Infinite rate mutually catalytic branching driven by α-stable Lévy processes

Dissertation zur Erlangung des Grades
„Doktor der Naturwissenschaften“

am Fachbereich Physik, Mathematik und Informatik
an der Johannes Gutenberg-Universität in Mainz

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Mainz, den 29. September 2016

1. Berichterstatter:
2. Berichterstatter:
Datum der mündlichen Prüfung: 19. Dezember 2016

D77 - Mainzer Dissertation

Abstract

The main objective of the present dissertation is to investigate an infinite rate mutually catalytic branching model in one colony, as introduced in [KM10], where the driving Brownian motions are replaced by spectrally positive α-stable Lévy processes. To this end, in the first part we examine the exit measure Q^α of the first quadrant $[0,\infty)^2$ of spectrally positive stable processes. Surprisingly, the exit measure of such processes coincides with the one of ρ-correlated Brownian motions with the special choice of $\rho = -\cos\left(\frac{\pi}{\alpha}\right)$ for the correlation parameter.
This identity is proved by making use of certain Fredholm-type integral equations for the density functions of Q^α, which trace back the exit measure of the first quadrant to the exit measure of the upper half-plane. These integral equations are then shown to determine uniquely the density functions of Q^α. The result can be generalised to the case where the y-axis is rotated by an angle $\zeta \in [0, \pi/2)$.

In the second part of the dissertation, we define a Markov process Z which, in analogy to [KM10], can be understood as a mutually catalytic branching process with infinite branching rate (α-IMUB). This is done by giving an explicit expression for the transition semigroup in terms of Q^α. A strong construction as well as a Trotter-type construction is given for that process.

We finally show weak convergence of the finite branching rate processes to the α-IMUB Z when the branching rate tends to infinity. The right topologisation of the pathspace of càdlàg functions is the Meyer-Zheng pseudo-path topology, which we introduce in Chapter 4.
The dissertation also contains an introductory chapter on Lévy processes with an emphasis on stable processes as well as the exit positions of two-dimensional correlated Brownian motions exiting from the wedge and the half-plane.

Zusammenfassung

Das Hauptanliegen der vorliegenden Dissertation ist die Einführung eines Modells zu wechselseitig katalytischem Verzweigen in einer Kolonie, analog zu [KM10], wenn die treibenden Brownschen Bewegungen durch spektral positive α-stabile Lévy Prozesse ersetzt werden. Hierzu untersuchen wir im ersten Teil der Arbeit das Austrittsmaß Q^α des ersten Quadranten $[0,\infty)^2$ für spektral positive α-stabile Prozesse. Interessanterweise fällt das Austrittsmaß für solche Prozesse mit demjenigen für ρ-korrelierte Brownsche Bewegungen zusammen, mit der speziellen Wahl $\rho = -\cos\left(\frac{\pi}{\alpha}\right)$ für den Korrelationsparameter.
Diese Identität wird mit Hilfe bestimmter Fredholmscher Integralgleichungen für die Dichtefunktionen von Q^α bewiesen, die das Austrittsmaß des ersten Quadranten auf das Austrittsmaß der oberen Halbebene zurückführen. Diese Integralgleichungen legen, wie gezeigt wird, die Dichtefunktionen von Q^α eindeutig fest. Das Ergebnis kann auf den Fall, dass die y-Achse um den Winkel $\zeta \in [0, \pi/2)$ gedreht ist verallgemeintert werden.

Im zweiten Teil der Dissertation definieren wir einen Markovprozess Z, der, in Analogie zu [KM10], als wechselseitig katalytischer Verzweigungsprozess mit unendlicher Verzweigungsrate (α-IMUB) aufgefasst werden kann. Dies geschieht durch Angabe der Übergangshalbgruppe als Integraloperator bezüglich Q^α. Im Anschluss geben wir eine starke Konstruktion sowie eine Trotter-Produkt Konstruktion für den eingeführten Prozess an.

Abschließend zeigen wir, dass die Prozesse mit endlicher Verzweigungsrate schwach gegen den α-IMUB Z konvergieren, falls die Verzweigungsrate gegen unendlich strebt. Zu diesem Zweck topologisieren wir den Pfadraum der càdlàg Funktionen durch die Meyer-Zheng pseudo-path Topologie, die ebenfalls in Kapitel 4 vorgestellt wird.
Die Dissertation enthält darüber hinaus ein einführendes Kapitel über Lévy Prozesse mit Schwerpunkt auf stabilen Prozessen, sowie über Austrittsorte aus dem Kegel und der Halbebene von zweidimensionalen korrelierten Brownschen Bewegungen.

Introduction

Motivation and main results

Mutually catalytic branching processes (mcb processes for short), which have their origin in the fields of interacting particle systems and measure-valued diffusion processes, have been an object of intensive research for about twenty years. A cornerstone in the development of mutually catalytic branching was the celebrated work by Dawson and Perkins [DP98]. Therein, two basic types of mcb processes are treated, being distinguished by their state space. On the one hand, there is the continuous-space mutually catalytic branching process which is introduced as a collection of stochastic partial differential equations driven by space-time white noises. In the present work we will not study this process but rather focus on the second class of mcb processes, living on a countable site-space S. Here, the model is characterised by a system of ordinary stochastic differential equations with drift mechanism given by the q-matrix of a Markov chain on S. We explain this model in more detail.
For a given finite or infinite discrete set S of loci, at each site $k \in S$ there are located two distinguished types of particles with masses given by $Y^i(k) \geq 0$ for $i = 1, 2$. The process Y now evolves randomly in time according to a certain stochastic differential equation. The branching rate of particle i at time t at site k is proportional to the mass $Y^{2-i}(k)$ of the respective other particle at the same site. Gene flow between sites is given through the q-matrix $\mathcal{A}$ of a Markov chain on S. Formally, this reads as

$$\begin{aligned} \mathrm{d}Y_t^1 &= (\mathcal{A}Y_t^1)(k)\mathrm{d}t + \sqrt{\gamma Y_t^1 Y_t^2}\,\mathrm{d}W_t^1(k), \\ \mathrm{d}Y_t^2 &= (\mathcal{A}Y_t^2)(k)\mathrm{d}t + \sqrt{\gamma Y_t^1 Y_t^2}\,\mathrm{d}W_t^2(k). \end{aligned} \tag{1}$$

Here, $W^i(k)$ are independent Brownian motions for $i = 1, 2$, $k \in S$ and $\gamma > 0$ is the branching rate parameter.
Although we are talking about *branching* processes, it is not hard to see that the defining property, i.e. the branching property, breaks down in the present setting. Roughly speaking, a Markov process X fulfils the branching property if X started in $x + y$ evolves like the sum of two independent copies of X started in x and y respectively. One of the reasons why mcb can nonetheless be traced mathematically is the famous self-duality of mcb. This duality relation is due to Leonid Mytnik and was first introduced in [Myt98]. In particular, the self-duality can be used to show weak uniqueness of the solutions of (1) and also allows for the study of the long-time

behaviour of the process. It could be shown already in [DP98] that, depending on the recurrence or transience of the migration matrix $\mathcal{A}$, there is a dichotomy between segregation and coexistence of types. In the first case this leads to the formation of clusters of particles of the same type.

We now explain two ideas which lead to further developments based on the original setting treated in [DP98]. See also the overview article [DM11].
A first step in extending mcb arose from the study of quantitative cluster growth in the recurrent case. We, for simplicity, restrict ourselves in this example to the case where S is a singleton. Due to a space-time scaling relation of mcb, see equation (1.4) in [KM10], it turned out to be benificial in this context to examine the process which, in some sense, results from letting the branching rate γ tend to infinity. Morally, this has, by the mentioned scaling relation, the same effect as speeding up time, so that when started in some arbitrary point in $[0,\infty)^2$ mcb will immediatly hit the axes $E = [0,\infty)^2 \backslash (0,\infty)^2$. Thus, the resulting process, now called infinite rate mutually catalytic branching or IMUB, is a pure jump process on E. In order to understand the dynamics of this infinite rate process, it is crucial to know the distribution Q of a two-dimensional Brownian motion stopped on first hitting E. This approach was rigorously followed by Klenke and Mytnik in the series of papers [KM10], [KM12a] and [KM12b]. There they showed that the self-duality carries over to the infinite rate process living on E. They were also able to give a Trotter-type construction of the resulting process and describe it as a solution to jump-type stochastic differential equations. In the present work, we will basically follow the road of [KM10], where they lay down the foundations of a wider exploration by concentrating on the special case of S being a singleton.
Another possibility of generalising (1) is dropping the independence assumption for the driving Brownian motions. This means, for some $\rho \in [-1,1]$ we assume for the driving Brownian motions at sites $k \neq l$ for $i,j = 1,2$ the following correlation structure:

$$\begin{aligned} \mathbf{E}\left[W_t^1(k)W_t^2(k)\right] &= \rho t \quad \text{and} \\ \mathbf{E}\left[W_t^i(k)W_t^j(l)\right] &= 0. \end{aligned}$$

This approach was first followed in [Reb95] and then studied in [EF04], where they showed that a modified self-duality relation also holds in the correlated noise case. The resulting processes are called symbiotic branching processes (msb or SBM$^\rho$). Also, for msb it is possible to let the branching rate γ tend to infinity and in this way receive a new process called infinite rate symbiotic branching, generally denoted by SBM$_\infty$. A good reference on the mentioned developments and also a nice overview of the subject is [DM11], where also a striking interpretation of SBM$_\infty$ as a generalised voter processes is presented. This leads to interesting connections between well-known models from population genetics.

However, in the present work we proceed into another direction. As Brownian motion, which is the driving randomness for all mutually catalytic branching models which we

mentioned so far, is itself a member of the bigger family of stable Lévy processes, we examine the question whether it is possible to obtain similar results as in [KM10] when replacing the driving Brownian motion in (1) by independent spectrally positive α-stable Lévy processes X for any $\alpha \in (1,2]$. The restriction to spectrally positive Lévy processes is due to the close connection between these processes and continuous state branching processes. This relation is usually referred to as Lamperti transformation, see e.g. [Kyp14] chapter 12. For further information about continuous state branching processes [Gal12] is a good introduction.
Mathematically speaking, we examine the equations

$$\begin{aligned} \mathrm{d}Y_t^1 &= c(\theta_1 - Y_t^1)\mathrm{d}t + \left(\gamma Y_t^1 Y_t^2\right)^{1/\alpha} \mathrm{d}X_t^1, \\ \mathrm{d}Y_t^2 &= c(\theta_2 - Y_t^2)\mathrm{d}t + \left(\gamma Y_t^1 Y_t^2\right)^{1/\alpha} \mathrm{d}X_t^2, \end{aligned} \tag{2}$$

for some $c \geq 0$, $\theta = (\theta_1, \theta_2) \in [0,\infty)^2$ and $\gamma > 0$. Unfortunately, we were not able to find an adequate self-duality like the one for mutually catalytic or symbiotic branching also in the case of (2). Therefore, the question of uniquenes of solutions to (2) is still open.
We were, however, able to compute the harmonic measure Q^α of X on exiting the first quadrant. This is done by deriving integral equations for the one-dimensional density functions of Q^α and showing that the densities are uniquely determined by these equations. As the equations only depend on the exit distributions of X from the upper half-plane and these are equal for α-stable spectrally positive processes and ρ-correlated Brownian motion with $\rho = -\cos\left(\frac{\pi}{\alpha}\right)$, we receive, as a consequence, that $Q^\alpha = Q^\rho$, where Q^ρ is as in [DM11] equation (2.1). With a little more effort, it is also possible to compute the generalised harmonic measure $Q^{\alpha,\zeta}$, which results from the case where the angle between x- and y-axis is $\zeta \in [\pi/2, \pi)$. Based on Q^α it is now a simple exercise to proceed along the lines of [KM10] to construct a process Z living on E, given by its transition semigroup, which is a reasonable analogue to IMUB in the case where Brownian motion is replaced by more general α-stable Lévy processes.

That Z is indeed the weak limit of solutions to (2) as $\gamma \to \infty$, is shown in Chapter 4.2. Our proof relies fundamentally on the self duality for SBM$^\rho$, where $\rho = -\cos\left(\frac{\pi}{\alpha}\right)$. We start by showing that for any sequence $\gamma_n \to \infty$, any family of solutions $(Y^n)_n$ to (2) is tight in the Meyer-Zheng pseudo-path topology. This is basically a consequence of uniform first moment bounds for $(Y_t^n)_n$ for all $t > 0$. We then show that any weak limit point Y is concentrated on E almost surely, $\mathbf{E}\left[Y^1 Y^2\right] = 0$, and give a characterisation of the law of Y in terms of the expectation of Y_t evaluated at a certain class of test functions which is measure determining on E. As we explicitly know the semigroup of Z, it is then easy to show that the expectations of Z_t and Y_t, applied to these test functions coincide for almost all $t \geq 0$, which proves that Z_t is the unique weak limit popint of $(Y_t^n)_n$ for almost all t. As both Z and Y are càdlàg, the laws of Y and Z have to coincide.

Acknowledgement

For reasons of privacy protection, the acknowledgements are left blank in the electronic version of this thesis.

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Chapter 1

Preliminaries

In this first chapter, we give a rough overview over the mathematical objects used troughout this work. We start by recalling the most fundamental facts about Lévy processes and then state some more specific results which we will need in later chapters. In particular, we examine the class of non-negative Lévy processes known as subordinators and deal with the method of subordination, i.e., time-changing Lévy processes by an independent subordinator. We continue by introducing the class of stable (more precisely: strictly stable) processes, the family of Lévy processes which satisfy a certain space-time scaling relation, similar to the well-known scaling property for Brownian motion. We conclude this chapter with a rough treatment of correlated Brownian motion, where we, in particular, give the form of the exit distribution of correlated Brownian motion from the (possibly rotated) upper half-plane and the wedge of angle ζ.
This chapter is written in the style of a summary so there will be in general no proofs given. Instead, we give references to corresponding results in the literature. Most of the presented content can be found in [Kyp14], which is an excellent introduction to the subject.

1.1 Some notes on Lévy processes

In this section, we collect basic facts about Lévy processes including their connection with infinite divisible probability laws and the representation in terms of characteristic functions. Further, we introduce the notion of subordinators and explain how to construct such processes as level-crossing times of special Lévy processes. We then present how we can specify the characteristic exponent of a Lévy process time changed by an independent subordinator.
Most of the results can be found in any standard textbook on Lévy processes, see for example the monographs [Kyp14] or [Ber98]. We will generally stick to the notation in [Kyp14].

1.1.1 The Lévy-Khintchine formula

We begin by recalling the definition of a d-dimensional Lévy process.

Definition 1.1 *A stochastic process $(X_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ for some $d \in \mathbb{N}$, defined on some filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbf{P})$, is called a Lévy process if the following holds:*

1. *$X_0 = 0$ a.s. and X has almost surely càdlàg paths,*
2. *$X_t - X_s$ is independent of $\sigma(X_u : u \leq s)$ for all $0 \leq s \leq t$,*
3. *$X_t - X_s \stackrel{d}{=} X_{t-s}$ for all $0 \leq s \leq t$.*

Here, the French abbreviation càdlàg stands for right continuous with limits from the left. Looking at the definition, it becomes clear why another common name for Lévy processes, which is widely used in older literature, is *processes with stationary and independent increments*. It should be clear that in the definition of a Lévy process there is always a filtered probability space fixed in the background.
For a given Lévy process X on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, we will use the notation $\mathbf{P}_x$ for the probability measure given by the law of $X + x$ under $\mathbf{P}$. This is the distribution of X when started in x. Obviously, $\mathbf{P}_0 = \mathbf{P}$.
Unless otherwise stated, we always assume $d = 1$, which is the case that we will usually be concerned with.

A useful tool when dealing with Lévy processes is their characteristic function respectively their characteristic exponent. These functions completely determine the finite dimensional distributions and thus, because Lévy processes have càdlàg paths, also determine the law of the process.

Definition 1.2 *Let X be a d-dimensional Lévy process. We call the function $\Psi_X : \mathbb{R}^d \to \mathbb{C}$,*

$$\Psi_X(\theta) := -\log \mathbf{E}\left[e^{i\langle\theta, X_1\rangle}\right]$$

the characteristic exponent of X.

When there is no danger of confusion, we will usually write Ψ instead of Ψ_X.
It is easy to see that, due to the independent and stationary increments, the following relation involving general $t \geq 0$ holds, see e.g. [Kyp14] equation (1.3),

$$\mathbf{E}\left[e^{i\langle\theta, X_t\rangle}\right] = e^{-t\Psi(\theta)}. \tag{1.1}$$

We see that a Lévy process is completely determined by the law of the process at time 1, represented by the characteristic exponent. It is thus natural to ask which probability distributions can appear as law of X_1 for an arbitrary Lévy process X. This leads us to the concept of *infinite divisibility*.

Definition 1.3 *A probability law μ is called infinite divisible if for every $n \in \mathbb{N}$ there is a probability law μ^n such that μ is the n-fold convolution of μ^n:*

$$\mu = \mu_1^n * \ldots * \mu_n^n,$$

where $\mu_1^n, \ldots, \mu_n^n$ are n copies of μ^n.

The answer to the above question then is, see e.g. [Kyp14] Theorem 1.3 and Theorem 1.6, a probability law μ is the law of X_1 for some Lévy process X if and only if μ is infinite divisible.
We give a first simple yet important example.

Example 1.4 *The Cauchy distribution* Cau_a *with parameter $a > 0$ is infinitely divisible, see [Kle13] Example 16.2. We call the Lévy process Y with $\mathcal{L}(Y_1) = \mathrm{Cau}_a$ Cauchy process with parameter a. We then have $\Psi_Y(\theta) = a|\theta|$. So, for any $t > 0$, Y_t is also Cauchy distributed with parameter $a \cdot t$. We explicitly know the one-dimensional marginals of Y, namely*

$$\mathbf{P}\left[Y_t \in ds\right] = \frac{1}{\pi} \frac{at}{(at)^2 + s^2} ds. \tag{1.2}$$

The concept of infinite divisibility is also intimately connected to the next result. We now cite the most basic identity about Lévy processes, cf. e.g. [Kyp14] Theorem 1.6.

Theorem 1.5 (Lévy-Khintchine formula in one dimension) *Suppose that X is a real-valued Lévy process. Then, there exist $a \in \mathbb{R}$, $b \geq 0$ and a measure ν on $\mathbb{R}\backslash\{0\}$ satisfying the integrability condition $\int_{-\infty}^{\infty} (1 \wedge x^2) \nu(dx) < \infty$ such that for all $\theta \in \mathbb{R}$*

$$\Psi_X(\theta) = ai\theta + \frac{b}{2}\theta^2 + \int_{-\infty}^{\infty} \left(1 - e^{i\theta x} + i\theta x \mathbb{1}_{\{|x|<1\}}\right) \nu(dx).$$

The triplet (a, b, ν) is called characteristic triplet and is uniquely determined by X.
On the other hand, for any given a, b, ν as above, there exists a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbf{P})$ and a real-valued Lévy process X on this space, such that X has characteristic triplet (a, b, ν).

A first important consequence is that, if X^1 and X^2 are two independent Lévy processes on $\mathbb{R}$, the processes $X^1 + X^2$ and cX^1 with $c \in \mathbb{R}$ are also Lévy processes.
A measure concentrated on $\mathbb{R}\backslash\{0\}$ which satisfies the above mentioned integrability condition is called a *Lévy measure.* Of course, there is also a multidimensional version of the Lévy-Khintchine formula, see for example [Ber98] Theorem 1. For simplicity, we concentrate on the one-dimensional case, which will be sufficient for all our needs. It shall also be noted that the truncation function $x\mathbb{1}_{\{|x|<1\}}$ is arbitrary to some extent and can be replaced by other functions satisfying certain conditions. In this case, the 'diffusion' term b and the Lévy measure ν stay the same and just the 'drift' a has to be modified. See in this context, for example, the remarks at the end of [Kyp14] Chapter 2.5.

The Lévy-Khintchine formula gives us a good intuition of what a Lévy process actually is. It becomes apparent that we can build any given Lévy process X as the sum of three independent processes, coming from three types of generic Lévy processes. For this purpuse, we re-write the Lévy-Khintchine formula in the following form

$$\begin{aligned}\Psi_X(\theta) = \left(ai\theta + \frac{b}{2}\theta^2\right) + \int_{|x|\geq 1}\left(1 - e^{i\theta x}\right)\nu(\mathrm{d}x) \\ + \int_{|x|<1}\left(1 - e^{i\theta x} + i\theta x\right)\nu(\mathrm{d}x).\end{aligned} \tag{1.3}$$

We see that the first term on the right-hand side of the above formula is no more than the characteristic exponent of a (scaled) Brownian motion with drift.
The first integral is known to be the characteristic exponent of a compound Poisson process with intensity $\lambda := \nu([1,\infty))$ and jump distribution $\frac{\nu}{\lambda}$, assumed that $\lambda > 0$. The reader should note that λ is finite due to the fact that ν is a Lévy measure. Furthermore, all jumps are of total magnitude greater than or equal to one.
The last integral is a bit more fancy. We fix $\varepsilon \in (0,1)$ and write as above

$$\begin{aligned}\int_{|x|<1}\left(1 - e^{i\theta x} + i\theta x\right)\nu(\mathrm{d}x) = \int_{|x|<\varepsilon}\left(1 - e^{i\theta x} + i\theta x\right)\nu(\mathrm{d}x) \\ + \int_{\varepsilon\leq|x|<1}\left(1 - e^{i\theta x} + i\theta x\right)\nu(\mathrm{d}x).\end{aligned} \tag{1.4}$$

The second integral on the right-hand side corresponds to a compound Poisson process Y^ε with drift, again since $\nu([\varepsilon,1)) < \infty$. It can be shown, see [Kyp14] Lemma 2.9 that the Y^ε are square-integrable martingales with L^2-norm uniformly bounded in ε. Iterating this procedure, now with $\int_{|x|<\varepsilon}$ instead of $\int_{|x|<1}$, it becomes reasonable to think of the first integral on the right-hand side of (1.4) as corresponding to an (possibly infinite) sum of independent compound Poisson processes with jump sizes which tend to zero but increase in intensity, such that the limiting process is a square-integrable martingale and has characteristic exponent given by the second integral on the right-hand side of (1.3).
The above considerations and also the exact way in which the limit should be understood can be found in detail in [Kyp14] Chapter 2.5.

Next, we state the strong Markov property for Lévy processes, cf. for example Proposition I.6 of [Ber98].

Theorem 1.6 (Strong Markov property) *Let X be a Lévy process and τ be a stopping time such that $\mathbf{P}[\tau < \infty] > 0$. Then conditionally on $\{\tau < \infty\}$, the process $(X_{\tau+t} - X_\tau)_{t\geq 0}$ is independent of $\mathcal{F}_\tau$ and has the same distribution as X.*

Another important definition which we will frequently use is the following.

Definition 1.7 *A one-dimensional Lévy process with characteristic triplet (a,b,ν) is called spectrally positive if $\nu((-\infty,0)) = 0$.*

Morally, this means that spectrally positive processes are the ones which never jump downwards. A Lévy process X is called spectrally negative if $-X$ is spectrally positive and spectrally one-sided if it is either spectrally negative or positive.
We next introduce the Laplace exponent for spectrally negative processes.

Definition 1.8 *Let X be a spectrally negative Lévy process. For $q \geq 0$ and $t \geq 0$ we define the Laplace exponent of X by*

$$\mathbf{E}\left[e^{qX_t}\right] = e^{t\psi(q)}.$$

In the sequel, we implicitly exclude the trivial cases where either X is the negative of a subordinator or a pure drift process. Then $\psi(q) < \infty$ for all $q \geq 0$ and we even have

$$\mathbf{E}\left[e^{qX_t}\right] = e^{t\psi(q)} \quad \text{for all } q \in \mathbb{C} \text{ with } \operatorname{Re} q \geq 0. \tag{1.5}$$

For such q the following important identity holds,

$$\psi(q) = -\Psi(-iq). \tag{1.6}$$

The justification of this can be found in detail in [Ber98] Chapter VII, p. 187 ff.

1.1.2 Subordinators

We now introduce the class of subordinators and present a way of obtaining such processes as level-crossing times of certain Lévy processes. Thereafter, we introduce the method of subordination. For more information about subordinators see also [Ber99].

Definition 1.9 *A one-dimensional Lévy process X is called a subordinator if X is non-decreasing.*

Obviously, a Lévy process X is a subordinator if and only if X_1 is supported in $[0, \infty)$. It is also important to recognise a subordinator from its characteristic triplet. The following useful lemma is taken from [Kyp14], Lemma 2.14.

Lemma 1.10 *A one-dimensional Lévy process X with characteristic triplet (a, b, ν) is a subordinator if and only if $\nu\left((-\infty, 0)\right) = 0$, $\int_{(0,\infty)} (1 \wedge x)\, \nu(dx) < \infty$, $b = 0$ and*

$$\delta := -\left(a + \int_{(0,1)} x\, \nu(dx)\right) \geq 0.$$

As X_1 has values in $[0, \infty)$, the law of a subordinator is completely determined by its Laplace exponent.

Definition 1.11 *Let X be a subordinator. The function $\psi : [0, \infty) \to [0, \infty)$ defined by*

$$\mathbf{E}\left[e^{-qX_t}\right] = e^{-t\psi(q)}$$

is called Laplace exponent of X.

It is important to keep in mind that we defined the term *Laplace exponent* twice for different kinds of processes, as is also common in literature. So, it depends on the context which definition is meant.

Well-known examples of subordinators are compound Poisson processes with jump-measure concentrated on $(0, \infty)$ or Gamma processes. For these and other examples see [Kyp14] Chapter 1.2. Another important class of subordinators are the so-called level-crossing times. These processes will play an important role in Section 2.2.
From now on, always assume that X is a one-dimensional Lévy process. We define the following non-negative random variables, called level-crossing times, which are stopping times according to [Kyp14], Theorem 3.3.

Definition 1.12 *Let X be a one-dimensional Lévy process. For $x \geq 0$ we define the (not necessarily finite) stopping times*

$$\tau_x^+(X) = \tau_x^+ := \inf_{t>0}\{X_t > x\} \quad \text{and}$$
$$\tau_x(X) = \tau_x := \inf_{t>0}\{X_t < -x\}.$$

As an important consequence of the spatial homogeneity of X, we see that we can also understand τ as

$$\tau_x(X) \overset{d}{=} \inf_{t>0}\{X_t + x < 0\},$$

i.e. the first time the process started at x becomes negative. These two definitions shall be used synonymously. We will always apply the one which is more convenient in the given context.
It is clear that if X is spectrally negative, then we have $X_{\tau_x^+} = x$ almost surely. If X is, however, spectrally positive, we get $X_{\tau_x} = x$ almost surely, see e.g. [Kyp14] (2.24).
We have the following important relation. See for example [Kyp14] Theorem 3.12 or [NCY05] Lemma 1.1.

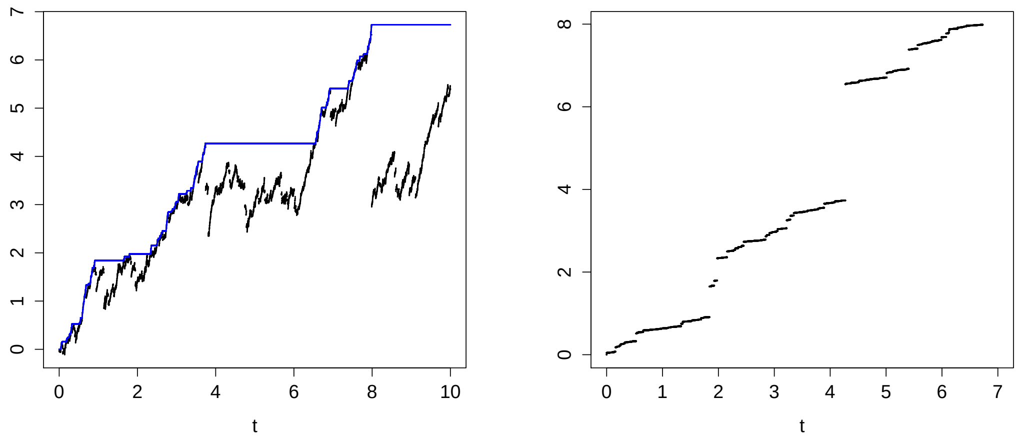
Lemma 1.13 *Let X be a spectrally negative Lévy process with $\mathbf{E}[X_1] \geq 0$ and Laplace exponent ψ. The process $(\tau_x^+)_{x\geq 0}$ is a Lévy subordinator with Laplace exponent given by*

$$\mathbf{E}\left[e^{-q\tau_x^+}\right] = e^{-\Phi(q)x},$$

where $q \geq 0$ and $\Phi(q)$ is the largest root of $\psi(q) = q$, i.e., the right inverse of ψ.

It is important to note that $(\tau_x)_{x\geq 0}$ is càdlàg almost surely. Figure 1.1 visualises the method of obtaining subordinators as level-crossing times of a certain class of Lévy processes.

Figure 1.1: Subordinators as level-crossing times



(a) A spectrally negative Lévy process together with its maximum process

(b) The corresponding level-crossing subordinator

We conclude this first part by presenting a method for building new Lévy processes from a given one and an independent subordinator. This method will play a crucial role in Section 2.2. More information about the cited result can be found in [Ber99] around Proposition 8.6 and [Kyp14] Lemma 2.15. For a multidimensional version of Lemma 1.14 see [Sat99] Theorem 30.1.

Lemma 1.14 *Let X be a Lévy process with characteristic exponent Ψ and let σ be an independent subordinator with Laplace exponent ψ. The process Y defined by*

$$Y_t := X_{\sigma_t}, \qquad \text{for } t \geq 0$$

is again a Lévy process with characteristic exponent given by

$$\Psi_Y(\theta) = \psi\left(\Psi(\theta)\right), \qquad \text{for } \theta \in \mathbb{R}.$$

In view of the well-definedness of $\psi \circ \Psi$, it is important to make two more considerations. First, the reader should notice that for any given Lévy process X, the real part of its characteristic exponent by the Lévy-Khintchine formula satisfies

$$\operatorname{Re}\Psi(\theta) = \frac{b}{2}\theta^2 + \int_{-\infty}^{\infty} (1 - \cos(\theta x))\, \nu(\mathrm{d}x) \geq 0,$$

due to the fact that $b \geq 0$ and $1 - \cos(\theta x) \geq 0$. So, the function Ψ maps $\mathbb{R}$ to $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$.
Second, for the characteristic exponent Ψ of a subordinator σ, we get from the Lévy-Khintchine formula and Lemma 1.10

$$\Psi(\theta) = -i\delta\theta + \int_0^{\infty} (1 - e^{i\theta x})\, \nu(\mathrm{d}x), \tag{1.7}$$

for $\theta \in \mathbb{R}$. This function can be analytically continued to the set

$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}.$$

Let to this end $z \in \mathbb{H}$ and $x \geq 0$. Note that

$$|1 - e^{izx}| \leq 2 \wedge (|z|x)\,,$$

which is due to that for any complex u we have $(1 - e^{iu}) = -iu \int_0^1 e^{itu}\mathrm{d}t$ and $|e^{itu}| \leq 1$ for $t \in (0,1)$. This is true only for $\operatorname{Im} u \geq 0$. We have thus shown that the integral

$$\int_0^\infty (1 - e^{i\theta x})\,\nu(\mathrm{d}x)$$

is well defined whenever $\theta \in \mathbb{C}$ with $\operatorname{Im}\theta \geq 0$ and we can define Ψ for each $\theta \in \mathbb{H}$ by (1.7). We also see, as in the case $\theta \in \mathbb{R}$, that for all $\theta \in \mathbb{H}$ and $t \geq 0$

$$\mathbf{E}\left[e^{i\theta\sigma_t}\right] = e^{-t\Psi(\theta)}.$$

This shows that we can also extend the Laplace exponent ψ of σ and get the relation

$$\psi(\theta) = \Psi(i\theta) = \delta\theta + \int_0^\infty (1 - e^{-\theta x})\,\nu(\mathrm{d}x), \tag{1.8}$$

for all $\theta \in \mathbb{C}_+$. This explains the way in which the above lemma should be understood.

1.2 Stable processes

In this section, we collect some facts about the class of Lévy processes which is particularly important for us. These are the so-called stable processes. After giving the definition, we state some basic results and then take a closer look at the important subclass of strictly stable processes. In particular, we give some explicit results for this subclass using the methods introduced in the previous section.
More information about stable random variables and stable processes can be found, for example, in [ST94], [Jan11], [Zol86] or also [Nol], to name just a few.

1.2.1 General properties

We begin with the definition of stability.

Definition 1.15 *A real-valued random variable X is said to be stable if for all $n \in \mathbb{N}$ there are $a_n > 0$ and $b_n \in \mathbb{R}$, such that*

$$X_1 + \ldots + X_n \stackrel{d}{=} a_n X + b_n,$$

where $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) copies of X.
X is said to be strictly stable if $b_n = 0$ for all n.

It can be shown that, if X is stable, then, in fact, $a_n = n^{1/\alpha}$ for some $\alpha \in (0, 2]$, see e.g. [Kle13] Theorem 16.22. The parameter α is called index of stability. It is easy to see from the definition that any stable random variable is infinitely divisible and therefore, for any stable random variable Z, we can find a Lévy process X with $\mathcal{L}(X_1) = \mathcal{L}(Z)$. Such a process X is called a *stable Lévy process.*

We now give the characteristic triplet of a general stable process X. First, note that in the case $\alpha = 2$, X is a Gaussian random variable and therefore we have $\nu = 0$. If $\alpha < 2$, we have $b = 0$, so there is no Gaussian part, and the Lévy measure is given by

$$\nu(\mathrm{d}x) = \begin{cases} c_1 x^{-1-\alpha}\mathrm{d}x, & \text{if } x > 0, \\ c_2 |x|^{-1-\alpha}\mathrm{d}x, & \text{if } x < 0, \end{cases} \tag{1.9}$$

for some $c_1, c_2 \geq 0$ with $c_1 + c_2 > 0$. The parameter a from the Lévy-Khintchine formula is uniquely determined by the law of X and can attain any real value.

Stable random variables with index $\alpha < 2$ are heavy tailed, i.e., they do not have finite second moments, as we may see from the Lévy measure, cf. [ST94] Property 1.2.16. For the connection between moments and Lévy measure for general Lévy processes, see also Theorem 25.3 in [Sat99]. We have the following relation for an α-stable random variable Z with $\alpha \in (0, 2)$

$$\mathbf{E}\left[|Z|^p\right] < \infty \Leftrightarrow p < \alpha. \tag{1.10}$$

For the tail behaviour of the stable distribution, we get for some constant $C > 0$

$$\lim_{\lambda \to \infty} \lambda^\alpha \mathbf{P}\left[Z > \lambda\right] = C, \tag{1.11}$$

cf. [ST94] Property 1.2.15.

1.2.2 Strictly stable processes

From now on, we will restrict ourselves to the case of strictly stable processes. We will apply to the common but a bit dangerous convention that, by saying stable, we actually mean strictly stable if we do not explicitly state something else. The reason for concentrating on this subclass is that strictly stable processes fulfil the following essential scaling property. The Lévy process X is strictly stable with index $\alpha \in (0, 2]$ if and only if, for all $\lambda > 0$ and $t \geq 0$,

$$X_{\lambda t} \stackrel{d}{=} \lambda^{1/\alpha} X_t. \tag{1.12}$$

Strict stability can also be stated in terms of the characteristic exponent Ψ_X of X. Equationa (1.12) is equivalent to

$$\Psi_X(\lambda\theta) = \lambda^\alpha \Psi_X(\theta) \tag{1.13}$$

for all $\theta \in \mathbb{R}$ and $\lambda > 0$.

For completeness, we state the characteristic exponent of a general (strictly) α-stable process, cf. [Kyp14] (1.13).

$$\Psi(\theta) = \begin{cases} c|\theta|^{\alpha}\left(1 - i\beta\tan\left(\frac{\pi\alpha}{2}\right)\operatorname{sgn}\theta\right) & \text{for } \alpha \in (0,1)\cup(1,2) \\ c|\theta| + i\eta\theta & \text{for } \alpha = 1, \end{cases} \tag{1.14}$$

where $c > 0$, $\beta \in [-1,1]$ and $\eta \in \mathbb{R}$. It is clear that the family of strictly stable processes is parameterised by three parameters. It shall be noted that there are several different ways of parametrising general stable distributions which is a constant source of confusion. See in this context also [Nol].

We now turn to the specific choice of strictly stable processes, which will be the object of interest in the rest of this work. We introduce this process by the explicit presentation of its characteristic exponent. Set for every $\alpha \in (1,2)$ and $\theta \in \mathbb{R}$

$$\Psi(\theta) := -\left(-i\theta\right)^{\alpha}. \tag{1.15}$$

That Ψ is indeed the characteristic exponent of a Lévy process will follow from the Lévy-Khintchine representation of Ψ which will be computed below. It is immediate from (1.13) that a Lévy process with this characteristic exponent is strictly stable with index α.
As it will be of great importance to correctly deal with (1.15), we take the time to carefully compute the parameters in the representations (1.14) and the Lévy-Khintchine formula. We have for $\theta \in \mathbb{R}$

$$\begin{aligned}(-i\theta)^{\alpha} &= |\theta|^{\alpha} e^{-i\frac{\pi\alpha}{2}\operatorname{sgn}(\theta)} \\ &= |\theta|^{\alpha}\left(\cos\left(-\operatorname{sgn}(\theta)\frac{\pi\alpha}{2}\right) + i\sin\left(-\operatorname{sgn}(\theta)\frac{\pi\alpha}{2}\right)\right) \\ &= |\theta|^{\alpha}\left(\cos\left(\frac{\pi\alpha}{2}\right) - \operatorname{sgn}(\theta) i \sin\left(\frac{\pi\alpha}{2}\right)\right) \\ &= |\theta|^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)\left(1 - \operatorname{sgn}(\theta) i\tan\left(\frac{\pi\alpha}{2}\right)\right).\end{aligned}$$

Note that indeed $\ln\left(e^{-i\pi/2\operatorname{sgn}(\theta)}\right) = -i\pi/2\operatorname{sgn}(\theta)$ for the main branch ln of the complex logarithm. We thus have

$$-\left(-i\theta\right)^{\alpha} = -|\theta|^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)\left(1 - \operatorname{sgn}(\theta) i\tan\left(\frac{\pi\alpha}{2}\right)\right). \tag{1.16}$$

The parameter choice in (1.14) now can be read off,

$$\begin{aligned} c &= -\cos\left(\frac{\pi\alpha}{2}\right) > 0 \quad \text{and} \\ \beta &= 1. \end{aligned}$$

That $\beta = 1$ already follows from the fact that our process is spectrally positive, see e.g. [Jan11] (3.12).
For getting the Lévy-Khintchine representation, for $\alpha \in (1,2)$ we evaluate the following integral. See also [Kyp14] Exercise 1.4. We have

$$\int_0^{\infty}\left(1 - e^{i\theta x} + i\theta x\right) x^{-1-\alpha}\mathrm{d}x = -\Gamma(-\alpha)\left(-i\theta\right)^{\alpha},$$

for all $\theta \in \mathbb{R}$. Actually, the above integral identity can be extended to complex values. Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) \leq 0$, it holds

$$\int_0^\infty \left(e^{zx} - 1 - zx\right) x^{-1-\alpha} \mathrm{d}x = \Gamma(-\alpha)\,(-z)^\alpha\,. \tag{1.17}$$

This means, we have for the Lévy-Khintchine representation

$$\begin{aligned} a &= \frac{1}{\Gamma(-\alpha)(\alpha-1)}, \\ b &= 0, \\ \nu(\mathrm{d}h) &= \frac{1}{\Gamma(-\alpha)} h^{-\alpha-1} \mathbb{1}_{\{h>0\}} \mathrm{d}h. \end{aligned}$$

The rest of this section will be used to derive some basic properties of Lévy processes X with characteristic exponent given by (1.15). We directly see, e.g. with [Kle13] Theorem 15.31, that $\mathbf{E}[X_1] = 0$ and therefore the process X is a martingale. Furthermore, note the important identity

$$\mathbf{P}[X_t \geq 0] = 1 - \frac{1}{\alpha}, \tag{1.18}$$

which is independent of t due to the scaling property, see [Zol86] Section 2.6. In the literature, the quantity $\mathbf{P}[X_t \geq 0]$ is often referred to as the positivity parameter ρ. We state the following results in form of lemmas in order to make later referencing more comfortable. We begin with a remark on the long-time behaviour of X.

Lemma 1.16 *For every Lévy process X with characteristic exponent given by (1.15) we have*

$$\limsup_{t\to\infty} X_t = -\liminf_{t\to\infty} X_t = \infty \quad a.s.$$

Proof. This is a direct consequence of Theorem 7.1 and equation (1.18) of [Kyp14]. □

From the above lemma, we see that X oscillates almost surely. This has a nice effect on the stopping times τ and τ^+ which we introduced in Definition 1.12.

Corollary 1.17 *Let X be a Lévy process with characteristic exponent given by (1.15). For any $x \geq 0$ let τ^+ and τ be as in Definition 1.12. Then almost surely*

$$\tau_x < \infty \quad \text{and} \quad \tau_x^+ < \infty.$$

We are even able to give the Laplace exponents of τ^+ and τ.

Example 1.18 *Let X be a Lévy process with characteristic exponent given by (1.15). Then, $-X$ is obviously spectrally negative with characteristic exponent*

$$\Psi_{-X}(\theta) = \Psi_X(-\theta) = -(i\theta)^\alpha, \quad \text{for } \theta \in \mathbb{R}.$$

By (1.6), the Laplace exponent ψ of $-X$ is given by

$$\psi(q) = -\Psi_{-X}(-iq) = q^{\alpha}, \quad \textit{for } q \geq 0.$$

With Lemma 1.13, we see that τ^+ is a Lévy subordinator with Laplace exponent given by

$$\psi_{\tau^+}(q) = q^{1/\alpha}, \quad \textit{for } q \geq 0.$$

Here, obviously $q^{1/\alpha}$ is the right inverse to q^{α}. This means that τ^+ is a stable subordinator with index $1/\alpha \in \left(\frac{1}{2}, 1\right)$.
Returning to X, we see that $\tau_x(X) = \tau_x^+(-X)$. So, also τ is a stable subordinator with index $1/\alpha$. More precisely, $(\tau_x)_{x\geq 0}$ is a Lévy subordinator with Laplace exponent given by

$$\psi_{\tau}(q) = q^{1/\alpha}, \quad \textit{for } q \geq 0.$$

We conclude this section by studying the regularity of 0 for the open sets $(-\infty, 0)$ and $(0, \infty)$. For a definition of regularity see [Kyp14] Definition 6.4. By [Kyp14] Theorem 6.5 and the fact that X has unbounded variation, we get the following result.

Lemma 1.19 *For X a Lévy process with characteristic exponent given by (1.15), we have almost surely*

$$\tau_0 = 0 \quad \textit{and} \quad \tau_0^+ = 0.$$

Remark 1.20 *In this section we focused on the case where the stability index α is in $(1, 2)$, leaving out the case $\alpha \in (0, 1)$. This is due to the fact that in the latter case the only spectrally positive α-stable processes are subordinators and therefore the stopping times τ_x are all infinite with probability one. So, these processes cannot be used to construct processes as in [KM10].*

1.3 Correlated Brownian motion

In this section, we deal with a two-dimensional Lévy process which is particularly important for us when calculating the precise law of a two-dimensional stable process exiting from the first quadrant. This process is the two-dimensional correlated Brownian motion. We first give a precise definition of the term and then examine the connection between correlated Brownian motion and a two-dimensional Brownian motion without correlation. We then state the precise laws of a correlated Brownian motion when exiting the first quadrant $[0, \infty)^2$ and the upper half-plane $\mathbb{R} \times [0, \infty)$. After this, we also look at generalisations of the above situations, where we allow for the y-axis to be rotated by a certain angle ζ. As a general rule, we will always denote correlated Brownian motions by B and uncorrelated Brownian motions by W.

1.3.1 Correlation structure

We recall the definition of the upper complex half-plane

$$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}. \tag{1.19}$$

In many situations it will be convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$.

Definition 1.21 *Let $\rho \in [-1, 1]$. By ρ-correlated Brownian motion we mean a two-dimensional continuous Lévy process $B = (B^1, B^2)$ with $\mathbf{E}\left[B_t^i\right] = 0$ for all $i = 1, 2$ and $t \geq 0$ and covariance structure*

$$\mathbf{E}\left[B_t^i B_t^j\right] = \begin{cases} t & \textit{for } i = j \\ \rho t & \textit{for } i \neq j. \end{cases}$$

It is important to realise that there is a close relation between correlated Brownion motion and Brownian motion without correlation, as we will explain in the sequel. This method of 'subtracting out' the correlation is also practical to simulate correlated processes.
Let (B^1, B^2) be a ρ-correlated Brownian motion for some $\rho \in (-1, 1)$, starting from $(x_1, x_2) \in \mathbb{R}^2$. We define the two-dimensional process W by

$$W^1 := \frac{\rho B^2 - B^1}{\sqrt{1 - \rho^2}} \quad \text{and} \quad W^2 := B^2. \tag{1.20}$$

Then, W is an uncorrelated two-dimensional Brownian motion with starting point

$$\left(\frac{\rho x_2 - x_1}{\sqrt{1 - \rho^2}}, x_2\right) \in \mathbb{R}^2.$$

This can be seen by directly computing the covariances. It is important to note that we would have received the same results by making the choice

$$W^1 = \frac{B^1 - \rho B^2}{\sqrt{1 - \rho^2}}.$$

It is also possible to proceed the other way round. Let W be a two-dimensional uncorrelated Brownian motion starting from $(x_1, x_2) \in \mathbb{R}^2$ and $\rho \in [-1, 1]$. We then define the two-dimensional process B by

$$B^1 := \rho W^2 - \sqrt{1 - \rho^2} W^1 \quad \text{and} \quad B^2 := W^2. \tag{1.21}$$

Then, B is a ρ-correlated Brownian motion with start in $(\rho x_2 - \sqrt{1 - \rho^2} x_1, x_2)$.

1.3.2 Correlated Brownian motion exiting first quadrant and upper half-plane

We now state a result about correlated Brownian motion exiting from the first quadrant. The exit measure has one-dimensional Lebesgue densities on the axes

$$\begin{aligned} E_1 &:= \{(r,0) \in \mathbb{R}^2 : r \geq 0\} \quad \text{and} \\ E_2 &:= \{(0,r) \in \mathbb{R}^2 : r \geq 0\}, \end{aligned}$$

given by the following equations, see [DM11] equation (2.16) or in [BDE11] the proof of Theorem 5.1. Here,

$$\tau := \inf_{t>0} \left\{B_t^1 B_t^2 = 0\right\}$$

and $(u,v) \in (0,\infty)^2$ denotes the starting point.

$$\begin{aligned} \mathbf{P}_{u,v}\left[B_\tau^1 = 0, B_\tau^2 \in \mathrm{d}r\right] &= \frac{1}{\pi\sqrt{1-\rho^2}^{\pi/\theta}} \frac{\frac{\pi}{\theta} r^{\pi/\theta-1} z_2}{z_2^2 + \left(\left(\frac{r}{\sqrt{1-\rho^2}}\right)^{\pi/\theta} + z_1\right)^2} \mathrm{d}r \quad \text{and} \\ \mathbf{P}_{u,v}\left[B_\tau^1 \in \mathrm{d}r, B_\tau^2 = 0\right] &= \frac{1}{\pi\sqrt{1-\rho^2}^{\pi/\theta}} \frac{\frac{\pi}{\theta} r^{\pi/\theta-1} z_2}{z_2^2 + \left(\left(\frac{r}{\sqrt{1-\rho^2}}\right)^{\pi/\theta} - z_1\right)^2} \mathrm{d}r, \end{aligned}$$

where $z_1 = z_1(u,v)$ and $z_2 = z_2(u,v)$ are defined as

$$\begin{aligned} z_1 &:= \left(u^2 + \frac{(v-u\rho)^2}{1-\rho^2}\right)^{\frac{\pi}{2\theta}} \cos\left(\frac{\pi}{\theta}\left(\arctan\left(\frac{v-u\rho}{\sqrt{1-\rho^2}u}\right) + \arctan\left(\frac{\rho}{\sqrt{1-\rho^2}}\right)\right)\right), \\ z_2 &:= \left(u^2 + \frac{(v-u\rho)^2}{1-\rho^2}\right)^{\frac{\pi}{2\theta}} \sin\left(\frac{\pi}{\theta}\left(\arctan\left(\frac{v-u\rho}{\sqrt{1-\rho^2}u}\right) + \arctan\left(\frac{\rho}{\sqrt{1-\rho^2}}\right)\right)\right), \end{aligned}$$

cf. [DM11] equation (2.17). The parameter θ is defined by

$$\begin{aligned} \theta &= \frac{\pi}{2} + \arctan\left(\frac{\rho}{\sqrt{1-\rho^2}}\right) \\ &= \arccos(-\rho). \end{aligned}$$

We denote the exit measure from $[0,\infty)^2$ of a two dimensional ρ-correlated Brownian motion B by Q^ρ,

$$Q^\rho := \mathcal{L}(B_\tau).$$

Imitating the proof of the above result, see e.g. in [BDE11] the proof of Theorem 5.1, we can also explicitly compute where a ρ-correlated Brownian motion leaves $\mathbb{H}$ when starting from $(0,t) \in \mathbb{H}$ for some $t > 0$. To this end, we make use of the well-known fact that the exit law from $\mathbb{H}$ of a two-dimensional Brownian motion W is Cauchy distributed, cf. e.g. [Kle13] Exercise 25.4.1. In particular, for $v > 0$ and $u, r \in \mathbb{R}$, we have

$$\begin{aligned} \mathbf{P}_{(u,v)}\left[W^1_{\tau^2(W)} \leq r\right] &= \frac{1}{\pi}\int_{-\infty}^{r-u} \frac{v}{v^2+x^2}\,\mathrm{d}x \\ &= \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{r-u}{v}\right), \end{aligned} \tag{1.22}$$

where

$$\tau^2(W) = \inf_{t>0}\{W_t^2 = 0\}$$

is the first exit time of W from $\mathbb{H}$. We have the following.

Lemma 1.22 *Let $\rho \in (-1,1)$ and B be a ρ-correlated Brownian motion starting at $(0,t)$ for some $t > 0$. Then, for all $A \in \mathcal{B}(\mathbb{R})$,*

$$\mathbf{P}_{(0,t)}[B_{\tau^2} \in A] = \mathbf{P}\left[Y_t - \rho t \in A\right],$$

where Y is a Cauchy process with parameter $\sqrt{1-\rho^2}$.

Proof. Define the two-dimensional Lévy process W as in (1.20), then, as already mentioned, W is a two-dimensional Brownian motion without correlation and with starting point

$$\left(\frac{\rho t}{\sqrt{1-\rho^2}}, t\right).$$

Let $\tau^2(B)$ and $\tau^2(W)$ be the first exit times from $\mathbb{H}$ of B and W respectively. Then, $\tau^2(B) = \tau^2(W)$ almost surely, as the second component is the same for both processes. We then get for $r \in \mathbb{R}$ with (1.22)

$$\begin{aligned} \mathbf{P}_{(0,t)}\left[B^1_{\tau^2(B)} \leq r\right] &= \mathbf{P}_{\left(\frac{\rho t}{\sqrt{1-\rho^2}}, t\right)}\left[-\sqrt{1-\rho^2}W^1_{\tau^2(W)} \leq r\right] \\ &= \mathbf{P}_{\left(\frac{\rho t}{\sqrt{1-\rho^2}}, t\right)}\left[W^1_{\tau^2(W)} \geq -\frac{r}{\sqrt{1-\rho^2}}\right] \\ &= \frac{1}{\pi}\int_{\frac{-r-\rho t}{\sqrt{1-\rho^2}}}^{\infty} \frac{t}{t^2+x^2}\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\infty}^{\frac{r+\rho t}{\sqrt{1-\rho^2}}} \frac{t}{t^2+x^2}\,\mathrm{d}x. \end{aligned}$$

On the other hand, if Y is a Cauchy process with parameter $\sqrt{1-\rho^2}$, we have already seen in Example 1.4 that

$$\begin{aligned} \mathbf{P}\left[Y_t - \rho t \leq r\right] &= \frac{1}{\pi}\int_{-\infty}^{r+t\rho} \frac{t\sqrt{1-\rho^2}}{\left(t\sqrt{1-\rho^2}\right)^2 + x^2}\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\infty}^{\frac{r+\rho t}{\sqrt{1-\rho^2}}} \frac{t}{t^2+x^2}\,\mathrm{d}x \end{aligned}$$

and the lemma is proved. □

We want to stress that, due to the spatial homegeneity of the Lévy process B, we can easily compute the exit law starting at any other point $x \in \mathbb{H}$ by simply replacing A by $A - x_1$, namely

$$\mathbf{P}_x\left[B_{\tau^2} \in A\right] = \mathbf{P}_{(0,x_2)}\left[B_{\tau^2} \in A - x_1\right].$$

This is, for any $x_1 \in \mathbb{R}$ and $x_2 > 0$, the law of the stopped process $\mathcal{L}_x(B_{\tau^2})$ is a Cauchy distribution with scale parameter $x_2\sqrt{1-\rho^2}$ and median $x_1 - \rho x_2$.

1.3.3 Correlated Brownian motion exiting the rotated upper half-plane and the wedge of angle ζ

We now proceed with generalising the results from the last section to the case when we allow the y-axis to be turned by a certain angle $\zeta - \pi$, where for convenience ζ is restricted to $(\pi/2, \pi)$. By identifying the connection to uncorrelated Brownian motion, we will show that this again leads to a Cauchy distribution and give the parameters in terms of ρ, ζ and the starting point x. We conclude this section by stating the precise form of the densities of the exit measure $Q^{\rho,\zeta}$ of a ρ-correlated Brownian motion exiting the wedge of angle ζ.
Although the ideas are basically the same as in Section 1.3.2, the calculations become a bit more involved.

We fix $\zeta \in [0, \pi]$ and $\rho \in (-1, 1)$. Recall that B denotes a ρ-correlated Brownian motion. Set

$$\beta^\zeta := (\cos(\zeta), \sin(\zeta)) \tag{1.23}$$

and define the rotated half-plane by

$$\mathbb{H}^\zeta := \left\{x \in \mathbb{R}^2 : \arg(x) \in [\zeta - \pi, \zeta]\right\},$$

where naturally $\arg(x) = \arg(x_1 + ix_2)$. Then, obviously, $\mathbb{H} = \mathbb{H}^\pi$. We denote the wedge of angle ζ with $\mathbb{W}(\zeta)$,

$$\mathbb{W}(\zeta) := \{x \in \mathbb{R}^2 : \arg(x) \in [0, \zeta]\} = \mathbb{H}^\pi \cap \mathbb{H}^\zeta.$$

See Figure 1.2 for a clarification. We furthermore define for fixed ρ the shearing operator $M : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$Mx := \begin{pmatrix} \frac{1}{\sqrt{1-\rho^2}} & \frac{-\rho}{\sqrt{1-\rho^2}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1 - \rho x_2}{\sqrt{1-\rho^2}} \\ x_2 \end{pmatrix}.$$

As we have seen in Section 1.3.1, the stochastic process

$$W := MB = \begin{pmatrix} \frac{B^1 - \rho B^2}{\sqrt{1-\rho^2}} \\ B^2 \end{pmatrix}$$

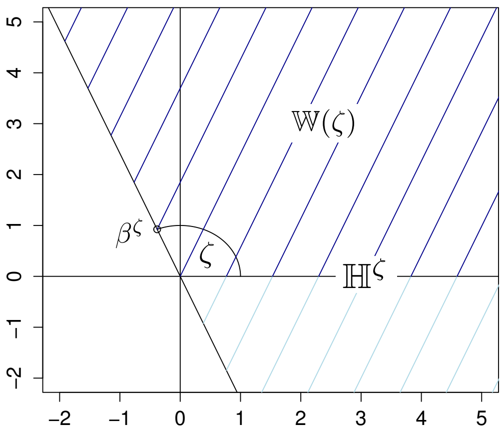


Figure 1.2: The objects $\zeta, \beta^\zeta, \mathbb{W}(\zeta)$ and $\mathbb{H}^\zeta$ for $\zeta = \frac{5}{8}\pi$

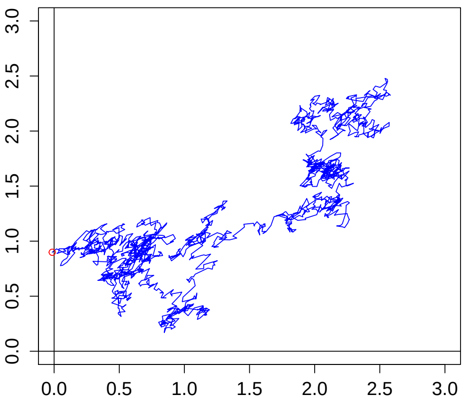
is an uncorrelated Brownian motion with starting point MB_0. It is an easy exercise to verify that for $\zeta \in (0, \pi)$ and $\rho \in (-1, 1)$ we have the relation

$$x \in \mathbb{H}^\zeta \quad \text{if and only if} \quad Mx \in \mathbb{H}^{\bar{\zeta}},$$

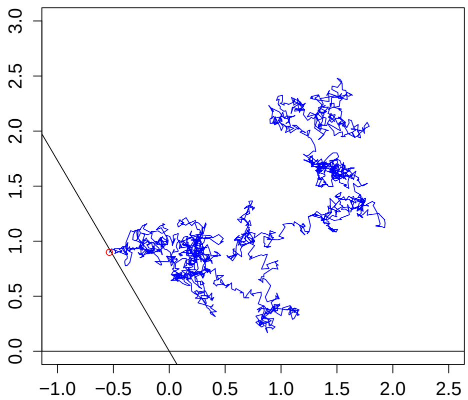
where

$$\bar{\zeta} := \arg\left(M\beta^\zeta\right) = \frac{\pi}{2} - \arctan\left(\frac{\cos(\zeta) - \rho\sin(\zeta)}{\sin(\zeta)\sqrt{1-\rho^2}}\right) \in (0, \pi).$$

Figure 1.3: Illustration with $\rho = 0.5$ and $\zeta = \frac{\pi}{2}$.



(a) The ρ-correlated BM B while leaving $\mathbb{W}(\zeta)$



(b) The uncorrelated BM MB while leaving $\mathbb{W}(\bar{\zeta})$

As the second coordinate is invariant under M, we see from this that

$$x \in \mathbb{W}(\zeta) \quad \text{if and only if} \quad Mx \in \mathbb{W}(\bar{\zeta}).$$

We define further, for given B started at $x \in \mathbb{H}^\zeta$ with ζ as above, the stopping times

$$\sigma^\zeta := \sigma_x^\zeta(B) := \inf_{s>0}\{B_s \notin \mathbb{H}^\zeta\}, \tag{1.24}$$

$$\tau^\zeta := \tau_x^\zeta(B) := \inf_{s>0}\{B_s \notin \mathbb{W}(\zeta)\} = \sigma_x^\zeta(B) \wedge \tau_{x_2}(B^2). \tag{1.25}$$

Note that if $\zeta = \frac{\pi}{2}$ then this definition coincides with τ from Section 1.3.2. Obviously, we have $\sigma^\zeta < \infty$ almost surely and, as B is continuous,

$$B_{\sigma^\zeta} \in \partial\mathbb{H}^\zeta = \mathbb{R}\beta^\zeta.$$

We also get that $\tau^\zeta < \infty$ almost surely and that

$$B_{\tau^\zeta} \in \partial\mathbb{W}(\zeta) = [0,\infty)\beta^\zeta \cup [0,\infty) \times \{0\}.$$

We now have the following important result. The restriction on the range of ζ is due to convenience, as in this case $\sin(\zeta) > 0$. However, this restriction could easily be removed. For a visualisation of the idea, see Figure 1.3.

Lemma 1.23 *Let $\zeta \in (\pi/2, \pi)$, $\rho \in (-1,1)$ and $x \in \mathbb{H}^\zeta$. Let further B be a ρ-correlated Brownian motion starting at x and set $W := MB$. Then, we have almost surely for all $r \in \mathbb{R}$ and $t > 0$*

$$B_{\sigma^\zeta} \in (-\infty, r]\beta^\zeta \Leftrightarrow W_{\sigma^{\bar{\zeta}}} \in (-\infty, cr]\beta^{\bar{\zeta}} \tag{1.26}$$

and

$$B_{\tau^\zeta} \in [0,t)\beta^\zeta \Leftrightarrow W_{\tau^{\bar{\zeta}}} \in [0,ct)\beta^{\bar{\zeta}}, \tag{1.27}$$

$$B_{\tau^\zeta} \in [0,t) \times \{0\} \Leftrightarrow W_{\tau^{\bar{\zeta}}} \in \left[0, \frac{t}{\sqrt{1-\rho^2}}\right) \times \{0\}, \tag{1.28}$$

where

$$\begin{aligned} c := \|M\beta^\zeta\|_2 &= \frac{\sqrt{1-2\rho\sin(\zeta)\cos(\zeta)}}{\sqrt{1-\rho^2}} \\ &= \sqrt{\frac{1-\rho\sin(2\zeta)}{1-\rho^2}} \in (0,\infty). \end{aligned}$$

Proof. All equivalences are to be understood almost surely. First, note that

$$c\beta^{\bar{\zeta}} = M\beta^\zeta = \begin{pmatrix} \frac{\cos(\zeta)-\rho\sin(\zeta)}{\sqrt{1-\rho^2}} \\ \sin(\zeta) \end{pmatrix}.$$

We set $\bar{x} := Mx = W_0$. As $B \in \mathbb{H}^\zeta \Leftrightarrow W \in \mathbb{H}^{\bar{\zeta}}$ and $B \in \mathbb{W}(\zeta) \Leftrightarrow W \in \mathbb{W}(\bar{\zeta})$, we also have for the stopping times

$$\begin{aligned} \sigma_x^\zeta(B) &= \sigma_{\bar{x}}^{\bar{\zeta}}(W) \quad \text{and} \\ \tau_x^\zeta(B) &= \tau_{\bar{x}}^{\bar{\zeta}}(W). \end{aligned} \tag{1.29}$$

Furthermore, we get by the linearity of M, for r and t like above and $s \geq 0$,

$$\begin{aligned} B_s = r\beta^\zeta &\Leftrightarrow W_s = cr\beta^{\bar{\zeta}} \quad \text{and} \\ B_s = (t, 0) &\Leftrightarrow W_s = t\left(\frac{1}{\sqrt{1-\rho^2}}, 0\right). \end{aligned}$$

Together with (1.29) this proves the assertion. $\square$

Corollary 1.24 *Let $\zeta \in [\pi/2, \pi)$, $\rho \in (-1, 1)$ and $x \in \mathbb{H}^\zeta$. Let B be a ρ-correlated Brownian motion started from x. There exists a probability density function (p.d.f.) $h^\zeta : \mathbb{R} \to [0, \infty)$, such that for all $r \in \mathbb{R}$*

$$\mathbf{P}_x\left[B_{\sigma^\zeta} \in (-\infty, r]\beta^\zeta\right] = \int_{-\infty}^r h^\zeta(t)\,dt.$$

More precisely, $h^\zeta(t) = \frac{1}{\pi}\frac{s}{s^2+(t-x_0)^2}$ is the density function of a Cauchy distribution with scale parameter s and median x_0 given by

$$\begin{aligned} s &= \frac{\sqrt{1-\rho^2}(x_1\sin(\zeta) - x_2\cos(\zeta))}{1 - 2\rho\sin(\zeta)\cos(\zeta)} \quad \textit{and} \\ x_0 &= \frac{x_1(\cos(\zeta) - \rho\sin(\zeta)) + x_2(\sin(\zeta) - \rho\cos(\zeta))}{1 - 2\rho\sin(\zeta)\cos(\zeta)}. \end{aligned}$$

Proof. We first mention that the case $\zeta = \frac{\pi}{2}$ directly follows from Lemma 1.22 and the considerations thereafter. Here, the roles of x_1 and x_2 have to be exchanged, as we consider hitting the y-axis.
Set $W := MB$. Then, as mentioned above, W is an uncorrelated Brownian motion started from $\bar{x} = Mx \in \mathbb{H}^{\bar{\zeta}}$. We now rotate the whole picture by $-\bar{\zeta}$ so that the line $\mathbb{R}\beta^{\bar{\zeta}}$ is isometrically mapped to the x-axis. This means, we define the rotation matrix

$$\begin{aligned} A_{-\bar{\zeta}} &= \begin{pmatrix} \cos(\bar{\zeta}) & \sin(\bar{\zeta}) \\ -\sin(\bar{\zeta}) & \cos(\bar{\zeta}) \end{pmatrix} = \frac{1}{\sqrt{1+a^2}}\begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix} \\ &= \frac{1}{\sqrt{1-2\rho\sin(\zeta)\cos(\zeta)}}\begin{pmatrix} \cos(\zeta) - \rho\sin(\zeta) & \sin(\zeta)\sqrt{1-\rho^2} \\ -\sin(\zeta)\sqrt{1-\rho^2} & \cos(\zeta) - \rho\sin(\zeta) \end{pmatrix}, \end{aligned}$$

where

$$a := \frac{\cos(\zeta) - \rho\sin(\zeta)}{\sin(\zeta)\sqrt{1-\rho^2}}.$$

Then, $\bar{W} := AW$ is again an uncorrelated Brownian motion starting from

$$y := A_{-\bar{\zeta}}\bar{x} = \frac{1}{\sqrt{1-2\rho\sin(\zeta)\cos(\zeta)}} \begin{pmatrix} \frac{x_1(\cos(\zeta)-\rho\sin(\zeta))+x_2(\sin(\zeta)-\rho\cos(\zeta))}{\sqrt{1-\rho^2}} \\ \cos(\zeta)x_2 - \sin(\zeta)x_1 \end{pmatrix}.$$

It is important to note that the rotation maps $\mathbb{H}^{\bar{\zeta}}$ to the lower half-plane $\mathbb{H}^0$, which explains why we have to consider $-y_2 \geq 0$ instead of y_2. We end up with

$$\begin{aligned} \mathbf{P}_x\left[B_{\sigma^\zeta} \in (-\infty, r]\beta^\zeta\right] &= \mathbf{P}_{\bar{x}}\left[W_{\sigma^{\bar{\zeta}}} \in (-\infty, cr]\beta^{\bar{\zeta}}\right] \\ &= \mathbf{P}_y\left[\bar{W}^1_{\tau^2} \in (-\infty, cr] \times \{0\}\right] \\ &= \frac{1}{\pi}\int_{-\infty}^{cr-y_1} \frac{-y_2}{y_2^2+t^2}\mathrm{d}t \\ &= \frac{1}{\pi}\int_{-\infty}^{r} \frac{-cy_2}{y_2^2+(ct-y_1)^2}\mathrm{d}t. \end{aligned}$$

The claim now follows by setting $x_0 = y_1/c$ and $s = -y_2/c$. □

Corollary 1.25 *Let $\zeta \in [\pi/2, \pi)$, $\rho \in (-1,1)$ and $x \in \mathbb{W}(\zeta)$. Let further B be a ρ-correlated Brownian motion starting from $x \in \mathbb{W}(\zeta)$. There exist density functions $f_x^\zeta, \bar{f}_x^\zeta : [0,\infty) \to [0,\infty)$, such that for all $t > 0$*

$$\begin{aligned} \mathbf{P}_x\left[B_{\tau^\zeta} \in [0,t)\beta^\zeta\right] &= \int_0^t \bar{f}_x^\zeta(s)\,ds \quad \textit{and} \\ \mathbf{P}_x\left[B_{\tau^\zeta} \in [0,t) \times \{0\}\right] &= \int_0^t f_x^\zeta(s)\,ds. \end{aligned}$$

The functions $f^\zeta, \bar{f}^\zeta$ are given by

$$\begin{aligned} \bar{f}_x^\zeta(t) &= \frac{1}{\pi}\frac{rt^{r-1}\bar{z}_2}{\bar{z}_2^2+(t^r+\bar{z}_1)^2} \quad \textit{and} \\ f_x^\zeta(t) &= \frac{1}{\pi}\frac{rt^{r-1}z_2}{z_2^2+(t^r-z_1)^2}, \end{aligned}$$

where

$$r := \frac{\pi}{\bar{\zeta}} = \frac{\pi}{\frac{\pi}{2} - \arctan\left(\frac{\cos(\zeta)-\rho\sin(\zeta)}{\sin(\zeta)\sqrt{1-\rho^2}}\right)} \in (1,\infty)$$

and the parameters $\bar{z}, z$ are defined as

$$\bar{z} := \left(\frac{Mx}{c}\right)^r \qquad z := \left(\sqrt{1-\rho^2}Mx\right)^r,$$

i.e., for $i = 1,2$,

$$\begin{aligned} \bar{z}_1 &= \left(\frac{x_1^2+x_2^2-2\rho x_1x_2}{1-2\rho\sin(\zeta)\cos(\zeta)}\right)^{\frac{r}{2}}\cos(r\varphi), \\ \bar{z}_2 &= \left(\frac{x_1^2+x_2^2-2\rho x_1x_2}{1-2\rho\sin(\zeta)\cos(\zeta)}\right)^{\frac{r}{2}}\sin(r\varphi), \\ z_1 &= \left(x_1^2+x_2^2-2\rho x_1x_2\right)^{\frac{r}{2}}\cos(r\varphi), \\ z_2 &= \left(x_1^2+x_2^2-2\rho x_1x_2\right)^{\frac{r}{2}}\sin(r\varphi), \end{aligned}$$

with

$$\varphi = \arg(Mx) = \frac{\pi}{2} - \arctan\left(\frac{x_1 - \rho x_2}{x_2\sqrt{1-\rho^2}}\right).$$

Proof. The case $\zeta = \frac{\pi}{2}$ is considered already in [DM11]. See also the subsequent remark.

With Lemma 1.23, we know that for the uncorrelated Brownian motion $W := MB$ we have

$$\mathbf{P}_x\left[B_{\tau^\zeta} \in [0,t)\beta^\zeta\right] = \mathbf{P}_{\bar{x}}\left[W_{\tau^{\bar{\zeta}}} \in [0,ct)\beta^{\bar{\zeta}}\right].$$

We now apply the conformal map $x \mapsto x^r$, interpreted as a map $\mathbb{C} \to \mathbb{C}$, to the process W. We then obtain, due to the conformal invariance of Brownian motion, again a Brownian motion started at $\bar{x}^r$, which we will denote by $\bar{W} := W^r$. We see that with φ as above

$$y := \bar{x}^r = |\bar{x}|^r\left(\cos(r\varphi), \sin(r\varphi)\right),$$

where

$$|\bar{x}| = \frac{\sqrt{x_1^2 + x_2^2 - 2\rho x_1 x_2}}{\sqrt{1-\rho^2}}.$$

Note that $x_2 > 0$. It is also important to note that $x \mapsto x^r$ maps the wedge $\mathbb{W}(\bar{\zeta})$ to the upper half-plane $\mathbb{H}$. We already know that the exit distribution from $\mathbb{H}$ is Cauchy. So, we compute

$$\begin{aligned}
\mathbf{P}_{\bar{x}}\left[W_{\tau^{\bar{\zeta}}} \in [0,ct)\beta^{\bar{\zeta}}\right] &= \mathbf{P}_y\left[\bar{W}^1_{\tau^2} \in (-(ct)^r, 0] \times \{0\}\right] \\
&= \frac{1}{\pi}\int_{-(ct)^r - y_1}^{-y_1} \frac{y_2}{y_2^2 + s^2}\mathrm{d}s \\
&= \frac{1}{\pi}\int_0^t \frac{r c^r s^{r-1} y_2}{y_2^2 + (c^r s^r + y_1)^2}\mathrm{d}s.
\end{aligned}$$

Now, the first equality follows with $\bar{z}_i = \frac{y_i}{c^r}$.

We furthermore get from Lemma 1.23

$$\mathbf{P}_x\left[B_{\tau^\zeta} \in [0,t) \times \{0\}\right] = \mathbf{P}_{\bar{x}}\left[W_{\tau^{\bar{\zeta}}} \in \left[0, \frac{t}{\sqrt{1-\rho^2}}\right) \times \{0\}\right].$$

With the same notation as above,

$$\begin{aligned}
\mathbf{P}_x\left[B_{\tau^\zeta} \in [0,t) \times \{0\}\right] &= \mathbf{P}_y\left[\bar{W}^1_{\tau^2} \in \left[0, \left(\frac{t}{\sqrt{1-\rho^2}}\right)^r\right) \times \{0\}\right] \\
&= \frac{1}{\pi}\int_{-y_1}^{\left(t/\sqrt{1-\rho^2}\right)^r - y_1} \frac{y_2}{y_2^2 + s^2}\mathrm{d}s \\
&= \frac{1}{\pi}\int_0^t \frac{r y_2 s^{r-1}\left(1-\rho^2\right)^{\frac{-r}{2}}}{y_2^2 + \left(s^r(1-\rho^2)^{\frac{-r}{2}} - y_1\right)^2}\mathrm{d}s.
\end{aligned}$$

Therefore, the claim follows with $z_i = y_i(1-\rho^2)^{\frac{r}{2}}$. ∎

Remark 1.26 *For $\zeta = \frac{\pi}{2}$ we have the well-known result from [DM11] equation (2.16). Note in this context that for $x_1, x_2 > 0$*

$$\arctan\left(\frac{x_2 - \rho x_1}{x_1\sqrt{1-\rho^2}}\right) + \arctan\left(\frac{\rho}{\sqrt{1-\rho^2}}\right) = \frac{\pi}{2} - \arctan\left(\frac{x_1 - \rho x_2}{x_2\sqrt{1-\rho^2}}\right).$$

Chapter 2

On the exit measure of an SPMI leaving the first quadrant

In this chapter, we examine the exit measure Q^α of a two-dimensional spectrally positive α-stable martingale with independent coordinates (SPMI process for short), while exiting from the first quadrant $[0,\infty)^2$. After accurately defining the probability measure Q^α, we prove an important scaling relation which stems from the scaling property of stable processes and, thereafter, derive moment bounds for Q^α. We then take a closer look at an SPMI exiting from the upper half-plane $\mathbb{H}$. By the results presented in the first chapter, we can explicitly give the distribution of the exit points, which will be crucial for the considerations in Chapter 3.
In the second part of this chapter, we generalise the most important results to the case where the y-axis is rotated by a certain angle. To this end, we give the distribution of X on first exiting the rotated upper half-plane $\mathbb{H}^\zeta$.
Let $\alpha \in (1,2]$ and recall that we defined

$$E = [0,\infty)^2 \backslash (0,\infty)^2.$$

For $x \in \mathbb{R}^2$ we generally mean by $|x|$ the 1-norm, $|x| = |x_1| + |x_2|$.

2.1 Definition of Q^α and first properties

This section is devoted to introduce the exit measure Q^α of special α-stable Lévy processes while exiting from the first quadrant $[0,\infty)^2$. After fixing notation, we first show some basic properties of Q^α and then provide moment estimates and a special scaling relation which is ascribed to the scaling relation for stable processes. We end the section by defining the harmonic measure $Q^{\alpha,\zeta}$ of an SPMI while leaving the wedge of angle $\zeta \in [\pi/2,\pi)$. The methods we use are quite robust, so it should be possible to transfer most of them to more general processes.

2.1.1 Definition of Q^α

Definition 2.1 *An $\mathbb{R}^2$-valued stochastic process $X = (X^1, X^2)$, where X^1, X^2 are independent, α-stable, spectrally positive martingales starting from the origin, is called an SPMI process.*

Another equivalent way of defining an SPMI process X is via the characteristic exponent of its independent coordinates, which is given by (1.15),

$$\Psi_{X^j}(\theta) = -\left(-i\theta\right)^\alpha,$$

and uniquely determines the law of X.
We now introduce the first exit times of an SPMI from the first quadrant.

Definition 2.2 *Let X be an SPMI and $x \in [0,\infty)^2$. We set*

$$\tau_x = \tau_x(X) := \inf_{t>0}\{X_t \in E - x\} = \tau_{x_1}(X^1) \wedge \tau_{x_2}(X^2).$$

The stopping times $\tau(X^i)$ are to be understood like in Definition 1.12. By the considerations following Definition 1.12, it is clear that, as the X^i creep downwards, $X_{\tau_x} + x \in E$.
It is important to note that, due to the spatial homogeneity of the Lévy process X, we can more conveniently think of τ_x as the first time that an SPMI starting from $x \in [0,\infty)^2$ hits the axes E. Depending on the context, we will use both equivalent definitions. To keep notation as simple as possible, we define an auxiliary process D as follows. Here, for $x, y \in \mathbb{R}^2$, we mean by $x \leq y$ that $x_i \leq y_i$ for both $i = 1, 2$.

Definition 2.3 *For X an SPMI and $x \in [0,\infty)^2$ let*

$$D_x := X_{\tau_x} + x \in E.$$

We further define the σ-field $\mathcal{F}^D_x$ by

$$\mathcal{F}^D_x := \sigma\left(D_y : y \leq x\right).$$

We are now ready to intoduce our main object, the exit measure Q^α.

Definition 2.4 *Let X be an SPMI of index $\alpha \in (1, 2]$ and let $x \in [0,\infty)^2$. We define the probability measure Q^α_x on $(E, \mathcal{B}(E))$ by*

$$Q^\alpha_x := \mathcal{L}\left(D_x\right).$$

This is for all $A \in \mathcal{B}(E)$,

$$Q^\alpha_x(A) = \mathbf{P}\left[D_x \in A\right] = \mathbf{P}\left[X_{\tau_x} + x \in A\right].$$

We will occasionally suppress the indices α and x. By Corollary 1.17, it is clear that X will almost surely hit E in finite time, so Q^α is indeed a well-defined probability measure on $(E, \mathcal{B}(E))$. From Lemma 1.19 we see, as $\tau_x(X) \leq \tau_{x_i}(X^i)$ for $i = 1, 2$, that $Q^\alpha_x = \delta_x$ for all $x \in E$.

We continue by giving a series of simple results about Q^α. In the subsequent lemma we understand the space of all probability measures on $(E, \mathcal{B}(E))$ as topologised by the topology of weak convergence.

Lemma 2.5 *The mapping $x \mapsto Q^\alpha_x$ is continuous: for every $\varepsilon > 0$ and $x \in [0,\infty)^2$ and every bounded and continuous function $f : E \to \mathbb{R}$ there is a $\delta > 0$ such that, for all $y \in [0,\infty)^2$ with $|x-y| < \delta$, we have*

$$\left| \int_E f(z) Q^\alpha_x(dz) - \int_E f(z) Q^\alpha_y(dz) \right| < \varepsilon.$$

Proof. Due to the Portemanteau theorem, see [Kle13] Theorem 13.16, it is enough to show the statement for all bounded Lipschitz functions f. Let $f : E \to \mathbb{R}$ thus be Lipschitz with constant $L > 0$ and bounded in absolute value by some $M > 0$. Let further X be an SPMI started from the origin and fix $\varepsilon > 0$.
We show that we can even find δ independent of x. Suppose for now we have already shown that

$$\begin{gathered} \text{there is a } \delta_1 > 0 \text{ such that } \mathbf{P}\left[\left| X_{\tau_x} - X_{\tau_y} \right| \geq \frac{\varepsilon}{4L} \right] \leq \frac{\varepsilon}{4M} \\ \text{for all } x, y \text{ with } |x-y| < \delta_1. \end{gathered} \tag{2.1}$$

For $x, y \in [0,\infty)^2$ we define

$$A := A_{x,y} := \left\{ \left| X_{\tau_x} - X_{\tau_y} \right| < \frac{\varepsilon}{4L} \right\}.$$

Then, for $\delta := \delta_1 \wedge \frac{\varepsilon}{4L} > 0$ and all $|x-y| < \delta$,

$$\begin{aligned} \left| \int_E f(z) Q^\alpha_x(\mathrm{d}z) - \int_E f(z) Q^\alpha_y(\mathrm{d}z) \right| &= \left| \mathbf{E}[f(X_{\tau_x} + x)] - \mathbf{E}[f(X_{\tau_y} + y)] \right| \\ &\leq \mathbf{E}\left[\mathbb{1}_A \left| f(X_{\tau_x} + x) - f(X_{\tau_y} + y) \right| \right] \\ &\quad + \mathbf{E}\left[\mathbb{1}_{A^c} \left| f(X_{\tau_x} + x) - f(X_{\tau_y} + y) \right| \right] \\ &\leq \frac{\varepsilon}{2} \mathbf{P}[A] + 2M \mathbf{P}[A^c] \\ &\leq \frac{\varepsilon}{2} + 2M \frac{\varepsilon}{4M} = \varepsilon. \end{aligned}$$

The first part of the second inequality is valid since on A we have

$$\left| X_{\tau_x} + x - (X_{\tau_y} + y) \right| \leq \left| X_{\tau_x} - X_{\tau_y} \right| + |x-y| \leq \frac{\varepsilon}{2L}$$

and f is Lipschitz. The last inequality is a consequence of (2.1). Thus, the statement is shown once we proved (2.1).
To this end, first note that for $C > 0$, $z \in E$ and $x \in [0,\infty)^2$ we have

$$\mathbf{P}\left[\tau_{x+z}(X) > C \right] \leq \mathbf{P}\left[\tau_{x_1}(X^1) > C \right] + \mathbf{P}\left[\tau_{x_2}(X^2) > C \right]. \tag{2.2}$$

This follows without loss of generality for $z = (0, r)$ by

$$\tau_{x+z} = \tau_{(x_1, x_2+r)} = \tau_{x_1} \wedge \tau_{x_2+r} \leq \tau_{x_1}$$

and therefore

$$\mathbf{P}\left[\tau_{x+z} > C\right] \leq \mathbf{P}\left[\tau_{x_1} > C\right].$$

We want to stress that it is enough to show the claim for $y \geq x$, as the general case follows by the triangle inequality with $z := x \wedge y$. Let thus $y \in [0,\infty)^2$ with $y \geq x$. Then, also $\tau_y \geq \tau_x$ and we get with the strong Markov property

$$\begin{aligned} \mathbf{P}\left[|\tau_y - \tau_x| > C\right] &= \int_E \mathbf{P}\left[|\tau_y - \tau_x| > C | X_{\tau_x} + x = z\right] \mathbf{P}\left[X_{\tau_x} + x \in \mathrm{d}z\right] \\ &= \int_E \mathbf{P}\left[\tau_{y-x+z} > C\right] \mathbf{P}\left[X_{\tau_x} + x \in \mathrm{d}z\right] \\ &\leq \int_E \left(\mathbf{P}\left[\tau_{y_1-x_1} > C\right] + \mathbf{P}\left[\tau_{y_2-x_2} > C\right]\right) \mathbf{P}\left[X_{\tau_x} + x \in \mathrm{d}z\right] \\ &= \mathbf{P}\left[\tau_{y_1-x_1} > C\right] + \mathbf{P}\left[\tau_{y_2-x_2} > C\right] \quad \xrightarrow{y \downarrow x} 0. \end{aligned} \tag{2.3}$$

Here we used the fact that τ_{x_i} is right-continuous almost surely with $\tau_0 = 0$ and therefore also $\tau_{x_i} \xrightarrow{\text{prob.}} 0$ as $x_i \downarrow 0$.
We furthermore get from Doob's martingale inequality and the scaling relation for stable processes that for all $C > 0$

$$\mathbf{P}\left[\sup_{t \leq T} |X_t^i| \geq C\right] \leq \frac{\mathbf{E}[|X_T^i|]}{C} \leq \frac{T^{1/\alpha}\mathbf{E}[|X_1^i|]}{C} \quad \xrightarrow{T \downarrow 0} 0. \tag{2.4}$$

Note that X^i is a martingale with $\mathbf{E}[|X_1^i|] < \infty$. Thus, $|X^i|$ is a non-negative submartingale, see [Kle13] Theorem 9.35.
We come to the proof of (2.1). First, choose $T > 0$ small enough such that for $i = 1, 2$,

$$\mathbf{P}\left[\sup_{t \leq T} |X_t^i| \geq \frac{\varepsilon}{8L}\right] \leq \frac{\varepsilon}{16M}.$$

This is possible due to (2.4). Then, choose $\delta_1 > 0$ such that for all x, y with $y \geq x$ and $|x - y| < \delta_1$ we have

$$\mathbf{P}\left[|\tau_x - \tau_y| > T\right] \leq \frac{\varepsilon}{8M}.$$

This is possible due to (2.3). Putting everything together, we get for $x \leq y$ with

$|x-y|<\delta_1$

$$\begin{aligned}
\mathbf{P}\left[|X_{\tau_x}-X_{\tau_y}|\geq\frac{\varepsilon}{4L}\right]=&\mathbf{P}\left[|X_{\tau_x}-X_{\tau_y}|\geq\frac{\varepsilon}{4L}\text{ and }|\tau_x-\tau_y|>T\right]\\
&+\mathbf{P}\left[|X_{\tau_x}-X_{\tau_y}|\geq\frac{\varepsilon}{4L}\text{ and }|\tau_x-\tau_y|\leq T\right]\\
\leq&\frac{\varepsilon}{8M}+\mathbf{P}\left[\sup_{t\leq T}|X_{\tau_x+t}-X_{\tau_x}|\geq\frac{\varepsilon}{4L}\right]\\
=&\frac{\varepsilon}{8M}+\mathbf{P}\left[\sup_{t\leq T}|X_t|\geq\frac{\varepsilon}{4L}\right]\\
\leq&\frac{\varepsilon}{8M}+\mathbf{P}\left[\sup_{t\leq T}|X_t^1|\geq\frac{\varepsilon}{8L}\right]+\mathbf{P}\left[\sup_{t\leq T}|X_t^2|\geq\frac{\varepsilon}{8L}\right]\\
=&\frac{\varepsilon}{4M}.
\end{aligned}$$

□

It is important to note that, as E is locally compact and Polish, the space $\mathcal{M}_f(E)$ of all finite measures on $(E,\mathcal{B}(E))$ together with the topology induced by weak convergence is also Polish, see [Kle13] Remark 13.14 (iii). It follows that in $\mathcal{M}_f(E)$ continuity and the generally weaker notion of sequential continuity are the same. We thus can read the above lemma equivalently as: whenever $x_n\to x$, we have $Q^\alpha_{x_n}\xrightarrow{w}Q^\alpha_x$.

2.1.2 Scaling properties of Q^α

Let X denote an SPMI of index $\alpha\in(1,2)$. We come to prove the Markov property for the process D.

Lemma 2.6 *For all $x,y\in[0,\infty)^2$ the Markov property holds,*

$$\mathbf{P}\left[D_{x+y}\in A|\mathcal{F}^D_x\right]=Q^\alpha_{y+D_x}(A),$$

for all $A\in\mathcal{B}(E)$.

Proof. Let $x,y\in[0,\infty)^2$ and $A\in\mathcal{B}(E)$. Let X be an SPMI process with generated filtration $\mathcal{F}^X$ defined by

$$\mathcal{F}^X_t:=\sigma\left(X_s:s\leq t\right)\quad\text{for }t>0.$$

We start with a preliminary consideration. For $z\in E-x$ we trivially have $z+x+y\in[0,\infty)^2$. Now, due to the spatial homogeneity of X, for $x'\in[0,\infty)^2$ with $x'\geq x$ (then $x'+z\in[0,\infty)^2$),

$$\mathbf{P}_z[X_{\tau_{x'}}\in A-x']=\mathbf{P}_0[X_{\tau_{x'+z}}\in A-(x'+z)]=Q^\alpha_{x'+z}(A).$$

Recall that we denote by $\mathbf{P}_z$ the law of $X+z$ under $\mathbf{P}$. Furthermore, by the strong Markov property of X, see Theorem 1.6, we get that for $t\geq0$ and $A'\subset\mathbb{R}^2$ measurable,

$$\begin{aligned}
\mathbf{P}_0\left[X_{\tau_x+t}\in A'|\mathcal{F}^X_{\tau_x}\right]&=\mathbf{P}_0\left[X_{\tau_x+t}-X_{\tau_x}\in A'-X_{\tau_x}|\mathcal{F}^X_{\tau_x}\right]\\
&=\mathbf{P}_{X_{\tau_x}}[X_t\in A'].
\end{aligned}$$

Altogether, for $z := X_{\tau_x} \in E - x$ with $x' = x + y$,

$$\begin{aligned}\mathbf{P}_0\left[X_{\tau_{x+y}} \in A - (x+y) | \mathcal{F}^X_{\tau_x}\right] &= \mathbf{P}_{X_{\tau_x}}\left[X_{\tau_{x+y}} \in A - (x+y)\right] \\ &= Q^\alpha_{D_x+y}(A).\end{aligned}$$

Note that $\tau_{x+y} \geq \tau_x$. Next, for all $x \in [0,\infty)^2$, it is

$$\mathcal{F}^D_x \subset \mathcal{F}^X_{\tau_x}.$$

This is due to that for all $y \leq x$ we have $\tau_y \leq \tau_x$ and therefore also $\mathcal{F}_{\tau_y} \subset \mathcal{F}_{\tau_x}$. In particular, we get that $D_y = X_{\tau_y} + y$ is $\mathcal{F}^X_{\tau_x}$-measurable for all $y \leq x$. So, by the tower property, we get

$$\begin{aligned}\mathbf{P}\left[D_{x+y} \in A | \mathcal{F}^D_x\right] &= \mathbf{E}\left[\mathbf{P}_0\left[X_{\tau_{x+y}} \in A - (x+y) | \mathcal{F}^X_{\tau_x}\right] | \mathcal{F}^D_x\right] \\ &= \mathbf{E}\left[Q^\alpha_{D_x+y}(A) | \mathcal{F}^D_x\right] \\ &= Q^\alpha_{D_x+y}(A).\end{aligned}$$

□

The following important lemma derives a scaling property of Q^α from the one of X. We decided to give the proof of the first part in great detail, as the result is particularly important for us and the idea of the proof will be used again later.

Lemma 2.7 *Let $x, y \in [0,\infty)^2$ and $c > 0$. Then, for all $A \in \mathcal{B}(E)$ and $f : E \to \mathbb{R}$ bounded and measurable, we have*

i) $Q^\alpha_x(A) = Q^\alpha_{cx}(cA)$,

ii) $\int_E f(cy) Q^\alpha_x(dy) = \int_E f(y) Q^\alpha_{cx}(dy)$,

iii) The mapping $x \mapsto Q^\alpha_x(A)$ is measurable for all $A \in \mathcal{B}(E)$,

iv) $\int_E Q^\alpha_{cz+y}(A) Q^\alpha_x(dz) = Q^\alpha_{cx+y}(A)$ *for all $A \in \mathcal{B}(E)$.*

Proof. *i)* Let X be an SPMI of index α and $c > 0$. We define the process

$$Y_t := \frac{1}{c} X_{c^\alpha t}.$$

Obviously, Y^1 and Y^2 are independent, $Y_0 = 0$ and Y has almost surely càdlàg paths. Let now $0 \leq s \leq t$. Then, $X_{c^\alpha t} - X_{c^\alpha s}$ is independent of $\sigma(X_u : u \leq c^\alpha s) = \sigma(Y_u : u \leq s)$. Hence, also Y has independent increments. Furthermore, for s and t as above,

$$\begin{aligned}Y_t - Y_s &= \frac{1}{c} X_{c^\alpha t} - \frac{1}{c} X_{c^\alpha s} \\ &\overset{d}{=} \frac{1}{c} X_{c^\alpha (t-s)} \\ &= Y_{t-s}.\end{aligned}$$

Therefore, Y is a Lévy process. We also get, by using (1.1) and (1.13), for the characteristic function of Y^j with $j = 1, 2$

$$\begin{aligned}\mathbf{E}\left[e^{i\theta Y_1^j}\right] &= \mathbf{E}\left[\exp\left(i\frac{\theta}{c}X_{c^\alpha}^j\right)\right] \\ &= \exp\left(-c^\alpha \Psi_{X^j}\left(\frac{\theta}{c}\right)\right) \\ &= \exp\left(-\Psi_{X^j}(\theta)\right),\end{aligned}$$

where $\theta \in \mathbb{R}$. This shows that also Y is an SPMI process.
Next, note that we have

$$\begin{aligned}\tau_x(Y) &= \inf_{t>0}\{Y_t \in E - x\} \\ &= \inf_{t>0}\{X_{c^\alpha t} \in E - cx\} \\ &= c^{-\alpha}\inf_{t>0}\{X_t \in E - cx\} \\ &= c^{-\alpha}\tau_{cx}(X).\end{aligned}$$

From this we get that for all $x \in [0,\infty)$ and $A \in \mathcal{B}(E)$

$$\begin{aligned}\left\{Y_{\tau_x(Y)} + x \in A\right\} &= \left\{\frac{1}{c}X_{c^\alpha c^{-\alpha}\tau_{cx}(X)} + x \in A\right\} \\ &= \left\{X_{\tau_{cx}(X)} + cx \in cA\right\}.\end{aligned}$$

As $\mathcal{L}\left(Y_{\tau(Y)}\right) = \mathcal{L}\left(X_{\tau(X)}\right)$, for $x \in [0,\infty)^2$ and $A \in \mathcal{B}(E)$ we end up with

$$\begin{aligned}Q_x^\alpha(A) &= \mathbf{P}\left[Y_{\tau_x(Y)} + x \in A\right] \\ &= \mathbf{P}\left[X_{\tau_{cx}(X)} + cx \in cA\right] \\ &= Q_{cx}^\alpha(cA).\end{aligned}$$

ii) This is trivial.
iii) We define

$$\mathcal{H} := \{A \in \mathcal{B}(E) : x \mapsto Q_x^\alpha(A) \text{ is measurable}\}.$$

Then, obviously, $\emptyset \in \mathcal{H}$ and with $A \in \mathcal{H}$ also $A^c \in \mathcal{H}$. Now let $A_1, A_2, \ldots \in \mathcal{H}$ be pairwise disjoint. Hence, for $x \in [0,\infty)^2$,

$$Q_x^\alpha\left(\dot{\bigcup_{i\in\mathbb{N}}} A_i\right) = \sum_{i=1}^\infty Q_x^\alpha(A_i) = \sup_{n\in\mathbb{N}}\sum_{i=1}^n Q_x^\alpha(A_i).$$

Then, also $\dot{\bigcup}_{i\in\mathbb{N}} A_i \in \mathcal{H}$. This means, $\mathcal{H}$ is a Dynkin system.
For $A \subset E$ open we have with Lemma 2.5 and the Portemanteau theorem, cf. [Kle13] Theorem 13.16 p. 254, that $x \mapsto Q_x^\alpha(A)$ is lower semi-continuous and therefore measurable. By Dynkin's π-λ theorem, see [Kle13] Theorem 1.19, we get

$$\mathcal{B}(E) = \sigma(\mathcal{O}) = \delta(\mathcal{O}) \subset \mathcal{H} \subset \mathcal{B}(E).$$

In particular, we have

$$\mathcal{B}(E) = \mathcal{H},$$

which is the assertion. Here, $\mathcal{O}$ denotes the collection of all open subsets of E, which is trivially a π-system. By $\delta(\mathcal{O})$ we mean the Dynkin system generated by $\mathcal{O}$.
iv) It is immediate from *iii)* that for $z \in [0,\infty)^2$ the mapping $z \mapsto Q^\alpha_{z+y}(A)$ is measurable for all $y \in [0,\infty)^2$ and $A \in \mathcal{B}(E)$. So,

$$\begin{aligned}\int_E Q^\alpha_{cz+y}(A) Q^\alpha_x(\mathrm{d}z) &= \int_E Q^\alpha_{z+y}(A) Q^\alpha_{cx}(\mathrm{d}z) \quad (\text{by } ii))\\ &= \mathbf{E}[Q^\alpha_{D_{cx}+y}(A)]\\ &= \mathbf{E}\left[\mathbf{P}\left[D_{cx+y} \in A | \mathcal{F}^D_{cx}\right]\right] \quad (\text{by Lemma 2.6})\\ &= \mathbf{P}\left[D_{cx+y} \in A\right] = Q^\alpha_{cx+y}(A).\end{aligned}$$

□

Remark 2.8 *It follows from the scaling property that the family of harmonic measures Q^α_x, $x \in (0,\infty)^2$ is completely determined by its values on the line $\{(a,\varepsilon) : \varepsilon > 0\}$ for any $a > 0$. This is due to the relation*

$$Q^\alpha_x(A) = Q^\alpha_{\left(a, \frac{x_2}{x_1} a\right)}\left(\frac{a}{x_1} A\right). \tag{2.5}$$

It is therefore enough to concentrate on studying $Q^\alpha_\varepsilon := Q^\alpha_{(1,\varepsilon)}$ for $\varepsilon > 0$.

2.1.3 Moment estimates

In this section, we give moment bounds for Q^α. In particular, we show that the exit measure Q^α has finite first moments. Our proofs make use of robust arguments for which we do not need to know the exact form of the distribution of Q^α, but rather deal with the scaling property.
As always, we denote the 1-norm in $\mathbb{R}^2$ by $|\cdot|$. Let X be an SPMI of index α. As mentioned in Remark 2.8, we concentrate on the case where $x = (1,\varepsilon)$.
The following lemma gives a good bound for the expectation of $\tau(X)$. The reader should note that the proof fails for $\alpha = 2$. In this case, i.e., when X is a Brownian motion, then, of course, $\mathbf{E}[\tau_x] = \infty$ for all $x \in (0,\infty)^2$.

Lemma 2.9 *Let $\alpha \in (1,2)$. There is a positive constant C, independent of ε, such that for all $\varepsilon > 0$*

$$\mathbf{E}\left[\tau_{(1,\varepsilon)}(X)\right] = \mathbf{E}\left[\tau_1(X^1) \wedge \tau_\varepsilon(X^2)\right] \le C\varepsilon.$$

Proof. Note that $\tau(X) \ge 0$. We make use of the following identity for non-negative random variables $Y \ge 0$ and positive real numbers $p > 0$,

$$\mathbf{E}[Y^p] = p \int_0^\infty t^{p-1} \mathbf{P}[Y > t]\,\mathrm{d}t. \tag{2.6}$$

We begin with a few considerations, concerning constants that we will use later on. First, by making use of the asymptotic relation (1.11), we see that, due to that we deal with probabilities and that $x^{-1/\alpha} \to \infty$ as $x \to 0$, there is C_1 such that for *all* $x \geq 0$

$$\mathbf{P}\left[\tau_1(X^2) > x\right] \leq C_1 x^{-1/\alpha}.$$

So, we have

$$\mathbf{P}\left[\tau_\varepsilon(X^2) > t\right] = \mathbf{P}\left[\tau_1(X^2) > \varepsilon^{-\alpha} t\right] \leq \varepsilon C_1 t^{-1/\alpha}.$$

Furthermore, if we set $p := 1 - \frac{1}{\alpha} \in \left(0, \frac{1}{2}\right)$ in (2.6), we have

$$C_2 = \int_0^\infty t^{-1/\alpha} \mathbf{P}\left[\tau_1(X^1) > t\right] \mathrm{d}t = \frac{\alpha}{\alpha - 1} \mathbf{E}\left[\tau(X^1)^{1-1/\alpha}\right],$$

which is finite by (1.10), as $1 - 1/\alpha < 1/\alpha$ for $\alpha \in (1, 2)$.
Thus, by using the scaling property (1.12) for the stable process τ^{X^i} and the estimate for the tail behaviour, we end up with

$$\begin{aligned}
\mathbf{E}\left[\tau_1(X^1) \wedge \tau_\varepsilon(X^2)\right] &= \int_0^\infty \mathbf{P}\left[\tau_1(X^1) \wedge \tau_\varepsilon(X^2) > t\right] \mathrm{d}t \\
&= \int_0^\infty \mathbf{P}\left[\tau_1(X^1) > t\right] \mathbf{P}\left[\tau_\varepsilon(X^2) > t\right] \mathrm{d}t \\
&\leq C_1 \varepsilon \int_0^\infty t^{-1/\alpha} \mathbf{P}\left[\tau_1(X^1) > t\right] \mathrm{d}t = C_1 C_2 \varepsilon.
\end{aligned}$$

□

Remark 2.10 *As for $a, b, c \geq 0$ it is $ca \wedge cb = c(a \wedge b)$, we get*

$$\mathbf{E}\left[\tau_{cx}\right] = \mathbf{E}\left[\tau_{cx_1} \wedge \tau_{cx_2}\right] = \mathbf{E}\left[c^\alpha \tau_{x_1} \wedge c^\alpha \tau_{x_2}\right] = c^\alpha \mathbf{E}\left[\tau_x\right].$$

Lemma 2.11 *Let X be an SPMI of index α and $x \in (0, \infty)^2$. Then, for any $p \in [1, \alpha)$,*

$$\int_E |y|^p Q_x^\alpha(dy) = \mathbf{E}\left[|X_{\tau_x} + x|^p\right] < \infty.$$

Proof. We write shortly τ_{x_i} for $\tau_{x_i}(X^i)$. First, consider the moments of $\tau := \tau(X)$. With Remark 2.10 and Lemma 2.9, we see that for all $x \in (0, \infty)^2$

$$\mathbf{E}\left[\tau_x\right] = x_1^\alpha \mathbf{E}\left[\tau_{(1, x_2/x_1)}\right] < \infty.$$

Next, for $i = 1, 2$ we compute the quadratic variation process of X^i,

$$[X^i]_t = \int_0^t \int_0^\infty h^2 N_p^i(\mathrm{d}h, \mathrm{d}s) = \sum_{s \leq t} (\Delta X_s^i)^2 < \infty \quad \text{a.s.},$$

cf. [App09] (4.16), page 257. Here, the integrals are with respect to a Poisson random measure N_p^i with intensity measure given by

$$\nu(\mathrm{d}h, \mathrm{d}s) = \frac{1}{\Gamma(-\alpha)} h^{-\alpha-1} \mathbb{1}_{\{h>0\}} \mathrm{d}h \otimes \mathrm{d}s,$$

where $\frac{1}{\Gamma(-\alpha)}h^{-\alpha-1}\mathbb{1}_{\{h>0\}}\mathrm{d}h$ is the Lévy measure of X^i. See [Kyp14] Chapter 2.2 for an introduction to the subject. We will, however, come back to this in Chapter 4.2. The process $[X^i]$ is a Lévy process because the increments are independent and stationary. Obviously, $[X^i]$ is a pure jump process and also a subordinator, see also [Kal02] Exercise 26.11. We can compute the Lévy measure of $[X^i]$, $\bar{\nu}$, as follows. For all $0 \leq a \leq b$ we have

$$\begin{aligned}\bar{\nu}([a,b]) &= \nu([\sqrt{a},\sqrt{b}]) \\ &= \frac{1}{\Gamma(-\alpha)}\int_{\sqrt{a}}^{\sqrt{b}} h^{-\alpha-1}\mathrm{d}h \\ &= \frac{1}{2\Gamma(-\alpha)}\int_a^b h^{-\frac{\alpha}{2}-1}\mathrm{d}h.\end{aligned}$$

This is, $[X^i]$ is an $\alpha/2$-stable Lévy subordinator, see also [Kal02] Exercise 26.12.
We easily see, by making use of (1.10) that $[X^i]_t$ has finite $p' := 1/2+\varepsilon'$-th moments for all $\varepsilon' \in (0, \frac{\alpha-1}{2})$.
It is important to note that $[X^i]_t$ is a subordinator and is therefore monotonously nondecreasing. We can thus estimate $[X^i]_\tau$ from above, by considering the bigger discrete stopping time $\bar{\tau}$ defined as

$$\bar{\tau} := \inf\{n \in \mathbb{N} : \tau < n\}.$$

Then, obviously, we have $\bar{\tau} \geq \tau$ and $\bar{\tau} - \tau \leq 1$. Moreover, $\bar{\tau}$ is a stopping time with respect to the same filtration as τ, as we easily see from the relation

$$\{\bar{\tau} \leq t\} = \{\bar{\tau} \leq k\} = \{\tau < k\},$$

where we have for all $t \geq 0$ the unique representation $t = k + r$ for some $k \in \mathbb{N}_0$ and $r \in (0,1]$. Note that τ has finite mean and so the same is true for $\bar{\tau}$,

$$\mathbf{E}[\bar{\tau}] = \mathbf{E}[\bar{\tau} - \tau] + \mathbf{E}[\tau] \leq 1 + \mathbf{E}[\tau] < \infty.$$

Due to the fact that $\bar{\tau}$ is integer-valued and because of the monotonicity of the variation process and the stationarity of increments, we can estimate the stopped process from above by a randomly stopped sum as follows,

$$[X^i]_\tau \leq [X^i]_{\bar{\tau}} = \sum_{k=1}^{\bar{\tau}} \bar{X}_k,$$

where the $\bar{X}_k$ are i.i.d. with

$$\bar{X}_k \stackrel{d}{=} \int_0^1 \int_0^\infty h^2 N_p^i(\mathrm{d}h, \mathrm{d}s) = [X^i]_1.$$

By equation (1.2) of [GJ86], we can derive from the existence of p'-th moments of the $\bar{X}_k$ and the finite mean of $\bar{\tau}$ that

$$\mathbf{E}\left[[X^i]_\tau^{p'}\right] \leq \mathbf{E}\left[[X^i]_{\bar{\tau}}^{p'}\right] = \mathbf{E}\left[\left(\sum_{k=1}^{\bar{\tau}} \bar{X}_k\right)^{p'}\right] \leq \mathbf{E}[\bar{\tau}]\mathbf{E}\left[\bar{X}^{p'}\right] < \infty. \tag{2.7}$$

Note in this context that $p' \in (0,1)$.
We now apply the Burkholder-Davis-Gundy inequality, [Kal02] Theorem 26.12, to finish our proof. Let $\varepsilon \in (0, \alpha - 1)$, $p := 1 + \varepsilon$ and $p' := p/2$. Because $X_{\tau_x} + x \in E$, we have

$$\mathbf{E}\left[|X_{\tau_x} + x|^p\right] = \mathbf{E}\left[\left(X^1_{\tau_x} + x_1\right)^p\right] + \mathbf{E}\left[\left(X^2_{\tau_x} + x_2\right)^p\right].$$

Note that for $i = 1, 2$, due to the optional stopping theorem, see e.g. [Kle13] Exercise 21.1.3, $M^i_t := X^i_{\tau \wedge t}$ is a right-continuous martingale with $M_0 = 0$. We set

$$M^{i,*} := \sup_{t \geq 0} |M^i_t| = \sup_{t \leq \tau_x} |X^i_t|.$$

Now, as $\tau_x < \infty$ almost surely, we have $[M^i]_\infty = [X^i]_{\tau_x}$, see also [Kal02] Theorem 26.6. We now can pick a constant c, such that for $i = 1, 2$

$$\begin{aligned}
\mathbf{E}\left[\left|X^i_{\tau_x} + x_i\right|^p\right] &\leq 2^{p-1}\left(\mathbf{E}\left[\left|X^i_{\tau_x}\right|^p\right] + x_i^p\right) \\
&\leq 2^{p-1}\left(\mathbf{E}\left[\left(M^{i,*}\right)^p\right] + x_i^p\right) \\
&\leq c2^{p-1}\left(\mathbf{E}\left[[M^i]^{p/2}_\infty\right] + x_i^p\right) \\
&= c2^{p-1}\left(\mathbf{E}\left[[X^i]^{p'}_{\tau_x}\right] + x_i^p\right).
\end{aligned}$$

This is finite due to (2.7) and the lemma is proved. □

At this point, we cannot show that the given moment bound is sharp. However, as a consequence of Corollary 3.3 below, we will see that indeed $\mathbf{E}\left[|X_{\tau_x} + x|^p\right] = \infty$ if $\varepsilon = \alpha - 1$.

Remark 2.12 *In the proof of Lemma 2.11 we actually showed that for some $p > 1$ and all $x \in (0,\infty)^2$ there is a finite constant $C = C(x,p)$ such that*

$$\mathbf{E}\left[\left(\sup_{t \leq \tau_x} |X^i_t|\right)^p\right] \leq C.$$

Then, trivially,

$$\sup_{t \geq 0} \mathbf{E}\left[\left|X^i_{t \wedge \tau_x}\right|^p\right] \leq \mathbf{E}\left[\left(\sup_{t \leq \tau_x} |X^i_t|\right)^p\right] \leq C < \infty.$$

Now, with [Kle13] Corollary 6.21, we see that $\left(X^i_t\right)_{t \leq \tau_x}$ is uniformly integrable and therefore, by the optional sampling theorem e.g. in the version of [Ber98] p.4, we have

$$\mathbf{E}\left[X^i_{\tau_x}\right] = \mathbf{E}\left[X^i_0\right] = 0.$$

2.1.4 Definition of $Q^{\alpha,\zeta}$

We now introduce for $\zeta \in [\pi/2, \pi)$ the exit measure $Q^{\alpha,\zeta}$ of an SPMI exiting from the wedge of angle ζ. As we will not use $Q^{\alpha,\zeta}$ to construct more sophisticated processes as for Q^{α} in Chapter 4, it suffices to give only the definition of the probability measure $Q^{\alpha,\zeta}$. However, all possible information on $Q^{\alpha,\zeta}$ will be available, once we calculated the exact form of the densities of $Q^{\alpha,\zeta}$ as a consequence of Theorem 3.1.

Let $\zeta \in [\pi/2, \pi)$. Recall the definition $\beta^{\zeta} = (\cos(\zeta), \sin(\zeta))$ from Section 1.3.3. We define the rotated positive y-axis

$$E_2^{\zeta} := [0, \infty)\beta^{\zeta}$$

and let

$$E^{\zeta} := E_1 \cup E_2^{\zeta} = \partial\mathbb{W}(\zeta),$$

the coordinate axes with rotated y-axis. For $x \in \mathbb{W}(\zeta)$ and X an SPMI starting at x, recall the stopping times

$$\tau_x^{\zeta}(X) = \inf_{s>0}\{X_s \in E^{\zeta}\}.$$

Definition 2.13 *Let X be an SPMI of index α for some $\alpha \in (1, 2]$ and let $\zeta \in [\pi/2, \pi)$. We define for $x \in \mathbb{W}(\zeta)$ the probability measure $Q^{\alpha,\zeta}$ on $(E^{\zeta}, \mathcal{B}(E^{\zeta}))$ by*

$$Q_x^{\alpha,\zeta}(A) := \mathbf{P}\left[X_{\tau_x^{\zeta}} + x \in A\right]$$

for $A \in \mathcal{B}(E^{\zeta})$.

We directly see that for $\zeta = \frac{\pi}{2}$ we have $E^{\zeta} = E$ and $Q^{\alpha,\zeta} = Q^{\alpha}$. As $\tau_x^{\zeta} \leq \tau_{x_2}(X^2)$ and the second stopping time is finite almost surely, we also have that $\tau_x^{\zeta} < \infty$ almost surely. Furthermore, due to the spectral positivity of X^1 and X^2, it is clear that X has to hit E^{ζ} continuously, i.e.,

$$\mathbf{P}\left[X_{\tau_x^{\zeta}} \in E^{\zeta} - x\right] = 1$$

for all $x \in [0, \infty)^2$. So, $Q^{\alpha,\zeta}$ is indeed a well-defined probability measure on E^{ζ}.

2.2 SPMI exiting the upper half-plane

We move on to a key observation of this chapter, which is stated in the subsequent lemma. Roughly speaking, we can show that an SPMI while escaping from the upper half-plane has the same distribution as a certain Cauchy random variable. Considering the half-plane instead of the first quadrant greatly simplifies the situation, since now we deal with a Lévy process subordinated by *independent* stopping times. Although this simpler problem is not exactly what we are interested in, it will serve as a door opener to our original problem, as with this result we will be able to derive integral

equations for the densities of Q^α in Theorem 3.1.
For an SPMI $X = (X^1, X^2)$ and $r \geq 0$, we recall the definition of $\tau_r^i := \tau_r(X^i)$, cf. Definition 1.12,

$$\tau_r^i = \inf_{t>0}\{X_t^i \notin [-r, \infty)\}.$$

Lemma 2.14 *Let X be an SPMI of index $\alpha \in (1, 2]$, starting at the origin. For all $r \geq 0$ we have*

$$X^1_{\tau_r^2} \stackrel{d}{=} Y_r + r\cos\Big(\frac{\pi}{\alpha}\Big),$$

where Y is a Cauchy process with parameter $\sin\big(\frac{\pi}{\alpha}\big)$.

Proof. We already know from Section 1.2.2 that, for $\alpha \in (1, 2)$, X^1 has characteristic function

$$\mathbf{E}\left[e^{i\theta X_t^1}\right] = \exp\left(t\int_0^\infty (e^{i\theta x} - 1 - i\theta x)\frac{1}{\Gamma(-\alpha)}x^{-\alpha-1}\mathrm{d}x\right) = e^{t(-i\theta)^\alpha},$$

for $\theta \in \mathbb{R}$ and $t \geq 0$. This means, the characteristic exponent of X^1 is given by

$$\Psi_{X^1}(\theta) = -(-i\theta)^\alpha.$$

From Example 1.18 we know that $(\tau_r^2)_{r\geq 0}$ is a Lévy subordinator with Laplace exponent $\psi(\lambda) = \lambda^{1/\alpha}$, which means by Definition 1.11

$$\mathbf{E}\left[e^{-\lambda\tau_r^2}\right] = e^{-r\psi(\lambda)} = e^{-r\lambda^{1/\alpha}}, \quad \text{for } \lambda \geq 0.$$

Obviously, τ^2 is independent of X^1. From Lemma 1.14 now follows that the subordinated process $\left(X^1_{\tau_r^2}\right)_{r\geq 0}$ is a Lévy process with characteristic exponent Ψ,

$$\mathbf{E}\left[\exp\left(i\theta X^1_{\tau_r^2}\right)\right] = \exp\left(-r\Psi(\theta)\right),$$

for $r \geq 0$ and $\theta \in \mathbb{R}$, where Ψ is given by

$$\Psi(\theta) = \psi\left(\Psi_{X^1}(\theta)\right) = \left(-(-i\theta)^\alpha\right)^{1/\alpha}.$$

Note that, by the considerations following Lemma 1.14, the above expressions are indeed well-defined.
We are able to bring the characteristic exponent Ψ to a more recognisable form. First, note that for all $\lambda > 0$ and $\theta \in \mathbb{R}$:

$$\Psi(\lambda\theta) = \left(-\left(-i\lambda\theta\right)^\alpha\right)^{1/\alpha} = \lambda\left(-\left(-i\theta\right)^\alpha\right)^{1/\alpha} = \lambda\Psi(\theta).$$

We see that the subordinated process is strictly stable with index 1. We are therefore able to write the exponent in the form given by (1.14). For $\theta \in \mathbb{R}$, it is

$$-i\theta = |\theta|e^{-i\frac{\pi}{2}\operatorname{sgn}(\theta)}.$$

In order to compute the α-th respectively $1/\alpha$-th power, we make use of the main branch of the complex logarithm ln with values in $\mathbb{R} \times i(-\pi, \pi]$. We have for general powers

$$z^a = e^{a \ln(z)}, \quad \text{where } a, z \in \mathbb{C}.$$

Thus, we compute for $\alpha \in (1, 2]$

$$\begin{aligned} (-i\theta)^\alpha &= |\theta|^\alpha e^{-i\frac{\pi}{2}\alpha \operatorname{sgn}(\theta)} \quad \text{and} \\ -(-i\theta)^\alpha &= |\theta|^\alpha e^{-i\frac{\pi}{2}\alpha \operatorname{sgn}(\theta) + i\pi}. \end{aligned}$$

Note that for $\alpha \in (1, 2]$

$$\pi - \frac{\pi}{2}\alpha \operatorname{sgn}(\theta) \in \begin{cases} [0, \frac{\pi}{2}), & \text{if } \theta > 0 \\ (\frac{3}{2}\pi, 2\pi], & \text{if } \theta < 0. \end{cases}$$

Hence,

$$\begin{aligned} \ln\left(e^{i\pi(1-\frac{\alpha}{2}\operatorname{sgn}(\theta))}\right) &= \begin{cases} i\pi(1 - \frac{\alpha}{2}), & \text{if } \theta > 0 \\ i\pi(1 + \frac{\alpha}{2}) - 2\pi i, & \text{if } \theta < 0, \end{cases} \\ &= i\pi\left(1 - \frac{\alpha}{2}\right)\operatorname{sgn}(\theta). \end{aligned}$$

Summing up, we get

$$\begin{aligned} \left(-(-i\theta)^\alpha\right)^{1/\alpha} &= |\theta| e^{\frac{1}{\alpha} i\pi \operatorname{sgn}(\theta)(1-\frac{\alpha}{2})} \\ &= |\theta| e^{i \operatorname{sgn}(\theta)(\frac{\pi}{\alpha} - \frac{\pi}{2})} \\ &= |\theta| \left[\cos\left(\operatorname{sgn}(\theta)\left(\frac{\pi}{\alpha} - \frac{\pi}{2}\right)\right) + i \sin\left(\operatorname{sgn}(\theta)\left(\frac{\pi}{\alpha} - \frac{\pi}{2}\right)\right)\right] \\ &= |\theta| \cos\left(\frac{\pi}{\alpha} - \frac{\pi}{2}\right) + i \operatorname{sgn}(\theta)|\theta| \sin\left(\frac{\pi}{\alpha} - \frac{\pi}{2}\right) \\ &= |\theta| \sin\left(\frac{\pi}{\alpha}\right) - i\theta \cos\left(\frac{\pi}{\alpha}\right). \end{aligned}$$

Note that $c := \sin\left(\frac{\pi}{\alpha}\right) \in (0, 1]$ for $\alpha \in (1, 2]$, so we are in the allowed parameter range, cf. (1.14). Lévy processes $\bar{X}_t$ on $\mathbb{R}$ with characteristic exponent Ψ as given above,

$$\mathbf{E}\left[e^{i\theta \bar{X}_t}\right] = e^{-t\Psi(\theta)} = \exp\left(-t \sin\left(\frac{\pi}{\alpha}\right)|\theta| + i\theta t \cos\left(\frac{\pi}{\alpha}\right)\right),$$

are well known. It is

$$\bar{X}_t \overset{d}{=} Y_t + \cos\left(\frac{\pi}{\alpha}\right) t,$$

where Y is a Cauchy process with parameter $\sin\left(\frac{\pi}{\alpha}\right) > 0$ and start in 0. $\square$

For $\alpha = 2$, i.e. the case of a Brownian motion, we have $\sin\left(\frac{\pi}{\alpha}\right) = 1$ and $\cos\left(\frac{\pi}{\alpha}\right) = 0$, so we get the characteristic exponent of a standard Cauchy distribution. This we already know from (1.22).

If we set $\alpha = 1$, the resulting exponent is the one of a delta-mass at -1, which means that the process is a completely deterministic drift at negative unit speed. This results from the fact that $-(-i\theta)^\alpha = i\theta$, which means for $\alpha = 1$ each coordinate of our SPMI would be a deterministic drift at negative unit speed.
For the convenience of the reader, we recall the density function of Y_t for $t > 0$ from the above lemma. It is

$$\begin{aligned}\mathbf{P}\left[Y_t \in \mathrm{d}x\right] &= \frac{1}{\sin\left(\frac{\pi}{\alpha}\right) t\pi} \frac{1}{1+\left(\frac{x}{\sin\left(\frac{\pi}{\alpha}\right)t}\right)^2}\mathrm{d}x \\ &= \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi}\frac{t}{\left(\sin\left(\frac{\pi}{\alpha}\right)t\right)^2+x^2}\mathrm{d}x.\end{aligned} \tag{2.8}$$

We have for an SPMI X and $z \in \mathbb{R} \times (0,\infty)$ the explicit density

$$\mathbf{P}_z\left[X^1_{\tau^2} \in \mathrm{d}x\right] = \frac{1}{\pi}\frac{\sin\left(\frac{\pi}{\alpha}\right)z_2}{\left(\sin\left(\frac{\pi}{\alpha}\right)z_2\right)^2 + \left(x - \left(z_1 + z_2\cos\left(\frac{\pi}{\alpha}\right)\right)\right)^2}\mathrm{d}x \tag{2.9}$$

and the distribution function

$$\mathbf{P}_z\left[X^1_{\tau^2} \le x\right] = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x - \left(z_1 + z_2\cos\left(\frac{\pi}{\alpha}\right)\right)}{z_2\sin\left(\frac{\pi}{\alpha}\right)}\right). \tag{2.10}$$

Remark 2.15 *Using Lemma 2.14, we can explicitly compute that*

$$\mathbf{P}\left[X^1_{\tau^2_t} \ge 0\right] = 1 - \frac{1}{\alpha},$$

which is independent of t. This is, however, something that we already know from (1.18) and the fact that τ^2 and X^1 are independent.

We want at this point draw attention to the following simple equality, which will be used silently at several points henceforth. For all $\alpha \in (1,2]$, it is

$$\arctan\left(\frac{\cos\left(\frac{\pi}{\alpha}\right)}{\sin\left(\frac{\pi}{\alpha}\right)}\right) = \frac{\pi}{2} - \frac{\pi}{\alpha}. \tag{2.11}$$

2.3 SPMI exiting the rotated half-plane $\mathbb{H}^\zeta$

We now examine the exit-point distribution of an SPMI X of index $\alpha \in (1,2]$, while leaving the rotated half-plane of angle $\zeta \in (\pi/2, \pi)$. The restriction on the value of ζ is needed, since, if $\zeta \in (0, \pi/2)$, X leaves the rotated half-plane of angle ζ by jumping over the boundary. The cases $\zeta = \frac{\pi}{2}, \pi$ are already known from Section 2.2 and shall be excluded for convenience reasons.
As now we do not deal with subordinating a Lévy process by an independent subordinator, computing the exit distribution is a bit more sophisticated than in Section 2.2. We use the notation from Section 1.3.3.

2.3.1 Preliminary results

We begin with some simple considerations concerning the starting point. As we shall see, it is enough to concentrate on the case where the SPMI process starts in $(0,1)$. To this end, first notice that, in case we start in some arbitrary point $x \in \mathbb{H}^\zeta$, the probability of hitting a subset $A \subset \beta^\zeta\mathbb{R}$ is, by the spatial homogeneity of X, given by

$$\mathbf{P}_x\left[X_{\sigma^\zeta} \in A\right] = \mathbf{P}_{(0,x_2-\tan(\zeta)x_1)}\left[X_{\sigma^\zeta} \in A - (x_1, \tan(\zeta)x_1)\right].$$

The stopping time σ^ζ is as in (1.24). So, it suffices to concentrate on starting points located on the y-axis. We want to stress that, for our range of ζ, we have

$$\mathbb{H}^\zeta = \{x \in \mathbb{R}^2 : x_2 \geq \tan(\zeta)x_1\}.$$

Next, we fix an SPMI X with start in zero. We denote for $t \geq 0$ with $G_t = G_t^\zeta$ the set

$$G_t^\zeta := \beta^\zeta\mathbb{R} - (0,t).$$

We now want to understand σ_t^ζ from (1.24) as the first hitting time of X in G_t^ζ,

$$\sigma_t^\zeta := \inf_{s>0}\{X_s \in G_t^\zeta\}.$$

Let $A_{-\zeta}$ be the rotation by $-\zeta$,

$$A_{-\zeta} = \begin{pmatrix} \cos(\zeta) & \sin(\zeta) \\ -\sin(\zeta) & \cos(\zeta) \end{pmatrix}.$$

We then have the following important observation. Note that in the subsequent lemma Y_t can be interpreted as the unique real-valued process given by $X_{\sigma_t^\zeta} + (0,t) = Y_t\beta^\zeta$. We also want to mention that the second coordinate of

$$A_{-\zeta}\left(X_{\sigma_t^\zeta} + (0,t)\right)$$

is zero for all $t \geq 0$.

Lemma 2.16 *Let X be an SPMI of index α starting at zero. The process $(Y_t)_{t\geq 0}$, given by the first coordinate*

$$Y_t := \left[A_{-\zeta}\left(X_{\sigma_t^\zeta} + (0,t)\right)\right]^1 \in \mathbb{R},$$

is a Lévy process.

Proof. Obviously, as X starts at zero, $\sigma_0^\zeta = 0$ almost surely. So, also $Y_0 = 0$.
We can write σ_t^ζ as

$$\sigma_t^\zeta = \inf_{s>0}\{\cos(\zeta)X_s^2 - \sin(\zeta)X_s^1 = -t\cos(\zeta)\}.$$

Note that the process $\cos(\zeta)X^2 - \sin(\zeta)X^1$ is again, as $\cos(\zeta) < 0$ and $\sin(\zeta) > 0$, a spectrally negative α-stable Lévy process. This readily follows from (1.13) and the fact that X^1 and X^2 are independent. So, the process $(\sigma_t^\zeta)_{t\geq 0}$ is an $1/\alpha$-stable Lévy process and is càdlàg in particular. From this observation we also get the justification that

$$\sigma_t^\zeta < \infty \quad \text{almost surely.}$$

Now, as the composition of two càdlàg processes is again càdlàg, it follows that Y has almost surely càdlàg paths.
As the σ_t^ζ are stopping times, we get by the strong Markov property of X that, for all $s, t \geq 0$, $X_{\sigma_t^\zeta + s} - X_{\sigma_t^\zeta}$ is independent of $X_{\sigma_t^\zeta}$ and

$$X_{\sigma_t^\zeta + s} - X_{\sigma_t^\zeta} \stackrel{d}{=} X_s.$$

Therefore, as $\sigma_{t+s}^\zeta(X) \geq \sigma_t^\zeta(X)$ almost surely,

$$Y_{t+s} - Y_t = A_{-\zeta}\left(X_{\sigma_{t+s}^\zeta} - X_{\sigma_t^\zeta}\right) \stackrel{d}{=} A_{-\zeta}\left(X_{\sigma_s^\zeta}\right) = Y_s$$

and $Y_{t+s} - Y_t$ is independent of Y_t. So, we see that Y is a Lévy process. □

The next step is to show that Y is (strictly) 1-stable.

Lemma 2.17 *For all $c \geq 0$ and Y as in Lemma 2.16, we have*

$$Y_c \stackrel{d}{=} cY_1.$$

Proof. Let X be an SPMI of index α with start in zero. Then we have, as in the proof of Lemma 2.7 that the process $(Z_t)_{t\geq 0}$, given by

$$Z_t := cX_{c^{-\alpha}t},$$

is also an SPMI of index α starting at zero. Thus, also $\mathcal{L}(X) = \mathcal{L}(Z)$. We then have for all $t \geq 0$

$$\mathcal{L}\left(X_{\sigma_t^\zeta(X)}\right) = \mathcal{L}\left(Z_{\sigma_t^\zeta(Z)}\right).$$

Furthermore, we have the almost sure relation

$$\begin{aligned}\sigma_c^\zeta(Z) &= \inf_{s>0}\{Z_s \in G_c^\zeta\} = \inf_{s>0}\{cX_{c^{-\alpha}s} \in G_c^\zeta\}\\ &= \inf_{s>0}\{X_{c^{-\alpha}s} \in G_1^\zeta\} = c^\alpha \inf_{s>0}\{X_s \in G_1^\zeta\}\\ &= c^\alpha \sigma_1^\zeta(X).\end{aligned}$$

Note that $cG_1^\zeta = G_c^\zeta$. Now, almost surely

$$Z_{\sigma_c^\zeta(Z)} = cX_{c^{-\alpha}c^\alpha\sigma_1^\zeta(X)} = cX_{\sigma_1^\zeta(X)}.$$

Therefore, the claim follows as

$$\begin{aligned}\mathcal{L}(Y_c) &= \mathcal{L}\left(A_{-\zeta}\left(Z_{\sigma_c^\zeta(Z)} + (0,c)\right)\right)\\ &= \mathcal{L}\left(A_{-\zeta}\left(cX_{\sigma_1^\zeta(X)} + (0,c)\right)\right)\\ &= \mathcal{L}\left(cA_{-\zeta}\left(X_{\sigma_1^\zeta(X)} + (0,1)\right)\right)\\ &= \mathcal{L}(cY_1).\end{aligned}$$

□

We know that the family of strictly 1-stable random variables is parameterised by two parameters, the scale parameter $s > 0$ and the median $x_0 \in \mathbb{R}$. The above lemma gives us the justification why it is enough to concentrate on the starting point $(0, 1)$. When we start at some arbitrary point $(0, t)$ on the y-axis, the law of X_σ is again a Cauchy distribution. As Y is a Lévy process, the scale parameter is in this case given by $t \cdot s$ and median by $t \cdot x_0$.
We want to stress that the distribution of Y_1 has a one-dimensional Lebesgue density, which is given by

$$\mathbf{P}\left[Y_1 \in A\right] = \int_A \frac{1}{\pi} \frac{s}{s^2 + (t - x_0)^2} \mathrm{d}t, \tag{2.12}$$

for all $A \in \mathcal{B}(\mathbb{R})$. In the next section, we compute the scale and median for the distribution of Y_1, which uniquely determines the distribution of X_{σ^ζ} for all possible starting points $x \in \mathbb{H}^\zeta$.

2.3.2 Computing the parameters of Y_1

The aim of this subsection is to compute the parameters x_0 and s from (2.12), which fully characterise the distribution of Y_1. To this end, it is enough to find two non-equivalent equations involving this distribution. For given $\zeta \in (\pi/2, \pi)$, the objects with a bar refer to the corresponding objects with angle $\bar{\zeta} := \frac{3}{2}\pi - \zeta$, which is the reflection at $\frac{3}{4}\pi$.

We start with the probability p of exiting the upper half-plane through $(-\infty, 0] \times \{0\}$, when started in some point on the half-line $(0, \infty)\beta^\zeta$. We always use the short notation $\tau^2 = \tau(X^2)$. For $t > 0$, we get with (2.10)

$$\begin{aligned} p &:= \mathbf{P}_{t\beta^\zeta}\left[X^1_{\tau^2} \in (-\infty, 0]\right] = \frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{\cos(\zeta) + \cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)}{\sin(\zeta)\sin\left(\frac{\pi}{\alpha}\right)}\right),\\ \bar{p} &:= \mathbf{P}_{t\beta^{\bar{\zeta}}}\left[X^1_{\tau^2} \in (-\infty, 0]\right] = \frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{\sin(\zeta) + \cos\left(\frac{\pi}{\alpha}\right)\cos(\zeta)}{\cos(\zeta)\sin\left(\frac{\pi}{\alpha}\right)}\right).\end{aligned} \tag{2.13}$$

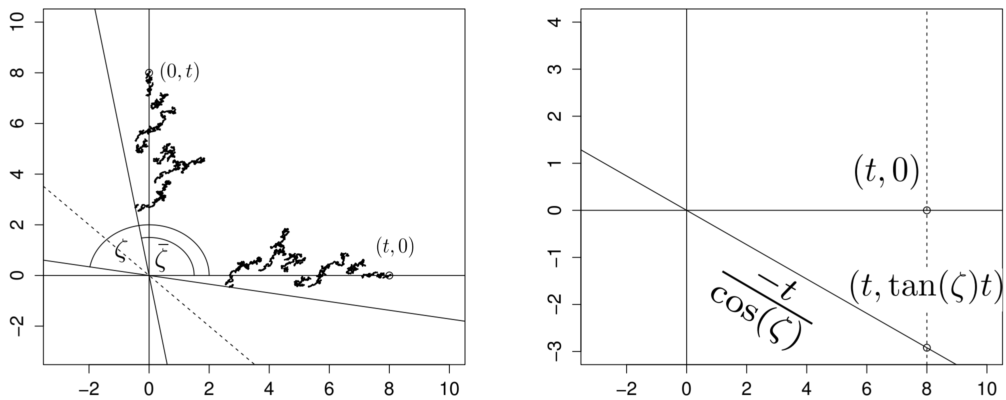
We see that p is independent of t.
Next, we compute the probability q of exiting the rotated upper half-plane through

$[0,\infty)\beta^\zeta$, when started at some point on the y-axis. This quantity can be expressed through the distribution of Y_1 and therefore in terms of the parameters s and x_0 from (2.12),

$$\begin{aligned} q &= \mathbf{P}_{(0,t)}\left[X_{\sigma_0^\zeta} \in [0,\infty)\beta^\zeta\right] = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x_0}{s}\right), \\ \bar{q} &= \mathbf{P}_{(0,t)}\left[X_{\sigma_0^{\bar{\zeta}}} \in [0,\infty)\beta^{\bar{\zeta}}\right] = \mathbf{P}_{(t,0)}\left[X_{\sigma^\zeta} \in (-\infty,0]\beta^\zeta\right] \\ &= \mathbf{P}_{(0,-t\tan(\zeta))}\left[X_{\sigma_0^\zeta} \in [-t/\cos(\zeta),\infty)\,\beta^\zeta\right] \\ &= \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{1-\sin(\zeta)x_0}{\sin(\zeta)s}\right). \end{aligned} \tag{2.14}$$

Note that $\sigma_0^\zeta = \sigma^\zeta$. In order to compute $\bar{q}$, we first used the fact that $\bar{q}$ can also be computed, due to the symmetry of X, by letting ζ fix, starting in $(t,0)$ and then computing the probability of exiting the rotated half-plane trough $(-\infty,0]\beta^\zeta$. Secondly, we made use of the spatial homogeneity of X to transform the point $(t,0)$ to the y-axis together with the fact that Y_t is a Cauchy process. See Figure 2.1 for a clarification. It is an important fact that q is independent of $t>0$, which could also have been verified by making use of the scaling property of X.

Figure 2.1: Transformations used to compute $\bar{q}$



(a) Exchanging X^1 and X^2 (b) Transforming $(t,0)$ to $(0,-t\tan(\zeta))$

Furthermore, we consider the probability of hitting the rotated y-axis before hitting the x-axis,

$$\begin{aligned} p_1 &:= \mathbf{P}_{(0,1)}\left[X_{\tau^\zeta} \in [0,\infty)\beta^\zeta\right] \quad \text{and} \\ \bar{p}_1 &:= \mathbf{P}_{(0,1)}\left[X_{\tau^{\bar{\zeta}}} \in [0,\infty)\beta^{\bar{\zeta}}\right], \end{aligned}$$

where τ^ζ is as in (1.25). We know from Remark 2.15 that $\mathbf{P}_{(0,1)}\left[X^1_{\tau^2} \in (-\infty,0]\right] = \frac{1}{\alpha}$. By making a decomposition after the first hitting-time of $[0,\infty)\beta^\zeta$, which has to be hit by an SPMI before it can hit $(-\infty,0]\times\{0\}$, we get the relation

$$\begin{aligned}\frac{1}{\alpha}&=\int_0^\infty \mathbf{P}_{(0,1)}\left[X_{\tau^\zeta}\in \mathrm{d}(\beta^\zeta t)\right]\mathbf{P}_{t\beta^\zeta}\left[X^1_{\tau^2}\in(-\infty,0]\times\{0\}\right]\\&=\int_0^\infty \mathbf{P}_{(0,1)}\left[X_{\tau^\zeta}\in \mathrm{d}(\beta^\zeta t)\right]p\\&=p_1p.\end{aligned}$$

Similarly, we get

$$\frac{1}{\alpha}=\bar{p}_1\bar{p}.$$

The parameter p_1 can be expressed in terms of q and $\bar{q}$. To this end, first count all paths of X, which exit the rotated half-plane $\mathbb{H}^\zeta$ through $[0,\infty)\beta^\zeta$ and then subtract the paths which first hit the positive x-axis and from there go on to exit $\mathbb{H}^\zeta$ through $[0,\infty)\beta^\zeta$,

$$\begin{aligned}p_1&=q-\int_0^\infty \mathbf{P}_{(0,1)}\left[X_{\tau^\zeta}\in \mathrm{d}t\times\{0\}\right]\mathbf{P}_{(t,0)}\left[X_{\sigma^\zeta}\in[0,\infty)\beta^\zeta\right]\\&=q-\int_0^\infty \mathbf{P}_{(0,1)}\left[X_{\tau^\zeta}\in \mathrm{d}t\times\{0\}\right](1-\bar{q})\\&=q-(1-p_1)(1-\bar{q}).\end{aligned}$$

The same way, we receive

$$\bar{p}_1=\bar{q}-(1-\bar{p}_1)(1-q).$$

Inserting the equations into each other, we end up with

$$\begin{aligned}q&=\frac{p_1}{p_1+\bar{p}_1-p_1\bar{p}_1}=\frac{\bar{p}}{\bar{p}+p-\frac{1}{\alpha}}\quad\text{and}\\ \bar{q}&=\frac{\bar{p}_1}{p_1+\bar{p}_1-p_1\bar{p}_1}=\frac{p}{p+\bar{p}-\frac{1}{\alpha}}.\end{aligned}$$

Recall that we already found explicit formulas for p and $\bar{p}$, so we can explicitly compute q and $\bar{q}$. To this end, note that we have for $\alpha\in(1,2)$

$$\begin{aligned}&\arctan\left(\frac{\cos(\zeta)+\cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)}{\sin(\zeta)\sin\left(\frac{\pi}{\alpha}\right)}\right)+\arctan\left(\frac{\sin(\zeta)+\cos\left(\frac{\pi}{\alpha}\right)\cos(\zeta)}{\cos(\zeta)\sin\left(\frac{\pi}{\alpha}\right)}\right)\\&=-\left(\arctan\left(\frac{\sin\left(\frac{\pi}{\alpha}\right)\left(1+2\cos\left(\frac{\pi}{\alpha}\right)\cos(\zeta)\sin(\zeta)\right)}{2\cos\left(\frac{\pi}{\alpha}\right)^2\sin(\zeta)\cos(\zeta)+\cos\left(\frac{\pi}{\alpha}\right)}\right)+\pi\right)\\&=-\frac{\pi}{\alpha}.\end{aligned}\tag{2.15}$$

The first equality is no more than the addition formula fot the arctangent,

$$\arctan(x)+\arctan(y)=\pi\,\mathrm{sgn}(x)+\arctan\left(\frac{x+y}{1-xy}\right)\quad\text{for } xy>1,$$

where we used that, for $\alpha \in (1,2)$,

$$\begin{aligned}&\frac{\cos(\zeta)+\cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)}{\sin(\zeta)\sin\left(\frac{\pi}{\alpha}\right)}\cdot\frac{\sin(\zeta)+\cos\left(\frac{\pi}{\alpha}\right)\cos(\zeta)}{\cos(\zeta)\sin\left(\frac{\pi}{\alpha}\right)}\\&=1+\frac{2\cos\left(\frac{\pi}{\alpha}\right)^2}{\sin\left(\frac{\pi}{\alpha}\right)^2}+\frac{\cos\left(\frac{\pi}{\alpha}\right)}{\sin\left(\frac{\pi}{\alpha}\right)^2\cos(\zeta)\sin(\zeta)}>1.\end{aligned}$$

In the second equality, we used that for $\alpha \in (1,2)$,

$$\arctan\left(\tan\left(\frac{\pi}{\alpha}\right)\right)=\frac{\pi}{\alpha}-\pi.$$

It is immediate that (2.15) also holds for $\alpha = 2$. So, we have for all $\alpha \in (1,2]$

$$p+\bar{p}-\frac{1}{\alpha}=1.$$

We infer that $p=\bar{q}$ and $\bar{p}=q$. Now, we have two equations for the two unknown parameters s and x_0. Inserting this to equation (2.13), together with (2.14), we end up with

$$\begin{aligned}\frac{1-\sin(\zeta)x_0}{\sin(\zeta)s}&=\frac{-\cos(\zeta)-\cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)}{\sin(\zeta)\sin\left(\frac{\pi}{\alpha}\right)}\quad\text{and}\\\frac{x_0}{s}&=\frac{-\sin(\zeta)-\cos\left(\frac{\pi}{\alpha}\right)\cos(\zeta)}{\cos(\zeta)\sin\left(\frac{\pi}{\alpha}\right)}.\end{aligned}$$

So, we get that

$$\begin{aligned}s&=\frac{-\cos(\zeta)\sin\left(\frac{\pi}{\alpha}\right)}{1+2\cos\left(\frac{\pi}{\alpha}\right)\cos(\zeta)\sin(\zeta)}\quad\text{and}\\x_0&=\frac{\sin(\zeta)+\cos\left(\frac{\pi}{\alpha}\right)\cos(\zeta)}{1+2\cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)\cos(\zeta)}.\end{aligned}\tag{2.16}$$

We thus get the following lemma.

Lemma 2.18 *Let $\zeta \in (\pi/2,\pi)$, $\alpha \in (1,2]$ and $x \in \mathbb{H}^\zeta$. Let X be an SPMI of index α with start in x. We have for all $r \in \mathbb{R}$*

$$\mathbf{P}_x\left[X_{\sigma^\zeta}\in(-\infty,r]\beta^\zeta\right]=\frac{1}{\pi}\int_{-\infty}^{r}\frac{s}{s^2+(t-x_0)^2}\,dt,$$

where

$$\begin{aligned}s&=\frac{\sin\left(\frac{\pi}{\alpha}\right)(x_1\sin(\zeta)-x_2\cos(\zeta))}{1+2\cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)\cos(\zeta)}\geq 0\quad\textit{and}\\x_0&=\frac{x_1(\cos(\zeta)+\cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta))+x_2(\sin(\zeta)+\cos\left(\frac{\pi}{\alpha}\right)\cos(\zeta))}{1+2\cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)\cos(\zeta)}.\end{aligned}$$

Proof. We set $\bar{s}$ and $\bar{x}_0$ for parameters of the law of X_σ when starting in $(0,1)$, as given in (2.16). We know, as Y from Lemma 2.16 is a Lévy process that, when X is started at $(0,t)$ for some $t>0$, the resulting distribution is again Cauchy with parameters $t\bar{s}$ and $t\bar{x}_0$. We now transform the starting point to the y-axis,

$$\begin{aligned}\mathbf{P}_x\left[X_{\sigma^\zeta}\in(-\infty,r]\beta^\zeta\right]&=\mathbf{P}_{(0,x_2-\tan(\zeta)x_1)}\left[X_{\sigma^\zeta}\in\left(-\infty,r-\frac{x_1}{\cos(\zeta)}\right]\beta^\zeta\right]\\ =&\frac{1}{\pi}\int_{-\infty}^{r}\frac{(x_2-\tan(\zeta)x_1)\bar{s}}{((x_2-\tan(\zeta)x_1)\bar{s})^2+\left(t-\frac{x_1}{\cos(\zeta)}-(x_2-\tan(\zeta)x_1)\bar{x}_0\right)^2}\mathrm{d}t.\end{aligned}$$

The claim thus follows with

$$\begin{aligned}s&=(x_2-\tan(\zeta)x_1)\bar{s}\quad\text{and}\\ x_0&=\frac{x_1}{\cos(\zeta)}+\bar{x}_0(x_2-\tan(\zeta)x_1).\end{aligned}$$

□

Remark 2.19 *Note that if $\zeta=\frac{\pi}{2}$ the above formulas are still correct, as we readily get by using (2.8). On the other hand, if we let $\zeta\to\pi$, we end up with $s=\sin\left(\frac{\pi}{\alpha}\right)x_2$ and $x_0=-(x_2\cos\left(\frac{\pi}{\alpha}\right)+x_1)$. The reason for the negative sign of x_0 is that $\beta^\pi=(-1,0)$.*

2.3.3 First intersection of two α-stable processes

We end this section by presenting a nice by-product of the above computations. We are able to give the distribution of the position where two spectrally one-sided α-stable Lévy processes first meet.

Corollary 2.20 *Let X and Y be two independent one-dimensional Lévy processes with characteristic exponents given for $\theta\in\mathbb{R}$ by*

$$\begin{aligned}\Psi_X(\theta)&=-(i\theta)^\alpha\quad\text{and}\\ \Psi_Y(\theta)&=-(-i\theta)^\alpha,\end{aligned}$$

for some $\alpha\in(1,2]$. Let $X_0=0$, $Y_0=y>0$ and $a\in[0,\infty)$. If we set

$$\tau:=\inf_{s>0}\{X_s=aY_s\},$$

then X_τ is Cauchy distributed with scale s and median x_0, given by

$$\begin{aligned}x_0&=\frac{ya^2\left(\frac{1}{a}-\cos\left(\frac{\pi}{\alpha}\right)\right)}{1+a^2-2a\cos\left(\frac{\pi}{\alpha}\right)},\\ s&=\frac{ya^2\sin\left(\frac{\pi}{\alpha}\right)}{1+a^2-2a\cos\left(\frac{\pi}{\alpha}\right)}.\end{aligned}$$

Proof. Note that, as X is spectrally negative, Y is spectrally positive and $X_0<Y_0$, the two processes have to meet continuously and, as τ is $1/\alpha$-stable, they also have to meet in finite time. We now set

$$\zeta:=\operatorname{arccot}(-a)\in\left[\frac{\pi}{2},\pi\right).$$

The process $Z := (-X, Y)$ is then an SPMI started from $(0, y)$ and $\tau = \sigma^\zeta(Z) =: \sigma$. We get for all $r \in \mathbb{R}$

$$\begin{aligned}
\mathbf{P}\left[X_\tau \in (-\infty, r]\right] &= \mathbf{P}_{(0,y)}\left[Z^1_\sigma \in [-r, \infty)\right] \\
&= \mathbf{P}_{(0,y)}\left[Z_\sigma \in \left(-\infty, \frac{-r}{\cos(\zeta)}\right] \beta^\zeta\right] \\
&= \frac{1}{\pi}\int_{-\infty}^{r} \frac{-\frac{\bar{s}}{\cos(\zeta)}}{\bar{s}^2 + \left(\frac{t}{\cos(\zeta)} + \bar{x}_0\right)^2} \mathrm{d}t.
\end{aligned}$$

Here the objects $\bar{s}$ and $\bar{x}_0$ refer to the corresponding objects from Lemma 2.18 with starting point $(0, y)$. The claim now follows with

$$\begin{aligned}
s &= -\cos(\zeta)\bar{s} \quad \text{and} \\
x_0 &= -\cos(\zeta)\bar{x}_0.
\end{aligned}$$

$\square$

Chapter 3

Explicit computation of $Q^{\alpha,\zeta}$

In this chapter, we prove a theorem, which gives a surprising relation between the exit measures of the first quadrant, possibly with rotated y-axis, of SPMI processes and ρ-correlated Brownian motion initiating from the same point. Here, ρ depends only on the stability parameter α.
Our method mainly relies on deriving certain Fredholm-type integral equations for the densities of $Q^{\alpha,\zeta}$. By the use of simple functional analytic arguments, namely by showing that the induced integral operators are contractions on $L^1[0,\infty)$, we show that these equations uniquely determine the densities of $Q^{\alpha,\zeta}$. That we indeed find a solution for the mentioned equations, is due to the fact that the exit measures from the upper half-plane coincide for an α-SPMI and a ρ-correlated Brownian motion with the special choice of $\rho = -\cos\left(\frac{\pi}{\alpha}\right)$.
The reason for considering the case where the y-axis is rotated, is that, in the case of Brownian motion, considering the exit distribution from the wedge results in adding correlation between the coordinates, as we saw in Section 1.3.2. Although correlation can not be defined for stable processes, due to the lack of second moments, our approach might be useful for investigations in this direction.
We begin this chapter by stating and proving the theorem. Then, as a series of corollaries, we give the precise form of the density functions of the measures Q^{α} and $Q^{\alpha,\zeta}$. In the second part, for the special case $\zeta = \frac{\pi}{2}$, we present some methods of using only the integral equations to derive certain properties of the density functions, without being able to solve the equations explicitly.
Also in this chapter, let $E = [0,\infty)^2 \backslash (0,\infty)^2$, $\zeta \in [\pi/2, \pi)$ and $\alpha \in (1,2]$. By $L^1[0,\infty)$ we denote the Banach space of integrable functions $h : [0,\infty) \to \mathbb{R}$ with corresponding norm $\|h\|_1 = \int_0^\infty |h(s)| \mathrm{d}s$.

3.1 The main theorem

The following theorem is one of the key results of the present work. The assumptions are obviously stronger than needed. That is why we are sure that Theorem 3.1 can be generalised to a wider class of stochastic processes or hitting times. In particular, the restriction to Lévy processes is only due to convenience reasons. However, we stated the theorem in this rather simple form, as it is sufficient for our needs and already in this form has quite interesting consequences.

3.1.1 Theorem 3.1 and implications

We recall for a given two-dimensional process X the definitions of the stopping times σ, τ and τ^2,

$$\begin{aligned}\sigma = \sigma^\zeta(X) &= \inf_{s>0}\{X_s \in \mathbb{R}\beta^\zeta\},\\ \tau^2 = \tau^2(X) &= \inf_{s>0}\{X_s^2 = 0\},\\ \tau = \tau^\zeta(X) &= \inf_{s>0}\{X_s \notin \mathbb{W}(\zeta)\} = \sigma \wedge \tau^2.\end{aligned}$$

We recall furthermore that, for $\zeta \in \left[\frac{\pi}{2}, \pi\right)$,

$$\begin{aligned}\beta^\zeta &= (\cos(\zeta), \sin(\zeta)),\\ \mathbb{H}^\zeta &= \left\{x \in \mathbb{R}^2 : \arg(x) \in [\zeta - \pi, \zeta]\right\},\\ \mathbb{W}(\zeta) &= \left\{x \in \mathbb{R}^2 : \arg(x) \in [0, \zeta]\right\}.\end{aligned}$$

For any set A, we write $\mathring{A}$ for the interior of A.

Theorem 3.1 *Let $\zeta \in [\pi/2, \pi)$. Let X and Y be two-dimensional Lévy processes such that $\sigma^\zeta(X), \sigma^\zeta(Y) < \infty$ almost surely with respect to $\mathbf{P}_x$ for all $x \in \mathbb{H}^\zeta$ and $\tau^2(X), \tau^2(Y) < \infty$ almost surely with respect to $\mathbf{P}_y$ for all $y \in \mathbb{H}$. Assume furthermore $\mathbf{P}_x\left[X_\sigma \in \mathbb{R}\beta^\zeta\right] = \mathbf{P}_y\left[X_{\tau^2} \in \mathbb{R} \times \{0\}\right] = 1$ and for all $t > 0$*

$$\begin{aligned}\mathcal{L}_{(t,0)}(X_\sigma) &= \mathcal{L}_{(t,0)}(Y_\sigma) \quad \textit{and}\\ \mathcal{L}_{t\beta^\zeta}(X_{\tau^2}) &= \mathcal{L}_{t\beta^\zeta}(Y_{\tau^2}).\end{aligned} \tag{3.1}$$

If $\mathcal{L}_x(X_\sigma)$ has a one-dimensional probability density function (p.d.f.) h_x^ζ on $\beta^\zeta\mathbb{R}$ for all $x \in \mathring{\mathbb{H}}^\zeta$ and $\mathcal{L}_x(X_{\tau^2})$ has a one-dimensional p.d.f. h_x on $\mathbb{R} \times \{0\}$ for all $x \in \mathring{\mathbb{H}}$ with

$$\sup_{x \in (0,\infty)\times\{0\}} \int_0^\infty h_x^\zeta(s)ds \le C_1 \quad \textit{and} \sup_{x \in (0,\infty)\beta^\zeta} \int_0^\infty h_x(s)ds \le C_2 \tag{3.2}$$

for some constants $C_1, C_2 \le 1$ with $C_1 \wedge C_2 < 1$. Then, also $\mathcal{L}_x(X_\tau)$ and $\mathcal{L}_x(Y_\tau)$ have one-dimensional Lebesgue densities on E_1, E_2^ζ for $x \in \mathring{\mathbb{W}}(\zeta)$ and

$$\mathcal{L}_x(X_\tau) = \mathcal{L}_x(Y_\tau)$$

for all $x \in \mathbb{W}(\zeta)$.

The proof of Theorem 3.1 will be given further down as a series of lemmas. We first want to state some important consequences of the theorem. As a special case, we get a result which is of particular importance for us.

Theorem 3.2 *Let $\alpha \in (1,2]$, $\zeta \in [\pi/2, \pi)$ and $x \in \mathbb{W}(\zeta)$. Let X be an SPMI of index α. Then, we have*

$$\mathcal{L}_x(X_\tau) = \mathcal{L}_x(Y_\tau),$$

where Y is a ρ-correlated Brownian motion with correlation parameter

$$\rho = -\cos\Big(\frac{\pi}{\alpha}\Big) \in [0,1).$$

The densities f_x^ζ on E_1 and $\bar{f}_x^\zeta$ on E_2^ζ of $\mathcal{L}_x(X_\tau) = Q_x^{\alpha,\zeta}$ are given by

$$\begin{aligned} \bar{f}_x^\zeta(t) &= \frac{1}{\pi}\frac{rt^{r-1}\bar{z}_2}{\bar{z}_2^2 + (t^r + \bar{z}_1)^2} \quad \text{and} \\ f_x^\zeta(t) &= \frac{1}{\pi}\frac{rt^{r-1}z_2}{z_2^2 + (t^r - z_1)^2}, \end{aligned}$$

where

$$r := \frac{\pi}{\frac{\pi}{2} - \arctan\left(\frac{\cos(\zeta) + \cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)}{\sin(\zeta)\sin\left(\frac{\pi}{\alpha}\right)}\right)},$$

$$\begin{aligned} \bar{z}_1 &:= \left(\frac{x_1^2 + x_2^2 + 2\cos\left(\frac{\pi}{\alpha}\right)x_1x_2}{1 + 2\cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)\cos(\zeta)}\right)^{\frac{r}{2}} \cos(r\varphi), \\ \bar{z}_2 &:= \left(\frac{x_1^2 + x_2^2 + 2\cos\left(\frac{\pi}{\alpha}\right)x_1x_2}{1 + 2\cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)\cos(\zeta)}\right)^{\frac{r}{2}} \sin(r\varphi), \\ z_1 &:= \left(x_1^2 + x_2^2 + 2\cos\Big(\frac{\pi}{\alpha}\Big)x_1x_2\right)^{\frac{r}{2}} \cos(r\varphi), \\ z_2 &:= \left(x_1^2 + x_2^2 + 2\cos\Big(\frac{\pi}{\alpha}\Big)x_1x_2\right)^{\frac{r}{2}} \sin(r\varphi) \end{aligned}$$

and

$$\varphi = \frac{\pi}{2} - \arctan\left(\frac{x_1 + \cos\left(\frac{\pi}{\alpha}\right)x_2}{x_2\sin\left(\frac{\pi}{\alpha}\right)}\right).$$

Proof. By Lemma 2.18, we have that the one-dimensional Lebesgue density of X_σ on $\mathbb{R}\beta^\zeta$ is for all $x \in \mathbb{H}^\zeta$ and $r \in \mathbb{R}$ given by

$$h_x^\zeta(r) = \frac{\mathrm{d}}{\mathrm{d}r}\mathbf{P}_x\left[X_\sigma \in (-\infty, r]\beta^\zeta\right] = \frac{1}{\pi}\frac{s}{s^2 + (r - x_0)^2},$$

where

$$\begin{aligned} s &= \frac{\sin\left(\frac{\pi}{\alpha}\right)(x_1\sin(\zeta) - x_2\cos(\zeta))}{1 + 2\cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)\cos(\zeta)} \quad \text{and} \\ x_0 &= \frac{x_1(\cos(\zeta) + \cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)) + x_2(\sin(\zeta) + \cos\left(\frac{\pi}{\alpha}\right)\cos(\zeta))}{1 + 2\cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)\cos(\zeta)}. \end{aligned}$$

By Corollary 1.24, we get that this is also the density of Y_σ on $\mathbb{R}\beta^\zeta$, which shows the first equality of (3.1). Furthermore, we have for $t > 0$

$$\begin{aligned}\int_0^\infty h^\zeta_{(t,0)}(r)\mathrm{d}r &= \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x_0}{s}\right) \\ &= \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{\cos(\zeta) + \cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)}{\sin\left(\frac{\pi}{\alpha}\right)\sin(\zeta)}\right).\end{aligned}$$

This shows the first equality of (3.2) with

$$C_1 := \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{\cos(\zeta) + \cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)}{\sin\left(\frac{\pi}{\alpha}\right)\sin(\zeta)}\right) < 1.$$

Note that, for our range of α and ζ, we always have $\sin\left(\frac{\pi}{\alpha}\right)\sin(\zeta) > 0$.
By Lemma 2.14, we have that the one-dimensional Lebesgue densities on $\mathbb{R} \times \{0\}$ of X_{τ^2} are for any $r \in \mathbb{R}$ given by

$$h_x(r) = \frac{\mathrm{d}}{\mathrm{d}r}\mathbf{P}_x\left[X_{\tau^2} \in (-\infty, r] \times \{0\}\right] = \frac{1}{\pi}\frac{x_2\sin\left(\frac{\pi}{\alpha}\right)}{(x_2\sin\left(\frac{\pi}{\alpha}\right))^2 + (r - (x_1 + \cos\left(\frac{\pi}{\alpha}\right)x_2))^2}$$

and by Lemma 1.22, we know that this is also the density of Y_{τ^2}. So, the second part of (3.1) is shown. Next, for $t > 0$ we have

$$\begin{aligned}\int_0^\infty h_{t\beta^\zeta}(r)\mathrm{d}r &= \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{\cos(\zeta)t + \cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)t}{\sin(\zeta)t\sin\left(\frac{\pi}{\alpha}\right)}\right) \\ &= \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{\cos(\zeta) + \cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)}{\sin(\zeta)\sin\left(\frac{\pi}{\alpha}\right)}\right).\end{aligned}$$

By setting

$$C_2 := \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{\cos(\zeta) + \cos\left(\frac{\pi}{\alpha}\right)\sin(\zeta)}{\sin(\zeta)\sin\left(\frac{\pi}{\alpha}\right)}\right) < 1,$$

the second part of (3.2) is shown.
With Corollary 1.25 and Theorem 3.1 the claim follows. $\square$

Yet another special case of Theorem 3.2, where $\zeta = \frac{\pi}{2}$, is the precise form of the densities of Q^α. This generalises the known results on the hitting distribution Q on E of a Brownian motion, see e.g. [KM10] (2.5), to the wider class of spectrally positive stable processes. The parameters z_1, z_2 in the subsequent Lemma allow for a nice probabilistic interpretation, as we will see in Remark 3.8.

Corollary 3.3 *For $\alpha \in (1, 2]$ and X an SPMI of index α started at $x \in \mathring{\mathbb{W}}\left(\frac{\pi}{2}\right) = (0,\infty)^2$, the densities $\bar{f}_x$ on E_2 and f_x on E_1 of Q^α are given for $t > 0$ by*

$$\begin{aligned}\bar{f}_x(t) &= \frac{1}{\pi}\frac{\alpha t^{\alpha-1}z_2}{z_2^2 + (t^\alpha + z_1)^2} \quad \textit{and} \\ f_x(t) &= \frac{1}{\pi}\frac{\alpha t^{\alpha-1}z_2}{z_2^2 + (t^\alpha - z_1)^2},\end{aligned}$$

where

$$z_1 = \left(x_1^2 + x_2^2 + 2\cos\left(\frac{\pi}{\alpha}\right)x_1x_2\right)^{\alpha/2} \cos\left(\alpha\left(\frac{\pi}{2} - \arctan\left(\frac{x_1 + \cos\left(\frac{\pi}{\alpha}\right)x_2}{x_2\sin\left(\frac{\pi}{\alpha}\right)}\right)\right)\right),$$

$$z_2 = \left(x_1^2 + x_2^2 + 2\cos\left(\frac{\pi}{\alpha}\right)x_1x_2\right)^{\alpha/2} \sin\left(\alpha\left(\frac{\pi}{2} - \arctan\left(\frac{x_1 + \cos\left(\frac{\pi}{\alpha}\right)x_2}{x_2\sin\left(\frac{\pi}{\alpha}\right)}\right)\right)\right).$$

Proof. With (2.11), we have $r = \alpha$ and the claim follows. □

3.1.2 Proof of Theorem 3.1

Lemma 3.4 *Let X and ζ be as in Theorem 3.1. Then, for all $x \in \mathring{\mathbb{W}}(\zeta)$, $\mathcal{L}_x(X_\tau)$ has one-dimensional Lebesgue densities on E_1 and E_2^ζ. This means, there are functions $f_x^\zeta, \bar{f}_x^\zeta : [0,\infty) \to [0,\infty)$ such that for all $A \in \mathcal{B}(E_1)$ with $\bar{A} := A\beta^\zeta \in \mathcal{B}(E_2^\zeta)$*

$$\mathbf{P}_x\left[X_\tau \in \bar{A}\right] = \int_A \bar{f}_x^\zeta(s)\,ds \quad \textit{and}$$

$$\mathbf{P}_x\left[X_\tau \in A\right] = \int_A f_x^\zeta(s)\,ds.$$

Proof. In order to prove the existence of density functions, we make use of the Radon-Nikodym theorem. To this end, let $\bar{A} = \beta^\zeta A \subset E_2^\zeta$ be of Lebesgue measure zero. Here we mean the one-dimensional Lebesgue measure on the line $\mathbb{R}\beta^\zeta$. Then, obviously, also $A \subset [0,\infty)$ has zero Lebesgue measure. We make a decomposition of $\{X_\sigma \in \bar{A}\}$ as the disjoint union of the set where the paths of X hit the x-axis before hitting $\bar{A}$ and the set where the paths of X hit $\bar{A}$ without having hit the x-axis before,

$$\{X_\sigma \in \bar{A}\} = \{X_\sigma \in \bar{A} \text{ and } \inf_{s<\sigma} X_s^2 \le 0\} \,\dot\cup\, \{X_\sigma \in \bar{A} \text{ and } \inf_{s<\sigma} X_s^2 > 0\}.$$

So, we get for the probabilities

$$\begin{aligned}
\mathbf{P}_x\left[X_\tau \in \bar{A}\right] &= \mathbf{P}_x\left[X_\sigma \in \bar{A} \text{ and } \inf_{s<\sigma} X_s^2 > 0\right] \\
&\le \mathbf{P}_x\left[X_\sigma \in \bar{A}\right] \\
&= \int_A h_x^\zeta(s)\mathrm{d}s = 0.
\end{aligned}$$

By the Radon-Nikodym theorem, see [Kle13] Corollary 7.34, $\mathcal{L}_x(X_\tau)|_{E_2^\zeta}$ possesses a one-dimensional Lebesgue density.

On the other hand, we have, again by path decomposition for $A \subset \mathcal{B}([0,\infty))$ of Lebesgue measure zero,

$$\begin{aligned}
\{X_{\tau^2} \in A \times \{0\}\} =& \{X_{\tau^2} \in A \times \{0\} \text{ and } X_s \in \mathbb{R}\beta^\zeta \text{ for some } s \in [0,\tau^2)\} \\
&\dot\cup\, \{X_{\tau^2} \in A \times \{0\} \text{ and } X_s \notin \mathbb{R}\beta^\zeta \text{ for all } s \in [0,\tau^2)\}.
\end{aligned}$$

As above,

$$\begin{aligned}
\mathbf{P}_x\left[X_\tau \in A\right] &= \mathbf{P}_x\left[X_{\tau^2} \in A \times \{0\} \text{ and } X_s \notin \mathbb{R}\beta^\zeta \text{ for all } s \in [0,\tau^2)\right] \\
&\le \mathbf{P}_x\left[X_{\tau^2} \in A \times \{0\}\right] \\
&= \int_A h_x(s)\mathrm{d}s = 0.
\end{aligned}$$

Again by the Radon-Nikodym theorem, we get the existence of the density on E_1. $\square$

It is important to note that, as X and Y from Theorem 3.1 satisfy the same conditions, we also get the existence of densities for both X and Y. The following two results show that the density functions of X and Y actually have to coincide.

Lemma 3.5 *Let X, Y and ζ be as in Theorem 3.1. Then, the density functions from Lemma 3.4 for both X and Y satisfy the following integral equations for all $x \in \mathring{\mathbb{W}}(\zeta)$ and Lebesgue almost all $t > 0$*

$$f_x^\zeta(t) = h_x(t) - \int_0^\infty \bar{f}_x^\zeta(s) h_{s\beta^\zeta}(t)\,ds, \tag{3.3}$$

$$\bar{f}_x^\zeta(t) = h_x^\zeta(t) - \int_0^\infty f_x^\zeta(s) h^\zeta_{(s,0)}(t)\,ds. \tag{3.4}$$

Proof. We first consider X. Like in Lemma 3.4, we see that for $t > 0$

$$\begin{aligned}\{X_{\tau^2} \in [0,t) \times \{0\}\} = &\{X_{\tau^2} \in [0,t) \times \{0\} \text{ and } X_s \in \mathbb{R}\beta^\zeta \text{ for some } s \in [0,\tau^2)\} \\ &\dot\cup \{X_{\tau^2} \in [0,t) \times \{0\} \text{ and } X_s \notin \mathbb{R}\beta^\zeta \text{ for all } s \in [0,\tau^2)\}.\end{aligned}$$

We take a closer look at the first set on the right-hand side. By making a path-decomposition of X after the position of first hitting E_2^ζ and using the strong Markov property, we receive

$$\begin{aligned}&\mathbf{P}_x\left[X_{\tau^2} \in [0,t) \times \{0\} \text{ and } X_s \in \mathbb{R}\beta^\zeta \text{ for some } s \in [0,\tau^2)\right] \\ &= \int_0^\infty \bar{f}_x^\zeta(s) \mathbf{P}_{s\beta^\zeta}\left[X_{\tau^2} \in [0,t)\right] \mathrm{d}s.\end{aligned}$$

So, for $t > 0$, we end up with

$$\begin{aligned}\int_0^t f_x^\zeta(s)\mathrm{d}s &= \mathbf{P}_x\left[X_\tau \in [0,t) \times \{0\}\right] \\ &= \mathbf{P}_x\left[X_{\tau^2} \in [0,t) \times \{0\}\right] - \int_0^\infty \bar{f}_x^\zeta(s)\mathbf{P}_{s\beta^\zeta}\left[X_{\tau^2} \in [0,t)\right]\mathrm{d}s \\ &= \int_0^t h_x(s)\mathrm{d}s - \int_0^\infty \bar{f}_x^\zeta(s) \int_0^t h_{s\beta^\zeta}(r)\mathrm{d}r\,\mathrm{d}s \\ &= \int_0^t h_x(s)\mathrm{d}s - \int_0^t \int_0^\infty \bar{f}_x^\zeta(s) h_{s\beta^\zeta}(r)\mathrm{d}s\,\mathrm{d}r.\end{aligned}$$

The last equality is Fubini's theorem and justified by the non-negativity of $\bar{f}$ and h. By the well-known fact that $\int_0^t g_1(s)\mathrm{d}s = \int_0^t g_2(s)\mathrm{d}s$ implies $g_1(s) = g_2(s)$ for Lebesgue almost all $s \in [0,t]$, (3.3) follows in the case of X.
Next, we have as above

$$\begin{aligned}\{X_\sigma \in [0,t)\beta^\zeta\} = &\{X_\sigma \in [0,t)\beta^\zeta \text{ and } \inf_{s<\sigma} X_s^2 \le 0\} \\ &\dot\cup \{X_\sigma \in [0,t)\beta^\zeta \text{ and } \inf_{s<\sigma} X_s^2 > 0\}.\end{aligned}$$

Again, by path-decomposing X after first hitting E_1, we get

$$\mathbf{P}_x\left[X_\sigma \in [0,t)\beta^\zeta \text{ and } \inf_{s<\sigma} X_s^2 \leq 0\right] = \int_0^\infty f_x^\zeta(r)\mathbf{P}_{(0,r)}\left[X_\sigma \in [0,t)\beta^\zeta\right]\mathrm{d}r$$

and therefore,

$$\begin{aligned}\int_0^t \bar{f}_x^\zeta(s)\mathrm{d}s &= \mathbf{P}_x\left[X_\tau \in [0,t)\beta^\zeta\right] \\ &= \mathbf{P}_x\left[X_\sigma \in [0,t)\beta^\zeta\right] - \int_0^\infty f_x^\zeta(r)\mathbf{P}_{(0,r)}\left[X_\sigma \in [0,t)\beta^\zeta\right]\mathrm{d}r \\ &= \int_0^t h_x^\zeta(s)\mathrm{d}s - \int_0^\infty f_x^\zeta(r)\int_0^t h_{(r,0)}^\zeta(s)\mathrm{d}s\,\mathrm{d}r \\ &= \int_0^t h_x^\zeta(s)\mathrm{d}s - \int_0^t\int_0^\infty f_x^\zeta(r)h_{(r,0)}^\zeta(s)\mathrm{d}r\,\mathrm{d}s.\end{aligned}$$

As above, from this we get (3.4) for X.
We now want to stress that, as X and Y are Lévy processes, we also receive from (3.1) that

$$\begin{aligned}\mathcal{L}_x(X_\sigma) &= \mathcal{L}_x(Y_\sigma) \quad \text{and} \\ \mathcal{L}_x(X_{\tau^2}) &= \mathcal{L}_x(Y_{\tau^2})\end{aligned}$$

for all $x \in \mathbb{W}(\zeta)$. So, as h and h^ζ are identical for X and Y, we can replace X by Y in the above computations. Hence, we immediately see that the densities of Y also satisfy the integral equations (3.3) and (3.4) and the lemma is proved. □

Lemma 3.6 *Let h and h^ζ be one-dimensional probability density functions on $\mathbb{R}\times\{0\}$ and $\beta^\zeta\mathbb{R}$ respectively, which fulfil (3.2). Then, for all $x \in \mathring{\mathbb{W}}(\zeta)$, there is at most one pair of functions $f_x^\zeta, \bar{f}_x^\zeta \in L^1[0,\infty)$ which satisfies (3.3) and (3.4).*

Proof. We define the linear operators $K_1, K_2 : L^1[0,\infty) \to L^1[0,\infty)$ by

$$\begin{aligned}K_1 f(t) &:= \int_0^\infty f(s)h_{s\beta^\zeta}(t)\mathrm{d}s \quad \text{and} \\ K_2 f(t) &:= \int_0^\infty f(s)h_{(s,0)}^\zeta(t)\mathrm{d}s.\end{aligned}$$

We then have for $f \in L^1[0,\infty)$ with the second part of (3.2)

$$\begin{aligned}\|K_1 f\|_1 &= \int_0^\infty \left|\int_0^\infty f(s)h_{s\beta^\zeta}(t)\mathrm{d}s\right|\mathrm{d}t \\ &\leq \int_0^\infty\int_0^\infty |f(s)|h_{s\beta^\zeta}(t)\mathrm{d}s\,\mathrm{d}t \\ &= \int_0^\infty |f(s)|\int_0^\infty h_{s\beta^\zeta}(t)\mathrm{d}t\,\mathrm{d}s \\ &\leq C_2\|f\|_1.\end{aligned}$$

So, we see that K_1 indeed maps L^1 to L^1 and that K_1 is continuous with operator norm $\|K_1\| \le C_1 \le 1$. The same is true for K_2, as we see with

$$\|K_2 f\|_1 \le C_1 \|f\|_1.$$

As $C_1 \wedge C_2 < 1$, either K_1 or K_2 is in fact a contraction.
We can now re-write (3.3) and (3.4)

$$f_x^\zeta = h_x - K_1 \bar{f}_x^\zeta, \tag{3.5}$$
$$\bar{f}_x^\zeta = h_x^\zeta - K_2 f_x^\zeta. \tag{3.6}$$

Inserting the equations into each other leads to

$$(\mathrm{Id} - K_1 K_2)\, f_x^\zeta = h_x - K_1 h_x^\zeta,$$
$$(\mathrm{Id} - K_2 K_1)\, \bar{f}_x^\zeta = h_x^\zeta - K_2 h_x.$$

Here, $\mathrm{Id} : L^1 \to L^1$ is the identity operator. Note that by the operator norm inequality, we have

$$\|K_1 K_2\| \le \|K_1\| \|K_2\| = C_2 C_1 < 1$$

and the same way $\|K_2 K_1\| < 1$. Now, due to the fact that $L^1[0,\infty)$ is a Banach space, we get with [Kre99] Theorem 2.9 that $\mathrm{Id} - K_1 K_2$ has a bounded inverse operator, mapping $L^1[0,\infty)$ to $L^1[0,\infty)$. This operator is given by the Neumann series,

$$(\mathrm{Id} - K_1 K_2)^{-1} = \sum_{j=0}^{\infty} (K_1 K_2)^j.$$

The same is true when changing the roles of K_1 and K_2. We can now write down the functions f_x^ζ and $\bar{f}_x^\zeta$,

$$f_x^\zeta = (\mathrm{Id} - K_1 K_2)^{-1} \left(h_x - K_1 h_x^\zeta \right) \quad \text{and}$$
$$\bar{f}_x^\zeta = (\mathrm{Id} - K_2 K_1)^{-1} \left(h_x^\zeta - K_2 h_x \right),$$

which uniquely determines f_x^ζ and $\bar{f}_x^\zeta$ through an explicit representation in terms of h_x, h_x^ζ and the K_i, see also [Kre99] Theorem 2.10.
Note that $h_x|_{[0,\infty)}$ and $h_x^\zeta|_{[0,\infty)\beta^\zeta}$, as density functions of sub-probability measures, certainly belong to $L^1[0,\infty)$. $\square$

Finally, as a consequence of the considerations made so far, Theorem 3.1 is fully proved.

3.2 Direct implications of the integral representations

In Theorem 3.1, in order to compute the law of X_τ, it was crucial to find a corresponding process Y with the same hitting distributions on $\mathbb{R}\beta^\zeta$ and $\mathbb{R} \times \{0\}$, for which we also know the distribution of Y_τ. We now show that, even if we are not able to find such a process Y, there are already some interesting results which we can deduce from the integral equations (3.3) and (3.4).

To keep notation simple, we concentrate on the case $\zeta = \frac{\pi}{2}$ and $x = (1, \varepsilon)$ for some $\varepsilon > 0$, see also Remark 2.8. We shortly write f_ε for $f^{\pi/2}_{(1,\varepsilon)}$ and respectively $\bar{f}_\varepsilon$. The integral equations (3.3) and (3.4) then get a simpler form. The following holds for Lebesgue almost all $t \geq 0$

$$f_\varepsilon(t) = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi}\left(\frac{\varepsilon}{\left(\varepsilon\sin\left(\frac{\pi}{\alpha}\right)\right)^2 + \left(t - 1 - \varepsilon\cos\left(\frac{\pi}{\alpha}\right)\right)^2} - \int_0^\infty \bar{f}_\varepsilon(s)k(t,s)\mathrm{d}s\right), \tag{3.7}$$

$$\bar{f}_\varepsilon(t) = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi}\left(\frac{1}{\sin\left(\frac{\pi}{\alpha}\right)^2 + \left(t - \varepsilon - \cos\left(\frac{\pi}{\alpha}\right)\right)^2} - \int_0^\infty f_\varepsilon(s)k(t,s)\mathrm{d}s\right), \tag{3.8}$$

where for all $s, t \geq 0$

$$k(t,s) = \frac{s}{s^2 + t^2 - 2st\cos\left(\frac{\pi}{\alpha}\right)}. \tag{3.9}$$

First, we compute the total masses of Q^α on E_1 and E_2. The idea is quite similar to the one we used to compute p_1 in Section 2.3.2.

Lemma 3.7 *For all* $\varepsilon > 0$

$$Q^\alpha_{(1,\varepsilon)}\left(\{0\} \times [0,\infty)\right) = \int_0^\infty \bar{f}_\varepsilon(s)ds = \frac{\alpha}{2} - \frac{\alpha}{\pi}\arctan\left(\frac{1 + \varepsilon\cos\left(\frac{\pi}{\alpha}\right)}{\varepsilon\sin\left(\frac{\pi}{\alpha}\right)}\right).$$

Proof. Let X be an SPMI, $\bar{X}$ an independent copy of X and Y a Cauchy process with parameter $\sin\left(\frac{\pi}{\alpha}\right)$.

Recall that X^1 and τ^2 are independent. Note that, by (1.18),

$$\mathbf{P}_0\left[X^1_r \geq 0\right] = 1 - \frac{1}{\alpha}, \quad \text{for all } r > 0.$$

Hence, for all $s > 0$ we get

$$\begin{aligned}
\mathbf{P}_{(0,s)}\left[X^1_{\tau^2} \in [0,\infty)\right] &= \int_0^\infty \mathbf{P}_0\left[X^1_r \in [0,\infty)\right]\mathbf{P}_{(0,s)}\left[\tau^2 \in \mathrm{d}r\right] \\
&= \left(1 - \frac{1}{\alpha}\right)\int_0^\infty \mathbf{P}_{(0,s)}\left[\tau^2 \in \mathrm{d}r\right] \\
&= 1 - \frac{1}{\alpha}.
\end{aligned}$$

Hence, analogously to the computation in Lemma 3.5, by decomposing after first hitting the y-axis, we get with Lemma 2.14

$$\begin{aligned}
\int_0^\infty \bar{f}_\varepsilon(s)\mathrm{d}s =& Q^\alpha_{(1,\varepsilon)}\left(\{0\} \times [0,\infty)\right) \\
=& 1 - Q^\alpha_{(1,\varepsilon)}\left([0,\infty) \times \{0\}\right) \\
=& 1 - \mathbf{P}\left[Y_\varepsilon + \varepsilon\cos\Big(\frac{\pi}{\alpha}\Big) \in [-1,\infty)\right] + \int_0^\infty \bar{f}_\varepsilon(s)\mathbf{P}_{(0,s)}\left[\bar{X}^1_{\bar{\tau}^2} \in [0,\infty)\right]\mathrm{d}s \\
=& 1 - \mathbf{P}\left[Y_\varepsilon \in \left[-1 - \varepsilon\cos\Big(\frac{\pi}{\alpha}\Big), \infty\right)\right] + \left(1 - \frac{1}{\alpha}\right)\int_0^\infty \bar{f}_\varepsilon(s)\mathrm{d}s.
\end{aligned}$$

Rearranging the terms, we receive

$$\int_0^\infty \bar{f}_\varepsilon(s)\mathrm{d}s = \alpha\left(1 - \mathbf{P}\left[Y_\varepsilon \in \left[-1-\varepsilon\cos\left(\frac{\pi}{\alpha}\right),\infty\right)\right]\right). \tag{3.10}$$

We compute the probability on the right-hand side, note that $\varepsilon\sin\left(\frac{\pi}{\alpha}\right) > 0$,

$$\begin{aligned}\mathbf{P}\left[Y_\varepsilon \in \left[-1-\varepsilon\cos\left(\frac{\pi}{\alpha}\right),\infty\right)\right] &= \frac{1}{\pi}\int_{-1-\varepsilon\cos\left(\frac{\pi}{\alpha}\right)}^\infty \frac{\varepsilon\sin\left(\frac{\pi}{\alpha}\right)}{\left(\varepsilon\sin\left(\frac{\pi}{\alpha}\right)\right)^2+t^2}\mathrm{d}t \\ &= \frac{1}{2\pi}\left[2\arctan\left(\frac{1+\varepsilon\cos\left(\frac{\pi}{\alpha}\right)}{\varepsilon\sin\left(\frac{\pi}{\alpha}\right)}\right)+\pi\right].\end{aligned}$$

By inserting this into (3.10), the claim follows. □

Remark 3.8 *By the scaling relation of Q^α, we get the total mass on E_2 for general starting point $x \in (0,\infty)^2$,*

$$Q_x^\alpha(E_2) = Q^\alpha_{\left(1,\frac{x_2}{x_1}\right)}(E_2) = \frac{\alpha}{2} - \frac{\alpha}{\pi}\arctan\left(\frac{x_1+x_2\cos\left(\frac{\pi}{\alpha}\right)}{x_2\sin\left(\frac{\pi}{\alpha}\right)}\right).$$

With this quantity we get a nice representation of the density parameters z_1, z_2 in Corollary 3.3, namely

$$\begin{aligned} z_1 &= \left(x_1^2+x_2^2+2\cos\left(\frac{\pi}{\alpha}\right)x_1x_2\right)^{\alpha/2}\cos\left(\pi Q_x^\alpha(E_2)\right) \quad \textit{and} \\ z_2 &= \left(x_1^2+x_2^2+2\cos\left(\frac{\pi}{\alpha}\right)x_1x_2\right)^{\alpha/2}\sin\left(\pi Q_x^\alpha(E_2)\right).\end{aligned}$$

Next, we show that the densities of Q^α can be assumed to be continuous.

Lemma 3.9 *For every $x \in (0,\infty)^2$ the solutions f_x and $\bar{f}_x$ of (3.3) and (3.4) for $\zeta = \frac{\pi}{2}$ can be chosen to be continuous.*

Proof. We first prove the result for $x = (1,\varepsilon)$ with $\varepsilon > 0$. We just show the continuity of the right-hand side of (3.7), as the corresponding result for $\bar{f}_\varepsilon$ is completely analogue. Obviously,

$$t \mapsto \frac{\varepsilon}{\left(\varepsilon\sin\left(\frac{\pi}{\alpha}\right)\right)^2+\left(t-1-\varepsilon\cos\left(\frac{\pi}{\alpha}\right)\right)^2}$$

is continuous in $t \geq 0$, so it is enough to show that the integral part

$$t \mapsto \int_0^\infty \bar{f}_\varepsilon(s)\frac{s}{s^2+t^2-2st\cos\left(\frac{\pi}{\alpha}\right)}\mathrm{d}s,$$

which is independent of the choice of $\bar{f}_\varepsilon$, is continuous in $t \geq 0$. We use [Kle13] Theorem 6.27 p. 142 and thus have to show

i) for every $t \in [0,\infty)$ the mapping $s \mapsto \bar{f}_\varepsilon(s)\frac{s}{s^2+t^2-2st\cos\left(\frac{\pi}{\alpha}\right)}$ is in L^1, *ii*) for every $s \in (0,\infty)$ the mapping $t \mapsto \bar{f}_\varepsilon(s)\frac{s}{s^2+t^2-2st\cos\left(\frac{\pi}{\alpha}\right)}$ is continuous in all $[0,\infty)$,

iii) there is $h \in L^1$, $h \geq 0$ such that $\left|\bar{f}_\varepsilon(s)\frac{s}{s^2+t^2-2st\cos\left(\frac{\pi}{\alpha}\right)}\right| \leq h(s)$ for every $t \geq 0$.

i) Note that, for almost all $t \in [0,\infty)$, both $f_\varepsilon(t) \in [0,\infty)$ and

$$\frac{\varepsilon}{\left(\varepsilon\sin\left(\frac{\pi}{\alpha}\right)\right)^2 + \left(t-1-\varepsilon\cos\left(\frac{\pi}{\alpha}\right)\right)^2} \in (0,\infty).$$

So, by (3.7) and the simple fact that the integrand is non-negative, we also have *i*).
ii) This follows because for all $s \in (0,\infty)$ the mapping

$$t \mapsto \frac{s}{s^2+t^2-2st\cos\left(\frac{\pi}{\alpha}\right)}$$

is continuous in all $t \in [0,\infty)$.
iii) From *i*) we have in particular that

$$\int_0^\infty \frac{\bar{f}_\varepsilon(s)}{s}\mathrm{d}s = \frac{\varepsilon}{\left(\varepsilon\sin\left(\frac{\pi}{\alpha}\right)\right)^2 + \left(1+\varepsilon\cos\left(\frac{\pi}{\alpha}\right)\right)^2} - \frac{\pi}{\sin\left(\frac{\pi}{\alpha}\right)}f_\varepsilon(0) \in [0,\infty).$$

Let thus $h(s) := \frac{\bar{f}_\varepsilon(s)}{s} \in L^1$. Then, we readily have *iii*) as $t^2 - 2st\cos\left(\frac{\pi}{\alpha}\right) \geq 0$ and so

$$\frac{s}{s^2+t^2-2st\cos\left(\frac{\pi}{\alpha}\right)} \leq \frac{1}{s}.$$

Therefore, we are done with the case $x = (1,\varepsilon)$.
Let now $x \in (0,\infty)^2$ be arbitrary. By Lemma 2.7 *i*) we get a scaling relation for the density functions

$$\begin{aligned}\int_0^t f_x(s)\mathrm{d}s &= Q_x^\alpha\left([0,t]\times\{0\}\right)\\ &= Q^\alpha_{\left(1,\frac{x_2}{x_1}\right)}\left(\left[0,\frac{t}{x_1}\right]\times\{0\}\right)\\ &= \int_0^{t/x_1} f_{\frac{x_2}{x_1}}(s)\mathrm{d}s\\ &= \frac{1}{x_1}\int_0^t f_{\frac{x_2}{x_1}}\left(\frac{s}{x_1}\right)\mathrm{d}s.\end{aligned}$$

So, we also have that for every $T \geq 0$

$$f_x(T) = \frac{\mathrm{d}}{\mathrm{d}t}Q_x^\alpha\left([0,t]\times\{0\}\right)|_{t=T} = \frac{1}{x_1}f_{\frac{x_2}{x_1}}\left(\frac{T}{x_1}\right).$$

We have just seen that $f_{\frac{x_2}{x_1}}$ has a continuous version and so then has f_x. It should be noted that in the calculation above we made use of the substitution rule, and hence of the continuity, just in the special case of $f_{\frac{x_2}{x_1}}$, for which continuity was shown in the first part of the proof. □

Chapter 4

α-stable infinite rate mutually catalytic branching

In this last chapter, we finally come to introduce the α-stable infinite rate mutually catalytic branching process in one colony, which generalises the approach followed in [KM10]. We begin the first part by giving the definition of the process by stating its transition semigroup which will be shown to be a Feller semigroup. We then give a strong construction in terms of a given SPMI process, in analogy to [KM10] Theorem 1.6. The third section of the first part is devoted to present a Trotter-type construction of α-IMUB in one colony. It shall be stressed that all the results in this first section only rely on some very general properties of Q^α. The exact form is not needed in order to derive the mentioned results.
The second part of this chapter is dedicated to verifying that the α-IMUB Z is indeed the weak limit point of the finite γ processes, i.e. solutions to equation (2) from the introduction. We start by explaining what we mean by a solution to (2). Then, we show tightness of the family of solutions $(Y^n)_n$ to (2), where the branching rates γ_n tend to infinity. We regard the Y^n as probability measures on the space of càdlàg paths, topologised by the Meyer-Zheng topology, which we will introduce in Section 4.2.1. Thereafter, we give a characterisation of the law of any weak limit point in terms of a certain class of test functions and explain why this, indeed, proves convergence. Our proof relies on the self-duality relation for ρ-correlated Brownian motion with $\rho = -\cos(\frac{\pi}{\alpha})$.
Continue to let $\alpha \in (1, 2]$ and $E = [0, \infty)^2 \backslash (0, \infty)^2$.

4.1 Characterisation of α-IMUB

4.1.1 Definition of α-IMUB

In this section, we give the definition of a stochastic process living on E, which we call α-IMUB. To this end, we will define the operator semigroup which we will verify to be Feller-Dynkin in the sense of [RW00]. From this, we in particular get the existence of a nice version of our process. We do our best to stick to the notation of [RW00]

Chapter 3.
We first introduce the class of test functions for the transition semigroup. The restriction to functions having a unique limit at infinity enables us to extend these functions to the one-point compactification of E.

Definition 4.1 *Let*

$$\mathcal{C}_0 := \mathcal{C}_0(E) := \left\{ f : E \to \mathbb{R} \text{ continuous} \mid \lim_{r\to\infty} f(r,0) = \lim_{r\to\infty} f(0,r) = 0 \right\}.$$

$\mathcal{C}_0$ equipped with the uniform norm $\|\cdot\| := \|\cdot\|_\infty$ is a complete normed vector space. Recall Q^α from Definition 2.4.

Definition 4.2 *Fix $c > 0$ and $\theta \in [0,\infty)^2$. For $t \geq 0$ and $x \in E$ we define the transition kernel*

$$p_t(x,\cdot) := p_t^{\alpha,c,\theta}(x,\cdot) := Q^\alpha_{e^{-ct}x+(1-e^{-ct})\theta}.$$

Furthermore, we define the operator semigroup $(S_t)_{t\geq 0}$ on $\mathcal{C}_0$ by

$$S_t f(x) := \int_E f(y) p_t(x, dy).$$

The following lemma ensures that this definition makes sense, see also [RW00] Definition 6.5, page 241.

Lemma 4.3 $(S_t)_{t\geq 0}$ *is a Feller-Dynkin semigroup on E, i.e.,*

i) $S_t : \mathcal{C}_0 \to \mathcal{C}_0$ *for all* $t \geq 0$.

ii) For all $t \geq 0$ and $f \in \mathcal{C}_0$ with $0 \leq f \leq 1$, we have $0 \leq S_t f \leq 1$.

iii) $S_s S_t = S_{s+t}$ *for all $s, t \geq 0$ and $S_0 = \mathrm{Id}$, the identity operator on $\mathcal{C}_0$.*

iv) For all $f \in \mathcal{C}_0$, we have $\lim_{t\downarrow 0} \|S_t f - f\| = 0$.

Proof. i) Let $f \in \mathcal{C}_0$ and $t \geq 0$. First, note that $S_t f$ maps E to $\mathbb{R}$, as f is bounded and $p_t(x,\cdot)$ is a probability measure on E for all $x \in E$. Now, let $(x_n)_n$ be a sequence in E with limit $x \in E$. Then,

$$y_n := e^{-ct}x_n + (1-e^{-ct})\theta \xrightarrow{n\to\infty} e^{-ct}x + (1-e^{-ct})\theta =: y. \tag{4.1}$$

Using Lemma 2.5, we see, as f is bounded,

$$S_t f(x_n) = \int_E f(z) p_t(x_n, \mathrm{d}z) = \int_E f(z) Q^\alpha_{y_n}(\mathrm{d}z) \xrightarrow{n\to\infty} \int_E f(z) Q^\alpha_y(\mathrm{d}z) = S_t f(x),$$

which shows that $S_t f$ is continuous.
We show that

$$\lim_{r\to\infty} S_t f(r,0) = 0.$$

To this end, fix $\varepsilon > 0$ and let $R > 1$ big enough such that $|f(x,0)| < \frac{\varepsilon}{2}$ for all $x > R-1$. For a given sequence $(x_n)_n \subset (0,\infty) \times \{0\}$ with $\lim_{n\to\infty} x_n^1 = \infty$, let y_n be defined as in (4.1). Then,

$$0 < y_n^1 = e^{-ct}x_n^1 + (1-e^{-ct})\theta_1 \xrightarrow{n\to\infty} \infty.$$

Without loss of generality, we assume $y_n^1 > R$ for all n. Now, let $0 < K < \infty$ such that $|f| < K$. Note that for $a, K \in (0,\infty)$ we have

$$a\{z \in E : z_1 \leq K\} = \{z \in E : z_1 \leq aK\},$$

and therefore,

$$\begin{aligned} \frac{R}{y_n^1}\{z_1 \leq R-1\} &= \left\{z_1 \leq \frac{R}{y_n^1}(R-1)\right\} \\ &\subset \{z_1 \leq (R-1)\}, \end{aligned}$$

as $y_n^1 > R$. Here, we understand all appearing sets as subsets of E. Furthermore,

$$\frac{R}{y_n^1} y_n \xrightarrow{n\to\infty} (R,0).$$

Altogether, by making use of Lemma 2.7 *i*),

$$\begin{aligned} p_t(x_n, \{z_1 \leq R-1\}) = Q^\alpha_{y_n}(\{z_1 \leq R-1\}) &= Q^\alpha_{\frac{R}{y_n^1}y_n}\left(\frac{R}{y_n^1}\{z_1 \leq R-1\}\right) \\ &\leq Q^\alpha_{\frac{R}{y_n^1}y_n}(\{z_1 \leq R-1\}). \end{aligned}$$

With Lemma 2.5 and the Portemanteau theorem,

$$\limsup_{n\to\infty} Q^\alpha_{\frac{R}{y_n^1}y_n}(\{z_1 \leq R-1\}) \leq Q^\alpha_{(R,0)}(\{z_1 \leq R-1\}) = 0.$$

Therefore,

$$p_t(x_n, \{z_1 \leq R-1\}) \xrightarrow{n\to\infty} 0.$$

We can thus choose N big enough such that for all $n > N$,

$$p_t(x_n, \{z_1 \leq R-1\}) \leq \frac{\varepsilon}{2K}.$$

Then, for all $n > N$,

$$\begin{aligned} |S_t f(x_n)| &= \left| \int_E f(z) p_t(x_n, \mathrm{d}z) \right| \\ &\leq \int_{\{z_1 \leq R-1\}} |f(z)| p_t(x_n, \mathrm{d}z) + \int_{\{z_1 > R-1\}} |f(z)| p_t(x_n, \mathrm{d}z) \\ &\leq K p_t(x_n, \{z_1 \leq R-1\}) + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence,

$$\lim_{r\to\infty} S_t f(r,0) = 0.$$

The limit on the other axis is completely analogous.
ii) This follows directly from the fact that the Q_x^α are probability measures on E.
iii) We show that $(p_t)_t$ satisfies the Chapman-Kolmogorov equation. Let $s, t \geq 0$, $A \in \mathcal{B}(E)$ and $x \in E$. Note that we get measurability of the map $x \mapsto p_t(x, A)$ from Lemma 2.7 *iii*), as the transformation $x \mapsto e^{-ct}x + (1 - e^{-ct})\theta$ is continuous and therefore measurable. Now, with Lemma 2.7 *iv*),

$$\begin{aligned}
\int_E p_s(y, A)p_t(x, \mathrm{d}y) &= \int_E Q^\alpha_{e^{-cs}y+(1-e^{-cs})\theta}(A) Q^\alpha_{e^{-ct}x+(1-e^{-ct})\theta}(\mathrm{d}y) \\
&= Q^\alpha_{e^{-cs}(e^{-ct}x+(1-e^{-ct})\theta)+(1-e^{-cs})\theta}(A) \\
&= Q^\alpha_{e^{-c(s+t)}x+(1-e^{-c(s+t)})\theta}(A) = p_{s+t}(x, A).
\end{aligned}$$

So, we have $S_tS_s = S_{s+t} = S_sS_t$ for all $s, t \geq 0$.
Now assume $x \in E$ and $f \in \mathcal{C}_0$. Then,

$$S_0 f(x) = \int_E f(z)Q^\alpha_x(\mathrm{d}z) = \int_E f(z)\delta_x(\mathrm{d}z) = f(x).$$

This means that S_0 is the identity operator on $\mathcal{C}_0$.
iv) It is easy to see that $(S_t)_t$ is a Markov semigroup on $\mathcal{C}_0$ as defined in [RW00] III.3, p. 231. In our case, Condition (3.2) *iv*) of [RW00] follows with monotone convergence. Therefore, with [RW00] Lemma III.6.7 p. 241, all there is to show is that for all $f \in \mathcal{C}_0$ and $x \in E$ we have

$$S_t f(x) \to f(x), \qquad \text{as } t \downarrow 0.$$

Note that with $t \downarrow 0$ also $x(t) := e^{-ct}x + (1 - e^{-ct})\theta \to x$. So, with Lemma 2.5,

$$S_t f(x) = \int_E f(y)p_t(x, \mathrm{d}y) = \int_E f(y)Q^\alpha_{x(t)}(\mathrm{d}y) \xrightarrow{t\downarrow 0} \int_E f(y)Q^\alpha_x(\mathrm{d}y) = f(x).$$

□

Remark 4.4 *We want to mention that it is an easy consequence of Lemma 4.3 iv) that the assertion is also valid in the more general case, where t does not decrease monotonically,*

$$\textit{for all } f \in \mathcal{C}_0, \textit{ we have } \lim_{t\to 0,\, t\geq 0} \|S_t f - f\| = 0.$$

By E_∂ we denote the one-point compactification of E.

Lemma 4.5 *There is a strong Markov process Z with values in E_∂, càdlàg paths and transition semigroup given by S, i.e.,*

$$\mathbf{E}_x[f(Z_t)] = S_t f(x) \quad \textit{for all } x \in E,\ f \in \mathcal{C}_0.$$

Proof. This follows from [RW00] Chapter 3.7-3.9. $\square$

Definition 4.6 *A Markov process $(Z_t)_{t\geq 0}$ on E with transition semigroup S and càdlàg paths is called α stable infinite rate mutually catalytic branching process (α-IMUB).*

4.1.2 Strong construction

In this section, we give a strong construction for α-IMUB in terms of a given SPMI X. See in this context also [KM10] Theorem 1.6 and Section 4.
In the whole section, let X be an SPMI started at 0 and let D as in Definition 2.3. Let furthermore $\theta \in [0,\infty)^2$ and $c > 0$ be given.

Theorem 4.7 *For $x \in E$, we define the process Z on E starting at x by*

$$Z_t := Z_t^{c,\theta} := e^{-ct} D_{x+(e^{ct}-1)\theta}.$$

Then, Z is a Markov process on E with càdlàg paths and transition probabilities

$$p_t(x,\cdot) = \mathbf{P}_x[Z_t \in \cdot] = Q^{\alpha}_{e^{-ct}x+(1-e^{-ct})\theta}.$$

This is, Z is an α-IMUB.

The proof will be given as a series of lemmas.

Lemma 4.8 *The process $t \mapsto D_{x+(e^{ct}-1)\theta}$ has almost surely càdlàg paths.*

Proof. As we have already seen, the processes $\tau^i(X^i)$ have càdlàg paths. Therefore, also $\tau = \tau^1 \wedge \tau^2$ has càdlàg paths. Now, we have for $t_0 \in [0,\infty)$ with $t \downarrow t_0$

$$x^i + (e^{ct}-1)\theta^i \downarrow x^i + (e^{ct_0}-1)\theta^i, \quad \text{for } i = 1,2$$

and for $t_0 > 0$ with $t \uparrow t_0$ also

$$x^i + (e^{ct}-1)\theta^i \uparrow x^i + (e^{ct_0}-1)\theta^i, \quad \text{for } i = 1,2.$$

As the τ^i are monotonically increasing, we get for all t_0

$$\tau_{x+(e^{ct}-1)\theta} \downarrow \tau_{x+(e^{ct_0}-1)\theta}, \quad \text{for } t \downarrow t_0$$

and that $\lim_{t\uparrow t_0} \tau_{x+(e^{ct}-1)\theta}$ exists. Therefore, as X is càdlàg,

$$D_{x+(e^{ct}-1)\theta} = X_{\tau_{x+(e^{ct}-1)\theta}} + x + (e^{ct}-1)\theta$$

is càdlàg. $\square$

Lemma 4.9 *The process Z from Theorem 4.7 satisfies*

$$\mathbf{P}_x[Z_t \in A] = Q^{\alpha}_{e^{-ct}x+(1-e^{-ct})\theta}(A),$$

for all $x \in E$ and $A \in \mathcal{B}(E)$.

Proof.

$$\begin{aligned}\mathbf{P}_x[Z_t \in A] &= \mathbf{P}[e^{-ct} D_{x+(e^{ct}-1)\theta} \in A] \\ &= \mathbf{P}[D_{x+(e^{ct}-1)\theta} \in e^{ct} A] \\ &= Q^{\alpha}_{x+(e^{ct}-1)\theta}(e^{ct} A) \\ &= Q^{\alpha}_{e^{ct}(e^{-ct}x+(1-e^{-ct})\theta)}(e^{ct} A) \\ &= Q^{\alpha}_{e^{-ct}x+(1-e^{-ct})\theta}(A),\end{aligned}$$

where the last equality is due to Lemma 2.7 *i*). □

Lemma 4.10 *The process* $(Z_t)_{t\geq 0}$ *with start at* $x \in E$ *has the Markov property with respect to the filtration*

$$\mathcal{F}_t := \mathcal{F}^D_{x+(e^{ct}-1)\theta}.$$

This is, for $s, t \geq 0$,

$$\mathbf{P}_x[Z_{t+s} \in A|\mathcal{F}_t] = p_s(Z_t, A) = Q^{\alpha}_{e^{-cs}Z_t+(1-e^{-cs})\theta}(A), \quad \mathbf{P}_x\text{-almost surely.}$$

Proof. By Lemma 2.6, we have

$$\begin{aligned}\mathbf{P}_x[Z_{t+s} \in A|\mathcal{F}_t] &= \mathbf{P}\left[D_{x+(e^{c(t+s)}-1)\theta} \in e^{c(t+s)} A|\mathcal{F}^D_{x+(e^{ct}-1)\theta}\right] \\ &= \mathbf{P}\left[D_{x+(e^{ct}-1)\theta+e^{ct}(e^{cs}-1)\theta} \in e^{ct}e^{cs} A|\mathcal{F}^D_{x+(e^{ct}-1)\theta}\right] \\ &= Q^{\alpha}_{e^{ct}(e^{cs}-1)\theta+D_{x+(e^{ct}-1)\theta}}(e^{ct}e^{cs}A) \\ &= Q^{\alpha}_{e^{-cs}Z_t+(1-e^{-cs})\theta}(A) \\ &= p_s(Z_t, A).\end{aligned}$$

□

4.1.3 Trotter construction for α-IMUB

Another way of constructing the α-IMUB process is the so-called Trotter-type construction. We refer to the works [KO10] and [Oel08] for more information about the subject. The idea behind the construction is quite simple. For $x \in E$ and $\varepsilon > 0$, first consider for the time period of length ε the process without any random fluctuation. This means, during the interval $[0, \varepsilon)$, Z^{ε} consists only of a deterministic drift. The reason for proceeding this way is that, as Z starts at E, the noise term, given by

$$\int_0^t \left(\gamma Y^1_s Y^2_s\right)^{1/\alpha} \mathrm{d}X^i_s,$$

is absent at time zero, which explains why the movement is dominated by the deterministic drift in a short time period.
Next, choose a random point $z \in E$, sampled by the probability distribution $Q^{\alpha}_{Z^{\varepsilon}_{\varepsilon-}}$. This is due to that, by the scaling property of the driving process X, sending the

branching rate to infinity has the same effect as speeding up time in the noise term. This results in an immediate jump to the boundary, which explains the discontinuity in the construction.
In the succeeding interval the same procedure is repeated, now with starting point z. And so on.

We begin this section with the definition of the approximating processes Z^ε and thereafter show the weak convergence to an α-IMUB Z, as $\varepsilon \downarrow 0$.
We stick to the notation from the last section. In particular, $|\cdot|$ is, depending on the context, either the 1-norm in $\mathbb{R}^2$ or the absolute value in $\mathbb{R}$.

Construction of the approximating processes:

By $D(E) := D_{[0,\infty)}(E)$ we denote the space of all càdlàg functions $[0,\infty) \to E$, equipped with the Skorohod metrik d.
Let $x \in E$ and $\varepsilon > 0$. We construct the process $Z^\varepsilon = (Z^{1,\varepsilon}, Z^{2,\varepsilon})$, with values in $[0,\infty)^2$ starting at x, successively on $[k\varepsilon, (k+1)\varepsilon)$, for $k \in \mathbb{N}_0$.

$k = 0$: On $[0,\varepsilon)$, Z^ε is defined as the unique solution of the deterministic differential equation

$$\mathrm{d}Z_t^{\varepsilon,i} = c(\theta_i - Z_t^{\varepsilon,i})\mathrm{d}t, \quad \text{for } i = 1,2.$$

This means, as $Z_0^\varepsilon = x$,

$$Z_t^{\varepsilon,i} = \theta_i + (x_i - \theta_i)e^{-ct}, \quad \text{for } i = 1,2.$$

$k-1 \mapsto k$: Assume that Z^ε is already constructed on $[0,k\varepsilon)$. We construct Z^ε on $[k\varepsilon,(k+1)\varepsilon)$. At time $k\varepsilon$ the process has a discontinuity, choose $z \in E$ randomly, distributed according to the probability measure

$$Q^\alpha_{Z^\varepsilon_{k\varepsilon-}} = Q^\alpha_{\theta + (Z^\varepsilon_{(k-1)\varepsilon} - \theta)e^{-c\varepsilon}}.$$

Now, define $Z^\varepsilon_{k\varepsilon} := z$. On $[k\varepsilon,(k+1)\varepsilon)$, Z^ε is continuous with

$$\begin{aligned} Z_t^\varepsilon &= \theta + (Z^\varepsilon_{k\varepsilon} - \theta)e^{-c(t-k\varepsilon)} \\ &= e^{-c(t-k\varepsilon)} Z^\varepsilon_{k\varepsilon} + (1 - e^{-c(t-k\varepsilon)})\theta, \quad t \in [k\varepsilon,(k+1)\varepsilon). \end{aligned}$$

Obviously, Z^ε is càdlàg. Note that Z^ε is a time inhomogeneous Markov process. The discrete-time process $Y^\varepsilon := (Z^\varepsilon_{k\varepsilon})_{k\in\mathbb{N}_0}$ is, however, a time homogeneous Markov chain on E with transition probabilities $p^\varepsilon(x,\cdot) = Q^\alpha_{\theta+(x-\theta)e^{-c\varepsilon}}$.

Now, we show that the processes Z^ε converge weakly to an α-IMUB process Z, as $\varepsilon \downarrow 0$. To this end, let $\varepsilon_n \downarrow 0$ be a decreasing sequence of positive numbers. We define $Z^n := Z^{\varepsilon_n}$ and $Y^n := Y^{\varepsilon_n}$, with the same notation as above. In particular, $Z_0^n = x$ for all n. Furthermore, define the continuous time càdlàg processes $\bar{Z}^n$ by

$$\bar{Z}_t^n := Y^n_{\lfloor t/\varepsilon_n \rfloor}.$$

Then, we readily get $\bar{Z}^n_t = Z^n_{\lfloor t/\varepsilon_n \rfloor}$ and $\bar{Z}^n_t \in E$ for all $n \in \mathbb{N}_0$, $t \geq 0$.

Theorem 4.11 *The processes Z^n converge weakly in $D\left([0,\infty)^2\right)$ to an α-IMUB process Z, as $n \to \infty$.*

In the previous theorem, the topologisation of $D\left([0,\infty)^2\right)$ is given by the Skorohod metric, see e.g. [EK86] Chapter 3.5. As a preparation for the proof, we first show that $\bar{Z}^n$ converges weakly to Z ($\bar{Z}^n \Rightarrow Z$) and that the Markov chains Y^n are bounded on finite time intervals.
The proof of the subsequent Lemma crucially relies on a result from [EK86], which will be cited for the convenience of the reader. Note that $\hat{\mathcal{C}}$ in the notation of [EK86] is our class $\mathcal{C}_0$.

Theorem 4.12 *Let E be locally compact and separable. For $n = 1, 2, \ldots$, let $\mu_n(x, \Gamma)$ be a transition function on $E \times \mathcal{B}(E)$ such that T_n, defined by*

$$T_n f(x) = \int_E f(y)\mu_n(x, dy),$$

satisfies $T_n : \mathcal{C}_0 \to \mathcal{C}_0$. Suppose that S_t is a Feller-Dynkin semigroup on $\mathcal{C}_0$. Let $\varepsilon_n > 0$ satisfy $\lim_{n\to\infty} \varepsilon_n = 0$ and suppose that for every $f \in \mathcal{C}_0$,

$$\lim_{n\to\infty} T^{\lfloor t/\varepsilon_n \rfloor} f = S_t f, \quad t \geq 0.$$

For each $n \geq 1$, let $(Y^n_k)_k$ be a Markov chain on E with transition function $\mu_n(x, \Gamma)$, and suppose Y^n_0 has limiting distribution $\nu \in \mathcal{P}(E)$. Define X^n by $X^n_t = Y^n_{\lfloor t/\varepsilon_n \rfloor}$. Then, there is a Markov process X corresponding to S_t with initial distribution ν and sample paths in $D(E)$, and $X^n \Rightarrow X$.

Proof. See [EK86] Theorem 4.2.6, page 168. □

Lemma 4.13 *The laws of $\bar{Z}^n$ converge weakly to the law of Z as probability measures on $D(E)$.*

Proof. For $f \in \mathcal{C}_0$ define

$$T_n f(x) := \int_E f(y) Q^\alpha_{xe^{-c\varepsilon_n} + (1-e^{-c\varepsilon_n})\theta}(\mathrm{d}y) = S_{\varepsilon_n} f(x),$$

where S is the transition semigroup of α-IMUB from Definition 4.2. By Lemma 4.3, we have $T_n f \in \mathcal{C}_0$. Let $t \geq 0$ and define $k_n := \lfloor t/\varepsilon_n \rfloor \in \mathbb{N}_0$. Then, by the Chapman-Kolmogorov equation,

$$T_n^{\lfloor t/\varepsilon_n \rfloor} f = T_n^{k_n} f = S^{k_n}_{\varepsilon_n} f = S_{k_n \varepsilon_n} f.$$

Note that as $n \to \infty$, $k_n \varepsilon_n \to t$ and $k_n \varepsilon_n \leq t$. Therefore, by the continuity of S, cf. Lemma 4.3 *vi*), we have strongly in $\mathcal{C}_0$

$$\lim_{n\to\infty} T_n^{\lfloor t/\varepsilon_n \rfloor} f = \lim_{s \to t} S_s f = S_t f.$$

See also Remark 4.4. Recall that Y^n is a Markov chain on E with transition probabilities $Q^{\alpha}_{xe^{-c\varepsilon_n}+(1-e^{-c\varepsilon_n})\theta}$. With Theorem 4.12, we get that there exists a Markov process Z on E with transition semigroup S, this is an α-IMUB, such that the laws of $\bar{Z}^n$ converge weakly to the law of Z, as probability measures on $D(E)$. $\square$

Lemma 4.14 *Let $T \geq 0$ and $\varepsilon > 0$ be given. Let $\varepsilon_n \downarrow 0$ and define $k_n := \lfloor T/\varepsilon_n \rfloor \in \mathbb{N}_0$. Then, there is a $K > 0$, such that for all $n \in \mathbb{N}_0$*

$$\mathbf{P}\left[|Y^n_k| \geq K \textit{ for some } k = 0, \ldots, k_n\right] < \varepsilon.$$

Proof. For all $n \in \mathbb{N}_0$, we define the Markov chain $\bar{Y}^n_k := e^{c\varepsilon_n k} Y^n_k$ for $k = 0, 1, \ldots$. Then, as seen in Remark 2.12,

$$\begin{aligned}
\mathbf{E}\left[\bar{Y}^n_{k+1} | \bar{Y}^n_k\right] &= e^{c\varepsilon_n (k+1)} \mathbf{E}\left[Y^n_{k+1} | Y^n_k\right] \\
&= e^{c\varepsilon_n (k+1)} \left(\theta + \left(Y^n_k - \theta\right) e^{-c\varepsilon_n}\right) \\
&= \bar{Y}^n_k + (e^{c\varepsilon_n} - 1)\theta e^{c\varepsilon_n k} \geq \bar{Y}^n_k
\end{aligned}$$

for all $k \geq 0$. Therefore, the coordinates $\bar{Y}^{n,i}$ of $\bar{Y}^n$ are submartingales and are also non-negative. Furthermore, for all $N \in \mathbb{N}_0$,

$$\begin{aligned}
\mathbf{E}\left[\bar{Y}^n_N\right] &= x + (e^{c\varepsilon_n} - 1)\theta \sum_{l=0}^{N-1} e^{c\varepsilon_n l} \\
&= x + \theta \left(e^{c\varepsilon_n N} - 1\right),
\end{aligned}$$

as one readily gets from induction and the tower property of conditional expectation. With Doob's inequality applied to the coordinates of $\bar{Y}^n$, see e.g. [Kle13] Theorem 11.2, p.218, we have for every $K > 0$, $N \in \mathbb{N}_0$ and $n \in \mathbb{N}$

$$\begin{aligned}
&\mathbf{P}\left[|\bar{Y}^n_k| \geq K \text{ for some } k = 0, \ldots, N\right] \\
&\leq \sum_{i=1}^{2} \mathbf{P}\left[\bar{Y}^{n,i}_k \geq \frac{K}{2} \text{ for some } k = 0, \ldots, N\right] \\
&\leq \frac{2}{K}\left(\mathbf{E}\left[\bar{Y}^{n,1}_N\right] + \mathbf{E}\left[\bar{Y}^{n,2}_N\right]\right) \\
&= \frac{2}{K}\left(x_1 + x_2 + (e^{c\varepsilon_n N} - 1)(\theta_1 + \theta_2)\right).
\end{aligned} \tag{4.2}$$

Next, as $k_n \varepsilon_n \to T$,

$$\left(e^{c\varepsilon_n k_n} - 1\right) \xrightarrow{n \to \infty} \left(e^{cT} - 1\right).$$

In particular, there is a $C > 0$ such that $\left(e^{c\varepsilon_n k_n} - 1\right) < C$ for all $n \in \mathbb{N}$. Now choose K big enough such that for this C

$$\frac{2}{K}\left((x_1 + x_2) + C(\theta_1 + \theta_2)\right) < \varepsilon.$$

Then, for all $n \in \mathbb{N}$ with (4.2),

$$\begin{aligned}\mathbf{P}\left[|Y_k^n| \geq K \text{ for some } k = 0, \dots, k_n\right] &= \mathbf{P}\left[|\bar{Y}_k^n| \geq e^{c\varepsilon_n k} K \text{ for some } k = 0, \dots, k_n\right] \\ &\leq \mathbf{P}\left[|\bar{Y}_k^n| \geq K \text{ for some } k = 0, \dots, k_n\right] \\ &\leq \frac{2}{K}\Big((x_1 + x_2) + (e^{c\varepsilon_n k_n} - 1)(\theta_1 + \theta_2)\Big) \\ &\leq \frac{2}{K}\left((x_1 + x_2) + C(\theta_1 + \theta_2)\right) < \varepsilon.\end{aligned}$$

□

Proof of Theorem 4.11:

With Slutzky's theorem, see [Kle13] Theorem 13.18, p. 255, by Lemma 4.13 it is enough to show that $d(Z^n, \bar{Z}^n) \to 0$ stochastically in the Skorohod metric d on $D([0,\infty)^2)$. Let thus $\varepsilon, \delta > 0$ be given. Assume, for convenience, $\varepsilon < 2$. We show that there is an $N \in \mathbb{N}$ such that

$$\mathbf{P}[d(\bar{Z}^n, Z^n) > \varepsilon] < \delta, \quad \text{for all } n \geq N. \tag{4.3}$$

To this end, choose $T \in (0, \infty)$ such that $\int_T^\infty e^{-u} \mathrm{d}u = \frac{\varepsilon}{2}$. Let $k_n := \lfloor T/\varepsilon_n \rfloor \in \mathbb{N}_0$ and $K > 0$ big enough such that, with Lemma 4.14,

$$\mathbf{P}\left[|Y_k^n| \geq K \text{ for some } k = 0, \dots, k_n\right] < \delta, \quad \text{for all } n.$$

Finally, choose N big enough such that

$$T(K + \theta_1 + \theta_2)(1 - e^{-c\varepsilon_n}) < \frac{\varepsilon}{2}, \quad \text{for all } n \geq N.$$

We set

$$B_n := \{\omega : |Y_k^n(\omega)| \geq K \text{ for some } k = 0, \dots, k_n\}.$$

Then, for all $n \geq N$ on B_n^c, see [EK86] (5.2) page 117,

$$\begin{aligned}d(\bar{Z}^n, Z^n) &= \inf_{\lambda \in \Lambda}\left[\gamma(\lambda) \vee \int_0^\infty e^{-u} d(\bar{Z}^n, Z^n, \lambda, u)\mathrm{d}u\right] \\ &\leq \int_0^\infty e^{-u} \sup_{t \geq 0}\left(|\bar{Z}_{t\wedge u}^n - Z_{t\wedge u}^n| \wedge 1\right) \mathrm{d}u \\ &\leq \frac{\varepsilon}{2} + \int_0^T \sup_{t \geq 0} |\bar{Z}_{t\wedge u}^n - Z_{t\wedge u}^n| \mathrm{d}u \\ &\leq \frac{\varepsilon}{2} + \int_0^T \sup_{t \in [0,T]} |\bar{Z}_t^n - Z_t^n| \mathrm{d}u \\ &= \frac{\varepsilon}{2} + T \sup_{t \in [0,T]} |\bar{Z}_t^n - Z_t^n| \\ &\leq \frac{\varepsilon}{2} + T \sup_{k=0,\dots,k_n} |Y_k^n - \theta|(1 - e^{-c\varepsilon_n}) \\ &\leq \frac{\varepsilon}{2} + T(K + \theta_1 + \theta_2)(1 - e^{-c\varepsilon_n}) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

The second to last inequality follows from

$$\begin{aligned}|\bar{Z}_t^n - Z_t^n| &= |Y_k^n - \theta - (Y_k^n - \theta)e^{-ct}| \\ &= (1 - e^{-ct})|Y_k^n - \theta| \\ &\leq (1 - e^{-c\varepsilon_n})|Y_k^n - \theta|,\end{aligned}$$

where $t \in [0,T]$ and $k := \lfloor t/\varepsilon_n \rfloor$. Note that $Z_t^n = \bar{Z}_t^n = Y_k^n$ for $t = l\varepsilon_n$, $l \in \mathbb{N}_0$ and $\bar{Z}^n$ is piecewise constant, whereas Z^n moves away from $\bar{Z}^n$ monotonically by drifting, as indicated above.
In the first inequality, we set $\lambda(t) := t$. Then, plainly, $\lambda \in \Lambda$ with $\gamma(\lambda) = 0$. For the convenience of the reader, we give the definition of Λ and γ from [EK86]. By increasing in the subsequent definition, we always mean strictly increasing.

$$\begin{aligned}\gamma(\lambda) &:= \sup_{s>t\geq 0}\left|\log \frac{\lambda(s)-\lambda(t)}{s-t}\right|, \quad \text{for } \lambda : [0,\infty) \to [0,\infty) \quad \text{increasing}, \\ \Lambda &:= \{\lambda : [0,\infty) \to [0,\infty)\} | \, \lambda \text{ is increasing, bijective and Lipschitz } \}.\end{aligned}$$

The above inequality shows that $B_n^c \cap \{d(\bar{Z}^n, Z^n) > \varepsilon\} = \emptyset$ for all $n \geq N$. So, for $n \geq N$ with N as above, we have

$$\mathbf{P}[\{d(\bar{Z}^n, Z^n) > \varepsilon\}] = \mathbf{P}[\{d(\bar{Z}^n, Z^n) > \varepsilon\} \cap B_n] \leq \mathbf{P}[B_n] < \delta.$$

This proves Theorem 4.11.

4.2 Convergence of the finite γ processes

So far, we constructed a process Z with values in E, which is a candidate for the limit process of solutions to (2), as $\gamma \to \infty$. In the case where $\alpha = 2$, convergence was proved in [KM10]. In this section, we show that the α-IMUB Z is indeed the unique limit point for any choice of solutions to the SDEs (2), as $\gamma \to \infty$. Our approach makes use of the self-duality relation for mutually symbiotic branching processes, which was introduced in the case of $\rho = 0$ in [Myt98] and extended to general $\rho \in (-1,1)$ in [EF04]. We will discuss this self-duality in Chapter 4.2.3.
Until the end of this chapter, let $\alpha \in (1,2)$ be fixed. We write $\Rightarrow$ for distributional convergence of random variables and denote the stochastic integral $\int Y_s \mathrm{d}X_s$ by $Y \cdot X$. Furthermore, we continue to write $|x| = |x_1| + |x_2|$ for $x \in \mathbb{R}^2$.

4.2.1 Definition of the finite γ processes

We begin this section by giving a precise definiton of the finite γ processes and then state some general results about the Meyer-Zheng pseudo-path topology, which we will use later on. See [Kur91] or [MZ84] for more information about the subject.
We want to stress that the assumptions we make concerning the existence of solutions to (2) are obviously stronger than needed. However, we did not want to bother too

much about technical details, but rather quickly come to the investigation of the self-duality relation for stable processes.
At the end of this section, we state our main convergence result. For more information about stochastic integrals with respect to point processes we refer to Chapter II.3 of [IW89].

Let X be an SPMI of index α for some $\alpha \in (1,2)$. Let further $c \geq 0$ and $\theta \in [0,\infty)^2$ be given. We assume that, for all branching rates $\gamma > 0$ and starting points $y \in [0,\infty)^2$, the pair of stochastic differential equations

$$\begin{aligned} \mathrm{d}Y_t^1 &= c(\theta_1 - Y_t^1)\mathrm{d}t + \left(\gamma Y_t^1 Y_t^2\right)^{1/\alpha} \mathrm{d}X_t^1, \\ \mathrm{d}Y_t^2 &= c(\theta_2 - Y_t^2)\mathrm{d}t + \left(\gamma Y_t^1 Y_t^2\right)^{1/\alpha} \mathrm{d}X_t^2, \end{aligned} \tag{4.4}$$

has a weak solution with $Y_0 = y$. To be more precise, we assume the existence of a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, an $\mathbb{F}$-adapted SPMI X of index α and an $\mathbb{F}$-adapted càdlàg process Y with values in $[0,\infty)^2$, such that almost surely for all $t \geq 0$

$$\begin{aligned} Y_t^1 &= y_1 + \int_0^t c(\theta_1 - Y_s^1)\mathrm{d}s + \int_0^t \left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} \mathrm{d}X_s^1, \\ Y_t^2 &= y_2 + \int_0^t c(\theta_2 - Y_s^2)\mathrm{d}s + \int_0^t \left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} \mathrm{d}X_s^2. \end{aligned} \tag{4.5}$$

As usual, we mean by Y_{t-}^i the left-continuous process given by

$$Y_{t-}^i := \begin{cases} \lim_{s \uparrow t} Y_s^i, & \text{if } t > 0, \\ Y_0^i, & \text{if } t = 0. \end{cases}$$

We denote by ΔX^i the process

$$\Delta X_t^i = X_t^i - X_{t-}^i,$$

which represents the jumps of X^i. Furthermore, we define for $i = 1,2$, $A \in \mathcal{B}((0,\infty))$ and $t > 0$ the $\bar{\mathbb{N}}_0$-valued random variable

$$N_p^i(A \times [0,t]) := \#\left\{s \leq t : \Delta X_s^i \in A\right\}.$$

As X is an SPMI process, the random measures N_p^1 and N_p^2 are independent Poisson point processes on $[0,\infty) \times [0,\infty)$ with intensity measure (or compensator)

$$\hat{N}_p^i(\mathrm{d}h, \mathrm{d}t) = \frac{1}{\Gamma(-\alpha)} \mathbb{1}_{\{h>0\}} h^{-\alpha-1} \mathrm{d}h \otimes \mathrm{d}t,$$

see for example [IW89] Chapter I.9 or [Kyp14] Chapter 2.2. We define furthermore for $t > 0$ and $U \in \mathcal{B}((0,\infty))$ with $\int_U h^{-\alpha-1}\mathrm{d}h < \infty$ the compensated random measure

$$\widetilde{N}_p^i(t,U) := N_p^i(t,U) - \hat{N}_p^i(t,U).$$

The stochastic processes $t \mapsto \widetilde{N}_p^i(t, U)$ are $\mathbb{F}$-martingales, null at zero. To be more precise, the processes are compensated Poisson processes. We may write

$$X_t^i = \int_0^t \int_0^\infty h \widetilde{N}_p^i(\mathrm{d}h, \mathrm{d}s),$$

where the integral on the right-hand side is understood as the sum of a compensated compound Poisson process of jumps greater than one and the L^2-martingale of small jumps. See [Kyp14] Chapter 2.4 for more information.
We have to impose an additional assumption to the process Y, which is due to the fact that we need the stochastic integrals $\int \left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} \mathrm{d}X_s^i$ to be proper martingales and not just local martingales. The condition is

$$\mathbf{E}\left[\int_0^t \left(\gamma Y_s^1 Y_s^2\right)^{2/\alpha} \mathrm{d}s\right] < \infty \quad \text{for all } t. \tag{4.6}$$

As mentioned in [App09] Example 4.3.8, from this assumption we already get that the integrals with respect to the X^i in (4.5) are well-defined. As another important consequence, we get for any $0 \leq p \leq 2/\alpha$ that

$$\mathbf{E}\left[\int_0^t \left(\gamma Y_s^1 Y_s^2\right)^{p} \mathrm{d}s\right] < \infty \quad \text{for all } t. \tag{4.7}$$

We want to stress that assumption (4.6) is most probably too strong. In order to prove our result without having explicitly shown the existence of a solution to (4.4), we need to, however, impose this assumption due to technical reasons.
With the considerations made so far, it is clear that the stochastic integrals in (4.5) have to be understood in the following sense,

$$\begin{aligned}\int_0^t \left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} \mathrm{d}X_s^i &= \int_0^t \int_0^\infty h \left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} \widetilde{N}_p^i(\mathrm{d}h, \mathrm{d}s) \\ &= \int_0^t \int_0^1 h \left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} \widetilde{N}_p^i(\mathrm{d}h, \mathrm{d}s) \\ &\quad + \int_0^t \int_1^\infty h \left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} \widetilde{N}_p^i(\mathrm{d}h, \mathrm{d}s).\end{aligned}$$

Then,

$$\begin{aligned}\int_0^t \int_1^\infty h \left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} \widetilde{N}_p^i(\mathrm{d}h, \mathrm{d}s) &= \int_0^t \int_1^\infty h \left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} N_p^i(\mathrm{d}h, \mathrm{d}s) \\ &\quad - \frac{1}{(\alpha-1)\Gamma(-\alpha)} \int_0^t \left(\gamma Y_s^1 Y_s^2\right)^{1/\alpha} \mathrm{d}s.\end{aligned}$$

is a martingale due to [IW89] equation (II.3.8). The first integral on the right-hand side is no more than an integral with respect to a compound Poisson process and therefore to be understood as a pathwise Lebesgue-Stieltjes integral. On the other hand,

$$\int_0^t \int_0^1 h \left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} \widetilde{N}_p^i(\mathrm{d}h, \mathrm{d}s)$$

is an integral with respect to the L^2-martingale represented by the compensated jumps of X^i of magnitude smaller than one. Due to (4.6) and the considerations in [IW89] p. 63, this process is also an L^2-martingale, null at zero.

We now discuss in more detail the way we want to topologise the space $D = D([0,\infty)^2)$ of all càdlàg functions $f : [0,\infty) \to [0,\infty)^2$. As the normal topologisation, which is given by the Skorohod metric, is, at least in the case of Brownian motion, too strong to show convergence, we choose a weaker topology, i.e., the topology induced by convergence in measure, also called the Meyer-Zheng pseudo-path topology. See [Kur91] Section 4 or [MZ84] for more information about the subject. To be more precise, for two Borel measurable functions $f, g : [0,\infty) \to [0,\infty)^2$, we define the equivalence relation

$$f \sim g :\Leftrightarrow \int_0^\infty \mathbb{1}_{f(s)\neq g(s)} \mathrm{d}s = 0.$$

Let

$$M = M_{[0,\infty)^2}[0,\infty) = \{f : [0,\infty) \to [0,\infty)^2 \text{ Borel measurable}\}/_\sim$$

the space of equivalence classes of such functions. Convergence in measure on M is then metrified by the metric

$$d(f,g) := \int_0^\infty e^{-t} \left(1 \wedge |f(t) - g(t)|\right) \mathrm{d}t. \tag{4.8}$$

Of course, if f and g are càdlàg, then $d(f,g) = 0$ implies $f = g$. We can thus understand D as a subset of M. In general, however, D is not complete with respect to d. In the sequel, let M always be equipped with the topology induced by the metric d. The remainder of this chapter will be concerned with the proof of the following theorem.

Theorem 4.15 *For any given sequences $\gamma_n \to \infty$ and $y^n \to z \in E$ with $\gamma_n > 0$ and $y^n \in [0,\infty)^2$, let Y^n be a $D([0,\infty)^2)$-valued solution of (4.4) with branching rate γ_n and starting point y^n, driven by some SPMI X^n. Let furthermore Z be an α-IMUB starting at z. Then, we have weakly in the Meyer-Zheng topology*

$$Y^n \Rightarrow Z, \quad \text{as } n \to \infty$$

and for Lebesgue almost all $t \in [0,\infty)$ for the one-dimensional marginals

$$Y^n_t \Rightarrow Z_t, \quad \text{as } n \to \infty,$$

weakly as probability measures on $[0,\infty)^2$.

4.2.2 Tightness of $(Y^n)_n$

We start with simple, nonetheless important first moment bounds for the finite γ processes.

Lemma 4.16 *Let Y be a solution to (4.4) for some given $\gamma > 0$ and starting point $y \in [0, \infty)^2$. We have for $i = 1, 2$ and $t \geq 0$*

$$\mathbf{E}\left[Y_t^i\right] \leq y_i + c\theta_i t, \tag{4.9}$$

$$c\,\mathbf{E}\left[\int_0^t Y_s^i\, ds\right] \leq y_i + c\theta_i t. \tag{4.10}$$

Proof. Note that we have for $i = 1, 2$

$$Y_t^i + c\int_0^t Y_s^i \mathrm{d}s = y_i + c\theta_i t + \int_0^t \left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} \mathrm{d}X_s^i.$$

With the considerations from the last section,

$$\mathbf{E}\left[Y_t^i\right] + c\mathbf{E}\left[\int_0^t Y_s^i \mathrm{d}s\right] = y_i + c\theta_i t.$$

In particular, as $Y^i \geq 0$, for all $t \geq 0$

$$\mathbf{E}\left[Y_t^i\right] = \mathbf{E}\left[|Y_t^i|\right] \leq y_i + c\theta_i t$$
$$c\mathbf{E}\left[\int_0^t Y_s^i \mathrm{d}s\right] = c\mathbf{E}\left[\left|\int_0^t Y_s^i \mathrm{d}s\right|\right] \leq y_i + c\theta_i t.$$

This is (4.9) and (4.10).

□

As an easy consequence of the above lemma, we get tightness of the sequence $(Y^n)_n \subset M$. In order to prove this result, we have to introduce the conditional variation $V(Y)$ of an $\mathbb{F}$-adapted càdlàg process Y, see [Kur91] equation (1.1).

$$V_t(Y) := \sup \mathbf{E}\left[\sum_i \left|\mathbf{E}\left[Y_{t_{i+1}} - Y_{t_i} | \mathcal{F}_{t_i}\right]\right|\right],$$

where the supremum is taken over all partitions $0 = t_1 < t_2 < \ldots < t_m = t$ of the interval $[0, t]$.

Lemma 4.17 *Let Y^n be as in Theorem 4.15. The sequence $(Y^n)_n$ is tight in M and any weak limit point Y admits a càdlàg version.*

Proof. We apply [Kur91] Theorem 5.8. This is, we have to show that for all $t > 0$ and $i = 1, 2$

$$C^i(t) := \sup_n \left(V_t(Y^{n,i}) + \mathbf{E}\left[Y_t^{n,i}\right]\right) < \infty.$$

To this end, we write for $n \in \mathbb{N}$ and $t \geq 0$

$$\begin{aligned} Y_t^n &= y^n + \int_0^t c(\theta - Y_s^n)\mathrm{d}s + \int_0^t \left(\gamma_n Y_{s-}^{n,1} Y_{s-}^{n,2}\right)^{1/\alpha} \mathrm{d}X_s^n \\ &=: y^n + A_t^n + M_t^n. \end{aligned}$$

It is clear from the definition of V_t that

$$V_t(Y^{n,i}) = V_t(A^{n,i} + M^{n,i}) \leq V_t(A^{n,i}) + V_t(M^{n,i}).$$

Hence, it is enough to consider $V_t(A^{n,i})$ and $V_t(M^{n,i})$ separately. We first consider the conditional variation of M^n. As mentioned in [MZ84] p. 358, as M^n is a martingale,

$$V_t(M^{n,i}) = \sup_{s\leq t} \mathbf{E}\left[|M_s^{n,i}|\right].$$

We have that

$$\begin{aligned}
|M_t^{n,i}| &= \left|Y_t^{n,i} - y_i^n - c\int_0^t (\theta_i - Y_s^{n,i})\mathrm{d}s\right| \\
&\leq Y_t^{n,i} + y_i^n + c\theta_i t + c\int_0^t Y_s^{n,i}\mathrm{d}s.
\end{aligned}$$

So, with Lemma 4.16,

$$\mathbf{E}\left[|M_t^{n,i}|\right] \leq 3(y_i^n + c\theta_i t).$$

Hence,

$$V_t(M^{n,i}) \leq 3(y_i^n + c\theta_i t) < \infty.$$

We now consider A^n. Note that by the conditional version of Jensen's inequality, see [Kle13] Theorem 8.20, we have for any integrable random variable Y and any σ-field $\mathcal{G}$

$$\mathbf{E}\left[|\mathbf{E}\left[Y|\mathcal{G}\right]|\right] \leq \mathbf{E}\left[|Y|\right].$$

With this, we get for any partition $0 = t_1 < \ldots < t_m = t$ and $i = 1, 2$

$$\begin{aligned}
\sum_{j=1}^{m-1} \mathbf{E}\left[\left|\mathbf{E}\left[A_{t_{j+1}}^{n,i} - A_{t_j}^{n,i}|\mathcal{F}_{t_j}\right]\right|\right] &\leq \sum_{j=1}^{m-1} \mathbf{E}\left[\left|A_{t_{j+1}}^{n,i} - A_{t_j}^{n,i}\right|\right] \\
&= \sum_{j=1}^{m-1} \mathbf{E}\left[\left|\int_{t_j}^{t_{j+1}} c(\theta_i - Y_s^{n,i})\mathrm{d}s\right|\right] \\
&\leq \sum_{j=1}^{m-1} \left(c\theta_i(t_{j+1} - t_j) + c\mathbf{E}\left[\int_{t_j}^{t_{j+1}} Y_s^{n,i}\mathrm{d}s\right]\right) \\
&= c\theta_i t + c\mathbf{E}\left[\int_0^t Y_s^{n,i}\right]\mathrm{d}s \\
&\leq 2c\theta_i t + y_i^n.
\end{aligned}$$

As y_i^n is convergent, for all t there exists a constant C^i, independent of n and the partition, such that

$$\sum_{j=1}^{m-1} \mathbf{E}\left[\left|\mathbf{E}\left[A_{t_{j+1}}^{n,i} - A_{t_j}^{n,i}|\mathcal{F}_{t_j}\right]\right|\right] \leq C^i.$$

Therefore,

$$\sup_n V_t(A^{n,i}) \leq C^i.$$

We have thus shown that $\sup_n V_t(Y^{n,i}) < \infty$. As $\mathbf{E}\left[Y_t^{n,i}\right]$ is bounded uniformly in n, we get with [Kur91] Theorem 5.8 that $(Y^n)_n$ is relatively compact and therefore tight in $M_{[0,\infty)^2}[0,\infty)$ and any weak limit point Y has a càdlàg version. □

The reader should note that the preceding lemma was proved by showing the uniform boundedness of the conditional variations

$$\sup_n V_t(Y^{n,i}) < \infty.$$

From this we also get the convergence of the finite-dimensional marginals for a subset $I \subset [0,\infty)$ of full Lebesgue measure. The following result is taken from [MZ84].

Lemma 4.18 *Let $(Y^n)_{n\in\mathbb{N}}$ and Y be càdlàg processes on $[0,\infty)^2$ such that*

$$Y^n \Rightarrow Y$$

weakly in the Meyer-Zheng pseudo-path topology. There is a set $I \subset [0,\infty)$ of full Lebesgue measure such that, along a subsequence, for any $l \in \mathbb{N}$ and any choice $(t_1,\dots,t_l) \in I^l$

$$\left(Y_{t_1}^{n_k},\dots,Y_{t_l}^{n_k}\right) \Rightarrow \left(Y_{t_1},\dots,Y_{t_l}\right),$$

as $k \to \infty$, weakly as probability measures on $\left([0,\infty)^2\right)^l$.

Proof. This is a special case of Theorem 5 of [MZ84]. □

4.2.3 Characterisation of the limit points

The following result is basically a modification of [Kal92] Proposition 4.4 a). Although the statement may look trivial and the proof of [Kal92] Proposition 4.4 a) actually does not use that the stochastic integrals are driven by the same stable process, we decided to give the proof in full detail, as the proof in [Kal92] is quite involved.
Recall that for any random process Y the $[0,\infty]$-valued random variable Y^* is defined as

$$Y^* = \sup_{s\in[0,\infty)} |Y_s|.$$

Lemma 4.19 *Let $(Y^n)_n$ be as in Theorem 4.15 and $i \in \{1,2\}$. If for given $t \geq 0$*

$$\int_0^t \left(Y_{s-}^{n,1}Y_{s-}^{n,2}\right)^{1/\alpha} dX_s^{n,i} \xrightarrow{n\to\infty} 0 \quad \textit{in } L^1,$$

then

$$\int_0^t Y_s^{n,1}Y_s^{n,2} ds \xrightarrow{n\to\infty} 0 \quad \textit{in probability.}$$

Proof. We define for $n \in \mathbb{N}$ and $s \geq 0$ the processes

$$V_s^n := \mathbb{1}_{[0,t]}(s) \left(Y_{s-}^{n,1} Y_{s-}^{n,2}\right)^{1/\alpha}.$$

Then V^n is left-continuous with limits from the right and, plainly,

$$(V^n \cdot X^{n,i})^* = \sup_{0 \leq r \leq t} \left| \int_0^r \left(Y_{s-}^{n,1} Y_{s-}^{n,2}\right)^{1/\alpha} \mathrm{d}X_s^{n,i} \right|.$$

Let $\varepsilon > 0$ be given. With Doob's inequality and the assumed L^1-convergence, we get

$$\mathbf{P}\left[(V^n \cdot X^{n,i})^* > \varepsilon\right] \leq \frac{\mathbf{E}\left[\left|\int_0^t \left(Y_{s-}^{n,1} Y_{s-}^{n,2}\right)^{1/\alpha} \mathrm{d}X_s^{n,i}\right|\right]}{\varepsilon} \xrightarrow{n\to\infty} 0. \tag{4.11}$$

Hence, $(V^n \cdot X^{n,i})^*$ converges to zero in probability.
Let X be a one-dimensional Lévy process with characteristic exponent given by $-(-i\theta)^\alpha$. We view X as a random variable on the Skorohod space $D[0,\infty)$ and write $\mathbf{P}_X$ for the law of X. Let the space of all left-continuous functions $f : [0,\infty) \to [0,\infty)$ with limits from the right, càglàd for short, be denoted by $\widetilde{D}[0,\infty)$. Then, $\widetilde{D}[0,\infty)$ is also Polish and we may view the processes V^n as random variables on $\widetilde{D}[0,\infty)$. We define the probability space Ω by

$$\Omega := D[0,\infty) \times \left(\widetilde{D}[0,\infty)\right)^{\mathbb{N}}$$

and denote by κ^n a regular version of the conditional expectation of V^n given X^n,

$$\kappa^n(x,\cdot) := \mathbf{E}\left[V^n \in \cdot | X^n = x\right].$$

Furthermore, we define a probability measure $\mathbf{P}$ on Ω by the following relation. Let for given $l \in \mathbb{N}$, $A \subset D[0,\infty)$ be measurable and $B_k \subset \widetilde{D}[0,\infty)$ be measurable for all $k = 1,\dots,l$. We set

$$\mathbf{P}\left[A \times B_1 \times \ldots \times B_l\right] := \int_A \kappa^1(x,B_1)\ldots\kappa^l(x,B_l)\mathbf{P}_X(\mathrm{d}x).$$

Let the processes $\hat{V}^n$ be defined as the n-th coordinate of Ω,

$$\hat{V}^n := \pi_n.$$

Clearly, we can think of $X = \pi_0$. Then, for all $A \subset D[0,\infty)$ and $B \subset \widetilde{D}[0,\infty)$ measurable,

$$\begin{aligned}
\mathbf{P}\left[(X,\hat{V}^n) \in A \times B\right] &= \int_A \kappa^n(x,B)\mathbf{P}_X(\mathrm{d}x) \\
&= \int_A \kappa^n(x,B)\mathbf{P}_{X^n}(\mathrm{d}x) \\
&= \mathbf{P}\left[(X^n,V^n) \in A \times B\right].
\end{aligned}$$

In particular,

$$\mathbf{P}\left[\hat{V}^n \in B\right] = \mathbf{P}\left[V^n \in B\right].$$

This means, for all $n \in \mathbb{N}$,

$$\begin{aligned}(X, \hat{V}^n) &\stackrel{d}{=} (X^n, V^n) \quad \text{and} \\ V^n &\stackrel{d}{=} \hat{V}^n.\end{aligned}$$

From this we get

$$\hat{V}^n \cdot X \stackrel{d}{=} V^n \cdot X^n$$

and hence with (4.11) we see that $(\hat{V}^n \cdot X)^*$ converges to zero in probability. Therefore, with [Kal92] Proposition 4.4 a),

$$\int_0^\infty (\hat{V}_s^n)^\alpha \mathrm{d}s \xrightarrow{n\to\infty} 0 \quad \text{in probability}$$

and thus also

$$\int_0^t Y_s^{n,1} Y_s^{n,2} \mathrm{d}s = \int_0^\infty (V_s^n)^\alpha \mathrm{d}s \xrightarrow{n\to\infty} 0 \quad \text{in probability.}$$

This is the assertion. □

The next result ensures that any weak limit point Y of the sequence $(Y^n)_n$ is almost surely concentrated on the boundary E of the first quadrant.

Lemma 4.20 *Let $(Y^n)_n$ be as in Theorem 4.15. The càdlàg version Y of any weak limit point of $(Y^n)_n$ almost surely satisfies for all $t \geq 0$*

$$Y_t^1 Y_t^2 = 0.$$

Proof. As Y^n is a solution to (4.4), we have for all $n \in \mathbb{N}$ and $i = 1, 2$,

$$\int_0^t \left(\gamma_n Y_{s-}^{n,1} Y_{s-}^{n,2}\right)^{1/\alpha} \mathrm{d}X_s^{n,i} = Y_t^{n,i} - y_i^n - c\theta_i t + c \int_0^t Y_s^{n,i} \mathrm{d}s.$$

From Lemma 4.16 we get the existence of a constant $C = C(t)$, independent of n, such that for all n

$$\mathbf{E}\left[\left|\int_0^t \left(\gamma_n Y_{s-}^{n,1} Y_{s-}^{n,2}\right)^{1/\alpha} \mathrm{d}X_s^{n,i}\right|\right] \leq C.$$

As $\gamma_n \to \infty$, we thus have

$$\mathbf{E}\left[\left|\int_0^t \left(Y_{s-}^{n,1} Y_{s-}^{n,2}\right)^{1/\alpha} \mathrm{d}X_s^{n,i}\right|\right] \to 0. \tag{4.12}$$

This means that the random variables $\int_0^t \left(Y_{s-}^{n,1}Y_{s-}^{n,2}\right)^{1/\alpha} \mathrm{d}X_s^{n,i}$ converge to zero in L^1. With Lemma 4.19 we get that

$$\int_0^t Y_s^{n,1}Y_s^{n,2}\mathrm{d}s \xrightarrow{n\to\infty} 0 \quad \text{in probability.}$$

Thus, also

$$\int_0^t \left(Y_s^{n,1}Y_s^{n,2} \wedge 1\right) \mathrm{d}s \xrightarrow{n\to\infty} 0 \quad \text{in probability}$$

and after changing over to a suitable subsequence $(n_k)_k$, we may assume that

$$\int_0^t \left(Y_s^{n_k,1}Y_s^{n_k,2} \wedge 1\right) \mathrm{d}s \xrightarrow{k\to\infty} 0 \quad \text{almost surely.}$$

Therefore,

$$\mathbf{E}\left[\int_0^t \left(Y_s^{n_k,1}Y_s^{n_k,2} \wedge 1\right) \mathrm{d}s\right] \xrightarrow{k\to\infty} 0.$$

On the other hand, by weak convergence of $(Y_s^{n_k})_k$ to Y_s for almost all $s \geq 0$ and as the function

$$(x_1, x_2) \mapsto x_1x_2 \wedge 1$$

is bounded and continuous on $[0,\infty)^2$, we receive

$$\begin{aligned}
\mathbf{E}\left[\int_0^t \left(Y_s^1Y_s^2 \wedge 1\right) \mathrm{d}s\right] &= \int_0^t \mathbf{E}\left[\left(Y_s^1Y_s^2 \wedge 1\right)\right] \mathrm{d}s \\
&= \int_0^t \lim_{k\to\infty} \mathbf{E}\left[\left(Y_s^{n_k,1}Y_s^{n_k,2} \wedge 1\right)\right] \mathrm{d}s \\
&= \lim_{k\to\infty} \int_0^t \mathbf{E}\left[\left(Y_s^{n_k,1}Y_s^{n_k,2} \wedge 1\right)\right] \mathrm{d}s \\
&= \lim_{k\to\infty} \mathbf{E}\left[\int_0^t \left(Y_s^{n_k,1}Y_s^{n_k,2} \wedge 1\right) \mathrm{d}s\right] = 0.
\end{aligned}$$

Note that $Y^{n,i}, Y^i \geq 0$, which justifies the application of Fubini's theorem. Due to the dominated convergence theorem, exchanging the limit and the integral is justified. Dominated convergence also ensures measurability of the mapping $s \mapsto \mathbf{E}\left[Y_s^1Y_s^2 \wedge 1\right]$. As $\int_0^t \left(Y_s^1Y_s^2 \wedge 1\right) \mathrm{d}s \geq 0$, we get that almost surely for all t

$$\int_0^t \left(Y_s^1Y_s^2 \wedge 1\right) \mathrm{d}s = 0$$

and thus, almost surely for all $t \geq 0$ and almost all $s \in [0,t]$,

$$Y_s^1Y_s^2 = 0. \tag{4.13}$$

As Y is càdlàg, the assertion follows. $\square$

We move on to characterise the distribution of the limit process Y in terms of its expectation evaluated at a certain class of test functions. To this end, we introduce the lozenge product, which is the same as in the case of ρ-correlated Brownian motion, where $\rho = -\cos\left(\frac{\pi}{\alpha}\right)$.

Definition 4.21 *For $x, y, x', y' \in [0,\infty)^2$ and $\alpha \in (1,2]$ let*

$$\begin{aligned} x \diamond y := &-\sqrt{1+\cos\left(\frac{\pi}{\alpha}\right)}\,(x_1+x_2)\,(y_1+y_2) \\ &+ i\sqrt{1-\cos\left(\frac{\pi}{\alpha}\right)}\,(x_1-x_2)\,(y_1-y_2)\,. \end{aligned} \tag{4.14}$$

Set furthermore

$$F(x,y) := F_\alpha(x,y) := \exp(x \diamond y) \tag{4.15}$$

and

$$H(x,x',y,y') := F(x,y)F(x',y').$$

It is important to note that $|F(x,y)| \leq 1$ and that $F(x,y) = F(y,x)$ for all $x, y \in [0,\infty)^2$. We have the following result.

Lemma 4.22 *For all $z \in E$ and $y \in [0,\infty)^2$, we have*

$$\int_E F(x,z)Q^\alpha_y(dx) = F(y,z).$$

Proof. We set $\rho = -\cos\left(\frac{\pi}{\alpha}\right)$. Then, as seen in Corollary 3.3, $Q^\alpha = Q^\rho$. Let B be a ρ-correlated Brownian motion and τ be the first hitting time of B in E. By equation (4.8) of [BDE11] p.271, we have for all $x, y \in [0,\infty)^2$

$$\mathbf{E}_x\left[F(B_\tau, y)\right] = \mathbf{E}_y\left[F(B_\tau, x)\right].$$

In particular, for $z \in E$, as in this case $Q^\alpha_z = \delta_z$,

$$\begin{aligned} \int_E F(x,z)Q^\rho_y(\mathrm{d}x) &= \mathbf{E}_y\left[F(B_\tau, z)\right] \\ &= \mathbf{E}_z\left[F(B_\tau, y)\right] \\ &= F(y,z). \end{aligned}$$

□

We are now able to evaluate the semigroup of Z at the functions F from above.

Corollary 4.23 *Let Z be an α-IMUB as defined in Definition 4.6. For all $y, z \in E$ and $t \geq 0$,*

$$\mathbf{E}_y\left[F(Z_t, z)\right] = F(y, e^{-ct}z)F(\theta, (1-e^{-ct})z).$$

Proof. With p_t as in Definition 4.2, we get by using Lemma 4.22

$$\begin{aligned}\mathbf{E}_y\left[F(Z_t,z)\right] &= \int_E F(x,z)p_t(y,\mathrm{d}x)\\ &= \int_E F(x,z)Q^{\alpha}_{e^{-ct}y+(1-e^{-ct})\theta}(\mathrm{d}x)\\ &= F(e^{-ct}y+(1-e^{-ct})\theta,z)\\ &= F(e^{-ct}y,z)F((1-e^{-ct})\theta,z)\\ &= F(y,e^{-ct}z)F(\theta,(1-e^{-ct})z).\end{aligned}$$

□

Before proving a similar result for the Y^n, we give another auxiliary result. We want to stress that if one could show that the Y^i_t have any finite p-th moment for $p>1$, the statement would readily follow from the fact that $e^{ct}Y^i_t$ is a nonnegative submartingale.

Lemma 4.24 *For all $T\geq 0$ and $i=1,2$, the random variable*

$$S := \sup_{0\leq t\leq T} Y^i_t$$

satisfies $\mathbf{E}\left[S\right]<\infty$.

Proof. We consider the martingales

$$M^i_t := \int_0^t \left(\gamma Y^1_{s-}Y^2_{s-}\right)^{1/\alpha}\mathrm{d}X^i_s.$$

We re-write M^i by separating the small and the large jumps of X^i. For readability, we drop the dependence on i. The subsequent calculations are valid for both $M=M^1,M^2$.

$$\begin{aligned}M_t &= \int_0^t\int_0^1 h\left(\gamma Y^1_{s-}Y^2_{s-}\right)^{1/\alpha}\widetilde{N}_p(\mathrm{d}h,\mathrm{d}s)+\int_0^t\int_1^\infty h\left(\gamma Y^1_{s-}Y^2_{s-}\right)^{1/\alpha}\widetilde{N}_p(\mathrm{d}h,\mathrm{d}s)\\ &=: I^1_t+I^2_t.\end{aligned}$$

Note that, by assumption (4.6) and the considerations made in [IW89] p.63, we have that I^1 is an L^2-martingale. Hence, for any $T\geq 0$, $\mathbf{E}\left[\left|I^1_T\right|^2\right]<\infty$. With Doob's martingale inequality, [Kle13] Theorem 11.2, we have for any $x>0$

$$\mathbf{P}\left[\sup_{0\leq t\leq T}\left|I^1_t\right|>x\right]\leq\frac{\mathbf{E}\left[\left|I^1_T\right|^2\right]}{x^2}.$$

Therefore,

$$\begin{aligned}\mathbf{E}\left[\sup_{0\leq t\leq T}\left|I^1_t\right|\right] &= \int_0^\infty \mathbf{P}\left[\sup_{0\leq t\leq T}\left|I^1_t\right|>x\right]\mathrm{d}x\\ &\leq 1+\int_1^\infty \mathbf{P}\left[\sup_{0\leq t\leq T}\left|I^1_t\right|>x\right]\mathrm{d}x \qquad (4.16)\\ &\leq 1+\mathbf{E}\left[\left|I^1_T\right|^2\right]<\infty.\end{aligned}$$

On the other hand, by using (4.7) with $p = 1/\alpha$, we see that I^2 is a stochastic integral with respect to a compensated compound Poisson process. With [IW89] (II.3.8), we can re-write I^2,

$$I_t^2 = \int_0^t \int_1^\infty h\left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} N_p(\mathrm{d}h, \mathrm{d}s) - \int_0^t \int_1^\infty h\left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} \hat{N}_p(\mathrm{d}h, \mathrm{d}s).$$

Thus,

$$\left|I_t^2\right| \leq \int_0^t \int_1^\infty h\left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} N_p(\mathrm{d}h, \mathrm{d}s) + \int_0^t \int_1^\infty h\left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} \hat{N}_p(\mathrm{d}h, \mathrm{d}s).$$

As $h\left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} \geq 0$, both summands are non-decreasing functions in t. Hence,

$$\begin{aligned}\sup_{0\leq t\leq T} \left|I_t^2\right| \leq &\int_0^T \int_1^\infty h\left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} N_p(\mathrm{d}h, \mathrm{d}s)\\ &+ \int_0^T \int_1^\infty h\left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha} \hat{N}_p(\mathrm{d}h, \mathrm{d}s).\end{aligned}$$

We end up with

$$\mathbf{E}\left[\sup_{0\leq t\leq T} \left|I_t^2\right|\right] \leq 2\int_0^T \int_1^\infty h\mathbf{E}\left[\left(\gamma Y_{s-}^1 Y_{s-}^2\right)^{1/\alpha}\right] \hat{N}_p(\mathrm{d}h, \mathrm{d}s) < \infty. \tag{4.17}$$

By combining (4.16) and (4.17), we receive

$$\mathbf{E}\left[\sup_{0\leq t\leq T} \left|M_t^i\right|\right] \leq \mathbf{E}\left[\sup_{0\leq t\leq T} \left|I_t^1\right|\right] + \mathbf{E}\left[\sup_{0\leq t\leq T} \left|I_t^2\right|\right] < \infty.$$

We come to the proof that $S \in L^1$. By the integral representation of Y^i, we get for $s \geq 0$

$$Y_s^i \leq y_i + c\theta_i s + c\int_0^s Y_r^i \mathrm{d}r + |M_s^i|.$$

Therefore,

$$\sup_{s\leq T} Y_s^i \leq y_i + c\theta_i T + c\int_0^T Y_r^i \mathrm{d}r + \sup_{s\leq T} |M_s^i|.$$

□

Before stating the next result, we give the definition of two important classes of functions from [IW89] Chapter II.3. In the following definition, we consider functions

$$f : [0, \infty) \times E \times \Omega \to \mathbb{R}.$$

By predictability we mean $(\mathcal{F}_t)$-predictability in the sense of [IW89] Definition (II).3.3.

$$F_p^1 := \left\{ f | f \text{ is predictable and for all } t > 0,\ \mathbf{E}\left[\int_0^t \int_E |f(s, h, \cdot)|\nu(\mathrm{d}h)\mathrm{d}s\right] < \infty \right\},$$

$$F_p^2 := \left\{ f | f \text{ is predictable and for all } t > 0,\ \mathbf{E}\left[\int_0^t \int_E f(s, h, \cdot)^2\nu(\mathrm{d}h)\mathrm{d}s\right] < \infty \right\}.$$

The following theorem states a key result of this chapter. The reader should note that the right-hand side of equation (4.18) is equal to $\mathbf{E}_y\left[F(U_t,z)\right]$, where U is the unique $D([0,\infty)^2)$-valued solution to (4.4) with $\alpha = 2$, where X is replaced by a correlated Brownian motion B with correlation factor $\rho = -\cos\left(\frac{\pi}{\alpha}\right)$. Of course, the hinted identity is only valid for $z \in E$.

Theorem 4.25 *Let $\gamma > 0$, $\theta \in [0,\infty)^2$ and $y \in [0,\infty)^2$. For any solution Y of (4.4) and for all $t \geq 0$ and $z \in E$, we have*

$$\mathbf{E}_y\left[F(Y_t,z)\right] = F(y, e^{-ct}z)F(\theta,(1-e^{-ct})z). \tag{4.18}$$

Proof. Set for $z \in [0,\infty)^2$ with $z \neq (0,0)$,

$$a := a(z) := -\sqrt{1+\cos\Big(\frac{\pi}{\alpha}\Big)}(z_1+z_2) + i\sqrt{1-\cos\Big(\frac{\pi}{\alpha}\Big)}(z_1-z_2) \in \mathbb{C}.$$

Then, with $a_1 := a$ and $a_2 := \bar{a}$,

$$x \diamond z = a_1x_1 + a_2x_2$$

and thus

$$F(Y_t,z) = \exp(a_1Y_t^1 + a_2Y_t^2).$$

We first show that for all $y \in [0,\infty)^2$ and $z \in E$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathbf{E}_y\left[F(Y_t,z)\right] = F(y,z)\left[c(\theta-y)\diamond z\right]. \tag{4.19}$$

From the existence of a solution to (4.4), we know that there is a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbf{P})$, a Poisson point process N_p on $E\times[0,\infty)$ with intensity measure

$$\hat{N}_p(\mathrm{d}h,\mathrm{d}t) = \nu(\mathrm{d}h)\otimes\mathrm{d}t,$$

where

$$\nu\left(\mathrm{d}(h_1,h_2)\right) = \begin{cases} \frac{1}{\Gamma(-\alpha)}h_1^{-\alpha-1}\mathrm{d}h_1, & \text{on } h_2 = 0 \\ \frac{1}{\Gamma(-\alpha)}h_2^{-\alpha-1}\mathrm{d}h_2, & \text{on } h_1 = 0 \end{cases},$$

such that for all $t \in [0,\infty)$

$$Y_t = Y_0 + \int_0^t c(\theta - Y_s)\mathrm{d}s + \int_0^t h\left(\gamma Y_{s-}^1Y_{s-}^2\right)^{1/\alpha}\widetilde{N}_p(\mathrm{d}h,\mathrm{d}s).$$

We set for $i = 1,2$, $h \in E$, $s \geq 0$ and $\omega \in \Omega$

$$g^i(s,h,\omega) := \mathbb{1}_{[0,1)}\left(h_i\left(\gamma Y_{s-}^1(\omega)Y_{s-}^2(\omega)\right)^{1/\alpha}\right)h_i\left(\gamma Y_{s-}^1(\omega)Y_{s-}^2(\omega)\right)^{1/\alpha},$$

$$f^i(s,h,\omega) := \mathbb{1}_{[1,\infty)}\left(h_i\left(\gamma Y_{s-}^1(\omega)Y_{s-}^2(\omega)\right)^{1/\alpha}\right)h_i\left(\gamma Y_{s-}^1(\omega)Y_{s-}^2(\omega)\right)^{1/\alpha}.$$

Hence, for $i = 1, 2$,

$$\begin{aligned} Y_t^i =& Y_0^i + \int_0^t c(\theta_i - Y_s^i)\mathrm{d}s - \int_0^t \int_E f^i(s, h, \cdot)\nu(\mathrm{d}h)\mathrm{d}s \\ &+ \int_0^t \int_E f^i(s, h, \cdot)N_p(\mathrm{d}h, \mathrm{d}s) + \int_0^t \int_E g^i(s, h, \cdot)\widetilde{N}_p(\mathrm{d}h, \mathrm{d}s). \end{aligned}$$

Note that by (4.7) we have for almost all t that $\left(\gamma Y_t^1 Y_t^2\right)^{1/\alpha} < \infty$. Therefore,

$$\begin{aligned} \int_0^t \int_E f^i(s, h, \cdot)\nu(\mathrm{d}h)\mathrm{d}s &= \frac{1}{(\alpha - 1)\Gamma(-\alpha)} \int_0^t \gamma Y_s^1 Y_s^2 \mathrm{d}s, \\ \int_0^t \int_E g^i(s, h, \cdot)^2 \nu(\mathrm{d}h)\mathrm{d}s &= \frac{1}{(2 - \alpha)\Gamma(-\alpha)} \int_0^t \gamma Y_s^1 Y_s^2 \mathrm{d}s. \end{aligned}$$

With (4.7) applied to $p = 1 < 2/\alpha$, we see that

$$\mathbf{E}\left[\int_0^t \gamma Y_s^1 Y_s^2 \mathrm{d}s\right] < \infty, \quad \text{for all } t.$$

Hence, $f^i \in F_p^1$ and $g^i \in F_p^2$. As in [IW89] p. 66, we have that

$$A_t^i := \int_0^t c(\theta_i - Y_s^i)\mathrm{d}s - \int_0^t \int_E f^i(s, h, \cdot)\nu(\mathrm{d}h)\mathrm{d}s$$

is a continuous, $\mathbb{F}$-adapted process with $A_0^i = 0$. As $Y^i \geq 0$, the processes $t \mapsto \int_0^t Y_s^i \mathrm{d}s$ and $t \mapsto \int_0^t Y_s^1 Y_s^2 \mathrm{d}s$ are non-decreasing and thus of bounded variation. Furthermore,

$$f^i(s, h, \omega)g^j(s, h, \omega) = 0 \quad \text{for all } h, s, \omega, i, j.$$

It is also immediate that $g^i \leq 1$ is bounded. We apply Itô's formula, Theorem 5.1 in [IW89] Chapter II, to the analytic function $F_z(x) := F(x, z)$. We suppress the dependence on ω of $f=(f^1,f^2)$ and $g=(g^1,g^2)$ in the subsequent calculations.

$$
\begin{aligned}
&F_z(Y_t)-F_z(Y_0)\\
=&\sum_{i=1}^2\int_0^t a_iF_z(Y_s)\left[c(\theta_i-Y_s^i)-\int_E f^i(s,h)\nu(\mathrm{d}h)\right]\mathrm{d}s\\
&+\int_0^t\int_E\left(F_z\left(Y_{s-}+f(s,h)\right)-F_z(Y_{s-})\right)N_p(\mathrm{d}h,\mathrm{d}s)\\
&+\int_0^t\int_E\left(F_z\left(Y_{s-}+g(s,h)\right)-F_z(Y_{s-})\right)\widetilde{N}_p(\mathrm{d}h,\mathrm{d}s)\\
&+\int_0^t\int_E\left(F_z\left(Y_s+g(s,h)\right)-F_z(Y_s)-\sum_{i=1}^2 a_ig^i(s,h)F_z(Y_s)\right)\nu(\mathrm{d}h)\mathrm{d}s\\
=&\sum_{i=1}^2\int_0^t a_iF_z(Y_s)c(\theta_i-Y_s^i)\mathrm{d}s\\
&+\int_0^t\int_E\left(F_z\left(Y_{s-}+f(s,h)\right)-F_z(Y_{s-})\right)\widetilde{N}_p(\mathrm{d}h,\mathrm{d}s)\\
&+\int_0^t\int_E\left(F_z\left(Y_{s-}+g(s,h)\right)-F_z(Y_{s-})\right)\widetilde{N}_p(\mathrm{d}h,\mathrm{d}s)\\
&+\int_0^t\int_E\left(F_z\left(Y_s+g(s,h)\right)-F_z(Y_s)-\sum_{i=1}^2 a_ig^i(s,h)F_z(Y_s)\right)\nu(\mathrm{d}h)\mathrm{d}s\\
&+\int_0^t\int_E\left(F_z\left(Y_s+f(s,h)\right)-F_z(Y_s)-\sum_{i=1}^2 a_if^i(s,h)F_z(Y_s)\right)\nu(\mathrm{d}h)\mathrm{d}s
\end{aligned}
\tag{4.20}
$$

Note that

$$
\begin{aligned}
&\int_0^t\int_E\left|F_z\left(Y_{s-}+f(s,h)\right)-F_z(Y_{s-})\right|\nu(\mathrm{d}h)\mathrm{d}s\\
\le&\sum_{i=1,2}\int_0^t\int_0^\infty\left|F_z\left(f(s,h)\right)-1\right|\frac{1}{\Gamma(-\alpha)}h_i^{-\alpha-1}\mathrm{d}h_i\mathrm{d}s\\
\le&\frac{4}{\Gamma(-\alpha)}\int_{(\gamma Y_{s-}^1Y_{s-}^2)^{-1/\alpha}}^\infty h^{-\alpha-1}\mathrm{d}h\mathrm{d}s\\
=&\frac{4}{\alpha\Gamma(-\alpha)}\int_0^t\gamma Y_s^1Y_s^2\mathrm{d}s.
\end{aligned}
$$

Therefore, $F_z\left(Y_{s-}+f(s,h)\right)-F_z(Y_{s-})$ is in F_p^1. Thus, the process

$$\int_0^t\int_E\left(F_z\left(Y_{s-}+f(s,h)\right)-F_z(Y_{s-})\right)\widetilde{N}_p(\mathrm{d}h,\mathrm{d}s)$$

is a martingale, as was noted in [IW89] (II.3.8). Furthermore, as was shown in [IW89] p. 73, the process

$$\int_0^t\int_E\left(F_z\left(Y_{s-}+g(s,h)\right)-F_z(Y_{s-})\right)\widetilde{N}_p(\mathrm{d}h,\mathrm{d}s)$$

is an L^2 martingale, due to the boundedness of g. We set

$$\begin{aligned}\widetilde{M}_t :&= \int_0^t \int_E \left[F_z\left(Y_{s-} + h\left(\gamma Y^1_{s-}Y^2_{s-}\right)^{1/\alpha}\right) - F_z(Y_{s-})\right] \widetilde{N}_p(\mathrm{d}h, \mathrm{d}s) \\ &= \int_0^t \int_E \left(F_z\left(Y_{s-} + f(s,h)\right) - F_z(Y_{s-})\right) \widetilde{N}_p(\mathrm{d}h, \mathrm{d}s) \\ &\quad + \int_0^t \int_E \left(F_z\left(Y_{s-} + g(s,h)\right) - F_z(Y_{s-})\right) \widetilde{N}_p(\mathrm{d}h, \mathrm{d}s).\end{aligned} \tag{4.21}$$

Then, $\widetilde{M}$ is a martingale, null at zero and the right-hand side of (4.20) equals

$$\begin{aligned}&a_1 \int_0^t F_z(Y_s)c(\theta_1 - Y^1_s)\mathrm{d}s + a_2 \int_0^t F_z(Y_s)c(\theta_2 - Y^2_s)\mathrm{d}s + \widetilde{M}_t \\ &+ \sum_{i=1}^2 \int_0^t F_z(Y_s) \int_0^\infty \left[\exp\left(h_i a_i \left(\gamma Y^1_s Y^2_s\right)^{1/\alpha}\right) - 1 - h_i a_i \left(\gamma Y^1_s Y^2_s\right)^{1/\alpha}\right] \frac{h_i^{-\alpha-1}}{\Gamma(-\alpha)} \mathrm{d}h_i \mathrm{d}s.\end{aligned}$$

As $Y_s \in [0,\infty)^2$ for all s,

$$\mathrm{Re}\left(a_i \left(\gamma Y^1_s Y^2_s\right)^{1/\alpha}\right) \leq 0$$

and with equation (1.17) we see that

$$\begin{aligned}&\sum_{i=1}^2 \int_0^t F_z(Y_s) \int_0^\infty \left[\exp\left(h_i a_i \left(\gamma Y^1_s Y^2_s\right)^{1/\alpha}\right) - 1 - h_i a_i \left(\gamma Y^1_s Y^2_s\right)^{1/\alpha}\right] \frac{h_i^{-\alpha-1}}{\Gamma(-\alpha)} \mathrm{d}h_i \mathrm{d}s \\ &= \int_0^t F_z(Y_s)\gamma Y^1_s Y^2_s \left[(-a_1)^\alpha + (-a_2)^\alpha\right] \mathrm{d}s.\end{aligned}$$

The reader should note that

$$-a = \left(2\left(z_1^2 + z_2^2 + 2\cos\left(\frac{\pi}{\alpha}\right) z_1 z_2\right)\right)^{1/2} \left(\cos(A), \sin(A)\right),$$

where

$$\begin{aligned}|a| &= \left(2\left(z_1^2 + z_2^2 + 2\cos\left(\frac{\pi}{\alpha}\right) z_1 z_2\right)\right)^{1/2} \geq 0 \quad \text{and} \\ A &= \arg(-a) = -\arctan\left(\tan\left(\frac{\pi}{2\alpha}\right)\left(\frac{z_1 - z_2}{z_1 + z_2}\right)\right).\end{aligned}$$

Thus,

$$(-a_1)^\alpha + (-a_2)^\alpha = 2\,\mathrm{Re}\left((-a)^\alpha\right) = 2|a|^\alpha \cos(\alpha A) = 0,$$

as for any $z \in E$ we have

$$A = \pm\frac{\pi}{2\alpha}.$$

We thus get from (4.20),

$$F_z(Y_t) - F_z(Y_0) = a_1 \int_0^t F_z(Y_s)c(\theta_1 - Y^1_s)\mathrm{d}s + a_2 \int_0^t F_z(Y_s)c(\theta_2 - Y^2_s)\mathrm{d}s + \widetilde{M}_t.$$

As $|F_z(Y_s)| \leq 1$ for all s and $\mathbf{E}\left[\int_0^t Y_s^i \mathrm{d}s\right]$ is bounded for given $t > 0$, the application of Fubini's theorem is justified and from the above we infer

$$\mathbf{E}\left[F_z(Y_t)\right] = F_z(Y_0) + a_1 \int_0^t \mathbf{E}\left[F_z(Y_s)c(\theta_1 - Y_s^1)\right] \mathrm{d}s + a_2 \int_0^t \mathbf{E}\left[F_z(Y_s)c(\theta_2 - Y_s^2)\right] \mathrm{d}s.$$

With Lemma 4.24, we see that for $t \in [0,1]$ and $i = 1, 2$,

$$Y_t^i \leq \sup_{0 \leq s \leq 1} Y_s^i \in L^1.$$

By dominated convergence and the right-continuity of Y_t^i, we get that all integrands in the above equation are actually continuous at zero. Taking derivatives with respect to t in $t = 0$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathbf{E}\left[F_z(Y_t)\right] = F_z(Y_0)\left(a_1 c(\theta_1 - Y_0^1) + a_2 c(\theta_2 - Y_0^2)\right).$$

Hence, for $y \in [0, \infty)^2$ and $z \in E$,

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathbf{E}_y\left[F_z(Y_t)\right] = F(y,z)\left(a_1 c(\theta_1 - y_1) + a_2 c(\theta_2 - y_2)\right) = F(y,z)\left[c(\theta - y) \diamond z\right].$$

This is (4.19).

In order to finish the proof, we make use of a duality argument, i.e., we have to find an auxiliary Markov process $\widetilde{Y}$ with values in $E \times [0, \infty)^2$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathbf{E}_{\widetilde{y}}\left[H((y,\theta), \widetilde{Y}_t)\right] = H((y,\theta), \widetilde{y})(c(\theta - y) \diamond \widetilde{y}(1))$$

for $y \in [0, \infty)^2$ and $\widetilde{y} \in E \times [0, \infty)^2$. This means with (4.19),

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathbf{E}_{\widetilde{y}}\left[H((y,\theta), \widetilde{Y}_t)\right] = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathbf{E}_y\left[H((Y_t,\theta), \widetilde{y})\right],$$

for all $y \in [0, \infty)^2$ and $\widetilde{y} = (\widetilde{y}(1), \widetilde{y}(2)) \in E \times [0, \infty)^2$. By verifying certain additional technical assumptions to be specified below, we get from [EK86] Corollary 4.13 that for all $t \geq 0$

$$\mathbf{E}_y\left[H((Y_t,\theta), \widetilde{y})\right] = \mathbf{E}_{\widetilde{y}}\left[H((y,\theta), \widetilde{Y}_t)\right].$$

For given $\widetilde{y}(1) \in E$ and $\widetilde{y}(2) \in [0, \infty)^2$, we define the deterministic Markov process $\widetilde{Y}$ with values in $E \times [0, \infty)^2$ by

$$\begin{aligned} \widetilde{Y}_t(1) &:= e^{-ct}\widetilde{y}(1) \quad \text{and} \\ \widetilde{Y}_t(2) &:= \widetilde{y}(2) + \widetilde{y}(1)\left(1 - e^{-ct}\right). \end{aligned}$$

Note that for $y \in [0, \infty)^2$ and any differentiable function $f : [0, \infty) \to [0, \infty)^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}F(y, f(t)) = \left(a_1(y)f_1'(t) + a_2(y)f_2'(t)\right)F(y, f(t)) = \left(\nabla f(t) \diamond y\right)F(y, f(t)).$$

Recall that for $x(1), x(2), y(1), y(2) \in [0,\infty)^2$ with $x = (x(1), x(2))$, $y = (y(1), y(2))$,

$$H(x,y) = F(x(1), y(1))F(x(2), y(2)).$$

Therefore, for $y, \theta \in [0,\infty)^2$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H((y,\theta), \widetilde{Y}_t) &= \frac{\mathrm{d}}{\mathrm{d}t} F(y, \widetilde{Y}_t(1))F(\theta, \widetilde{Y}_t(2)) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t} F(y, e^{-ct}\widetilde{y}(1))\right) F(\theta, \widetilde{Y}_t(2)) \\
&\quad + F(y, \widetilde{Y}_t(1)) \left(\frac{\mathrm{d}}{\mathrm{d}t} F\left(\theta, \widetilde{y}(2) + \widetilde{y}(1)(1-e^{-ct})\right)\right) \\
&= -ce^{-ct} \left(\widetilde{y}(1) \diamond y\right) H((y,\theta), \widetilde{Y}_t) \\
&\quad + ce^{-ct} \left(\widetilde{y}(1) \diamond \theta\right) H((y,\theta), \widetilde{Y}_t) \\
&= ce^{-ct} \left[\theta \diamond \widetilde{y}(1) - y \diamond \widetilde{y}(1)\right] H((y,\theta), \widetilde{Y}_t) \\
&= e^{-ct} \left[c(\theta - y) \diamond \widetilde{y}(1)\right] H((y,\theta), \widetilde{Y}_t) \\
&= \left[c(\theta - y) \diamond \widetilde{Y}_t(1)\right] H((y,\theta), \widetilde{Y}_t).
\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathbf{E}_{\widetilde{y}} \left[H((y,\theta), \widetilde{Y}_t)\right] = H((y,\theta), \widetilde{y}) \left(c(\theta - y) \diamond \widetilde{y}(1)\right).$$

For $y, \theta \in [0,\infty)^2$ and $\widetilde{y} \in E \times [0,\infty)^2$, we define the function

$$g(y,\theta,\widetilde{y}) := H((y,\theta), \widetilde{y}) \left[c(\theta - y) \diamond \widetilde{y}(1)\right].$$

We then have

$$\begin{aligned}
&H((Y_t,\theta), \widetilde{y}) - \int_0^t g(Y_s, \theta, \widetilde{y}) \mathrm{d}s \\
=&F(\theta, \widetilde{y}(2)) \left(F(Y_t, \widetilde{y}(1)) - \int_0^t F(Y_s, \widetilde{y}(1)) \left[c(\theta - Y_s) \diamond \widetilde{y}(1)\right] \mathrm{d}s\right).
\end{aligned}$$

With (4.20), we see that for the martingale $\widetilde{M_t}$ from (4.21) and $\widetilde{y}(1) \in E$,

$$F(Y_t, \widetilde{y}(1)) - \int_0^t F(Y_s, \widetilde{y}(1)) \left[c(\theta - Y_s) \diamond \widetilde{y}(1)\right] \mathrm{d}s = F(Y_0, \widetilde{y}(1)) + \widetilde{M_t}.$$

Hence, the process

$$H((Y_t,\theta), \widetilde{y}) - \int_0^t g(Y_s, \theta, \widetilde{y}) \mathrm{d}s$$

is a martingale for all $\widetilde{y} \in E \times [0,\infty)^2$ and $\theta \in [0,\infty)^2$. On the other hand, trivially,

$$\begin{aligned}
H((y,\theta), \widetilde{Y}_t) - \int_0^t g(y,\theta,\widetilde{Y}_s) \mathrm{d}s &= H((y,\theta), \widetilde{Y}_t) - \left(H((y,\theta), \widetilde{Y}_t) - H((y,\theta), \widetilde{Y}_0)\right) \\
&= H((y,\theta), \widetilde{y}).
\end{aligned}$$

This is certainly a (deterministic) martingale, as it is constant.
To finish the proof, we have to show that for all $T > 0$ there is an integrable random variable Γ_T, such that

$$\sup_{s,t\leq T} |g(Y_s, \theta, \widetilde{Y}_t)| \leq \Gamma_T. \tag{4.22}$$

Note that for $x, y \in [0,\infty)^2$,

$$|x \diamond y| \leq 3(x_1 + x_2)(y_1 + y_2).$$

With this, for $s, t \geq 0$,

$$\begin{aligned}
|g(Y_s, \theta, \widetilde{Y}_t)| &= |H(Y_s, \theta, \widetilde{Y}_t)|\,|c(\theta - Y_s) \diamond \widetilde{Y}_t(1)| \\
&\leq c|\theta \diamond \widetilde{Y}_t(1)| + c|Y_s \diamond \widetilde{Y}_t(1)| \\
&\leq 3ce^{-ct}(\theta_1 + \theta_2)(\widetilde{y}(1)_1 + \widetilde{y}(1)_2) + 3ce^{-ct}(Y_s^1 + Y_s^2)(\widetilde{y}(1)_1 + \widetilde{y}(1)_2) \\
&\leq 3c(\theta_1 + \theta_2)(\widetilde{y}(1)_1 + \widetilde{y}(1)_2) + 3c(Y_s^1 + Y_s^2)(\widetilde{y}(1)_1 + \widetilde{y}(1)_2) \\
&= C(\theta_1 + \theta_2) + C\sup_{s\leq T}(Y_s^1 + Y_s^2) =: \Gamma_T,
\end{aligned}$$

where we defined $C := 3c(\widetilde{y}(1)_1 + \widetilde{y}(1)_2)$. By Lemma 4.24, $\Gamma_T \geq 0$ has finite mean and therefore (4.22) is verified.
Hence, by [EK86] Corollary 4.13 p.195, we have that for all $t \geq 0$

$$\begin{aligned}
\mathbf{E}_y\left[H((Y_t, \theta), \widetilde{y})\right] &= \mathbf{E}_{\widetilde{y}}\left[H((y, \theta), \widetilde{Y}_t)\right] \\
&= F\left(y, e^{-ct}\widetilde{y}(1)\right) F\left(\theta, \widetilde{y}(2) + \widetilde{y}(1)(1 - e^{-ct})\right).
\end{aligned}$$

Setting $z = \widetilde{y}(1) \in E$ and dividing by $F(\theta, \widetilde{y}(2)) \neq 0$ leads to (4.18). □

4.2.4 Proof of Theorem 4.15

In [KM10] Corollary 2.4 it is shown that the space spanned by

$$\mathcal{F}_2 = \{F_2(\cdot, z) : z \in E\}$$

is measure determining on E. The proof can be copied word by word, so that we see that for any $\alpha \in (1, 2]$ the space spanned by

$$\mathcal{F}_\alpha := \{F_\alpha(\cdot, z) : z \in E\}$$

is also measure determining on E. Note further that, for all $t \geq 0$ and $z' \in E$, the mapping

$$[0,\infty)^2 \to \mathbb{C}, \quad x \mapsto F(x, z')$$

is continuous and bounded, as F is obviously continuous and bounded. By weak convergence of Y_t^n to Y_t for almost all $t \geq 0$, possibly along a subsequence, we get

$$\lim_{n\to\infty} \mathbf{E}_{y_n}\left[F(Y_t^n, z')\right] = \mathbf{E}_z\left[F(Y_t, z')\right].$$

We thus get for almost all t, by Theorem 4.25 and Corollary 4.23 ,

$$\begin{aligned}\mathbf{E}_z\left[F(Y_t,z')\right] &= \lim_{n\to\infty}\mathbf{E}_{y_n}\left[F(Y_t^n,z')\right]\\ &= \lim_{n\to\infty}F(y_n,e^{-ct}z')F(\theta,(1-e^{-ct})z')\\ &= F(z,e^{-ct}z')F(\theta,(1-e^{-ct})z')\\ &= \mathbf{E}_z\left[F(Z_t,z')\right].\end{aligned}$$

As $Y_t \in E$ for all t, this proves that

$$Y_t \stackrel{d}{=} Z_t$$

for almost all $t \geq 0$ and as both Y and Z are càdlàg, the laws of Y and Z coincide. This is Theorem 4.15.

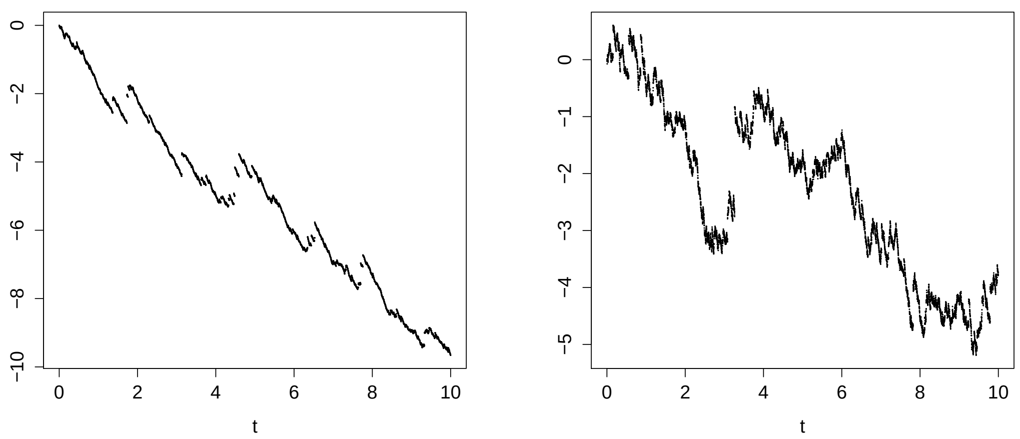
Appendix A

Simulating stable processes in R

As we made intensive use of simulations at each step of this work in order to get a feeling about the behaviour of the examined objects, we want to give a rough treatment of how to simulate stable random variables and processes in the programming language **R**.
We use the **R** package **stabledist** which provides quantile-, density- and distribution functions as well as a random generator in the **R** typical form.
In order to plot the path segment of an α-stable process X on $[0, \Delta n]$ for some $n \in \mathbb{N}$ and Δ small, we use the fact that the increments of X are independent and identically distributed together with the fact that, by stability, we have $X_\Delta \stackrel{d}{=} \Delta^{1/\alpha} X_1$. So all we need to do is to produce n independent copies of X_1, called $X^{(1)}, \ldots, X^{(n)}$. These are stable random variables with characteristic exponent given by (1.14) which also determins the parameters. Then, we have to plot the sequence $(k\Delta)_{k=0,\ldots,n}$ against $\left(\Delta^{1/\alpha} \sum_{i=1}^{k}\right)_{k=0,\ldots,n}$ to get the desired result.
The following **R** code shows how to use this method to plot an approximation to a stable process. The output is shown in Figure A. The 1 as a last entry in *rstable* gives the parametrisation, cf. also *?rstable*. The specific choice of this parameter ensures that we use the same parametrisation as in (1.14).
For a general treatment on how to generate stable random variables see [ST94] Chapter 1.7.

```
alpha<-1.6  # index of stability (to be chosen)

beta<-1     # spectrally positive
delta<-0    # centered
gamma<- (-cos(alpha*pi/2))^(1/alpha) # scaling

step<-0.001 # the fineness of the grid
T<-10       # time horizon
start<-0

t<-seq(0,T,step) #generating the time steps
```

```
x<-rstable(T/step,alpha,beta,gamma,delta,1) #generating the randomness
x<-step^(1/alpha)*x # scaling to the correct time interval
x<-c(start,x) #adding the starting point

plot(t,cumsum(x),cex=.1,ylab="")
```



(a) $\alpha = 1.3$ (b) $\alpha = 1.7$

Figure A.1: Two examples of Lévy processes with characteristic exponent $-(-i\theta)^{\alpha}$

Bibliography

[App09] D. Applebaum. *Lévy Processes and Stochastic Calculus.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2009.

[BDE11] J. Blath, L. Döring, and A. Etheridge. On the moments and the interface of the symbiotic branching model. *Ann. Probab.*, 39(1):252–290, 01 2011.

[Ber98] J. Bertoin. *Lévy Processes.* Cambridge Tracts in Mathematics. Cambridge University Press, 1998.

[Ber99] J. Bertoin. Subordinators: Examples and applications. In Pierre Bernard, editor, *Lectures on Probability Theory and Statistics*, volume 1717 of *Lecture Notes in Mathematics*, pages 1–91. Springer Berlin Heidelberg, 1999.

[DM11] L. Döring and L. Mytnik. Mutually catalytic branching processes and voter processes with strength of opinion. *arXiv:1109.6106v2*, 2011.

[DP98] D. A. Dawson and E. A. Perkins. Long-time behavior and coexistence in a mutually catalytic branching model. *The Annals of Probability*, 26(3):1088–1138, 1998.

[EF04] A.M. Etheridge and K. Fleischmann. Compact interface property for symbiotic branching. *Stochastic Processes and their Applications*, 114(1):127 – 160, 2004.

[EK86] S.N. Ethier and T.G. Kurtz. *Markov processes: characterization and convergence.* Wiley series in probability and mathematical statistics. Probability and mathematical statistics. Wiley, 1986.

[Gal12] J.F.L. Gall. *Spatial Branching Processes, Random Snakes and Partial Differential Equations.* Lectures in Mathematics. ETH Zürich. Birkhäuser Basel, 2012.

[GJ86] A. Gut and S. Janson. Converse results for existence of moments and uniform integrability for stopped random walks. *The Annals of Probability*, 14(4):pp. 1296–1317, 1986.

[IW89] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes.* Kodansha scientific books. North-Holland, 1989.

[Jan11] S. Janson. Stable distributions. *arXiv:1112.0220*, 2011.

[Kal92] O. Kallenberg. Some time change representations of stable integrals, via predictable transformations of local martingales. *Stochastic Processes and their Applications*, 40(2):199 – 223, 1992.

[Kal02] O. Kallenberg. *Foundations of Modern Probability*. Applied probability. Springer, 2002.

[Kle13] A. Klenke. *Probability Theory: A Comprehensive Course*. Universitext. Springer London, 2013.

[KM10] A. Klenke and L. Mytnik. Infinite rate mutually catalytic branching. *Ann. Probab.*, 38(4):1690–1716, 07 2010.

[KM12a] A. Klenke and L. Mytnik. Infinite rate mutually catalytic branching in infinitely many colonies: construction, characterization and convergence. *Probability Theory and Related Fields*, 154(3-4):533–584, 2012.

[KM12b] A. Klenke and L. Mytnik. Infinite rate mutually catalytic branching in infinitely many colonies: The longtime behavior. *Ann. Probab.*, 40(1):103–129, 01 2012.

[KO10] A. Klenke and M. Oeler. A trotter-type approach to infinite rate mutually catalytic branching. *Ann. Probab.*, 38(2):479–497, 03 2010.

[Kre99] R. Kress. *Linear Integral Equations*. Number Bd. 82 in Applied Mathematical Sciences. Springer New York, 1999.

[Kur91] T. G. Kurtz. Random time changes and convergence in distribution under the meyer-zheng conditions. *Ann. Probab.*, 19(3):1010–1034, 07 1991.

[Kyp14] A. E. Kyprianou. *Fluctuations of Lévy Processes with Applications*. Universitext. Springer-Verlag Berlin Heidelberg, 2 edition, 2014.

[Myt98] L. Mytnik. Uniqueness for a mutually catalytic branching model. *Probability Theory and Related Fields*, 112(2):245–253, 1998.

[MZ84] P. A. Meyer and W. A. Zheng. Tightness criteria for laws of semimartingales. *Annales de l'I.H.P. Probabilités et statistiques*, 20(4):353–372, 1984.

[NCY05] S. Nok Chiu and C. Yin. Passage times for a spectrally negative lévy process with applications to risk theory. *Bernoulli*, 11(3):511–522, 06 2005.

[Nol] J. P. Nolan. Stable distributions, models for heavy tailed data. `http://academic2.american.edu/~jpnolan/stable/chap1.pdf`.

[Oel08] M. Oeler. Mutually catalytic branching at infinite rate. `https://publications.ub.uni-mainz.de/theses/frontdoor.php?source_opus=1819`, 2008. Dissertation.

[Reb95] J. A. Rebholz. A skew-product representation for the generator of a two sex population model. In Alison Etheridge, editor, *Stochastic Partial Differential Equations*, pages 230–240. Cambridge University Press, 1995. Cambridge Books Online.

[RW00] L.C.G. Rogers and D. Williams. *Diffusions, Markov Processes, and Martingales: Volume 1, Foundations*. Cambridge Mathematical Library. Cambridge University Press, 2000.

[Sat99] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.

[ST94] G. Samoradnitsky and M.S. Taqqu. *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Stochastic Modeling Series. Taylor & Francis, 1994.

[Zol86] V.M. Zolotarev. *One-dimensional Stable Distributions*. Translations of mathematical monographs. American Mathematical Society, 1986.