncated at an R dependent, shrinking truncation radius a_R. It turns out that, by the inverse-square shape of K,

$$\sup_{|x|<R} |\overline{V} - V^{(a_R)}| = \mathcal{O}(a_R^{-2} \log R), \quad \mathbb{P}\text{-a.s. }, \ R \to \infty. \tag{2.18}$$

This result will follow from Corollary 5.3 below. Due to (2.17) we will only consider $V^{(a)}$ in the remaining part of this section.

Computation of the Local Eigenvalues In order to determine the amount of potential the Brownian particle can collect in a region $D \subset \mathbb{R}^3$, we have to control the variational formula $\mathscr{V}(D, V^{(a)}_{\mathcal{P}\cap D})$, cf. (1.11). Here $V^{(a)}_{\mathcal{P}\cap D}(x) := \sum_{y\in\mathcal{P}\cap D} \mathbb{1}_{\{|x-y|\le a\}}|x-y|^{-2}$. Given the Hardy inequality (1.66), it is clear that $\mathscr{V}(D, V^{(a)}_{\mathcal{P}\cap D}) = 0$ when $\theta\omega(D) \le \frac{1}{8}$. In order to get a non-zero contribution out of $\mathscr{V}(D, V^{(a)}_{\mathcal{P}\cap D})$, it is therefore necessary to have at least $\lfloor \frac{1}{8\theta} \rfloor + 1$ Poisson points in the domain D. We are now going to argue that the optimal regions consist of exactly $\lfloor \frac{1}{8\theta} \rfloor + 1$ points, which should be as close together as possible. For this purpose we have to obtain an upper bound on $\mathscr{V}(D, V^{(a)}_{\mathcal{P}\cap D})$ and a lower bound on -for technical reasons- $\mathscr{V}(D, V_{\mathcal{P}\cap D})$ with $V_{\mathcal{P}\cap D}(x) := \sum_{y\in\mathcal{P}\cap D} |x-y|^{-2}$ being the non-truncated inverse-square potential associated to the Poisson points in D. The randomness of the cloud $\mathcal{P} \cap D$ is of no importance at this point and we think of $\mathcal{Y} = \mathcal{P} \cap D$ as of a deterministic point constellation.

We begin with the lower bound on $\mathscr{V}(D, V_{\mathcal{Y}})$. For simplicity, assume that $D = B_r(0)$ for some $r > 0$. We define an auxiliary potential as

$$\widetilde{V}(x) := |x|^{-2} \wedge 1.$$

When all $y \in \mathcal{Y}$ are sufficiently close to 0, $\widetilde{V}$ is a good lower bound on $V_{\mathcal{Y}}$ in the sense that there exists (by the assumption $\theta\omega(D) > \frac{1}{8}$) a $\theta' > \frac{1}{8}$ such that for all $x \in \mathbb{R}^d$

$$\theta V_{\mathcal{Y}}(x) \ge \theta' \widetilde{V}(x),$$

cf. Lemma 3.24. With the same method with which it can be shown that the constant 4 is optimal in Hardy's inequality, i.e. inserting a sequence of explicit functions into the inequality, we can show that $\mathscr{V}(D, \widetilde{V})$ is strictly positive, see Lemma 3.23. It will then follow from Brownian scaling and a Fourier decomposition that, for $x \in D$,

$$\mathbb{E}_x \left[\exp\left(\int_0^t \theta V_{\mathcal{Y}}(W_s)\, \mathrm{d}s \right) \mathbb{1}_{\{\tau_{D^c} > t\}} \right] \ge c_1 \exp\left(c_2 t r^{-2}\right), \tag{2.19}$$

see Lemma 3.25. The constant c_2 will depend on $\#\mathcal{Y}$ but will not increase when $\#\mathcal{Y}$ gets large. Therefore, the Brownian particle should search for a ball containing exactly $\lfloor \frac{1}{8\theta} \rfloor + 1$ Poisson points and which has minimal radius (and which is not too far away). Any additional Poisson point, which necessarily would increase the radius, is expendable.

We will now illustrate this phenomenon further by discussing the upper bound. Similarly as in the argument used in Proposition 1.4, we can compute $\mathscr{V}(\mathbb{R}^3, V^{(a)}_{\mathcal{P}\cap D})$ by dividing $\mathbb{R}^3$ into various subsets $D_1, ..., D_n$ and by separately computing the 'eigenvalues' $\lambda_i := \mathscr{V}(D_i, \lambda_{\mathcal{Y}\cap D_i})$, using a partition of unity $\psi_1, ..., \psi_n$ with supp $\psi_i \subset D_i$. This produces an error of order $\max_{i=1}^n |\nabla\psi_i|_2$, i.e. of order $\max_{i=1}^n \operatorname{diam}(D_i)^{-2}$. When all D_i are chosen such that $\#\mathcal{Y} \cap D_i \leq \lfloor \frac{1}{8\theta} \rfloor$, then for all $i = 1, \dots, n$ we have $\lambda_i = 0$ and thus only the error term is nontrivial. Therefore, the best upper bound we can get on $\mathscr{V}(\mathbb{R}^3, V^{(a)}_{\mathcal{P}\cap D})$ this way is of order H^{-2} with

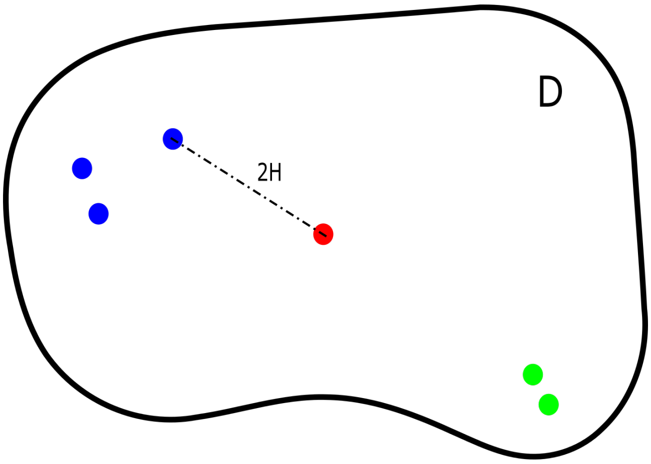

Figure 2.2: Let $\theta \in (\frac{1}{32}, \frac{1}{24}]$, i.e. $k_\theta + 1 = 4$. Then the value of $\mathscr{V}(D, V_{D\cap\mathcal{P}})$ does not depend on how close the three blue points are together (or the two green points). Instead the distance between the group of blue points and the red point determines $\mathscr{V}(D, V_{D\cap\mathcal{P}})$, as the red point completes the blue group to a $k_\theta + 1$ group.

$$H := \inf\left\{\rho > 0 \colon \begin{array}{c} \text{one of the connected components of } \mathcal{U}^{(\rho)}_{\mathcal{Y}} \\ \text{contains more than } \lfloor \frac{1}{8\theta} \rfloor \text{ elements of } \mathcal{Y} \end{array}\right\}$$

and $\mathcal{U}^{(\rho)}_{\mathcal{Y}} := \bigcup_{y\in\mathcal{Y}} B_\rho(y)$ being the ρ-neighbourhood of $\mathcal{Y}$. In the case $\#\mathcal{Y} \cap D = \lfloor \frac{1}{8\theta} \rfloor + 1$ we have

$$H = \inf\left\{\rho > 0 \colon \mathcal{U}^{(\rho)}_{\mathcal{Y}} \text{ is connected}\right\}.$$

If the cloud $\mathcal{Y}$ contains a group of $\lfloor \frac{1}{8\theta} \rfloor$ points lying close to each other, or even many of such groups, this alone does not produce a large value of $\mathscr{V}(\mathbb{R}^3, V^{(a)}_{\mathcal{P}\cap D})$. For the size of $\mathscr{V}(\mathbb{R}^3, V^{(a)}_{\mathcal{P}\cap D})$ it is crucial *how close a* $(\lfloor \frac{1}{8\theta} \rfloor + 1)$*-th point lies to some group of* $\lfloor \frac{1}{8\theta} \rfloor$ *points*. Thus, also the upper bound convinces us, that small balls with exactly $\lfloor \frac{1}{8\theta} \rfloor + 1$ will be the most desirable regions. The rigorous computation of this upper bound in Proposition 3.19 will result in a *multipolar Hardy inequality*.

How close will $k_\theta + 1$ Poisson Points be located? Let $R_n \uparrow \infty$ be a sequence of increasing radii. Given the upper and lower bounds on local eigenvalues that we discussed above, we have to determine how small we can choose the radii $r_n > 0$ such that (with high $\mathbb{P}$-probability) the large balls $B_{R_n}(0)$ contain at least one tiny ball $B_{r_n}(x)$ of radius r_n such that $\omega(B_{r_n}(x)) = k_\theta + 1$. This is an elementary combinatorial problem which we will treat in Section 5.1. Essentially, we just need to compute two quantities: the probability $\mathbb{P}[\omega(B_{r_n}(0)) = k_\theta + 1] = \mathrm{Poi}_{|B_{r_n}|}(k_\theta + 1)$ to find $k_\theta + 1$ points in a specific ball (e.g. around the origin) and the rough number of balls of radius r_n that are needed to cover $B_{R_n}(0)$, i.e. $\sim c\frac{R_n^3}{r_n^3}$. In order that $\mathrm{Poi}_{|B_{r_n}|}(k_\theta+1)\frac{R_n^3}{r_n^3}$ does not vanish as $n \uparrow \infty$, r_n has to be at least of rough order $R_n^{-\frac{1}{k_\theta}}$. What *rough* means for our purposes depends on whether we want to let the event $\{\exists x \in B_{R_n}(0)\colon \omega(B_{r_n}(x)) = k_\theta + 1\}$ happen

- with probability going to 1, (in the proof of the lower bound in Theorem 2.4)
- $\mathbb{P}$-a.s. infinitely often (in the proof of the lower bound in Theorem 2.7) or
- $\mathbb{P}$-a.s. for eventually all n (in the proof of the lower bound in Theorem 2.7).

For the proofs of the upper bounds we distinguish between the cases where the event $\{\exists x \in B_{R_n}(0)\colon \omega(B_{r_n}(x)) = k_\theta + 1\}$

- happens with probability going to 0 (Theorem 2.4),
- $\mathbb{P}$-a.s. does *not* happen for eventually all n (Theorem 2.7) or
- $\mathbb{P}$-a.s. does *not* happen for infinitely many n (Theorem 2.8).

The latter case will be the only one a bit more difficile (due to independence issues in the corresponding Borel-Cantelli argument).

The Lower Bound The proof of the lower bound of the large time asymptotic in Theorem 2.15 is now straight forward and follows essentially the same strategy as used for (1.72). Until time $0 < t_0 < t$ the Brownian particle moves to a ball U of radius roughly $R(t)^{-\frac{1}{k_\theta}}$ (this is however much smaller than the radius considered in the proof of (1.72)), which contains $k_\theta + 1$ Poisson points and is located not further away than distance $R(t)$ from the origin. Then it stays in some slightly larger ball $\widetilde{U}$ until time t. On the exponential scale, the costs for reaching this ball are $-\frac{R(t)^2}{2t_0}$. As a lower bound on the Feynman-Kac functional we then have by (2.19)

$$\mathbb{E}_{W_{t_0}}\left[\exp\left(\int_0^{t-t_0} V_{\mathcal{P}\cap\widetilde{U}}(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{\tau_{\widetilde{U}^c} > t-t_0\}}\right] \geq c_1 \exp\left(c_2(t-t_0)R(t)^{\frac{2}{k_\theta}}\right).$$

Thus the contribution of the sketched strategy is roughly

$$\exp\left(-\frac{R(t)^2}{2t_0} + c_2(t-t_0)R(t)^{\frac{2}{k_\theta}}\right).$$

We optimize over $R(t)$ and t_0 and get $R(t) = t^{\frac{k_\theta}{k_\theta-1}}$ and hence the correct exponential scale $t^{\frac{k+1}{k-1}}$.

A Path Decomposition In contrast to the proof of the lower bound we have to take into account all possible Brownian paths for the upper bound, not only those following a specific strategy. Therefore, we use a *path decomposition* technique which works as follows.

Let $R \in (0,\infty)$, $\alpha \in (\frac{1}{k_\theta+1}, \frac{1}{k_\theta})$ and $a_R = R^{-\alpha}$. Assume without restriction that the truncation radius a is smaller than 1. By $\widetilde{\mathcal{P}}_R := \mathcal{P} \cap B_{R+1}$ we denote the set of Poisson points inside the ball B_{R+1}. We then consider the $2a_R$- neighbourhood $\mathcal{U}^{(2a_R)}_{\widetilde{\mathcal{P}}_R} = \bigcup_{y\in\widetilde{\mathcal{P}}_R} B_{2a_R}(y)$ of $\widetilde{\mathcal{P}}_R$ which consists of connected components $\mathcal{C}$, each containing at most $k_\theta + 1$ Poisson points of $\widetilde{\mathcal{P}}_R$ ($\mathbb{P}$-a.s. eventually as $R \uparrow \infty$). Write

$$\lambda_{\max} = \max_{\mathcal{C} \text{ component of } \mathcal{U}^{(2a_R)}_{\widetilde{\mathcal{P}}_R}} \mathscr{V}(\mathcal{C}, V^{(a)}_{\mathcal{C}\cap\widetilde{\mathcal{P}}_R})$$

for the maximal 'eigenvalue' of $\frac{1}{2}\Delta + V^{(a)}_{\mathcal{C}\cap\widetilde{\mathcal{P}}}$ in one of these components. According to our previous computations, $\lambda_{\max}$ should be of order $R^{\frac{2}{k_\theta}}$. Taking some $\gamma > \lambda_{\max}$, we have the crucial identity

$$\mathbb{E}_0\Big[\exp\Big(\int_0^t V^{(a)}(W_s)\,\mathrm{d}s\Big)\mathbb{1}_{\{\tau^{\mathrm{c}}_{B_{R/2}} < t \le \tau_{B^{\mathrm{c}}_R}\}}\Big] = \mathrm{e}^{t\gamma}\mathbb{E}_0\Big[\exp\Big(\int_0^t V^{(a)}(W_s) - \gamma\,\mathrm{d}s\Big)\mathbb{1}_{\{\tau^{\mathrm{c}}_{B_{R/2}} < t \le \tau_{B^{\mathrm{c}}_R}\}}\Big].$$

The factor $\mathrm{e}^{t\gamma}$ should contain all of the amount of potential that can be absorbed until time t, provided (W_s) does not leave the ball B_R (but does leave the ball $B_{R/2}$). We want to argue that the contribution of the remaining expectation consists only in the probabilistic costs of leaving the ball $B_{R/2}$.

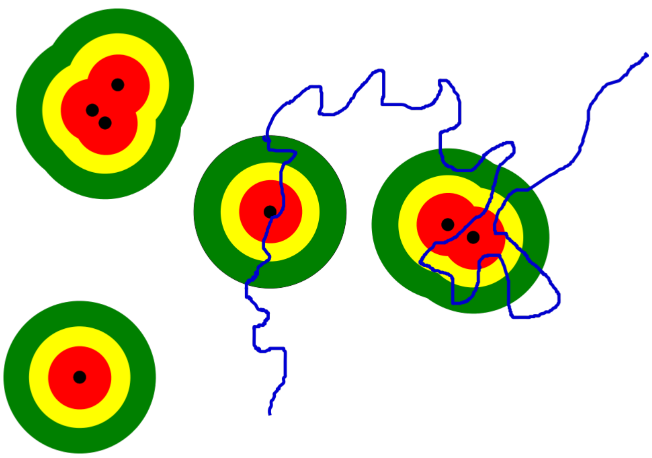

Figure 2.3: We decompose the Brownian paths according to their excursions into the neighbourhoods of the Poisson points. The stopping time $\hat{\tau}_n$ denotes the first exit time from the *green* neighbourhood after the n-th entrance into the *yellow* neighbourhood, which occurs at time $\check{\tau}_n$. Outside of the red area we have $V^{(a)}(W_s) - \gamma \le 0$. After entering the yellow annulus, the Brownian particle has to pay a fee of order $\mathrm{e}^{-c\sqrt{\gamma}(\text{diameter of the yellow annulus})}$ to enter the red *high potential* area.

First, we replace $V^{(a)}$ by the further truncated potential $V^{(a_R)}$ which vanishes outside of $\mathcal{U}^{(2a_R)}_{\widetilde{\mathcal{P}}_R}$. The error of this substitution is controlled by γ, since, for $R \to \infty$,

$$V^{(a)}(x) - V^{(a_R)}(x) \leq \sup_{x \in B_R(0)} (\# B_a(x) \cap \mathcal{P}) R^{2\alpha} \leq \mathcal{O}(\log R) R^{2\alpha} \ll R^{\frac{2}{k_\theta}}.$$

Contributions to to the remaining integral $\int_0^t V^{(a_R)}(W_s) - \gamma \, \mathrm{d}s$ are only produced while (W_s) visits $\mathcal{U}^{(2a_R)}_{\widetilde{\mathcal{P}}_R}$. By considering two auxiliary neighbourhoods $\mathcal{U}^{(3a_R)}_{\widetilde{\mathcal{P}}_R}$ and $\mathcal{U}^{(5a_R)}_{\widetilde{\mathcal{P}}_R}$ of $\widetilde{\mathcal{P}}_R$, we can describe the subsequent excursions near the Poisson point set via two sequences of stopping times $\check{\tau}_n$ and $\hat{\tau}_n$, defined recursively, setting $\check{\tau}_0 = \hat{\tau}_0 = 0$, as

$$\begin{aligned} \check{\tau}_{n+1} &:= \begin{cases} \infty & \text{if } \hat{\tau}_n := \infty, \\ \inf\left\{ t > \hat{\tau}_n \colon \ W_t \in \overline{\mathcal{U}^{(3a_R)}_{\widetilde{\mathcal{P}}_R}} \right\} & \text{otherwise,} \end{cases} \\ \hat{\tau}_{n+1} &= \begin{cases} \infty & \text{if } \check{\tau}_{n+1} = \infty, \\ \inf\left\{ t > \check{\tau}_{n+1} \colon \ W_t \notin \mathcal{U}^{(5a_R)}_{\widetilde{\mathcal{P}}_R} \right\} & \text{otherwise.} \end{cases} \end{aligned} \tag{2.20}$$

To control

$$\mathbb{E}_0 \left[\exp\left(\int_{\check{\tau}_n}^{t \wedge \hat{\tau}_n} V^{(a_R)}(W_s) - \gamma \, \mathrm{d}s \right) \mathbb{1}_{\{\tau_{B^c_{R/2}} < t \leq \tau_{B^c_R}, \check{\tau}_n < t\}} \right],$$

we distinguish between the cases $\hat{\tau}_n < t$ and $\hat{\tau}_n \geq t$. The second case corresponds to the situation where (W_s) stays at least until time t inside of that component $\mathcal{C}_n$ of $\mathcal{U}^{(5a_R)}_{\widetilde{\mathcal{P}}_R}$, which it is visiting at time $\check{\tau}_n$. In this situation, we apply the standard Feynman-Kac eigenvalue bound

$$\mathbb{E}_{W_{\check{\tau}_n}} \left[\exp\left(\int_{\check{\tau}_n}^{t} V^{(a_R)}_{\mathcal{P} \cap \mathcal{C}_n}(W_s) - \gamma \, \mathrm{d}s \right) \mathbb{1}_{\{\tau_{\mathcal{C}_n^c} > t\}} \right] \leq \exp\left(t \left(\mathscr{V}(\mathcal{C}_n, V^{(a_R)}_{\mathcal{P} \cap \mathcal{C}_n}) - \gamma \right) \right) \leq 1.$$

In the first case, (W_s) will leave $\mathcal{C}_n$ before time t. Hence, we have to apply a *stopped* Feynman-Kac formula corresponding to the time-independent Schrödinger equation where (W_s) is killed when exiting $\mathcal{C}_n$, cf. Proposition 3.14. We will control the stopped Feynman-Kac functional $\mathbb{E}_{W_{\check{\tau}_n}} \left[\exp\left(\int_0^{\tau_{\mathcal{C}_n^c}} V^{(a_R)}_{\mathcal{P} \cap \mathcal{C}_n}(W_s) - \gamma \, \mathrm{d}s \right) \right]$ using the resolvent $\left[\frac{1}{2}\Delta + V^{(a_R)}_{\mathcal{P} \cap \mathcal{C}_n} - \gamma \right]^{-1}$, resulting in the bound

$$\mathbb{E}_{W_{\check{\tau}_n}} \left[\exp\left(\int_0^{\tau_{\mathcal{C}_n^c}} V^{(a_R)}_{\mathcal{P} \cap \mathcal{C}_n}(W_s) - \gamma \, \mathrm{d}s \right) \right] \leq c \left(1 + \frac{\gamma + \theta(5a_R)^{-2}}{\gamma - \lambda_{\max}(\mathcal{C}_n, V^{(a)}_{\mathcal{P} \cap \mathcal{C}_n})} \right).$$

Both of the two bounds neither depend on where (W_s) enters $\mathcal{C}_n$, nor on which specific component of $\mathcal{U}^{(5a_R)}_{\widetilde{\mathcal{P}}_R}$ is represented by $\mathcal{C}_n$. Rather, the bound we will obtain on

$$\mathbb{E}_0 \left[\exp\left(\int_0^{t} V^{(a_R)}(W_s) - \gamma \, \mathrm{d}s \right) \mathbb{1}_{\{\tau_{B^c_{R/2}} < t \leq \tau_{B^c_R}\}} \right]$$

does only depend on the number of excursions into $\overline{\mathcal{U}^{(3a_R)}_{\widetilde{\mathcal{P}}_R}}$, i.e. on $E_t := \#\{n \colon \check{\tau}_n < t\}$. We will therefore decompose the Feynman-Kac functional, and especially the probabilistic costs of leaving $B_{R/2}$, according to the value $N \in \mathbb{N}$ of E_t.

During N excursions, (W_s) can move at most a distance of $N(k_\theta + 1)8a_R$ towards the boundary of the ball $B_{R/2}$ which it has to leave until time t. When N is small, (W_s) still has to travel a long distance $R - N(k_\theta + 1)8a_R$ outside of the components, which allows us to count large enough probabilistic costs of order $\exp\left(-\frac{(R-N(k_\theta+1)8a_R)^2}{2t}\right)$. When N gets too large these costs will vanish. However, the Brownian particle has to pay a small but sufficiently high price for each excursion: Every time (W_s) enters $\overline{\mathcal{U}^{(3a_R)}_{\widetilde{\mathcal{P}}_R}}$ and leaves $\mathcal{U}^{(5a_R)}_{\widetilde{\mathcal{P}}_R}$ afterwards it has to pay at least the costs of travelling the distance $2a_R$. This way, we are able to take into account sufficiently high probabilistic costs, that control the growth of $\mathrm{e}^{t\lambda_{\max}}$, for any value of E_t: either the costs of leaving a large ball until time t, or the costs of a huge number of excursions.

The Upper Bound Having settled the path decomposition, also the upper bounds can be derived in a straight forward way. In contrast to the lower bound's proof, it is not sufficient to choose a single radius $R(t)(\approx t^{\frac{k_\theta}{k_\theta-1}})$ and let the Brownian motion search for a region of high potential inside $B_{R(t)}$. While regions much closer than $R(t)$ will not yield sufficiently concentrated Poisson points (=high enough potential) and regions much further away will be too costly to reach until time t, we have to take into account regions *a little bit nearer or further away*, i.e. which are located at distance $t^{\frac{k_\theta}{k_\theta-1}}\ell(t)^{\pm 1}$ with $\ell(t)$ slowly varying at infinity.

Structure of the Proof Except for the beginning of Section 3, the rest of this work is devoted to the proof of our main theorems. In Section 3, we will give the lower and upper bounds on the local principal eigenvalues and derive upper and lower bounds on (stopped and time-dependent) Feynman-Kac functionals. In Section 4, we establish the path decomposition. In Section 5.1, the asymptotics of small distances in the Poisson cloud and the corresponding eigenvalue asymptotics are derived. Then with the help of all these preparations the proofs of the main theorems are completed in the last section.

2.3 Open Problems

We leave the following open problems to future study.

In (2.4)-(2.5), it might be feasible to extend the tightness result to a convergence in distribution. A possible way might be to write

$$t^{-\frac{2}{k-1}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right] \stackrel{d}{=} tX_t$$

where the random variable X_t should only depend on the Poisson point process ω and describe the energy produced by the -at time t- optimal constellation of $k+1$ points. The estimates coming from our technique of proof for the upper and lower bounds would have to be sharper to prove such a result. The problem then could be reduced to the convergence of X_t.

The following question has a similar flavour. In (2.9), does $\mathbb{P}$-a.s. convergence to a deterministic constant hold?

Further questions include those of *localization* and *aging*.

- Does the Brownian particle indeed travel to the optimal region used for the lower bound in Theorem 2.4 with probability converging to one under the quenched Gibbs measure $\overline{\mathbb{Q}}_{t,\omega}$?
- How long is a specific region optimal before it is replaced by a better one?

For the first question one would desire an answer in the spirit of the pinning effect stated in Theorem 1.5.

3 Schrödinger Semigroups, Hardy Inequality and Bounds on the Feynman-Kac Functional

As already suggested in Section 1, a substantial part of the study of Brownian motion in a Poisson potential is based on the interplay with spectral properties of Schrödinger operators due to the classical Feynman-Kac formula. We collect all results about Schrödinger operators and semigroups required for the proofs of our main results. However, it turns out that the spectral properties of the multipolar inverse-square Schrödinger operator

$$\frac{1}{2}\Delta + c \sum_{y \in \mathcal{Y}} |x-y|^{-2} \tag{3.1}$$

makes up an interesting object in its own right. The multipolar inverse-square potential appears as a critical potential not included in the Kato class but lying just on the border of it. We thus will make a few more remarks on the analysis of (3.1) than absolutely necessary. Note that throughout this section we consider a deterministic point cloud $\mathcal{Y}$.

We will proceed as follows. First, we briefly state the connection between the parabolic Anderson problem, Schrödinger semigroups and the Feynman-Kac functionals (time dependent and stopped) for Kato class potentials. We then comment on Hardy-type inequalities and inverse-square Schrödinger operators. Finally, we apply these analytic results to prove the upper and lower bounds on the Feynman-Kac functional we need for the proof of our main theorems.

Literature: There is a huge body of literature concerning Schrödinger operators, semigroups, random Schrödinger operators and their relation to Brownian motion, of which we mention only a few. For an introduction to basic semigroup theory cf. [Dav81; Paz83; EN91] and for the special case of Schrödinger semigroups cf. the review articles [Sim82; Sim00] (and many references therein). A major reference to random Schrödinger operators is the book by Carmona and Lacroix, [CL12]. For the interplay with Brownian motion see [Sim80; CZ95] and the first chapter of [Szn98]. Literature concerning the inverse-square case will be given at the point where it appears.

3.1 Brownian Motion, Parabolic Anderson Problem and Schrödinger Semigroups

The study of the Feynman-Kac functional $\mathbb{E}_x\big[\exp(\int_0^t q(W_s)\,\mathrm{d}s)\big]$ intersects the fields of probability, semigroup theory and partial differential equations. We will recall this connection briefly and state all results needed to prove of our main theorems. This section may easily be skipped by the reader familiar with the field.

3.1.1 Notation

We are going to introduce some notation we use in this section and throughout the thesis. Define the family of non-empty, locally finite subsets of $\mathbb{R}^d$,

$$\mathscr{Y} = \{\mathcal{Y} \subset \mathbb{R}^d\colon\ \mathcal{Y} \neq \emptyset,\ \#K \cap \mathcal{Y} < \infty\ \forall \text{ compact } K \subset \mathbb{R}^d\}, \tag{3.2}$$

and the family of non-empty, finite subsets

$$\mathscr{Y}_f = \{\mathcal{Y} \in \mathscr{Y}\colon\ \#\mathcal{Y} < \infty\}. \tag{3.3}$$

Note that the support $\mathcal{P} = \{x \in \mathbb{R}^d\colon \omega(\{x\}) = 1\}$ of the Poisson point process ω almost surely belongs to $\mathscr{Y}$. For $\mathcal{Y} \in \mathscr{Y}$ and $a \in (0,\infty]$ satisfying either $\mathcal{Y} \in \mathscr{Y}_f$ or $a < \infty$, let

$$V_{\mathcal{Y}}^{(a)}(x) = \sum_{y \in \mathcal{Y}} \frac{\mathbb{1}_{\{|x-y|<a\}}}{|x-y|^2}, \quad x \in \mathbb{R}^d \setminus \mathcal{Y}. \tag{3.4}$$

Note that, for $a < \infty$, $V_{\mathcal{P}}^{(a)} = V^{(a)}$ as defined in (2.10). For $\mathcal{Y} \in \mathscr{Y}_f$, we write $V_{\mathcal{Y}} = V_{\mathcal{Y}}^{(\infty)}$.

We denote the volume of a Borel measurable subset $D \subset \mathbb{R}^d$ by $|D|$ and the the entrance time of Brownian motion in it by $\tau_D := \inf\{t \geq 0\colon W_t \in D\}$. By $B_r(x) = \{y \in \mathbb{R}^d\colon |x-y| < r\}$ we denote the open ball with radius $r \in (0,\infty)$ around $x \in \mathbb{R}^d$ with respect to the standard euclidean norm $|\cdot|$; in the case $x = 0$ we abbreviate $B_r = B_r(0)$. For $D \subset \mathbb{R}^d$ open, we write $B_r(D) = \{x \in \mathbb{R}^d\colon \exists y \in D, |x-y| < r\}$ for the r-neighbourhood of D.

Let $\emptyset \neq D \subset \mathbb{R}^d$. We introduce several function spaces.

- $C^n(D)$: space of n-times continuously differentiable functions on D, $n \in \mathbb{N}_0 \cup \{\infty\}$
- $C_c^n(D)$: compactly supported functions in $C^n(D)$
- $L^p(D)$: usual L^p-space on D with respect to the Lebesgue measure, $p \in [1,\infty]$
- $L^1_{loc}(D)$: space of locally integrable functions on D
- $W^{n,p}(D)$: Sobolev space, closure of $C^\infty(D)$ with respect to the Sobolev norm $\|f\|_{W^{n,p}(D)} = \sum_{|\alpha|\leq n} \|D^\alpha f\|_p$, where $|\alpha| = \alpha_1 + \cdots + \alpha_d$ for all multindices $\alpha \in \mathbb{N}_0^d$ and $D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$, $f \in C^\infty(D)$, $n, p \in \mathbb{N}$
- $W_0^{n,p}$: Sobolev spaces with zero-boundary condition, the closure of $C_c^\infty(D)$ with respect to $\|\cdot\|_{W^{n,p}(D)}$

- Abbreviate $H^n(D) := W^{n,2}(D)$ and $H^n_0(D) := W^{n,2}_0$

When $A\colon L^p(D) \to L^{p'}(D)$ is a bounded linear operator we denote its norm by $\|A\|_{p,p'}$ or by $\|A\|_p$ provided $p = p'$. By $\langle \cdot, \cdot \rangle$ we denote the canonical inner product on $L^2(D)$.

3.1.2 Parabolic Anderson Problem and Time-Independent Schrödinger Equation

Let $\emptyset \neq D \subset \mathbb{R}^d$ denote either a bounded open domain in $\mathbb{R}^d$ or the whole space $\mathbb{R}^d$. For $p, p' \in (0,\infty)$, we write $\Delta\colon L^p(D) \to L^{p'}(D)$, $f \mapsto \sum_{i=1}^d \frac{\partial^2 f}{\partial^2 x_i}$ for the (weak) Laplacian acting on $L^p(D)$ with domain chosen either as $\mathcal{D}(\Delta) = \mathcal{D}^{(p,p')}(\Delta) = \{f \in W^{1,p}(D)\colon \Delta f \in L^{p'}(D)\}$ or $\mathcal{D}(\Delta) = \mathcal{D}^{(p,p')}_0(\Delta) = \{f \in W^{1,p}_0\colon \Delta f \in L^{p'}(D)\}$. Given a potential $q \in L^1_{loc}$ on D, the *Schrödinger operator* is formally written as

$$\mathcal{H} = \mathcal{H}^{(q)} = \frac{\Delta}{2} + q. \tag{3.5}$$

It depends on the regularity properties of q, whether $\mathcal{H}$ can be well defined.

Definition 3.1. *Let $u_0\colon \mathbb{R}^d \to \mathbb{R}$. The initial-boundary value problem*

$$\partial_t u(t,x) = \frac{\Delta}{2} u(t,x) + q(x)u(t,x), \quad (t,x) \in [0,\infty) \times D \tag{3.6}$$

$$u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial D \tag{3.7}$$

$$u(0,x) = u_0(x), \quad x \in D \tag{3.8}$$

is called ***parabolic Anderson problem with initial data*** *u_0.*

In the case $D = \mathbb{R}^d$ (3.6)-(3.8) reduces to the initial-value problem

$$\partial_t u(t,x) = \frac{\Delta}{2} u(t,x) + q(x)u(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^d \tag{3.9}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^d. \tag{3.10}$$

Since we are especially interested in the initial condition $u_0 \equiv 1 \notin W^{1,p}_0$, we have to take into account solutions $u(t,x)$ to (3.6)-(3.8) in a weaker than the classical sense, i.e. when u is differentiable in time, twice differentiable in space and satisfies (3.6),

$$\lim_{\substack{x \to x_0,\\ t \downarrow 0}} u(t,x) = u_0(x_0), \quad x_0 \in D \qquad \text{and} \qquad \lim_{\substack{x \to x_0,\\ t \downarrow t_0}} u(t,x) = 0, \quad x_0 \in \partial D. \tag{3.11}$$

We call u a *mild solution* to (3.6)-(3.8), if

$$\int_0^t \int_D \tilde{p}^D_{t-s}(x,y)|q(y)u(s,y)|\,\mathrm{d}y\,\mathrm{d}s < \infty, \quad x \in D,\ t > 0 \tag{3.12}$$

and

$$u(t,x) = \int_D \tilde{p}^D_t(x,y)u_0(y)\,\mathrm{d}y + \int_0^t \int_D \tilde{p}^D_{t-s}(x,y)q(y)u(s,y)\,\mathrm{d}y\,\mathrm{d}s, \quad x \in D,\ t > 0. \tag{3.13}$$

Here $p_t^D(x,y) = \mathbb{P}_x[\tau_{D^c} > t, W_t \in dy]$ denotes the transition density of Brownian motion killed when leaving D, cf. [CZ95, Chapter 2]. Problem (3.6)-(3.8) is an example for the homogeneous abstract Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t), \quad u(0) = x, \tag{3.14}$$

where A is a linear operator on a Banach space $(X, \|\cdot\|_X)$ and $x \in X$ the initial data. Usually, $u \in X$ is usually called a mild solution to (3.14) if $\int_0^t u(s)\,\mathrm{d}s \in \mathcal{D}(A)$ and u satisfies the integral equation

$$u(t) = A\int_0^t u(s)\,\mathrm{d}s + x,$$

see e.g. [EN91, Definition 6.3]. If A is the generator of a strongly continuous semigroup T_t, the unique mild solution to (3.14) is given by $T_t x$ and if $u_0 \in \mathcal{D}(A)$, this solution is classical. In the next section we present a class of potentials q for which the Schrödinger operator $\frac{1}{2}\Delta + q$ satisfies this condition. However, our characterization (3.12)-(3.13) of a mild solution makes sense for more general potentials q and is consistent with the abstract definition of a mild solution u to the inhomogeneous, possibly nonlinear evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + f(t,u(t)), \quad u(0) = x, \tag{3.15}$$

which demands $u(t) = T_t x + \int_0^t T_{t-s} f(s,u(s))\,\mathrm{d}s$, see [Paz83, Definition 6.1]. We need this slightly more general concept to include the Schrödinger operator with inverse-square potential. I.e. we only require the existence of the semigroup generated by the Laplacian, putting $A = \frac{1}{2}\Delta$ and $f(t,u(t)) = qu(t)$ in (3.15). On the other hand, both notions of a mild solution to (3.14) and (3.15) are consistent if the right-hand side of (3.15) in fact has the form $Bu(t)$ and B generates a C_0-semigroup.

Furthermore, we have to introduce the stationary version of (3.6)-(3.8).

Definition 3.2. *Let $f\colon \partial D \to \mathbb{R}$. The time independent problem*

$$\frac{\Delta}{2}u(x) + q(x)u(x) = 0, \qquad x \in D, \tag{3.16}$$

$$u(x) = f(x), \qquad x \in \partial D. \tag{3.17}$$

is called ***Dirichlet boundary value problem for the Schrödinger equation****.*

As usual, we call a function $u\colon [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ a *weak solution* to (3.16), if $u(t,\cdot) \in L^1_{loc}(D)$, $qu(t,\cdot) \in L^1_{loc}(D)$ for all $t > 0$ and for all $\phi \in C_c^\infty(D)$

$$\int_D u(x)\Delta\phi(x)\,\mathrm{d}x = -2\int_D q(x)u(x)\phi(x)\,\mathrm{d}x. \tag{3.18}$$

3.1.3 The Kato Class K_d

We introduce a class of potentials q which yields sufficiently nice properties of the Schrödinger operator $\mathcal{H}^{(q)}$.

Definition 3.3. *Let $d \geq 1$. The **Kato class** K_d is defined as*

$$K_d = \Big\{ q\colon \mathbb{R}^d \to \mathbb{R} \text{ Borel measurable, } \lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \mathbb{E}_x \Big[\int_0^t |q(W_s)| \, \mathrm{d}s \Big] = 0 \Big\} \tag{3.19}$$

The class K_d^{loc} is the family of all Borel measurable $q\colon \mathbb{R}^d \to \mathbb{R}$, such that for any bounded $D \subset \mathbb{R}^d$, $q\mathbb{1}_D \in K_d$.

We have the following purely analytic characterization of K_d in terms of the Green kernel (i.e. the integral kernel of $(\Delta)^{-1}$),

$$g(x) = \begin{cases} |x|^{2-d}, & d \geq 3, \\ \log \frac{1}{|x|}, & d = 2. \end{cases} \tag{3.20}$$

A Borel measurable function $q\colon \mathbb{R}^d \mapsto \mathbb{R}$ is in K_d if and only if

$$\lim_{\varepsilon \downarrow 0} \left[\sup_{x\in\mathbb{R}^d} \int_{|y-x|\leq\varepsilon} |g(y-x)q(y)| \, \mathrm{d}y \right] = 0 \tag{3.21}$$

When $d = 1$, $q \in K_d$ is characterized by $\sup_{x\in\mathbb{R}} \int_{[x-1,x+1]} |q(y)| \, \mathrm{d}y < \infty$. We refer to [Szn98, Exercise 1, Section 1.2].

Note that $L^\infty(\mathbb{R}^d) \subset K_d$. Checking (3.21) we observe, that for $d \geq 2$ neither the inverse-square potential $q(x) = |x-y|^{-2}$ nor the multipolar inverse-square potential $q(x) = \sum_{y\in\mathcal{Y}} |x-y|^{-2}$ (with some $\mathcal{Y} \in \mathscr{Y}_f$) is included in K_d, whereas for any $\varepsilon > 0$ and $y \in \mathbb{R}^d$, $q(x) = |x-y|^{-(2-\varepsilon)}$ and even $q(x) = |x-y|^{-2}[-\log|x-y|]^{-\alpha}$ with $\alpha > 1$ are Kato class potentials. Thus the inverse-square potential is critical with a singularity that is just slightly too strong. A nice probabilistic proof of this fact can be found in [Szn98, Example 2.3, Section 1].

From (3.21) we obtain, cf. [CZ95, Proposition 3.1], that if $q \in K_d$, then

$$\sup_{x\in\mathbb{R}^d} \int_{|x-y|\leq 1} |q(y)| \, \mathrm{d}y < \infty. \tag{3.22}$$

and thus, $K_d^{loc} \subset L^1_{loc}$. It might come as a surprise that the uniform integrability condition (3.19) is sufficient to get uniform exponential integrability, which makes the Kato class the right class for our purposes, cf. [Szn98, Theorem 1.2.2] .

Proposition 3.4. *Let $q \in K_d$. Then there exist constants A, B s.t. for all $t \geq 0$ and $x \in \mathbb{R}^d$*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}_x \left[\exp\left(\int_0^t |q(W_s)| \, \mathrm{d}s \right) \right] \leq A \exp(Bt). \tag{3.23}$$

3.1.4 Schrödinger Semigroups with Kato Class Potentials

The spectral properties of the Laplacian as well as of Schrödinger operators have been studied extensively. In quantum mechanics, only selfadjoint Schrödinger operators are interpreted as observables and provide unique solutions to the time dependent Schrödinger equation. It is a main question, what properties of the potential-free Hamiltonian Δ can be transferred to $\mathcal{H}^{(q)}$, e.g. (essential-) selfadjointness, composition of the essential and discrete spectrum or the decay of the eigenfunctions. A lot of answers are obtained by regarding $\mathcal{H}^{(q)}$ as a perturbation of Δ which works nicely when q is relatively bounded with respect to the Laplacian, (e.g., when $q \in K_d$). For more general information about these topics we refer the reader to the review article [Sim00].

Here we collect basic properties of Schrödinger operators with Kato class potentials and their corresponding semigroups.

Proposition 3.5. *Let $q \in K_d$ and $p, p' \in [1, \infty)$. The following statements hold true.*

a) *The Schrödinger operator $\mathcal{H}^{(q)} \colon L^p(D) \to L^{p'}(D)$ is well defined with domain*

$$\mathcal{D}(\mathcal{H}) = \mathcal{D}^{(p,p')}(\mathcal{H}^{(q)}) = \left\{ f \in W_0^{1,p}(D) \colon \mathcal{H}^{(q)} f \in L^{p'}(D) \right\}. \tag{3.24}$$

In the case $p = p' = 2$, $(H^{(q)}, \mathcal{D}(\mathcal{H}^{(q)}))$ is selfadjoint.

b) *Suppose D is bounded and $p = p' = 2$. Then $(\mathcal{H}^{(q)}, \mathcal{D}(\mathcal{H}^{(q)}))$ has purely discrete spectrum* $\operatorname{spec}(\mathcal{H}^{(q)}) = \operatorname{spec}_{disc}(\mathcal{H}^{(q)})$ *consisting of eigenvalues*

$$\infty > \lambda_1 (= \lambda_{\max}(D, q)) > \lambda_2 \geq \ldots \geq \lambda_n \geq \ldots > -\infty. \tag{3.25}$$

There exists an orthonormal basis of $L^2(D)$ consisting of corresponding eigenfunctions $e_1, e_2, \ldots$ s.t. for all $f \in \mathcal{D}(\mathcal{H}^{(q)})$

$$\mathcal{H}^{(q)} f = \sum_{i=1}^{\infty} \lambda_i \langle e_i, f \rangle e_i. \tag{3.26}$$

c) *Let $p = p' = 2$ and $D = \mathbb{R}^d$. If for any $\varepsilon > 0$, q is decomposable as $q = q_1 + q_2$, s.t. $q_1 \in L^2(\mathbb{R}^d)$ and $q_2 \in L^\infty(D)$ with $\|q_2\| < \varepsilon$, then* $\operatorname{spec}_{ess}(\mathcal{H}^{(q)}) = \operatorname{spec}_{ess}(\Delta) = (-\infty, 0]$.

Proof. The selfadjointness is derived in the proof of [CZ95, Proposition 3.29]. In the case $D = \mathbb{R}^d$, the result goes back to Kato, [Kat72]. The description of the spectrum in b) can be found in [CZ95, p.125]. For statement c) see [HS96, Corollary 14.10]. □

Next we have, cf. [CZ95, Theorem 3.28]:

Lemma 3.6. *Let $q \in K_d$. Then $\mathcal{D}^{2,2}(\mathcal{H}^{(q)})$ is dense in $H_0^1(D)$ with respect to the Sobolev norm $\|f\|_{H^1(D)}$.*

Using Lemma 3.6 and the spectral theorem for selfadjoint operators, we have the crucial Rayleigh-Ritz variational formula:

Corollary 3.7. *It holds true that*

$$\lambda_{\max}(D,q) = \mathscr{V}(D,q) = \sup\left\{\int_D \left[-\frac{1}{2}|\nabla\phi|^2 + q\phi^2\right] \mathrm{d}x,\ \phi \in C_c^\infty(D), \|\phi\|_2 = 1\right\}. \quad (3.27)$$

Remark 3.8. When $D \subset D'$ and $q \le q'$, we have $\mathscr{V}(D,q) \le \mathscr{V}(D',q')$.

Proposition 3.9. *Let $q \in K_d$. The following statements hold true.*

a) The Schrödinger operator $(\mathcal{H}^{(q)}, \mathcal{D}(\mathcal{H}^{(q)}))$ is the generator of a strongly continuous semigroup $T_t = T_t^{(q)}$ on $L^2(D)$.

b) For any $1 \le p \le p' \le \infty$ and any $t \ge 0$, T_t is bounded from $L^p(D)$ to $L^{p'}(D)$ and there exist constants c_1, c_2 independent of p such that

$$\|T_t\|_p \le \|T_t\|_\infty \le c_1 \,\mathrm{e}^{c_2 t}. \quad (3.28)$$

for all $t \ge 0$. Especially, T_t is also a C_0-semigroup acting on $L^p(D)$ for any $1 \le p \le \infty$.

c) The spectrum of T_t acting on $L^p(D)$ is p - independent for any $t \ge 0$.

Proof. Cf. Theorems 3.17 and 3.27 in [CZ95]. □

Corollary 3.10. *For any $u_0 \in L^1(D)$ and $q \in K_d$, there exists a unique mild solution to* (3.6)-(3.8) *given by $u(t,x) = [T_t^{(q)} u_0](x)$.*

Proposition 3.11. *Suppose D is bounded and $q \in K_d$. For $t > 0$, T_t is compact with purely discrete spectrum satisfying* $\operatorname{spec}(T_t)\backslash\{0\} = \mathrm{e}^{t\,\operatorname{spec}(\mathcal{H})}$. *There exists an orthonormal basis of corresponding eigenfunctions $e_1, e_2, \ldots$ such that for all $f \in L^2(D)$*

$$T_t f = \sum_{i=1}^{\infty} \mathrm{e}^{t\lambda_i}\langle e_i, f\rangle e_i. \quad (3.29)$$

Especially, for all $t \ge 0$, we have $\|T_t\|_2 = \mathrm{e}^{t\lambda_{\max}}$ and

$$\langle T_t f, f\rangle \ge \mathrm{e}^{t\lambda_1}\langle e_1, f\rangle^2. \quad (3.30)$$

Proof. This follows from the results on $\mathcal{H}$ through functional calculus. For the compactness see [CZ95, Proposition 3.15] and for the purely discrete spectrum see [CZ95, p.125]. □

Denote the resolvent of $\mathcal{H}^{(q)}$ at $\gamma > \lambda_{\max}$ by $\mathcal{R}_\gamma = [\mathcal{H} - \gamma\,\mathrm{Id}]^{-1}$. Then we have the standard bound

$$\|\,\mathcal{R}_\gamma(A)\|_2 \le \frac{1}{\gamma - \lambda_{\max}}. \quad (3.31)$$

The next result about L^p regularity shows to what extent Schrödinger semigroups with Kato class potentials are 'smoothing', c.f. [Sim82, Theorem B.1.1] and the remarks thereafter.

Proposition 3.12. *Let $q_- \in K_d(\mathbb{R}^d)$ and $q_+ \in K_d^{loc}(\mathbb{R}^d)$ and $1 \le p \le p' \le \infty$. Then for any $A \ge$* $\sup\operatorname{spec}(\mathcal{H})$ *there exists $C < \infty$ independent of p and p' such that, writing $\gamma = \frac{1}{4}(p^{-1} - p'^{-1})$,*

$$\|\,\mathrm{e}^{t\mathcal{H}}\|_{p,p'} \le Ct^{-\gamma}\exp(At). \quad (3.32)$$

3.1.5 Feynman-Kac Formula and Stopped Feynman-Kac Formula

Proposition 3.13 (Feynman-Kac formula). *Let $q \in K_d$ and $u_0 \in L^2(D)$. Then the unique mild solution to the initial-boundary-value problem* (3.6)-(3.8) *can be written as*

$$u(t,x) = \mathbb{E}_x\Big[\exp\Big(\int_0^t q(W_s)\,\mathrm{d}s\Big)\,\mathbb{1}_{\{\tau_{D^c}\geq t\}}u_0(W_t)\Big], \quad (t,x) \in (0,\infty)\times D. \tag{3.33}$$

In other words, $(u(t,\cdot))_{t\geq 0}$ forms a strongly continuous semigroup on $L^2(D)$ with generator $\mathcal{H}^{(q)}$.

Proof. The result can be found in many references, see e.g. [Sim80, Theorem A.2.7.], where three different approaches are sketched. See also [CZ95, Theorem 3.27]. □

We also need a *stopped Feynman-Kac formula* providing the solution to the boundary value problem (3.16)-(3.17). It is a classical result from potential theory, that the Dirichlet problem with W_t being killed when exiting D has a unique solution provided D is *regular*, i.e. if $\mathbb{P}_x[\tau_{D^c} = 0] = 1$ for all $x \in \partial D$. Given a bounded domain $D \subset \mathbb{R}^d$ (therefore satisfying $\tau_D < \infty$ $\mathbb{P}_x$-a.s., $x \in D$) and a Borel measurable $q\colon \mathbb{R}^d \to \mathbb{R}$, the integral $\int_0^{\tau_{D^c}} q(W_s)\,\mathrm{d}s$ is well defined a.s. if $q \in K_d$ due to (3.23) and Jensen's inequality. If $f\colon \partial D \to \mathbb{R}$ is Borel measurable, the function

$$u_f(x) = u(D,q,f,x) = \mathbb{E}_x\left[\exp\left(\int_0^{\tau_{D^c}} q(W_s)\,\mathrm{d}s\right) f(W_{\tau_{D^c}})\right] \tag{3.34}$$

is well defined, Borel measurable and positive, but may take the value ∞. If $q \in K_d$, then u_1 is bounded on D and $(D;q)$ is called *gaugeable*.

Proposition 3.14 (Stopped Feynman-Kac formula). *Suppose $D \subset \mathbb{R}^d$ is bounded, $f\colon \partial D \to \mathbb{R}$ is bounded and Borel measurable, $q \in K_d^{loc}$ and $(D;q)$ is gaugeable. Then u_f is a weak solution to the time independent Schrödinger equation* (3.16). *Furthermore, if D is regular and f is continuous on ∂D, then u_f is the unique weak solution to the boundary value problem* (3.16)-(3.17).

Proof. This is part of Theorem 4.7 in [CZ95]. □

3.2 Inverse-Square Potentials and Hardy-Type Inequalities

3.2.1 Hardy Inequality

Hardy-type inequalities are a crucial tool in our study, because they allow us to control the principal eigenvalues of Schrödinger operators with (multipolar) inverse-square potential. In general, Hardy-type inequalities are applied in the theory of partial differential equations, the calculus of variation and various other fields in geometry and physics. In its original version, which appeared back in 1920 in a note on Hilbert's double series criteria, cf. [Har20], Hardy's inequality reads

$$\int_a^\infty \left(\frac{F(x)}{x}\right)^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_a^\infty f(x)^p \,\mathrm{d}x, \tag{3.35}$$

where $a > 0, p > 1$ and $F(x) = \int_a^\infty f(t)\,\mathrm{d}t$ for some arbitrary (Riemann-integrable) function $f > 0$. Here $\left(\frac{p}{p-1}\right)^p$ is in fact the optimal constant, although Hardy did not prove this fact in his original publication. We refer to [KMP06] for an interesting treatment of the historical background between 1906 and 1922. Until now, (3.35) has been generalized in various ways, including a d-dimensional version:

Lemma 3.15. *For $d \geq 2$, $1 \leq p < \infty$ and all $u \in H^1(\mathbb{R}^d)$,*

$$\left(\frac{|d-p|}{p}\right)^p \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p}\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x. \tag{3.36}$$

The constant $\left(\frac{|d-p|}{p}\right)^p$ is optimal in the sense that for all $\varepsilon > 0$ there is a $u \in H^1(\mathbb{R}^d)$ such that (3.36) *is violated when $\left(\frac{|d-p|}{p}\right)^p$ is replaced by $\left(\frac{|d-p|}{p}\right)^p + \varepsilon$.*

The above lemma has been proved in various ways. As a brief illustration in the case $p = 2$, note that for $u \in H^1(\mathbb{R}^d)$ and $\alpha \in (0, \infty)$, using the product formula and Green's identity,

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^d} \left|\nabla u + \alpha \frac{x}{|x|^2} u\right|^2 \mathrm{d}x = \int_{\mathbb{R}^d} |\nabla u|^2 + \alpha\left(\nabla u^2\right)\frac{x}{|x|^2} + \alpha^2 \frac{u^2}{|x|^2}\,\mathrm{d}x \qquad (3.37)\\ &= \int_{\mathbb{R}^d} |\nabla u|^2 + \alpha u^2 \left(\nabla \cdot \frac{x}{|x|^2}\right) + \alpha^2 \frac{u^2}{|x|^2}\,\mathrm{d}x = \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x + (\alpha^2 - (d-2)\alpha) \int_{\mathbb{R}^d} \frac{u^2}{|x|^2}\,\mathrm{d}x \end{aligned}$$

which shows (3.36) with $\alpha := \frac{d-2}{2}$. To obtain the optimality of α^2, insert in (3.36) the functions

$$u_M(x) := \begin{cases} M^{(d-2)/2}, & \text{if } 0 \leq |x| \leq M^{-1}, \\ |x|^{-(d-2)/2}, & \text{if } M^{-1} \leq |x| \leq M, \\ \frac{2M-|x|}{M^{d/2}}, & \text{if } M \leq |x| \leq 2M, \\ 0, & \text{if } 2M \leq |x|. \end{cases} \tag{3.38}$$

Note that an equivalent formulation of (3.36) is given in terms of the variational formula (3.27) with $\mathcal{Y} = \{0\}$, i.e. by

$$\mathscr{V}\left(\mathbb{R}^d, \tfrac{1}{2}\left(\tfrac{|d-p|}{p}\right)^p V_{\{0\}}\right) \leq 0. \tag{3.39}$$

Further generalizations include Hardy inequalities for bounded domains with Dirichlet and Neumann boundary conditions. The amount of literature in the field has become extensive; for an overview and further references cf. [Dav99; Psa11; BEL15]. For our purposes, the study of Hardy inequalities with more than one singularity, i.e. of multipolar Hardy inequalities, is crucial.

3.2.2 Multipolar Hardy Inequality

Let $d \geq 2$ and let $\mathcal{Y} \in \mathscr{Y}_f$ be a finite set of singularities with cardinality $M = \#\mathcal{Y}$. Let $V_{\mathcal{Y}}$ be the corresponding multipolar inverse-square potential (cf. (3.4)). It follows from (3.36) without effort that for all $\mu \leq \frac{(d-2)^2}{4M}$

$$\mu \int_{\mathbb{R}^d} |u|^2 V_{\mathcal{Y}} \,\mathrm{d}x \leq \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x \quad \text{for all } u \in H^1(\mathbb{R}^d), \tag{3.40}$$

but this estimate can be improved significantly. In the bipolar case, where due to translation and rotation invariance there is only one geometric constellation, $\mathcal{Y} = \{-y, y\}, y \in \mathbb{R}^d \setminus \{0\}$, we have

Proposition 3.16 (Bipolar Hardy inequality)**.** *There exists a nonincreasing function $K\colon (2,\infty) \to [0,\infty)$ with $\lim_{t\downarrow 2} K(t) = \infty$ and $K(t) = 0$ for $t \geq 2\sqrt{2}$, such that for $d \geq 3$ and $\mu \in (0, (d-2)^2/4)$*

$$\mu \int_{\mathbb{R}^d} |u|^2 \left(\frac{1}{|x-y|^2} + \frac{1}{|x+y|^2} \right) \mathrm{d}x \leq \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x + K\big(\tfrac{d-2}{\sqrt{\mu}}\big) |y|^{-2} \int_{\mathbb{R}^d} |u|^2 \,\mathrm{d}x. \tag{3.41}$$

for all $u \in H^1(\mathbb{R}^d)$.

Proof. Cf. [BDE08, Theorem 7]. □

The function K, which comes as an upper bound on the optimal constant in (3.41), is given explicitly in [BDE08]. In the case, where $\mathbb{R}^d$ is replaced by a bounded domain with two singularities on the boundary, the optimal constant in the analogue of (3.41) has been obtained in [CZ13].

In [BDE08], the authors rely on two methods to prove inequalities like (3.41) and (3.42). The *expansion of the square method* is essentially based on a generalization of the identity (3.37), while the *IMS-method* (for Ismagilov, Morgan-Simon, Sigal) uses an appropriate partition of unity to localize each singularity similarly as in Proposition 1.4. The latter method easily explains the additional term of order $|y|^{-2}\|u\|_{L^2}^2$, as the L^2-norms of the gradients of a partition separating y and $-y$ grow like $|y|^{-2}$ as $y \downarrow 0$. The situation with more than two poles is more complex, since then there is an infinite number of possible geometric constellations for $\mathcal{Y}$. The scale $|y|$ has to be replaced by a quantity depending on the pairwise distances of the poles in $\mathcal{Y}$, e.g.

$$\widetilde{\Gamma}^{-2} := \sum_{y \neq y' \in \mathcal{Y}} |y - y'|^{-2}.$$

This leads to the following result (cf. [BDE08] Theorem 10.)

Proposition 3.17 (Multipolar Hardy inequality)**.** *For any $M \geq 2$, there exists a nonincreasing function $K_M\colon (2,\infty) \to [0,\infty)$ with $\lim_{t\downarrow 2} K_M(t) = \infty$ and $K(t) = 0,\ t \geq 2\sqrt{M}$, such that for $d \geq 3$ and $\mu \in (0, (d-2)^2/4)$*

$$\mu \int_{\mathbb{R}^d} |u|^2 V_{\mathcal{Y}} \,\mathrm{d}x \leq \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x + K_M\big(\tfrac{d-2}{\sqrt{\mu}}\big)\widetilde{\Gamma}^{-2} \int_{\mathbb{R}^d} |u|^2 \,\mathrm{d}x \tag{3.42}$$

for any $u \in H^1(\mathbb{R}^d)$ and any $\mathcal{Y} \in \mathscr{Y}_f$ with $\#\mathcal{Y} = M$.

An immediate consequence of Proposition 3.17 is the following

Corollary 3.18. *Let $d \geq 3$ and $\mu \in (0, h_d)$. For all $\mathcal{Y} \in \mathscr{Y}_f$, we have*

$$\mathscr{V}(\mathbb{R}^d, \mu V_{\mathcal{Y}}) < \infty. \tag{3.43}$$

The estimate (3.42) may be improved in many ways. The scale $\widetilde{\Gamma}^{-2}$ is appropriate when the pairwise distances in $\mathcal{Y}$ are of the same order. However, in the situation with $k \leq \frac{(d-2)^2}{4\mu}$ particles being close to each other and $M-k$ particles further away, the small distances between the former do not contribute (using (3.40) and the IMS-method). Thus, we can expect to get a better estimate replacing $\widetilde{\Gamma}^{-2}$ with a quantity which does not take into account the distances between each element of $\mathcal{Y}$ and its $\lfloor \frac{(d-2)^2}{4\mu} \rfloor - 1$ closest neighbours. Using the methods from [BDE08] and recalling $h_d := \frac{(d-2)^2}{8}$, we obtain

Proposition 3.19. *Let $d \geq 3$, $M \geq 2$, $\theta \in \left(\frac{h_d}{M}, \frac{h_d}{(M-1)}\right]$ and*

$$\Gamma := \inf \left\{ r > 0 \colon \bigcup_{y \in \mathcal{Y}} B_r(y) \text{ is connected} \right\}. \tag{3.44}$$

Then

$$\mathscr{V}(\mathbb{R}^d, \theta V_{\mathcal{Y}}) \leq \frac{M(\pi^2 + 3\theta)}{2\Gamma^2}. \tag{3.45}$$

Proof. Fix $r \in (0, \Gamma)$ and choose $\widehat{\mathcal{Y}} \subset \mathcal{Y}$ such that $\widehat{\mathcal{Y}} \neq \emptyset$, $N := \#\widehat{\mathcal{Y}} \leq \lfloor M/2 \rfloor$,

$$\bigcup_{y \in \widehat{\mathcal{Y}}} B_r(y) \text{ is connected and } \bigcup_{y \in \widehat{\mathcal{Y}}} B_r(y) \ \cap \bigcup_{y \in \mathcal{Y} \setminus \widehat{\mathcal{Y}}} B_r(y) = \emptyset, \tag{3.46}$$

i.e., $\bigcup_{y \in \widehat{Y}} B_r(y)$ is a connected component of $\bigcup_{y \in \mathcal{Y}} B_r(y)$ containing at most half of the points of $\mathcal{Y}$. Define a partition of unity (cf. the definition after Theorem 1 in [BDE08]) with 2 terms as follows. Set

$$J(t) := \begin{cases} 0, & t \in [0, 1/2], \\ -\cos(\pi t), & t \in [1/2, 1], \\ 1, & t \geq 1, \end{cases} \tag{3.47}$$

put $J_1(x) := \prod_{y \in \widehat{\mathcal{Y}}} J(|x-y|/r)$ and $J_2(x) := [1 - J_1(x)^2]^{1/2}$. By Lemma 2 in [BDE08], we have, for all $u \in H^1(\mathbb{R}^d)$,

$$Q[u] := \int_{\mathbb{R}^d} \left\{ \theta V_{\mathcal{Y}}(x) u(x)^2 - |\nabla u(x)|^2 \right\} \mathrm{d}x = \sum_{i=1}^{2} Q[J_i u] + \int_{\mathbb{R}^d} u(x)^2 \sum_{i=1}^{2} |\nabla J_i(x)|^2 \,\mathrm{d}x. \tag{3.48}$$

Note that, by (3.46) and the definition of J_1, J_2,

$$V_{\mathcal{Y}}(x) J_2(x)^2 \leq V_{\widehat{\mathcal{Y}}}(x) J_2(x)^2 + \frac{M-N}{r^2} \quad \forall\, x \in \mathbb{R}^d \setminus \widehat{\mathcal{Y}} \tag{3.49}$$

while, for all $x \notin \widetilde{\mathcal{Y}} := \mathcal{Y} \setminus \widehat{\mathcal{Y}}$,

$$\begin{aligned} V_{\mathcal{Y}}(x) J_1(x)^2 &= \left\{ V_{\widetilde{\mathcal{Y}}}(x) + \sum_{y \in \widehat{\mathcal{Y}}} \frac{\mathbb{1}_{\{|x-y| \geq r/2\}}}{|x-y|^2} \right\} J_1(x)^2 \\ &\leq V_{\widetilde{\mathcal{Y}}}(x) J_1(x)^2 + \frac{N}{r^2} \sup_{t \geq 1/2} \frac{J(t)^2}{t^2} \leq V_{\widetilde{\mathcal{Y}}}(x) J_1(x)^2 + \frac{2N}{r^2} \end{aligned} \tag{3.50}$$

since $\sup_{t \geq 1/2} J(t)^2/t^2 = \sup_{t \in [1/2,1]} \cos(\pi t)^2/t^2 < 2$ (see the proof of Lemma 3 in [BDE08]). Applying (3.40), we obtain

$$\sum_{i=1}^{2} Q[J_i u] \leq \theta \frac{M+N}{r^2} \|u\|^2_{L^2(\mathbb{R}^d)} \quad \forall\, u \in H^1(\mathbb{R}^d). \tag{3.51}$$

Next we claim that

$$\sum_{i=1}^{2} |\nabla J_i(x)|^2 = \frac{|\nabla J_1(x)|^2}{1 - J_1(x)^2} \leq N \frac{\pi^2}{r^2} \quad \forall\, x \in \mathbb{R}^d. \tag{3.52}$$

Indeed, first note that, using the identity $\cos(a + \pi/2) = \sin(a)$,

$$\frac{|\nabla J_1(x)|^2}{1 - J_1(x)^2} \leq \frac{\pi^2}{r^2} \sup_{\eta \in (0,\pi/2)^N} F(\eta) \tag{3.53}$$

where, for $\eta = (\eta_1, \ldots, \eta_N) \in (0, \pi/2)^N$,

$$F(\eta) := \left(1 - \prod_{i=1}^{N} \sin(\eta_i)^2 \right)^{-1} \left(\sum_{i=1}^{N} \cos(\eta_i) \prod_{j \neq i} \sin(\eta_j) \right)^2. \tag{3.54}$$

Let us show that $\sup_{\eta \in (0,\pi/2)^N} F(\eta) = N$. To this end, first extend F to $[0, \pi/2]^N$ by noting that it is well defined and continuous on $\{\eta \in [0, \pi/2]^N \colon \min_i \eta_i < \pi/2\}$ and that $F(\eta) \to N$ as $\eta \to (\pi/2, \ldots, \pi/2)$. Now, if $\min_i \eta_i = 0$, then $F(\eta) \leq 1 < N$. Moreover, if $\min_i \eta_i < \max_i \eta_i = \pi/2$, by ignoring the coordinates that are equal to $\pi/2$ we may rewrite F in the same form as (3.54) with N substituted by some $1 \leq N' \leq N - 1$. Reasoning by induction, we reduce to the case where F assumes its maximum at a point $\bar{\eta} \in (0, \pi/2)^N$. In this case write F in the form $F = f/g$ where

$$f(\eta) := \left(\sum_{i=1}^{N} \cot(\eta_i) \right)^2, \quad g(\eta) := \prod_{i=1}^{N} \csc(\eta_i)^2 - 1. \tag{3.55}$$

Fix $1 \leq k \leq N$ such that $\cot(\bar{\eta}_k) = \max_i \cot(\bar{\eta}_i)$ and denote by ∂_k the partial derivative with respect to the k-th coordinate. Since $\bar{\eta}$ is a critical point for F we have $\partial_k F(\bar{\eta}) = 0$ and hence

$$F(\bar{\eta}) = \frac{\partial_k f(\bar{\eta})}{\partial_k g(\bar{\eta})} < \sum_{i=1}^{N} \frac{\cot(\bar{\eta}_i)}{\cot(\bar{\eta}_k)} \leq N. \tag{3.56}$$

Thus $\sup_{\eta\in(0,\pi/2)^N} F(\eta) = N$, proving (3.52) and, as a consequence,

$$\int_{\mathbb{R}^d} u(x)^2 \sum_{i=1}^{2} |\nabla J_i(x)|^2 \,\mathrm{d}x \le \frac{\lfloor M/2\rfloor \pi^2}{r^2} \|u\|^2_{L^2(\mathbb{R}^d)} \quad \forall\, u \in H^1(\mathbb{R}^d). \tag{3.57}$$

This together with (3.48) and (3.51) gives

$$\mathscr{V}(\mathbb{R}^d, \theta V_y) \le \frac{\lfloor M/2\rfloor \pi^2 + \theta(M+N)}{r^2}, \tag{3.58}$$

and we conclude (3.45) by letting $r \uparrow \Gamma$. □

3.2.3 Regularity Properties of Inverse-Square Schrödinger Operators

Inverse-square potentials

The inverse-square potentials $c|x|^{-2}$ do not belong to the Kato class K_d and thus standard results discussed in Section 3.1 cannot be applied. The criticality of the potential $c|x|^{-2}$ comes from the fact that, due to the same scaling property, it is *not* a lower order perturbation of the Laplacian, which would preserve main spectral properties. Following [AGG06], we define, for $\lambda > 0$, the "scaling" operator $U_\lambda \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $f \mapsto \lambda^{d/2} f(\lambda\cdot)$, normalized such that U_λ is unitary. Note that U_λ is invertible with inverse $U_\lambda^{-1} = U_{\lambda^{-1}}$ and denote the multiplication operator with respect to $|x|^{-p}$ by M_p. We have for any $f \in \mathcal{D}(\Delta + c|x|^{-p}) := C_c^\infty(\mathbb{R}^d \setminus \{0\})$

$$U_\lambda^{-1} \Delta U_\lambda = \lambda^2 \Delta \quad \text{and} \quad U_\lambda^{-1} M_p U_\lambda = \lambda^p M_p \tag{3.59}$$

hinting at the fact that when $p < 2$, M_p is relatively bounded with respect to Δ. However, in the case $p = 2$, writing $A_c := \Delta + \frac{c}{|x|^2}$, we have

$$U_\lambda^{-1} A_c U_\lambda = \lambda^2 A_c \tag{3.60}$$

which yields that $\operatorname{spec}(A_c) = \lambda \operatorname{spec}(A_c)$ for all $\lambda > 0$. In view of the variation $\mathscr{V}(\mathbb{R}^d, c|x|^{-2})$ related to Hardy's inequality, we can only have $\operatorname{spec}(A_c) = (-\infty, 0]$ or $\operatorname{spec}(A_c) = \mathbb{R}$, depending on whether $c \le (d-2)^2/4$ or not. In the latter case, there cannot exist an extension of $(A_c, C_c^\infty(\mathbb{R}^d \setminus \{0\}))$ which generates a C_0-semigroup and thus the corresponding Cauchy problem (3.9)-(3.10) is ill posed. When $c \le (d-2)^2/4$ there exists a selfadjoint extension of $(A_c, C_c^\infty(\mathbb{R}^d \setminus \{0\}))$ generating a strongly continuous contraction semigroup and this extension is unique if and only if $c \le (d-2)^2/4 - 1$, cf. [RS75, p.160-161,186], [Kat72; Sim73; Kal+74; BG84a]. In [AGG06] and [BG84b], this semigroup - respectively the solution to the corresponding Cauchy problem - is constructed going the detour of considering the truncated operators $\Delta + \left(\frac{c}{|x|^2} \wedge n\right)$ and letting $n \uparrow \infty$. We also have to use this crutch in the proofs of Lemma 3.21 and Lemma 3.22 below. Whereas in the Kato class case it was possible to define this semigroup on each L^p - space and we even have the L^p smoothing property (3.32), we have to be careful in the inverse-square case (cf. [AGG06], Theorem 3.5 and Proposition 3.7):

Proposition 3.20. *Let $1 < p < \infty$. If $c \leq (d-2)^2\left(\frac{p-1}{2p^2}\right)$, then $(A_c, C_c^\infty(\mathbb{R}^d \setminus \{0\}))$ has an extension on $L^p(\mathbb{R}^d)$ generating a C_0 - semigroup $T_t^{p,c} := \lim_{k\to\infty} \mathrm{e}^{t(B(p)+cV_k)}$ acting on $L^p(\mathbb{R}^d)$. If $c > (d-2)^2\left(\frac{p-1}{2p^2}\right)$, then $(A_c, C_c^\infty(\mathbb{R}^d \setminus \{0\}))$ has no extension on $L^p(\mathbb{R}^d)$ which generates a C_0-semigroup acting on $L^p(\mathbb{R}^d)$.*

Note that at the critical value $c = \frac{1}{4}$ we can only get an L^2-semigroup. Let us remark that this fact forces us to choose an auxiliary function between (3.63) and (3.64) below, since$|x|^{-2} \notin L^2(D)$.

Multipolar Inverse Square Potential

Let $\mathcal{Y} = \{y_1, ..., y_n\} \subset \mathbb{R}^d$, $a \in \mathbb{R}^n$ and $A_a = \frac{1}{2}\Delta + c\sum_{i=1}^n a_i|y_i - \cdot|^2$ the multipolar inverse-square Schrödinger operator. Together with the corresponding multipolar Hardy inequality the properties of operators as A_a or more general differential operators with inverse-square potential have been the object of an intense study during the last decade. We refer to [FMT07; FT06; FFK17] and the references therein. It turns out, cf. [FMT07, Theorem 1.1], that for some coefficient a, there exists a configuration of poles $\mathcal{Y}$ such that A_a is positive definite if and only if a satisfies

$$a_i < \frac{(d-2)^2}{4} \text{ for all } i \in \{1, \ldots n\}, \text{ and } \sum_{i=1}^n a_i < \frac{(d-2)^2}{4}. \tag{3.61}$$

In the same reference, the questions on (essential-) selfadjointness and the composition of the spectrum of A_c are discussed. In general, these questions are again related to the geometric constellation of the poles through (3.45). However, in the case of two poles we have the stronger result that under (3.61), the positivity of A_c holds for all configurations $\{y_1, y_2\}$. Finally, note that the statement of Proposition 3.20 should be easily transferable to the multipolar case, combining the proof methods used there and a multipolar Hardy inequality.

3.3 Upper and Lower Spectral Bounds

Having recalled general results on Feynman-Kac functionals and Schrödinger semigroups, we are now ready to give the concrete estimates that will be used in the proofs of the path expansion and the lower bounds of the main theorems.

3.3.1 Upper Bounds

Let $D' \subset D \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) be non-empty, open and bounded with D bounded. Let $\mathcal{Y} \subset D$ be finite, and denote its cardinality by $M := \#\mathcal{Y}$. We give next an upper L^1 - bound for the stopped Feynman-Kac functional appearing in (3.34),

Lemma 3.21. *Fix $\theta \in (0,\infty)$. There exists a constant $c = c(d) \in (0,\infty)$ not depending on θ, D or $\mathcal{Y}$ such that, for all $a \in (0,\infty]$ and any $\gamma > \mathscr{V}(D,\theta V_{\mathcal{Y}}^{(a)})$,*

$$\int_{D'} \mathbb{E}_x\left[\exp\left(\int_0^{\tau_{D^c}} (\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma)\,\mathrm{d}s\right)\right]\mathrm{d}x \le |D'| + c\sqrt{|D||D'|}\frac{\gamma+(M^2+\theta)\,\mathrm{dist}(D^c,\mathcal{Y})^{-2}}{\gamma-\mathscr{V}(D,\theta V_{\mathcal{Y}}^{(a)})} \tag{3.62}$$

Proof. Fix $\gamma > \mathscr{V}(D,\theta V_{\mathcal{Y}}^{(a)})$. For $m \in \mathbb{N}$, let $F_m = \min(V_{\mathcal{Y}}^{(a)}, m)$. According to Proposition 3.14, $u_m(x) = \mathbb{E}_x\left[\exp\left(\int_0^{\tau_{D^c}} \theta F_m(W_s) - \gamma\,\mathrm{d}s\right)\right]$ is the unique weak solution to the boundary value problem

$$\begin{aligned}\left(\frac{\Delta}{2}+\theta F_m-\gamma\right)u(x)&=0, \quad x\in D\\ u(x)&=1, \quad x\in\partial D.\end{aligned} \tag{3.63}$$

Abbreviate $\delta = \mathrm{dist}(D^c,\mathcal{Y})$, take $g\colon \mathbb{R}\to[0,1]$ smooth such that $g(r)=0$ for $r\le 1/2$ and $g(r)=1$ for $r\ge 1$, and put $\phi(x) := \prod_{y\in\mathcal{Y}} g(|x-y|/\delta)$. It is straightforward to check that $\phi \in C^2(\mathbb{R}^d)$ with $0\le\phi\le 1$ on D, $\phi\equiv 1$ on D^c, and that there exists a constant $c = c(d) \in (1,\infty)$, not depending on D or $\mathcal{Y}$, such that $|\Delta\phi| \le cM^2\delta^{-2}$ and $\phi V_{\mathcal{Y}} \le c\delta^{-2}$ uniformly on $\mathbb{R}^d$. Then $v_m(x) = u_m(x)-\phi(x)$ is the unique solution to

$$\begin{aligned}\left(\frac{\Delta}{2}+\theta F_m-\gamma\right)v(x)&=-\left(\frac{\Delta}{2}+\theta F_m-\gamma\right)\phi(x), \quad x\in D\\ v(x)&=0, \quad x\in\partial D.\end{aligned} \tag{3.64}$$

Write $\mathcal{R}_\gamma^{(m)}$ for the resolvent of $\frac{\Delta}{2}+\theta F_m$ at γ. Then $v_m = -\mathcal{R}_\gamma^{(m)}\left(\frac{\Delta}{2}+\theta F_m-\gamma\right)\phi$ and

$$\begin{aligned}\|v_m\|_{L^1(D')} &= \left\langle \left|-\mathcal{R}_\gamma^{(m)}\left(\frac{\Delta}{2}+\theta F_m-\gamma\right)\phi\right|, \mathbb{1}_{D'}\right\rangle_{L^2(D)}\\ &\le \sqrt{|D'|}\left\|\mathcal{R}_\gamma^{(m)}\right\|_2\left\|\left(\frac{\Delta}{2}+\theta F_m-\gamma\right)\phi\right\|_{L^2(D)}\\ &\le \sqrt{|D'|}\frac{\gamma+c(M^2+\theta)\delta^{-2}}{\gamma-\lambda_{\max}(D,\theta F_m)}\sqrt{|D|}\end{aligned} \tag{3.65}$$

by the bound (3.31) on the resolvent and the pointwise bounds on ϕ, $\Delta\phi$ and $V_{\mathcal{Y}}^{(a)}\phi$. Noting now that, since for all $m\in\mathbb{N}$ $F_m \le V_{\mathcal{Y}}^{(a)}$, $\lambda_{\max}(D,\theta F_m) = \mathscr{V}(D,\theta F_m) \le \mathscr{V}(D,\theta V_{\mathcal{Y}}^{(a)})$ by Remark 3.8, we obtain

$$\|u_m\|_{L^1(D')} \le \|v_m\|_{L^1(D')} + \|\phi\|_{L^1(D')} \le c\sqrt{|D'||D|}\frac{\gamma+(M^2+\theta)\delta^{-2}}{\gamma-\mathscr{V}(D,\theta V_{\mathcal{Y}}^{(a)})} + |D'|. \tag{3.66}$$

Finally, by monotone convergence,

$$\|u\|_{L^1(D')} = \lim_{m\to\infty}\|u_m\|_{L^1(D')}, \tag{3.67}$$

and since the bound in (3.66) does not depend on m, this yields (3.62). □

Using the L^1-bound from above, we derive two pointwise estimates needed in the proof of the path expansion in the next section.

Lemma 3.22. *Fix $x \in D\backslash\mathcal{Y}$ and set $\varepsilon_x = \frac{1}{2}\operatorname{dist}(x,\mathcal{Y})$. Assume that $0 < a < \varepsilon_x$ and $\gamma > \lambda_{\max}(D,\theta V_{\mathcal{Y}}^{(a)})$, and let $c = c(d)$ be the constant from Lemma 3.21. Then*

$$\mathbb{E}_x\left[\exp\int_0^{\tau_{D^c}}(\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma)\,\mathrm{d}s\right] \le 2 + c\sqrt{\frac{|D|}{|B_{\varepsilon_x}|}}\frac{\gamma+(M^2+\theta)\operatorname{dist}(D^c,\mathcal{Y})^{-2}}{\gamma-\mathscr{V}(D,\theta V_{\mathcal{Y}}^{(a)})}. \tag{3.68}$$

Moreover, for all $t \in (0,\infty)$,

$$\mathbb{E}_x\left[\mathbb{1}_{\{\tau_{D^c}>t\}}\exp\int_0^t(\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma)\,\mathrm{d}s\right] \le 2 + \sqrt{\frac{|D|}{|B_{\varepsilon_x}|}}\left(1+c\frac{\gamma+(M^2+\theta)\operatorname{dist}(D^c,\mathcal{Y})^{-2}}{\gamma-\mathscr{V}(D,\theta V_{\mathcal{Y}}^{(a)})}\right). \tag{3.69}$$

Proof. Fix $0 < r < \varepsilon_x$ and abbreviate $I_s^t := \mathrm{e}^{\int_s^t(\theta V_{\mathcal{Y}}^{(a)}(W_u)-\gamma)\,\mathrm{d}u}$. We begin with the proof of (3.68). Since $V_{\mathcal{Y}}^{(a)} \equiv 0$ on $B_{\varepsilon_x}(x)$, using the strong Markov property we may write

$$\mathbb{E}_x\left[I_0^{\tau_{D^c}}\right] \le 1 + \mathbb{E}_x\left[\mathbb{1}_{\{\tau_{B_r(x)^c}<\tau_{D^c}\}}I_{\tau_{B_r(x)^c}}^{\tau_{D^c}}\right] \le 1 + \mathbb{E}_x\left[\mathbb{E}_{W_{\tau_{\partial B_r(x)}}}\left[I_0^{\tau_{D^c}}\right]\right]. \tag{3.70}$$

Since $W_{\tau_{\partial B_r(x)}}$ is uniformly distributed on the sphere $\partial B_r(x)$,

$$\mathbb{E}_x\left[I_0^{\tau_{D^c}}\right] \le 1 + \frac{1}{\sigma_d r^{d-1}}\int_{\partial B_r(x)}\mathbb{E}_z\left[I_0^{\tau_{D^c}}\right]\sigma(\mathrm{d}z), \tag{3.71}$$

where σ denotes surface measure on $\partial B_r(x)$ and σ_d is the area of the d-dimensional unit sphere. Multiplying both sides of (3.71) by $\sigma_d r^{d-1}$ and integrating over r between 0 and ε_x leads to

$$|B_{\varepsilon_x}|\left(\mathbb{E}_x\left[I_0^{\tau_{D^c}}\right]-1\right) \le \int_{B_{\varepsilon_x}(x)}\mathbb{E}_z\left[I_0^{\tau_{D^c}}\right]\mathrm{d}z. \tag{3.72}$$

Now apply the L^1-bound from Lemma 3.21 to the r.h.s. with $D' = B_{\varepsilon_x}(x)$, which gives

$$\int_{B_{\varepsilon_x}(x)}\mathbb{E}_z\left[I_0^{\tau_{D^c}}\right]\mathrm{d}z \le |B_{\varepsilon_x}|\left\{1 + c\sqrt{\frac{|D|}{|B_{\varepsilon_x}|}}\left(\frac{\gamma+(M^2+\theta)\operatorname{dist}(D^c,\mathcal{Y})^{-2}}{\gamma-\mathscr{V}_{\max}(D,\theta V_{\mathcal{Y}}^{(a)})}\right)\right\}. \tag{3.73}$$

This yields (3.68), and we continue with the proof of (3.69). Again, by the strong Markov property and since $V_{\mathcal{Y}}^{(a)} \equiv 0$ on $B_{\varepsilon_x}(x)$,

$$\mathbb{E}_x\left[I_0^t\mathbb{1}_{\{\tau_{D^c}>t\}}\right] \le 1 + \mathbb{E}_x\left[\mathrm{e}^{-\gamma\tau_{\partial B_r(x)}}\mathbb{1}_{\{\tau_{\partial B_r(x)}<t\}}\mathbb{E}_{W_{\tau_{\partial B_r(x)}}}\left[I_0^{t-s}\mathbb{1}_{\{\tau_{D^c}>t-s\}}\right]_{s=\tau_{\partial B_r(x)}}\right]. \tag{3.74}$$

Split the event $\{\tau_{D^c} > t-s\}$ according to whether $\tau_{D^c} > t$ or not to write, using $\gamma \ge 0$, $V_{\mathcal{Y}}^{(a)} \ge 0$,

$$I_0^{t-s}\mathbb{1}_{\{\tau_{D^c}>t-s\}} = \mathrm{e}^{s\gamma}\,\mathrm{e}^{\int_0^{t-s}\theta V_{\mathcal{Y}}^{(a)}(W_s)\,\mathrm{d}s-t\gamma}\mathbb{1}_{\{\tau_{D^c}>t-s\}} \le \mathrm{e}^{s\gamma}\left\{I_0^t\mathbb{1}_{\{\tau_{D^c}>t\}}+I_0^{\tau_{D^c}}\right\}. \tag{3.75}$$

Substituting this back into (3.74), we obtain

$$\begin{aligned}\mathbb{E}_x\left[I_0^t \mathbb{1}_{\{\tau_{D^c}>t\}}\right] &\leq 1+\mathbb{E}_x\left[\mathbb{E}_{W_{\tau_{\partial B_r(x)}}}\left[I_0^t \mathbb{1}_{\{\tau_{D^c}>t\}}+I_0^{\tau_{D^c}}\right]\right] \\ &= 1+\frac{1}{\sigma_d r^{d-1}}\left\{\int_{\partial B_r(x)} \mathbb{E}_z\left[I_0^t \mathbb{1}_{\{\tau_{D^c}>t\}}\right]\sigma(\mathrm{d}z)+\int_{\partial B_r(x)} \mathbb{E}_z\left[I_0^{\tau_{D^c}}\right]\sigma(\mathrm{d}z)\right\}.\end{aligned} \tag{3.76}$$

With the same calculation as between (3.71) and (3.72), we obtain

$$|B_{\varepsilon_x}|\left(\mathbb{E}_x\left[I_0^t \mathbb{1}_{\{\tau_{D^c}>t\}}\right]-1\right) \leq \int_{B_{\varepsilon_x}(x)} \mathbb{E}_z\left[I_0^t \mathbb{1}_{\{\tau_{D^c}>t\}}\right]\mathrm{d}z+\int_{B_{\varepsilon_x}(x)} \mathbb{E}_z\left[I_0^{\tau_{D^c}}\right]\mathrm{d}z. \tag{3.77}$$

To bound the first integral in (3.77), write $(T_t^{(m)})_{m\in\mathbb{N}}$ for the Schrödinger semigroup associated with the potential $(V_{\mathcal{Y}}^{(a)}(W_s)\wedge m)$ as given by Proposition 3.9. Note that, for all $m\in\mathbb{N}$,

$$\begin{aligned}\int_{B_{\varepsilon_x}(x)} \mathbb{E}_z\left[\mathrm{e}^{\int_0^t(\theta(V_{\mathcal{Y}}^{(a)}(W_s)\wedge m)-\gamma)\,\mathrm{d}s}\mathbb{1}_{\{\tau_{D^c}>t\}}\right]\mathrm{d}z &= \mathrm{e}^{-t\gamma}\langle \mathbb{1}_{B_{\varepsilon_x}(x)}, T_t^{(m)}\mathbb{1}_D\rangle_{L^2(D)} \\ &\leq \mathrm{e}^{-t\gamma}\|\mathbb{1}_{B_{\varepsilon_x}(x)}\|_{L^2(D)}\|T_t^{(m)}\|_2\|\mathbb{1}_D\|_{L^2(D)} \\ &\leq \mathrm{e}^{-t\gamma}\sqrt{|B_{\varepsilon_x}||D|}\,\mathrm{e}^{t\lambda_{\max}(D,\theta V_{\mathcal{Y}}^{(a)}\wedge m)} \leq \sqrt{|B_{\varepsilon_x}||D|},\end{aligned} \tag{3.78}$$

where we have used the Cauchy-Schwarz inequality, the spectral bound from Proposition 3.11 and that, for all $m\in\mathbb{N}$, $\lambda_{\max}(D,\theta V_{\mathcal{Y}}\wedge m)\leq \mathscr{V}(D,\theta V_{\mathcal{Y}})\leq\gamma$ by Remark 3.8. Letting $m\to\infty$ we obtain by monotone convergence

$$\int_{B_{\varepsilon_x}(x)} \mathbb{E}_z\left[\mathrm{e}^{\int_0^t(\theta(V_{\mathcal{Y}}^{(a)}(W_s))-\gamma)\,\mathrm{d}s}\mathbb{1}_{\{\tau_{D^c}>t\}}\right]\mathrm{d}z \leq \sqrt{|B_{\varepsilon_x}||D|}. \tag{3.79}$$

Collecting (3.77), (3.79) and (3.73), we finish the proof of (3.69). □

3.3.2 Lower Bound

We derive a lower L^1 - bound on the Feynman-Kac functional in (3.33) with $q=V_{\mathcal{Y}}$. Let $d\geq 3$ and define the truncated potential

$$\widetilde{V}(x) := \begin{cases}1, \text{ if } |x|\leq 1,\\ |x|^{-2}, \text{ else.}\end{cases} \tag{3.80}$$

The following is a version of the variation (1.68) with $|x|^{-2}$ replaced by $\widetilde{V}$.

Lemma 3.23. *For any $\varepsilon>0$, there exists $K_\varepsilon\in[1,\infty)$ such that, for all $K\geq K_\varepsilon$,*

$$\sup_{g\in H_0^1(B_K),\|g\|_{L^2(B_K)}=1} (h_d+\varepsilon)\int_{B_K} g^2(x)\widetilde{V}(x)\,\mathrm{d}x-\frac{1}{2}\|\nabla g\|_{L^2(B_K)}^2>0. \tag{3.81}$$

Proof. Taking, for $n \in \mathbb{N}$,

$$\tilde{g}_n(x) := \begin{cases} 1 & \text{when} \quad |x| \le 1, \\ |x|^{-(d-2)/2} & \text{when} \quad 1 < |x| \le n, \\ n^{-d/2}(2n - |x|) & \text{when} \quad n < |x| \le 2n, \\ 0 & \text{when} \quad |x| > 2n, \end{cases} \tag{3.82}$$

it follows by an explicit calculation that, for all $K > 2n$, $\tilde{g}_n \in H_0^1(B_K)$ and

$$(h_d + \varepsilon) \frac{\int_{B_K} \tilde{g}_n^2(x) \widetilde{V}(x)\, \mathrm{d}x}{\frac{1}{2} \int_{B_K} |\nabla \tilde{g}_n(x)|^2\, \mathrm{d}x} \ge (1 + \varepsilon/h_d) \left(1 - \frac{c}{\log n}\right) \tag{3.83}$$

for some constant $c \in (0, \infty)$. Letting $g_n := \tilde{g}_n / \|\tilde{g}_n\|_{L^2(B_K)}$, we obtain

$$(h_d + \varepsilon) \int_{B_K} g_n^2(x) \widetilde{V}(x)\, \mathrm{d}x - \frac{1}{2} \|\nabla g_n\|^2_{L^2(B_K)} \ge \varepsilon (4h_d)^{-1} \|\nabla g_n\|^2_{L^2(B_K)} > 0 \tag{3.84}$$

for n large and $K > 2n$. □

Let $\mathcal{Y} \subset \mathbb{R}^d$ be finite, $M = \#\mathcal{Y} \ge 2$ and $\theta \in (\frac{h_d}{M}, h_d]$.
We define

$$\delta_\star = \delta_\star(d, M, \theta) := \frac{1}{6} \left(1 - \frac{h_d}{\theta M}\right). \tag{3.85}$$

When $|y|$ is sufficiently small for $y \in \mathcal{Y}$, $\widetilde{V}$ is a proper lower bound on $V_\mathcal{Y}$:

Lemma 3.24. *If $|y| \le \delta_\star$ for all $y \in \mathcal{Y}$, then*

$$\theta V_\mathcal{Y}(x) \ge \left(h_d + \tfrac{\theta M - h_d}{2}\right) \widetilde{V}(x) \quad \forall\, x \in \mathbb{R}^d \setminus \mathcal{Y}. \tag{3.86}$$

Proof. The inequality $|x-y|^2 \le |x|^2 + 2|x||y| + |y|^2$ and $|y| \le \delta_\star$ lead to $\theta V_\mathcal{Y}(x) \ge \frac{\theta M}{|x|^2 + 2|x|\delta_\star + \delta_\star^2}$. In the case $|x| \le 1$, this is bounded from below by $\frac{\theta M}{(\delta_\star+1)^2}$, and in the case $|x| \ge 1$, by $\frac{\theta M}{(\delta_\star+1)^2}|x|^{-2}$. Thus we have $\theta V_\mathcal{Y}(x) \ge \frac{\theta M}{(\delta_\star+1)^2}\widetilde{V}(x)$ for all $x \in \mathbb{R}^d \setminus \mathcal{Y}$. Now note that $(\delta_\star + 1)^{-2} \ge 1 - 3\delta_\star$ since the inequality holds for $\theta = \frac{h_d}{M}$ as well as for $\theta = h_d$, and the difference of the terms on both sides does not have a critical point in $[h_d, Mh_d]$ as a function of θM. □

The following is the key lemma to obtain a lower bound on the total mass.

Lemma 3.25. *There exist constants $K \ge 6$ and $c_1, c_2 > 0$ depending on d, M, θ such that, for any $a \in (0, \infty)$ and any $x \in \mathbb{R}^d \setminus \mathcal{Y}$ such that $\mathcal{Y} \subset B_a(x)$,*

$$\int_{B_{Ka}(x)} \mathbb{E}_z \left[\mathrm{e}^{\int_0^t \theta V_\mathcal{Y}(W_s)\, \mathrm{d}s} \mathbb{1}\{\tau_{B_{Ka}(x)^c} > t\} \right] \mathrm{d}z \ge c_1 a^d\, \mathrm{e}^{c_2 t a^{-2}} \quad \forall\, t \ge 0. \tag{3.87}$$

Proof. By translation invariance, we may suppose that $x = 0$ and $\mathcal{Y} \subset B_a$. Set $b = \delta_\star/a$, $K = K_\star/\delta_\star$, where $K_\star$ is given by Lemma 3.23 with $\varepsilon := \frac{\theta M - h_d}{2}$, and write

$$\int_{B_{Ka}} \mathbb{E}_z \left[\mathrm{e}^{\int_0^t \theta V_{\mathcal{Y}}(W_s)\, \mathrm{d}s} \mathbb{1}\{\tau_{B^c_{Ka}} > t\} \right] \mathrm{d}z = b^{-d} \int_{B_{K_\star}} \mathbb{E}_{z/b} \left[\mathrm{e}^{\int_0^t \theta V_{\mathcal{Y}}(W_s)\, \mathrm{d}s} \mathbb{1}\{\tau_{B^c_{Ka}} > t\} \right] \mathrm{d}z. \quad (3.88)$$

By Brownian scaling, the integrand in the right-hand side of (3.88) equals

$$\mathbb{E}_z \left[\mathrm{e}^{\int_0^t \theta V_{\mathcal{Y}}(b^{-1} W_{b^2 s})\, \mathrm{d}s} \mathbb{1}\{\tau_{B^c_{K_\star}} > b^2 t\} \right] = \mathbb{E}_z \left[\mathrm{e}^{\int_0^{b^2 t} \theta V_{b\mathcal{Y}}(W_s)\, \mathrm{d}s} \mathbb{1}\{\tau_{B^c_{K_\star}} > b^2 t\} \right] \quad (3.89)$$

where $b\mathcal{Y} := \{by\colon\ y \in \mathcal{Y}\}$. Since $|y| \le \delta_\star$ for all $y \in b\mathcal{Y}$, (3.89) is at least

$$\mathbb{E}_z \left[\mathrm{e}^{\int_0^{b^2 t} (h_d + \varepsilon) \widetilde{V}(W_s)\, \mathrm{d}s} \mathbb{1}_{\{\tau_{B^c_{K_\star}} > b^2 t\}} \right] \quad (3.90)$$

by Lemma 3.23. Now the Fourier expansion bound (3.30) (with $f = \mathbb{1}_{B_{K_\star}}$) gives the lower bound

$$\int_{B_{K_\star}} \mathbb{E}_z \left[\mathrm{e}^{\int_0^{b^2 t} (h_d + \varepsilon) \widetilde{V}(W_s)\, \mathrm{d}s} \mathbb{1}\{\tau_{B^c_{K_\star}} > b^2 t\} \right] \mathrm{d}z \ge \mathrm{e}^{b^2 t \widetilde{\lambda}_{\max}} \|e_1\|^2_{L^1(B_{K_\star})} \quad (3.91)$$

where $\widetilde{\lambda}_{\max} = \lambda_{\max}(B_{K_\star}, (h_d + \varepsilon)\widetilde{V})$ and e_1 is an eigenfunction to $\widetilde{\lambda}_{\max}$ normalized so that $\|e_1\|_{L^2(B_{K_\star})} = 1$. Now (3.87) follows with $c_1 = \delta_\star^{-d} \|e_1\|^2_{L^1(B_{K_\star})}$ and $c_2 = \delta_\star^2 \lambda_{\max}$ which is strictly positive as $\widetilde{\lambda}_{\max} = \mathscr{V}(B_{K_\star}, (h_d + \varepsilon)\widetilde{V}) > 0$ by Lemma 3.23. $\square$

4 Path Expansion

We provide the key upper bound for the contribution to the Feynman-Kac formula of Brownian paths that leave a large ball. This is achieved by means of a path expansion technique that splits the Brownian path according to its excursions into some carefully chosen neighbourhood of the Poisson cloud.

4.1 The Path Expansion Theorem

We begin by introducing the notation needed to formulate the main theorem of this section. In $\mathbb{R}^d$, $d \in \mathbb{N}$, we fix a (deterministic) point cloud $\mathcal{Y} \in \mathscr{Y}_{\mathrm{f}}$ with finitely many elements (cf. (3.3)) and define, for $r > 0$, $\mathcal{U}_{\mathcal{Y}}^{(r)}$ as the r-neighbourhood of $\mathcal{Y}$ with respect to the euclidean norm, i.e.

$$\mathcal{U}_{\mathcal{Y}}^{(r)} := \bigcup_{y \in \mathcal{Y}} B_r(y). \tag{4.1}$$

By $\mathscr{C}_{\mathcal{Y}}^{(r)}$ we denote the family of connected components of $\mathcal{U}_{\mathcal{Y}}^{(r)}$. Now for a fixed parameter $\theta \in (0, h_d)$, a fixed truncation level $a \in (0, r)$ and $\mathcal{C} \in \mathscr{C}_{\mathcal{Y}}^{(r)}$, we write

$$N_{\mathcal{C}} := \#(\mathcal{Y} \cap \mathcal{C}), \qquad \lambda_{\mathcal{C}} := \mathscr{V}(\mathcal{C}, \theta V_{\mathcal{Y}}^{(a)}) = \mathscr{V}(\mathcal{C}, \theta V_{\mathcal{Y} \cap \mathcal{C}}^{(a)}). \tag{4.2}$$

The maxima of these quantities over $\mathcal{C} \in \mathscr{C}_{\mathcal{Y}}^{(r)}$ are written as

$$N_{\mathcal{Y}}^{(r)} := \max_{\mathcal{C} \in \mathscr{C}_{\mathcal{Y}}^{(r)}} N_{\mathcal{C}}, \qquad \Lambda_{\mathcal{Y}}^{(\theta,a,r)} := \max_{\mathcal{C} \in \mathscr{C}_{\mathcal{Y}}^{(r)}} \lambda_{\mathcal{C}}. \tag{4.3}$$

Note that $\Lambda_{\mathcal{Y}}^{(\theta,a,r)} < \infty$ by the multipolar Hardy inequality (3.42).

We are now ready to state

Theorem 4.1. *For each $d \in \mathbb{N}$, there exist constants $K \in [1, \infty)$ and $c, c_* \in (0, \infty)$ such that the following holds. Let $\mathcal{Y} \in \mathscr{Y}_f$, $\theta > 0$, $a > 0$ and $r > 4a$. For $\gamma > \Lambda_{\mathcal{Y}}^{(\theta,a,r)} \vee 0$, let*

$$\begin{aligned} L = L(\mathcal{Y}, \theta, a, r, \gamma) &:= K \Big(N_{\mathcal{Y}}^{(r)}\Big)^{5/2} \left(\frac{r}{a}\right)^{\frac{d}{2}} \left(2 + \frac{\gamma + \theta r^{-2}}{\gamma - \Lambda_{\mathcal{Y}}^{(\theta,a,r)}}\right), \\ \varrho = \varrho(\mathcal{Y}, \theta, a, r, \gamma) &:= L \exp\left\{-a c_* \sqrt{\gamma}\right\}. \end{aligned} \tag{4.4}$$

Assume that $\varrho \le 1/2$. Then, for all $R \ge 8rN_{\mathcal{Y}}^{(r)}$ and all $t > 0$,

$$\sup_{z\in(\mathcal{U}_{\mathcal{Y}}^{(r)})^{\mathrm{c}}} \mathbb{E}_z\left[\exp\left(\int_0^t \{\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma\}\,\mathrm{d}s\right)\mathbb{1}_{\{\tau_{B_R^{\mathrm{c}}(z)}\le t\}}\right] \le 2KL\left\{\frac{R}{r}\,\mathrm{e}^{-\frac{cR^2}{t}}+\varrho^{\frac{R}{4rN_{\mathcal{Y}}^{(r)}}}\right\}. \tag{4.5}$$

The proof of Theorem 4.1 is the content of the following section.

4.2 Proof of Theorem 4.1

We start with some auxiliary results that will be needed in the following, and that will allow us to identify the constants in the statement of Theorem 4.1. The first lemma provides standard bounds for d-dimensional Brownian motion.

Lemma 4.2. *There exist $K_* = K_*(d) \in [1,\infty)$ and $c_* = c_*(d) \in (0,\infty)$ such that*

$$\mathbb{P}_0\left[\sup_{0\le s\le t}|W_s| > R\right] \le K_*\,\mathrm{e}^{-\frac{c_*R^2}{t}} \quad \text{for all } t, R > 0, \tag{4.6}$$

and

$$\mathbb{E}_0\left[\mathrm{e}^{-u\tau_{B_a^{\mathrm{c}}}}\right] \le K_*\,\mathrm{e}^{-c_*a\sqrt{u}} \quad \text{for all } a, u > 0. \tag{4.7}$$

Proof. This follows from union bounds and standard estimates for one-dimensional Brownian motion, e.g. Exercise 2.18 and Remark 2.22 in [MP10]). □

Next we want to reformulate the results obtained in Lemma 3.22 such that they may be applied directly in the setting of Theorem 4.1. We choose our starting point x not too close to $\mathcal{Y}$, i.e., more than $2a$ away, in order to apply the pointwise bounds (3.68) and (3.69) properly.

Lemma 4.3. *There exists a constant $K_1 \in [1,\infty)$ such that, for all $\mathcal{Y} \in \mathscr{Y}_f$, $\theta \in (0,\infty)$, $a \in (0,\infty)$, $r > 2a$, $\mathcal{C} \in \mathscr{C}_{\mathcal{Y}}^{(r)}$, $\gamma > \lambda_{\mathcal{C}} \vee 0$ and $x \in \mathcal{C}\setminus\mathcal{U}_{\mathcal{Y}}^{(2a)}$,*

$$\mathbb{E}_x\left[\exp\left(\int_0^{\tau_{\mathcal{C}^{\mathrm{c}}}}\left(\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma\right)\mathrm{d}s\right)\right] \le K_1 N_{\mathcal{C}}^{5/2}\left(\frac{r}{a}\right)^{\frac{d}{2}}\left(1+\frac{\gamma+\theta r^{-2}}{\gamma-\lambda_{\mathcal{C}}}\right) \tag{4.8}$$

and

$$\sup_{t\ge 0}\mathbb{E}_x\left[\exp\left(\int_0^t\left(\theta V_{\mathcal{Y}}^{(a)}(W_s)-\gamma\right)\mathrm{d}s\right)\mathbb{1}_{\{\tau_{\mathcal{C}^{\mathrm{c}}}>t\}}\right] \le K_1 N_{\mathcal{C}}^{5/2}\left(\frac{r}{a}\right)^{\frac{d}{2}}\left(2+\frac{\gamma+\theta r^{-2}}{\gamma-\lambda_{\mathcal{C}}}\right). \tag{4.9}$$

Proof. Apply (3.68) and (3.69) from Lemma 3.22 with $D = \mathcal{C}$, noting that $V_{\mathcal{Y}}^{(a)}(x) = V_{\mathcal{Y}\cap\mathcal{C}}^{(a)}(x)$ for $x \in \mathcal{C}\setminus\mathcal{Y}$. Then use $|\mathcal{C}| \le |B_1| N_{\mathcal{C}} r^d$ and $N_{\mathcal{C}} \ge 1$. □

Now combine the probabilistic costs to leave a ball of radius a in the sense of (4.7) with the bounds (4.8) and (4.9), now starting at a distance of more than $3a$ away from the cloud $\mathcal{Y}$ and of more than a away from $\partial\mathcal{C}$:

Corollary 4.4. *For any $\mathcal{Y} \in \mathscr{Y}_f$, $\theta \in (0,\infty)$, $a \in (0,\infty)$, $r > 4a$, $\mathcal{C} \in \mathscr{C}_{\mathcal{Y}}^{(r)}$, $\gamma > \lambda_{\mathcal{C}} \vee 0$ and $x \in \mathcal{C} \cap \mathcal{U}_{\mathcal{Y}}^{(r-a)} \setminus \mathcal{U}_{\mathcal{Y}}^{(3a)}$,*

$$\mathbb{E}_x\left[\exp\left(\int_0^{\tau_{\mathcal{C}^c}} \left(\theta V_{\mathcal{Y}}^{(a)}(W_s) - \gamma\right) \mathrm{d}s\right)\right] \le K_* K_1 N_{\mathcal{C}}^{5/2} \left(\frac{r}{a}\right)^{\frac{d}{2}} \mathrm{e}^{-c_* a\sqrt{\gamma}} \left(1 + \frac{\gamma + \theta r^{-2}}{\gamma - \lambda_{\mathcal{C}}}\right) \quad (4.10)$$

where K_*, c_* are as in Lemma 4.2 and K_1 as in Lemma 4.3.

Proof. Use the strong Markov property of W at the exit time of $B_a(x)$ and apply Lemma 4.3 and (4.7). □

With these results in place, we may identify the constants K, c in Theorem 4.1 as

$$K := 2(K_*)^2 K_1, \qquad c := \frac{c_*}{16}, \quad (4.11)$$

where K_*, c_* are as in Lemma 4.2 and K_1 as in Lemma 4.3. Now fix $\mathcal{Y} \in \mathscr{Y}_{\mathrm{f}}$, $\theta > 0$, $a > 0$, $r > 4a$ and $\gamma > \Lambda_{\mathcal{Y}}^{(\theta,a,r)}$. In the following, we take K, c as in (4.11) and let L, ϱ be defined by (4.4). Note that the terms on the right-hand sides of (4.8) and (4.9) and the term on the right-hand side of (4.10) are bounded from above by

$$\frac{L}{2K_*} \quad \text{and} \quad \frac{\rho}{2K_*}, \quad (4.12)$$

respectively.

The core of the proof of Theorem 4.1 is a decomposition of the Brownian path according to its excursions to and from neighbourhoods of $\mathcal{Y}$, which are marked by the following stopping times. Let $\check{\tau}_0 = \hat{\tau}_0 := 0$ and, recursively for $n \ge 0$,

$$\begin{aligned} \check{\tau}_{n+1} &:= \begin{cases} \infty & \text{if } \hat{\tau}_n = \infty, \\ \inf\left\{t > \hat{\tau}_n \colon\ W_t \in \overline{\mathcal{U}_{\mathcal{Y}}^{(3a)}}\right\} & \text{otherwise,} \end{cases} \\ \hat{\tau}_{n+1} &= \begin{cases} \infty & \text{if } \check{\tau}_{n+1} = \infty, \\ \inf\left\{t > \check{\tau}_{n+1} \colon\ W_t \notin \mathcal{U}_{\mathcal{Y}}^{(r)}\right\} & \text{otherwise.} \end{cases} \end{aligned} \quad (4.13)$$

For $t \ge 0$, define the random number of excursions of (W_s) into $\overline{\mathcal{U}_{\mathcal{Y}}^{(3a)}}$ until time t by

$$E_t := \inf\{n \ge 0 \colon\ \check{\tau}_{n+1} > t\}. \quad (4.14)$$

Abbreviate, for $0 \le s_1 \le s_2 \le \infty$,

$$I_{s_1}^{s_2} := \exp\left\{\int_{s_1}^{s_2} \left(\theta V_{\mathcal{Y}}^{(a)}(W_s) - \gamma\right) \mathrm{d}s\right\}. \quad (4.15)$$

We begin with a version of (4.5) where we do not demand that W leaves the ball of radius R until time t but that it (re)enters the neighbourhood $\overline{\mathcal{U}_{\mathcal{Y}}^{(3a)}}$ exactly n times before that time.

Lemma 4.5. *For all $n \in \mathbb{N}_0$,*

$$\sup_{x \notin \mathcal{U}_{\mathcal{Y}}^{(r)}} \sup_{t \geq 0} \mathbb{E}_x \left[I_0^t \mathbb{1}_{\{E_t = n\}} \right] \leq \varrho^n. \tag{4.16}$$

Proof. We will prove (4.16) by induction on n. The case $n = 0$ is simple since then $V_{\mathcal{Y}}^{(a)}(W_s) = 0$ for all $0 \leq s \leq t$.

To treat the case $n = 1$, fix $x \notin \mathcal{U}_{\mathcal{Y}}^{(r)}$ and $t > 0$. Now there are two possible cases: either $\check{\tau}_1 \leq t < \hat{\tau}_1$ or $\hat{\tau}_1 \leq t < \check{\tau}_2$. Take $\check{\mathcal{C}}_1 \in \mathscr{C}_{\mathcal{Y}}^{(r)}$ such that $W_{\check{\tau}_1} \in \check{\mathcal{C}}_1$. Using the Markov property, $\gamma > \Lambda_{\mathcal{Y}}^{(\theta,a,r)}$ and Lemma 4.3 together with (4.12), we may bound, $\mathbb{P}_x$-a.s. on the event $\{\check{\tau}_1 \leq t\}$,

$$\begin{aligned} \mathbb{E}_x \left[I_{\check{\tau}_1}^t \mathbb{1}_{\{\check{\tau}_1 \leq t < \hat{\tau}_1\}} \,\middle|\, \check{\tau}_1, (W_s)_{s \leq \check{\tau}_1} \right] &= \mathbb{E}_{W_{\check{\tau}_1}} \left[I_0^{t-s} \mathbb{1}_{\{\tau_{\mathcal{C}^c} > t-s\}} \right]_{s = \check{\tau}_1, \mathcal{C} = \check{\mathcal{C}}_1} \\ &\leq L/(2K_*) \end{aligned} \tag{4.17}$$

and, using that $V_{\mathcal{Y}}^{(a)}(W_s) = 0$ for all $s \in [\hat{\tau}_1, t]$ when $\hat{\tau}_1 \leq t < \check{\tau}_2$, Corollary 4.4 and (4.12),

$$\begin{aligned} \mathbb{E}_x \left[I_{\check{\tau}_1}^t \mathbb{1}_{\{\hat{\tau}_1 \leq t < \check{\tau}_2\}} \,\middle|\, \check{\tau}_1, (W_s)_{s \leq \check{\tau}_1} \right] &\leq \mathbb{E}_x \left[I_{\check{\tau}_1}^{\hat{\tau}_1} \,\middle|\, \check{\tau}_1, (W_s)_{s \leq \check{\tau}_1} \right] \\ &= \mathbb{E}_{W_{\check{\tau}_1}} \left[I_0^{\tau_{\mathcal{C}^c}} \right]_{\mathcal{C} = \check{\mathcal{C}}_1} \\ &\leq \varrho/(2K_*) < L/(2K_*). \end{aligned} \tag{4.18}$$

Since $r > 4a$ and $x \notin \mathcal{U}_{\mathcal{Y}}^{(r)}$, $\check{\tau}_1 \geq \tau_{B_a^c(x)}$ and thus

$$\mathbb{E}_x \left[I_0^{\check{\tau}_1} \mathbb{1}_{\{\check{\tau}_1 \leq t\}} \right] \leq \mathbb{E}_0 \left[e^{-\gamma \tau_{B_a^c}} \right] \leq K_* \, e^{-c_* a \sqrt{\gamma}} \tag{4.19}$$

by Lemma 4.2 and (4.12). This together with (4.17)–(4.18) gives

$$\begin{aligned} \mathbb{E}_x \left[I_0^t \mathbb{1}_{\{E_t = 1\}} \right] &= \mathbb{E}_x \left[I_0^{\check{\tau}_1} \mathbb{1}_{\{\check{\tau}_1 \leq t\}} \mathbb{E}_x \left[I_{\check{\tau}_1}^t \mathbb{1}_{\{E_t = 1\}} \,\middle|\, \check{\tau}_1, (W_s)_{s \leq \check{\tau}_1} \right] \right] \\ &\leq L \, e^{-c_* a \sqrt{\gamma}} = \varrho \end{aligned} \tag{4.20}$$

by (4.4), concluding the case $n = 1$.

Suppose now by induction that (4.16) has been shown for some $n \geq 1$. If $E_t = n + 1$, then $\hat{\tau}_1 \leq t$ and we can write

$$\begin{aligned} \mathbb{E}_x \left[I_0^t \mathbb{1}_{\{E_t = n+1\}} \right] &= \mathbb{E}_x \left[I_0^{\hat{\tau}_1} \mathbb{1}_{\{\hat{\tau}_1 \leq t\}} \mathbb{E}_{W_{\hat{\tau}_1}} \left[I_0^{t-s} \mathbb{1}_{\{E_{t-s} = n\}} \right]_{s = \hat{\tau}_1} \right] \\ &\leq \varrho^n \mathbb{E}_x \left[I_0^{\hat{\tau}_1} \mathbb{1}_{\{\hat{\tau}_1 \leq t\}} \right] \leq \varrho^{n+1}/(2K_*) \end{aligned} \tag{4.21}$$

by the induction hypothesis, (4.19) and (4.4). This concludes the proof. □

The next result is the key lemma for the proof of Theorem 4.1. We now add again the demand $\{\tau_{B_R^c(z)} \leq t\}$.

Lemma 4.6. *For all $R > 0$, $n \in \mathbb{N}_0$, $x \in (\mathcal{U}_{\mathcal{Y}}^{(r)})^{\mathrm{c}}$ and $t \geq 0$,*

$$\mathbb{E}_x \left[I_0^t \mathbb{1}_{\{E_t = n, \tau_{B_R^{\mathrm{c}}(x)} \leq t\}} \right] \leq 2L\varrho^{(n-1)^+} \, \mathbb{P}_x \left[\sup_{0 \leq s \leq t} |W_s - x| > R - 2N_{\mathcal{Y}}^{(r)} nr \right]. \tag{4.22}$$

Proof. Fix $R, t > 0$ and $x \in (\mathcal{U}_{\mathcal{Y}}^{(r)})^{\mathrm{c}}$. As in Lemma 4.5, we prove (4.22) by induction on n. The case $n = 0$ is easy, since then $V_{\mathcal{Y}}^{(a)}(W_s) = 0$ for all $0 \leq s \leq t$. Define the events

$$\mathcal{E}_u^n := \left\{ \sup_{0 \leq s \leq u} |W_s - x| \geq R - 2nrN_{\mathcal{Y}}^{(r)} \right\}, \quad n \in \mathbb{N}_0, \; u \geq 0. \tag{4.23}$$

For $n = 1$, consider first the case $\tau_{B_R^{\mathrm{c}}(x)} \leq \hat{\tau}_1$. We claim that, on this event, $\mathcal{E}_{\check{\tau}_1}^1$ occurs. Indeed, if $\tau_{B_R^{\mathrm{c}}(x)} \leq \check{\tau}_1$, this is clear, and if $\check{\tau}_1 < \tau_{B_R^{\mathrm{c}}(x)} \leq \hat{\tau}_1$, then $|W_{\check{\tau}_1} - x| > R - 2rN_{\mathcal{Y}}$ as the diameter of any component $\mathcal{C} \in \mathscr{C}_{\mathcal{Y}}^{(r)}$ is bounded by $2rN_{\mathcal{Y}}^{(r)}$. Thus

$$\begin{aligned} \mathbb{E}_x \left[I_0^t \mathbb{1}_{\{\tau_{B_R^{\mathrm{c}}(x)} \leq \hat{\tau}_1, E_t = 1\}} \right] &\leq \mathbb{E}_x \left[\mathbb{1}_{\mathcal{E}_{\check{\tau}_1}^1 \cap \{\check{\tau}_1 \leq t\}} \mathbb{E}_x \left[I_{\check{\tau}_1}^t \mathbb{1}_{\{E_t = 1\}} \mid \check{\tau}_1, (W_s)_{s \leq \check{\tau}_1} \right] \right] \\ &\leq L \mathbb{P}_x \left[\mathcal{E}_t^1 \right] \end{aligned} \tag{4.24}$$

by (4.17)–(4.18) above. If $t \geq \tau_{B_R^{\mathrm{c}}(x)} > \hat{\tau}_1$, then $\hat{\tau}_1 < t < \check{\tau}_2$, and thus

$$\mathbb{E}_x \left[I_0^t \mathbb{1}_{\{\hat{\tau}_1 < \tau_{B_R^{\mathrm{c}}(x)} \leq t, E_t = 1\}} \right] \leq \mathbb{E}_x \left[I_0^{\hat{\tau}_1} \mathbb{1}_{\{\hat{\tau}_1 \leq t\}} \mathbb{P}_{W_{\hat{\tau}_1}} \left[\mathcal{E}_{t-s}^{(0)} \right]_{s = \hat{\tau}_1} \right]. \tag{4.25}$$

Now note that, since $\check{\tau}_1 \leq \hat{\tau}_1$ and $|W_{\hat{\tau}_1} - W_{\check{\tau}_1}| \leq 2rN_{\mathcal{Y}}$,

$$\mathbb{P}_{W_{\hat{\tau}_1}} \left[\mathcal{E}_{t-s}^0 \right]_{s = \hat{\tau}_1} \leq \mathbb{P}_{W_{\check{\tau}_1}} \left[\mathcal{E}_{t-s}^1 \right]_{s = \check{\tau}_1}. \tag{4.26}$$

Hence (4.25) is at most

$$\mathbb{E}_x \left[\mathbb{1}_{\{\check{\tau}_1 \leq t\}} \mathbb{P}_{W_{\check{\tau}_1}} \left(\mathcal{E}_{t-s}^1 \right)_{s = \check{\tau}_1} \mathbb{E}_{W_{\check{\tau}_1}} \left[I_0^{\tau_{\mathcal{C}^{\mathrm{c}}}} \right]_{\mathcal{C} = \check{\mathcal{C}}_1} \right] \leq \frac{\varrho}{2} \, \mathbb{P}_x \left[\mathcal{E}_t^1 \right] < L \, \mathbb{P}_x \left[\mathcal{E}_t^1 \right] \tag{4.27}$$

by Corollary 4.4, (4.12) and (4.4). Collecting (4.24)–(4.27), we conclude the case $n = 1$.

Assume by induction that (4.22) holds for some $n \geq 1$. There are two possible cases: either $\tau_{B_R^{\mathrm{c}}(x)} \leq \hat{\tau}_1$ or not. In the first case, we conclude as before that $\mathcal{E}_{\check{\tau}_1}^1$ occurs. Then we may write

$$\begin{aligned} \mathbb{E}_x \left[I_0^t \mathbb{1}_{\{E_t = n+1, \tau_{B_R^{\mathrm{c}}(x)} \leq \hat{\tau}_1\}} \right] &\leq \mathbb{E}_x \left[\mathbb{1}_{\mathcal{E}_{\check{\tau}_1}^1 \cap \{\check{\tau}_1 \leq t\}} \mathbb{E}_{W_{\check{\tau}_1}} \left[I_0^{\tau_{\mathcal{C}^c}} \mathbb{E}_{W_{\tau_{\mathcal{C}^c}}} \left[I_0^{t-s} \mathbb{1}_{\{E_t = n\}} \right]_{s = \tau_{\mathcal{C}^c}} \right]_{\mathcal{C} = \check{\mathcal{C}}_1} \right] \\ &\leq \varrho^n \frac{\varrho}{2} \, \mathbb{P}_x \left[\mathcal{E}_t^1 \right] < L\varrho^n \, \mathbb{P}_x \left[\mathcal{E}_t^1 \right] \end{aligned} \tag{4.28}$$

by Lemma 4.5, Corollary 4.4, (4.12) and (4.4). Now consider the case $\hat{\tau}_1 < \tau_{B_R^{\mathrm{c}}(x)}$ and write

$$\begin{aligned} \mathbb{E}_x \left[I_0^t \mathbb{1}_{\{E_t = n+1, \hat{\tau}_1 < \tau_{B_R^{\mathrm{c}}(x)} \leq t\}} \right] &= \mathbb{E}_x \left[I_0^{\hat{\tau}_1} \mathbb{1}_{\{\hat{\tau}_1 \leq t\}} \mathbb{E}_{W_{\hat{\tau}_1}} \left[I_0^{t-s} \mathbb{1}_{\{E_{t-s} = n, \tau_{B_R^{\mathrm{c}}(x)} \leq t-s\}} \right]_{s = \hat{\tau}_1} \right] \\ &\leq 2L\varrho^{n-1} \mathbb{E}_x \left[I_0^{\hat{\tau}_1} \mathbb{1}_{\{\hat{\tau}_1 \leq t\}} \mathbb{P}_{W_{\hat{\tau}_1}} \left[\mathcal{E}_{t-s}^n \right]_{s = \hat{\tau}_1} \right] \end{aligned} \tag{4.29}$$

by the induction hypothesis. Reasoning as for (4.26), we see that

$$\mathbb{P}_{W_{\hat{\tau}_1}}\left[\mathcal{E}^n_{t-s}\right]_{s=\hat{\tau}_1} \le \mathbb{P}_{W_{\check{\tau}_1}}\left[\mathcal{E}^{n+1}_{t-s}\right]_{s=\check{\tau}_1}. \tag{4.30}$$

Hence (4.29) is at most

$$\begin{aligned} &2L\varrho^{n-1}\mathbb{E}_x\left[\mathbb{1}_{\{\check{\tau}_1\le t\}}\mathbb{P}_{W_{\check{\tau}_1}}\left[\mathcal{E}^{n+1}_{t-s}\right]_{s=\check{\tau}_1}\mathbb{E}_{\check{\tau}_1}\left[I_0^{\tau_{\mathcal{C}^c}}\right]_{\mathcal{C}=\check{\mathcal{C}}_1}\right] \\ \le\, &2L\varrho^{n-1}(\varrho/2)\mathbb{P}_x\left[\mathcal{E}^{n+1}_t\right] \end{aligned} \tag{4.31}$$

by Corollary 4.4. Combining (4.28) and (4.31) we conclude the induction step. □

We are now ready to finish the

Proof of Theorem 4.1. Fix $x \in (\mathcal{U}^{(r)}_{\mathcal{Y}})^c$ and write

$$\begin{aligned} \mathbb{E}_x\left[I_0^t\mathbb{1}_{\{\tau_{B^c_R(x)}\le t\}}\right] &= \sum_{n=0}^{\infty}\mathbb{E}_x\left[I_0^t\mathbb{1}_{\{\tau_{B^c_R(x)}\le t, E_t=n\}}\right] \\ &\le 2L\sum_{n=0}^{\infty}\varrho^{(n-1)^+}\mathbb{P}_0\left[\sup_{0\le s\le t}|W_s| \ge R - 2N^{(r)}_{\mathcal{Y}}nr\right] \end{aligned} \tag{4.32}$$

by Lemma 4.6 and the translation invariance of Brownian motion. Split the sum in (4.32) according to whether $4N^{(r)}_{\mathcal{Y}}(n-1)r \ge R$ or not to obtain

$$\begin{aligned} \frac{1}{2L}\mathbb{E}_x\left[I_0^t\mathbb{1}_{\{\tau_{B^c_R(x)}\le t\}}\right] &\le 2\varrho^{\frac{R}{4rN^{(r)}_{\mathcal{Y}}}} + \left(\frac{R}{4rN^{(r)}_{\mathcal{Y}}}+2\right)\mathbb{P}_0\left[\sup_{0\le s\le t}|W_s| \ge \tfrac{1}{4}R\right] \\ &\le K\left\{\varrho^{\frac{R}{4rN^{(r)}_{\mathcal{Y}}}} + \frac{R}{r}\,\mathrm{e}^{-\frac{cR^2}{t}}\right\} \end{aligned} \tag{4.33}$$

using $\varrho \le 1/2$, $R \ge 8rN^{(r)}_{\mathcal{Y}}$, Lemma 4.2 and (4.11). This concludes the proof. □

5 Small Distances in Poisson Clouds and Eigenvalue Asymptotics

We collect some elementary results concerning the probability that the Poisson cloud $\mathcal{P}_R = \mathrm{supp}(\omega) \cap B_R(0)$ contains (or does not contain) a given number of n points, all lying in some (small) ball $B_r(x)$ with $x \in B_R$ and $r > 0$. Moreover, we obtain some results on the asymptotic behaviour as R goes to infinity and r goes to zero. With the help of Proposition 3.19, this will allow us to control the growth of the principal eigenvalue $\Lambda_{\mathcal{Y}}^{(\theta,a,r)}$ appearing in Theorem 4.1 (with $\mathcal{Y} = \mathcal{P}_R$).

5.1 Small Distances in Poisson Clouds

Throughout this section fix $d \in \mathbb{N}$. For $D \subset \mathbb{R}^d$ we use the notation $B_r(D) = \{x + y \colon x \in D, y \in B_r(x)\}$. Recall that we denote the volume of the d-dimensional unit ball with respect to the euclidean norm by $\omega_d = |B_1| = |\{x \in \mathbb{R}^d \colon |x| \leq 1\}|$.

We begin with an upper bound on the probability to find at least one small ball of radius r containing $n+1$ elements of the cloud $\mathcal{P} \cap D$, $D \subset \mathbb{R}^d$.

Lemma 5.1. *For any measurable bounded $D \subset \mathbb{R}^d$, any $r \in (0, \infty)$ and any $n \in \mathbb{N}$,*

$$\mathbb{P}\Big[\exists \text{ distinct } y_0, \dots, y_n \in \mathcal{P} \colon\ y_0 \in D, \max_{1 \leq i \leq n} |y_i - y_{i-1}| \leq r\Big] \leq |D| \frac{(\omega_d r^d)^n}{(n+1)!} \tag{5.1}$$

and

$$\mathbb{P}\Big[\exists\, x \in D \colon\ \omega(B_r(x)) \geq n+1\Big] \leq |B_r(D)| \frac{(\omega_d (2r)^d)^n}{(n+1)!}. \tag{5.2}$$

Proof. Since (5.2) follows from (5.1) (with D, r substituted by $B_r(D), 2r$), it is enough to prove the latter. Note that, if $y_0 \in D$ and $|y_i - y_{i-1}| < r$ for $1 \leq i \leq n$, then $\{y_0, \dots, y_n\} \subset D_n := B_{nr}(D)$. Let $(X_i)_{i \geq 0}$ be i.i.d. random vectors, each uniformly distributed in D_n. Note that, for any fixed $N \in \mathbb{N}$, $D_n \cap \mathcal{P}$ has the same distribution as $\{X_1, \dots, X_N\}$ under its conditional law given that $\omega(D_n) = N$. Estimate with a union bound

$$\begin{aligned}
&\mathbb{P}\left[\exists \text{ distinct } j_0, \dots, j_n \in \{1, \dots, N\} \colon X_{j_0} \in D, \max_{1 \le i \le n} |X_{j_i} - X_{j_{i-1}}| \le r\right] \\
&\le \binom{N}{n+1} \mathbb{P}\left[X_0 \in D, \max_{1 \le i \le n} |X_i - X_{i-1}| \le r\right] \\
&= \binom{N}{n+1} \frac{1}{|D_n|^{n+1}} \int_D \mathrm{d}x_0 \int_{B_r(x_0)} \mathrm{d}x_1 \cdots \int_{B_r(x_{n-1})} \mathrm{d}x_n = \binom{N}{n+1} \frac{|D||B_r|^n}{|D_n|^{n+1}}. \qquad (5.3)
\end{aligned}$$

Since $\#\mathcal{P} \cap D_n$ has a $\mathrm{Poi}_{|D_n|}$ distribution, splitting the left-hand side of (5.1) according to whether $\#\mathcal{P} \cap D = N \ge n+1$ and using (5.3), we get the bound

$$\sum_{N=n+1}^{\infty} \binom{N}{n+1} \frac{|D||B_r|^n}{|D_n|^{n+1}} \frac{|D_n|^N}{N!} \mathrm{e}^{-|D_n|} = |D| \frac{|B_r|^n}{(n+1)!} = |D| \frac{(\omega_d r^d)^n}{(n+1)!}, \qquad (5.4)$$

as advertised. $\square$

The first statement of the next lemma provides a lower bound on the probability that there is at least one small ball of radius r containing $n+1$ elements of the cloud $\mathcal{P} \cap D$. The second statement treats several domains $D^{(1)}, \dots, D^{(m)}$ and several radii $r_1, \dots, r_m$ simultaneously and provides a lower bound on the probability that, for all $i = 1, \dots, m$, no small ball of radius r_i containing $n+1$ elements of the cloud $\mathcal{P} \cap D^{(i)}$ exists.

Lemma 5.2. *Let $D \subset \mathbb{R}^d$ be bounded and measurable, $n \in \mathbb{N}$ and $r \in (0, \infty)$. Then*

$$\mathbb{P}\left[\exists x \in D \colon \omega(B_r(x)) = n+1\right] \ge 1 - \exp\left(-\mathrm{Poi}_{\omega_d r^d}(\{n+1\}) \# B_r(D) \cap 2r\mathbb{Z}^d\right). \qquad (5.5)$$

Furthermore, let $m, n \in \mathbb{N}$, $D^{(1)}, ..., D^{(m)} \subset \mathbb{R}^d$ be bounded and measurable and $r_1, ..., r_m \in (0, \infty)$. Then

$$\mathbb{P}\left[\bigcap_{i=1}^m \bigcap_{x \in D^{(i)}} \{\omega(B_{r_i}(x)) \le n\}\right] \ge \prod_{i=1}^m \mathrm{Poi}_{\omega_d((\sqrt{d}+1)r_i)^d}(\{0, ..., n\})^{\# B_{(\sqrt{d}+1)r_i}(D^{(i)}) \cap 2r_i\mathbb{Z}^d}. \qquad (5.6)$$

Proof. We begin with the poof of (5.5) and estimate

$$\begin{aligned}
&\mathbb{P}\left[\forall x \in D \colon \omega\left(B_r(x)\right) \ne n+1\right] \le \mathbb{P}\left[\forall x \in D \cap 2r\mathbb{Z}^d \colon \omega\left(B_r(x)\right) \ne n+1\right] \qquad (5.7) \\
&= \left(1 - \mathbb{P}\left[\omega\left(B_r\right) = n+1\right]\right)^{\# D \cap 2r\mathbb{Z}^d} \le \exp\left(-\mathbb{P}\left[\omega\left(B_r\right) = n+1\right] \# D \cap 2r\mathbb{Z}^d\right), \qquad (5.8)
\end{aligned}$$

where the equality follows from the independence of the family $(\omega\left(B_r(x)\right))_{x \in D \cap 2r\mathbb{Z}^d}$ and the second inequality from the estimate $(1-x) \le \exp(-x)$. Since $\mathbb{P}\left[\omega\left(B_r\right) = n+1\right] = \mathrm{Poi}_{\omega_d r^d}(\{n+1\})$ we obtain (5.5) and continue with the proof of (5.6). Write $\tilde{r}_i = (\sqrt{d}+1)r_i$ and $\widetilde{D}^{(i)} = B_{(\sqrt{d}+1)r_i}(D^{(i)})$. Note that

$$\begin{aligned}
&\mathbb{P}\left[\bigcap_{i=1}^m \bigcap_{x \in D^{(i)}} \{\omega(B_{r_i}(x)) \le n\}\right] \ge \mathbb{P}\left[\bigcap_{i=1}^m \bigcap_{x \in \widetilde{D}^{(i)} \cap 2r_i\mathbb{Z}^d} \{\omega\left(B_{\tilde{r}_i}(x)\right) \le n\}\right] \qquad (5.9) \\
&\ge \prod_{i=1}^m \prod_{x \in \widetilde{D}^{(i)} \cap 2r_i\mathbb{Z}^d} \mathbb{P}\left[\omega\left(B_{\tilde{r}_i}(x)\right) \le n\right] = \prod_{i=1}^m \mathbb{P}[\omega(B_{\tilde{r}_i}) \le n]^{\# \widetilde{D}^{(i)} \cap 2r_i\mathbb{Z}^d}, \qquad (5.10)
\end{aligned}$$

which gives the desired bound provided we can justify the second inequality. The family $\left\{\omega\left(B_{\tilde{r}_i}(x)\right), x \in \widetilde{D}^{(i)} \cap 2r_i\mathbb{Z}^d, i = 1, ..., m\right\}$ is clearly not independent, but, following [CR11, Lemma 3.1], it is *associated*, where a family of real random variables $X_1, ..., X_m$ is called associated if

$$\operatorname{Cov}[f(X), g(X)] \geq 0 \tag{5.11}$$

for any bounded measurable functions $f, g\colon \mathbb{R}^m \to \mathbb{R}$ which are non-increasing in each component. Applying (5.11) recursively with $f_j(x) = \prod_{i=1}^{j} \mathbb{1}_{\{x_i \leq n\}}$ and $g_j(x) := \mathbb{1}_{\{x_{j+1} \leq n\}}$, $j = m-1, m-2, ..., 1$, we derive

$$\mathbb{P}[\forall i = 1, ..., m\colon X_i \leq n] \geq \prod_{i=1}^{m} \mathbb{P}[X_i \leq n]. \tag{5.12}$$

This finally yields (5.10). $\square$

The following corollary is needed for the lower bounds in Theorems 2.4 and 2.15.

Corollary 5.3. *Fix $n \in \mathbb{N}$ and let $R(t), r(t) \in (0, \infty)$ satisfy $r(t) \to 0$, $R(t)r(t)^n \to \infty$ as $t \to \infty$. Then*

$$\lim_{t\to\infty} \mathbb{P}\left[\exists\, x \in B_{R(t)}\colon \omega\left(B_{r(t)}(x)\right) = n+1\right] = 1. \tag{5.13}$$

Proof. This follows from (5.5) since there exist constants $c_1 (= \omega_d/2^d)$ and $c_2 (= \omega_d^{n+1}/(n+1)!) \in (0, \infty)$, such that

$$\# B_{R(t)} \cap 2r(t)\mathbb{Z}^d \sim c_1 \frac{R(t)^d}{r(t)^d} \quad \text{and} \quad \operatorname{Poi}_{\omega_d r(t)^d}(\{n+1\}) \sim c_2 r(t)^{d(n+1)}, \quad t \to \infty. \tag{5.14}$$

$\square$

Similarly we can show that the maximal number of Poisson points in the a-neighbourhoods of points $x \in B_R$ grows at most logarithmically in R $\mathbb{P}$-a.s. when a is fixed. This asymptotic will be needed to control the midrange interactions between W and $\mathcal{P}$, cf. (6.2) below.

Corollary 5.4. *For any $a \in (0, \infty)$,*

$$\lim_{R\to\infty} (\log R)^{-1} \sup_{x\in B_R} \omega(B_a(x)) = 0 \quad \mathbb{P}\text{-a.s.} \tag{5.15}$$

Proof. Fix $a \in (0, \infty)$ and $l \in \mathbb{N}$. By (5.2) there exists a constant $c \in (0, \infty)$ such that for all $n \geq 2l$

$$\mathbb{P}\left[\sup_{x\in B_{\mathrm{e}^{n+1}}} \omega(B_a(x)) \geq \lfloor n/l \rfloor\right] \leq c(\mathrm{e}^{n+1})^d \frac{\left((2a)^d\omega_d\right)^{\lfloor n/l\rfloor}}{(\lfloor n/l\rfloor)!} \leq c \frac{\left(2^d\omega_d a^d\, \mathrm{e}^{2dl}\right)^{\lfloor n/l\rfloor}}{(\lfloor n/l\rfloor)!}. \tag{5.16}$$

The r.h.s. is summable with respect to n due to the convergence of the exponential series. Thus, the Borel-Cantelli lemma yields that $\mathbb{P}$-a.s. eventually $\sup_{x\in B_{\mathrm{e}^{n+1}}} \omega(B_a(x)) \leq \lfloor n/l \rfloor$. For $R \in (0, \infty)$, take $n_R \in \mathbb{N}$ s.t. $\mathrm{e}^{n_R} \leq R \leq \mathrm{e}^{n_R+1}$. Then

$$\limsup_{R\to\infty} (\log R)^{-1} \sup_{x\in B_R} \omega(B_a(x)) \leq \limsup_{n\to\infty} n_R^{-1} \sup_{x\in B_{\mathrm{e}^{n_R+1}}} \omega(B_a(x)) \leq l^{-1} \quad \mathbb{P}\text{-a.s.} \tag{5.17}$$

We complete the proof letting $l \downarrow 0$, noting that $(\log R)^{-1} \sup_{x\in B_R} \omega(B_a(x) \geq 0$ for $R > 1$. $\square$

In the last three lemmata of this section, we derive various almost sure asymptotics concerning the random variable $\sup_{x\in B_R}\omega(B_r(x))$. We begin with a characterization of its limes superior depending on how fast we let $R\uparrow\infty$ and $r\downarrow 0$.

Lemma 5.5. *Let $k\in\{2,3,...\}$ and let $\ell\colon(0,\infty)\to(1,\infty)$ be slowly varying at infinity. Take $r(t)=t^{-\frac{1}{k-1}}\ell(t)^{-\frac{1}{d(k-1)}}$ and $R(t)=1+t^{\frac{k}{k-1}}\ell(t)^{\frac{1}{d(k-1)}}$. Then*

$$\limsup_{t\to\infty}\sup_{x\in B_{R(t)}}\omega\Big(B_{r(t)}(x)\Big)\begin{cases}\le k & \mathbb{P}\text{-a.s.,}\quad \text{if } \int_1^\infty\frac{\mathrm{d}t}{t\ell(t)}<\infty,\\ \ge k+1 & \mathbb{P}\text{-a.s.,}\quad \text{if } \int_1^\infty\frac{\mathrm{d}t}{t\ell(t)}=\infty.\end{cases}\tag{5.18}$$

Moreover, recall the notation for $N^{(r)}_{\mathcal{Y}}$ in (4.3) and let $\alpha\in\big(\frac{1}{k},\infty\big)$. Then

$$\limsup_{R\to\infty}\sup_{x\in B_R}\omega(B_{R^{-\alpha}}(x))\le k\quad\mathbb{P}\text{-a.s.}\tag{5.19}$$

and

$$\limsup_{R\to\infty}N^{(R^{-\alpha})}_{\mathcal{P}\cap B_R}\le k\quad\mathbb{P}\text{-a.s.}\tag{5.20}$$

Remark 5.6. Our choice of $R(t)$ is explained by the fact that, in the proof of the upper bound in Theorem 2.16, we will let the Brownian motion (W_s) stay inside the ball of radius $t^{\frac{k}{k-1}}\ell(t)^{\frac{1}{d(k-1)}}$ around the origin until time t. For $0\le s\le t$, the value of $V^{(a)}(W_s)$ then may also depend on the elements of the Poisson cloud lying outside of this ball, but not further away form it than distance a. When the truncation radius a is smaller than 1, it therefore suffices to consider the Poisson field in $B_{R(t)}$.

Remark 5.7. Using that ℓ is slowly varying at infinity and thus

$$\lim_{r\to\infty}\frac{\ell(\lambda r)}{\ell(r)}\overset{r\to\infty}{\longrightarrow}1\tag{5.21}$$

uniformly over λ in compact subsets of $(0,\infty)$, cf. [BGT02, Theorem 1.2.1], we can translate the integrability condition in (5.18) into a summability condition via the following condensation test. Applying (5.21) uniformly over $\lambda\in[\frac{1}{2},2]$ we note that there exists a constant $c_\ell\in(1,\infty)$ s.t.

$$\sup_{r\in(1,\infty),\lambda\in[\frac{1}{2},2]}\frac{\ell(\lambda r)}{\ell(r)}\le c_\ell\tag{5.22}$$

and thus

$$\frac{1}{2c_\ell}\sum_{n=0}^{\infty}\ell(2^n)^{-1}\le\sum_{n=0}^{\infty}\int_{2^n}^{2^{n+1}}2^{-(n+1)}\frac{\ell(2^n)^{-1}}{c_\ell}\,\mathrm{d}r\le\int_1^\infty\frac{\mathrm{d}r}{r\ell(r)}\tag{5.23}$$

$$\le\sum_{n=0}^{\infty}\int_{2^n}^{2^{n+1}}2^{-n}c_\ell\ell(2^n)^{-1}\,\mathrm{d}r=c_\ell\sum_{n=0}^{\infty}\ell(2^n)^{-1}.\tag{5.24}$$

Proof of Lemma 5.5. We begin the with the upper bound in (5.18). Set $t_n = 2^n$ and let c_ℓ be as in (5.22). By (5.2) we have for $n \in \mathbb{N}$

$$\mathbb{P}\left[\sup_{x\in B_{c_\ell R(t_{n+1})}} \omega\Big(B_{c_\ell r(t_n)}(x)\Big) \geq k+1\right] \leq \omega_d^{k+1}\Big((2c_\ell)^{k+1} R(t_{n+1}) r(t_n)^k\Big)^d \tag{5.25}$$

$$\leq const\ \ell(t_n)^{-1}. \tag{5.26}$$

The r.h.s is summable with respect to n due to (5.23)-(5.24) and thus by the Borel-Cantelli lemma $\mathbb{P}$-a.s. eventually $\sup_{x\in B_{c_\ell R(t_{n+1})}} \omega\Big(B_{c_\ell r(t_n)}(x)\Big) \leq k$. Now, for $t > 0$, choose $n = n(t) \in \mathbb{N}$ such that $t_n \leq t < t_{n+1}$. Then $R(t) \leq c_\ell R(t_{n+1})$ and $r(t) \leq c_\ell r(t_n)$. Hence

$$\limsup_{t\to\infty} \sup_{x\in B_{R(t)}} \omega\Big(B_{r(t)}(x)\Big) \leq \limsup_{n\to\infty} \sup_{x\in B_{c_\ell R(t_{n+1})}} \omega\Big(B_{c_\ell r(t_n)}(x)\Big) \leq k \quad \mathbb{P}\text{a.s.} \tag{5.27}$$

which shows the upper bound in (5.18).

To obtain (5.19), take any $\ell\colon (0,\infty) \to (1,\infty)$ slowly varying at infinity and satisfying the integrability condition in (5.18), and write $t = t(R) = R^{\frac{k-1}{k}}$. Then we have $R \leq R(t)$ and for R sufficiently large $R^{-\alpha} \leq r(t)$, as $\alpha > \frac{1}{k}$ and ℓ is slowly varying. Therefore

$$\limsup_{R\to\infty} \sup_{x\in B_R} \omega(B_{R^{-\alpha}}(x)) \leq \limsup_{t\to\infty} \sup_{x\in B_{R(t)}} \omega\Big(B_{r(t)}(x)\Big) \leq k \quad \mathbb{P}\text{-a.s.}$$

which gives (5.19). The proof of (5.20) works out.

We continue with the lower bound in (5.18). Set

$$A_n = B_{R(t_n)-r(t_n)} \setminus B_{R(t_{n-1})+r(t_n)}, \quad n \in \mathbb{N}. \tag{5.28}$$

As there exist constants $c_1, c_2 \in (0,\infty)$ such that

$$\# A_n \cap 2r(t_n)\mathbb{Z}^d \sim c_1 \frac{R(t_n)^d}{r(t_n)^d} \quad \text{and} \quad \mathrm{Poi}_{\omega_d r(t_n)^d}(\{k+1\}) \sim c_2 r(t_n)^{d(k+1)}, \quad n\to\infty \tag{5.29}$$

and as $\lim_{x\downarrow 0} \frac{1-\exp(-x)}{x} = 1$, (5.5) ensures the existence of $c \in (0,\infty)$ such that

$$\mathbb{P}\left[\sup_{x\in A_n} \omega(B_{r(t_n)}(x)) \geq k+1\right] \geq c\ell(t_n)^{-1} \quad \text{for all } n \in \mathbb{N}.$$

Using (5.23)-(5.24) and our assumption on ℓ we get

$$\sum_{n=1}^{\infty} \mathbb{P}\left[\sup_{x\in A_n} \omega(B_{r(t_n)}(x)) \geq k+1\right] = \infty. \tag{5.30}$$

As the family $\{\sup_{x\in A_n} \omega(B_{r(t_n)}(x)), n \in \mathbb{N}\}$ is independent, the Borel-Cantelli lemma yields

$$\limsup_{n\to\infty} \sup_{x\in A_n} \omega(B_{r(t_n)}(x)) \geq k+1 \quad \mathbb{P}\text{-a.s.} \tag{5.31}$$

This implies the lower bound in (5.18) and completes our proof. $\square$

In the last two lemmata we characterize the limes inferior of $\sup_{x\in B_{R(t)}} \omega(B_{r(t)}(x))$ as $t\to\infty$.

Lemma 5.8. *Fix $k\in\mathbb{N}$ and let $R(t), r(t)\in(0,\infty)$ satisfy $r(t)\downarrow 0, R(t)\uparrow\infty$ as $t\to\infty$ and $(R(t)r(t)^k)^d\ge c\log\log t$ for all large enough t with a constant $c>\frac{2^d(k+1)!}{\omega_d^{k+2}}$. Furthermore assume that for any $\varepsilon>0$ there exists $a\in(1,\infty)$ such that $\sup_{n\in\mathbb{N}}\frac{R(a^n)}{R(a^{n+1})}\ge 1-\varepsilon$. Then*

$$\liminf_{t\to\infty}\sup_{x\in B_{R(t)}}\omega\Big(B_{r(t)}(x)\Big)\ge k+1\quad \mathbb{P}\text{-a.s.} \tag{5.32}$$

Proof. Choose $\varepsilon>0$ such that $c(1-\varepsilon)^d>\frac{2^d(k+1)!}{\omega_d^{k+2}}$, $a\in(1,\infty)$ such that $\sup_{n\in\mathbb{N}}\frac{R(a^n)}{R(a^{n+1})}\ge 1-\varepsilon$ and set $t_n=a^n$. By (5.5), (5.14) and the assumptions on R and r we have

$$\begin{aligned}\log\mathbb{P}\Big[\sup_{x\in B_{R(t_n)}}\omega\Big(B_{r(t_{n+1})}(x)\Big)\le k\Big] &\le -\operatorname{Poi}_{\omega_d r(t_{n+1})^d}(\{k+1\})\#B_{R(t_n)}\cap 2r(t_{n+1})\mathbb{Z}^d \\ &\sim -\frac{\omega_d^{k+2}}{2^d(k+1)!}R(t_n)^d r(t_{n+1})^{dk}\le -\frac{\omega_d^{k+2}c(1-\epsilon)^d}{2^d(k+1)!}\log\log t_{n+1}.\end{aligned} \tag{5.33}$$

as $n\to\infty$. As $\frac{\omega_d^{k+2}c(1-\epsilon)^d}{2^d(k+1)!}>1$, the exponential of the sequence on the right-hand side is summable with respect to n. Thus an application of the Borel-Cantelli lemma implies that a.s. eventually in n, $\sup_{x\in B_{R(t_n)}}\omega\Big(B_{r(t_{n+1})}(x)\Big)\ge k+1$. Now, for $t>0$, choose $n\in\mathbb{N}$ such that $t_n\le t<t_{n+1}$. Then $R(t)\ge R(t_n)$ and $r(t)\ge r(t_{n+1})$ and hence

$$\liminf_{R\to\infty}\sup_{x\in B_{R(t)}}\omega\Big(B_{r(t)}(x)\Big)\ge\liminf_{n\to\infty}\sup_{x\in B_{R(t_n)}}\omega\Big(B_{r(t_{n+1})}(x)\Big)\ge k+1\quad\mathbb{P}\text{-a.s.}\qquad\square$$

Lemma 5.9. *Fix $k\in\{2,3,\dots\}$, let $\tilde\ell(t):=\log\log t$ and define for $t\in(\mathrm{e},\infty)$ and $n\in\mathbb{N}_0$*

$$R(t)=t^{\frac{k}{k-1}}\tilde\ell(t)^{-\frac{1}{d(k-1)}},\quad r_n(t)=c_n t^{-\frac{1}{k-1}}\tilde\ell(t)^{\frac{1}{d(k-1)}},\quad z(t)=\lfloor\log_2(\log(t)^{\frac{2}{d(k-1)}})\rfloor \tag{5.34}$$

with a sequence $(c_n)_{n\in\mathbb{N}_0}\in(0,\infty)^{\mathbb{N}_0}$ satisfying $c_n\downarrow 0$ as $n\to\infty$ and

$$\sum_{n=0}^{\infty}(2^n c_n^k)^d<1/(kC_k)\quad\text{with}\quad C_k:=\frac{(\sqrt{d}+1)^{d(k+1)}\omega_d^{k+2}}{2^d(k+1)!}. \tag{5.35}$$

Then

$$\liminf_{t\to\infty}\max_{n=0}^{z(t)}\sup_{x\in B_{2^nR(t)+1}}\omega\Big(B_{r_n(t)}(x)\Big)\le k\quad\mathbb{P}\text{-a.s.} \tag{5.36}$$

Proof. Fix $\rho_k\in(\frac{k-1}{k},1)$ such that $\sum_{n=0}^{\infty}(2^nc_n^k)^d\le(1-\rho_k)/C_k$, which is possible by (5.35). Take any $t_0\in(\mathrm{e},\infty)$. Then $\exp(\rho_k\tilde\ell(t_0))>1$ and $z(t_0)\ge 0$, and we define a growing sequence $(t_j)_{j\in\mathbb{N}_0}$ recursively by

$$t_j=t_{j-1}\exp\{\rho_k\tilde\ell(t_{j-1})\},\quad j\in\mathbb{N}. \tag{5.37}$$

For $j \in \mathbb{N}$ and $n \in \mathbb{N}_0$ set, abbreviating $R_n(t) = 2^n R(t) + 1$,

$$A_{j,n} := B_{R_n(t_j)} \setminus B_{R(t_{j-1})}, \qquad X_j := \max_{n=0}^{z(t_j)} \sup_{x \in B_{R_n(t_j)}} \omega(B_{r_n(t_j)}(x)),$$

$$\hat{X}_j := \max_{n=0}^{z(t_j)} \sup_{x \in A_{j,n}} \omega(B_{r_n(t_j)}(x)), \qquad \check{X}_j := \sup_{x \in B_{R(t_{j-1})}} \omega(B_{r_0(t_j)}(x)).$$

Note that $X_j = \max(\check{X}_j, \hat{X}_j)$. Thus it will be sufficient to show that $\mathbb{P}$-a.s. both

$$\limsup_{j\to\infty} \check{X}_j \le k \quad \text{and} \quad \liminf_{j\to\infty} \hat{X}_j \le k. \tag{5.38}$$

To obtain the first inequality, note that by (5.2) there exists a constant $c \in (0,\infty)$ such that for all $j \in \mathbb{N}$

$$\mathbb{P}[\check{X}_j \ge k+1] \le c\left(R(t_{j-1}) r_1(t_j)^k\right)^d = c c_0^{dk} \tilde{\ell}(t_{j-1})^{-\frac{1}{k-1}} \tilde{\ell}(t_j)^{\frac{k}{k-1}} \mathrm{e}^{-\frac{dk\rho_k}{k-1}\tilde{\ell}(t_{j-1})} \tag{5.39}$$

$$\le c c_0^{dk} \exp\left(-\tfrac{dk\rho_k}{k-1}\tilde{\ell}(t_{j-1}) + \tfrac{k}{k-1}\log(\tilde{\ell}(t_j))\right) \tag{5.40}$$

$$\le c c_0^{dk} \exp\left(-\tfrac{dk\rho_k}{k-1}\tilde{\ell}(t_{j-1}) + \tfrac{k}{k-1}\tilde{\ell}(2t_{j-1})\right) \tag{5.41}$$

$$\le c' \exp\left(-(\tfrac{dk\rho_k}{k-1} - \tfrac{k}{k-1})\tilde{\ell}(t_{j-1})\right). \tag{5.42}$$

Note that $\frac{dk\rho_k}{k-1} - \frac{k}{k-1} > 1$ and that for any $\alpha > 1$ there exists $\delta \in (0,\infty)$ such that

$$\infty > \int_{t_1}^{\infty} \frac{1}{t} \exp(-\alpha\tilde{\ell}(t))\,\mathrm{d}t = \sum_{j=1}^{\infty} \int_{t_j}^{t_{j+1}} \frac{1}{t}\exp(-\alpha\tilde{\ell}(t))\,\mathrm{d}t \tag{5.43}$$

$$\ge \sum_{j=1}^{\infty} \frac{t_{j+1}-t_j}{t_{j+1}} \exp(-\alpha\tilde{\ell}(t_{j+1})) \ge \delta \sum_{j=1}^{\infty} \exp(-\alpha\tilde{\ell}(t_{j+1})). \tag{5.44}$$

Hence, the term in (5.42) is summable with respect to j and the Borel-Cantelli lemma yields that

$$\mathbb{P}[\check{X}_j > k \ \text{ infinitely often}] = 0. \tag{5.45}$$

Therefore we have shown the first inequality in (5.38) and continue with the second one. Note that for all $j \in \mathbb{N}$,

$$\log \mathbb{P}[\hat{X}_j \le k] \ge \log \mathbb{P}[X_j \le k] \tag{5.46}$$

$$\ge \sum_{n=0}^{z(t)} \left(\# B_{R_n(t_j)+(\sqrt{d}+1)r_n(t_j)} \cap 2r_n(t_j)\mathbb{Z}^d\right) \log\left(\mathrm{Poi}_{\omega_d((\sqrt{d}+1)r_n(t_j))^d}(\{0,\dots,k\})\right)$$

where the second inequality follows from (5.6). Furthermore there exist $\varepsilon_i(t) \in (0,1)$ with $\varepsilon_i(t) \downarrow 0$ as $t \to \infty$, ($i = 1,2,3$), such that for all $n \in \mathbb{N}$ and large j

$$\# B_{R_n(t_j)+(\sqrt{d}+1)r_n(t_j)} \cap 2r_n(t_j)\mathbb{Z}^d \le (\omega_d + \varepsilon_1(t_j))\left(2^{n-1}\frac{R(t_j)}{r_n(t_j)}\right)^d, \tag{5.47}$$

$$\mathrm{Poi}_{\omega_d((\sqrt{d}+1)r_n(t_j))^d}(\{0,...,k\}) = 1 - \sum_{i=k+1}^{\infty} \mathrm{Poi}_{\omega_d((\sqrt{d}+1)r_n(t_j))^d}(\{i\}) \tag{5.48}$$

$$\geq 1 - \left(\tfrac{((\sqrt{d}+1)^d\omega_d)^{k+1}}{(k+1)!} + \varepsilon_2(t_j)\right) r_n(t_j)^{d(k+1)} \tag{5.49}$$

and, using that $\frac{\log(1-x)}{-x} \downarrow 1$ as $x \downarrow 0$,

$$\log \mathrm{Poi}_{\omega_d((\sqrt{d}+1)r_n(t_j))^d}(\{0,...,k\}) \geq -(1+\varepsilon_3(t_j)) \left(\tfrac{((\sqrt{d}+1)^d\omega_d)^{k+1}}{(k+1)!} + \varepsilon_2(t_j)\right) r_n(t_j)^{d(k+1)}. \tag{5.50}$$

Thus there exists $\varepsilon(t) \downarrow 0$ $(t \to \infty)$ such that for large j the right-hand side of (5.46) is not smaller than

$$\sum_{n=0}^{z(t)} -(1+\varepsilon(t_j))C_k \left(r_n(t_j)^{k+1} \frac{2^n R(t_j)}{r_n(t_j)}\right)^d = -(1+\varepsilon(t_j))C_k \sum_{n=0}^{z(t)} \left(2^n R(t_j) r_n(t_j)^k\right)^d \tag{5.51}$$

$$= -\tilde{\ell}(t_j)(1+\varepsilon(t_j))C_k \sum_{n=0}^{z(t)} (2^n c_n^k)^d \geq -(1-\rho_k)\tilde{\ell}(t_j). \tag{5.52}$$

Since

$$\infty = \int_{t_1}^{\infty} \frac{1}{t} \exp(-\tilde{\ell}(t))\,\mathrm{d}t = \sum_{j=1}^{\infty} \int_{t_j}^{t_{j+1}} \frac{1}{t} \exp(-\tilde{\ell}(t))\,\mathrm{d}t \tag{5.53}$$

$$\leq \sum_{j=1}^{\infty} \frac{t_{j+1}-t_j}{t_j} \exp(-\tilde{\ell}(t_j)) \leq \sum_{j=1}^{\infty} \exp(-(1-\rho_k)\tilde{\ell}(t_j)) \tag{5.54}$$

we get by (5.46)-(5.54) that

$$\sum_{j=1}^{\infty} \mathbb{P}[\hat{X}_j \leq k] = \infty. \tag{5.55}$$

Now note that there exists an $j_0 \in \mathbb{N}$ s.t. both the families $(\hat{X}_{2j},\ j \in \{j_0, j_0+1, ...\})$ and $(\hat{X}_{2j-1},\ j \in \{j_0, j_0+1, ...\})$ are independent, as for large j we have

$$R(t_{j+1}) - r_0(t_{j+1}) \geq R_{z(t_j)}(t_j) + r_{z(t_j)}(t_j)$$

and hence the members of each of the two families do only depend on the Poisson cloud in pairwise disjoint subsets. Thus we get by the Borel-Cantelli lemma, that

$$\mathbb{P}[\hat{X}_j \leq k \text{ infinitely often}] = 1, \tag{5.56}$$

yielding the second statement in (5.38). This completes our proof. □

5.2 Eigenvalue Asymptotics

Let $d \geq 3$, $\theta \in (0, \frac{h_d}{2}]$ and $k = k_\theta = \lfloor \frac{h_d}{\theta} \rfloor$. In order to make use of the upper bound given in Theorem 4.1, we study the $\mathbb{P}$-a.s. asymptotics of $\Lambda_{\mathcal{Y}}^{(\theta,a,r)}$ defined in (4.3) as $R \to \infty$, where

$\mathcal{Y} = \mathcal{P}_R$, $a = R^{-\alpha}$ with $\alpha \in (\frac{1}{k+1}, \infty)$ and $r = 5a$. For the proof of the following results we combine the multipolar Hardy inequality (3.45) and the Poissonian asymptotics stated in the previous section.

Recall definitions (4.1)-(4.2). Note that for $s > 0$

$$\{\Lambda_{\mathcal{P}_R}^{(\theta,a,r)} > s\} = \{\Lambda_{\mathcal{P}_R}^{(\theta,a,r)} > s\} \cap \{N_{\mathcal{P}_R}^{(r)} \geq k+1\} \tag{5.57}$$

$$\subset \Big(\{\Lambda_{\mathcal{P}_R}^{(\theta,a,r)} > s\} \cap \{N_{\mathcal{P}_R}^{(r)} = k+1\}\Big) \cup \{N_{\mathcal{P}_R}^{(r)} \geq k+2\}. \tag{5.58}$$

The equality holds, since $\lambda_{\mathcal{C}} = 0$ for each component $\mathcal{C} \in \mathscr{C}_{\mathcal{P}_R}^{(r)}$ with $\{N_{\mathcal{C}} \leq k\}$ due to the multipolar Hardy inequality (3.40). The second event in (5.58) can be controlled by

$$\{N_{\mathcal{P}_R}^{(r)} \geq k+2\} \subset \big\{\exists x \in B_R \colon \omega\big(B_{b_k r}(x)\big) \geq k+2\big\} \tag{5.59}$$

with $b_k := k+1$. To control the first event in (5.58) we use the multipolar Hardy inequality (3.45). Write for $\mathcal{C} \in \mathscr{C}_{\mathcal{P}_R}^{(r)}$

$$\widetilde{\Gamma}(\mathcal{C}) := \inf\left\{\rho > 0 \colon \bigcup_{y \in \mathcal{P}_R \cap \mathcal{C}} B_\rho(y) \text{ is connected}\right\}.$$

Then by (3.45),

$$\begin{aligned}
&\{\Lambda_{\mathcal{P}_R}^{(\theta,a,r)} > s\} \cap \{N_{\mathcal{P}_R}^{(r)} = k+1\} \\
&\subset \{\exists \mathcal{C} \in \mathscr{C}_{\mathcal{P}_R}^{(r)} \colon \lambda_{\mathcal{C}} > s \text{ and } N_{\mathcal{C}} = k+1\} \\
&\subset \left\{\exists \mathcal{C} \in \mathscr{C}_{\mathcal{P}_R}^{(r)} \colon \tfrac{\tilde{c}_{mp}}{\widetilde{\Gamma}(\mathcal{C})^2} > s \text{ and } N_{\mathcal{C}} = k+1\right\} \\
&\subset \left\{\exists \text{ distinct } y_0, y_1, \ldots, y_k \in \mathcal{P}_R \, \forall i \in \{0, \ldots, k\} \, \exists j \in \{0, \ldots, k\} \colon 0 < |y_i - y_j| < 2\sqrt{\tilde{c}_{\mathrm{mp}} s^{-1}}\right\} \\
&\subset \big\{\exists x \in B_R \colon \omega\big(B_{c_{\mathrm{mp}} s^{-1/2}}(x)\big) \geq k+1\big\},
\end{aligned}$$

where

$$\tilde{c}_{mp} := \frac{(k+1)(\pi^2 + 3\theta)}{2}, \quad c_{\mathrm{mp}} := k\sqrt{\tilde{c}_{mp}}. \tag{5.60}$$

Combining these results, we get

$$\{\Lambda_{\mathcal{P}_R}^{(\theta,a,r)} > s\} \subset \Big(\big\{\exists x \in B_R \colon \omega\big(B_{c_{\mathrm{mp}} s^{-1/2}}(x)\big) \geq k+1\big\} \cup \big\{\exists x \in B_R \colon \omega\big(B_{b_k r}(x)\big) \geq k+2\big\}\Big). \tag{5.61}$$

With this inclusion at hand, we derive several results on the eigenvalue asymptotics from the results in Section 5.1.

Lemma 5.10. *Let $0 < a < r < R < \infty$ and $\theta \in (0, \frac{h_d}{2}]$. There exists a constant $c \in (0, \infty)$, depending only on θ and d, such that for all $s \in (0, \infty)$*

$$\mathbb{P}[\Lambda_{\mathcal{P}_R}^{(\theta,a,r)} > s] \leq cR^d \left(s^{-\frac{d}{2}k} + r^{d(k+1)}\right). \tag{5.62}$$

Proof. Using (5.61) we get

$$\begin{aligned}
&\mathbb{P}[\Lambda_{\mathcal{P}_R}^{(\theta,a,r)} > s] \\
&\le \mathbb{P}\Big[\exists x \in B_R \colon \omega\Big(B_{c_{\mathrm{mp}}s^{-1/2}}(x)\Big) \ge k+1\Big] + \mathbb{P}\Big[\exists x \in B_R \colon \omega\Big(B_{b_k r}(x)\Big) \ge k+2\Big] \\
&\le \Big(2^d\omega_d\Big)^{k+1}\Big(Rc_{\mathrm{mp}}^k s^{-k/2}\Big)^d + \Big(2^d\omega_d\Big)^{k+2}\Big(R(b_k r)^{k+1}\Big)^d
\end{aligned} \tag{5.63}$$

where the second inequality follows from (5.2). This shows (5.62). □

An immediate consequence of this bound is the following

Corollary 5.11. *Let $\alpha \in (\frac{1}{k+1}, \infty)$, $\alpha' \in (\frac{1}{k}, \infty)$ and $\beta(R) \overset{R\to\infty}{\longrightarrow} \infty$. Writing $\Lambda_R := \Lambda_{\mathcal{P}_R}^{(\theta,R^{-\alpha},5R^{-\alpha})}$, we have*

$$R^{-\frac{2}{k}}\beta(R)^{-1}\Lambda_R \overset{R\to\infty}{\longrightarrow} 0 \quad \textit{in probability} \tag{5.64}$$

and

$$\limsup_{R\to\infty} \frac{\Lambda_R}{R^{2\alpha'}} \le 0 \quad \mathbb{P}\text{-a.s.} \tag{5.65}$$

Proof. The proof of (5.64) is straightforward using (5.62). With respect to (5.65), note that for any $n \in \mathbb{N}$, (5.61) yields

$$\begin{aligned}
\{\Lambda_R > n^{-1}R^{2\alpha'}\} \subset \{&\exists x \in B_R \colon \omega(B_{c_{\mathrm{mp}}\sqrt{n}R^{-\alpha'}}(x)) \ge k+1\} \\
&\cup \{\exists x \in B_R \colon \omega(B_{b_k R^{-\alpha}}(x)) \ge k+2\}.
\end{aligned} \tag{5.66}$$

Thus (5.19) shows that $\limsup_{R\to\infty} \frac{\Lambda_R}{R^{2\alpha'}} \le \frac{1}{n}$ $\mathbb{P}$-a.s. Now let $n \uparrow \infty$ to get (5.65). □

Lemma 5.12. *Let $\ell\colon (0,\infty) \to (1,\infty)$ be slowly varying at infinity and satisfy $\int_1^\infty \frac{\mathrm{d}r}{r\ell(r)} < \infty$. Set $R(t) := t^{\frac{k}{k-1}}\ell(t)^{\frac{1}{d(k-1)}} + 1$, $r(t) = R(t)^{-\alpha}$ for some $\frac{1}{k+1} < \alpha < \infty$ and $\Lambda_t := \Lambda_{\mathcal{P}_{R(t)}}^{(\theta,r(t),5r(t))}$. Then we have*

$$\limsup_{t\to\infty} t^{-\frac{2}{k-1}}\ell(t)^{-\frac{2}{d(k-1)}}\Lambda_t \le 0 \quad \mathbb{P}\text{-a.s.} \tag{5.67}$$

Proof. For any $n \in \mathbb{N}$, (5.61) yields

$$\begin{aligned}
\{\Lambda_t > n^{-1}t^{\frac{2}{k-1}}\ell(t)^{\frac{2}{d(k-1)}}\} \subset \{&\exists x \in B_{R(t)} \colon \omega(B_{c_{\mathrm{mp}}\sqrt{n}t^{-\frac{1}{k-1}}\ell(t)^{-\frac{1}{d(k-1)}}}(x)) \ge k+1\} \\
&\cup \{\exists x \in B_{R(t)} \colon \omega(B_{b_k r(t)}(x)) \ge k+2\}.
\end{aligned} \tag{5.68}$$

With probability 1, both events on the r.h.s eventually do not happen due to (5.18) and (5.19). Therefore, we obtain $\limsup_{t\to\infty} t^{-\frac{2}{k-1}}\ell(t)^{-\frac{2}{d(k-1)}}\Lambda_t \le \frac{1}{n}$ $\mathbb{P}$-a.s. Now let $n \uparrow \infty$ to get (5.65). □

For the formulation of the next lemma we use the following notation. Let C_k be the constant from (5.35), $\tilde{c}_2 \in (0,\infty)$ another fixed constant, which we will specify in (6.50) below, and

$$C^{\inf}(k,d) := \left(kC_k c_{\mathrm{mp}}^{kd}\left(3\frac{2^d}{\tilde{c}_2^{\frac{d}{2}}}\right)\right)^{\frac{2}{d(k-1)}}. \tag{5.69}$$

Furthermore, let $z(t)$, $\tilde{\ell}(t)$ and $R(t)$ be defined as in Lemma 5.9. Finally let $\alpha \in (\frac{1}{k+1}, \infty)$, $a_n(t) = (2^n R(t))^{-\alpha}$ and

$$\Lambda_{t,n} := \Lambda^{(\theta, a_n(t), 5a_n(t))}_{\mathcal{P}_{2^n R(t)+1}} (n \in \mathbb{N}_0), \;\; \Theta_0(t) = t\Lambda_{t,0}, \;\; \Theta_n(t) = t\Lambda_{t,n} - \tilde{c}_2 4^{n-1} t^{\frac{k+1}{k-1}} \tilde{\ell}(t)^{-\frac{2}{d(k-1)}} \, (n \geq 1).$$

Lemma 5.13. *We have*

$$\liminf_{t\to\infty} t^{-\frac{k+1}{k-1}} \tilde{\ell}(t)^{\frac{2}{d(k-1)}} \max_{n=0}^{z(t)} \Theta_n(t) \leq C^{\inf}(k,d) \quad \mathbb{P}\text{-a.s.} \tag{5.70}$$

Proof. Fix $C > C^{\inf}(k,d)$ and let $r_n(t)$ be defined as in Lemma 5.9 with

$$c_0 = c_{\mathrm{mp}} C^{-1/2} \quad \text{and} \quad c_n = c_{\mathrm{mp}} \left(C + \tilde{c}_2 4^{n-1} \right)^{-\frac{1}{2}}, \quad n \geq 1.$$

By (5.61) we get

$$\begin{aligned}
&\Big\{ \max_{n=0}^{z(t)} \Theta_n(t) \leq C t^{\frac{k+1}{k-1}} \tilde{\ell}(t)^{-\frac{2}{d(k-1)}} \Big\} \\
&= \bigcap_{n=0}^{z(t)} \Big\{ \Lambda_{t,n} \leq t^{\frac{2}{k-1}} \tilde{\ell}(t)^{-\frac{2}{d(k-1)}} c_{\mathrm{mp}}^2 c_n^{-2} \Big\} \\
&\supset \bigcap_{n=0}^{z(t)} \big(\{ \nexists x \in B_{2^n R(t)+1} \colon \omega(B_{r_n(t)}(x)) \geq k+1 \} \cap \{ \nexists x \in B_{2^n R(t)+1} \colon \omega(B_{b_k a_n(t)}(x)) \geq k+2 \} \big) \\
&\supset \Big\{ \max_{n=0}^{z(t)} \sup_{x \in B_{2^n R(t)+1}} \omega(B_{r_n(t)}(x)) \leq k \Big\} \cap \Big\{ \sup_{x \in B_{2^{z(t)} R(t)+1}} \omega(B_{b_k a_0(t)}(x)) \leq k+1 \Big\}.
\end{aligned}$$

The first event on the r.h.s $\mathbb{P}$-a.s. does happen infinitely often by (5.36) and the second event does happen eventually by (5.18). Since $C > C^{\inf}(k,d)$ was arbitrary, this yields (5.70). Note, that we may apply (5.36) since, setting $n_0 = \lfloor \log \frac{4C}{\tilde{c}_2} / \log 4 \rfloor$,

$$\begin{aligned}
c_{\mathrm{mp}}^{-kd} \sum_{n=0}^{\infty} \left(2^n c_n^k \right)^d &= C^{-\frac{dk}{2}} + \sum_{n=1}^{\infty} \frac{2^{dn}}{\left(C + \tilde{c}_2 (4^{(n-1)}) \right)^{\frac{dk}{2}}} && (5.71) \\
&= C^{-\frac{dk}{2}} \left\{ 1 + \sum_{n=1}^{\infty} \left(2^{-\frac{2n}{k}} + \frac{\tilde{c}_2}{4C} 4^{n\frac{k-1}{k}} \right)^{-\frac{dk}{2}} \right\} && (5.72) \\
&\leq C^{-\frac{dk}{2}} \left\{ 1 + \sum_{n=1}^{n_0} 2^{dn} + \sum_{n=n_0+1}^{\infty} \left(\frac{\tilde{c}_2}{4C} 4^{n\frac{k-1}{k}} \right)^{-\frac{dk}{2}} \right\} && (5.73) \\
&= C^{-\frac{dk}{2}} \left\{ 2^{dn_0} \frac{1 - 2^{-d(n_0+1)}}{1 - 2^{-d}} + \left(\frac{4C}{\tilde{c}_2} \right)^{\frac{dk}{2}} \frac{2^{-d(k-1)(n_0+1)}}{1 - 2^{-d(k-1)}} \right\} && (5.74) \\
&\leq C^{-\frac{dk}{2}} \left\{ \frac{2}{1 - 2^{-d}} \left(\frac{4C}{\tilde{c}_2} \right)^{\frac{d}{2}} \right\} \leq 2^d 3 C^{-\frac{d(k-1)}{2}} \tilde{c}_2^{-\frac{d}{2}} < \frac{1}{c_{\mathrm{mp}}^{kd} k C_k}.
\end{aligned}$$

□

6 Proofs of the Main Theorems

We are now ready to prove our main theorems. We first study the difference between the renormalized inverse-square potential $\overline{V}$ and the truncated potential $V^{(a)}$. In the subsequent section, we will prove all of the upper bounds in our main theorems, including the finiteness stated in Theorems 2.1 and 2.13. We continue with the proofs of the lower bounds. The last section is devoted to the proofs of Theorems 2.3 and 2.14.

6.1 Truncation of Poisson Potentials

In the following lemma we control the error that occurs when the renormalized potential $\overline{V}$ is substituted by the truncated and non-renormalized potential $V^{(a)}$ as defined in (3.4).

Lemma 6.1. *Let $a \in (0,\infty)$ and let $D \subset \mathbb{R}^3$ be a bounded set. Then*

$$\sup_{x\in D} |\overline{V}(x) - V^{(a)}(x)| < \infty \quad \mathbb{P}\text{-}a.s. \tag{6.1}$$

Moreover, we have for all $R \mapsto a_R \in (0,a]$,

$$\lim_{R\to\infty} \frac{a_R^2}{\log R} \sup_{|x|\le R} |\overline{V}(x) - V^{(a_R)}(x)| = 0 \quad \mathbb{P}\text{-}a.s., \tag{6.2}$$

and (6.2) *extends to general dimension $d \ge 3$ provided we replace $\overline{V}$ by $V^{(a)}$.*

Proof. Let $\alpha\colon \mathbb{R}^+ \to [0,1]$ be a smooth truncation function with $\alpha(\lambda) = 1$ on $[0,1]$, $\alpha(\lambda) = 0$ for $\lambda \ge 3$ and $-1 \le \alpha'(\lambda) \le 0$. Decompose $\overline{V} = \overline{V}_1 + \overline{V}_2$ by setting

$$\overline{V}_1(x) := \int_{\mathbb{R}^3} \frac{1-\alpha(a^{-1}|x-y|)}{|x-y|^2}[\omega(\mathrm{d}y) - \mathrm{d}y], \quad \overline{V}_2(x) := \int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|x-y|)}{|x-y|^2}[\omega(\mathrm{d}y) - \mathrm{d}y]. \tag{6.3}$$

Note that $\overline{V}_1$ exactly matches $\overline{V}_{a,\varepsilon}$ in (3.5) of Section 3.2. in [CR11] with $\varepsilon = 1$. Thus by (3.6) in [CR11],

$$\sup_{x\in D} \overline{V}_1(x) < \infty \quad \mathbb{P}\text{-a.s.} \tag{6.4}$$

while, by Lemma 3.3 taken in the same reference

$$\lim_{R\to\infty} (\log R)^{-1} \sup_{|x|\le R} |\overline{V}_1(x)| = 0 \quad \mathbb{P}\text{-a.s.} \tag{6.5}$$

Furthermore, since the integrand in the definition of $\overline{V}_2$ is in $L^1(\mathbb{R}^3)$, we may separate the integration in terms of $\omega(\mathrm{d}y)$ and $\mathrm{d}y$ using Lemma 1.9, i.e.

$$\overline{V}_2(x) = \int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|x-y|)}{|x-y|^2}\omega(\mathrm{d}y) - \int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|x-y|)}{|x-y|^2}\,\mathrm{d}y. \tag{6.6}$$

The second integral above is a finite constant independent of x. For the first integral, we get

$$\int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|x-y|)}{|x-y|^2}\omega(\mathrm{d}y) = V^{(b)}(x) + \int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|x-y|)}{|x-y|^2}\mathbb{1}_{\{|x-y|\geq b\}}\omega(\mathrm{d}y) \tag{6.7}$$

for any truncation level $b \in (0,a]$. Now note that, since $\alpha(\lambda) = 0$ for $\lambda \geq 3$,

$$\sup_{x\in D}\int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|x-y|)}{|x-y|^2}\mathbb{1}_{\{|x-y|\geq b\}}\omega(\mathrm{d}y) \leq b^{-2}\sup_{x\in D}\omega(B_{3a}(x)) < \infty \quad \mathbb{P}\text{-a.s.} \tag{6.8}$$

Combining the decomposition of $\overline{V}$ and (6.4)-(6.8) with $b = a$, we obtain (6.1). With respect to (6.2), note that Corollary 5.4 yields that

$$\lim_{R\to\infty}(\log R)^{-1}\sup_{|x|\leq R}\omega(B_{3a}(x)) = 0 \quad \mathbb{P}\text{-a.s.} \tag{6.9}$$

Thus (6.5) and (6.3)–(6.9) with $D = B_R(0)$ and $b = a_R$ in (6.8) yield (6.2). In general dimension $d \geq 3$, we have

$$\sup_{|x|\leq R}|V^{(a)}(x) - V^{(a_R)}(x)| \leq a_R^{-2}\sup_{|x|\leq R}\omega(B_a(x)),$$

and therefore the supplement to (6.2) follows again from Corollary 5.4. $\square$

6.2 Upper Bounds

We first introduce some notation that will be used in all of the following proofs of the upper bounds.

Let $d \geq 3$ be arbitrary in general, but $d = 3$ whenever we treat the renormalized potential $\overline{V}$. Throughout this section, $\theta \in (0, \frac{h_d}{2}]$ will be fixed. Recall that $k = k_\theta := \lfloor \frac{h_d}{\theta} \rfloor$. For any $t \in (0,\infty)$ we will use a sequence of growing radii $R_n(t)_{n\in\mathbb{N}}$ chosen as

$$R_n(t) = (2^{n-1}t)^{\frac{k}{k-1}} \quad \text{(in the proof of Theorems 2.1,2.13, 2.4, and 2.15)}, \tag{6.10}$$

$$R_n(t) = (2^{n-1}t)^{\frac{k}{k-1}}\ell(2^{n-1}t)^{\frac{1}{d(k-1)}} \quad \text{(in the proof of Theorems 2.7 and 2.16)}, \tag{6.11}$$

$$R_n(t) = 2^{n-1}t^{\frac{k}{k-1}}\tilde{\ell}(t)^{-\frac{1}{d(k-1)}} \quad \text{(in the proof of Theorems 2.8 and 2.17)}, \tag{6.12}$$

where $n \in \mathbb{N}$, $\ell\colon (0,\infty) \to (1,\infty)$ is a slowly varying function satisfying the integrability criteria in Theorem 2.7 and $\tilde{\ell}(t) = \log\log t$.

We fix some arbitrary $\alpha = \alpha_k \in (\frac{1}{k+1}, \frac{1}{k})$ and introduce, according to the choice of $R_n(t)$,

$$a_n(t) = R_n(t)^{-\alpha} \wedge 1, \quad r_n(t) = 5a_n(t), \quad \widetilde{R}_n(t) = R_n(t) + 1, \quad R_0(t) = 8(k+1)r_1(t)$$

as well as the hitting times

$$\tau_n = \tau_n(t) = \tau_{B^c_{R_n(t)}} = \inf\{s \geq 0 \colon W_s \notin B_{R_n(t)}\}, \quad n \in \mathbb{N}_0. \tag{6.13}$$

Finally, we will use the following two error terms. Fix some arbitrary $a \in (0, \infty)$ and set

$$S_n(t) := \sup_{x \in B_{R_n(t)}} |V^{(a)}(x) - V^{(a_n(t))}(x)| \tag{6.14}$$

and

$$\widetilde{S}_n(t) := \sup_{x \in B_{R_n(t)}} |\overline{V}(x) - V^{(a)}(x)|. \tag{6.15}$$

Both $S_n(t) \leq \omega(B_{R_n(t)+a}) a_n(t)^{-2}$ and $\widetilde{S}_n(t)$ are finite $\mathbb{P}$-a.s., the latter by (6.1).

We now start with the proof of the finiteness result of Theorem 2.1 (and with the corresponding version for $V^{(a)}$ stated in Theorem 2.13).

6.2.1 Proof of Theorems 2.1 and 2.13

Proof. In order to prove Theorem 2.1, it will be sufficient to show that for each $\theta \in (0, \frac{h_d}{2}]$ and each $y \in \mathbb{R}^d$, $\mathbb{P}$-almost surely for all $x \in B_1(y) \setminus \mathcal{P}$ and all $t \geq 0$, we have

$$\mathbb{E}_x \left[\exp\left(\theta \int_0^t \overline{V}(W_s)\, \mathrm{d}s \right) \right] < \infty. \tag{6.16}$$

Theorem 2.1 then follows from (6.16) by the union bound

$$\mathbb{P}[\exists x \in \mathbb{R}^d \setminus \mathcal{P} \colon \text{(6.16) does not hold}] \leq \sum_{y \in \mathbb{Q}^d} \mathbb{P}[\exists x \in B_1(y) \setminus \mathcal{P} \colon \text{(6.16) does not hold}] = 0.$$

Furthermore, by the homogeneity of the Poisson field ω, we may take $y = 0$ in (6.16) without loss of generality. The same holds true when $\overline{V}$ is substituted by $V^{(a)}$, and we begin with the proof of (6.16) given this replacement.

We fix $a \in (0, \infty)$. In order to show the a.s. statement for all $t \in (0, \infty)$, it will be sufficient to show it for $t \in [t_-, t_+]$ with $0 < t_- < t_+ < \infty$ fixed but arbitrary. Recall the definitions of $\mathcal{U}_y^{(r)}$ and $N_y^{(r)}$ in (4.1) and (4.3). Define a family of random variables by

$$\zeta_t(x) := 0 \vee \sup\left\{ n \in \mathbb{N} \colon x \in \mathcal{U}_{\mathcal{P}}^{(r_n(t))} \text{ or } R'_{n-1}(t) < 8 r_n(t) N_{\mathcal{P}_{\widetilde{R}_n(t)}}^{(r_n(t))} \text{ or } N_{\mathcal{P}_{\widetilde{R}_n(t)}}^{(r_n(t))} \geq k+2 \right\} + 1 \tag{6.17}$$

with $R'_{n-1}(t) := R_{n-1}(t) - 1$. Note that

$$\mathbb{P}[\forall\, t \in [t_-, t_+]\ \forall x \in B_1 \setminus \mathcal{P} \colon \zeta_t(x) < \infty] = 1,$$

since, as $n \to \infty$, $r_n(t) \leq r_n(t_-) \to 0$, $R_n(t) \geq R_n(t_-) \to \infty$ and

$$\limsup_{n \to \infty} N_{\mathcal{P}_{\widetilde{R}_n(t)}}^{(r_n(t))} \leq \limsup_{n \to \infty} N_{\mathcal{P}_{\widetilde{R}_n(t_+)}}^{(r_n(t_-))} \leq k+1 \quad \mathbb{P}\text{-a.s.} \tag{6.18}$$

by (5.20). Note that the event with probability one in (6.18) does not depend on t, but only on t_- and t_+.

We decompose the expectation in (6.16) according to the largest ball $B_{R_n(t)}$ that the Brownian motion has left until time t and split $V^{(a)} = V^{(a_n(t))} + (V^{(a)} - V^{(a_n(t))})$ as in Lemma 6.1. For all $x \in B_1 \setminus \mathcal{P}$ we have,

$$\mathbb{E}_x\left[\exp\left(\theta\int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right] = \mathbb{E}_x\left[\exp\left(\theta\int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{t<\tau_{\zeta_t(x)}\}}\right] \tag{6.19}$$
$$+\sum_{n=\zeta_t(x)+1}^{\infty}\mathbb{E}_x\left[\exp\left(\theta\int_0^t [V^{(a_n(t))} + (V^{(a)} - V^{(a_n(t))})](W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{\tau_{n-1}\le t<\tau_n\}}\right].$$

On the event $\{t < \tau_{\zeta_t(x)}\}$ we have $V^{(a)}(W_s) = V^{(a)}_{\mathcal{P}_{R_{\zeta_t(x)}(t)+a}}(W_s)$ for $0 \le s \le t$, and since $\mathcal{P}_{R_{\zeta_t(x)}(t)+a}$ is finite $\mathbb{P}$-a.s. we can apply Corollary 3.18 and Remark 3.8 to obtain

$$\mathscr{V}\left(B_{R_{\zeta_t(x)}(t)+a}, \theta V^{(a)}_{\mathcal{P}_{R_{\zeta_t(x)}(t)+a}}\right) < \infty,$$

where the variation $\mathscr{V}$ was defined in (1.11).

Therefore, the first expectation on the right hand side of (6.19) is finite $\mathbb{P}$-a.s. by an application of (3.69). On the other hand, the series on the r.h.s of (6.19) is bounded by

$$\sum_{n=\zeta_t(x)+1}^{\infty}\exp\left(\theta t S_n(t)\right)\mathbb{E}_x\left[\exp\left(\theta\int_0^t V^{(a_n(t))}_{\mathcal{P}_{\widetilde{R}_n(t)}}(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{t\le\tau_{B_{R'_{n-1}(t)}}(x)^c\}}\right], \tag{6.20}$$

using that on the event $\{t < \tau_n\}$, $V^{(a_n(t))}(W_s) = V^{(a_n(t))}_{\mathcal{P}_{\widetilde{R}_n(t)}}(W_s)$ for $0 \le s \le t$. We wish next to apply the upper bound (4.5) to the expectations appearing in the series in (6.20), with the parameters R, r, a, $\mathcal{Y}$ and γ of Section 4 chosen as follows:

$$\mathcal{Y} = \mathcal{P}_{\widetilde{R}_n(t)},\quad R = R'_{n-1}(t),\quad r = r_n(t),\quad a = a_n(t),\quad \gamma = \gamma_n(t) = \kappa\max(\Lambda_{\mathcal{Y}}^{(\theta,a,r)}, R_n(t)^{\frac{2}{k}}),$$

where $\kappa \in (2,\infty)$ is chosen such that, with ϱ as in (4.4), uniformly in $x \in B_1 \setminus \mathcal{P}$,

$$\sup_{n\ge\zeta_t(x)}\varrho(\mathcal{P}_{\widetilde{R}_n(t)}, \theta, a_n(t), r_n(t), \gamma_n(t)) \le \tfrac{1}{2}. \tag{6.21}$$

This is possible since, by the definition of $\zeta_t(x)$ and our choice of parameters, L in (4.4) is bounded by a deterministic constant uniformly over $n \ge \zeta_t(x)$.

Also note that $\gamma_n(t) < \infty$ due to the multipolar Hardy inequality (3.45) and that, for all $n \ge \zeta_t(x)$, $x \notin \mathcal{U}^{(r_n(t))}_{\mathcal{P}_{\widetilde{R}_n(t)}}$ and $R'_{n-1}(t) \ge 8r_n(t)N^{(r_n(t))}_{\mathcal{P}_{\widetilde{R}_n(t)}}$. We may therefore apply (4.5) to the expectations in (6.20) obtaining, for $n \ge \zeta_t(x)+1$,

$$\mathbb{E}_x\left[\mathrm{e}^{\int_0^t \theta V^{(a_n(t))}_{\mathcal{P}_{\widetilde{R}_n(t)}}(W_s)-\gamma_n(t)\,\mathrm{d}s}\mathbb{1}_{\{t\le\tau_{B_{R'_{n-1}(t)}}(x)^c\}}\right]$$
$$\le 2KL\left\{\frac{R'_{n-1}(t)}{r_n(t)}\,\mathrm{e}^{-\frac{cR'_{n-1}(t)^2}{t}} + \exp\left(\left(\log L - c_\star a_n(t)\sqrt{\gamma_n(t)}\right)\left(\frac{R'_{n-1}(t)}{4r_n(t)(k+1)}\right)\right)\right\}$$
$$\le \tilde{c}_1\exp\left(-\tilde{c}_2(2^n t)^{\frac{k+1}{k-1}}\right) \tag{6.22}$$

for some constants $\tilde{c}_1, \tilde{c}_2 \in (0,\infty)$ not depending on n, t or x, using that $\gamma_n(t) \geq \kappa R_n(t)^{\frac{2}{k}}$. Hence, we get that (6.20) is bounded by

$$\sum_{n=2}^{\infty} \tilde{c}_1 \exp\left(\theta t S_n(t) + \gamma_n(t)t - \tilde{c}_2(2^n t)^{\frac{k+1}{k-1}}\right) \tag{6.23}$$

which is a.s. finite for all $(t,x) \in [t_-,t_+] \times B_1 \setminus \mathcal{P}$ since, by (6.2) and (5.65) respectively, we have the almost sure bounds $S_n(t) = \mathcal{O}(a_n(t)^{-2} \log R_n(t))$, $\gamma_n(t) = o(R_n(t)^{2\beta})$ for any $\beta > \frac{1}{k}$, and the events of probability 1 where these hold depend only on t_-, t_+ (actually, only on t_-). This finishes the proof in the case of the truncated potential $V^{(a)}$ and we continue with the proof of (6.16) in the case of the renormalized potential $\overline{V}$. Decomposing $\overline{V} = V^{(a_n(t))} + (V^{(a)} - V^{(a_n(t))}) + (\overline{V} - V^{(a)})$ yields similarly as (6.19)-(6.20)

$$\begin{aligned}\mathbb{E}_x\left[\exp\left(\theta \int_0^t \overline{V}(W_s)\,\mathrm{d}s\right)\right] &\leq \exp(\theta t \widetilde{S}_{\zeta_t})\mathbb{E}_x\left[\exp\left(\theta\int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{t<\tau_{\zeta_t(x)+1}\}}\right] \\ &+ \sum_{n=\zeta_t(x)+1}^{\infty} \exp\left(\theta t(S_n(t)+\widetilde{S}_n(t))\right)\mathbb{E}_x\left[\exp\left(\theta\int_0^t V_{\mathcal{P}_{\widetilde{R}_n(t)}}^{(a_n(t))}(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{t\leq\tau_{B_{R'_{n-1}(t)}}(x)^c\}}\right].\end{aligned} \tag{6.24}$$

Then by (6.22) the series (6.24) is bounded by

$$\sum_{n=2}^{\infty} \tilde{c}_1 \exp\left(\theta t(S_n(t)+\widetilde{S}_n(t)) + \gamma_n(t)t - \tilde{c}_2(2^n t)^{\frac{k+1}{k-1}}\right) \tag{6.25}$$

which converges, since, in addition to the asymptotics stated after (6.23), by (6.2) $\widetilde{S}_n(t) = \mathcal{O}(\log(R_n(t)))$ $\mathbb{P}$-a.s. ☐

Remark 6.2. It follows from our proof that the statement of Theorem 2.1 is also true with $\overline{V}$ being replaced by $|\overline{V}|$, since

$$||\overline{V}(x)| - V^{(a)}(x)| \leq |\overline{V}(x) - V^{(a)}(x)| \tag{6.26}$$

as $V^{(a)}$ is nonnegative.

6.2.2 Upper Bound in Theorems 2.4 and 2.15

Proof of the upper bound in Theorems 2.4 and 2.15. We begin with the proof of (2.14). Write $\zeta_t := \zeta_t(0)$ with $\zeta_t(x)$ defined as in (6.17). By (6.19)-(6.23), we have already established the bound

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right] \leq \mathbb{E}_0\left[\exp\left(\theta\int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\mathbb{1}_{\{t<\tau_{\zeta_t}\}}\right] \tag{6.27}$$

$$+ \sum_{n=2}^{\infty} \tilde{c}_1 \exp\left(\theta t S_n(t) + \gamma_n(t)t - \tilde{c}_2(2^n t)^{\frac{k+1}{k-1}}\right). \tag{6.28}$$

Note that for all large enough t, $\zeta_t = 1$ $\mathbb{P}$-a.s. For these t, we may bound the expectation on the r.h.s. of (6.27) by

$$\mathrm{e}^{\theta t S_1(t)} \mathbb{E}_0\left[\mathrm{e}^{\theta \int_0^t V^{(a_1(t))}_{\mathcal{P}_{\widetilde{R}_1(t)}}(W_s)\,\mathrm{d}s} \mathbb{1}_{\{t\le\tau_1\}}\right] \le \mathrm{e}^{\theta t S_1(t)} \left\{\mathbb{E}_0\left[\mathrm{e}^{\theta \int_0^t V^{(a_1(t))}_{\mathcal{P}_{\widetilde{R}_1(t)}}(W_s)\,\mathrm{d}s} \mathbb{1}_{\{t<\tau_1\}}\right]\right. \tag{6.29}$$

$$\left. + \mathrm{e}^{t\gamma_1(t)}\, \mathbb{E}_0\left[\mathrm{e}^{\theta \int_0^t V^{(a_1(t))}_{\mathcal{P}_{\widetilde{R}_1(t)}}(W_s) - \gamma_1(t)\,\mathrm{d}s} \mathbb{1}_{\{\tau_1\le t\}}\right]\right\}. \tag{6.30}$$

The first expectation on the r.h.s vanishes as $t\to\infty$ since for large t, $\omega(B_{R_0(t)+a_1(t)}) = 0$ $\mathbb{P}$-a.s. and $\lim_{t\to\infty}\mathbb{P}[t<\tau_1] = 0$. By another application of (4.5) as between (6.20)-(6.22), the second expectation on the r.h.s in (6.29) is bounded by

$$2KL\left\{\frac{R_0(t)}{r_1(t)}\,\mathrm{e}^{-\frac{cR_0(t)^2}{t}} + \exp\left(\left(\log L - c_\star a_1(t)\sqrt{\gamma_1(t)}\right)\left(\frac{R_0(t)}{4r_1(t)(k+1)}\right)\right)\right\} \tag{6.31}$$

which converges, using $\gamma_1(t) \ge \kappa R_1(t)^{2/k}$, to some deterministic constant as $t\to\infty$. Hence, there exists a constant $\tilde{c}_3 \in (0,\infty)$ such that for large enough t

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left(\theta\int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right]\\
&\le \tilde{c}_3\exp(\theta t S_1(t) + t\gamma_1(t)) + \sum_{n=2}^{\infty}\tilde{c}_1\exp\left(\theta t S_n(t) + \gamma_n(t)t - \tilde{c}_2(2^n t)^{\frac{k+1}{k-1}}\right)\\
&\le \tilde{c}_3\log t \max_{2\le n\le\lfloor\log(t)\rfloor}\left\{\mathrm{e}^{\theta t S_1(t)+t\gamma_1(t)},\ \exp\left(\theta t S_n(t) + \gamma_n(t)t - \tilde{c}_2(2^n t)^{\frac{k+1}{k-1}}\right)\right\} &(6.32)\\
&\quad + \tilde{c}_1\sum_{\lfloor\log(t)\rfloor}^{\infty}\exp\left(\theta t S_n(t) + \gamma_n(t)t - \tilde{c}_2(2^n t)^{\frac{k+1}{k-1}}\right). &(6.33)
\end{aligned}$$

As $t\to\infty$, the remaining series in (6.33) converges to 0 $\mathbb{P}$-a.s., as $S_n(t) = \mathcal{O}(a_n(t)^{-2}\log R_n(t))$ $\mathbb{P}$-a.s. by (6.2) and $\gamma_n(t) = o(R_n(t)^{2\beta})$ for any $\beta > \frac{1}{k}$ $\mathbb{P}$-a.s. by (5.65). Hence, it remains to show that

$$\beta(t)^{-1}t^{-\frac{k+1}{k-1}}\log\left[\log t \max_{1\le n\le\lfloor\log(t)\rfloor}\exp\left(\theta t S_n(t) + \gamma_n(t)t - \tilde{c}_2(2^n t)^{\frac{k+1}{k-1}}\right)\right] \xrightarrow{t\to\infty} 0 \tag{6.34}$$

in probability. By (6.2),

$$\beta(t)^{-1}t^{-\frac{k+1}{k-1}}\log\log t + \beta(t)^{-1}t^{-\frac{k+1}{k-1}}\max_{1\le n\le\lfloor\log(t)\rfloor}\theta t S_n(t) \xrightarrow{t\to\infty} 0 \quad \mathbb{P}\text{-a.s.} \tag{6.35}$$

Furthermore, we have

$$\beta(t)^{-1}t^{-\frac{k+1}{k-1}}t\gamma_1(t) \xrightarrow{t\to\infty} 0 \quad \text{in probability} \tag{6.36}$$

by (5.64). To control the remaining term, fix $\varepsilon > 0$ and estimate with a union bound

$$\begin{aligned}
&\mathbb{P}\left[t^{-\frac{k+1}{k-1}}\max_{2\le n\le\lfloor\log(t)\rfloor}\left(\gamma_n(t)t - \tilde{c}_2(2^n t)^{\frac{k+1}{k-1}}\right) > \varepsilon\beta(t)\right] &(6.37)\\
&\le \sum_{n=2}^{\infty}\mathbb{P}\left[\gamma_n(t) > t^{\frac{2}{k-1}}\left(\varepsilon\beta(t) + \tilde{c}_2(2^n)^{\frac{k+1}{k-1}}\right)\right]. &(6.38)
\end{aligned}$$

Now note that, since $\beta(t) \to \infty$, when t is large enough, it is impossible to have $\gamma_n(t) = \kappa R_n(t)^{2/k}$ if $\gamma_n(t)$ satisfies the inequality in (6.38). Thus in this case $\gamma_n(t) = \kappa\, \Lambda^{(\theta, a_n(t), r_n(t))}_{\mathcal{P}_{\widetilde{R}_n(t)}}$, and applying (5.62) we obtain for (6.38) the bound, for some constants $c, c' \in (0, \infty)$,

$$c \sum_{n=2}^{\infty} \widetilde{R}_n(t)^d \Big[\kappa^{\frac{dk_\theta}{2}} t^{-\frac{dk}{k-1}} \Big(\varepsilon\beta(t) + \tilde{c}_2 (2^n)^{\frac{k+1}{k-1}} \Big)^{-\frac{dk}{2}} + r_n(t)^{d(k+1)} \Big] \tag{6.39}$$

$$\leq c' \sum_{n=2}^{\infty} \left(\frac{(2^n)^{\frac{2}{k-1}}}{\varepsilon\beta(t) + \tilde{c}_2 (2^n)^{1+\frac{2}{k-1}}} \right)^{\frac{dk}{2}} + c' \sum_{n=2}^{\infty} \widetilde{R}_n(t)^{d(1-\alpha(k_\theta+1))} \overset{t\to\infty}{\longrightarrow} 0. \tag{6.40}$$

Together with (6.35) this shows (6.34), which completes the proof of (2.14). The proof of (2.5) is similar. Use the bound (6.25) instead of (6.23) and note that by (6.2) $\widetilde{S}_n(t)$ is asymptotically negligible compared to the bound on $S_n(t)$. ☐

6.2.3 Upper Bound in Theorems 2.7 and 2.16

Proof. Let $R_n(t) = (2^{n-1}t)^{\frac{k}{k-1}} \ell(2^{n-1}t)^{\frac{1}{d(k-1)}}$ for a fixed function $\ell\colon (0,\infty) \to (1,\infty)$ slowly varying at infinity and satisfying

$$\int_1^\infty \frac{\mathrm{d}r}{r\ell(r)} < \infty. \tag{6.41}$$

Adapt all parameters from the beginning of this section accordingly. We carry out the steps (6.19)-(6.23) analogously for our new choice of $R_n(t)$, setting

$$\gamma_n(t) := \max\Big(\Lambda^{(\theta, a_n(t), r_n(t))}_{\mathcal{P}_{\widetilde{R}_n(t)}}, (2^{n-1}t)^{\frac{2}{k-1}} \Big).$$

Instead of (6.22) we use the bound

$$\begin{aligned}
&\mathbb{E}_0 \left[\mathrm{e}^{\int_0^t \theta V^{(a_n(t))}_{\mathcal{P}_{\widetilde{R}_n(t)}}(W_s) - \gamma_n(t)\,\mathrm{d}s} \mathbb{1}_{\{t \leq \tau_{B_{R_{n-1}(t)}(x)^c}\}} \right] \\
&\leq 2KL \left\{ \frac{R_{n-1}(t)}{r_n(t)} \mathrm{e}^{-\frac{cR_{n-1}(t)^2}{t}} + \exp\left(\Big(\log L - c_\star a_n(t) \sqrt{\gamma_n(t)} \Big) \Big(\frac{R_{n-1}(t)}{4r_n(t)(k+1)} \Big) \right) \right\} \\
&\leq \tilde{c}_1 \exp\Big(-\tilde{c}_2 R_{n-1}(t) \sqrt{\tilde{\gamma}_n(t)} \Big)
\end{aligned} \tag{6.42}$$

with $\tilde{\gamma}_n(t) := \min\{\gamma_n(t),\, R_{n-1}(t)^2/t^2\}$ and constants $\tilde{c}_1, \tilde{c}_2 \in (0,\infty)$. We proceed as between (6.27) and (6.33). For large t, we obtain, as the analogue of (6.32), the upper bound

$$\begin{aligned}
&\mathbb{E}_0 \left[\exp\Big(\theta \int_0^t V^{(a)}(W_s)\,\mathrm{d}s \Big) \right] \\
&\leq \tilde{c}_3 \exp(\theta t S_1(t) + t\gamma_1(t)) + \sum_{n=2}^{\infty} \tilde{c}_1 \exp\Big(\theta t S_n(t) + \gamma_n(t)t - \tilde{c}_2 \sqrt{\tilde{\gamma}_n(t)} R_{n-1}(t) \Big).
\end{aligned} \tag{6.43}$$

Using (5.67) and (6.2) we have for $n \geq 2$,

$$\theta t S_n(t) + \gamma_n(t)t - \tilde{c}_2 \sqrt{\tilde{\gamma}_n(t)} R_{n-1}(t) = \theta t S_n(t) + \sqrt{\tilde{\gamma}_n(t)} R_{n-1}(t) \Big(o(1) 2^{-n} - \tilde{c}_2 \Big) \tag{6.44}$$

$$= -\sqrt{\tilde{\gamma}_n(t)} R_{n-1}(t) \Big(\tilde{c}_2 - o(1) \Big), \quad t \to \infty. \tag{6.45}$$

This implies that the series on the right-hand side of (6.43) vanishes as $t \to \infty$. Furthermore, using again (5.67) and (6.2), we have

$$\theta t S_1(t) + t\gamma_1(t) = o\left(t^{\frac{k+1}{k-1}} \ell(t)^{\frac{2}{d(k-1)}}\right), \quad t \to \infty. \tag{6.46}$$

Collecting (6.43)-(6.46), we have established

$$\limsup_{t\to\infty} t^{-\frac{k+1}{k-1}} \ell(t)^{-\frac{2}{d(k-1)}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right] = 0.$$

The proof of the upper bound in (2.8) now follows from the same supplement as at the end of the proof of (2.5). $\square$

6.2.4 Upper Bound in Theorems 2.8 and 2.17

Proof. We are going to show that

$$\liminf_{t\to\infty} \log\log(t)^{\frac{2}{d(k-1)}} t^{-\frac{k+1}{k-1}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right] \le C^{\inf}(k,d) \tag{6.47}$$

with the constant $C^{\inf}(k,d)$ defined in (5.69). Set $\tilde{\ell}(t) = \log\log t$, $z(t) = \lfloor \log_2(\log(t)^{\frac{2}{d(k-1)}})\rfloor$, $R_n(t) = 2^{n-1} t^{\frac{k}{k-1}} \tilde{\ell}(t)^{-\frac{1}{d(k-1)}}$ and

$$\gamma_n(t) := \max\left(\Lambda^{(\theta, a_n(t), r_n(t))}_{\mathcal{P}_{\tilde{R}_n(t)}}, \tilde{c}^2 (R_{n-1}(t))^2/t^2\right)$$

where

$$\tilde{c} := \frac{1}{2} \min\left(\sqrt{c}, \frac{c_\star}{20(k+1)}\right) \tag{6.48}$$

with the constants c and $c_\star$ coming from Theorem 4.1.
Moreover, define $\widetilde{\Theta}_1(t) := t\Lambda^{(\theta, a_1(t), r_1(t))}_{\mathcal{P}_{\tilde{R}_1(t)}}$ and, for $n \ge 2$,

$$\widetilde{\Theta}_n(t) := t\gamma_n(t) - \tilde{c}_2 R_{n-1}(t)^2/t. \tag{6.49}$$

with

$$\tilde{c}_2 = \frac{1}{2} \min\left(c, \frac{\tilde{c} c_\star}{20(k+1)}\right). \tag{6.50}$$

Our proof now works similarly as the previous proof of the upper bound in Theorem 2.16. As the analogue of (6.42), we obtain the bound

$$\mathbb{E}_0\left[\mathrm{e}^{\int_0^t \theta V^{(a_n(t))}_{\mathcal{P}_{\tilde{R}_n(t)}}(W_s) - \gamma_n(t)\,\mathrm{d}s} \mathbb{1}_{\{t \le \tau_{B_{R_{n-1}(t)}}(x)^{\mathrm{c}}\}}\right] \le \tilde{c}_1 \exp\left(-\tilde{c}_2 \frac{R_{n-1}(t)^2}{t}\right), \quad n \ge 2, \tag{6.51}$$

for some $\tilde{c}_1 \in (0,\infty)$.

The same way we have derived (6.43), we obtain that there exists some constant $\tilde{c}_3 \in (0,\infty)$ such that for large t

$$\mathbb{E}_0\left[\exp\left(\theta\int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right] \tag{6.52}$$

$$\leq \sum_{n=1}^{\infty} \tilde{c}_3 \exp\left(\theta t S_n(t) + \widetilde{\Theta}_n(t)\right) \tag{6.53}$$

$$\leq \tilde{c}_3 z(t) \max_{n=1}^{z(t)} \exp\left(\theta t S_n(t) + \widetilde{\Theta}_n(t)\right) + \tilde{c}_3 \sum_{n=z(t)+1}^{\infty} \exp\left(\theta t S_n(t) + \widetilde{\Theta}_n(t)\right). \tag{6.54}$$

Note that for all $n \geq 2$ and large t for which $\gamma_n(t) = \tilde{c}^2(R_{n-1}(t))^2/t^2$ we have

$$\widetilde{\Theta}_n(t) \leq \left(\tilde{c}^2 - \tilde{c}_2\right)(R_{n-1}(t))^2/t \tag{6.55}$$

with $\left(\tilde{c}^2 - \tilde{c}_2\right) < 0$. Together with (5.67) and (6.2) this implies that

$$\limsup_{t\to\infty} \sum_{n=z(t)+1}^{\infty} \exp\left(\theta t S_n(t) + \widetilde{\Theta}_n(t)\right) = 0 \quad \mathbb{P}\text{-a.s.} \tag{6.56}$$

Using (6.2) once again, we have

$$\begin{aligned}&\liminf_{t\to\infty} \log\log(t)^{\frac{2}{d(k-1)}} t^{-\frac{k+1}{k-1}} \log\left\{z(t)\max_{n=1}^{z(t)} \exp\left(\theta t S_n(t) + \widetilde{\Theta}_n(t)\right)\right\}\\ &\quad = \liminf_{t\to\infty} \log\log(t)^{\frac{2}{d(k-1)}} t^{-\frac{k+1}{k-1}} \max_{n=1}^{z(t)} \widetilde{\Theta}_n(t)\end{aligned} \tag{6.57}$$

and it remains to argue that (6.57) is bounded from above by $C^{\inf}(k,d)$. Because of (6.55) it suffices to take the maximum in (6.57) only over those $2 \leq n \leq z(t)$ for which $\gamma_n(t) = \Lambda_{\mathcal{P}_{\widetilde{R}_n(t)}}^{(\theta,a_n(t),r_n(t))}$. In this case we may identify $\widetilde{\Theta}_n(t)$ with $\Theta_{n-1}(t)$ defined before Lemma 5.13. Thus, Lemma 5.13 allows us to conclude.

The proof of the upper bound in (2.8) once again follows from the same supplement as at the end of the proof for (2.5). □

6.3 Lower Bounds

As in the previous section on the upper bounds, we let $d \geq 3$ be arbitrary but restrict ourselves to the case $d = 3$ whenever we treat the renormalized potential $\overline{V}$. We write $k = k_\theta := \lfloor \frac{h_d}{\theta} \rfloor$ for $\theta \in (0, \frac{h_d}{2}]$.

6.3.1 Key Lemma for the Lower Bounds

The following lemma is needed for the proof of all the lower bounds in our main theorems.

Lemma 6.3. *Fix $\theta \in (0, h_d/2]$, $a \in (0,\infty)$ and let $R(t), r(t) \in (0,\infty)$ satisfy $r(t) \to 0$, $r(t) >> t^{-1}$ and $R(t) \to \infty$ with $R(t) = \mathcal{O}(t^\alpha)$ for some $\alpha < \infty$, as $t \to \infty$. Let c_2 be as given by Lemma 3.25 and*

$$A_t := \left\{ \exists x \in B_{R(t)} \colon \omega\left(B_{r(t)}(x)\right) = k+1 \right\}. \tag{6.58}$$

Then on A_t

$$\log \mathbb{E}_0\left[\exp\left(\theta \int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right] \geq -\sqrt{2c_2}\frac{R(t)}{r(t)} + \frac{c_2 t}{r(t)^2} - \mathcal{O}(t), \quad \text{as } t \to \infty, \tag{6.59}$$

and the same holds true with $V^{(a)}$ being replaced by $\overline{V}$.

Proof. On A_t, we can choose $x_t \in B_{R(t)}$ such that $\omega\left(B_{r(t)}(x_t)\right) = k+1$ and set $\mathcal{P}_t := \{y \in B_{r(t)}(x_t) \colon \omega(\{y\}) = 1\}$. Clearly $V^{(a)} \geq V^{(a)}_{\mathcal{P}_t}$, and $V_{\mathcal{P}_t} - V^{(a)}_{\mathcal{P}_t} \leq \omega(\mathcal{P}_t)a^{-2} = (k+1)a^{-2}$. Thus

$$V^{(a)} \geq V_{\mathcal{P}_t} - c_0 \quad \text{where } c_0 := \frac{k+1}{a^2}. \tag{6.60}$$

Hence,

$$\log \mathbb{E}_0\left[\exp\left(\theta \int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right] \geq -\theta c_0 t + \log \mathbb{E}_0\left[\exp\left(\theta \int_0^t V_{\mathcal{P}_t}(W_s)\,\mathrm{d}s\right) \mathbb{1}_{\{\tau_t \geq t\}}\right]. \tag{6.61}$$

where $\tau_t := \tau_{B^c_{2R(t)}}$. Now, for any $0 \leq t_0 \leq t$,

$$\mathbb{E}_0\left[\mathrm{e}^{\theta \int_0^t V_{\mathcal{P}_t}(W_s)\,\mathrm{d}s} \mathbb{1}_{\{\tau_t \geq t\}}\right] \geq \mathbb{E}_0\left[\mathrm{e}^{\theta \int_{t_0}^t V_{\mathcal{P}_t}(W_s)\,\mathrm{d}s} \mathbb{1}_{\{\tau_t \geq t,\ W_s \in B_{Kr(t)}(x_t) \text{ for } t_0 \leq s \leq t\}}\right] \tag{6.62}$$

with the constant $K \geq 6$ coming from Lemma 3.25 as well. Denote the probability density of a Brownian motion started at 0 by $p_t(y) = (2\pi t)^{-d/2}\,\mathrm{e}^{-|y|^2/(2t)}$ and the law of a Brownian bridge moving from x to y in time s by $\mathbb{P}^s_{x,y}$. Applying the Markov property and setting $\widetilde{R}(t) := R(t) + Kr(t)$, we obtain

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left(\theta \int_0^t V_{\mathcal{P}_t}(W_s)\,\mathrm{d}s\right) \mathbb{1}_{\{\tau_t \geq t\}}\right] && (6.63)\\
&\geq \int_{B_{Kr(t)}(x_t)} p_{t_0}(y)\mathbb{P}^{t_0}_{0,y}[\tau_t \geq t_0]\mathbb{E}_y\left[\mathrm{e}^{\theta \int_0^{t-t_0} V_{\mathcal{P}_t}(W_s)\,\mathrm{d}s} \mathbb{1}_{\{\tau_{B_{Kr(t)}(x_t)} \geq t-t_0\}}\right] \mathrm{d}y \\
&\geq C(2\pi t)^{-\frac{d}{2}}\,\mathrm{e}^{-\frac{\widetilde{R}(t)^2}{2t_0}} \int_{B_{Kr(t)}(x_t)} \mathbb{E}_y\left[\mathrm{e}^{\theta \int_0^{t-t_0} V_{\mathcal{P}_t}(W_s)\,\mathrm{d}s} \mathbb{1}_{\{\tau_{B_{Kr(t)}(x_t)} \geq t-t_0\}}\right] \mathrm{d}y, && (6.64)
\end{aligned}$$

using that by classical estimates we have $\mathbb{P}^{t_0}_{0,y}[\tau_t \geq t_0] \geq C > 0$ uniformly in y, t_0 and t. The integral above can be identified with the integral on the left-hand side of (3.87) with $a = r(t), x = x_t$, implying, that (6.64) is not smaller than

$$c_1 C(2\pi t)^{-\frac{d}{2}} r(t)^d \exp\left(-\frac{\widetilde{R}(t)^2}{2t_0} + c_2(t-t_0)\big(r(t)^{-2}\big)\right). \tag{6.65}$$

We maximize the exponent over $t_0 \in (0, t)$ and obtain

$$t_0 = \frac{\widetilde{R}(t)r(t)}{\sqrt{2c_2}},$$

which yields

$$\log \mathbb{E}_0 \left[e^{\theta \int_0^t V_{\mathcal{P}_t}(W_s)\, ds} \right] \geq -\sqrt{2c_2}\frac{\widetilde{R}(t)}{r(t)} + \frac{c_2 t}{r(t)^2} + \log\left(cC_1 (2\pi t)^{-\frac{d}{2}} r(t)^d \right). \tag{6.66}$$

Thus (6.59) follows from (6.61) and (6.66). As $\overline{V} \geq V^{(a)} - \left|\overline{V} - V^{(a)}\right|$, the proof of (6.59) in the version with $\overline{V}$ passes through analogously with an additional term

$$-\theta t \sup_{x \in B_{2R(t)}} \left|\overline{V} - V^{(a)}\right| \tag{6.67}$$

added on the r.h.s. of the bound (6.61), which is of order $o(t)$ by Lemma 6.1. □

6.3.2 Completion of the Proofs for the Lower Bounds

With Lemma 6.3 at hand we are now ready to complete the proof of the lower bounds in our main theorems. As the statement of Lemma 6.3 is the same for $\overline{V}$ and $V^{(a)}$, we will only execute the proofs for Theorems 2.15, 2.16 and 2.17. The proofs in the case of Theorems 2.4, 2.7 and 2.8 literally are the same provided $V^{(a)}$ is replaced by $\overline{V}$.

Proof of (2.4) **and** (2.13)

Proof. We may assume without loss of generality that

$$\beta(t) = t^{o(1)}, \quad \text{as } t \to \infty. \tag{6.68}$$

Fix $N \in \mathbb{N}$ and set

$$R(t) = \sqrt{N\beta(t)^{-1}} t^{\frac{k}{k-1}}, \quad r(t) = \sqrt{c_2 (8N)^{-1} \beta(t)} t^{-\frac{1}{k-1}} \tag{6.69}$$

with the constant c_2 from Lemma 3.25. By Lemma 6.3 we have on the event A_t as defined in (6.58)

$$\beta(t) t^{-\frac{k+1}{k-1}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t V^{(a)}(W_s)\, ds \right) \right] \geq 4N - o(1), \quad \text{as } t \to \infty. \tag{6.70}$$

By Corollary 5.3 we know that $\lim_{t\to\infty} \mathbb{P}[A_t] = 1$. Hence,

$$\lim_{t\to\infty} \mathbb{P}\left[\beta(t) t^{-\frac{k+1}{k-1}} \log \mathbb{E}_0 \left[\exp\left(\theta \int_0^t V^{(a)}(W_s)\, ds \right) \right] \geq N \right] = 1 \tag{6.71}$$

and as $N \in \mathbb{N}$ can be chosen arbitrarily large, this completes the proof of (2.13). □

Proof of the Lower Bounds in (2.8) **and** (2.15)

Proof. Let $\ell\colon (0,\infty)\to(1,\infty)$ be slowly varying at infinity and satisfy

$$\int_1^\infty \frac{\mathrm{d}r}{r\ell(r)} = \infty. \tag{6.72}$$

Fix $N\in\mathbb{N}$ and set $R(t)=\sqrt{N}\ell(t)^{\frac{1}{d(k-1)}}t^{\frac{k}{k-1}}$, $r(t)=\sqrt{c_2(8N)^{-1}}\ell(t)^{-\frac{1}{dk-1}}t^{-\frac{1}{k-1}}$ with the constant c_2 from Lemma 3.25. Using Lemma 6.3 we obtain on A_t,

$$\log\mathbb{E}_0\left[\mathrm{e}^{\theta\int_0^t V^{(a)}(W_s)\,\mathrm{d}s}\right] \ge 4Nt^{\frac{k+1}{k-1}}\ell(t)^{\frac{2}{d(k-1)}} - \mathcal{O}(t). \tag{6.73}$$

Together with (6.61), this shows that on A_t

$$t^{-\frac{k+1}{k-1}}\ell(t)^{-\frac{2}{d(k-1)}}\log\mathbb{E}_0\left[\exp\left(\theta\int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right]\ge N \tag{6.74}$$

provided t is large enough. By (5.18) we know that $\mathbb{P}\left[\limsup_{t\to\infty}A_t\right]=1$ and thus

$$\limsup_{t\to\infty} t^{-\frac{k+1}{k-1}}\ell(t)^{-\frac{2}{d(k-1)}}\log\mathbb{E}_0\left[\exp\left(\theta\int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right]\ge N \quad \mathbb{P}\text{-a.s.} \tag{6.75}$$

Finally let $N\uparrow\infty$. □

Proof of the Lower Bounds in (2.9) **and** (2.16)

Proof. Let $\varepsilon>0$ be fixed but arbitrary, define

$$c_{\inf}=c_{\inf}(\varepsilon):=2^d(k+1)!/\omega_d^{k+2}+\varepsilon, \qquad \tilde{c}:=\frac{\sqrt{2c_2}}{k+1}$$

with the constant c_2 from Lemma 3.25. Further recall that $\tilde{\ell}(t)=\log\log t$ and take

$$R(t)=(c_{\inf}\tilde{\ell}(t))^{-\frac{1}{d(k-1)}}(\tilde{c}t)^{\frac{k}{k-1}}, \quad r(t)=(c_{\inf}\tilde{\ell}(t))^{\frac{1}{d(k-1)}}(\tilde{c}t)^{-\frac{1}{k-1}}.$$

Using Lemma 6.3 we obtain on A_t

$$\log\mathbb{E}_0\left[\mathrm{e}^{\theta\int_0^t V_{\mathcal{P}_t}(W_s)\,\mathrm{d}s}\right]\ge C_{\inf}(k,\varepsilon)t^{\frac{k+1}{k-1}}\tilde{\ell}(t)^{-\frac{2}{d(k-1)}}-\mathcal{O}(t),$$

where

$$C_{\inf}(k,\varepsilon)=c_2\frac{k-1}{k+1}\left(\frac{\tilde{c}}{\sqrt[d]{c_{\inf}}}\right)^{\frac{2}{k-1}}. \tag{6.76}$$

Therefore (5.32) shows that

$$\mathbb{P}\left[\liminf_{t\to\infty}t^{-\frac{k+1}{k-1}}\tilde{\ell}(t)^{\frac{2}{d(k-1)}}\log\mathbb{E}_0\left[\exp\left(\theta\int_0^t V^{(a)}(W_s)\,\mathrm{d}s\right)\right]\ge C_{\inf}(k,\varepsilon)\right] \tag{6.77}$$

$$\ge\mathbb{P}\left[\liminf_{t\to\infty}A_t\right]=1. \tag{6.78}$$

Letting $\varepsilon\downarrow 0$ we have shown (2.9) with $C_{\inf}(k)=C_{\inf}(k,0)$. □

6.4 Proof of Theorems 2.3 and 2.14

Proof. We begin with the proof of Theorem 2.14 and fix $a \in (0, \infty)$. Note that for each $N \in \mathbb{N}$ the truncated potential $F_N := V^{(a)} \wedge N$ is bounded and thus included in the Kato class. Write $p_t(x) = (2\pi t)^{-d/2} \exp(-|x|^2/(2t))$, $t \in (0, \infty)$, $x \in \mathbb{R}^d$, for the Gaussian transition probability. Then by Proposition 3.13 the unique mild solution to (1.2) with $q = F_N$ is given by $u_N(t,x) = \mathbb{E}_x \left[\exp(\int_0^t F_N(W_s)\, \mathrm{d}s)\right]$, i.e.

$$u_N(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x) F_N(y) u_N(s,y)\, \mathrm{d}y\, \mathrm{d}s, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d \setminus \mathcal{P}. \tag{6.79}$$

Letting $N \uparrow \infty$ on both sides and using the monotone convergence theorem, we see that (6.79) does still hold true with both sides of the equation being finite, when u_N is replaced by $u^{(a)}$ and F_N by $V^{(a)}$, using that $u(t,x)$ is finite for all $(t,x) \in (0,\infty) \times \mathbb{R}^d \setminus \mathcal{P}$ almost surely by Theorem 2.13. We have therefore checked (3.12)-(3.13) for $q = V^{(a)}$ and $u(t,x) = \mathbb{E}_x \left[\exp(\int_0^t V^{(a)}(W_s)\, \mathrm{d}s)\right]$. Hence $u^{(a)}$ is indeed a mild solution to (1.2).

We continue deriving Theorem 2.3. First note that, setting $\bar{v}_\theta(t,x) = \mathbb{E}_x[\exp \theta \int_0^t |\overline{V}|(W_s)\, \mathrm{d}s]$,

$$\int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x) |\overline{V}|(y) \bar{v}_\theta(s,y)\, \mathrm{d}y\, \mathrm{d}s < \infty, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d \setminus \mathcal{P}, \tag{6.80}$$

again using a truncation $|\overline{V}| \wedge N$, monotone convergence and Remark 6.2.

Moreover, we have for all $(t,x) \in (0,\infty) \times \mathbb{R}^d \setminus \mathcal{P}$

$$\begin{aligned}
\bar{u}_\theta(t,x) &= \mathbb{E}_x \left[1 + \int_0^t \overline{V}(W_s) \exp\left(\int_s^t \overline{V}(W_u)\, \mathrm{d}u\right) \mathrm{d}s\right] \\
&= 1 + \int_0^t \mathbb{E}_x \left[\overline{V}(W_s) \exp\left(\int_s^t \overline{V}(W_u)\, \mathrm{d}u\right)\right] \mathrm{d}s \\
&= 1 + \int_0^t \mathbb{E}_x \left[\overline{V}(W_s) \bar{u}_\theta(t-s,y)\right] \mathrm{d}s \\
&= 1 + \int_0^t \int_{\mathbb{R}^d} p_s(y-x) \overline{V}(y) \bar{u}_\theta(t-s,y)\, \mathrm{d}y\, \mathrm{d}s \\
&= 1 + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x) \overline{V}(y) \bar{u}_\theta(s,y)\, \mathrm{d}y\, \mathrm{d}s,
\end{aligned}$$

where we have used for the first equality that by the fundamental theorem of calculus

$$\exp\left(\int_0^t \overline{V}(W_s)\, \mathrm{d}s\right) = 1 + \int_0^t \overline{V}(W_s) \exp\left(\int_s^t \overline{V}(W_s)\, \mathrm{d}u\right) \mathrm{d}s, \tag{6.81}$$

for the second equality Fubini's theorem justified by (6.80) and for the third equality the Markov property of W. This completes the proof. □

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