Functional equations of polylogarithms in motivic cohomology

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Contents

Summary

For an infinite field F, we study the integral relationship between the Bloch group $B_2(F)$ and the higher Chow group $CH^2(F,3)$ by proving some relations corresponding to the functional equations of the dilogarithm. As a second result, the groups involved in Suslin's exact sequence

$$0 \to Tor_1^{\mathbb{Z}}(F^\times, F^\times)^\sim \to CH^2(F,3) \to B_2(F) \to 0$$

are identified with homology groups of the cycle complex $Z^2(F,\bullet)$ computing Bloch's higher Chow groups.

Using these results, we give explicit cycles in motivic cohomology generating the integral motivic cohomology groups of some specific number fields and determine whether a given cycle in the Chow group already lives in one of the other groups of Suslin's sequence. In principle, this enables us to find a presentation of the codimension two Chow group of an arbitrary number field.

Finally, we also prove some relations in the higher Chow groups of codimension three modulo 2-torsion coming from relations in the higher Bloch group $B_3(F) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$. Further, we can prove a series of relations in $CH^3(\mathbb{Q}(\zeta_p), 5)$ for a primitive p^{th} root of unity ζ_p. This relates the higher Bloch group $B_3(F) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ and the motivic cohomology group $CH^3(F,5) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ of a number field F.

Zusammenfassung

Wir untersuchen den Zusammenhang zwischen der Blochgruppe $B_2(F)$ und der höheren Chowgruppe $CH^2(F, 3)$ mit ganzzahligen Koeffizienten für einen unendlichen Körper F, indem wir die Gültigkeit von Relationen in der Chowgruppe beweisen, die zu charakterisierenden Funktionalgleichungen des Dilogarithmus korrespondieren. Als weiteres Ergebnis können wir die Gruppen, welche durch Suslins kurze exakte Sequenz

$$0 \to Tor_1^{\mathbb{Z}}(F^\times, F^\times)^\sim \to CH^2(F, 3) \to B_2(F) \to 0$$

in Beziehung stehen, mit Homologiegruppen des Zykelkomplexes $Z^2(F, \bullet)$, der die höheren Chowgruppen berechnet, identifizieren.

Mit diesen Ergebnissen geben wir explizite Zykel in der motivischen Kohomologie von ausgewählten Zahlkörpern an, die ihre höheren Chowgruppen in Kodimension zwei erzeugen. Außerdem bestimmen wir, ob ein gegebener Zykel in der Chowgruppe schon in einer der anderen Gruppen aus Suslins Sequenz enthalten ist.

Letztlich beweisen wir die Gültigkeit einiger Relationen in den höheren Chowgruppe in Kodimension drei mit Koeffizienten in $\mathbb{Z}\left[\frac{1}{2}\right]$ von unendlichen Körpern, die von definierenden Relationen in der höheren Blochgruppe $B_3(F) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ kommen. Wir können sogar eine Serie von Relationen in $CH^3(\mathbb{Q}(\zeta_p), 5)$ für eine primitive p. Einheitswurzel ζ_p beweisen. Durch diese Relationen bringen wir die höhere Blochgruppe $B_3(F) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ und die motivische Kohomologiegruppe $CH^3(F, 5) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ eines Zahlkörpers F in Verbindung.

Introduction

In modern mathematics progress is very often made by a clever combination of seemingly unrelated branches of classical themes or by viewing old problems in the light of some recent development in some other area of mathematical investigation.

As a specific example of such progress, this thesis deals with motivic cohomology. This theory grew out of Grothendieck's vision to merge algebraic topology and what later became called arithmetic algebraic geometry to generalize the notion of singular homology of topological spaces to arbitrary schemes over $\mathsf{Spec}(\mathbb{Z})$. This should produce some sort of "universal" or "underlying" cohomology theory of which the more familiar theories to be found in algebraic or arithmetic geometry, e. g. (algebraic) De Rham, ℓ-adic, étale, crystalline cohomology, or in number theory, e. g. Galois cohomology, are just so-called realizations. In other words, Grothendieck envisioned a theory encapsulating all the information about the cohomology of a scheme, but revealing only part of its information when one studies one of its more well-known realizations.

The idea can roughly be described as follows: Decompose an arbitrary scheme over some base into its building blocks, which one can think of as an affine open cover or smooth parts or irreducible components, and associate to these building blocks so-called "motives". The – yet to define – cohomology groups of those motives should contain all the information that is available.

But as one knows, arbitrary schemes can be terribly complicated, and standard methods in algebraic geometry did not suffice for translating topological notions into the category of schemes over some base. In section 16 of "En guise d'Avant-Propos" of his famous "Récoltes et Semailles", Grothendieck describes his way of thinking of motives:

"... Contrairement à ce qui se passait en topologie ordinaire, on se trouve donc placé là devant une abondance déconcertante des théories cohomologiques différentes. On avait l'impression très nette qu'en un sens, qui restait d'abord très flou, toutes ces theories devaient ≪revenir au même≫, qu'elles ≪donnaient les mêmes résultats≫. C'est pour parvenir à exprimer cette intuition de ≪paranté≫ entre théories cohomologiques différentes, que j'ai dégagé la notion de ≪motif≫ associé à une variété algébrique.
Par ce terme j'étends suggérer qu'il s'agit du ≪motif commun≫ (ou de la raison commune) sous-jacent à cette multitude d'invariants cohomologiques différentes associés à la variété, à l'aide de la multitude de toutes les théories cohomologiques possibiles à priori. [...]"

Further, he explains that motivic cohomology should be the basic motive from which other cohomology theories are just facets:

"Ainsi, le motif associé à une variété algébrique constituerait l'invariante cohomologique ≪ultime≫, ≪par excellence≫, dont tous les autres (associeés aux différentes théories cohomologiques possible) se déduiraient, comme autant d'≪incarnations≫ musicales, ou de ≪réalisations≫ différentes. Toutes les propriétés essentielles de ≪la cohomologie≫ de la variété se ≪liraient≫ (ou ≪s'entendraient≫) déjà sur le motif correspondant, de sorte que les propriétés et structures familières sur les invariants cohomologiques particularisés (l-adique ou cristallins, par exemple), seraient simplement le fidèle reflet des propriétés et structures internes au motif."

According to Grothendieck, motives can be considered as geometrical objects which link geometric and arithmetic properties of algebraic varieties:

"Dans ma vision des motifs, ceux-ci constituent une sorte de ≪cordon≫ très caché et très délicat, reliant les propriétés algébro-géométriques d'une variété algébrique, à des propriétés de nature ≪arithmétique≫ incarnées par son motif. Ce dernier peut être considéré comme un objet de nature ≪géométrique≫ dans son esprit même, mais où les propriétés ≪arithmétiques≫ subordonnées à la géométrie se trouvent, pour ainsi dire, ≪mises à nu≫."

The theory of motives is still very incomplete and relies on very deep conjectures on algebraic cycles, the so-called standard conjectures. Either the description of motives or of the concrete realization functors is far from being understood up to now. One way to describe motives is to think of them as objects in a yet to be constructed category. This is very abstract and complicated. We do not go into the details of this line of development. The other possibility to describe them is by finding a cohomology theory with the expected "universal" properties envisioned by Grothendieck. This is the path we follow in this thesis: We will compute motivic cohomology in some cases explicitly, where some concrete information is available.

At the beginning of the 1980s A. Beĭlinson, P. Deligne and S. Lichtenbaum pursued Grothendieck's ideas for smooth schemes X over a field k, and described a "universal" or a "motivic" cohomology theory with coefficients in an abelian group A as a family of functors

$$H^{p,q}(\bullet, A) : Sm/k \longrightarrow Ab, \quad X \mapsto H^{p,q}_{\mathcal{M}}(X, A)$$

indexed by p and q satisfying several formal properties. The different approaches of a concrete description of these functors had advantages as well as disadvantages compared to the other ones, but they were all rather abstract. Nowadays there are several more or less explicit constructions for the cohomology groups conjectured above given by S. Bloch, E. Friedlander, A. Suslin, V. Voevodsky et al. which are known to be isomorphic at least for schemes over a field of characteristic zero.

In the present thesis, we examine motivic cohomology groups for schemes consisting of just one point, namely $\mathsf{Spec}(F)$ for a number field F, and give some arithmetic applications. This is the most easy case where an explicit description of these groups is possible because of their relations to other, better understood, groups. The motivation behind studying motivic cohomology groups naturally has two aims. At first, one hopes to achieve some understanding of the theory by computing some easy examples, and secondly, one would like to gain new, or more refined, information compared to known theories by making use of comparison theorems.

An example of the latter motivation stems from algebraic K–theory: Generalizing Atiyah's topological K–theory, D. Quillen gave the definition of higher algebraic K–groups. Because of their arithmetic significance, e. g. in deep conjectures in algebraic number theory such as the Beĭlinson conjectures about special values of L-functions or the Bloch-Kato conjecture in arithmetic geometry, it is a major goal of modern number theory to compute higher K–groups for number fields explicitly. Their crucial relation to the motivic cohomology of a number field is given by

$$gr_{\gamma}^{q} K_{2q-1}(F) \otimes \mathbb{Q} \cong H_{\mathcal{M}}^{1,q}(\mathsf{Spec}(F), \mathbb{Z}) \otimes \mathbb{Q} \cong H_{\mathcal{M}}^{1,q}(\mathsf{Spec}(F), \mathbb{Q}),$$

where gr_{γ}^{q} denotes the graded piece of weight q of the γ-filtration in K–theory, and where the last isomorphism is a general property of motivic cohomology of number fields. One has to mention that Beĭlinson originally defined motivic cohomology by this equality!

Therefore, the study of motivic cohomology groups of a number field serves at least two purposes, namely acquiring knowledge about the conjectured universal cohomology theory for schemes in the baby case of a scheme consisting of one point only, and on the other hand gaining new methods of calculating the algebraic K–groups of a field: If it is at least possible to compute the rational motivic cohomology groups of a field, one may add them up to computing the corresponding rational K–group. Unfortunately, it is not possible to approach the integral K–groups in the same way, but with the help of a certain spectral sequence

$$CH^{-q}(X, -p-q) \Rightarrow K_{-p-q}(X)$$

for an equidimensional scheme over a field k, one can at least in principle compute the integral K–groups as well.

Since we are interested in explicit calculations in motivic cohomology, we shall use S. Bloch's candidate for motivic cohomology, the higher Chow groups. These offer for some purposes the most concrete description in terms of cohomology groups of certain algebraic cycles in projective space over a field. The higher Chow groups of a number field are conjectured to possess another remarkable property: They might be described – at least rationally – by the so-called Bloch groups. These groups are roughly subquotients of free abelian groups over the non-zero elements of a number field with integer coefficients modulo some relations coming from universal functional equations of polylogarithms. This is another great example of the merging of two seemingly completely different branches of mathematics. Therefore, one can – and we will – use the theory of polylogarithms to gain information on the theory of motives. This is done in the following way: There are computer algorithms producing lots of elements in the Bloch groups – at least in low degree – which can be mapped to elements in the algebraic K–group of the corresponding field or equivalently to the motivic cohomology group. In case we had a good understanding of these groups and especially their relationship, we could use computer power to find explicit elements or generators in motivic cohomology.

In order to make the rational isomorphism between the higher Chow groups and Bloch groups explicit, S. Müller-Stach and H. Gangl have investigated a concrete map ρ_2 being believed to induce the isomorphism $\overline{\rho}_2 \otimes \mathbb{Q} : B_2(F) \otimes \mathbb{Q} \xrightarrow{\cong} CH^2(F,3) \otimes \mathbb{Q}$ in codimension two. These authors and J. Zhao have also provided strong evidence for a map in codimension three to be a candidate inducing the conjectured isomorphism.

The purpose of this thesis is to refine their results to the integral setting. We shall prove universal relations in the integral higher Chow group $CH^2(F,3)$ for infinite fields F coming from the defining relations of the Bloch group $B_2(F)$ of these fields. With these relations we obtain an explicit presentation of the higher Chow groups in codimension two for some number fields.

Some results are also obtained in codimension three, but mostly up to 2-torsion, because there are no appropriate cycles avaliable in $CH^3(F,5)$ which can be used to extend the results of [GMS99] and [Zha07]. There are only partial results on relations involving cyclotomic elements corresponding to the trilogarithm function evaluated at roots of unity.

Let us now take a closer look at the different chapters:

After surveying motivic cohomology of a field, chapter 1 collects basic definitions and properties of the dilogarithm and trilogarithm functions accompanied with their single-valued variants. We introduce the main objects of this thesis, i. e. the Bloch group $B_2(F)$ of a field F as well as its generalization, S. Bloch's higher Chow groups and their main properties as far as needed. After that we shall also recall a generalized Abel – Jacobi map due to M. Kerr, J. D. Lewis and S. Müller-Stach from higher Chow groups to Deligne – Beĭlinson cohomology which helps detecting torsion cycles. With the help of the injectivity of this map we can conclude that cycles with nonzero (or non-torsion) image in Deligne – Beĭlinson cohomology must have been nonzero (or non-torsion) in the higher Chow group. There is nothing new in this chapter except, of course, the presentation.

Chapter 2 presents the results in codimension two. Applying a result of E. Nart, we can slightly simplify computations in the higher Chow group: There are acyclic subcomplexes of the complex computing the higher Chow groups. In the sequel, we shall work in one particular quotient only. Then we set up some fundamental relations needed to prove Abel's five-term relation in the Chow group in several versions. As corollaries, we also get an inversion relation. Therefore, we obtain a presentation of the higher Chow groups of an arbitrary field F with generators that depend on the field F and the relations just proved. In some examples, we determine these generators concretely, but it is clear how one can use this approach in general to find a presentation of the Chow group of an arbitrary infinite field F. The rest of the chapter is devoted to exploring the difference between the two choices of an acyclic subcomplex to be divided out at the beginning. It turns out that one can divide out two subcomplexes: both of them are acyclic. But one cannot divide out both of them simultaneously without modifying the resulting motivic cohomology group. It turns out that dividing out both of them leads to the Bloch group, whereas the difference is related to the Tor-group of Suslin's exact sequence: We close the chapter with an explicit description of the Bloch group and the group $Tor(F^{\times}, F^{\times})^{\sim}$ in terms of algebraic cycles.

Chapter 3 uses and extends the ideas of chapter 2 to prove the Kummer – Spence – relation, a series of distribution relations, an inversion relation, and Goncharov's relation in $CH^3(F,5) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ under the assumption of the integral Beĭlinson– Soulé vanishing conjecture in codimension two. But even under this strong assertion, the technical problems are much harder in this codimension three Chow groups. Another problem one has to face is that there are no suitable cycles in the integral higher Chow group.

We work around this problem by tensoring with $\mathbb{Z}\left[\frac{1}{2}\right]$, i. e. neglecting 2-torsion. But it turns out that some of the cycles we would like to use cannot be contained in the image of a map $\mathbb{Z}[F^{\times}] \to Z^3(F,5)$ inducing the rational isomorphism $B_3(F) \otimes \mathbb{Q} \xrightarrow{\cong} CH^3(F,5) \otimes \mathbb{Q}$.

From the present point of view we cannot prove much beyond what is done in the present thesis in codimenson three, but as we will see there is still a lot to do.

Chapter 1

Motivic cohomology of a field

1.1 What is motivic cohomology of a field?

Having glimpsed at the theory of motivic cohomology in the introduction, it is time to be a little more concrete. For us, motivic cohomology will be the Zariski hypercohomology of certain (motivic) complexes of sheaves. This approach is due to Beĭlinson.

We will only consider nonsingular varieties in this thesis. So let X for the moment be a smooth projective variety over a field k. It was A. Beĭlinson who noticed that the algebraic K–theory of X can be re-indexed to give a universal cohomology theory with coefficients in $\mathbb{Q}$: If one isolates a graded piece $gr^q_\gamma K_{2q-p}(X)$ of weight q of the γ-filtration in algebraic K–theory and defines this to be the universal cohomology group $H^{p,q}_{\mathcal{M}}(X, \mathbb{Q})$, then $H^{\bullet,\bullet}_{\mathcal{M}}(\bullet, \mathbb{Q})$ gives rise to a candidate for the conjectured cohomology theory.

In order to define an integral version of this theory, Beĭlinson considered truncations of Zariski direct images of the étale sheaves $\mu_m^{\otimes i}$ and their hypercohomology groups. He also noted that these satisfy the formal axioms of a Weil cohomology theory, which led him to his famous conjectures asserting the integral motivic cohomology for a smooth variety X over a field be given by the Zariski hypercohomology $H^{p,q}_{\mathcal{M}}(X, \mathbb{Z}) := \mathbb{H}^p_{Zar}(X, \mathbb{Z}(q)_X)$ of certain natural chain complexes of sheaves $\mathbb{Z}(\bullet)_X$, where we shall drop the subscript if the context is clear, subject to the following conditions:

- $\mathbb{Z}(0)_X$ is the constant sheaf $\mathbb{Z}_X$, and $\mathbb{Z}(i)_X = 0$ for $i < 0$.
- $\mathbb{Z}(1)_X = \mathcal{O}_X^\times[-1]$, i. e. the sheaf $\mathcal{O}_X^\times$ understood as a complex of sheaves concentrated in degree 1
- For a strictly Hensel local scheme S over a field k and an integer ℓ prime to $char(k)$, one has
$$H^{p,q}(S, \mathbb{Z}/\ell\mathbb{Z}) = \begin{cases} \mu_\ell^{\otimes q}(S), & p = 0, q > 0, \\ 0, & else, \end{cases}$$
where $\mu_\ell(S)$ denotes the group of ℓ^{th} roots of unity in S.

- $H^{n,n}_{\mathcal{M}}(\mathsf{Spec}(k), \mathbb{Z}) \cong K^M_n(k)$ so that motivic cohomology agrees with Milnor K–theory for fields k.
- $H^{2n,n}_{\mathcal{M}}(X, \mathbb{Z}) \cong CH^n(X)$, where $CH^n(X)$ denotes the classical Chow group of codimension n cycles on a smooth projective X modulo rational equivalence.
- For a smooth X over a field there should be a spectral sequence

$$E_2^{p,q} = H^{p,q}_{\mathcal{M}}(X, \mathbb{Z}) \Rightarrow K_{2q-p}(X)$$

giving rise to a rational isomorphism

$$H^{p,q}_{\mathcal{M}}(X, \mathbb{Z}) \otimes \mathbb{Q} \cong gr^q_\gamma K_{2q-p}(X) \otimes \mathbb{Q}.$$

There is a rather explicit construction of these motivic cohomology groups due to Spencer Bloch, namely his higher Chow groups, which we shall introduce later. In our case, where $X = \mathsf{Spec}(F)$ consists of just the generic point, the computation of Zariski hypercohomology of $\mathbb{Z}(\bullet)_X$ boils down to a computation of the "ordinary" homology, i. e. no hyperhomology of a certain complex. This simplifies concrete computations enormously.

According to Beĭlinson's formula $H^{p,q}_{\mathcal{M}}(X, \mathbb{Z}) \otimes \mathbb{Q} \cong gr^q_\gamma K_{2q-p}(X) \otimes \mathbb{Q}$, we can also use the knowledge of the algebraic K–groups of a number field to determine the motivic cohomology groups abstractly: As one can see in an article by C. Weibel [Wei05], the algebraic K–groups of local and global fields are abstractly well-known. So, having determined the appropriate graded piece of the γ-filtration, one knows the corresponding motivic cohomology group. If one is interested in finding an explicit set of generators in terms of algebraic cycles – the geometric objects from Grothendieck's intuition of the nature of motives – for this group, then one can find them by checking the order of lots of elements until an element of the correct order is found. This can be done by a regulator map to some suitable cohomology theory, but this is a tedious task.

There is a more conceptional approach to this problem of finding generators due to S. Bloch, Don Zagier et al. who study the so-called Bloch groups $B_m(F)$ of a field F. These groups are defined inductively and can – in principle – be described rather easily as the kernel of an explicit map modulo some relations coming from functional equations of the m^{th} polylogarithm. These groups have the advantage that one can use a computer to produce many elements in short time. The general conjecture predicts that these groups are isomorphic – at least modulo torsion – to certain motivic cohomology groups of a field. The easiest case is given by $m = 2$: One considers the map

$$\beta_2 : \mathbb{Z}[F^\times] \to \Lambda^2 F^\times, [a] \mapsto a \wedge (1-a)$$

and defines

$$B_2(F) := \frac{ker\,\beta_2}{\left\langle [x] + [y] + [1-xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right] \right\rangle}, \qquad x, y \in F^\times, xy \neq 1.$$

The five-term relation in the denominator characterizes a single-valued version of the dilogarithm function more or less uniquely in the sense that every measurable function on $\mathbb{C}$ satisfying this relation is a multiple of this single-valued dilogarithm [Blo]. Unfortunately, despite some progress in low degrees, it is not clear at all how the higher Bloch groups can be presented explicitly in codimension greater than two. Even in codimension three, there is only some agreement about what relations to divide out for the definition. Therefore our approach to motivic cohomology via Bloch groups appears to be at most partly promising. The next problem one has to face is the unknown integral relation between the Bloch groups and the higher Chow group. The only result is a theorem of Suslin in codimension two, where he uses (stability results for) the group homology of $GL(F)$ to prove that there is an exact sequence

$$0 \to Tor_1^{\mathbb{Z}}(F^\times, F^\times)^\sim \to CH^2(F,3) \to B_2(F) \to 0,$$

where $B_2(F)$ denotes the Bloch group in the sense of Suslin defined as the kernel of an explicit map modulo a relation corresponding to the five-term relation for the dilogarithm, and where the tilde indicates a nontrivial $\mathbb{Z}/2\mathbb{Z}$ – extension of the Tor-group. In codimension three, one would have to investigate $H_3(GL(F), \mathbb{Z})$ for a number field F, which seems to be hard work. But nevertheless, it is worth exploring the connection between Bloch groups and higher Chow groups, because there is a lot of beauty in this approach.

A more formal approach to motivic cohomology of a field was initiated by A. Goncharov. He defines certain complexes whose cohomology is supposed to calculate the rational motivic cohomology. Unfortunately, there will be no opportunity to use this and subsequent ideas of his to compute Chow groups or motivic cohomology groups explicitly.

The structure of the present chapter is the following. In the next section, we collect some basic facts about polylogarithms and Bloch groups. Starting with the dilogarithm and its single-valued variant, the Bloch Wigner dilogarithm, we focus on the Bloch group and its significance in geometry and arithmetic, we then concentrate on the trilogarithm where the state of knowledge is located somewhere between the completely explicit description of the dilogarithm and the classical Bloch group and the general, abstract picture. We shall collect some facts about the trilogarithm and the Bloch group $B_3(F)$. In the next subsection, the general picture is sketched. The main conjecture is due to D. Zagier. Abstractly, the Bloch groups $B_m(F)$ and their relation to the algebraic K–groups $K_{2m-1}(F)$ are reviewed. This leads to a conjectured expression of the value of the Dedekind ζ-function of a number field F at the integer m in terms of the m^{th} polylogarithm evaluated at elements of the Bloch- resp. the K–group of F.

After this excursion to polylogarithms, our approach to motivic cohomology, the higher Chow groups, will be explored at the end of this chapter, where we list some properties and finally a certain regulator map from the higher Chow groups to Deligne – Beĭlinson cohomology which is needed to detect torsion and non-torsion cycles in the Chow groups.

1.2 Polylogarithms and Bloch groups

1.2.1 The dilogarithm and number theory

The dilogarithm is defined by the power series

$$Li_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \qquad z \in \mathbb{C}, \quad |z| < 1.$$

This definition generalizes the Taylor expansion for the usual logarithm around 1, namely

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} =: Li_1(z).$$

By analytic continuation, the dilogarithm can be extended to a function

$$Li_2(z) = -\int_0^z \log(1-t)\frac{dt}{t}, \qquad z \in \mathbb{C} \setminus [1,\infty).$$

This is a multivalued function with monodromy $2\pi i \log|z|$ on crossing the branch cut $[1,\infty)$.

One only knows a few examples, where the value of the dilogarithm can be given in closed form. These are

$$Li_2(0) = 0, \qquad Li_2(1) = \frac{\pi^2}{6},$$
$$Li_2(-1) = -\frac{\pi^2}{12}, \qquad Li_2(\frac{1}{2}) = \frac{\pi^2}{12} - \frac{1}{2}\log^2(2),$$
$$Li_2\left(\frac{3-\sqrt{5}}{2}\right) = \frac{\pi^2}{15} - \log^2\left(\frac{1+\sqrt{5}}{2}\right), \qquad Li_2\left(\frac{-1+\sqrt{5}}{2}\right) = \frac{\pi^2}{10} - \log^2\left(\frac{1+\sqrt{5}}{2}\right),$$
$$Li_2\left(\frac{1-\sqrt{5}}{2}\right) = -\frac{\pi^2}{15} + \frac{1}{2}\log^2\left(\frac{1+\sqrt{5}}{2}\right), \quad Li_2\left(\frac{-1-\sqrt{5}}{2}\right) = -\frac{\pi^2}{10} + \frac{1}{2}\log^2\left(\frac{1+\sqrt{5}}{2}\right).$$

The dilogarithm satisfies a number of functional equations. There are the elementary ones like the so-called reflection properties

$$Li_2(z) = -Li_2(\frac{1}{z}) - Li_2(1) - \frac{1}{2}\log^2(z), \quad z \in \mathbb{C} \setminus [1,\infty),$$
$$Li_2(z) = -Li_2(1-z) + Li_2(1) - \log(z)\log(1-z), \quad z \in \mathbb{C} \cap]0,1[$$

which give rise to the following remarkable fact: The six functions

$$Li_2(z),\ Li_2(1-z),\ Li_2\left(\frac{1}{z}\right),\ Li_2\left(\frac{1}{1-z}\right),\ Li_2\left(\frac{z-1}{z}\right),\ Li_2\left(\frac{z}{z-1}\right)$$

are equal modulo elementary functions, i. e. constants, logarithms and products of logarithms.

In the following, we would like to call these elementary functions to be "of lower order" compared to the dilogarithm or more generally to the m^{th} polylogarithm. So if an identity of m^{th} polylogarithms is said to hold up to terms of lower order, we neglect all n^{th} polylogarithms and products of these for $n < m$.

The next series of functional equation of the dilogarithm is given by the distribution property

$$Li_2(z^n) = n \sum_{\zeta^n=1} Li_2(\zeta z), \qquad |z| < 1, n \in \mathbb{N}.$$

But the most important relation is given by the five-term relation found and rediscovered by Abel, Kummer, Spence and others, which can be stated in the following form:

$$Li_2(x) + Li_2(y) + Li_2\left(\frac{1-x}{1-xy}\right) + Li_2(1-xy) + Li_2\left(\frac{1-y}{1-xy}\right)$$
$$= \frac{\pi^2}{6} - \log(x)\log(1-x) - \log(y)\log(1-y) + \log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{1-y}{1-xy}\right)$$

for $x, y \in \mathbb{C} \setminus \{0,1\}$, $xy \neq 1$. There are several other forms of this relation to be found in literature, but up to 2-torsion they are all equivalent modulo the reflection properties of the dilogarithm mentioned above.

Further, there is a six-term relation of the form

$$Li_2(x) + Li_2(y) + Li_2(z) = \frac{1}{2}\left[Li_2\left(-\frac{xy}{z}\right) + Li_2\left(-\frac{yz}{x}\right) + Li_2\left(-\frac{zx}{y}\right)\right]$$

for $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ due to Kummer and Newman.

In order to remove the multivalued character of the dilogarithm, one can consider the function $Li_2(z) + i\arg(1-z)\log|z|$, where we choose the branch of arg lying between $-\pi$ and π. This function turns out to be continuous. But even more is true, when one considers its imaginary part: There is a single-valued variant, called the Bloch – Wigner dilogarithm, namely

$$D(z) := P_2(z) := \mathrm{Im}(Li_2(z) + \log|z|\log(1-z))$$

for $z \in \mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}$ satisfying (beside others) the following properties:

- D is real analytic on $\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}$, and continuous on the compactified complex plane when one sets $D(0) = D(1) = D(\infty) = 0$.

- All of the functional equations valid for the ordinary dilogarithm can be expressed in terms of the Bloch – Wigner dilogarithm, and then all lower order terms disappear. For example, we have an exact 6-fold symmetry

$$D(x) = -D(1-x) = D\left(\frac{1}{1-x}\right) = -D\left(\frac{x}{x-1}\right) = D\left(1-\frac{1}{x}\right) = -D\left(\frac{1}{x}\right)$$

and there is the so-called clean five-term relation

$$D(x) + D(y) + D(1-xy) + D\left(\frac{1-x}{1-xy}\right) + D\left(\frac{1-y}{1-xy}\right) = 0, \quad x, y \in \mathbb{C}, xy \neq 1.$$

Remark 1.2.1. If we define

$$\tilde{D}(z_0, \dots, z_3) := D\left(\frac{z_0 - z_2}{z_0 - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}\right), \qquad z_i \in \mathbb{P}^1_{\mathbb{C}},$$

then the five-term relation can be written in the most symmetric form

$$\sum_{i=0}^{4} (-1)^i \tilde{D}(z_0, \dots, \hat{z}_i, \dots, z_4), \qquad z_i \in \mathbb{P}^1_{\mathbb{C}}.$$

In terms of hyperbolic geometry, this curiosity can be interpreted geometrically:

Let $\mathcal{H}^3$ be the Lobachevsky space, which we can think of (according to the half-space model) as $\mathbb{C} \times \mathbb{R}_+$ and let $I(z_0, \dots, z_3)$ denote the ideal geodesic simplex with vertices at points $z_0, \dots, z_3$ of $\overline{\partial\mathcal{H}^3} \cong \mathbb{P}^1_{\mathbb{C}}$. Thus, the vertices are at infinity; but nevertheless, the hyperbolic volume of such an simplex is finite. In particular, it is given by

$$vol(I(z_0, \dots, z_3)) = \tilde{D}(z_0, \dots, z_3).$$

By a Möbius transformation, three points can be chosen equal to $z_0 = 0, z_1 = 1, z_2 = \infty$, and one obtains the special case of the expression above: $vol(I(0, 1, \infty, z)) = D(z)$. The geometrically well-known fact that the "signed volume" $\sum_{i=0}^{4} (-1)^i vol(I(z_0, \dots, \hat{z}_i, \dots, z_4))$ vanishes is equivalent via the expression for the volume of such a tetrahedron to the most symmetric form of the five-term relation.

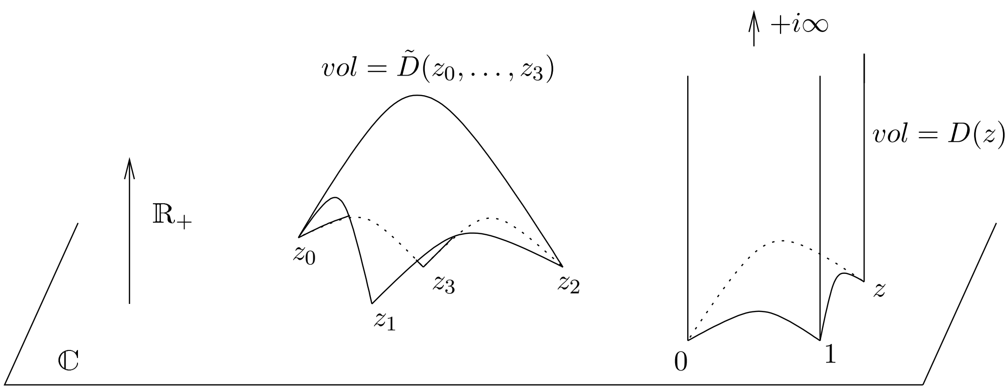

Figure 1.1: Volumes of hyperbolic tetrahedra

Unfortunately, this geometric interpretation of functional equations for polylogarithms is no longer valid in higher dimensions because of general properties of hyperbolic n-spaces (cf. the remarks at the end of [GZ00, sect. 2]). ◊

Let us now recall the Bloch groups and their connection to algebraic K–theory or motivic cohomology. This is where polylogarithms play the important role. S. Bloch and D. Zagier initiated an approach to an explicit presentation of algebraic K–groups in term of so called Bloch groups which are defined as subquotients of free abelian groups on the invertible elements of a number field F modulo specific relations coming from functional equations of the polylogarithmic functions. This approach links the rather complicated algebraic K–theory to the not less complicated but more explicit theory of polylogarithms: We state some notions from Suslin's article [Sus91] keeping an eye on Bloch's original definition in [Blo], some new aspects in Dupont and Sah [DS82] as well as Lichtenbaum's comparison in [Lic89].

We shall not stress too many details of algebraic K–theory. The facts to be remembered are that this theory associates to fields or more general rings – such as discrete valuation rings or local rings – a sequence of abelian groups indexed by nonnegative integers. These groups are defined in a highly nonconstructive way, but nevertheless serve as special invariants attached to these fields or rings. The first one for example, K_0, is known as the classical Grothendieck group of isomorphism classes of finitely generated, projective modules over a given ring regarded as monoid under direct summation, and K_1 includes the units in a given ring. If the reader wants to, he might as well take

$$K_n(F) \otimes \mathbb{Q} \cong \bigoplus_{\substack{q \leq n \\ 2q-p=n}} H^{p,q}_{\mathcal{M}}(F, \mathbb{Z}) \otimes \mathbb{Q}$$

as the definition of the higher K–groups, although historically K–theory was invented before motivic cohomology. In general, the algebraic K–groups of rings can be complicated, but in case of a number field, the following result tells us that the $\mathbb{Q}$-rank of these K–groups is especially easy to describe:

Theorem 1.2.2 (Borel [Bor77])**.** *For a number field F with r_1 real and r_2 pairs of conjugate complex embeddings $F \hookrightarrow \mathbb{C}$ we have*

$$rk(K_n(F) \otimes \mathbb{Q}) = \begin{cases} 1, & n = 0, \\ 0, & n \geq 2 \ \textit{even}, \\ r_1 + r_2 - 1, & n = 1, \\ r_2, & n = 2m-1, m \ \textit{even}, \\ r_1 + r_2, & n = 2m-1, m \ \textit{odd}. \end{cases}$$

There are two important filtrations defined on the algebraic K–groups of infinite fields. We shall quickly recall them for the sake of completeness. Both of them are linked via the so-called rank conjecture due to Suslin (unpublished). The work of Gangl and Müller-Stach heavily relies on the validity of this conjecture as we will see later in remark 1.3.10.

First we define an increasing filtration (cf. explanations in [Hai94]), the so-called γ-

filtration on $K_n(F)\otimes\mathbb{Q}$ by

$$F_\gamma^m K_n(F)\otimes\mathbb{Q} := \oplus_{j=m}^n K_n^{(j)}(F)\otimes\mathbb{Q},$$

where $K_n^{(j)}(F)\otimes\mathbb{Q}$ denotes the j^{th} eigenspace of the Adams operation Ψ^k on $K_n(F)\otimes\mathbb{Q}$, i. e. the part of the rational K–group where Ψ^k acts via multiplication with k.

If we interpret $K_n(F)\otimes\mathbb{Q}$ as the primitive part of $H_n(GL(F),\mathbb{Q})$, then it makes sense to define another increasing filtration $F_\bullet^{rank}$ on $K_n(F)\otimes\mathbb{Q}$ by

$$F_r^{rank}K_n(F)\otimes\mathbb{Q} := \mathrm{Im}\,(H_n(GL_r(F),\mathbb{Q})\hookrightarrow H_n(GL(F),\mathbb{Q}))\bigcap K_n(F)\otimes\mathbb{Q}$$

for $r\geq 1$. The content of the rank conjecture is that this filtration is the complement of the γ-filtration in an algebraic K–group of an infinite field:

Conjecture 1.2.3. *Let F be an infinite field. For $n\geq 1$ and all $r\geq 0$ there is a direct sum decomposition*

$$K_n(F)\otimes\mathbb{Q} = F_r^{rank}K_n(F)\otimes\mathbb{Q}\bigoplus F_\gamma^{r+1}K_n(F)\otimes\mathbb{Q}.$$

Remark 1.2.4. This conjecture has been settled for number fields by Borel and Yang [BY94], and there has been some progress on arbitrary infinite fields by Gerdes, Suslin, de Jeu et al.. ◊

Closely related to the algebraic K_3 of a field is its Bloch group, which we are going to introduce now. Let us denote by $\mathbb{Z}[F^\times]$ the free abelian group with basis $\{[a], a\in F^\times\setminus\{1\}\}$ and then consider the $\mathbb{Z}$-linear homomorphism

$$\beta_2:\mathbb{Z}[F^\times]\to\Lambda^2F^\times, [a]\mapsto a\wedge(1-a)$$

defined on the generators. By a well-known theorem of Matsumoto on the presentation of the second Milnor K–group $K_2^M(F)$ by symbols we have that $coker(\beta_2)=K_2^M(F)\cong K_2(F)$ up to 2-torsion again.

Definition 1.2.5 ([Sus91])**.** The Scissors' congrence group $\mathfrak{p}(F)$ is given by the quotient of $\mathbb{Z}[F^\times]$ by a variant of the five-term relation, namely

$$[a]-[b]+\left[\frac{b}{a}\right]-\left[\frac{1-b}{1-a}\right]+\left[\frac{1-b^{-1}}{1-a^{-1}}\right].$$

The Bloch group of Suslin is defined via the short exact sequence

$$0\to B_2(F)\to\mathfrak{p}(F)\xrightarrow{\lambda}F^\times\wedge F^\times\to K_2^M(F)\to 0,$$

where $\lambda([a]):=a\otimes(1-a)$.

Remark 1.2.6. This definition is not unique in the following way. There are several slightly inequivalent definitions found in literature. We do not recall their history and the developments which led to changes, but instead just mention some variants:

- Dupont and Sah show in [DS82] that the five-term relation implies $2([a]+[\frac{1}{a}])=0$ if $char(F)\neq 2$, and that the five-term relation together with the inversion relation imply $6([a]+[1-a])=0$.

- There are variants of this definition, where the set of generators is extended to include $0,1$ and where more relations are introduced. Precise descriptions can be found in [Blo], [Zag91], [DS82], and for an overview compare [Lic89]. But as shown in loc. cit., [DS82], and [Cat96, Proposition 2] they all differ by at most 6-torsion.

- Considering an algebraically closed field or ignoring torsion, all of these definitions are equivalent as one knows again from [DS82, App. A] and [Cat96, Prop. 2].

- The above exact sequence also holds in greater generality, e. g. for semilocal rings with infinite residue fields (cf.[EVG00])

- In order to generalize this definition later on, we shall mention that an important variant of this definition is given in [GZ00]:

$$B_2(F) := \frac{\mathcal{A}_2(F)}{\mathcal{C}_2(F)} := \frac{ker(\beta_2)}{\langle \text{five-term relation}\rangle}.$$

◊

The Bloch group (in the way we introduced it) satisfies a number of interesting properties among which are the following:

Theorem 1.2.7 ([Sus91])**.** *Let F be an arbitrary infinite field.*

- *We have an exact sequence*

$$0 \to Tor_1^{\mathbb{Z}}(F^\times, F^\times)^\sim \to K_3^{ind}(F) \to B_2(F) \to 0, \tag{1.2.1}$$

  where $Tor_1^{\mathbb{Z}}(F^\times,F^\times)^\sim$ is the unique non-trivial extension of $Tor_1^{\mathbb{Z}}(F^\times,F^\times)$ by $\mathbb{Z}/2\mathbb{Z}$, and $K_3^{ind}(F) := Coker(K_3^M(F)\to K_3(F))$ denotes the indecomposable part of algebraic K–theory.

- *The element $c_F := [x]+[1-x] \in B_2(F)$ is independent of the choice of $x\in F^\times - \{1\}$.*

- *c_F is at most 6-torsion. If $\sqrt{-1}\in F$, then c_F is 3-torsion. If a primitive 3^{rd} root of unity ζ_3 is contained in F, then c_F is 2-torsion, and if both $\sqrt{-1},\zeta_3\in F$, then $c_F=0$.*

- *In general: If F is formally real, the order of c_F is equal to six. In particular, $B_2(\mathbb{Q})$ is a cyclic group of order 6 generated by the element $c_{\mathbb{Q}}$.*

- *The Bloch group is rationally invariant.*

- *For $F=\overline{F}$, the Bloch group is uniquely divisible.*

- *As remarked in [EVG00], C. Weibel has computed $B_2(k)$ in the variant of [GZ00] in case of a finite field k with more than five elements and has also obtained a rational isomorphism to (Quillen's) algebraic K–group $K_3(k)$.*

So obviously $K_3^{ind}(F) \otimes \mathbb{Q} \simeq B_2(F) \otimes \mathbb{Q}$ abstractly. The next interesting question for us is how to relate the two groups explicitly, because knowing this, one can find elements in the Bloch group mapping to a basis of the K–group. Bloch and others contributed to some results in this direction leading to a famous conjecture in the general case. Roughly, we will introduce a regulator map from K–theory to a lattice in $\mathbb{R}^n$ which can be expressed in terms of polylogarithms for $K_3(F)$ and $K_5(F)$, but conjecturally in any degree. To shed some light on this circle of ideas, we recall some number theory and the relevant constructions. The main reference will be [GZ00].

The main ingredient of the reasoning will be to express a particular ζ-value of the number field in question by special values of the dilogarithm: We will see that this ζ-value can be expressed by elements in a corresponding algebraic K–group (this is a theorem of Borel) and by Suslin's exact sequence and some work of Bloch and Zagier, we can express the ζ-value mentioned above in terms of the dilogarithm evaluated at elements in the Bloch group corresponding to the elements in the algebraic K–group. But in general, one does not know these elements. Knowing the relationship between the Bloch group and the algebraic K–group, we can use the explicit description of the Bloch group to obtain explicit elements in the K–group. Let us start with a classical example:

Example 1.2.8. *Consider a number field F with $\mathcal{O}_F$ as ring of integers, then the Dedekind ζ-function of F is defined as*

$$\zeta_F(s) := \prod_{\substack{\mathfrak{p} \subset \mathcal{O}_F \\ \mathfrak{p} \text{ prime}}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1} = \sum_{0 \neq \mathfrak{a} \subset \mathcal{O}_F} \frac{1}{N(\mathfrak{a})^s}.$$

In general, we have the well-known class number formula also by Dedekind: Let r_1 be the number of real embeddings of F, and r_2 be the number of (conjugate) pairs of complex ones. Further set $n := r_1 + 2r_2$, and let R_F be the regulator, h_F the class number, ω_F be the number of roots of unity in F, and Δ_F be the discriminant of F. Then

$$\lim_{s \to 1} (s-1)\zeta_F(s) = \frac{2^{r_1+r_2} \pi^{r_2} R_F h_F}{\omega_F |\Delta_F|^{1/2}}.$$

In the case of a imaginary quadratic number field $F = \mathbb{Q}(\sqrt{-d})$ with discriminant d there is a slightly easier expression, namely Humbert's formula

$$\zeta_F(2) = \frac{4\pi^2}{d\sqrt{d}} Vol(\mathcal{H}^3 / SL_2(\mathcal{O}_F)),$$

where $\mathcal{H}^3$ is the hyperbolic 3-space on which $SL_2(\mathcal{O}_F)$ acts as a discrete group of symmetries. By the insight of Lobachevsky, the volume of a general hyperbolic tetrahedron can be expressed *as a combination of a fixed number of values of the Bloch – Wigner dilogarithm introduced above [Thu], cf. remark 1.2.1.* ♦

So we can conclude that for an imaginary quadratic number field F the value $\zeta_F(2)$ can be expressed in terms of the Bloch – Wigner dilogarithm and some irrational multiples of a power of π. A similar result holds for more general number fields as well: For a number field F with just one pair of conjugate complex embeddings, the value of $\zeta_F(2)$ can be expressed as sums over products of Bloch – Wigner dilogarithms with algebraic arguments. More precisely, there exists an element $\xi \in \overline{\mathbb{Q}}$ such that $\zeta_F(2) = |\Delta_F|^{-1/2}\pi^2 D(\xi)$.

Example 1.2.9. *Consider the example from [Zag07]:*

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{4\pi^2}{3\cdot 7^{3/2}}\left(2D\left(\left[\frac{1+\sqrt{-7}}{2}\right]\right) + D\left(\left[\frac{-1+\sqrt{-7}}{4}\right]\right)\right)$$

is a comparably easy expression of that kind. For convenience, we demonstrate that the argument of the Bloch – Wigner dilogarithm is also contained in the Bloch group $B_2(\mathbb{Q}(\sqrt{-7}))$: We have to show that $2\cdot\left[\frac{1+\sqrt{-7}}{2}\right] + \left[\frac{-1+\sqrt{-7}}{4}\right] \in ker(\beta_2)$ or in other words that

$$2\cdot\frac{1+\sqrt{-7}}{2}\wedge\left(1-\frac{1+\sqrt{-7}}{2}\right) + \frac{-1+\sqrt{-7}}{4}\wedge\left(1-\frac{-1+\sqrt{-7}}{4}\right) = 0.$$

But if we abbreviate $\eta := \frac{1-\sqrt{-7}}{2}$ and $\mu := \frac{-1-\sqrt{-7}}{2}$, then we can see that

$$2\cdot\frac{1+\sqrt{-7}}{2}\wedge\left(1-\frac{1+\sqrt{-7}}{2}\right) = 2\cdot\frac{1+\sqrt{-7}}{2}\wedge\frac{1-\sqrt{-7}}{2} = 2(-\mu)\wedge\eta,$$
$$\frac{-1+\sqrt{-7}}{4}\wedge\left(1-\frac{-1+\sqrt{-7}}{4}\right) = \frac{-1+\sqrt{-7}}{4}\wedge\frac{5-\sqrt{-7}}{4} = \frac{1}{\mu}\wedge\frac{\eta^2}{\mu},$$

but further

$$2(-\mu)\wedge\eta + \frac{1}{\mu}\wedge\frac{\eta^2}{\mu} = \mu^2\wedge\eta - \mu\wedge\eta^2 = 2\mu\wedge\eta - 2\mu\wedge\eta = 0.$$

So we see that $2\left[\frac{1+\sqrt{-7}}{2}\right] + \left[\frac{-1+\sqrt{-7}}{4}\right] \in ker(\beta_2)$. The numerical equality of both expressions for the ζ-value above can be checked by computer. ♦

This is just an instance of the following theorem (cf. [GZ00, Sect. 1]):

Theorem 1.2.10. *Let F be a number field with r_1 real and r_2 pairs of complex embeddings. Then*

- *The Bloch group $B_2(F)$ is finitely generated of rank r_2.*
- *Let $\xi_1, \ldots, \xi_{r_2}$ be a $\mathbb{Q}$-basis of $B_2(F)\otimes\mathbb{Q}$ and $\sigma_1, \ldots, \sigma_{r_2}$ be a set of complex embeddings*

(none of which are conjugate) of F into $\mathbb{C}$ such that $[F:\mathbb{C}] = n := r_1 + 2r_2$. Then

$$\zeta_F(2) \sim_{\mathbb{Q}^\times} |\Delta_F|^{-1/2}\pi^{2n} \det\left(D\left(\sigma_i(\xi_j)\right)_{1\le i,j\le r_2}\right),$$

where $\sim_{\mathbb{Q}^\times}$ denotes "proportional up to unit in $\mathbb{Q}$".

This theorem is the first step towards a general conjecture due to Bloch and Zagier which can be stated roughly by saying that the elements in the Bloch group come from elements in algebraic K–theory. We shall state this conjecture in section 1.2.3 after we have collected the right notions to understand its whole content.

Notation . Let F be a number field with r_1 real and r_2 pairs of complex embeddings and consider the algebraic K–group $K_{2m-1}(F)$. Then we define the dimension of the target $\mathbb{R}$-vector space of th Borel regulator as

$$n_\mp := \begin{cases} r_1 + r_2, & m \text{ odd}, \\ r_2, & m \text{ even}. \end{cases}$$

Another fundamental tool in algebraic K–theory is the existence of regulator mappings from algebraic K–theory to Deligne cohomology. We shall define Deligne – Beĭlinson cohomology later in section 1.3.2, but here just mention that because there is a natural inclusion $K_{2m-1}(F) \hookrightarrow K_{2m-1}(\mathbb{C})$ for every number field F and $m \ge 1$, these regulators can be interpreted as maps

$$K_{2m-1}(\mathbb{C}) \to \mathbb{C}/\mathbb{A}(m),$$

where $\mathbb{A} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ and $\mathbb{A}(n) := (2\pi i)^n\mathbb{A}$ for some $n > 0$. The first regulators are given by

$$\log : K_1(\mathbb{C}) \cong \mathbb{C}^\times \to \mathbb{C}/\mathbb{Z}(1), \qquad \text{and} \qquad \log|\cdot| : K_1(\mathbb{C}) \cong \mathbb{C}^\times \to \mathbb{C}/\mathbb{R}(1) \cong \mathbb{R}.$$

In the following we give a rough idea how to construct a regulator maps

$$K_3(\mathbb{C}) \to \mathbb{C}/\mathbb{Q}(2), \qquad \text{resp.} \qquad K_3(\mathbb{C}) \to \mathbb{C}/\mathbb{R}(2) \cong \mathbb{R}(1).$$

The second one is due to Borel, and we will explain the so-called Borel regulator in general since this is not more difficult than the special case concerning K_3. The reader is advised to refer to [Rap88] for more details: Let $H_c^\bullet(G, \mathbb{R})$ be the continuous cohomology of a Lie group G with $\mathbb{R}$–coefficients. Then one knows that

$$H_c^\bullet(GL(\mathbb{C}), \mathbb{R}) = \Lambda_{\mathbb{R}}^\bullet(u_1, u_3, \ldots),$$

where u_{2n-1} are certain topological cohomology classes which can be lifted to elements $c_n \in H_c^{2n-1}(GL(\mathbb{C}), \mathbb{R}(n-1))$, the so-called Borel classes. Considered as functionals on homology, these classes induce a map

$$\mathsf{reg}_{n,\mathbb{C}} : K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q} \to \mathbb{R}(n-1),$$

given by the composition

$$K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q} := \pi_{2n-1}(BGL(\mathbb{C})^+) \otimes \mathbb{Q} \overset{\text{Hurewicz}}{\longrightarrow} H_{2n-1}(GL(\mathbb{C}), \mathbb{Q}) \xrightarrow{c_n} \mathbb{R}(n-1).$$

So we define the Borel regulator

$$\mathsf{reg}^{Borel}_{m,F} : K_{2m-1}(F) \otimes \mathbb{Q} \to (\mathbb{Z}^{Hom(F,\mathbb{C})} \otimes \mathbb{R}(m-1))^+ = \mathbb{R}(m-1)^{n_\mp},$$

where $(\cdot)^+$ denotes the subspace invariant with respect to complex conjugation, by the composition

$$\mathsf{reg}^{Borel}_{m,F} : K_{2m-1}(F) \otimes \mathbb{Q} \to \bigoplus_{Hom(F,\mathbb{C})} (K_{2m-1}(\mathbb{C}) \otimes \mathbb{Q}) \overset{\prod \mathsf{reg}_{m,\mathbb{C}}}{\longrightarrow} \mathbb{R}(m-1)^{n_\mp}$$

mapping $K_{2m-1}(F)/\text{torsion}$ isomorphically onto a cocompact lattice $\mathsf{Reg}_{m,F}$, whose covolume is a rational multiple of $|\Delta_F|^{1/2}\zeta_F(m)/\pi^{mn_\mp}$ [Bor77]. In other words:

Theorem 1.2.11. *Let F be a number field of degree n with r_1 real and r_2 pairs of conjugate, complex embeddings denoted by $\sigma_1, \dots, \sigma_{r_1+r_2}$. Then there are elements $\gamma_i \in K_{2m-1}(F) \otimes \mathbb{Q}$ and a constant $c \in \mathbb{Q}^\times$ such that*

$$\zeta_F(m) = c \cdot \frac{\pi^{mn_\mp}}{|\Delta_F|^{\frac{1}{2}}} \det\big(\mathsf{reg}_{m,F}(\sigma_j(\gamma_i))\big)_{i,j} \qquad \textit{with } 1 \le i \le n_\pm,\ 1 \le j \le r_1 + r_2.$$

Thus, we conjecture a connection between algebraic K–groups and Bloch groups. Indeed:

Theorem 1.2.12 (Bloch [Blo], Suslin [Sus86])**.** *For each number field F there exists a map $\phi : K_3(F) \to B_2(F)$. Further, both groups are canonically isomorphic up to torsion.*

Moreover, by the isomorphism of the theorem the Borel regulator map on $K_3(F)$ corresponds to the Bloch – Wigner dilogarithm on $B_2(F)$.

Remark 1.2.13. The Borel regulator in algebraic K–theory introduced above can be seen as a natural generalization of the Dirichlet regulator from standard algebraic number theory. Additionally, there is another regulator map due to Beĭlinson [Bei85] from the algebraic K–theory of algebraic varieties X over $\mathbb{R}$ to Deligne – Beĭlinson cohomology, which in case $X = \mathsf{Spec}(\mathcal{O})$ for the ring of integers $\mathcal{O}$ of a number field F is of the form:

$$\mathsf{reg}^{Bei}_{m,F} : K_{2m-1}(\mathcal{O}) \otimes \mathbb{Q} \longrightarrow H^1_{\mathcal{D}}(\mathsf{Spec}(\mathcal{O}), \mathbb{R}(m)).$$

But as Rapoport demonstrated, the Borel regulator and the Beĭlinson regulator coincide up to a non-zero rational factor in case of a number field (cf. [Rap88]). Moreover, in [Gil01] the precise relationship

$$\mathsf{reg}^{Borel}_{m,F} = 2 \cdot \mathsf{reg}^{Bei}_{m,F}$$

is proved. Therefore, we shall not introduce the definition of the Beĭlinson regulator formula.

This fact shows that the Borel regulator and its properties can be used in our case to detect "interesting" cycles in algebraic K–theory generating the K–groups or the motivic cohomology groups of a number field: Via the coincidence of the two regulators we have a more explicit description of the Beĭlinson regulator. On the contrary: If we could not use the proportionality of the Beĭlinson regulator and the Borel regulator with its connection to the polylogarithm, our approach to motivic cohomology via the Bloch groups would possibly lead astray. We will come back to these regulators and some explicit expressions for computing them in section 1.3.2. ♢

So we arrive at the following commutative diagram combining the results of Bloch, Suslin, and Zagier:

$$\begin{array}{ccc} B_2(F)\otimes\mathbb{Q} & \xrightarrow{\cong} & K_3(F)\otimes\mathbb{Q} \\ & {\scriptstyle (D\circ\sigma_1,\dots,D\circ\sigma_{r_2})}\searrow \quad \swarrow{\scriptstyle \mathsf{reg}^{Borel}_{2,F}} & \\ & \mathbb{R}^{r_2}. & \end{array} \tag{1.2.2}$$

The work of Müller-Stach and Gangl [GMS99] was an attempt to make the top isomorphism explicit using the description of K–theory for a field by motivic cohomology.

Example 1.2.14. *Finding elements in the Bloch group of a number field which can be used as arguments of the Bloch – Wigner dilogarithm to express $\zeta_F(2)$ in the sense of the theorem above, is equivalent – via Bloch's map inducing the isomorphism in the previous theorem – to giving a $\mathbb{Q}$-basis of the indecomposable part of $K_3(F)$: Let $F=\mathbb{Q}(\zeta_\ell)$, where ℓ is an odd prime and ζ_ℓ an ℓ^{th} root of unity. Then one verifies that $\ell\cdot[\zeta_\ell^i]\in B_2(F)$ for all i. Bloch shows in [Blo] that $\{\ell\cdot[\zeta_\ell],\dots,\ell\cdot[\zeta_\ell^{\frac{\ell-1}{2}}]\}$ maps to a basis of $K_3(F)\otimes\mathbb{Q}$.*

Further, using the formula $D(\zeta_\ell^i)=\mathrm{Im}(\sum_{m=1}^{\infty}\frac{\zeta_\ell^{mi}}{m^2})$, he also computes that the lattice in $\mathbb{R}^{\frac{\ell-1}{2}}$ generated by the vectors

$$\left(D(\ell\zeta_\ell^i),\dots,D(\ell\zeta_\ell^{\frac{i(\ell-1)}{2}})\right),\qquad i=1,\dots,\frac{\ell-1}{2},$$

has a volume equal to

$$2^{\frac{1-\ell}{2}}\ell^{\frac{3(\ell-1)}{4}}\prod_{\chi}|L(2,\chi)|,$$

where χ runs through the odd Dirichlet characters of F and $L(s,\chi)$ denotes the Dirichlet L-function corresponding to χ. ♦

The second regulator map, $K_3(\mathbb{C})\to\mathbb{C}/\mathbb{Q}(2)$, which is also denoted by c_2, can be described via the multivalent dilogarithm function Li_2. This is unpublished work of Bloch and Wigner (cf. [DS82]): For $x\in\mathbb{C}$ such that $|x-\frac{1}{2}|<\frac{1}{2}$ define

$$\rho(x)=\frac{1}{2}\left[\log(x)\wedge\log(1-x)+2\pi i\wedge\frac{1}{2\pi i}\left(Li_2(1-x)-Li_2(x)-\frac{\pi^2}{6}\right)\right]\in\Lambda^2\mathbb{C}$$

with all logarithms taken to be the principal branches. One shows that this function is invariant under monodromy, and therefore extends to a single-valued function

$$\rho : \mathbb{C} \setminus \{0, 1\} \to \Lambda^2\mathbb{C}.$$

As explained in [Hai94], this function induces a homomorphism $H_3(SL_2(\mathbb{C})) \to \mathbb{C}/\mathbb{Q}(2)$ and completes the following commutative diagram

$$\begin{array}{ccc} K_3(\mathbb{C}) & \xrightarrow{c_2} & \mathbb{C}/\mathbb{Q}(2) \\ \downarrow{\scriptstyle proj} & \overset{\rho}{\nearrow} & \\ gr^2_\gamma K_3(\mathbb{C}), & & \end{array}$$

where c_2 denotes a Borel class. One finally shows that the map ρ constructed above extends to the whole K–group inducing the regulator c_2 on all of $K_3(\mathbb{C})$.

As a further variant of the dilogarithm with no special use for us in the following, one should mention the Rogers dilogarithm $L(x)$. It is defined as

$$L(x) := Li_2(x) + \frac{1}{2}\log(x)\log(1-x), \qquad 0 < x < 1,$$

together with its extension to the whole of $\mathbb{R}$ by setting $L(0) := 0, L(1) := \frac{\pi^2}{6}$ and

$$L(x) := \begin{cases} \frac{\pi^2}{3} - L(x^{-1}), & x > 1, \\ -L\left(\frac{x}{x-1}\right), & x < 0. \end{cases}$$

This function is not continuous at infinity since one computes

$$\lim_{x\to+\infty} L(x) = 2L(1) = \frac{\pi^2}{3}, \qquad \lim_{x\to-\infty} L(x) = -L(1) = -\frac{\pi^2}{6}.$$

But if one considers the function only modulo $\pi^2/2$, then the resulting function $\overline{L}(x) := L(x) \,(\mathrm{mod} \frac{\pi^2}{2})$ is monotone, increasing, continuous, real-valued on the real numbers, and even real analytic except for 0 and 1, where it is continuous only. Comparable to the Bloch – Wigner dilogarithm, the modified Rogers dilogarithm also satisfies "clean" functional equations, in particular the reflection properties for $x \in \mathbb{R}$:

$$\overline{L}(x) + \overline{L}(1-x) - \overline{L}(1) = 0, \qquad \overline{L}(x) + \overline{L}(x^{-1}) + \overline{L}(1) = 0$$

and the five-term relation for $x, y \in \mathbb{R}, xy \neq 1$:

$$\overline{L}(x) + \overline{L}(y) + \overline{L}\left(\frac{1-x}{1-xy}\right) + \overline{L}(1-xy) + \overline{L}\left(\frac{1-y}{1-xy}\right) = 0.$$

The special values of the dilogarithm mentioned at the beginning of this subsection also clean up: E. g. one simply has

$$L\left(\frac{1}{2}\right)=\frac{\pi^2}{12},\qquad L\left(\frac{3-\sqrt{5}}{2}\right)=\frac{\pi^2}{15}.$$

These special values of the Rogers dilogarithm are closely linked to torsion elements in the Bloch group $B_2(\overline{\mathbb{Q}})$. In fact, an element $[a]\in B_2(\overline{\mathbb{Q}})$ is known to be torsion if and only if its Bloch – Wigner dilogarithm in all complex embeddings vanishes (reflecting the fact that the image of this function is a lattice in a suitable real vector space detecting "the free part" of the Bloch group of a number field) and its Rogers dilogarithm in all its real embeddings is a rational multiple of π^2. By an elementary consideration (cf. [Zag07]) one can show that the only numbers $\alpha_i\in\mathbb{C}\setminus\{0,1\}$ for which $[\alpha_i]$ lies in $B_2(\mathbb{C})$ are

$$\alpha_i\in\left\{-1,\frac{1}{2},2,\frac{\pm1\pm\sqrt{5}}{2},\frac{3\pm\sqrt{5}}{2}\right\}.$$

1.2.2 The trilogarithm and number theory

Let us now recall the first generalization of the facts from the preceding subsection: We consider the classical trilogarithm function

$$Li_3(z)=\sum_{m=1}^{\infty}\frac{z^m}{m^3},\qquad z\in\mathbb{C},\quad |z|<1,$$

which can be extended by analytic continuation to a covering of $\mathbb{P}^1_{\mathbb{C}}-\{0,1,\infty\}$:

$$Li_3(z):=\int_0^z Li_2(t)\frac{dt}{t}.$$

There is also a single-valued version of the trilogarithm given by

$$P_3(z):=Re\left(Li_3(z)-Li_2(z)\log|z|+\frac{1}{3}Li_1(z)\log^2|z|\right).$$

Functional equations for the trilogarithm are very complicated in general and involve lots of terms of lower order, i. e. products of dilogarithms and elementary functions. Among the easier ones are an inversion relation

$$Li_3(z)=Li_3(z^{-1})-\frac{1}{6}\log^3(z)-\frac{\pi^2}{6}\log(z)$$

a series of distribution relations

$$Li_3(z^n)=n^2\sum_{\zeta^n=1}Li_3(\zeta z)$$

and the three-term relation

$$Li_3(z)+Li_3\left(\frac{1}{1-z}\right)+Li_3\left(1-\frac{1}{z}\right)=-Li_3(1)+\frac{1}{6}\log^3(z)+\frac{\pi^2}{6}\log(z)-\frac{1}{2}\log^2(z)\log(1-z).$$

On the other hand there is a very general functional equation for the trilogarithm which is given in terms of the single-valued function P_3 (cf. remarks in [Gan03, sect.3]). Denote by $cr(a,b,c,d)$ the cross ratio of four points $a,b,c,d \in \mathbb{P}^1_{\mathbb{C}}$ defined by $cr(a,b,c,d) := \frac{a-c}{a-d}\cdot\frac{b-d}{b-c}$:

Theorem 1.2.15 (Wojtkowiak)**.** *Let $\phi : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ be a rational function, $A_i, B_j, C_k, D_l \in \mathbb{P}^1_{\mathbb{C}}$, and finally $\{i,j,k,l\} \in \{1,2\}$. Then, denoting by $deg(\phi)$ the degree of ϕ, we have*

$$\sum_{i,j,k,l}(-1)^{i+j+k+l}\left(\sum_{\alpha_i,\beta_j,\gamma_k,\delta_l}P_3(cr(\alpha_i,\beta_j,\gamma_k,\delta_l))-deg(\phi)\cdot P_3(cr(A_i,B_j,C_k,D_l))\right)=0,$$

where $\alpha_i,\beta_j,\gamma_k,\delta_l$ run through the preimages of A_i,B_j,C_k,D_l respectively, with multiplicities.

On the contrary, Goncharov [Gon95a] found a functional equation of the trilogarithm in three variables $\alpha_1,\alpha_2,\alpha_3$ in terms of configuration spaces. His equation can be stated in the following form, where we use the shorthand notation $\beta_i = 1-\alpha_i+\alpha_i\alpha_{i-1}$, $i=1,2,3$:

Theorem 1.2.16. *Let $\gamma(\alpha_1,\alpha_2,\alpha_3) \in \mathbb{Z}[\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3)]$ be a formal linear combination given by*

$$\gamma(\alpha_1,\alpha_2,\alpha_3)=\sum_{i=1}^{3}\left(\left[\frac{1}{\alpha_i}\right]+[\beta_i]+\left[\frac{\alpha_i\alpha_{i-1}}{\beta_i}\right]+\left[\frac{\beta_i}{\beta_{i+1}\alpha_{i+2}}\right]+\left[\frac{\beta_i\alpha_{i+1}}{-\beta_{i+1}}\right]\right)$$
$$+\left[\frac{-1}{\alpha_1\alpha_2\alpha_3}\right]-\sum_{i=1}^{3}\left(\left[\frac{\beta_i}{\alpha_{i-1}}\right]+\left[\frac{\beta_i}{\beta_{i+1}\alpha_i\alpha_{i-1}}\right]+[1]\right).$$

Then $P_3(\gamma(\alpha_1,\alpha_2,\alpha_3))=0$.

This relation will be referred to as Goncharov's relation. As shown in [Gan03], Goncharov's relation is a special case of Wojtkowiak's relation.

Remark 1.2.17. There is a generalization to polylogarithms of a real variable, namely

$$L_n(x)=\sum_{j=0}^{n-1}\frac{(-\log|x|)^j}{j!}Li_{n-j}(x)+\frac{(-\log|x|)^{n-1}}{n!}\log|1-x|, \qquad |x|\le 1,$$

and

$$L_n(x)=(-1)^{n-1}L_n\left(\frac{1}{x}\right), \qquad |x|>1.$$

◊

Searching for a generalization of the relationship between $B_2(F)$ and $K_3(F)$ from the previous subsection, one would like to define some higher Bloch group $B_3(F)$ rationally mapping to (a part of) the algebraic $K_5(F)$. This is not so easy, and we defer it to the next subsection as it better fits the context there. More precisely, there is an inductive process to construct higher Bloch groups which are conjectured to be isomorphic rationally to a graded part of algebraic K–groups. We just introduce a certain map:

Definition 1.2.18. Consider the map

$$\beta_3 : \mathbb{Z}[F^\times] \to B_2(F) \otimes F^\times, \qquad [x] \mapsto \{x\}_2 \otimes x, \qquad [1] \mapsto 0,$$

where $\{x\}_2$ denotes the image of x in $B_2(F)$. The kernel of this map will be denoted by $\mathcal{A}_3(F)$.

Remark 1.2.19. A naive generalization of the Bloch group, $B_3(F)$, would be to consider the quotient of $\mathcal{A}_3(F)$ by a group of suitable relations. But as we will see in the next subsection, the resulting group does not map to the right regulator lattice in a real vector space. This definition would only suffice for $F = \mathbb{Q}$ or F not real. ♢

Nevertheless, let us discuss a group of suitable relations. It turns out that for general number fields, we have to modify the definition of $\mathcal{A}_3(F)$ before dividing out the group of relations yet to be defined in order to obtain a suitable Bloch group. We need some preparation (cf. [Gon95b]):

Let V^3 be a three dimensional vector space over F. For a fixed volume form ω and six vectors $l_1, \dots, l_6$ in general position in V^3, set $\Delta(l_i, l_j, l_k) := \langle \omega, l_i \wedge l_j \wedge l_k \rangle \in F^\times$ for pairwise distinct i, j, k. For a complex-valued function $f : V^3 \to \mathbb{C}$ let further

$$\mathsf{Alt}_6(f(l_1, \dots, l_6)) := \sum_{\sigma \in S_6} (-1)^{sgn(\sigma)} f(l_{\sigma(1)}, \dots, l_{\sigma(6)}).$$

Then one sets

$$r_3(l_1, \dots, l_6) := \mathsf{Alt}_6 \left\{ \frac{\Delta(l_1, l_2, l_4)\Delta(l_2, l_3, l_5)\Delta(l_3, l_1, l_6)}{\Delta(l_1, l_2, l_5)\Delta(l_2, l_3, l_6)\Delta(l_3, l_1, l_4)} \right\} \in \mathbb{Z}[F^\times].$$

One can show [Gon95b] that this quantity does not depend on the length of the vectors l_i and in consequence is a generalized cross ratio of 6 points on the projective plane.

Theorem 1.2.20. *For any* 7 *points* $(z_1, \dots, z_7)$ *in generic position in* $\mathbb{P}^2_{\mathbb{C}}$ *we have*

$$\sum_{i=1}^{7} (-1)^i P_3(r_3(z_1, \dots, \hat{z}_i, \dots, z_7)) = 0.$$

Further, if one chooses $x \in \mathbb{P}^2_{\mathbb{C}}$, *then the function* $c_5(g_0, \dots, g_5) := P_3(r_3(g_0 x, \dots, g_5 x))$ *defined for all* $g_i \in GL_3(\mathbb{C})$ *such that* $(g_0 x, \dots, g_5 x)$ *is in general position, is a measurable* 5*-cocycle representing a nontrivial cohomology class of the group* $GL_3(\mathbb{C})$.

Definition 1.2.21. We let $\mathcal{C}_3(F)$ be the subgroup of $\mathbb{Z}[F^\times]$ generated by the relations

$$[z] + [z^{-1}] = 0, \quad [z] + \left[\frac{1}{1-z}\right] + \left[1 - \frac{1}{z}\right] = [1], \qquad z \in F^\times,$$

and $\sum_{i=1}^{7}(-1)^i r_3(z_1, \ldots, \hat{z}_i, \ldots, z_7) = 0$ for generic configurations of 7 points in the projective plane over F, where we set $[0] = [\infty] = 0$.

Remark 1.2.22. The relation $\sum_{i=1}^{7}(-1)^i r_3(z_1, \ldots, \hat{z}_i, \ldots, z_7) = 0$ is a more symmetric variant of Goncharov's relation from theorem 1.2.16. ♢

There is a result due to Goncharov generalizing theorem 1.2.7, which was conjectured by Zagier:

Theorem 1.2.23 (Goncharov [Gon95b]). *For any number field F with r_1 real embeddings $\sigma_1, \ldots, \sigma_{r_1}$ and r_2 pairs of complex ones $\sigma_{r_1+1}, \overline{\sigma_{r_1+1}} \ldots, \sigma_{r_1+r_2}, \overline{\sigma_{r_1+r_2}}$, there exist elements $\xi_1, \ldots, \xi_{r_1+r_2} \in ker(\beta_3) \otimes \mathbb{Q}$ such that*

$$\zeta_F(3) = \pi^{3r_2} |\Delta_F|^{-1/2} \det \left(P_3(\sigma_j(\xi_i))_{1 \le i,j \le r_1+r_2}\right).$$

Example 1.2.24. *The following is due to Zagier:*

$$\zeta_{\mathbb{Q}(\sqrt{5})}(3) = \frac{24}{25\sqrt{5}} \cdot P_3(1) \cdot \left(P_3\left(\frac{1+\sqrt{5}}{2}\right) - P_3\left(\frac{1-\sqrt{5}}{2}\right)\right).$$

This and many more numerical examples can be found in [Zag91]. ♦

Rationally, one knows (cf. [Gon94]) that there are maps

$$gr^{rank}_{3-i} K_{6-i}(F) \otimes \mathbb{Q} \to H^i(B_F(3) \otimes \mathbb{Q}),$$

where $gr^{rank}_{\bullet}$ denotes the graded quotient of the rank filtration, expected to be isomorphism. This is known for $i = 3$.

1.2.3 Generalizations

The classical m^{th} polylogarithm function is defined by

$$Li_m(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^m}, \qquad (m = 1, 2, \ldots; \quad z \in \mathbb{C}, |z| < 1).$$

It is a function which can be analytically continued to a multivalued function on $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ by using the integral expression

$$Li_m(z) = \int_0^z Li_{m-1}(t) \frac{dt}{t}, \qquad Li_1(z) = -\log(1-z).$$

Many of the conjectures on polylogarithms, however, require a single-valued variant $P_m(z)$ of $Li_m(z)$, which is known to exist [Ram86]. Concretely, one uses the expression which can be found in [GZ00]:

$$P_m(z) := \begin{cases} \mathrm{Re}_m \left(\sum_{r=0}^{m} \frac{2^r B_r}{r!} Li_{m-r}(z) \log^r |z| \right), & |z| < 1, \\ (-1)^{m-1} P_m(z^{-1}), & |z| \geq 1, \end{cases}$$

where Re_m denotes "the real part of" for m odd and "the imaginary part of" for m even. Moreover, B_r is the r^{th} Bernoulli number ($B_0 = 1, B_1 = -1/2, B_2 = 1/6, \ldots$).

Remark 1.2.25. For $m \geq 2$, the functions $P_m(z)$ can be extended to the whole complex plane: They are real analytic on $\mathbb{C} - \{0, 1\}$ and can be extended continuously to the whole of $\mathbb{P}^1_{\mathbb{C}}$ by setting $P_m(0) = P_m(\infty) = 0$ and $P_m(1) = \begin{cases} \zeta(m), & m \text{ odd}, \\ 0, & \text{else}. \end{cases}$ ♢

There is strong evidence for the following conjectures, which we will assume in the following, because it lies at the heart of the definition of the higher Bloch groups:

Conjecture 1.2.26.
- *Every P_m, $m \geq 2$, satisfies a nontrivial functional equation in several variables with constant coefficients. We define a functional equation to be trivial, if it can be derived from the inversion relation $P_m(z) = (-1)^{m-1} P_m(z^{-1})$ and one of the distribution relations. $P_m(z^n) = n^{m-1} \sum_{\zeta^n=1} P_m(\zeta z)$ with $n \in \mathbb{N}$.*
- *Every functional equation of the higher polylogarithm reflects some structure of the corresponding Bloch- or K–group [Zag91].*

Remark 1.2.27. For the dilogarithm, this non-trivial relation in question is the five-term relation, since as shown by Wojtkowiak (cf. the remarks in [Gan03, sect. 2]), any functional equation for the dilogarithm with $\mathbb{C}$-rational expressions in one variable as arguments can be written as a sum of five-term relations.

For the trilogarithm Goncharov's relation seems to be this universal relation mentioned in the conjecture. In general the search for such a characterizing functional equation is a major goal in the theory of polylogarithms.

For higher polylogarithms, i. e. up to degree 7, the results on functional equations are surveyed in [Gan03]. The general case is widely believed. ♢

Let us now see what is known and conjectured about the relationship between $K_{2m-1}(F)$ and $\zeta_F(m)$ on the one hand and the m^{th} polylogarithm on the other hand for $m > 3$: Two aspects of this relationship are of current interest, one initiated by Zagier [Zag91], the other one by Goncharov [Gon95a]. We will quickly review both of them.

In [Zag91], Zagier has given candidates for higher degree Bloch groups ($m \geq 3$). These are also defined as kernels of an explicit map β_m analogous to the above β_2 modulo implicit relations coming from characterizing functional equations of the m^{th} polylogarithm. Let us review this construction.

The explicit map under consideration is the following:

$$\beta_m = \beta_{m,F} : \mathbb{Z}[F] \to \mathsf{Sym}^{m-2} F^\times \otimes \Lambda^2 F^\times, [z] \mapsto \begin{cases} z^{\otimes(m-2)} \otimes (z \wedge (1-z)), & z \neq 0, 1 \\ 0 & z = 0, \text{ or } z = 1. \end{cases}$$

It is easy to see that β_2 coincides with our previous map. Unfortunately, as mentioned in [GZ00] (cf. also [Zag91]), it is not enough to consider the whole kernel of this map. One needs another condition. In order to formulate it, we need to observe that if we choose $\xi := \sum n_i[a_i] \in ker\beta_m$, then for every homomorphism $\phi : F^\times \to \mathbb{Z}$ the element $\iota_\phi(\xi) := \sum n_i \phi(a_i)[a_i] \in ker\beta_{m-1} \subset \mathcal{A}_{m-1}$. So the following definition makes sense:

$$\mathcal{A}_m(F) := \left\{ \xi \in \mathbb{Z}[F] \,\middle|\, \iota_\phi(\xi) \in \mathcal{C}_{m-1}(F) \quad \forall \phi \in \mathrm{Hom}(F^\times, \mathbb{Z}) \right\} .$$

The problem with the definition of $\mathcal{C}_m(F)$ is that one does not know explicitly all necessary functional equations for higher polylogarithms. Therefore, we shall just give a numerically usable working definition for these groups:

$$\mathcal{C}_m(F) := \{ \xi \in \mathcal{A}_m(F) \mid P_m(\sigma(\xi)) = 0 \quad \forall \text{ embeddings } \sigma : F \hookrightarrow \mathbb{C} \} .$$

But in order to give a more precise definition of the groups containing the necessary relations to be divided out, one must introduce a homotopic or function field definition of these groups, which is also valid for arbitrary number fields (cf. [GZ00]):

$$\mathcal{C}^*_m(F) := \{ \xi(1) - \xi(0) \mid \xi(t) \in \mathcal{A}_m(F(t)) \} .$$

As remarked in [GZ00], an element $\xi(t) \in \mathbb{Z}[F(t)]$ lies in $\ker(\beta_{m,F(t)})$ if and only if $t \mapsto P_m(\xi(t))$ is the constant map. Therefore $\mathcal{C}^*_m(F)$ is the subgroup of $\mathcal{C}_m(F)$ generated by specializations of the functional equations to values in F. But caution:

Remark 1.2.28. One has to note that although obviously $\mathcal{C}^*_m(F) \subseteq \mathcal{C}_m(F)$, the reverse inclusion is known only for $m = 2$ (cf. [GZ00, Sect. 2]). ◊

Definition 1.2.29. Let F be a number field. Then the n^{th} higher Bloch group for $n \geq 3$ is defined as the quotient

$$\mathcal{B}_n(F) = \frac{\mathcal{A}_n(F)}{\mathcal{C}_n(F)}.$$

The Bloch groups are constructed in a way that the following picture should be true: For a number field F let us also define the map $P_{m,F} : \mathcal{B}_m(F) \to \mathbb{R}^{n_\mp}$ by composing the inclusion $\mathcal{B}_m(F) \to \mathcal{B}_m(\mathbb{R}^{r_1}) \times \mathcal{B}_m(\mathbb{C}^{r_2})$ with the polylogarithmic function P_m applied in every coordinate.

Conjecture 1.2.30. *For a number field F and for every $m \geq 1$ there is a rational isomorphism between the Bloch groups $B_m(F) \otimes \mathbb{Q}$ and $K_{2m-1}(F) \otimes \mathbb{Q}$ which makes the following* *natural generalization of diagram 1.2.2 commute:*

$$\begin{array}{ccc} \mathcal{B}_m(F)\otimes\mathbb{Q} & \hookrightarrow & \mathcal{B}_m(\mathbb{R})^{r_1}\times\mathcal{B}_m(\mathbb{C})^{r_2} \\ \cong\downarrow & \searrow^{P_{m,F}} & \downarrow(P_m,\ldots,P_m) \\ K_{2m-1}(F)\otimes\mathbb{Q} & \xrightarrow[\mathsf{reg}_{m,F}]{} & \mathbb{R}^{n_\mp}. \end{array} \tag{1.2.3}$$

More precisely one hopes for the following generalization of Borel's results on the image of the regulator and the ζ-value at a special integer:

Conjecture 1.2.31 (Zagier)**.** *Let F be a number field. Recall the definition of $n_\mp$ from notation: The image of the modified m^{th} polylogarithm $P_{m,F}:\mathcal{B}_m(F)\to\mathbb{R}^{n_\mp}$ is commensurable with the Borel regulator lattice $\mathsf{Reg}_{m,F}\subset\mathbb{R}^{n_\mp}$. In particular,*

- *the higher Bloch groups $\mathcal{B}_m(F)=\mathcal{A}_m(F)/\mathcal{C}_m(F)$ are finitely generated of rank $n_\mp$.*
- *Let $\xi_1,\ldots,\xi_{n_\mp}$ be a $\mathbb{Q}$-basis of $\mathcal{B}_m(F)\otimes\mathbb{Q}$ and $\sigma_1,\ldots,\sigma_{n_\mp}$ run through all embeddings not conjugate. Then*

$$\zeta_F(m)\sim_{\mathbb{Q}^\times}|\Delta_F|^{-\frac{1}{2}}\pi^{mn_\pm}\det\left(P_m(\sigma_i(\xi_j))_{1\le i,j\le n_\mp}\right).$$

Example 1.2.32. *By definition $\zeta_\mathbb{Q}(2n)=P_{2n}(1)=\zeta(2n)\sim_{\mathbb{Q}^\times}\pi^{2n}$. The only general result apart from the theorems above in this direction is the Klingen-Siegel theorem. For totally real fields F, one has*

$$\zeta_F(2n)\sim_{\mathbb{Q}^\times}\frac{\pi^{2r_1n}}{|\Delta_F|^{\frac{1}{2}}}.$$

Note also that for $n=1$ we recover the classical Dedekind formula we encountered before:

$$Res_{s\to1}(\zeta_F(s))=\frac{\pi^{r_2}\cdot2^{r_1+r_2}\cdot h_F}{w\cdot|\Delta_F|^{\frac{1}{2}}}R_F,$$

where h_F is the class number of F, R_F the classical regulator known from standard number theory, and w the number of roots of unity in F. ♦

We close this section with an overview of a more general and conceptional approach to Bloch groups, namely via Goncharov's motivic complexes presented in [Gon95a]. Note that with $\mathcal{G}_n(F):=\mathbb{Z}[F^\times]/\mathcal{C}_n^*(F)$ the definition of $\mathcal{A}_m(F)$ can be written as

$$\mathcal{A}_m(F)=ker\left(\mathbb{Z}[F]\xrightarrow{\iota}F^\times\otimes\mathbb{Z}[F]/\mathcal{C}_{m-1}^*(F)\right),$$

where ι is defined by $[x]\mapsto x\otimes[x]$ and $[0]\mapsto 0$. Since $\beta_m\mathcal{C}_m^*(F)=0$, we can also write

$$\mathcal{B}_m(F)=\ker\left(\mathcal{G}_m(F)\xrightarrow{\beta_m}\mathcal{G}_{m-1}(F)\otimes F^\times\right)$$

with a map β_m induced by ι. Note that $\mathcal{G}_2(F)=B_2(F)$ and that $\mathcal{G}_3(F)=\mathcal{B}_3(F)$. More generally, if we define

$$\mathcal{R}_n(F) := \{\{\infty\}, \{0\}\} \cup \{\xi(0) - \xi(1) \mid |\xi(t) \in \mathcal{A}_n(F(t))\},$$

then Goncharov shows that $\beta_m(\mathcal{R}_m(F)) = 0$ so that one gets a map

$$\beta_m : \mathcal{G}_m(F) \to \mathcal{G}_{m-1}(F) \otimes F^\times, m \geq 3, \qquad \beta_2 : \mathcal{G}_2(F) \to \Lambda^2 F^\times,$$

where we have $\beta_k(x_1 \wedge \ldots \wedge x_{m-k} \otimes [x]) = x_1 \wedge \ldots \wedge x_m \wedge x \otimes [x]$ except for the last degree, where β_2 is given by $\beta(x_1 \wedge \ldots \wedge x_{m-2} \otimes [x]) = x_1 \wedge \ldots \wedge x_{m-2} \wedge x \wedge (1-x)$.

Additionally, we have $\beta_m \circ \beta_{m-1} = 0$ for all n so that there is a complex

$$\Gamma(m, F) : \mathcal{G}_m(F) \stackrel{\beta_m}{\to} F^\times \otimes \mathcal{G}_{m-1}(F) \stackrel{\beta_{m-1}}{\to} \Lambda^2 F^\times \otimes \mathcal{G}_{m-2}(F) \stackrel{\beta_{m-2}}{\to} \ldots$$
$$\ldots \stackrel{\beta_3}{\to} \Lambda^{m-2} F^\times \otimes \mathcal{G}_2(F) \stackrel{\beta_2}{\to} \Lambda^m F^\times,$$

The cohomology at the first degree is immediately recognized as $\mathcal{B}_m(F)$. In general, one has the following expectation:

Conjecture 1.2.33 (Goncharov [Gon95b])**.** *The rational motivic cohomology of a field can be computed as the homology of Goncharov's complex:* $H^i\Gamma(n, F) \otimes \mathbb{Q} \simeq gr_n^\gamma K_{2n-i}(F) \otimes \mathbb{Q}$.

Further, the composition $gr_n^\gamma K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q} \to H^1(\Gamma(n, \mathbb{C})) \otimes \mathbb{Q} \stackrel{P_{n,\mathbb{C}}}{\longrightarrow} \mathbb{R}$ *is a nonzero rational multiple of the Borel regulator.*

Remark 1.2.34. As a consequence of this conjecture, one would obtain a description

$$K_m(F) \otimes \mathbb{Q} = \bigoplus_n H^{2n-m}(\Gamma(n, F) \otimes \mathbb{Q}),$$

which would lead to a generalization of the description of Milnor K–groups in terms of symbols.

This has been achieved in the case $n = 3$, where Goncharov used his complex to prove Zagier's conjecture on $\zeta_F(3)$. But as already mentioned, it is not clear whether his relations generate all functional equations of the trilogarithm. They just generate a common version of the Bloch group $B_3(F)$. ◊

Remark 1.2.35. Although Zagier's conjecture is still open, there is a partial result due to Beĭlinson– Deligne and later de Jeu (cf. remarks in [Gon95b]) stating that for a number field F, there is a map

$$l_n : ker(\beta_n) \otimes \mathbb{Q} \to K_{2n-1} \otimes \mathbb{Q}$$

such that for any $\sigma : F \hookrightarrow \mathbb{C}$, one has $\mathsf{reg}_{n,F}^{Borel}(\sigma \circ l_n(y)) = P_n(\sigma(y))$ for $y \in F$. ◊

Remark 1.2.36. In summary, one expects Goncharov's complexes to be the weight n motivic cohomology complexes conjectured by Beĭlinson and Lichtenbaum. In the next sections, we present a different approach to motivic cohomology of a number field via Bloch's higher Chow groups, which are more practical for concrete computations, which is our ultimate goal. $\Diamond$

1.3 Motivic cohomology via Bloch's higher Chow groups

In order to study the motivic cohomology of a number field, we choose Bloch's candidate, the higher Chow groups. This offers some advantages. First of all, they can be described rather easily though they may be hard to compute in general. But since we want to find an integral presentation of motivic cohomology, the higher Chow groups have very good properties, namely the comparison map inducing an isomorphism for smooth schemes X:

$$gr_\gamma^p K_n(X) \otimes \mathbb{Z}\left[\frac{1}{(n-1)!}\right] \simeq CH^p(X,n) \otimes \mathbb{Z}\left[\frac{1}{(n-1)!}\right].$$

In some sense, the relationship between Bloch's Chow groups and the graded parts of algebraic K–groups is better than a priori asserted in the conjectural framework of motivic cohomology, because there only a rational isomorphism was expected. As mentioned in the introduction, there is also a spectral sequence abutting to K–theory as we will see in theorem 1.3.4.

A second advantage of this approach is that there is a regulator map from higher Chow groups to Deligne – Beĭlinson cohomology. This regulator map can be seen to generalize and refine Borel's regulator map from K–theory. We will give an overview of these matters later.

Our aim is to generalize ideas of H. Gangl, S. Müller-Stach and J. Zhao to present higher Chow groups, i. e. motivic cohomology and therefore algebraic K–groups of a number field explicitly in terms of fractional algebraic cycles modulo certain relations coming from functional equations of polylogarithms. So let us describe Bloch's candidates for motivic cohomology in some detail.

1.3.1 Bloch's higher Chow groups

Since there are many good expositions in the literature, we will only consider the cubical version keeping in mind that Levine [Lev94] established a quasi-isomorphism to the "original" simplicial version due to S. Bloch [Blo86].

We let

$$\square_F^n = (\mathbb{P}_F^1 \setminus \{1\})^n$$

with coordinates $(z_1, \ldots, z_n)$ be the algebraic standard cube with 2^n faces of codimension 1,

$$\partial\square_F^n = \bigcup_{i=1}^n \{(z_1, \ldots, z_n) \in \square_F^n \,|\, z_i \in \{0, \infty\}\},$$

and faces of codimension k,

$$\partial^k \square_F^n = \bigcup_{i_1<\ldots<i_k} \{(z_1,\ldots,z_n) \in \square_F^n \mid z_{i_1},\ldots,z_{i_k} \in \{0,\infty\}\}.$$

Definition 1.3.1. For a smooth quasi-projective variety X over F, we now let $Z^p(X,n) = c^p(X,n)/d^p(X,n)$ be the quotient of the free abelian group $c^p(X,n)$ generated by integral closed algebraic subvarieties of codimension p in $X \times \square_F^n$ which are admissible (i. e. meeting all faces of all codimensions in codimension p again – or not at all) modulo the subgroup $d^p(X,n)$ of degenerate cycles (i. e. pull-backs of coordinate projections $\square^n \to \square^{n-1}$).

These groups form a simplicial abelian group:

$$\ldots Z^r(X,3) \overset{\to}{\underset{\to}{\to}} Z^r(X,2) \overset{\to}{\to} Z^r(X,1) \rightrightarrows Z^r(X,0).$$

Definition 1.3.2. Bloch's higher Chow groups $CH^r(X,n)$ are the homology groups of the above complex, where the boundary is given by

$$\partial = \sum_i (-1)^{i-1}(\partial_i^0 - \partial_i^\infty),$$

and $\partial_i^0, \partial_i^\infty$ denote the restriction maps to the faces $z_i = 0$ resp. $z_i = \infty$:

$$CH^p(X,n) := \pi_n(Z^p(X,\bullet)) = H_n(Z^p(X,\bullet),\partial).$$

Theorem 1.3.3 (Friedlander, Voevodsky, [FV99])**.** *Assume that k admits resolution of singularities and let X be a smooth quasi-projective variety over k. Then Bloch's higher Chow groups are isomorphic to the motivic cohomology groups:*

$$CH^p(X,n) \simeq H_{\mathcal{M}}^{2p-n,p}(X,\mathbb{Z}).$$

These groups satisfy several important properties including the following (for the proofs see [Blo86], [Blo94b], [Tot82]):

- Functoriality: The groups $CH^\bullet(X,\bullet)$ are covariant for proper maps and contravariant for flat maps. Further, they are contravariant for all maps between smooth affine schemes X and Y.
- Products: If X is smooth, there is a product

$$CH^p(X,r) \otimes CH^q(X,s) \to CH^{p+q}(X,r+s).$$

- Homotopy invariance: For any equidimensional scheme X over a field F, we have

$$CH^\bullet(X,n) \simeq CH^\bullet(X \times \mathbb{A}_F^1,n).$$

- Localization: If X is quasi–projective over a field F and if $W \subset X$ is a closed subvariety of pure codimension r, then one has a localization sequence:

$$\ldots \to CH^{\bullet - r}(W, n) \to CH^{\bullet}(X, n) \to CH^{\bullet}(X - W, n) \to \ldots.$$

The relevant facts and conjectures for the groups $CH^p(F, n)$ for an arbitrary infinite field F are the following. Note that most of them are properties expected for a motivic cohomology theory by Beĭlinson, Lichtenbaum et al. :

Theorem 1.3.4 ([BL95], [Lev94])**.** *Let X be a smooth, quasi-projective variety of dimension d over a field F. Let further $gr^q_\gamma K_n(X)$ be the q^{th} piece of the weight filtration of Quillen's K–theory of X. Then*

$$gr^q_\gamma K_n(X) \otimes \mathbb{Z}\left[\frac{1}{(n+d-1)!}\right] \simeq CH^q(X, n) \otimes \mathbb{Z}\left[\frac{1}{(n+d-1)!}\right].$$

Moreover, for an equidimensional scheme X over a field there is a spectral sequence

$$CH^{-q}(X, -p-q) \Rightarrow K_{-p-q}(X)$$

for $p, q \in \mathbb{Z}$ abutting to K–theory.

Theorem 1.3.5 ([EVMS02])**.** *If R is an essentially smooth, semi-local F–algebra over an infinite field F, then there is a surjective morphism from Milnor K–theory to the higher Chow groups:*

$$K_n^M(R) \twoheadrightarrow CH^n(R, n), \qquad n \geq 1.$$

Several people ([Blo86], [Nar89], [Tot82], and [Sus86]) have contributed to an explicit calculation of these groups.

Theorem 1.3.6. *1. When X is a smooth variety, then*

$$CH^1(X, n) = H^{2-n,1}_{\mathcal{M}}(X, \mathbb{Z}) = \begin{cases} Pic(X), & n = 0, \\ \Gamma(X, \mathcal{O}_X^\times), & n = 1, \\ 0, & n \geq 2. \end{cases}$$

2. For $X = \mathsf{Spec}(F)$ for a number field F and $p \geq 1$ we have

$$CH^p(F, n) = H^{2p-n,p}_{\mathcal{M}}(F, \mathbb{Z}) = \begin{cases} 0, & p > n, \\ K_p^M(F), & p = n, \\ ? & p < n. \end{cases}$$

The last entry is only known in very few cases, e. g.

$$CH^2(F, 3) \simeq K_3^{ind}(F) := coker(K_3^M(F) \to K_3(F))$$

is known by [Sus86] and later [BL95]. But one at least hopes for the following vanishing conjectures:

Conjecture 1.3.7 (Beĭlinson, Soulé)**.** *If X is smooth, then $CH^p(X,n)\otimes\mathbb{Q} = 0$ for $2 \leq 2p \leq n$. Equivalently, $H^{i,p}_{\mathcal{M}}(X,\mathbb{Z}) \otimes \mathbb{Q} = 0$ if $i < 0$, or if $i = 0$ and $p > 0$.*

Remark 1.3.8. This conjecture is still an open problem for $p \geq 2$. Additionally, there is also an integral version of this conjecture, which is also much less believed to be true, though it is true for $p < 2$ as well. ♢

Conjecture 1.3.9. *If F satisfies the rank conjecture of Suslin, e. g. if F is a number field, then the higher Chow groups $CH^p(F, p+q), q \geq 0$ are generated by linearly embedded cycles, i. e. cycles which can be parametrized via products of Möbius transforms in each coordinate.*

Remark 1.3.10. The latter conjecture is a theorem of Gerdes for $q = 0, 1$ [Ger91], and due to unpublished work of Elbaz-Vincent, for $CH^3(F, 5) \otimes \mathbb{Q}$ as well. This is the crucial fact for the work of Gangl, Müller-Stach and Zhao. Note in particular that we do not know whether $CH^3(F, 5)$ is generated by linearly embedded cycles before tensoring with the rationals. ♢

The approach of Gangl, Müller-Stach, and Zhao to relating Bloch groups and higher Chow groups explicitly consists of starting with a certain fractional cycle in the Chow group which can be transformed via reparametrization or addition of boundaries into a sum of other cycles yielding a relation between the starting cycle and some new ones. In this way, the authors prove the universal relations for the polylogarithms which appear in the definition of the corresponding Bloch groups.

Having proved all of the relations occurring in the definition of the Bloch group, one obtains a map $B_m(F) \to CH^m(F, 2m-1)$. This is very technical and up to now only possible modulo torsion. Then one needs to show that this map indeed induces an isomorphism rationally, as conjectured by Zagier.

The following special fractional cycles $C_a^{(m)}$ play the crucial role in all that follows. By arguments presented in the next subsection, these cycles "correspond" to the m^{th} polylogarithm for $m \geq 2$ in a way that the map

$$\mathbb{Z}[F^\times] \to Z^2(F, 3), \qquad [a] \mapsto C_a^{(2)}$$

is supposed to induce the rational isomorphism between the Bloch group and the higher Chow group. This fits the general philosophy that polylogarithmic elements generate motivic cohomology groups of number fields or – more general – Shimura varieties.

We now come to the definition: Given a rational map $\phi : (\mathbb{P}^1_F)^n \to (\mathbb{P}^1_F)^m$, let

$$Z_\phi := \phi_*((\mathbb{P}^1_F)^n) \cap \square^m$$

be the cycle associated to ϕ in the sense of [Ful98, sect. 1.4]. Then let $x = (x_1, \ldots, x_n)$ and define

$$[\phi_1(x), \ldots, \phi_m(x)] := Z_{(\phi_1(x),\ldots,\phi_m(x))}.$$

We will use this notation in the rest of this work.

Definition 1.3.11. For $a \in F^\times$ and every $n \geq 2$ we set

$$C_a^{(n)} := \left[1 - \frac{a}{x_{n-1}}, 1 - \frac{x_{n-1}}{x_{n-2}}, 1 - \frac{x_{n-2}}{x_{n-3}}, \ldots, 1 - \frac{x_2}{x_1}, 1 - x_1, x_{n-1}, x_{n-2}, \ldots, x_1\right] \in Z^n(F, 2n-1).$$

1.3.2 The Abel – Jacobi map

As we have seen, the higher Chow groups are rationally a refinement of Quillen's algebraic K–theory. In particular, modulo torsion they are isomorphic to graded pieces of the K–groups. So it is natural to ask for a refined regulator map from these groups to a suitable cohomology theory which specializes the Borel regulator. Such a regulator map was proposed by Bloch in [Blo94a]. The suitable cohomology theory for us is Deligne – Beĭlinson cohomology $H^\bullet_{\mathcal{D}}(X, \mathbb{Z}(\bullet))$ and the corresponding map

$$CH^p(X, n) \to H^{2p-n}_{\mathcal{D}}(X, \mathbb{Z}(p)),$$

is seen to be a generalization of Griffiths' Abel – Jacobi map.

In this section we shall introduce a regulator map due to Kerr, Lewis, and Müller-Stach [KLMS06] suitable for our purposes from Bloch's higher Chow groups to Deligne – Beĭlinson cohomology generalizing Beĭlinson's regulator map, i. e. with a real-valued part coinciding with the Beĭlinson regulator.

Therefore we start by surveying the basic definition of Deligne – Beĭlinson cohomology $H^\bullet_{\mathcal{D}}(X, \mathbb{Z}(\bullet))$ for a smooth (quasi-)projective variety X as in [EV88] and then introduce the Abel – Jacobi map

$$\Phi_{p,n} : CH^p(X, n) \to H^{2p-n}_{\mathcal{D}}(X, \mathbb{Z}(p)) \tag{1.3.1}$$

of Lewis, Müller-Stach and Kerr [KLMS06].

Remark 1.3.12. We shall work in the analytic topology and define Ω^k_X to be the sheaf of holomorphic k-forms on X. In contrast, we let $\Omega^{p,q}_{X^\infty}$ be the sheaf of $\mathcal{C}^\infty$ (p,q)-forms on X and ${}'\Omega^{p,q}_{X^\infty}$ be the sheaf of distributions over $\Omega^{-p,-q}_{X^\infty}$. If we are only interested in the total degree, we set $\Omega^k_{X^\infty} := \oplus_{p+q=k}\Omega^{p,q}_{X^\infty}$ as well as $\mathcal{D}(\Omega^k_{X^\infty}) := \oplus_{p+q=k}{}'\Omega^{p,q}_{X^\infty}$. Finally, cohomology groups without subscript denote Betti cohomology groups. ♢

Definition 1.3.13. Let $X/\mathbb{C}$ be a smooth projective variety of complex dimension m and let $\mathbb{A} \subseteq \mathbb{R}$ be a subring. Then the Deligne complex is defined as

$$\mathbb{A}_{\mathcal{D}}(p) : \mathbb{A}(p) \to \underbrace{\mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{p-1}_X}_{=:\Omega^{\bullet<p}_X},$$

where $\mathbb{A}(p) := (2\pi i)^p\mathbb{A}$. (Naive) Deligne cohomology is defined as the analytic hypercohomology of this complex:

$$H^i_{\mathcal{D}}(X, \mathbb{A}(p)) := \mathbb{H}^i_{an}(\mathbb{A}_{\mathcal{D}}(p)).$$

We also need an expanded definition of Deligne cohomology for smooth quasiprojective varieties: Let Z be of that kind with a good compactification $\overline{Z}$ with a normal crossing divisor (NCD) E. Also recall that if $\mu : A^\bullet \to B^\bullet$ is a morphism of complexes, then the cone complex is given by

$$Cone(A^\bullet \xrightarrow{\mu} B^\bullet) := A^\bullet[1] \oplus B^\bullet,$$

where the differential $\delta : A^{\bullet+1} \oplus B^\bullet \to A^{\bullet+2} \oplus B^\bullet$ is given by $\delta(a,b) = (-da, \mu(a) + db)$.

Definition 1.3.14. Write $C_\bullet(\overline{Z}, \mathbb{A}(p))$ for the complex of singular C^∞-chains in $\overline{Z}$ with coefficients in $\mathbb{A}(p)$, and let ${}'C^i := C_{-1}$. Then define

$${}'C^\bullet(\overline{Z}, E, \mathbb{A}(p)) := {}'C^\bullet(\overline{Z}, \mathbb{A}(p)) / {}'C^\bullet_E(\overline{Z}, \mathbb{A}(p)),$$

where ${}'C^\bullet_E(\overline{Z}, \mathbb{A}(p))$ is the subcomplex of chains supported on E. Deligne homology ${}'H^\bullet_{\mathcal{D}}(Z, \mathbb{A}(p))$, as defined in [Jan88], is given by the cohomology of the cone complex

$$Cone\left({}'C^\bullet(\overline{Z}, E, \mathbb{A}(p)) \oplus \left(\bigoplus_{i+j=\bullet, i\geq p} {}'\Omega_{\overline{Z}^\infty}\langle E\rangle(\overline{Z})\right) \xrightarrow{\epsilon - l} {}'\Omega^\bullet_{\overline{Z}^\infty}\langle E\rangle(\overline{Z})\right)[-1],$$

where ϵ and l are natural maps of complexes as defined in [Jan88]. In addition, we set $\Omega^\bullet_{\overline{Z}}\langle E\rangle := \Omega^\bullet_{\overline{Z}}(\log E)$ to be the de Rham complex of meromorphic forms on $\overline{Z}$, holomorphic on $U := \overline{Z} - E$, with at most logarithmic poles along E. Further $\Omega^\bullet_{\overline{Z}^\infty}\langle E\rangle := \Omega^\bullet_{\overline{Z}}\langle E\rangle \otimes_{\Omega^\bullet_{\overline{Z}}} \Omega^\bullet_{\overline{Z}^\infty}$. At last, ${}'\Omega^\bullet_{\overline{Z}^\infty}\langle E\rangle := \Omega^\bullet_{\overline{Z}}\langle E\rangle \otimes_{\Omega^\bullet_{\overline{Z}}} {}'\Omega^\bullet_{\overline{Z}^\infty}$.

Via Poincaré duality, we can now also define Deligne cohomology as the dual of Deligne homology just introduced. For further details we refer to [KLMS06] and references therein. The key point of this complicated construction, which we only roughly sketched, is that for our purposes, namely computing the Deligne – Beĭlinson cohomology of the quasiprojective variety $\square^n$ as target space of the Abel – Jacobi map of [KLMS06], we can concretely describe the cohomology groups in terms of those for a smooth projective variety. Then one can make use of the following machinery:

There is a short exact sequence

$$0 \to \Omega_X^{\bullet<p}[-1] \to \mathbb{A}_{\mathcal{D}}(p) \to \mathbb{A}(p) \to 0,$$

which in turn (with some work) gives another short exact sequence

$$0 \to \frac{H^{2p-n-1}(X,\mathbb{C})}{F^pH^{2p-n-1}(X,\mathbb{C}) + H^{2p-n-1}(X,\mathbb{Z}(p))} \to H^{2p-n}_{\mathcal{D}}(X,\mathbb{Z}(p)) \to H^{2p-n}(X,\mathbb{Z}(p)) \bigcap F^pH^{2p-n-1}(X,\mathbb{C}) \to 0.$$

Now let

$$CH^p_{\text{hom}}(X,n) := \ker\{CH^p(X,n) \to H^{2p-n}_{\mathcal{D}}(X,\mathbb{Z}(p)) \to H^{2p-n}(X,\mathbb{Z}(p))\}$$

be the group of cycles in $CH^p(X,n)$ homologically equivalent to zero. Then we have an induced regulator map

$$\Phi_{p,n} : CH^p_{\mathrm{hom}}(X,n) \to \frac{H^{2p-n-1}(X,\mathbb{C})}{F^pH^{2p-n-1}(X,\mathbb{C}) + H^{2p-n-1}(X,\mathbb{Z}(p))} =: J^{p,n}(X),$$

where $J^{p,n}(X)$ is called the generalized intermediate Jacobian. Topologically these Jacobians are isomorphic to some $\mathbb{C}^m/\mathbb{Z}^n$ for $n \leq 2m$. In order to make this map explicit, we need to recall currents on smooth varieties.

Definition 1.3.15. An d–current on the quasiprojective variety X of complex dimension m is a section of the sheaf $'\mathcal{D}^d_{X^\infty} := \mathcal{D}(\Omega^{2m-d}_{X^\infty})$ of distributions of $\mathcal{C}^\infty$ forms on X.

We will associate to a given meromorphic function $f \in \mathbb{C}(X)$ the oriented $(2m-1)$-chain $T_f := \overline{f^{-1}(\mathbb{R}^-)}$ as in [KLMS06, 5.1]. The orientation is chosen so that $\partial T_f = (f) = |f_0| - |f_\infty|$. We are interested a particular current on $\square^n$: Let

$$T^n := T_{z_1} \cap \ldots \cap T_{z_n} \qquad \text{a topological } n\text{–chain}$$

and

$$\begin{aligned} R^n = R(z_1,\ldots,z_n) := {} & \log(z_1)d\log(z_2) \wedge \ldots \wedge d\log(z_n) \\ & + (-1)^{n-1}\big(2\pi i \log(z_2)d\log(z_3) \wedge \ldots \wedge d\log(z_n) \cdot \delta_{T_{z_1}} + \ldots \\ & \ldots + (2\pi i)^{n-1}\log(z_n) \cdot \delta_{T_{z_1}\cap\ldots\cap T_{z_{n-1}}}\big) \in {}'\mathcal{D}^{n-1}_{\square^n_F}. \end{aligned}$$

Remark 1.3.16. Via pullback, one can pretend that T^n, R^n be currents on $X \times \square^n_F$. But then one has to make sure that the cycle class $\mathcal{Z} \in CH^p(X,n)$ is in real good position, i. e. intersects the faces of the real n-cube in an admissible way. ♢

Proposition 1.3.17 ([KLMS06], Sect. 5.7)**.** *The Abel – Jacobi map $\Phi_{p,n}$ from* (1.3.1) *for $X = \mathsf{Spec}(F)$ is given in the following form for a cycle class $[\mathcal{Z}] \in CH^p(F, 2p-1)$:*

$$\frac{1}{(-2\pi i)^{p-1}} \int_{\mathcal{Z}} R^{2p-1} \in \mathbb{C}/\mathbb{Z}(p) \cong H^1_{\mathcal{D}}(\mathsf{Spec}(F), \mathbb{Z}(p)).$$

Example 1.3.18. *Consider a cycle class $[\mathcal{Z}] \in CH^2(F,3)$ intersecting the real 3–cube in an admissible way. Then we have*

$$-\Phi_{2,3}[\mathcal{Z}] = \int_{\mathcal{Z}\cap T_{z_1}} \log(z_2)d\log(z_3) + 2\pi i \sum_{p\in\mathcal{Z}\cap T_{z_1}\cap T_{z_2}} \log(p) \tag{1.3.2}$$

as image of the cycle class. ♦

Remark 1.3.19. As argued in [KLMS06], Sect. 5.7, one can show that every element in the Bloch group $B_2(\mathbb{C})$ can be completed to a higher Chow cycle $[\mathcal{Z}]$ by adding decomposable cycles which do not contribute to the regulator of the element in question. The real part of $\Phi_{2,3}$ then agrees with the Borel regulator. $\Diamond$

Remark 1.3.20. By the injectivity of the Beĭlinson regulator for number fields, checking whether a given cycle class is torsion amounts to checking this property for the image under the Abel – Jacobi map. However, this check might be a bit technical, since one has to provide real transversality of the currents involved. $\Diamond$

Chapter 2

Explicit computations in codimension two

In this chapter we prove universal relations in $CH^2(F, 3)$ for arbitrary number fields F. As mentioned in the introduction, D. Zagier initiated a program (extending Bloch's results on K_3 [Blo]) studying the complicated algebraic K–groups of a number field using their relation to higher Bloch groups. Modulo torsion, the Bloch groups as introduced before are conjectured to be isomorphic to the corresponding motivic cohomology groups, which in turn are isomorphic to a graded piece of the algebraic K–groups. Integrally, Zagier's program should amount to the search for an analogous result to Suslin's exact sequence relating the classical Bloch group and the indecomposable part of K_3.

The present chapter is based on the following ideas: Gangl and Müller-Stach in [GMS99] use the (generalized) Totaro cycles of [BK95] corresponding to the m^{th} polylogarithm to prove universal relations in $CH^m(F, 2m - 1)$ for $m = 2, 3$ coming from functional equations of the polylogarithms. In this way they construct a homomorphism $\overline{\rho}_2 : B_2(F) \to CH^2(F, 3)$ which is supposed, in view of [Sus91], [Sus86], to induce an isomorphism

$$\overline{\rho}_2 \otimes \mathbb{Q} : B_2(F) \otimes \mathbb{Q} \xrightarrow{\cong} K_3^{ind}(F) \otimes \mathbb{Q} \cong CH^2(F, 3) \otimes \mathbb{Q},$$

where $K_3^{ind}(F)$ denotes the indecomposable part of Quillen's algebraic K–group $K_3(F)$ defined as the quotient of $K_3(F)$ by the Milnor K–group $K_3^M(F)$. In order that this map to be an isomorphism modulo torsion, one has to work out an inverse to it. Gangl and Müller-Stach have made some remarks on these matters in [GMS99] and also have some work on this in progress.

The new feature in this chapter is that we verify the relations in question integrally and refine the results known modulo torsion: First of all, we prove some auxiliary relations between cycles occurring in the complex computing the Chow group in codimension two. Then, we are ready to prove refined versions of the relations Gangl and Müller-Stach proved in [GMS99]. This already suffices to find a presentation of the integral motivic cohomology groups of number fields. We give some explicit generators in some interesting cases.

After that we obtain an identification of the other groups of Suslin's exact sequence with certain homology groups of a subcomplex of the complex computing motivic cohomology. In this way, we can determine, whether a given cycle in $CH^2(F,3)$ already lives in one of the subgroups.

At the end of the chapter we obtain some more symmetric relations in the quotient $C^2(F,3)/\partial C^2(F,4)$ reflecting the symmetry in some particular functional equations of the dilogarithm.

2.1 The setup

Our aim is to prove some universal relations in $CH^2(F,3)$ for an arbitrary infinite field. Unlike Gangl/Müller-Stach we do not use the alternating cycles of Bloch and Kříž since they are only defined over $\mathbb{Q}$, but simply the classical Totaro cycles [Tot82] mentioned at the end of section 1.3.1:

$$C_a := C_a^{(2)} := Z_{(1-\frac{a}{x},1-x,x)} = \left[1-\frac{a}{x}, 1-x, x\right] \in Z^2(F,3).$$

To simplify our computations in the quotient $Z^2(F,3)/\partial Z^2(F,4)$, we divide out several acyclic subcomplexes of $Z^2(F,\bullet)$ consisting of cycles with a constant coordinate entry on the right- or left-hand side:

Lemma 2.1.1. *The following subcomplexes of $Z^2(F,\bullet)$ are acyclic:*

$$Z'(F,\bullet) := \ldots \to Z^1(F,1)\otimes Z^1(F,3) \to Z^1(F,1)\otimes Z^1(F,2) \to Z^1(F,1)\otimes \partial Z^1(F,2) \to 0$$

resp.

$$Z''(F,\bullet) := \ldots \to Z^1(F,3)\otimes Z^1(F,1) \to Z^1(F,2)\otimes Z^1(F,1) \to \partial Z^1(F,2)\otimes Z^1(F,1) \to 0.$$

Proof. First, one checks that both of them are complexes. This follows at once from the fact that both complexes are truncated versions of $Z^1(F,\bullet)$ tensored with $Z^1(F,1)$, which does not change homology.

The acyclicity of both subcomplexes is essentially the acyclicity result of Nart [Nar89], who explicitly constructs a contracting homotopy. His proof carries over literally to our integral setting. Again the acyclicity is not changed by tensoring the whole subcomplex by $Z^1(F,1)$. □

Remark 2.1.2. As we will see in section 2.4, dividing out both of them at once does change the homology. ◊

From now on, we define

$$C^2(F,\bullet) := Z^2(F,\bullet)/Z'(F,\bullet),$$

and use this quotient for explicit computations since the homology $H_3(C^2(F,\bullet))$ still computes $CH^2(F,3)$.

Note that there are some easy cycles in the Chow group consisting of just one element, the cyclotomic elements: C_1 is the most easy one, because $\partial(C_1) = (0,1) \in Z^2(F,2)$, which just vanishes by definition. We shall demonstrate this kind of computations done throughout this thesis in order to make the reader familiar with them: As $C_a = \left[1 - \frac{a}{x}, 1-x, x\right]$, we have

$$\partial C_a = (1-a, a) \in Z^2(F,2),$$

where the symbol on the right-hand side corresponds to the restriction of C_a to the zero of $1-a/x$ intersected with $\square^2$. Note that restricting C_a to zeros and poles of the other coordinate functions does not contribute to the boundary since these points (i. e. boundaries) are not contained in $\square^2$.

More generally nC_{ζ_n} for a primitive n^{th} root of unity – if it is contained in the field F: $\partial(C_{\zeta_n}) = (1-\zeta_n, \zeta_n)$ and since we divide out the subgroup $Z^1(F,1) \otimes \partial Z^1(F,2)$ in degree two of $Z^2(F,\bullet)$, where the right factor consists of formal sums of the form $\sum_i n_i([a_i b_i] - [a_i] - [b_i])$ with $n_i \in \mathbb{Z}$ and $a_i, b_i \in F^\times$ for all i, it immediately follows, that

$$\partial(nC_{\zeta_n}) = n\partial(C_{\zeta_n}) = 0 \in Z^2(F,2)/(Z^1(F,1) \otimes \partial Z^1(F,2)).$$

When it comes to finding explicit generators for the higher Chow groups of concrete cyclotomic fields, these $n \cdot C_{\zeta_n}$ or combinations of them will be the canonical candidates.

These elements are called cyclotomic elements also because of their image in Deligne – Beĭlinson cohomology: We see from formula (1.3.2) and the fact that $\log(1) = 0$ that

$$\Phi_{2,3}[C_1] = \int_0^1 \log(1-x) d\log(x) = Li_2(1) = \frac{\pi^2}{6}$$

since $T_{1-1/x} \cap T_{1-x} = \{1\}$. By the same argument as above, we see that nC_{ζ_n} is a nullhomologous cycle in $CH^2(F,3)$ corresponding via the regulator map to the function $n \cdot Li_2(\zeta_n)$.

Let us now set up the basic tools for performing calculations with algebraic cycles in $C^2(F,\bullet)$: splitting cycles with a product of two rational functions in one coordinate.

Remark 2.1.3. One has to care especially for the admissibility condition mentioned in the survey on higher Chow groups (definition 1.3.1). In our setting, a fractional cycle $\mathcal{Z} = [f(x), g(x), h(x)]$ is called admissible if and only if the following holds: Every zero or pole occurring more than once among the divisors of f, g or h has to be in the preimage of 1 of one of the other functions.

This is easy to see: an admissible cycle must intersect each face of the algebraic n-cube properly and so no coordinate may have a zero or pole of order greater than one. It also may not meet the intersection of two faces of the n-cube. But the condition for any cycle stated above on the zeros and poles of a fractional cycle assures that the cycles with "bad" intersection behavior are empty because one of the coordinates is equal to one.

In this and in the next chapter, we have basically either used integral versions of the cycles Gangl and Müller-Stach used, which certainly does not change admissibility, or checked the admissibility of all cycles occurring and will not stress this point any more. The general relations in this section are of course always valid for admissible cycles only. We note that the admissibility condition is very strict. It is very easy to obtain cycles which turn out to be not admissible. But relations involving inadmissible cycles are useless. This is what causes the most severe problems in proving relations in the cubical higher Chow groups.

◊

Notation . For a rational function f, we denote by $div(f)$ its divisor, i. e. its zeros and poles together with their multiplicities.

Proposition 2.1.4. *Let f, g, h_1, h_2 be rational functions of one variable x such that all cycles occurring are admissible. Then the following identities hold in $C^2(F,3)/\partial C^2(F,4)$.*

$$\begin{aligned}
&[h_1(x)h_2(x), f(x), g(x)] = [h_1(x), f(x), g(x)] + [h_2(x), f(x), g(x)] \\
&\quad - \sum_{x_0 \in div(f)} \pm \left[\frac{z - h_1(x_0)h_2(x_0)}{z - h_1(x_0)}, z, g(x_0)\right] + \sum_{x_0 \in div(g)} \pm \left[\frac{z - h_1(x_0)h_2(x_0)}{z - h_1(x_0)}, z, f(x_0)\right], \\
&[f(x), h_1(x)h_2(x), g(x)] = [f(x), h_1(x), g(x)] + [f(x), h_2(x), g(x)] \\
&\quad + \sum_{x_0 \in div(f)} \pm \left[\frac{z - h_1(x_0)h_2(x_0)}{z - h_1(x_0)}, z, g(x_0)\right], \\
&[f(x), g(x), h_1(x)h_2(x)] = [f(x), g(x), h_1(x)] + [f(x), g(x), h_2(x)] \,.
\end{aligned}$$

Proof. Each relation is the boundary of one of the following terms: $\left[\frac{z-h_1(x)h_2(x)}{z-h_1(x)}, z, f(x), g(x)\right]$, $\left[f(x), \frac{z-h_1(x)h_2(x)}{z-h_1(x)}, z, g(x)\right]$, $\left[f(x), g(x), \frac{z-h_1(x)h_2(x)}{z-h_1(x)}, z\right]$. One just has to keep in mind that terms with a constant in the left coordinate are being divided out. □

Before going on to proving something with these rules, let us examine the different types of terms occurring in our complex $C^2(F, \bullet)$. Note also that in addition to terms of the form $[f(x), g(x), c]$ for rational functions f, g and a constant c as encountered in the proposition, there is another kind of terms with a constant coordinate possible, namely $[f(x), c, g(x)]$.

Proposition 2.1.5. *Any admissible term of the form $[f(x), c, g(x)] \in C^2(F,3)$ for some Möbius transformations f, g and a constant $c \in F^\times$ can be expressed as a sum of terms of the form*

$$Z(a,c) := \left[1 - \frac{1-a}{x}, c, 1-x\right] = \left[\frac{x-a}{x-1}, c, x\right]$$

and terms with a constant in the right coordinate.

Proof. Consider a generic admissible term $\mathcal{Z} := [f(x), c, g(x)]$. Under the reparametrization $x \mapsto g^{-1}(x)$ such a $\mathcal{Z}$ is mapped onto the still admissible term $[f(g^{-1}(x)), c, x]$. Then by invoking the above proposition sufficiently often, one can factor $f(g^{-1}(x))$ into terms of the form $\frac{a_i - x}{1-x}$. Note that the denominator guarantees admissibility. Again reparametrizing $x \mapsto 1 - x$, we produce several terms of the form $Z(a_i, c)$ for certain constants a, c and other terms with a constant on the right. □

Our next goal is to show that terms with a constant coordinate on the right from proposition 2.1.4 can be expressed via the $Z(a, b)$ as well.

Remark 2.1.6. The main motivation behind these considerations is based on computing the images of terms with a constant in the middle or on the right under the map inducing the Abel – Jacobi map from section 1.3.2: Pick constants $a, b \in F^\times$ such that there is an $n \in \mathbb{N}$ satisfying $n\partial Z(a, b) = n(b, a) + n(a, b) = 0$. One computes:

$$-n\Phi_{2,3}[Z(a,b)] = n \int_0^{1-a} \log(b) d \log(1-x) = n \log(a) \log(b).$$

On the other hand let $\mathcal{Z} = [f(x), g(x), c]$ be admissible and such that $n\partial\mathcal{Z} = 0$. Then by computing the Abel – Jacobi map, one notices that only the last term survives:

$$-n\Phi_{2,3}[\mathcal{Z}] = 2\pi i n \sum_{p \in \mathcal{Z} \cap T_{f(x)} \cap T_{g(x)}} \log(c)(p).$$

From the theory of the dilogarithm function, one recognizes terms of the first kind as being the correction term of the Rogers dilogarithm eliminating the multivalent character of Li_2, while the ones of the second kind correspond to the monodromy of the dilogarithm crossing the branch $[1, \infty)$. For this reason, we will sometimes refer to both of these ones as "monodromy terms" or "lower order terms". ♢

Remark 2.1.7. The appearance of those terms in proposition 2.1.4 is the reason for the main technical problems in the sequel. These "lower order terms" appear as soon as one tries to prove polylogarithmic identities in the higher Chow groups and make concrete computations with the relations quite tedious. One needs some way of eliminating them to simplify computations. In codimension two, we were quite successful in eliminating technical problems, but already in codimension three, we would still need more additional assumptions in order to simplify the expressions. ♢

Let us return to the relation between the different terms and some more technical lemmas.

Lemma 2.1.8. *The following relation holds in $C^2(F,3)/\partial C^2(F,4)$ for rational functions f and g provided all of the terms are admissible:*

$$[f(x), g(x), c] = -[f(x), c, g(x)] + \sum_{x_0 \in div(f)} \pm Z(c, g(x_0)).$$

Proof. The relation is just the boundary of $\left[f(x), \frac{z-c}{z-1}, g(x), z\right] \in C^2(F, 4)$. □

So both kinds of cycles with one constant are related to each other. In particular, one trivially has

$$\left[\frac{x-a}{x-b}, x, c\right] = -\left[\frac{x-a}{x-b}, c, x\right] + Z(c,a) - Z(c,b).$$

Favorably, the whole right hand side should be expressible in terms of the $Z(a,b)$. Indeed:

Lemma 2.1.9. *Let $a,b,c \in F^\times$. Then there is a cycle $\mathcal{W} \in C^2(F,4)$ whose boundary gives rise to the following relation in $C^2(F,3)/\partial C^2(F,4)$ provided all the terms are admissible:*

$$[f(x), c, g(x)] = -\left[\frac{1}{f(x)}, c, g(x)\right] + \sum_{x_0 \in div(g)} \pm \left(\left[\frac{z-1}{z-f(x_0)}, c, z\right] + Z(c, f(x_0)) \right).$$

Proof. Set $\mathcal{W} := -\left[\frac{z-1}{z-f(x)}, z, c, g(x)\right]$, compute its boundary to be

$$[f(x), c, g(x)] + \left[\frac{1}{f(x)}, c, g(x)\right] + \sum_{x_0 \in div(g)} \pm \left[\frac{z-1}{z-f(x_0)}, z, c\right]$$

and then use the lemma above. □

Corollary 2.1.10. *With all assumptions of the lemma we have in $C^2(F,3)/\partial C^2(F,4)$:*

$$\left[\frac{x-a}{x-b}, c, x\right] = Z(c, \frac{a}{b}),$$
$$Z(a,c) = Z(c,a).$$

Proof. One applies the lemma to calculate

$$\left[\frac{x-a}{x-b}, c, x\right] = -\left[\frac{x-b}{x-a}, c, x\right] + \left[\frac{x-1}{x-\frac{a}{b}}, c, x\right] + Z(c, \frac{a}{b}),$$

dividing the numerator and denominator of first coordinate of the second term by b, we have

$$= -\left[\frac{x-b}{x-a}, c, x\right] + \left[\frac{xb-b}{xb-a}, c, x\right] + Z(c, \frac{a}{b}),$$

substituting $x \mapsto xb^{-1}$ in the second term gives

$$= -\left[\frac{x-b}{x-a}, c, x\right] + \left[\frac{x-b}{x-a}, c, xb^{-1}\right] + Z(c, \frac{a}{b}).$$

Splitting the second term in the last coordinate gives the first term and one term with two constant coordinates, i. e. a negligible term. So

$$\left[\frac{x-a}{x-b}, c, x\right] = Z(c, \frac{a}{b}).$$

For the second assertion use the lemma to obtain

$$Z(a,c) = \left[\frac{x-a}{x-1}, c, x\right] = -\left[\frac{x-1}{x-a}, c, x\right] + \left[\frac{x-1}{x-a}, c, x\right] + Z(c,a) = Z(c,a). \qquad \square$$

In summary,

$$\left[\frac{x-a}{x-b}, x, c\right] = Z(c,a) - Z(c,b) - Z(c,\frac{a}{b}). \tag{2.1.1}$$

Trivially, one also derives $\left[\frac{x-a}{x-b}, x, c\right] = \left[\frac{x-a}{x-\frac{a}{b}}, x, c\right]$ by comparing the corresponding right-hand sides. Note finally that

$$\left[\frac{x-a}{x-b}, x, c\right] + \left[\frac{x-b}{x-a}, x, c\right] = \left[\frac{x-1}{x-\frac{a}{b}}, x, c\right].$$

Corollary 2.1.11. *Let $a, b, c \in F^\times$ such that all cycles occurring are admissible, and assume that F contains an n^{th} primitive root of unity ζ_n. Then the following relations hold in $C^2(F,3)/\partial C^2(F,4)$:*

$$n\left[\frac{x-a}{x-b}, x, \zeta_n\right] = n\left(Z(a,\zeta_n) - Z(\frac{a}{b},\zeta_n) - Z(b,\zeta_n)\right) = 0.$$

Proof. The relation follows trivially from equation (2.1.1) and proposition 2.1.4. $\square$

Example 2.1.12. *So we can express the rules for splitting cycles entirely from proposition 2.1.4 in terms of the Z–cycles:*

$$\begin{aligned}
&[h_1(x), h_2(x), f(x)g(x)] = [h_1(x), h_2(x), f(x)] + [h_1(x), h_2(x), g(x)]\\
&[h_1(x), f(x)g(x), h_2(x)] = [h_1(x), f(x), h_2(x)] + [h_1(x), g(x), h_2(x)]\\
&\qquad + \sum_{x_0 \in div(h_1)} \pm \left(Z(h_2(x_0), f(x_0)g(x_0)) - Z(h_2(x_0), f(x_0)) - Z(h_2(x_0), g(x_0))\right)
\end{aligned}$$

and

$$\begin{aligned}
&[f(x)g(x), h_1(x), h_2(x)] = [f(x), h_1(x), h_2(x)] + [g(x), h_1(x), h_2(x)]\\
&\qquad + \sum_{x_0 \in div(h_2)} \pm \left(Z(h_1(x_0), f(x_0)g(x_0)) - Z(h_1(x_0), f(x_0)) - Z(h_1(x_0), g(x_0))\right)\\
&\qquad - \sum_{x_0 \in div(h_1)} \pm \left(Z(h_2(x_0), f(x_0)g(x_0)) - Z(h_2(x_0), f(x_0)) - Z(h_2(x_0), g(x_0))\right).
\end{aligned}$$

We shall not use this variant any more, because one does not gain any new information compared to proposition 2.1.4. It does show that the "defect" of a term to be multiplicative in one coordinate is given by a sum $Z(a,bc) - Z(a,b) - Z(a,c)$ for some constants $a, b, c \in F^\times$ or by a monodromy term of the dilogarithm in the sense of remark 2.1.6. ♦

In the sequel, we will make use of one more trick: Permuting coordinate entries:

Lemma 2.1.13. *Let f, g, h be rational functions in one variable, and let all of the cycles be admissible. Then the following identities hold in $C^2(F,3)/\partial C^2(F,4)$:*

$$\begin{aligned}
[f(x), g(x), h(x)] &= -[f(x), h(x), g(x)] + \sum_{x_0 \in div(f)} \pm Z(h(x_0), g(x_0)), \\
[f(x), g(x), h(x)] &= -[g(x), f(x), h(x)] + \sum_{x_0 \in div(g)} \pm Z(f(x_0), h(x_0)), \\
[f(x), g(x), h(x)] &= -[h(x), g(x), f(x)] + \sum_{x_0 \in div(f)} \pm Z(h(x_0), g(x_0)) \\
&\quad - \sum_{x_0 \in div(g)} \pm Z(f(x_0), h(x_0)) + \sum_{x_0 \in div(h)} \pm Z(f(x_0), g(x_0)).
\end{aligned}$$

Proof. Compute the boundary of $\left[f(x), \frac{z-g(x)}{z-1}, h(x), z\right]$ for the first, $\left[\frac{z-f(x)}{z-1}, g(x), z, h(x)\right]$ for the second, and $\left[\frac{z-f(x)}{z-1}, g(x), h(x), z\right]$ and $\left[\frac{z-g(x)}{z-1}, h(x), z, f(x)\right]$ for the last relation. □

Example 2.1.14. *The following relations hold in $C^2(F,3)/\partial C^2(F,4)$:*

$$\left[1 - \frac{a}{x}, x, 1-x\right] = -C_a + Z(a, 1-a).$$

So we can define

$$\tilde{C}_a := \left[1 - \frac{a}{x}, 1-x, x\right] - \left[1 - \frac{a}{x}, x, 1-x\right],$$

and see that $\tilde{C}_a = 2C_a - Z(a, 1-a)$, in particular $\tilde{C}_1 = 2C_1$. We shall use this variant of terms in $C^2(F,3)/\partial C^2(F,4)$ later on. They play the role of the Rogers dilogarithm: The image of $\tilde{C}_a$ under the map inducing the Abel – Jacobi map (1.3.2) is easily seen to be $2Li_2(a) + \log(a)\log(1-a)$, which is equal to twice the Rogers dilogarithm. Expressing relations for the dilogarithm in terms of these elements eliminates terms of lower order. ♦

Remark 2.1.15. At this point, one can see that if we had chosen a different permutation of words in the definition of C_a, the difference would be at worst a term of the form $Z(a, 1-a)$. But in the sequel we will mainly specialize relations to roots of unity. This leads to all lower order terms vanishing – at least after multiplying with an integer big enough. Therefore, a different permutation of the coordinates of the C_a-terms leads to slightly different relations, but up to torsion they clearly coincide. ♢

2.2 Relations

With the aid of proposition 2.1.4 we are now ready to mimic the proofs of several relations as in [GMS99]. The ideas will be more or less the same: Starting with a suitable reparametrization of a Totaro cycle, we break this term up into pieces which can be identified or at least glued together to other Totaro cycles giving a relation between certain Totaro cycles and some lower order terms. Throughout this section, F denotes an arbitrary infinite field.

Proposition 2.2.1. *For $a \in F^\times - \{1\}$ the following identity holds in $C^2(F,3)/\partial C^2(F,4)$:*

$$C_a + C_{1-a} - C_1 = Z(a, 1-a). \qquad (2.2.1)$$

Proof. One easily calculates

$$\begin{aligned} C_a &= \left[\frac{x-a}{x}, 1-x, x\right] = \left[\frac{x-a}{x-1}, 1-x, x\right] + \left[\frac{x-1}{x}, 1-x, x\right] \\ &= \left[1 - \frac{1-a}{x}, x, 1-x\right] + C_1. \end{aligned}$$

Next one makes use of lemma 2.1.13 to see that this is equivalent to

$$C_a = -C_{1-a} + Z(1-a, a) + C_1.$$

The claim follows from corollary 2.1.10. □

Remark 2.2.2. Notice the analogy with

$$Li_2(z) + Li_2(1-z) - Li_2(1) = -\log(z)\log(1-z), \qquad z \in \mathbb{C} \setminus [1, \infty),$$

a functional equation of the dilogarithm. ♢

Remark 2.2.3. If we rewrite this relation in terms of $\tilde{C}_a$ instead of C_a, then we see that

$$\begin{aligned} \tilde{C}_a &= \left[\frac{x-a}{x}, 1-x, x\right] - \left[\frac{x-a}{x}, x, 1-x\right] \\ &= \left[\frac{x-a}{x-1}, 1-x, x\right] + \left[\frac{x-1}{x}, 1-x, x\right] - \left[\frac{x-a}{x-1}, x, 1-x\right] - \left[\frac{x-1}{x}, x, 1-x\right] \\ &= -\tilde{C}_{1-a} + \tilde{C}_1. \end{aligned}$$

This was what we expected: Earlier we remarked that the image of $\tilde{C}_a$ under the map inducing the Abel – Jacobi map is given by the Rogers dilogarithm whereas C_a just gives the ordinary dilogarithm. ♢

The following distribution relations are more interesting:

Proposition 2.2.4. *Let $a \in F^\times$ and assume F contains a primitive n^{th} root of unity ζ_n. Then the following relation holds in $C^2(F,3)/\partial C^2(F,4)$:*

$$nC_{a^n} = n^2 \sum_{j=1}^{n} C_{\zeta_n^j a} + 2 \sum_{i=2}^{n} \left[\frac{z - \prod_{j=1}^{i}(1-\zeta_n^j a)}{z - (1-\zeta_n^i a)}, z, a^n\right]. \qquad (2.2.2)$$

Proof. We prove the formula for $n = 2$, the general case follows by repetition of similar arguments, mainly proposition 2.1.4: First note that $2C_{a^2} = \left[1 - \left(\frac{a}{z}\right)^2, 1 - z^2, z^2\right]$ by the push-forward property of algebraic cycles (cf.[Ful98, 1.4]).

$$\begin{aligned}2C_{a^2} &= \left[1-\left(\frac{a}{z}\right)^2, 1-z^2, z^2\right] = \left[1-\frac{a}{z}, 1-z^2, z^2\right] + \left[1+\frac{a}{z}, 1-z^2, z^2\right] \\ &= 2\left[1-\frac{a}{z}, 1-z, z^2\right] + 2\left[1+\frac{a}{z}, 1-z, z^2\right] \\ &\quad + \left[\frac{z-(1-a^2)}{z-(1-a)}, z, a^2\right] + \left[\frac{z-(1-a^2)}{z-(1+a)}, z, a^2\right] \\ &= 4C_a + 4C_{-a} + \left[\frac{z-(1-a^2)}{z-(1-a)}, z, a^2\right] + \left[\frac{z-(1-a^2)}{z-(1+a)}, z, a^2\right],\end{aligned} \tag{2.2.3}$$

and the assertion in case $n = 2$ follows from the fact that in the quotient $C^2(F,3)/\partial C^2(F,4)$ we have $[\frac{z-a}{z-b}, c, z] = [\frac{z-a}{z-\frac{a}{b}}, c, z]$ for $a, b, c \in F^\times$ and so

$$2C_{a^2} = 4C_a + 4C_{-a} + 2\left[\frac{z-(1-a^2)}{z-(1-a)}, z, a^2\right].$$

For general n we have $nC_{a^n} = \left[1-\left(\frac{a}{z}\right)^n, 1-z^n, z^n\right]$. Then one uses proopositionn 2.1.4 to split each coordinate into linear factors. The result follows by reparametrizing appropriately. □

Remark 2.2.5. Note that no Z-terms occur: The corresponding functional equation for the dilogarithm also has no lower order terms. The other terms in proposition 2.2.4 with a constant in the right coordinate reflect the monodromy of the dilogarithm depending on the choice of a. ♢

Now let us turn to the five-term relation. We shall prove it with various simplifications of the lower order terms. The first one is a refined relation of the one Gangl and Müller-Stach proved. After that, one can use the relations from the previous subsection and some symmetry considerations to simplify it modulo 2-torsion.

Proposition 2.2.6. *Let $a, b \in F^\times - \{1\}$ such that $a \neq b, 1-b$. Then the following relation holds in $C^2(F,3)/\partial C^2(F,4)$:*

$$\begin{aligned}V_{a,b} :=& C_{\frac{a(1-b)}{b(1-a)}} - C_{\frac{1-b}{1-a}} + C_{1-b} - C_{\frac{a}{b}} + C_a - Z\left(\frac{1}{b}, \frac{1}{1-a}\right) - \left[\frac{z-1}{z-b}, z, 1-b\right] \\ &+ \left[\frac{z-\frac{b-a}{b(1-a)}}{z-\frac{b-a}{1-a}}, z, \frac{1-b}{1-a}\right] + \left[\frac{z-1}{z-(1-a)}, z, a\right] + \left[\frac{z-\frac{b-a}{b(1-a)}}{z-\frac{b-a}{b}}, z, \frac{a}{b}\right] = 0.\end{aligned} \tag{2.2.4}$$

Proof. We mimic the proof of [GMS99] making use of the basic proposition 2.1.4. We start with the following reparametrization $t \mapsto \frac{a(1-t)}{t(1-a)}$ of $C_{\frac{a(1-b)}{b(1-a)}}$:

$$\begin{aligned}
C_{\frac{a(1-b)}{b(1-a)}} &= \left[\frac{b-t}{b(1-t)}, \frac{t-a}{t(1-a)}, \frac{a(1-t)}{t(1-a)}\right] \\
&= \left[\frac{b-t}{b(1-t)}, \frac{t-a}{t(1-a)}, \frac{a}{t}\right] + \left[\frac{b-t}{b(1-t)}, \frac{t-a}{t(1-a)}, \frac{1-t}{1-a}\right] \\
&= \left[\frac{b-t}{1-t}, \frac{t-a}{t(1-a)}, \frac{1-t}{1-a}\right] + \left[\frac{z-1}{z-\frac{1}{b}}, z, \frac{1}{1-a}\right] + \left[\frac{b-t}{b(1-t)}, \frac{t-a}{t}, \frac{a}{t}\right] \\
&\quad + \left[\frac{b-t}{b(1-t)}, \frac{1}{1-a}, \frac{a}{t}\right] + \left[\frac{z-\frac{b-a}{b(1-a)}}{z-\frac{b-a}{b}}, z, \frac{a}{b}\right] - \left[\frac{z-1}{z-(1-a)}, z, a\right]. \\
&= \left[\frac{b-t}{1-t}, \frac{t-a}{1-a}, \frac{1-t}{1-a}\right] + \left[\frac{b-t}{1-t}, \frac{1}{t}, \frac{1-t}{1-a}\right] + \left[\frac{b-t}{b}, \frac{t-a}{t}, \frac{a}{t}\right] \\
&\quad + \left[\frac{1}{1-t}, \frac{t-a}{t}, \frac{a}{t}\right] + \left[\frac{b-t}{b(1-t)}, \frac{1}{1-a}, \frac{a}{t}\right] + \left[\frac{z-\frac{b-a}{b(1-a)}}{z-\frac{b-a}{1-a}}, z, \frac{1-b}{1-a}\right] \\
&\quad + \left[\frac{z-1}{z-\frac{1}{b}}, z, \frac{1}{1-a}\right] + \left[\frac{z-\frac{b-a}{b(1-a)}}{z-\frac{b-a}{b}}, z, \frac{a}{b}\right] - \left[\frac{z-1}{z-(1-a)}, z, a\right]
\end{aligned}$$

we obtain after some inversions

$$\begin{aligned}
C_{\frac{a(1-b)}{b(1-a)}} &= C_{\frac{1-b}{1-a}} - C_{1-b} + \left[\frac{b-t}{1-t}, \frac{1}{t}, \frac{1}{1-a}\right] + C_{\frac{a}{b}} - C_a \\
&\quad + \left[\frac{b-t}{b(1-t)}, \frac{1}{1-a}, \frac{a}{t}\right] + \left[\frac{z-\frac{b-a}{b(1-a)}}{z-\frac{b-a}{1-a}}, z, \frac{1-b}{1-a}\right] + \left[\frac{z-1}{z-b}, z, 1-b\right] \\
&\quad + \left[\frac{z-1}{z-\frac{1}{b}}, z, \frac{1}{1-a}\right] + \left[\frac{z-\frac{b-a}{b(1-a)}}{z-\frac{b-a}{b}}, z, \frac{a}{b}\right] - \left[\frac{z-1}{z-(1-a)}, z, a\right].
\end{aligned}$$

At the end of the proof we also show that

$$\left[\frac{b-t}{1-t}, \frac{1}{t}, \frac{1}{1-a}\right] + \left[\frac{b-t}{b(1-t)}, \frac{1}{1-a}, \frac{a}{t}\right] = Z\left(\frac{1}{b}, \frac{1}{1-a}\right) - \left[\frac{z-1}{z-\frac{1}{b}}, z, \frac{1}{1-a}\right]$$

to conclude

$$\begin{aligned}
C_{\frac{a(1-b)}{b(1-a)}} &= C_{\frac{1-b}{1-a}} - C_{1-b} + C_{\frac{a}{b}} - C_a + Z\left(\frac{1}{b}, \frac{1}{1-a}\right) + \left[\frac{z-1}{z-b}, z, 1-b\right] \\
&\quad - \left[\frac{z-\frac{b-a}{b(1-a)}}{z-\frac{b-a}{1-a}}, z, \frac{1-b}{1-a}\right] - \left[\frac{z-\frac{b-a}{b(1-a)}}{z-\frac{b-a}{b}}, z, \frac{a}{b}\right] - \left[\frac{z-1}{z-(1-a)}, z, a\right].
\end{aligned}$$

So it remains to show the following:

$$\left[\frac{b-t}{b(1-t)}, \frac{1}{1-a}, \frac{a}{t}\right] = \left[\frac{b-t}{1-t}, \frac{1}{1-a}, \frac{1}{t}\right] - \left[\frac{z-1}{z-\frac{1}{b}}, z, \frac{1}{1-a}\right]$$
$$= -\left[\frac{b-t}{1-t}, \frac{1}{z}, \frac{1}{1-a}\right] + Z\left(\frac{1}{b}, \frac{1}{1-a}\right) - \left[\frac{z-1}{z-\frac{1}{b}}, z, \frac{1}{1-a}\right].$$

Adding the other term $\left[\frac{b-t}{1-t}, \frac{1}{t}, \frac{1}{1-a}\right]$ gives the desired result. □

Remark 2.2.7. Using the $\tilde{C}_a$-terms instead of C_a, we can again get rid of the Z-term, but for the price of several more terms with a constant in the right coordinate. We shall use the relation just stated and only simplify the extra terms a little in the following corollaries. ♢

Corollary 2.2.8. *For $a \in F^\times - \{1\}$ terms of the form $\left[\frac{z-1}{z-a}, z, 1-a\right]$ are 2-torsion in $C^2(F,3)/\partial C^2(F,4)$.*

Proof. Compute $0 = V_{a,b} - V_{1-b,1-a}$ for $a, b \in F^\times - \{1\}$, $a \neq b, 1-b$ to obtain

$$2\left[\frac{z-1}{z-a}, z, 1-a\right] = 2\left[\frac{z-1}{z-(1-b)}, z, b\right].$$

Specializing $b = -1$, we obtain the result for $a \neq -1, 2$. But for $a = 2$ the result holds by corollary 2.1.9, and for $a = -1$, one concludes by proposition 2.1.4

$$2\left[\frac{z-1}{z+1}, z, 2\right] = \left[\frac{z-1}{z+1}, z, 2\right] - \left[\frac{z-1}{z+1}, z, \frac{1}{2}\right] \overset{(2.1.1)}{=} 2Z(-1,2) - 2Z\left(-1, \frac{1}{2}\right)$$

and by the symmetry of $Z(a,b)$ and again (2.1.1)

$$= 2\left[\frac{z-1}{z-2}, z, -1\right].$$

But this expression vanishes again by corollary 2.1.11. □

Corollary 2.2.9. *Let $a, b \in F^\times - \{1\}$ such that $a \neq b, 1-b$. Then the following relation holds in $C^2(F,3)/\partial C^2(F,4)$:*

$$\begin{aligned} 0 = C_{\frac{a(1-b)}{b(1-a)}} &- C_{\frac{1-b}{1-a}} + C_{1-b} - C_{\frac{a}{b}} + C_a - Z(b, 1-a) \\ &- \left[\frac{z-1}{z-b}, z, 1-b\right] + \left[\frac{z-\frac{b-a}{b(1-a)}}{z-\frac{b-a}{1-a}}, z, \frac{1-b}{1-a}\right] + \left[\frac{z-1}{z-(1-a)}, z, a\right] \\ &+ \left[\frac{z-1}{z-a}, z, 1-b\right] + \left[\frac{z-\frac{b-a}{b(1-a)}}{z-\frac{b-a}{b}}, z, \frac{a}{b}\right]. \end{aligned} \tag{2.2.5}$$

Proof. Use the proposition and the fact that

$$Z\left(\frac{1}{b}, \frac{1}{1-a}\right) = Z(b, 1-a) + \left[\frac{z-1}{z-a}, z, 1-b\right]. \qquad \square$$

Corollary 2.2.10. *Let $a, b \in F^\times - \{1\}$ such that $a \neq b, 1-b$. Then the following relation holds in $C^2(F,3)/\partial C^2(F,4)$:*

$$\begin{aligned} 2V'(a,b) &:= 2C_{\frac{a(1-b)}{b(1-a)}} - 2C_{\frac{1-b}{1-a}} + 2C_{1-b} - 2C_{\frac{a}{b}} + 2C_a - 2Z(b, 1-a) \\ &+ 2\left[\frac{z - \frac{b-a}{b(1-a)}}{z - \frac{b-a}{1-a}}, z, \frac{1-b}{1-a}\right] + 2\left[\frac{z-1}{z-a}, z, 1-b\right] + 2\left[\frac{z - \frac{b-a}{b(1-a)}}{z - \frac{b-a}{b}}, z, \frac{a}{b}\right] = 0. \end{aligned} \tag{2.2.6}$$

Corollary 2.2.11. *Let $a, b \in F^\times - \{1\}$, $a \neq b, 1-b$. Then*

$$2\left[\frac{z-1}{z-a}, z, 1-b\right] = 2\left[\frac{z-1}{z-(1-b)}, z, a\right] \quad \text{in } C^2(F,3)/\partial C^2(F,4).$$

Proof. Compute $2V'(a,b) - 2V'(1-b, 1-a)$. $\square$

Now we come to an inversion formula, which will be valuable afterwards.

Proposition 2.2.12. *For $c = a/b \in F^\times$ such that $a, b \neq 1, a \neq b, 1-b, \frac{b}{b-1}$, the following inversion relation holds in $C^2(F,3)/\partial C^2(F,4)$:*

$$\begin{aligned} 2\left(C_c + C_{\frac{1}{c}} - 2C_1\right) &= Z(a, 1-a) + Z(b, 1-b) + Z\left(\frac{1}{a}, 1 - \frac{1}{a}\right) + Z\left(\frac{1}{b}, 1 - \frac{1}{b}\right) \\ &- Z\left(\frac{1}{b}, \frac{1}{1-a}\right) - Z\left(b, \frac{a}{a-1}\right) - Z\left(\frac{1}{a}, \frac{1}{1-b}\right) - Z\left(a, \frac{b}{b-1}\right) \\ &- \left[\frac{z - \frac{a-b}{1-b}}{z - \frac{a-b}{a(1-b)}}, z, \frac{b(1-a)}{a(1-b)}\right] - \left[\frac{z-1}{z-(1-a)}, z, a\right] - \left[\frac{z - \frac{a-b}{1-b}}{z - \frac{b-a}{b}}, z, \frac{a}{b}\right] + \left[\frac{z-1}{z-b}, z, 1-b\right] \end{aligned} \tag{2.2.7}$$

plus the additional monodromy terms

$$\begin{aligned} &- \left[\frac{z - \frac{a-b}{a(1-b)}}{z - \frac{a-b}{1-b}}, z, \frac{1-a}{1-b}\right] - \left[\frac{z-1}{z-(1-\frac{1}{a})}, z, \frac{1}{a}\right] - \left[\frac{z - \frac{a-b}{a(1-b)}}{z - \frac{a-b}{a}}, z, \frac{b}{a}\right] + \left[\frac{z-1}{z-\frac{1}{b}}, z, 1 - \frac{1}{b}\right] \\ &- \left[\frac{z - \frac{b-a}{1-a}}{z - \frac{b-a}{b(1-a)}}, z, \frac{a(b-1)}{b(a-1)}\right] - \left[\frac{z-1}{z-(1-b)}, z, b\right] - \left[\frac{z - \frac{a-b}{1-a}}{z - \frac{a-b}{a}}, z, \frac{b}{a}\right] + \left[\frac{z-1}{z-a}, z, 1-a\right] \\ &- \left[\frac{z - \frac{b-a}{b(1-a)}}{z - \frac{b-a}{1-a}}, z, \frac{1-b}{1-a}\right] - \left[\frac{z-1}{z-(1-\frac{1}{b})}, z, \frac{1}{b}\right] - \left[\frac{z - \frac{b-a}{b(1-a)}}{z - \frac{b-a}{b}}, z, \frac{a}{b}\right] + \left[\frac{z-1}{z-\frac{1}{a}}, z, 1 - \frac{1}{a}\right]. \end{aligned}$$

Proof. This is proven just as in the proof of [GMS99, Thm. 2.4]. One has only to collect all extra terms. $\square$

If we use our results modulo 2-torsion, then we can improve the result:

Proposition 2.2.13. *For $c = a/b \in F^\times$ such that $a, b \neq 1, a \notin \{b, 1-b, \frac{b}{b-1}\}$, the following inversion relation holds in the quotient $C^2(F,3)/\partial C^2(F,4)$:*

$$
\begin{aligned}
0 &= 4\left(C_c + C_{\frac{1}{c}} - 2C_1\right) - 2Z(b, 1-a) - 2Z(a, 1-b) \\
&- 2Z\left(\frac{1}{b}, 1-\frac{1}{a}\right) - 2Z\left(\frac{1}{a}, 1-\frac{1}{b}\right) + 2Z(b, 1-b) + 2Z\left(\frac{1}{b}, 1-\frac{1}{b}\right) \\
&+ 2\left[\frac{z-\frac{b-a}{b(1-a)}}{z-\frac{b-a}{1-a}}, z, \frac{1-b}{1-a}\right] + 2\left[\frac{z-1}{z-a}, z, 1-b\right] + 2\left[\frac{z-\frac{b-a}{b(1-a)}}{z-\frac{b-a}{b}}, z, \frac{a}{b}\right] \\
&+ 2\left[\frac{z-\frac{a-b}{a(1-b)}}{z-\frac{a-b}{1-b}}, z, \frac{1-a}{1-b}\right] + 2\left[\frac{z-1}{z-b}, z, 1-a\right] + 2\left[\frac{z-\frac{a-b}{a(1-b)}}{z-\frac{a-b}{a}}, z, \frac{b}{a}\right] \\
&+ 2\left[\frac{z-\frac{a-b}{a-1}}{z-\frac{a-b}{b(a-1)}}, z, \frac{a(1-b)}{b(1-a)}\right] + 2\left[\frac{z-1}{z-\frac{1}{a}}, z, 1-\frac{1}{b}\right] + 2\left[\frac{z-\frac{a-b}{a-1}}{z-\frac{a-b}{a}}, z, \frac{b}{a}\right] \\
&+ 2\left[\frac{z-\frac{b-a}{b-1}}{z-\frac{b-a}{a(b-1)}}, z, \frac{b(1-a)}{a(1-b)}\right] + 2\left[\frac{z-1}{z-\frac{1}{b}}, z, 1-\frac{1}{a}\right] + 2\left[\frac{z-\frac{b-a}{b-1}}{z-\frac{b-a}{b}}, z, \frac{a}{b}\right].
\end{aligned}
\tag{2.2.8}
$$

Proof. Compute $2V'(a,b) + 2V'(\frac{1}{a}, \frac{1}{b})$ and subtract $(2C_b + 2C_{1-b} - 2C_1 - Z(b, 1-b)) + (2C_{\frac{1}{b}} + 2C_{1-\frac{1}{b}} - 2C_1 - 2Z(\frac{1}{b}, 1-\frac{1}{b})$ and then add the corresponding expression with a and b interchanged. Finally use (2.1.1) when needed. □

Remark 2.2.14. Using corollary 2.2.11, this formula can still be improved noting that in $C^2(F,3)/\partial C^2(F,4)$ we have

$$\left[\frac{z-1}{z-\frac{1}{a}}, z, 1-\frac{1}{b}\right] = \left[\frac{z-1}{z-a}, z, 1-b\right] - \left[\frac{z-1}{z-a}, z, -b\right]$$

for every allowed combination of a, b. This cancels several of the extra terms in the middle columns but gives new ones. So we do not pursue this here.

It is also possible to use the multiplicativity of the Z-terms (2.1.1) in order to have one Z-term, $Z(ab, ab)$, only at the end resembling the functional equation of the dilogarithm, but this again produces several terms with a constant in the right coordinate. Therefore, we shall not make this explicit. ◊

2.3 Application to number fields

In this section, we use the relations from the preceding section in combination with the Abel–Jacobi map introduced earlier to find explicit generators for the Chow groups of some number fields. The strategy is the following:

According to the general philosophy of cyclotomic elements generating motivic cohomology groups, we shall try to use the relations from the last section to find (cyclotomic) elements in the Chow group which are n-torsion if the Chow group has a summand of order n. With the help of the Abel – Jacobi map, in particular with its injectivity, we can test if the order of a given cycle is at least n. The combination of both results assures that the order of the cycle in question is exactly n so that it is a generator of a summand of order n of the Chow group.

Note that by Bloch's result presented in example 1.2.14, we already know cycles generating the free part of the codimension two Chow groups of cyclotomic fields. The interesting task is to find generators of the torsion part.

Luckily, the orders of the algebraic K_3 of local and global fields are well-known abstractly. Let us recall some key results from [Wei05]:

Theorem 2.3.1. *[Wei05, Thm. 0.1] Let F be a number field with r_1 real and r_2 conjugate pairs of complex embeddings, and let $\mathcal{O}_S$ be the ring of S-integers for some multiplicative subset S of prime ideals in the ring of integers $\mathcal{O}_F$ of F. Then $K_3(\mathcal{O}_S) \cong K_3(F)$. Further,*

$$K_3(F) \cong \begin{cases} \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_2(F)\mathbb{Z}, & F \textit{ is totally imaginary,} \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/2w_2(F)\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{r_1-1}, & F \textit{ has a real embedding,} \end{cases}$$

where the integer $w_2(F)$ is defined as follows: Let $\overline{F}$ be a separable closure of F and $\mathcal{G} = Gal(\overline{F}/F)$ be the absolute Galois group. The abelian group μ of all roots of unity in $\overline{F}$ is known to be a $\mathcal{G}$-module with the action

$$\mathcal{G} \times \mu \to \mu, (g, \zeta) \mapsto g(\zeta).$$

We write $\mu(2)$ for the abelian group μ made into a $\mathcal{G}$-module by letting $g \in \mathcal{G}$ act on μ as $\zeta \mapsto g^2(\zeta)$. If F is a global or local field, then it is proved in [Wei05, cor. 2.3.1] that the group $\mu(2)^{\mathcal{G}}$ on invariants is a finite group with its order denoted by $w_2(F)$.

This quantity can be computed explicitly using the results cited in [Wei05]. First of all, according to proposition 2.3 in loc. cit., there is a decomposition:

$$w_2(F) = \prod_{\ell \text{ prime}} w_2^{(\ell)}(F), \qquad w_2^{(\ell)}(F) := \max\{\ell^\nu | Gal(F(\mu_{\ell^\nu})/F) \text{ has exponent } 2\},$$

with $w_2^{(\ell)}(F) := \ell^\infty$ in case there is no maximum. According to corollary 2.3.1, loc. cit., for a global or local field F, the numbers $w_2(F)$ are finite for all i. Thus one only needs to determine the factors $w_2^{(\ell)}(F)$ which are not equal to 1 to determine the whole number $w_2(F)$. This can be done with the help of the following results. Recall that a field F is called non-exceptional if $Gal(F(\mu_{2^\nu})/F)$ is cyclic for every ν and exceptional otherwise.

Proposition 2.3.2. *[Wei05, prop. 2.7] Fix a prime $\ell \neq 2$, and let F be a field of characteristic $\neq \ell$. Let s be maximal such that $F(\mu_\ell)$ contains a primitive ℓ^sth root of unity. Then if $r = [F(\mu_\ell) : F]$ and $t = \log_\ell(2)$, the numbers $w_2^{(\ell)} := w_2^{(\ell)}(F)$ are:*

1. *if* $\mu_\ell \in F$ *then* $w_2^{(\ell)} = \ell^{s+t}$,

2. *if* $\mu_\ell \notin F$ *and* $2 \equiv 0\,(r)$ *then* $w_2^{(\ell)} = \ell^{s+t}$,

3. *if* $\mu_\ell \notin F$ *and* $2 \not\equiv 0\,(r)$ *then* $w_2^{(\ell)} = 1$.

Proposition 2.3.3. *[Wei05, prop. 2.8] Let F be a field of characteristic $\neq 2$. Let s be maximal such that $F(\sqrt{-1})$ contains a primitive 2^ath root of unity. Then the 2-primary numbers $w_2^{(2)} := w_2^{(2)}(F)$ are:*

1. *if* $\sqrt{-1} \in F$ *then* $w_2^{(2)} = 2^{s+1}$,

2. *if* $\sqrt{-1} \notin F$, *then* $w_2^{(2)} = \begin{cases} 2^{s+1}, & F \text{ is exceptional,} \\ 2^s, & F \text{ is non-exceptional.} \end{cases}$

As remarked in [Wei05, p.7], the integer $w_2(F)$ is always divisible by 24. Real quadratic number fields are especially easy: One can show for every square-free integer $D > 0$ that $w_2(\mathbb{Q}(\sqrt{D})) = 24k$ with $k = D$ in case $D \in \{2, 5\}$ and $k = 1$ else. This is why we chose these two extraordinary number fields among our explicit examples. The result is an easy consequence of reinterpreting $w_2(F)$ as $w_2(F) = 2\prod_{\ell \text{ prime}} \ell^{n(\ell)}$, where $n(\ell)$ is equal to the maximal non-negative integer such that F contains $\mathbb{Q}(\zeta_{\ell^n} + \bar{\zeta}_{\ell^n})$.

In Weibel's article it is remarked that Bass and Tate have proved the isomorphism

$$K_3^M(F) \cong (\mathbb{Z}/2\mathbb{Z})^{r_2},$$

and by results of Merkurjev and Suslin one knows that this group injects into the Quillen K–group so that one can in principle compute the indecomposable K_3 of a given number field abstractly. We are now going to determine explicit generators of the K–groups, i. e. of higher Chow groups of some number fields, especially for some of those, whose torsion part is not of order 24; this is already covered by the first example, which we are going to discuss next.

2.3.1 $CH^2(\mathbb{Q}, 3)$

Let us start with an easy example. By the work of Lee and Szczarba [LS76] and Suslin [Sus86] we know that $CH^2(\mathbb{Q}, 3) \cong K_3^{ind}(\mathbb{Q}) \cong (\mathbb{Z}/48\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/24\mathbb{Z}$.

There are the cyclotomic elements C_1 and C_{-1} in the Chow group. We already know from the distribution relation for $n = 2$ that $2C_1 = -4C_{-1} \in CH^2(\mathbb{Q}, 3)$. Now we use the inversion relation to obtain a second relation between C_1 and C_{-1}. For this we specialize by setting $b = -a$. Noting that for each five-term relation the last two extra terms are 2-torsion, we use twice (2.2.7) and forget these terms at once:

$$\begin{aligned} 8C_{-1} - 8C_1 = {} & 2Z\Big(-a, \frac{a}{a-1}\Big) + 2Z\Big(-\frac{1}{a}, \frac{1}{1-a}\Big) + 2Z\Big(\frac{1}{a}, \frac{1}{1+a}\Big) + 2Z\Big(a, \frac{a}{1+a}\Big) \\ & + 2Z\Big(-\frac{1}{a}, 1+\frac{1}{a}\Big) + 2Z\Big(\frac{1}{a}, 1-\frac{1}{a}\Big) + 2Z(-a, 1+a) + 2Z(a, 1-a) \\ & - 2\left[\frac{z-\frac{2a}{1+a}}{z-\frac{2}{1+a}}, z, \frac{1-a}{1+a}\right] - 2\left[\frac{z-\frac{2}{1+a}}{z-\frac{2a}{1+a}}, z, \frac{1-a}{1+a}\right] \\ & - 2\left[\frac{z-\frac{2a}{a-1}}{z-\frac{2}{1-a}}, z, \frac{1+a}{1-a}\right] - 2\left[\frac{z-\frac{2}{1-a}}{z-\frac{2a}{a-1}}, z, \frac{1+a}{1-a}\right]. \end{aligned} \tag{2.3.1}$$

Since the admissible cycle $\left[\frac{z-1}{z-\frac{x-a}{x-b}}, z, x, c\right] \in C^2(F,4)$ bounds to

$$\left[\frac{x-a}{x-b}, x, c\right] + \left[\frac{x-b}{x-a}, x, c\right] - \left[\frac{x-1}{x-\frac{a}{b}}, x, c\right] = 0 \in C^2(F,3)/\partial C^2(F,4),$$

we can further simplify:

$$\begin{aligned} 8C_{-1} - 8C_1 = {} & 2Z\Big(-a, \frac{a}{a-1}\Big) + 2Z\Big(-\frac{1}{a}, \frac{1}{1-a}\Big) + 2Z\Big(\frac{1}{a}, \frac{1}{1+a}\Big) + 2Z\Big(a, \frac{a}{1+a}\Big) \\ & + 2Z\Big(-\frac{1}{a}, 1+\frac{1}{a}\Big) + 2Z\Big(\frac{1}{a}, 1-\frac{1}{a}\Big) + 2Z(-a, 1+a) + 2Z(a, 1-a) \\ & + 2\left[\frac{z-1}{z-a}, z, 1+a\right] - 2\left[\frac{z-1}{z+a}, z, 1-a\right]. \end{aligned} \tag{2.3.2}$$

Unfortunately, we do not have any information about the orders of the terms on the right-hand side in general. But we have freedom of choice for the parameter a. If we choose the constants of the extra terms on the right hand side resp. one of the constants in the Z–terms to be a root of unity, we can control the order of the extra terms. In order to be able to choose suitable constants, the following theorem due to Levine helps:

Proposition 2.3.4 ([Lev89], cor. 4.6)**.** *Let E be an arbitrary field and F an extension of E. Then the map $K_3^{ind}(E) \to K_3^{ind}(F)$ induced by the inclusion $E \hookrightarrow F$ is injective.*

Since we know that $K_3^{ind}(E) \cong CH^2(E,3)$, this result can be applied in the following way: Returning to equation (2.2.8), we specialize further by setting $a = i := \sqrt{-1}$ to deduce a second relation between C_1 and C_{-1}. Combining this relation with $2C_1 = -4C_{-1}$ from the distribution relation, we obtain a relation of the form $nC_1 = 0 \in C^2(F,3)/\partial C^2(F,4)$ for some n. This n is an upper bound of the order of $C_1 \in CH^2(\mathbb{Q}(i),3)$. This bound cannot be lower in $CH^2(\mathbb{Q},3)$ by the proposition above.

$$\begin{aligned}8C_{-1}-8C_1 = 2Z\Big(i,\frac{1+i}{2}\Big)+2Z\Big(-i,\frac{1-i}{2}\Big)+2Z(-i,1+i)+2Z(i,1-i)\\ +2\left[\frac{z-1}{z-i},z,1+i\right]-2\left[\frac{z-1}{z+i},z,1-i\right].\end{aligned}$$

Since terms of the form $[\bullet,\bullet,\zeta]$ for a primitive n-th root of unity are n-torsion, and since $2\left[\frac{z-1}{z-i},z,1+i\right]-2\left[\frac{z-1}{z+i},z,1-i\right]=2\left[\frac{z-1}{z-i},z,\frac{1+i}{1-i}\right]$, the following terms survive multiplication by 2:

$$16C_{-1}-16C_1 = 4Z\Big(i,\frac{1+i}{2}\Big)+4Z\Big(-i,\frac{1-i}{2}\Big)+4Z(-i,1+i)+4Z(i,1-i).$$

Now we make use of (2.1.1) several times to obtain:

$$16C_{-1}-16C_1 = 4Z(i,1)+4Z(-i,1)=0.$$

So if we combine Levine's result with the relation $2C_1=-4C_{-1}$ coming from the distribution relation, we immediately see that

$$24C_1 = 48C_{-1} = 0 \in CH^2(\mathbb{Q}(i),3),$$

and because of the injectivity proved in Levine's theorem, it is clear that C_1 must be 24-torsion in $CH^2(\mathbb{Q},3)$ as well. Computing the image of C_1 under the Abel – Jacobi map from equation (1.3.2) to be $Li_2(1)=\frac{\pi^2}{6}$, which is an element of order 24 in $H^1_{\mathcal{D}}(\mathsf{Spec}(\mathbb{Q}),\mathbb{Z}(2))=\mathbb{C}/(4\pi^2\mathbb{Z})$. Since we also know that the order of the Chow group is exactly 24, this makes C_1 a generator of $CH^2(\mathbb{Q},3)$. Thus we have proved:

Proposition 2.3.5. *The group $CH^2(\mathbb{Q},3)\cong\mathbb{Z}/24\mathbb{Z}$ is generated by the cycle $C_1\in CH^2(\mathbb{Q},3)$.*

2.3.2 $CH^2(\mathbb{Q}(i),3)$

Proposition 2.3.6. *The torsion part of the group $CH^2(\mathbb{Q}(i),3)\cong\mathbb{Z}\oplus\mathbb{Z}/24\mathbb{Z}$ (cf. [Wei05]) is generated by the cycle $C_1\in CH^2(\mathbb{Q}(i),3)$ The free part is generated by $4C_i$ or equivalently by $4C_{-i}$.*

Proof. The first part is clear by what we have seen in the last paragraph and the second assertion follows from the fact that $\partial(C_i)=(1-i,i)$ so that not C_i but $4C_i\in CH^2(\mathbb{Q},3)$ and further that the image of $4C_i$ under the Abel – Jacobi map has a non-vanishing imaginary part indicating that $4C_i$ is non-torsion in the Chow group. The same reasoning applies to C_{-i}. □

Remark 2.3.7. The group $CH^2(\mathbb{Q}(\zeta_3),3)\cong\mathbb{Z}^2\oplus\mathbb{Z}/24\mathbb{Z}$ is treated analogously: By Levine's result C_1 can be chosen as generator of the torsion part of $CH^2(\mathbb{Q}(\zeta_3),3)$.

The free part of the Chow group is generated by $3C_{\zeta_3}$ respectively $3C_{\zeta_3^2}$ as can be seen by computing the Abel – Jacobi map again. $\diamondsuit$

Remark 2.3.8. Consulting Weibel's article again, especially propositions 2.7 and 2.8, one can compute the numbers $w_2(F)$ for number fields rather easily and one can also see that the number very often is equal to 24. The theorem of Levine 2.3.4 combined with the Abel – Jacobi map of Kerr, Lewis and Müller-Stach implies that whenever the torsion part of $CH^2(F,3)$ for a number field F has order 24, it is generated by C_1. $\diamondsuit$

2.3.3 $CH^2(\mathbb{Q}(\zeta_5), 3)$

A number field with a more interesting Chow group is treated in this example: We are looking for a generator of the torsion part of the second higher Chow group with order equal to 120. We use the distribution relation for the fifth roots of unity:

$$5C_1 = 25C_1 + 25C_{\zeta_5} + 25C_{\zeta_5^2} + 25C_{\bar{\zeta}_5^2} + 25C_{\bar{\zeta}_5}$$

and using that $24C_1 = 0$

$$-150(C_{\zeta_5} + C_{\bar{\zeta}_5})) = 150(C_{\zeta_5^2} + C_{\bar{\zeta}_5^2})).$$

Now we use the inversion relation (2.2.7) with $a = \zeta_5^3, b = \zeta_5^2$: We can already simplify the relation a little: Terms with a fifth root of unity in the right coordinate are 5-torsion, while terms of the form $\left[\frac{z-1}{z-a}, z, 1-a\right]$ are 2-torsion. Note also that $(\frac{1-\zeta_5^3}{1-\zeta_5^2})^{10} = 1$. So we multiply the entire relation by 10 and get

$$\begin{aligned}20C_{\zeta_5} + 20C_{\bar{\zeta}_5} - 16C_1 = 10\,\big(Z(\zeta_5^3, 1-\zeta_5^3) + Z(\zeta_5^2, 1-\zeta_5^2) + Z(\bar{\zeta}_5^3, 1-\bar{\zeta}_5^3)\\ +Z(\bar{\zeta}_5^2, 1-\bar{\zeta}_5^2) - Z\Big(\bar{\zeta}_5^2, \frac{1}{1-\zeta_5^3}\Big) - Z\Big(\zeta_5^2, \frac{\zeta_5^3}{\zeta_5^3-1}\Big) - Z\Big(\bar{\zeta}_5^3, \frac{1}{1-\zeta_5^2}\Big) - Z\Big(\zeta_5^3, \frac{\zeta_5^2}{\zeta_5^2-1}\Big)\Big).\end{aligned}$$

Remark 2.3.9. Note that we have already settled the admissibility of $Z(a,b)$ for arbitrary $a, b \in F^\times$. So one does not need to care when specializing to some primitive roots of unity. $\diamondsuit$

Now one applies equation (2.1.1) several times and keeps in mind that terms with an n^{th} root of unity in the rightmost coordinate are n-torsion:

$$\begin{aligned}20C_{\zeta_5} + 20C_{\bar{\zeta}_5} - 16C_1 = 10Z\Big(\bar{\zeta}_5^2, \frac{1-\zeta_5^3}{1+\zeta_5^3}\Big) + 10Z\Big(\zeta_5^2, \frac{1+\zeta_5^2}{1-\zeta_5^2}\Big)\\ + 10Z\Big(\bar{\zeta}_5^3, \frac{1-\zeta_5^2}{1+\zeta_5^2}\Big) + 10Z\Big(\zeta_5^3, \frac{1+\zeta_5^3}{1-\zeta_5^3}\Big).\end{aligned}$$

Let us use (2.1.1) again to see that $Z\Big(\bar\zeta_5^3, \frac{1-\zeta_5^2}{1+\zeta_5^2}\Big) = Z\Big(\zeta_5^2, \frac{1-\zeta_5^2}{1+\zeta_5^2}\Big) + Z\Big(-1, \frac{1-\zeta_5^2}{1+\zeta_5^2}\Big)$ and analogously for $\bar\zeta_5^2$ in the first coordinate:

$$\begin{aligned} &= -10Z\Big(-1, \frac{1-\zeta_5^3}{1+\zeta_5^3}\Big) - 10Z\Big(-1, \frac{1-\zeta_5^2}{1+\zeta_5^2}\Big) \\ &= -10Z\Big(-1, \frac{(1-\zeta_5^3)(1+\bar\zeta_5^3)}{(1-\bar\zeta_5^3)(1+\zeta_5^3)}\Big) \\ &= -10Z(-1,1) = 0. \end{aligned}$$

Putting this together, we conclude that $60(C_{\zeta_5} + C_{\bar\zeta_5}) = 0 \in C^2(F,3)/\partial C^2(F,4)$ so that in the end

$$-300(C_{\zeta_5} + 60C_{\bar\zeta_5}) = 300(C_{\zeta_5^2} + 60C_{\bar\zeta_5^2}) = 0.$$

Remembering that $C_{\zeta_5} \notin CH^2(\mathbb{Q}(\zeta_5),3)$, but $5C_{\zeta_5} \in CH^2(\mathbb{Q}(\zeta_5),3)$, we can deduce that the higher Chow cycle $5(C_{\zeta_5} + C_{\bar\zeta_5})$ or equivalently $5(C_{\zeta_5^2} + C_{\bar\zeta_5^2})$ is 60-torsion.

Using the Abel – Jacobi map (1.3.2) we compute the image of this cycle in Deligne – cohomology $H^1_{\mathcal{D}}(\mathsf{Spec}(\mathbb{Q}(\zeta_5)), \mathbb{Z}(2)) \cong \mathbb{C}/4\pi^2\mathbb{Z}$ to be $\pi^2/15$ so that the order of this cycle is exactly 60. Remembering that $C_1 \in CH^2(\mathbb{Q}(\zeta_5),3)$ is of order 24, we deduce analogously to the former examples:

Proposition 2.3.10. *The torsion part of the group* $CH^2(\mathbb{Q}(\zeta_5),3) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/120\mathbb{Z}$ *is generated by* $C_1 + 5(C_{\zeta_5} + C_{\bar\zeta_5}) \in CH^2(\mathbb{Q}(\zeta_5),3)$*. The free part is generated by* $5C_{\zeta_5}$ *and* $5C_{\zeta_5^2}$*.*

Remark 2.3.11. The generators of the free part of the Chow group are not unique in the following sense: Note that the inversion relation implies that Totaro cycles with conjugate pairs of roots of unity as their arguments are equal modulo torsion. Therefore we can also choose $5C_{\zeta_5^3}$ and $5C_{\zeta_5^4}$ as generators. Only the independence of the roots of unity is important. ◊

Remark 2.3.12. By standard class field theory one knows that $\mathbb{Q}(\sqrt{5}) \hookrightarrow \mathbb{Q}(\zeta_5)$ is the maximal real subfield. One may ask for a generator of the higher Chow group of the former field coming from a generator of the higher Chow group of the latter field.

Unfortunately, there seems to be no canonical way of constructing such a generator. One knows by the theorem of Levine or the formal property of the higher Chow groups to allow Galois descent that taking Galois conjugates of the generators of $CH^2(\mathbb{Q}(\zeta_5),3)$ and pushing them forward to $CH^2(\mathbb{Q}(\sqrt{5}),3)$ via the canonical inclusion $\mathbb{Q}(\sqrt{5}) \hookrightarrow \mathbb{Q}(\zeta_5)$ that there is a linear combination of cycles in $CH^2(\mathbb{Q}(\sqrt{5}),3)$ generating this group equivalent to the ones that come descent from $CH^2(\mathbb{Q}(\zeta_5),3)$, but there is no obvious way of constructing them.

For example, one can make use of the Abel – Jacobi map once again to check that the image of the cycle

$$(C_{\frac{1}{2}(\sqrt{5}-1)} - C_{\frac{1}{4}(\sqrt{5}-1)^2}) \in CH^2(\mathbb{Q}(\sqrt{5}),3)$$

in Deligne – Beĭlinson cohomology is equal to $\pi^2/30$ showing that the order of this cycle is at least 120. In accordance with the result above on the cyclotomic field containing this quadratic field, its order must be equal to 120 turning it into a generator of $CH^2(\mathbb{Q}(\sqrt{5}),3)$. However, without the aid of the relations among special values of the dilogarithm from [Lew82] and the regulator map of [KLMS06] to determine the order of explicit elements in the Chow group, there would be no way of detecting cycles of this kind. ◊

2.3.4 $CH^2(\mathbb{Q}(\zeta_8), 3)$

As a last example we consider another cyclotomic field, namely $\mathbb{Q}(\zeta_8)$, containing three quadratic subfields. In order to find a generator of its Chow group, we shall start with a distribution relation again:

$$8C_1 = 64\left(C_{\zeta_8} + C_i + C_{\zeta_8^3} + C_{-1} + C_{\zeta_8^5} + C_{-i} + C_{\zeta_8^7} + C_1\right).$$

Using the distribution relation for the fourth roots of unity, i. e.

$$4C_1 = 16(C_i + C_{-1} + C_{-i} + C_1),$$

we deduce – using $24C_1 = 0$ again:

$$\begin{aligned} -8C_1 &= 64\left(C_{\zeta_8} + C_{\zeta_8^3} + C_{\zeta_8^5} + C_{\zeta_8^7}\right) \\ -192(C_{\zeta_8} + C_{\zeta_8^7}) &= 192(C_{\zeta_8^3} + C_{\zeta_8^5}). \end{aligned}$$

Let us now try to relate the terms in brackets using the inversion relation and some auxiliary relations between terms with a constant on the right. Note in particular that we already multiplied by 8 in order to kill torsion terms:

$$\begin{aligned} 16C_{\zeta_8} + 16C_{\zeta_8^7} - 8C_1 &= 8Z(\zeta_8^n, 1-\zeta_8^n) + 8Z\big(\zeta_8^{n-1}, 1-\zeta_8^{n-1}\big) + 8Z\big(\bar\zeta_8^n, 1-\bar\zeta_8^n\big) \\ &+ 8Z\big(\bar\zeta_8^{n-1}, 1-\bar\zeta_8^{n-1}\big) - 8Z\left(\bar\zeta_8^{n-1}, \frac{1}{1-\zeta_8^n}\right) - 8Z\left(\zeta_8^{n-1}, \frac{1}{1-\bar\zeta_8^n}\right) - 8Z\left(\bar\zeta_8^n, \frac{1}{1-\zeta_8^{n-1}}\right) \\ &- 8Z\left(\zeta_8^n, \frac{1}{1-\bar\zeta_8^{n-1}}\right) - 8\left[\frac{z-1}{z-\zeta_8^n}, z, 1-\zeta_8^{n-1}\right] - 8\left[\frac{z-1}{z-\zeta_8^{n-1}}, z, 1-\zeta_8^n\right]. \end{aligned}$$

Now let us simplify the terms above:

$$\begin{aligned} 16C_{\zeta_8}+16C_{\zeta_8^7} - 8C_1 &= -8Z\big(\zeta_8^n, (1-\zeta_8^n)(1-\bar\zeta_8^{n-1})\big) - 8Z\big(\bar\zeta_8^n, (1-\bar\zeta_8^n)(1-\zeta_8^{n-1})\big) \\ &- 8Z\big(\zeta_8^{n-1}, (1-\zeta_8^n)(1-\bar\zeta_8^{n-1})\big) - 8Z\big(\bar\zeta_8^{n-1}, (1-\bar\zeta_8^n)(1-\zeta_8^{n-1})\big) \\ &- 8\left[\frac{z-1}{z-\zeta_8^n}, z, 1-\zeta_8^{n-1}\right] - 8\left[\frac{z-1}{z-\zeta_8^{n-1}}, z, 1-\zeta_8^n\right] \\ &= -8Z\big(\zeta_8^n, (1-\zeta_8^n)(1-\bar\zeta_8^{n-1})\big) - 8Z\big(\bar\zeta_8^n, (1-\zeta_8^n)(1-\bar\zeta_8^{n-1})\big) - 8Z\big(\zeta_8^n, \bar\zeta_8\big) \\ &- 8Z\big(\zeta_8^{n-1}, (1-\zeta_8^n)(1-\bar\zeta_8^{n-1})\big) - 8Z\big(\bar\zeta_8^{n-1}, (1-\zeta_8^n)(1-\bar\zeta_8^{n-1})\big) \\ &- 8Z(\bar\zeta_8^{n-1}, \bar\zeta_8) - 8\left[\frac{z-1}{z-\zeta_8^n}, z, 1-\zeta_8^{n-1}\right] - 8\left[\frac{z-1}{z-\zeta_8^{n-1}}, z, 1-\zeta_8^n\right] \end{aligned}$$

and further

$$= -8Z(\zeta_8^n, \bar{\zeta}_8) - 8Z(\bar{\zeta}_8^{n-1}, \bar{\zeta}_8).$$

The last step follows from the fact that

$$-8Z(\zeta_8^n, (1-\zeta_8^n)(1-\bar{\zeta}_8^{n-1})) - 8Z(\bar{\zeta}_8^n, (1-\zeta_8^n)(1-\bar{\zeta}_8^{n-1})) \\ = 8\left[\frac{z-1}{z-\zeta_8^n}, z, (1-\zeta_8^n)(1-\bar{\zeta}_8^{n-1})\right] = 8\left[\frac{z-1}{z-\zeta_8^n}, z, 1-\bar{\zeta}_8^{n-1}\right]$$

and analogously for the other term. Now it is easy to see that

$$0 = 8Z(\zeta_8^n, 1) = 64Z(\zeta_8^n, \zeta_8^m)$$

since extra terms are 8-torsion. So we conclude

$$128C_{\zeta_8} + 128C_{\bar{\zeta}_8} + 16C_1 = 0$$

or in other words

$$394(C_{\zeta_8} + C_{\bar{\zeta}_8}) = 0.$$

Again, not C_{ζ_8}, but $8C_{\zeta_8} \in CH^2(\mathbb{Q}(\zeta_8), 3)$ so that the cycle $8(C_{\zeta_8} + C_{\bar{\zeta}_8}) = 8(C_{\zeta_8^3} + C_{\bar{\zeta}_8^3})$ is 48-torsion.

Again invoking a regulator argument as in the above case, we calculate the image of this cycle in $H^1_{\mathcal{D}}(\mathsf{Spec}(\mathbb{Q}(\zeta_5)), \mathbb{Z}(2)) \cong \mathbb{C}/4\pi^2\mathbb{Z}$ to be $\pi^2/12$ so that the order of this cycle is exactly 48.

Proposition 2.3.13. *The torsion part of the group $CH^2(\mathbb{Q}(\zeta_8), 3) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/48\mathbb{Z}$ is generated by the elements $\{8(C_{\zeta_8} + C_{\bar{\zeta}_8})\} \in CH^2(\mathbb{Q}(\zeta_5), 3)$.*

Remark 2.3.14. One knows that $\mathbb{Q}(\zeta_8)$ contains three quadratic subfields, namely $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-2})$, and $\mathbb{Q}(i)$. It is easy to see that $4(C_i + C_{-i}) \in CH^2(\mathbb{Q}(\zeta_8), 3)$ already lives in $CH^2(\mathbb{Q}(i), 3)$ and also generates this group. Unfortunately, it is far more complicated to find generators of the other two quadratic subfields.

The idea of constructing generators is the following: One considers $Gal(\mathbb{Q}(\zeta_8) \,|\, \mathbb{Q}(\sqrt{\pm 2}))$ and take Galois conjugates of the generators of the higher Chow group of the cyclotomic field. A descent argument as above ensures that there is a cycle in the quadratic subfield generating its higher Chow group. In practice, this is not constructive. But with the help of [Lew82] again, one finds that

$$2C_{(\sqrt{2}-1)^4} - 12C_{(\sqrt{2}-1)^2} + 8C_{\sqrt{2}-1} = C_1.$$

Indeed, half of the left hand side is a cycle in $CH^2(\mathbb{Q}(\sqrt{2}), 3)$ having order 48, i. e. a generator of the Chow group.

A clever linear combination of terms evaluated at algebraic arguments certainly gives a generator of $CH^2(\mathbb{Q}(\sqrt{-2}), 3)$. But this is not the intention of this section. ◊

Remark 2.3.15. The above results can be explained in the following way: As one knows from [Bor77] and [Wei05], the group $K_3(F)$ for a number field F with r_1 real and r_2 pairs of conjugate complex places is given by $K_3^{ind}(F) \cong \mathbb{Z}^{r_2} \oplus T$, where T is a torsion group. It is also proved in that article that the torsion part of the K–group of a cyclotomic field $\mathbb{Q}(\zeta_p)$, p an odd prime, is the same as the one of the maximal real subfield $\mathbb{Q}(\zeta_p + \overline{\zeta}_p)$. Thus in order to compute generators for the codimension two Chow group of a cyclotomic field, it suffices to compute generators for the maximal real subfield. But for those subfields, which we considered in the examples, there are canonical candidates expected to generate the Chow group. ♢

2.3.5 A remark on $\mathbb{Q}_p$

Since our techniques apply to infinite fields, it is worth also considering p-adic fields. The idea is quite simple. From [Wei05, Sect.5], one knows that if E is a local field with residue field $\mathbb{F}_q$ of characteristic p, then the Milnor K–groups are uncountable, uniquely divisible abelian groups for $n \geq 3$. Further, they are summands of the Quillen K–groups. These in turn can be described rather well, at least abstractly.

Proposition 2.3.16. *[Wei05, Prop. 5.3] If $i > 0$ there is a summand of $K_3(E)$ isomorphic to $K_3(\mathbb{F}_q) \cong \mathbb{Z}/w_2(\mathbb{F}_q)\mathbb{Z}$, where*

$$w_2(\mathbb{F}_q) = \begin{cases} 24, & q = 2, q = 3, \\ q^2 - 1, & else. \end{cases}$$

The complementary summand is uniquely ℓ-divisible for every prime $\ell \neq p$, i. e., a $\mathbb{Z}_{(p)}$-module.

Since we are interested in the indecomposable part of $K_3(E)$, we have to determine the Milnor K-group $K_3^M(E)$. But as remarked in [Wei05, Sect. 5], this is only known to be an uncountable, uniquely divisible abelian group. Thus we see that

$$K_3^{ind}(E) \cong \mathbb{Z}/w_2(E)\mathbb{Z} \oplus M,$$

where M denotes the quotient of a $\mathbb{Z}_{(p)}$-module by the Milnor K-group. Further, $w_2(E)$ is divisible by 24.

Proposition 2.3.17. *Let $F = \mathbb{Q}_p$ for $p = 2, 3, 5$. Then there is a summand in the group $CH^2(F, 3)$ generated by $C_1 \in CH^2(F, 3)$ subject to the usual relations: the distribution relations* (2.2.2)*, the five-term relation* (2.2.4)*, and relation* (2.2.1)*.*

Proof. From the discussion above, the Chow group $CH^2(F, 3)$ of the fields in question contains a finite direct summand of order 24. We know that $C_1 \in CH^2(\mathbb{Q}, 3)$ is of order 24 and by Levine's theorem (proposition 2.3.4) above, its order cannot decrease in any extension. □

For the other p-adic fields, one might speculate that cyclotomic elements comparable to the generators for the Chow groups of cyclotomic fields generate the Chow group. Unfortunately, there is no explicit regulator available in order to check whether certain elements are nontrivial in the Chow group of a p-adic field. The Abel – Jacobi map used in this thesis for number fields might not suffice as p-adic fields cannot be canonically embedded into $\mathbb{C}$. A more natural approach seems to be to embed p-adic fields into $\mathbb{C}_p$ and to prove a new regulator formula in the p-adic setting – perhaps using Coleman integrals. But this is another direction of research.

Therefore, we cannot and will not say anything nontrivial about Chow groups of p-adic fields.

2.4 Detecting the Bloch group

Let us briefly return to the general setup. Our computations so far took place in the complex $Z^2(F,\bullet)/(Z^1(F,1)\otimes Z^1(F,\bullet-1))$. We could have used the complex

$$Z^2(F,\bullet)/(Z^1(F,\bullet-1)\otimes Z^1(F,1))$$

as well, but we already claimed that we could not divide out both subcomplexes at one time because the resulting quotient would not compute $CH^2(F,3)$. Let us shed some more light on this in the present section:

The basic proposition 2.1.4 used for proving relations in the Chow group, has a more simple form in the quotient $Z^2(F,\bullet)/\big(Z^1(F,1)\otimes Z^1(F,\bullet-1)+Z^1(F,\bullet-1)\otimes Z^1(F,1)\big)$:

$$\begin{aligned}
[h_1(x)h_2(x),f(x),g(x)] &= [h_1(x),f(x),g(x)]+[h_2(x),f(x),g(x)]\\
[f(x),h_1(x)h_2(x),g(x)] &= [f(x),h_1(x),g(x)]+[f(x),h_2(x),g(x)]\\
[f(x),g(x),h_1(x)h_2(x)] &= [f(x),g(x),h_1(x)]+[f(x),g(x),h_2(x)]\,.
\end{aligned}\tag{2.4.1}$$

This follows at once since we divide out the terms with a constant on the right hand side. Thus the relations derived in the last paragraph simplify with only the Z-terms surviving apart from the C-terms. Concretely:

Proposition 2.4.1. *Let $a,b,\zeta\in F^\times$ subject to the conditions $\zeta^n=1$, $a,b\neq 0$, $a\neq b,1-b$, then the following relations hold in the quotient*

$$\big(Z^2(F,3)/\big(Z^1(F,1)\otimes Z^1(F,2)+Z^1(F,2)\otimes Z^1(F,1)\big)\big)/\partial Z^2(F,4):$$

$$\begin{aligned}
nC_{a^n}-n^2\sum_{\zeta^n=1}C_{\zeta a} &= 0,\\
C_a+C_{1-a}-C_1 &= Z(a,1-a),\\
C_{\frac{a(1-b)}{b(1-a)}}-C_{\frac{1-b}{1-a}}+C_{\frac{a}{b}}-C_{1-b}+C_a &= Z(1-a,b),
\end{aligned}$$

and

$$2\left(C_{\frac{a}{b}}+C_{\frac{b}{a}}-2C_1\right)=Z(a,1-a)+Z(b,1-b)+Z\left(\frac{1}{a},1-\frac{1}{a}\right)$$
$$+Z\left(\frac{1}{b},1-\frac{1}{b}\right)-Z(b,1-a)-Z\left(\frac{1}{b},1-\frac{1}{a}\right)$$
$$-Z(a,1-b)-Z\left(\frac{1}{a},1-\frac{1}{b}\right).$$

Proof. These relations are proved exactly as in the preceding section but using the above equations (2.4.1) instead. Note that in the last relation, we used the proof of (2.2.6) but starting with the five-term relation (2.2.5). □

Using the basic relations (2.4.1) above, several manipulations of the additional terms in those relations are possible, which allows us to put them into the following form reflecting the functional equations for the dilogarithm even better: We have the variant of (2.1.1):

$$Z(ab,c)=Z(c,ab)=\left[1-\frac{c}{x},ab,1-x\right]=\left[1-\frac{c}{x},a,1-x\right]+\left[1-\frac{c}{x},b,1-x\right]$$
$$=Z(c,a)+Z(c,b)=Z(a,c)+Z(b,c).$$

Using these new facts, the last relation from above can be transformed into the following shape:

$$2\left(C_{\frac{a}{b}}+C_{\frac{b}{a}}-2C_1\right)=Z(ab,ab).$$

We can even go one step further picking up the terms $\tilde{C}_a$ introduced earlier as being equal to $2C_a-Z(a,1-a)$ in the quotient $C^2(F,3)/\partial C^2(F,4)$ or even in

$$\left(Z^2(F,3)/\left(Z^1(F,1)\otimes Z^1(F,2)+Z^1(F,2)\otimes Z^1(F,1)\right)\right)/\partial Z^2(F,4)$$

giving rise to the "homogeneous" relations

$$n\tilde{C}_{a^n}-n^2\sum_{\zeta^n=1}\tilde{C}_{\zeta a}=0,$$
$$\tilde{C}_a+\tilde{C}_{1-a}-\tilde{C}_1=0,$$
$$\tilde{C}_{\frac{a(1-b)}{b(1-a)}}-\tilde{C}_{\frac{1-b}{1-a}}+\tilde{C}_{\frac{a}{b}}-\tilde{C}_{1-b}+\tilde{C}_a=0,$$
$$2\left(\tilde{C}_{\frac{a}{b}}+\tilde{C}_{\frac{b}{a}}-2\tilde{C}_1\right)=0.$$

The proof of these relations consists of repeating the steps of the proofs of [GMS99] once more noting that the Z-terms vanish as well. In the same way as in loc. cit., we can deduce from these relations that $6\tilde{C}_1=-12\tilde{C}_{-1}=0$ in the quotient. But since

$$\tilde{C}_1=\left[1-\frac{1}{x},1-x,x\right]-\left[1-\frac{1}{x},x,1-x\right]=2C_1$$

according to lemma 2.1.13, this is equivalent to $12C_1 = 24C_{-1} = 0$ in the quotient. This gives rise to the following variant of the homogeneous relations in

$$\left(Z^2(F,3)/\left(Z^1(F,1)\otimes Z^1(F,2)+Z^1(F,2)\otimes Z^1(F,1)\right)\right)/\partial Z^2(F,4):$$

$$n\tilde{C}_{a^n} - n^2\sum_{\zeta^n=1}\tilde{C}_{\zeta a} = 0, \tag{2.4.2}$$

$$6\left(\tilde{C}_a + \tilde{C}_{1-a}\right) = 0, \tag{2.4.3}$$

$$\tilde{C}_{\frac{a(1-b)}{b(1-a)}} - \tilde{C}_{\frac{1-b}{1-a}} + \tilde{C}_{\frac{a}{b}} - \tilde{C}_{1-b} + \tilde{C}_a = 0, \tag{2.4.4}$$

$$6\left(\tilde{C}_{\frac{a}{b}} + \tilde{C}_{\frac{b}{a}}\right) = 0. \tag{2.4.5}$$

Thus we can prove the first result of this section which is essentially a variant of [Cat96, Proposition 5]:

Proposition 2.4.2. *Consider the quotient of the free abelian group $\mathcal{P}_F$ of symbols $\tilde{C}_{a_i}$ in*

$$\left(Z^2(F,3)/(Z^1(F,1)\otimes Z^1(F,2)+Z^1(F,2)\otimes Z^1(F,1))\right)/\partial Z^2(F,4)$$

by relation (2.4.4). Then the map

$$\phi : Z^2(F,3)\to \mathbb{Z}[F^\times], \tilde{C}_a \mapsto [a],$$

induces an isomorphism of groups

$$\phi : \mathcal{P}_F\otimes\mathbb{Z}\left[\frac{1}{6}\right]\to B_2(F)\otimes\mathbb{Z}\left[\frac{1}{6}\right].$$

If $F=\overline{F}$, then both groups are integrally isomorphic.

Proof. The proof in [Cat96] can be transferred almost literally: If we allow coefficients in $\mathbb{Z}[\frac{1}{6}]$ in $\mathcal{P}_F$, our variant of the five term relation (2,4,4) used to define $\mathcal{P}_F$ can be transformed into Cathelineau's relation by combining it with the relations (2.4.3) and (2.4.5) Then one just copies the proof from [Cat96]. ☐

Going one step further, we should be able to recover the Tor-group of Suslin's exact sequence (1.2.1) in terms of a certain homology group:

Lemma 2.4.3. *There is a short exact sequence*

$$0\to G\to H_3(Z^2(F,\bullet)/(Z'(F,\bullet)+Z''(F,\bullet)))\to H_3(Z^2(F,\bullet)/Z'(F,\bullet))\to 0,$$

where $G\subset H_2(Z^2(F,\bullet))$ is given by $G:=\{\sum n_{a,b}[a]\otimes_{\mathbb{Z}}[b]\,\big|\, a,b\in\partial Z^1(F,2), n_{a,b}\in\mathbb{Z}\}$.

Proof. Consider the short exact sequence

$$\begin{aligned} 0 \to & Z''(F,\bullet)/\left(Z'(F,\bullet)\cap Z''(F,\bullet)\right) \to Z^2(F,\bullet)/Z'(F,\bullet) \to \\ & \to Z^2(F,\bullet)/\left(Z'(F,\bullet)+Z''(F,\bullet)\right) \to 0 \end{aligned} \tag{2.4.6}$$

inducing the long exact sequence in homology

$$\begin{aligned} \ldots \to H_3\left(\frac{Z'(F,\bullet)}{Z'(F,\bullet)\cap Z''(F,\bullet)}\right) \to CH^2(F,3) \to \\ \to H_3\left(\frac{Z^2(F,\bullet)}{Z'(F,\bullet)+Z''(F,\bullet)}\right) \to H_2\left(\frac{Z'(F,\bullet)}{Z'(F,\bullet)\cap Z''(F,\bullet)}\right) \to \ldots. \end{aligned} \tag{2.4.7}$$

Further, consider the short exact sequence

$$0 \to Z'(F,\bullet)\cap Z''(F,\bullet) \to Z'(F,\bullet) \to Z'(F,\bullet)/Z'(F,\bullet)\cap Z''(F,\bullet) \to 0$$

with associated long exact sequence

$$\begin{aligned} \ldots \to \underbrace{H_3(Z'(F,\bullet))}_{=0} \to & H_3(Z'(F,\bullet)/Z'(F,\bullet)\cap Z''(F,\bullet)) \to \\ & \to H_2(Z'(F,\bullet)\cap Z''(F,\bullet)) \to \underbrace{H_2(Z'(F,\bullet))}_{=0} \to \ldots, \end{aligned}$$

where the vanishing of the homology groups is just the acyclicity result of Nart [Nar89]. Now $Z'(F,\bullet)\cap Z''(F,\bullet)$ in degree two is given by $\partial Z^1(F,2)\otimes\partial Z^1(F,2)$. Obviously, all elements of the intersection have vanishing boundary. Further, none of them is the boundary of an element in degree three, because $Z'(F,3)\cap Z''(F,3)$ consists of terms with a constant on the left and on the right, i. e. which are pull-backs of a coordinate projection, and so vanish by the definition of the complex $Z^2(F,3)$. □

Can one describe the elements of G more explicitly? The condition $[a]\in\partial Z^1(F,2)$ implies that if $a=a_1a_2\in F^\times$, then $[a]-[a_1]-[a_2]=0\in\partial Z^1(F,2)$. Thus we can write elements in $\partial Z^1(F,2)\otimes\partial Z^1(F,2)$ in the form

$$\sum n_{a_1,a_2,b_1,b_2}[a_1a_2]\otimes[b_1b_2], \quad a_1,a_2,b_1,b_2\in\mathbb{Z}[F^\times]$$

with the property

$$\sum n_{a_1,a_2,b_1,b_2}\left([a_1a_2]-[a_1]-[a_2]\right)\otimes\left([b_1b_2]-[b_1]-[b_2]\right)=0.$$

There is a connection to the group $Tor_1^{\mathbb{Z}}(F^\times,F^\times)^\sim$, the unique nontrivial extension of the Tor-group by $\mathbb{Z}/2\mathbb{Z}$. So let us describe this group more explicitly. Choose an injective resolution of $F^\times$ of the form

$$I_\bullet : \mathbb{Z}[F^\times\times F^\times] \to \mathbb{Z}[F^\times] \to F^\times \to 0,$$

where the first map is given by $([a,b]) \mapsto [a]+[b]-[ab]$. Then

$$Tor_1^{\mathbb{Z}}(F^\times, F^\times) = H_1(I_\bullet \otimes F^\times) = \frac{ker(\mathbb{Z}[F^\times] \otimes F^\times \to F^\times \otimes F^\times)}{im(\mathbb{Z}[F^\times \times F^\times] \otimes F^\times \to \mathbb{Z}[F^\times] \otimes F^\times)}.$$

So there should be a bijective morphism between cycles $\mathcal{Z} \in Z^2(F,3)$ and elements of the Tor–group, which are given by $\sum n_{a_1,a_2,b}[a_1a_2] \otimes [b]$ with $a_1, a_2, b \in F^\times$ and $n_{a_1,a_2,b} \in \mathbb{Z}$ subject to the condition

$$\sum n_{a_1,a_2,b}([a_1a_2] - [a_1] - [a_2]) \otimes [b] = 0.$$

Lemma 2.4.4. *The map*

$$Z^2(F,3) \ni \sum n_{a_1,a_2,b} \left[\frac{z-a_1a_2}{z-a_1}, z, b\right] \mapsto \sum n_{a_1,a_2,b}[a_1a_2] \otimes [b] \in \mathbb{Z}[F^\times] \otimes F^\times$$

with $a_1, a_2, b \in F^\times$ induces an isomorphism

$$H := \left\{X \in Z^2(F,3) \mid |X = \sum n_{a_1,a_2,b} \left[\frac{z-a_1a_2}{z-a_1}, z, b\right], \partial X = 0\right\} \to Tor_1^{\mathbb{Z}}(F^\times, F^\times)$$

Proof. The condition $\partial X = 0$ is equivalent to the condition

$$\sum n_{a_1,a_2,b} \left(([a_1a_2] - [a_1] - [a_2]\right) \otimes [b] = 0 \in Z^2(F,2).$$

Thus follows that there is a one-to-one correspondence between elements in $X \in H$ and elements in the Tor-group. □

Therefore we are led to the following proposition

Proposition 2.4.5. *The group $Tor_1^{\mathbb{Z}}(F^\times, F^\times)^\sim$ can be identified with the group*

$$G \subset H_3\left(\frac{Z^1(F,\bullet-1) \otimes Z^1(F,1)}{(Z^1(F,\bullet-1) \otimes Z^1(F,1)) \cap (Z^1(F,1) \otimes Z^1(F,\bullet-1))}\right)$$

from lemma 2.4.3.

Proof. In $Z^2(F,3) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ one can identify terms of the form $Z(a,b)$ for admissible choices of $a, b \in F^\times$ with terms of the form $[\frac{z-a}{z-1}, z, b]$. To see this, we copy ideas from [GMS99] and originally from [BK95]: Let $S_2 \cong \mathbb{Z}/2\mathbb{Z}$ be the symmetric group with two elements acting on $Z(F,\bullet)$ by permuting the rightmost two coordinates. Now define $sgn : S_n \to \mathbb{Z}/2\mathbb{Z}$ to be the unique non-trivial character and also define the idempotent

$$\mathsf{Alt}_2 : Z^2(F,\bullet) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \to Z^2(F,\bullet) \otimes \mathbb{Z}\left[\frac{1}{2}\right], \qquad Z \mapsto \frac{1}{2}\sum_{g \in S_2} sgn(g)g(Z).$$

One easily checks that the differential ∂ commutes with Alt_2 so that there is a complex $Z(F,\bullet) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ with a differential induced by $\mathsf{Alt}_2 \circ \partial$.

Then it is easy to see that in $Z^2(F,3) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ we have

$$Z(a,b) = \left[\frac{x-a}{x-1}, x, b\right],$$

because they just differ by a permutation of the two right coordinates. But these terms are n-torsion in case b is an n^{th} root of unity. In other words, modulo 2-torsion, elements of the form $\sum n_{a_1,a_2,b}(Z(a_1a_2,b) - Z(a_1,b) - Z(a_2,b)) \in G$ are contained in H from the lemma above, i. e. in $Tor(F^\times, F^\times)$. G is the non-trivial $\mathbb{Z}/2\mathbb{Z}$ – extension of $Tor(F^\times, F^\times)$. □

With these results, one can immediately determine whether a given cycle in $CH^2(F,3)$ of a number field is already contained in $B_2(F)$ or the Tor-group.

2.5 More symmetric relations

Unfortunately the relations derived in section 2.2 do not reflect the symmetries of their arguments. This means that given a functional equation of the dilogarithm in n terms whose arguments are invariant under the action of the full symmetric group of order n by translation, this invariance is not preserved among the extra terms occurring in the corresponding relation among the Totaro cycles. Consider the five-term relation (2.2.4) for example. We cannot write it in the form

$$\sum_{i=1}^{5} (nC_{q_i} + mR(q_i)) = 0,$$

where $\{q_i\}_{i=1}^5$ denote the five arguments of the Totaro cycles, and $R(q_i)$ denotes some linear combination of terms in $C^2(F,3)/\partial C^2(F,4)$ with a constant coordinate somewhere depending on the argument q_i.

In this section we obtain cycles in $C^2(F,3)/\partial C^2(F,4)$ symmetric in their arguments whose image under the Abel – Jacobi map restricted to the real part coincides with functional equations of the dilogarithm. For this we make use of the following proposition [KLMS06, p. 18]:

Proposition 2.5.1. *Given any element $\sum_i m_i[a_i] \in B_2(\mathbb{C})$, then $\sum_i m_i C_{a_i} \in Z^2(F,3)$ for some number field F may be completed to a higher Chow cycle $\mathcal{Z}$ by adding "decomposable" elements in $Z^1(F,1) \wedge Z^1(F,2)$. The real part of the image of $\mathcal{Z}$ under the Abel – Jacobi map is then computed by $\sum_i m_i D(a_i) \in \mathbb{R}$, where D denotes the Bloch – Wigner dilogarithm.*

Thus if we start with sum of Totaro cycles corresponding to a functional equation of the dilogarithm and complete these cycles to a higher Chow cycle in a symmetric way, then the image of this higher Chow cycle under the Abel – Jacobi map vanishes by assumption, and our higher Chow cycle as well. In summary, we obtain a symmetric relation in $C^2(F,3)/\partial C^2(F,4)$.

This leads to a very general symmetric relation in $C^2(F,3)/\partial C^2(F,4)$ reflecting a general functional equation of the dilogarithm.

Let us look at some motivating examples: We let $F \subset \mathbb{C}$ be an infinite field as usual, and consider the following functional equation of the dilogarithm. We write $A := a^2 - a + 1$ for some $a \in F^\times$. Then

$$Li_2\left(\frac{1}{A}\right) + Li_2\left(\frac{(1-a)^2}{A}\right) + Li_2\left(\frac{a^2}{A}\right) = \text{ lower order terms}$$

or equivalently

$$D\left(\frac{1}{A}\right) + D\left(\frac{(1-a)^2}{A}\right) + D\left(\frac{a^2}{A}\right) = 0.$$

This is a special case of functional equations of the dilogarithm whose arguments can be expressed as roots of the equation $x^n(x-1)^m = t$ for some nonnegative integers n, m and a constant $t \in F^\times$, namely

$$x^2(x-1) = t, \qquad t = -\frac{a^2(1-a)^2}{A^3}.$$

If we set

$$q_1(a) := \sqrt{A}, \quad q_2(a) := \sqrt{\frac{A}{a^2}}, \quad q_3(a) := \sqrt{\frac{A}{(1-a)^2}},$$

then the above relation can be written in the form

$$\sum_{i=1}^{3} Li_2\left(\frac{1}{q_i^2}\right) = \text{ lower order terms} \qquad \Longleftrightarrow \qquad \sum_{i=1}^{3} D\left(\frac{1}{q_i^2}\right) = 0.$$

One notes that the symmetric group S_3 generated by $a \mapsto 1-a$ and $a \mapsto \frac{1}{a}$ permutes the q_i but leaves t invariant. We can now show:

Proposition 2.5.2. *Let $a \in F^\times - \{1, -\zeta_3, -\zeta_3^2\}$. Denote by $\{q_i\}_{i=1}^3$ as above the roots of the equation $x^2(x-1) = -\frac{a^2(1-a)^2}{A^3}$. Then – counting indices modulo 3:*

$$\sum_{i=1}^{3}\left(C_{\frac{1}{q_i^2}} - \left[\frac{z + \frac{1}{q_{i+1}q_{i+2}}}{z - \frac{1}{q_{i+1}}}, z, \frac{1}{q_i^2}\right] + 2Z\left(-\frac{1}{q_i}, \frac{1}{q_{i+1}}\right)\right) = 0 \in C^2(F,3)/\partial C^2(F,4). \qquad (2.5.1)$$

Proof. One easily computes the vanishing boundary of the relation above by summing up the boundaries of the different summands and using the fact that the boundaries live in $Z^1(F,2)/(Z^1(F,1) \otimes \partial Z^1(F,2))$:

$$\partial(C_{\frac{1}{q_i^2}}) = \left(-\frac{1}{q_{i+1}q_{i+2}}, \frac{1}{q_i^2}\right)$$

$$\partial\left(\left[\frac{z + \frac{1}{q_{i+1}q_{i+2}}}{z - \frac{1}{q_{i+1}}}, z, \frac{1}{q_i^2}\right]\right) = \left(-\frac{1}{q_{i+1}q_{i+2}}, \frac{1}{q_i^2}\right) - \left(\frac{1}{q_{i+1}}, \frac{1}{q_i^2}\right) - \left(-\frac{1}{q_{i+2}}, \frac{1}{q_i^2}\right)$$

$$= \left(-\frac{1}{q_{i+1}q_{i+2}}, \frac{1}{q_i^2}\right) - 2\left(\frac{1}{q_{i+1}}, -\frac{1}{q_i}\right) - 2\left(-\frac{1}{q_{i+2}}, \frac{1}{q_i}\right)$$

and further

$$2\partial Z\left(\frac{1}{q_{i+1}}, -\frac{1}{q_i}\right) = 2\left(\frac{1}{q_{i+1}}, -\frac{1}{q_i}\right) + 2\left(-\frac{1}{q_i}, \frac{1}{q_{i+1}}\right).$$

This already proves the result by the proposition quoted above: The image of this cycle under the Abel – Jacobi map is given by

$$\sum_{i=1}^{3} D\left(\frac{1}{q_i^2}\right) = 0$$

since it is just a functional equation of the dilogarithm. □

Remark 2.5.3. This proof shows that in order to prove a relation in the Chow group we only have to show that a given linear combination of terms in $Z^2(F,3)$ is a higher Chow cycle. If it is by construction only a completed relation corresponding to a functional equation of the dilogarithm, then this cycle automatically vanishes in the Chow group. ♢

Proposition 2.5.4. *With the same assumptions as above, the following relation holds in the quotient $C^2(F,3)/\partial C^2(F,4)$:*

$$0 = \sum_{i=1}^{3}\left(2C_{-\frac{q_i^2}{q_{i+1}q_{i+2}}} + \left[\frac{z+\frac{1}{q_{i+1}q_{i+2}}}{z-\frac{1}{q_{i+1}}}, z, \frac{1}{q_i^2}\right] + Z\left(\frac{1}{q_i^2}, \frac{1}{q_i^2}\right)\right.$$
$$\left. -2Z\left(\frac{1}{q_i^2}, -\frac{q_i^2}{q_{i+1}q_{i+2}}\right) - 2Z\left(-\frac{1}{q_i}, \frac{1}{q_{i+1}}\right)\right).$$

Proof. This is a consequence of the relation above and proposition 2.2.1. □

Remark 2.5.5. The extra terms needed to complete the Totaro cycles to a Chow cycle appear to be rather ad hoc. But if we rewrite these relations in terms of the antisymmetric $\tilde{C}_a$, then they become rather analogous. ♢

Proposition 2.5.6. *Let $a \in F^\times$ such that $a \notin \{1, -\zeta_3, -\zeta_3^2\}$, and let again $A := a^2 - a + 1$. Set again*

$$q_1(a) := \sqrt{A}, \quad q_2(a) := \sqrt{\frac{A}{a^2}}, \quad q_3(a) := \sqrt{\frac{A}{(1-a)^2}}.$$

Then the following relations hold in $C^2(F,3)/\partial C^2(F,4)$:

$$0 = \sum_{i=1}^{3}\left(\tilde{C}_{\frac{1}{q_i^2}} - \left[\frac{z+\frac{1}{q_{i+1}q_{i+2}}}{z-\frac{1}{q_{i+1}}}, z, \frac{1}{q_i^2}\right] + \left[\frac{z-q_i^2}{z-q_i}, z, \frac{1}{q_{i+1}q_{i+2}}\right]\right) \tag{2.5.2}$$

$$0 = \sum_{i=1}^{3}\left(\tilde{C}_{-\frac{q_{i+1}q_{i+2}}{q_i^2}} + \left[\frac{z+\frac{q_{i+1}q_{i+2}}{q_i^2}}{z+q_{i+1}q_{i+2}}, z, \frac{1}{q_i^2}\right] - \left[\frac{z-q_i^2}{z-q_i}, z, -\frac{q_{i+1}q_{i+2}}{q_i^2}\right] + \left[\frac{z+q_{i+1}q_{i+2}}{z+q_{i+1}}, z, \frac{1}{q_i^2}\right]\right). \tag{2.5.3}$$

Proof. As above, one only needs to compute the boundaries keeping in mind that $\partial\tilde{C}_a = (1-a,a)-(a,1-a)$. □

Even the five-term relation can be simplified and written much more elegantly in the following way:

Proposition 2.5.7. *Let $a, b \in F^\times$ be two distinct elements of a number field such that $a \neq b$ and $ab \neq 1$. Let further be*

$$q_1 := a, \quad q_2 := b, \quad q_3 := 1-ab, \quad q_4 := \frac{1-a}{1-ab}, \quad q_5 := \frac{1-b}{1-ab}$$

such that the five-term relation for the Bloch – Wigner dilogarithm, $\sum_{i=1}^5 D(q_i) = 0$, holds. Then the following relation holds in $C^2(F,3)/\partial C^2(F,4)$:

$$0 = \sum_{i=1}^{5} \left(\tilde{C}_{q_i} - \left[\frac{z - q_{i-2}q_{i+2}}{z - q_{i-2}}, z, q_i \right] \right). \tag{2.5.4}$$

Proof. Making use of the identity $1 - q_i = (q_{i-1}^{-1} - 1)(q_{i+1}^{-1} - 1) = q_{i-2}q_{i+2}$, we compute the following boundaries in $Z^2(F,2)/Z^1(F,1) \otimes \partial Z^1(F,2)$, which sum up to zero:

$$\begin{aligned} \partial(\tilde{C}_{q_i}) &= (1-q_i, q_i) - (q_i, 1-q_i) = (q_{i-2}q_{i+2}, q_i) - (q_i, q_{i-2}q_{i+2}) \\ &= (q_{i-2}q_{i+2}, q_i) - (q_i, q_{i+2}) - (q_i, q_{i-2}), \\ \partial \left[\frac{z - q_{i-2}q_{i+2}}{z - q_{i-2}}, z, q_i \right] &= (q_{i-2}q_{i+2}, q_i) - (q_{i-2}, q_i) - (q_{i+2}, q_i). \end{aligned}$$

Again we use the Abel – Jacobi map to determine that the image vanishes. The image of the terms $\tilde{C}_{q_i}$ vanishes because it is given by the Rogers dilogarithm, which satisfies a "clean" functional equation without extra terms. □

More generally, we can prove the following theorem which can be seen as a step towards an explicit description of a map $\overline{\rho}_2' : B_2(F) \longrightarrow \frac{CH^2(F,3)}{im(Tor(F^\times, F^\times)^\sim)}$, namely we show how to complete a rather general relation coming from a functional equation of the dilogarithm with a strong inherent symmetry to a higher Chow cycle preserving the symmetry:

Proposition 2.5.8. *Consider an infinite field F and a relation for the Bloch – Wigner dilogarithm of the form $s \cdot \sum_{i=1}^m D(a_i) = 0$, $s \in \mathbb{N}$ such that there is a (not necessarily unique) factorization of a_i and $1-a_i$ into elements $p_\mu \in F^\times$:*

$$a_i = \pm \prod_{j=1}^{k} p_{\sigma^i(j)}^{n(\sigma^i(j))} \qquad \text{and} \qquad 1 - a_i = \pm \prod_{j'=1}^{k'} p_{\sigma^i(j')}^{n(\sigma^i(j'))}$$

for $\sigma \in G$, where G denotes a finite groups which acts on the set $\{a_i\}$ via permutation, and where there are numbers $k, m \in \mathbb{N}$; $s, n(j) \in \mathbb{Z}$. Then we have the following relation in

$C^2(F,3)/\partial C^2(F,4)$:

$$
\begin{aligned}
0 = s \cdot \sum_{i=1}^{m} \Bigg(& \tilde{C}_{a_i} \\
& + \sum_{j=2}^{k} \left[\frac{z \mp \prod_{\ell=1}^{j} p_{\sigma^i(\ell)}^{n(\sigma^i(\ell))}}{z - p_{\sigma^i(j)}^{n(\sigma^i(j))}}, z, \pm \prod_{j'} p_{\sigma^i(j')}^{n(\sigma^i(j'))} \right] + \sum_{\substack{j \text{ with} \\ n(\sigma^i(j))>1}} \left[\frac{z - p_{\sigma^i(j)}^{n(\sigma^i(j))}}{z - p_{\sigma^i(j)}}, z, \pm \prod_{j'} p_{\sigma^i(j')}^{n(\sigma^i(j'))} \right] \\
& \mp \sum_{j'=1}^{k'} \left[\frac{z \mp \prod_{\ell=1}^{j'} p_{\sigma^i(\ell)}^{n(\sigma^i(\ell))}}{z - p_{\sigma^i(j')}^{n(\sigma^i(j'))}}, z, \pm \prod_{j} p_{\sigma^i(j)}^{n(\sigma^i(j))} \right] \mp \sum_{\substack{j' \text{ with} \\ n(\sigma^i(j'))>1}} \left[\frac{z - p_{\sigma^i(j')}^{n(\sigma^i(j'))}}{z - p_{\sigma^i(j')}}, z, \pm \prod_{j} p_{\sigma^i(j)}^{n(\sigma^i(j))} \right] \Bigg)
\end{aligned}
\tag{2.5.5}
$$

Proof. A straightforward computation as before shows that the boundaries of the lower order terms cancel each other except for those which are canceled by the boundaries of the Totaro cycles. Then again invoking the proposition from [KLMS06, p. 18] we see that we obtained a relation in $C^2(F,3)/\partial C^2(F,4)$ since by assumption we started with a non-trivial element in the Bloch group. □

Remark 2.5.9. The results in this section aim at two targets. First, they show that we can prove more symmetric relations in $C^2(F,3)/\partial C^2(F,4)$. But still this is not so easy: We only showed that they exist. The ultimate goal would be to start with a reparametrization of a symmetric fractional cycle in $C^2(F,4)$, which bounds the five-term relation as stated above. This would be a beautiful extension of the proofs of [GMS99]. Here we just give the results without finding a way via reparametrizations and breaking up cycles in order to somehow derive these symmetric relations.

On the other hand, one would like to have an explicit map inducing the isomorphism

$$B_2(F) \to \frac{CH^2(F,3)}{im(Tor_1^{\mathbb{Z}}(F^\times, F^\times)^\sim)}$$

and coinciding with $\overline{\rho_2} \otimes \mathbb{Q}$ from [GMS99] rationally. But this again seems still far away. The functional equations we started with in the proposition are surely contained in the Bloch group, more precisely the arguments of the dilogarithms involved. The proposition above associates to these special elements in the Bloch group a higher Chow cycle contained in $\frac{CH^2(F,3)}{im(Tor_1^{\mathbb{Z}}(F^\times,F^\times)^\sim)}$. But as we have mentioned, our cycle is not unique. Further, in order to define a map inducing the above isomorphism, we would need to know how to associate a cycle to every element in the Bloch group in a unique way. It is not clear how to extend our result to the whole Bloch group. But still, it might be a starting point. ◊

Chapter 3

Explicit computations in codimension three

In Gangl's and Müller-Stach's paper and also in Zhao's work there was some progress on a map

$$\rho_3 : \mathbb{Z}[F^\times] \to Z^3(F,5), \quad [a] \mapsto C_a^{(3)}$$

being supposed to induce an isomorphism

$$\overline{\rho}_3 \otimes \mathbb{Q} : B_3(F) \otimes \mathbb{Q} \xrightarrow{\cong} CH^3(F,5) \otimes \mathbb{Q}$$

based on the one discussed for codimension two Chow groups and the corresponding Bloch group.

In Zhao's recent work [Zha07], the proof of the main relations of a variant of the higher Bloch group $B_3(F)$, namely Goncharov's relation, has been completed. The proofs of a distribution relation and an inversion relation were already settled modulo the three term relation in [GMS99]. These results together with a yet to be found proof of the three-term relation would imply the welldefinedness of the map in question.

Unfortunately, the complexity of the formulas is much higher integrally because of torsion effects. Additionally, the relation between the higher Bloch group and the higher Chow group is not known: There is no exact sequence comparable to the one of Suslin. Neither does one know much about the Bloch group itself. It is not clear whether it is finitely generated. Proving relations in the Chow group is possible by copying the proofs of Gangl, Müller-Stach and Zhao. Of course, there are many lower order terms corresponding to monodromy behavior of the trilogarithm. These terms make the relations in the integral setup seem far too complicated to be used for finding generators of the Chow groups of some number fields.

But before trying to find relations, one has to look for cycles. It is not so easy to produce cycles in the Chow group corresponding to the trilogarithm or its special values at cyclotomic arguments. It seems to be very hard to produce anything beyond cyclotomic elements. Maybe this is enough for some purposes, but then one cannot use the proofs of the relations from literature, since they are useless for the special cyclotomic elements.

The first section of this chapter explores the setup in which computations take place. We set up some new auxiliary relations needed for proving relations in the integral Chow group in codimension three, and which are an extension of those used by Gangl, Müller-Stach and Zhao. Under the assumption of the integral version of the Beĭlinson– Soulé vanishing conjecture 1.3.7 in codimension two, there are also be several acyclic subcomplexes to be divided out. Additionally, we obtain some technical lemmas used for computations in the Chow group. We note that the integral Beĭlinson-Soulé conjecture is a daring assumption whose justification is not clear at all. We close the introduction with an discussion of special cyclotomic elements derived by M. Kerr from a general procedure sketched in [Neu88] and the proof a series of distribution relations in $C^3(F,5)/\partial C^3(F,6) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$.

In the following section, we give a proof of the Kummer – Spence – relation using the strategy of [GMS99] in $C^3(F,5)/\partial C^3(F,6) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$. In section three we sketch the integral version of the proof of an inversion relation. In the fourth section we demonstrate a proof of Goncharov's relation in $C^3(F,5)/\partial C^3(F,6) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ using the strategy of [Zha07].

Remark 3.0.10. Also note that we have to care for the admissibility of all of our cycles. Luckily, this was done before by Gangl, Müller-Stach and Zhao. Nevertheless, we checked the admissibility of all terms and do not comment on this any more. ♢

3.1 The setup

In this section we generalize the results of the previous section to the more complicated situation. In order to compute the codimension three Chow group of the spectrum of an infinite field F, we need to compute the homology in degree five of the cycle complex $Z^3(F,\bullet)$ introduced in section 1.3.1. We are interested in special cycles parametrized by constants in $F^\times$, which are supposed to generate the whole Chow group. In particular, we consider the following cycles due to Bloch and Kříž [BK95]

$$C_a := C_a^{(3)} = \left[1-\frac{a}{y}, 1-\frac{y}{x}, 1-x, y, x\right] \in Z^3(F,5)$$

generalizing the Totaro cycles in $Z^2(F,3)$. Note that we cannot use the alternating cycles of [BK95]. since they can only compute the rational higher Chow groups. Finding the right cycles in the Chow group is a rather delicate problem which we will address below.

First of all, we divide out some subcomplexes of $Z^3(F,\bullet)$ to simplify computations as in the last chapter:

Proposition 3.1.1. *Assume the integral Beĭlinson– Soulé vanishing conjecture in the form $CH^2(F,n) = 0$ for $n \geq 4$. Then the subcomplex generated of $Z^3(F,\bullet)$ by the following three* *subcomplexes is acyclic:*

$$\begin{aligned} D_1(F,\bullet):\ \ldots &\to Z^1(F,1)\otimes Z^2(F,5)\to Z^1(F,1)\otimes Z^2(F,4)\to Z^1(F,1)\otimes \partial Z^2(F,4)\to 0,\\ D_2(F,\bullet):\ \ldots &\to Z^2(F,2)\otimes Z^1(F,3)\to Z^2(F,2)\otimes Z^1(F,2)\to Z^2(F,2)\otimes \partial Z^1(F,2)\to 0,\\ D_3(F.\bullet):\ \ldots &\to Z^2(F,3)\otimes Z^1(F,3)\to Z^2(F,3)\otimes Z^1(F,2)\to Z^2(F,3)\otimes \partial Z^1(F,2)\to 0. \end{aligned}$$

Proof. It is clear that $D_i(F,\bullet), i=1,2$ are constructed to be complexes. $D_3(F,\bullet+1)$ is only a well-defined complex in the quotient $Z^3(F,\bullet+1)/(D_1(F,\bullet)+D_2(F,\bullet+1)$ as $\partial Z^2(F,3))\subset Z^2(F,2)$. The first complex is acyclic assuming the Beĭlinson– Soulé vanishing conjecture integrally in codimension two. This implies that $D_2(F,\bullet+1)\subset D_1(F,\bullet)$ is also acyclic. $D_3(F,\bullet)$ is acyclic if one uses the theorem of Nart [Nar89] proving that $Z^1(F,\bullet)$ is acyclic. □

Remark 3.1.2. One really has to stress the point that the integral version of the Beĭlinson–Soulé conjecture for number fields is probably to restrictive. It is by no means clear that this assumption is justified. Nevertheless it does not suffice for our purposes of finding explicit elements in the higher Chow groups in codimension three. ♢

Thus, let us denote the quotient

$$\frac{Z^3(F,\bullet)}{D_1(F,\bullet)+D_2(F,\bullet)+D_3(F,\bullet)}$$

by $C^3(F,\bullet)$ in accordance with the previous section. From the preceding discussion follows that this complex still computes $CH^3(F,5)$. Terms contained in one of the acyclic subcomplexes will be called negligible.

Now, we come back to the question which cycles to choose as generators for computations in the Chow group: Observe that $\partial C_a=[1-\frac{a}{x},1-x,a,x]-[1-\frac{a}{x},1-x,x,x]$: In particular $\partial C_1=[1-\frac{1}{x},1-x,x,x]$, which is not known to be zero. So one first has to find some cycles in the Chow group whose image under the Abel – Jacobi map is a trilogarithm. This is a torsion effect because now the C-terms do not live in the alternating cycle complex $C^3(F,\bullet)$ of [GMS99], any more. One can only observe that

$$C'_a := [1-\frac{a}{y},1-\frac{y}{x},1-x,y,x]-[1-\frac{a}{y},1-\frac{y}{x},1-x,x,y]$$

has boundary equal to $[1-\frac{a}{x},1-x,a,x]-[1-\frac{a}{x},1-x,x,a]$. So $\partial C'_1=0$ at least. But then C_{ζ_n} has boundary $[1-\frac{\zeta_n}{x},1-x,\zeta_n,x]-[1-\frac{\zeta_n}{x},1-x,x,\zeta_n]$. One can show that the latter vanishes in $C^3(F,\bullet)$ since it lives in $Z^2(F,3)\otimes\partial Z^1(F,2)$, but this does not hold a priori for the former one.

Remark 3.1.3. From this discussion we can see that it is absolutely hopeless to prove relations in the integral Chow group $CH^3(F,5)$, because the C-terms, i. e. Boch – Kříž cycles, involved are mostly not contained in the Chow group. To have more sensible relations, we use the ideas from the proof of proposition 2.4.5 and let the symmetric group S_2 act via permutation of the two rightmost coordinates of an element of $Z(F,\bullet)$. To define this action, we need to consider $Z(F,\bullet) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$, whose homology computes $CH^3(F,5) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$. Thus, we shall investigate relations in the Chow group modulo 2-torsion. $\Diamond$

Remark 3.1.4. Reading [Neu88] carefully, one also recognizes Beĭlinson's construction of cyclotomic elements $\tilde{\ell}_{a,b}(w) \in K_{2m-1}(\mathbb{Q}(\zeta_n)) \otimes \mathbb{Q}$ for some n^{th} root of unity w in motivic cohomology. In order to have the desired properties according to his conjectures, it is necessary that $c \cdot n^{m-1}\tilde{\ell}_{a,b}(w) \in K_{2m-1}(\mathbb{Q}(\zeta_n)) \otimes \mathbb{Q}$ or equivalently via Zagier's conjecture $\in B_m(\mathbb{Q}(\zeta_n)) \otimes \mathbb{Q}$. The constant c is not important, but the exponent, $m-1$, of n is.

In our case, we have special"cyclotomic elements" $n \cdot C_{\zeta_n^i} \in CH^3(F,5) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$. But they are not in accordance with Beĭlinson's construction: Considered as the images of the map $B_3(\mathbb{Q}(\zeta_n)) \to CH^3(\mathbb{Q}(\zeta_n),5)$ due to Bloch and Kříž [BK95], our elements have no preimages in the Bloch group because of the well-known fact that $n^{m-1}[\zeta_n] \in B_m(\mathbb{Q}(\zeta_n))$. $\Diamond$

Remark 3.1.5. M. Kerr explained to me his special cyclotomic elements in the higher Chow group $CH^3(\mathsf{Spec}(\mathbb{Q}(\zeta_\ell)),5) \otimes \mathbb{Q}$ given by

$$\begin{aligned} Z(\zeta_\ell) := &-\frac{1}{\ell^2}\left[\frac{x}{x-1}, \frac{y}{y-1}, 1-\zeta_\ell xy, x^\ell, y^\ell\right] \\ &-\frac{1}{2\ell^2}\left[\frac{x}{x-1}, \frac{1}{1-\zeta_\ell x}, \frac{(y-x^\ell)(y-x^{-\ell})}{(y-1)^2}, x^\ell y, x^{-\ell}y\right], \end{aligned} \tag{3.1.1}$$

which are part of an inductive procedure sketched in [Neu88] to construct cyclotomic elements in motivic cohomology generalizing the Bloch and Kříž – cycles. One checks that these cycles are contained in the rational Bloch group $B_3(\mathbb{Q}(\zeta_\ell)) \otimes \mathbb{Q}$, and one further shows that their image in Deligne – Beĭlinson cohomology is given by

$$\Phi_{3,5}(Z(\zeta_\ell)) = Li_3(\zeta_\ell).$$

The fact that the $Z(\zeta_\ell)$ are contained in the (rational) Bloch group is important for using them as cyclotomic elements in motivic cohomology coming from the higher Bloch group. More precisely from the definition one immediately observes that $\ell^2 Z(\zeta_\ell) \in B_3(\mathbb{Q}(\zeta_\ell)) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$, but $\ell Z(\zeta_\ell) \notin B_3(\mathbb{Q}(\zeta_\ell)) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ as requested by Beĭlinson [Bei85]. $\Diamond$

Remark 3.1.6. From a more conceptional point of view, one might interpret these problems concerning the right cycles in the following way: The Bloch – Kříž cycles $C_a^{(3)}$ in codimension three are expected to generate the whole Chow group of a number field modulo torsion. But it seems as if they do not generate the whole group integrally as well. As we will see below in proposition 3.1.26, Kerr's cycles at least satisfy a distribution relation in $C^3(F,5)/\partial C^3(F,6)$ without extra terms. This is a big advantage over the Bloch – Kříž cycles, which demonstrates their importance.

Note that it is also possible that we still need more complicated generators of the torsion part of the Chow group than the ones of Kerr. $\Diamond$

Now one needs some rules equivalent to proposition 2.1.4.

Proposition 3.1.7. *Let*

$$[f_1(y), f_2(x,y), f_3(x), f_4(y), f_5(x)]$$

be an admissible cycle in projective 5*-space, where all* f_i *are rational functions and* f_2 *is a product of fractional linear transformations when considered as a function in the variable* y*. Then the following relations hold in* $C^3(F,5)/\partial C^3(F,6)$*:*

- *Suppose additionally that one can write* $f_1(y) = g(y)\cdot h(y)$ *for some rational functions* g *and* h*. Then*

$$\begin{aligned}
&[g(y)h(y), f_2(x,y), f_3(x), f_4(y), f_5(x)] = \\
&\quad [g(y), f_2(x,y), f_3(x), f_4(y), f_5(x)] + [h(y), f_2(x,y), f_3(x), f_4(y), f_5(x)] \\
&\quad - \sum_{div(f_2)} \pm \left[\frac{z-g(y(x))h(y(x))}{z-g(y(x))}, z, f_3(x), f_4(y(x)), f_5(x)\right] \\
&\quad + \sum_{div(f_3)} \pm \left[\frac{z-g(y)h(y)}{z-g(y)}, z, f_2(x_0,y), f_4(y), f_5(x_0)\right] \\
&\quad - \sum_{div(f_4)} \pm \left[\frac{z-g(y_0)h(y_0)}{z-g(y_0)}, z, f_2(x,y_0), f_3(x), f_5(x)\right] \\
&\quad + \sum_{div(f_5)} \pm \left[\frac{z-g(y)h(y)}{z-g(y)}, z, f_2(x_0,y), f_3(x_0), f_4(y),\right].
\end{aligned} \tag{3.1.2}$$

- *Suppose that one can write* $f_2(x,y) = g(x,y)\cdot h(x,y)$ *for some rational functions* g, h*. Then*

$$\begin{aligned}
&[f_1(y), g(x,y)h(x,y), f_3(x), f_4(y), f_5(x)] = \\
&\quad [f_1(y), g(x,y), f_3(x), f_4(y), f_5(x)] + [f_1(y), h(x,y), f_3(x), f_4(y), f_5(x)] \\
&\quad + \sum_{div(f_1)} \pm \left[\frac{z-g(x,y_0)h(x,y_0)}{z-g(x,y_0)}, z, f_3(x), f_4(y_0), f_5(x)\right] \\
&\quad - \sum_{div(f_3)} \pm \left[f_1(y), \frac{z-g(x_0,y)h(x_0,y)}{z-g(x_0,y)}, z, f_4(y), f_5(x_0)\right] \\
&\quad - \sum_{div(f_5)} \pm \left[f_1(y), \frac{z-g(x_0,y)h(x_0,y)}{z-g(x_0,y)}, z, f_3(x_0), f_4(y)\right].
\end{aligned} \tag{3.1.3}$$

- *Suppose that one can write* $f_3(x) = g(x)\cdot h(x)$ *for some rational functions* g *and* h*. Then*

$$\begin{aligned}
&[f_1(y), f_2(x,y), g(x)h(x), f_4(y), f_5(x)] = \\
&\quad [f_1(y), f_2(x,y), g(x), f_4(y), f_5(x)] + [f_1(y), f_2(x,y), h(x), f_4(y), f_5(x)] \\
&\quad - \sum_{div(f_1)} \pm \left[f_2(x,y_0), \frac{z-g(x)h(x)}{z-g(x)}, z, f_4(y_0), f_5(x)\right] \\
&\quad + \sum_{div(f_2)} \pm \left[f_1(y(x)), \frac{z-g(x)h(x)}{z-g(x)}, z, f_4(y(x)), f_5(x)\right] \\
&\quad + \sum_{div(f_5)} \pm \left[f_1(y), \frac{z-g(x_0)h(x_0)}{z-g(x_0)}, z, f_3(x_0), f_4(y)\right].
\end{aligned} \tag{3.1.4}$$

- *Suppose that one can write* $f_4(y) = g(y) \cdot h(y)$ *for some rational functions* g *and* h. *Then*

$$\begin{aligned}
&[f_1(y), f_2(x,y), f_3(x), g(y)h(y), f_5(x)] = \\
&\quad [f_1(y), f_2(x,y), f_3(x), g(y), f_5(x)] + [f_1(y), f_2(x,y), f_3(x), h(y), f_5(x)] \\
&\quad + \sum_{div(f_1)} \pm \left[f_2(x, y_0), f_3(x), \frac{z - g(y_0)h(y_0)}{z - g(y_0)}, z, f_5(x) \right] \\
&\quad - \sum_{div(f_2)} \pm \left[f_1(y(x)), f_3(x), \frac{z - g(y(x))h(y(x))}{z - g(y(x))}, z, f_5(x) \right] \\
&\quad + \sum_{div(f_3)} \pm \left[f_1(y), f_2(x_0, y), \frac{z - g(y)h(y)}{z - g(y)}, z, f_5(x_0) \right] \\
&\quad - \sum_{div(f_5)} \pm \left[f_1(y), f_2(x_0, y), f_3(x_0), \frac{z - g(y)h(y)}{z - g(y)}, z \right].
\end{aligned} \tag{3.1.5}$$

- *Suppose that one can write* $f_5(x) = g(x) \cdot h(x)$ *for some rational functions* g *and* h. *Then*

$$\begin{aligned}
&[f_1(y), f_2(x,y), f_3(x), f_4(y), g(x)h(x)] = \\
&\quad [f_1(y), f_2(x,y), f_3(x), f_4(y), g(x)] + [f_1(y), f_2(x,y), f_3(x), f_4(y), h(x)] \\
&\quad - \sum_{div(f_1)} \pm \left[f_2(x, y_0), f_3(x), f_4(y_0), \frac{z - g(x)h(x)}{z - g(x)}, z \right] \\
&\quad + \sum_{div(f_2)} \pm \left[f_1(y(x)), f_3(x), f_4(y(x)), \frac{z - g(x)h(x)}{z - g(x)}, z \right].
\end{aligned} \tag{3.1.6}$$

The choice of the sign depends on the divisor: zeros get positive signs, poles negative ones.

Proof. Compute the boundaries of $\left[\frac{z-g(y)h(y)}{z-g(y)}, z, f_2(x,y), f_3(x), f_4(y), f_5(x)\right]$, $\left[f_1(y), \frac{z-g(x,y)h(x,y)}{z-g(x,y)}, z, f_3(x), f_4(y), f_5(x)\right]$, $\left[f_1(x), f_2(x,y), \frac{z-g(x)h(x)}{z-g(x)}, z, f_4(y), f_5(x)\right]$, $\left[f_1(x), f_2(x,y), f_3(x), \frac{z-g(y)h(y)}{z-g(y)}, z, f_5(x)\right]$, and $\left[f_1(x), f_2(x,y), f_3(x), f_4(y), \frac{z-g(x)h(x)}{z-g(x)}, z\right]$. □

Let now $(g_j)_{j \in I}$ be rational functions in zero, one or two variables. Then a term in $C^3(F,5)/\partial C^3(F,6) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ with a superscript i and and a subscript $(\pm, g_1; \pm, g_2; \ldots, \pm, g_{|I|})$ on the right indicates that in addition there is also another term with the rational function $g_1, \ldots, g_{|I|}$ in place of the i^{th} coordinate and with the corresponding sign, i. e. the sign left of the function.

Example 3.1.8.

$$\begin{aligned}\left[1-\frac{a}{x},1-\frac{y}{x},1-x,y,x\right]^1_{(+,1+\frac{a}{x})} &= \left[1-\frac{a}{x},1-\frac{y}{x},1-x,y,x\right]+\left[1+\frac{a}{x},1-\frac{y}{x},1-x,y,x\right]\\ &= C_a + C_{-a},\\ \left[1-\frac{a}{x},1-\frac{y}{x},\frac{1-x}{x},y,x\right]^3_{(+,1-x;+,\frac{1}{x})} &= \left[1-\frac{a}{x},1-\frac{y}{x},\frac{1-x}{x},y,x\right]\\ &+\left[1-\frac{a}{x},1-\frac{y}{x},1-x,y,x\right]+\left[1-\frac{a}{x},1-\frac{y}{x},\frac{1}{x},y,x\right].\end{aligned}$$

Now we shall look at some special cycles giving rise to useful relations:

Lemma 3.1.9. *Let*

$$\mathcal{Z} := \left[f(x),g(x),\frac{z-h_1(x)h_2(x)}{z-h_1(x)},z,\frac{y-h_3(x)h_4(x)}{y-h_3(x)},y\right] \in C^3(F,6)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$$

be a surface with rational functions f, g, h_1, h_2 in one variable chosen to guarantee the admissibility of $\mathcal{Z}$. Then $\mathcal{Z}$ gives rise to the following relation in $C^3(F,5)/\partial C^3(F,6)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$:

$$\begin{aligned}&\left[f(x),g(x),h_1(x)h_2(x),\frac{y-h_3(x)h_4(x)}{y-h_3(x)},y\right]^3_{(-,h_1;-,h_2)} =\\ &-\left[f(x),g(x),\frac{y-h_1(x)h_2(x)}{y-h_1(x)},y,h_3(x)h_4(x)\right]^5_{(+,h_3;+,h_4)}.\end{aligned}$$

Proof. Compute the boundary of $\mathcal{Z}$. □

Corollary 3.1.10. *With the same assumptions on the rational functions as above the following relation holds in $C^3(F,5)/\partial C^3(F,6)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$:*

$$\left[f(x),g(x),\frac{y-h_1(x)}{y-1},y,h_2(x)h_3(x)\right]^5_{(+,h_2;+,h_3)} = 0.$$

An analogous result holds for the third and fifth coordinate interchanged.

Proof. Set $h_2(x)=1$ and change the numbering:

$$\begin{aligned}&\left[f(x),g(x),\frac{y-h_1(x)}{y-1},y,h_2(x)h_3(x)\right]^5_{(-,h_2;-,h_3)} =\\ &-\left[f(x),g(x),h_1(x),\frac{y-h_2(x)h_3(x)}{y-h_2(x)},y\right]^3_{(+,1;+,h_1)} = 0.\end{aligned}$$

By symmetry the second assertion follows easily. □

With this, one obtains:

Proposition 3.1.11. *Let f, g, h_1, h_2 be rational functions in one variable such that the following terms are admissible. Then the following relation is satisfied in $C^3(F,5)/\partial C^3(F,6)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$:*

$$\left[f(x), g(x), \frac{y-h_1(x)h_2(x)}{y-h_1(x)}, y, h(x)\right] = \left[f(x), g(x), h(x), \frac{y-h_1(x)h_2(x)}{y-h_1(x)}, y\right].$$

Proof. Consider the element

$$\mathcal{Z} := \left[f(x), g(x), \frac{z-h(x)}{z-1}, \frac{y-h_1(x)h_2(x)}{y-h_1(x)}, y, z\right] \in C^3(F,6)$$

and compute its boundary. The assertion follows from the corollary. □

We can derive two more easy special cases of this proposition:

Corollary 3.1.12. *With the same assumptions as in the lemma, the following relations are satisfied in $C^3(F,5)/\partial C^3(F,6)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$:*

$$\left[f(x), g(x), h_1(x)h_2(x), \frac{z-h_1(x)h_2(x)}{z-h_1(x)}, z\right]^3_{(-,h_1;-,h_2)} = 0.$$

The same relation holds if the product is placed in the fifth coordinate.

Let g be a product of linear transformations considered as rational function in one of the variables, and let $a, b \in F^\times$. Then

$$[f(x), g(x), h(x), ab, h_1(x)]^4_{(-,a;-,b)} = 0.$$

The same holds if the constants are placed in the third or fifth coordinate.

Proof. For the first claim combine lemma 3.1.9 and proposition 3.1.11. The second claim is proved combining corollary 3.1.10 and proposition 3.1.11. □

Remark 3.1.13. Let us introduce one more special notation. In the following, lots of sums of the form $\sum_{div(g)} \mathcal{Z}$ occur where $\mathcal{Z} \in C^3(F,5)/\partial C^3(F,6)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$. This means that depending on which of the variables x_0 or y_0 occurs in $\mathcal{Z}$, we let this variable run through the divisors of the rational function g. ◊

We continue by proving the following generalization of [GMS99, Lem. 2.8]

Lemma 3.1.14. *Let $f_i, i = 1,3,4,5; g, h$ be rational functions in one variable. Let further be f_2 be a product of linear transformations of the form $(a_1x+b_1y+c_1)/(a_2x+b_2y+c_2)$. Set $y = y(x)$ to be the solution of $f_2(x,y) = 0, \infty$ respectively. Assume that one has $f_4(y(x)) = f_5(x) = g(x)h(x)$ and that $g(y(x)) = g(x)$ or $g(y(x)) = h(x)$. Then the following relations* *hold in* $C^3(F,5)/\partial C^3(F,6)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$ *for admissible terms:*

$$\begin{aligned}&2\mathcal{Z}(f_1,f_2,f_3,f_4,f_5)=\\&\mathcal{Z}(f_1,f_2,f_3,f_4,g)+\mathcal{Z}(f_1,f_2,f_3,f_4,h)+\mathcal{Z}(f_1,f_2,f_3,g,f_5)+\mathcal{Z}(f_1,f_2,f_3,h,f_5)\\&-\sum_{div(f_1)}\pm\left[f_2(x,y_0),f_3(x),f_4(y_0),\frac{z-f_5(x)}{z-g(x)},z\right]+\sum_{div(f_3)}\pm\left[f_1(y),f_2(x_0,y),\frac{z-f_4(y)}{z-g(y)},z,f_5(x_0)\right]\\&-\sum_{div(f_5)}\pm\left[f_1(y),f_2(x_0,y),f_3(x_0),\frac{z-f_4(y)}{z-g(y)},z\right]\end{aligned}$$

and further

$$\begin{aligned}&2\mathcal{Z}(f_1,f_2,f_3,f_4,f_5)=\\&2\mathcal{Z}(f_1,f_2,f_3,g,h)+2\mathcal{Z}(f_1,f_2,f_3,g,g)+2\mathcal{Z}(f_1,f_2,f_3,h,g)+2\mathcal{Z}(f_1,f_2,f_3,h,h)\\&-\sum_{div(f_1)}\pm\left(\left[f_2(x,y_0),f_3(x),f_4(y_0),\frac{z-f_5(x)}{z-g(x)},z\right]-\left[f_2(x,y_0),f_3(x),g(y_0),\frac{z-f_4(x)}{z-g(x)},z\right]^3_{(+,h(y_0))}\right)\\&-\sum_{div(f_3)}\pm\left(\left[f_1(y),f_2(x_0,y),\frac{z-f_5(y)}{z-g(y)},z,g(x_0)\right]^5_{(+,h(x_0))}-\left[f_1(y),f_2(x_0,y),\frac{z-f_4(y)}{z-g(y)},z,f_5(x_0)\right]\right)\\&-\sum_{div(f_5)}\pm\left[f_1(y),f_2(x_0,y),f_3(x_0),\frac{z-f_4(y)}{z-g(y)},z\right]\\&+\sum_{div(g),div(h)}\pm\left[f_1(y),f_2(x_0,y),f_3(x_0),\frac{z-f_5(y)}{z-g(y)},z\right].\end{aligned}$$

Proof. For the first relation compute the boundary of the following term in $C^3(F,6)$:

$$\left[f_1(y),f_2(x,y),f_3(x),f_4(y),\frac{z-f_5(x)}{z-g(x)},z\right]-\left[f_1(y),f_2(x,y),f_3(x),\frac{z-f_4(y)}{z-g(y)},z,f_5(x)\right].$$

The second relation is proved by computing the boundary of

$$\left[f_1(y),f_2(x,y),f_3(x),g(y),\frac{z-f_4(x)}{z-g(x)},z\right]+\left[f_1(y),f_2(x,y),f_3(x),h(y),\frac{z-f_4(x)}{z-g(x)},z\right]$$

and subtracting the boundary of

$$\left[f_1(y),f_2(x,y),f_3(x),\frac{z-f_5(y)}{z-g(y)},z,g(x)\right]+\left[f_1(y),f_2(x,y),f_3(x),\frac{z-f_5(y)}{z-g(y)},z,h(x)\right].$$

Now one invokes the lemma from above. □

Remark 3.1.15. In case $f_4=f_5$ in the above lemma, then the second relation simplifies drastically:

$$\begin{aligned}&\mathcal{Z}(f_1,f_2,f_3,f_4,f_5)=\\&\mathcal{Z}(f_1,f_2,f_3,g,h)+\mathcal{Z}(f_1,f_2,f_3,g,g)+\mathcal{Z}(f_1,f_2,f_3,h,g)+\mathcal{Z}(f_1,f_2,f_3,h,h),\end{aligned}\tag{3.1.7}$$

which is proved by computing the boundary of

$$\left[f_2(x), f_3(x), \frac{z-ab}{z-a}, z, \frac{y-g(x)h(x)}{y-g(x)}, y\right] \in C^3(F,6):$$

One obtains

$$0=\left[f_2(x), f_3(x), ab, \frac{y-g(x)h(x)}{y-g(x)}, y\right]^3_{(-,a;-,b)} + \left[f_2(x), f_3(x), \frac{z-ab}{z-a}, z, g(x)h(x)\right]^5_{(-,g(x);-,h(x))}.$$

But the latter terms vanish in $C^3(F,5)/\partial C^3(F,6) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ because of the second relation of corollary 3.1.12. The result follows. ♢

In order to extend Zhao's proof of Goncharov's relation [Zha07] in our setting, we still need some more preparation:

Proposition 3.1.16. *For some rational functions $f_i, i=1,3,4,5; g,h$ in one variable and p,q rational functions of two variables satisfying the following: The only non-constant solution of $p(x,y)=0$ and $p(x,y)=\infty$ resp. of $q(x,y)=0$ and $q(x,y)=0$ is $y(x)=x$. Then the following relations hold in $C^3(F,5)/\partial C^3(F,6)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$:*

$$\begin{aligned}
&[f_1(y), p(x,y), g(x)h(x), f_4(y), f_5(x)] + [f_1(y), q(x,y), g(x)h(x), f_5(y), f_4(x)]\\
&\quad= [f_1(y), p(x,y), g(x), f_4(y), f_5(x)]^3_{(+,h(x))} + [f_1(y), q(x,y), g(x), f_5(y), f_4(x)]^3_{(+,h(x))}\\
&\quad - \sum_{div(f_1)} \pm \left(\left[p(x,y_0), \frac{z-g(x)h(x)}{z-g(x)}, z, f_4(y_0), f_5(x)\right]\right.\\
&\quad\quad + \left.\left[q(x,y_0), \frac{z-g(x)h(x)}{z-g(x)}, z, f_5(y_0), f_4(x)\right]\right)
\end{aligned}$$

and

$$\begin{aligned}
&[f_1(y), p(x,y), f_3(x), g(y)h(y), f_5(x)] + [f_1(y), q(x,y), f_3(x), f_4(y), g(x)h(x)]\\
&\quad= [f_1(y), p(x,y), f_3(x), g(y), f_5(x)]^4_{(+,h(y))} + [f_1(y), q(x,y), f_3(x), f_4(y), g(x)]^5_{(+,h(x))}\\
&\quad + \sum_{div(f_1)} \pm \left[q(x,y_0), f_3(x), f_4(y_0), \frac{z-g(x)h(x)}{z-g(x)}, z\right]\\
&\quad + \sum_{div(f_3)} \pm \left[f_1(y), p(x_0,y), \frac{z-g(y)h(y)}{z-g(y)}, z, f_5(x_0)\right]\\
&\quad - \sum_{div(f_5)} \pm \left[f_1(y), p(x_0,y), f_3(x_0), \frac{z-g(y)h(y)}{z-g(y)}, z\right].
\end{aligned}$$

Proof. Compute the boundary of

$$\left[f_1(y), p(x,y), \frac{z-g(x)h(x)}{z-g(x)}, z, f_4(y), f_5(x)\right] + \left[f_1(y), q(x,y), \frac{z-g(x)h(x)}{z-g(x)}, z, f_5(y), f_4(x)\right]$$

for the first relation and the one of

$$\left[f_1(y), p(x,y), f_3(x), \frac{z-g(y)h(y)}{z-g(y)}, z, f_5(x)\right] - \left[f_1(y), q(x,y), f_3(x), f_4(y), \frac{z-g(x)h(x)}{z-g(x)}, z\right]$$

for the second. $\square$

Corollary 3.1.17. *In case $g(x)=\alpha \in F^{\times}$ and $h(x)=f_3(x)$ we obtain a special case of the proposition:*

$$\begin{aligned}
[f_1(y), & p(x,y), \alpha f_3(x), f_4(y), f_5(x)] + [f_1(y), q(x,y), \alpha f_3(x), f_5(y), f_4(x)] \\
&= [f_1(y), p(x,y), \alpha, f_4(y), f_5(x)]^3_{(+,f_3(x))} + [f_1(y), q(x,y), \alpha, f_5(y), f_4(x)]^3_{(+,f_3(x))} \\
&\quad - \sum_{div(f_1)} \pm \left(\left[p(x,y_0), \frac{z-\alpha f_3(x)}{z-\alpha}, z, f_4(y_0), f_5(x)\right]\right. \\
&\qquad \left. + \left[q(x,y_0), \frac{z-\alpha f_3(x)}{z-g(x)}, z, f_5(y_0), f_4(x)\right]\right).
\end{aligned}$$

We will have to make use of the above proposition in case the first coordinate is a product as well:

Proposition 3.1.18. *With the same assumptions as above the following identity of terms in $C^3(F,5)/\partial C^3(F,6) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ holds:*

$$\begin{aligned}
&[g(y)h(y), p(x,y), f_3(x), f_4(y), f_5(x)] + [g(y)h(y), q(x,y), f_3(x), f_5(y), f_4(x)] \\
&= [g(y), p(x,y), f_3(x), f_4(y), f_5(x)]^1_{(+,h(y))} + [g(y), q(x,y), f_3(x), f_5(y), f_4(x)]^1_{(+,h(y))} \\
&+ \sum_{div(f_3)} \pm \left(\left[\frac{z-g(y)h(y)}{z-g(y)}, z, p(x_0,y), f_4(y), f_5(x_0)\right] + \left[\frac{z-g(y)h(y)}{z-g(y)}, z, q(x_0,y), f_5(y), f_4(x_0)\right]\right) \\
&- \sum_{div(f_4)} \pm \left(\left[\frac{z-g(y_0)h(y_0)}{z-g(y_0)}, z, p(x,y_0), f_3(x), f_5(x)\right] - \left[\frac{z-g(y)h(y)}{z-g(y)}, z, q(x_0,y), f_3(x_0), f_5(y)\right]\right) \\
&+ \sum_{div(f_5)} \pm \left(\left[\frac{z-g(y)h(y)}{z-g(y)}, z, p(x_0,y), f_3(x_0), f_4(y)\right] - \left[\frac{z-g(y_0)h(y_0)}{z-g(y_0)}, z, q(x,y_0), f_3(x), f_4(x)\right]\right) \\
&- \sum_{div(p)} \left[\frac{z-g(y(x))h(y(x))}{z-g(y(x))}, z, f_3(x), f_4(y(x)), f_5(x)\right] \\
&- \sum_{div(q)} \left[\frac{z-g(y(x))h(y(x))}{z-g(y(x))}, z, f_3(x), f_5(y(x)), f_4(x)\right].
\end{aligned}$$

Proof. The expression

$$\left[\frac{z-g(y)h(y)}{z-g(y)}, z, p(x,y), f_3(x), f_4(y), f_5(x)\right] + \left[\frac{z-g(y)h(y)}{z-g(y)}, z, q(x,y), f_3(x), f_5(y), f_4(x)\right]$$

has the desired relation as its boundary. $\square$

Corollary 3.1.19. *In case $g(y)=\alpha \in F^{\times}$ and $h(y)=f_1(y)$ this proposition amounts to the* *following relation in* $C^3(F,5)/\partial C^3(F,6)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$*:*

$$\begin{aligned}&[\alpha f_1(y),p(x,y),f_3(x),f_4(y),f_5(x)]+[\alpha f_1(y),q(x,y),f_3(x),f_5(y),f_4(x)]\\&\quad=[f_1(y),p(x,y),f_3(x),f_4(y),f_5(x)]+[f_1(y),q(x,y),f_3(x),f_5(y),f_4(x)]\end{aligned}$$

and lower order terms

$$\begin{aligned}&+\sum_{div(f_3)}\pm\left(\left[\frac{z-\alpha f_1(y)}{z-\alpha},z,p(x_0,y),f_4(y),f_5(x_0)\right]+\left[\frac{z-\alpha f_1(y)}{z-\alpha},z,q(x_0,y),f_5(y),f_4(x_0)\right]\right)\\&-\sum_{div(f_4)}\pm\left(\left[\frac{z-\alpha f_1(y_0)}{z-\alpha},z,p(x,y_0),f_3(x),f_5(x)\right]-\left[\frac{z-\alpha f_1(y)}{z-\alpha},z,q(x_0,y),f_3(x_0),f_5(y)\right]\right)\\&+\sum_{div(f_5)}\pm\left(\left[\frac{z-\alpha f_1(y)}{z-\alpha},z,p(x_0,y),f_3(x_0),f_4(y)\right]-\left[\frac{z-\alpha f_1(y_0)}{z-\alpha},z,q(x,y_0),f_3(x),f_4(x)\right]\right)\\&-\sum_{div(p)}\left[\frac{z-\alpha f_1(y(x))}{z-\alpha},z,f_3(x),f_4(y(x)),f_5(x)\right]\\&-\sum_{div(q)}\left[\frac{z-\alpha f_1(y(x))}{z-\alpha},z,f_3(x),f_5(y(x)),f_4(x)\right].\end{aligned}$$

The next and last proposition will be helpful for Zhao's proof:

Proposition 3.1.20. *Let* $s,t,u,v\in F$ *be constants chosen to guarantee admissibility in all of the terms involved. Then the following identity hold in* $C^3(F,5)/\partial C^3(F,6)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$*:*

$$2\left[\frac{u-y}{v-y},1-\frac{y}{x},\frac{1-sx}{1-tx},y,x\right]=2C_{us}-2C_{vs}-2C_{ut}+2C_{vt}$$

plus terms of lower order:

$$\begin{aligned}&+2\left[1-\frac{u}{x},\frac{z-1}{z-(1-tx)},z,u,x\right]+2\left[1-\frac{v}{x},\frac{z-1}{z-(1-tx)},z,v,x\right]\\&-2\left[1-\frac{u}{x},\frac{z-\frac{1-sx}{1-tx}}{z-(1-sx)},z,u,x\right]-2\left[1-\frac{v}{x},\frac{z-\frac{1-sx}{1-tx}}{z-(1-sx)},z,v,x\right]\\&-2\left[\frac{z-1}{z-(1-\frac{v}{y})},z,1-sy,y,\frac{1}{s}\right]+2\left[\frac{z-1}{z-(1-\frac{v}{y})},z,1-ty,y,\frac{1}{t}\right]\end{aligned}$$

plus more terms of lower order

$$\begin{aligned}&+2\left[\frac{z-\frac{u-y}{v-y}}{z-(1-\frac{u}{y})},z,1-sy,y,\frac{1}{s}\right]-2\left[\frac{z-\frac{u-y}{v-y}}{z-(1-\frac{u}{y})},z,1-ty,y,\frac{1}{t}\right]\\&+R(u,s)-R(u,t)-R(v,s)+R(v,t),\end{aligned}$$

where $R(u,s)$ *is defined by*

$$R(u,s):=2\left[1-\frac{us}{y},1-\frac{y}{x},1-x,\frac{1}{s},\frac{x}{s}\right]+2\left[1-\frac{us}{y},1-\frac{y}{x},1-x,\frac{y}{s},\frac{1}{s}\right]$$

plus more extra terms

$$-\left[1-\frac{us}{x},1-x,u,\frac{z-\frac{x}{s}}{z-x},z\right]+\left[1-\frac{us}{x},1-x,\frac{z-u}{z-us},z,\frac{x}{s}\right]$$
$$-\left[1-\frac{us}{x},1-x,us,\frac{z-\frac{x}{s}}{z-x},z\right]^3_{(-,\frac{1}{s})}+\left[1-\frac{us}{x},1-x,\frac{z-u}{z-us},z,x\right]^5_{(+,\frac{1}{s})}$$
$$+\left[1-\frac{us}{x},1-x,\frac{x}{s},\frac{z-\frac{x}{s}}{x},z\right]-\left[1-\frac{us}{x},1-x,\frac{z-\frac{x}{s}}{z-x},z,\frac{x}{s}\right]$$
$$+\left[1-\frac{us}{x},1-x,x,\frac{z-\frac{x}{s}}{z-x},z\right]^3_{(+,\frac{1}{s})}-\left[1-\frac{us}{x},1-x,\frac{z-\frac{x}{s}}{z-x},z,x\right]^5_{(-,\frac{1}{s})}.$$

The right-hand side except for the C-terms will be denoted by $R_1(s,t,u,v)$.

Proof. The proof first makes use of proposition 3.1.7 two times. The first and the third coordinate have to be split and some resulting terms inverted. In the end, we will be able to identify the C-terms of the claim. Let us start by splitting the first coordinate of the given term:

$$\left[\frac{u-y}{v-y},1-\frac{y}{x},\frac{1-sx}{1-tx},y,x\right]=\left[1-\frac{u}{y},1-\frac{y}{x},\frac{1-sx}{1-tx},y,x\right]^1_{(+,\frac{y}{y-v})}$$
$$+\left[\frac{z-\frac{u-y}{v-y}}{z-(1-\frac{u}{y})},z,1-sy,y,\frac{1}{s}\right]-\left[\frac{z-\frac{u-y}{v-y}}{z-(1-\frac{u}{y})},z,1-ty,y,\frac{1}{t}\right].$$

The second term now has to be inverted:

$$\left[1-\frac{u}{y},1-\frac{y}{x},\frac{1-sx}{1-tx},y,x\right]^1_{(+,\frac{y}{y-v})}=\left[1-\frac{u}{y},1-\frac{y}{x},\frac{1-sx}{1-tx},y,x\right]^1_{(-,1-\frac{v}{y})}$$
$$-\left[\frac{z-1}{z-(1-\frac{v}{y})},z,1-sy,y,\frac{1}{s}\right]+\left[\frac{z-1}{z-(1-\frac{v}{y})},z,1-ty,y,\frac{1}{t}\right].$$

Then we split the third coordinate in the same way to obtain

$$\left[1-\frac{u}{y},1-\frac{y}{x},\frac{1-sx}{1-tx},y,x\right]^1_{(-,1-\frac{v}{y})}$$
$$=\left[1-\frac{u}{y},1-\frac{y}{x},1-sx,y,x\right]^3_{(+,\frac{1}{1-tx})}-\left[1-\frac{v}{y},1-\frac{y}{x},1-sx,y,x\right]^3_{(-,\frac{1}{1-tx})}$$
$$-\left[1-\frac{u}{x},\frac{z-\frac{1-sx}{1-tx}}{z-(1-sx)},z,u,x\right]-\left[1-\frac{v}{x},\frac{z-\frac{1-sx}{1-tx}}{z-(1-sx)},z,v,x\right].$$

Now we shall invert some coordinates again:

$$\left[1-\frac{u}{y},1-\frac{y}{x},1-sx,y,x\right]^3_{(+,\frac{1}{1-tx})}-\left[1-\frac{v}{y},1-\frac{y}{x},1-sx,y,x\right]^3_{(-,\frac{1}{1-tx})}$$

which is the same as

$$= \left[1-\frac{u}{y}, 1-\frac{y}{x}, 1-sx, y, x\right]^3_{(-,1-tx)} - \left[1-\frac{v}{y}, 1-\frac{y}{x}, 1-sx, y, x\right]^3_{(+,1-tx)}$$
$$+ \left[1-\frac{u}{x}, \frac{z-1}{z-(1-tx)}, z, u, x\right] + \left[1-\frac{v}{x}, \frac{z-1}{z-(1-tx)}, z, v, x\right].$$

After the obvious reparametrizations we have to get rid of the constants in the fourth and fifth coordinate. For this, we shall use proposition 3.1.14:

$$2\left[1-\frac{us}{y}, 1-\frac{y}{x}, 1-x, \frac{y}{s}, \frac{x}{s}\right]^1_{(-,1-\frac{ut}{y})} - 2\left[1-\frac{vs}{y}, 1-\frac{y}{x}, 1-x, \frac{y}{s}, \frac{x}{s}\right]^1_{(+,1-\frac{vt}{y})}$$
$$= 2C_{us} - 2C_{ut} - 2C_{vs} + 2C_{vt}$$
$$+ R(u,s) - R(u,t) - R(v,s) + R(v,t),$$

where the R-terms are there to denote all the extra terms coming from an application of the second formula of proposition 3.1.14. Adding everything up gives the asserted formula. □

Corollary 3.1.21. *With the same assumptions as in the proposition, the following identity holds as well:*

$$2\left[\frac{1-uy}{1-vy}, 1-\frac{x}{y}, \frac{s-x}{t-x}, y, x\right] = 2C_{us} - 2C_{vs} - 2C_{ut} + 2C_{vt}$$
$$-2\left[1-\frac{s}{x}, \frac{1-ux}{1-vx}, \frac{z-x}{z-1}, s, x\right] + 2\left[1-\frac{t}{x}, \frac{1-ux}{1-vx}, \frac{z-x}{z-1}, t, x\right]$$
$$-2\left[\frac{s-y}{t-y}, 1-uy, \frac{z-\frac{1}{u}}{z-1}, y, z\right] + 2\left[\frac{s-y}{t-y}, 1-vy, \frac{z-\frac{1}{v}}{z-1}, y, z\right]$$
$$+2\left[\frac{s-y}{t-y}, 1-\frac{y}{x}, \frac{1-ux}{1-vx}, y, x\right]$$
$$+2\left[z, 1-ux, \frac{z-\frac{s-x}{t-x}}{z-1}, \frac{1}{u}, x\right] - 2\left[z, 1-vx, \frac{z-\frac{s-x}{t-x}}{z-1}, \frac{1}{v}, x\right]$$
$$-2\left[\frac{z-\frac{1-uy}{1-vy}}{z-1}, 1-\frac{s}{y}, z, y, s\right] + 2\left[\frac{z-\frac{1-uy}{1-vy}}{z-1}, 1-\frac{t}{y}, z, y, t\right] + R_1(s,t,u,v).$$

The right-hand side except for the C-terms will be denoted by $R_2(s,t,u,v)$.

Proof. We reduce to the proposition: Taking the boundary of $[\frac{z-\frac{1-uy}{1-vy}}{z-1}, 1-\frac{x}{y}, \frac{s-x}{t-x}, z, y, x]$ gives

$$\left[\frac{1-uy}{1-vy}, 1-\frac{x}{y}, \frac{s-x}{t-x}, y, x\right] = \left[1-\frac{x}{y}, \frac{s-x}{t-x}, \frac{1-uy}{1-vy}, y, x\right]$$
$$-\left[\frac{z-\frac{1-uy}{1-vy}}{z-1}, 1-\frac{s}{y}, z, y, s\right] + \left[\frac{z-\frac{1-uy}{1-vy}}{z-1}, 1-\frac{t}{y}, z, y, t\right].$$

Taking the boundary of $[z, 1-\frac{x}{y}, \frac{z-\frac{s-x}{t-x}}{z-1}, \frac{1-uy}{1-vy}, y, x]$ further transforms the first term on the right-hand side into

$$\left[1-\frac{x}{y}, \frac{s-x}{t-x}, \frac{1-uy}{1-vy}, y, x\right] = -\left[\frac{s-x}{t-x}, 1-\frac{x}{y}, \frac{1-uy}{1-vy}, y, x\right] + \left[z, 1-ux, \frac{z-\frac{s-x}{t-x}}{z-1}, \frac{1}{u}, x\right] - \left[z, 1-vx, \frac{z-\frac{s-x}{t-x}}{z-1}, \frac{1}{v}, x\right].$$

After changing the roles of x, y, we invert the right two coordinates in the first term on the right-hand side: Take the boundary of $[\frac{s-y}{t-y}, 1-\frac{y}{x}, \frac{1-ux}{1-vx}, \frac{z-x}{z-1}, y, z]$:

$$\begin{aligned}-\left[\frac{s-y}{t-y}, 1-\frac{y}{x}, \frac{1-ux}{1-vx}, x, y\right] = &\left[\frac{s-x}{t-x}, \frac{1-ux}{1-vx}, \frac{z-x}{z-1}, x, z\right] \\ &-\left[1-\frac{s}{x}, \frac{1-ux}{1-vx}, \frac{z-x}{z-1}, s, x\right] + \left[1-\frac{t}{x}, \frac{1-ux}{1-vx}, \frac{z-x}{z-1}, t, x\right] \\ &-\left[\frac{s-y}{t-y}, 1-uy, \frac{z-\frac{1}{u}}{z-1}, y, z\right] + \left[\frac{s-y}{t-y}, 1-vy, \frac{z-\frac{1}{v}}{z-1}, y, z\right] \\ &+\left[\frac{s-y}{t-y}, 1-\frac{y}{x}, \frac{1-ux}{1-vx}, y, x\right].\end{aligned}$$

Now apply the proposition to the last term on the right-hand side. □

Remark 3.1.22. Note that already these auxiliary relations appear very complicated with all the extra terms. It seems to be almost impossible to find a combination of these terms which vanishes. Further we do not have any results as to the orders of these terms as we had for the extra terms in $CH^2(F, 3)$. ♢

Let us, nevertheless, prove a first relation with these rules: A distribution relation.

Proposition 3.1.23. *If F contains a primitive n^{th} root of unity ζ. Then every $a \in F^\times$ gives rise to a distribution relation in $C^3(F, 5)/\partial C^3(F, 6) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$:*

$$\begin{aligned}n^2 C_{a^n} = n^4 \sum_{i=1}^{n} C_{\zeta a} + n \sum_{i=2}^{n} \left[\frac{z-\prod_{j=2}^{i}(1-\zeta^j \frac{a}{x})}{z-(1-\frac{a}{x})}, z, 1-x^n, a^n, x^n\right] \\ - n\sum_{i=2}^{n}\left[1-\frac{a}{x}, \frac{z-\prod_{j=2}^{i}(1-\zeta^j x)}{z-(1-\zeta^i x)}, a^n, x^n\right] - n^3 \sum_{i,j=2}^{n}\left[1-\zeta^i\frac{a}{x}, 1-x, \zeta^i a, \frac{z-x^j}{z-x}, z\right].\end{aligned}$$

Proof. We just prove the case $n = 2$:

$$4C_{a^2} = \left[1-\left(\frac{a}{y}\right)^2, 1-\left(\frac{y}{x}\right)^2, 1-x^2, y^2, x^2\right] = \left[1-\frac{a}{y}, 1-\left(\frac{y}{x}\right)^2, 1-x^2, y^2, x^2\right]^1_{(+,1+\frac{a}{y})},$$

which we split piece by piece in the following:

$$
\begin{aligned}
&= 2\left[1-\frac{a}{y}, 1-\frac{y}{x}, 1-x^2, y^2, x^2\right]^1_{(+,1+\frac{a}{y})} + 2\left[\frac{z-(1-\frac{a^2}{x^2})}{z-(1-\frac{a}{x})}, z, 1-x^2, a^2, x^2\right] \\
&= 4\left[1-\frac{a}{y}, 1-\frac{y}{x}, 1-x, y^2, x^2\right]^1_{(+,1+\frac{a}{y})} - 2\left[1-\frac{a}{x}, \frac{z-(1-x^2)}{z-(1-x)}, z, a^2, x^2\right]^1_{(+,1+\frac{a}{x})} \\
&\quad + 2\left[\frac{z-(1-\frac{a^2}{x^2})}{z-(1-\frac{a}{x})}, z, 1-x^2, a^2, x^2\right] \\
&= 8\left[1-\frac{a}{y}, 1-\frac{y}{x}, 1-x, y, x^2\right]^1_{(+,1+\frac{a}{y})} - 4\left[1-\frac{a}{x}, 1-x, \frac{z-x^2}{z-x}, z, x^2\right]^1_{(+,1+\frac{a}{x})} \\
&\quad - 2\left[1-\frac{a}{x}, \frac{z-(1-x^2)}{z-(1-x)}, z, a^2, x^2\right]^1_{(+,1+\frac{a}{x})} + 2\left[\frac{z-(1-\frac{a^2}{x^2})}{z-(1-\frac{a}{x})}, z, 1-x^2, a^2, x^2\right].
\end{aligned}
$$

Then finally

$$
\begin{aligned}
4C_{a^2} &= 16C_a + 16C_{-a} \\
&\quad - 8\left[1-\frac{a}{x}, 1-x, a, \frac{z-x^2}{z-x}, z\right]^3_{(-,-a)} + 8\left[1-\frac{a}{x}, 1-x, x, \frac{z-x^2}{z-x}, z\right]^1_{(+,1+\frac{a}{x})} \\
&\quad - 4\left[1-\frac{a}{x}, 1-x, \frac{z-x^2}{z-x}, z, x^2\right]^1_{(+,1+\frac{a}{x})} \\
&\quad - 2\left[1-\frac{a}{x}, \frac{z-(1-x^2)}{z-(1-x)}, z, a^2, x^2\right]^1_{(+,1+\frac{a}{x})} + 2\left[\frac{z-(1-\frac{a^2}{x^2})}{z-(1-\frac{a}{x})}, z, 1-x^2, a^2, x^2\right].
\end{aligned}
$$

According to proposition 3.1.11, the second and the third of the extra terms vanish in the quotient $C^3(F,5)/\partial C^3(F,6) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ proving the claim for $n = 2$. □

Now we can easily deduce

Lemma 3.1.24. $C_{-1} = -\frac{3}{4}C_1 \in CH^3(\mathbb{Q}, 5) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$.

Proof. First note that $\partial C_1 = 0$ and $2\partial(C_{-1}) = 0$ in the Chow group. Then one uses the distribution relation for $n = 2$ to obtain $4C_1 = 16C_1 + 16C_{-1} \in C^3(F,5)/\partial C^3(F,6) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$. Dividing by 4 proves the assertion. □

Remark 3.1.25. As we have discussed in the introduction to this chapter, this result is in some way very weak. The homology of $C^3(F,\bullet) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ in fact only computes the Chow group $CH^3(F,5) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$, and $\ell C_{\zeta_\ell} \notin B_3(F)$, not even in $B_3(F) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$. Nevertheless, one can use M. Kerr's cycles (3.1.1) and a similar proof of the distribution relation. This will be a relation without extra terms in the Chow group: ◊

Proposition 3.1.26. *Let* $Z(\zeta_\ell) \in CH^3(\mathbb{Q}(\zeta_\ell), 5)$ *be the cycles from* (3.1.1). *Then the fol-*

lowing relation holds in $CH^3(\mathbb{Q}(\zeta_\ell),5)$:

$$2n^2\ell^2 Z(\zeta_\ell^n) = \sum_{i=1}^{n} 2n^4\ell^2 Z(\zeta_\ell^i).$$

Proof. We start in the case $n=2$ with a reparametrization of $8\ell^2 Z(\zeta_\ell^2)$, namely

$$\begin{aligned}
-8\ell^2 Z(\zeta_\ell^2) &= 2\left[\frac{x^2}{x^2-1},\frac{y^2}{y^2-1},1-\zeta_\ell^2x^2y^2,x^{2\ell},y^{2\ell}\right]\\
&\quad+\left[\frac{x^2}{x^2-1},\frac{1}{1-\zeta_\ell^2x^2},\frac{(y^2-x^{2\ell})(y^2-x^{-2\ell})}{(y^2-1)^2},x^{2\ell}y^2,x^{-2\ell}y^2\right]\\
&= 4\left[\frac{x}{x-1},\frac{y^2}{y^2-1},1-\zeta_\ell^2x^2y^2,x^{2\ell},y^{2\ell}\right]\\
&\quad+2\left[\frac{x}{x-1},\frac{1}{1-\zeta_\ell^2x^2},\frac{(y^2-x^{2\ell})(y^2-x^{-2\ell})}{(y^2-1)^2},x^{2\ell}y^2,x^{-2\ell}y^2\right]
\end{aligned}$$

as one obtains by computing the boundaries of $2[\frac{z-\frac{x^2}{x^2-1}}{z-\frac{x}{x-1}},z,\frac{y^2}{y^2-1},1-\zeta_\ell^2x^2y^2,x^{2\ell},y^{2\ell}]$ resp. the one of $[\frac{z-\frac{x^2}{x^2-1}}{z-\frac{x}{x-1}},z,\frac{1}{1-\zeta_\ell^2x^2},\frac{(y^2-x^{2\ell})(y^2-x^{-2\ell})}{(y^2-1)^2},x^{2\ell}y^2,x^{-2\ell}y^2]$ with the aid of proposition 3.1.7, and by the obvious reparametrizations. Note that all extra terms immediately cancel each other.

$$\begin{aligned}
&= 8\left[\frac{x}{x-1},\frac{y}{y-1},1-\zeta_\ell^2x^2y^2,x^{2\ell},y^{2\ell}\right]\\
&\quad+2\left[\frac{x}{x-1},\frac{1}{1-\zeta_\ell x},\frac{(y^2-x^{2\ell})(y^2-x^{-2\ell})}{(y^2-1)^2},x^{2\ell}y^2,x^{-2\ell}y^2\right]^2_{(+,\frac{1}{1+\zeta_\ell x})}
\end{aligned}$$

as one obtains as above by computing the boundaries of $4[\frac{x}{x-1},\frac{z-\frac{y^2}{y^2-1}}{z-\frac{y}{y-1}},z,1-\zeta_\ell^2x^2y^2,x^{2\ell},y^{2\ell}]$ resp. $2[\frac{x}{x-1},\frac{z-\frac{1}{1-\zeta_\ell^2x^2}}{z-\frac{1}{1-\zeta_\ell x}},z,\frac{(y^2-x^{2\ell})(y^2-x^{-2\ell})}{(y^2-1)^2},x^{2\ell}y^2,x^{-2\ell}y^2]$.

$$\begin{aligned}
&= 8\left[\frac{x}{x-1},\frac{y}{y-1},1-\zeta_\ell xy,x^{2\ell},y^{2\ell}\right]^3_{(+,1-\zeta_\ell xy)}\\
&\quad+4\left[\frac{x}{x-1},\frac{1}{1-\zeta_\ell x},\frac{(y-x^{\ell})(y-x^{-\ell})}{(y-1)^2},x^{2\ell}y^2,x^{-2\ell}y^2\right]^2_{(+,\frac{1}{1+\zeta_\ell x})}
\end{aligned}$$

by computing the boundaries of $8[\frac{x}{x-1},\frac{y}{y-1},\frac{z-(1-\zeta_\ell^2x^2y^2)}{z-(1-\zeta_\ell^2 xy)},z,x^{2\ell},y^{2\ell}]$ resp. the one of the expressions $4[\frac{x}{x-1},\frac{1}{1-\zeta_\ell x},\frac{z-\frac{(y^2-x^{2\ell})(y^2-x^{-2\ell})}{(y^2-1)^2}}{z-\frac{(y-x^{\ell})(y-x^{-\ell})}{(y-1)^2}},z,x^{2\ell}y^2,x^{-2\ell}y^2]^2_{(+,\frac{1}{1+\zeta_\ell x})}$.

$$\begin{aligned}
&= 16\left[\frac{x}{x-1},\frac{y}{y-1},1-\zeta_\ell xy,x^{\ell},y^{2\ell}\right]^3_{(+,1+\zeta_\ell xy)}\\
&\quad+8\left[\frac{x}{x-1},\frac{1}{1-\zeta_\ell x},\frac{(y-x^{\ell})(y-x^{-\ell})}{(y-1)^2},x^{\ell}y,x^{-2\ell}y^2\right]^2_{(+,\frac{1}{1+\zeta_\ell x})}
\end{aligned}$$

by computing the boundaries of $8[\frac{x}{x-1}, \frac{y}{y-1}, 1-\zeta_\ell x, \frac{z-x^{2\ell}}{z-x^\ell}, z, y^{2\ell}]^3_{(+,1+\zeta_\ell x)}$ resp. of the expressions $4[\frac{x}{x-1}, \frac{1}{1-\zeta_\ell x}, \frac{(y-x^\ell)(y-x^{-\ell})}{(y-1)^2}, \frac{z-x^{2\ell}y^2}{z-x^\ell y}, z, x^{-2\ell}y^2]^2_{(+,\frac{1}{1+\zeta_\ell x})}$.

$$= 32\left[\frac{x}{x-1}, \frac{y}{y-1}, 1-\zeta_\ell xy, x^\ell, y^\ell\right]^3_{(+,1+\zeta_\ell xy)}$$
$$+ 8\left[\frac{x}{x-1}, \frac{1}{1-\zeta_\ell x}, \frac{(y-x^\ell)(y-x^{-\ell})}{(y-1)^2}, x^\ell y, x^{-\ell}y\right]^2_{(+,\frac{1}{1+\zeta_\ell x})}$$

by computing the boundaries of $16[\frac{x}{x-1}, \frac{y}{y-1}, 1-\zeta_\ell x, x^\ell, \frac{z-y^{2\ell}}{z-y^\ell}, z]^3_{(+,1+\zeta_\ell x)}$ resp. of the expressions $4[\frac{x}{x-1}, \frac{1}{1-\zeta_\ell x}, \frac{(y-x^\ell)(y-x^{-\ell})}{(y-1)^2}, x^\ell y, \frac{z-x^{-2\ell}y^2}{z-x^{-\ell}y}, z]^2_{(+,\frac{1}{1+\zeta_\ell x})}$.

But the right-hand side is just equal to $-16\left(2\ell^2 Z(\zeta_\ell) + 2\ell^2 Z(-\zeta_\ell)\right)$. Whence the claim in general follows by a repetition of similar arguments. □

Corollary 3.1.27. *Let $F \supset \mathbb{Q}$ be any number field. Then the following relation holds in $CH^3(F,5)$:*

$$-96Z(1) = 128Z(-1).$$

Proof. Since -1 is contained in any number field, the cycles $Z(1)$ and $Z(-1)$ are defined in the Chow group. Then the claim is an easy application of the proposition for $\ell = n = 2$. □

Remark 3.1.28. Unfortunately, we cannot go much beyond this distribution relations for Kerr's cycles with computations in the integral Chow groups. This is because the defining relations of the Bloch group $B_3(F)$ are too complicated to choose all the cycles involved among the cyclotomic elements. Alternatively, we would need some cycles defined over $\mathbb{Z}$ representing the trilogarithm for arbitrary arguments in $F^\times$: These are not yet available.

We can only reprove relations from [GMS99] and [Zha07] in $C^3(F,5)/\partial C^3(F,6) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ knowing that this will only provide a map $B_3(F) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \to CH^3(F,5) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$. ♢

3.2 The Kummer – Spence – relation modulo 2-torsion

In this section, we mimic the steps of Gangl's and Müller-Stach's proof of the Kummer – Spence – relation in $C^3(F,5)/\partial C^3(F,6) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$. The steps are the same as in the both authors took, but there are many more extra terms to consider because of proposition 3.1.7. We have to remark – as pointed out in a preliminary version of [Zha07] – that there is a slight flaw in Gangl's and Müller-Stach's proof of this relation, because the proof uses an inversion relation which the authors derive from the Kummer – Spence relation. We fix this little problem as suggested by Zhao. In consequence, our relation differs from the original relation in [GMS99] by several applications of the inversion relation to be derived in the next section. Our result is the following:

Proposition 3.2.1. *Let $a, b \in F^{\times}$ subject to the conditions $a, b \neq 0, 1$ and $a \neq b, 1-b$. Let further*

$$\begin{aligned}\mathcal{KS}(a,b) := C_{\frac{b(1-b)}{a(1-a)}} + C_{\frac{a(1-b)}{b(1-a)}} - C_{\frac{ab}{(1-a)(1-b)}} - 2C_{\frac{b}{a}} - 2C_{\frac{b}{1-a}} \\ - 2C_{\frac{1-b}{a}} - 2C_{\frac{1-b}{1-a}} + 2C_{\frac{1}{a}} + 2C_{\frac{1}{1-a}} + 2C_{\frac{b}{b-1}}.\end{aligned}$$

Then the following relation holds in $C^3(F,5)/\partial C^3(F,6) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$.

$$\begin{aligned}
4\mathcal{KS}(a,b) = {} & 4 \sum_{y_0=b,1-b} \left[\frac{z - \left(1-\frac{y_0}{x}\right)\left(1-\frac{y_0}{1-x}\right)}{z-\left(1-\frac{y_0}{x}\right)}, z, \left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right), y_0, x \right] \\
& - 4 \sum_{x_0=a,1-a} \left[\left(1-\frac{b}{y}\right)\left(1-\frac{b}{1-y}\right), \frac{z-\left(1-\frac{y_0}{x}\right)\left(1-\frac{y_0}{1-x}\right)}{z-\left(1-\frac{y_0}{x}\right)}, z, y, x_0 \right] \\
& + \left[1-\frac{b(1-b)}{y(1-y)}, 1-\frac{y(1-y)}{x(1-x)}, 1-\frac{x(1-x)}{a(1-a)}, \frac{1}{a(1-a)}, x(1-x) \right] \\
& + \left[1-\frac{b(1-b)}{y(1-y)}, 1-\frac{y(1-y)}{x(1-x)}, 1-\frac{x(1-x)}{a(1-a)}, y(1-y), \frac{1}{a(1-a)} \right] \\
& - 4\left[\frac{y-(1-b)}{y-1}, \frac{x-1}{x}, \left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right), y, x \right] \\
& - 4\left[\frac{z-\left(1-\frac{(1-b)}{1-x}\right)\frac{x-1}{x}}{z-\frac{x-1}{x}}, z, \left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right), 1-b, x \right] \\
& + 4 \sum_{x_0=a,1-a} \left[\frac{y-(1-b)}{y-1}, \frac{z-\left(1-\frac{y}{1-x_0}\right)\frac{x_0-1}{x_0}}{z-\frac{x_0-1}{x_0}}, z, y, x_0 \right] \\
& + 4\left[\frac{(1-b)-x}{1-x}, \frac{x-a}{1-a}\cdot\frac{x-(1-a)}{ax}, b, \frac{z-1}{z-x}, z \right] \\
& + 4\left[\frac{y-(1-b)}{y-1}, \frac{x}{x-1}, \left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right), y, x \right] \\
& + 4\left[\frac{z-\left(1-\frac{(1-b)}{x}\right)\frac{x}{x-1}}{z-\frac{x}{x-1}}, z, \left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right), 1-b, x \right] \\
& - 4 \sum_{x_0=a,1-a} \left[\frac{y-(1-b)}{y-1}, \frac{z-\left(1-\frac{y}{x_0}\right)\frac{x_0}{x_0-1}}{z-\frac{x_0}{x_0-1}}, z, y, x_0 \right] \\
& - \left[\frac{z-\left(1-\frac{b}{1-ay}\right)}{z+\frac{ay}{1-ay}}, z, 1-y, ay, a \right] - \left[\frac{z-\left(1-\frac{b}{1-(1-a)y}\right)}{z+\frac{(1-a)y}{1-(1-a)y}}, z, 1-y, (1-a)y, a \right] \\
& + 4 \sum_{x_0=a,1-a} \left[\frac{z-\frac{(1-b)-y}{(1-b)(1-y)}}{z-\frac{(1-b)-y}{1-y}}, z, \frac{y-x_0}{1-x_0}, \frac{1-y}{y}, \frac{1-x_0}{x_0} \right]
\end{aligned}$$

and some more terms

$$
\begin{aligned}
&+4\left[\frac{z-\frac{(1-b)-x}{(1-b)(1-x)}}{z-\frac{(1-b)-x}{1-x}}, z, \frac{x-a}{1-a}\cdot\frac{x-(1-a)}{a}, \frac{b}{1-b}, \frac{1-x}{x}\right]\\
&-4\sum_{x_0=a,1-a}\left[\frac{(1-b)-y}{(1-b)(1-y)}, \frac{z-\frac{y-x_0}{y(1-x_0)}}{z-\frac{y-x_0}{1-x_0}}, z, \frac{1-y}{y}, \frac{1-x_0}{x_0}\right]\\
&+4\left[\frac{(1-b)-x}{(1-b)(1-x)}, \frac{z-\frac{x-a}{1-a}\cdot\frac{x-(1-a)}{a}}{z-\frac{x-a}{1-a}}, z, \frac{b}{1-b}, \frac{1-x}{x}\right]\\
&+4\left[\frac{(1-b)-y}{(1-b)(1-y)}, \frac{y-x}{y(1-x)}, \frac{x-a}{x(1-a)}, \frac{1-y}{y}, \frac{a}{1-a}\right]\\
&+4\left[\frac{(1-b)-y}{(1-b)(1-y)}, \frac{y-x}{y(1-x)}, \frac{x-a}{x(1-a)}, \frac{a}{1-a}, \frac{1-x}{x}\right]\\
&-4\left[\frac{(1-b)-x}{(1-b)(1-x)}, \frac{z-\frac{x-(1-a)}{xa}}{z-\frac{x-(1-a)}{a}}, z, \frac{b}{1-b}, \frac{1-x}{x}\right]-4\left[\frac{(1-b)-x}{(1-b)(1-x)}, \frac{z-\frac{x-a}{x(1-a)}}{z-\frac{x-a}{1-a}}, z, \frac{b}{1-b}, \frac{1-x}{x}\right]\\
&+4\left[\frac{(1-b)-y}{(1-b)(1-y)}, \frac{y-x}{y(1-x)}, \frac{x-(1-a)}{xa}, \frac{1-y}{y}, \frac{1-a}{a}\right]\\
&+4\left[\frac{(1-b)-y}{(1-b)(1-y)}, \frac{y-x}{y(1-x)}, \frac{x-(1-a)}{xa}, \frac{1-a}{a}, \frac{1-x}{x}\right]\\
&+4\left[1-\frac{b}{ay}, 1-\frac{y}{x}, 1-x, a, x\right]+4\left[1-\frac{b}{ay}, 1-\frac{y}{x}, 1-x, y, a\right]\\
&+4\left[1-\frac{b}{(1-a)y}, 1-\frac{y}{x}, 1-x, 1-a, x\right]+4\left[1-\frac{b}{(1-a)y}, 1-\frac{y}{x}, 1-x, y, 1-a\right]\\
&+4\left[1-\frac{1-b}{ay}, 1-\frac{y}{x}, 1-x, a, x\right]+4\left[1-\frac{1-b}{ay}, 1-\frac{y}{x}, 1-x, y, a\right]\\
&-4\left[1-\frac{1}{ay}, 1-\frac{y}{x}, 1-x, a, x\right]-4\left[1-\frac{1}{ay}, 1-\frac{y}{x}, 1-x, y, a\right]\\
&+4\left[1-\frac{1-b}{(1-a)y}, 1-\frac{y}{x}, 1-x, 1-a, x\right]+4\left[1-\frac{1-b}{(1-a)y}, 1-\frac{y}{x}, 1-x, y, 1-a\right]\\
&-4\left[1-\frac{1}{(1-a)y}, 1-\frac{y}{x}, 1-x, 1-a, x\right]-4\left[1-\frac{1}{(1-a)y}, 1-\frac{y}{x}, 1-x, y, 1-a\right]\\
&-4\left[\frac{z-1}{z-\frac{-ay}{1-ay}}, z, 1-y, ay, a\right]-4\left[\frac{z-1}{z-\frac{-(1-a)y}{1-(1-a)y}}, z, 1-y, (1-a)y, 1-a\right]\\
&-4\sum_{y_0=b,1-b}\left[1-\frac{y_0}{x}, \frac{z-\left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right)}{z-\left(1-\frac{x}{a}\right)}, z, y_0, x\right]\\
&+4\sum_{x_0=a,1-a}\left[\frac{z-\left(1-\frac{1-b}{1-y}\right)\left(1-\frac{1-b}{y}\right)}{z-\left(1-\frac{1-b}{y}\right)}, z, 1-\frac{y}{x_0}, y, x_0\right].
\end{aligned}
$$

Proof. We proceed in several steps along the lines of the proof in [GMS99]. The new ingredients are proposition 3.1.7 and in particular propositions 3.1.11 and 3.1.14, which help to show the vanishing of several terms occurring in the course of the proof.

1^{st} **step:** Let us start with an application of (3.1.7):

$$\begin{aligned}
4C_{\phi_1(b)} &= \left[1-\frac{b(1-b)}{y(1-y)}, 1-\frac{y(1-y)}{x(1-x)}, 1-\frac{x(1-x)}{a(1-a)}, \frac{y(1-y)}{a(1-a)}, \frac{x(1-x)}{a(1-a)}\right] \\
&= \left[1-\frac{b(1-b)}{y(1-y)}, 1-\frac{y(1-y)}{x(1-x)}, 1-\frac{x(1-x)}{a(1-a)}, y(1-y), x(1-x)\right] \\
&+ \left[1-\frac{b(1-b)}{y(1-y)}, 1-\frac{y(1-y)}{x(1-x)}, 1-\frac{x(1-x)}{a(1-a)}, \frac{1}{a(1-a)}, x(1-x)\right] \\
&+ \left[1-\frac{b(1-b)}{y(1-y)}, 1-\frac{y(1-y)}{x(1-x)}, 1-\frac{x(1-x)}{a(1-a)}, y(1-y), \frac{1}{a(1-a)}\right].
\end{aligned}$$

Here, and in the next splitting, we use (3.1.7) again:

$$\begin{aligned}
&= \left[1-\frac{b(1-b)}{y(1-y)}, 1-\frac{y(1-y)}{x(1-x)}, 1-\frac{x(1-x)}{a(1-a)}, y, x\right] \\
&+ \left[1-\frac{b(1-b)}{y(1-y)}, 1-\frac{y(1-y)}{x(1-x)}, 1-\frac{x(1-x)}{a(1-a)}, 1-y, x\right] \\
&+ \left[1-\frac{b(1-b)}{y(1-y)}, 1-\frac{y(1-y)}{x(1-x)}, 1-\frac{x(1-x)}{a(1-a)}, y, 1-x\right] \\
&+ \left[1-\frac{b(1-b)}{y(1-y)}, 1-\frac{y(1-y)}{x(1-x)}, 1-\frac{x(1-x)}{a(1-a)}, 1-y, 1-x\right] \\
&+ \left[1-\frac{b(1-b)}{y(1-y)}, 1-\frac{y(1-y)}{x(1-x)}, 1-\frac{x(1-x)}{a(1-a)}, \frac{1}{a(1-a)}, x(1-x)\right] \\
&+ \left[1-\frac{b(1-b)}{y(1-y)}, 1-\frac{y(1-y)}{x(1-x)}, 1-\frac{x(1-x)}{a(1-a)}, y(1-y), \frac{1}{a(1-a)}\right].
\end{aligned}$$

Now we chose another parametrization for the first four types of terms to obtain eight terms of the same shape:

$$\begin{aligned}
&= 4\left[\left(1-\frac{b}{y}\right)\left(1-\frac{b}{1-y}\right), \left(1-\frac{y}{x}\right)\left(1-\frac{y}{1-x}\right), \left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right), y, x\right] \\
&+ \left[1-\frac{b(1-b)}{y(1-y)}, 1-\frac{y(1-y)}{x(1-x)}, 1-\frac{x(1-x)}{a(1-a)}, \frac{1}{a(1-a)}, x(1-x)\right] \\
&+ \left[1-\frac{b(1-b)}{y(1-y)}, 1-\frac{y(1-y)}{x(1-x)}, 1-\frac{x(1-x)}{a(1-a)}, y(1-y), \frac{1}{a(1-a)}\right].
\end{aligned}$$

Then one splits the second coordinate:

$$\begin{aligned}
4C_{\phi_1(b)} &= 4\left[\left(1-\frac{b}{y}\right)\left(1-\frac{b}{1-y}\right), 1-\frac{y}{x}, \left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right), y, x\right]^2_{(+,1-\frac{x}{1-y})} \\
&+ 4\sum_{y_0=b,1-b}\left[\frac{z-\left(1-\frac{y_0}{x}\right)\left(1-\frac{y_0}{1-x}\right)}{z-\left(1-\frac{y_0}{x}\right)}, z, \left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right), y_0, x\right] \\
&- 4\sum_{x_0=a,1-a}\left[\left(1-\frac{b}{y}\right)\left(1-\frac{b}{1-y}\right), \frac{z-\left(1-\frac{y}{x_0}\right)\left(1-\frac{y}{1-x_0}\right)}{z-\left(1-\frac{y}{x_0}\right)}, z, y, x_0\right]
\end{aligned}$$

plus extra terms

$$+\left[1-\frac{b(1-b)}{y(1-y)},1-\frac{y(1-y)}{x(1-x)},1-\frac{x(1-x)}{a(1-a)},\frac{1}{a(1-a)},x(1-x)\right]$$
$$+\left[1-\frac{b(1-b)}{y(1-y)},1-\frac{y(1-y)}{x(1-x)},1-\frac{x(1-x)}{a(1-a)},y(1-y),\frac{1}{a(1-a)}\right].$$

2^{nd} **step:** The next step consists of re-starting again and performing several manipulations with $C_{\phi_2(b)}$. The results from above are kept for later. We consider

$$\begin{aligned}C_{\phi_2(b)} &= \left[\frac{b-y}{b(1-y)},\frac{y-x}{y(1-x)},\frac{x-a}{x(1-a)},\frac{a(1-y)}{y(1-a)},\frac{a(1-x)}{x(1-a)}\right]\\ &= \left[\frac{b-y}{b(1-y)},\frac{y-x}{y(1-x)},\frac{x-a}{x(1-a)},\frac{1-y}{y},\frac{1-x}{x}\right]\\ &\quad+\left[\frac{b-y}{b(1-y)},\frac{y-x}{y(1-x)},\frac{x-a}{x(1-a)},\frac{1-y}{y},\frac{a}{1-a}\right]+\left[\frac{b-y}{b(1-y)},\frac{y-x}{y(1-x)},\frac{x-a}{x(1-a)},\frac{a}{1-a},\frac{1-x}{x}\right]\end{aligned}$$

and split the third coordinate by applying proposition 3.1.7:

$$\begin{aligned}&= \left[\frac{b-y}{b(1-y)},\frac{y-x}{y(1-x)},\frac{x-a}{1-a},\frac{1-y}{y},\frac{1-x}{x}\right]^3_{(+,\frac{1}{x})}\\ &\quad-\left[\frac{b-x}{b(1-x)},\frac{z-\frac{x-a}{x(1-a)}}{z-\frac{x-a}{1-a}},z,\frac{1-b}{b},\frac{1-x}{x}\right]\\ &\quad+\left[\frac{b-y}{b(1-y)},\frac{y-x}{y(1-x)},\frac{x-a}{x(1-a)},\frac{1-y}{y},\frac{a}{1-a}\right]+\left[\frac{b-y}{b(1-y)},\frac{y-x}{y(1-x)},\frac{x-a}{x(1-a)},\frac{a}{1-a},\frac{1-x}{x}\right].\end{aligned}$$

But notice that

$$2\left[\frac{b-y}{b(1-y)},\frac{y-x}{y(1-x)},\frac{1}{x},\frac{1-y}{y},\frac{1-x}{x}\right]=2\left[1+\frac{1-b^{-1}}{y},1-\frac{y}{x},1+x,y,x\right]=2C_{1-b^{-1}}$$

by an application of (3.1.7) with $g(x)=g(y)=-1$ and using the corollary 3.1.12.

3^{rd} **step:** We copy these manipulations for $C_{\phi_3(b)}$ arriving at:

$$\begin{aligned}2C_{\phi_3(b)} &= 2\left[\frac{b-y}{b(1-y)},\frac{y-x}{y(1-x)},\frac{x-(1-a)}{a},\frac{1-y}{y},\frac{1-x}{x}\right]+2C_{1-b^{-1}}\\ &\quad-2\left[\frac{b-x}{b(1-x)},\frac{z-\frac{x-(1-a)}{xa}}{z-\frac{x-a}{1-a}},z,\frac{b}{1-b},\frac{1-x}{x}\right]\\ &\quad+2\left[\frac{b-y}{b(1-y)},\frac{y-x}{y(1-x)},\frac{x-(1-a)}{xa},\frac{1-y}{y},\frac{1-a}{a}\right]\\ &\quad+2\left[\frac{b-y}{b(1-y)},\frac{y-x}{y(1-x)},\frac{x-(1-a)}{xa},\frac{1-a}{a},\frac{1-x}{x}\right].\end{aligned}$$

4^{th} **step:** Let us add $2C_{\phi_2(1-b)}$ and $2C_{\phi_3(1-b)}$, by gluing them in the third coordinate:

$$2C_{\phi_2(1-b)} + 2C_{\phi_3(1-b)} = 2\left[\frac{(1-b)-y}{(1-b)(1-y)}, \frac{y-x}{y(1-x)}, \frac{x-a}{1-a}\cdot\frac{x-(1-a)}{a}, \frac{1-y}{y}, \frac{1-x}{x}\right]$$
$$+2\left[\frac{(1-b)-x}{(1-b)(1-x)}, \frac{z-\frac{x-a}{1-a}\cdot\frac{x-(1-a)}{a}}{z-\frac{x-a}{1-a}}, z, \frac{1-b}{b}, \frac{1-x}{x}\right] + 4C_{\frac{b}{b-1}}$$

plus more terms with a constant coordinate

$$-2\left[\frac{(1-b)-x}{(1-b)(1-x)}, \frac{z-\frac{x-a}{x(1-a)}}{z-\frac{x-a}{1-a}}, z, \frac{b}{1-b}, \frac{1-x}{x}\right]$$
$$+2\left[\frac{(1-b)-y}{(1-b)(1-y)}, \frac{y-x}{y(1-x)}, \frac{x-a}{x(1-a)}, \frac{1-y}{y}, \frac{a}{1-a}\right]$$
$$+2\left[\frac{(1-b)-y}{(1-b)(1-y)}, \frac{y-x}{y(1-x)}, \frac{x-a}{x(1-a)}, \frac{a}{1-a}, \frac{1-x}{x}\right]$$
$$-2\left[\frac{(1-b)-x}{(1-b)(1-x)}, \frac{z-\frac{x-(1-a)}{xa}}{z-\frac{x-(1-a)}{a}}, z, \frac{b}{1-b}, \frac{1-x}{x}\right]$$
$$+2\left[\frac{(1-b)-y}{(1-b)(1-y)}, \frac{y-x}{y(1-x)}, \frac{x-(1-a)}{xa}, \frac{1-y}{y}, \frac{1-a}{a}\right]$$
$$+2\left[\frac{(1-b)-y}{(1-b)(1-y)}, \frac{y-x}{y(1-x)}, \frac{x-(1-a)}{xa}, \frac{1-a}{a}, \frac{1-x}{x}\right].$$

For the moment, we will just focus on the first term keeping all of the terms with a constant coordinate in mind. We proceed by splitting the second coordinate:

$$2\left[\frac{(1-b)-y}{(1-b)(1-y)}, \frac{y-x}{y(1-x)}, \frac{x-a}{1-a}\cdot\frac{x-(1-a)}{a}, \frac{1-y}{y}, \frac{1-x}{x}\right]$$
$$=2\left[\frac{(1-b)-y}{(1-b)(1-y)}, \frac{y-x}{1-x}, \frac{x-a}{1-a}\cdot\frac{x-(1-a)}{a}, \frac{1-y}{y}, \frac{1-x}{x}\right]^2_{(+,\frac{1}{y})}$$
$$+2\left[\frac{z-\frac{(1-b)-x}{(1-b)(1-x)}}{z-\frac{b-x}{1-x}}, z, \frac{x-a}{1-a}\cdot\frac{x-(1-a)}{a}, \frac{b}{1-b}, \frac{1-x}{x}\right]$$
$$-2\sum_{x_0=a,1-a}\left[\frac{(1-b)-y}{(1-b)(1-y)}, \frac{z-\frac{y-x_0}{y(1-x_0)}}{z-\frac{y-x_0}{1-x_0}}, z, \frac{1-y}{y}, \frac{1-x_0}{x_0}\right]$$

and again split the first coordinate, where the second term – the one with $\frac{1}{1-b}$ in the left coordinate – vanishes in $C^3(F,5)/\partial C^3(F,6)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$:

$$2\left[\frac{(1-b)-y}{(1-b)(1-y)}, \frac{y-x}{1-x}, \frac{x-a}{1-a}\cdot\frac{x-(1-a)}{a}, \frac{1-y}{y}, \frac{1-x}{x}\right]$$
$$=2\left[\frac{(1-b)-y}{1-y}, \frac{y-x}{1-x}, \frac{x-a}{1-a}\cdot\frac{x-(1-a)}{a}, \frac{1-y}{y}, \frac{1-x}{x}\right]$$
$$+2\sum_{x_0=a,1-a}\left[\frac{z-\frac{(1-b)-y}{(1-b)(1-y)}}{z-\frac{(1-b)-y}{1-y}}, z, \frac{y-x_0}{1-x_0}, \frac{1-y}{y}, \frac{1-x_0}{x_0}\right].$$

Now let us split the last two coordinates:

$$\begin{aligned}
2&\left[\frac{(1-b)-y}{1-y},\frac{y-x}{1-x},\frac{x-a}{1-a}\cdot\frac{x-(1-a)}{a},\frac{1-y}{y},\frac{1-x}{x}\right]\\
&=2\left[\frac{(1-b)-y}{1-y},\frac{y-x}{1-x},\frac{x-a}{1-a}\cdot\frac{x-(1-a)}{a},1-y,1-x\right]\\
&\quad+2\left[\frac{(1-b)-y}{1-y},\frac{y-x}{1-x},\frac{x-a}{1-a}\cdot\frac{x-(1-a)}{a},\frac{1}{y},1-x\right]\\
&\quad+2\left[\frac{(1-b)-y}{1-y},\frac{y-x}{1-x},\frac{x-a}{1-a}\cdot\frac{x-(1-a)}{a},1-y,\frac{1}{x}\right]\\
&\quad+2\left[\frac{(1-b)-y}{1-y},\frac{y-x}{1-x},\frac{x-a}{1-a}\cdot\frac{x-(1-a)}{a},\frac{1}{y},\frac{1}{x}\right].
\end{aligned}$$

Now note that twice the first term is equal to $2T_1$ as is [GMS99]. Further, for twice the second we have

$$=-2T_3-2\sum_{x_0=a,1-a}\left[\frac{(1-b)-y}{1-y},\frac{y-x_0}{1-x_0},\frac{z-1}{z-y},z,x_0\right],$$

while twice the third one is

$$=-2T_4+2\left[\frac{(1-b)-x}{1-x},\frac{x-a}{1-a}\cdot\frac{x-(1-a)}{ax},b,\frac{z-1}{z-x},z\right].$$

The fourth at last is

$$=2T_2-2\sum_{x_0=a,1-a}\left[\frac{(1-b)-y}{1-y},\frac{y-x_0}{1-x_0},\frac{z-1}{z-y},z,\frac{1}{x_0}\right].$$

Now we go on with one more manipulation:

$$\begin{aligned}
2T_2&=2\left[\frac{y-(1-b)}{y-1},1-\frac{y}{x},\left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right),y,x\right]\\
&\quad+2\left[\frac{y-(1-b)}{y-1},\frac{x}{x-1},\left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right),y,x\right]\\
&\quad+2\left[\frac{z-\left(1-\frac{1-b}{x}\right)\frac{x}{x-1}}{z-\frac{x}{x-1}},z,\left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right),1-b,x\right]\\
&\quad-2\sum_{x_0=a,1-a}\left[\frac{y-(1-b)}{y-1},\frac{z-\left(1-\frac{y}{x_0}\right)\frac{x_0}{x_0-1}}{z-\frac{x_0}{x_0-1}},z,y,x_0\right].
\end{aligned}$$

Let us denote the first term by $2T_2'$. Analogously denote the first term on the right-hand side by $2T_4'$:

$$\begin{aligned}
2T_4&=2\left[\frac{y-(1-b)}{y-1},1-\frac{y}{1-x},\left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right),y,x\right]\\
&\quad+2\left[\frac{y-(1-b)}{y-1},\frac{x-1}{x},\left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right),y,x\right]
\end{aligned}$$

and further

$$+2\left[\frac{z-\left(1-\frac{1-b}{1-x}\right)\frac{x-1}{x}}{z-\frac{x-1}{x}}, z, \left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right), 1-b, x\right]$$
$$-2\sum_{x_0=a,1-a}\left[\frac{y-(1-b)}{y-1}, \frac{z-\left(1-\frac{y}{1-x_0}\right)\frac{x_0-1}{x_0}}{z-\frac{x_0-1}{x_0}}, z, y, x_0\right].$$

Now we have the last two steps:

$$2T_1+2T_2' = 2Z_1 - 2\sum_{x_0=a,1-a}\left[\frac{z-\left(1-\frac{b}{y}\right)\left(\frac{y-(1-b)}{y-1}\right)}{z-\left(1-\frac{b}{y}\right)}, z, 1-\frac{y}{x_0}, y, x_0\right]$$

as well as

$$2T_3+2T_4' = 2Z_2 - 2\sum_{x_0=a,1-a}\left[\frac{z-\left(1-\frac{b}{y}\right)\left(\frac{y-(1-b)}{y-1}\right)}{z-\left(1-\frac{b}{y}\right)}, z, 1-\frac{y}{1-x_0}, y, x_0\right].$$

In summary: $2C_{\phi_2(1-b)}+2C_{\phi_3(1-b)} = 2Z_1-2Z_2+4C_{\frac{b}{b-1}} \pm \ldots$, where $\ldots$ should denote terms of lower order, i. e. with one constant entry. Adding $8C_{\phi_1(b)}$ as well, we arrive at

$$8\sum_{i=1}^{3} C_{\phi_i(b)} = 16Z_1 + 16C_{\frac{b}{b-1}} \pm \ldots.$$

The last step: Split Z_1 from [GMS99]:

$$\begin{aligned}
2Z_1 &= 2\left[\left(1-\frac{b}{1-y}\right)\left(1-\frac{b}{y}\right), 1-\frac{y}{x}, \left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right), y, x\right]\\
&= 2\left[1-\frac{b}{1-y}, 1-\frac{y}{x}, \left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right), y, x\right]\\
&\quad + 2\left[1-\frac{b}{y}, 1-\frac{y}{x}, \left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right), y, x\right]\\
&\quad + 2\sum_{x_0=a,1-a}\left[\frac{z-\left(1-\frac{b}{1-y}\right)\left(1-\frac{b}{y}\right)}{z-\left(1-\frac{b}{y}\right)}, z, 1-\frac{y}{x_0}, y, x_0\right]\\
&= 2\left[1-\frac{b}{1-y}, 1-\frac{y}{x}, 1-\frac{x}{a}, y, x\right] + 2\left[1-\frac{b}{1-y}, 1-\frac{y}{x}, 1-\frac{x}{1-a}, y, x\right]\\
&\quad + 2\left[1-\frac{b}{y}, 1-\frac{y}{x}, 1-\frac{x}{a}, y, x\right] + 2\left[1-\frac{b}{y}, 1-\frac{y}{x}, 1-\frac{x}{1-a}, y, x\right]\\
&\quad - 2\sum_{y_0=b,1-b}\left[1-\frac{y_0}{x}, \frac{z-\left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right)}{z-\left(1-\frac{x}{a}\right)}, z, y_0, x\right]
\end{aligned}$$

and further

$$+2\sum_{x_0=a,1-a}\left[\frac{z-\left(1-\frac{b}{1-y}\right)\left(1-\frac{b}{y}\right)}{z-\left(1-\frac{b}{y}\right)},z,1-\frac{y}{x_0},y,x_0\right].$$

Let us identify some terms:

$$\begin{aligned}
2Z_1 = {}& 2C_{\frac{b}{a}}+2\left[1-\frac{b}{ay},1-\frac{y}{x},1-x,a,x\right]+2\left[1-\frac{b}{ay},1-\frac{y}{x},1-x,y,a\right]\\
&+2C_{\frac{b}{1-a}}+2\left[1-\frac{b}{(1-a)y},1-\frac{y}{x},1-x,1-a,x\right]\\
&+2\left[1-\frac{b}{(1-a)y},1-\frac{y}{x},1-x,y,1-a\right]\\
&+2\left[1-\frac{b}{1-ay},1-\frac{y}{x},1-x,ay,ax\right]\\
&+2\left[1-\frac{b}{1-(1-a)y},1-\frac{y}{x},1-x,(1-a)y,(1-a)x\right]\\
&-2\sum_{y_0=b,1-b}\left[1-\frac{y_0}{x},\frac{z-\left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right)}{z-\left(1-\frac{x}{a}\right)},z,y_0,x\right]\\
&+2\sum_{x_0=a,1-a}\left[\frac{z-\left(1-\frac{b}{1-y}\right)\left(1-\frac{b}{y}\right)}{z-\left(1-\frac{b}{y}\right)},z,1-\frac{y}{x_0},y,x_0\right]\\
={}& 2C_{\frac{b}{a}}+2\left[1-\frac{b}{ay},1-\frac{y}{x},1-x,a,x\right]+2\left[1-\frac{b}{ay},1-\frac{y}{x},1-x,y,a\right]\\
&+2C_{\frac{b}{1-a}}+2\left[1-\frac{b}{(1-a)y},1-\frac{y}{x},1-x,1-a,x\right]\\
&+2\left[1-\frac{b}{(1-a)y},1-\frac{y}{x},1-x,y,1-a\right]\\
&+2\left[1-\frac{1-b}{ay},1-\frac{y}{x},1-x,ay,ax\right]+2\left[-\frac{ay}{1-ay},1-\frac{y}{x},1-x,ay,ax\right]\\
&+2\left[1-\frac{1-b}{(1-a)y},1-\frac{y}{x},1-x,(1-a)y,(1-a)x\right]\\
&+2\left[-\frac{(1-a)y}{1-(1-a)y},1-\frac{y}{x},1-x,(1-a)y,(1-a)x\right]\\
&-2\left[\frac{z-(1-\frac{b}{1-ay})}{z+\frac{ay}{1-ay}},z,1-y,ay,a\right]-2\left[\frac{z-(1-\frac{b}{1-(1-a)y})}{z+\frac{(1-a)y}{1-(1-a)y}},z,1-y,(1-a)y,a\right]\\
&-2\sum_{y_0=b,1-b}\left[1-\frac{y_0}{x},\frac{z-\left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right)}{z-\left(1-\frac{x}{a}\right)},z,y_0,x\right]
\end{aligned}$$

plus the following extra terms

$$+2\sum_{x_0=a,1-a}\left[\frac{z-\left(1-\frac{b}{1-y}\right)\left(1-\frac{b}{y}\right)}{z-\left(1-\frac{b}{y}\right)},z,1-\frac{y}{x_0},y,x_0\right].$$

Now, we have to invert some terms:

$$\begin{aligned}
2Z_1 = {} & 2C_{\frac{b}{a}}+2\left[1-\frac{b}{ay},1-\frac{y}{x},1-x,a,x\right]+2\left[1-\frac{b}{ay},1-\frac{y}{x},1-x,y,a\right]\\
&+2C_{\frac{b}{1-a}}+2\left[1-\frac{b}{(1-a)y},1-\frac{y}{x},1-x,1-a,x\right]\\
&+2\left[1-\frac{b}{(1-a)y},1-\frac{y}{x},1-x,y,1-a\right]\\
&+2\left[1-\frac{1-b}{ay},1-\frac{y}{x},1-x,ay,ax\right]-2\left[1-\frac{1}{ay},1-\frac{y}{x},1-x,ay,ax\right]\\
&+2\left[1-\frac{1-b}{(1-a)y},1-\frac{y}{x},1-x,(1-a)y,(1-a)x\right]\\
&-2\left[1-\frac{1}{(1-a)y},1-\frac{y}{x},1-x,(1-a)y,(1-a)x\right]\\
&-2\left[\frac{z-1}{z-\frac{-ay}{1-ay}},z,1-y,ay,a\right]-2\left[\frac{z-1}{z-\frac{-(1-a)y}{1-(1-a)y}},z,1-y,(1-a)y,1-a\right]\\
&-2\sum_{y_0=b,1-b}\left[1-\frac{y_0}{x},\frac{z-\left(1-\frac{x}{a}\right)\left(1-\frac{x}{1-a}\right)}{z-\left(1-\frac{x}{a}\right)},z,y_0,x\right]\\
&+2\sum_{x_0=a,1-a}\left[\frac{z-\left(1-\frac{b}{1-y}\right)\left(1-\frac{b}{y}\right)}{z-\left(1-\frac{b}{y}\right)},z,1-\frac{y}{x_0},y,x_0\right]\\
&-2\left[\frac{z-(1-\frac{b}{1-ay})}{z+\frac{ay}{1-ay}},z,1-y,ay,a\right]-2\left[\frac{z-(1-\frac{b}{1-(1-a)y})}{z+\frac{(1-a)y}{1-(1-a)y}},z,1-y,(1-a)y,a\right].
\end{aligned}$$

Finally:

$$\begin{aligned}
2Z_1 = {} & 2C_{\frac{b}{a}}+2C_{\frac{b}{1-a}}+2C_{\frac{1-b}{a}}+2C_{\frac{1-b}{1-a}}-2C_{\frac{1}{a}}-2C_{\frac{1}{1-a}}\\
&+2\left[1-\frac{b}{ay},1-\frac{y}{x},1-x,a,x\right]+2\left[1-\frac{b}{ay},1-\frac{y}{x},1-x,y,a\right]\\
&+2\left[1-\frac{b}{(1-a)y},1-\frac{y}{x},1-x,1-a,x\right]+2\left[1-\frac{b}{(1-a)y},1-\frac{y}{x},1-x,y,1-a\right]\\
&+2\left[1-\frac{1-b}{ay},1-\frac{y}{x},1-x,a,x\right]+2\left[1-\frac{1-b}{ay},1-\frac{y}{x},1-x,y,a\right]\\
&-2\left[1-\frac{1}{ay},1-\frac{y}{x},1-x,a,x\right]-2\left[1-\frac{1}{ay},1-\frac{y}{x},1-x,y,a\right]\\
&+2\left[1-\frac{1-b}{(1-a)y},1-\frac{y}{x},1-x,1-a,x\right]+2\left[1-\frac{1-b}{(1-a)y},1-\frac{y}{x},1-x,y,1-a\right]
\end{aligned}$$

plus the rest

$$
\begin{aligned}
&-2\left[1-\frac{1}{(1-a)y},1-\frac{y}{x},1-x,1-a,x\right]-2\left[1-\frac{1}{(1-a)y},1-\frac{y}{x},1-x,y,1-a\right]\\
&-2\left[\frac{z-1}{z-\frac{-ay}{1-ay}},z,1-y,ay,a\right]-2\left[\frac{z-1}{z-\frac{-(1-a)y}{1-(1-a)y}},z,1-y,(1-a)y,1-a\right]\\
&-2\sum_{y_0=b,1-b}\left[1-\frac{y_0}{x},\frac{z-(1-\frac{x}{a})\left(1-\frac{x}{1-a}\right)}{z-\left(1-\frac{x}{a}\right)},z,y_0,x\right]\\
&+2\sum_{x_0=a,1-a}\left[\frac{z-\left(1-\frac{b}{1-y}\right)\left(1-\frac{b}{y}\right)}{z-\left(1-\frac{b}{y}\right)},z,1-\frac{y}{x_0},y,x_0\right]\\
&-2\left[\frac{z-(1-\frac{b}{1-ay})}{z+\frac{ay}{1-ay}},z,1-y,ay,a\right]-2\left[\frac{z-(1-\frac{b}{1-(1-a)y})}{z+\frac{(1-a)y}{1-(1-a)y}},z,1-y,(1-a)y,a\right].
\end{aligned}
$$

Then the assertion follows. □

3.3 An inversion relation modulo 2-torsion

The above variant of the Kummer – Spence – relation can now be used in the same way as Gangl and Müller-Stach did in their paper [GMS99] to derive an inversion relation:

Proposition 3.3.1. *With the assumptions of the previous section, an inversion of the following shape holds in* $C^3(F,5)/\partial C^3(F,6)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$*:*

$$32(C_a + C_{\frac{1}{a}}) = \textit{ lower order terms.}$$

Proof. Let us consider $\langle x\rangle := C_x - C_{\frac{1}{x}}$ and further $A := 1-\frac{1}{a}$, $B := 1-\frac{1}{b}$. We explain the strategy of the proof and just give the resulting formula. The intermediate expressions are not valuable.

First, we subtract the Kummer – Spence – relation with arguments $(a, 1-b)$ from the one with arguments (a, b). This gives the expression (cf. [GMS99])

$$4\mathcal{KS}(b,a) - 4\mathcal{KS}(b,1-a) = 4\langle A/B\rangle + 4\langle AB\rangle + 8\langle 1/A\rangle$$

plus pages of extra terms with a constant somewhere. Similarly, we have

$$4\mathcal{KS}(a,b) - 4\mathcal{KS}(a,1-b) = 4\langle B/A\rangle + 4\langle BA\rangle + 8\langle 1/B\rangle \tag{3.3.1}$$

plus pages of extra terms with a constant somewhere. Adding both expressions gives

$$8\langle AB\rangle - 16\langle A\rangle - 16\langle B\rangle = \text{lower order terms.}$$

Then one substitutes $A \mapsto B/A, B \mapsto BA$ and rewrites the expression in the form

$$16\langle B/A\rangle + 16\langle BA\rangle - 8\langle B^2\rangle = \text{lower order terms}$$

and uses the distribution relation for $n = 2$ and 3.3.1 to find

$$\begin{aligned}16\langle B/A\rangle + 16\langle BA\rangle - 8\langle B^2\rangle &= \text{lower order terms}\\ -32\langle 1/B\rangle - 8\langle B^2\rangle &= \text{lower order terms}\\ -32\langle 1/B\rangle &= \text{lower order terms}\end{aligned}$$

as was to be proven. □

3.4 The Goncharov – relation modulo 2-torsion

In this section, our aim is to prove the Goncharov – relation in $C^3(F,5)/\partial C^3(F,6) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$. We do this in the same way as Zhao did in his article [Zha07], but again making use of the appropriate rules settled in the first section. Further, we prove the relation making use of the inversion relation from the last section. In the proof we shall not use the inversion relation, but in order to arrive at the relation given in the proposition, one has to use the inversion relation.

We decided not to write down the explicit shape of the relation. This is very complicated and takes a lot of space. One may criticize this missing formula, but our justification is that one does not gain any information by seeing the whole relation explicitly. If one had further arguments to simplify the extra terms, one could simplify the steps in the proof and write down explicitly an easier relation.

Remark 3.4.1. We use Zhao's notation [Zha07] freely. The interested reader should keep Zhao's article at hand in order to understand all our steps. We also note that there is some overlap with our notation from the last chapter. For example, we introduced some elements $Z(a,b) \in C^2(F,3)/\partial C^2(F,4)$. Zhao also defines elements in $Z^3(F,5$ denoted by $Z(a,b)$. But this should not cause too much confusion, because we do not use the notation from the last chapter in the following. ♢

Define for $a \in F^\times - \{1\}$ the expression $\mathcal{T}(a) := C_a + C_{1-a} + C_{1-\frac{1}{a}} \in C^3(F,5)/\partial C^3(F,6) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$. Copying ideas from Gangl and Müller-Stach [GMS99], we can show that $\mathcal{T}(a) - \mathcal{T}(b) =$ lower order terms for $a, b \neq 0, 1$. Thus, both expression are equal modulo lower order terms. As Zhao did, we set $\eta := \mathcal{T}(a)$. Our result is the following:

Proposition 3.4.2. *For any combination of parameters $a, b, c \in \mathbb{P}^1_F$ the following relation* *holds in* $C^3(F,5)/\partial C^3(F,6)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$*:*

$$\mathcal{R}(a,b,c) := C_{-abc} + \bigoplus_{cyc(a,b,c)} \Big(C_{ac-a+1} + C_{\frac{ca-a+1}{ca}} - C_{\frac{ca-a+1}{c}} + C_{\frac{a(bc-c+1)}{a-1-ac}} + C_{\frac{bc-c+1}{b(ca-a+1)}} + C_c - C_{\frac{bc-c+1}{bc(ca-a+1)}} - \eta \Big) = \text{ lower order terms,}$$

where $cyc(a,b,c)$ *denotes a cyclic permutation of the three variables provided none of the terms in* $\mathcal{R}(a,b,c)$ *except for* η *is equal to* C_1 *or not admissible.*

Proof. We shall start with recalling some abbreviations of [Zha07]: Let $f(x) := x, A(x) := (ax-a+1)/a$, and $B(x) := bx-x+1$ for some constants $a,b\in F$. Denote further by $k(x) := B(x)/abxA(x)$ and $l(y) := 1-k(c)/k(y)$, and let lastly be $\mu := -(ab-b+1)/a$.

Let us start with

$$C_{k(c)} := \left[1-\frac{k(c)}{y}, 1-\frac{y}{x}, 1-x, y, x\right].$$

1^{st} **step:** By lemma 3.1.14, we can write

$$\begin{aligned} 8C_{k(c)} = 8&\left[1-\frac{k(c)}{y}, 1-\frac{y}{x}, 1-x, \frac{aby}{\mu}, \frac{abx}{\mu}\right] \\ &- 8\left[1-\frac{k(c)}{y}, 1-\frac{y}{x}, 1-x, \frac{ab}{\mu}, x\right] - 8\left[1-\frac{k(c)}{y}, 1-\frac{y}{x}, 1-x, y, \frac{ab}{\mu}\right]. \end{aligned}$$

With the reparametrization $(x,y)\mapsto(k(x),k(y))$ this can be written as

$$\begin{aligned} = 2&\left[l(y), 1-\frac{k(y)}{k(x)}, 1-k(x), \frac{B(y)}{\mu y A(y)}, \frac{B(x)}{\mu x A(x)}\right] \\ &- 8\left[1-\frac{k(c)}{y}, 1-\frac{y}{x}, 1-x, \frac{ab}{\mu}, x\right] - 8\left[1-\frac{k(c)}{y}, 1-\frac{y}{x}, 1-x, y, \frac{ab}{\mu}\right]. \end{aligned}$$

The next step consists of expressing the first term in another way: We let

$$Z(f_1,f_2) := \left[l(y), 1-\frac{k(y)}{k(x)}, 1-k(x), f_1(y), f_2(x)\right]$$

so that

$$\begin{aligned} 8C_{k(c)} = 2&Z\left(\frac{B}{\mu f A}, \frac{B}{\mu f A}\right) \\ &- 8\left[1-\frac{k(c)}{y}, 1-\frac{y}{x}, 1-x, \frac{ab}{\mu}, x\right] - 8\left[1-\frac{k(c)}{y}, 1-\frac{y}{x}, 1-x, y, \frac{ab}{\mu}\right]. \end{aligned}$$

In the sequel, we often have to determine the zeros and poles of $l(y)$. As one can check, the zeros are $y = c$ and $y = \frac{c-1-ac}{a(bc-c+1)}$, whereas the pole is at $y = \frac{1}{1-b}$. Note further that the pole of l is a zero of B. So when we write $\sum_{y_0}$ in the following, we shall mean one term with $y_0 = c$ and another term with y_0 equal to the second zero of l. Something similar applies for $1 - k(y)/k(x)$ as well: The zeros are $y(x) = x$ and $y(x) = -A(x)/B(x)$ whereas the pole is $y(x) = 1 - 1/a$. Again, the pole does not produce any extra terms because the resulting term will be 2-torsion. We shall simply write $\sum_{y_x}$ for the sum over the zeros:

$$
\begin{aligned}
8C_{k(c)} = {} & -2Z\left(\frac{\mu f A}{B}, \frac{B}{\mu f A}\right) + 2\sum_{y_x}\left[1 - \frac{k(c)}{k(y_x)}, 1 - k(x), \frac{z-1}{z - \frac{\mu y_x A(y_x)}{B(y_x)}}, z, \frac{B(x)}{\mu x A(x)}\right] \\
& - 8\left[1 - \frac{k(c)}{y}, 1 - \frac{y}{x}, 1 - x, \frac{ab}{\mu}, x\right] - 8\left[1 - \frac{k(c)}{y}, 1 - \frac{y}{x}, 1 - x, y, \frac{ab}{\mu}\right]
\end{aligned}
$$

and further

$$
\begin{aligned}
= {} & 2Z\left(\frac{\mu f A}{B}, \frac{\mu f A}{B}\right) + 2\sum_{y_0}\left[1 - \frac{k(y_0)}{k(x)}, 1 - k(x), \frac{\mu y_0 A(y_0)}{B(y_0)}, \frac{z-1}{z - \frac{B(x)}{\mu x A(x)}}, z\right] \\
& + 8\left[1 - \frac{k(c)}{y}, 1 - \frac{y}{x}, 1 - x, \frac{ab}{\mu}, x\right] + 8\left[1 - \frac{k(c)}{y}, 1 - \frac{y}{x}, 1 - x, y, \frac{ab}{\mu}\right].
\end{aligned}
$$

In the same way as in [Zha07], the first expression can be split using lemma 3.1.14 again with the following result:

$$
\begin{aligned}
8C_{k(c)} = {} & 8Z(A, A) + 2\sum_{y_0}\left[1 - \frac{k(y_0)}{k(x)}, 1 - k(x), \frac{\mu y_0 A(y_0)}{B(y_0)}, \frac{z-1}{z - \frac{B(x)}{\mu x A(x)}}, z\right] \\
& + 8\left[1 - \frac{k(c)}{y}, 1 - \frac{y}{x}, 1 - x, \frac{ab}{\mu}, x\right] + 8\left[1 - \frac{k(c)}{y}, 1 - \frac{y}{x}, 1 - x, y, \frac{ab}{\mu}\right].
\end{aligned}
$$

Before going on, we shall recall the Zhao's notation: We set

$$
\begin{aligned}
\alpha &= \frac{bc-c}{bc-c+1}, & \delta &= \frac{1}{b}, & v(x) &= \frac{abx+1}{aA(x)}, \\
g(x) &= \frac{B(x)}{(b-1)x}, & h(x) &= (b-1)x & p_4(x,y) &= \frac{\mu(x-y)}{A(y)B(x)}, \\
q_4(x,y) &= \frac{y-x}{A(y)}, & s_4(x,y) &= \frac{(b-1)(y-x)}{B(y)}, & r_4 &= \frac{(b-1)(y-x)}{xB(y)}, \\
w_4(x,y) &= \frac{y-x}{B(x)(y-1)}, & l_1(y) &= 1 - \frac{y}{c}, & l_2(y) &= \frac{a(bc-c+1)y + ca - a + 1}{(ca-a+1)B(y)}.
\end{aligned}
$$

We also need the following terms:

$$
\begin{aligned}
Y_1 &:= \left[l_1(y), \frac{y-x}{A(y)}, \frac{abx+1}{ax-a+1}, (1-b)y, \frac{B(x)}{(b-1)x}\right],, \\
Y_2 &:= \left[l_1(y), \frac{(y-x)(ab-b+1)}{aA(y)B(x)}, \frac{abx+1}{ax-a+1}, \frac{B(y)}{(b-1)y}, (1-b)x\right]
\end{aligned}
$$

and

$$\begin{aligned}
Y_3 &:= \left[l_2(y), \frac{y-x}{A(y)}, \frac{abx+1}{ax-a+1}, (1-b)y, \frac{(b-1)x}{B(x)}\right],\\
Y_4 &:= \left[l_2(y), \frac{(y-x)(ab-b+1)}{aA(y)B(x)}, \frac{abx+1}{ax-a+1}, \frac{(b-1)y}{B(y)}, (1-b)x\right].
\end{aligned}$$

2^{nd} **step:** Let us start with some preliminary considerations: Zhao defined four terms $Z_1, \ldots, Z_4$ which he used in his proof:

$$\begin{aligned}
Z_1\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) &:= \left[l(y), q_4(x,y), \frac{x-1}{x}, \frac{b-1}{\mu}A(y), \frac{b-1}{\mu}A(x)\right],\\
Z_2\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) &:= \left[l(y), \frac{A(y)}{y}\left(1-\frac{\mu x}{A(y)B(x)}\right), \frac{x-1}{x}, \frac{b-1}{\mu}A(y), \frac{b-1}{\mu}A(x)\right],\\
Z_3\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) &:= \left[l(y), q_4(x,y), \delta v(x), \frac{b-1}{\mu}A(y), \frac{b-1}{\mu}A(x)\right],\\
Z_4\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) &:= \left[l(y), \frac{A(y)}{y}\left(1-\frac{\mu x}{A(y)B(x)}\right), \delta v(x), \frac{b-1}{\mu}A(y), \frac{b-1}{\mu}A(x)\right].
\end{aligned}$$

We shall now throw away some terms with constants by using proposition 3.1.7 and lemma 3.1.14:

$$\begin{aligned}
2Z_1\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) &= 2Z_1(A,A)\\
&\quad + 2\left[l(y), q_4(x,y), \frac{x-1}{x}, \frac{b-1}{\mu}, A(x)\right] + 2\left[l(y), q_4(x,y), \frac{x-1}{x}, A(y), \frac{b-1}{\mu}\right],\\
2Z_3\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) &= 2Z_3(A,A)\\
&\quad + 2\left[l(y), q_4(x,y), \delta v(x), \frac{b-1}{\mu}, A(x)\right] + 2\left[l(y), q_4(x,y), \delta v(x), A(y), \frac{b-1}{\mu}\right],\\
2Z_2\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) &= 2\left[l(y), 1-\frac{\mu x}{A(y)B(x)}, \frac{x-1}{x}, \frac{b-1}{\mu}A(y), \frac{b-1}{\mu}A(x)\right]^2_{(+,\frac{A(y)}{y})}\\
&\quad + 2\sum_{y_0}\left[\frac{z-\frac{A(y_0)}{y_0}\left(1-\frac{\mu x}{A(y_0)B(x)}\right)}{z-\frac{A(y_0)}{y_0}}, z, \frac{x-1}{x}, \frac{b-1}{\mu}A(y_0), \frac{b-1}{\mu}A(x)\right]\\
&\quad - 2\left[l(y), \frac{z-\frac{A(y)}{y}\left(1-\frac{\mu}{A(y)b}\right)}{z-\frac{A(y)}{y}}, z, \frac{b-1}{\mu}A(y), \frac{1-b}{ab-b+1}\right]\\
&\quad + 2\left[l(y), \frac{z-1}{z-\frac{A(y)}{y}}, z, \frac{1}{1-a}, \frac{b-1}{\mu}A(y)\right]\\
2Z_4\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) &= 2\left[l(y), \frac{A(y)}{y}\left(1-\frac{\mu x}{A(y)B(x)}\right), v(x), \frac{b-1}{\mu}A(y), \frac{b-1}{\mu}A(x)\right]^3_{(+,\frac{1}{b})}\\
&\quad - 2\sum_{y_0}\left[\frac{A(y_0)}{y_0}\left(1-\frac{\mu x}{A(y_0)B(x)}\right), \frac{z-\delta v(x)}{z-\frac{1}{b}}, z, \frac{x-1}{x}, \frac{b-1}{\mu}A(y_0), \frac{b-1}{\mu}A(x)\right].
\end{aligned}$$

Now we break up $Z\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big)$:

$$Z\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) = \Big[l(y), q_4(x,y), 1-k(x), \frac{b-1}{\mu}A(y), \frac{b-1}{\mu}A(x)\Big]^2_{(+,\frac{A(y)}{x}(1-\frac{\mu x}{A(y)B(x)}))},$$

where the first term will be denoted by $Z'\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big)$ and the second by $Z''\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big)$

$$+\sum_{y_0}\left[\frac{z-(1-\frac{k(y_0)}{k(x)})}{z-\frac{y_0-x}{A(y_0)}}, z, 1-k(x), \frac{b-1}{\mu}A(y_0), \frac{b-1}{\mu}A(x)\right]$$
$$-\left[l(y), \frac{z-(1-k(y))}{z-\frac{y-1}{A(y)}}, z, \frac{b-1}{\mu}A(y), \frac{1-b}{ab-b+1}\right]$$
$$-\left[l(y), \frac{z-(1-k(y))}{z-\frac{y+\frac{1}{ab}}{A(y)}}, z, \frac{b-1}{\mu}A(y), \frac{b-1}{b}\right].$$

Further, we see that

$$2Z'\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) = 2\left[l(y), q_4(x,y), \delta v(x), \frac{b-1}{\mu}A(y), \frac{b-1}{\mu}A(x)\right]^3_{(+,\frac{x-1}{x})}$$
$$-2\sum_{y_0}\left[\frac{y_0-x}{A(y_0)}, \frac{z-(1-k(x))}{z-\frac{x-1}{x}}, z, \frac{b-1}{\mu}A(y_0), \frac{b-1}{\mu}A(x)\right],$$

where the first term is recognized to be $2Z_1\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big)$ while the second is just equal to $2Z_3\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big)$. In rather the same way, we imitate the reasoning of [Zha07] to see

$$2Z''\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) = \left[l(y), p_4(x,y), 1-k(x), \frac{1}{g(y)}, \frac{b-1}{\mu}A(x)\right]$$
$$+\left[l(y), 1-\frac{y}{x}, 1-k(x), \frac{b-1}{\mu}A(y), \frac{1}{g(x)}\right]$$

which is equal to

$$= 2Z_2\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) + 2Z_4\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big)$$
$$+2\sum_{y_0}\left(\left[\frac{\mu(x-y_0)}{A(y_0)B(x)}, \frac{z-(1-k(x))}{z-\frac{x-1}{x}}, z, \frac{(b-1)y_0}{B(y_0)}, \frac{b-1}{\mu}A(x)\right]\right.$$
$$\left.+\left[1-\frac{y_0}{x}, \frac{z-(1-k(x))}{z-\frac{x-1}{x}}, z, \frac{b-1}{\mu}A(y_0), \frac{1}{g(x)}\right]\right)$$
$$+\left[l(y), p_4(x,y), \frac{1-k(x)}{1-\delta v(x)}, \frac{1}{g(y)}, \frac{b-1}{\mu}A(x)\right]$$
$$+\left[l(y), 1-\frac{y}{x}, \frac{1-k(x)}{1-\delta v(x)}, \frac{b-1}{\mu}A(y), \frac{1}{g(x)}\right].$$

Now we can quote from [Zha07] again that $2Z\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) = 2\sum_{i=1}^4 Z_i\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big)$ plus lower order terms and so we finally see

$$\begin{aligned}8C_{k(c)} &= 8Z\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) + 2\sum_{y_0}\left[1-\frac{k(y_0)}{k(x)}, 1-k(x), \frac{\mu y_0 A(y_0)}{B(y_0)}, \frac{z-1}{z-\frac{B(x)}{\mu x A(x)}}, z\right]\\ &+ 8\left[1-\frac{k(c)}{y}, 1-\frac{y}{x}, 1-x, \frac{ab}{\mu}, x\right] + 8\left[1-\frac{k(c)}{y}, 1-\frac{y}{x}, 1-x, y, \frac{ab}{\mu}\right]\\ &- 8\left[l(y), 1-\frac{k(y)}{k(x)}, 1-k(x), \frac{b-1}{\mu}, A(x)\right] - 8\left[l(y), 1-\frac{k(y)}{k(x)}, 1-k(x), A(y), \frac{b-1}{\mu}\right]\end{aligned}$$

which by means of the relations derived at the beginning of this step is the same as

$$= \sum_{i=1}^4 Z_i\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) + \text{lower order terms.}$$

Remark 3.4.3. Note our abbreviation. Lower order terms are again those with a constant in one of the coordinates and terms which can be written as a product of two terms, i. e. are contained in $Z^2(F,3) \wedge Z^1(F,2)$. ♢

3^{rd} **step:** Let us proceed by considering the reparametrization $\rho_x : x \mapsto -A(x)/B(x)$ and

$$\begin{aligned}&\rho_x Z_2\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) + \rho_x Z_4\Big(\frac{b-1}{\mu}A, \frac{b-1}{\mu}A\Big) =\\ &\quad\left[l(y), q_4(x,y), v(x), \frac{b-1}{\mu}A(y), \frac{1}{g(x)}\right] + \left[l(y), p_4(x,y), v(x), \frac{1}{g(y)}, \frac{b-1}{\mu}A(x)\right].\end{aligned}$$

By lemma 3.1.14, we deduce

$$\begin{aligned}&= \left[l(y), q_4(x,y), v(x), \frac{A(y)}{-\mu y}, \frac{1}{g(x)}\right]^4_{(+,-h(y))} + \left[l(y), p_4(x,y), v(x), \frac{1}{g(y)}, \frac{A(x)}{-\mu x}\right]^5_{(+,-h(x))}\\ &\quad - \sum_{y_0}\left[\frac{\mu(x-y_0)}{A(y_0)B(x)}, v(x), \frac{(b-1)y_0}{B(y_0)}, \frac{z-\frac{b-1}{\mu}A(x)}{z-(1-b)x}, z\right]\\ &\quad + \left[l(y), \frac{y+\frac{1}{ab}}{A(y)}, \frac{z-\frac{b-1}{\mu}A(y)}{z-(1-b)y}, z, \frac{1-b}{ab-b+1}\right] - \left[l(y), \frac{y}{A(y)}, \frac{1}{1-a}, \frac{z-\frac{b-1}{\mu}A(y)}{z+h(y)}, z\right]\end{aligned}$$

leading to the following expression for the first four terms on the right-hand side:

$$\begin{aligned}&\left[l(y), q_4(x,y), v(x), \frac{A(y)}{-\mu y}, \frac{1}{g(x)}\right]^4_{(+,-h(y))} + \left[l(y), p_4(x,y), v(x), \frac{1}{g(y)}, \frac{A(x)}{-\mu x}\right]^5_{(+,-h(x))}\\ &= X_1 - X_2\\ &\quad - \sum_{y_0}\left[\frac{y_0-x}{A(y_0}, v(x), (1-b)y_0, \frac{z-1}{z-g(x)}, z\right]\\ &\quad + \left[l(y), \frac{aby+1}{ay-a+1}, \frac{z-1}{z-k(y)}, z, \frac{ab-b+1}{1-b}\right] + \left[l(y), \frac{(ab-b+1)y}{ay-a+1}, \frac{1}{1-a}, \frac{z-1}{z-g(y)}, z\right].\end{aligned}$$

Then we put X_2 in a different way:

$$
\begin{aligned}
2X_2 &= 2\,[l(y), q_4(x,y), v(x), -h(y), g(x)] + 2\,[l(y), p_4(x,y), v(x), g(y), -h(x)] \\
&+ 2\sum_{y_0}\left[\frac{\mu(x-y_0)}{A(y_0)B(x)}, v(x), \frac{B(y_0)}{(b-1)y_0}.\frac{z-(1-b)x}{z+1}, z\right] \\
&- 2\left[l(y), \frac{y+\frac{1}{ab}}{A(y)}, \frac{z-(b-1)y}{z+1}, z, \frac{ab-b+1}{1-b}\right] - 2\left[l(y), \frac{y}{A(y)}, \frac{1}{1-a}, \frac{z-(b-1)y}{z+1}, z\right].
\end{aligned}
$$

We now want to compute X_1: First define

$$\tilde{Z}(f_1, f_2) := [l(y), p_4(x,y), v(x), f_1(y), f_2(x)]$$

so that one can write

$$
\begin{aligned}
X_1 &= \tilde{Z}\Big(\frac{(b-1)f}{B}, \frac{A}{-\mu f}\Big) + \tilde{Z}\Big(\frac{A}{-\mu f}, \frac{(b-1)f}{B}\Big) \\
&+ \left[l(y), \frac{B(x)}{-\mu}, v(x), \frac{A(y)}{-\mu y}, \frac{1}{g(x)}\right] + \sum_{y_0}\left[\frac{z-\frac{y_0-x}{A(y_0)}}{z+\frac{B(x)}{\mu}}, z, v(x), \frac{A(y_0)}{-\mu y_0}, \frac{1}{g(x)}\right] \\
&- \left[l(y), \frac{z-\frac{y+\frac{1}{ab}}{A(y)}}{z+\frac{1}{b}}, z, \frac{A(y)}{-\mu y}, \frac{1-b}{ab-b+1}\right] - \left[l(y), \frac{z-\frac{y}{A(y)}}{z-\frac{1-b}{\mu}}, z, \frac{1}{1-a}, \frac{A(y)}{-\mu y}\right] \\
&+ \left[l(y), \frac{z-1}{z+1}, z, \frac{A(y)}{-\mu y}, \frac{(1-a)(1-b)}{ab-b+1}\right].
\end{aligned}
$$

We proceed by finding

$$
\begin{aligned}
Z_3\Big(\frac{A}{f}, \frac{A}{f}\Big)) &= \left[l(y), q_4(x,y), v(x), \frac{A(y)}{y}, \frac{A(x)}{x}\right] \\
&+ \left[l(y), q_4(x,y), v(x), \frac{-1}{\mu}, \frac{A(x)}{x}\right] + \left[l(y), q_4(x,y), v(x), \frac{A(y)}{y}, \frac{-1}{\mu}\right]
\end{aligned}
$$

and the first term is seen to be equal to

$$
\begin{aligned}
&\tilde{Z}\Big(\frac{A}{f}, \frac{A}{f}\Big) - \left[l(y), \frac{-\mu}{B(x)}, v(x), \frac{A(y)}{y}, \frac{A(x)}{x}\right] \\
&- \sum_{y_0}\left[\frac{z-\frac{\mu(x-y_0)}{A(y_0)B(x)}}{z-\frac{y_0-x}{A(y_0)}}, z, v(x), \frac{A(y_0)}{y_0}, \frac{A(x)}{x}\right] \\
&+ \left[l(y), \frac{z-\frac{1+aby}{ay-a+1}}{z-\frac{1+aby}{(ay-a+1)b}}, z, \frac{A(y)}{y}, ab-b+1\right] - \left[l(y), \frac{z+\frac{\mu y}{A(y)}}{z+\mu}, z, \frac{1}{1-a}, \frac{A(y)}{y}\right].
\end{aligned}
$$

It is easy to see that $Z_3\big(\frac{A}{B}, \frac{A}{B}\big) = \tilde{Z}\big(\frac{A}{B}, \frac{A}{B}\big)$, but further note that

$$
\begin{aligned}
Z_3\Big(\frac{f}{B}, \frac{f}{B}\Big) &= \left[l(y), \frac{y-x}{yB(x)}, v(x), \frac{y}{B(y)}, \frac{x}{B(x)}\right] \\
&+ \left[l(y), \frac{y-x}{yB(x)}, v(x), \frac{y}{B(y)}, 1-b\right] + \left[l(y), \frac{y-x}{yB(x)}, v(x), 1-b, \frac{x}{B(x)}\right],
\end{aligned}
$$

where the first term on the right-hand side again can be written as

$$\begin{aligned}
&\tilde{Z}\Big(\frac{f}{B},\frac{f}{B}\Big)-\Big[l(y),\frac{-\mu y}{A(y)},v(x),\frac{y}{B(y)},\frac{x}{B(x)}\Big]\\
&-\sum_{y_0}\left[\frac{z-\frac{\mu(x-y_0)}{A(y_0)B(x)}}{z-\frac{-\mu y_0}{A(y_0)}},z,v(x),\frac{y_0}{B(y_0)},\frac{x}{B(x)}\right]\\
&+\left[l(y),\frac{z+\frac{1+aby}{ay-a+1}}{z-\frac{-\mu y}{A(y)}},z,\frac{y}{B(y)},\frac{-1}{ab-b+1}\right]+\left[l(y),\frac{z+\frac{\mu y}{A(y)}}{z+\frac{\mu y}{A(y)}},z,\frac{1}{1-a},\frac{y}{B(y)}\right].
\end{aligned}$$

With the help of this and lemma 3.1.14, one computes

$$\begin{aligned}
X_1&=\tilde{Z}\Big(\frac{A}{B},\frac{A}{B}\Big)-\tilde{Z}\Big(\frac{A}{-\mu f},\frac{A}{-\mu f}\Big)-\tilde{Z}\Big(\frac{(b-1)f}{B},\frac{(b-1)f}{B}\Big)+\text{lower order terms}\\
&=\tilde{Z}\Big(\frac{A}{B},\frac{A}{B}\Big)-\tilde{Z}\Big(\frac{1}{-\mu},\frac{A}{f}\Big)-\tilde{Z}\Big(\frac{A}{f},\frac{1}{-\mu}\Big)-\tilde{Z}\Big(b-1,\frac{(b-1)f}{B}\Big)-\tilde{Z}\Big(\frac{(b-1)f}{B},b-1\Big)\\
&\quad-\tilde{Z}\Big(\frac{A}{f},\frac{A}{f}\Big)-\tilde{Z}\Big(\frac{f}{B},\frac{f}{B}\Big)+\text{lower order terms}
\end{aligned}$$

and expresses the $\tilde{Z}$–terms via the Z–terms as above.

Let us now decompose X_2: With the notations of [Zha07], we find

$$\begin{aligned}
X_2&=\sum_{i=1}^{4}Y_i\\
&+\left[\frac{z-l(y)}{z-l_1(y)},z,\frac{y+1/ab}{A(y)},-h(y),\frac{ab-b+1}{1-b}\right]-\left[\frac{z-l(y)}{z-l_1(y)},z,\frac{y}{A(y)},\frac{1}{1-a},-h(y)\right]\\
&+\left[\frac{z-l(y)}{z-l_1(y)},z,\frac{1+aby}{ay-a+1},g(y),\frac{1-b}{ab}\right]+\left[\frac{z-l(y)}{z-l_1(y)},z,\frac{\mu y}{A(y)},\frac{1}{1-a},g(y)\right].
\end{aligned}$$

4^{th} **step:** Our aim is to compute Y_1+Y_2: Again, we follow the notation of [Zha07]:

$$\begin{aligned}
&[\alpha l_1,q_4,\delta v,gh,gh]+[\alpha l_1,s_4,\delta v,gh,gh]\\
&\quad=[\alpha l_1,q_4,\delta v,g,gh]^4_{(+,h)}+[\alpha l_1,s_4,\delta v,gh,g]^5_{(+,h)}\\
&\quad+\left[\frac{c-x}{A(c)},\delta v(x),B(c),\frac{z-B(x)}{z-(b-1)x},z\right]+\left[\alpha l_1(y),\frac{(b-1)(1+aby)}{ab(by-y+1)},\frac{z-B(y)}{z-(b-1)y},z,\frac{ab-b+1}{ab}\right]\\
&\quad-\left[\alpha l_1(y),\frac{(b-1)(ay-a+1)}{a(by-y+1)},\frac{z-B(y)}{z-(b-1)y},z,\frac{ab-b+1}{a}\right].
\end{aligned}$$

Then we derive step by step:

$$\begin{aligned}
&[\alpha l_1,q_4,\delta v,gh,g]+[\alpha l_1,s_4,\delta v,g,gh]\\
&\quad=[\alpha l_1,q_4,\delta v,gh,g]+[\alpha l_1,r_4,\delta v,g,gh]^2_{(+,\frac{1}{x})}+\left[\frac{z-\frac{(b-1)(c-x)}{xB(c)}}{z-\frac{1}{x}},z,\delta v(x),\frac{B(c)}{(b-1)c},B(x)\right]
\end{aligned}$$

and

$$-\left[\alpha l_1(y), \frac{z-\frac{(b-1)(aby+1)}{B(y)}}{z+ab}, z, g(y), \frac{ab-b+1}{ab}\right] + \left[\alpha l_1(y), \frac{z-\frac{(b-1)(ay-a+1)}{a(by-y+1)}}{z-\frac{a}{a-1}}, z, g(y), \frac{ab-b+1}{a}\right]$$

and getting rid of the constants in the middle of the first two terms, we see that

$$\begin{aligned}
&[\alpha l_1, q_4, \delta v, gh, g] + [\alpha l_1, r_4, \delta v, g, gh] \\
&\quad = [\alpha l_1, q_4, v, gh, g]^3_{(+,\delta)} + [\alpha l_1, r_4, v, g, gh]^3_{(+,\delta)} \\
&\quad\quad - \left[\frac{c-x}{A(c)}, \frac{z-\delta v(x)}{z-\delta}, z, B(c), g(x)\right] - \left[\frac{(b-1)(c-x)}{B(c)}, \frac{z-\delta v(x)}{z-\delta}, z, \frac{B(c)}{(b-1)c}, B(x)\right].
\end{aligned}$$

Now we again replace the second coordinate in one of the terms:

$$\begin{aligned}
&[\alpha l_1, q_4, v, gh, g] + [\alpha l_1, r_4, v, g, gh] \\
&\quad = [\alpha l_1, q_4, v, gh, g] + [\alpha l_1, w_4, v, g, gh]^2_{(-,\frac{(b-1)(y-1)B(x)}{xB(y)})} \\
&\quad\quad - \left[\frac{z-\frac{(b-1)(c-x)}{xB(c)}}{z-\frac{c-x}{B(x)(c-1)}}, z, v(x), \frac{B(c)}{(b-1)c}, B(x)\right] + \left[\alpha l_1(y), \frac{z-\frac{(1-b)(aby+1}{B(y)}}{z-\frac{1+aby}{(y-1)(ab-b+1)}}, z, g(y), \frac{ab-b+1}{ab}\right] \\
&\quad\quad - \left[\alpha l_1(y), \frac{z-r_4(\frac{a-1}{a},y)}{z-w_4(\frac{a-1}{a},y)}, z, g(y), \frac{ab-b+1}{a}\right].
\end{aligned}$$

A complicated step consists of getting rid of the constants in the first coordinate:

$$\begin{aligned}
&[\alpha l_1, q_4, v, gh, g] + [\alpha l_1, w_4, v, g, gh] = [l_1, q_4, v, gh, g] + [l_1, w_4, v, g, gh] \\
&\quad + \left[\frac{z-\alpha l_1(y)}{z-\alpha}, z, \frac{1+aby}{(ay-a+1)b}, B(y), \frac{ab-b+1}{1-b}\right] \\
&\quad + \left[\frac{z-\alpha l_1(y)}{z-\alpha}, z, \frac{1+aby}{(ab-b+1)(y-1)}, g(y), \frac{ab-b+1}{ab}\right] \\
&\quad - \left[\frac{z-\alpha l_1(y)}{z-\alpha}, z, \frac{ay-a+1}{(ab-b+1)(y-1)}, g(y), \frac{ab-b+1}{a}\right] \\
&\quad - \left[\frac{z-1}{z-\alpha}, z, \frac{a(bx-x+1)}{ab-b+1}, v(x), g(x)\right] \\
&\quad - \left[\frac{z-\alpha l_1(y)}{z-\alpha}, z, \frac{ay}{ay-a+1}, \frac{1}{1-a}, B(y)\right] - \left[\frac{z-1}{z-\alpha}, z, \frac{1}{b}, v(x), B(x)\right].
\end{aligned}$$

We go on:

$$\begin{aligned}
&[l_1, w_4, v, g, gh] = [l_1, p_4, v, g, gh]^2_{(-,\frac{A(y)}{\mu(1-y)})} \\
&\quad - \left[\frac{z-w_4(x,c)}{z-p_4(x,c)}, z, v(x), \frac{B(c)}{(b-1)c}, B(x)\right] + \left[l_1(y), \frac{z-w_4(-1/ab,y)}{z-p_4(-1/ab,y)}, z, g(y), \frac{ab-b+1}{ab}\right] \\
&\quad - \left[l_1(y), \frac{z-w_4(\frac{a-1}{a},y)}{z-p_4(\frac{a-1}{a},y)}, z, g(y), \frac{ab-b+1}{a}\right].
\end{aligned}$$

Then we go on by splitting the product gh:

$$\begin{aligned}&[l_1,q_4,v,gh,g]+[l_1,p_4,v,g,gh]=[l_1,q_4,v,h,g]^4_{(+,g)}+[l_1,q_4,v,g,h]^5_{(+,g)}\\ &\quad+\left[\frac{\mu(x-c)}{A(c)B(x)},v(x),\frac{B(c)}{(b-1)c},\frac{z-B(x)}{z-g(x)},z\right]-\left[l_1(y),\frac{1+aby}{(ay-a+1)b},\frac{z-B(y)}{z-g(y)},z,\frac{ab-b+1}{1-b}\right]\\ &\quad-\left[l_1(y),\frac{y}{A(y)},\frac{1}{1-a},\frac{z-B(y)}{z-g(y)},z\right]\end{aligned}$$

and finally see that

$$\begin{aligned}&[l_1,q_4,v,h,g]^4_{(+,g)}+[l_1,p_4,v,g,h]^5_{(+,g)}\\ &\quad=[l_1,p_4,v,g,h]+[l_1,p_4,v,h,g]+2[l_1,p_4,v,g,g]-[l_1,q_4/p_4,v,g,g]\\ &\quad-\left[\frac{z-\frac{c-x}{A(c)}}{z+\frac{\mu}{B(x)}},z,v(x),g(c),g(x)\right]-\left[l_1(y),\frac{z-1}{z+1},z,g(y),\frac{ab-b+1}{(1-a)(1-b)}\right]\\ &\quad+\left[l_1(y),\frac{z-q_4(-1/ab,y)}{z-p_4(-1/ab,y)},z,g(y),\frac{ab-b+1}{1-b}\right]+\left[l_1(y),\frac{z-q_4(0,y)}{z-p_4(0,y)},z,\frac{1}{1-a},g(y)\right].\end{aligned}$$

Now we perform something similar with two other terms, which we have also seen before:

$$\begin{aligned}&[\alpha l_1,q_4,\delta v,gh,h]+[\alpha l_1,s_4,\delta v,h,gh]=[l_1,q_4,\delta v,gh,h]+[l_1,s_4,\delta v,h,gh]\\ &\quad+\left[\frac{z-\alpha l_1(y)}{z-\alpha},z,\frac{1+aby}{(ay-a+1)b},B(y),\frac{ab-b+1}{ab}\right]^3_{(-,\frac{(b-1)(1+aby)}{ab(by-y+1)})}\end{aligned}$$

plus more terms

$$\begin{aligned}&+\left[\frac{z-\alpha l_1(y)}{z-\alpha},z,\frac{(b-1)(1+aby)}{abB(y)},-h(y),\frac{ab-b+1}{ab}\right]\\ &-\left[\frac{z-\alpha l_1(y)}{z-\alpha},z,\frac{(b-1)(ay-a+1)}{(by-y+1)a},-h(y),\frac{ab-b+1}{a}\right]\\ &-\left[\frac{z-1}{z-\alpha},z,\frac{a(bx-x+1)}{ab-b+1},\delta v(x),(b-1)x\right]+\left[\frac{z-\alpha l_1(y)}{z-\alpha},z,\frac{ay}{ay-a+1},\frac{\delta}{1-a},B(y)\right].\end{aligned}$$

The second term on the right-hand side is equal to

$$\begin{aligned}&[l_1,s_4,\delta v,h,gh]=[l_1,q_4,\delta v,h,gh]^2_{(-,\frac{B(y)}{(b-1)A(y)})}\\ &\quad-\left[\frac{z-\frac{(b-1)(c-x)}{B(c)}}{z-\frac{c-x}{A(c)}},z,\delta v(x),(b-1)c,B(x)\right]-\left[l_1(y),\frac{z-s_4(\frac{a-1}{a},y)}{z-1},z,-h(y),\frac{ab-b+1}{a}\right]\\ &\quad+\left[l_1(y),\frac{z-s_4(-1/ab,y)}{z-q_4(-1/ab,y)},z,-h(y),\frac{ab-b+1}{ab}\right]\end{aligned}$$

and so

$$[l_1, q_4, \delta v, gh, h] + [l_1, q_4, \delta v, h, gh] = [l_1, q_4, \delta v, g, h] + [l_1, q_4, \delta v, h, g] + 2[l_1, q_4, \delta v, h, h]$$
$$+ \left[\frac{c-x}{A(c)}, \delta v(x), (b-1)c, \frac{z-B(x)}{z-(b-1)x}, z\right] + \left[l_1(y), \frac{aby+1}{b(ay-a+1)}, \frac{z-B(y)}{z-(b-1)y}, z, \frac{1-b}{ab}\right]$$
$$- \left[l_1(y), q_4(0,y), \frac{1}{1-a}, \frac{z-B(y)}{z-h(y)}, z\right].$$

Getting rid of the constants in the first two terms, we find

$$[l_1, q_4, \delta v, g, h] + [l_1, q_4, \delta v, h, g] = [l_1, q_4, v, g, h]^3_{(+,\delta)} + [l_1, q_4, \delta v, h, g]^3_{(+,\delta)}$$
$$- \left[\frac{c-x}{A(c)}, \frac{z-\delta v(x)}{z-\delta}, z, \frac{bc-c+1}{(b-1)c}, h(x)\right] - \left[\frac{c-x}{A(c)}, \frac{z-\delta v(x)}{z-\delta}, z, (b-1)c, g(x)\right].$$

Finally:

$$[l_1, q_4, \delta v, g, h] = [l_1, p_4, \delta v, g, h]^2_{(-,\frac{-\mu}{B(x)})}$$
$$+ \left[\frac{z-q_4(x,c)}{z-p_4(x,c)}, z, v(x), g(c), h(x)\right] - \left[l_1(y), \frac{z-q_4(\frac{-1}{ab},y)}{z-p_4(\frac{-1}{ab},y)}, z, g(y), \frac{1-b}{ab}\right]$$
$$- \left[l_1(y), \frac{z-q_4(0,y)}{z-p_4(0,y)}, z, \frac{1}{1-a}, g(y)\right].$$

Summing up all together, we see that

$$Y_1 + Y_2 = [\alpha l_1, q_4, \delta v, gh, gh] - [l_1, p_4, v, g, g] - [l_1, q_4, \delta v, h, h] + \text{lower order terms}.$$

5^{th} **step:** We shall now compute $Y_3 + Y_4$. Additionally, we set $y_2 := l_2^{-1}(0)$:

$$2(Y_3+Y_4) = 2[l_2(y), q_4(x,y), v(x), h(y), g(x)] + 2[l_2(y), p_4(x,y), v(x), g(y), h(x)]$$
$$= -2[l_2(y), q_4(x,y), v(x), h(y), \frac{1}{g(x)}] - 2[l_2(y), p_4(x,y), v(x), \frac{1}{g(y)}, h(x)]$$
$$- 2\left[\frac{y_2-x}{A(y_2)}, v(x), (b-1)y_2, \frac{z-1}{z-g(x)}, z\right] - 2\left[l_2(y), \frac{aby+1}{ay-a+1}, \frac{z-1}{z-g(y)}, z, \frac{1-b}{ab}\right]$$
$$+ 2\left[l_2(y), \frac{(ab-b+1)y}{ay-a+1}, \frac{1}{1-a}, \frac{z-1}{z-g(y)}, z\right].$$

Now since $-2[l_2(y), q_4(x,y), v(x), h(y), \frac{1}{g(x)}] - 2[l_2(y), p_4(x,y), v(x), \frac{1}{g(y)}, h(x)]$ is just as well $-2[l_2(y), q_4(x,y), v(x), -h(y), \frac{1}{g(x)}] - 2[l_2(y), p_4(x,y), v(x), \frac{1}{g(y)}, -h(x)]$, we can proceed by finding

$$-2\left[l_2(y), q_4(x,y), v(x), -h(y), \frac{1}{g(x)}\right] - 2\left[l_2(y), p_4(x,y), v(x), \frac{1}{g(y)}, -h(x)\right]$$
$$= +2\left[l_2(y), q_4(x,y), \frac{1}{v(x)}, -h(y), \frac{1}{g(x)}\right] + 2\left[l_2(y), p_4(x,y), \frac{1}{v(x)}, \frac{1}{g(y)}, -h(x)\right]$$
$$+ 2\left[q_4(x,y_2), \frac{z-1}{z-v(x)}, z, -h(y_2), \frac{1}{g(x)}\right] + 2\left[p_4(x,y_2), \frac{z-1}{z-v(x)}, z, -h(y_2), \frac{1}{g(x)}\right]$$

and the first two terms can be written as

$$
\begin{aligned}
&2\left[l_2(y),\frac{ab(y-x)}{1+aby},\frac{1}{v(x)},-h(y),\frac{1}{g(x)}\right]+2\left[l_2(y),\frac{(ab-b+1)(y-x)}{(1+aby)B(x)},\frac{1}{v(x)},\frac{1}{g(y)},-h(x)\right]\\
&-2\left[l_2(y),\frac{1}{v(y)},\frac{1}{v(x)},\frac{1}{g(y)},-h(x)\right]-2\left[l_2(y),\frac{1}{v(y)},\frac{1}{v(x)},-h(y),\frac{1}{g(x)}\right]\\
&-2\left[\frac{z-\frac{y_2-x}{A(y_2)}}{z-\frac{aA(y_2)}{aby_2+1}},z,\frac{1}{v(x)},(1-b)y_2,\frac{1}{g(x)}\right]\\
&-2\left[l_2(y),\frac{z-\frac{y+1/ab}{A(y)}}{z-\frac{1}{v(y)}},z,-h(y),\frac{1-b}{ab-b+1}\right]+2\left[l_2(y),\frac{z-\frac{y}{A(y)}}{z-\frac{1}{v(y)}},z,1-a,-h(y)\right]\\
&-2\left[\frac{z-\frac{\mu(x-y_2)}{A(y_2)B(x)}}{z-\frac{1}{v(y_2)}},z,\frac{1}{v(x)},\frac{ab}{ab-b+1},-h(x)\right]\\
&-2\left[l_2(y),\frac{z+\frac{1+aby}{ay-a+1}}{z-\frac{aA(y)}{aby+1}},z,\frac{1}{g(y)},\frac{b-1}{ab}\right]+2\left[l_2(y),\frac{z-1}{z-\frac{1}{v(y)}},z,\frac{1}{g(y)},\frac{(1-a)(1-b)}{-a}\right]\\
&-2\left[l_2(y),\frac{z+\frac{\mu y}{A(y)(1-b)}}{z-\frac{(ab-b+1)y}{(aby+1)(1-b)}},z,1-a,-h(y)\right].
\end{aligned}
$$

But now we again rewrite the first two terms to find

$$
\begin{aligned}
&2\left[l_2(y),\frac{ab(y-x)}{1+aby},\frac{1}{v(x)},-h(y),\frac{1}{g(x)}\right]+2\left[l_2(y),\frac{(ab-b+1)(y-x)}{(1+aby)B(x)},\frac{1}{v(x)},\frac{1}{g(y)},-h(x)\right]\\
&=-\tau_{a,c}(Y_1+Y_2)\\
&\quad-\left[\frac{(ab-b+1)(y_2-x)}{(1+aby_2)B(x)},\frac{1}{v(x)},-\frac{(ac-a+1)(1-b)}{ab-b+1},\frac{z-1}{z+h(x)},z\right]\\
&\quad+\left[l_2(y),\frac{aby}{1+aby},1-a,\frac{z-1}{z+h(y)},z\right].
\end{aligned}
$$

6^{th} **step:** Let us perform the following decompositions:

$$
\begin{aligned}
Z_3(A,A)&=[l_1(y)\varepsilon_1(A),q_4(x,y),v(x),A(y),A(x)]^1_{(+,l_2(y)\varepsilon_2(A))}\\
&\quad+\left[\frac{z-l(y)}{z-\varepsilon_1(A)l_1(y)},z,\frac{y+1/ab}{A(y)},A(y),-\frac{ab-b+1}{ab}\right]\\
&=T_3(A)+T_4(A)+\left[\frac{z-l_y}{z-\varepsilon_1(A)l_1(y)},z,\frac{y+1/ab}{A(y)},A(y),\frac{-1-ab+b}{ab}\right]\\
Z_3\left(\frac{A}{B},\frac{A}{B}\right)&=T_1(f)+T_2(f)\\
2Z_3\left(\frac{A}{f},\frac{A}{f}\right)&=2\left[l(y),\frac{y-x}{yB(x)},\frac{x-1}{x},\frac{y}{A(y)},\frac{x}{A(x)}\right]\\
&\quad+2\left[l(y),\frac{y-x}{yB(x)},\frac{x-1}{x},\frac{-1}{\mu},\frac{x}{A(x)}\right]+2\left[l(y),\frac{y-x}{yB(x)},\frac{x-1}{x},\frac{y}{A(y)},\frac{-1}{\mu}\right].
\end{aligned}
$$

We concentrate on the first term:

$$\begin{aligned}
&\left[l(y), \frac{y-x}{yB(x)}, \frac{x-1}{x}, \frac{y}{A(y)}, \frac{x}{A(x)}\right] = -\left[l(y), \frac{y-x}{yB(x)}, \frac{x-1}{x}, \frac{A(y)}{y}, \frac{x}{A(x)}\right] \\
&\quad + \left[l(x), \frac{x-1}{x}, \frac{z-1}{z-\frac{x}{A(x)}}, z, \frac{x}{A(x)}\right] - \left[l(y), \frac{(1-y)b}{y}, \frac{z-1}{z-\frac{y}{A(y)}}, z, \frac{-1(ab+a-1}{(1-b)^2}\right] \\
&\quad - \left[l(y), \frac{y-1}{yb}, \frac{z-1}{z-\frac{A(y)}{y}}, z, \frac{a}{1-a}\right] - \left[l(y), \frac{x-1}{x}, \frac{z-1}{z-\frac{y}{A(y)}}, z, \frac{a}{ab-b+1}\right] \\
&= \left[l(y), \frac{y-x}{yB(x)}, \frac{x-1}{x}, \frac{A(y)}{y}, \frac{A(x)}{x}\right] \\
&\quad - \sum_{y_0} \left[\frac{y_0-x}{y_0B(x)}, \frac{x-1}{x}, \frac{A(y_0)}{y_0}, \frac{z-1}{z-\frac{x}{A(x)}}, z\right] \\
&\quad - \left[l(y), 1-b, \frac{z-1}{z-\frac{y}{A(y)}}, z, \frac{-1(ab+a-1}{(1-b)^2}\right] - \left[l(y), \frac{y-1}{yb}, \frac{z-1}{z-\frac{A(y)}{y}}, z, \frac{a}{1-a}\right].
\end{aligned}$$

Then, we get rid of a superfluous term:

$$\begin{aligned}
&\left[l(y), \frac{y-x}{yB(x)}, \frac{x-1}{x}, \frac{A(y)}{y}, \frac{A(x)}{x}\right] = \left[l(y), 1-\frac{x}{y}, \frac{x-1}{x}, \frac{A(y)}{y}, \frac{A(x)}{x}\right]^2_{(+,\frac{1}{B(x)})} \\
&\quad + \sum_{y_0} \left[\frac{z-\frac{y_0-x}{y_0B(x)}}{z-(1-\frac{x}{y_0})}, z, \frac{x-1}{x}, \frac{A(y_0)}{y_0}, \frac{A(x)}{x}\right] - \left[l(y), \frac{z-\frac{(y-1)b}{y}}{z-b}, z, \frac{A(y)}{y}, \frac{1}{a}\right]
\end{aligned}$$

and further

$$+ \left[l(y), \frac{z-\frac{y-(1-\frac{1}{a})}{yB(1-\frac{1}{a})}}{z-(1-\frac{a-1}{ay}}, z, \frac{1}{1-a}, \frac{A(y)}{y}\right].$$

Now the first term can be transformed:

$$\begin{aligned}
&\left[l(y), 1-\frac{x}{y}, \frac{x-1}{x}, \frac{A(y)}{y}, \frac{A(x)}{x}\right] = \left[l(y), 1-\frac{x}{y}, \frac{(1-a)(x-1)}{x}, \frac{A(y)}{y}, \frac{A(x)}{x}\right]^3_{(-,1-a)} \\
&\quad + \sum_{y_0} \left[1-\frac{x}{y_0}, \frac{z-\frac{(1-a)(x-1)}{x}}{z-(1-a)}, z, \frac{A(y_0)}{y_0}, \frac{A(x)}{x}\right] \\
&= T_1\left(\frac{A}{f}\right) + T_2\left(\frac{A}{f}\right) + \left[\frac{z-l(y)}{z-l_1(y)}, z, 1-\frac{1}{y}, \frac{A(y)}{y}, \frac{a-1}{a}\right] \\
&\quad - \left[l(y), 1-\frac{x}{y}, 1-a, \frac{A(y)}{y}, \frac{A(x)}{x}\right] + \sum_{y_0} \left[1-\frac{x}{y_0}, \frac{z-\frac{(1-a)(x-1)}{x}}{z-(1-a)}, z, \frac{A(y_0)}{y_0}, \frac{A(x)}{x}\right].
\end{aligned}$$

Now we use the abbreviation $\rho_{x,y} := \rho_y \circ \rho_x$ and look at

$$\begin{aligned}\rho_{x,y}[l_1, p_4, v, g, g] = &- \left[\varepsilon_2(A)l_2(y), 1-\frac{x}{y}, \frac{x-1}{x}, A(y), \frac{1}{A(x)}\right]\\
&+ \left[\varepsilon_2(A)l_2(y), 1-\frac{x}{y}, \frac{x-1}{x}, \frac{\mu}{b-1}, \frac{1}{A(x)}\right] + \left[\varepsilon_2(A)l_2(y), 1-\frac{x}{y}, \frac{x-1}{x}, \frac{1}{A(y)}, \frac{\mu}{b-1}\right]\\
&+ \left[\varepsilon_2(A)l_2(x), \frac{x-1}{x}, \frac{z-1}{z-A(x)}, z, \frac{1}{A(x)}\right] - \left[\varepsilon_2(A)l_2(y), \frac{y-1}{y}, \frac{z-1}{z-A(y)}, z, a\right]\\
&- \left[\varepsilon_2(A)l_2(y), 1-\frac{a-1}{ay}, \frac{1}{1-a}, \frac{z-1}{z-A(y)}, z\right]\end{aligned}$$

and with one more inversion we find

$$\begin{aligned}&= T_2(A)\\
&- \left[1-\frac{x}{y_2}, \frac{x-1}{x}, A(y_2), \frac{z-1}{z-A(x)}, z\right] + \left[\varepsilon_2(A)l_2(x), \frac{x-1}{x}, A(x), \frac{z-1}{z-A(x)}, z\right]\\
&+ \left[\varepsilon_2(A)l_2(y), 1-\frac{x}{y}, \frac{x-1}{x}, \frac{\mu}{b-1}, \frac{1}{A(x)}\right] + \left[\varepsilon_2(A)l_2(y), 1-\frac{x}{y}, \frac{x-1}{x}, \frac{1}{A(y)}, \frac{\mu}{b-1}\right]\\
&- \left[\varepsilon_2(A)l_2(y), \frac{y-1}{y}, \frac{z-1}{z-A(y)}, z, \frac{a}{1-a}\right]\\
&- \left[\varepsilon_2(A)l_2(y), 1-\frac{a-1}{ay}, \frac{1}{1-a}, \frac{z-1}{z-A(y)}, z\right].\end{aligned}$$

Lastly, we see

$$\begin{aligned}2[l_1, q_4, \delta v, h, h] &= 2\,[l_1(y), q_4(x,y), \delta v(x), y, x]\\
&+ 2\,[l_1(y), q_4(x,y), \delta v(x), b-1, x] + 2\,[l_1(y), q_4(x,y), \delta v(x), y, b-1]\end{aligned}$$

and the first term is seen to be

$$\begin{aligned}2\,[l_1(y), q_4(x,y), \delta v(x), y, x] &= 2T_3(f) + \left[l_1(y), \frac{A(y)}{y}, \delta v(x), y, x\right]\\
&+ 2\left[\frac{z-\frac{c-x}{A(c)}}{z-\frac{A(c)}{c}}, z, \delta v(x), c, x\right] - 2\left[l_1(y), \frac{z-\frac{y+1/ab}{A(y)}}{z-\frac{A(y)}{y}}, z, y, -\frac{1}{ab}\right]\\
&+ 2\left[l_1(y), \frac{z-1}{z-\frac{A(y)}{y}}, z, y, 1-\frac{1}{a}\right].\end{aligned}$$

7^{th} **step:** As in [Zha07] the last step consists of a splitting of $C_{k(c)}$. We proceed in several steps:

$$\begin{aligned}T_1(f) &= \left[1-\frac{y}{c}, 1-\frac{x}{y}, 1-\frac{1}{x}, y, x\right] = \left[1-\frac{1}{cy}, 1-\frac{y}{x}, 1-x, \frac{1}{y}, \frac{1}{x}\right]\\
&= -\left[1-\frac{1}{cy}, 1-\frac{y}{x}, 1-x, y, \frac{1}{x}\right] + \left[1-\frac{1}{cx}, 1-x, \frac{z-1}{z-x}, z, \frac{1}{x}\right]\end{aligned}$$

and so

$$T_1(f) = C_{\frac{1}{c}} - \left[1 - \frac{1}{cy}, 1-x, \frac{1}{c}, \frac{z-1}{z-x}, z\right].$$

Since the last two steps, i. e. inverting the two right coordinates occur several times in the following, we shall denote the last term with $Z(c)$ and keep in mind that

$$\left[l_1(y), 1-\frac{x}{y}, 1-\frac{1}{x}, y, x\right] = C_{\frac{1}{c}} - \left[1-\frac{1}{cy}, 1-x, \frac{1}{c}, \frac{z-1}{z-x}, z\right].$$

$$\begin{aligned} T_2(f) &= \left[\frac{y_2-y}{y_2 B(y)}, 1-\frac{x}{y}, 1-\frac{1}{x}, y, x\right] = \left[1-\frac{y}{y_2}, 1-\frac{x}{y}, 1-\frac{1}{x}, y, x\right]^1_{(+,\frac{1}{B(y)})} \\ &= C_{\frac{1}{y_2}} - Z(y_2) - \left[B(y), 1-\frac{x}{y}, 1-\frac{1}{x}, y, x\right] \\ &= C_{\frac{1}{y_2}} - Z(1/y_2) - C_{1-b} + Z(1-b). \end{aligned}$$

$$\begin{aligned} T_3(f) = \left[1-\frac{y}{c}, 1-\frac{x}{y}, \frac{abx+1}{abA(x)}, y, x\right] &= \left[1+\frac{1}{abcy}, 1-\frac{y}{x}, \frac{x-1}{x(b-ab)+1}, \frac{1}{y}, \frac{1}{x}\right] \\ + \left[1+\frac{1}{abcy}, 1-\frac{y}{x}, \frac{x-1}{x(b-ab)+1}, -\frac{1}{ab}, \frac{1}{x}\right] &+ \left[1+\frac{1}{abcy}, 1-\frac{y}{x}, \frac{x-1}{x(b-ab)+1}, \frac{1}{y}, -\frac{1}{ab}\right]. \end{aligned}$$

We split the first term in the third coordinate and obtain

$$\begin{aligned} &\left[1+\frac{1}{abcy}, 1-\frac{y}{x}, \frac{x-1}{x(b-ab)+1}, \frac{1}{y}, \frac{1}{x}\right] = \left[1+\frac{1}{abcy}, 1-\frac{y}{x}, 1-x, \frac{1}{y}, \frac{1}{x}\right]^3_{(+,\frac{1}{1-x(ab-b)})} \\ &\left[+\frac{1}{abcx}, \frac{z-\frac{x-1}{x(b-ab)+1}}{z-(1-x)}, z, \frac{-1}{abc}, \frac{1}{x}\right] \end{aligned}$$

which is equal to

$$\begin{aligned} &= C_{-\frac{1}{abc}} - Z\left(-\frac{1}{abc}\right) - \left[1+\frac{1}{abcy}, 1-\frac{y}{x}, 1-x(ab-b), \frac{1}{y}, \frac{1}{x}\right] \\ &\quad + \left[1+\frac{1}{abcx}, \frac{z-1}{z-(1-x(ab-b))}, z, -\frac{1}{abc}, \frac{1}{x}\right] \end{aligned}$$

Now we turn to the third term:

$$\begin{aligned} &\left[1+\frac{1}{abcy}, 1-\frac{y}{x}, 1-(ab-b)x, \frac{1}{y}, \frac{1}{x}\right] \\ &= \left[1+\frac{a-1}{acy}, 1-\frac{y}{x}, 1-x, \frac{1}{y}, \frac{1}{x}\right] + \left[1+\frac{a-1}{acy}, 1-\frac{y}{x}, 1-x, ab-b, \frac{1}{x}\right] \\ &\quad + \left[1+\frac{a-1}{acy}, 1-\frac{y}{x}, 1-x, \frac{1}{y}, ab-b\right]. \end{aligned}$$

And the first term is seen to be

$$\left[1+\frac{a-1}{acy},1-\frac{y}{x},1-x,\frac{1}{y},\frac{1}{x}\right]=C_{\frac{1-a}{ac}}-Z\left(\frac{ac}{a-1}\right)=C_{1-\frac{ac-a+1}{ac}}-Z\left(\frac{ac}{a-1}\right).$$

A similar reasoning can be applied to $\tau_{a,c}T_3(f)$, where $\tau_{a,c}:=\left(\frac{ab-b+1}{b(a-1)},\frac{ab-b+1}{ac-a+1}\right)$. Keep in mind that $ab-b+1=-a\mu$:

$$\begin{aligned}\tau_{a,c}T_3(f)&=\left[1-\frac{ab-b+1}{ca-a+1}y,1-\frac{x}{y},1+\frac{a}{-a\mu x-1},y,x\right]\\&=\left[1-\frac{ab-b+1}{(ca-a+1)y},1-\frac{y}{x},\frac{1-\frac{(1-a)x}{-a\mu}}{1-\frac{x}{-a\mu}},\frac{1}{y},\frac{1}{x}\right]\\&=\left[1-\frac{1}{(ca-a+1)y},1-\frac{y}{x},\frac{1-(1-a)x}{1-x},\frac{1}{y},\frac{1}{x}\right]\\&\quad+\left[1-\frac{1}{(ca-a+1)y},1-\frac{y}{x},\frac{1-(1-a)x}{1-x},-a\mu,\frac{1}{x}\right]\\&\quad+\left[1-\frac{1}{(ca-a+1)y},1-\frac{y}{x},\frac{1-(1-a)x}{1-x},\frac{1}{y},-a\mu\right].\end{aligned}$$

The first term can be split:

$$\begin{aligned}&\left[1-\frac{1}{(ca-a+1)y},1-\frac{y}{x},\frac{1-(1-a)x}{1-x},\frac{1}{y},\frac{1}{x}\right]=\left[1-\frac{1}{(ca-a+1)y},1-\frac{y}{x},1-(1-a)x,\frac{1}{y},\frac{1}{x}\right]^3_{(+,\frac{1}{1-x})}\\&\quad+\left[1-\frac{1}{(ca-a+1)y},\frac{z-\frac{(1-a)x}{1-x}}{z-(1-a)x},z,\frac{1}{y},\frac{1}{x}\right]\\&=\left[1-\frac{1-a}{(ca-a+1)y},1-\frac{y}{x},1-x,\frac{1-a}{y},\frac{1-a}{x}\right]-\left[1-\frac{1}{(ca-a+1)y},1-\frac{y}{x},1-x,\frac{1}{y},\frac{1}{x}\right]\\&\quad+\left[1-\frac{1}{(ca-a+1)x},\frac{z-1}{z-(1-x)},z,ca-a+1,\frac{1}{x}\right]\\&\quad+\left[1-\frac{1}{(ca-a+1)y},1-\frac{y}{x},\frac{1}{1-x},\frac{1}{y},\frac{1}{x}\right]+\left[1-\frac{1}{(ca-a+1)y},\frac{z-\frac{(1-a)x}{1-x}}{z-(1-a)x},z,\frac{1}{y},\frac{1}{x}\right]\\&=C_{\frac{1-a}{ca-a+1}}-Z\left(\frac{ca-a+1}{1-a}\right)-C_{\frac{1}{ca-a+1}}+Z(ca-a+1)\\&\quad+\left[1-\frac{1-a}{(ca-a+1)y},1-\frac{y}{x},1-x,1-a,\frac{1}{x}\right]+\left[1-\frac{1-a}{(ca-a+1)y},1-\frac{y}{x},1-x,\frac{1}{y},1-a\right]\\&\quad+\left[1-\frac{1}{(ca-a+1)x},\frac{z-1}{z-(1-x)},z,ca-a+1,\frac{1}{x}\right]\\&\quad+\left[1-\frac{1}{(ca-a+1)y},1-\frac{y}{x},\frac{1}{1-x},\frac{1}{y},\frac{1}{x}\right]+\left[1-\frac{1}{(ca-a+1)y},\frac{z-\frac{(1-a)x}{1-x}}{z-(1-a)x},z,\frac{1}{y},\frac{1}{x}\right].\end{aligned}$$

Let us now turn to $T_2(A)$. We silently use the reparametrization $x_i\mapsto x_i+\frac{a-1}{a}$ for $x_i=x,y$:

$$T_2(A)=\left[\frac{\frac{cb-c+1}{\mu c}-y}{\frac{b-1}{ac\mu}-y},1-\frac{y}{x},\frac{1-\frac{x}{a}}{1-\frac{1-a}{a}x},\frac{1}{y},\frac{1}{x}\right]$$

and further

$$\begin{aligned}
T_2(A) = &-\left[\frac{\frac{cab-ca+a}{a\mu c}-y}{\frac{b-1}{ac\mu}-y}, 1-\frac{y}{x}, \frac{1-\frac{x}{a}}{1-\frac{1-a}{a}x}, y, \frac{1}{x}\right] + \left[\frac{\frac{cb-c+1}{\mu c}-x}{\frac{b-1}{ac\mu}-x}, \frac{1-\frac{x}{a}}{1-\frac{1-a}{a}x}, \frac{z-1}{z-x}, z, \frac{1}{x}\right] \\
&-\left[\frac{\frac{cb-c+1}{\mu c}-y}{\frac{b-1}{ac\mu}-y}, 1-ay, \frac{z-1}{z-y}, z, \frac{1}{a}\right] + \left[\frac{\frac{cb-c+1}{\mu c}-y}{\frac{b-1}{ac\mu}-y}, 1-\frac{a}{a-1}y,, \frac{z-1}{z-y}, z, \frac{a}{a-1}\right] \\
= &\left[\frac{\frac{cb-c+1}{\mu c}-y}{\frac{b-1}{ac\mu}-y}, 1-\frac{y}{x}, \frac{1-\frac{x}{a}}{1-\frac{1-a}{a}x}, y, x\right] \\
&-\left[1-\frac{cab-ca+a}{a\mu cx}, \frac{1-\frac{x}{a}}{1-\frac{1-a}{a}x}, \frac{\mu c}{cb-c+1}, \frac{z-1}{z-x}, z\right] \\
&+\left[1-\frac{b-1}{ac\mu x}, \frac{1-\frac{x}{a}}{1-\frac{1-a}{a}x}, \frac{ac\mu}{b-1}, \frac{z-1}{z-x}, z\right] \\
&-\left[\frac{\frac{cb-c+1}{\mu c}-y}{\frac{b-1}{ac\mu}-y}, 1-ay, \frac{z-1}{z-y}, z, \frac{1}{a}\right] + \left[\frac{\frac{cb-c+1}{\mu c}-y}{\frac{b-1}{ac\mu}-y}, 1-\frac{a}{a-1}y, \frac{z-1}{z-y}, z, \frac{a}{a-1}\right].
\end{aligned}$$

We are now in the position to apply proposition 3.1.20 to obtain

$$\begin{aligned}
&\left[\frac{\frac{cb-c+1}{\mu c}-y}{\frac{b-1}{ac\mu}-y}, 1-\frac{y}{x}, \frac{1-\frac{x}{a}}{1-\frac{1-a}{a}x}, y, x\right] \\
&= C_{-\frac{cb-c+1}{c(ab-b+1)}} - C_{\frac{1-b}{ab-b+1}} - C_{\frac{(cb-c+1)(a-1)}{c(ab-b+1)}} + C_{\frac{(1-b)(a-1)}{ab-b+1}} + \text{lower order terms.}
\end{aligned}$$

The same reasoning also applies to $\tau_{a,c}T_2(A)$. Let us set

$$\beta := \frac{1-b-abc+2bc-c+ab+ab^2c-b^2c}{b(a-ac-1)}, \qquad \gamma := \frac{ab^2-b^2+2b-ab-1}{-ab}.$$

Then we have

$$\begin{aligned}
\tau_{a,c}T_2(A) &= \left[\frac{\beta-y}{\gamma-y}, 1-\frac{y}{x}, \frac{1-\frac{ab-b}{ab-b+1}x}{1-\frac{-1}{ab-b+1}x}, \frac{1}{y}, \frac{1}{x}\right] \\
= &\left[\frac{\beta-y}{\gamma-y}, 1-\frac{y}{x}, \frac{1-\frac{ab-b}{ab-b+1}x}{1-\frac{-1}{ab-b+1}x}, y, x\right] \\
&-\left[1-\frac{\beta}{x}, \frac{1-\frac{ab-b}{ab-b+1}x}{1-\frac{-1}{ab-b+1}x}, \frac{1}{\beta}, \frac{z-1}{z-x}, z\right] + \left[1-\frac{\gamma}{x}, \frac{1-\frac{ab-b}{ab-b+1}x}{1-\frac{-1}{ab-b+1}x}, \frac{1}{\gamma}, \frac{z-1}{z-x}, z\right] \\
&-\left[\frac{\beta-y}{\gamma-y}, 1-\frac{ab-b+1}{ab-b}y, \frac{z-1}{z-y}, z, \frac{ab-b}{ab-b+1}\right] \\
&+\left[\frac{\beta-y}{\gamma-y}, 1+(ab-b+1)y, \frac{z-1}{z-y}, z, \frac{-1}{ab-b+1}\right]
\end{aligned}$$

and applying 3.1.20 to the first term again, we conclude

$$\tau_{a,c}T_2(A) = C_{1-\frac{c(ab-b+1}{ca-a+1}} - C_{1-\frac{ab-b+1}{a}} - C_{\frac{bc-c+1}{b(ca-a+1)}} + C_{1-\frac{ab-(1-b)}{ab}} + \text{ lower order terms.}$$

Using the reparametrization from above we have:

$$\begin{aligned}
T_3(A) &= \left[1-\frac{a}{(ca-a+1)y}, 1-\frac{y}{x}, 1+\frac{\mu}{b}x, \frac{1}{y}, \frac{1}{x}\right]\\
&= \left[1+\frac{ab-b+1}{b(ca-a+1)y}, 1-\frac{y}{x}, 1-x, \frac{-\mu}{by}, \frac{-\mu}{bx}\right]\\
&= C_{-\frac{ab-b+1}{b(ca-a+1)}} - Z\left(-\frac{b(ca-a+1)}{ab-b+1}\right)\\
&\quad + \left[1+\frac{ab-b+1}{b(ca-a+1)y}, 1-\frac{y}{x}, 1-x, \frac{-\mu}{by}, \frac{1}{x}\right] + \left[1+\frac{ab-b+1}{b(ca-a+1)y}, 1-\frac{y}{x}, 1-x, \frac{1}{y}, \frac{-\mu}{b}\right]
\end{aligned}$$

and

$$\begin{aligned}
T_4(A) &= \left[\frac{\frac{cab-ca+a}{cab-cb+c}+y}{\frac{ab-a}{ab-b+1}+y}, 1-\frac{y}{x}, 1+\frac{ab-b+1}{ab}x, \frac{1}{y}, \frac{1}{x}\right] = \left[\frac{\frac{bc-c+1}{bc}+y}{\frac{b-1}{b}+y}, 1-\frac{y}{x}, 1-x, \frac{-\mu}{by}, \frac{-\mu}{bx}\right]\\
&= \left[\frac{\frac{bc-c+1}{bc}+y}{\frac{b-1}{b}+y}, 1-\frac{y}{x}, 1-x, \frac{1}{y}, \frac{1}{x}\right]\\
&\quad + \left[\frac{\frac{bc-c+1}{bc}+y}{\frac{b-1}{b}+y}, 1-\frac{y}{x}, 1-x, \frac{-\mu}{b}, \frac{1}{x}\right] + \left[\frac{\frac{bc-c+1}{bc}+y}{\frac{b-1}{b}+y}, 1-\frac{y}{x}, 1-x, \frac{1}{y}, \frac{-\mu}{b}\right],
\end{aligned}$$

where the first term is seen to be

$$\begin{aligned}
&\left[\frac{\frac{bc-c+1}{bc}+y}{\frac{b-1}{b}+y}, 1-\frac{y}{x}, 1-x, \frac{1}{y}, \frac{1}{x}\right] = \left[\frac{\frac{bc-c+1}{bc}+y}{-y}, 1-\frac{y}{x}, 1-x, \frac{1}{y}, \frac{1}{x}\right]^1_{(+,\frac{1}{\frac{b-1}{b}+y})}\\
&= C_{\frac{bc-c+1}{bc}} + Z\left(\frac{bc}{bc-c+1}\right) - C_{1-\frac{1}{b}} + Z\left(\frac{b}{b-1}\right).
\end{aligned}$$

For the next two identifications we start with another reparametrization, namely $x_i \mapsto \frac{1-a}{ax_i-a}$ for $x_i = x, y$. With this we show

$$\begin{aligned}
T_1\left(\frac{A}{f}\right) &= \left[\frac{(ca-a+1)-acy}{ac-acy}, \frac{x-y}{x-1}, 1-ax, y, x\right]\\
&= \left[\frac{(ca-a+1)-acy}{ac-acy}, 1-\frac{y}{x}, 1-ax, y, x\right]^2_{(+,\frac{x}{1-x})}\\
&\quad + \left[\frac{z-\frac{x-(1-\frac{a-1}{ac})}{x-1}}{z-\frac{x}{1-x}}, z, 1-ax, 1-\frac{a-1}{ac}, x\right] - \left[\frac{(ca-a+1)-acy}{ac-acy}, \frac{z-\frac{1-ay}{1-a}}{z-(1-ay)}, z, y, \frac{1}{a}\right].
\end{aligned}$$

The first term will now be treated in the following way:

$$\begin{aligned}
&\left[\frac{(ca-a+1)-acy}{ac-acy}, 1-\frac{y}{x}, 1-ax, y, x\right] = \left[\frac{(ca-a+1)-cy}{ac-cy}, 1-\frac{y}{x}, 1-x, y, x\right]\\
&\quad + \left[\frac{(ca-a+1)-cy}{ac-cy}, 1-\frac{y}{x}, 1-x, a, x\right] + \left[\frac{(ca-a+1)-cy}{ac-cy}, 1-\frac{y}{x}, 1-x, y, a\right]
\end{aligned}$$

so that we can split the first coordinate

$$\left[\frac{(ca-a+1)-acy}{ac-acy}, 1-\frac{y}{x}, 1-x, y, x\right] = C_{\frac{ca-a+1}{c}} + Z\left(\frac{c}{ca-a+1}\right) - C_a + Z\left(\frac{1}{a}\right).$$

Analogously, we obtain

$$\begin{aligned}
T_2\left(\frac{A}{f}\right) &= \left[\frac{\frac{-c(ab-b+1)}{ca-a+1}+y}{-\frac{ab-b+1}{a}+y}, 1-\frac{y}{x}, 1-ax, y, x\right]^2_{(+,\frac{x}{1-x})} \\
&+ \left[\frac{z-\frac{x-\frac{c(ab+b+1)}{ca-a+1}}{x-1}}{z-\frac{x}{1-x}}, z, 1-ax, \frac{c(ab+b+1)}{ca-a+1}, x\right] \\
&- \left[\frac{z-\frac{x-\frac{ab-b+1}{a}}{x-1}}{z-\frac{x}{1-x}}, z, 1-ax, \frac{ab-b+1}{a}, x\right] - \left[\frac{-\frac{c(ab+b+1)}{ca-a+1}+y}{-\frac{ab-b+1}{a}+y}, \frac{z-\frac{1-ay}{1-a}}{z-(1-ay)}, z, y, \frac{1}{a}\right]
\end{aligned}$$

so that finally the first term is equal to

$$\begin{aligned}
&C_{\frac{ac(ab+b+1)}{ca-a+1}} + Z\left(\frac{ca-a+1}{ac(ab+b+1)}\right) - C_{ab-b+1} + Z\left(\frac{1}{ab-b+1}\right) \\
&+ \left[\frac{\frac{-ac(ab-b+1)}{ca-a+1}+y}{-(ab-b+1)+y}, 1-\frac{y}{x}, 1-x, a, x\right] + \left[\frac{\frac{-ac(ab-b+1)}{ca-a+1}+y}{-(ab-b+1)+y}, 1-\frac{y}{x}, 1-x, y, a\right].
\end{aligned}$$

For the final two identifications we make use of the reparametrization $x_i \mapsto \frac{x_i-1}{b-1}$ for $x_i = x, y$ and obtain

$$\begin{aligned}
T_1(B) &= \left[1-\frac{y}{bc-c+1}, \frac{a(x-y)}{ab-b+1-ay}, \frac{x-\frac{ab-b+1}{ab}}{x-\frac{ab-b+1}{a}}, y, x\right] \\
&= \left[1-\frac{y}{bc-b+1}, \frac{x-y}{-y}, \frac{x-\frac{ab-b+1}{ab}}{x-\frac{ab-b+1}{a}}, y, x\right]^2_{(+,\frac{-ay}{ab-b+1-ay})} \\
&+ \left[\frac{z+\frac{x-(bc-c+1)}{\mu+bc-c+1}}{z-\left(1-\frac{x}{bc-c+1}\right)}, z, \frac{x-\frac{ab-b+1}{ab}}{x-\frac{ab-b+1}{a}}, bc-c+1, x\right] \\
&- \left[1-\frac{y}{bc-c+1}, \frac{z-\frac{\frac{ab-b+1}{b}-ay}{ab-b+1-ay}}{z-\frac{\frac{ab-b+1}{ab}-y}{-y}}, z, y, \frac{ab-b+1}{ab}\right] \\
&+ \left[1-\frac{y}{bc-c+1}\, \frac{z-1}{z-\frac{\frac{ab-b+1}{a}-y}{-y}}, z, y, \frac{ab-b+1}{a}\right] \\
&- \left[1-\frac{y}{bc-c+1}, \frac{z-\frac{ay}{ab-b+1-ay}}{z-1}, z, \frac{1}{b}, y\right].
\end{aligned}$$

Now the first term is transformed:

$$\begin{aligned}
&\left[1-\frac{y}{bc-b+1},\frac{x-y}{-y},\frac{x-\frac{ab-b+1}{ab}}{x-\frac{ab-b+1}{a}},y,x\right]\\
&=\left[1-\frac{1}{(bc-b+1)y},1-\frac{y}{x},\frac{1-\frac{ab-b+1}{ab}x}{1-\frac{ab-b+1}{a}x},\frac{1}{y},\frac{1}{x}\right]\\
&=\left[1-\frac{1}{(bc-b+1)y},1-\frac{y}{x},1-\frac{ab-b+1}{ab}x,\frac{1}{y},\frac{1}{x}\right]^3_{(+,\frac{1}{1-\frac{ab-b+1}{a}x})}\\
&\quad-\left[1-\frac{1}{(bc-c+1)x},\frac{z-\frac{x-\frac{ab-b+1}{ab}}{x-\frac{ab-b+1}{a}}}{z-(x-\frac{ab-b+1}{ab})},z,bc-c+1,\frac{1}{x}\right],
\end{aligned}$$

where the first term on the right-hand side is

$$\begin{aligned}
&\left[1-\frac{ab-b+1}{ab(bc-c+1)y},1-\frac{y}{x},1-x,\frac{ab-b+1}{aby},\frac{ab-b+1}{abx}\right]\\
&=C_{\frac{ab-b+1}{ab(bc-c+1)}}-Z\left(\frac{ab(bc-c+1)}{ab-b+1}\right)\\
&\quad+\left[1-\frac{ab-b+1}{ab(bc-c+1)y},1-\frac{y}{x},1-x,\frac{ab-b+1}{ab},\frac{1}{x}\right]\\
&\quad+\left[1-\frac{ab-b+1}{ab(bc-c+1)y},1-\frac{y}{x},1-x,\frac{1}{x},\frac{ab-b+1}{ab}\right],
\end{aligned}$$

whereas the second one can be written as

$$\begin{aligned}
&-\left[1-\frac{1}{(bc-b+1)y},1-\frac{y}{x},1-\frac{ab-b+1}{a}x,\frac{1}{y},\frac{1}{x}\right]\\
&\quad+\left[1-\frac{1}{(bc-c+1)x},\frac{z-1}{z-(1-\frac{ab-b+1}{a}x)},z,bc-c+1,\frac{1}{x}\right]\\
&=\left[1-\frac{ab-b+1}{ay(bc-c+1)},1-\frac{y}{x},1-x,\frac{ab-b+1}{ay},\frac{ab-b+1}{ax}\right]\\
&=C_{\frac{ab-b+1}{a(bc-c+1)}}-Z\left(\frac{a(bc-c+1)}{ab-b+1}\right)\\
&\quad+\left[1-\frac{ab-b+1}{ay(bc-c+1)},1-\frac{y}{x},1-x,\frac{ab-b+1}{a},\frac{1}{x}\right]\\
&\quad+\left[1-\frac{ab-b+1}{ay(bc-c+1)},1-\frac{y}{x},1-x,\frac{1}{x},\frac{ab-b+1}{a}\right]\\
&\quad+\left[1-\frac{1}{(bc-c+1)x},\frac{z-1}{z-(1-\frac{ab-b+1}{a}x)},z,bc-c+1,\frac{1}{x}\right].
\end{aligned}$$

Analogously, we treat the last case:

$$\begin{aligned}
\tau_{a,c}T_1(B) &= \left[1-\frac{ab-b+1}{a(bc-c+1)}y, \frac{(y-x)(ab-b+1)}{y(ab-b+1)-ab}, \frac{(x-2)(ab-b+1)+a}{x(ab-b+1)-ab}, y, x\right] \\
&= \left[1+\frac{\mu}{bc-c+1}y, \frac{y-x}{y+\frac{b}{\mu}}, \frac{x-(2-\mu)}{x+\frac{b}{\mu}}, y, x\right] \\
&= \left[1+\frac{\mu}{bc-c+1}y, \frac{y-x}{y}, \frac{x-(2-\mu)}{x+\frac{b}{\mu}}, y, x\right]^2_{(+,\frac{y}{y+\frac{b}{\mu}})} \\
&\quad + \left[\frac{z-\frac{(x-\frac{\mu}{bc-c+1})(ab-b+1)}{ab-\frac{\mu}{bc-c+1}(ab-b+1)}}{z-\frac{\frac{\mu}{bc-c+1}+x}{\frac{\mu}{bc-c+1}-\frac{b}{\mu}}}, z, \frac{x-(2-\mu)}{x+\frac{b}{\mu}}, -\frac{a(bc-c+1)}{ab-b+1}, x\right] \\
&\quad - \left[1+\frac{\mu}{bc-c+1}y, \frac{z-\frac{(y-(2-\mu))(ab-b+1)}{y(ab-b+1)-ab}}{z-\frac{(2-\mu)-y}{y}}, z, y, (2-\mu)\right] \\
&\quad + \left[1+\frac{\mu}{bc-c+1}y, \frac{z-\frac{(y+\frac{b}{\mu})(ab-b+1)}{y(ab-b+1)-ab}}{z-\frac{\frac{b}{\mu}-y}{y}}, z, y, -\frac{b}{\mu}\right] \\
&\quad + \left[1+\frac{\mu}{bc-c+1}y, \frac{z-\frac{y}{y+\frac{b}{\mu}}}{z-1}, z, \frac{2-\mu}{-\frac{b}{\mu}}, y\right].
\end{aligned}$$

Then we again find

$$\begin{aligned}
&\left[1+\frac{\mu}{bc-c+1}y, \frac{y-x}{y}, \frac{x-(2-\mu)}{x+\frac{b}{\mu}}, y, x\right] = \left[1+\frac{\mu}{(bc-c+1)y}, 1-\frac{y}{x}, \frac{1-(2-\mu)x}{1+\frac{b}{\mu}x}, \frac{1}{y}, \frac{1}{x}\right] \\
&\quad = \left[1+\frac{\mu}{(bc-c+1)y}, 1-\frac{y}{x}, 1-(2-\mu)x, \frac{1}{y}, \frac{1}{x}\right]^3_{(+,\frac{1}{1+\frac{b}{\mu}x})} \\
&\qquad - \left[1+\frac{\mu}{(bc-c+1)x}, \frac{z-\frac{1-(2-\mu)x}{1+\frac{b}{\mu}x}}{z-(1-(2-\mu)x)}, z, \frac{bc-c+1}{-\mu}, \frac{1}{x}\right],
\end{aligned}$$

where the first term is transformed into

$$\begin{aligned}
&C_{\frac{\mu(\mu-2)}{(bc-c+1)}} - Z\left(\frac{(bc-c+1)}{\mu(\mu-2)}\right) \\
&\quad + \left[1+\frac{\mu(2-\mu)}{(bc-c+1)y}, 1-\frac{y}{x}, 1-x, 2-\mu, \frac{1}{x}\right] \\
&\quad + \left[1+\frac{\mu(2-\mu)}{(bc-c+1)y}, 1-\frac{y}{x}, 1-x, \frac{1}{x}, 2-\mu\right],
\end{aligned}$$

and the second term from above into

$$
\begin{aligned}
&-\left[1+\frac{\mu}{(bc-c+1)y}, 1-\frac{y}{x}, 1+\frac{b}{\mu}x, \frac{1}{y}, \frac{1}{x}\right]\\
&\quad+\left[1+\frac{\mu}{(bc-c+1)x}, \frac{z-1}{z-(1+\frac{b}{\mu}x)}, z, -\frac{bc-c+1}{\mu}, \frac{1}{x}\right]\\
&=-C_{1-\frac{b}{bc-c+1}}+Z\left(\frac{b(bc-c+1)}{b(bc-c+1)-\mu^2}\right)\\
&\quad-\left[1+\frac{b}{(bc-c+1)y}, 1-\frac{y}{x}, 1-x, -\frac{b}{\mu y}, \frac{1}{x}\right]+\left[1+\frac{b}{(bc-c+1)y}, 1-\frac{y}{x}, 1-x, \frac{1}{y}, -\frac{b}{\mu}\right]\\
&\quad-\left[1+\frac{\mu}{(bc-c+1)x}, \frac{z-1}{z-(1+\frac{b}{\mu}x)}, z, -\frac{bc-c+1}{\mu}, \frac{1}{x}\right].
\end{aligned}
$$

The result follows. □

Remark 3.4.4. In summary, we extended Zhao's proof of the Goncharov relation to be valid in the quotient $C^3(F,5)/\partial C^3(F,6)\otimes\mathbb{Z}\left[\frac{1}{2}\right]$ by also computing all of the extra terms which vanish rationally. Unfortunately, those are far too many to use the relations for practical computations in the Chow group. For this reason we did not write it down explicitly in closed form. Since our proof follows Zhao rather close, one may also consult the original article for more details. ♢

Concluding remarks

In this thesis we obtained a presentation of the integral higher Chow groups of number fields explicitly. The importance of knowing explicit elements in motivic cohomology is apparent in the conjectures of Beĭlinson and – concentrating on number fields only – Zagier. More generally, a presentation of the algebraic K–groups of number fields in more accessible terms than the very definition is one of the goals of modern number theory or arithmetic algebraic geometry.

As we have explained, the approach to motivic cohomology and regulator maps via Bloch groups and the theory of polylogarithms is a fruitful way – at least in case one is just interested in describing the free part of motivic cohomology groups of number fields. So it was natural to extend this approach to acquire information on the torsion part as well. This is what was the goal of the present work.

In chapter two we obtained several results concerning the codimension two Chow groups of a number field. With our methods we could find explicit cycles which can be used in connection with the relations we proved in these groups to write down an explicit presentation for some concrete number fields. We could see that the relations in the Chow groups are a little more complicated than they are without considering torsion. In principle, our methods suffice to compute a set of generators of the codimension two Chow groups of an arbitrary number field. Unfortunately, our methods do not suffice for p-adic fields because of a missing regulator map into a suitably well-understood cohomology group.

A disadvantage of this approach certainly is that one needs to know the abstract structure of the algebraic K–group of the number field in question in order to know which order the generator has to have.

The natural next step to codimension three Chow groups of a number field is already unequally harder. We could not use integral versions of the cycles supposed to generate the rational Chow group in this codimension since they are not elements in the integral Chow group! Neglecting 2-torsion solves this problem, but it is not clear which role in the interplay of the Bloch group and the higher Chow group these cycles take. We also had to deal with an exploding complexity of the relations to be proved in the motivic cohomology groups which makes it impossible from the present point of view to obtain explicit elements in codimension three. Successfully, there are cycles which represent the so-called cyclotomic elements in motivic cohomology. But we could only prove easy relations in the higher Chow groups not sufficing for an explicit presentation.

This is where a further investigation should begin. One needs to know more and "better" cycles living in the integral Chow group, which fulfill relations of the type valid in the corresponding Bloch group with fewer extra terms than ours. In addition one needs to know the validity of the integral Beĭlinson– Soulé vanishing conjecture for the motivic cohomology of number fields on which our computations rely. As a different line of development, one may try to deduce explicit formulas for a p-adic regulator in order to find cycles generating the codimension two Chow groups of a p-adic field. This involves p-adic polylogarithms and p-adic analysis, but the results might be interesting not only because the K–theory of p-adic fields is not known in many cases. Thus, knowing explicit elements might help to determine the orders of torsion parts of the K–groups.

So it seems as if we explored the limit of what can be said about the approach to integral higher Chow groups of number fields via Bloch groups up to now. To get new results, one either has to affirm deep conjectures in the theory of motives, or one needs better cycles for which the desired relations coming from the Bloch groups hold without too many extra terms so that one can extract explicit information about the generators of the integral higher Chow groups.

Vita

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