Lookdown-Constructions of Symbiotic Branching Processes

Dissertation
zur Erlangung des Grades
"Doktor
der Naturwissenschaften"

am Fachbereich Physik, Mathematik und Informatik
der Johannes Gutenberg-Universität
in Mainz

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geb. in Bayreuth

Mainz, den 25. November, 2019

Prüfung am 6. Februar 2020

Abstract

A symbiotic diffusion is a solution to the system of stochastic differential equations

$$\begin{cases} \mathrm{d}Y_t &= \sqrt{bY_tZ_t}\ \mathrm{d}W_t^1, \\ \mathrm{d}Z_t &= \sqrt{bY_tZ_t}\ \mathrm{d}W_t^2, \end{cases}$$

where W^1 and W^2 are correlated Brownian motions with constant correlation coefficient $\rho \in [-1,1]$. We are concerned with the construction of so-called lookdown representations for symbiotic diffusions and their discrete mass analoga. A lookdown representation is a particle model where the particles, representing families or lineages, are assigned levels that evolve in time and govern the reproductive dynamics. Lookdown representations carry genealogical information. We study models, where the levels take non discrete values in $\mathbb{R}^+$. This kind of lookdown construction was introduced for the Dawson-Watanabe process by Kurtz and Rodrigues in 2011 (see [KR11]).

The construction in [KR11] relies on a deterministic evolution of the levels. We modify their approach insofar as the level motion will be random or deterministic only given the level configuration of the partner population. We explore possible birth and death mechanisms that allow for coupling of the branching events in both populations. In the discrete mass setting, we construct lookdown representations for the whole range of $\rho \in [-1,1]$. In contrast to the Kurtz-Rodrigues model, continuity of the level paths is lost and, in general, only right continuity remains. In the diffusive limit however, the discontinuous paths converge to conditional geometric Brownian motions. We construct lookdown representations of symbiotic diffusions for $\rho \in [0,1)$ as weak limits of discrete mass models. For $\rho = 0$ (the mutually catalytic case) we also give an explicit construction.

Acknowledgement

I want to thank my supervisors for their trust in my dissertation project, for their good advice and long discussions.

I thank my colleagues at the AG Stochastik in Mainz. I am grateful for help, moral support and friendship.

And finally I thank my family. This dissertation project would not have been possible without the support of my wife and our children.

Contents

1. Introduction

Lookdown processes are probabilistic particle representations for the genealogies of branching processes (with or without spatial motion) and for their diffusive limits (for example Feller's branching diffusion or the Dawson-Watanabe super Brownian motion). The particles reproduce and die, and they may exhibit spatial movement in some appropriate space, often interpreted as evolution of *types*. The main feature is that each particle carries a *level* that governs the birth dynamics and the particle's time of death. Lookdown representations often exhibit a nested coupling property that allows for elegant constructions of diffusive limits and limiting superprocesses. More importantly, they provide (and retain in the limit) genealogical information on the corresponding branching process.

Since the original "lookdown construction" of the Fleming-Viot measure-valued diffusion, introduced by Donnelly and Kurtz in [DK96], this particular flavour of Poisson constructions was applied to a variety of genealogical population models and particle systems. In 1999, Donnelly and Kurtz published two articles concerning their lookdown approach. With respect of the importance of the Fleming-Viot model for applications in genetics and population biology, they extended their representation of this process in [DK99a] to incorporate various selection and recombination mechanisms. In [DK99b] they introduced a "modified lookdown construction" that provides a unified approach to a number of particle systems with different birth and death mechanisms, including linear and branching models. This modified lookdown approach allows for the construction of the Dawson-Watanabe superprocess, as well as the Fleming-Viot superprocess.

The Donnelly-Kurtz models have in common that the levels of the particles take discrete values. We sketch the main idea for case of the modified lookdown from [DK99b] with a simple branching birth and death mechanism: Let $N_t = N_0 + N_t^b - N_t^d$ denote the total population size at time t, where N_t^b is the number of births up to time t and N_t^d is the number of deaths up to time t. We choose the dynamics such that the total mass process N_t is a simple critical Galton-Watson process. That is, N^b and N^d both have jumps of size $+1$ at rate N_t and the jumps come independently. The particles' spatial locations evolve like Brownian motions between birth and death events.

Donnelly and Kurtz consider two particle representations: a classical model (I) and their modified lookdown model (II). In the classical model, when a birth event happens, one particle is chosen at random and gives birth to one offspring at the same location. In case of a death event a particle is chosen at random to die. Labelling the particles arbitrarily, we denote the particles positions by $\left(Y_t^1, Y_t^2, \ldots, Y_t^{N_t}\right)$. The population at time t can be represented by the empirical measure

$$Z_t^I := \sum_{i=1}^{N_t} \delta_{Y_t^i}$$

since the enumeration was arbitrary.

In the modified lookdown model (II) the particles carry levels $1, 2, \ldots, N_t$. Let X_t^i denote the spatial position of the particle with level i at time t. At the time of a death event, the particle with the highest level dies. In case of a birth event at time t, the levels in the population change: Two levels $1 \leq i < j \leq N_t$ are chosen at random. The particle with level i is the parent and gives birth to a new particle with level j. The newborn particle inherits the spatial position of its parent. All levels above j are incremented by one. Levels below j remain unchanged (see Figure 1.1).

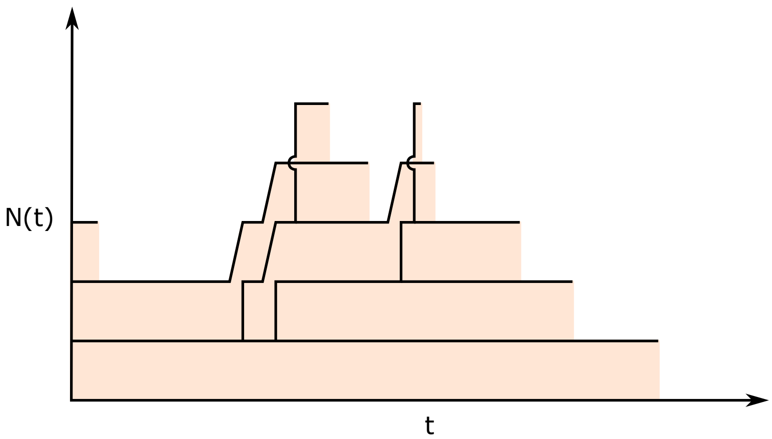

Figure 1.1.: The modified Donnelly-Kurtz lookdown construction of a simple critical Galton-Watson process. The name "lookdown construction" stems from the fact that if one traces the ancestral line of one particle backwards in time, one has to look down for the particle's parent, when reaching its birth event.

Let $(X_0^1, X_0^2, \ldots, X_0^{N_0})$ have the same (exchangeable) distribution as $(Y_0^1, Y_0^2, \ldots, Y_0^{N_0})$. Define the empirical measure

$$Z_t^{II} := \sum_{i=1}^{N_t} \delta_{X_t^i}.$$

Theorem 1.1 in [DK99b] states that Z^I and Z^{II} have the same distribution, and for all $t \geq 0$, $(X_t^1, X_t^2, \ldots, X_t^{N_t})$ is exchangeable. The authors show that under the proper rescaling of the mass process, $P_t^n := \frac{N_{n \cdot t}}{n}$, the lookdown representation converges: Let N^n and $X^{i,n}$ correspond to the n-th stage of rescaling. The sequence of processes $\left(X^{1,n}, X^{2,n}, \ldots, X^{N_0^n,n}\right)$ converges to $(X^1, X^2, \ldots)$ as $n \to \infty$ and the sequence of empirical measures

$$\frac{1}{n} \sum_{i=1}^{N_t^n} \delta_{X_t^{i,n}}$$

converges in distribution to

$$P_t \cdot \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \delta_{X_t^i},$$

where P is the limiting mass process.

In 2011 Kurtz and Rodrigues introduced a new type of lookdown construction in [KR11]. The levels of the particles in this representation of the Dawson-Watanabe superprocess are not discrete but take values in $\mathbb{R}^+$. (In this thesis, $\mathbb{R}^+$ denotes the nonnegative numbers $[0,\infty)$.) The model is described in detail for the nonspatial case in Section 2.1. The particles' levels govern the birth dynamics in a way that is essentially a continuous analogue of the birth mechanism in the discrete-level lookdown constructions: Particles produce offspring with higher level than their own at rate $2\,\mathrm{d}t\,\mathrm{d}u$, where t is time and u the offspring's level coordinate. Hence a particle with low level produces more offspring than a high level particle, and the newborn's level is in a sense "uniformly distributed" above the parent's level. The death mechanism is fundamentally different from the one in the discrete-level constructions: The level of a particle changes continuously in time, driven by an ordinary differential equation (ODE). A given level goes to infinity in finite time and when it does, the corresponding particle dies. Since the levels are all driven by the same ODE, the levels cannot overtake. The order of a given set of levels does not change and the highest level particle will die first. The same is true for the discrete level constructions, but there the deaths are triggered by an external process. In the Kurtz-Rodrigues model, the remaining lifetime of a particle is a deterministic function of its level. Since the dynamics are tuned in such a way, that levels never overtake, the construction carries genealogical information in a similar way as the Donnelly-Kurtz lookdown does: If we look for a particle's parent, we have to "look down" at particles with a lower level (see Figure 1.2). The particles with the lowest levels will have the majority of the offspring and they will die last. They are the "progenitors" of the population. As in the Donnelly-Kurtz model, particles are not interpreted as individuals of the branching populations, but as lineages. One major benefit of the continuous level construction is that the mass of the represented branching diffusion (or Dawson-Watanabe superprocess in the spatial setting) is inherent in the level configuration since it is the asymptotic level density. In the Donnelly-Kurtz models in contrast the mass has rather the character of an auxiliary process that drives births and deaths at the particle stage.

The key for the construction in [KR11] is the so-called Markov Mapping Theorem (see Theorem A.15 in [KR11] or Theorem A.5.1 in this thesis) that Kurtz first introduced in [Kur98]. It gives conditions under which the projection of a Markov process retains the Markov property. The heuristics behind the Markov Mapping Theorem can be described as follows: The level dynamics are made in such a way that, at every time $t > 0$, the levels in an interval $[a, b]$ are independent and uniformly distributed, given the total mass in $[a, b]$. But since the distribution of the level configuration is known, one can "integrate out" the level information of the particles, retaining the total particle mass. The level dynamics are tuned in such a way that the mass process performs as desired. Note that the "uniformity" of the level system is not obvious, but the Markov Mapping Theorem establishes this property.

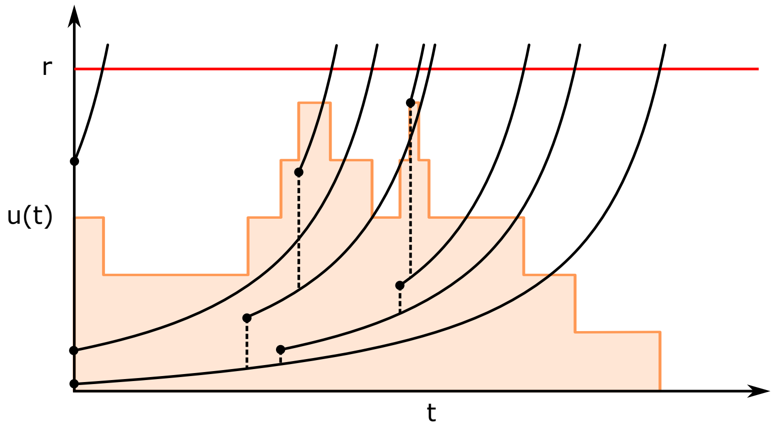

Figure 1.2.: The Kurtz-Rodrigues model, restricted to a level interval $[0, r]$, is a lookdown representation of a simple critical Galton-Watson process (orange). Compare the order of levels with Figure 1.1.

A recent example for particle representations that feature the lookdown philosophy is the construction of a spatial Fleming-Viot process and other genealogical models by Etheridge and Kurtz in [EK19]. The construction also relies heavily on the Markov Mapping Theorem. The authors introduce a toolbox of birth and death mechanisms in the continuous level setting, including multiple deaths, one for one replacement and event based mechanisms. The different birth and death mechanisms and the appropriate level dynamics can be combined as independent "building blocks" in order to assemble a particle representation. It is convenient to describe the individual mechanisms in terms of Markov generators since the combination of these mechanisms simply amounts to adding the generators.

The multiple-death mechanism in [EK19] plays an essential role in our models. It stands halfway between the discrete level dynamics in [DK99b] and the continuous dynamics in [KR11]. Levels take values in $\mathbb{R}^+$, so the death mechanism may be combined with birth or death mechanisms determined by the other building blocks in [EK19]. But the multiple death events are triggered by an external Poisson process, similarly to the death events in the discrete level models. In case of a multiple death event high level particles are killed, and the levels of the survivors jump upwards "to fill the gaps", thus conserving the uniform distribution of levels, without changing the order of levels. Though the levels take values in $\mathbb{R}^+$, their order performs precisely a pure death dynamics in the sense of the discrete level model described above.

In this thesis we will adapt the continuous-level lookdown approach introduced in [KR11] and apply some of the ideas from [EK19] to obtain level representations of *symbiotic branching systems*. We distinguish a continuous mass setting and a discrete mass setting. Our models do not exhibit a spatial structure, so we deal with diffusions or discrete mass branching processes. A symbiotic branching system is a bivariate process $(Y_t, Z_t)_{t\geq 0}$. The marginals Y and Z model the masses of two branching populations that feature "symbiotic" interaction. I.e., the branching rate of Y_t is proportional to Z_t and vice versa, and the pair (Y, Z) is correlated with a constant correlation coefficient. Note that the established term "symbiotic" could be misleading and should not be interpreted in a strict biological sense.

In the diffusive setting (Y, Z) is a solution to the system of stochastic differential equations (SDE)

$$\begin{cases} \mathrm{d}Y_t &= \sqrt{bY_tZ_t}\, \mathrm{d}W_t^1 \\ \mathrm{d}Z_t &= \sqrt{bY_tZ_t}\, \mathrm{d}W_t^2, \end{cases}$$

where W^1 and W^2 are correlated Brownian motions with constant correlation coefficient ρ. The case "$\rho = 0$" is referred to as the *mutually catalytic* case. It is introduced in [DP98] (Uniqueness is proven in [Myt98].) The "$\rho \in (-1, 1)\backslash\{0\}$" cases are introduced in [EF04] (The extremal case $\rho = -1$ is known as Wright-Fisher-Model and is not covered in this thesis).

In the discrete mass setting, we denote the symbiotic masses by $(M, N) := (M_t, N_t)_{t\geq 0}$ instead of (Y, Z). In [EF04], the authors propose a particle model for a discrete mass symbiotic branching process: For every pair, consisting of one M-individual and one N-individual, branching events happen at constant rate. In case of such an event, one of the following alternatives happens: Either the M-particle or the N-particle branches alone (i.e. it dies or gives birth), or a joint (simultaneous) branching event happens that causes correlation of M and N. Positive correlation may be introduced to the system by allowing for a simultaneous birth event in which the M-particle and the N-particle give birth. Positive correlation may also be realized by a simultaneous death event, where both the M- and the N-particle die. Negative correlation is realized by allowing for a birth-death event, where one particle gives birth and the other particle dies. We adopt the "building blocks"-idea from [EK19] and provide a toolbox of birth and death mechanisms that allow for discrete mass symbiotic branching similar to the Etheridge-Fleischmann particle model.

In Chapter 2, for the reader's convenience, we briefly present the continuous-level lookdown construction from [KR11]. Then we construct a level representation of a mutually catalytic diffusion from two independent Kurtz-Rodrigues models via a series of time changes. In Chapter 3, we introduce a toolbox of discrete mass branching mechanisms from which level representations of discrete mass symbiotic branching processes will be assembled (for any correlation coefficient $\rho \in [-1, 1]$). Chapter 4 is dedicated to a lookdown construction of Feller's branching diffusion where the continuous level birth mechanism of the Kurtz-Rodrigues construction is combined with a "killing" mechanism similar to the multiple death mechanism in [EK19]. In Chapter 5, the ideas (and the calculations) from Chapter 4 are used to construct a lookdown representation for a symbiotic diffusion with $\rho \in (0, 1)$ as a weak limit of models that are similar to those of Chapter 3. In Chapter 6, we give some insight into the interplay of the low levels in our constructions (namely our construction of Feller's branching diffusion) and the total mass process.

2. Lookdown-construction of mutually catalytic diffusions

We call a solution $(Y, Z) = (Y_t, Z_t)_{t\geq 0}$ to the stochastic system

$$\begin{cases} \mathrm{d}Y_t &= \sqrt{bY_tZ_t}\ \mathrm{d}W_t^1 \\ \mathrm{d}Z_t &= \sqrt{bY_tZ_t}\ \mathrm{d}W_t^2, \end{cases} \tag{2.1}$$

where W^1 and W^2 are independent Brownian motions and $b > 0$, a *mutually catalytic branching diffusion* (sometimes a *pair of mutually catalytic branching diffusions*). If the Brownian motions are correlated with constant correlation coefficient $\rho \neq 0$, we call the system *symbiotic*. A solution Y to the stochastic differential equation

$$\mathrm{d}Y_t = \sqrt{bY_t}\ \mathrm{d}W_t + cY_t\ \mathrm{d}t, \tag{2.2}$$

where W is a Brownian motion, we call Feller's branching diffusion with drift or, in the case $c = 0$, we call it simply Feller's branching diffusion.

The Lookdown construction of a pair of mutually catalytic diffusions is quite straightforward, if one has the Kurtz-Rodrigues construction for the Feller's branching diffusion at hand. In the next section we introduce the latter shortly. Then we apply a certain type of generator calculation to a heuristically manifest candidate for the desired level representation. This kind of calculation forms the basis of the Kurtz-Rodrigues construction and we use it a lot, both to construct level representations rigorously and to gain heuristic insights into the behaviour of a given level system. Finally, in Section 2.3, we construct the level representation via a time change of two independent Kurtz-Rodrigues representations of Feller's diffusion.

2.1. Review of the Kurtz-Rodrigues construction

We consider Model 2.2 in [KR11]. It consists of one population of particles, all living in the same place. So we do not have a spatial structure. The particles are numbered arbitrarily. Let $r > 0$, $b \geq 0$ and $c \in \mathbb{R}$. Every particle i has a level $U^{i,r} = (U_t^{i,r})_{t\geq 0}$ that evolves according to the differential equation $\dot{u} = bu^2 - cu$. The starting levels are i.i.d. uniformly distributed in $[0, r]$. When a level reaches r, the corresponding particle dies. Every particle i gives birth at instantaneous rate $2b(r - U_t^{i,r})$ and the offspring gets a level, that is uniformly distributed in $[U_t^{i,r}, r]$. The number of particles in this model form a Galton-Watson process with birth rate rb and death rate $rb - c$. The model has the property that at any time $t \geq 0$, conditioned on the number of particles alive, the levels are i.i.d. uniformly distributed in $[0, r]$.

This model exhibits a form of consistency that allows for coupling of models with increasing branching rates and thus passing to the almost sure limit of the level dynamics for $r \to \infty$. Envision the dynamics for $r < \infty$ the following way: For every particle i we have a rate $2b$ Poisson point process on the space $[0, \infty)^2$, representing "time $\times$ level". We draw the death threshold r and the particle's level trajectory in the picture (cf. Figure 2.1).

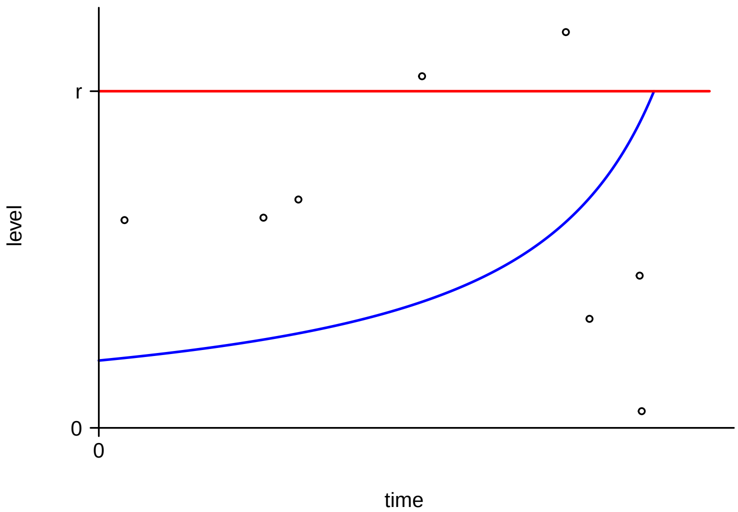

Figure 2.1.: Graphical representation of the birth and death mechanism of one particle in the model of [KR11]. Blue: the particle's level trajectory, red: the threshold at which the particle dies.

Whenever the particle's level passes below one Poisson point that is lower than r, one offspring is born. The level-coordinate of the corresponding point is the offspring's level. If the threshold of death is increased to $r' > r$, more births are occurring, but the level trajectory and the offspring corresponding to points below r are not affected. For every $r > 0$, the models are coupled and letting $r \to \infty$, the level systems converge pathwise. The limiting model (Model 2.3 in [KR11]) features the property that at any time $t \geq 0$, the levels are a Poisson point process with intensity Y_t, and $Y = (Y_t)_{t\geq 0}$ solves (2.2). In other words, the level system is a genealogical representation of Feller's branching diffusion with drift. We call it the *Kurtz-Rodrigues representation* (c.f. Model 2.3 in [KR11]).

At this point, we fix some notation and give the formal setting of the Kurtz-Rodrigues representation. The models we consider in this thesis are Markov processes with state space $\mathcal{S}_E$, the space of locally finite counting measures on a metric space E, endowed with the vague topology. E varies from model to model. For all spaces E that we consider in this thesis, $\mathcal{S}_E$ is Polish (in fact it is for all locally compact second countable Hausdorff spaces E; see Theorem A2.3 in [Kal02]). For the Kurtz-Rodrigues representation we have $E = \mathbb{R}^+$. A state at time t can be written as

$$U_t = \sum_{i=1}^{m} \delta_{U_t^i},$$

where U_t^i is called the level of the particles i at time t. By leaving out the time parameter, we denote the path of a process: $U^i := (U_t^i)_{t\geq 0}$, etc. Since we have no space in our models, we will not distinguish sharply the notions "level of a particle" and "particle". Depending on the context we will abuse a vector notation and we write $U_t = (U_t^1, U_t^2, \ldots)$. Recall that the indexing is arbitrary.

We characterize the Kurtz-Rodrigues model by its generator. We use test functions of the form

$$f(u) := \exp\left(\int \log g(x)\ u(\mathrm{d}x)\right),$$

where the function $g : \mathbb{R}^+ \to [0,1]$ is continuously differentiable and there is an $r_g > 0$, such that $g(x) = 1$ for $x \geq r_g$. For simplicity's sake, we write $f(u) = \prod_i g(u_i)$.

The Kurtz-Rodrigues representation is characterized by the generator

$$A_{KR} f(u) := f(u) \sum_i \left[2b \int_{u_i}^{\infty} (g(x) - 1)\ \mathrm{d}x + \left(bu_i^2 - cu\right) \cdot \frac{g'(u_i)}{g(u_i)} \right], \tag{2.3}$$

with domain

$$\mathcal{D}_{KR} := \left\{ f(u) := \prod_i g(u_i)\ :\ 0 \leq g \leq 1 \in C^1(\mathbb{R}),\ g(x) = 1 \text{ for } x \geq r_g \right\}.$$

The first term of A_{KR} characterizes the birth mechanism: Consider a parent particle U^i. In any level-interval $[a_1, a_2]$, where $U_t^i \leq a_1 < a_2$, offspring is born at instantaneous rate $2b(a_2 - a_1)$ and the offspring's level is uniformly distributed in $[a_1, a_2]$. Thus in any timespan the particle U^i gives birth to infinitely many offspring. The second term characterizes the movement of the particles (more precisely: the movement of the particles' levels). They move according to the differential equation $\dot{u} = bu^2 - cu$.

In [KR11] it is proven that if U is a solution to the martingale problem given by (2.3), then

$$Y := \limsup_{r\to\infty} \frac{U[0,r)}{r}$$

exists and solves (2.2). This result is obtained by applying the Markov Mapping Theorem A.5 in [KR11] (see also Theorem A.5.1). We use the same approach extensively in this thesis. The procedure is as follows: The level system is represented as a counting measure, so in the Kurtz-Rodrigues case the state space is $S = \mathcal{S}_{\mathbb{R}^+}$. Let $\gamma : S \to S_0$ be a projection to a "smaller" space S_0 and let α be a stochastic kernel $S_0 \to S$ satisfying $\alpha(y, \gamma^{-1}(y)) = 1$. In the Kurtz-Rodrigues case $\gamma(U) = \limsup_{r\to\infty} \frac{U[0,r)}{r}$ and $\alpha(y, \cdot)$ is the distribution of a Poisson point process on $\mathbb{R}^+$ with intensity y. Additional to some technical conditions the authors show that A_{KR} and the generator

$$C_{FD}\hat{f}(y) := by \cdot \frac{\mathrm{d}^2}{\mathrm{d}y^2}\hat{f}(y) + cy \cdot \frac{\mathrm{d}}{\mathrm{d}y}\hat{f}(y)$$

of Feller's branching diffusion with drift c satisfy the "intertwining relation"

$$C_{FD}\int \alpha(\cdot, \mathrm{d}u)\ f(u) = \int \alpha(\cdot, \mathrm{d}u)\ A_{KR}f(u).$$

The Markov Mapping Theorem then gives the above assertion. More precisely, it states the following: Let $\tilde{Y}$ be a solution to C_{FD}. Then there exists a solution U to the martingale problem for A_{KR} such that the asymptotic mass density process Y is a solution to the martingale problem given by C_{FD} with the same distribution as $\tilde{Y}$. Uniqueness of the martingale problem for A_{KR} implies uniqueness of the martingale problem for C_{FD}. And finally $\mathbf{P}\left[U_t \in \Gamma \mid \mathcal{F}_t^Y\right] = \alpha(Y_t, \Gamma)$ for all $t \geq 0$ and any measurable $\Gamma \subset \mathcal{S}_{\mathbb{R}}^+$. In other words, U_t is a conditional Poisson point process with intensity Y_t. We denote by $U^{(1)} := \min(U^i)_i$ the lowest level and by $U^{(k)} = \min\left\{U^i\ :\ U^i > U^{(k-1)}\right\}$ we denote the k-th lowest level.

2.2. Heuristics for the level representation of mutually catalytic branching diffusions

Our goal for this chapter is to construct a genealogical lookdown representation for a mutually catalytic system (Y, Z) (see (2.1)), similar to the Kurtz-Rodrigues representation. Heuristically, we may think of the marginal Y as a conditional Feller branching diffusion (without drift), whose branching rate is proportional to Z and vice versa. One might reason that a level representation with state space $\mathcal{S}_{\mathbb{R}+} \times \mathcal{S}_{\mathbb{R}+}$, that is obtained from a pair of Kurtz-Rodrigues constructions, where we adapt the birth rates and the differential equations accordingly, should do the trick.

Let

$$\mathcal{D} := \left\{g \in \mathcal{C}^1(\mathbb{R}^+)\ :\ 0 \leq g \leq 1, \text{there is } 0 < r_g < \infty \text{ such that } g(x) = 1 \text{ for } x \geq r_g\ \right\}.$$

We consider test functions $\mathcal{S}_{\mathbb{R}+} \times \mathcal{S}_{\mathbb{R}+} \to \mathbb{R}$ of the form

$$f(u,v) := f_1(u)f_2(v) := \exp\left(\int \log g_1(x)\ u(\mathrm{d}x)\right) \cdot \exp\left(\int \log g_2(x)\ v(\mathrm{d}x)\right)$$

where $g_1, g_2 \in \mathcal{D}$. We use the convention $\exp(\log(0)) = 0$. Call this class of test functions $\mathcal{D}_{MC}$.

For $u \in \mathcal{S}_{\mathbb{R}+}$, write

$$\lambda(u) := \limsup_{r\to\infty} \frac{1}{r}u([0,r))$$

for the asymptotic mass density. Note that the asymptotic density exists (almost surely), if u is a Poisson point process.

We define formally the generator

$$\begin{aligned}A_{MC}f(u,v) &:= A^{(1)}f(u,v) + A^{(2)}f(u,v)\\ &:= f(u,v)\sum_i \lambda(v)\left[2b\int_{u_i}^{\infty}\mathrm{d}x\ (g_1(x)-1) + bu_i^2\,\frac{g_1'(u_i)}{g_1(u_i)}\right]\\ &\quad + f(u,v)\sum_j \lambda(u)\left[2b\int_{v_j}^{\infty}\mathrm{d}x\ (g_2(x)-1) + bv_j^2\,\frac{g_2'(v_j)}{g_2(v_j)}\right].\end{aligned}$$

Note that the map $\lambda : \mathcal{S}_{\mathbb{R}^+} \to \mathbb{R}^+$ is not continuous. So the image of A_{MC} contains noncontinuous functions and the Markov Mapping Theorem A.5 in [KR11] is not applicable. We choose a different approach for the construction, relying on random time changes (see Section 2.3).

Note that in [KN11], one can find a version of the Markov Mapping theorem that allows for generators that map not only to noncontinuous functions (c.f. Corollary 3.3 in [KN11]). We did not take this approach, but we used the generator calculations as a tool to check, if our ideas are going in the right direction. (Behind the scenes we used this tool repeatedly.)

We give some heuristics, why the intertwining relation may be used as a kind of "test calculation": If the level-dynamics is set up in such a way that the knowledge of masses at time t does not provide information about the levels, then "forgetting" the level-information does not ruin the Markov property. Recall that α is a stochastic kernel $\mathbb{R}^+ \to \mathcal{S}_{\mathbb{R}^+}$ with $\alpha(y,\cdot)$ being the law of a Poisson point process with intensity y. Define

$$\alpha(y,z;\mathrm{d}u,\mathrm{d}v) := \alpha(y,\mathrm{d}u)\otimes\alpha(z,\mathrm{d}v).$$

Let the level-process (U,V) be a Markov process, adapted to the filtration $\mathcal{F}$, with generator $(A_{MC},\mathcal{D}_{MC})$ and let $\mathcal{F}^{Y,Z}$ be the filtration generated by the mass-densities

$$(Y_t,Z_t) := \left(\limsup_{r\to\infty}\frac{1}{r}U_t[0,r),\ \limsup_{r\to\infty}\frac{1}{r}V_t[0,r)\right)$$

of the process. For $f\in\mathcal{D}_{MC}$, the process

$$X_t := \mathbf{E}\left[f(U_t,V_t)\mid\mathcal{F}_t^{Y,Z}\right] - \int_0^t \mathbf{E}\left[A_{MC}f(U_s,V_s)\mid\mathcal{F}_s^{Y,Z}\right]\mathrm{d}s$$

is an $\mathcal{F}$-martingale. Since $\mathcal{F}^{Y,Z}\subset\mathcal{F}$ is a subfiltration, X is an $\mathcal{F}^{Y,Z}$-martingale and we have for $s \le t$ and $B \in \mathcal{F}_s^{Y,Z}$

$$\begin{aligned}
0 &= \mathbf{E}\left[(X_t - X_s)\mathbb{1}_B\right] \\
&= \mathbf{E}\left[\left(f(U_t,V_t) - f(U_s,V_s) - \int_s^t A_{MC} f(U_z,V_z)\,\mathrm{d}z\right)\mathbb{1}_B\right] \\
&= \mathbf{E}\left[\left(\mathbf{E}\left[f(U_t,V_t) \mid \mathcal{F}_t^{Y,Z}\right] - \mathbf{E}\left[f(U_s,V_s) \mid \mathcal{F}_s^{Y,Z}\right]\right.\right. \\
&\qquad \left.\left. - \int_s^t \mathbf{E}\left[A_{MC} f(U_z,V_z) \mid \mathcal{F}_z^{Y,Z}\right] \mathrm{d}z\right)\mathbb{1}_B\right].
\end{aligned} \tag{2.4}$$

Assume now that $\alpha(Y_t, Z_t; \mathrm{d}u, \mathrm{d}v)$ is the distribution of the levels at every time $t > 0$, given $\mathcal{F}_t^{Y,Z}$. (This is not obvious; the property is a consequence of the Markov Mapping Theorem.) We have then

$$\begin{aligned}
\mathbf{E}\left[f(U_t,V_t) \mid \mathcal{F}_t^{Y,Z}\right] &= \int \alpha(Y_t, Z_t; \mathrm{d}u, \mathrm{d}v)\ f(u,v) \quad \text{and} \\
\mathbf{E}\left[A_{MC} f(U_t,V_t) \mid \mathcal{F}_t^{Y,Z}\right] &= \int \alpha(Y_t, Z_t; \mathrm{d}u, \mathrm{d}v)\ A_{MC} f(u,v)
\end{aligned}$$

and Equation (2.4) shows that the mass-process (Y, Z) solves the martingale problem for the generator

$$C_{MC} \int \alpha(y, z; \mathrm{d}u, \mathrm{d}v)\ f(u,v) := \int \alpha(y, z; \mathrm{d}u, \mathrm{d}v)\ A_{MC} f(u,v) \tag{2.5}$$

In the following calculation we integrate out the level information to check if the intertwining relation (2.5) holds for the generator

$$C_{MC} \hat{f}(y,z) = yz\left(b \cdot \frac{\partial^2}{\partial y^2}\hat{f}(y,z) + b \cdot \frac{\partial^2}{\partial z^2}\hat{f}(y,z)\right).$$

The calculation below should be understood to be formal.

Calculation 2.2.1. Define

$$\beta_{g_i} := \int_0^\infty (1 - g_i(x))\ \mathrm{d}x, \qquad i = 1, 2. \tag{2.6}$$

Recall that $1 - g_i$ has compact support. Hence the integrals in (2.6) are well-defined. If X is a Poisson process with intensity measure μ, then for $F \ge 0$ its Laplace transform is given by

$$\mathbf{E}\left[e^{-\int F\ \mathrm{d}X}\right] = \exp\left(\int \left(e^{-F(x)} - 1\right)\ \mu(\mathrm{d}x)\right)$$

(see Lemma A.1.1). In terms of our α and with $f_i(u) = \exp\left(\int \log g_i(x)\ u(\mathrm{d}x)\right)$ this reads

$$\int \alpha(y, \mathrm{d}u)\, f_i(u) = e^{-y\beta_i}, \qquad i = 1, 2.$$

So we have

$$\begin{aligned}
\hat{f}(y,z) &:= \int \alpha(y, z; \mathrm{d}u, \mathrm{d}v)\ f(u,v) \\
&= \int \alpha(y, \mathrm{d}u)\, f_1(u) \cdot \int \alpha(z, \mathrm{d}v)\, f_2(v) \\
&= e^{-y\beta_{g_1}} \cdot e^{-z\beta_{g_2}}.
\end{aligned}$$

And using (A.5) in Lemma A.1.1, we obtain for the birth part

$$\begin{aligned}
&\int \alpha(y, \mathrm{d}u) \int \alpha(z, \mathrm{d}v)\, f(u,v) \sum_i 2b\lambda(v) \int_{u_i}^{\infty} (g_1(x) - 1)\ \mathrm{d}x \\
&\quad = \int \alpha(z, \mathrm{d}v)\, \lambda(v) f_2(v) \cdot \int \alpha(y, \mathrm{d}u) \prod_k g_1(u_k) \sum_i 2b \int_{u_i}^{\infty} (g_1(x) - 1)\ \mathrm{d}x \\
&\quad = 2byze^{-y\beta_{g_1}} e^{-z\beta_{g_2}} \int_0^{\infty} g_1(x) \int_x^{\infty} (g_1(s) - 1)\ \mathrm{d}s\ \mathrm{d}x.
\end{aligned} \tag{2.7}$$

For the movement part we calculate, again using (A.5) and then partial integration,

$$\begin{aligned}
&\int \alpha(y, \mathrm{d}u) \int \alpha(z, \mathrm{d}v)\, f(u,v) \sum_i \lambda(v) b u_i^2 \cdot \frac{g_1'(u_i)}{g_1(u_i)} \\
&\quad = yze^{-y\beta_{g_1}} e^{-z\beta_{g_2}} \int_0^{\infty} bx^2 g_1'(x)\ \mathrm{d}x \\
&\quad = yze^{-y\beta_{g_1}} e^{-z\beta_{g_2}} \left(bx^2 g_1(x) \Big|_0^{r_{g_1}} - \int_0^{r_{g_1}} 2bx g_1(x)\ \mathrm{d}x \right) \\
&\quad = yze^{-y\beta_{g_1}} e^{-z\beta_{g_2}} \left(\int_0^{r_{g_1}} 2bx\ \mathrm{d}x - \int_0^{r_{g_1}} 2bx g_1(x)\ \mathrm{d}x \right) \\
&\quad = yze^{-y\beta_{g_1}} e^{-z\beta_{g_2}} \int_0^{\infty} 2bx(1 - g_1(x))\ \mathrm{d}x.
\end{aligned} \tag{2.8}$$

Using lemma A.2.1, we get

$$\begin{aligned}
\int_0^{\infty} x(g_1(x) - 1)\ \mathrm{d}x &= \int_0^{r_{g_1}} \int_0^{x} (g_1(x) - 1)\ \mathrm{d}s\ \mathrm{d}x \\
&= \int_0^{r_{g_1}} \int_s^{r_{g_1}} (g_1(x) - 1)\ \mathrm{d}x\ \mathrm{d}s \\
&= \int_0^{\infty} \int_x^{\infty} (g_1(s) - 1)\ \mathrm{d}s\ \mathrm{d}x.
\end{aligned} \tag{2.9}$$

We combine (2.7), (2.8) and (2.9) to obtain

$$\begin{aligned}
&\int \alpha(y, \mathrm{d}u) \int \alpha(z, \mathrm{d}v)\, A^{(1)} f(u,v) \\
&\quad = yze^{-y\beta_{g_1}} e^{-z\beta_{g_2}} 2b \int_0^{\infty} (g_1(x) - 1) \int_x^{\infty} (g_1(s) - 1)\ \mathrm{d}s\ \mathrm{d}x.
\end{aligned}$$

Observe that

$$2 \int_0^{r_{g_1}} (g_1(x) - 1) \int_z^{r_{g_1}} (g_1(s) - 1)\ \mathrm{d}s\ \mathrm{d}z = \left(\int_0^{r_{g_1}} (1 - g_1(x))\ \mathrm{d}x \right)^2.$$

Hence we obtain

$$\int \alpha(y_1, \mathrm{d}u) \int \alpha(y_2, \mathrm{d}v)\, A^{(1)} f(u,v) = yzbe^{-y\beta_{g_1}} e^{-z\beta_{g_2}} \beta_{g_1}^2.$$

The same calculation for $A^{(2)}$ gives us for $\hat{f}(y,z) = e^{-y\beta_{g_1}}e^{-z\beta_{g_2}}$ the intertwining relation

$$\begin{aligned}\int \alpha(y,\mathrm{d}u)\int \alpha(z,\mathrm{d}v)\,Af(u,v) &= yzbe^{-y\beta_{g_1}}e^{-z\beta_{g_2}}\beta_{g_1}^2 + yzbe^{-y\beta_{g_1}}e^{-z\beta_{g_2}}\beta_{g_1}^2 \\ &= yz\left(b\cdot\frac{\partial^2}{\partial y^2}\hat{f}(y,z) + b\cdot\frac{\partial^2}{\partial z^2}\hat{f}(y,z)\right) \\ &= C_{MC}\int \alpha(y,\mathrm{d}u)\int \alpha(z,\mathrm{d}v)\,f(u,v).\end{aligned}$$

$\diamondsuit$

2.3. Construction via time change

While Calculation 2.2.1 gives the intertwining relation (2.5), the Markov Mapping Theorem A.5 in [KR11] is not applicable. Instead we construct the lookdown representation for the mutual catalytic diffusion "by hand" via successive time changes starting from two independent Kurtz-Rodrigues representations of Feller's branching diffusion.

Because of technical reasons we stop the system, as soon as one of its mass densities falls below some threshold $\delta > 0$. Define the generator

$$A^{\delta}_{MC}f(u,v) := \mathbb{1}_{(\delta,\infty)}(\lambda(u)\wedge\lambda(v))\cdot A_{MC}f(u,v).$$

Theorem 2.3.1. *Let $(\check{Y},\check{Z})$ be a mutually catalytic branching diffusion, i.e., a solution to (2.1). Let $\tau := \{s \geq 0 \;:\; \check{Y}_s \wedge \check{Z}_s \leq \delta\}$ and define $(\check{Y}^{\delta}_t, \check{Z}^{\delta}_t) := (\check{Y}_{t\wedge\tau}, \check{Z}_{t\wedge\tau})$. The local $D_{\mathcal{S}_{\mathbb{R}+}\times\mathcal{S}_{\mathbb{R}+}}[0,\infty)$-martingale problem $(A^{\delta}_{MC},\mathcal{D}_{MC})$ has a solution (U^{δ},V^{δ}) with continuous mass density processes*

$$(Y^{\delta},\, Z^{\delta}) := \Big(\limsup_{r\to\infty}\frac{1}{r}U^{\delta}([0,r)),\, \limsup_{r\to\infty}\frac{1}{r}V^{\delta}([0,r))\Big).$$

We have

$$(Y^{\delta},Z^{\delta}) \stackrel{d}{=} (\check{Y}^{\delta},\check{Z}^{\delta}).$$

Remark 2.3.2. *Note that the construction does not yield uniqueness of the level representation.*

Proof. Let $(\tilde{U},\tilde{V})$ be a pair of two independent Kurtz-Rodrigues representations for Feller's branching diffusion. Its mass density process

$$(\tilde{Y},\, \tilde{Z}) := \Big(\limsup_{r\to\infty}\frac{1}{r}\tilde{U}([0,r)),\, \limsup_{r\to\infty}\frac{1}{r}\tilde{V}([0,r))\Big)$$

is a weak solution to the stochastic system

$$\begin{cases} \mathrm{d}\tilde{Y}_t &= \sqrt{2b\tilde{Y}_t}\,\mathrm{d}W_t^1, \\ \mathrm{d}\tilde{Z}_t &= \sqrt{2b\tilde{Z}_t}\,\mathrm{d}W_t^2, \end{cases}$$

where W^1 and W^2 are independent Brownian motions. Let $(\tilde{\mathcal{F}}_t^U)_{t\geq 0}$ and $(\tilde{\mathcal{F}}_t^V)_{t\geq 0}$ be right-continuous filtrations that $\tilde{U}$ and $\tilde{V}$ are adapted to. Define the stopping times

$$\tau^Y := \inf\{s \geq 0 \,:\, \tilde{Y}_s \leq \delta\},$$
$$\tau^Z := \inf\{s \geq 0 \,:\, \tilde{Z}_s \leq \delta\},$$

and denote the stopped processes by $\tilde{U}_t^\delta := \tilde{U}_{t\wedge\tau^Y}$, $\tilde{V}_t^\delta := \tilde{V}_{t\wedge\tau^Z}$, $\tilde{Y}_t^\delta := \tilde{Y}_{t\wedge\tau^Y}$ and $\tilde{Z}_t^\delta := \tilde{Z}_{t\wedge\tau^Z}$. Denote by

$$A_{KR}^\delta f(u) := \mathbb{1}_{(\delta,\infty)}(\lambda(u)) \cdot A_{KR} f(u)$$

the generator of the stopped Kurtz-Rodrigues dynamics. Hence, for $f \in \mathcal{D}_{KR}$,

$$\tilde{M}_t^{U,f} := f(\tilde{U}_t^\delta) - f(\tilde{U}_0^\delta) - \int_0^t A_{KR}^\delta f(\tilde{U}_s^\delta)\ \mathrm{d}s,$$
$$\tilde{M}_t^{V,f} := f(\tilde{V}_t^\delta) - f(\tilde{V}_0^\delta) - \int_0^t A_{KR}^\delta f(\tilde{V}_s^\delta)\ \mathrm{d}s$$

are martingales with respect to $\tilde{\mathcal{F}}^U$ or $\tilde{\mathcal{F}}^V$ respectively.

Define random time changes $t \mapsto \sigma_t^Y$ and $t \mapsto \sigma_t^Z$ by the right inverse of the quadratic variations $\langle \tilde{Y}^\delta \rangle_t$ and $\langle \tilde{Z}^\delta \rangle_t$, i.e.

$$\sigma_t^Y := \inf\left\{ s \geq 0 \,:\, 2b\int_0^s \tilde{Y}_h^\delta\ \mathrm{d}h \geq t \right\},$$
$$\sigma_t^Z := \inf\left\{ s \geq 0 \,:\, 2b\int_0^s \tilde{Z}_h^\delta\ \mathrm{d}h \geq t \right\}.$$

Define the time changed level dynamics $\hat{U}_t^\delta := \tilde{U}_{\sigma_t^Y}^\delta$ and $\hat{V}_t^\delta := \tilde{V}_{\sigma_t^Z}^\delta$. By the rule for differentiation of inverse functions, the time change σ_t^Y solves the equation

$$\sigma_t^Y = \int_0^t \frac{1}{2b\tilde{Y}_{\sigma_s^Y}^\delta}\ \mathrm{d}s.$$

Hence σ_t^Y is bounded, $\sigma_t^Y \leq \frac{t}{2b\delta}$, and by the Optional Sampling Theorem (cf. Theorem 7.29 in [Kal02]) we obtain that

$$\begin{aligned} \tilde{M}_{\sigma_t^Y}^{U,f} &= f(\tilde{U}_{\sigma_t^Y}^\delta) - f(\tilde{U}_0^\delta) - \int_0^{\sigma_t^Y} A_{KR}^\delta f(\tilde{U}_s^\delta)\ \mathrm{d}s \\ &= f(\tilde{U}_{\sigma_t^Y}^\delta) - f(\tilde{U}_0^\delta) - \int_0^t A_{KR}^\delta f(\tilde{U}_{\sigma_s^Y}^\delta) \cdot (\sigma^Y)'_s\ \mathrm{d}s \\ &= f(\hat{U}_t^\delta) - f(\hat{U}_0^\delta) - \int_0^t A_{KR}^\delta f(\hat{U}_s^\delta) \cdot \frac{1}{2b\hat{Y}_s^\delta}\ \mathrm{d}s \end{aligned}$$

is an $(\tilde{\mathcal{F}}^U_{\sigma^Y_t})$-martingale. An analogous consideration holds for $\tilde{M}^{V,f}_{\sigma^Z_t}$. Each of the time changed level dynamics $\hat{U}^\delta$ and $\hat{V}^\delta$ solve the martingale problem for

$$A^\delta_{BM} f(u) := \frac{1}{2b\lambda(u)} A^\delta_{KR}$$

By a similar argument (or, alternatively, by the Dubins-Schwarz Theorem 18.4 in [Kal02]) the mass densities

$$\begin{aligned}\hat{Y}^\delta &:= \limsup_{r\to\infty} \frac{1}{r}\hat{U}^\delta([0,r)),\\ \hat{Z}^\delta &:= \limsup_{r\to\infty} \frac{1}{r}\hat{V}^\delta([0,r))\end{aligned}$$

are stopped Brownian motions (hence the label "*BM*"). Recall that $\hat{U}^\delta$ and $\hat{V}^\delta$ are defined on the same probability space. The generator of the joint process $\left(\hat{U}^\delta, \hat{V}^\delta\right)$ is

$$A^\delta_{BM} f(u,v) := f_2(v) A^\delta_{BM} f_1(u) + f_1(u) A^\delta_{BM} f_2(v),$$

where $f \in \mathcal{D}_{MC}$.

Define the filtration $\hat{\mathcal{F}}$ by $\hat{\mathcal{F}}_t := \tilde{\mathcal{F}}^U_{\sigma^Y_t} \vee \tilde{\mathcal{F}}^V_{\sigma^Z_t}$. We apply a second time change to obtain the level representation of the mutually catalytic diffusion. Define

$$\sigma_t := \inf\left\{ s \geq 0 \,:\, \int_0^s \frac{1}{2b\hat{Y}^\delta_h \hat{Z}^\delta_h} \,\mathrm{d}h \geq t \right\}$$

and

$$\begin{aligned}\left(U^\delta, V^\delta\right)_t &:= \left(\hat{U}^\delta, \hat{V}^\delta\right)_{\sigma_t},\\ \left(Y^\delta, Z^\delta\right)_t &:= \left(\hat{Y}^\delta, \hat{Z}^\delta\right)_{\sigma_t}.\end{aligned}$$

We check that $\left(U^\delta, V^\delta\right)$ solves the local martingale problem for A^δ_{MC}: We introduce an upper threshold $K > \delta$ for the mass densities. Define the $\mathcal{F}$-stopping time

$$\tau_K := \inf\left\{ s \geq 0 \,:\, Y^\delta_s \vee Z^\delta_s \geq K \right\}.$$

Note that σ_t solves the equation

$$\begin{aligned}\sigma_t &= 2b\int_0^t \hat{Y}^\delta_{\sigma_s} \hat{Z}^\delta_{\sigma_s} \,\mathrm{d}s\\ &= 2b\int_0^t Y^\delta_s Z^\delta_s \,\mathrm{d}s.\end{aligned}$$

Hence we have

$$\sigma_{t\wedge\tau_K} \leq 2bK^2 t$$

and, by the Optional Sampling Theorem, the process

$$\begin{aligned} M^f_{t\wedge\tau_K} &:= f(\hat{U}^\delta_{\sigma_{t\wedge\tau_K}}, \hat{V}^\delta_{\sigma_{t\wedge\tau_K}}) - f(\hat{U}^\delta_0, \hat{V}^\delta_0) - \int_0^{\sigma_{t\wedge\tau_K}} A^\delta_{BM} f(\hat{U}^\delta_s, \hat{V}^\delta_s)\ \mathrm{d}s \\ &= f(U^\delta_{t\wedge\tau_K}, V^\delta_{t\wedge\tau_K}) - f(U^\delta_0, V^\delta_0) - \int_0^{t\wedge\tau_K} A^\delta_{BM} f(U^\delta_s, V^\delta_s) \cdot 2bY^\delta_s Z^\delta_s\ \mathrm{d}s \end{aligned}$$

is a martingale. Finally we argue that $\tau_K \to \infty$ almost surely: Let

$$\hat{\tau}_K := \inf\left\{ s \geq 0 \,:\, \hat{Y}^\delta_s \vee \hat{Z}^\delta_s \geq K \right\}$$

and observe that the inverse of the time change σ is

$$\sigma_t^{-1} = \int_0^t \frac{1}{2b\hat{Y}^\delta_h \hat{Z}^\delta_h}\ \mathrm{d}h.$$

We have for $t < \infty$

$$\begin{aligned} \lim_{K\to\infty} \mathbf{P}\left[\tau_K \leq t\right] &= \lim_{K\to\infty} \mathbf{P}\left[\hat{\tau}_K \leq \sigma_t^{-1}\right] \\ &\leq \lim_{K\to\infty} \mathbf{P}\left[\hat{\tau}_K \leq \frac{t}{2b\delta^2}\right] = 0, \end{aligned}$$

since $\hat{Y}^\delta$ and $\hat{Z}^\delta$ are Brownian motions, stopped when they hit δ. This implies

$$\lim_{K\to\infty} \tau_K = \infty \quad \text{a.s.}$$

and (U^δ, V^δ) solves the local martingale problem for A^δ_{MC}.

An analogous consideration yields that $(Y^\delta, Z^\delta) \stackrel{d}{=} (\check{Y}^\delta, \check{Z}^\delta)$. □

3. Lookdown-construction of symbiotic branching particle systems

The construction in Chapter 2 gives us the solution to Equation (2.1) where the driving Brownian motions are independent. In the following chapters, we are interested in models where W^1 and W^2 are correlated, with constant correlation coefficient ρ. In this setting, things get more complicated. The mechanism we choose to introduce the correlation is the timing of births and deaths. Letting particles die simultaneously in both populations, or give birth simultaneously in both populations, should lead to positive correlation. And by killing particles in one of the populations when births happen in the other population, we should end up with a negatively correlated model. But for the moment it is not clear how to implement these simultaneous events for the continuous mass model described in Chapter 2. In Chapter 5 a continuous mass model with simultaneous deaths in both populations is introduced.

We take a step backwards and construct discrete mass particle systems (U^r, V^r) with levels in the interval $[0, r)$, for which the masses (M^r, N^r) are symbiotic discrete mass branching systems. I.e. (M^r, N^r) is a càdlàg Markov process with values in $\mathbb{N}_0 \times \mathbb{N}_0$, where certain birth-death events happen at rates that are proportional to $M_t^r \cdot N_t^r$. Let $a, a_1, a_2, b, b_1, b_2, c \in \mathbb{R}^+$ and $s_1, s_2, d_1, d_2 \in \mathbb{N}$. On the mass-stage the following birth-death events are implemented:

Mutually catalytic deaths: M^r is decreased by one at rate $a_1 M_t^r N_t^r$ and, independently, N^r is decreased by one at rate $a_2 M_t^r N_t^r$.

Mutually catalytic births: M^r is increased by s_1 at rate $b_1 M_t^r N_t^r$ and, independently, N^r is increased by s_2 at rate $b_2 M_t^r N_t^r$.

Positively correlated symbiotic births: M^r is increased by s_1 and simultaneously N^r is increased by s_2 at rate $b M_t^r N_t^r$.

Positively correlated symbiotic deaths: M^r is decreased by d_1 and simultaneously N^r is decreased by d_2 at rate $a M_t^r N_t^r$.

Negatively correlated symbiotic births and deaths: M^r is increased by s_1 and simultaneously N^r is decreased by d_2 at rate $c M_t^r N_t^r$ (or vice versa).

(It should be understood that all parameters may be chosen independently for the different mechanisms.)

On the level-stage our models are similar to the Model 2.2 in [KR11] with $r < \infty$. But their behaviour for large r is seriously more difficult to handle as they lack a property that turned out to be convenient in the original model of Kurtz and Rodrigues. The Kurtz-Rodrigues construction features the strong notion of consistency mentioned in the previous chapter: The models for $r' < \infty$ can be obtained by restricting the $(r = \infty)$-model to $[0, r']$. This allows for the construction of the $(r = \infty)$-model as an almost sure limit of the simpler $r < \infty$ models. We will see that the level systems in this chapter lack this property.

We fix some notation, this time for the $r < \infty$ setting. In the discrete mass case, a state can be written as

$$(U^r, V^r)_t := \Big(\sum_i \delta_{U_t^{i,r}}, \sum_j \delta_{V_t^{j,r}}\Big),$$

where $(U^{i,r})_i, (V^{j,r})_j \in [0, r]$ are the levels of respective populations. So the state space is $\mathcal{S}_{[0,r]} \times \mathcal{S}_{[0,r]}$. Recall that the enumeration is arbitrary and we denote by $U^{(k),r}, V^{(k),r}$ the k-th lowest level of the respective population. Denote the masses of the particle systems by $M_t^r := U_t^r([0, r))$ and $N_t^r := V_t^r([0, r))$. Depending on the context we will abuse notation and write $U_t^r = \Big(U_t^{1,r}, \ldots, U_t^{M_t^r,r}\Big)$ and $V_t^r = \Big(V_t^{1,r}, \ldots, V_t^{N_t^r,r}\Big)$.

Let $\alpha_r(m, \mathrm{d}u)$ be the distribution of a Poisson point process on $[0, r]$, conditioned to have mass m. When we use the vector notation for U^r and V^r the definition of α_r looks a bit ambiguous: In this vector setting $\alpha_r(m, \mathrm{d}u)$ should be understood as joint distribution of m independent, uniformly distributed random variables on $[0, r]$. We denote the product measure

$$\alpha_r(m, n; \mathrm{d}u, \mathrm{d}v) := \alpha_r(m, \mathrm{d}u) \otimes \alpha_r(n, \mathrm{d}v).$$

At time $t = 0$, the levels are independent and uniformly distributed given (M_0, N_0),

$$\mathbf{P}\left[(U_0^r, V_0^r) \in \Gamma \mid M_0 = m, N_0 = n\right] = \alpha_r(m, n; \Gamma) \tag{3.1}$$

for measurable $\Gamma \subset \mathcal{S}_{[0,r]} \times \mathcal{S}_{[0,r]}$.

The paths of the individual levels are càdlàg and between jumps the dynamics are driven by a system of ordinary differential equations. The motion of the levels makes sure that the uniform distribution property (3.1) stays true for all $t \geq 0$. The speed of a U-level depends on the V-levels and vice versa,

$$\begin{aligned} \dot{U}_t^{i,r} &= \sum_{j=1}^{N_t^r} F_1(U_t^{i,r}, V_t^{j,r}), \quad i = 1, \ldots, M_t^r, \\ \dot{V}_t^{j,r} &= \sum_{i=1}^{M_t^r} F_2(U_t^{i,r}, V_t^{j,r}), \quad j = 1, \ldots, N_t^r, \end{aligned}$$

where $F_k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, $k = 1, 2$ are to be defined for the different birth ant death mechanisms in Section 3.1.

When a level reaches the upper bound r, the corresponding particle dies and its level is removed from the ensemble. We will also introduce "killing" of particles by letting the levels of one population jump simultaneously, e.g. when death is triggered by an external Poisson point process. The highest levels jump to r and thus to their death, the other levels are scaled up by the a common factor, thus preserving the uniform distribution.

Reproduction is controlled by conditional Poisson processes, where the birth rate of a particle i at time $t > 0$ depends on its level $U_t^{i,r}$ and on the level configuration V_t^r. The same is true for the V-particles with reversed roles.

We use test functions of the form $f(u, v) = f_1(u) \cdot f_2(v)$ with

$$f_1(u) = \exp\left(\int \log\left(g_1(x)\right) u(\mathrm{d}x)\right) \quad \text{and} \quad f_2(v) = \exp\left(\int \log\left(g_2(x)\right) v(\mathrm{d}x)\right).$$

The functions $g_1, g_2 : \mathbb{R}^+ \to [0, 1]$ are continuously differentiable with $g_1(x) = g_2(x) = 1$ for $x > r$. We often write $f_k(u) = \prod_i g_k(u_i)$, $k = 1, 2$. We call this class of test functions $\mathcal{D}^r_{SBDM}$ (for *symbiotic branching of discrete masses*).

Furthermore, we define λ_k by

$$e^{-\lambda_k} = \frac{1}{r}\int_0^r g_k(z)\, dz, \qquad k = 1, 2$$

and we let

$$\begin{aligned}\hat{f}(m, n) &:= \int \alpha_r(m, n; \mathrm{d}u, \mathrm{d}v)\, f(u, v)\\ &= e^{-\lambda_1 m} e^{-\lambda_2 n}.\end{aligned}$$

Note that, by letting $r \to \infty$ with $\frac{M_0}{r} \to Y_0$ and $\frac{N_0}{r} \to Y_0$, heuristically we should pass to corresponding continuous mass models, as for the Kurtz-Rodrigues-representation. Note that the state spaces $\mathcal{S}_{[0,r]}$, provided with the topology of vague convergence, is Polish (cf. [Kal17], Theorem 4.2).

3.1. Birth and death mechanisms

The generators of the Markov models can be assembled from components, representing birth and death mechanisms, that allow for a variety of population models. Following the ideas in [EK19] we give a "toolbox of building blocks", that can be used to establish mutually catalytic branching as well as different kinds of symbiotic branching.

The calculations in the following subsections should be understood to be formal. They are used in Section 3.2 to apply the Markov Mapping Theorem A.5 in [KR11] and characterize our models. The dynamics in the models are chosen in such a way, that at any time $t > 0$, the distribution of the levels, given the masses $M_t^r = m$ and

$N_t^r = n$, is $\alpha_r(m, n; \mathrm{d}u, \mathrm{d}v)$. Apart from the building blocks "mutually catalytic dying" (Section 3.1.1) and "positive symbiotic dying" (Section 3.1.4) it is not obvious that this distribution property holds. It is the Markov Mapping Theorem that ensures the property. The models are "reverse engineered", such that using α_r to average out the level-information, leads to the desired models for the masses (M^r, N^r). Note that the uniform distribution α_r is chosen for convenience. Other distributions would also work and could be better adapted to different situations.

3.1.1. Mutually catalytic deaths

The building block for mutually catalytic deaths and the building block for mutually catalytic births (see Section 3.1.2) are based on the ideas in Section 2.1 of [KR11]. The model given by the following level dynamic is a representation of a mutually catalytic pure death process, i.e., the marginals M^r and N^r of the mass-process (M^r, N^r) are uncorrelated pure death processes where the death rates of M^r and N^r at time $t \geq 0$ are proportional to $M_t^r \cdot N_t^r$.

Let $a_1, a_2 \geq 0$. We start with positive masses $(M_0^r, N_0^r) = (m_0, n_0) \in \mathbb{N} \times \mathbb{N}$ and independent, uniformly distributed levels, $(U_0^r, V_0^r) \sim \alpha_r(m_0, n_0; \,\cdot\,, \,\cdot\,)$. The particles do not generate offspring. The levels move upwards continuously, where the movement is given by $F_1(u_i, v_j) = a_1 u_i$ and $F_2(u_i, v_j) = a_2 v_j$. I.e., the levels solve the system of conditional ordinary differential equations

$$\dot{U}_t^{i,r} = \sum_{j=1}^{N_t^r} F_1(U_t^{i,r}, V_t^{j,r}) = a_1 N_t^r U_t^{i,r}, \quad i \in \{1, 2, \ldots, M_t^r\},$$
$$\dot{V}_t^{j,r} = \sum_{i=1}^{M_t^r} F_2(U_t^{i,r}, V_t^{j,r}) = a_2 M_t^r V_t^{j,r}, \quad j \in \{1, 2, \ldots, N_t^r\}.$$

Recall that the domain of the generator is chosen in such a way that a particle dies when its level reaches the threshold r. Define $m := u\left([0, r)\right)$ and $n := v\left([0, r)\right)$. The generator of this process is

$$A_{md}^r f(u, v) = f(u, v) \left[\sum_{i=1}^{m} a_1 n u_i \frac{g_1'(u_i)}{g_1(u_i)} + \sum_{j=1}^{n} a_2 m v_j \frac{g_2'(v_j)}{g_2(v_j)} \right] \tag{3.2}$$

The division by $g_1(u_i)$ and $g_2(v_j)$ in (3.2) is there to cancel out the corresponding factors in $f(u, v)$,

$$f(u, v)/g_1(u_i) = f_2(v) \cdot \prod_{\substack{k=1 \\ k \neq i}}^{m} g_1(u_k), \qquad f(u, v)/g_2(v_j) = f_1(u) \cdot \prod_{\substack{k=1 \\ k \neq j}}^{n} g_2(v_k)$$

and should be understood this way if the denominator is zero.

Note that if the V-mass $N^r \equiv n$ was fixed, the U-population would be in the situation of Model 2.1 in [KR11] and the time of survival for each particle was exponentially distributed.

We presume the conditional level distribution α_r and check that averaging out the levels leads to the desired mass-dynamics.

Calculation 3.1.1. Recall that λ_k is defined by $\frac{1}{r}\int_0^r g_k(z)\,dz = e^{-\lambda_k}$ for $k=1,2$ and thus

$$\hat{f}(m,n) = \int \alpha_r(m,\mathrm{d}u)\int \alpha_r(n,\mathrm{d}v) f(u,v) = e^{-\lambda_1 m}e^{-\lambda_2 n}.$$

Using partial integration and $g_1(r)=1$, we have

$$\begin{aligned}
&\int \alpha_r(m,\mathrm{d}u)\int \alpha_r(n,\mathrm{d}v)\, f(u,v)\sum_{i=1}^m a_1 n u_i \frac{g_1'(u_i)}{g_1(u_i)}\\
&\quad = a_1 m n e^{-\lambda_1(m-1)}e^{-\lambda_2 n}\frac{1}{r}\int_0^r z g_1'(z)\ \mathrm{d}z\\
&\quad = a_1 m n e^{-\lambda_1(m-1)}e^{-\lambda_2 n}\left(\frac{1}{r} z g_1(z)\Big|_0^r - \frac{1}{r}\int_0^r g_1(z)\ \mathrm{d}z\right)\\
&\quad = a_1 m n e^{-\lambda_1(m-1)}e^{-\lambda_2 n}\left(1-e^{-\lambda_1}\right).
\end{aligned}$$

An analogous calculation for the second summand of A^r_{md} yields the intertwining relation

$$\begin{aligned}
C^r_{md}\hat{f}(m,n) &:= \int \alpha_r(m,\mathrm{d}u)\int \alpha_r(n,\mathrm{d}v)\, A^r_{md} f(u,v)\\
&= a_1 mn(\hat{f}(m-1,n)-\hat{f}(m,n)) + a_2 mn(\hat{f}(m,n-1)-\hat{f}(m,n)).
\end{aligned}$$

The generator C^r_{md} belongs to a mutually catalytic death process, as portrayed above. $\Diamond$

3.1.2. Mutually catalytic births

The next building block is "mutually catalytic births". A model given by the level dynamics of this block is a representation of a mutually catalytic pure birth process. It is easy to allow for multiple simultaneous births within each of the populations separately. So we include multiple births per subpopulation in this building block although they are not needed later. The model that allows for birth events that generate U- and V-offspring simultaneously is more complicated. It is introduced in Section 3.1.3.

The marginals M^r and N^r of the mass-process (M^r, N^r) are uncorrelated pure birth-processes, where the birth rates are proportional to $M^r_t N^r_t$ at every time $t \geq 0$. For the corresponding model with one population, i.e. a pure birth process, compare Section 3.4 from [EK19].

Every pair of a U-particle and a V-particle may give birth to U-particles or V-particles. As stated above, this building block allows simultaneous births of U-particles and simultaneous births of V-particles, but no simultaneous births of a U- and a V-particle.

Let $b_1, b_2 \geq 0$. Again we start with positive masses $(M_0^r, N_0^r) = (m_0, n_0) > 0$ and independent, uniformly distributed levels on $[0, r]$,

$$(U_0^r, V_0^r) \sim \alpha_r(m_0, n_0;\ \cdot\ ,\ \cdot\ ).$$

The pair $(U^{i,r}, V^{j,r})$ gives birth to s_1 U-children at rate

$$b_1 \frac{s_1 + 1}{r^{s_1}} (r - U_t^{i,r})^{s_1},$$

and, independently (i.e., driven by an independent conditional Poisson process), to s_2 V-children at rate

$$b_1 \frac{s_2 + 1}{r^{s_2}} (r - V_t^{j,r})^{s_2}.$$

The levels of the U-offspring are independent, uniformly distributed on $[U_t^{i,r}, r]$, and the levels of the V-offspring are independent, uniformly distributed on $[V_t^{j,r}, r]$.

Since the offspring levels are always above their respective parent's level, the birth mechanism alone would lead to an accumulation of levels in the upper region of the interval $[0, r]$. It is necessary that the levels move downwards to maintain the uniform distribution. We choose a continuous movement given by

$$F_1(u_i, v_j) = F_1(u_i) = b_1 \left[\frac{(r - u_i)^{s_1+1}}{r^{s_1}} - (r - u_i) \right], \tag{3.3}$$

$$F_2(u_i, v_j) = F_2(v_j) = b_2 \left[\frac{(r - v_j)^{s_2+1}}{r^{s_2}} - (r - v_j) \right], \tag{3.4}$$

i.e., the system of ordinary differential equations

$$\begin{aligned} \dot{U}_t^{i,r} &= \sum_{j=1}^{N_t^r} F_1(U_t^{i,r}) = b_1 N_t^r \left[\frac{(r - U_t^{i,r})^{s_1+1}}{r^{s_1}} - \left(r - U_t^{i,r}\right) \right], \\ \dot{V}_t^{j,r} &= \sum_{i=1}^{M_t^r} F_2(V_t^{j,r}) = b_2 M_t^r \left[\frac{(r - V_t^{j,r})^{s_2+1}}{r^{s_2}} - \left(r - V_t^{j,r}\right) \right]. \end{aligned}$$

Recall that $m = u([0, r))$ and $n = v([0, r))$. The generator for the model is

$$\begin{aligned} A_{mb}^r f(u, v) &= A_{mb}^{1,r} f(u, v) + A_{mb}^{2,r} f(u, v) \\ &= f(u, v) \sum_{i=1}^{m} \sum_{j=1}^{n} \left[b_1 \frac{s_1 + 1}{r^{s_1}} \int_{u_i}^{r} \mathrm{d}x_1 \cdots \int_{u_i}^{r} \mathrm{d}x_{s_1} \left(\prod_{k=1}^{s_1} g_1(x_k) - 1 \right) + F_1(u_i) \frac{g_1'(u_i)}{g_1(u_i)} \right] \\ &\quad + f(u, v) \sum_{i=1}^{m} \sum_{j=1}^{n} \left[b_2 \frac{s_2 + 1}{r^{s_2}} \int_{v_j}^{r} \mathrm{d}y_1 \cdots \int_{v_j}^{r} \mathrm{d}y_{s_2} \left(\prod_{k=1}^{s_2} g_2(y_k) - 1 \right) + F_2(v_j) \frac{g_2'(v_j)}{g_2(v_j)} \right]. \end{aligned}$$

Calculation 3.1.2. Again we check that averaging out the level information, using the conditional distribution $\alpha_r(m, n, \mathrm{d}u, \mathrm{d}v)$, leads to the desired mass-model, i.e., the generator

$$C_{mb}^r \hat{f}(m, n) := \int \alpha_r(m, \mathrm{d}u) \int \alpha_r(n, \mathrm{d}v)\, A_{mb}^r f(u, v)$$

belongs to a mutually catalytic birth process.

For the birth-part of $A^{1,r}_{mb}$, we calculate

$$\begin{aligned}
&\int \alpha_r(m,\mathrm{d}u)\int \alpha_r(n,\mathrm{d}v)\, f(u,v)\sum_{i=1}^m\sum_{j=1}^n b_1\frac{s_1+1}{r^{s_1}}\int_{u_i}^r \mathrm{d}x_1\cdots\int_{u_i}^r \mathrm{d}x_{s_1}\Big(\prod_{k=1}^{s_1} g_1(x_k)-1\Big)\\
&\quad = b_1 n e^{-\lambda_2 n}e^{-\lambda_1(m-1)}\sum_{i=1}^m \frac{s_1+1}{r^{s_1+1}}\int_0^r \mathrm{d}u_i\, g_1(u_i)\int_{u_i}^r \mathrm{d}x_1\cdots\int_{u_i}^r \mathrm{d}x_{s_1}\Big(\prod_{k=1}^{s_1} g_1(x_k)-1\Big)\\
&\quad = b_1 mn e^{-\lambda_2 n}e^{-\lambda_1(m-1)}\Big(e^{-\lambda_1(s_1+1)}-\frac{s_1+1}{r^{s_1+1}}\int_0^r \mathrm{d}x\, g_1(x)(r-x)^{s_1}\Big).
\end{aligned} \tag{3.5}$$

For the last step in (3.5), note that the offspring-levels are greater than the parent's level. Let $\mathcal{U}_1,\mathcal{U}_2,\dots,\mathcal{U}_{s_1+1}$ be independent, uniformly on $[0,r]$ distributed random variables, then

$$\begin{aligned}
&\frac{s_1+1}{r^{s_1+1}}\int_0^r \mathrm{d}u_i\, g_1(u_i)\int_{u_i}^r \mathrm{d}x_1\cdots\int_{u_i}^r \mathrm{d}x_{s_1}\prod_{k=1}^{s_1} g_1(x_k)\\
&\qquad = (s_1+1)\cdot \mathbf{E}\Big[\prod_{k=1}^{s_1+1} g_1(\mathcal{U}_k)\cdot \mathbb{1}\Big\{\mathcal{U}_1=\min_k \mathcal{U}_k\Big\}\Big]\\
&\qquad = \mathbf{E}\Big[\prod_{k=1}^{s_1+1} g_1(\mathcal{U}_k)\Big]\\
&\qquad = \Big(\frac{1}{r}\int_0^r \mathrm{d}u_i\, g_1(u_i)\Big)^{s_1+1}.
\end{aligned} \tag{3.6}$$

The calculation above fits our heuristics for the factor (s_1+1) in the birth rate: Looking backwards in time, birth events are coalescence events. If we have no level information and (s_1+1) particles are alive, the coalescence happens at overall rate $b_1(s_1+1)$ (backwards in time). In the model, where we do have level information, the same should apply, but only the particle with the lowest level can be the parent. Hence the birth rate (forwards in time) has to be adapted accordingly.

We continue with the motion-part of $A^{1,r}_{mb}$. With partial integration and using the fact $F_1(0)=F_1(r)=0$ we obtain

$$\begin{aligned}
&\int \alpha_r(m,\mathrm{d}u)\int \alpha_r(n,\mathrm{d}v)\, f(u,v)\sum_{i=1}^m\sum_{j=1}^n F_1(u_i)\frac{g_1'(u_i)}{g_1(u_i)}\\
&\quad = n e^{-\lambda_2 n}\sum_{i=1}^m\int \alpha_r(m,\mathrm{d}u)\,\frac{f_1(u)}{g_1(u_i)}F_1(u_i)g_1'(u_i)\\
&\quad = mn e^{-\lambda_2 n}e^{-\lambda_1(m-1)}\frac{1}{r}\int_0^r \mathrm{d}z\; F_1(z)g_1'(z)\\
&\quad = -mn e^{-\lambda_2 n}e^{-\lambda_1(m-1)}\frac{1}{r}\int_0^r \mathrm{d}z\; F_1'(z)g_1(z)\\
&\quad = b_1 mn e^{-\lambda_2 n}e^{-\lambda_1(m-1)}\Big(\frac{s_1+1}{r^{s_1+1}}\int_0^r \mathrm{d}z\, g_1(z)(r-z)^{s_1}-\frac{1}{r}\int_0^r \mathrm{d}z\, g_1(z)\Big).
\end{aligned} \tag{3.7}$$

Combining equations (3.5) and (3.7) we obtain

$$\int \alpha_r(m, \mathrm{d}u) \int \alpha_r(n, \mathrm{d}v)\, A^{1,r}_{mb} f(u,v) = b_1 mn e^{-\lambda_2 n} \left(e^{-\lambda_1 m + s_1} - e^{-\lambda_1 m} \right).$$

The analogous calculation for $A^{2,r}_{mb}$ gives the intertwining relation for A^r_{mb} and the generator of a mutually catalytic pure birth process:

$$C^r_{mb} \hat{f}(n,m) = b_1 mn \left(\hat{f}(m+s_1, n) - \hat{f}(m,n) \right) + b_2 mn \left(\hat{f}(m, n+s_2) - \hat{f}(m,n) \right).$$

$\Diamond$

3.1.3. Positively correlated symbiotic births

Using the building blocks A^r_{md} and A^r_{mb} only models can be assembled, where M^r and N^r are uncorrelated. To introduce correlation, we adapt the ideas for the particle representation of symbiotic branching systems in [EF04]. We want mechanisms in our toolbox that allow for simultaneous branching events in the U- and in the V-population. This building block allows for simultaneous births of U- and V-particles, thus introducing positive correlation (see also simultaneous death events in Section 3.1.4).

Let $s_1, s_2 \in \mathbb{N}$. We start with independent, uniformly distributed levels on $[0,r]$: Let $(M^r_0, N^r_0) = (m_0, n_0) \in \mathbb{N} \times \mathbb{N}$ and $(U^r_0, V^r_0) \sim \alpha_r(m_0, n_0, \,\cdot\,, \,\cdot\,)$. The simultaneous births are achieved the following way: Each pair of a U-particle i and a V-particle j gives simultaneously birth to s_1 U-particles and s_2 V-twins at rate

$$b \frac{(s_1+1)(s_2+1)}{r^{s_1+s_2}} (r-u_i)^{s_1} (r-v_j)^{s_2}.$$

The levels of the U-offspring are independent and uniformly distributed on $[U^{i,r}_t, r]$, the levels of the V-offspring on $[V^{j,r}_t, r]$ (see Figure 3.1). Again the levels perform continuous movement, given by a system of ordinary differential equations with random coefficients. Unlike in the Sections 3.1.1 and 3.1.2, the movement of the U-particles does not only depend on the number of V-particles but also on their levels. The movement is given by

$$\dot{U}^{i,r}_t = \sum_{j=1}^{N^r_t} F_1(U^{i,r}_t, V^{j,r}_t),$$
$$\dot{V}^{j,r}_t = \sum_{i=1}^{M^r_t} F_2(U^{i,r}_t, V^{j,r}_t),$$

where

$$F_1(u_i, v_j) = b \frac{(s_2+1)(r-v_j)^{s_2} + r^{s_2}}{2r^{s_2}} \left[\frac{(r-u_i)^{s_1+1}}{r^{s_1}} - (r-u_i) \right],$$
$$F_2(u_i, v_j) = b \frac{(s_1+1)(r-u_i)^{s_1} + r^{s_1}}{2r^{s_1}} \left[\frac{(r-v_j)^{s_2+1}}{r^{s_2}} - (r-v_j) \right].$$

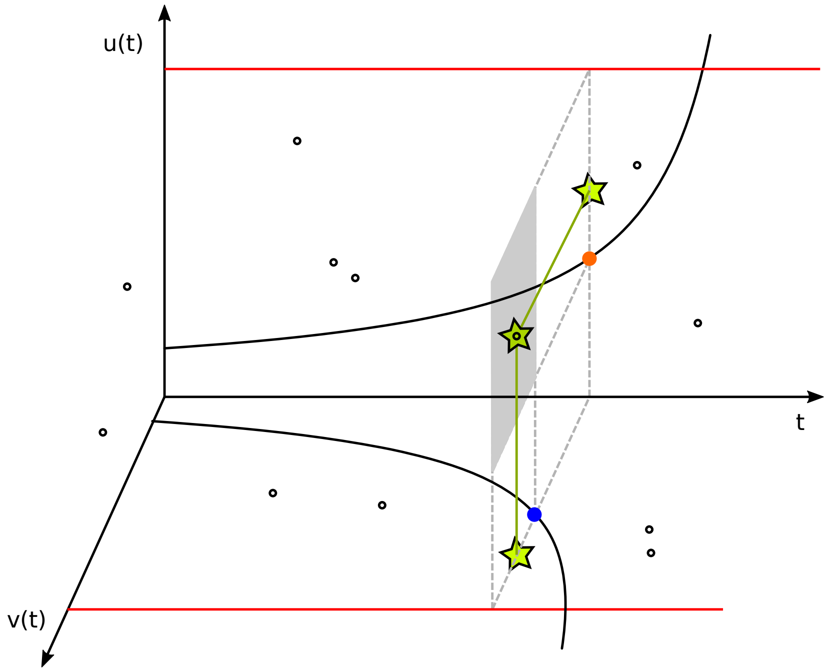

Figure 3.1.: Graphical representation of the positively correlated symbiotic birth mechanism for $s_1 = s_2 = 1$: For every pair of a U- and a V-particle exists a conditional Poisson point process on $\mathbb{R}^+ \times [0, r] \times [0, r]$ (time $\times$ U-level $\times$ V-level). The U- and the V-particle span a "window". When the window hits one of the Poisson points as the levels move along their trajectories, a new U- and a new V-particle is born.

Writing $m = u([0, r))$ and $n = v([0, r))$, the corresponding generator is

$$
\begin{aligned}
&A^r_{psb} f(u, v) \\
&\quad = f(u, v) \sum_{i=1}^{m} \sum_{j=1}^{n} \Bigg[b \frac{(s_1 + 1)(s_2 + 1)}{r^{s_1+s_2}} \int\limits_{[u_i,r]^{s_1}} \mathrm{d}x \int\limits_{[v_j,r]^{s_2}} \mathrm{d}y \left(\prod_{k=1}^{s_1} g_1(x_k) \prod_{l=1}^{s_2} g_2(y_l) - 1 \right) \\
&\qquad\qquad + F_1(u_i, v_j) \frac{g_1'(u_i)}{g_1(u_i)} + F_2(u_i, v_j) \frac{g_2'(v_j)}{g_2(v_j)} \Bigg].
\end{aligned}
$$

Calculation 3.1.3. Use equation (3.6) to calculate for the reproduction part of the generator

$$
\begin{aligned}
&\int \alpha_r(m,\mathrm{d}u)\int \alpha_r(n,\mathrm{d}v)\, f(u,v)\sum_{i=1}^{m}\sum_{j=1}^{n} b\frac{(s_1+1)(s_2+1)}{r^{s_1+s_2}}\\
&\qquad\qquad \times \int_{[u_i,r]^{s_1}}\mathrm{d}x\int_{[v_j,r]^{s_2}}\mathrm{d}y\left(\prod_{k=1}^{s_1} g_1(x_k)\prod_{l=1}^{s_2} g_2(y_l)-1\right)\\
&= b\frac{(s_1+1)(s_2+1)}{r^{s_1+s_2}}e^{-\lambda_1(m-1)}e^{-\lambda_2(n-1)}\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^r \mathrm{d}u_i\int_0^r \mathrm{d}v_j g_1(u_i)g_2(v_j)\\
&\qquad\qquad \times \int_{[u_i,r]^{s_1}}\mathrm{d}x\int_{[v_j,r]^{s_2}}\mathrm{d}y\left(\prod_{k=1}^{s_1} g_1(x_k)\prod_{l=1}^{s_2} g_2(y_l)-1\right)\\
&= bmne^{-\lambda_1(m-1)}e^{-\lambda_2(n-1)}\Bigg(e^{-\lambda_1(s_1+1)}e^{-\lambda_2(s_2+1)}\\
&\qquad -\frac{(s_1+1)(s_2+1)}{r^{s_1+s_2+2}}\int_0^r \mathrm{d}z_1\int_0^r \mathrm{d}z_2 g_1(z_1)g_2(z_2)(r-z_1)^{s_1}(r-z_2)^{s_2}\Bigg).
\end{aligned}
\tag{3.8}
$$

Note that

$$
F_1(0,v_j)=F_2(u_i,0)=F_1(r,v_j)=F_2(u_i,r)=0. \tag{3.9}
$$

Observe that

$$
\begin{aligned}
\partial_1 F_1(u_i,v_j) &= b\frac{(s_2+1)(r-v_j)^{s_2}+r^{s_2}}{2r^{s_2}}\left[1-\frac{(r-u_i)^{s_1}(s_1+1)}{r^{s_1}}\right]\\
&= \frac{b}{2}\Bigg(1-\frac{(s_1+1)(s_2+1)}{r^{s_1+s_2}}(r-u_i)^{s_1}(r-v_j)^{s_2}\\
&\qquad +\frac{(s_2+1)(r-v_j)^{s_2}r^{s_1}}{r^{s_1+s_2}}-\frac{(s_1+1)(r-u_i)^{s_1}r^{s_2}}{r^{s_1+s_2}}\Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_2 F_2(u_i,v_j) &= \frac{b}{2}\Bigg(1-\frac{(s_1+1)(s_2+1)}{r^{s_1+s_2}}(r-u_i)^{s_1}(r-v_j)^{s_2}\\
&\qquad +\frac{(s_1+1)(r-u_i)^{s_1}r^{s_2}}{r^{s_1+s_2}}-\frac{(s_2+1)(r-v_j)^{s_2}r^{s_1}}{r^{s_1+s_2}}\Bigg),
\end{aligned}
$$

hence

$$
\partial_1 F_1(u_i,v_j)+\partial_2 F_2(u_i,v_j) = b\left(1-\frac{(s_1+1)(s_2+1)}{r^{s_1+s_2}}(r-u_i)^{s_1}(r-v_j)^{s_2}\right). \tag{3.10}
$$

Applying partial integration, Equation (3.9) and Equation (3.10) we obtain for the movement part

$$\begin{aligned}
&\int \alpha_r(m, \mathrm{d}u) \int \alpha_r(n, \mathrm{d}v)\, f(u,v) \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} \left(F_1(u_i, v_j) \frac{g_1'(u_i)}{g_1(u_i)} + F_2(u_i, v_j) \frac{g_2'(v_j)}{g_2(v_j)} \right) \\
&\quad = mne^{-\lambda_1(m-1)} e^{-\lambda_2(n-1)} \frac{1}{r^2} \int_0^r \mathrm{d}z_1 \int_0^r \mathrm{d}z_2 (F_1(z_1,z_2) g_1'(z_1) g_2(z_2) \\
&\qquad\qquad + F_2(z_1,z_2) g_2'(z_2) g_1(z_1)) \\
&\quad = -mne^{-\lambda_1(m-1)} e^{-\lambda_2(n-1)} \frac{1}{r^2} \int_0^r \mathrm{d}z_1 \int_0^r \mathrm{d}z_2 g_1(z_1) g_2(z_2) (\partial_1 F_1(z_1,z_2) + \partial_2 F_2(z_1,z_2)) \\
&\quad = -bmne^{-\lambda_1(m-1)} e^{-\lambda_2(n-1)} \Bigg(e^{-\lambda_1} e^{-\lambda_2} \\
&\qquad\qquad - \frac{(s_1+1)(s_2+1)}{r^{s_1+s_2+2}} \int_0^r \mathrm{d}z_1 \int_0^r \mathrm{d}z_2 g_1(z_1) g_2(z_2) (r-z_1)^{s_1} (r-z_2)^{s_2} \Bigg).
\end{aligned} \tag{3.11}$$

Adding (3.8) and (3.11) gives us the intertwining relation for the desired generator

$$\begin{aligned}
C_{psb}\hat{f}(m,n) &:= \int \alpha_r(m, \mathrm{d}u) \int \alpha_r(n, \mathrm{d}v)\ A_{psb} f(u,v) \\
&= bmn \left(e^{-\lambda_1(m+s_1)} e^{-\lambda_2(n+s_2)} - e^{-\lambda_1 m} e^{-\lambda_2 n} \right) \\
&= bmn \left(\hat{f}(m+s_1, n+s_2) - \hat{f}(m,n) \right).
\end{aligned}$$

$\diamondsuit$

3.1.4. Positively correlated symbiotic deaths

The second mechanism in our toolbox that introduces positive correlation is simultaneous deaths of U- and V-particles. This is inspired by Model 3.2 in [EK19]. We tried to transfer this mechanism to the continuous mass case in a meaningful manner. Note that some changes have to be made in order to achieve this task (see Chapter 5).

This death mechanism is fundamentally different from the mutually catalytic deaths in Section 3.1.1, in that a conditional Poisson process triggers the death events. Hence we call this type of deaths "killing" of particles. The levels do not move continuously, but they jump. Fix $a > 0$ and $d_1, d_2 \in \mathbb{N}$. Jumps occur at rate $aM_t^r N_t^r$. For a level configuration U let

$$\varphi_k^r(U) := \inf\{s > 0 : \#\{i : s \cdot U^i \geq r\} \geq d_k\} \qquad \text{for} \qquad k = 1, 2. \tag{3.12}$$

When a jump occurs at time $t > 0$, the levels of both populations are multiplied by the respective factors $\varphi_1^r(U_t^r)$ and $\varphi_2^r(V_t^r)$. The jump kills d_1 U-particles with highest levels and d_2 V-particles, provided there are enough particles alive. The d_k-highest level, $k = 1, 2$, is shifted onto the death threshold r, and the levels above it are shifted beyond r. Recall that $\inf \emptyset = \infty$. If there are not enough particles in one of the subpopulations, all of this subpopulation's particles die. Note that as before, the particles with the highest levels will die next. The generator for this dynamics is

$$A^r_{psd}f(u,v) = amn\left[\left(\prod_{i=1}^{m} g_1(\varphi_1^r(u)u_i)\right)\left(\prod_{j=1}^{n} g_2(\varphi_2^r(v)v_j)\right) - f(u,v)\right].$$

The lowest of the dying particles is put on r. If the levels were independent, uniformly distributed on $[0,r]$ before the jump, the remaining particles after the jump are also independent, uniformly distributed on $[0,r]$.

Note that, if the simultaneous death mechanism is implemented for $d_1 > 1$ or $d_2 > 1$, the corresponding mass process is technically not a branching process.

Calculation 3.1.4. Let $d = d_1 \wedge m$ be the number of particles to be killed. Let $\mathcal{U}_1, \dots, \mathcal{U}_m \sim \mathrm{Unif}_{[0,r]}$ be independent, uniform random variables. Denote by $\mathcal{U}_{(k)}$ the k-th order statistics. The k lowest $\mathcal{U}$'s are uniformly distributed below $\mathcal{U}_{(k+1)}$,

$$\mathcal{L}\left(\mathcal{U}_1, \dots, \mathcal{U}_k \,\middle|\, \mathcal{U}_1, \dots, \mathcal{U}_k \le \mathcal{U}_{(k+1)}\right) = \mathrm{Unif}_{[0,\mathcal{U}_{(k+1)}]^k}.$$

Hence, conditioned on $\left\{\mathcal{U}_i < \mathcal{U}_{(k)}\right\}$, we have $\frac{r}{\mathcal{U}_{(k+1)}}\mathcal{U}_i \sim \mathrm{Unif}_{[0,r]}$. Since $g_1(x) = 1$ for $x \ge r$, we have

$$\mathbf{E}\left[\prod_{i=1}^{m} g\left(\frac{r}{\mathcal{U}_{(m-d+1)}}\mathcal{U}_i\right)\right] = \mathbf{E}\left[\prod_{i=1}^{m-d} g\left(\mathcal{U}_i\right)\right]$$

and thus

$$\begin{aligned} C^r_{psd}\hat{f}(n,m) &= \int \alpha_r(m, \mathrm{d}u) \int \alpha_r(n, \mathrm{d}v) A^r_{psd} f(u,v,m,n) \\ &= amn(\hat{f}(m-(d_1 \wedge m), n-(d_2 \wedge n)) - \hat{f}(n,m)). \end{aligned}$$

$\Diamond$

3.1.5. Negatively correlated symbiotic births and deaths

For negative correlation, we want to have births in one population and deaths in the other population simultaneously. We achieve this by using the conditional Poisson point processes that trigger the births of U-particles to kill V-particles. Let $b \ge 0$. For this model the marginal M^r of the mass process (M^r, N^r) is a pure birth process with rate $bM^r_t N^r_t$ and the marginal N^r is a pure death process with the same rate. Both marginals are perfectly correlated; that is, when M^r jumps, N^r jumps simultaneously.

Every pair of particles $(U^{i,r}, V^{j,r})$ has s_1 U-offspring at rate

$$b\frac{s_1+1}{r^{s_1}}(r - U^{i,r}_t)^{s_1}.$$

The offspring levels are independent and uniformly distributed on $[U_t^{i,r}, r]$. If a birth event happens, d_2 V-particles are killed, i.e., the V-levels are multiplied by $\varphi_2^r(V_t^r)$ (see (3.12)). If there are enough V-particles alive, d_2 of them are killed. Otherwise all remaining v-particles die. Apart from these jumps, the V-levels do not change.

The movement of the U-particles, which compensates for the accumulation of levels in the upper region of the interval $[0, r]$, is continuous and is characterized by F_1 in equation (3.3). As before let $m = u\,([0, r))$ and $n = v\,([0, r))$. The generator of this model is

$$\begin{aligned}
&A^r_{nsbu} f(u, v) = \\
&\sum_{i=1}^m \sum_{j=1}^n b\frac{s_1+1}{r^{s_1}}(r-u_i)^{s_1}\left[f_1(u)\int_{u_i}^r \mathrm{d}x_1 \cdot\cdot \int_{u_i}^r \mathrm{d}x_{s_1}\left(\prod_{k=1}^{s_1}\frac{1}{r-u_i}g_1(x_k)\right)\left(\prod_{l=1}^n g_2(\varphi_2^r(v)v_l)\right)\right. \\
&\qquad\qquad \left. - f(u,v)\right] + f(u,v)\sum_{i=1}^m\sum_{j=1}^n F_1(u_i)\frac{g_1'(u_i)}{g_1(u_i)}.
\end{aligned}$$

We obtain the generator of the model with negative symbiotic V-births by switching the roles of the U- and the V-population,

$$A^r_{nsbv} f(u, v) = A^r_{nsbu} f(v, u).$$

Calculation 3.1.5. Using equation (3.6) and

$$\int \alpha_r(n, \mathrm{d}v) \prod_{l=1}^n g_2(\varphi_2^r(v)v_l) = e^{-\lambda_2(n-(d_2\wedge n))},$$

(cf. calculation 3.1.4) we obtain for the birth and death part of the generator

$$\begin{aligned}
&\int \alpha_r(m, \mathrm{d}u)\int \alpha_r(n, \mathrm{d}v)\sum_{i=1}^m\sum_{j=1}^n b\frac{s_1+1}{r^{s_1}}(r-u_i)^{s_1} \\
&\quad\times\left[f_1(u)\int_{u_i}^r \mathrm{d}x_1 \cdots \int_{u_i}^r \mathrm{d}x_{s_1}\left(\prod_{k=1}^{s_1}\frac{1}{r-u_i}g_1(x_k)\right)\left(\prod_{l=1}^n g_2(\varphi_2^r(v)v_l)\right) - f(u,v)\right] \\
&= bmne^{-\lambda_1(m-1)}\left[\frac{s_1+1}{r^{s_1+1}}\int_0^r \mathrm{d}z\, g_1(z)\int_{u_i}^r \mathrm{d}x_1\cdots\int_{u_i}^r \mathrm{d}x_{s_1}\prod_{k=1}^{s_1} g_1(x_k)\right. \\
&\qquad\qquad \left.\times\int \alpha_r(n,\mathrm{d}v)\prod_{l=1}^n g_2(\varphi_2^r(v)v_l) - e^{-\lambda_2 n}\frac{s_1+1}{r^{s_1+1}}\int_0^r \mathrm{d}x\,(r-x)^{s_1}g_1(x)\right] \\
&= bmne^{-\lambda_1(m-1)}\left[e^{-\lambda_1(s_1+1)}e^{-\lambda_2(n-(d_2\wedge n))} - e^{-\lambda_2 n}\frac{s_1+1}{r^{s_1+1}}\int_0^r \mathrm{d}x\,(r-x)^{s_1}g_1(x)\right].
\end{aligned}$$

With equation (3.7) we obtain the intertwining relation for A^r_{nsbu} and

$$\begin{aligned}
C^r_{nsbu}\hat{f}(m,n) &= \int \alpha_r(m,\mathrm{d}u)\int\alpha_r(n,\mathrm{d}v)\,A^r_{nsbu}f(u,v) \\
&= bmn\left(\hat{f}(m+s_1, n-(d_2\wedge n)) - \hat{f}(m,n)\right),
\end{aligned}$$

the generator of a negatively correlated symbiotic birth-death process, where d_2 V-particles die whenever a U-birth event happens. The same applies for A^r_{nsbv} and C^r_{nsbv}, where

$$C^r_{nsbv}\hat{f}(m,n) = bmn\left(\hat{f}(m-(d_1\wedge m), n+s_2) - \hat{f}(m,n)\right).$$

$\Diamond$

3.2. Application of the Markov Mapping Theorem

In this section we apply the Markov Mapping Theorem from [KR11] to check that models, assembled from the building blocks in Section 3.1, are indeed genealogical representations of symbiotic branching particle systems. The reader can find the Markov Mapping Theorem in Section A.5.

Consider a generator

$$\begin{aligned}
A^r &:= A^r_{md} + A_{mb} + A^r_{psb} + A^r_{psd} + A^r_{nsbu} + A^r_{nsbv},\\
\mathcal{D}^r_{SBDM} &:= \Big\{ f(u,v) := f_1(u)f_2(v) := \prod_{i=1}^m g_1(u_i)\cdot\prod_{j=1}^n g_2(v_j)\ :\ 0\le g_1,g_2\le 1\in C^1(\mathbb{R}^+),\\
&\qquad\qquad g_1(x)=g_2(x)=1 \text{ for } x\ge r\Big\},
\end{aligned}$$

that is assembled from the generators of Section 3.1. It should be understood that the parameters, which govern the birth and death dynamics for the different mechanisms, shall be chosen independently for each building block.

Lemma 3.2.1. *Let $m = u([0,r))$ and $n = v([0,r))$. For each $f\in\mathcal{D}^r_{SBDM}$, there exists $c_f>0$ such that for all $m,n\in\mathbb{N}$*

$$|A^r f(u,v)| \le m\cdot n\cdot c_f.$$

Proof. We have $0\le g_1,g_2\le 1$, $g_1(r)=g_2(r)=1$ and $u,v\le r$. Since g_k, $k=1,2$, are continuously differentiable, the derivatives g_k' take a maximal value on $[0,r]$. We check the statement for each of the operators in Section 3.1:

(i) Mutually catalytic deaths:

$$\begin{aligned}
|A^r_{md}f(u,v)| &= \left| f(u,v)\left[\sum_{i=1}^m a_1 n u_i \frac{g_1'(u_i)}{g_1(u_i)} + \sum_{j=1}^n a_2 m v_j \frac{g_2'(v_j)}{g_2(v_j)}\right]\right|\\
&\le mn\left(a_1 r\left\|g_1'\right\|_\infty + a_2 r\left\|g_2'\right\|_\infty\right).
\end{aligned}$$

(Recall that the division by $g_1(u_i)$ and $g_2(v_j)$ removes one respective factor from the product $f(u,v)$.)

(ii) Mutually catalytic births:

$$
\begin{aligned}
\left|A^r_{mb} f(u,v)\right| &= \left|A^1_{mb} f(u,v) + A^2_{mb} f(u,v)\right| \\
&\le \left|A^1_{mb} f(u,v)\right| + \left|A^2_{mb} f(u,v)\right|
\end{aligned}
$$

and

$$
\begin{aligned}
\left|A^{1,r}_{mb} f(u,v)\right| &= \Bigg| f(u,v)\cdot b_1 n \sum_{i=1}^{m} \Bigg[\frac{s_1+1}{r^{s_1}} \int_{u_i}^{r} \mathrm{d}x_1 \cdots \int_{u_i}^{r} \mathrm{d}x_{s_1} \left(\prod_{k=1}^{s_1} g_1(x_k) - 1 \right) \\
&\qquad + \left(\frac{(r-u_i)^{s_1+1}}{r^{s_1}} - (r-u_i) \right) \frac{g_1'(u_i)}{g_1(u_i)} \Bigg] \Bigg| \\
&\le mnb_1 \left[(s_1+1) + r \left\| g_1' \right\|_\infty \right], \\
\left|A^{2,r}_{mb} f(u,v)\right| &\le mnb_2 \left[(s_2+1) + r \left\| g_2' \right\|_\infty \right].
\end{aligned}
$$

(iii) Positively correlated symbiotic births:

$$
\begin{aligned}
&\left|A^r_{psb} f(u,v)\right| \\
&= \Bigg| f(u,v) \sum_{i=1}^{m} \sum_{j=1}^{n} \Bigg[b \frac{(s_1+1)(s_2+1)}{r^{s_1+s_2}} \int\limits_{[u_i,r]^{s_1}} \mathrm{d}x \int\limits_{[v_j,r]^{s_2}} \mathrm{d}y \left(\prod_{k=1}^{s_1} g_1(x_k) \prod_{l=1}^{s_2} g_2(y_l) - 1 \right) \\
&\qquad\qquad + F_1(u_i,v_j) \frac{g_1'(u_i)}{g_1(u_i)} + F_2(u_i,v_j) \frac{g_2'(v_j)}{g_2(v_j)} \Bigg] \Bigg|,
\end{aligned}
$$

where

$$
\begin{aligned}
\left|F_1(u_i,v_j)\right| &= \left| b \frac{(s_2+1)(r-v_j)^{s_2} + r^{s_2}}{2r^{s_2}} \left[\frac{(r-u_i)^{s_1+1}}{r^{s_1}} - (r-u_i) \right] \right| \\
&\le rb(s_2+2), \\
\left|F_2(u_i,v_j)\right| &\le rb(s_1+2).
\end{aligned}
$$

Hence

$$
\left|A^r_{psb} f(u,v)\right| \le mnb\Big[(s_1+1)(s_2+1) + r(s_2+2) \left\| g_1' \right\|_\infty + r(s_1+2) \left\| g_2' \right\|_\infty \Big].
$$

(iv) Positively correlated symbiotic deaths:

$$
\begin{aligned}
\left|A^r_{psd} f(u,v)\right| &= \left| amn \left[\left(\prod_{i=1}^{m} g_1(\varphi_1^r(u) u_i) \right) \left(\prod_{j=1}^{n} g_2(\varphi_2^r(v) v_j) \right) - f(u,v) \right] \right| \\
&\le amn.
\end{aligned}
$$

(v) Negatively correlated symbiotic births and deaths:

$$\begin{aligned}|A^r_{nsbu}f(u,v)| &= \Bigg| \sum_{i=1}^{m}\sum_{j=1}^{n} b\frac{s_1+1}{r^{s_1}} f(u,v)\int_{u_i}^{r}\mathrm{d}x_1\cdots\int_{u_i}^{r}\mathrm{d}x_{s_1} \\ &\quad \times \left(\left(\prod_{k=1}^{s_1} g_1(x_k)\right)\frac{f_2(\varphi_2^r(v)v)}{f_2(v)} - 1\right) \\ &\quad + f(u,v)\sum_{i=1}^{m}\sum_{j=1}^{n} b\left[\frac{(r-u_i)^{s_1+1}}{r^{s_1}} - (r-u_i)\right]\frac{g_1'(u_i)}{g_1(u_i)}\Bigg| \\ &\le mnb\left((s_1+1) + r\left\|g_1'\right\|_\infty\right).\end{aligned}$$

□

Theorem 3.2.2. *Let $\mu_0 \in \mathcal{M}_1(\mathbb{N}_0\times\mathbb{N}_0)$ and define $\nu_0 := \int \alpha_r(y,z;\ \cdot\ )\ \mu_0(\mathrm{d}y,\mathrm{d}z)$. There exists a unique solution (U^r,V^r) of the the $D_{\mathcal{S}_{[0,r]}\times\mathcal{S}_{[0,r]}}[0,\infty)$-martingale problem for (A^r,ν_0) such that $(M^r,N^r) := (U([0,r)),V([0,r)))$ is a solution of the $D_{\mathbb{N}_0\times\mathbb{N}_0}[0,\infty)$-martingale problem for (C^r,μ_0), where*

$$C^r = C^r_{md} + C^r_{mb} + C^r_{psb} + C^r_{psd} + C^r_{nsbu} + C^r_{nsbv}.$$

For every $t\ge 0$, $\Gamma\in\mathcal{B}(\mathcal{S}_{[0,r]}\times\mathcal{S}_{[0,r]})$, we have

$$\mathbf{P}\left[(U^r_t,V^r_t)\in\Gamma \,\middle|\, \mathcal{F}^{M^r,N^r}_t\right] = \alpha_r(M^r_t,N^r_t;\Gamma). \tag{3.13}$$

Proof. Existence of (U^r,V^r) and the distribution property (3.13) follow from the Markov Mapping Theorem A.5.1 with $\psi(u,v) := m{\cdot}n$ and $\gamma(u,v) := (u([0,r)),v([0,r)))$ where we use the notation of said theorem.

Uniqueness of the level system is straightforward (see, for example, Problem 28 in Chapter 4.11 of [EK86]). The "motion part" of the dynamics, without births or other jumps, is given piecewise by ordinary differential equations. Uniqueness of the motion then follows from uniqueness of the solutions to the differential equations. We obtain uniqueness of the level dynamics up to the first time one of the masses hit some fixed $\tilde{m}$ by a perturbation result (see Theorem 4.10.3 from [EK86]). Letting $\tilde{m}\to\infty$, we obtain uniqueness of the whole process. We omit the details. □

4. Lookdown-construction of Feller's branching diffusion with multiple deaths

In this chapter, we will give a lookdown construction for Feller's branching diffusion as a weak limit of models that use a jump-death mechanism, similar to Model 3.1.4. But, while in Model 3.1.4 a fixed number of particles is killed per death event, we allow for a random number of casualties in this model. This construction can be seen as a preparation for the construction of a lookdown representation of a symbiotic diffusion with positive correlation in Chapter 5.

Before we start with the construction of the level representation of Feller's branching diffusion in Section 4.1, we give heuristics, why we use this special jump-death mechanism in the level representation of a symbiotic branching diffusion. Recall that symbiotic diffusions are solutions of the system

$$\begin{cases} \mathrm{d}Y_t &= \sqrt{bY_tZ_t}\,\mathrm{d}W_t^1, \\ \mathrm{d}Z_t &= \sqrt{bY_tZ_t}\,\mathrm{d}W_t^2, \end{cases}$$

where W^1 and W^2 are correlated Brownian motions with constant correlation coefficient ρ; i.e.

$$\langle W^1, W^2\rangle_t = \rho \cdot t.$$

We tried to carry the symbiotic discrete mass systems of Chapter 3 to a meaningful weak limit for $r \to \infty$. Let us first consider the simultaneous birth model 3.1.3 and observe why this approach does not lead to the desired result. Consider a family $(U^r, V^r)_r$ of level representations that perform the simultaneous birth dynamics given by A^r_{psb} and assume for the moment weak convergence for $r \to \infty$. The limiting process (U, V) takes states in $\mathcal{S}_{\mathbb{R}+} \times \mathcal{S}_{\mathbb{R}+}$, topologized by test functions that have compact support. Now consider a particle U^i, that gives birth at time t and fix a compact set $K_1 \times K_2 \subset \mathbb{R}^+ \times \mathbb{R}^+$ with $U^i \in K_1$. Only finitely many V-particles are in K_2 at time t, so the V-particle, that gives birth at the same time t is almost surely not in K_2, due to the "uniform" distribution of the particles. This means that, in the topology of the state space of our limiting system, we are not able to identify the "birth partner" of a given birth event and therefore cannot hope to see correlation of the masses when we forget the level information.

We do not have this problem for the models with simultaneous deaths, however. The particles that die simultaneously do not simply vanish without affecting the rest of the population, but all particles jump. The simultaneous death event induces a decrease in the particle density everywhere in $\mathbb{R}^+$, not only "at infinity". A natural starting point for the construction of a level representation of symbiotic branching diffusions is a discrete mass model, built from the blocks A^r_{mb} (*mutually catalytic births*) and A^r_{psd} (*positively correlated symbiotic deaths*) in Chapter 3. But we encounter a first problem: While the consistency in the Kurtz-Rodrigues setting means that the discrete mass models for $r_1 < r_2 < \dots$ are coupled and the $r \to \infty$ model can be obtained as an almost sure limit, the models in Chapter 3 do not feature this type of consistency. For illustration, consider a very simple mutually catalytic death model (U^r, V^r), given by A^r_{md}. For $r' < r$, the restriction $(U^r|_{[0,r']}, V^r|_{[0,r']})$ does not solve the martingale problem for $A^{r'}_{md}$, since the death rates in the restricted system are too high. Hence we constructed the representation as a weak limit in Chapter 5.

There is another hoop we have to jump through, that we illustrate below by faulty heuristics for the level dynamics in the simpler case of one population. We start with a level system U^r characterized by the the generator

$$\begin{aligned}\tilde{A}^r f(u) = f(u) \sum_{i=1}^m \left[2b \int_{u_i}^r (g(x) - 1)\ \mathrm{d}x + \left(b u_i^2 - b r u_i \right) \frac{g'(u_i)}{g(u_i)} \right] \\ + rbm \left(\prod_{i=1}^m g\left(\frac{r}{\max_j u_j} u_i \right) - f(u) \right),\end{aligned}$$

$$\tilde{\mathcal{D}}^r := \left\{ f(u) := \prod_{i=1}^m g(u_i)\ :\ 0 \le g \le 1 \in C^2(\mathbb{R}), g(x) = 1 \text{ for } x \ge r \right\}.$$

Without proof we note that the system is a lookdown representation with "killing" for a critical Galton-Watson process with birth and death rate rb. (To apprehend this, compare the first term of $\tilde{A}^r$ with the U-dynamics given by the generator A^r_{mb} with $s_1 = 1$ and set $N^r \equiv 1$. The second term kills the highest level particle at a rate that fits the first term.) We want to understand the movement of the levels for large r, thus we have to sort out the interplay of the numerous small jumps, induced by the killing, and the linear part of the differential equation. The problem is that the jump factor $\frac{r}{\max_j U_t^{j,r}}$ depends on the highest level, and its fate as $r \to \infty$ is unclear. Define $Y_t^r := \frac{M_t^r}{r}$. The levels at time t are i.i.d. uniformly on $[0, r]$ given M_t^r. Hence $r - \max_j U_t^{j,r}$ is approximately exponential distributed with rate Y_t^r for large r. Furthermore the level $U_t^{i,r}$ of one fixed particle at a given time t is asymptotically independent of $r - \max_j U_t^{j,r}$. This leads to the idea to treat the jump factor as a random variable

$$\varphi := \frac{r}{r - \mathcal{E}},$$

where $\mathcal{E} \sim \mathrm{Exp}_{Y_t^r}$ is conditionally independent of U^r. We suppose (but we did not prove) that the resulting models do indeed converge to a level representation for the Feller's branching diffusion. However, we changed the model fundamentally in the course of our heuristic considerations: In the model we started with, the highest particle is put onto the death threshold r when a death event happens. Thus precisely one particle is killed per death event, and the jump factor is inherent in the system. In the new model the jump factor is random, hence a random number of particles is killed: Multiplication by φ in the i-th death event, amounts to killing all particles in the interval $[r - \mathcal{E}_i, r]$, where $\mathcal{E}_i$, $i = 1, 2, \ldots$ are independent, conditionally exponential random variables. Informal calculations showed that for $r \to \infty$ the model with the random jump factor has a higher variance than the discrete mass models, where only one particle is killed per death event. Hence it is not a reasonable candidate for the limit.

We adapt the discrete mass models and implement a death mechanism that uses the ideas from above but is slightly simpler (the "death region" is deterministic, conditioned on the population size). In this chapter we introduce the new death mechanism in a one population-model that converges (weakly) to a representation of Feller's branching diffusion. In this (simpler) setting we establish tightness and study the limiting dynamics. In Chapter 5 we will be able to "recycle" a lot of the work and construct a lookdown representation for a positively correlated symbiotic diffusion.

4.1. A birth-death process with jump induced deaths

We introduce a level representation of a birth-death process, where death events are triggered by a conditional Poisson process. At a death event, particles of a whole region near r are killed. In the limit, only the expectation and the variance of the number of particles killed per event, should matter. The simplest setup we could think of is a "death zone" that is deterministic given the total mass.

Consider one population U^r of particles with levels in $[0, r]$, so the state space is $\mathcal{S}_{[0,r]}$. We use again the vector notation when appropriate. Recall that $(U^{i,r})_i$ is an arbitrary enumeration of the atoms, while $U^{(k),r}$ denotes the k-th lowest level. Let M^r be the number of particles and $Y^r := \frac{M^r}{r}$ the mass density.

At time $t = 0$, the levels are uniformly distributed on $[0, r]$ given M_0^r. Particle $U^{i,r}$ generates one offspring at rate $b(r - U_t^{i,r})$. The offspring's level is uniformly distributed above the parent's level. The continuous movement of the particles is chosen in such a way that it compensates for the accumulation of particles in the higher region of $[0, r]$, due to births. The particles move according to the differential equation

$$\dot{U}_t^{i,r} = b\left(U_t^{i,r}\right)^2 - brU_t^{i,r}.$$

These birth dynamics are a non-spatial version of Model 3.4 in [EK19]. A particle is considered dead when its level exceeds r. Fix a constant $0 < c < M_0^{(r)}$ and assume $M_t^r > c$. The deaths are event based, and when a death event happens at time t all particles in the interval $\left[r(1 - \frac{c}{M_t^r}),\ r\right]$ are killed. The levels below the death threshold are scaled up, such that they are uniformly distributed on $[0, r]$. To this end all levels are multiplied by $\frac{1}{1 - c/M_t^r}$.

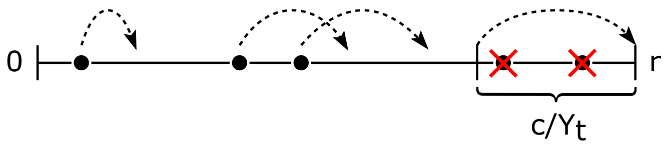

Figure 4.1.: In a death event at time t particles in a "death zone" of width $\frac{c}{Y_t^r} = \frac{c \cdot r}{M_t^r}$ are killed. The remaining part of the interval is scaled up.

Since the birth dynamics are set up in such a way that, conditioned on M_t^r, U_t^r is a Poisson point process on $[0, r]$ for every $t \geq 0$, the number of individuals killed during a single death event is binomially distributed, $U_t^r[r(1 - \frac{c}{M_t^r}),\ r] \sim \mathrm{Bin}(M_t^r, \frac{c}{M_t^r})$. If $M_t^r \leq c$ all remaining particles are killed in the next death event.

In order to keep the model simple, we choose to stay in the critical setting and the death events are triggered at rate rbM_t^r/c. Define

$$\varphi(U^r) := \mathbb{1}_{(c,\infty)}(M^r) \cdot \frac{1}{1 - c/M^r} + \mathbb{1}_{[0,c]}(M^r) \cdot \frac{r}{\min U^r}.$$

The generator of the process is

$$\begin{aligned} A_{BDjd}^r f(u) = f(u) \sum_{i=1}^{m} \left[2b \int_{u_i}^{r} (g(x) - 1) \ \mathrm{d}x + (bu_i^2 - bru_i) \frac{g'(u_i)}{g(u_i)} \right] \\ + \frac{rbm}{c} \left(\prod_{i=1}^{m} g\left(\varphi(u) \cdot u_i\right) - f(u) \right), \end{aligned} \tag{4.1}$$

$$\mathcal{D}_{BDjd}^r := \left\{ f(u) := \prod_{i=1}^{m} g(u_i) \ : \ 0 \leq g \leq 1 \in C^2(\mathbb{R}), g(x) = 1 \text{ for } x \geq r \right\}.$$

BDjd stands for "birth death process with jump induced deaths". Recall that $\alpha_r(m, \cdot)$ is the distribution of a Poisson point process on $[0, r]$, conditioned to have mass m.

Theorem 4.1.1. *Let $\mu_0 \in \mathcal{M}_1(\mathbb{N})$ and define $\nu_0 := \int \alpha_r(y,\ \cdot\ )\ \mu_0(\mathrm{d}y)$. There exists a unique solution U^r of the the $D_{\mathcal{S}_{[0,r]}}[0,\infty)$-martingale problem for (A_{BDjd}^r, ν_0) such that $M^r := U([0, r))$ is a solution of the $D_{\mathbb{N}_0}[0,\infty)$-martingale problem for (C_{BDjd}^r, μ_0), where*

$$\begin{aligned} C_{BDjd}^r \hat{f}(m) =& rbm \left(\hat{f}(m+1) - \hat{f}(m) \right) \\ &+ \mathbb{1}_{(c,\infty)}(m) \cdot \frac{rbm}{c} \left(\sum_{i=0}^{m} \binom{m}{i} \left(\frac{c}{m} \right)^i \left(1 - \frac{c}{m} \right)^{m-i} \hat{f}(m-i) - \hat{f}(m) \right) \\ &+ \mathbb{1}_{[0,c]}(m) \cdot \frac{rbm}{c} \left(\hat{f}(0) - \hat{f}(m) \right). \end{aligned}$$

For every $t \geq 0$ and $\Gamma \in \mathcal{B}(\mathcal{S}_{[0,r]})$, we have

$$\mathbf{P}\left[U_t^r \in \Gamma \ \middle|\ \mathcal{F}_t^{M^r} \right] = \alpha_r(M_t^r, \Gamma).$$

Proof. The proof is very similar to the proof of Theorem 3.2.2. We check the intertwining relation

$$\int \alpha_r(m, \mathrm{d}u)\ A^r_{BDjd} f(u) = C^r_{BDjd} \int \alpha_r(m, \mathrm{d}u)\ f(u). \tag{4.2}$$

Defining

$$e^{-\lambda} := \frac{1}{r} \int_0^r g(z)\, dz,$$

we have

$$\hat{f}(m) := \int \alpha_r(m, \mathrm{d}u)\ f(u) = e^{-\lambda m}.$$

By Calculation 3.1.2 for Model 3.1.2 (mutually catalytic births), the relation

$$\begin{aligned}
&\int \alpha_r(m, \mathrm{d}u)\ f(u) \sum_{i=1}^{m} \left[2b \int_{u_i}^{\infty} (g(x) - 1)\ \mathrm{d}x + (bu_i^2 - bru_i) \frac{g'(u_i)}{g(u_i)} \right] \\
&\qquad = rbm \left(\hat{f}(m+1) - \hat{f}(m) \right)
\end{aligned}$$

is obvious (Consider the U-marginal of Model 3.1.2 for $s_1 = 1$ and set $N^r \equiv 1$). Assume $m > c$, then $\varphi(u) = \frac{1}{1 - c/m}$. For the death part, we obtain

$$\begin{aligned}
&\int \alpha_r(m, \mathrm{d}u)\ \frac{rbm}{c} \left(\prod_{i=1}^{m} g\left(\varphi(u) \cdot u_i\right) - f(u) \right) \\
&\quad = \int \alpha_r(m, \mathrm{d}u)\ \frac{rbm}{c} \left[\prod_{i=1}^{m} \left(\mathbb{1}_{[0,r)}(\varphi(u)u_i) \cdot g(\varphi(u)u_i) + \mathbb{1}_{[r,\infty)}(\varphi(u)u_i) \right) - f(u) \right] \\
&\quad = \frac{rbm}{c} \left[\left(\frac{1}{r} \int_0^{\frac{r}{\varphi(u)}} \mathrm{d}z\ g(\varphi(u)z) + \frac{1}{r} \int_{\frac{r}{\varphi(u)}}^{r} \mathrm{d}z \right)^m - \hat{f}(m) \right] \\
&\quad = \frac{rbm}{c} \left[\left(\frac{1}{r\varphi(u)} \int_0^r \mathrm{d}z\ g(z) + \frac{c}{m} \right)^m - \hat{f}(m) \right] \\
&\quad = \frac{rbm}{c} \left[\left(\left(1 - \frac{c}{m}\right) e^{-\lambda} + \frac{c}{m} \right)^m - \hat{f}(m) \right] \\
&\quad = \frac{rbm}{c} \left(\sum_{i=0}^{m} \binom{m}{i} \left(\frac{c}{m}\right)^i \left(1 - \frac{c}{m}\right)^{m-i} \hat{f}(m-i) - \hat{f}(m) \right).
\end{aligned} \tag{4.3}$$

If $m \le c$, then $\varphi(u) = \frac{r}{\min u}$ and obviously

$$\int \alpha_r(m, \mathrm{d}u)\ \frac{rbm}{c} \left(\prod_{i=1}^{m} g\left(\varphi(u) \cdot u_i\right) - f(u) \right) = \frac{rbm}{c} \left(\hat{f}(0) - \hat{f}(m) \right).$$

Hence the relation (4.2) is valid and the assertion of the theorem follows from the Markov Mapping Theorem A.5.1 with $\psi(u) := m$ and $\gamma(u) := u([0, r))$, where we use the notation of said theorem.

Uniqueness of the level system follows from uniqueness of the system that performs only the continuous motion. By Theorem 4.10.3 in [EK86], we have uniqueness of the level dynamics until the first time the mass hits some fixed $\tilde{m}$. Letting $m \to \infty$, we obtain uniqueness of the whole process. □

Note that the level system given by A^r_{BDjd} does not feature the consistency property of the Kurtz-Rodrigues model.

Before we proceed to proving convergence of the level dynamics, we try to lay our hands on the limit heuristically and check if its mass density is indeed Feller's branching diffusion. We write $y = \frac{m}{r}$. Expanding $\frac{1}{1-c/m} = 1 + \sum_{k=1}^{\infty} \left(\frac{c}{yr}\right)^k$ in a power series and performing a Taylor approximation, we obtain

$$
\begin{aligned}
&\frac{rbm}{c}\left(\prod_{i=1}^{m} g\left(\tau \cdot u_i\right) - f(u)\right) \\
&\quad = \frac{r^2 by}{c}\Bigg(f(u)\sum_{i=1}^{m} u_i \frac{g'(u_i)}{g(u_i)} \cdot (\tau - 1) + \frac{1}{2} f(u) \sum_{i=1}^{m} u_i^2 \frac{g''(u_i)}{g(u_i)} \cdot (\tau-1)^2 \\
&\qquad + \frac{1}{2} f(u) \sum_{i=1}^{m} \sum_{\substack{j=1\\ j\neq i}}^{m} u_i u_j \frac{g'(u_i) g'(u_j)}{g(u_i)g(u_j)} \cdot (\tau - 1)^2 + o(r^{-2})\Bigg) \\
&\quad = \frac{r^2 by}{c}\Bigg(f(u)\sum_{i=1}^{m} u_i \frac{g'(u_i)}{g(u_i)} \cdot \left(\frac{c}{yr} + \frac{c^2}{y^2 r^2}\right) + \frac{1}{2} f(u) \sum_{i=1}^{m} u_i^2 \frac{g''(u_i)}{g(u_i)} \cdot \frac{c^2}{y^2 r^2} \\
&\qquad + \frac{1}{2} f(u) \sum_{i=1}^{m} \sum_{\substack{j=1\\ j\neq i}}^{m} u_i u_j \frac{g'(u_i) g'(u_j)}{g(u_i)g(u_j)} \cdot \frac{c^2}{y^2 r^2} + o(r^{-2})\Bigg).
\end{aligned}
$$

When we plug this in the generator A^r_{BDjd}, the "$-bru_i$"-term of the differential equation is cancelled out and we obtain a candidate for the particle representation of a Feller's branching diffusion with jump deaths,

$$
\begin{aligned}
A_{FDjd} f(u) = f(u) \Bigg[&\sum_i 2b \int_{u_i}^{\infty} (g(x) - 1)\ \mathrm{d}x + \sum_i \left(bu_i^2 + \frac{bc}{y} u_i\right) \frac{g'(u_i)}{g(u_i)} \\
&+ \frac{1}{2} \sum_i \frac{bc}{y} u_i^2 \frac{g''(u_i)}{g(u_i)} + \frac{1}{2} \sum_{i \neq j} \frac{bc}{y} u_i u_j \frac{g'(u_i) g'(u_j)}{g(u_i) g(u_j)} \Bigg], \qquad (4.4)
\end{aligned}
$$

$$
\mathcal{D}_{FDjd} := \left\{ f(u) := \prod_i g(u_i)\ :\ 0 \le g \le 1 \in C^2(\mathbb{R}), g(x) = 1 \text{ for } x \ge r_g \right\}.
$$

Note that the mapping $u \mapsto y$ is not continuous in $\mathcal{S}_{\mathbb{R}^+}$ endowed with the vague topology. Hence we cannot use the Markov Mapping Theorem A.5 of [KR11], because $A_{FDjd} f$ is not continuous. In a test calculation we check that the intertwining relation holds and that our heuristic arguments lead to the desired mass dynamics.

Calculation 4.1.2. Recall that $\alpha(y, \cdot)$ is the distribution of of a Poisson point process on $[0, \infty]$ with intensity y. Define

$$
\beta_g := \int_0^{\infty} (1 - g(x))\ \mathrm{d}x.
$$

Then we have for $f \in \mathcal{D}_{FDjd}$

$$
\hat{f}(y) := \int \alpha(y, \mathrm{d}u)\ f(u) = e^{-y\beta_g}
$$

(see Lemma A.1.1). See the lookdown representation of Feller's branching diffusion in [KR11] (or Equation 2.3) to apprehend

$$\int \alpha(y, \mathrm{d}u) f(u) \sum_i \left[2b \int_{u_i}^{\infty} (g(x) - 1)\ \mathrm{d}x + b u_i^2 \frac{g'(u_i)}{g(u_i)} \right] = bye^{-y\beta_g} \beta_g^2. \tag{4.5}$$

The remaining part of the generator is associated to perfectly correlated, conditional geometric Brownian motions. Using Lemma A.1.1, we obtain

$$\begin{aligned} &\int \alpha(y, \mathrm{d}u) f(u) \left[\sum_i \frac{bc}{y} u_i \frac{g'(u_i)}{g(u_i)} + \frac{1}{2} \sum_i \frac{bc}{y} u_i^2 \frac{g''(u_i)}{g(u_i)} + \frac{1}{2} \sum_{i \neq j} \frac{bc}{y} u_i u_j \frac{g'(u_i) g'(u_j)}{g(u_i) g(u_j)} \right] \\ &\qquad = bce^{-\beta_g y} \left[\int_0^{\infty} x g'(x)\ \mathrm{d}x + \int_0^{\infty} \frac{1}{2} x^2 g''(x)\ \mathrm{d}x + \frac{y}{2} \left(\int_0^{\infty} x g'(x)\ \mathrm{d}x \right)^2 \right]. \end{aligned} \tag{4.6}$$

Integration by parts gives us

$$\int_0^{\infty} x g'(x)\ \mathrm{d}x = \int_0^{r_g} x g'(x)\ \mathrm{d}x = r_g - \int_0^{r_g} g(x)\ \mathrm{d}x = \int_0^{\infty} (1 - g(x))\ \mathrm{d}x = \beta_g \tag{4.7}$$

and

$$\int_0^{\infty} x^2 g''(x)\ \mathrm{d}x = \int_0^{r_g} x^2 g''(x)\ \mathrm{d}x = -2 \int_0^{r_g} x g'(x)\ \mathrm{d}x = -2\beta_g. \tag{4.8}$$

Putting (4.5), (4.6), (4.7) and (4.8) together, we obtain with $\hat{f}(y) := e^{-y\beta_g}$

$$\begin{aligned} C_{FDjd} \hat{f}(y) &:= \int \alpha(y, \mathrm{d}u) A_{FDjd} f(u) = b \left(1 + \frac{c}{2} \right) y \beta_g^2 e^{-y\beta_g} \\ &= b \left(1 + \frac{c}{2} \right) y \hat{f}''(y). \end{aligned}$$

This is the generator of Feller's branching diffusion with branching rate $b\left(1 + \frac{c}{2}\right)$. $\Diamond$

4.2. Tightness of $(U^r, Y^r)_r$

Let U^r be the level representation given by A^r_{BDjd} (see (4.1)). Let $k \in \mathbb{N}$ and denote by

$$X^r = (X^{1,r}, X^{2,r}, \dots, X^{k+1,r}) := (U^{(1),r}, U^{(2),r}, \dots, U^{(k),r}, Y^r)$$

the $\mathbb{R}^{k+1}$-valued process consisting of the k lowest levels and the mass density process. Recall that $\mathcal{S}_{\mathbb{R}^+}$ is the space of locally finite counting measures on $\mathbb{R}^+$, endowed with the vague topology. First we prove $D_{\mathbb{R}^{k+1}}[0, \infty)$-tightness of the family $(X^r)_r$, and then we "lift" this result to $D_{\mathcal{S}_{\mathbb{R}^+}}[0, \infty)$-tightness of $(U^r)_r$.

We expect tightness only until shortly before extinction. Let $\delta > 0$ be some threshold and let

$$\tau^r := \inf\{s \geq 0 : Y^r_s \leq \delta\}$$

be the first time when the mass density coordinate hits δ or jumps below δ. Define the stopped processes $U_t^{r,\delta} := U_{t\wedge\tau^r}^r$, $Y_t^{r,\delta} := Y_{t\wedge\tau^r}^r$ and $X^{r,\delta} := X_{t\wedge\tau^r}^r$. In the following we assume $0 < c < r\delta$ such that the "strip of death" does not extend into the negatives. We will prove a compact containment condition for the k lowest particles. Note that, when a new particle with rank k or smaller is born, it is pigeon-holed in $U^{(1,\ldots,k),r}$. Since the level of a given particle does explode, the "rearrangement" that happens when a new low level particle is born is crucial for the containment.

The idea of the proof is the following: Given $K > 0$, we choose a partition of $[0, T]$ that is coarse enough for the probability of the event that $U^{(k),r}$ is above $\frac{K}{2}$ at one of the nodes to be small. (We use the "uniformity property" of the levels at fixed times for this.) But then, given that $U^{(k),r}$ is below $\frac{K}{2}$ at *every node* of the partition, the event $\left\{\sup_{0\le s\le T} U_s^{(k),r} > K\right\}$ implies one of two possibilities: Either a particle came down from K to $\frac{K}{2}$ between two nodes, or a new particle was born in $[0, K]$ (and ended up below $\frac{K}{2}$). We can bound these ascent and descent probabilities.

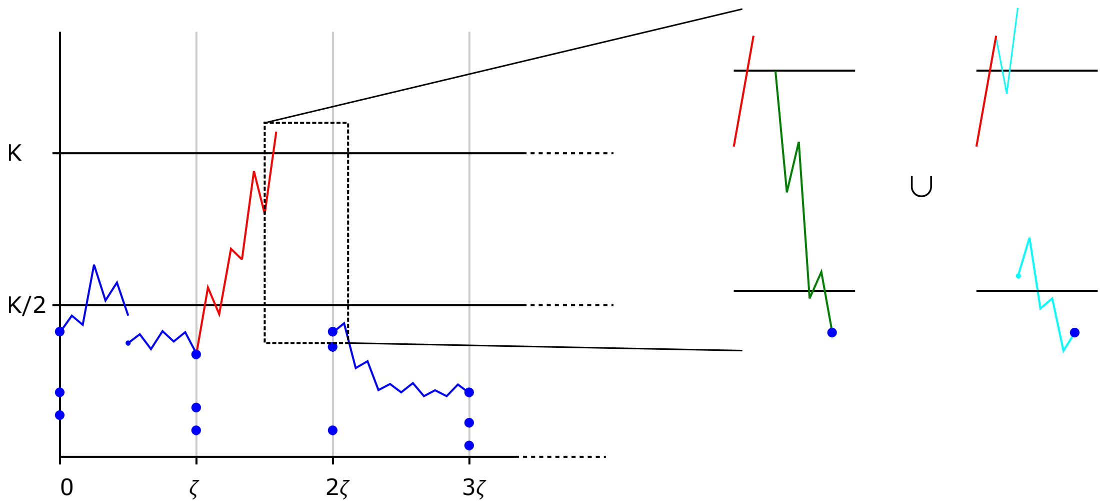

Figure 4.2.: On the event, that all k lowest particles are below $\frac{K}{2}$ at the nodes of a partition of $[0, T]$, the following has to happen if $\sup_{0\le s\le T} U_s^{(k),r} > K$: First $U^{(k),r}$ has to ascend from $\frac{K}{2}$ to K (red), then either $U^{(k),r}$ descends to $\frac{K}{2}$ without negative jumps (green) or a birth happens below K (light blue).

Lemma 4.2.1. *Set the birth rate of the Model given by A_{BDjd}^r in (4.1) to $b = 1$. Let $k \in \mathbb{N}$, $0 < K < r$ and $\zeta > 0$.*

(i) For any start configuration $\tilde{u} \in \mathcal{S}_{[0,r]}$ satisfying $\tilde{u}^{(k)} < \frac{K}{2}$, we have

$$\mathbf{P}_{\tilde{u}}\left[\sup_{s\le\zeta} U_s^{(k),r,\delta} \ge K\right] \le 4\zeta\frac{r^2c\delta}{(r\delta - c)^2} \cdot \left[1 - \zeta\left(2K + \frac{2rc}{r\delta - c}\right)\right]^{-2}.$$

(ii) For any start configuration $\hat{u} \in \mathcal{S}_{[0,r]}$ satisfying $\hat{u}^{(k)} > K$, we have

$$\mathbf{P}_{\hat{u}}\left[\inf_{s\le\zeta} U_s^{(k),r,\delta} \le \frac{K}{2}\right] \le 4\zeta\frac{r^2c\delta}{(r\delta-c)^2}\cdot\left[1-\zeta\left(4K+\frac{2rc}{r\delta-c}\right)\right]^{-2} + 16\zeta\frac{r^2c\delta}{(r\delta-c)^2} + 2\zeta K(k-1)$$

Proof. We drop the δ in our notation, $U^{(k),r} = U^{(k),r,\delta}$, $Y^r = Y^{r,\delta}$, and agree on the assumption that $Y^r > \delta$. We want to trace the particle $U^{(k),r}$. Define the processes

$$\begin{aligned} J_t^{(k),r,+} &:= \sum_{\substack{0<s\le t\\ \Delta U_s^{(k),r}>0}} \Delta U_s^{(k),r} \text{ and}\\ J_t^{(k),r,-} &:= \sum_{\substack{0<s\le t\\ \Delta U_s^{(k),r}<0}} \Delta U_s^{(k),r}, \end{aligned}$$

consisting of the upward jumps and downward jumps of $U^{(k),r}$, respectively. We recall the death-jump dynamics of U^r: The particles jump upwards simultaneously at rate $\frac{r^2Y^r}{c}$. At the time s of such a jump, the levels are multiplied by $\frac{r}{r-c/Y_s^r}$, so the increment of $U^{(k),r}$ is $\frac{c}{rY_s^r-c}U_s^{(k),r}$. Adding and subtracting the compensator of the upward jumps, we decompose $U^{(k),r}$ into a predictable bounded variation part, a compensated jump-type martingale $M^{(k),r}$ with start $M_0^{(k),r} = 0$ and the downward jumps,

$$\begin{aligned} U_t^{(k),r} &= U_0^{(k),r} + \int_0^t \mathrm{d}s\ \left((U_s^{(k),r})^2 - rU_s^{(k),r}\right) + J_t^{(k),r,+} + J_t^{(k),r,-}\\ &= U_0^{(k),r} + \int_0^t \mathrm{d}s\ \left((U_s^{(k),r})^2 - rU_s^{(k),r} + \frac{r^2Y_s^r}{c}\cdot\frac{c}{rY_s^r-c}U_s^{(k),r}\right) + M_t^{(k),r} + J_t^{(k),r,-}\\ &= U_0^{(k),r} + \int_0^t \mathrm{d}s\ \left((U_s^{(k),r})^2 + \frac{rc}{rY_s^r-c}\cdot U_s^{(k),r}\right) + M_t^{(k),r} + J_t^{(k),r,-}. \end{aligned} \tag{4.9}$$

In order to estimate how far the level $U^{(k),r}$ moves, we will need the quadratic variation of the martingale part:

$$\begin{aligned} \langle M^{(k),r}\rangle_t &= \int_0^t \mathrm{d}s\ \frac{r^2Y_s^r}{c}\left(\frac{c}{rY_s^r-c}U_s^{(k),r}\right)^2\\ &= \int_0^t \mathrm{d}s\ \frac{r^2cY_s^r}{(rY_s^r-c)^2}\left(U_s^{(k),r}\right)^2\\ &\le \int_0^t \mathrm{d}s\ \frac{r^2c\delta}{(r\delta-c)^2}\left(U_s^{(k),r}\right)^2, \end{aligned}$$

where the last inequality is true for $r > c/\delta$.

(i) We turn to the ascent probability. Let $\tilde{u} \in \mathcal{S}_{[0,r]}$ be a start configuration that satisfies $\tilde{u}^{(k)} < \frac{K}{2}$. Define $\tau := \inf\left\{s > 0 : U_s^{(k),r} \geq K\right\}$. Using Chebyshev's inequality we estimate

$$\begin{aligned}
\mathbf{P}_{\tilde{u}}\left[\sup_{s\leq\zeta} U_s^{(k),r} \geq K\right] &= \mathbf{P}_{\tilde{u}}\left[U_{\zeta\wedge\tau}^{(k),r} \geq K\right] \\
&= \mathbf{P}_{\tilde{u}}\left[U_0^{(k),r} + \int_0^{\zeta\wedge\tau} \mathrm{d}s\ \left((U_s^{(k),r})^2 + \frac{rc}{rY_s^r - c}\cdot U_s^{(k),r}\right) + M_{\zeta\wedge\tau}^{(k),r} + J_{\zeta\wedge\tau}^{(k),r,-} \geq K\right] \\
&\leq \mathbf{P}_{\tilde{u}}\left[\frac{K}{2} + \zeta\left(K^2 + \frac{rc}{r\delta - c}\cdot K\right) + M_{\zeta\wedge\tau}^{(k),r} \geq K\right] \\
&= \mathbf{P}_{\tilde{u}}\left[M_{\zeta\wedge\tau}^{(k),r} \geq \frac{K}{2} - \zeta\left(K^2 + \frac{rc}{r\delta - c}\cdot K\right)\right] \\
&\leq \frac{\mathbf{E}_{\tilde{u}}\left[\langle M^{(k),r}\rangle_{\zeta\wedge\tau}\right]}{\left[\frac{K}{2} - \zeta\cdot\left(K^2 + \frac{rc}{r\delta-c}\cdot K\right)\right]^2} \\
&\leq \frac{\zeta\frac{r^2c\delta}{(r\delta-c)^2}K^2}{\left[\frac{K}{2} - \zeta\cdot\left(K^2 + \frac{rc}{r\delta-c}\cdot K\right)\right]^2} \\
&= 4\zeta\frac{r^2c\delta}{(r\delta-c)^2}\cdot\left[1 - 2\zeta\left(K + \frac{rc}{r\delta - c}\right)\right]^{-2}.
\end{aligned} \tag{4.10}$$

(ii) Next we estimate the probability that a particle starting above K descends to $\frac{K}{2}$ in the time interval $[0,\zeta]$. Let $\hat{u}$ be a start configuration that satisfies $\hat{u}^{(k)} > K$. Define the event B_K^r that at least one of the $k-1$ lowest particles gives birth to an offspring below the threshold K in the time interval $[0,\zeta]$ and decompose on this event:

$$\mathbf{P}_{\hat{u}}\left[\inf_{s\leq\zeta} U_s^{(k),r} \leq \frac{K}{2}\right] \leq \mathbf{P}_{\hat{u}}\left[\left\{\inf_{s\leq\zeta} U_s^{(k),r} \leq \frac{K}{2}\right\} \cap (B_K^r)^c\right] + \mathbf{P}_{\hat{u}}\left[B_K^r\right].$$

Each particle with level u_i gives birth to an offspring with level in $[a,b]$, $u_i < a < b$ with rate $2(b-a)$. So the probability that one of the $(k-1)$ lowest particles gives birth to a child below K during a time interval of length ζ can be bounded by

$$\mathbf{P}\left[B_K^r\right] \leq 1 - e^{-2\zeta K(k-1)} \leq 2\zeta K(k-1). \tag{4.11}$$

On the event that no particle with a level below K is born in the time interval of length ζ, the particle $U^{(k),r}$ does not *jump* downwards across K. Therefore it hits K if it starts above K and moves down to $\frac{K}{2}$. Since U^r is a strong Markov process, we obtain

$$\mathbf{P}_{\hat{u}}\left[\left\{\inf_{s\leq\zeta} U_s^{(k),r} \leq \frac{K}{2}\right\} \cap (B_K^r)^c\right] \leq \sup_{\hat{u}'\in\Theta_r} \mathbf{P}_{\hat{u}'}\left[\left\{\inf_{s\leq\zeta} U_s^{(k),r} \leq \frac{K}{2}\right\} \cap (B_K^r)^c\right],$$

where

$$\Theta_r := \left\{\hat{u} \in \mathcal{S}_{[0,r]}\ :\ \hat{u}^{(k)} = K,\ \hat{u}([0,r)) \geq \delta r\right\}.$$

Next, we decompose on the event that the particle $U^{(k),r}$ ascends to $2K$ before it descends to $\frac{K}{2}$: Let $\hat{u}' \in \Theta_r$. Then

$$\begin{aligned}
&\mathbf{P}_{\hat{u}'}\left[\left\{\inf_{s\le\zeta} U_s^{(k),r} \le \frac{K}{2}\right\} \cap (B_K^r)^c\right] \\
&\quad\le \mathbf{P}_{\hat{u}'}\left[\left\{\inf_{s\le\zeta} U_s^{(k),r} \le \frac{K}{2}\right\} \cap \left\{\sup_{s\le\zeta} U_s^{(k),r} < 2K\right\} \cap (B_K^r)^c\right] + \mathbf{P}_{\hat{u}'}\left[\sup_{s\le\zeta} U_s^{(k),r} \ge 2K\right].
\end{aligned} \tag{4.12}$$

Similarly as in (4.10), we estimate the ascent probability. Let

$$\tau' := \inf\left\{s > 0 : U_s^{(k),r} \ge 2K\right\}.$$

Then

$$\begin{aligned}
\mathbf{P}_{\hat{u}'}&\left[\sup_{s\le\zeta} U_s^{(k),r} \ge 2K\right] \le \mathbf{P}_{\hat{u}'}\left[U_{\zeta\wedge\tau'}^{(k),r} \ge 2K\right] \\
&= \mathbf{P}_{\hat{u}'}\left[U_0^{(k),r} + \int_0^{\zeta\wedge\tau'} \mathrm{d}s\, \left((U_s^{(k),r})^2 + \frac{rc}{rY_s^r - c}\cdot U_s^{(k),r}\right) + M_{\zeta\wedge\tau'}^{(k),r} + J_{\zeta\wedge\tau'}^{(k),r,-} \ge 2K\right] \\
&\le \mathbf{P}_{\hat{u}'}\left[K + \zeta\left((2K)^2 + \frac{rc}{r\delta - c}\cdot 2K\right) + M_{\zeta\wedge\tau'}^{(k),r} \ge 2K\right] \\
&\le \frac{\mathbf{E}_{\hat{u}'}\left[\langle M^{(k),r}\rangle_{\zeta\wedge\tau'}\right]}{\left[K - \zeta\left(4K^2 + \frac{rc}{r\delta-c}\cdot 2K\right)\right]^2} \\
&\le \frac{\zeta\frac{r^2c\delta}{(r\delta-c)^2}(2K)^2}{\left[K - \zeta\left(4K^2 + \frac{rc}{r\delta-c}\cdot 2K\right)\right]^2} \\
&= 4\zeta\frac{r^2c\delta}{(r\delta-c)^2}\cdot\left[1 - 2\zeta\left(2K + \frac{rc}{r\delta-c}\right)\right]^{-2}.
\end{aligned} \tag{4.13}$$

Now we turn to the descent probability, i.e. the first summand of the right hand side of (4.12). On the event that the level does not reach $2K$, we can stop at τ' without changing the descent probability and, on the event that no birth happens, there are no downward jumps. With

$$\tau'' := \inf\left\{s > 0 \ : \ U_s^{(k),r} \le \frac{K}{2}\right\}$$

we obtain

$$
\begin{aligned}
&\mathbf{P}_{\hat{u}'}\left[\left\{\inf_{s\leq\zeta} U_s^{(k),r} \leq \frac{K}{2}\right\} \cap \left\{\sup_{s\leq\zeta} U_s^{(k),r} < 2K\right\} \cap (B_K^r)^c\right] \\
&\quad \leq \mathbf{P}_{\hat{u}'}\left[\left\{\inf_{s\leq\zeta} U_{s\wedge\tau'}^{(k),r} \leq \frac{K}{2}\right\} \cap (B_K^r)^c\right] \\
&\quad \leq \mathbf{P}_{\hat{u}'}\left[\left\{U_{s\wedge\tau'\wedge\tau''}^{(k),r} \leq \frac{K}{2}\right\} \cap (B_K^r)^c\right] \\
&\quad \leq \mathbf{P}_{\hat{u}'}\left[U_0^{(k),r} + \int_0^{\zeta\wedge\tau'\wedge\tau''} \mathrm{d}s \; \left((U_s^{(k),r})^2 + \frac{rc}{rY_s^r - c} \cdot U_s^{(k),r}\right) + M_{\zeta\wedge\tau'\wedge\tau''}^{(k),r} \leq \frac{K}{2}\right] \\
&\quad \leq \mathbf{P}_{\hat{u}'}\left[M_{\zeta\wedge\tau'\wedge\tau''}^{(k),r} \leq -\frac{K}{2}\right] \\
&\quad \leq \mathbf{P}_{\hat{u}'}\left[\left|M_{\zeta\wedge\tau'\wedge\tau''}^{(k),r}\right| \geq \frac{K}{2}\right] \\
&\quad \leq \frac{4\mathbf{E}_{\hat{u}'}\left[\langle M^{(k),r}\rangle_{\zeta\wedge\tau'\wedge\tau''}\right]}{K^2} \\
&\quad \leq 16\zeta \frac{r^2 c\delta}{(r\delta - c)^2}.
\end{aligned}
\tag{4.14}
$$

We collect equations (4.13), (4.14) and (4.11) and obtain

$$
\begin{aligned}
\mathbf{P}_{\hat{u}}\left[\inf_{s\leq\zeta} U_s^{(k),r} \leq \frac{K}{2}\right] &\leq \mathbf{P}_{\hat{u}}\left[\left\{\inf_{s\leq\zeta} U_s^{(k),r} \leq \frac{K}{2}\right\} \cap (B_K^r)^c\right] + \mathbf{P}\left[B_K^r\right] \\
&\leq 4\zeta \frac{r^2 c\delta}{(r\delta - c)^2} \cdot \left[1 - 2\zeta\left(2K + \frac{rc}{r\delta - c}\right)\right]^{-2} \\
&\qquad + 16\zeta \frac{r^2 c\delta}{(r\delta - c)^2} + 2\zeta K(k-1).
\end{aligned}
$$

□

Note that the mass densities Y^r are not exactly Galton-Watson processes, since several particles may die simultaneously. Nonetheless Y^r converges to Feller's branching diffusion as $r \to \infty$.

Proposition 4.2.2. *The mass densities $(Y^r)_r$ are C-tight.*

Proof. This is a simple application of Kolmogorov's moment criterion for C-tightness and can be done similarly as in [Kle06], 21.9, pp. 460. We omit the details. □

In Section 4.3.1 we show that any limit point of $(Y^r)_r$ has the semimartingale characteristics of Feller's branching diffusion.

Theorem 4.2.3. *The family* $\left(X^{r,\delta}\right)_r$ *is tight in* $D_{\mathbb{R}^{k+1}}[0,\infty)$.

Proof. By Proposition 4.2.2 $(Y^r)_r$ is C-tight. By Corollary VI.3.33 in [JS03] (see Lemma A.3.3) the family $(X^{r,\delta})_r$ is tight if the process $(U^{(1),r,\delta}, U^{(2),r,\delta}, \ldots, U^{(k),r,\delta})$ is tight in $D_{\mathbb{R}^k}[0,\infty)$. We suppress the δ in our notation, $U^{(i),r} := U^{(i),r,\delta}$ and set $b = 1$ for convenience.

We use Aldous' Tightness Criterion (see Theorem 16.10 in [Bil99]). I.e., we show the following two conditions:

(i) For all $T > 0$

$$\lim_{K\to\infty} \limsup_{r\to\infty} \mathbf{P}\left[\sup_{0\le s\le T} U_s^{(k),r} > K\right] = 0$$

holds.

(ii) For each ϵ, η, T there exists γ_0 and r_0 such that, if $\gamma \le \gamma_0$ and $r \ge r_0$, and if τ is a discrete X^r-stopping time satisfying $\tau \le T$, then

$$\mathbf{P}\left[\max_{i=1,\ldots,k} \left|U_{\tau+\gamma}^{(i),r} - U_{\tau}^{(i),r}\right| \ge \epsilon\right] \le \eta.$$

First, we check the compact containment condition (i). Let $T > 0$ and $K > \frac{4}{\delta}\log(\frac{2}{T})$. Let $0 < \zeta < 2\zeta < \ldots < T$ be a partition of $[0,T]$. Define the events that the k lowest levels are below $\frac{K}{2}$ at the nodes of the partition,

$$A_{j,K}^r := \left\{U_{j\zeta}^{(k),r} < \frac{K}{2}\right\} \quad \text{and} \quad A_K^r := \bigcap_{j=0}^{T/\zeta} A_{j,K}^r.$$

Then we have

$$\mathbf{P}\left[\sup_{0\le s\le T} U_s^{(k),r} > K\right] \le \mathbf{P}\left[\left\{\sup_{0\le s\le T} U_s^{(k),r} > K\right\} \cap A_K^r\right] + \mathbf{P}\left[(A_K^r)^c\right]. \tag{4.15}$$

First we treat the probability that one or more of the k lowest particles is above K at the nodes of the partition. Since the levels of the whole system are independent and identically distributed in $[0,r]$ at times $j\cdot\zeta$, the number of particles below $\frac{K}{2}$ is binomially distributed. Choose mesh size $\zeta = T/\left\lfloor Te^{\frac{\delta K}{4}} - 1\right\rfloor$. Then $\frac{T}{\zeta} + 1 \le Te^{\frac{\delta K}{4}}$

and we obtain

$$
\begin{aligned}
\limsup_{r\to\infty} \mathbf{P}\left[(A_K^r)^c\right] &\le \limsup_{r\to\infty} \sum_{j=0}^{T/\zeta} \mathbf{P}\left[(A_{j,K}^r)^c\right] \\
&\le \limsup_{r\to\infty} \left(\frac{T}{\zeta}+1\right) \sum_{i=0}^{k-1} \mathrm{Bin}_{r\delta,\frac{K}{2r}}(i) \\
&= \left(\frac{T}{\zeta}+1\right) \sum_{i=0}^{k-1} \mathrm{Poi}_{\frac{\delta K}{2}}(i) \qquad (4.16)\\
&= \left(\frac{T}{\zeta}+1\right) e^{-\frac{\delta K}{2}} \sum_{i=0}^{k-1} \frac{\delta^i K^i}{2^i} \cdot \frac{1}{i!} \\
&\le T e^{-\frac{\delta K}{4}} \sum_{i=0}^{k-1} \delta^i K^i.
\end{aligned}
$$

In order to decompose the first summand of (4.15), as depicted in Figure 4.2, define the U^r-stopping times

$$
\tau_j := \inf\left\{ s > j\zeta : U_s^{(k),r} \ge K \right\}.
$$

Using the Markov property of U^r,

$$
\begin{aligned}
\mathbf{P}\left[\left\{\sup_{0\le s\le T} U_s^{(k),r} > K\right\} \cap A_K^r\right] &\le \sum_{j=0}^{T/\zeta-1} \mathbf{P}\left[\{\tau_j \le (j+1)\zeta\} \cap A_K^r\right] \\
&\le \sum_{j=0}^{T/\zeta-1} \mathbf{P}\left[\{\tau_j \le (j+1)\zeta\} \cap A_{j,K}^r \cap A_{j+1,K}^r\right] \\
&= \sum_{j=0}^{T/\zeta-1} \mathbf{E}\left[\mathbf{E}\left[\mathbb{1}_{\{\tau_j \le (j+1)\zeta\}\cap A_{j,K}^r} \cdot \mathbb{1}_{A_{j+1,K}^r} \mid \mathcal{F}_{\tau_j}^{U^r}\right]\right] \\
&= \sum_{j=0}^{T/\zeta-1} \mathbf{E}\left[\mathbb{1}_{\{\tau_j \le (j+1)\zeta\}\cap A_{j,K}^r} \cdot \mathbf{E}\left[\mathbb{1}_{A_{j+1,K}^r} \mid \mathcal{F}_{\tau_j}^{U^r}\right]\right] \\
&\le \sum_{j=0}^{T/\zeta-1} \mathbf{E}\left[\mathbb{1}_{\{\tau_j \le (j+1)\zeta\}\cap A_{j,K}^r} \cdot \mathbf{P}_{U_{\tau_j}^r}\left[\inf_{s\le\zeta} U_s^{(k),r} \le \frac{K}{2}\right]\right].
\end{aligned} \qquad (4.17)
$$

The set of level configurations

$$
\Theta_r := \left\{ \tilde{u} \in \mathcal{S}_{[0,r]} \ : \ \tilde{u}^{(k)} < \frac{K}{2},\ \tilde{u}([0,r)) \ge \delta r \right\}
$$

contains all states $U^r_{j\zeta}$ can be in, given $A^r_{j,K}$. We obtain

$$\begin{aligned}\mathbf{P}\left[\{\tau_j \le (j+1)\zeta\} \cap A^r_{j,K}\right] &\le \mathbf{P}\left[\tau_j \le (j+1)\zeta \mid A^r_{j,K}\right] \\ &\le \sup_{\tilde{u}\in\Theta_r} \mathbf{P}_{\tilde{u}}\left[\tau_0 \le \zeta\right] \\ &= \sup_{\tilde{u}\in\Theta_r} \mathbf{P}_{\tilde{u}}\left[\sup_{s\le\zeta} U^{(k),r}_s \ge K\right].\end{aligned} \tag{4.18}$$

Recall that $U^{(k),r}$ exceeds K by jumping. On the event $\{\tau_j \le (j+1)\zeta\}$, the set of level configurations

$$\hat{\Theta}_r := \left\{\hat{u} \in \mathcal{S}_{[0,r]} \;:\; \hat{u}^{(k)} \ge K,\; \hat{u}([0,r)) \ge \delta r\right\}$$

contains all states $U^r_{\tau_j}$ can be in, and we have

$$\mathbf{P}_{U^r_{\tau_j}}\left[\inf_{s\le\zeta} U^{(k),r}_s \le \frac{K}{2}\right] \le \sup_{\hat{u}\in\hat{\Theta}_r} \mathbf{P}_{\hat{u}}\left[\inf_{s\le\zeta} U^{(k),r}_s \le \frac{K}{2}\right] \tag{4.19}$$

on the $\{\tau_j \le (j+1)\zeta\}$. Plug (4.19) and (4.18) in the bound (4.17) to obtain

$$\begin{aligned}\mathbf{P}\left[\left\{\sup_{0\le s\le T} U^{(k),r}_s > K\right\} \cap A^r_K\right] \le \frac{T}{\zeta} &\sup_{\tilde{u}\in\Theta_r} \mathbf{P}_{\tilde{u}}\left[\sup_{s\le\zeta} U^{(k),r}_s \ge K\right] \\ &\times \sup_{\hat{u}\in\hat{\Theta}_r} \mathbf{P}_{\hat{u}}\left[\inf_{s\le\zeta} U^{(k),r}_s \le \frac{K}{2}\right].\end{aligned} \tag{4.20}$$

But Lemma 4.2.1 gives us

$$\limsup_{r\to\infty} \sup_{\tilde{u}\in\Theta_r} \mathbf{P}_{\tilde{u}}\left[\sup_{s\le\zeta} U^{(k),r}_s \ge K\right] \le \frac{4c}{\delta}\cdot\zeta + o\left(\zeta\right) \tag{4.21}$$

and

$$\limsup_{r\to\infty} \sup_{\hat{u}\in\hat{\Theta}_r} \mathbf{P}_{\hat{u}}\left[\inf_{s\le\zeta} U^{(k),r}_s \le \frac{K}{2}\right] \le 20\zeta\frac{c}{\delta} + 2\zeta K(k-1) + o(\zeta) \tag{4.22}$$

for $\zeta \to 0$.

Now we can piece together the compact containment condition: Plug equations (4.21) and (4.22) in (4.20) to obtain

$$\limsup_{r\to\infty} \mathbf{P}\left[\left\{\sup_{0\le s\le T} U^{(k),r}_s > K\right\} \cap A^r_K\right] \le \zeta\cdot\frac{4Tc}{\delta}\left(\frac{20c}{\delta} + 2K(k-1)\right) + o(\zeta)$$

and with (4.15) and (4.16) we have

$$\begin{aligned}\limsup_{r\to\infty} \mathbf{P}\left[\sup_{0\le s\le T} U^{(k),r}_s > K\right] &\le \zeta\cdot\frac{4Tc}{\delta}\left(\frac{20c}{\delta} + (k-1)2K\right) + Te^{-\frac{\delta K}{4}}\sum_{i=0}^{k-1}\delta^i K^i + o(\zeta) \\ &\xrightarrow{K\to\infty} 0\end{aligned}$$

since $\zeta\cdot K \xrightarrow{K\to\infty} 0$ according to prerequisites.

Next we verify the structure condition (ii). Note that the uniform distribution property of the particle representation holds only for $\mathcal{F}^Y$-stopping times. But condition *(ii)* has to be true for discrete $\mathcal{F}^X$-stopping times. Thus we have to trace the levels "by hand", as we did for condition *(i)*.

Let $\epsilon, T > 0$ and $\tau < T$ an $\mathcal{F}^X$ stopping time that takes only finitely many values. Proceeding similarly as in (4.9), we add and subtract the compensator of the upward jumps, thus obtaining a martingale representation for the increment after τ,

$$\begin{aligned}
\left|U^{(i),r}_{\tau+\gamma} - U^{(i),r}_{\tau}\right| &= \left|\int_0^\gamma \mathrm{d}s \ \left((U^{(i),r}_{\tau+s})^2 - rU^{(i),r}_{\tau+s}\right) + \sum_{\substack{0<s\le\gamma \\ \left|\Delta U^{(i),r}_{\tau+s}\right|>0}} \Delta U^{(i),r}_{\tau+s}\right| \\
&\le \left|\int_0^\gamma \mathrm{d}s \ \left((U^{(i),r}_{\tau+s})^2 - rU^{(i),r}_{\tau+s}\right) + \sum_{\substack{0<s\le\gamma \\ \Delta U^{(i),r}_{\tau+s}>0}} \Delta U^{(i),r}_{\tau+s}\right| + \left|\sum_{\substack{0<s\le\gamma \\ \Delta U^{(i),r}_{\tau+s}<0}} \Delta U^{(i),r}_{\tau+s}\right| \\
&= \left|\int_0^\gamma \mathrm{d}s \ \left((U^{(i),r}_{\tau+s})^2 + \frac{rc}{rY^r_{\tau+s} - c} \cdot U^{(i),r}_{\tau+s}\right) + M^{(i),r}_\gamma\right| + \left|\sum_{\substack{0<s\le\gamma \\ \Delta U^{(i),r}_{\tau+s}<0}} \Delta U^{(i),r}_{\tau+s}\right|.
\end{aligned}$$

We will use the compact containment condition (i) and decompose on the event, that $U^{(k),r}$ stays below a threshold $K > 0$. Let $r_0 = 2\frac{c}{\delta}$ and let $\eta > 0$ be arbitrarily small. Because of the compact containment condition *(i)* we find K such that for $r > r_0$

$$\mathbf{P}\left[\sup_{0\le s\le T} U^{(k),r}_s \ge K\right] \le \frac{\eta}{3k}. \tag{4.23}$$

For $\gamma < \gamma' := \frac{\epsilon}{2}\left(K^2 + \frac{2c}{\delta}K\right)^{-1}$ and $r > r_0$, we have

$$\int_0^\gamma \mathrm{d}s \ \left|(U^{(i),r}_{\tau+s})^2 + \frac{rc}{rY^r_{\tau+s} - c} \cdot U^{(i),r}_{\tau+s}\right| \le \gamma \cdot \left(K^2 + \frac{2c}{\delta}K\right) < \frac{\epsilon}{2} \tag{4.24}$$

on the event $\left\{\sup_{0\le s<T} U^{(k),r}_s \le K\right\}$.

Proceeding as above, we will use the quadratic variation of the martingale $M^{(i),r}$ to bound its increment. To this end we compute

$$\begin{aligned}
\langle M^{(i),r}\rangle_t &= \int_0^t \mathrm{d}s \ \frac{r^2 c Y^r_{\tau+s}}{\left(rY^r_{\tau+s} - c\right)^2}\left(U^{(i),r}_{\tau+s}\right)^2 \\
&\le \int_0^t \mathrm{d}s \ \frac{r^2 c\delta}{(r\delta - c)^2}\left(U^{(i),r}_{\tau+s}\right)^2 \\
&\le \int_0^t \mathrm{d}s \ \frac{4c}{\delta}\left(U^{(i),r}_{\tau+s}\right)^2.
\end{aligned}$$

The last inequality holds for $r > r_0 = 2\frac{c}{\delta}$. Define

$$\sigma := \inf \left\{ s \geq 0 : U^{(i),r}_{\tau+s} \geq K \right\}.$$

We obtain for $r > r_0 > \frac{c}{\delta}$ and $\gamma < \gamma'' := \frac{\eta}{3k} \cdot \left(\frac{16c}{\delta\epsilon^2} K^2 \right)^{-1}$

$$\begin{aligned}
\mathbf{P}\left[\left\{ \left| M^{(i),r}_{\gamma} \right| > \frac{\epsilon}{2} \right\} \cap \left\{ \sup_{0 \leq s < T} U^{(k),r}_s \leq K \right\} \right] &\leq \mathbf{P}\left[\left| M^{(i),r}_{\gamma \wedge \sigma} \right| > \frac{\epsilon}{2} \right] \\
&\leq \frac{4\mathbf{E}\left[\langle M^{(i),r} \rangle_{\gamma \wedge \sigma} \right]}{\epsilon^2} \\
&\leq \gamma \cdot \frac{16c}{\delta \epsilon^2} K^2 \\
&< \frac{\eta}{3k},
\end{aligned} \tag{4.25}$$

using Chebyshev's inequality.

Finally, on the event $\left\{ \sup_{0 \leq s < T} U^{(k),r}_s \leq K \right\}$ the instantaneous rate of downward jumps at time t is bounded by $(k-1)2K$. Let $\gamma < \gamma''' := \frac{1}{2(k-1)K} \cdot \frac{\eta}{3k}$. Then the probability that $U^{(i),r}$ jumps downwards can be estimated by

$$\begin{aligned}
&\mathbf{P}\left[\left\{ \left| \sum_{\substack{0 \leq s < \gamma \\ \Delta U^{(i),r}_{\tau+s} < 0}} \Delta U^{(i),r}_{\tau+s} \right| > 0 \right\} \cap \left\{ \sup_{0 \leq s < T} U^{(k),r}_s \leq K \right\} \right] \\
&\qquad \leq 1 - e^{-2(k-1)K\gamma} \leq 2(k-1)K\gamma < \frac{\eta}{3k}.
\end{aligned} \tag{4.26}$$

Combining equations (4.23), (4.24), (4.25) and (4.26), we obtain for $\gamma < \min(\gamma', \gamma'', \gamma''')$

$$\begin{aligned}
\mathbf{P}\left[\left| U^{(i),r}_{\tau+\gamma} - U^{(i),r}_{\tau} \right| > \epsilon \right] &\leq \mathbf{P}\left[\left\{ \left| U^{(i),r}_{\tau+\gamma} - U^{(i),r}_{\tau} \right| > \epsilon \right\} \cap \left\{ \sup_{0 \leq s < T} U^{(k),r}_s \leq K \right\} \right] \\
&\qquad + \mathbf{P}\left[\sup_{0 \leq s < T} U^{(k),r}_s > K \right] \\
&\leq \mathbf{P}\left[\left\{ \int_0^{\gamma} \mathrm{d}s \left| (U^{(i),r}_{\tau+s})^2 + \frac{rc}{rY^r_{\tau+s} - c} \cdot U^{(i),r}_{\tau+s} \right| > \frac{\epsilon}{2} \right\} \cap \left\{ \sup_{0 \leq s < T} U^{(i),r}_s \leq K \right\} \right] \\
&\qquad + \mathbf{P}\left[\left\{ \left| M^{(i),r}_{\gamma} \right| > \frac{\epsilon}{2} \right\} \cap \left\{ \sup_{0 \leq s < T} U^{(k),r}_s \leq K \right\} \right] \\
&\qquad + \mathbf{P}\left[\left\{ \left| \sum_{\substack{0 \leq s < \gamma \\ \Delta U^{(i),r}_{\tau+s} < 0}} \Delta U^{(i),r}_{\tau+s} \right| > 0 \right\} \cap \left\{ \sup_{0 \leq s < T} U^{(k),r}_s \leq K \right\} \right] \\
&\qquad + \mathbf{P}\left[\sup_{0 \leq s < T} U^{(k),r}_s > K \right] \\
&\leq 0 + 3 \cdot \frac{\eta}{3k} = \frac{\eta}{k}.
\end{aligned}$$

This implies condition *(ii)*. $\square$

In the next step we "lift" the tightness of $(U^{(1,\ldots,k),r,\delta})_r$ to the case of the measure valued family $(U^{r,\delta})_r$. We show that, for fixed $C > 0$ and for k large enough, the level $U^{(k),r,\delta}$ stays above C. Hence a testfunction with compact support "sees" only the k lowest levels.

Recall that $\mathcal{S}_{\mathbb{R}+}$ is the space of locally finite measures on $[0,\infty)$, equipped with the topology of vague convergence and let C_c^+ be the set of non negative, continuous functions on $[0,\infty)$ with compact support.

Theorem 4.2.4. *There exists $r_0 > 0$ such that the family $\left(U^{r,\delta}, Y^{r,\delta}\right)_{r\geq r_0}$ is tight in $D_{\mathcal{S}_{\mathbb{R}+}\times\mathbb{R}^+}[0,\infty)$.*

Remark 4.2.5. *By Prohorov's Theorem, $\left(U^{r_n,\delta}, Y^{r_n,\delta}\right)_n$ is weakly relatively sequentially compact for any sequence $r_n \to \infty$.*

Proof. Again we confine ourselves to the level coordinate $U^{r,\delta}$, drop the δ in our notation and set $b = 1$. Theorem 16.27 in [Kal02], p. 324, states that it is enough to show tightness of $U^r(f) := \int f \, \mathrm{d}U^r$ in $D_{\mathbb{R}^+}[0,\infty)$ for every $f \in C_c^+$. Let $T > 0$, $f \in C_c^+$, $C \geq \sup\operatorname{supp} f$ and let $r > 4C$. We show that for k large enough, the k-th lowest level stays above the threshold C (with high probability) and we deduce tightness of $(U^r(f))_r$ from tightness of $(U^{(1),r},\ldots,U^{(k),r})_r$.

We use similar arguments as in the proof of Theorem 4.2.3. We choose a partition $0 < \zeta < 2\zeta < \ldots < T$ of $[0,T]$, such that the probability of the k-th level being above $2C$ at every node of the partition, is high. Let

$$A^r_{i,k} := \left\{U^{(k),r}_{i\zeta} > 2C\right\} \quad \text{and} \quad A^r_k := \bigcap_{i=0}^{T/\zeta} A^r_{i,k}.$$

By compact containment for any $\epsilon > 0$, we can choose $L_\epsilon > 0$ such that

$$\limsup_{r\to\infty} \mathbf{P}\left[\sup_{0\leq s\leq T} Y^r_s > L_\epsilon\right] < \epsilon.$$

We have

$$\begin{aligned}
\limsup_{r\to\infty} \mathbf{P}\left[(A_k^r)^c\right] &\le \limsup_{r\to\infty} \mathbf{P}\left[(A_k^r)^c \cap \left\{\sup_{0\le s<T} Y_s^r \le L_\epsilon\right\}\right] + \epsilon \\
&\le \limsup_{r\to\infty} \sum_{j=0}^{T/\zeta} \mathbf{P}\left[\left(A_{j,k}^r\right)^c \cap \left\{Y_{j\zeta}^r \le L_\epsilon\right\}\right] + \epsilon \\
&\le \limsup_{r\to\infty} \sum_{j=0}^{T/\zeta} \mathbf{P}\left[\left(A_{j,k}^r\right)^c \;\middle|\; Y_{j\zeta}^r \le L_\epsilon\right] + \epsilon \\
&\le \limsup_{r\to\infty} \left(\frac{T}{\zeta}+1\right) \mathrm{Bin}_{\lceil rL_\epsilon\rceil, \frac{2C}{r}}\left(\{k, k+1, \ldots\}\right) + \epsilon \\
&= \left(\frac{T}{\zeta}+1\right) \mathrm{Poi}_{2L_\epsilon C}\left(\{k, k+1, \ldots\}\right) + \epsilon \\
&= \left(\frac{T}{\zeta}+1\right) e^{-2L_\epsilon C} \sum_{i=k}^{\infty} \frac{(2L_\epsilon C)^i}{i!} + \epsilon \\
&\le 2\left(\frac{T}{\zeta}+1\right) e^{-2L_\epsilon C} \frac{(2L_\epsilon C)^k}{k!} + \epsilon.
\end{aligned}$$

Choose mesh size $\zeta = T \cdot \left\lfloor Te^k \right\rfloor^{-1}$ and let $\epsilon \to 0$ to obtain

$$\limsup_{r\to\infty} \mathbf{P}\left[(A_k^r)^c\right] \xrightarrow{k\to\infty} 0. \tag{4.27}$$

Now define the U^r-stopping times

$$\tau_j := \inf\left\{s > j\zeta : U_s^{(k),r} \le C\right\}.$$

Using the strong Markov property of U^r, we obtain

$$\begin{aligned}
\mathbf{P}\left[\left\{\inf_{0\le s\le T} U_s^{(k),r} \le C\right\} \cap A_k^r\right] &\le \sum_{j=0}^{T/\zeta-1} \mathbf{P}\left[\{\tau_j \le (j+1)\zeta\} \cap A_{j,k}^r \cap A_{j+1,k}^r\right] \\
&= \sum_{j=0}^{T/\zeta-1} \mathbf{E}\left[\mathbb{1}_{\{\tau_j\le(j+1)\zeta\}\cap A_{j,k}^r} \cdot \mathbf{E}\left[\mathbb{1}_{A_{j+1,k}^r} \mid \mathcal{F}_{\tau_j}^{U^r}\right]\right] \\
&\le \sum_{j=0}^{T/\zeta-1} \mathbf{E}\left[\mathbb{1}_{\{\tau_j\le(j+1)\zeta\}\cap A_{j,k}^r} \cdot \mathbf{P}_{U_{\tau_j}^r}\left[\sup_{s\le\zeta} U_s^{(k),r} \ge 2C\right]\right].
\end{aligned}$$

Given $A_{j,k}^r$, the set of level configurations

$$\Theta_r := \left\{\tilde{u} \in \mathcal{S}_{[0,r]} \;:\; \tilde{U}^{(k)} \ge 2C,\ \delta r \le \tilde{u}([0,r)) \le L_\epsilon r\right\}$$

contains all states $U_{j\zeta}^r$ can be in, and, on $\{\tau_j \le (j+1)\zeta\}$,

$$\Theta_r' := \left\{\tilde{u} \in \mathcal{S}_{[0,r]} \;:\; \tilde{u}^{(k)} \le C,\ \delta r \le \tilde{u}([0,r)) \le L_\epsilon r\right\}$$

contains all states $U^r_{\tau_j}$ can be in. With completely analogous arguments as in the proof of Theorem 4.2.3 (cf. (4.18) and (4.19)) we obtain

$$\begin{aligned}\mathbf{P}\left[\left\{\inf_{0\le s\le T} U_s^{(k),r}\le C\right\}\cap A_k^r\right]\le \frac{T}{\zeta}\sup_{\tilde u\in\Theta_r}\mathbf{P}_{\tilde u}\left[\inf_{s\le\zeta}U_s^{(k),r}\le C\right]\\ \times\sup_{\tilde u'\in\Theta_r'}\mathbf{P}_{\tilde u'}\left[\sup_{s\le\zeta}U_s^{(k),r}\ge 2C\right].\end{aligned}\tag{4.28}$$

Now Lemma 4.2.1 (ii) gives us

$$\limsup_{r\to\infty}\sup_{\tilde u\in\Theta_r}\mathbf{P}_{\tilde u}\left[\inf_{s\le\zeta}U_s^{(k),r}\le C\right]\le\zeta\cdot\left(\frac{20c}{\delta}+(k-1)4C\right)+o(\zeta)\tag{4.29}$$

and Lemma 4.2.1 (i) gives us

$$\limsup_{r\to\infty}\sup_{\tilde u'\in\Theta_r'}\mathbf{P}_{\tilde u'}\left[\sup_{s\le\zeta}U_s^{(k),r}\ge 2C\right]\le\zeta\cdot\frac{4c}{\delta}+o\left(\zeta\right).\tag{4.30}$$

Plug (4.29) and (4.30) in (4.28) and we obtain with (4.27)

$$\begin{aligned}\limsup_{r\to\infty}\mathbf{P}\left[\inf_{0\le s\le T}U_s^{(k),r}\le C\right]&\le\limsup_{r\to\infty}\left(\mathbf{P}\left[\left\{\inf_{0\le s\le T}U_s^{(k),r}\le C\right\}\cap A_k^r\right]+\mathbf{P}\left[(A_k^r)^c\right]\right)\\&\xrightarrow{k\to\infty}0.\end{aligned}\tag{4.31}$$

On the event $\left\{\inf_{0\le s\le T}U_s^{(k),r}>C\right\}$, we have

$$U^r(f)=\sum_{i=1}^k f(U^{(i),r}).$$

Since $\left(U^{(1),r},\dots,U^{(k),r}\right)_r$ is tight, so is $\left(\sum_{i=1}^k f(U^{(i),r})\right)_r$ because of Lemma A.4.1 and the Continuous Mapping Theorem (see for example Theorem 2.7 in [Bil99]). Thus we find a compact set $\tilde K\subset D_{\mathbb{R}^+}[0,T)$ such that

$$\sup_{r>r_0}\mathbf{P}\left[\sum_{i=1}^k f(U^{(i),r})\notin\tilde K\right]<\frac{\epsilon}{2}$$

and because of (4.31) there is $r_0>0$ and $k\in\mathbb{N}$ such that

$$\begin{aligned}\sup_{r>r_0}\mathbf{P}\left[U^r(f)\notin\tilde K\right]&\le\sup_{r>r_0}\mathbf{P}\left[\left\{U^r(f)\notin\tilde K\right\}\cap\left\{\inf_{0\le s\le T}U_s^{(k),r}>C\right\}\right]\\&\quad+\sup_{r>r_0}\mathbf{P}\left[\inf_{0\le s\le T}U_s^{(k),r}\le C\right]\\&\le\sup_{r>r_0}\mathbf{P}\left[\sum_{i=1}^k f(U^{(i),r})\notin\tilde K\right]+\sup_{r>r_0}\mathbf{P}\left[\inf_{0\le s\le T}U_s^{(k),r}\le C\right]\\&\le\frac{\epsilon}{2}+\frac{\epsilon}{2}.\end{aligned}$$

□

4.3. The dynamics of the level system U^δ in the limit

In the typical two-step procedure for proving weak convergence of $X^{r,\delta}$ for $r \to \infty$, the second step would be to prove convergence of the finite dimensional distributions. We substitute the second step by the approach described in [JS03], where the limit is identified by its semimartingale characteristics (see below). To this end we use Theorem IX.2.4 in [JS03] (see Theorem A.4.2 in the appendix). Our candidate for the limit is given by the martingale problem (4.4), or to be more precise, it is $X^\delta = (U^{(1,\dots,k),\delta}, Y^\delta)$, where $U^{(1,\dots,k)}$ are the k lowest particles of the system given by (4.4) and Y is Feller's branching diffusion.

Here lies a problem for the characterization of $X^{r,\delta}$ and therefore the characterization of X^δ: While Y^r is the Galton-Watson-like process, characterized by the generator C^r_{BDjd}, we do not know the joint law of Y^r and $U^{(1,\dots,k),r}$. The lowest levels $U^{(1,\dots,k),r}$, representing the most persistent ancestors of the population, provide information on the mass density Y^r. Nevertheless our primary interest at this point is the level dynamics in the limit, and we do know the level dynamics in the r-th process, conditioned on Y^r. So one way to deal with the problem is to forgo the explicit characterization of the mass density coordinate, but use Y^r implicitly in the characterization of the level dynamics. The mass density process Y^r has the role of an auxiliary process that converges jointly with $U^{(1,\dots,k),r}$ and provides us with a way to write down the semimartingale characteristics of $U^{(1,\dots,k),r}$.

In Chapter 6 we construct models $\tilde{X}^r$ where we retain a little more information from the Poisson construction of the full level system and that allow for a joint characterization of the lowest levels and the mass density.

We fix the notation for the calculations below. Let $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a stochastic basis. Let $\mathcal{O}$ be the optional σ-field, i.e., the σ-field on $\Omega \times \mathbb{R}_+$ that is generated by all càdlàg adapted processes. Define

$$\tilde{\Omega} := \Omega \times \mathbb{R}_+ \times \mathbb{R}^k, \quad \tilde{\mathcal{O}} := \mathcal{O} \otimes \mathcal{B}(\mathbb{R}^k).$$

Let μ be a random measure in the sense of [JS03], Definition II.1.3. I.e., $\mu(\omega; \mathrm{d}t, \mathrm{d}x)$ is a nonnegative measure on $(\mathbb{R}_+ \times \mathbb{R}^k, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^k))$ for every $\omega \in \Omega$. Let W an $\tilde{\mathcal{O}}$-measurable function on $\tilde{\Omega}$. Define

$$W * \mu_t(\omega) := \int_{[0,t]\times\mathbb{R}^k} W(\omega; s, x)\, \mu(\omega; ds, dx)$$

if

$$\int_{[0,t]\times\mathbb{R}^k} |W(\omega; s, x)|\ \mu(\omega; ds, dx) < \infty.$$

A truncation function is a continuous bounded function that is the identity around 0.

We use $h(x) = (\tilde{h}(x_i))_i$, where

$$\tilde{h}(x) := \begin{cases} x & \text{for } |x| < 1 \\ x \cdot (2 - |x|) & \text{for } 1 < |x| \leq 2 \\ 0 & \text{for } 2 < |x| \,. \end{cases} \tag{4.32}$$

Consider a k-dimensional special semimartingale Z. Let $\Delta Z_t := Z_t - Z_{t-}$ be the process of the jumps of Z and define the empirical jump measure

$$\mu(\omega; dt, dx) := \sum_{s\colon\, \Delta Z_s(\omega) \neq 0} \delta_{(s, \Delta Z_s(\omega))}(dt, dx).$$

We write $\nu(\omega; dt, dx)$ for its predictable compensator (cf. [JS03], Theorem II.1.8).

Write

$$\begin{aligned} \hat{Z}(h)_t &:= \sum_{s \leq t} \Big[\Delta Z_s - h(\Delta Z_s) \Big] \text{ and} \\ Z(h)_t &:= Z_t - \hat{Z}(h)_t \end{aligned}$$

for the sum of big jumps up to time t and the process minus the big jumps respectively.

Let $\tilde{B}_t := Z_t - \sum_{s \leq t} \Delta Z_s$ be the process without jumps. Since $\Delta Z(h)_t = h(\Delta Z_t)$ we have

$$Z(h)_t = \tilde{B}_t + \Delta Z(h)_t = \tilde{B}_t + h * \mu_t.$$

Assume that the martingale part of Z is of the "pure jump"-type (cf. [JS03], Definition I.4.11) and that Z has no predictable jumps. We obtain then the canonical decomposition $Z(h) = M(h) + B(h)$ with martingale part

$$M(h)_t := h * (\mu - \nu)_t$$

and predictable finite variation part

$$B(h)_t = \tilde{B}_t + h * \nu_t. \tag{4.33}$$

We use the notion of semimartingale characteristics as introduced in Chapter II of [JS03]. The semimartingale Z is characterized by three processes $(B, \tilde{C}, \nu)$ where B is the predictable bounded variation part of the canonical semimartingale decomposition, $\tilde{C}$ is the covariation of the martingale part and ν is the predictable compensator of the empirical jump measure. Note that $B = B(h)$ and $\tilde{C} = \tilde{C}(h)$ depend on the truncation function h. In [JS03] the identifier C is reserved for the covariation of the continuous martingale part (which is independent of h), whereas $\tilde{C}$ is called "modified" third characteristic. We keep this notation to provide consistency with the literature.

If the continuous martingale part of Z is zero and if Z has no predictable jumps, the modified second characteristic $\tilde{C}^{ij}(h)$ is defined in terms of the third characteristic ν,

$$\begin{aligned}\tilde{C}^{ij}(h)_t &= \langle M^i(h), M^j(h)\rangle_t \\ &= \langle h^i * (\mu-\nu), h^j * (\mu-\nu)\rangle_t \\ &= C^{ij}_t + h^i h^j * \nu_t + \sum_{s\le t} \Delta B^i_s \Delta B^j_s \\ &= h^i h^j * \nu_t.\end{aligned} \tag{4.34}$$

4.3.1. The mass density coordinate $Y^{\delta,r}$ in the limit $r\to\infty$

Before we turn to the dynamics of the low levels, we attend to the mass density coordinate.

Recall that we denote by "$\Rightarrow$" weak convergence of processes in Skorohod topology. We will use the following lemma frequently without notice.

Lemma 4.3.1. *Let $(Z^r)_r$ be tight in $D_\mathbb{R}[0,\infty)$ and $a_r \xrightarrow{r\to\infty} 0$. Then*

$$a_r Z^r \Rightarrow 0 \quad \textit{for } r\to\infty.$$

Proof. The family $(a_r Z^r)_r$ is tight in $D_\mathbb{R}[0,\infty)$ and by compact containment of Z^r

$$a_r Z^r \xrightarrow{\text{f.d.d.}} 0.$$

$\square$

Let $\delta > 0$ and let Y^δ be a limit point of $(Y^{r,\delta})_r$. We suppress the subsequence notation.

Theorem 4.3.2. *Let $\check{Y}$ be Feller's branching diffusion. I.e., $\check{Y}$ solves the SDE*

$$\mathrm{d}\check{Y}_t = \sqrt{b(2+c)\check{Y}_t}\,\mathrm{d}W_t, \tag{4.35}$$

W being Brownian motion. Let $\tau(\delta) := \inf\left\{t\ge 0:\ \check{Y}_t \le \delta\right\}$ and $\check{Y}^\delta_t := \check{Y}_{t\wedge\tau(\delta)}$. Then

$$Y^\delta \overset{d}{=} \check{Y}^\delta.$$

Proof. Assume $\frac{c}{r\delta} \le 1$. We drop the δ in our notation. The r-th processes Y^r are special semimartingales. Let $(B^{Y,r}, \tilde{C}^{Y,r}, \nu^{Y,r})$ be the characteristics of Y^r. Since Y^r is piecewise constant and Y^r has no predictable jumps, the first and second characteristics can be written in terms of the jump measure $\nu^{Y,r}$ (cf. (4.33) and (4.34)).

Birth events happen at instantaneous rate $r^2 b Y^r_t$ and in case of such an event one particles with mass $\frac{1}{r}$ is born. When a death event happens at time t, all particles in a "strip of death" $[r - c/Y_t^r,\, r]$ are killed. Since, given Y_t^r, the particles are uniformly distributed on $[0, r]$, the number of particles killed is binomially distributed with parameters $n_t^r := rY_t^r$ and $p_t^r := \frac{c}{rY_t^r}$. The death events happen at rate $\frac{r^2 bY_t^r}{c}$. The distribution of a jump at time t is

$$\mathrm{B}_{r,Y_t^r}(\mathrm{d}x) := \sum_{i=0}^{n_t^r} \binom{n_t^r}{i} (p_t^r)^i (1 - p_t^r)^{n_t^r - i} \delta_{-\frac{i}{r}}(\mathrm{d}x).$$

The compensator of the jump measure is

$$\nu^{Y,r}(\omega; \mathrm{d}t, \mathrm{d}x) = r^2 bY_t^r \ \mathrm{d}t \cdot \delta_{\frac{1}{r}}(\mathrm{d}x) + \frac{r^2 bY_t^r}{c} \mathrm{d}t \cdot \mathrm{B}_{r,Y_t^r}(\mathrm{d}x).$$

(Note that $\nu^{Y,r}$ is random; we omit the notation of ω below.) Since Y^r is piecewise constant, we have

$$\tilde{C}_t^{Y,r} = h^2 * \nu_t^{Y,r}$$

and

$$B^{Y,r}(h)_t = h * \nu_t^{Y,r}.$$

Let $(B^Y, \tilde{C}^Y, \nu^Y)$ be the semimartingale characteristics of Feller's branching diffusion (4.35),

$$B_t^Y = 0, \quad \tilde{C}_t^Y = \int_0^t \mathrm{d}s\, b\,(2 + c)\, Y_s, \quad \nu^Y(\mathrm{d}t, \mathrm{d}x) = 0.$$

We apply Theorem IX.2.4 in [JS03] (see also Theorem A.4.2 in the appendix). It is enough to show

$$(Y^r, B^{Y,r}, \tilde{C}^{Y,r}) \Rightarrow (Y, B^Y, \tilde{C}^Y)$$

and

$$(Y^r, g * \nu^{Y,r}) \Rightarrow (Y, g * \nu^Y)$$

for all nonnegative continuous bounded functions $g : \mathbb{R} \to \mathbb{R}$ which are 0 around 0. Note that we are talking about weak convergence in $D_{\mathbb{R}^d}[0, \infty)$ with $d = 3$ or $d = 2$ respectively.

First we address B^Y. Let

$$(M_3^r)_t := n_t^r(n_t^r - 1)(n_t^r - 2)(p_t^r)^3 + 3n_t^r(n_t^r - 1)(p_t^r)^2 + n_t^r p_t^r$$

the third moment of Bin_{n^r,p^r}. By the Continuous Mapping Theorem (c.f. Theorem 13.25 in [Kle06]) $(M_3^r)_r$, $(Y^r M_3^r)_r$, etc. converge. We have

$$\begin{aligned}
\frac{r^2 bY_t^r}{c} \int \mathrm{B}_{r,Y_t^r}(\mathrm{d}x) h(x) &= \frac{r^2 bY_t^r}{c} \int \mathrm{Bin}_{n_t^r,p_t^r}(\mathrm{d}x)\ h\left(-\frac{x}{r}\right) \\
&= -\frac{rbY_t^r}{c} \int \mathrm{Bin}_{n_t^r,p_t^r}(\mathrm{d}x)\ x + R_t^r \\
&= -rbY_t^r + R_t^r,
\end{aligned}$$

where

$$\begin{aligned}
R^r &= \frac{r^2 bY^r}{c}\int \mathrm{Bin}_{n^r,p^r}(\mathrm{d}x)\ \left(\frac{x}{r} + h\left(-\frac{x}{r}\right)\right)\\
&= \frac{r^2 bY^r}{c}\int \mathrm{Bin}_{n^r,p^r}(\mathrm{d}x)\ \mathbb{1}_{(r,\infty)}(x)\left(\frac{x}{r} + h\left(-\frac{x}{r}\right)\right)\\
&\le \frac{r^2 bY^r}{c}\int \mathrm{Bin}_{n^r,p^r}(\mathrm{d}x)\ \mathbb{1}_{(r,\infty)}(x)\frac{x}{r}\\
&\le \frac{rn^r bY^r}{c}\int \mathrm{Bin}_{n^r,p^r}(\mathrm{d}x)\ \mathbb{1}_{(r,\infty)}(x)\\
&\le \frac{n^r bY^r}{r^2 c}M_3^r\\
&\Rightarrow 0
\end{aligned} \tag{4.36}$$

using Markov's inequality for third moments of the (conditional) binomial distribution and Lemma 4.3.1. Since the map $x \mapsto \int_0^\cdot x(s)\,\mathrm{d}s$ is a Skorohod-continuous functional (Lemma A.4.1), we obtain

$$\begin{aligned}
B^{Y,r} &= h * \nu^{Y,r}\\
&= \int_0^\cdot \mathrm{d}s\ \left(br^2 Y_s^r h\left(\frac{1}{r}\right) - rbY_s^r + R_s^r\right)\\
&\Rightarrow 0.
\end{aligned}$$

We turn to the quadratic Variation $\tilde{C}^Y$. We have

$$\begin{aligned}
\frac{r^2 bY_t^r}{c}\int \mathrm{B}_{r,Y_t^r}(\mathrm{d}x)\ h(x)^2 &= \frac{r^2 bY_t^r}{c}\int \mathrm{Bin}_{n_t^r,p_t^r}(\mathrm{d}x)\ h\left(\frac{x}{r}\right)^2\\
&= \frac{bY_t^r}{c}\int \mathrm{Bin}_{n_t^r,p_t^r}(\mathrm{d}x)\ x^2 + R_t^r\\
&= bY_t^r\left(1 + c - \frac{c}{rY_t^r}\right) + R_t^r,
\end{aligned}$$

where

$$\begin{aligned}
|R^r| &\le \frac{r^2 bY_s^r}{c}\int \mathrm{Bin}_{n^r,p^r}(\mathrm{d}x)\ \left|h\left(-\frac{x}{r}\right)^2 - \left(\frac{x}{r}\right)^2\right|\\
&= \frac{r^2 bY_s^r}{c}\int \mathrm{Bin}_{n^r,p^r}(\mathrm{d}x)\ \mathbb{1}_{(r,\infty)}(x)\left|h\left(-\frac{x}{r}\right)^2 - \left(\frac{x}{r}\right)^2\right|\\
&\le \frac{bY_s^r}{c}\int \mathrm{Bin}_{n^r,p^r}(\mathrm{d}x)\ \mathbb{1}_{(r,\infty)}(x)x^2\\
&\le \frac{(n^r)^2\, bY_s^r}{c}\int \mathrm{Bin}_{\hat{n}^r,\hat{p}^r}(\mathrm{d}x)\ \mathbb{1}_{(r,\infty)}(x)\\
&\le \frac{(n^r)^2\, bY_s^r}{r^3 c}M_3^r\\
&\Rightarrow 0
\end{aligned}$$

with Markov's inequality for third moments of the conditional binomial distribution. By Lemma A.4.1, we obtain

$$\begin{aligned}\tilde{C}^{Y,r} &= h^2 * \nu^{Y,r} \\ &= \int_0^{\cdot} \mathrm{d}s \left(br^2 Y_s^r h\left(\frac{1}{r}\right)^2 + bY_s^r\left(1 + c - \frac{c}{rY_s^r}\right) + R_s^r\right) \\ &\Rightarrow \int_0^{\cdot} \mathrm{d}s\, b\,(2+c)\,Y_s.\end{aligned}$$

We turn to the compensator of the jump measure. Let $g : \mathbb{R} \to [0,\infty)$ be a nonnegative continuous bounded function that is 0 in a neighbourhood of 0. Let $C_g, \epsilon_g > 0$ such that $g \le C_g$ and $g(x) = 0$ for $|x| < \epsilon_g$. Similarly as above, we obtain

$$\begin{aligned}\frac{r^2 bY^r}{c} \cdot \int \mathrm{B}_{r,Y^r}(\mathrm{d}x)\, g(x) &= \frac{r^2 bY^r}{c} \int \mathrm{Bin}_{n^r,p^r}(\mathrm{d}x)\, g\left(-\frac{x}{r}\right) \\ &\le \frac{r^2 bC_g Y^r}{c} \int \mathrm{Bin}_{n^r,p^r}(\mathrm{d}x)\, \mathbb{1}_{(r\epsilon_g,\infty)}(x) \\ &\le \frac{bC_g Y^r}{r\epsilon_g^3 c} M_3^r \\ &\Rightarrow 0.\end{aligned} \tag{4.37}$$

By Lemma A.4.1, we have

$$\begin{aligned}g * \nu^{Y,r} &= \int_0^{\cdot} \mathrm{d}s \left(r^2 bY_s^r g\left(\frac{1}{r}\right) + \frac{r^2 bY_s^r}{c} \cdot \int \mathrm{B}_{r,Y_s^r}(\mathrm{d}x)\, g(x)\right) \\ &\Rightarrow 0.\end{aligned} \tag{4.38}$$

By Lemma A.3.3, there is a subsequence $(r') \subset (r)$ such that for $r' \to \infty$

$$\begin{aligned}(Y^{r'}, B^{Y,r'}, \tilde{C}^{Y,r'}) &\Rightarrow (Y, B^Y, \tilde{C}^Y) \quad \text{and} \\ (Y^{r'}, g * \nu^{Y,r'}) &\Rightarrow (Y, g * \nu^Y),\end{aligned}$$

and $(Y^r)_r$ converges to the same limit as the subsequence $(Y^{r'})_{r'}$. □

4.3.2. The low levels $U^{(1,\dots,k),r,\delta}$ in the limit $r \to \infty$

We characterize the k lowest levels in the limit, $U^{(1,\dots,k),\delta}$, as a semimartingale via its semimartingale characteristics. The procedure is similar to the proof of Theorem 4.3.2, but the mass density processes $(Y^r)_r$ act as auxiliary processes that allow us to write down the characteristics of $(U^{(1,\dots,k),r,\delta})_r$. Write 0_q for $(0,\dots,0) \in \mathbb{R}^q$.

Theorem 4.3.3. *Let $X^\delta = (U^{(1,\dots,k),\delta}, Y^\delta)$ be the limit of a convergent subsequence of the family $\left(X^{r,\delta}\right)_r = \left(U^{(1,\dots,k),r,\delta}, Y^{r,\delta}\right)_r$. Then $U^{(1,\dots,k),\delta}$ is a stopped semimartingale* *with characteristic triplet $(B, \tilde{C}, \nu)$. Define $\tau := \inf\{s : Y_s \leq \delta\}$. The jump measure of $U^{(1,\dots,k),\delta}$ is*

$$\nu(\mathrm{d}t, \mathrm{d}x) = \mathbb{1}_{\{t<\tau\}} \sum_{q=2}^{k} 2b(q-1) \left(U_t^{(q),\delta} - U_t^{(q-1),\delta}\right) \mathrm{d}t \cdot \delta_{0_{q-1}}(\mathrm{d}x_1, \dots, \mathrm{d}x_{q-1})$$
$$\times \operatorname{Unif}_{[U_t^{(q-1),\delta} - U_t^{(q),\delta}, 0]}(\mathrm{d}x_q) \cdot \delta_{U_t^{(q,\dots,k-1),\delta} - U_t^{(q+1,\dots,k),\delta}}(\mathrm{d}x_{q+1}, \dots, \mathrm{d}x_k).$$

For the bounded variation part we have

$$B_t^i = \int_0^{t\wedge\tau} \mathrm{d}s \, \left(b\left(U_s^{(i),\delta}\right)^2 + \frac{bc}{Y_s} U_s^{(i),\delta}\right) + h^i * \nu_t \quad \text{for } i = 1, \dots, k,$$

and for the covariation of the martingale part we have

$$\tilde{C}_t^{ij} = \int_0^{t\wedge\tau} \mathrm{d}s \, \frac{bc}{Y_s} U_s^{(i),\delta} U_s^{(j),\delta} + h^i h^j * \nu_t \quad \text{for } i, j \in \{1, \dots, k\}.$$

In other words, between jumps and up to time τ the k lowest levels are a system of perfectly correlated, conditional geometric Brownian motions. $U^{(1,\dots,k),\delta}$ solves

$$\mathrm{d}U_t^{(i),\delta} = \left(b\left(U_t^{(i),\delta}\right)^2 + \frac{bc}{Y_t} U_t^{(i),\delta}\right) \mathrm{d}t + \sqrt{\frac{bc}{Y_t}} \cdot U_t^{(i),\delta} \, \mathrm{d}W_s \quad \text{for } i = 1, \dots, k,$$

with W being the same Brownian motion for all coordinates i.

Remark 4.3.4. *The path of a single particle of the full system is continuous. The jumps of $U^{(1,\dots,k),r,\delta}$ are attributed to the fact that we trace the k lowest particles and newborn particles with low enough levels have to be pigeon-holed.*

We will use the following lemma several times.

Lemma 4.3.5.

$$U^{(1,\dots,k),r,\delta} \frac{c}{rY^{r,\delta} - c} \Rightarrow 0,$$

for $r \to \infty$.

Proof. Tightness follows from tightness of $(X^{r,\delta})_r$. Convergence of the finite dimensional distributions follows from $Y_t^r \geq \delta$ and $U_t^{r,\delta} \sim \alpha_r(rY^{r,\delta}, \cdot)$ given Y_t^r for all $t > 0$. □

For readability's sake, we suppress the subsequence notation below.

Proof of Theorem 4.3.3. Recall that $\tau^r = \inf\{s \geq 0 : Y_s^r \leq \delta\}$. As we did before, we drop the δ in the notation; we write X^r, B^r, etc. instead of $X^{r,\delta}, B^{r,\delta}$, etc. Let $(B^r, \tilde{C}^r, \nu^r)$ be the semimartingale characteristics of $U^{(1,\dots,k),r}$. Note that the continuous martingale part of $U^{(1,\dots,k),r}$ is zero and $U^{(1,\dots,k),r}$ has no predictable jumps. We constitute ν^r as a sum

$$\nu^r(\omega; dt, dx) = \nu^{D,r}(\omega; dt, dx) + \nu^{B,r}(\omega; dt, dx),$$

where $\nu^{D,r}$ compensates the jumps induced by death events, $\nu^{B,r}$ compensates the jumps induced by births among the lowest k levels. Note that the compensator ν^r is random since the jumps depend on $U^{(1,\ldots,k),r}$ and the auxiliary process Y^r, but the ω is omitted below.

Death jumps happen at time t with instantaneous rate $\frac{r^2bY_t^r}{c}$ and the height of the jump for a particle with level u^i is $u^i \cdot \left(\frac{r}{r-c/Y_{t-}^r} - 1\right) = u^i \cdot \frac{c}{rY_{t-}^r - c}$. The compensator for these jumps is

$$\nu^{D,r}(\mathrm{d}t, \mathrm{d}x) = \mathbb{1}_{\{t<\tau^r\}} \frac{r^2bY_t^r}{c} \,\mathrm{d}t \cdot \delta_{U_t^{(1,\ldots,k),r} \cdot \frac{c}{rY_t^r - c}}(\mathrm{d}x).$$

In the full level system, every particle u^i gives birth with rate $2b(r-u^i)$. The process $U^{(1,\ldots,k),r}$ displays only the birth of particles that are among the k lowest. Offspring with rank q can only have a parent with rank smaller than q. Since the offspring's level is uniform above the parent's level, offspring is born into the rank q at rate

$$2b(q-1)\left(U_t^{(q),r} - U_t^{(q-1),r}\right).$$

If such a birth happens at time t, the q-th level $U_t^{(q),r}$ is the level of the newborn. Thus $U^{(q),r}$ performs a uniformly distributed downward jump,

$$\Delta U_t^{(q),r} \sim \mathrm{Unif}_{[U_{t-}^{(q-1),r} - U_{t-}^{(q),r}, 0]}.$$

The coordinates $U^{(1,\ldots,q-1),r}$ do not jump if a rank q particle is born and the coordinates $U^{(q+1,\ldots,k),r}$ jump down as the particles' ranks shift up,

$$\begin{aligned}
\Delta U_t^{(1,\ldots,q-1),r} &= 0_{q-1},\\
\Delta U_t^{(q+1,\ldots,k),r} &= U_{t-}^{(q,\ldots,k-1),r} - U_{t-}^{(q+1,\ldots,k),r}.
\end{aligned}$$

The compensator for the birth jumps is

$$\begin{aligned}
\nu^{B,r}(\mathrm{d}t, \mathrm{d}x) = \mathbb{1}_{\{t<\tau^r\}} \sum_{q=2}^{k} 2b(q-1)\left(U_t^{(q),r} - U_t^{(q-1),r}\right) \mathrm{d}t \cdot \delta_{0_{q-1}}(\mathrm{d}x_1, \ldots, \mathrm{d}x_{q-1})\\
\times \mathrm{Unif}_{[U_t^{(q-1),r} - U_t^{(q),r}, 0]}(\mathrm{d}x_q) \cdot \delta_{U_t^{(q,\ldots,k-1),r} - U_t^{(q+1,\ldots,k),r}}(\mathrm{d}x_{q+1}, \ldots, \mathrm{d}x_k).
\end{aligned}$$

We apply Theorem IX.2.4 and Remark IX.2.21 in [JS03] (cf. Theorem A.4.2), i.e., we take a convergent subsequence $(U^{(1,\ldots,k),r})_r$ with limit $U^{(1,\ldots,k)}$ and show

$$(U^{(1,\ldots,k),r}, Y^r, B^r, \tilde{C}^r) \Rightarrow (U^{(1,\ldots,k)}, Y, B, \tilde{C}) \quad \text{and} \tag{4.39}$$

$$(U^{(1,\ldots,k),r}, Y^r, g * \nu^r) \Rightarrow (U^{(1,\ldots,k)}, Y, g * \nu) \tag{4.40}$$

for all nonnegative continuous bounded functions $g : \mathbb{R}^k \to [0,\infty)$ which are 0 around 0.

Below we will use the Continuous Mapping Theorem repeatedly without mentioning it explicitly every time (c.f. Theorem 13.25 in [Kle06]). For instance, with X^r converging, so does $X^{i,r} \cdot X^{k+1,r} = U^{(i),r} \cdot Y^r$, since the application of a continuous function is Skorohod-continuous (see Lemma A.4.1), etc.

Use Skorohod's Representation Theorem (see for example Theorem 4.30 in [Kal02]) and the continuity of Y to apprehend $\tau^r \xrightarrow{\text{stoch.}} \tau$. This implies $\mathbb{1}_{\{s<\tau^r\}} \Rightarrow \mathbb{1}_{\{s<\tau\}}$.

Convergence of the third characteristics. Let us start with Condition (4.40), the convergence of jump measures. For the births among the lowest levels, we obtain

$$\begin{aligned}
g * \nu^{B,r} &= \sum_{q=2}^{k} \int_0^{\cdot} \mathrm{d}s\ \mathbb{1}_{\{s<\tau^r\}} 2b(q-1) \left(U_s^{(q),r} - U_s^{(q-1),r}\right) \\
&\quad \times \frac{1}{U_s^{(q),r} - U_s^{(q-1),r}} \int_{U_s^{(q-1),r} - U_s^{(q),r}}^{0} \mathrm{d}z\ g\left(0_{q-1}, z, U_s^{(q,\dots,k-1),r} - U_s^{(q+1,\dots,k),r}\right) \\
&\Rightarrow \sum_{q=2}^{k} \int_0^{\cdot} \mathrm{d}s\ \mathbb{1}_{\{s<\tau\}} 2b(q-1) \left(U_s^{(q)} - U_s^{(q-1)}\right) \\
&\quad \times \frac{1}{U_s^{(q)} - U_s^{(q-1)}} \int_{U_s^{(q-1)} - U_s^{(q)}}^{0} \mathrm{d}z\ g\left(0_{q-1}, z, U_s^{(q,\dots,k-1)} - U_s^{(q+1,\dots,k)}\right) \\
&= g * \nu
\end{aligned} \tag{4.41}$$

because of Lemma A.4.1 and the Continuous Mapping Theorem.

By Lemma 4.3.5, we have

$$U^{(1,\dots,k),r} \frac{c}{rY^r - c} \Rightarrow 0$$

and therefore, since g is zero around zero,

$$\frac{r^2 b Y_t^r}{c} \cdot g\left(U_t^{(1,\dots,k),r} \frac{c}{rY_t^r - c}\right) \Rightarrow 0.$$

By Lemma A.4.1 and the Continuous Mapping Theorem we have

$$g * \nu^{D,r} = \int_0^{\cdot} \mathrm{d}s\ \mathbb{1}_{\{s<\tau^r\}} \frac{r^2 b Y_s^r}{c} g\left(U_s^{(1,\dots,k),r} \frac{c}{rY_s^r - c}\right) \Rightarrow 0. \tag{4.42}$$

Equations (4.41) and (4.42) give us $g * \nu^r \Rightarrow g * \nu$. Since $g * \nu$ and Y are continuous, we have joint convergence

$$(U^{(1,\dots,k),r}, Y^r, g * \nu^r) \Rightarrow (U^{(1,\dots,k)}, Y, g * \nu)$$

by Lemma A.3.3.

Convergence of the first characteristics. Since the particles in the r-th model move according to the differential equation $\dot{u} = bu^2 - bru$ between jumps, we have

$$\begin{aligned}\tilde{B}_t^{i,r} &:= U_t^{(i),r} - \sum_{s \le t} \Delta U_s^{(i),r} \\ &= \int_0^t \mathrm{d}s \, \mathbb{1}_{\{s<\tau^r\}} \left(b \left(U_s^{(i),r} \right)^2 - brU_s^{(i),r} \right)\end{aligned}$$

for $i = 1, \dots, k$. Define $H^{i,r} := \left\{ \left\| U^{(i),r} \frac{c}{rY^r - c} \right\|_\infty < 1 \right\}$. Since weak convergence in $D_{\mathbb{R}^+}[0,\infty)$ to a continuous path implies weak convergence with respect to the uniform topology, Lemma 4.3.5 gives us $\mathbf{P}\left[(H^{i,r})^c\right] \to 0$ for $r \to \infty$. Hence we have

$$\begin{aligned} r^2 \mathbb{1}(H^{i,r})^c &\Rightarrow 0 \quad \text{and} \\ r^2 Y^r \mathbb{1}(H^{i,r})^c &\Rightarrow 0, \end{aligned}$$

and thus

$$\begin{aligned} &\mathbb{1}(H^{i,r})^c \cdot \left(\frac{r^2 bY^r}{c} \tilde{h} \left(U^{(i),r} \frac{c}{rY^r - c} \right) - brU^{(i),r} \right) \\ &\qquad \le \mathbb{1}(H^{i,r})^c \cdot \left(\frac{r^2 bY^r}{c} + br^2 \right) \\ &\qquad \Rightarrow 0. \end{aligned}$$

We obtain

$$\begin{aligned} &\frac{r^2 bY^r}{c} \tilde{h} \left(U^{(i),r} \frac{c}{rY^r - c} \right) - brU^{(i),r} \\ &\qquad = \left(\mathbb{1}H^{i,r} + \mathbb{1}(H^{i,r})^c \right) \cdot \left(\frac{r^2 bY^r}{c} \tilde{h} \left(U^{(i),r} \frac{c}{rY^r - c} \right) - brU^{(i),r} \right) \\ &\qquad \Rightarrow \frac{bc}{Y} U^{(i)}. \end{aligned} \tag{4.43}$$

Hence we have for the bounded variation part

$$\begin{aligned} B^{i,r} &= \tilde{B}^{i,r} + h^i * \nu^r \\ &= \tilde{B}^{i,r} + h^i * \nu^{D,r} + h^i * \nu^{B,r} \\ &= \int_0^\cdot \mathrm{d}s \, \mathbb{1}_{\{s<\tau^r\}} \left(b \left(U_s^{(i),r} \right)^2 - brU_s^{(i),r} + \frac{r^2 bY_s^r}{c} \tilde{h} \left(U_s^{(i),r} \frac{c}{rY_s^r - c} \right) \right) \\ &\quad + h^i * \nu^{B,r} \\ &\Rightarrow \int_0^\cdot \mathrm{d}s \, \mathbb{1}_{\{s<\tau\}} \left(b \left(U_s^{(i)} \right)^2 + \frac{bc}{Y_s} U_s^{(i)} \right) + h^i * \nu. \end{aligned}$$

Convergence of the second characteristics. Finally, we address the covariation $\tilde{C}$. We have

$$\tilde{C}_t^{ij,r} = h^i h^j * \nu_t^r = h^i h^j * (\nu^{D,r} + \nu^{B,r})_t.$$

Again, using Lemma 4.3.5 and Lemma A.4.1,

$$\begin{aligned}
h^i h^j * \nu^{D,r} &= \int_0^\cdot \mathrm{d}s\, \mathbb{1}_{\{s<\tau^r\}} \frac{r^2 b Y_s^r}{c} \tilde{h}\left(U_s^{(i),r} \frac{c}{rY_s^r - c}\right) \cdot \tilde{h}\left(U_s^{(j),r} \frac{c}{rY_s^r - c}\right) \\
&= \int_0^\cdot \mathrm{d}s\, \mathbb{1}_{\{s<\tau^r\}} \frac{r^2 b Y_s^r}{c} \Bigg(\mathbb{1}_{H^{i,r}} \mathbb{1}_{H^{j,r}} \cdot \frac{c^2}{(rY_s^r - c)^2} U_s^{(i),r} U_s^{(j),r} \\
&\qquad + (1 - \mathbb{1}_{H^{i,r}} \mathbb{1}_{H^{j,r}}) \cdot \tilde{h}\left(U_s^{(i),r} \frac{c}{rY_s^r - c}\right) \cdot \tilde{h}\left(U_s^{(j),r} \frac{c}{rY_s^r - c}\right) \Bigg) \\
&\Rightarrow \int_0^\cdot \mathrm{d}s\, \mathbb{1}_{\{s<\tau\}} \frac{bc}{Y_s} U_s^{(i)} U_s^{(j)}.
\end{aligned} \tag{4.44}$$

Furthermore, we have

$$h^i h^j * \nu^{B,r} \Rightarrow h^i h^j * \nu.$$

Lemma A.3.3 gives us the simultaneous convergence

$$(U^{(1,\dots,k),r}, Y^r, B^r, \tilde{C}^r) \Rightarrow (U^{(1,\dots,k)}, Y, B, \tilde{C}).$$

□

5. Lookdown-construction of symbiotic branching diffusions with positive correlation

In this chapter we finalize the construction of a level representation of a symbiotic diffusion. Recall that a symbiotic diffusion is a solution of the system

$$\begin{cases} \mathrm{d}Y_t &= \sqrt{bY_tZ_t}\,\mathrm{d}W_t^1 \\ \mathrm{d}Z_t &= \sqrt{bY_tZ_t}\,\mathrm{d}W_t^2, \end{cases} \tag{5.1}$$

where W^1 and W^2 are correlated Brownian motions with constant correlation coefficient ρ (i.e. $\langle W^1, W^2\rangle_t = \rho \cdot t$). In our case we will construct a system with $0 < \rho < 1$. We recapitulate briefly, why the lookdown construction of Feller's branching diffusion with jump-induced deaths in Chapter 4 provides us with a handle that enables the coupling of two subpopulations: The death mechanism is not intrinsic in the system's deterministic motion. On the r-th stage it is triggered by an external Poisson process. In the continuous mass limit this becomes manifest in the geometric Brownian motion of the levels. Heuristically, a series of time changes as in the construction of the mutually catalytic system in Chapter 2 and coupling the noise that drives the level motion should lead to a symbiotic level system. But these heuristics hold a problem: The geometric Brownian motion of the levels arises in the weak construction of the level system in Chapter 4. So we cannot simply assume that the noise that drives the level motion in two separate level representations is coupled. Instead the noise coupling has to be established in the discrete mass models. Hence we take a direct approach, where we start with a discrete mass system similar to those in Chapter 3 and then pass to a weak limit. We may trace back most of the technical computations to the implementations in Chapter 4.

5.1. A symbiotic birth-death process with jump induced deaths

We are concerned with level representations of discrete mass systems. The level systems we construct in this section take values in $\mathcal{S}_{[0,r]} \times \mathcal{S}_{[0,r]}$, where $\mathcal{S}_{[0,r]}$ is the space of locally finite counting measures on $[0, r]$.

Let $c > 0$ and $0 \leq \eta \leq 1$. Recall the definition of the (corrected) jump factor from

Section 4.1,

$$\varphi(u) := \mathbb{1}_{(c,\infty)}(u([0,r))) \cdot \frac{1}{1-c/u([0,r))} + \mathbb{1}_{[0,c]}(u([0,r))) \cdot \frac{r}{\min_i u_i},$$

and denote by $m := u([0,r))$ and $n := v([0,r))$ the masses of u and v. We consider the martingale problem

$$\begin{aligned}
A^r_{sBDjd} f(u,v) &= f(u,v) \left(\sum_{i=1}^m \left[\frac{2bn}{r} \int_{u_i}^r (g_1(x)-1)\, \mathrm{d}x + \left(\frac{bn}{r} u_i^2 - bnu_i \right) \frac{g_1'(u_i)}{g_1(u_i)} \right] \right. \\
&\quad \left. + \sum_{j=1}^n \left[\frac{2bm}{r} \int_{v_j}^r (g_2(x)-1)\, \mathrm{d}x + \left(\frac{bm}{r} v_j^2 - bmv_j \right) \frac{g_2'(v_j)}{g_2(v_j)} \right] \right) \\
&\quad + (1-\eta) \frac{bmn}{c} \left(f_2(v) \prod_{i=1}^m g_1\left(\varphi(u) \cdot u_i\right) - f(u,v) \right) \\
&\quad + (1-\eta) \frac{bmn}{c} \left(f_1(u) \prod_{j=1}^n g_2\left(\varphi(v) \cdot v_j\right) - f(u,v) \right) \\
&\quad + \eta \frac{bmn}{c} \left(\prod_{i=1}^m g_1\left(\varphi(u) \cdot u_i\right) \prod_{j=1}^n g_2\left(\varphi(v) \cdot v_j\right) - f(u,v) \right), \\
\mathcal{D}^r_{sBDjd} &:= \left\{ f(u,v) := f_1(u) f_2(v) := \prod_{i=1}^m g_1(u_i) \prod_{j=1}^n g_2(v_j) \;:\; 0 \le g_1, g_2 \le 1 \in C^2(\mathbb{R}), \right. \\
&\qquad \left. g_1(x) = g_2(x) = 1 \text{ for } x \ge r \right\}.
\end{aligned}$$

The label *sBDjd* stands for "symbiotic birth-death process with jump induced deaths". The model consists of two populations that perform the mutually catalytic birth mechanism from Section 3.1.2 on the one hand. On the other hand the model exhibits a death mechanism, that is similar to the building block "Positively correlated symbiotic deaths" in Section 3.1.4 but kills particles in a "strip of death", similar to the model in Section 4.1. The births in both subpopulations are triggered by independent Poisson processes. The death events are partially coupled.

Recall the definition

$$\alpha_r(m,n;\mathrm{d}u,\mathrm{d}v) := \alpha_r(m,\mathrm{d}u) \otimes \alpha_r(n,\mathrm{d}u),$$

where $\alpha_r(m,\cdot)$ is the distribution of a Poisson point process on $[0,r]$, conditioned to have mass m.

Theorem 5.1.1. *Let $\mu_0 \in \mathcal{M}_1(\mathbb{N}_0 \times \mathbb{N}_0)$ and define $\nu_0 := \int \alpha_r(m,n;\,\cdot\,)\,\mu_0(\mathrm{d}m,\mathrm{d}n)$. There exists an unique solution (U^r, V^r) of the $D_{\mathcal{S}_{[0,r]}\times\mathcal{S}_{[0,r]}}[0,\infty)$-martingale problem for (A^r_{sBDjd}, ν_0). This solution has the property that $(M^r, N^r) := (U([0,r)), V([0,r)))$*

is a solution of the $D_{\mathbb{N}_0\times\mathbb{N}_0}[0,\infty)$-martingale problem for (C^r_{sBDjd},μ_0), where

$$
\begin{aligned}
C^r_{sBDjd}\hat f(m,n) &= bmn\left(\hat f(m+1,n)-\hat f(m,n)\right)+bnm\left(\hat f(m,n+1)-\hat f(m,n)\right)\\
&+\mathbb{1}_{(c,\infty)}(m)\cdot(1-\eta)\frac{bmn}{c}\left(\hat f_2(n)\sum_{i=0}^m\binom{m}{i}\left(\frac{c}{m}\right)^i\left(1-\frac{c}{m}\right)^{m-i}\hat f_1(m-i)-\hat f(m,n)\right)\\
&+\mathbb{1}_{[0,c]}(m)\cdot(1-\eta)\frac{bmn}{c}\left(\hat f(0,n)-\hat f(m,n)\right)\\
&+\mathbb{1}_{(c,\infty)}(n)\cdot(1-\eta)\frac{bmn}{c}\left(\hat f_1(m)\sum_{j=0}^n\binom{n}{j}\left(\frac{c}{n}\right)^j\left(1-\frac{c}{n}\right)^{n-j}\hat f_2(n-j)-\hat f(m,n)\right)\\
&+\mathbb{1}_{[0,c]}(n)\cdot(1-\eta)\frac{bmn}{c}\left(\hat f(m,0)-\hat f(m,n)\right)\\
&+\eta\frac{bmn}{c}\Bigg[\left(\mathbb{1}_{(c,\infty)}(m)\cdot\sum_{i=0}^m\binom{m}{i}\left(\frac{c}{m}\right)^i\left(1-\frac{c}{m}\right)^{m-i}\hat f_1(m-i)+\mathbb{1}_{[0,c]}(m)\cdot\hat f_1(0)\right)\\
&\times\left(\mathbb{1}_{(c,\infty)}(n)\cdot\sum_{j=0}^n\binom{n}{j}\left(\frac{c}{n}\right)^j\left(1-\frac{c}{n}\right)^{n-j}\hat f_2(n-j)+\mathbb{1}_{[0,c]}(n)\cdot\hat f_2(0)\right)-\hat f(m,n)\Bigg].
\end{aligned}
$$

For every $t\geq 0$ and $\Gamma\in\mathcal{B}(\mathcal{S}_{[0,r]}\times\mathcal{S}_{[0,r]})$, we have

$$\mathbf{P}\left[(U^r_t,V^r_t)\in\Gamma\;\middle|\;\mathcal{F}^{M^r,N^r}_t\right]=\alpha_r(M^r_t,N^r_t;\Gamma).$$

Remark 5.1.2. *The distinction of the cases "$m\leq c$", "$m>c$", etc. makes the generator C^r_{sBDjd} rather unwieldy. Since in the following sections we stop the process when one of the mass densities $\frac{M^r}{r}$ or $\frac{N^r}{r}$ falls below some threshold $\delta>0$, and since we are interested in the limit for $r\to\infty$, we may assume "$m>c$ and $n>c$" for our purposes.*

Proof. The proof goes along the same lines as the proofs of Theorems 3.2.2 and 4.1.1. In order to apply the Markov Mapping Theorem, we check the intertwining relation

$$\int\alpha_r(m,n;\mathrm{d}u,\mathrm{d}v)\;A^r_{sBDjd}f(u,v)=C^r_{sBDjd}\int\alpha_r(m,n;\mathrm{d}u,\mathrm{d}v)\;f(u,v).\tag{5.2}$$

Recall the definitions

$$e^{-\lambda_i}:=\frac{1}{r}\int_0^r g_i(z)\,dz,\quad i=1,2,$$

and

$$\hat f(m,n):=\int\alpha_r(m,n;\mathrm{d}u,\mathrm{d}v)\;f(u,v)=e^{-\lambda_1 m}e^{-\lambda_2 n}.$$

By Calculation 3.1.2 for Model 3.1.2 we have

$$
\begin{aligned}
&\int\alpha_r(m,n;\mathrm{d}u,\mathrm{d}v)\;f(u,v)\left(\sum_{i=1}^m\left[\frac{2bn}{r}\int_{u_i}^r(g_1(x)-1)\;\mathrm{d}x+\left(\frac{bn}{r}u_i^2-bnu_i\right)\frac{g_1'(u_i)}{g_1(u_i)}\right]\right.\\
&\qquad\qquad\left.+\sum_{j=1}^n\left[\frac{2bm}{r}\int_{v_j}^r(g_2(x)-1)\;\mathrm{d}x+\left(\frac{bm}{r}v_j^2-bmv_j\right)\frac{g_2'(v_j)}{g_2(v_j)}\right]\right)\\
&\quad=bmn\left(\hat f(m+1,n)-\hat f(m,n)\right)+bmn\left(\hat f(m,n+1)-\hat f(m,n)\right).
\end{aligned}
$$

Assume $m > c$, then $\varphi(u) = \frac{1}{1-c/m}$. As in Equation (4.3) we obtain

$$\begin{aligned}\int \alpha_r(m,n;\mathrm{d}u,\mathrm{d}v)\,(1-\eta)\frac{bmn}{c}&\left(f_2(v)\prod_{i=1}^{m} g_1\left(\varphi(u)\cdot u_i\right) - f(u,v)\right)\\ &= (1-\eta)\frac{bmn}{c}\left(\hat{f}_2(n)\sum_{i=0}^{m}\binom{m}{i}\left(\frac{c}{m}\right)^i\left(1-\frac{c}{m}\right)^{m-i}\hat{f}_1(m-i) - \hat{f}(m,n)\right).\end{aligned}$$

If $m \le c$ we have

$$\begin{aligned}\int \alpha_r(m,n;\mathrm{d}u,\mathrm{d}v)\,(1-\eta)\frac{bmn}{c}&\left(f_2(v)\prod_{i=1}^{m} g_1\left(\varphi(u)\cdot u_i\right) - f(u,v)\right)\\ &= (1-\eta)\frac{bmn}{c}\left(\hat{f}(0,n) - \hat{f}(m,n)\right).\end{aligned}$$

Similar calculations for the remaining jump terms and cases give us the intertwining relation (5.2). The assertion of the theorem follows from the Markov Mapping Theorem A.5.1 with $\psi(u,v) := m\cdot n$ and $\gamma(u,v) := (u([0,r)), v([0,r)))$, where we use the notation of said theorem.

Similarly as before, uniqueness of the level system follows from uniqueness of the system that performs only the continuous motion (no death events, no birth events). By Theorem 4.10.3 in [EK86], we have uniqueness of the level dynamics until the first time either the M^r or N^r hits some fixed $\tilde{m}$. Letting $\tilde{m} \to \infty$, we obtain uniqueness of the whole process. □

5.2. Tightness

In order to establish tightness of the level representation we stop the process when one of the mass densities $Y^r := \frac{1}{r}M^r$ or $Z^r := \frac{1}{r}N^r$ hits or falls below some level $\delta > 0$. Define

$$\tau_\delta^r := \inf\{s \ge 0 : Y_s^r \wedge Z_s^r \le \delta\}$$

and denote by

$$\begin{aligned}U_t^{r,\delta} &:= U_{t\wedge\tau_\delta^r}^r,\\ Y_t^{r,\delta} &:= Y_{t\wedge\tau_\delta^r}^r\end{aligned}$$

and

$$\begin{aligned}V_t^{r,\delta} &:= V_{t\wedge\tau_\delta^r}^r,\\ Z_t^{r,\delta} &:= Z_{t\wedge\tau_\delta^r}^r\end{aligned}$$

the stopped processes.

In this section we proof tightness of the joint process $(U^{r,\delta}, V^{r,\delta}, Y^{r,\delta}, Z^{r,\delta})$. Similarly as in Section 4.2 we show $D_{\mathbb{R}^{2k+2}}[0,\infty)$-tightness of the family

$$\left(\bar{X}^{r,\delta}\right)_r := \left(U^{(1,\dots,k),r,\delta}, V^{(1,\dots,k),r,\delta}, Y^{r,\delta}, Z^{r,\delta}\right)_r$$

first, and then we lift the result to the measure valued case.

The marginals $U^{r,\delta}$ and $V^{r,\delta}$ can be transformed to level representations that solve the martingale problem for A^r_{BDjd} in Section 4.1 (see (4.1)) via a time change that has an inverse with increments that are bounded in probability. This allows us to trace back the tightness of $\bar{X}^{r,\delta}$ to the tightness of the family $X^{r,\delta}$ in Section 4.2.

Theorem 5.2.1. *Assume* $\sup_r \mathbf{E}\left[Y_0^r\right], \sup_r \mathbf{E}\left[Z_0^r\right] < \infty$. *The family* $\left(\bar{X}^{r,\delta}\right)_r$ *is tight in* $D_{\mathbb{R}^{2k+2}}[0,\infty)$.

Proof. Define the random time changes

$$\sigma_t^{U,r} := \inf\left\{s \geq 0 \;:\; \int_0^s Z_h^{r,\delta}\ \mathrm{d}h \geq t\right\},$$
$$\sigma_t^{V,r} := \inf\left\{s \geq 0 \;:\; \int_0^s Y_h^{r,\delta}\ \mathrm{d}h \geq t\right\}.$$

Time changes $\sigma^{U,r}$ and $\sigma^{V,r}$ solve the equations

$$\sigma_t^{U,r} = \int_0^t \frac{1}{Z^{r,\delta}_{\sigma_s^{U,r}}}\ \mathrm{d}s,$$
$$\sigma_t^{V,r} = \int_0^t \frac{1}{Y^{r,\delta}_{\sigma_s^{V,r}}}\ \mathrm{d}s,$$

hence they are both bounded by $\frac{t}{\delta}$. Observe that, by the Optional Sampling Theorem (see Theorem 7.29 in [Kal02]), each of the processes

$$\tilde{U}_t^r := U^r_{\sigma_t^{U,r}},$$
$$\tilde{V}_t^r := V^r_{\sigma_t^{V,r}},$$

solves the martingale problem for A^r_{BDjd} (before they are stopped at time τ^r_δ). I.e. $\tilde{U}_t^{r,\delta}$ and $\tilde{V}_t^{r,\delta}$ are level representations for the discrete mass birth-death processes with jump induced deaths in Chapter 4.

Write $\tilde{Y}^{r,\delta} := \frac{1}{r}\tilde{U}^{r,\delta}([0,r))$ and $\tilde{Z}^{r,\delta} := \frac{1}{r}\tilde{V}^{r,\delta}([0,r))$. Now consider the random time changes

$$\tau_t^{U,r} := \inf\left\{s \geq 0 \;:\; \int_0^s \frac{1}{\tilde{Z}_h^{r,\delta}}\ \mathrm{d}h \geq t\right\},$$
$$\tau_t^{V,r} := \inf\left\{s \geq 0 \;:\; \int_0^s \frac{1}{\tilde{Y}_h^{r,\delta}}\ \mathrm{d}h \geq t\right\}.$$

Note that $\sigma^{U,r}$ and $\sigma^{V,r}$ are inverse to $\tau^{U,r}$ and $\tau^{U,r}$. Hence we have

$$\left(U_t^{(1,\ldots,k),r,\delta}, V_t^{(1,\ldots,k),r,\delta}\right) = \left(\tilde{U}^{(1,\ldots,k),r,\delta}_{\tau_t^{U,r}}, \tilde{V}^{(1,\ldots,k),r,\delta}_{\tau_t^{V,r}}\right).$$

By the rule for differentiation of inverse functions, the time changes $\tau^{U,r}$ and $\tau^{V,r}$ solve the equations

$$\tau_t^{U,r} = \int_0^t \tilde{Z}^{r,\delta}_{\tau_s^{U,r}} \, \mathrm{d}s,$$
$$\tau_t^{V,r} = \int_0^t \tilde{Y}^{r,\delta}_{\tau_s^{V,r}} \, \mathrm{d}s.$$

The processes $\tilde{Y}^{r,\delta}$ and $\tilde{Z}^{r,\delta}$ are nonnegative martingales. By the maximum inequality for martingales (cf. Proposition 7.15 in [Kal02]) we have for $a, T > 0$

$$a \cdot \mathbf{P}\left[\sup_{t\le T} \tilde{Z}_t^{r,\delta} \ge a\right] \le \mathbf{E}\left[\tilde{Z}_0^{r,\delta}\right] = \mathbf{E}\left[Z_0^{r,\delta}\right],$$

where the right hand side is bounded in r by assumption. Hence, for any $\epsilon > 0$, there exists $K > 0$ such that

$$\liminf_{r\to\infty} \mathbf{P}\left[\sup_{s,t\in[0,T]} \left|\tau_t^{U,r} - \tau_s^{U,r}\right| \le K \cdot |t-s|\right] \ge 1-\epsilon.$$

The same holds for $(\tau^{V,r})_r$. The families $\left(\tilde{U}^{(1,\ldots,k),r,\delta}\right)_r$ and $\left(\tilde{V}^{(1,\ldots,k),r,\delta}\right)_r$ are tight in $D_{\mathbb{R}^k}[0,\infty)$ by Theorem 4.2.3. By Lemma A.3.4, the processes $\left(U^{(1,\ldots,k),r,\delta}\right)_r$ and $\left(V^{(1,\ldots,k),r,\delta}\right)_r$ are tight in $D_{\mathbb{R}^k}[0,\infty)$.

The pair $(U^{(1,\ldots,k),r,\delta}, V^{(1,\ldots,k),r,\delta})$ is tight in $D_{\mathbb{R}^k}[0,\infty) \times D_{\mathbb{R}^k}[0,\infty)$ (endowed with the product topology) but not necessarily in $D_{\mathbb{R}^k\times\mathbb{R}^k}$ (endowed with the Skorohod topology; see Lemma A.3.6). Consider a limit point $(U^{(1,\ldots,k),\delta},\ V^{(1,\ldots,k),\delta})$. From the proof of Theorem 5.3.2 the semimartingale characteristics and therefore the discontinuities of the marginal $U^{(1,\ldots,k),\delta}$ and of the marginal $V^{(1,\ldots,k),\delta}$ are apparent (see also Remark 5.3.5): The jumps of $U^{(1,\ldots,k),\delta}$ and $V^{(1,\ldots,k),\delta}$ are due to births among the k lowest particles. In a time interval $[0,T]$, $T > 0$, these births are triggered by $2(k-1)$ orthogonal Poisson point processes with intensities that are bounded by

$$\sup_{0\le t\le T}\left(2bU_t^{(k),\delta} Z_t^\delta \vee 2bV_t^{(k),\delta} Y_t^\delta\right) < \infty.$$

Hence the relevant births almost surely do not happen simultaneously, and $U^{(1,\ldots,k),\delta}$ and $V^{(1,\ldots,k),\delta}$ almost surely do not jump simultaneously in $[0,T]$. By Lemma A.3.6 the family $\left(U^{(1,\ldots,k),r}, V^{(1,\ldots,k),r}\right)_r$ is tight in $D_{\mathbb{R}^{2k}}[0,\infty)$.

The family $\left(\tilde{Y}^{r,\delta}, \tilde{Z}^{r,\delta}\right)_r$ is C-tight and

$$Y_t^r = \tilde{Y}^r_{\tau_t^{U,r}},$$
$$Z_t^r = \tilde{Z}^r_{\tau_t^{V,r}}.$$

By Lemma A.3.4 $\left(Y^{r,\delta}\right)_r$ and $\left(Z^{r,\delta}\right)_r$ are C-tight. Hence by Lemma A.3.3 we have joint tightness of the family $\left(U^{(1,\ldots,k),r,\delta}, V^{(1,\ldots,k),r,\delta}, Y^{r,\delta}, Z^{r,\delta}\right)_r$. □

We lift $D_{\mathbb{R}^{2k+2}}[0,\infty)$-tightness of $(\bar{X}^{r,\delta})$ to the measure valued case. Again we may trace the result back to the level representation of Feller's branching diffusion in Chapter 4.

Theorem 5.2.2. *There exists $r_0 > 0$ such that the family $\left(U^{r,\delta}, V^{r,\delta}, Y^{r,\delta}, Z^{r,\delta}\right)_{r\geq r_0}$ is tight in $D_{\mathcal{S}_{\mathbb{R}^+}\times\mathcal{S}_{\mathbb{R}^+}\times\mathbb{R}^2}$.*

Remark 5.2.3. *By Prohorov's Theorem, $\left(U^{r_n,\delta}, V^{r_n,\delta}, Y^{r_n,\delta}, Z^{r_n,\delta}\right)_n$ is weakly relatively sequentially compact for any sequence $r_n \to \infty$.*

Proof. Since $\left(Y^{r,\delta}, Z^{r,\delta}\right)_{r\geq r_0}$ is C-tight, we confine ourselves to $\left(U^{r,\delta}, V^{r,\delta}\right)_{r\geq r_0}$ (see Remark A.3.7).

Recall the definition of the time changes $\tau^{U,r}$, $\tau^{V,r}$, $\sigma^{U,r}$ and $\sigma^{V,r}$ in the proof of Theorem 5.2.1. The processes $\tilde{U}^r_t := U^r_{\sigma^{U,r}_t}$ and $\tilde{V}^r := V^r_{\sigma^{V,r}_t}$ are solutions to the martingale problem given by A^r_{BDjd} (see (4.1)). In the proof of Theorem 4.2.4 it is shown that for every $f \in C_c^+(\mathbb{R}^+)$ the families $(\tilde{U}^{r,\delta}(f))_r := \int f \, \mathrm{d}U^{r,\delta}$ and $(\tilde{V}^{r,\delta}(f))_r$ are tight in $D_{\mathbb{R}^+}[0,\infty)$. By the same arguments as in the proof of Theorem 5.2.1 tightness of $(U^{r,\delta}(f))_r$ and $(V^{r,\delta}(f))_r$ follows. By Theorem 16.27 in [Kal02] this implies tightness of $(U^{r,\delta})_r$ and $(V^{r,\delta})_r$ in $D_{\mathcal{S}_{\mathbb{R}^+}}[0,\infty)$. For any limit points $U^\delta, V^\delta \in \mathcal{S}_{\mathbb{R}^+}$ and for any $T, K > 0$ we have

$$\sup_{t\leq T}\Big(U^\delta[0,K] \vee V^\delta[0,K]\Big) < \infty. \tag{5.3}$$

We interpret the pair $(U^{r,\delta}_t, V^{r,\delta}_t)$ as a measure $\Phi(U^{r,\delta}_t, V^{r,\delta}_t)$ on $\mathbb{R}^+ \times \{1,2\}$ by

$$\Phi(U^{r,\delta}_t, V^{r,\delta}_t)(A \times \{i\}) := \begin{cases} U^{r,\delta}_t(A) & \text{if } i = 1, \\ V^{r,\delta}_t(A) & \text{if } i = 2. \end{cases}$$

The map $\Phi : \mathcal{S}_{\mathbb{R}^+} \times \mathcal{S}_{\mathbb{R}^+} \to \mathcal{S}_{\mathbb{R}^+\times\{1,2\}}$ is bijective and both Φ and Φ^{-1} are continuous. Hence, by Lemma A.4.1, Φ and Φ^{-1} are continuous as maps of càdlàg paths. By the Continuous Mapping Theorem (cf. Theorem 13.25 in [Kle06]) we have tightness of $(U^{r,\delta}, V^{r,\delta})_r$ if and only if the family $(\Phi(U^{r,\delta}, V^{r,\delta}))_r$ is tight.

In order to prove tightness of $(\Phi(U^{r,\delta}, V^{r,\delta}))_r$, by Theorem 16.27 in [Kal02] it is enough to consider test functions $f \in C_c^+(\mathbb{R}^+ \times \{1,2\})$. I.e. we show tightness of

$$\Phi(U^{r,\delta}, V^{r,\delta})(f) := \int f \, \mathrm{d}\Phi(U^{r,\delta}, V^{r,\delta})$$

in $D_{\mathbb{R}^+}[0,\infty)$ for every $f \in C_c^+(\mathbb{R}^+ \times \{1,2\})$.

Any function $f \in C_c^+(\mathbb{R}^+ \times \{1,2\})$ can be written as

$$f(x,k) := \mathbb{1}_{\{1\}}(k) \cdot f_1(x) + \mathbb{1}_{\{2\}}(k) \cdot f_2(x),$$

where $f_1, f_2 \in C_c^+(\mathbb{R}^+)$. By Lemma A.3.6 the pair $(U^{r,\delta}(f_1), V^{r,\delta}(f_2))_r$ is tight in $D_{\mathbb{R}}[0,\infty) \times D_{\mathbb{R}}[0,\infty)$. Let $(U^\delta(f_1), V^\delta(f_2))$ be a limit point in this sense. By Theorem

5.3.2 and Remark 5.3.5 the discontinuities of $U^\delta(f_1)$ and $V^\delta(f_2)$ are due to births into the compact support of f_1 and f_2 respectively. Because of (5.3) the rate at which particles are born into a compact interval during the time interval $[0,T]$ is bounded. Hence there are only finitely many discontinuities up to time $T>0$ and they almost surely do not happen simultaneously, since the birth events are triggered by orthogonal Poisson point processes. By Lemma A.3.6, the pair $(U^{r,\delta}(f_1), V^{r,\delta}(f_2))_r$ is tight in $D_{\mathbb{R}^2}[0,\infty)$. This implies tightness of $(\Phi(U^{r,\delta}, V^{r,\delta}))_r$. $\square$

5.3. The level dynamics in the limit

We use the same techniques as in Section 4.3 to characterize the dynamics of the k lowest U-levels and the k lowest V-levels. First we determine the semimartingale characteristics of the mass process (Y^r, Z^r) in the limit $r\to\infty$ and verify that we obtain a symbiotic diffusion with correlation coefficient $\rho = \frac{\eta c}{2+c}$. Then we determine the semimartingale characteristics of $(U^{(1,\ldots,k),r}, V^{(1,\ldots,k),r})$ in the limit $r\to\infty$, using (Y^r, Z^r) as auxiliary process that enables us to write down the characteristics.

Define the truncation function $h(x) = (\tilde{h}(x_i))_i$, where

$$\tilde{h}(x) := \begin{cases} x & \text{for } |x| < 1 \\ x\cdot(2-|x|) & \text{for } 1 < |x| \le 2 \\ 0 & \text{for } 2 < |x|\,. \end{cases}$$

Let (Y^δ, Z^δ) be a limit point of $(Y^{r,\delta}, Z^{r,\delta})_r$. We suppress the subsequence notation.

Theorem 5.3.1. *Let $(\check{Y}, \check{Z})$ be a symbiotic diffusion with $\rho = \frac{\eta c}{2+c}$. I.e. $(\check{Y}, \check{Z})$ solves the SDE*

$$\begin{aligned} \mathrm{d}\check{Y}_t &= \sqrt{b(2+c)\check{Y}_t\check{Z}_t}\;\mathrm{d}W_t^1, \\ \mathrm{d}\check{Z}_t &= \sqrt{b(2+c)\check{Y}_t\check{Z}_t}\;\mathrm{d}W_t^2, \end{aligned} \tag{5.4}$$

where W^1, W^2 are correlated Brownian motions with correlation coefficient $\rho = \frac{\eta c}{2+c}$. Let $\tau := \inf\left\{t\ge 0:\ \check{Y}_t \wedge \check{Z}_t \le \delta\right\}$ and let $\left(\check{Y}_t^\delta, \check{Z}_t^\delta\right) := \left(\check{Y}_{t\wedge\tau}, \check{Z}_{t\wedge\tau}\right)$. Then

$$\left(Y^\delta, Z^\delta\right) \stackrel{d}{=} \left(\check{Y}^\delta, \check{Z}^\delta\right).$$

Proof. Assume $\frac{c}{r\delta} \le 1$. We drop the δ in our notation. The r-th processes (Y^r, Z^r) are special semimartingales. Denote by $(B^{(Y,Z),r}, \tilde{C}^{(Y,Z),r}, \nu^{(Y,Z),r})$ the characteristics of (Y^r, Z^r).

Particles have weight $\frac{1}{r}$. U-births and V-births happen independently at instantaneous rate $r^2bY_t^rZ_t^r$. At rate $\eta\frac{r^2bY_t^rZ_t^r}{c}$ death events happen that affect both subpopulations. In both populations additional death events happen independently at rate

$(1-\eta)\frac{r^2 bY_t^r Z_t^r}{c}$. In case of a death event at time t, in the affected subpopulation all particles in a "strip of death" $[r-c/Y_t^r,\, r]$ (or $[r-c/Z_t^r,\, r]$, respectively) are killed. The assumption $\frac{c}{r\delta}\le 1$ guarantees that the left border of the "strip of death" is not negative and always a binomially distributed number of particles is killed per subpopulation. If the U-population is affected by a death event, the number of U-particles killed is binomially distributed with parameters $n_t^{Y,r} := rY_t^r$ and $p_t^{Y,r} := \frac{c}{rY_t^r}$. If the V-population is affected, the parameters are $n_t^{Z,r} := rZ_t^r$ and $p_t^{Z,r} := \frac{c}{rZ_t^r}$.

Define

$$\begin{aligned}
\mathrm{B}_{r,Y_t^r}(\mathrm{d}x) &:= \sum_{i=0}^{n_t^{Y,r}} \binom{n_t^{Y,r}}{i} (p_t^{Y,r})^i (1-p_t^{Y,r})^{n_t^{Y,r}-i}\delta_{\left(-\frac{i}{r},0\right)}(\mathrm{d}x),\\
\mathrm{B}_{r,Z_t^r}(\mathrm{d}x) &:= \sum_{i=0}^{n_t^{Z,r}} \binom{n_t^{Z,r}}{i} (p_t^{Z,r})^i (1-p_t^{Z,r})^{n_t^{Z,r}-i}\delta_{\left(0,-\frac{i}{r}\right)}(\mathrm{d}x),\\
\mathrm{B}_{r,(Y_t^r,Z_t^r)}(\mathrm{d}x) &:= \sum_{i=0}^{n_t^{Y,r}} \sum_{j=0}^{n_t^{Z,r}} \binom{n_t^{Y,r}}{i} (p_t^{Y,r})^i (1-p_t^{Y,r})^{n_t^{Y,r}-i}\\
&\qquad \times \binom{n_t^{Z,r}}{j} (p_t^{Z,r})^j (1-p_t^{Z,r})^{n_t^{Z,r}-j}\delta_{\left(-\frac{i}{r},-\frac{j}{r}\right)}(\mathrm{d}x).
\end{aligned}$$

The compensator of the jump measure is

$$\begin{aligned}
\nu^{(Y,Z),r}(\mathrm{d}t,\mathrm{d}x) = r^2 bY_t^r Z_t^r\ \mathrm{d}t \cdot &\left(\delta_{\left(\frac{1}{r},0\right)}(\mathrm{d}x) + \delta_{\left(0,\frac{1}{r}\right)}(\mathrm{d}x)\right)\\
&+ (1-\eta)\frac{r^2 bY_t^r Z_t^r}{c}\ \mathrm{d}t\cdot\left(\mathrm{B}_{r,Y_t^r}(\mathrm{d}x) + \mathrm{B}_{r,Z_t^r}(\mathrm{d}x)\right)\\
&+ \eta\frac{r^2 bY_t^r Z_t^r}{c}\ \mathrm{d}t\cdot \mathrm{B}_{r,(Y_t^r,Z_t^r)}(\mathrm{d}x).
\end{aligned}$$

Since (Y^r, Z^r) is piecewise constant, we have

$$\tilde{C}_t^{(Y,Z),r} = h^2 * \nu_t^{(Y,Z),r}$$

and

$$B^{(Y,Z),r}(h)_t = h * \nu_t^{(Y,Z),r}.$$

Let $(B^{(Y,Z)}, \tilde{C}^{(Y,Z)}, \nu^{(Y,Z)})$ be the semimartingale characteristics of the symbiotic diffusion (5.4),

$$\begin{aligned}
B_t^{(Y,Z)} &= (0,0),\\
\tilde{C}_t^{(Y,Z),11} &= \tilde{C}_t^{(Y,Z),22} = \int_0^t \mathrm{d}s\ b\,(2+c)\,Y_s Z_s,\\
\tilde{C}_t^{(Y,Z),12} &= \tilde{C}_t^{(Y,Z),21} = \int_0^t \mathrm{d}s\ b(2+c) Y_s Z_s \cdot \frac{\eta c}{2+c}\\
\nu^{(Y,Z)}(\mathrm{d}t,\mathrm{d}x) &= 0.
\end{aligned}$$

In the following we show

$$\Big((Y^r, Z^r), B^{(Y,Z),r}, \tilde{C}^{(Y,Z),r}\Big) \Rightarrow \Big((Y, Z), B^{(Y,Z)}, \tilde{C}^{(Y,Z)}\Big)$$

and

$$\Big((Y^r, Z^r), g * \nu^{(Y,Z),r}\Big) \Rightarrow \Big((Y, Z), g * \nu^{(Y,Z)}\Big)$$

for all nonnegative continuous bounded functions $g : \mathbb{R}^2 \to \mathbb{R}$ which are 0 around 0. Theorem IX.2.4 in [JS03] (see also Theorem A.4.2 in the appendix) then gives us the assertion.

First we address $B^{(Y,Z)}$. We have

$$\begin{aligned}(1-\eta)\frac{r^2 b Y_t^r Z_t^r}{c}\int \mathrm{B}_{r,Y_t^r}(\mathrm{d}x)\; h(x) &= (1-\eta)\frac{r^2 b Y_t^r Z_t^r}{c}\int \mathrm{Bin}_{rY_t^r, \frac{c}{rY_t^r}}(\mathrm{d}x)\; h\Big(-\frac{x}{r}, 0\Big)\\ &= \Big(-(1-\eta) r b Y_t^r Z_t^r,\, 0\Big) + R_t^{1,r}.\end{aligned}$$

Further we have

$$(1-\eta)\frac{r^2 b Y_t^r Z_t^r}{c}\int \mathrm{B}_{r,Z_t^r}(\mathrm{d}x)\; h(x) = \Big(0,\, -(1-\eta) r b Y_t^r Z_t^r\Big) + R_t^{2,r}$$

and

$$\eta\frac{r^2 b Y_t^r Z_t^r}{c}\int \mathrm{B}_{r,(Y_t^r,Z_t^r)}(\mathrm{d}x)\; h(x) = \Big(-\eta r b Y_t^r Z_t^r,\, -\eta r b Y_t^r Z_t^r\Big) + R_t^{3,r}.$$

We have $R^{1,r}, R^{2,r}, R^{3,r} \Rightarrow 0$ by similar arguments as in the proof of Theorem 4.3.2 (see (4.36)). By Lemma A.4.1 and the Continuous Mapping Theorem, we have

$$\begin{aligned}B^{(Y,Z),r} &= h * \nu^{(Y,Z),r}\\ &= \int_0^\cdot \mathrm{d}s\; r^2 b Y_s^r Z_s^r \left(h\Big(\frac{1}{r},\, 0\Big) + h\Big(0,\, \frac{1}{r}\Big)\right)\\ &\quad + \int_0^\cdot \mathrm{d}s\; \left(\Big(-(1-\eta) r b Y_s^r Z_s^r,\, 0\Big) + \Big(0,\, -(1-\eta) r b Y_s^r Z_s^r\Big) + R_s^{1,r} + R_s^{2,r}\right)\\ &\quad + \int_0^\cdot \mathrm{d}s\; \left(\Big(-\eta r b Y_s^r Z_s^r,\, -\eta r b Y_s^r Z_s^r\Big) + R_s^{3,r}\right)\\ &\Rightarrow 0.\end{aligned}$$

We turn to the quadratic Covariation $\tilde{C}^{(Y,Z)}$. We have

$$\frac{r^2 b Y_t^r Z_t^r}{c}\int \mathrm{Bin}_{rY_t^r, \frac{c}{rY_t^r}}(\mathrm{d}x)\; \tilde{h}\left(-\frac{x}{r}\right)^2 = b Y_t^r Z_t^r \left(1 + c - \frac{c}{rY_t^r}\right) + R_t^r,$$

where $R^r \Rightarrow 0$ (cf. proof of Theorem 4.3.2). By Lemma A.4.1, we obtain

$$\begin{aligned}(\tilde{C}^{(Y,Z),r})_{11} &= h^1 h^1 * \nu^{(Y,Z),r}\\ &= \int_0^\cdot \mathrm{d}s\; \left(r^2 b Y_s^r Z_s^r \tilde{h}\Big(\frac{1}{r}\Big)^2 + \frac{r^2 b Y_s^r Z_s^r}{c}\int \mathrm{Bin}_{rY_s^r,\, \frac{c}{rY_s^r}}(\mathrm{d}x)\tilde{h}\Big(-\frac{x}{r}\Big)^2\right)\\ &= \int_0^\cdot \mathrm{d}s\; \left(r^2 b Y_s^r Z_s^r \tilde{h}\Big(\frac{1}{r}\Big)^2 + b Y^r Z^r \left(1 + c - \frac{c}{rY^r}\right) + R^r\right)\\ &\Rightarrow \int_0^\cdot \mathrm{d}s\; b\,(2+c)\, Y_s Z_s\end{aligned}$$

and, by an analogous calculation,

$$(\tilde{C}^{(Y,Z),r})_{22} \Rightarrow \int_0^{\cdot} \mathrm{d}s\, b\,(2+c)\, Y_s Z_s.$$

By similar arguments we obtain for $i=1, j=2$ and $j=1, i=2$

$$\begin{aligned}
(\tilde{C}^{(Y,Z),r})_{ij} &= h^i h^j * \nu^{(Y,Z),r} \\
&= \int_0^{\cdot} \mathrm{d}s\, \eta \frac{r^2 b Y_s^r Z_s^r}{c} \Big(\int \mathrm{Bin}_{rY_s^r, \frac{c}{rY_s^r}}(\mathrm{d}x_1)\, \tilde{h}\Big(-\frac{x_1}{r}\Big)\Big) \\
&\quad \times \Big(\int \mathrm{Bin}_{rZ_s^r, \frac{c}{rZ_s^r}}(\mathrm{d}x_2)\, \tilde{h}\Big(-\frac{x_2}{r}\Big)\Big) \\
&\Rightarrow \int_0^{\cdot} \mathrm{d}s\, \eta c b Y_s Z_s = \int_0^{\cdot} \mathrm{d}s\, b(2+c) Y_s Z_s \cdot \frac{\eta c}{2+c}
\end{aligned}$$

Finally, we obtain in a similar calculation as in the proof of Theorem 4.3.2 (see (4.37) and (4.38))

$$g * \nu^{(Y,Z),r} \Rightarrow 0,$$

for any $g : \mathbb{R}^2 \to [0,\infty)$ that is a nonnegative continuous bounded function, taking the value 0 in a neighbourhood of 0.

By Lemma A.3.3, there is a subsequence $(r') \subset (r)$ such that for $r' \to \infty$

$$\begin{aligned}
((Y^{r'}, Z^{r'}), B^{(Y,Z),r'}, \tilde{C}^{(Y,Z),r'}) &\Rightarrow ((Y,Z), B^{(Y,Z)}, \tilde{C}^{(Y,Z)}) \quad \text{and} \\
((Y^{r'}, Z^{r'}), g * \nu^{(Y,Z),r'}) &\Rightarrow (Y, g * \nu^{(Y,Z)}),
\end{aligned}$$

and $(Y^r, Z^r)_r$ converges to the same limit as the subsequence $(Y^{r'})_{r'}$. □

Similarly as in Section 4.3, we identify the low levels in the limit, $(U^{(1,\dots,k),\delta}, V^{(1,\dots,k),\delta})$, as a k-dimensional semimartingale, via its semimartingale characteristics. The mass density processes $(Y^{r,\delta}, Z^{r,\delta})_r$ act as auxiliary processes that allow us to write down the characteristics of the low levels.

Theorem 5.3.2. *Let $\bar{X}^\delta = \left(U^{(1,\dots,k),\delta}, V^{(1,\dots,k),\delta}, Y^\delta, Z^\delta\right)$ be the limit of a convergent subsequence of the family $\left(\bar{X}^{r,\delta}\right)_r$. Then $\left(U^{(1,\dots,k),\delta}, V^{(1,\dots,k),\delta}\right)$ is a stopped semimartingale with characteristic triplet $(B, \tilde{C}, \nu)$. Define $\tau := \inf\{s \geq 0 : Y_s \wedge Z_s \leq \delta\}$.*

The jump measure of $\left(U^{(1,\dots,k),\delta}, V^{(1,\dots,k),\delta}\right)$ *is*

$$\begin{aligned}\nu(\mathrm{d}t, \mathrm{d}x) = \mathbb{1}_{\{t<\tau\}} \Bigg[&\sum_{q=2}^{k} 2bZ_t^\delta (q-1) \left(U_t^{(q),\delta} - U_t^{(q-1),\delta}\right) \mathrm{d}t \cdot \delta_{0_{q-1}}(\mathrm{d}x_1, \dots, \mathrm{d}x_{q-1}) \\ &\times \mathrm{Unif}_{[U_t^{(q-1),\delta} - U_t^{(q),\delta}, 0]}(\mathrm{d}x_q) \cdot \delta_{U_t^{(q,\dots,k-1),\delta} - U_t^{(q+1,\dots,k),\delta}}(\mathrm{d}x_{q+1}, \dots, \mathrm{d}x_k) \\ &\times \delta_{0_k}(\mathrm{d}x_{k+1}, \dots, \mathrm{d}x_{2k}) \\ &+ \sum_{q=2}^{k} 2bY_t^\delta (q-1) \left(V_t^{(q),\delta} - V_t^{(q-1),\delta}\right) \mathrm{d}t \cdot \delta_{0_k}(\mathrm{d}x_1, \dots, \mathrm{d}x_k) \\ &\times \delta_{0_{q-1}}(\mathrm{d}x_{k+1}, \dots, \mathrm{d}x_{k+q-1}) \cdot \mathrm{Unif}_{[V_t^{(q-1),\delta} - V_t^{(q),\delta}, 0]}(\mathrm{d}x_{k+q}) \\ &\times \delta_{V_t^{(q,\dots,k-1),\delta} - V_t^{(q+1,\dots,k),\delta}}(\mathrm{d}x_{k+q+1}, \dots, \mathrm{d}x_{2k}) \Bigg].\end{aligned}$$

The bounded variation part is, for $i = 1, \dots, k$,

$$B_t^i = \int_0^{t\wedge\tau} \mathrm{d}s \, \left(bZ_s^\delta \left(U_s^{(i),\delta}\right)^2 + \frac{bcZ_s^\delta}{Y_s^\delta} U_s^{(i),\delta} \right) + h^i * \nu$$

and, for $i = k+1, \dots, 2k$,

$$B_t^i = \int_0^{t\wedge\tau} \mathrm{d}s \, \left(bY_s^\delta \left(V_s^{(i-k),\delta}\right)^2 + \frac{bcY_s^\delta}{Z_s^\delta} V_s^{(i-k),\delta} \right) + h^i * \nu.$$

For the covariation of the martingale part we have, for $i, j \in \{1, \dots, k\}$,

$$\tilde{C}_t^{ij} = \int_0^{t\wedge\tau} \mathrm{d}s \, \frac{bcZ_s^\delta}{Y_s^\delta} U_s^{(i),\delta} U_s^{(j),\delta} + h^i h^j * \nu_t,$$

for $i, j \in \{k+1, \dots, 2k\}$,

$$\tilde{C}_t^{ij} = \int_0^{t\wedge\tau} \mathrm{d}s \, \frac{bcY_s^\delta}{Z_s^\delta} V_s^{(i-k),\delta} V_s^{(j-k),\delta} + h^i h^j * \nu_t,$$

and, for $i \in \{1, \dots, k\}$ *and* $j \in \{k+1, \dots, 2k\}$,

$$\tilde{C}_t^{ij} = \int_0^{t\wedge\tau} \mathrm{d}s \, \eta bc U_s^{(i),\delta} V_s^{(j-k),\delta} + h^i h^j * \nu_t.$$

Remark 5.3.3. *In other words, between jumps and up to time* τ *we see the following dynamics:*

(i) The k *lowest levels of the* U*-population are a system of perfectly correlated, conditional geometric Brownian motions.* $U^{(1,\dots,k),\delta}$ *solves*

$$\mathrm{d}U_t^{(i),\delta} = \left(bZ_t^\delta \left(U_t^{(i),\delta}\right)^2 + \frac{bcZ_t^\delta}{Y_t^\delta} U_t^{(i),\delta} \right) \mathrm{d}t + \sqrt{\frac{bcZ_t^\delta}{Y_t^\delta}} \cdot U_t^{(i),\delta} \, \mathrm{d}W_s^1,$$

with W^1 *being the same Brownian motion for all coordinates* $1 \le i \le k$.

(ii) The same holds for the k lowest levels of the V-population. $V^{(1,\dots,k),\delta}$ solves

$$\mathrm{d}V_t^{(j),\delta} = \left(bY_t^\delta\left(V_t^{(j),\delta}\right)^2 + \frac{bcY_t^\delta}{Z_t^\delta}V_t^{(j),\delta}\right)\mathrm{d}t + \sqrt{\frac{bcY_t^\delta}{Z_t^\delta}}\cdot V_t^{(j),\delta}\,\mathrm{d}W_s^2,$$

with W^2 being the same Brownian motion for all coordinates $1 \leq j \leq k$.

(iii) The driving Brownian motions W^1 and W^2 - and therefore the continuous martingale parts of $U^{(i),\delta}$ and $V^{(j),\delta}$, $i, j \in \{1, \dots, k\}$ - correlated with correlation coefficient

$$\rho = \eta bc \cdot \sqrt{\frac{Y_t^\delta}{bcZ_t^\delta}} \cdot \sqrt{\frac{Z_t^\delta}{bcY_t^\delta}} = \eta.$$

Remark 5.3.4. *The path of a single particle of the full system is continuous. The jumps of $U^{(1,\dots,k),r,\delta}$ and $V^{(1,\dots,k),r,\delta}$ are attributed to the fact that we trace the k lowest particles and newborn particles with low enough levels have to be pigeon-holed.*

Proof of Theorem 5.3.2. We suppress the subsequence notation and we drop the δ in our notation. We denote by $(B^r, \tilde{C}^r, \nu^r)$ the semimartingale characteristics of $\left(U^{(1,\dots,k),r}, V^{(1,\dots,k),r}\right)$. We constitute ν^r as a sum

$$\nu^r(\omega; dt, dx) = \nu^{D,r}(\omega; dt, dx) + \nu^{B,r}(\omega; dt, dx),$$

where $\nu^{D,r}$ compensates the jumps induced by death events, $\nu^{B,r}$ compensates the jumps induced by births among the k lowest levels in both subpopulations.

Death events affect either one subpopulation alone or both subpopulations. At time t both death jumps of the U-population and death jumps of the V-population are triggered independently with instantaneous rate $(1-\eta)\frac{r^2bY_t^rZ_t^r}{c}$. Simultaneous death jumps of both populations happen at rate $\eta\frac{r^2bY_t^rZ_t^r}{c}$. In each case the affected particles are multiplied by the factor $\varphi(U^r) = \frac{r}{r-c/Y^r}$ or $\varphi(V^r) = \frac{r}{r-c/Z^r}$, respectively (the correction for the case "$rY^r < c$" is omitted; see Remark 5.1.2). So the height of the jump of a U-particle at level u^i is

$$u^i \cdot \left(\frac{r}{r - c/Y_{t-}^r} - 1\right) = u^i \cdot \frac{c}{rY_{t-}^r - c}.$$

(Analogous for a V-particle.) Recall that $\tau^r := \inf\{s : Y_s^r \wedge Z_s^r \leq \delta\}$. The compensator for the death jumps is

$$\begin{aligned}
\nu^{D,r}(\mathrm{d}t, \mathrm{d}x) = \mathbb{1}_{\{t<\tau^r\}}\Bigg(&(1-\eta)\frac{r^2bY_t^rZ_t^r}{c}\,\mathrm{d}t \cdot \delta_{\left(U_t^{(1,\dots,k),r}\cdot\frac{c}{rY_t^r-c},0\right)}(\mathrm{d}x)\\
&+ (1-\eta)\frac{r^2bY_t^rZ_t^r}{c}\,\mathrm{d}t \cdot \delta_{\left(0,V_t^{(1,\dots,k),r}\cdot\frac{c}{rZ_t^r-c}\right)}(\mathrm{d}x)\\
&+ \eta\frac{r^2bY_t^rZ_t^r}{c}\,\mathrm{d}t \cdot \delta_{\left(U_t^{(1,\dots,k),r}\cdot\frac{c}{rY_t^r-c},V_t^{(1,\dots,k),r}\cdot\frac{c}{rZ_t^r-c}\right)}(\mathrm{d}x)\Bigg).
\end{aligned}$$

For the birth induced jumps consider the U-population - analogous considerations hold for the V-particles. At time $t \geq 0$ every U-particle with level u^i gives birth at rate $2bZ_t^r(r-u^i)$. The process $U^{(1,\ldots,k),r}$ displays only the birth of particles that are among the k lowest. Except for role of the mass density Z_t^r in the rate, the birth mechanism is the same as in the particle representation for Feller's branching diffusion in Chapter 4. Write 0_q for $(0,\ldots,0) \in \mathbb{R}^q$. By analogous considerations as in the proof of Theorem 4.3.3 we obtain the compensator for the birth jumps:

$$\begin{aligned}
\nu^{B,r}(\mathrm{d}t,\mathrm{d}x) = \mathbb{1}_{\{t<\tau^r\}}\Big[&\sum_{q=2}^{k} 2bZ_t^r(q-1)\left(U_t^{(q),r}-U_t^{(q-1),r}\right)\mathrm{d}t\cdot\delta_{0_{q-1}}(\mathrm{d}x_1,\ldots,\mathrm{d}x_{q-1})\\
&\times \mathrm{Unif}_{[U_t^{(q-1),r}-U_t^{(q),r},0]}(\mathrm{d}x_q)\cdot\delta_{U_t^{(q,\ldots,k-1),r}-U_t^{(q+1,\ldots,k),r}}(\mathrm{d}x_{q+1},\ldots,\mathrm{d}x_k)\\
&\times \delta_{0_k}(\mathrm{d}x_{k+1},\ldots,\mathrm{d}x_{2k})\\
&+\sum_{q=2}^{k} 2bY_t^r(q-1)\left(V_t^{(q),r}-V_t^{(q-1),r}\right)\mathrm{d}t\cdot\delta_{0_k}(\mathrm{d}x_1,\ldots,\mathrm{d}x_k)\\
&\times \delta_{0_{q-1}}(\mathrm{d}x_{k+1},\ldots,\mathrm{d}x_{k+q-1})\cdot\mathrm{Unif}_{[V_t^{(q-1),r}-V_t^{(q),r},0]}(\mathrm{d}x_{k+q})\\
&\times \delta_{V_t^{(q,\ldots,k-1),r}-V_t^{(q+1,\ldots,k),r}}(\mathrm{d}x_{k+q+1},\ldots,\mathrm{d}x_{2k})\Big].
\end{aligned}$$

We apply Theorem IX.2.4 and Remark IX.2.21 in [JS03] (cf. Theorem A.4.2). I.e., for a convergent subsequence $\left(U^{(1,\ldots,k),r},V^{(1,\ldots,k),r},Y^r,Z^r\right)$ with limit $\left(U^{(1,\ldots,k)},V^{(1,\ldots,k)},Y,Z\right)$ we show

$$\left(U^{(1,\ldots,k),r},V^{(1,\ldots,k),r},Y^r,Z^r,B^r,\tilde{C}^r\right)\Rightarrow\left(U^{(1,\ldots,k)},V^{(1,\ldots,k)},Y,Z,B,\tilde{C}\right)$$

and

$$\left(U^{(1,\ldots,k),r},V^{(1,\ldots,k),r},Y^r,Z^r,g*\nu^r\right)\Rightarrow\left(U^{(1,\ldots,k)},V^{(1,\ldots,k)},Y,Z,g*\nu\right) \tag{5.5}$$

for all nonnegative continuous bounded functions $g:\mathbb{R}^{2k}\to[0,\infty)$ which are 0 around 0.

Observe that $\tau^r \xrightarrow{\text{stoch.}} \tau$ and $\mathbb{1}_{\{s<\tau^r\}}\Rightarrow\mathbb{1}_{\{s<\tau\}}$.

Convergence of the third characteristics. We turn to Condition (5.5). By the analogous arguments as in the proof of Theorem 4.3.3 (see (4.41) and (4.42) we have

$$g*\nu^{B,r}\Rightarrow g*\nu$$

and

$$g*\nu^{D,r}\Rightarrow 0.$$

Since $g*\nu$, Y and Z are continuous, we have joint convergence

$$\left(U^{(1,\ldots,k),r},V^{(1,\ldots,k),r},Y^r,Z^r,g*\nu^r\right)\Rightarrow\left(U^{(1,\ldots,k)},V^{(1,\ldots,k)},Y,Z,g*\nu\right)$$

by Lemma A.3.3.

Convergence of the first characteristics. Define for $i = 1, \dots, 2k$

$$\tilde{B}_t^{i,r} := \bar{X}^{i,r} - \sum_{s \le t} \Delta \bar{X}_s^{i,r}.$$

Between jumps, the particles in the r-th model move according to the differential equations $\dot{u} = bzu^2 - brzu$ and $\dot{v} = byv^2 - bryv$, respectively. Hence we have for $i = 1, \dots, k$

$$\tilde{B}_t^{i,r} = \int_0^t \mathrm{d}s\, \mathbb{1}_{\{s<\tau^r\}} \left(bZ_s^r \left(U_s^{(i),r} \right)^2 - brZ_s^r U_s^{(i),r} \right)$$

and, for $i = k+1, \dots, 2k$,

$$\tilde{B}_t^{i,r} = \int_0^t \mathrm{d}s\, \mathbb{1}_{\{s<\tau^r\}} \left(bY_s^r \left(V_s^{(i-k),r} \right)^2 - brY_s^r V_s^{(i-k),r} \right).$$

By analogous arguments as in the proof of Theorem 4.3.3 (see (4.43)) we obtain

$$\frac{r^2 bY^r Z^r}{c} \tilde{h}\Big(U^{(i),r} \frac{c}{rY^r - c} \Big) - brZ^r U^{(i),r} \Rightarrow \frac{bcZ}{Y} U^{(i)}$$

and

$$\frac{r^2 bY^r Z^r}{c} \tilde{h}\Big(V^{(i),r} \frac{c}{rZ^r - c} \Big) - brY^r V^{(i),r} \Rightarrow \frac{bcY}{Z} V^{(i)}.$$

Hence, for $i = 1, \dots, k$, we obtain for the bounded variation part

$$\begin{aligned} B^{i,r} &= \tilde{B}^{i,r} + h^i * \nu^{D,r} + h^i * \nu^{B,r} \\ &= \int_0^\cdot \mathrm{d}s\, \mathbb{1}_{\{s<\tau^r\}} \left(bZ_s^r \left(U_s^{(i),r} \right)^2 - brZ_s^r U_s^{(i),r} + \frac{r^2 bY^r Z^r}{c} \tilde{h}\Big(U^{(i),r} \frac{c}{rY^r - c} \Big) \right) \\ &\quad + h^i * \nu^{B,r} \\ &\Rightarrow \int_0^\cdot \mathrm{d}s\, \mathbb{1}_{\{s<\tau\}} \left(bZ_s \left(U_s^{(i)} \right)^2 + \frac{bcZ_s}{Y_s} U_s^{(i)} \right) + h^i * \nu \end{aligned}$$

and, for $i = 1 + k, \dots, 2k$, we have

$$B^{i,r} \Rightarrow \int_0^\cdot \mathrm{d}s\, \mathbb{1}_{\{s<\tau\}} \left(bY_s \left(V_s^{(i-k)} \right)^2 + \frac{bcY_s}{Z_s} V_s^{(i-k)} \right) + h^i * \nu.$$

Convergence of the second characteristics. Finally, we address the covariation $\tilde{C}$. Since $\bar{X}^r$ has no continuous martingale part, we have

$$\tilde{C}^{ij,r} = h^i h^j * \nu^r = h^i h^j * (\nu^{D,r} + \nu^{B,r}).$$

Using Lemma 4.3.5 and Lemma A.4.1 and by as similar argument as in the proof of Theorem 4.3.3 (see (4.44)) we have for $i, j \in \{1, \dots, k\}$

$$\begin{aligned} h^i h^j * \nu^{D,r} &= \int_0^\cdot \mathrm{d}s\, \mathbb{1}_{\{s<\tau^r\}} \frac{r^2 bY_s^r Z_s^r}{c} \tilde{h}\left(U_s^{(i),r} \frac{c}{rY_s^r - c} \right) \cdot \tilde{h}\left(U_s^{(j),r} \frac{c}{rY_s^r - c} \right) \\ &\Rightarrow \int_0^\cdot \mathrm{d}s\, \mathbb{1}_{\{s<\tau\}} \frac{bcZ_s}{Y_s} U_s^{(i)} U_s^{(j)}. \end{aligned}$$

For $i, j \in \{k+1, \dots, 2k\}$ we obtain

$$h^i h^j * \nu^{D,r} \Rightarrow \int_0^\cdot \mathrm{d}s\ \mathbb{1}_{\{s<\tau\}} \frac{bcY_s}{Z_s} V_s^{(i-k)} U_s^{(j-k)}.$$

And for $i \in \{1, \dots, k\}$ and $j \in \{k+1, \dots, 2k\}$ we obtain

$$\begin{aligned} h^i h^j * \nu^{D,r} &= \int_0^\cdot \mathrm{d}s\ \mathbb{1}_{\{s<\tau^r\}} \cdot \eta \frac{r^2 b Y_s^r Z_s^r}{c} \tilde{h}\left(U_s^{(i),r} \frac{c}{rY_s^r - c}\right) \cdot \tilde{h}\left(V_s^{(j-k),r} \frac{c}{rZ_s^r - c}\right) \\ &\Rightarrow \int_0^\cdot \mathrm{d}s\ \mathbb{1}_{\{s<\tau\}} \cdot \eta bc U_s^{(i)} V_s^{(j-k)}. \end{aligned}$$

Furthermore, we have for $i, j \in \{1, \dots, 2k\}$

$$h^i h^j * \nu^{B,r} \Rightarrow h^i h^j * \nu.$$

Lemma A.3.3 gives us the simultaneous convergence

$$\left(U^{(1,\dots,k),r}, V^{(1,\dots,k),r}, Y^r, Z^r, B^r, \tilde{C}^r\right) \Rightarrow \left(U^{(1,\dots,k)}, V^{(1,\dots,k)}, Y, Z, B, \tilde{C}\right).$$

$\square$

Remark 5.3.5. *Assume only tightness of* $\left(U^{r,\delta}, Y^{r,\delta}, Z^{r,\delta}\right)_r$ *and* $\left(V^{r,\delta}, Y^{r,\delta}, Z^{r,\delta}\right)_r$ *(but not joint tightness of* $\left(U^{r,\delta}, V^{r,\delta}, Y^{r,\delta}, Z^{r,\delta}\right)_r$*). Since the dynamics of* $U^{(1,\dots,k),r,\delta}$ *depend on* the mass $Z^{r,\delta}$ *but not on* the levels $V^{r,\delta}$*, one can determine the characteristics of the limit* $U^{(1,\dots,k),\delta}$ *alone. The same holds for* $V^{(1,\dots,k),\delta}$*. It is then apparent that the jumps of* $U^{(1,\dots,k),\delta}$ *and* $V^{(1,\dots,k),\delta}$ *are due to births among the* k *lowest particles. The Poisson point processes that trigger those births are orthogonal.*

6. Addendum: The interplay of low levels and the total mass

In this addendum we return to the level representation of Feller's branching diffusion from Chapter 4. Consider the process $X^\delta = (U^{(1,\dots,k),\delta}, Y^\delta)$, consisting of the k lowest levels and the mass density, stopped when the mass density hits $\delta > 0$. Originally, we planned to characterize the dynamics of X^δ as a semimartingale. As stated in the beginning of Section 4.3, we discontinued this approach because, at the r-th stage, we could not write the characteristics of the marginal Y^r in terms of $U^{(1,\dots,k),r}$ and Y^r. At first glance, it is not obvious to us that $X^r = (U^{(1,\dots,k),r}, Y^r)$ is even a semimartingale with respect to its own filtration. Working on these problems, we learned a lot about the interplay of the lowest levels and the mass density. We will illustrate these insights in this chapter. Note that the considerations in this chapter, in particular those in Section 6.3.1, are of a more informal nature.

Let U^r be the level representation of the birth-death process with jump-induced deaths, characterized by A^r_{BDjd} and let $M^r := U^r([0,r))$ be its total mass. One problem in characterizing the (joint) dynamics of $X^r = (U^{(1,\dots,k),r}, Y^r)$ is the following: When a particle is born below $U^{(k),r}$, the former k-th particle becomes the $k+1$-st particle and thus vanishes from our accounting. But we do know its position until it dies: We know the differential equation it follows between jumps; we know when it jumps, since *all* particles jump simultaneously; and we know the jump height. Since its position gives us information about the evolution of the total mass, the process X^r does not have the Markov property. In order to get back the Markov property, we should keep track of all particles that had rank k at some point in their past. We call these particles then *inactive progenitors*. The currently k lowest particles are called *active progenitors*. Note that the problem with the lost account of former progenitors does not occur if $k = 1$: The only active progenitor is never replaced.

6.1. The progenitor-level process $\check{U}^r$

In order to attain a process that consists of the active progenitors, the inactive progenitors and the total mass, we define, as a first step, the *progenitor-level process* $\check{U}^r$ by decomposing U^r in subpopulations that are, in some sense, "fuelled" by different progenitors. These subpopulations are numbered and, roughly speaking, the numbers code the age of the subpopulation. The state space of $\check{U}^r$ is $\mathcal{S}_{[0,r]\times\mathbb{N}_0}$, the space of locally finite counting measures on $[0, r] \times \mathbb{N}_0$. We may write

$$\check{U}^r := \sum_i \delta_{\check{U}^{i,r}} := \sum_i \delta_{(U^{i,r}, N^{i,r})}.$$

The first coordinate of an atom, $U^{i,r}$, is the level of the respective particle, the second coordinate, $N^{i,r}$, is the number of the subpopulation it belongs to. The subpopulation with number 0 has a special role and is called the *root*. The other subpopulations are called *extractions*.

The dynamics are as follows: At first, all particles belong to the root. When a new active progenitor is born, the formerly k-th particle becomes an inactive progenitor. This inactive progenitor and all root-particles above it, now form a new extraction. They change their subpopulation-number to 1. From now on these particles and their offspring are a birth-death process with a single progenitor and overall death rate M^r. Note that the levels of this newly formed extraction move essentially upwards (relatively to $U^{(1),r}$) and the extraction dies out when its lowest particle (the inactive progenitor) dies. At the time of the exodus of the new extraction the root contains only k particles (the active progenitors) which start to repopulate a new bulk. Whenever a new active progenitor is born, a new extraction is formed this way. The newest extraction always gets the subpopulation-number 1 and the other extractions move up one slot. This leads to a decomposition of the population in the root and a changing number of extractions. We characterize the system's dynamics by its generator. Since particles interact mostly within their respective subpopulation, it lends itself to use the vector notation that is already indicated in the paragraph above: For $n \in \mathbb{N}_0$, we define

$$\check{U}^r_n := \check{U}^r(\cdot \times \{n\}) = \sum_{i:\, N^{i,r}=n} \delta_{U^{i,r}},$$

the counting measure (on $[0, r]$) representing the n-th subpopulation. We write then

$$\check{U}^r = \left(\check{U}^r_0, \check{U}^r_1, \ldots\right).$$

The first entry $\check{U}^r_0$ houses the root, and the following entries $(\check{U}^r_n)_{n\geq 1}$ house the extractions. Let $\check{M}^r_n$ be the number of particles in the subpopulation n.

For each subpopulation we apply the same notational conventions that we used in the previous chapters for the whole population: We identify the counting measure $\check{U}^r_n = \sum_{i=1}^{\check{M}^r_n} \delta_{\check{U}^{i,r}_n}$ with the vector $(\check{U}^{1,r}_n, \ldots, \check{U}^{\check{M}_n,r}_n)$, when appropriate. The enumeration $(\check{U}^{i,r}_n)_i$ should be understood as arbitrary labelling within the subpopulation n. We write $\check{U}^{(k),r}_n$ for the k-th lowest particle in the subpopulation n.

For every $n \in \mathbb{N}_0$, let $0 \leq g_n \leq 1$ be differentiable, $\sup_n \|g'_n\|_\infty \leq 1$ and $g_n(x) = 1$ for $x \geq r$. The test functions have the form

$$f(u_0, u_1, \ldots) := \prod_{n\geq 0} f_n(u_n), \tag{6.1}$$

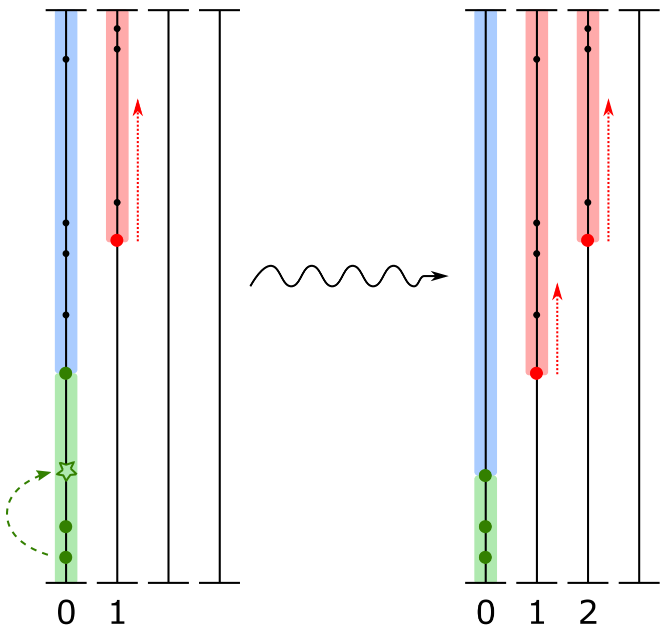

Figure 6.1.: The progenitor mass process consists of one root (in green and blue) and a changing number of extractions (in red). Each extraction consists of a branching bulk, fuelled by one progenitor. The root consists of a branching bulk (in blue) that is fuelled by several progenitors (in green). These progenitors may also give birth to a new progenitor, which leads to the formation of a new extraction, consisting of the former k-th lowest particle and the former bulk of the root.

where

$$\begin{aligned} f_n(u_n) &:= \exp\left(\int \mathrm{d}u_n \, \log g_n\right) \\ &= \prod_{i=1}^{u_n[0,r)} g_n(u_n^i). \end{aligned} \tag{6.2}$$

We denote this class of test functions by $\mathcal{D}^r_{PL}$ (the label PL stands for *progenitor-level process*).

For simplicity, we set $b = 1$ for the rest of this chapter. We constitute the generator of the progenitor-level process $\check{U}^r$ as a sum,

$$A^r_{PL} f(u_1, u_2, \dots) := (A^r_f + A^r_d + A^r_0 + A^r_e) f(u_1, u_2, \dots), \tag{6.3}$$

where the summands are the generators of the different mechanisms at work: The generator A^r_f models the formation of the extractions; A^r_d models the death-dynamics; A^r_0 characterizes the birth-dynamics in the subpopulation 0 (the root) and A^r_e the birth-dynamics in the extractions.

The formation of extractions. Let $\tau_1 < \tau_2 < \dots$ be the times at which offspring is born below the k-th particle, later referred to as *formation times.* At time zero (and up until τ_1), all particles are in the root,

$$(\check{U}_0^r)_t = U_t^r \quad \text{for all } 0 \leq t < \tau_1.$$

At time τ_i the $k+1$-st particle (the formerly k-th particle) becomes an inactive progenitor. It takes all particles from the bulk of the root and forms the extraction $\check{U}_1^r$,

$$(\check{U}_1^r)_{\tau_i} = (\check{U}_0^{(k,k+1,\dots),r})_{\tau_i-}.$$

The other extractions move up one slot,

$$(\check{U}_n^r)_{\tau_i} = (\check{U}_{n-1}^r)_{\tau_i-} \quad \text{for } n = 2, 3, \dots$$

Every particle $\check{U}_0^{(i),r}$ generates offspring at rate $2(r - \check{U}_0^{(i),r})$ that is uniformly distributed in $\left[\check{U}_0^{(i),r}, r\right]$. Offspring with rank q can only have a parent with rank smaller than q, so offspring is born into the rank q at time t at rate

$$2(q-1)\left((\check{U}_0^{(q),r})_t - (\check{U}_0^{(q-1),r})_t\right).$$

When there are less than k particles in the root, no extractions are alive and $\check{M}_0^r$ is the total mass. In this case the progenitors of the root are "filled up" before new extractions are formed.

For $u_n \in \mathcal{S}_{[0,r]}$, write $m_n := u_n[0,r)$. Write further $u = \sum_n u_n$ and $m := u[0,r)$. The formation dynamics described above is modelled by the generator

$$\begin{aligned} A_f^r f(u_0, u_1, \dots) := \mathbb{1}_{m \geq k}(u) \cdot \sum_{q=2}^{k} 2(q-1)(u_0^{(q)} - u_0^{(q-1)}) \\ \times \Bigg[\prod_{j=1}^{k-1} g_0(u_0^{(j)}) \cdot \frac{1}{u_0^{(q)} - u_0^{(q-1)}} \int_{u_0^{(q-1)}}^{u_0^{(q)}} g_0(x)\, \mathrm{d}x \cdot \prod_{j=k}^{m_0} g_1(u_0^{(j)}) \cdot \prod_{n \geq 2} f_n(u_{n-1}) \\ - f(u_0, u_1, \dots) \Bigg] \\ + \mathbb{1}_{m<k}(u) \cdot f(u_0, u_1, \dots) \sum_{i=1}^{m_0} 2 \int_{u_0^i}^{r} (g_0(x) - 1)\, \mathrm{d}x. \end{aligned} \tag{6.4}$$

Note that in the case $m < k$, we have $u_1 = u_2 = \dots = 0$.

The death dynamics. All particles are subject to the same death zone dynamics. So the upwards jumps in all extractions and the root are coupled. Recall that in the model of A_{BDjd}^r, particles jump at rate $\frac{rM_t^r}{c}$, and when doing so, their levels are multiplied by

$$\varphi(U^r) := \mathbb{1}_{(c,\infty)}(M^r) \cdot \frac{1}{1 - c/M^r} + \mathbb{1}_{[0,c]}(M^r) \cdot \frac{r}{\min U^r}. \tag{6.5}$$

This corresponds to a death zone of width $\frac{rc}{M_t^r}$.

The death dynamics are characterized by the generator

$$A_d^r f(u_0, u_1, \ldots) := \frac{rm}{c}\Big(f(\varphi(u)\cdot u_0, \varphi(u)\cdot u_1, \ldots) - f(u_0, u_1, \ldots)\Big). \tag{6.6}$$

Note that the jump factor $\varphi(u)$ depends on the mass of the *full* system.

The birth dynamics in the extractions. Particles that belong to an extraction generate offspring that belongs to the same extraction. The dynamics are given by the model A_{BDjd}^r: All particles move according to the differential equation

$$\dot{u} = u^2 - ru$$

and generate offspring at instantaneous rate $2(r - (\check{U}_n^{i,r})_t)$ and the offspring is placed uniformly at random above the parent.

The birth dynamics in the extractions is characterized by the generator

$$A_e^r f(u_0, u_1, \ldots) := f(u_0, u_1, \ldots) \sum_{n\geq 1} \sum_{i=1}^{m_n} \left(2\int_{u_n^i}^r (g_n(x) - 1)\,\mathrm{d}x + \Big((u_n^i)^2 - ru_n^i\Big)\frac{g_n'(u_n^i)}{g_n(u_n^i)}\right). \tag{6.7}$$

The birth dynamics in the root. Between formation times, the root behaves like an extraction with one difference: No particles are born below $\check{U}_0^{(k),r}$. The births below $\check{U}_0^{(k),r}$ are taken care of in the generator for the formation dynamics. Each of the particles $\check{U}_0^{(1,\ldots,k),r}$ gives birth at rate $r - \check{U}_0^{(k),r}$ to offspring with level that is uniform above $\check{U}_0^{(k),r}$. The offspring belongs to the root. Particles with rank $k+1$ or higher show the usual birth behaviour and all particles move according to the differential equation $\dot{u} = u^2 - ru$.

The birth dynamics in the root is characterized by

$$\begin{aligned} A_0^r f(u_0, u_1, \ldots) := f(u_0, u_1, \ldots)\Bigg(2k\int_{u_0^{(k)}}^r (g_0(x) - 1)\,\mathrm{d}x + \sum_{i=k+1}^{m_0} 2\int_{u_0^{(i)}}^r (g_0(x) - 1)\,\mathrm{d}x \\ + \sum_{i=1}^{m_0}\Big((u_0^i)^2 - ru_0^i\Big)\frac{g_0'(u_0^i)}{g_0(u_0^i)}\Bigg). \end{aligned} \tag{6.8}$$

Remark 6.1.1. *The process $\check{U}^r$ is not adapted to the filtration generated by U^r. By colouring the particles according to their affiliation to the different extractions, we gain genealogical information that is absent in the original system U^r. But the process $X^r = (U^{(1,\ldots,k),r}, Y^r)$ is adapted to both, the filtration generated by $\check{U}^r$ and the filtration generated by U^r.*

6.2. The progenitor-mass process $\check{X}^r$

Since we want to analyze the interplay of low levels and the mass, the next step is to "forget" the level information in the bulk of each subpopulation. We retain a projection $\check{X}^r$ of $\check{U}^r$ that consists of the active and inactive progenitors' levels and the mass of each subpopulation. We call $\check{X}^r$ the progenitor-mass process.

Note that, after forgetting the level information in the bulks, the extraction numbers of the progenitors are redundant information: The progenitors' levels are ordered,

$$\check{U}_0^{(1),r} \leq \ldots \leq \check{U}_0^{(k),r} \leq \check{U}_1^{(1),r} \leq \check{U}_2^{(1),r} \leq \ldots,$$

and we can infer the subpopulation number of a progenitor from its level's rank.

Since the number of extractions varies over time, we represent the progenitor-mass process as a measure valued process

$$\check{X}^r := \delta_{(\check{U}_0^{(1),r}, \check{M}_0^r)} + \ldots + \delta_{(\check{U}_0^{(k),r}, \check{M}_0^r)} + \sum_{n \geq 1} \delta_{(\check{U}_n^r, \check{M}_n^r)}.$$

The state space of $\check{X}^r$ is $\mathcal{S}_{[0,r] \times \mathbb{N}}$. Every atom represents a progenitor, where the first coordinate is the progenitor's level. This time the second coordinate is the mass of its subpopulation. When we write down the generator of $\check{X}^r$ or do generator calculations, we use an informal vector notation

$$\check{X}^r = \left(\check{U}_0^{(1,\ldots,k),r}, (\check{U}_n^{(1),r})_{n \geq 1}, (\check{M}_n^r)_{n \geq 0}\right).$$

In order to determine the dynamics of this projection, we apply the Markov Mapping theorem to $\check{U}^r$. Let $N \in \mathbb{N}$, $0 < u_0^{(1)} < \ldots < u_0^{(k)} < u_1^{(1)} < \ldots < u_N^{(1)} < r$, $m_0 \geq k$, and let

$$\check{x} = \delta_{(u_0^{(1)}, m_0)} + \ldots + \delta_{(u_0^{(k)}, m_0)} + \sum_{n=1}^{N} \delta_{(u_n^{(1)}, m_n)} \in \mathcal{S}_{[0,r) \times \mathbb{N}}.$$

Conditioned on $\{\check{X}^r = \check{x}\}$, the bulk-levels in extraction $n \geq 1$ are independent and uniformly distributed on $[\check{u}_n^{(1)}, r]$. The situation is similar for the root, but there the bulk-levels are uniformly distributed above the level $u_0^{(k)}$. Let $\mathcal{U}_0^i \sim \mathrm{Unif}_{[u_0^{(k)}, r]}$ and $\mathcal{U}_n^i \sim \mathrm{Unif}_{[u_n^{(1)}, r]}$, $i, n \in \mathbb{N}$, be independent, uniformly distributed random variables. Note that it is possible that there are no extractions alive ($N = 0$) or that there are only progenitors alive ($m_0 < k$). Using the vector notation for $\check{X}^r$, we define the probability kernel

$$\begin{aligned} \alpha_r(\check{x}; \cdot\,) &:= \alpha_r\Big(u_0^{(1,\ldots,k)}, (u_n^{(1)})_{1 \leq n \leq N}, (m_n)_{0 \leq n \leq N}; \cdot\,\Big) \\ &:= \mathcal{L}\Big(\delta_{u_0^{(1)}} + \ldots + \delta_{u_0^{(k \wedge m_0)}} + \sum_{i=k+1}^{m_0} \delta_{\mathcal{U}_0^i},\, \delta_{u_1^{(1)}} + \sum_{i=2}^{m_1} \delta_{\mathcal{U}_1^i},\, \ldots,\, \delta_{u_N^{(1)}} + \sum_{i=2}^{m_N} \delta_{\mathcal{U}_N^i}\Big). \end{aligned} \tag{6.9}$$

The generator calculations are similar to the ones in the preceding chapters, but quite lengthy. We exiled them to the Appendix (see A.6). We obtain the intertwining relation

$$\int \alpha_r\Big(u_0^{(1,\dots,k)}, (u_n^{(1)})_{1\le n\le N}, (m_n)_{0\le n\le N};\, \mathrm{d}u\Big)\; A^r_{PL} f(u_0,u_1,\dots)$$
$$= C^r_{PM} \int \alpha_r\Big(u_0^{(1,\dots,k)}, (u_n^{(1)})_{1\le n\le N}, (m_n)_{0\le n\le N};\, \mathrm{d}u\Big)\; f(u_0,u_1,\dots). \quad (6.10)$$

(PM stands for *progenitor-mass process*).

In order to write down the generator C^r_{PM}, we make some further definitions: Recall the definition of $\mathcal{D}^r_{PL}$ (see (6.1) and (6.2)). For $\tilde{u} \in [0,r)$ define

$$e^{-\lambda_n(\tilde{u})} := \frac{1}{r-\tilde{u}} \int_{\tilde{u}}^r g_n(z)\; \mathrm{d}z$$

and

$$\hat{f}_n(\tilde{u}, m) := g_n(\tilde{u}) e^{-\lambda_n(\tilde{u})(m-1)}. \quad (6.11)$$

Furthermore define, for $\tilde{u}^1 \le \ldots \le \tilde{u}^k$,

$$\hat{f}_0(\tilde{u}^1,\dots,\tilde{u}^k, m) := \prod_{i=1}^{k\wedge m} g_0(\tilde{u}^i)\cdot e^{-\lambda_0(\tilde{u}^k)((m-k)\vee 0)}. \quad (6.12)$$

(The truncation of the exponent deals with the case that less than k particles are alive.) Finally, we define, for a valid state

$$\check{x} = \delta_{(u_0^{(1)}, m_0)} + \ldots + \delta_{(u_0^{(k\wedge m_0)}, m_0)} + \sum_{n=1}^N \delta_{(u_n^{(1)}, m_n)} \in \mathcal{S}_{[0,r]\times\mathbb{N}}.$$

of $\check{X}^r$,

$$\begin{aligned}\hat{f}(\check{x}) &:= \int \alpha_r(\check{x};\, \mathrm{d}u)\; f(u)\\ &= \int \alpha_r\Big(u_0^{(1)},\dots,u_0^{(k)}, (u_n^{(1)})_{n\ge1}, (m_n)_{n\ge 0};\, \mathrm{d}u\Big)\; f(u_0,u_1,\dots)\\ &= \hat{f}_0(u_0^{(1)},\dots,u_0^{(k)}, m_0)\cdot \prod_{n\ge1} \hat{f}_n(u_n^{(1)}, m_n).\end{aligned} \quad (6.13)$$

(The product ranges over $\{n : m_n > 0\}$). When appropriate, we use the vector notation

$$\hat{f}(\check{x}) = \hat{f}\Big(u_0^{(1)},\dots,u_0^{(k)}, (u_n^{(1)})_{n\ge1}, (m_n)_{n\ge0}\Big).$$

Note that the vector notation may be misleading in some circumstances (also in the second line of (6.13)), because it suggests that there are at least k particles alive and should not be interpreted that way. Note that if there are only $m < k$ particles alive, we have $\hat{f}(\check{x}) = \prod_{i=1}^m g_0(u_0^{(i)})$.

The generator C^r_{PM} is composed of summands that correspond to the mechanisms described in Section 6.1,

$$C^r_{PM}\hat{f}\Big(u_0^{(1,\dots,k)},(u_n^{(1)})_{1\le n\le N},(m_n)_{0\le n\le N}\Big)$$
$$:= \Big(C^r_f + C^r_d + C^r_0 + C^r_e\Big)\hat{f}\Big(u_0^{(1,\dots,k)},(u_n^{(1)})_{1\le n\le N},(m_n)_{0\le n\le N}\Big).$$

As before, we write $u=\sum_{n\ge0}u_n$ and $m=u([0,r))$ and recall the definition of the jump factor

$$\varphi := \varphi(\check{x}) := \mathbb{1}_{(c,\infty)}(m)\cdot\frac{1}{1-c/m} + \mathbb{1}_{[0,c]}(m)\cdot\frac{r}{u_0^{(1)}}.$$

The formation of new extractions is modelled by

$$\begin{aligned}
&C^r_f\hat{f}\Big(u_0^{(1,\dots,k)},(u_n^{(1)})_{n\ge1},(m_n)_{n\ge0}\Big)\\
&:= \mathbb{1}_{[k,\infty)}(m)\cdot\sum_{q=2}^{k}2(q-1)(u_0^{(q)}-u_0^{(q-1)})\Bigg[\prod_{j=1}^{k-1}g_0(u_0^{(j)})\cdot\frac{1}{u_0^{(q)}-u_0^{(q-1)}}\int_{u_0^{(q-1)}}^{u_0^{(q)}}g_0(x)\,\mathrm{d}x\\
&\quad\times g_1(u_0^{(k)})e^{-\lambda_1(u_0^{(k)})(m_0-k)}\cdot\prod_{n\ge2}\hat{f}_n(u_{n-1}^{(1)},m_{n-1})-\hat{f}\Big(u_0^{(1,\dots,k)},(u_n^{(1)})_{n\ge1},(m_n)_{n\ge0}\Big)\Bigg]\\
&\quad+\mathbb{1}_{(0,k)}(m)\cdot\hat{f}_0(u_0^{(1,\dots,m)},m)\sum_{i=1}^{m}2\int_{u_0^{(i)}}^{r}\mathrm{d}x\,(g_0(x)-1).
\end{aligned}$$

Death events are modelled by

$$\begin{aligned}
&C^r_d\hat{f}\Big(u_0^{(1,\dots,k)},(u_n^{(1)})_{n\ge1},(m_n)_{n\ge0}\Big)\\
&= \mathbb{1}_{(c,\infty)}(m)\cdot\frac{rm}{c}\cdot\Bigg[\Bigg(\sum_{l=0}^{N_0}\binom{N_0}{l}p_0^l(1-p_0)^{N_0-l}\hat{f}_0\Big(\varphi u_0^{(1,\dots,k)},m_0-l\Big)\Bigg)\\
&\quad\times\prod_{n\ge1}\Bigg(\sum_{l=0}^{N_n}\binom{N_n}{l}p_n^l(1-p_n)^{N_n-l}\hat{f}_n\Big(\varphi u_n^{(1)},m_n-l\Big)\Bigg)-\hat{f}\Big(u_0^{(1,\dots,k)},(u_n^{(1)})_{n\ge1},(m_n)_{n\ge0}\Big)\Bigg]\\
&\quad+\mathbb{1}_{[0,c]}(m)\cdot\frac{rm}{c}\Big[\hat{f}(0)-\hat{f}\Big(u_0^{(1,\dots,k)},(u_n^{(1)})_{n\ge1},(m_n)_{n\ge0}\Big)\Big],
\end{aligned}\tag{6.14}$$

where the parameters fo the binomial distributions are

$$N_0 := (m_0-k)\vee 0,\qquad p_0 := \frac{rc}{(r-u_0^{(k)})m}\vee 1,$$
$$N_n := (m_n-1),\qquad p_n := \frac{rc}{(r-u_n^{(1)})m}\vee 1.$$

And the birth events are modelled by

$$\begin{aligned} &C_0^r \hat{f}\Big(u_0^{(1,\dots,k)}, (u_n^{(1)})_{n\geq 1}, (m_n)_{n\geq 0}\Big) \\ &\quad = \prod_{n\geq 1} \hat{f}_n(u_n^{(1)}, m_n) \cdot \Bigg[2k(r - u_0^{(k)}) \Big(\hat{f}_0(u_0^{(1,\dots,k)}, m_0+1) - \hat{f}_0(u_0^{(1,\dots,k)}, m_0)\Big) \\ &\qquad + (r - u_0^{(k)})(m_0 - k) \Big(\hat{f}_0(u_0^{(1,\dots,k)}, m_0+1) - \hat{f}_0(u_0^{(1,\dots,k)}, m_0)\Big) \\ &\qquad + \sum_{i=1}^{k} ((u_0^{(i)})^2 - r u_0^{(i)}) \frac{\mathrm{d}}{\mathrm{d}u_0^{(i)}} \hat{f}_0(u_0^{(1,\dots,k)}, m_1)\Bigg] \end{aligned} \tag{6.15}$$

and

$$\begin{aligned} &C_e^r \hat{f}\Big(u_0^{(1,\dots,k)}, (u_n^{(1)})_{n\geq 1}, (m_n)_{n\geq 0}\Big) \\ &\quad = \hat{f}_0(u_0^{(1,\dots,k)}, m_0) \sum_{n\geq 1} \prod_{\substack{l\geq 1\\ l\neq n}} \hat{f}_l(u_l^{(1)}, m_l) \cdot \Bigg[2(r - u_n^{(1)}) \Big(\hat{f}_n(u_n^{(1)}, m_n+1) - \hat{f}_n(u_n^{(1)}, m_n)\Big) \\ &\qquad + (r - u_n^{(1)})(m_n - 1) \Big(\hat{f}_n(u_n^{(1)}, m_n+1) - \hat{f}_n(u_n^{(1)}, m_n)\Big) \\ &\qquad + ((u_n^{(1)})^2 - r u_n^{(1)}) \cdot \frac{\mathrm{d}}{\mathrm{d}u_n^{(1)}} \hat{f}_n(u_n^{(1)}, m_n)\Bigg]. \end{aligned} \tag{6.16}$$

We summarize the dynamics briefly:

- At rate $2(q-1)((\check{U}_0^{(q),r})_t - (\check{U}_0^{(q-1),r})_t)$ a new root-progenitor with rank q is born at time t. There are two cases to consider. Case 1: $m_0 \geq k$. The progenitor with level $(\check{U}_0^{(k),r})_{t-}$ is removed from the root and the new progenitor is pigeon-holed into the root-progenitors. The mass of the root is set to k, and a new extraction is formed. The new extraction has the progenitor-level $(\check{U}_0^{(k),r})_{t-}$ and mass $(\check{M}_0^r)_{t-} - k + 1$. The new extraction gets the number 1 and the other extractions' numbers are incremented by one. Case 2: $m_0 < k$. When there are less than k particles left, the active progenitors are "filled up" before new extractions are formed.

- Death events happen at rate $\frac{rM_t^r}{c}$ (M^r is the total number of particles). All progenitors, whatever their extraction, are multiplied by $\varphi(U_t^r)$. This corresponds to death zone of width $\frac{rc}{m}$. Due to the uniform distribution of the bulk-particles, in each bulk a binomially distributed number of particles is killed. (Note that the bulk particles' levels are above their respective progenitors.) Note further that, if one or more progenitor levels are shifted beyond the threshold r, the mass argument of the test function in C_d^r has not to be reduced further, since it would make no difference (see (6.11) and (6.12)).

- Between formation times the bulk-mass of the root is a birth process with individual birth rate $r - (\check{U}_0^{(k),r})_t$ and immigration rate $2k(r - \check{U}_0^{(k),r})$, due to the k progenitors. The bulk mass of an extraction with number $n \geq 1$ is a birth process with individual rate $r - (\check{U}_n^{(1),r})_t$ and immigration rate $2(r - (\check{U}_n^{(1),r})_t)$.

- All progenitors move according to the differential equation $\dot{u} = u^2 - ru$.

With the formal calculations in Section A.6 at hand, we can apply the Markov Mapping Theorem. Let

$$\check{\mathcal{S}}^r_{\text{start}} := \Big\{ \delta_{(u_0^{(1)}, m_0)} + \ldots + \delta_{(u_0^{(k)}, m_0)} + \sum_{n=1}^{N} \delta_{(u_n, m_n)} \;:\; m_0 \geq k,\, m_n \in \mathbb{N}, N \in \mathbb{N}_0, \\ 0 < u_0^{(1)} < \ldots < u_0^{(k)} < u_1^{(1)} \ldots < u_N^{(1)} < r \Big\}.$$

be the set of valid initial states of the progenitor-mass process.

Theorem 6.2.1. *Let $\mu_0 \in \mathcal{M}_1(\mathcal{S}_{[0,r]\times\mathbb{N}})$ be a initial distribution concentrated on $\check{\mathcal{S}}^r_{start}$. Define $\nu_0 := \int \alpha_r(y, \,\cdot\,)\, \mu_0(\mathrm{d}y)$. There exists a solution $\check{U}^r$ of the the $D_{\mathcal{S}_{[0,r]\times\mathbb{N}_0}}[0,\infty)$-martingale problem for (A^r_{PL}, ν_0) (see (6.3)) such that*

$$\check{X}^r = \Big(\delta_{(\check{U}_0^{(1),r}, \check{M}_0^r)_t} + \ldots + \delta_{(\check{U}_0^{(k),r}, \check{M}_0^r)_t} + \sum_{n\geq 1} \delta_{(\check{U}_n^{(1),r}, \check{M}_n^r)_t} \Big)_{t\geq 0}$$

is a solution of the $D_{\mathcal{S}_{[0,r]\times\mathbb{N}}}[0,\infty)$-martingale problem for (C^r_{PM}, μ_0).

For every $t \geq 0$, $\Gamma \in \mathcal{B}(\mathcal{S}_{[0,r]\times\mathbb{N}_0})$ we have

$$\mathbf{P}\left[\check{U}^r_t \in \Gamma \;\middle|\; \mathcal{F}_t^{\check{X}^r} \right] = \alpha(\check{X}^r_t; \Gamma).$$

We call $\check{X}^r$ the progenitor-mass process.

Proof. The statement follows from calculations in A.6 and the Markov Mapping Theorem A.5.1 with $\psi(u) = \sum_n m_n$ (see Lemma A.6.1), and

$$\gamma(u) = \delta_{(u_0^{(1)}, m_0)} + \ldots + \delta_{(u_0^{(k)}, m_0)} + \sum_{n\geq 1} \delta_{(u_n^{(1)}, m_n)},$$

where we use the notation of said theorem.

Uniqueness of the progenitor-level process is obtained along the same lines as in the proof of Theorem 4.1.1. □

6.2.1. A few words on tightness of the progenitor-density process

We rescale the masses and define

$$\tilde{X}^r_t := \delta_{(\check{U}_0^{(1),r}, \check{Y}_0^r)_t} + \ldots + \delta_{(\check{U}_0^{(k),r}, \check{Y}_0^r)_t} + \sum_{n\geq 1} \delta_{(\check{U}_n^{(1),r}, \check{Y}_n^r)_t},$$

where $\check{Y}_n^r := \frac{\check{M}_n^r}{r}$. Similarly as we did earlier, we stop the processes shortly before the total mass vanishes completely. Let $\delta > 0$. Recall $\tau_\delta^r = \inf\{s : Y_s^r \leq \delta\}$ is the first time, when the total mass density coordinate falls below δ. Define the stopped processes $\check{U}_t^{r,\delta} := \check{U}_{t\wedge\tau_\delta^r}^r$ and $\tilde{X}^{r,\delta} := \tilde{X}_{t\wedge\tau_\delta^r}^r$.

We check briefly that the progenitor-level system $(\check{U}^{r,\delta})_r$ and the progenitor-density system $(\tilde{X}^{r\delta})_r$ are tight. We do not give complete proofs, but sketch the ideas.

Theorem 6.2.2. *There is an $r_0 > 0$ such that the family $\left(\check{U}^{r,\delta}\right)_{r\geq r_0}$ is tight in the Skorohod space $D_{\mathcal{S}_{\mathbb{R}^+\times\mathbb{N}}}[0,\infty)$.*

Sketch of proof. Recall that $U^{(i),r,\delta}$ is the *overall* i-th lowest level and let $N^{(i),r,\delta}$ be the subpopulation number of the corresponding particle. Denote by

$$\check{U}^{L,r,\delta} := \sum_{i=1}^{L} \delta_{(U^{(i),r,\delta},N^{(i),r,\delta})},$$

the process consisting of the L particles with the *overall* lowest levels and their subpopulation numbers. The highest extraction number in $\check{U}^{L,r,\delta}$ is at most $L-k$. Using this observation, one can easily adapt the proof of Theorem 4.2.3 and show that $\left(\check{U}^{L,r,\delta}\right)_{r\geq r_0}$ is tight in $D_{(\mathbb{R}^+)^L\times\mathbb{N}_0^L}[0,\infty)$. One can then proceed as in the proof of Theorem 4.2.4 and show tightness of the family $\left(\check{U}^{r,\delta}\right)_{r\geq r_0}$ in $D_{\mathcal{S}_{\mathbb{R}^+\times\mathbb{N}_0}}[0,\infty)$. □

Theorem 6.2.3. *The family $\left(\tilde{X}^{r,\delta}\right)_{r\geq r_0}$ is tight in $D_{\mathcal{S}_{\mathbb{R}^+\times\mathbb{R}^+}}[0,\infty)$.*

Sketch of proof. Tightness of $(Y^r)_r$ implies the compact containment condition for the mass $\check{Y}_n^r$ of the n-th extraction. The overall mass density Y^r has no drift, hence the drifts of the subpopulation densities $\check{Y}_n^r$ sum up to zero. For each n, the drift of $\check{Y}_n^r$ is bounded from above (cf. (6.18)). Since for any $T > 0$, there are only finitely many extractions alive at some point in $[0,T]$, the drift of $\check{Y}_n^r$ is also bounded from below. These considerations imply D-tightness of the family $(\check{Y}_n^r)_r$ in $D_{\mathbb{R}^+}[0,\infty)$. One may use then tightness of $\left(\check{U}^{r,\delta}\right)_{r\geq r_0}$ to show the assertion. □

Remark 6.2.4. *Note that progenitors' levels $(\check{U}_n^{(1),r,\delta})_r$, $n = 1,2,\dots$ do explode. They are not tight as $\mathbb{R}^+$-valued processes, but they are tight as atoms with regard to the vague topology.*

6.3. The progenitor-mass process as semimartingale

By stopping the process $\tilde{X}^r$ when the number of extractions exceeds some threshold $K \in \mathbb{N}$, we reduce the progenitor-density process to a finite dimensional setting. To this end we provide $K \in \mathbb{N}$ "slots" for extractions and agree on some dummy values for extractions that do not exist at a given time. Define

$$(\tilde{U}_n^{(i),r})_t := \begin{cases} (\check{U}_n^{(i),r})_t & \text{if the respective progenitor exists at time } t, \\ r & \text{if the respective progenitor does not exist at time } t, \end{cases}$$

and define

$$(\tilde{Y}_n^r)_t := \begin{cases} \frac{(\check{M}_n^r)_t}{r} & \text{if the respective extraction exists at time } t, \\ 0 & \text{if the respective extraction does not exist at time } t. \end{cases}$$

Let

$$\sigma_K^r := \inf\{s \geq 0 : \ \check{X}_s^r([0,r) \times \mathbb{N}) > K + k\}$$

be the first time, when more than K extractions exist. Define the process

$$\tilde{X}_t^{r,K} := \left(\tilde{U}_0^{(1,\ldots,k),r}, (\tilde{U}_n^{(1),r})_{1 \leq n \leq K}, (\tilde{Y}_n^r)_{0 \leq n \leq K}\right)_{t \wedge \sigma_K^r} \tag{6.17}$$

that takes values in $[0,r]^{k+K} \times (\mathbb{R}^+)^{K+1}$.

The process $\tilde{X}^{r,K}$ is a semimartingale. (We determine its semimartingale characteristics in Section A.7.) In Section 6.3.1 we will make some informal considerations concerning the dynamics of the limit of the progenitor-density process $\check{X}^{r,\delta}$. But first we revisit briefly the process $X^r = (U^{(1,\ldots,k),r}, Y^r)$ from Section 4.2. Note that σ_K^r is a stopping time with respect to the filtration, generated by X^r: Since the positions of former progenitors are adapted to the filtration $\mathcal{F}^{U^{(1,\ldots,k),r}}$, the formation- and extinction-times of extractions are $\mathcal{F}^{U^{(1,\ldots,k),r}}$-stopping times. We assert that for any $K > 0$ the stopped process $(X^r_{t \wedge \sigma_K^r})_{t \geq 0}$ is indeed a semimartingale with respect to its own filtration:

Theorem 6.3.1. *Let $K > 0$. The process $(X^r_{t \wedge \sigma_K^r})_t$ is a semimartingale with respect to its own filtration.*

Proof. The process $\tilde{X}^{r,K}$ is a semimartingale with respect to its own filtration $\mathcal{F}^{\tilde{X}^{r,K}}$. Since $Y^r_{t \wedge \sigma_K^r} = \sum_{n=0}^K (Y_n^r)_{t \wedge \sigma_K^r}$, the process $(X^r_{t \wedge \sigma_K^r})_t$ is a $\mathcal{F}^{\tilde{X}^{r,K}}$-semimartingale. The filtration generated by $(X^r_{t \wedge \sigma_K^r})_t$ is a subfiltration of $\mathcal{F}^{\tilde{X}^{r,K}}$, hence $(X^r_{t \wedge \sigma_K^r})_t$ is also a semimartingale with respect its own filtration by Stricker's Theorem (cf. Theorem II.4 in [Pro04]). $\square$

6.3.1. Informal considerations about the limit $\check{X}$ of the progenitor-density process

We stop the process $\tilde{X}^{r,K}$, when the overall mass density Y^r falls below some threshold $\delta > 0$. Recall the definition

$$\tau_\delta^r := \inf\{s : Y_s^r \leq \delta\}.$$

Define $\tilde{X}_t^{r,K,\delta} := \tilde{X}_{t\wedge\tau_\delta^r}^{r,K}$. The sequence $(\check{X}^{r,\delta})_r$ is tight, but, by Remark 6.2.4, the sequence of semimartingales $\tilde{X}^{K,r,\delta}$ is not. Hence the machinery of Theorem IX.2.4 in [JS03] cannot be put to use, in order to determine the dynamics in the limit. But we deem some considerations, regarding the bounded variation and the quadratic variation of the mass densities for "large r", insightful.

The following calculations are in no way rigorous and should be understood as an ad hoc approach to analyse the interplay of low levels and the mass density in the limit. In Section A.7 we determine the semimartingale characteristics of $\tilde{X}_t^{r,K,\delta}$. The characteristics are rather unwieldy, because of the "dummy values" for non existent extractions and because of the intricacies that occur, when progenitors are in the death zone. We refrain from these annoyances and consider benign situations.

For the rest of this section we drop the δ in our notation. Set $b = 1$ and assume $t < \tau_\delta^r \wedge \sigma_K^r$. Since we are interested in a "large r" scenario, we assume $r\delta > c, k$. (Hence we avoid situations where the death zone extends over the whole interval $[0, r]$, or, where active progenitors are dead.) We denote by

$$\nu^{\tilde{X},r}(\omega; dt, dx) = \nu_d^{\tilde{X},r}(\omega; dt, dx) + \sum_{n=0}^{K} \nu_n^{\tilde{X},r}(\omega; dt, dx) + \nu_f^{\tilde{X},r}(\omega; dt, dx),$$

the predictable compensator of the empirical jump measure of $\tilde{X}_t^{r,K}$ (see Section A.7). The measure $\nu_d^{\tilde{X},r}$ compensates the jumps induced by death events, $\nu_n^{\tilde{X},r}$ compensates the jumps induced by births in the bulk of the n-th subpopulation and $\nu_f^{\tilde{X},r}$ compensates the jumps induced by the genesis of new extractions (see (A.15), (A.16),(A.17) and (A.18)).

First we examine the bounded variation part of $\tilde{Y}_n^r$, $n \geq 1$. We assume that at time t the progenitor of the considered subpopulation is not in the death zone, $\tilde{U}_n^{(1),r} < r - c/Y_t^r$. We are not interested in the portion of the drift that is caused by the "shifting" of extractions after formation events. Hence we ignore the compensator of jumps that are to due to formations of new extractions. Let

$$N_t^{n,r} := r \cdot (\tilde{Y}_n^r)_t - 1 \quad \text{and} \quad p_t^{n,r} := \frac{c}{\left(r - (\tilde{U}_n^{(1),r})_t\right) \cdot Y_t^r}$$

and recall the definition of the truncation function $h(x) = (\tilde{h}(x_i))_i$ in (4.32). We have

$$
\begin{aligned}
B_t^{\tilde{Y}_n,r} &:= h^{k+K+1+n}(x) * (\nu_n^{\tilde{X},r} + \nu_d^{\tilde{X},r}) \\
&= \int_0^t \mathrm{d}s \ \left[\left(r - (\tilde{U}_n^{(1),r})_s\right) \left(r(\tilde{Y}_n^r)_s + 1\right) \tilde{h}\left(\frac{1}{r}\right) + \frac{r^2 Y_s^r}{c} \int \mathrm{Bin}_{N_s^{n,r}, p_s^{n,r}}(\mathrm{d}x) \ \tilde{h}\left(-\frac{x}{r}\right) \right] \\
&= \int_0^t \mathrm{d}s \ \left[\frac{\left(r - (\tilde{U}_n^{(1),r})_s\right) \left(r(\tilde{Y}_n^r)_s + 1\right)}{r} - \frac{r^2 Y_s^r}{c} \cdot \frac{c\left(r(\tilde{Y}_n^r)_s - 1\right)}{\left(r - (\tilde{U}_n^{(1),r})_s\right) \cdot Y_s^r} \cdot \frac{1}{r} + R_s^r \right] \\
&= \int_0^t \mathrm{d}s \ \left[\frac{2\left(r - (\tilde{U}_n^{(1),r})_s\right)}{r} + \frac{\left(r - (\tilde{U}_n^{(1),r})_s\right)\left(r(\tilde{Y}_n^r)_s - 1\right)}{r} - \frac{r\left(r(\tilde{Y}_n^r)_s - 1\right)}{\left(r - (\tilde{U}_n^{(1),r})_s\right)} + R_s^r \right] \\
&= \int_0^t \mathrm{d}s \ \left[\frac{2\left(r - (\tilde{U}_n^{(1),r})_s\right)}{r} + \frac{\left(r - (\tilde{U}_n^{(1),r})_s\right)^2 \left(r(\tilde{Y}_n^r)_s - 1\right) - r^2 \left(r(\tilde{Y}_n^r)_s - 1\right)}{r\left(r - (\tilde{U}_n^{(1),r})_s\right)} + R_s^r \right] \\
&= \int_0^t \mathrm{d}s \ \left[\frac{2\left(r - (\tilde{U}_n^{(1),r})_s\right)}{r} + \frac{\left(-2r(\tilde{U}_n^{(1),r})_s + (\tilde{U}_n^{(1),r})_s^2\right)\left(r(\tilde{Y}_n^r)_s - 1\right)}{r\left(r - (\tilde{U}_n^{(1),r})_s\right)} + R_s^r \right] \\
&= \int_0^t \mathrm{d}s \ \left(2 - 2(\tilde{U}_n^{(1),r})_s (\tilde{Y}_n^r)_s\right) + o(1)
\end{aligned}
\tag{6.18}
$$

for large r. In a very similar calculation we see that the bounded variation part of the root density is

$$
B_t^{\tilde{Y}_0,r} \approx \int_0^t \mathrm{d}s \ 2\left(k - (\tilde{U}_0^{(k),r})_s (\tilde{Y}_0^r)_s\right) \tag{6.19}
$$

for large r. At this point the non-Markovian nature of $(U^{(1,\dots,k),r}, Y^r)$ comes to light again: Consider a state, where no extractions are alive, $Y_t^r = (\tilde{Y}_0^r)_t$. The drift of Y^r is then solely given by (6.19),

$$
\begin{aligned}
\mathrm{d}B_t^{Y^r} &\approx \left(2k - 2(\tilde{U}_0^{(k),r})_t (\tilde{Y}_0^r)_t\right) \mathrm{d}t \\
&= \left(2k - 2U_t^{(k),r} Y_t^r\right) \mathrm{d}t.
\end{aligned}
$$

The first drift term, $2k$, is due to immigration, fuelled by the k progenitors of the root. The second term, $-2\tilde{U}_1^{(k),r} Y^r$, is due to the fact that the knowledge of low levels tells us, how long the process roughly persists. The second term forces the process to die out, as $U^{(k),r}$ explodes.

Let τ be the time of a new extraction's formation and assume there are no other extractions alive at that time. We obtain for a short time span $[\tau, \tau+\epsilon]$ (again ignoring the formation-term of the drift)

$$
\begin{aligned}
B_{\tau+\epsilon}^{Y^r} - B_\tau^{Y^r} &\approx \int_\tau^{\tau+\epsilon} \mathrm{d}s \ \left[\left(2k - 2(\tilde{U}_0^{(k),r})_s (\tilde{Y}_0^r)_s\right) + \left(2 - 2(\tilde{U}_1^{(1),r})_s (\tilde{Y}_1^r)_s\right) \right] \\
&\approx \epsilon \left[2k - 2U_\tau^{(k),r} Y_\tau^r + 2 \right].
\end{aligned}
$$

Hence, with the creation of the extraction, the drift increased by 2. As the new progenitor and its extraction fade away, the overall drift returns to normal. These considerations are in line with our intuition that the birth of a persistent family should lead to a positive drift in the overall mass density.

Let $n \geq 1$ and let

$$\tilde{C}^{n,r}_t := h^{k+K+1+n}(x)^2 * (\nu^{\tilde{X},r}_n + \nu^{\tilde{X},r}_d)$$

be the quadratic variation of the n-th extraction's mass density, where we ignore the jumps that are due to formations of new extractions. Again we want to explore informally, what happens with $\tilde{C}^{n,r}$ for "large r". Using

$$\begin{aligned}
\frac{r^2 Y^r_s}{c} \int \mathrm{Bin}_{N^{n,r}_s, p^{n,r}_s}(\mathrm{d}x)\, \tilde{h}\left(-\frac{x}{r}\right)^2 &= \frac{Y^r_s}{c} \int \mathrm{Bin}_{N^{n,r}_s, p^{n,r}_s}(\mathrm{d}x)\, x^2 + R^r_s \\
&= \frac{Y^r_s}{c} \cdot \left(N^{n,r}_s p^{n,r}_s \,(1 - p^{n,r}_s) + (N^{n,r}_s)^2\, (p^{n,r}_s)^2 \right) + R^r_s \\
&= \frac{Y^r_s}{c} \cdot \left(N^{n,r}_s p^{n,r}_s + \left((N^{n,r}_s)^2 - N^{n,r}_s \right) (p^{n,r}_s)^2 \right) + R^r_s \\
&= \frac{r\big(\tilde{Y}^r_n\big)_s - 1}{r - \big(\tilde{U}^{(1),r}_n\big)_s} + \frac{c\left((r\big(\tilde{Y}^r_n\big)_s)^2 - 3r\big(\tilde{Y}^r_n\big)_s + 2 \right)}{\left(r - \big(\tilde{U}^{(1),r}_n\big)_s \right)^2 Y^r_s} + R^r_s,
\end{aligned}$$

we obtain for large r

$$\begin{aligned}
\tilde{C}^{n,r}_t &= \int_0^t \mathrm{d}s\, \left[2\left(r - \big(\tilde{U}^{(1),r}_n\big)_s \right) + \left(r - \big(\tilde{U}^{(1),r}_n\big)_s \right) \left(r\big(\tilde{Y}^r_n\big)_s - 1 \right) \right] \tilde{h}\left(\frac{1}{r}\right)^2 \\
&\qquad + \int_0^t \mathrm{d}s\, \frac{r^2 Y^r_s}{c} \int \mathrm{Bin}_{N^{n,r}_s, p^{n,r}_s}(\mathrm{d}x)\, h\left(-\frac{x}{r}\right)^2 \\
&= \int_0^t \mathrm{d}s\, \left[\frac{2(r - \big(\tilde{U}^{(1),r}_n\big)_s)}{r^2} + \frac{(r - \big(\tilde{U}^{(1),r}_n\big)_s)(r\big(\tilde{Y}^r_n\big)_s - 1)}{r^2} \right. \\
&\qquad \left. + \frac{r\big(\tilde{Y}^r_n\big)_s - 1}{r - \big(\tilde{U}^{(1),r}_n\big)_s} + \frac{c\left((r\big(\tilde{Y}^r_n\big)_s)^2 - 3r\big(\tilde{Y}^r_n\big)_s + 2 \right)}{\left(r - \big(\tilde{U}^{(1),r}_n\big)_s \right)^2 Y^r_s} + R^r_s \right] \\
&= \int_0^t \mathrm{d}s\, \big(\tilde{Y}^r_n\big)_s \left(2 + \frac{c\big(\tilde{Y}^r_n\big)_s}{Y^r_s} \right) + o(1).
\end{aligned}$$

For the root we obtain in a similar calculation,

$$\tilde{C}^{0,r}_t \approx \int_0^t \mathrm{d}s\, \big(\tilde{Y}^r_0\big)_s \left(2 + \frac{c\big(\tilde{Y}^r_0\big)_s}{Y^r_s} \right).$$

So the volatility of an extraction's mass density depends on the relative density of the extraction. For large r we envision $\tilde{Y}^r_n$ as a conditional branching diffusion with branching rate $2 + \frac{c\tilde{Y}^r_n}{Y^r}$, which is high if the overall mass is low compared to $\tilde{Y}^r_n$.

For the covariation $\tilde{C}^{ij,r}$ of two extraction $\tilde{Y}_i^r$ and $\tilde{Y}_j^r$, $i, j \geq 1$ we obtain

$$\begin{aligned}
\tilde{C}^{ij,r} &:= h^{k+K+1+i}(x)h^{k+K+1+j}(x) * \nu^{\tilde{X},r} \\
&= \int_0^t \mathrm{d}s \frac{r^2 Y_s^r}{c} \int \mathrm{Bin}_{N_s^{i,r},p_s^{i,r}}(\mathrm{d}x_i)\ h\left(-\frac{x_i}{r}\right) \mathrm{Bin}_{N_s^{j,r},p_s^{j,r}}(\mathrm{d}x_j)\ h\left(-\frac{x_j}{r}\right) \\
&= \int_0^t \mathrm{d}s \left[\frac{Y_s^r}{c} N_s^{i,r} p_s^{i,r} N_s^{j,r} p_s^{j,r} + R_s^r\right] \\
&\approx \int_0^t \mathrm{d}s \frac{c\big(\tilde{Y}_i^r\big)_s \big(\tilde{Y}_j^r\big)_s}{Y_s^r}
\end{aligned}$$

We obtain the same result for covariations between root and extraction,

$$\tilde{C}^{0i,r} \approx \int_0^t \mathrm{d}s \frac{c\big(\tilde{Y}_1^r\big)_s \big(\tilde{Y}_i^r\big)_s}{Y_s^r}$$

In a kind of "test calculation" the covariations sum up correctly to the quadratic variation of the overall mass density (c.f. Theorem 4.3.2),

$$\begin{aligned}
\sum_{i,j} \tilde{C}^{ij,r} &\approx \int_0^t \mathrm{d}s\ \sum_i \left(\tilde{Y}_i^r \left(2 + \frac{c\big(\tilde{Y}_i^r\big)_s}{Y_s^r}\right) + \sum_{j:\ j \neq i} \frac{c\big(\tilde{Y}_i^r\big)_s \big(\tilde{Y}_j^r\big)_s}{Y_s^r} \right) \\
&= \int_0^t \mathrm{d}s\ \left[2Y_s^r + \sum_i \left(\frac{c\big(\tilde{Y}_i^r\big)_s^2}{Y_s^r} + \sum_{j:\ j \neq i} \frac{c\big(\tilde{Y}_i^r\big)_s \big(\tilde{Y}_j^r\big)_s}{Y_s^r} \right) \right] \\
&= \int_0^t \mathrm{d}s\ Y_s^r (2 + c).
\end{aligned}$$

A. Appendix

A.1. Poisson random measures

Let $(S, \mathcal{S})$ be a measurable space and ν a σ-finite measure on $\mathcal{S}$. The following is Lemma A.3 in [KR11].

Lemma A.1.1 (Poisson random measures process)**.** *Let ξ be a Poisson random measure with mean measure ν and $f \in L^1(\nu)$, then*

$$\mathbf{E}\left[e^{\int f(z)\,\xi(dz)}\right] = e^{\int (e^f - 1)\,d\nu}, \tag{A.1}$$

$$\mathbf{E}\left[\int f(z)\,\xi(dz)\right] = \int f d\nu, \tag{A.2}$$

$$\mathbf{Var}\left(\int f(z)\,\xi(dz)\right) = \int f^2\,d\nu. \tag{A.3}$$

Write $\xi = \sum_i \delta_{Z_i}$. For $g \geq 0$ with $\log(g) \in L^1(\nu)$ we have

$$\mathbf{E}\left[\prod_i g(Z_i)\right] = e^{\int (g-1)\,d\nu}. \tag{A.4}$$

If $hg, (g-1) \in L^1(\nu)$, then

$$\mathbf{E}\left[\sum_i h(Z_i) \prod_k g(Z_k)\right] = \int hg\,d\nu \cdot e^{\int (g-1)\,d\nu}, \tag{A.5}$$

$$\mathbf{E}\left[\sum_{i \neq j} h(Z_i)h(Z_j) \prod_k g(Z_k)\right] = \left(\int hg\,d\nu\right)^2 e^{\int (g-1)\,d\nu}. \tag{A.6}$$

A.2. Corollary to Fubini

We phrase the trick from equation (3.6) as a small "ready to use" lemma.

Lemma A.2.1. *Let $f : [a,b] \times [a,b] \to \mathbb{R}$ be an integrable function, then*

$$\int_a^b \left(\int_a^y f(x,y)\,\mathrm{d}x\right) \mathrm{d}y = \int_a^b \left(\int_x^b f(x,y)\,\mathrm{d}y\right) \mathrm{d}x.$$

Proof. Define $\Delta := \{(x, y) \in [a, b] \times [a, b] \ : \ x \leq y\}$. Then

$$\begin{aligned}\int_a^b \left(\int_a^y f(x,y) \,\mathrm{d}x \right) \mathrm{d}y &= \int_a^b \left(\int_a^b \mathbb{1}_\Delta(x,y) \cdot f(x,y) \,\mathrm{d}x \right) \mathrm{d}y \\ &= \int_a^b \left(\int_a^b \mathbb{1}_\Delta(x,y) \cdot f(x,y) \,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_a^b \left(\int_x^b f(x,y) \,\mathrm{d}y \right) \mathrm{d}x.\end{aligned}$$

□

A.3. Joint tightness in $D[0, \infty)$

The lemmas in this section are concerned with the question, under which circumstances $D[0, \infty)$-tightness of marginals imply tightness of the joint process.

Lemma A.3.1. *Let $x^{1,r}, x^{2,r}, \dots, x^{n,r} \in D_{\mathbb{R}}[0, \infty)$. The family $\left\{(x^{1,r}, x^{2,r}, \dots, x^{n,r})\right\}_r$ is relatively compact in $D_{\mathbb{R}^n}[0, \infty)$, if (and only if) $\left\{x^{i,r}\right\}_r$ and $\left\{x^{i,r} + x^{j,r}\right\}_r$ are relatively compact in $D_{\mathbb{R}}[0, \infty)$ for all $i, j = 1, \dots, n$.*

Proof. This is Problem 22 in Chapter 3 of [EK86]. □

Lemma A.3.2. *Let $X^{1,r}, X^{2,r}, \dots, X^{n,r}$ be tight processes in $D_{\mathbb{R}}[0, \infty)$. If $X^{i,r} + X^{j,r}$ is tight in $D_{\mathbb{R}}[0, \infty)$ for all $i, j = 1, \dots, n$, then $X^r := (X^{1,r}, X^{2,r}, \dots, X^{n,r})$ is tight in $D_{\mathbb{R}^n}[0, \infty)$.*

Proof. Let $\epsilon > 0$. Since $\left\{X^{i,r}\right\}_r$ and $\left\{X^{i,r} + X^{j,r}\right\}_r$ are tight, we find $K_i, K_{ij} \subset D_{\mathbb{R}}[0, \infty)$ compact, such that

$$\begin{aligned}\sup_r \mathbf{P}\left[X^{i,r} \notin K_i\right] &< \frac{\epsilon}{n^2} \quad \text{for } i = 1, \dots, n, \\ \sup_r \mathbf{P}\left[X^{i,r} + X^{j,r} \notin K_{i,j}\right] &< \frac{\epsilon}{n^2} \quad \text{for } i, j = 1, \dots, n \text{ and } i \neq j.\end{aligned}$$

Define the set

$$K := \left\{(x^1, x^2, \dots, x^n) \in D_{\mathbb{R}^n}[0, \infty) \ : \ x^i \in K_i, x^i + x^j \in K_{ij} \text{ for } i, j = 1, \dots, n, \ i \neq j\right\}.$$

For any sequence $\{(x^{1,r}, x^{2,r}, \dots, x^{n,r})\}_r \subset K$ the sequence of coordinates $\{x^{i,r}\}_r \subset K_i$ and the sequence of sums $\{(x^{i,r} + x^{j,r})\}_r \subset K_{ij}$ are contained in compact sets, thus they are relatively compact. Lemma A.3.1 states that $\{(x^{1,r}, x^{2,r}, \dots, x^{n,r})\}_r$ has a convergent subsequence in $D_{\mathbb{R}^n}[0, \infty)$. Hence K is relatively compact and $\overline{K}$ is compact in $D_{\mathbb{R}^n}[0,\infty)$. Furthermore we have

$$\begin{aligned}\sup_r \mathbf{P}\left[X^r \notin \overline{K}\right] &\le \sup_r \mathbf{P}\left[X^r \notin K\right] \\ &\le \sum_{i=1}^n \sup_r \mathbf{P}\left[X^{i,r} \notin K_i\right] + \sum_{\substack{i,j=1 \\ i\neq j}}^n \sup_r \mathbf{P}\left[X^{i,r} + X^{j,r} \notin K_{i,j}\right] \\ &< \epsilon.\end{aligned}$$

□

Lemma A.3.3. *If $(X^r)_r$ is a family of $D_{\mathbb{R}^{d_1}}[0,\infty)$-tight processes and $(Y^r)_r$ is a family of $C_{\mathbb{R}^{d_2}}[0,\infty)$-tight processes, then (X^r, Y^r) is tight in $D_{\mathbb{R}^{d_1+d_2}}[0,\infty)$.*

Proof. This is Corollary VI.3.33 in [JS03]. □

Lemma A.3.4. *Let $(X^n)_n$ be tight (C-tight) in $D_{\mathbb{R}^d}[0,\infty)$. Further let $(\tau_n(s))_s$ be random time changes with $\tau_n(0) = 0$ that are strictly increasing and continuous in s. Define*

$$Y^n_s := X^n_{\tau_n(s)}.$$

If for any $\epsilon, T > 0$ there is $K > 0$ such that

$$\liminf_{n\to\infty} \mathbf{P}\left[|\tau_n(t) - \tau_n(s)| \le K \cdot |t-s| \text{ for all } s,t \in [0,T]\right] \ge 1-\epsilon,$$

then $(Y^n)_n$ is tight (C-tight) in $D_{\mathbb{R}^d}[0,\infty)$.

Proof. Recall the definition of the modified modulus of continuity of a càdlàg path x,

$$w'_T(x,\rho) := \inf_{(I_k)>\rho} \max_k \sup_{s,t\in I_k} |x(t) - x(s)|,$$

where the infimum extends over all partitions of the interval $[0,T)$ into subintervals $I_k = [a,b)$ such that $b - a \ge \rho$. The family $(Y^n)_n$ is tight if and only if the following conditions hold (cf. Theorem 16.8 in [Bil99] or Theorem 16.10 in [Kal02]):

(i) For each $T > 0$,

$$\lim_{a\to\infty} \limsup_{n\to\infty} \mathbf{P}\left[\sup_{0\le t\le T} |Y^n_t| \ge a\right] = 0.$$

(ii) For each $T, \eta > 0$,

$$\lim_{\rho\to 0} \limsup_{n\to\infty} \mathbf{P}\left[w'_T(Y^n,\rho) \ge \eta\right] = 0.$$

Assume $(X^n)_n$ is tight. Let $a, T, \epsilon > 0$. Define the events

$$A^n_K := \{|\tau_n(t) - \tau_n(s)| \le K \cdot |t-s| \text{ for all } s,t \in [0,T]\}.$$

According to prerequisites there exists $K > 0$ such that

$$\limsup_{n\to\infty} \mathbf{P}\left[(A_K^n)^c\right] < \frac{\epsilon}{2}.$$

Since condition *(i)* holds for $(X^n)_n$ there exists $a > 0$ such that

$$\limsup_{n\to\infty} \mathbf{P}\left[\sup_{0\le t\le KT} |X_t^n| \ge a\right] < \frac{\epsilon}{2}.$$

Since τ_n is increasing and continuous, it maps $[0,T] \to [0,\tau_n(T)]$ one on one, and we have

$$\begin{aligned}
\limsup_{n\to\infty} \mathbf{P}\left[\sup_{0\le t\le T} |Y_t^n| \ge a\right] &= \limsup_{n\to\infty} \mathbf{P}\left[\sup_{0\le t\le T} \left|X_{\tau_n(t)}^n\right| \ge a\right] \\
&= \limsup_{n\to\infty} \mathbf{P}\left[\sup_{0\le t\le \tau_n(T)} |X_t^n| \ge a\right] \\
&\le \limsup_{n\to\infty} \mathbf{P}\left[(A_K^n)^c\right] + \limsup_{n\to\infty} \mathbf{P}\left[\left\{\sup_{0\le t\le \tau_n(T)} |X_t^n| \ge a\right\} \cap A_K^n\right] \\
&\le \frac{\epsilon}{2} + \limsup_{n\to\infty} \mathbf{P}\left[\sup_{0\le t\le KT} |X_t^n| \ge a\right] \\
&< \epsilon.
\end{aligned}$$

This implies condition *(i)* for $(Y^n)_n$.

We turn to the structure condition *(ii)*. Let $T, \eta, \epsilon > 0$ and let K, A_K^n be as above. Since condition *(ii)* holds for $(X^n)_n$ there exists $\rho > 0$ such that

$$\limsup_{n\to\infty} \mathbf{P}\left[w'_{KT}(X^n, K\rho) > \eta\right] < \frac{\epsilon}{2}.$$

We have

$$\begin{aligned}
w'_T(Y^n, \rho) &= \inf_{(I_k)>\rho} \max_k \sup_{s,t\in I_k} |Y_t^n - Y_s^n| \\
&= \inf_{(I_k)>\rho} \max_k \sup_{s,t\in \tau_n(I_k)} |X_t^n - X_s^n| \\
&= \inf_{(I'_k)\,:\,\tau_n^{-1}(I'_k)>\rho} \max_k \sup_{s,t\in I'_k} |X_t^n - X_s^n|,
\end{aligned}$$

where in the last line the infimum extends over all partitions of $[0, \tau_n(T))$ into subintervals $I'_k = [a,b)$ such that $\tau_n^{-1}(b) - \tau_n^{-1}(a) > \rho$.

We have

$$\{(I'_k) > K\rho\} \cap A_K^n \subseteq \left\{(I'_k)\ :\ \tau_n^{-1}(I'_k) > \rho\right\} \cap A_K^n.$$

Hence, on the event A_K^n,

$$\begin{aligned}
w'_T(Y^n, \rho) &\le \inf_{(I'_k)>K\cdot\rho} \max_k \sup_{s,t\in I'_k} |X_t^n - X_s^n| \\
&= w'_{\tau_n(T)}(X^n, K\rho) \\
&\le w'_{KT}(X^n, K\rho).
\end{aligned}$$

We obtain

$$\begin{aligned}\limsup_{n\to\infty}\mathbf{P}\left[w'_T(Y^n,\rho)>\eta\right] &\le \limsup_{n\to\infty}\mathbf{P}\left[(A^n_K)^c\right]+\limsup_{n\to\infty}\mathbf{P}\left[\{w'_T(Y^n,\rho)>\eta\}\cap A^n_K\right]\\ &\le \frac{\epsilon}{2}+\limsup_{n\to\infty}\mathbf{P}\left[w'_{KT}(X^n,K\rho)>\eta\right]\\ &< \epsilon.\end{aligned}$$

Assume now that $(X^n)_n$ is C-tight. Recall the definition of the (regular) modulus of continuity of a path x,

$$w_T(x,\rho):=\sup_{0\le a\le a+\rho\le T}\ \sup_{s,t\in[a,a+\rho]}|x(t)-x(s)|\,.$$

The family $(Y^n)_n$ is C-tight if and only if the condition *(i)* and the following condition hold (cf. Proposition VI.3.26 in [JS03]):

(iii) For each $T,\epsilon,\eta>0$ there are $n_0\in\mathbb{N}$ and $\rho>0$ with:

$$n\ge n_0\Rightarrow\mathbf{P}\left[w_T(Y^n,\rho)>\eta\right]\le\epsilon.$$

Let $K>0$ and A^n_K be as above. On the event A^n_K we have

$$\begin{aligned}w_T(Y^n,\rho) &= \sup_{0\le a\le a+\rho\le T}\ \sup_{s,t\in[a,a+\rho]}|Y^n_t-Y^n_s|\\ &= \sup_{0\le a\le a+\rho\le T}\ \sup_{s,t\in[\tau_n(a),\tau_n(a+\rho)]}|X^n_t-X^n_s|\\ &\le \sup_{0\le a\le a+K\rho\le T}\ \sup_{s,t\in[a,a+K\rho]}|X^n_t-X^n_s|\\ &= w_T(X^n,K\rho).\end{aligned}$$

Similarly as above this implies condition *(iii)* for $(Y^n)_n$, since condition *(iii)* holds for $(X^n)_n$. $\square$

Lemma A.3.5. *Let (E_1,d_1) and (E_2,d_2) be metric spaces. Let $(x_n)_n\subset D_{E_1}[0,\infty)$ and $(y_n)_n\subset D_{E_2}[0,\infty)$ be convergent with limits x and y. If x and y do not jump simultaneously, and, for any $T>0$, the joint path (x,y) has only finitely many jumps in $[0,T]$, then*

$$(x_n,y_n)\xrightarrow{n\to\infty}(x,y)\quad\text{in } D_{E_1\times E_2}[0,\infty).$$

Proof. Denote by Λ the set of all strictly increasing, continuous maps of $[0,\infty)$ onto itself. By definition of convergence in Skorohod space there are families $\lambda^x_n,\lambda^y_n\in\Lambda$ such that $(\lambda^x_n)_n,(\lambda^y_n)_n$ converge to the identity map uniformly, and, for each $T>0$

$$\begin{aligned}&\sup_{t\le T}d_1(x_n(\lambda^x_n(t)),\,x(t))\xrightarrow{n\to\infty}0,\\ &\sup_{t\le T}d_2(y_n(\lambda^y_n(t)),\,y(t))\xrightarrow{n\to\infty}0.\end{aligned}$$

Consider a discontinuity of x at time t. Since x and y do not jump simultaneously and there are only finitely many jumps of (x, y) in compact intervals, there exists $\epsilon > 0$ such that y is continuous on $(t-\epsilon, t+\epsilon)$. Considering also the jumps of y and choosing the neighbourhoods small enough, we obtain times $0 = t_1 < t_2 < \dots$ such that the consecutive intervals $I_k := [t_k, t_{k+1})$ have the following properties:

(a) Both x and y do not jump at the times $t_1, t_2, \dots$

(b) Both x and y are continuous on I_1.

(c) If there is a jump of x in I_k, then y is continuous on I_k and vice versa.

(d) Intervals, where both x and y are continuous, and intervals with a jump alternate.

We construct a series of time changes piecewise on $(I_k)_k$. Since $(\lambda_n^x)_n$ and $(\lambda_n^y)_n$ converge uniformly to the identity map, we may assume without loss of generality that $\lambda_n^x(t_k) < \lambda_n^y(t_{k+1})$ and $\lambda_n^y(t_k) < \lambda_n^x(t_{k+1})$. Define for $t \in I_{2k}$

$$\lambda_n(t) := \begin{cases} \lambda_n^x(t) & \text{if } x \text{ jumps on } I_{2k}, \\ \lambda_n^y(t) & \text{if } y \text{ jumps on } I_{2k}, \end{cases}$$

and interpolate linearly on the intervals with odd index. Then $(\lambda_n)_n \subset \Lambda$ and

$$\sup_{t\in[0,\infty)} |\lambda_n(t) - t| \leq \sup_{t\in[0,\infty)} |\lambda_n^x(t) - t| + \sup_{t\in[0,\infty)} |\lambda_n^y(t) - t| \xrightarrow{n\to\infty} 0.$$

If x is continuous on I_k, then

$$\sup_{t\in I_k} d_1\Big(x_n(\tilde{\lambda}_n(t)),\, x(t)\Big) \xrightarrow{n\to\infty} 0$$

for any series $(\tilde{\lambda}_n)_n \subset \Lambda$ that converges uniformly to the identity map. Hence, by construction, we have for any $T > 0$

$$\sup_{t\leq T}\Big(d_1(x_n(\lambda_n(t)),\, x(t)) \vee d_2(y_n(\lambda_n(t)),\, y(t))\Big) \xrightarrow{n\to\infty} 0.$$

$\square$

Lemma A.3.6. *For each $n \in \mathbb{N}$ let $(\Omega_n, \mathcal{A}_n, \mathbf{P}_n)$ be a probability space. Let X^n and Y^n be càdlàg processes, defined on Ω_n, with values in metric spaces (E_1, d_1) and (E_2, d_2). If $(X^n)_n$ and $(Y^n)_n$ are tight, then (X_n, Y_n) is tight in $D_{E_1}[0,\infty) \times D_{E_2}[0,\infty)$ endowed with the product topology. If any limit point (X, Y) of (X^n, Y^n) has almost surely the following properties:*

(i) The marginals X and Y do not jump simultaneously.

(ii) (X, Y) has only finitely many jumps in $[0, T]$ for any $T > 0$.

Then $(X^n, Y^n)_n$ is tight in $D_{E_1\times E_2}[0,\infty)$ endowed with the Skorohod topology.

Proof. Let $\epsilon > 0$. Since $(X^n)_n$ and $(Y^n)_n$ are tight, there exist for any $\epsilon > 0$ compact sets $K_X^\epsilon \subset D_{E_1}[0,\infty)$ and $K_Y^\epsilon \subset D_{E_2}[0,\infty)$ such that

$$\mathbf{P}_n\left[X^n \notin K_X^\epsilon\right] < \frac{\epsilon}{2},$$
$$\mathbf{P}_n\left[Y^n \notin K_Y^\epsilon\right] < \frac{\epsilon}{2}.$$

Define $K^\epsilon := K_X^\epsilon \times K_Y^\epsilon$, then

$$\mathbf{P}_n\left[(X^n, Y^n) \notin K^\epsilon\right] < \epsilon.$$

Hence (X_n, Y_n) is tight in $D_{E_1}[0,\infty) \times D_{E_2}[0,\infty)$ endowed with the product topology.

Define

$$C := \left\{(x,y) \in D_{E_1\times E_2}[0,\infty) \;:\; (x,y) \text{ has the properties } \textit{(i)} \text{ and } \textit{(ii)}\right\}.$$

Lemma A.3.5 states that for any sequence $((x_n,y_n))_n \subset D_{E_1\times E_2}[0,\infty)$ with $x_n \to x$, $y_n \to y$ and $(x,y) \in C$ we have convergence $(x_n,y_n) \to (x,y)$ in the Skorohod topology. In other words, the identity map between $D_{E_1}[0,\infty) \times D_{E_2}[0,\infty)$ endowed with the product topology and $D_{E_1\times E_2}[0,\infty)$ endowed with the Skorohod topology is continuous on C. The assertion follows by the Continuous Mapping Theorem (c.f. Theorem 13.25 in [Kle06]). □

Remark A.3.7. *If the limit point Y of $(Y^n)_n$ is continuous, then $(X^n, Y^n)_n$ is tight in $D_{E_1\times E_2}[0,\infty)$ endowed with the Skorohod topology. One may drop the assumption that only finitely many jumps occur in $[0,T]$.*

A.4. Weak convergence in $D[0,\infty)$

Lemma A.4.1. *(i) Let E and F be metric spaces, and let $f : E \to F$ be continuous. Then $x \mapsto f \circ x$ is a $D_E[0,\infty) \to D_F[0,\infty)$-continuous mapping.*

(ii) The mapping $f : x \mapsto \int_0^\cdot x(s)\,\mathrm{d}s$ is $D_\mathbb{R}[0,\infty) \to D_\mathbb{R}[0,\infty)$-continuous.

Proof. These are the Problems 13 and 26 in Chapter 3 of [EK86]. □

Theorem A.4.2. *Let $(Z^n)_n$ be a family of d-dimensional semimartingales with characteristics $(B^n, \tilde{C}^n, \nu^n)$. Let Z be the limit process, defined on the stochastic basis $(\Omega, \mathcal{A}, \mathcal{F}, \mathbf{P})$.*

(i) Assume $\mathcal{F}$ is the filtration generated by Z, and that the following conditions hold:

$$(Z^n, B^n, \tilde{C}^n) \Rightarrow (Z, B, \tilde{C}) \quad \textit{and} \tag{A.7}$$
$$(Z^n, g * \nu^n) \Rightarrow (Z, g * \nu) \tag{A.8}$$

for all nonnegative continuous bounded functions $g : \mathbb{R} \to \mathbb{R}$ which are 0 around 0. Then Z is a semimartingale on $(\Omega, \mathcal{A}, \mathcal{F}, \mathbf{P})$ with characteristics $(B, \tilde{C}, \nu)$.

(ii) Assume $\mathcal{F}$ is the filtration generated by (Z, Y), where Y is an auxiliary càdlàg process, and, for each n, let Y^n be an adapted càdlàg process, defined on the same stochastic basis as Z^n. If

$$(Z^n, Y^n, B^n, \tilde{C}^n) \Rightarrow (Z, Y, B, \tilde{C}) \quad \text{and} \tag{A.9}$$

$$(Z^n, Y^n, g * \nu^n) \Rightarrow (Z, Y, g * \nu), \tag{A.10}$$

for all nonnegative continuous bounded functions $g : \mathbb{R} \to \mathbb{R}$ which are 0 around 0, the assertion of (i) remains valid.

Proof. This is Theorem IX.2.4, p. 528, and Remark IX.2.21, p. 534, in [JS03]. In [JS03] it is proven that it is enough, if the Conditions (A.8) and (A.10) are true for a smaller class of functions g. □

A.5. The Markov Mapping Theorem

For convenience we give a literal quotation of the Markov Mapping Theorem A.15 in [KR11] and the accompanying entry:

Let (S, d) and (S_0, d_0) be complete, separable metric spaces, $B(S) \subset M(S)$ be the Banach space of bounded measurable functions on S, with $\|f\| = \sup_{x \in S} |f(x)|$ and $\overline{C}(S) \subset B(S)$ be the subspace of bounded continuous functions. An operator $A \subset B(S) \times B(S)$ is *dissipative* if $\|f_1 - f_2 - \epsilon(g_1 - g_2)\| \geq \|f_1 - f_2\|$ for all $(f_1, g_1), (f_2, g_2) \in A$ and $\epsilon > 0$; A is a *pre-generator* if A is dissipative and there are sequences of functions $\mu_n : S \to \mathcal{P}(S)$ and $\lambda_n : S \to [0, \infty)$ such that for each $(f, g) \in A$

$$g(x) = \lim_{n \to \infty} \lambda_n(x) \int_S (f(y) - f(x)) \mu_n(x, \mathrm{d}y)$$

for each $x \in S$. A is *graph separable* if there exists a countable subset $\{g_k\} \subset \mathcal{D}(A) \cap \overline{C}(S)$ such that the graph of A is contained in the bounded, pointwise closure of the linear span of $\{(g_k, Ag_k)\}$. [More precisely, we should say that there exists $\{(g_k, h_k)\} \subset A \cap \overline{C}(S) \times B(S)$ such that A is contained in the bounded pointwise closure of $\{(g_k, h_k)\}$, but typically A is single-valued, so we use the more intuitive notation Ag_k.] These two conditions are satisfied by essentially all operators A that might reasonably be thought to be generators of Markov processes. Note that A is graph separable if $A \subset L \times L$, where $L \subset B(S)$ is separable in the sup norm topology, for example, if S is locally compact, and L is the space of continuous functions vanishing at infinity.

A collection of functions $D \subset \overline{C}(S)$ is *separating* if $\nu, \mu \in \mathcal{P}(S)$ and $\int_S f \, \mathrm{d}\nu = \int_S f \, \mathrm{d}\mu$ for all $f \in D$ imply $\mu = \nu$.

For an S_0-valued, measurable process Y, $\hat{\mathcal{F}}_t^Y$ will denote the completion of the σ-algebra $\sigma(Y(0), \int_0^r h(Y(s)) \, \mathrm{d}s, r \leq t, h \in B(S_0))$. For almost every t, $Y(t)$ will be

$\hat{\mathcal{F}}_t^Y$-measurable, but in general, $\hat{\mathcal{F}}_t^Y$ does not contain $\mathcal{F}_t^Y = \sigma(Y(s) : s \le t)$. Let $\mathbf{T}^Y = \{t : Y(t) \text{ is } \hat{\mathcal{F}}_t^Y \text{ measurable}\}$. If Y is cadlag and has no fixed points of discontinuity [i.e., for every t, $Y(t) = Y(t-)$ a.s.], then $\mathbf{T}^Y = [0,\infty)$. $D_S[0,\infty)$ denotes the space of cadlag, S-valued functions with the Skorohod topology, and $M_S[0,\infty)$ denotes the space of Borel measurable functions, $x : [0,\infty) \to S$, topologized by convergence in Lebesgue measure.

Theorem A.5.1. *Let (S,d) and (S_0,d_0) be complete, separable metric spaces. Let $A \subset \overline{C}(S) \times C(S)$ and $\psi \in C(S)$, $\psi \ge 1$. Suppose that for each $f \in \mathcal{D}(A)$ there exists $c_f > 0$ such that*

$$|Af(x)| \le c_f \psi(x), \qquad x \in A,$$

and define $A_0 f(x) = Af(x)/\psi(x)$.

Suppose that A_0 is a graph-separable pre-generator, and suppose that $\mathcal{D}(A) = \mathcal{D}(A_0)$ is closed under multiplication and is separating. Let $\gamma : S \to S_0$ be Borel measurable, and let α be a transition function from S_0 into S [$y \in S_0 \to \alpha(y,\cdot) \in \mathcal{P}(S)$ is Borel measurable] satisfying $\int h \circ \gamma(z)\, \alpha(y, \mathrm{d}z) = h(y)$, $y \in S_0$, $h \in B(S_0)$, that is, $\alpha(y, \gamma^{-1}(y)) = 1$. Assume that $\tilde{\psi}(y) \equiv \int_S \psi(z)\, \alpha(y, \mathrm{d}z) < \infty$ for each $y \in S_0$, and define

$$C = \left\{ \left(\int_S f(z)\, \alpha(\cdot, \mathrm{d}z), \int_S Af(z)\, \alpha(\cdot, \mathrm{d}z) \right) : f \in \mathcal{D}(A) \right\}.$$

Let $\mu_0 \in \mathcal{P}(S_0)$, and define $\nu_0 = \int \alpha(y,\cdot)\, \mu_0(\mathrm{d}y)$.

(a) If $\tilde{Y}$ satisfies $\int_0^t \mathbf{E}\left[\tilde{\psi}(\tilde{Y}(s))\right] \mathrm{d}s < \infty$ for all $t \ge 0$, and $\tilde{Y}$ is a solution of the martingale problem for (C, μ_0), then there exists a solution X of the martingale problem for (A, ν_0) such that $\tilde{Y}$ has the same distribution on $M_{S_0}[0,\infty)$ as $Y = \gamma \circ X$. If Y and $\tilde{Y}$ are cadlag, then Y and $\tilde{Y}$ have the same distribution on $D_{S_0}[0,\infty)$.

(b) For $t \in \mathbf{T}^Y$,

$$\mathbf{P}\left[X(t) \in \Gamma \,|\, \hat{\mathcal{F}}_t^Y\right] = \alpha(Y(t), \Gamma), \qquad \Gamma \in \mathcal{B}(S).$$

(c) If, in addition, uniqueness holds for the martingale problem for (A, ν_0), then uniqueness holds for the $M_{S_0}[0,\infty)$-martingale problem for (C, μ_0). If $\tilde{Y}$ has sample paths in $D_{S_0}[0,\infty)$, then uniqueness holds for the $D_{S_0}[0,\infty)$-martingale problem for (C, μ_0).

(d) If uniqueness holds for the martingale problem for (A, ν_0), then Y restricted to $\mathbf{T}^Y$ is a Markov process.

A.6. Generator calculations for the progenitor-level system

In this section we do the generator calculations in order to check the intertwining relation (6.10). Let all notation be as in Chapter 6. Recall in particular the definition (and notation) of the class of test function $\mathcal{D}^r_{PL}$ (see (6.1) and (6.2)), the definition of the generator A^r_{PL} (see (6.3), (6.4), (6.6), (6.7) and (6.8)) and the definition of the kernel $\alpha_r(\check{x},\cdot)$, where $\check{x} \in \mathcal{S}_{[0,r]\times\mathbb{R}^+}$ is a valid state of $\check{X}^r$ (see (6.9)). Recall that

$$\begin{aligned}
\hat{f}(\check{x}) &:= \int \alpha_r(\check{x};\, \mathrm{d}u)\; f(u) \\
&= \int \alpha_r\Big(u_0^{(1)},\ldots,u_0^{(k)},(u_n^{(1)})_{n\geq 1},(m_n)_{n\geq 0};\, \mathrm{d}u\Big)\; f(u_0,u_1,\ldots) \\
&= \hat{f}_0(u_0^{(1)},\ldots,u_0^{(k)},m_0)\cdot\prod_{n\geq 1}\hat{f}_n(u_n^{(1)},m_n),
\end{aligned}$$

where

$$\begin{aligned}
\hat{f}_0(u_0^{(1)},\ldots,u_0^{(k)},m_0) &:= \prod_{i=1}^{k} g_0(u_0^{(i)})\cdot e^{-\lambda_0(u_0^{(k)})(m_0-k)\vee 0}, \\
\hat{f}_n(u_n^{(1)},m_n) &:= g_n(u_n^{(1)})e^{-\lambda_n(u_n^{(1)})(m_n-1)}.
\end{aligned}$$

We integrate the summands of $A^r_{PL} = A^r_f + A^r_d + A^r_0 + A^r_e$ separately. The formation part is straightforward,

$$\begin{aligned}
&C^r_f\hat{f}\Big(u_0^{(1,\ldots,k)},(u_n^{(1)})_{n\geq 1},(m_n)_{n\geq 0}\Big) := \int \alpha_r\Big(u_0^{(1,\ldots,k)},(u_n^{(1)})_{n\geq 1},(m_n)_{n\geq 0};\,\mathrm{d}u\Big)\; A^r_f f(u) \\
&= \int \alpha_r\Big(u_0^{(1)},\ldots,u_0^{(k)},(u_n^{(1)})_{n\geq 1},(m_n)_{n\geq 0};\,\mathrm{d}u\Big)\; \mathbb{1}_{m\geq k}(u)\cdot\sum_{q=2}^{k} 2(q-1)(u_0^{(q)}-u_0^{(q-1)}) \\
&\quad\times\Bigg[\prod_{j=1}^{k-1} g_0(u_0^{(j)})\cdot\frac{1}{u_0^{(q)}-u_0^{(q-1)}}\int_{u_0^{(q-1)}}^{u_0^{(q)}} g_0(x)\,\mathrm{d}x\prod_{j=k}^{m_0} g_1(u_0^{(j)})\cdot\prod_{n\geq 2} f_n(u_{n-1}) \\
&\qquad\qquad -f(u_0,u_1,\ldots)\Bigg] \\
&\quad+\mathbb{1}_{m<k}(u)\cdot f(u)\sum_{i=1}^{m} 2\int_{u_0^i}^{r}(g_0(x)-1)\,\mathrm{d}x \\
&= \mathbb{1}_{[k,\infty)}(m)\cdot\sum_{q=2}^{k} 2(q-1)(u_0^{(q)}-u_0^{(q-1)})\Bigg[\prod_{j=1}^{k-1} g_0(u_0^{(j)})\cdot\frac{1}{u_0^{(q)}-u_0^{(q-1)}}\int_{u_0^{(q-1)}}^{u_0^{(q)}} g_0(x)\,\mathrm{d}x \\
&\quad\times g_1(u_0^{(k)})e^{-\lambda_1(u_0^{(k)})(m_0-k)}\cdot\prod_{n\geq 2}\hat{f}_n(u_{n-1}^{(1)},m_{n-1}) - \hat{f}\Big(u_0^{(1,\ldots,k)},(u_n^{(1)})_{n\geq 1},(m_n)_{n\geq 0}\Big)\Bigg] \\
&\quad+\mathbb{1}_{(0,k)}(m)\cdot\hat{f}_0(u_0^{(1,\ldots,m)},m)\sum_{i=1}^{m} 2\int_{u_0^{(i)}}^{r}\mathrm{d}x\,(g_0(x)-1).
\end{aligned}$$

We turn to the generator of the death dynamics. Abbreviate the jump factor $\varphi := \varphi(u)$

(cf. (6.5)). We rewrite

$$\begin{aligned}
A_d^r f(u_0, u_1, \ldots) &= \frac{rm}{c}\Bigg(\prod_{n\geq 0}\prod_{i=1}^{m_n} g_n(\varphi u_n^i) - f(u_0, u_1, \ldots)\Bigg)\\
&= \frac{rm}{c}\Bigg(\prod_{n\geq 0}\prod_{i=1}^{m_n}\Big(\mathbb{1}_{[0,r)}(\varphi u_n^i) g_n(\varphi u_n^i) + \mathbb{1}_{[r,\infty)}(\varphi u_n^i)\Big) - f(u_1, u_2, \ldots)\Bigg).
\end{aligned}$$

Assume $u_0^{(k)} \leq r - \frac{rc}{m}$ and $u_n^{(1)} \leq r - \frac{rc}{m}$ for $n \geq 1$. I.e. the progenitors are not in the death zone. By definition of the jump factor we distinguish between the cases $m > c$ and $m \leq c$:

$$\begin{aligned}
&C_d^r \hat{f}\Big(u_0^{(1,\ldots,k)}, (u_n^{(1)})_{n\geq 1}, (m_n)_{n\geq 0}\Big) := \int \alpha_r\Big(u_0^{(1,\ldots,k)}, (u_n^{(1)})_{n\geq 1}, (m_n)_{n\geq 0}; \mathrm{d}u\Big)\ A_d^r f(u)\\
&= \mathbb{1}_{(c,\infty)}(m)\\
&\quad\times \frac{rm}{c}\Bigg[\prod_{j=1}^{k\wedge m} g_0(\varphi u_0^{(j)})\Bigg(\frac{1}{r - u_0^{(k)}}\int_{u_0^{(k)}}^{r} \mathrm{d}z\ \Big(\mathbb{1}_{[0,r)}(\varphi z) g_0(\varphi z) + \mathbb{1}_{[r,\infty)}(\varphi z)\Big)\Bigg)^{(m_0-k)\vee 0}\\
&\quad\times \prod_{n\geq 1} g_n(\varphi u_n^{(n)})\Bigg(\frac{1}{r - u_n^{(1)}}\int_{u_n^{(1)}}^{r} \mathrm{d}z\ \Big(\mathbb{1}_{[0,r)}(\varphi z) g_n(\varphi z) + \mathbb{1}_{[r,\infty)}(\varphi z)\Big)\Bigg)^{m_n-1}\\
&\qquad\qquad\qquad\qquad -\hat{f}\Big(u_0^{(1,\ldots,k)}, (u_n^{(1)})_{n\geq 1}, (m_n)_{n\geq 0}\Big)\Bigg]\\
&\quad+ \mathbb{1}_{[0,c]}(m)\cdot\frac{rm}{c}\Big[\hat{f}(0) - \hat{f}\Big(u_0^{(1,\ldots,k)}, (u_n^{(1)})_{n\geq 1}, (m_n)_{n\geq 0}\Big)\Big].
\end{aligned}$$

We have for $m > c$

$$\begin{aligned}
&\Bigg(\frac{1}{r - u_0^{(k)}}\int_{u_0^{(k)}}^{r} \mathrm{d}z\ \Big(\mathbb{1}_{[0,r)}(\varphi z) g_0(\varphi z) + \mathbb{1}_{[r,\infty)}(\varphi z)\Big)\Bigg)^{(m_0-k)\vee 0}\\
&\quad= \Bigg(\frac{1}{r - u_0^{(k)}}\int_{u_0^{(k)}}^{\frac{r}{\varphi}} g_0(\varphi z)\ \mathrm{d}z + \frac{1}{r - u_0^{(k)}}\int_{\frac{r}{\varphi}}^{r}\ \mathrm{d}z\Bigg)^{(m_0-k)\vee 0}\\
&\quad= \Bigg(\frac{1}{\varphi\cdot(r - u_0^{(k)})}\int_{\varphi u_0^{(k)}}^{r} g_0(z)\ \mathrm{d}z + \frac{r - r/\varphi}{r - u_0^{(k)}}\Bigg)^{(m_0-k)\vee 0}\\
&\quad= \Bigg(\frac{r - \varphi u_0^{(k)}}{\varphi\cdot(r - u_0^{(k)})} e^{-\lambda_0(\varphi u_0^{(k)})} + \frac{r - r/\varphi}{r - u_0^{(k)}}\Bigg)^{(m_0-k)\vee 0}\\
&\quad= \Bigg(\Bigg(1 - \frac{rc}{(r - u_0^{(k)})m}\Bigg) e^{-\lambda_0(\varphi u_0^{(k)})} + \frac{rc}{(r - u_0^{(k)})m}\Bigg)^{(m_0-k)\vee 0}\\
&\quad= \sum_{l=0}^{(m_0-k)\vee 0}\binom{m_0 - k}{l}\Bigg(\frac{rc}{(r - u_0^{(k)})m}\Bigg)^{l}\Bigg(1 - \frac{rc}{(r - u_0^{(k)})m}\Bigg)^{[(m_0-k)\vee 0]-l}\\
&\qquad\qquad\qquad\qquad\times e^{-\lambda_0(\varphi u_0^{(k)})[((m_0-k)\vee 0)-l]}
\end{aligned}$$

and, with a very similar calculation for $n \geq 1$,

$$\left(\frac{1}{r-u_n^{(1)}}\int_{u_n^{(1)}}^r \mathrm{d}z \ \left(\mathbb{1}_{[0,r)}(\varphi z)g_n(\varphi z)+\mathbb{1}_{[r,\infty)}(\varphi z)\right)\right)^{m_n-1}$$
$$=\sum_{l=0}^{m_n-1}\binom{m_n-1}{l}\left(\frac{rc}{(r-u_n^{(1)})m}\right)^l\left(1-\frac{rc}{(r-u_n^{(1)})m}\right)^{m_n-1-l} e^{-\lambda_n(\varphi u_n^{(1)})(m_n-1-l)}.$$

If a progenitor is in the death zone, all particles of its subpopulation are killed. We correct for this case by cutting the probability parameters off at 1 and obtain the form of C_d^r in (6.14).

We turn to the birth dynamics in the extractions,

$$C_e^r\hat{f}\left(u_0^{(1,\dots,k)},(u_n^{(1)})_{n\geq 1},(m_n)_{n\geq 0}\right):=\int \alpha_r\left(u_0^{(1,\dots,k)},(u_n^{(1)})_{n\geq 1},(m_n)_{n\geq 0};\mathrm{d}u\right)\ A_e^r f(u)$$
$$=\int \alpha_r\left(u_0^{(1,\dots,k)},(u_n^{(1)})_{n\geq 1},(m_n)_{n\geq 0};\mathrm{d}u\right)$$
$$\times f(u_0,u_1,\dots)\sum_{n\geq 1}\sum_{i=1}^{m_n}\left(2\int_{u_n^i}^r (g_n(x)-1)\ \mathrm{d}x+\left((u_n^i)^2-ru_n^i\right)\frac{g_n'(u_n^i)}{g_n(u_n^i)}\right).$$

We treat the extractions separately. Fix $n \geq 1$.

$$\int \alpha_r\left(u_0^{(1,\dots,k)},(u_{n'}^{(1)})_{n'\geq 1},(m_{n'})_{n'\geq 0};\mathrm{d}u\right)\ f(u_0,u_1,\dots)\sum_{i=1}^{m_n}2\int_{u_n^{(i)}}^r (g_n(x)-1)\ \mathrm{d}x$$
$$=\hat{f}_0(u_0^{(1,\dots,k)},m_0)\prod_{\substack{l\geq 1\\ l\neq n}}\hat{f}_l(u_l^{(1)},m_l)\Bigg[2g_n(u_n^{(1)})e^{-\lambda_n(u_n^{(1)})(m_n-1)}\int_{u_n^{(1)}}^r \mathrm{d}x\ (g_n(x)-1)$$
$$+(m_n-1)g_n(u_n^{(1)})e^{-\lambda_n(u_n^{(1)})(m_n-2)}\frac{2}{r-u_n^{(1)}}\int_{u_n^{(1)}}^r \mathrm{d}z\ g_n(z)\int_z^r \mathrm{d}x\ (g_n(x)-1)\Bigg]$$
$$=\hat{f}_0(u_0^{(1,\dots,k)},m_0)\prod_{\substack{l\geq 1\\ l\neq n}}\hat{f}_l(u_l^{(1)},m_l)\Bigg[2(r-u_n^{(1)})g_n(u_n^{(1)})e^{-\lambda_n(u_n^{(1)})(m_n-1)}\left(e^{-\lambda_n(u_n^{(1)})}-1\right)$$
$$+(m_n-1)g_n(u_n^{(1)})e^{-\lambda_n(u_n^{(1)})(m_n-2)}$$
$$\times\left((r-u_n^{(1)})e^{-2\lambda_n(u_n^{(1)})}-\frac{2}{r-u_n^{(1)}}\int_{u_n^{(1)}}^r \mathrm{d}z\ g_n(z)(r-z)\right)\Bigg]. \tag{A.11}$$

For the last equality we used Lemma A.2.1.

We turn to the movement part. For the progenitor we obtain

$$\begin{aligned}
&\int \alpha_r\Big(u_0^{(1,\dots,k)}, (u_{n'}^{(1)})_{n'\geq 1}, (m_{n'})_{n'\geq 0}; \mathrm{d}u\Big)\ f(u_0, u_1, \dots)((u_n^{(1)})^2 - r u_n^{(1)})\frac{g_n'(u_n^{(1)})}{g_n(u_n^{(1)})} \\
&\quad = \hat{f}_0(u_0^{(1,\dots,k)}, m_0) \cdot \prod_{\substack{l\geq 1\\ l\neq n}} \hat{f}_l(u_l^{(1)}, m_l) \cdot ((u_n^{(1)})^2 - r u_n^{(1)}) g_n'(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-1)}.
\end{aligned} \tag{A.12}$$

For the particles in the bulk we obtain, using partial integration,

$$\begin{aligned}
&\int \alpha_r\Big(u_0^{(1,\dots,k)}, (u_{n'}^{(1)})_{n'\geq 1}, (m_{n'})_{n'\geq 0}; \mathrm{d}u\Big)\ f(u_0, u_1, \dots)\sum_{i=2}^{m_n}((u_n^{(i)})^2 - r u_n^{(i)})\frac{g_n'(u_n^{(i)})}{g_n(u_n^{(i)})} \\
&= \hat{f}_0(u_0^{(1,\dots,k)}, m_0) \cdot \prod_{\substack{l\geq 1\\ l\neq n}} \hat{f}_l(u_l^{(1)}, m_l) \cdot (m_n - 1) g_n(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-2)} \\
&\qquad\qquad\qquad \times \frac{1}{r - u_n^{(1)}} \int_{u_n^{(1)}}^r \mathrm{d}z\ (z^2 - rz) g_n'(z) \\
&= \hat{f}_0(u_0^{(1,\dots,k)}, m_0) \cdot \prod_{\substack{l\geq 1\\ l\neq n}} \hat{f}_l(u_l^{(1)}, m_l) \cdot (m_n - 1) g_n(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-2)} \\
&\qquad \times \frac{1}{r - u_n^{(1)}} \Big(-((u_n^{(1)})^2 - r u_n^{(1)}) g_n(u_n^{(1)}) - \int_{u_n^{(1)}}^r \mathrm{d}z\ (2z - r) g_n(z) \Big) \\
&= \hat{f}_0(u_0^{(1,\dots,k)}, m_0) \cdot \prod_{\substack{l\geq 1\\ l\neq n}} \hat{f}_l(u_l^{(1)}, m_l) \cdot (m_n - 1) g_n(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-2)} \\
&\quad \times \frac{1}{r - u_n^{(1)}} \Big((r - u_n^{(1)}) u_n^{(1)} g_n(u_n^{(1)}) - \int_{u_n^{(1)}}^r \mathrm{d}z\ r g_n(z) + 2\int_{u_n^{(1)}}^r \mathrm{d}z\ (r - z) g_n(z) \Big) \\
&= \hat{f}_0(u_0^{(1,\dots,k)}, m_0) \cdot \prod_{\substack{l\geq 1\\ l\neq n}} \hat{f}_l(u_l^{(1)}, m_l) \cdot (m_n - 1) g_n(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-2)} \\
&\qquad \times \Big(u_n^{(1)} g_n(u_n^{(1)}) - r e^{-\lambda_n(u_n^{(1)})} + \frac{2}{r - u_n^{(1)}} \int_{u_n^{(1)}}^r \mathrm{d}z\ (r - z) g_n(z) \Big).
\end{aligned} \tag{A.13}$$

Combining (A.11), (A.12) and (A.13) we obtain for $n \geq 1$

$$
\begin{aligned}
&\int \alpha_r\Big(u_0^{(1,\dots,k)}, (u_{n'}^{(1)})_{n'\geq 1}, (m_{n'})_{n'\geq 0}; \mathrm{d}u\Big) \\
&\qquad \times f(u_0,u_1,\dots) \sum_{i=1}^{m_n} \left(2\int_{u_n^{(i)}}^r (g_n(x)-1)\;\mathrm{d}x + \Big((u_n^{(i)})^2 - r u_n^{(i)}\Big)\frac{g_n'(u_n^{(i)})}{g_n(u_n^{(i)})} \right) \\
&= \hat{f}_0(u_0^{(1,\dots,k)}, m_0)\cdot \prod_{\substack{l\geq 1\\ l\neq n}} \hat{f}_l(u_l^{(1)}, m_l)\cdot \Bigg[2(r-u_n^{(1)}) g_n(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-1)} \left(e^{-\lambda_n(u_n^{(1)})} - 1\right) \\
&\qquad + (m_n-1) g_n(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-2)} \Bigg((r-u_n^{(1)}) e^{-2\lambda_n(u_n^{(1)})} \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \frac{2}{r-u_n^{(1)}} \int_{u_n^{(1)}}^r \mathrm{d}z\; g_n(z)(r-z) \Bigg) \\
&\qquad + ((u_n^{(1)})^2 - r u_n^{(1)}) g_n'(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-1)} \\
&\qquad + (m_n-1) g_n(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-2)} \Bigg(u_n^{(1)} g_n(u_n^{(1)}) - r e^{-\lambda_n(u_n^{(1)})} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \frac{2}{r-u_n^{(1)}} \int_{u_n^{(1)}}^r \mathrm{d}z\; (r-z) g_n(z) \Bigg) \Bigg] \\
&= \hat{f}_0(u_0^{(1,\dots,k)}, m_0)\cdot \prod_{\substack{l\geq 1\\ l\neq n}} \hat{f}_l(u_l^{(1)}, m_l)\cdot \Bigg[2(r-u_n^{(1)}) \left(\hat{f}_n(u_n^{(1)}, m_n+1) - \hat{f}_n(u_n^{(1)}, m_n) \right) \\
&\qquad + (r-u_n^{(1)})(m_n-1)\hat{f}_n(u_n^{(1)}, m_n+1) + ((u_n^{(1)})^2 - r u_n^{(1)}) g_n'(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-1)} \\
&\qquad + u_n^{(1)}(m_n-1) g_n(u_n^{(1)}) \hat{f}_n(u_n^{(1)}, m_n-1) - r(m_n-1)\hat{f}_n(u_n^{(1)}, m_n) \Bigg] \\
&= \hat{f}_0(u_0^{(1,\dots,k)}, m_0)\cdot \prod_{\substack{l\geq 1\\ l\neq n}} \hat{f}_l(u_l^{(1)}, m_l)\cdot \Bigg[2(r-u_n^{(1)}) \left(\hat{f}_n(u_n^{(1)}, m_n+1) - \hat{f}_n(u_n^{(1)}, m_n) \right) \\
&\qquad + (r-u_n^{(1)})(m_n-1) \left(\hat{f}_n(u_n^{(1)}, m_n+1) - \hat{f}_n(u_n^{(1)}, m_n) \right) \\
&\qquad - u_n^{(1)}(m_n-1) \left(\hat{f}_n(u_n^{(1)}, m_n) - g_n(u_n^{(1)}) \hat{f}_n(u_n^{(1)}, m_n-1) \right) \\
&\qquad + ((u_n^{(1)})^2 - r u_n^{(1)}) g_n'(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-1)} \Bigg] \\
&= \hat{f}_0(u_0^{(1,\dots,k)}, m_0)\cdot \prod_{\substack{l\geq 1\\ l\neq n}} \hat{f}_l(u_l^{(1)}, m_l)\cdot \Bigg[2(r-u_n^{(1)}) \left(\hat{f}_n(u_n^{(1)}, m_n+1) - \hat{f}_n(u_n^{(1)}, m_n) \right) \\
&\qquad + (r-u_n^{(1)})(m_n-1) \left(\hat{f}_n(u_n^{(1)}, m_n+1) - \hat{f}_n(u_n^{(1)}, m_n) \right) \\
&\qquad + ((u_n^{(1)})^2 - r u_n^{(1)}) \cdot \frac{\mathrm{d}}{\mathrm{d}u_n^{(1)}} \hat{f}_n(u_n^{(1)}, m_n) \Bigg]
\end{aligned}
\tag{A.14}
$$

For the last equality, recall that $e^{-\lambda_n(u_n^{(1)})} = \frac{1}{r-u_n^{(1)}} \int_{u_n^{(1)}}^r g_n(x)\, \mathrm{d}x$. Hence we obtain

$$
\begin{aligned}
&((u_n^{(1)})^2 - r u_n^{(1)}) \cdot \frac{\mathrm{d}}{\mathrm{d}u_n^{(1)}} \hat{f}_n(u_n^{(1)}, m_n) = ((u_n^{(1)})^2 - r u_n^{(1)}) \cdot \frac{\mathrm{d}}{\mathrm{d}u_n^{(1)}} g_n(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-1)} \\
&= ((u_n^{(1)})^2 - r u_n^{(1)}) \\
&\quad \times \Bigg(g_n(u_n^{(1)})(m_n-1) e^{-\lambda_n(u_n^{(1)})(m_n-2)} \left(\frac{1}{(r-u_n^{(1)})^2} \int_{u_n^{(1)}}^r g_n(x)\, \mathrm{d}x - \frac{g_n(u_n^{(1)})}{r-u_n^{(1)}} \right) \\
&\qquad\qquad + g_n'(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-1)} \Bigg) \\
&= g_n(u_n^{(1)})(m_n-1) e^{-\lambda_n(u_n^{(1)})(m_n-2)} \left(-u_n^{(1)} e^{-\lambda_n(u_n^{(1)})} + u_n^{(1)} g_n(u_n^{(1)}) \right) \\
&\quad + ((u_n^{(1)})^2 - r u_n^{(1)}) g_n'(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-1)} \\
&= -u_n^{(1)}(m_n-1) \left(\hat{f}_n(u_n^{(1)}, m_n) - g_n(u_n^{(1)}) \hat{f}_n(u_n^{(1)}, m_n-1) \right) \\
&\quad + ((u_n^{(1)})^2 - r u_n^{(1)}) g_n'(u_n^{(1)}) e^{-\lambda_n(u_n^{(1)})(m_n-1)}.
\end{aligned}
$$

We obtain the form of C_e^r in (6.16) by summing over $n \geq 1$.

We take the birth-dynamics of the integrated root process to be apparent and we omit the generator calculation: If we ignore $U^{(1,\dots,k-1),r}$, the k-th progenitor $U_1^{(k)}$ and the bulk U_1 perform a birth dynamic with progenitor and the integrated process can be obtained as in (A.14). The remaining progenitors $U_1^{(1,\dots,k-1)}$ generate uniformly distributed offspring in $[U_1^{(k)}, r]$ with individual rate $2(r - U_1^{(k)})$, thus inducing immigration at total rate $2(k-1)(r - U_1^{(k)})$. This dynamics belong to the generator C_0^r in (6.15).

In order to apply the Markov Mapping Theorem we need the following result:

Lemma A.6.1. *For $u = \sum_i \delta_{(u^i, n^i)} \in \mathcal{S}_{[0,r]\times\mathbb{N}}$ define*

$$\psi(u) := m := \sum_n u_n([0,r)).$$

(Recall that $u_n := \sum_{i:\, n^i = n} \delta_{u^i}$.) For each $f \in \mathcal{D}_{PL}^r$ there exists $c_f > 0$ such that

$$|A_{PL}^r f(u)| \leq c_f \cdot \psi(u).$$

Proof. We have $0 \leq g_n \leq 1$, $g_n(r) = 1$ and $u_n \leq r$. We check the statement for each of the operators in section 6.1. We have

$$\begin{aligned}
\left|A_f^r f(u_0,u_1,\dots)\right| \le & \left| \sum_{q=2}^{k} 2(q-1)(u_0^{(q)}-u_0^{(q-1)}) \left[\prod_{j=1}^{k-1} g_0(u_0^{(j)}) \right.\right. \\
& \times \frac{1}{u_0^{(q)}-u_0^{(q-1)}} \int_{u_0^{(q-1)}}^{u_0^{(q)}} g_0(x)\ \mathrm{d}x \cdot \prod_{j=k}^{m_0} g_1(u_0^{(j)}) \cdot \prod_{n\ge 2} f_n(u_{n-1}) - f(u_0,u_1,\dots) \Bigg] \Bigg| \\
& + \left| f(u_0,u_1,\dots) \sum_{i=1}^{m_0} 2 \int_{u_0^i}^{r} (g_0(x)-1)\ \mathrm{d}x \right| \\
\le\ & k^2(r-1) + 2mr.
\end{aligned}$$

Furthermore we have

$$|A_d^r f(u_0,u_1,\dots)| = \frac{rm}{c}\Big| f(\varphi(u)\cdot u_0, \varphi(u)\cdot u_1,\dots) - f(u_0,u_1,\dots)\Big| \le \frac{rm}{c}.$$

Recall that the division by $g_1(u_1^i)$ in the operators A_0^r and A_e^r removes the respective factor from the product $f(u_1,u_2,\dots)$. Finally we have

$$\begin{aligned}
|A_0^r f(u_0,u_1,\dots)| = & \left| f(u_0,u_1,\dots) \left(2k \int_{u_0^{(k)}}^{r} (g_0(x)-1)\ \mathrm{d}x + \sum_{i=k+1}^{m_0} 2 \int_{u_0^{(i)}}^{r} (g_0(x)-1)\ \mathrm{d}x \right.\right. \\
& \left.\left. + \sum_{i=1}^{m_0} \Big((u_0^i)^2 - r u_0^i \Big) \frac{g_0'(u_0^i)}{g_0(u_0^i)} \right) \right| \\
& \le 2kr + 2(m_0-k)r + m_0 r^2
\end{aligned}$$

and

$$\begin{aligned}
|A_e^r f(u_0,u_1,\dots)| = & f(u_0,u_1,\dots) \sum_{n\ge 1} \sum_{i=1}^{m_n} \left(2\int_{u_n^i}^{r} (g_n(x)-1)\ \mathrm{d}x + \Big((u_n^i)^2 - r u_n^i \Big) \frac{g_n'(u_n^i)}{g_n(u_n^i)} \right) \\
& \le m(2r + r^2).
\end{aligned}$$

□

A.7. The semimartingale characteristics of the progenitor-mass process

Let all notation be as in Chapter 6. In this section we determine the semimartingale characteristics of $\tilde{X}^{r,K}$ (see Definition (6.17)). We stop the process $\tilde{X}^{r,K}$, when the overall mass density Y^r falls below some threshold $\delta > 0$. Recall the definition

$$\tau_\delta^r := \inf\{s : Y_s^r \le \delta\}.$$

Define $\tilde{X}_t^{r,K,\delta} := \tilde{X}_{t\wedge\tau_\delta^r}^{r,K}$. Recall the definition of the truncation function $h(x) = (\tilde{h}(x_i))_i$ in (4.32). Denote by $(B^{\tilde{X},r}, \tilde{C}^{\tilde{X},r}, \nu^{\tilde{X},r})$ be the semimartingale characteristics of $\tilde{X}^{r,K,\delta}$.

Since the continuous martingale part of $\tilde{X}^{r,K,\delta}$ is zero and $\tilde{X}^{r,K,\delta}$ has no predictable jumps, the modified second characteristic $\tilde{C}^{\tilde{X},r}(h)$ is defined in terms of the third characteristic $\nu^{\tilde{X},r}$,

$$(\tilde{C}^{\tilde{X},r})^{ij}(h) = h^i h^j * \nu^{\tilde{X},r}.$$

For the first characteristic we have

$$B^{\tilde{X},r}(h) = \tilde{B}^{\tilde{X},r} + h * \nu^{\tilde{X},r},$$

where $\tilde{B}^{\tilde{X},r,\delta} := \tilde{X}^{r,K,\delta} - \sum_{s\leq\cdot} \Delta\tilde{X}^{r,K,\delta}_s$ is the process without jumps.

We constitute the third characteristic of $\tilde{X}^{r,K,\delta}$, the predictable compensator $\nu^{\tilde{X},r}$ of the empirical jump measure, as a sum

$$\nu^{\tilde{X},r}(\omega; dt, dx) = \nu_d^{\tilde{X},r}(\omega; dt, dx) + \sum_{n=0}^{K} \nu_n^{\tilde{X},r}(\omega; dt, dx) + \nu_f^{\tilde{X},r}(\omega; dt, dx),$$

where $\nu_d^{\tilde{X},r}$ compensates the jumps induced by death events, $\nu_n^{\tilde{X},r}$ compensates the jumps induced by births in the bulk of the n-th subpopulation and $\nu_f^{\tilde{X},r}$ compensates the jumps induced by the genesis of new extractions.

We drop δ and K in our notation, $\tilde{X}^r = \tilde{X}^{r,K,\delta}$, $\tilde{Y}^r_n = \tilde{Y}^{r,K,\delta}_n$ etc. Since we are interested in the dynamics of $\check{X}^r$ for large r, we assume $r\delta > c, k$ and thus we omit the cases, where $[0, r]$ is completely in the "death zone" or where active progenitors are dead. We have then for the jump factor of the death mechanism

$$\varphi(U^r) = \frac{r}{r - c/Y^r}.$$

Because of our "dummy values" for extinct/non existent extractions we have to be careful to set the levels of dying progenitors onto the threshold r. Otherwise we would see continuous movement of dead progenitors due to the differential equation $\dot{u} = u^2 - ru$.

Jumps induced by death events Death jumps happen at instantaneous rate $\frac{r^2 b Y^r_t}{c}$. Consider a subpopulation, where the progenitor (or progenitors, in case of the root) is not in the death zone. A progenitor with level u performs a jump with height $u \cdot \left(\frac{r}{r-c/Y^r_t} - 1\right)$. All bulk particles in the interval $[r - c/Y^r_t,\, r]$ are killed. Since the bulk particles are uniformly distributed above the progenitor, the number of particles killed is binomially distributed. If a death jump happens at time t, the parameters of the binomial distribution for the root are

$$N^{0,r}_t := r \cdot (\tilde{Y}^r_0)_t - k \quad \text{and} \quad p^{0,r}_t := \frac{c}{\left(r - (\tilde{U}^{(k),r}_0)_t\right) \cdot Y^r_t},$$

and the parameters for the n-th extraction are

$$N^{n,r}_t := r \cdot (\tilde{Y}^r_n)_t - 1 \quad \text{and} \quad p^{n,r}_t := \frac{c}{\left(r - (\tilde{U}^{(1),r}_n)_t\right) \cdot Y^r_t}.$$

So the jump of $\tilde{Y}_n^r$ is $\mathrm{B}_{n,r,t}$-distributed, where

$$\mathrm{B}_{n,r,t}(\mathrm{d}x) := \sum_{j=0}^{N_t^{n,r}} \binom{N_t^{n,r}}{j} (p_t^{n,r})^j \, (1-p_t^{n,r})^{N_t^{n,r}-j} \, \delta_{-\frac{j}{r}}(\mathrm{d}x).$$

If the progenitor is in the death zone, it jumps onto the threshold r and all bulk particles are killed. Denote by $\tilde{m}_t^r := \max\{i=1,\dots,k : (\tilde{U}_0^{(i),r})_t < r - c/Y_t^r\}$ the number of progenitors in the root that are not in the death zone. The compensator for the death jumps is

$$\begin{aligned}
\nu_d^{\tilde{X},r}(\mathrm{d}t,\mathrm{d}x) &= \mathbb{1}_{\{t<\tau_\delta^r\wedge\sigma_K^r\}} \cdot \frac{r^2 b Y_t^r}{c}\,\mathrm{d}t \cdot \prod_{i=1}^{k} \delta_{\left(\left(\frac{r}{r-c/Y^r}\tilde{U}_0^{(i),r}\right)_t \wedge r\right) - \left(\tilde{U}_0^{(i),r}\right)_t}(\mathrm{d}x_i) \\
&\quad \times \prod_{n=1}^{K} \delta_{\left(\left(\frac{r}{r-c/Y^r}\tilde{U}_n^{(1),r}\right)_t \wedge r\right) - \left(\tilde{U}_n^{(1),r}\right)_t}(\mathrm{d}x_{k+n}) \\
&\quad \times \Big(\mathbb{1}_{\left\{\left(\tilde{U}_0^{(k),r}\right)_t < r - c/Y_t^r\right\}} \cdot \mathrm{B}_{0,r,t}(\mathrm{d}x_{k+K+1}) \\
&\qquad\qquad + \mathbb{1}_{\left\{(\tilde{U}_0^{(k),r})_t \geq r - c/Y_t^r\right\}} \cdot \delta_{-(\tilde{Y}_0^r)_t + \frac{1}{r}\tilde{m}_t^r}(\mathrm{d}x_{k+K+1})\Big) \\
&\quad \times \prod_{n=1}^{K} \Big(\mathbb{1}_{\left\{\left(\tilde{U}_n^{(1),r}\right)_t < r - c/Y_t^r\right\}} \cdot \mathrm{B}_{n,r,t}(\mathrm{d}x_{k+K+1+n}) \\
&\qquad\qquad + \mathbb{1}_{\left\{(\tilde{U}_n^{(1),r})_t \geq r - c/Y_t^r\right\}} \cdot \delta_{-(\tilde{Y}_n^r)_t}(\mathrm{d}x_{k+K+1+n})\Big).
\end{aligned} \tag{A.15}$$

Jumps induced by birth events in the bulks The compensator $\nu^{n,r}$ handles birth events happening in the bulk of subpopulation n. $\tilde{X}^{K,r}$ accounts those as mass increase.

There are two birth mechanisms at work: branching of the bulk particles and immigration fuelled by the progenitor(s). In the root the rate for the former birth type is $\left(r - \tilde{U}_0^{(k),r}\right)\left(r\tilde{Y}_0^r - k\right)$, and the rate for the latter birth type is $2k\left(r - \tilde{U}_0^{(k),r}\right)$. For the root this amounts to the compensator

$$\begin{aligned}
\nu_0^{\tilde{X},r}(\mathrm{d}t,\mathrm{d}x) = \mathbb{1}_{\{t<\tau_\delta^r\wedge\sigma_K^r\}} &\cdot \left(r - \tilde{U}_0^{(k),r}\right)\left(r\tilde{Y}_0^r + k\right)\mathrm{d}t \cdot \delta_{\frac{1}{r}}(\mathrm{d}x_{k+K+1}) \\
&\times \delta_0(\mathrm{d}x_1,\dots,\mathrm{d}x_{k+K},\mathrm{d}x_{k+K+2},\dots,\mathrm{d}x_{k+2K+1}),
\end{aligned} \tag{A.16}$$

In the n-th extraction births happen at rates $\left(r - \tilde{U}_n^{(1),r}\right)\left(r\tilde{Y}_n^r - 1\right)$ and $2\left(r - \tilde{U}_n^{(1),r}\right)$. The compensator for bulk births in the n-th extraction ($n \geq 1$) is

$$\begin{aligned}
\nu_n^{\tilde{X},r}(\mathrm{d}t,\mathrm{d}x) = \mathbb{1}_{\{t<\tau_\delta^r\wedge\sigma_K^r\}} &\cdot \left(r - (\tilde{U}_n^{(1),r})_t\right)\left(r(\tilde{Y}_n^r)_t + 1\right)\mathrm{d}t \cdot \delta_{\frac{1}{r}}(\mathrm{d}x_{k+K+n+1}) \\
&\times \delta_0(\mathrm{d}x_1,\dots,\mathrm{d}x_{k+K+n},\mathrm{d}x_{k+K+n+2},\dots,\mathrm{d}x_{k+2K+1}).
\end{aligned} \tag{A.17}$$

Jumps induced by formations of new extractions Although we do not use $\nu_f^{\tilde{X},r}$ in our considerations in Section 6.3.1, we specify it for the sake of completeness. Births among the k progenitors of the root lead to formation events as detailed in 6.1: If a new progenitor with rank q is born at time t it takes the place of the formerly q-th lowest particle, so $\tilde{U}_0^{(q),r}$ performs a uniformly distributed downward jump,

$$\Delta(\tilde{U}_0^{(q),r})_t \sim \mathrm{Unif}_{[(\tilde{U}_0^{(q-1),r}-\tilde{U}_0^{(q),r})_{t-},0]}.$$

The coordinates $\tilde{U}_0^{(1,\dots,q-1),r}$ don't jump and $\tilde{U}_0^{(q+1,\dots,k),r}$ jump down one rank,

$$\begin{aligned}
\Delta(\tilde{U}_0^{(1,\dots,q-1),r})_t &= 0,\\
\Delta(\tilde{U}_0^{(q+1,\dots,k),r})_t &= (\tilde{U}_0^{(q,\dots,k-1),r} - \tilde{U}_0^{(q+1,\dots,k),r})_{t-}.
\end{aligned}$$

All extractions move on one slot to make room for the newly formed extraction. Its progenitor has level $(\tilde{U}_0^{(k),r})_{t-}$ and its subpopulation mass is $(\tilde{Y}_0^r)_{t-} - \frac{k-1}{r}$. The jumps induced by the formations are compensated by

$$\begin{aligned}
\nu_f^{\tilde{X},r}(\mathrm{d}t,\mathrm{d}x) = \mathbb{1}_{\{t<\tau_\delta^r\wedge\sigma_K^r\}} \cdot \sum_{q=2}^{k} 2(i-1)\left((\tilde{U}_0^{(q),r})_t - (\tilde{U}_0^{(q-1),r})_t\right)\mathrm{d}t\\
\times \delta_0(\mathrm{d}x_1,\dots,\mathrm{d}x_{q-1}) \cdot \mathrm{Unif}_{[(\tilde{U}_0^{(q-1),r}-\tilde{U}_0^{(q),r})_t,0]}(\mathrm{d}x_q)\\
\times \delta_{(\tilde{U}_0^{(q,\dots,k-1),r}-\tilde{U}_0^{(q+1,\dots,k),r})_t}(\mathrm{d}x_{q+1},\dots,\mathrm{d}x_k)\\
\times \delta_{(\tilde{U}_0^{(k),r}-\tilde{U}_1^{(1),r})_t}(\mathrm{d}x_{k+1}) \cdot \prod_{n=2}^{K} \delta_{(\tilde{U}_{n-1}^{(1),r}-\tilde{U}_n^{(1),r})_t}(\mathrm{d}x_{k+n})\\
\times \delta_{\frac{k}{r}-(\tilde{Y}_0^r)_t}(\mathrm{d}x_{k+K+1}) \cdot \delta_{(\tilde{Y}_0^r)_t-\frac{k-1}{r}-(\tilde{Y}_1^r)_t}(\mathrm{d}x_{k+K+2})\\
\times \prod_{n'=2}^{K} \delta_{(\tilde{Y}_{n'-1}^r-\tilde{Y}_{n'}^r)_t}(\mathrm{d}x_{k+K+n'+1}).
\end{aligned} \tag{A.18}$$

References

[Bil99] Patrick Billingsley. *Convergence of Probability Measures (2nd edition)*. New York: Wiley, 1999.

[BB09] Matthias Birkner and Jochen Blath. "Measure-valued diffusions, general coalescents and population genetic inference". In: *Trends in Stochastic Analysis* 351 (2009).

[DP98] Donald A. Dawson and Edwin A. Perkins. "Long-time behaviour and coexistance in a mutually catalytic branching model". In: *The Annals of Probability* 26.3 (1998), pp. 1088–1138.

[DK96] Peter Donnelly and Thomas G. Kurtz. "A countable representation of the Fleming-Viot measure-valued diffusion". In: *The Annals of Probability* 24.2 (1996), pp. 698–742.

[DK99a] Peter Donnelly and Thomas G. Kurtz. "Genealogical processes for Fleming–Viot models with selection and recombination". In: *The Annals of Applied Probability* 9.4 (1999), pp. 1091–1148.

[DK99b] Peter Donnelly and Thomas G. Kurtz. "Particle representations for measure-valued population models". In: *The Annals of Probability* 27.1 (1999), pp. 166–205.

[Eth00] Alison M. Etheridge. *An Introduction to Superprocesses*. Providence: American Mathematical Society, 2000.

[EF04] Alison M. Etheridge and Klaus Fleischmann. "Compact interface property for symbiotic branching". In: *Stochastic Processes and their Applications* 114 (2004), pp. 127–160.

[EK19] Alison M. Etheridge and Thomas G. Kurtz. "Genealogical constructions of population models". In: *The Annals of Probability* 47.4 (2019), pp. 1827–1910.

[EK86] Stewart N. Ethier and Thomas G. Kurtz. *Markov Processes, Characterization and Convergence*. Hoboken, USA: Wiley, 1986.

[JS03] Jean Jacod and Albert N. Shiryaev. *Limit theorems for stochastic processes (2nd edition)*. Berlin: Springer, 2003.

[Kal02] Olaf Kallenberg. *Foundations of modern probability (2nd edn)*. New York: Springer, 2002.

[Kal17] Olaf Kallenberg. *Random Measures, Theory and Applications*. Cham, Switzerland: Springer, 2017.

[Kle06] Achim Klenke. *Wahrscheinlichkeitstheorie*. Berlin: Springer, 2006.

[Kur98] Thomas G. Kurtz. “Martingale problems for conditional distributions of Markov processes”. In: *Electronic Journal of Probability* 3.9 (1998), pp. 1–29.

[KN11] Thomas G. Kurtz and Giovanna Nappo. “The filtered martingale problem”. In: *The Oxford Handbook of nonlinear filtering*. Ed. by Dan Crisan and Boris Rozovskii. Oxford: Oxford University Press, 2011. Chap. 5, pp. 129–165.

[KR11] Thomas G. Kurtz and Eliane R. Rodrigues. “Poisson Representations of Branching Markov and Measure-Valued Branching Processes”. In: *The Annals of Probability* 39.3 (2011), pp. 939–984.

[Kyp14] Andreas E. Kyprianou. *Fluctuations of Lévy Processes with Applications (2nd edition)*. Berlin: Springer, 2014.

[Myt98] Leonid Mytnik. “Uniqueness for a mutually catalytic branching model”. In: *Probability Theory and Related Fields* 112 (1998), pp. 245–253.

[Mé82] Michel Métivier. *Semimartingales a Course on Stochastic Processes*. Berlin: Walter de Gruyter, 1982.

[Pro04] Philip E. Protter. *Stochastic Integration and Differential Equations*. Berlin: Springer, 2004.

[RY99] Daniel Revuz and Marc Yor. *Continuous Martingales and Brownian Motion*. Berlin: Springer, 1999.

[RP81] L. C. G. Rogers and J. W. Pitman. “Markov Functions”. In: *The Annals of Probability* 9.4 (1981), pp. 573–582.