Pseudodifferential analysis in Ψ^*-algebras on transmission spaces, infinite solving ideal chains and K-theory for conformally compact spaces

Dissertation
zur Erlangung des Grades
"Doktor
der Naturwissenschaften"

am Fachbereich "Physik, Mathematik und Informatik"
der Johannes Gutenberg-Universität
in Mainz

Jochen Alexander Ditsche

geb. in Düsseldorf

Mainz, den 9. August 2007

Tag der mündlichen Prüfung: 15. März 2008

(D77) Dissertation, Johannes Gutenberg-Universität Mainz

Summary

The present thesis is a contribution to the theory of algebras of pseudodifferential operators on singular settings. In particular, we focus on the b-calculus and the calculus on conformally compact spaces in the sense of Mazzeo and Melrose in connection with the notion of spectral invariant transmission operator algebras.

We summarize results given by Gramsch, Ueberberg and Wagner [46] and Lauter [60] on the construction of Ψ_0- and Ψ^*-algebras and the corresponding scales of generalized Sobolev spaces using commutators of certain closed operators and derivations.

In the case of a manifold with corners $\mathcal{Z}$ we construct a Ψ^*-completion $\mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ of the algebra of zero order b-pseudodifferential operators $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ in the corresponding C^*-closure $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \hookrightarrow \mathscr{L}(L^2(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}))$. The construction will also provide that localised to the (smooth) interior $\mathring{\mathcal{Z}}$ of $\mathcal{Z}$ the operators in the $\mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ can be represented as ordinary pseudodifferential operators.

In connection with the notion of solvable C^*-algebras - introduced by Dynin [32] - we calculate the length of the C^*-closure of $\Psi^0_{b,cl}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$ in $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$ by localizing $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ along the boundary face F using the (extended) indical familiy $I^{\mathcal{B}}_{F\mathcal{Z}}$. Moreover, we discuss how one can "localise" solving ideal chains in neighbourhoods U_p of arbitrary points $p \in \mathcal{Z}$. This localisation process will recover the singular structure of U_p; further, the induced length function $\mathfrak{l}_p$ is shown to be upper semi-continuous.

We give construction methods for Ψ^*- and C^*-algebras admitting only infinite long solving ideal chains. These algebras will first be realized as unconnected direct sums of (solvable) C^*-algebras and then refined such that the resulting algebras have arcwise connected spaces of one dimensional representations. In addition, we recall the notion of transmission algebras on manifolds with corners $(\mathcal{Z}_i)_{i\in\mathbb{N}}$ following an idea of Ali Mehmeti [3]. Thereby, we connect the underlying $\mathscr{C}^\infty$-function spaces using point evaluations in the smooth parts of the $\mathcal{Z}_i$ and use generalized Laplacians to generate an appropriate scale of Sobolev spaces. Moreover, it is possible to associate generalized (solving) ideal chains to these algebras, such that to every $n \in \mathbb{N}$ there exists an ideal chain of length n within the algebra.

Finally, we discuss the K-theory for algebras of pseudodifferential operators on conformally compact manifolds X and give an index theorem for these operators. In addition, we prove that the Dirac-operator associated to the metric of a conformally compact manifold X is not a Fredholm operator.

Contents

Conventions

- Sections are denoted by pairs of numbers like 3.4 and definitions, theorems, etc. by triples of numbers, e.g. theorem 3.4.5 in section 3.4.

  Equations are denoted by triples of numbers in parentheses like formula (3.4.5) in section 3.4.

- Unless otherwise indicated, Banach algebras and spaces are always considered over $\mathbb{C}$.

- Unless otherwise indicated, functions are assumed to be complex valued in this thesis.

- $\mathscr{S}(\mathbb{R}^n)$ denotes the *Schwartz space* on $\mathbb{R}^n$, i.e. the space of rapidly decreasing functions $\mathbb{R}^n \longrightarrow \mathbb{C}$.

- Let $\mathcal{E}$ and $\mathcal{F}$ be Banach or Fréchet spaces. Then $\mathscr{L}(\mathcal{E}, \mathcal{F})$ denotes the space of all continuous linear maps $\mathcal{E} \longrightarrow \mathcal{F}$ endowed with the usual operator norm.

- Let $\mathcal{T}$ be a topological space and let $\varphi, \psi : \mathcal{T} \longrightarrow \mathbb{C}$ be two mappings. We write $\varphi \prec \psi$, if $\psi \equiv 1$ on $\operatorname{supp} \varphi$; in particular $\varphi(x) = \varphi(x)\psi(x)$ holds for all $x \in \operatorname{supp} \varphi$.

- By $\overline{\mathbb{R}}_+$ we denote the positive half axis $[0, \infty[\subseteq \mathbb{R}$.

Introduction

The present thesis is a contribution to the theory of pseudodifferential operators on manifolds with corners, conformally compact spaces and transmission spaces in connection with spectral invariance:

(i) For a manifold with corners $\mathcal{Z}$ we show how to construct a Ψ^*-*completion* $\mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ of $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ using abstract construction concepts introduced by Gramsch, Ueberberg and Wagner [46]. The algebra $\mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ will be dense in the corresponding C^*-closure $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ of $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ and localized in the interior of $\mathcal{Z}$ the operators in $\mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ will be ordinary pseudodifferential operators. The algebra $\mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ itself will be contained in the space $\bigcap_s \mathscr{L}(\mathcal{H}^s_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}))$ of all operators of order 0 with respect to the scale $\mathcal{H}^s_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ of b-Sobolev spaces.

(ii) We give the definition of *solvable* Ψ^*-*algebras* and show that dense Ψ^*-subalgebras $\mathcal{A}$ of C^*-algebras $\mathcal{B}$ are solvable, provided they fulfil the operator theoretical condition that if $\mathcal{I} \trianglelefteq \mathcal{B}$ is a closed ideal in $\mathcal{B}$, then $\mathcal{I} \cap \mathcal{A}$ is dense in $\mathcal{I}$ (this will be called property E_0).

(iii) We calculate the length of the C^*-closure $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^l)$ of the parameter-dependent calculus $\Psi^0_{b,cl}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^l)$ of b-pseudodifferential operators on manifolds with corners F. To achieve this, we embed F into a larger manifold with corners $\mathcal{Z}$ and use the isomorphism $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)}) \cong \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})/\mathcal{I}_F$, where $\mathcal{I}_F$ denotes the kernel of the indical family associated to F.

(iv) We define the notion of the *local length* $\mathfrak{l}_p$ in $p \in \mathcal{Z}$ for certain classes of solvable C^*-algebras and show how to calculate it for $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ using the C^*-subalgebra $\mathcal{B}_\varphi(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$, where φ is an appropriate cut off function with $\varphi \equiv 1$ in a neighbourhood of p. It turns out that the local length $\mathfrak{l}_p$ recovers the codimension of a fixed neighbourhood of p and that $\mathfrak{l}_p$ is a upper semi-continuous function from $\mathcal{Z}$ to $\mathbb{N}$.

(v) We construct solvable C^*-algebras of b-pseudodifferential operators that admit no finite solving series. We realize this by using a direct sum of C^*-algebras $\mathcal{B}(\mathcal{Z}_i, {}^b\Omega^{\frac{1}{2}})$ where the $\mathcal{Z}_i$ are manifolds with corners of codimension i; the construction gives an unconnected algebra first, but we also give a definition such that the resulting space of one-dimensional representations is arcwise connected.

(vi) Following an idea of Ali Mehmeti [3], we define *transmission algebras* on families of manifolds with corners $\mathcal{Z}_i$ ($i \in \mathbb{N}$) that admit for each $n \in \mathbb{N}$ a *solving ideal chain* of length n. Thereby the interconnections are realized by an underlying function space with transmission, i.e. we identify $\mathscr{C}^\infty$-functions on appropriate neighbourhoods of the interior of the manifolds with corners $\mathcal{Z}_i$ and use constructing methods for Ψ^*-algebras, cf. [46], to get spectrally invariant algebras.

(vii) We compute the K-groups of $\mathcal{B}_0(X, {}^0\Omega^{\frac{1}{2}})$, the closure of the algebra of pseudodifferential operators of order zero $\Psi_0^0(X, {}^0\Omega^{\frac{1}{2}})$ on a conformally compact space X and give an index theorem for Fredholm elements of this calculus. Moreover, we discuss the *Dirac-operator* associated to this setting and prove that it is not a Fredholm operator.

Spectral invariance

The theory of pseudodifferential operators on manifolds $\mathcal{Z}$ with singularities often gives rise to an algebra $\Psi^0(\mathcal{Z})$ of operators of order zero, being almost always a Fréchet algebra due to its $\mathscr{C}^\infty$-structure. However, by passing over to its C^*-closure $\mathcal{B}(\mathcal{Z})$ in $\mathscr{L}(L^2(\mathcal{Z}))$ one looses (local) $\mathscr{C}^\infty$-properties such as pseudo- or micro-locality. To control this, it turned out that it is essential to construct algebras $\mathcal{A}^0(\mathcal{Z})$ of pseudodifferential operators of order 0 on the given singular space, which are symmetric Fréchet subalgebras of the corresponding C^*-closure $\mathcal{B}(\mathcal{Z})$ and satisfy the following crucial property of *spectral invariance* or *invariance under holomorphic functional calculus*

$$\mathcal{A}(\mathcal{Z}) \cap \mathscr{L}(L^2(\mathcal{Z}))^{-1} = \mathcal{A}(\mathcal{Z})^{-1}, \tag{1}$$

where $\mathcal{A}(\mathcal{Z})^{-1}$ resp. $\mathscr{L}(L^2(\mathcal{Z}))$ denotes the group of invertible elements in $\mathcal{A}(\mathcal{Z})$ resp. $\mathscr{L}(L^2(\mathcal{Z}))$. Algebras with these properties are called Ψ^**-algebras* in the sense of Gramsch [39]. These Ψ^*-properties yield a variety of remarkable (and sometimes unexpected) operator theoretical consequences: a Ψ^*-algebra $\mathcal{A}$ always has an open group of invertible elements (which is not true for an arbitrary Fréchet algebra) and there exists a holomorphic functional calculus, cf. Waelbroeck [112], for these algebras. Gramsch [39] also showed, that (1) plays an important role in the perturbation theory of Fredholm functions related to pseudodifferential analysis and has connections to non-abelian cohomology and to Oka's principle (cf. [42] and [44]). Note also, that it is possible to construct the *Green inverse* (*relative* resp. *pseudoinverse*) within the algebra. A slightly more complete overview will be given in chapter one; to see more interconnections and for the relevance of spectral invariance, we refer to [19], [28], [41], [44], [60], [65], [76], [97] and [100].

The proof of spectral invariance is often strongly connected to the characterization of pseudodifferential operators using commutator methods. Following an approach of Beals [14], Gramsch, Ueberberg and Wagner described in [46] a construction method for Ψ_0- and Ψ^*-algebras starting from closed derivations or closed operators using commutators.

Nowadays, many algebras of pseudodifferential operators have been shown to be Ψ^*-algebras; the interested reader is referred, for instance, to [9], [12], [24], [25], [39], [62], [66], [102], [103] and [110].

It is worth pointing out, that recently spectral invariance is also used in the context of *time frequency analysis* [47] and the *Novikov-conjecture* [71].

Submultiplicativity

Recall, that a Fréchet algebra $\mathcal{A}$ is called submultiplicative if the topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$ is given by a defining system of semi-norms $\{||\cdot||_k : k \in \mathbb{N}\}$ such that

$$||a \cdot b||_k \leq ||a||_k ||b||_k$$

holds for all $a, b \in \mathcal{A}$.

In the case of a commutative Fréchet algebra $\mathcal{A}$, it is a result by Mitiagin, Rolewicz and Żelazko [88] that submultiplicativity is equivalent to the property, that for every entire function $g(z) = \sum_{n=1}^{\infty} a_n z^n$ and every $x \in \mathcal{A}$ the series $g(x)$ is convergent. If $\mathcal{A}$ has an open group of invertible elements the inversion is continuous (cf. [113, p. 115]) and the holomorphic functional calculus of Waelbroeck is applicable, thus $\mathcal{A}$ is submultiplicative. However, in the non-commutative case it is still an open question whether every Ψ^*-algebra is submultiplicative or not. Note that there exist non-commutative Fréchet algebras with an open group of invertible elements which are not submultiplicative (see [115]).

A general approach for constructing submultiplicative Ψ^*-algebras was presented by Gramsch, Ueberberg and Wagner in [46], and the authors used this method to prove submultiplicativity for the *Hörmander classes* $\Psi^0_{\varrho,\delta}$ $(0 \leq \delta \leq \varrho \leq 1,\ \delta < 1)$. Moreover, Gramsch and Schrohe proved submultiplicativity for *Boutet de Monvel's* algebra and Baldus proved in [8] submultiplicativity of $\Psi(1, g)$ for all Hörmander metrics g.

Operatoralgebras on manifolds with corners

The calculus of b- or *totally characteristic pseudodifferential operators* was introduced by Melrose [83] in 1981; it turned out that the definition of b-pseudodifferential operators on manifolds with boundary naturally extends to more singular settings, namely to manifolds with corners.

Roughly speaking, manifolds with corners are locally of the form $\overline{\mathbb{R}}^k_+ \times \mathbb{R}^{n-1}$. This calculus of b-pseudodifferential operators was considered, for instance, by Melrose and Piazza [87], Melrose and Nistor [86], and Loya [74]. By taking the norm closure of b-pseudodifferential operators of order zero $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ in $\mathscr{L}(L^2(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}))$, we can attach a C^*-algebra $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ to the (singular) manifold with corners $\mathcal{Z}$.

Let us remark, that the algebra $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ can not be endowed with a topology making $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ into a topological algebra with an open group of invertible elements (see [60, Theorem 4.7.2] for the case for manifolds with boundary), although $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ can be realized as a symmetric subalgebra of $\mathscr{L}(L^2(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}))$. Thus instead of $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ itself, its C^*-closure $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ is often used to give a full description of certain aspects such as the Fredholm property.

Recall that it is a common feature of pseudodifferential calculi $\mathcal{B}$, that the Fredholm property can be characterized in the following sense: There exists a C^*-algebra $\mathcal{Q}$ and a homomorphism $\tau_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{Q}$ such that an operator $a \in \mathcal{B}$ is Fredholm if and only if its symbol $\tau_{\mathcal{B}}(a)$ is invertible in the algebra $\mathcal{Q}$.

For a b-pseudodifferential operator in $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ this joint symbol map is realized by the *principal symbol map*

$$ {}^b\sigma^{(0)}_{\mathcal{B}} : \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathscr{C}({}^bS^*\mathcal{Z}) \tag{2} $$

and the *indical families*

$$ I^{\mathcal{B}}_{F\mathcal{Z}} : \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^l) \tag{3} $$

where F is a face in $\mathcal{Z}$ of codimension l. The Fredholmness of an operator is thus characterized by the invertibility of the complete symbol $\tau_{\mathcal{B}}(a)$ given by

$$ ({}^b\sigma^{(0)}_{\mathcal{B}}(a), (I^{\mathcal{B}}_{F\mathcal{Z}}(a))_{F\in\mathcal{F}(\mathcal{Z})}) \in \mathscr{C}({}^bS^*\mathcal{Z}) \oplus \bigoplus_{F\in\mathcal{F}(\mathcal{Z})} \mathscr{C}_b(\mathbb{R}^l, \mathscr{L}(L^2(F, {}^b\Omega^{\frac{1}{2}}))). $$

Following an approach of Lauter [60, Chapter 6], we construct a dense Ψ^*-completion $\mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ of $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$, that on the one hand contains all the Fredholm inverses and on the other hand also respects certain $\mathscr{C}^\infty$-properties, for example, each $a \in \mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ extends to a bounded operator

$$ a : \mathcal{H}^s_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{H}^s_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}), $$

where $(\mathcal{H}^s_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}))_{s\in\mathbb{R}}$ denotes the scale of b-Sobolev spaces, and is an ordinary pseudodifferential operator if one localizes in coordinate charts supported in the smooth interior of $\mathcal{Z}$.

Operatoralgebras on conformally compact spaces

An open Riemannian manifold (X_0, g_0) is called *conformally compact space* if it is isometric to the interior of a compact manifold X with boundary ∂X, where X is endowed with the metric $g := \varrho^{-2}h$. Thereby h is a smooth metric on X and $\varrho : X \longrightarrow \overline{\mathbb{R}}_+$ is a boundary defining function. A conformally compact space (X_0, g_0) is a complete Riemannian manifold with negative sectional curvature outside a compact set by well known results.

In [65] Lauter developed an elliptic and Fredholm theory for pseudodifferential operators modelled along the geometry of conformally compact spaces following results of Mazzeo and Melrose [78]. Moreover, he gave a description of the C^*-completion of the algebra of operators of order 0. Thereby the analysis on (X_0, g_0) is in fact performed on X. To see that this is reasonable, note that the set of all smooth vector fields on X having bounded length with respect to the metric g is given by

$$ \mathcal{V}_0(X) := \left\{ V \in \mathscr{C}^\infty(X, TX) \,:\, V_{|\partial X} = 0 \right\}, $$

a characterization independent of the singular metric g. The metric g will be called 0-*metric*, these vector fields will be called 0-*vector fields* and their enveloping algebra of differential operators will be called 0-*differential operators on* X (cf. [78]) in the sequel.

Following [65] there are two invariants necessary to characterize the Fredholmness resp. compactness of a given pseudodifferential operator in the conformally compact setting:

the *principal symbol* and the *reduced normal operator*. The reduced normal operator - reflecting the behaviour of an operator near the boundary ∂X - can be regarded as an operator with values in $\mathcal{C}(S^*\partial X, \mathcal{B}_{b,c})$, where $\mathcal{B}_{b,c}$ denotes the C^*-algebra of b-c-type operators on the interval $[0,1]$[1]. If P is a 0-differential operator and $(x,y) \in \overline{\mathbb{R}}_+ \times \mathbb{R}^{n-1}_y$ are coordinates near the boundary of X, then P is given by $P = \sum_{j+|\alpha|\le m} a_{j,\alpha}(x,y)(x\partial_x)^j(x\partial_y)^\alpha$ with coefficients $a_{j,\alpha}$ smooth up to the boundary. The reduced normal operator of P is then

$$N(P)(y,\eta) = \sum_{j+|\alpha|\le m} a_{j,\alpha}(0,y) i^{|\alpha|} x^{|\alpha|} \eta^{|\alpha|} (x\partial_x)^j,$$

where $(y,\eta) \in T^*\partial X_{|\partial X} \cong \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$; this coincides with the definition given in [78] and [77] after using scale invariance.

K-theory

Following the fundamental work of Atiyah and Singer [7] there has been treatment of various kinds of index theorems for pseudodifferential operators on singular manifolds. And in some cases (see, for instance, [67] or [80]) it turned out that one can get an index formula on a manifold with boundary in terms of classes of full symbols using calculations on the (closed) double of X. We show that this is also true for the algebra $\mathcal{B}_0(X, {}^0\Omega^{\frac{1}{2}})$ of pseudodifferential operators on conformally compact spaces.

But first, let us stress that another important consequence of spectral invariance is, that a Ψ_0-algebra (resp. a Ψ^*-algebra) has the same K-theory as its norm closure (resp. C^*-closure). This has first been observed by Connes [22] using *Karoubi's density theorem* [55] (see also [56]), where results of Gramsch [39], [36] have been used and we will present a slightly generalization of this in the appendix A.1 of this thesis.

The exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{I}/\mathcal{K} \longrightarrow \mathcal{B}_0(X, {}^0\Omega^{\frac{1}{2}})/\mathcal{K} \longrightarrow \mathcal{B}_0(X, {}^0\Omega^{\frac{1}{2}})/\mathcal{I} \longrightarrow 0, \tag{4}$$

where $\mathcal{K}$ denotes the ideal of compact operators, $\mathcal{I}$ denotes the kernel of the reduced normal operator, yields that $\mathcal{B}_0(X, {}^0\Omega^{\frac{1}{2}})/\mathcal{I}$ is isomorphic to the range $\mathcal{B}(S^*\partial X)$ of the reduced normal operator[2]. Then the induced six term exact sequence is used to compute the K-groups of $\mathcal{B}$, if the left and the right quotient in (4) are well understood.

For this, we calculate the K-theory of $\mathcal{B}(S^*\partial X)$ using the short exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{B}(S^*\partial X) \longrightarrow \mathcal{B}(S^*\partial X)/\mathcal{J} \longrightarrow 0, \tag{5}$$

where the closed two-sided ideal $\mathcal{J}$ has a very special structure. Namely, it is isomorphic to $\mathcal{C}(S^*\partial X, \widehat{\mathcal{D}})$, where $\widehat{\mathcal{D}}$, a suitable ideal in the algebra of b-c-operators, has zero K-groups. Thus the K-groups of $\mathcal{B}(S^*\partial X)$ are isomorphic to the ones of $\mathcal{B}(S^*\partial X)/\mathcal{J}$ and the problem reduces to an analytic characterization of this quotient.

[1] Here b-c-type means, that the operators are of b-behaviour on the one end $\{0\}$ and of *cusp* behaviour on the other end $\{1\}$ of $[0,1]$; note, that we think of $[0,1]$ as the compactification of the half axis $[0,\infty[$.

[2] Which itself is a C^*-subalgebra of $\mathcal{C}(S^*\partial X, \mathcal{B}_{b,c})$.

Note that in [79] Schrohe, Melo and Nest used the same sequence (4) to compute the K-theory for *Boutet de Monvel's* algebra. And in fact, it turns out, that we could have used the K-theoretic techniques developed in [80] to calculate the K-groups of $\mathcal{B}(S^*\partial X)$ (after having done the analysis).

Solvable C^*-algebras and ideal chains

Following the definition of solvable C^*-algebras introduced by Dynin [32], Melrose and Nistor proved that the C^*-closure $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ (resp. the parameter depended version $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$, F a face in $\mathcal{Z}$) of $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ (resp. $\Psi^0_{b,cl}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$) is solvable. Lauter [63] then used a slightly different approach to calculate the length of this algebra making use of the full description of the *Jacobson topology* $\widehat{\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})}$, which he gave in [61].

Note, that if $\mathcal{Z}$ is a manifold with corners of $\dim \mathcal{Z} = m$, then a first measure of "how singular" the manifold is, is the codimesion of $\mathcal{Z}$, i.e. the largest codimension of a boundary face of $\mathcal{Z}$. It is then natural to ask, if one can recover this invariant by simply studying the C^*-algebra $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. It turns out, that the minimal length of the solving series is given in terms of this invariant: the length of the series equals m if $\operatorname{codim} \mathcal{Z} = m$ and $\operatorname{codim} \mathcal{Z} + 1$ in the other cases. The strong connection to the underlying geometry can also be seen in the definition of (one possibility for) the minimal solving series: it is given in terms of kernels of the symbol homomorphisms (2) and (3), descending the "singular hierarchy" of the faces of $\mathcal{Z}$. Let us also point out, that in the context of conformally compact spaces the solvability of $\mathcal{B}_0(X, {}^0\Omega^{\frac{1}{2}})$ with length two has been shown in [65, Corollary 7.5.3].

One of the main tools used for the calculation of the length of solving ideal chains is representation theory for C^*-algebras. This topic has been treated in several monographs and articles, see, for instance, [31] or [90]. In [61] Lauter developed a representation theory for Ψ^*-algebras. More precisely he used a result of Gramsch [40] on positive functionals to show that there is (under an additional condition E_0 resp. E_1) a continuous, bijective map $\Phi : \widehat{\mathcal{A}} \longrightarrow \widehat{\mathcal{B}}$, where $\mathcal{B}$ is the enveloping C^*-algebra of a Ψ^*-algebra $\mathcal{A}$ and $\widehat{\mathcal{A}}$ resp. $\widehat{\mathcal{B}}$ denotes the *spectrum* of $\mathcal{A}$ resp. $\mathcal{B}$, i.e. the set of all unitary equivalence classes of non-zero irreducible representations. Here the additional condition E_0 resp. E_1 is needed to provide that the Ψ^*-algebras in some sense behave well while taking intersections with closed two-sided C^*-ideals.

As *spectral invariance* generates this strong connection between Ψ^*-algebras and their C^*-closures, one expects that the notion of solving ideal chains for the C^*-algebra can be carried over to the dense Ψ^*-subalgebra. And indeed, it turns out, that it is possible to give a definition of solvable Ψ^*-algebras, such that if one takes a Ψ^*-algebra $\mathcal{A}$, where its C^*-closure $\mathcal{B}$ is solvable with length n (and $\mathcal{A}$ has property E_0 in $\mathcal{B}$), then $\mathcal{A}$ is solvable, too. Moreover the length of $\mathcal{A}$ is less or equal to the length of $\mathcal{B}$.

As we noted before, the length of the ideal chain measures how singular the manifold $\mathcal{Z}$ is. It is thus reasonable to ask, if there are C^*- resp. Ψ^*-algebras on singular settings such that there is no solving ideal chain of finite length. Using a direct sum construction, we prove that such algebras exist, where we use the result, that every solvable C^*-algebra

$\mathcal{B}$ induces a solving series for a closed two-sided ideal $\mathcal{I}$ of $\mathcal{B}$. Moreover, we construct algebras, such that the space of one-dimensional representations is arcwise connected.

Transmisson-spaces and -algebras

The concept of interaction or transmission spaces has been introduced by Ali Mehmeti [3, Chapter 4.3] resp. Ali Mehmeti and Nicaise [4]. To investigate, for instance, vibrations on cross-shaped networks, Ali Mehmeti used the *Friedrichs extension* to create "good" physically interpretable models. Thereby he considered products of appropriate function spaces and fixed the mutual influences by the choice of a subspace of this product, the so called *interaction space*. In [19, Section 4.7] Caps used this idea to prove well-posedness of the *Cauchy problem* for special time-dependent evolution equations of *Schrödinger* and (degenerate) diffusion type on networks.

We adept the approach of Ali Mehmeti to the case where the function spaces are given on families of manifolds with corners $\mathcal{Z}_i$, $i \in \mathbb{N}$. A suitable closed subspace is yielded by direct sums of the corresponding smooth function spaces identified over certain domains in the interior. This enables us to use on the one hand closed derivations to define a Ψ^*-algebra on the connected singular configuration spaces and gives on the other hand also the possibility to use generalized Laplacians - defined via the Friedrichs extension for a suitable positive selfadjoint operator - to get a scale of Sobolev spaces (in the sense of [23]) on which our operators act.

Moreover, by imposing an additional condition on the operators, we achieve, that there is a well defined $*$-morphism onto each face of the underlying manifolds with corners $\mathcal{Z}_i$. Thus, it is possible to define ideal chains

$$\mathcal{A} \supseteq \mathcal{I}_n \supseteq \mathcal{I}_{n-1} \supseteq \ldots \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_0 := \{0\}$$

of length n - that are slightly generalizations of solving ideal series - i.e. the quotients of the ideals are isomorphic to $\mathscr{C}_0(T_k, \mathcal{K}(H_k)$ for all quotients except $\mathcal{A}/\mathcal{I}_n$ and $\mathcal{I}_1/\{0\} = \mathcal{I}_1$.

Organisation of the text

CHAPTER 1 is concerned with the general concepts of spectrally invariant Fréchet algebras. In *section one* we recall the main definitions of Ψ_0- and Ψ^*-algebras. *Section two* then deals with how one can construct these algebras using (finite) sets of closed derivations or closed operators. In *section three* we discuss the special case of Ψ^*-subalgebras of $\mathscr{L}(\mathcal{H})$, where $\mathcal{H}$ denotes a Hilbert space and show how to define associated Sobolev spaces. As already mentioned, the methods of section two and three were first treated by Gramsch, Ueberberg and Wagner in [46], where they generalize concepts of Beals [14], Cordes [24], Coifman and Meyer [21] for the characterization of algebras of pseudodifferential operators to an abstract setting. Finally, in *section four* we use this construction method to define a Ψ^*-algebra of operators with order shift on a closed manifold.

Then in CHAPTER 2 we construct a Ψ^*-completion of $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ in $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ using the methods presented in chapter one. In *section one* we shortly review the main definitions and properties of the calculus of b-pseudodifferential operators on manifolds with corners. *Section two* is then concerned with the inductive construction of a Ψ^*-completion of $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. Hereby we use the construction given by Lauter [60] for manifolds with boundary as the beginning of the induction. The inductional step then uses the general principle, that $\mathscr{C}_b^\infty(\mathbb{R}^n, \mathcal{A})$ is a Ψ^*-algebra in $\mathscr{C}_b(\mathbb{R}^n, \mathcal{B})$, if $\mathcal{A}$ is a Ψ^*-algebra in $\mathcal{B}$. However, the resulting algebra $\mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ will still be to large, since $\mathcal{K}(L^2(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})) \hookrightarrow \mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. Thus in *section three*, we refine this algebra first by defining the following well-behaved Ψ^*-algebra, that acts on the scale of b-Sobolev spaces induced by the b-Laplacian Δ_b:

$$\begin{aligned}\mathcal{A}_1 := \big\{ a \in \mathscr{L}(L^2(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})) \,:\, a(D(\Lambda_b^\infty)) \subseteq D(\Lambda_b^\infty),& \\ \forall\, \nu \in \mathbb{N}\, \exists\, c_\nu \geq 0 : ||ad(\Lambda_b)^\nu(a)x|| \leq c_\nu ||x|| \big\}&.\end{aligned}$$

Then we impose additional commutator conditions to enable a *Beals-type* characterisation in the smooth part of the manifold with corners $\mathcal{Z}$, i.e. the operators in the resulting Ψ^*-algebra $\mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ are ordinary pseudodifferential operators if we localize in a coordinate chart that has support only in the the smooth interior $\mathring{\mathcal{Z}}$ of $\mathcal{Z}$.

In CHAPTER 3, we first present the main results in representation theory of C^*- and Ψ^*-algebras in *section one.* Then in *section two* we define the notion of solvable Ψ^*-algebras and prove, that a Ψ^*-algebra $\mathcal{A}$, which is dense and has property E_0 in a solvable C^*-algebra $\mathcal{B}$, is solvable, too. In *section three* we show how to calculate the length of $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^l)$. Thereby, we can restrict ourself to the case that F is a face of a manifold with corners $\mathcal{Z}$ of codimension l, i.e. we analyse the algebra $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$. We localise the algebra $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ along the boundary surface F to calculate the length of $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$ using the well-known characterization of the spectrum of $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$, cf. [64]. In particular the isomorphism

$$\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)}) \cong \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})/\mathcal{I}_F$$

is used, where $\mathcal{I}_F$ denotes the kernel of the indical family homomorphism

$$I^{\mathcal{B}}_{F\mathcal{Z}} : \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$$

corresponding to the face F. We then define the notion of local length for certain types of C^*-algebras in *section four.* The definition is given in a way that the underlying geometry of the manifold with corners $\mathcal{Z}$ gives rise to local ideal chains of pseudodifferential operators localized in open sets $U \subseteq \mathcal{Z}$. The length of such a chain is then calculated in detail, where we use a special solving series for $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$, which is due to Melrose and Nistor [86], and implicitly also given by Lauter in [62].

In CHAPTER 4 we first consider a C^*-algebra $\mathcal{B}$ that is defined by a C^*-direct sum construction of algebras $\mathcal{B}_k$ of pseudodifferential operators on manifolds with corners $\mathcal{Z}_k$, where the $\mathcal{B}_k$ (resp. the underlying manifold $\mathcal{Z}_k$) are chosen such that length of $\mathcal{B}_k$ is k. We prove, that $\mathcal{B}$ cannot be solvable with finite length in *section one.* In *section two* we discuss how one can connect the spaces of one-dimensional representation to get an example of an algebra with no finite solving series and an arcwise connected space of one-dimensional representations. Finally in *section three* we introduce the concept of transmission spaces and algebras following an idea of Ali Mehmeti [4]. Using the point evaluations in the interior of manifolds with corners $\mathcal{Z}_i$ ($i \in \mathbb{N}$), we define a *transmission space* of smooth functions. We then use commutator methods and the Friedrichs extension of a suitable Laplacian to define Ψ^*-algebras, that localized in the smooth interior of a single manifold $\mathcal{Z}_i$ behave like ordinary pseudodifferential operators. Moreover, by restricting our algebra further, i.e. by imposing that localized to a boundary face of a manifold $\mathcal{Z}_i$ the operators are b-pseudodifferential operators, we can define suitable replacements of indical families. The resulting ideals of kernels are then used to define solving ideal chains of arbitrary length.

Finally in CHAPTER 5 we treat the K-theory for C^*-algebras of pseudodifferential operators on conformally compact spaces. In *section one* we review the main properties of the algebra $\Psi_0^0(X, {}^0\Omega^{\frac{1}{2}})$ of pseudodifferential operators on a conformally compact space X. In particular, we introduce the main structure elements in 0-geometry and the reduced normal operator. In *section two* we discuss the main properties of the b-c-calculus on the interval $M := [0,1]$. Moreover, we compute the K-groups of the associated C^*-algebra $\mathcal{B}_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$ and discuss a certain subalgebra $\mathcal{D}$ which we will need in later sections. Then in *section three* we first analyse the range of the reduced normal operator using the sequence (5). Although we then could have used the methods given in [79] resp. [80] to compute the K-groups of $\mathcal{B}_0(X, {}^0\Omega^{\frac{1}{2}})$, we present a self-contained proof for this main result - a proof that also works in the setting of [79] resp. [80] - using only the following exact commutative diagram of abelian groups and morphisms:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{A}'_i \arrow[r, "f'_i"] & \mathcal{B}'_i \arrow[r, "g'_i"] & \mathcal{C}'_i \arrow[r, "h'_i"] & \mathcal{A}'_{i+1} \arrow[r] & \cdots . \\
\cdots \arrow[r] & \mathcal{A}_i \arrow[r, "f_i"] \arrow[u, "a_i"] & \mathcal{B}_i \arrow[r, "g_i"] \arrow[u, "b_i"] & \mathcal{C}_i \arrow[r, "h_i"] \arrow[u, "c_i"] & \mathcal{A}_{i+1} \arrow[r] \arrow[u, "a_{i+1}"] & \cdots
\end{tikzcd}$$

The K-theoretic result for $\mathcal{B}_0(X, {}^b, \Omega^{\frac{1}{2}})$ is then used in *section four* to give an index formula for Fredholm operators on conformally compact spaces. We prove, that the index of such an operator is given by the topological index map ind_t of the K_1 class of the full symbol τ_0 projected to $K_1(\mathcal{C}(T^*\mathring{X}))$. Then in *section five* we discuss the *Dirac operator* associated to the conformally compact metric g: Using [65], we give the explicit formula for the reduced normal operator of the Dirac operator associated to g and a condition, when this family is Fredholm. We use this to show that the Dirac operator on a conformally compact manifold is not a Fredholm operator on L^2 independent on the dimension of X, extending a result of Lott [73], where he proves this in the case $\dim X = 4k$ ($k \in \mathbb{N}$). Finally in *section six* we prove, that the K-theoretic results for the C^*-algebra $\mathcal{B}_0$ also hold for dense Ψ^*-completions of $\Psi_0^0(X, {}^0\Omega^{\frac{1}{2}})$.

During the final work of this thesis the author learned of a preprint of Albin and Melrose [2], where they consider families of 0-Fredholm operators on conformally compact spaces and also obtain an index map given in terms of the double of X.

Acknowledgements

Due to legal regulations the Johannes Gutenberg- University of Mainz does not allow to name the referees of this thesis or any persons in the acknowledgement in this online publication. Thus all these names have been withdrawn.

I am greatly indebted to my thesis advisor for valuable support during the last years and for drawing my interest on the topic of Ψ^*-algebras. In particular, I am grateful for many fruitful discussions and the hints he gave me concerning ideal chains with infinite length and transmission algebras.

It is also a pleasure to thank one of my advisor's former PHD students, who suggested studying the K-theory of pseudodifferential operators on a conformally compact manifold and for always answering my questions concerning the topic of pseudodifferential operators with great patience.

Moreover, I want to thank my host in Bucharest for his hospitality during my stay with him at the IMAR where some parts of this thesis have been developed. His suggestions and his experience in the field of K-theory and *Dirac*-operators yielded a great push forward in treating some parts of this thesis.

Furthermore I want to express my gratitude to the second referee of this thesis for the invitation to stay at the *Pennsylvania State University* at State College and for his warm hospitality during my time there. In addition I like to thank the DAAD for financial support during this stay.

I also wish my colleagues for our valuable discussions during the last years.

Schließlich möchte ich noch den Menschen danken, die mich stets begleitet und unterstützt haben und die immer für mich da waren, falls ich einmal guten Zuspruch benötigt habe: meiner Freundin, meinem Bruder und vor allem meinen Eltern.

Jochen Ditsche

Chapter 1

Spectrally invariant Fréchet algebras and commutator methods

This chapter is concerned with the general concepts that enable us to construct spectrally invariant Fréchet algebras. We recall the main definitions - in particular we define Ψ_0- and Ψ^*-algebras - and show how one can construct these algebras using (finite) sets of closed derivations or closed operators. This was first treated in [46] and was slightly generalized in [60]. Throughout this chapter we closely follow [60].

1.1 Basic results on Ψ^*-algebras

Let $\mathcal{A}$ be an algebra. In what follows we denote by $\mathcal{A}^{-1}$ the group of invertible elements in $\mathcal{A}$.

Definition 1.1.1 (Spectral invariance)**.** Let $\mathcal{B}$ be a Banach algebra with unit e and $\mathcal{A}$ a subalgebra of $\mathcal{B}$ such that $e \in \mathcal{A}$. $\mathcal{A}$ is called

(i) *locally spectral invariant* in $\mathcal{B}$, if there exists $\varepsilon > 0$ such that

$$\{a \in \mathcal{A} : ||e - a||_{\mathcal{B}} < \varepsilon\} \subseteq \mathcal{A}^{-1}$$

holds;

(ii) *spectrally invariant* in $\mathcal{B}$, if we have $\mathcal{A} \cap \mathcal{B}^{-1} = \mathcal{A}^{-1}$.

Remark 1.1.2. The notion of spectrally invariant algebras $\mathcal{A}$ - sometimes also called *full algebras* or *algèbre plaine* - goes back to the works of [18], [91],[112] and [113]; the pair $(\mathcal{A}, \mathcal{B})$ is often said to be a *Wiener-pair* (cf. [91, chapt III. pp. 203, 214, 310],[108]).

The following definition is due to Gramsch [39].

Definition 1.1.3 (Ψ^*-algebras)**.** Let $\mathcal{B}$ be an unital Banach algebra and $\mathcal{A}$ a subalgebra of $\mathcal{B}$ such that $e \in \mathcal{A}$. Then $\mathcal{A}$ is called

(i) Ψ_0-*algebra* in $\mathcal{B}$, if
   (a) $\mathcal{A}$ is locally spectral invariant in $\mathcal{B}$ and
   (b) $\mathcal{A}$ is endowed with a topology $\tau_{\mathcal{A}}$, such that $(\mathcal{A}, \tau_{\mathcal{A}}) \hookrightarrow \mathcal{B}$ is a continuously embedded Fréchet algebra;

(ii) Ψ^*-*algebra* in $\mathcal{B}$, if in addition to (i)

  (a) $\mathcal{B}$ is a C^*-algebra and

  (b) $\mathcal{A}$ is a symmetric Ψ_0-algebra in $\mathcal{B}$;

(iii) *submultiplicative* Ψ_0-, resp. Ψ^*-algebra, if the topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$ is generated by a submultiplictive family of seminorms $(q_j)_{j\in\mathbb{N}}$.

Remark 1.1.4.

(i) It is proven in [39, Lemma 5.3] that a dense locally spectral invariant subalgebra $\mathcal{A}$ of $\mathcal{B}$ is actually spectral invariant in $\mathcal{B}$. In particular, every Ψ^*-algebra $\mathcal{A}$ of a C^*-algebra $\mathcal{B}$ is spectrally invariant in $\mathcal{B}$. Moreover, it is shown that one can choose $\varepsilon = 1$ in the definition of Ψ_0-algebras.

(ii) If $\mathcal{A}$ is a submultiplicative Ψ_0-algebra, we can assume without loss of generality, that the system of seminorms $(p_j)_{j\in\mathbb{N}_0}$ generating the topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$ is increasing: If the system is not increasing, we can use the equivalent system of seminorms given by $p'_n := \max\limits_{j=0,\dots,n} p_j$ instead.

(iii) The class of (submultiplicative) Ψ_0-, resp. Ψ^*-algebras is stable with respect to countable intersection (see [60, p. 14] for instance).

(iv) Let $\mathcal{A}$ be a Fréchet algebra with an open group $\mathcal{A}^{-1}$ of invertible elements. Then the inversion $\mathcal{A}^{-1} \ni a \longmapsto a^{-1} \in \mathcal{A}$ is continuous (see [113]).

Example 1.1.5.

(i) Let Y be a m-dimensional (smooth) closed manifold and $l \geq 0$. Then the algebra of classical parameter-dependent zero order pseudodifferential operators $\Psi^0_{cl}(Y, \Omega^{\frac{1}{2}}, \mathbb{R}^l)$ is a Ψ^*-algebra in $\mathscr{C}_b(\mathbb{R}^l, \mathscr{L}(L^2(Y, \Omega^{\frac{1}{2}})))$ (in particular it is a Ψ^*-algebra in its C^*-closure $\mathcal{B}(Y, \Omega^{\frac{1}{2}}, \mathbb{R}^l)$, cf. [60, Theorem 4.2.23, Corollary 4.4.24], see also [104, Theorem 4.3.2]).

(ii) Let $\mathcal{A}$ be a Ψ^*-algebra in a C^*-algebra $\mathcal{B}$. Then $\mathscr{C}^\infty_b(\mathbb{R}^l, \mathcal{A})$ is a Ψ^*-algebra in the C^*-algebra $\mathscr{C}_b(\mathbb{R}^l, \mathcal{B})$ $(l \geq 0)$.

(iii) Let P be a compact manifold and $\mathcal{A}$ be a Ψ^*-algebra in a C^*-algebra $\mathcal{B}$. Then $\mathscr{C}^\infty(P, \mathcal{A})$ is a Ψ^*-algebra in $\mathscr{C}^\infty(P, \mathcal{B})$.

The following two propositions show that the Ψ^*-property is also stable with respect to taking preimages.

Proposition 1.1.6. *Let $\mathcal{B}$ and $\mathcal{D}$ be C^*-algebras, $(\mathcal{A}, (q_j)_{j\in\mathbb{N}_0}) \hookrightarrow \mathcal{B}$ be a (submultiplicative) Ψ^*-algebra in $\mathcal{B}$, $(\mathcal{C}, (p_j))_{j\in\mathbb{N}_0} \hookrightarrow \mathcal{D}$ be a (submultiplicative) Ψ^*-algebra in $\mathcal{D}$ and $\tau : \mathcal{B} \longrightarrow \mathcal{D}$ be a continuous $*$-homomorphism. We set*

$$\mathcal{A}_{\mathcal{C}} := \{a \in \mathcal{A} \,:\, \tau(a) \in \mathcal{C}\}$$

and define seminorms $\widehat{p}_j$ on $\mathcal{A}$ by $\widehat{p}_j(a) := p_j(\tau(a))$ for $a \in \mathcal{A}_\mathcal{C}$ and $j \in \mathbb{N}_0$. Then $(\mathcal{A}_\mathcal{C}, (q_j)_{j\in\mathbb{N}_0}, (\widehat{p}_j)_{j\in\mathbb{N}_0}) \hookrightarrow \mathcal{B}$ is a (submultiplicative) Ψ^*-algebra and $\tau_{|\mathcal{A}_\mathcal{C}} : \mathcal{A}_\mathcal{C} \longrightarrow \mathcal{C}$ is a continuous $*$-homomorphism.

Proof. Cf. [60, Lemma 2.1.10]: The closed graph theorem immediately yields the continuity of multiplication, inversion and $*$-operation in $\mathcal{A}_\mathcal{C}$. So it is left to prove spectral invariance: Let $a \in \mathcal{A}_\mathcal{C} \cap \mathcal{B}^{-1}$, then $a \in \mathcal{A}^{-1}$ using the spectral invariance of $\mathcal{A}$ in $\mathcal{B}$, i.e. $b := a^{-1} \in \mathcal{A}$. Moreover, $\tau(a) \in \mathcal{C} \cap \mathcal{D}^{-1} = \mathcal{C}^{-1}$ yields $\tau(b) = \tau(a)^{-1} \in \mathcal{C}$, i.e. $b \in \mathcal{A}_\mathcal{C}$. □

Proposition 1.1.7. *Let $\mathcal{B}$ be a C^*-algebra, and $\mathcal{A} \hookrightarrow \mathcal{B}$ be a Ψ^*-algebra. Furthermore, let Γ be an index set, $(\mathcal{D}_\gamma)_{\gamma\in\Gamma}$ be a family of unital C^*-algebras, and $\varphi_\gamma : \mathcal{B} \longrightarrow \mathcal{D}_\gamma$ resp. $\psi_\gamma : \mathcal{B} \longrightarrow \mathcal{D}_\gamma$ ($\gamma \in \Gamma$) be homomorphisms of C^*-algebras. Then:*

(i) *$\mathcal{B}_\Gamma := \{b \in \mathcal{B} : \varphi_\gamma(b) = \psi_\gamma(b)$ for all $\gamma \in \Gamma\} \subseteq \mathcal{B}$ is a C^*-subalgebra.*

(ii) *$\mathcal{A}_\Gamma := \mathcal{A} \cap \mathcal{B}_\gamma$ is a Ψ^*-algebra in $\mathcal{B}_\Gamma$.*

Proof. Cf. [60, Lemma 2.1.11]: (i) follows from the fact that φ_γ and ψ_γ are continuous $*$-algebra homomorphisms.
(ii) The continuous embedding $\mathcal{A} \hookrightarrow \mathcal{B}$ yields that $\mathcal{A}_\Gamma$ is a closed, symmetric subalgebra of $\mathcal{A}$. In particular it is a continuously embedded Fréchet subalgebra of $\mathcal{B}_\Gamma$. Finally

$$\mathcal{A}_\Gamma \cap \mathcal{B}_\Gamma^{-1} = \mathcal{A} \cap \mathcal{B}^{-1} \cap \mathcal{B}_\Gamma = \mathcal{A}^{-1} \cap \mathcal{B}_\Gamma = \mathcal{A}_\Gamma^{-1}$$

yields the spectral invariance of $\mathcal{A}_\Gamma$ in $\mathcal{B}_\Gamma$. □

Remark 1.1.8. The *Hörmander* classes $\Psi^0_{\varrho,\delta}(\mathbb{R}^n)$, i.e. the algebras of all zero order pseudodifferential operators with (ϱ, δ)-shift on $\mathbb{R}^n$, are submultiplicative Fréchet operator algebras with spectral invariance in $\mathscr{L}(L^2(\mathbb{R}^n))$. This result is well-known nowadays, but it has been a long way until this result has been completely proven. A lot of mathematicians including Hörmander, Seeley, Caldéron and Vaillancourt, Cordes, Fefferman, Boney and Chemin, Gramsch, Ueberberg, Schrohe and Wagner contributed to this important structure result.

Subsequent to the work [39] of Gramsch in 1984, many results for spectrally invariant algebras have been proven by various authors, and the theory is far from being completed.

Indeed, the Ψ^*-property has remarkable applications for instance in the development of Fréchet algebras in microlocal analysis, or for non-linear methods in the context of Banach- or C^*-settings (cf. [39], [53] and [54]).

Another important consequence is that a Ψ_0-algebra (resp. a Ψ^*-algebra) has the same K-theory as its norm closure (resp. C^*-closure), which has been observed in [22] using *Karoubi's density theorem* [55] (see also [56]) in connection with results in [39] and [36] (see also appendix A.1 of this thesis). Hence the dense embedding $\mathcal{A} \hookrightarrow \mathcal{B}$ of a Ψ^*-algebra $\mathcal{A}$ in a C^*-algebra $\mathcal{B}$ and the induced isomorphism in K-theory shows, that $\mathcal{A}$ is on the one hand large enough to preserve K-theory and on the other hand better related to the differential structure than the C^*-algebra. This has been used in [57] to prove a vanishing theorem for higher traces in cyclic cohomology for spectral projections. Furthermore, the authors of [57] give applications to the *quantum hall effect* and related *spectral gaps* of operators. The connection between dense Ψ^*-algebras $\mathcal{A}$ in C^*-algebras $\mathcal{B}$ is also stressed by representation theory: In [61] Lauter developed a representation theory for Ψ^*-algebras using a result due to Gramsch on positive functionals. Namely, he proved that there is a continuous, bijective map $\phi : \widehat{\mathcal{A}} \longmapsto \widehat{\mathcal{B}}$, where $\widehat{\mathcal{A}}$ resp. $\widehat{\mathcal{B}}$ denotes the spectrum of $\mathcal{A}$ resp. $\mathcal{B}$.

It is also an essential point in the theory of Ψ^*-algebras, that in Ψ^*-algebras $\mathcal{A}$ the Hilbert space Fredholm inverses are automatically included in $\mathcal{A}$. This gives rise to a rich *perturbation theory* in these Fréchet algebras for holomorphic Fredholm functions. As an example one gets an extension of the *Oka-principle* for holomorphic maps with values in complex Fréchet Lie-groups or in Fréchet manifolds of Fredholm and Semi-Fredholm operators in Ψ^*-algebras of pseudodifferential operators [40].

Since for Fréchet spaces and Ψ^*-algebras an appropriate implicit function theorem is not available, there had to be developed new explicit rational calculations for some infinite dimensional Fréchet manifolds which can be applied instead. It was shown in [39] that the sets of idempotent or relatively regular elements in Ψ^*algebras are *analytic locally rational Fréchet manifolds*.

As a contribution to additive, complex analytic cohomology theory, it was shown in [44] that there exists a decomposition theory for meromorphic Semi-Fredholm resolvents in Ψ^*-algebras where occurring singular parts have values in small ideals. In addition the authors gave results on the division problem for real analytic Fredholm functions and operator distributions in Ψ^*-algebras. Note that there is also a corresponding multiplicative decomposition for holomorphic Fredholm functions with values in $\mathcal{A}^{-1}$ on a *Stein manifold* [43].

A *holomorphic functional calculus* for complete locally convex algebras with continuous inversion has been introduced by Waelbroeck [112] in 1954 (even for the case of several variables). The holomorphic functional calculus for Ψ_0- and Ψ^*-algebras is a direct consequence of his work and has been used for example in [22], [38] or [57]. Moreover, it was shown in [59], that for any Hilbert space $\mathcal{H}$ the Ψ^*-algebra $\mathscr{L}(\mathcal{H})$ contains its *holomorphic functional calculus* in the sense of J. L. Taylor (see also [60] and [101]); thus Ψ^*-type algebras are also known as *smooth algebras* or *algebras stable under holomorphic calculus*. Moreover, this last result also applies in the setting of Ψ^*-valued $(n \times n)$-matrices. It was shown in [72] that any Jordan operator A in a Ψ^*-algebra $\mathcal{A} \subset \mathscr{L}(\mathcal{H})$ admits a Jordan decomposition within $\mathcal{A}$, which gives local similarity cross sections for A in $\mathcal{A}$.

In [10] it was shown, that on appropriate triples $(\mathcal{M}, g, M)$, where $\mathcal{M}$ denotes a in general non compact manifold $\mathcal{M}$ with metric g and weight function M on $T^*\mathcal{M}$, there exists a $S(M, g)$-pseudodifferential calculus. In particular, it was shown, that the algebra of order zero operators is a submultiplicative Ψ^*-algebra in $\mathscr{L}(L^2(\mathcal{M}))$. Using spectral invariance, the author also gives sufficient conditions for an operator in the $S(M, g)$-calculus to extend to a generator of a *Feller semi-group* (see also [9]).

It is worth pointing out, that development is under way concerning the L^p-theory based on the notion of Ψ_0- as well as algebras of $\mathscr{C}^\infty$-elements with respect to the group representations (cf. [34]).

Following a series of results of Schweitzer, Jolissant and de la Harpe it was pointed out in [20] that the notion of spectral invariance is an important tool in the work of Connes-

Moscovici on the *Novikov conjecture* as well as in Laffourges research on the *Baum-Connes* conjecture: For certain discrete groups G with length function $\mathfrak{l}$ the Schwarz-space $S_2^{\mathfrak{l}}(G)$ with respect to $\mathfrak{l}$ is a spectral invariant dense subalgebra of the reduced group C^*-algebra $C_r^*(G)$ (see [20] for more details).

Furthermore, in the most resent research on Ψ^*-algebras, there are approaches to Toeplitz operators, Ψ^*-algebras on infinite dimensional Hilbert space riggings and to continuous family groupoids:

As an approach to Ψ^*-algebras of Toeplitz operators on the *Segal-Bargmann space* $H^2(\mathbb{C}^n, \gamma)$, Bauer determined in [12] a class of vectorfields $\mathcal{Y}(\mathbb{C}^n)$ supported in cones $\mathcal{C} \subseteq \mathbb{C}^n$ such that for any finite subset $\mathcal{V} \subseteq \mathcal{Y}(\mathbb{C}^n)$ the Toeplitz projection is a smooth element in a Ψ_0-algebra constructed by commutator methods with respect to $\mathcal{V}$. Doing this he obtains localized Ψ_0- and Ψ^*-algebras $\mathcal{F}$ in the cones $\mathcal{C}$, that contain all Toeplitz operators T_f, where f is a smooth function that is bounded on $\mathbb{C}^n$ and has also bounded derivatives of all orders in a neighbourhood of $\mathcal{C}$. Moreover, the natural unitary group action on $H^2(\mathbb{C}^n, \gamma)$ induced by weighted shifts and unitary groups on $\mathbb{C}^n$ gives raise to a Ψ^*-algebra $\mathcal{A}$ of smooth elements in Toeplitz-C^*-algebras. Sufficient conditions on the symbol f of T_f to belong to $\mathcal{A}$ are given in terms of estimates on the corresponding *Berezin-transform* $\tilde{f}$.

In [66] Lauter, Monthubert and Nistor use commutator methods to construct algebras of pseudodifferential operators on *continuous family groupoids* $\mathcal{G}$. These algebras are closed under holomorphic functional calculus and contain the algebra of pseudodifferential operators of order zero on $\mathcal{G}$ as a dense subalgebra. Moreover, they get results for the structure of inverses of elliptic pseudodifferential operators on special classes of non-compact manifolds. They also introduce generalized cusp-calculi c_n $(n \geq 2)$ on manifolds with boundary resp. corners and embed these calculi in Ψ^*-algebras of operators with smooth kernels.

Höber defines in [51] Ψ^*-algebras in the context of (infinite dimensional) *Hilbert space riggings* $\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_-$ equipped with the corresponding Gaussian measure μ. Using the *Ornstein-Uhlenbeck operator* as Laplace operator, to generate a scale of Sobolev spaces, it is proven, that a large class of pseudodifferential operators considered by Albeverio and Dalecky in [1] is contained in the associated Ψ^*-algebra adapted to this configuration. Moreover, the author develops a symbolic calculus in the case of Laplacians with negative definite symbols and gives Fredholm-criteria in this case.

Now we present a theorem, which is due to Gramsch [40, Satz 5.6]; we will use it to construct ideal chains in Ψ^*-algebras out of ideal chains of the enveloping C^*-algebras.

Theorem 1.1.9. *Let $\mathcal{B}$ be a Banach algebra, $\mathcal{A}$ be a (submultiplicative) Ψ_0-algebra in $\mathcal{B}$, and $\mathcal{K} \subseteq \mathcal{B}$ be a closed, two-sided ideal in $\mathcal{B}$.*

(i) *$\mathcal{J} := \mathcal{A} \cap \mathcal{K}$ is a two-sided ideal in $\mathcal{A}$, which is closed in the topology $\tau_{\mathcal{A}}$ of $\mathcal{A}$, hence $\mathcal{A}/\mathcal{J}$ is a (submultiplicative) Fréchet algebra.*

(ii) *The map $j : \mathcal{A}/\mathcal{J} \longrightarrow \mathcal{B}/\mathcal{K} : a + \mathcal{J} \longmapsto a + \mathcal{K}$ is continuous and injective.*

(iii) *If $\mathcal{J}$ is dense in $\mathcal{K}$, then $j(\mathcal{A}/\mathcal{J})$ is a Ψ_0-algebra in the Banach algebra $\mathcal{B}/\mathcal{K}$.*

(iv) *If, in addition,* $\mathcal{A}$ *is a* Ψ^**-algebra in the* C^**-algebra* $\mathcal{B}$*, and* $\mathcal{J}$ *is dense in* $\mathcal{K}$*, then* $j(\mathcal{A}/\mathcal{J})$ *is a* Ψ^**-algebra in the* C^**-algebra* $\mathcal{B}/\mathcal{K}$*.*

For the convenience of the reader, let us sketch the proof:

Proof. (i) and (ii) are clear.
(iii) Denote by $q : \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{K}$ the canonical projection. Choose $\varepsilon > 0$ such that

$$\{a \in \mathcal{A} : ||e - a||_{\mathcal{B}} < \varepsilon\} \subseteq \mathcal{A}^{-1}$$

holds. Let $y := q(a) \in j(\mathcal{A}/\mathcal{J})$ be arbitrary with $a \in \mathcal{A}$ and

$$||y - q(e)||_{\mathcal{B}/\mathcal{K}} < \varepsilon.$$

By definition there exists $x \in \mathcal{K}$ with $q(x + a) = y$ and

$$||a + x - e|| < \varepsilon.$$

Using the density of $\mathcal{J}$ in $\mathcal{K}$ we can even find $x_0 \in \mathcal{J}$ such that

$$||a + x_0 - e||_{\mathcal{B}} < \varepsilon,$$

i.e. $(a + x_0) \in \mathcal{A}^{-1}$. We conclude $y^{-1} = q((a + x_0)^{-1}) \in j(\mathcal{A}/\mathcal{J})$, which completes the proof.
(iv) Closed two-sided ideals in C^*-algebras are symmetric; thus the algebra $j(\mathcal{A}/\mathcal{J})$ is a symmetric Ψ_0-algebra in $\mathcal{B}/\mathcal{K}$. □

1.2 Generating Ψ^*-algebras using closed operators

Let us state the main results and definitions concerning closed resp. closable operators. Hereby we follow [19]. If not stated otherwise, $\mathcal{E}$ always denotes a Banach space.

Definition 1.2.1. Let $\mathcal{D}(A) \leq \mathcal{E}$ be a subspace of $\mathcal{E}$ and $A : \mathcal{D}(A) \longrightarrow \mathcal{E}$ be a linear operator with domain $\mathcal{D}(A)$. Then A is called

(i) *densely defined*, if $\mathcal{D}(A)$ is dense in $\mathcal{E}$,

(ii) *closed*, if $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A)$ such that $x_n \xrightarrow{n\to\infty} x \in \mathcal{E}$ and $Ax_n \xrightarrow{n\to\infty} y \in \mathcal{E}$ implies that $x \in \mathcal{D}(A)$ and $Ax = y$,

(iii) *closable*, if $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A)$ such that $x_n \xrightarrow{n\to\infty} 0 \in \mathcal{E}$ and $Ax_n \xrightarrow{n\to\infty} y \in \mathcal{E}$ implies that $y = 0$.

In the case of (iii) there is a minimal closed extension $\overline{A}$ of A, called the closure of A.

Definition 1.2.2. Let $A : \mathcal{E} \supseteq \mathcal{D}(A) \longrightarrow \mathcal{E}$ be a linear operator. To given $j \in \mathbb{N}$ we define inductively

(i) $\mathcal{D}(A^0) := \mathcal{E}$, $\mathcal{D}(A^1) := \mathcal{D}(A)$ and

$$\mathcal{D}(A^{j+1}) := \left\{ x \in \mathcal{D}(A^j) \;:\; Ax \in \mathcal{D}(A^j) \right\};$$

(ii) $A^0 := Id$, $A^1 := A$ and $A^{j+1}x := A^j(Ax)$ for $x \in \mathcal{D}(A^{j+1})$;

(iii) $\mathcal{D}(A^\infty) := \bigcap_{j \in \mathbb{N}_0} \mathcal{D}(A^j)$.

Then $A^j : \mathcal{D}(A^j) \longrightarrow \mathcal{E}$ is a well defined linear operator for all $j \in \mathbb{N}_0$.

Definition 1.2.3. Let $\Lambda : \mathcal{E} \supseteq \mathcal{D}(\Lambda) \longrightarrow \mathcal{E}$ be a densely defined closed operator. We set

$$\begin{aligned} \mathcal{D}(\delta_\Lambda) := \{A \in \mathscr{L}(\mathcal{E}) \;:\; & A(\mathcal{D}(\Lambda)) \subseteq \mathcal{D}(\Lambda), \\ & \exists c \geq 0\, \forall x \in \mathcal{D}(\Lambda) : ||\Lambda A x - A \Lambda x|| \leq c||x||\}. \end{aligned}$$

Then to given $A \in \mathcal{D}(\delta_\Lambda)$ there exists an extension $\delta_\Lambda(A) \in \mathscr{L}(\mathcal{E})$ of $\Lambda A - A\Lambda : \mathcal{D}(\Lambda) \longrightarrow \mathcal{E}$. Moreover, we set

$$\mathcal{D}(\delta_\Lambda^{k+1}) := \left\{ A \in \mathcal{D}(\delta_\Lambda^k) \;:\; \delta_\Lambda(A) \in \mathcal{D}(\delta_\Lambda^k) \right\}$$

for $k \in \mathbb{N}$ and $\mathcal{D}(\delta_\Lambda^\infty) := \bigcap_{k \in \mathbb{N}} \mathcal{D}(\delta_\Lambda^k)$. Finally, we introduce the notation $\mathrm{ad}^0[\Lambda] = Id$, $\mathrm{ad}^1[\Lambda] := \mathrm{ad}[\Lambda]$ and $\mathrm{ad}^k[\Lambda] := \delta_\Lambda^k$ for $k \in \mathbb{N}$.

Proposition 1.2.4. *Let $\Lambda : \mathcal{E} \supseteq \mathcal{D}(\Lambda) \longrightarrow \mathcal{E}$ be a densely defined closed linear operator. The following conditions are equivalent:*

(i) $A \in \mathcal{D}(\delta_\Lambda^\infty)$,

(ii) $A(\mathcal{D}(\Lambda^\infty)) \subseteq \mathcal{D}(\Lambda^\infty)$ *and for all $k \in \mathbb{N}_0$ exists $a_k \geq 0$, such that $||\mathrm{ad}^k[\Lambda](A)x|| \leq a_k||x||$ for all $x \in \mathcal{D}(\Lambda^\infty)$.*

Proof. First, suppose that (i) holds. We show, that we have

$$\begin{aligned} A \in \mathcal{D}(\delta_\Lambda^k) \quad \Longrightarrow \quad & A(\mathcal{D}(\Lambda^j)) \subseteq \mathcal{D}(\Lambda^j) \text{ and } ||\mathrm{ad}^j[\Lambda](A)x|| \leq a_k||x|| \\ & \text{for all } x \in \mathcal{D}(\Lambda^k), \text{ where } a_k \geq 0 \text{ is suitable} \\ & \text{and } 1 \leq j \leq k \end{aligned}$$

for $A \in \mathscr{L}(\mathcal{E})$ and $k \in \mathbb{N}$. If $k = 1$ this is obviously true. Now, let $A \in \mathcal{D}(\delta_\Lambda^{k+1})$ and $x \in \mathcal{D}(\Lambda^{k+1})$ be arbitrary. Then $x, \Lambda x \in \mathcal{D}(\Lambda^k)$ and $A, \delta_\Lambda(A) \in \mathcal{D}(\delta_\Lambda^k)$ hold and therefore $Ax, A\Lambda x, \delta_\Lambda(A)x \in \mathcal{D}(\Lambda^k)$. This implies $\Lambda A x = \delta_\Lambda(A)x + A\Lambda x \in \mathcal{D}(\Lambda^k)$ and $Ax \in \mathcal{D}(\Lambda^{k+1})$. Moreover, we get

$$||\mathrm{ad}^{k+1}[\Lambda](A)x|| = ||\mathrm{ad}^k[\Lambda](\delta_\Lambda(A))x|| \leq a_k||\delta_\Lambda(A)x|| \leq a_k||x||,$$

for all $x \in \mathcal{D}(\Lambda^{k+1})$. This gives (ii).

Now we show, that (ii) implies (i). Again, this is done by induction. To fix notation, let us denote by $\widetilde{\delta}^k(A) \in \mathscr{L}(\mathcal{E})$ the extension of $\mathrm{ad}^k[\Lambda](A)$. To given $x \in \mathcal{D}(\Lambda)$ exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\Lambda^\infty)$ such that $x_n \xrightarrow{n\to\infty} x$ and $\Lambda x_n \xrightarrow{n\to\infty} \Lambda x$. This implies $Ax_n \xrightarrow{n\to\infty} Ax$ and

$$\Lambda A x_n = \widetilde{\delta}^1(A)x_n + A\Lambda x_n \xrightarrow{n\to\infty} \widetilde{\delta}(A)x + A\Lambda x.$$

But then $Ax \in \mathcal{D}(\Lambda)$ follows and therefore $A \in \mathcal{D}(\delta_\Lambda)$, where $\delta_\Lambda(A) = \widetilde{\delta}^1(A)$. To $A \in \mathcal{D}(\delta_\Lambda^k)$, $\delta_\Lambda(A) = \widetilde{\delta}(A)$ for $k \in \mathbb{N}_0$ and $x \in \mathcal{D}(\Lambda)$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\Lambda^\infty)$ with $x_n \xrightarrow{n\to\infty} x$ and $\Lambda x_n \xrightarrow{n\to\infty} \Lambda x$. This gives $\delta_\Lambda^k(A)x_n \xrightarrow{n\to\infty} \delta_\Lambda^k(A)x$ and

$$\begin{aligned}
\Lambda\delta_\Lambda^k(A)x_n &= \delta_\Lambda^k(A)\Lambda x_n + \Lambda(\operatorname{ad}^k[\Lambda])(A)x_n - (\operatorname{ad}^k[\Lambda])(A)\Lambda x_n \\
&= \delta_\Lambda^k(A)\Lambda x_n + (\operatorname{ad}^{k+1}[\Lambda])(A)x_n \\
&= \delta_\Lambda^k(A)\Lambda x_n + \widetilde{\delta}^{k+1}(A)x_n.
\end{aligned}$$

We get

$$\Lambda\delta_\Lambda^k(A)x_n \xrightarrow{n\to\infty} \delta_\Lambda^k(A)\Lambda x + \widetilde{\delta}^{k+1}(A)x,$$

which proves that $\delta_\Lambda^k(A)x \in \mathcal{D}(\Lambda)$ and $\Lambda\delta_\Lambda^k(A)x - \delta_\Lambda^k(A)\Lambda x = \widetilde{\delta}^{k+1}(A)x$ for $x \in \mathcal{D}(\Lambda)$. Consequently $\delta_\Lambda^k(A) \in \mathcal{D}(\delta_\Lambda)$ and $A \in \mathcal{D}(\delta_\Lambda^{k+1})$. This finishes the proof. □

In [46, Section 2] Gramsch, Ueberberg and Wagner developed methods to construct Ψ^*-algebras using closed operators and derivations. Lauter then used this in [62] to treat such constructions in C^*-, resp. Ψ^*-algebras.

Let us recall the definition of a derivation first:

Definition 1.2.5 (Derivation)**.** Let $\mathcal{D}(\delta)$ and $\mathcal{A}$ be algebras. A linear map $\delta : \mathcal{D}(\delta) \longrightarrow \mathcal{A}$ is called

(i) *derivation*, if $\delta(ab) = \delta(a)b + a\delta(b)$ holds;

(ii) $*$-*derivation*, if

  (a) δ is a derivation,

  (b) $\mathcal{D}(\delta)$ and $\mathcal{A}$ are endowed with a $*$-operation and

  (c) $\delta(a^*) = \delta(a)^*$ holds;

(iii) *anti-$*$-derivation*, if (ii) (a), (b) are fulfilled, but $\delta(a^*) = -\delta(a)^*$ holds instead of (c).

If in addition $\mathcal{D}(\delta)$ is a subalgebra of a Fréchet algebra $\mathcal{A}$, then δ is called *closed derivation*, if $\delta : \mathcal{D}(\delta) \longrightarrow \mathcal{A}$ defines a closed operator.

Notations 1.2.6. Let $\mathcal{B}$ be an unital C^*-algebra and assume that $(\mathcal{A}, (q_j)_{j\in\mathbb{N}_0})$ is a submultiplicative Ψ^*-algebra in $\mathcal{B}$. Moreover, let Δ be a finite set of closed derivations $\delta : \mathcal{A} \supseteq \mathcal{D}(\delta) \longrightarrow \mathcal{A}$ such that $e \in \mathcal{D}(\delta)$ holds. Define:

(i) $\Psi_0^\Delta := \mathcal{A}$ endowed with the seminorms $q_{0,j} := q_j$ $(j \in \mathbb{N}_0)$,

(ii) $\Psi_1^\Delta := \bigcap\limits_{\delta\in\Delta} \mathcal{D}(\delta)$,

(iii) $\Psi_n^\Delta := \left\{ a \in \Psi_{n-1}^\Delta \,:\, \delta(a) \in \Psi_{n-1}^\Delta \text{ for all } \delta \in \Delta \right\}$ $(n \geq 2)$,

(iv) $\Psi_\infty^\Delta := \bigcap_{n \in \mathbb{N}_0} \Psi_n^\Delta$,

where the system of seminorms on Ψ_n^Δ for $n \geq 1$ is given by

$$q_{n,j}(a) := q_{n-1,j}(a) + \sum_{\delta \in \Delta} q_{n-1,j}(\delta(a)),$$

for all $a \in \Psi_n^\Delta \subseteq \Psi_1^\Delta$ and $j \in \mathbb{N}_0$; Ψ_∞^Δ will be endowed with the system $(q_{n,j})_{n \in \mathbb{N}, j \in \mathbb{N}_0}$.

As a direct result of the construction above, we obtain (see [46] and [62, Proposition 2.4.3], [62, Corollary 2.4.4]):

Proposition 1.2.7.

(i) Ψ_n^Δ *is a subalgebra of* $\mathcal{A}$ *and* $q_{n,j}$ *defines a submultiplicative seminorm on* Ψ_n^Δ*,*

(ii) $\Delta_n : \Psi_{n-1}^\Delta \supseteq \Psi_n^\Delta \ni a \longmapsto \delta((a))_{\delta \in \Delta} \in \prod_{\delta \in \Delta} \Psi_{n-1}^\Delta$ *is a closed derivation* $(n \geq 1)$*,*

(iii) $(\Psi_n^\Delta, (q_{n,j})_{j \in \mathbb{N}_0}) \hookrightarrow \mathcal{A}$ *is a continuously embedded, submultiplicative Fréchet algebra,*

(iv) $(\Psi_\infty^\Delta, (q_{n,j})_{n \in \mathbb{N}, j \in \mathbb{N}_0}) \hookrightarrow \mathcal{A}$ *is a continuously embedded, submultiplicative Fréchet algebra.*

(v) Ψ_∞^Δ *is a submultiplicative* Ψ_0*-algebra in* $\mathcal{B}$ *and for each* $\delta \in \Delta$ *the map* $\delta : \Psi_\infty^\Delta \longrightarrow \Psi_\infty^\Delta$ *is continuous.*

Now, let each $\delta \in \Delta$ *be (in addition) a closed* $*$*-derivation with respect to the induced* $*$*-operation of* $\mathcal{B}$*. Then:*

(vi) Ψ_n^Δ *is a symmetric subalgebra of* $\mathcal{A}$ *with respect to the* $*$*-operation given on* $\mathcal{B}$ *and* Ψ_∞^Δ *is a submultiplicative* Ψ^**-algebra in* $\mathcal{B}$*.*

Proof. (i) We show, that Ψ_n^Δ is a subalgebra of $\mathcal{A}$ for all $n \in \mathbb{N}$: This is obvious for $n = 0, 1$, since $\mathcal{A}$ and $\mathcal{D}(\delta)$ with $\delta \in \Delta$ are algebras by definition. Fix $a, b \in \Psi_n^\Delta$ for some $n \geq 2$ and $\delta \in \Delta$. Then by induction we get

$$ab \in \Psi_{n-1}^\Delta \text{ and } \delta(ab) = \delta(a)b + a\delta(b) \in \Psi_{n-1}^\Delta,$$

thus $a, b \in \Psi_n^\Delta$ follows. By assumption $\Psi_0^\Delta = \mathcal{A}$ is submultiplicative and the inductional assumption gives for $a, b \in \Psi_n^\Delta$:

$$\begin{aligned} q_{n,j}(ab) &= q_{n-1,j}(ab) + \sum_{\delta \in \Delta} q_{n-1,j}(\delta(ab)) \\ &\leq q_{n-1,j}(a)q_{n-1,j}(b) \\ &\qquad + \sum_{\delta \in \Delta} \big(q_{n-1,j}(\delta(a))q_{n-1,j}(b) + q_{n-1,j}(a)q_{n-1,j}(\delta(b)) \big) \\ &\leq q_{n,j}(a)q_{n,j}(b). \end{aligned}$$

Using $\delta(e) = 0$ and $q_j(e) = 1$ for all $j \in \mathbb{N}_0$ we also get $q_{n,j}(e) = 1$ for all $n, j \in \mathbb{N}_0$.
(ii) We only consider the case $n = 1$; the general case follows then similarly by induction, hence it will be omitted. Assume

$$a_k \xrightarrow{\mathcal{A}} a \text{ and } \Delta_1 a_k \xrightarrow{\Pi\mathcal{A}} b = (b_\delta)_{\delta\in\Delta},$$

then $\mathcal{D}(\delta) \ni a_k \xrightarrow{\mathcal{A}} a$ and $\delta(a_k) \xrightarrow{\mathcal{A}} b_\delta$ for all $\delta \in \Delta$. Since $\delta : \mathcal{A} \supseteq \mathcal{D}(\delta) \longrightarrow \mathcal{A}$ is closed by assumption, it immediately follows that $a \in \mathcal{D}(\delta)$ and $b_\delta = \delta(a)$ for all $\delta \in \Delta$, i.e. $a \in \Psi_1^\Delta$ and $\Delta_1 a = b$.
(iii) This is immediate, since it only remains to prove completeness; but $(q_{n,j})_{j\in\mathbb{N}_0}$ induces the graph topology corresponding to the closed operator Δ_n.
(iv) This follows by (iii).
(v) In view of 1.1.1 (i), we have to prove that for each $a \in \Psi_\infty^\Delta$ with $||a||_\mathcal{B} < \varrho < 1$ it holds:

$$(e-a)^{-1} = \sum_{k=0}^{\infty} a^k \in \Psi_\infty^\Delta. \tag{1.2.1}$$

To prove this, we show, that for all $j \in \mathbb{N}_0$ there exist constants $c_{n,j}(a) > 0$ depending only on a, n and j such that one has for all $k \in \mathbb{N}_0$:

$$q_{n,j}(a^k) \le c_{n,j}(a) k^{2^n-1} \varrho^{k-2^n+1}. \tag{1.2.2}$$

The case $n = 0$ follows by [60, Lemma 2.1.8] using the continuous inversion in $\mathcal{A}$. Assume now, that (1.2.2) has already been proven for $n - 1 \ge 0$. Then the well-known formula

$$\delta(a^k) = \sum_{l=1}^{k} a^{l-1}\delta(a)a^{k-l}$$

yields

$$\begin{aligned} q_{n,j}(a^k) &= q_{n-1,j}(a^k) + \sum_{\delta\in\Delta} q_{n-1,j}(\delta(a^k)) \\ &\le q_{n-1,j}(a^k) + \sum_{\delta\in\Delta}\sum_{l=1}^{k} q_{n-1,j}(a^{l-1}) q_{n-1,j}(\delta(a)) q_{n-1,j}(a^{k-l}) \\ &\le c_{n,j}(a) k^{2^n-1} \varrho^{k-2^n-1} \end{aligned}$$

using the induction hypothesis. Thus the Neumann series (1.2.1) converges in Ψ_n^Δ for all $n \in \mathbb{N}_0$ and so it converges also in Ψ_∞^Δ. Finally, $\delta(a) \in \Psi_\infty^\Delta$ holds by construction and the continuity of $\delta : \Psi_\infty^\Delta \longrightarrow \Psi_\infty^\Delta$ is a consequence of the closed graph theorem.
(vi) The algebras Ψ_j^Δ are symmetric under the $*$-operation for $j = 0, 1$. Let $a \in \Psi_n^\Delta$ be arbitrary, then one has $a^* \in \Psi_{n-1}^\Delta$ and also $\Delta(a^*) = (\delta(a^*)) \in \Psi_{n-1}^\Delta$ by the inductional assumption. Thus it follows $a^* \in \Psi_n^\Delta$ $(n \in \mathbb{N})$ and Ψ_∞^Δ is also symmetric. The rest follows now by (i)-(v). $\square$

1.3 Generating Ψ^*-algebras by commutator methods

In what follows, we specialize to the case $\mathcal{B} = \mathscr{L}(\mathcal{H})$, where $\mathcal{H}$ denotes a Hilbert space.

Definition 1.3.1. Let $(\mathcal{A}, (q_j)_{j\in\mathbb{N}_0}) \hookrightarrow \mathscr{L}(\mathcal{H})$ be a submultiplicative Ψ^*-algebra; without loss of generality we assume $q_0 = ||\cdot||_{\mathscr{L}(\mathcal{H})}$. For a closed, densely defined operator $V : \mathcal{H} \supseteq \mathcal{D}(V) \longrightarrow \mathcal{H}$ we define

(i) $\mathcal{I}(V) := \{a \in \mathcal{A} \,:\, a(\mathcal{D}(V)) \subseteq \mathcal{D}(V)\}$,

(ii) $\mathrm{ad}\,[V](a) := Va - aV : \mathcal{D}(V) \longrightarrow \mathcal{H}$,

(iii) $\mathcal{B}(V)$ to be the set of all $a \in \mathcal{I}(V)$, such that $\mathrm{ad}\,[V](a)$ induces a bounded linear operator $\delta_V(a) \in \mathcal{A}$ and

(iv) $\mathcal{B}^*(V) := \{a \in \mathcal{B}(V) \,:\, a^* \in \mathcal{B}(V)\}$.

Lemma 1.3.2.

(i) *The operator $\delta_V : \mathcal{A} \supseteq \mathcal{B}(V) \longrightarrow \mathcal{A}$; $a \longmapsto \delta_V(a)$ defines a closed derivation.*

(ii) *If in addition $V : \mathcal{H} \supseteq \mathcal{D}(V) \longrightarrow \mathcal{H}$ is symmetric, then $i\delta_V : \mathcal{A} \supseteq \mathcal{B}^*(V) \longrightarrow \mathcal{H}$ is a closed $*$-derivation, i.e. δ_V defines a closed anti-$*$-derivation.*

For the proof of this lemma see also [62, Lemma 2.4.7].

Proof. (i) First we prove, that $\mathcal{B}(V)$ is an algebra: let $a, b \in \mathcal{B}(V)$ be arbitrary and $\varphi \in \mathcal{D}(V)$. Then

$$(\mathrm{ad}\,[V](ab))\,(\varphi) = (Va - aV)b(\varphi) + a(Vb - bV)(\varphi)$$

and thus the commutator $\mathrm{ad}\,[V](ab)$ extends to a bounded operator $\delta_V(ab) = \delta_A(a)b + a\delta_V(b)$, i.e. δ_V is a derivation and $ab \in \mathcal{B}(V)$ holds.

In order to show that δ_V is a closed derivation, let

$$\mathcal{B}(V) \ni a_k \xrightarrow{\mathcal{A}} a \in \mathcal{A} \text{ and } \mathcal{A} \ni \delta_V a_k \xrightarrow{\mathcal{A}} b \in \mathcal{A}.$$

Using the continuous embedding $\mathcal{A} \hookrightarrow \mathscr{L}(\mathcal{H})$ it follows for all $\varphi \in \mathcal{D}(V)$ that $\mathcal{D}(V) \ni a_k(\varphi) \longrightarrow a(\varphi)$ and

$$Va_k\varphi = (\delta_V a_k)\varphi + a_k V\varphi \xrightarrow{\mathcal{H}} b\varphi + aV\varphi.$$

Since V is a closed operator, we conclude $a\varphi \in \mathcal{D}(V)$ and $\mathrm{ad}\,[V]a(\varphi) = b(\varphi)$. Thus $a \in \mathcal{B}(V)$ and $\delta_V a = b \in \mathcal{A}$, i.e. δ_V is a closed derivation.

(ii) By definition $\mathcal{B}^*(V)$ is a symmetric subalgebra of $\mathcal{A}$. Let $\varphi, \psi \in \mathcal{D}(V)$ and $a \in \mathcal{B}$, then we get

$$\begin{aligned}
\langle \delta_V(a)\varphi \,|\, \psi\rangle &= \langle Va\varphi - aV\varphi \,|\, \psi\rangle \\
&= \langle \varphi \,|\, a^*V\psi - Va^*\psi\rangle \\
&= \langle \varphi \,|\, -\delta_V(a^*)\psi\rangle.
\end{aligned}$$

Now $\mathcal{D}(V)$ is dense by assumption, and we conclude that $\delta_A(a)^* = -\delta_A(a^*)$.

Finally, let us prove, that $\delta_V : \mathcal{B}^*(V) \longrightarrow \mathcal{A}$ is closed. To this end let $\mathcal{B}(V^*) \ni a_k \xrightarrow{\mathcal{A}} a \in \mathcal{A}$ and $\mathcal{A} \ni \delta_V(a_k) \xrightarrow{\mathcal{A}} b \in \mathcal{A}$. Then our previous discussion shows $a \in \mathcal{B}(V)$ and $\delta_V(a) = b$. The continuity of the $*$-operation in $\mathcal{A}$ now gives $a_k^* \xrightarrow{\mathcal{A}} a^*$ and thus

$$\delta_V(a_k^*) = -\delta_V(a_k)^* \xrightarrow{\mathcal{A}} b^*.$$

This proves $a \in \mathcal{B}^*(V)$ and we have finished the proof. $\square$

Definition 1.3.3. Let $\mathcal{E}$ be a Banach space and $\mathcal{V}$ be a finite set of densely defined closed operators $V : \mathcal{E} \supseteq \mathcal{D}(V) \longrightarrow \mathcal{E}$. We define

(i) $\mathcal{H}_{\mathcal{V}}^0 := \mathcal{E}$ endowed with norm $p_0 := ||\cdot||_{\mathcal{E}}$,

(ii) $\mathcal{H}_{\mathcal{V}}^1 := \bigcap\limits_{V \in \mathcal{V}} \mathcal{D}(V)$,

(iii) $\mathcal{H}_{\mathcal{V}}^n := \left\{\varphi \in \mathcal{H}_{\mathcal{V}}^{n-1} \,:\, V(\varphi) \in \mathcal{H}_{\mathcal{V}}^{n-1} \text{ for all } V \in \mathcal{V}\right\}$ $(n \geq 2)$,

(iv) $\mathcal{H}_{\mathcal{V}}^\infty := \bigcap\limits_{n \in \mathbb{N}} \mathcal{H}_{\mathcal{V}}^n$.

Hereby we endow $\mathcal{H}_{\mathcal{V}}^n$ with the norm

$$p_n(\varphi) := p_{n-1}(\varphi) + \sum_{V \in \mathcal{V}} p_{n-1}(V(\varphi)), \;\; (\varphi \in \mathcal{H}_{\mathcal{V}}^n)$$

and $\mathcal{H}_{\mathcal{V}}^\infty$ with the system of norms $(p_n)_{n \in \mathbb{N}_0}$.

Lemma 1.3.4.

(i) $\mathcal{V}_n : \mathcal{H}_{\mathcal{V}}^{n-1} \supseteq \mathcal{H}_{\mathcal{V}}^n \longrightarrow \prod\limits_{V \in \mathcal{V}} \mathcal{H}_{\mathcal{V}}^{n-1};\ \xi \longmapsto (V(\varphi))_{V \in \mathcal{V}}$ *is a closed operator,*

(ii) $(\mathcal{H}_{\mathcal{V}}^n, p_n)$ *is a Banach space,*

(iii) $(\mathcal{H}_{\mathcal{V}}^\infty, (p_n)_{n \in \mathbb{N}_0})$ *is a Fréchet space,*

(iv) *if* $\mathcal{E}$ *is a Hilbert space, there is an equivalent norm* $\widetilde{p}_n$ *on* $\mathcal{H}_{\mathcal{V}}^n$*, such that* $(\mathcal{H}_{\mathcal{V}}^n, \widetilde{p}_n)$ *is a Hilbert space and* $(H_{\mathcal{V}}^\infty, (p_n)_{n \in \mathbb{N}_0})$ *is a Fréchet Hilbert space.*

In what follows the spaces $\mathcal{H}_{\mathcal{V}}^n$*, resp.* $\mathcal{H}_{\mathcal{V}}^\infty$ *will be denoted as* $\mathcal{V}$*-Sobolev spaces.*

Proof. This follows using arguments similar to the proof of proposition 1.2.7, hence we will omit it. $\square$

Theorem 1.3.5. *Let* $\mathcal{H}$ *be a Hilbert space,* $(\mathcal{A}, (q_j)_{j \in \mathbb{N}_0})$ *a submultiplicative* Ψ^**-algebra in* $\mathscr{L}(\mathcal{H})$*,* $\mathcal{B}$ *a* C^**-algebra in* $\mathscr{L}(\mathcal{H})$ *such that* $\mathcal{A} \subseteq \mathcal{B}$ *and* $\mathcal{V}$ *be a finite set of closed densely defined symmetric operators* $V : \mathcal{H} \supseteq \mathcal{D}(V) \longrightarrow \mathcal{H}$*. Moreover, we denote by* $\Delta := \Delta_{\mathcal{V}} := \{\delta_V : V \in \mathcal{V}\}$ *the set of all closed anti-$*$-derivations* $\delta_V : \mathcal{C} \supseteq \mathcal{D}(\delta_V) \longrightarrow \mathcal{A}$ *and by* $\Psi_n^{\mathcal{V}} := \Psi_n^{\Delta}$ *resp.* $\Psi_\infty^{\mathcal{V}} := \Psi_\infty^{\Delta}$ *the scale of submultiplicative Fréchet algebras. Then:*

(i) $\Psi^{\mathcal{V}}_\infty \subseteq \Psi^{\mathcal{V}}_n \subseteq \mathcal{A} \subseteq \mathcal{B}$ *holds for all* $n \in \mathbb{N}$,

(ii) $(\Psi^{\mathcal{V}}_\infty, (q_{n,j})_{n\in\mathbb{N}, j\in\mathbb{N}_0}) \hookrightarrow \mathcal{B}$ *is a submultiplicative* Ψ^**-algebra,*

(iii) $\Psi^{\mathcal{V}}_n \times \mathcal{H}^n_{\mathcal{V}} \longrightarrow \mathcal{H}^n_{\mathcal{V}}$; $(a,\varphi) \longrightarrow a(\varphi)$ *is continuous and bilinear,*

(iv) $\Psi^{\mathcal{V}}_\infty \times \mathcal{H}^\infty_{\mathcal{V}} \longrightarrow \mathcal{H}^\infty_{\mathcal{V}}$; $(a,\varphi) \longmapsto a(\varphi)$ *is continuous and bilinear,*

(v) $\delta_V : \Psi^{\mathcal{V}}_\infty \longrightarrow \Psi^{\mathcal{V}}_\infty$ *is continuous.*

Proof. Cf. [62, Theorem 2.4.13]. It remains to prove (iii): To do this we show that for $n \in \mathbb{N}_0$ and for each pair $(a,\varphi) \in \Psi^{\mathcal{V}}_n \times \mathcal{H}^n_{\mathcal{V}}$ the following inequality holds:

$$p_n(a\varphi) \leq q_{n,0}(a)p_n(\varphi).$$

If $n = 0$, then one gets

$$p_0(a\varphi) = ||a\varphi||_{\mathcal{H}} \leq ||a||_{\mathscr{L}(\mathcal{H})}||\varphi||_{\mathcal{H}} = q_{0,0}(a)p_0(\varphi).$$

The induction step now is due to the following calculation:

$$\begin{aligned} p_n(a\varphi) &= p_{n-1}(a\varphi) + \sum_{V\in\mathcal{V}} p_{n-1}(Va\varphi) \\ &\leq q_{n-1,0}(a)p_{n-1}(\varphi) + \sum_{V\in\mathcal{V}} p_{n-1}\left(aV\varphi + (\delta_V a)\,\varphi\right) \\ &\leq q_{n,0}(a)p_n(\varphi) \end{aligned}$$

This finishes the proof. □

Often commutator estimates are well-known for differential operators and the following proposition shows how to get commutator estimates for the corresponding square roots:

Proposition 1.3.6. *Let $\mathcal{H}$ be a Hilbert space and $Q : \mathcal{D}(Q) \longrightarrow \mathcal{H}$ be a strictly positive, selfadjoint operator. Let $A : \mathcal{H}^\infty_Q \longrightarrow \mathcal{H}^\infty_Q$ be such that for $k, j \in \mathbb{N}_0$ there are constants $a_{2k,j} \geq 0$ with*

$$||Q^{2k}\mathrm{ad}^{\,j}(Q^2)(A)x||_{\mathcal{H}} \leq a_{2k,j}||Q^{2k+m+j}x||_{\mathcal{H}}$$

for $x \in \mathcal{H}^\infty_Q$. Then for $k \in \mathbb{Z}$, $j \in \mathbb{N}_0$ there are $c_{k,j} \geq 0$ with

$$||Q^k\mathrm{ad}^{\,j}(Q)(A)x||_{\mathcal{H}} \leq c_{k,j}||Q^{k+m}x||_{\mathcal{H}}$$

for $x \in \mathcal{H}^\infty_Q$.

Proof. See [19, Proposition 2.3.8]. □

Let us close this section with a result of Gramsch and Kalb [45] on abstract hypoellipticity or elliptic regularity. Recall, that the Fredholm inverse exists within the Ψ^*-algebra, thus they are an adequate substitute for the usual parametrix construction.

Let $\mathcal{C}$ be an unital C^*-algebra with $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{C} \subseteq \mathscr{L}(\mathcal{H})$, where $\mathcal{H}$ denotes a Hilbert space. Moreover, let $\Psi \subseteq \mathcal{C}$ be a Ψ^*-algebra in $\mathcal{C}$ and $\mathcal{A} \subseteq \Psi$ be a symmetric subalgebra. Suppose that there exists a family $(\mathcal{Q}_\gamma)_{\gamma\in\Gamma}$ of C^*-algebras together with unital $*$-morphisms

$$\tau_\gamma : \mathcal{A} \longrightarrow \mathcal{Q}_\gamma$$

such that $\sup\limits_{\gamma\in\Gamma} ||\tau_\gamma(a)||_{\mathcal{Q}_\gamma} < \infty$ and

$$\pi(a) \in (\mathcal{C}/\mathcal{K}(\mathcal{H}))^{-1} \Longleftrightarrow \tau_\gamma(a) \in \mathcal{Q}_\gamma^{-1} \text{ for all } \gamma \in \Gamma$$

holds for all $a \in \mathcal{A}$. Then the map

$$\tau : \mathcal{A} \longrightarrow \mathcal{Q} := \bigoplus_{\gamma\in\Gamma} \mathcal{Q}_\gamma;\ a \longmapsto (\tau_\gamma(a))_{\gamma\in\Gamma}$$

extends to a morphism $\tau_{\mathcal{B}} : \mathcal{B} := \overline{\mathcal{A} + \mathcal{K}(\mathcal{H})}^{\mathcal{C}} \longrightarrow \mathcal{Q}_{\mathcal{B}} := \overline{\mathrm{r}\,(\tau)}^{\mathcal{Q}}$ of C^*-algebras (see, for instance, [60, Proposition 2.5.4]). Moreover, $\mathcal{A}_\Psi := \overline{\mathcal{A}}^{\Psi}$ is a Ψ^*-algebra in $\mathcal{C}$ and we denote by τ_Ψ the restriction of $\tau_{\mathcal{B}}$ to $\mathcal{A}_\Psi$ and define $\mathcal{J} := \ker(\tau_\Psi) = \mathcal{A}_\Psi \cap \mathcal{K}(\mathcal{H})$. An algebra $\mathcal{A}$ satisfying all these assumptions is called an algebra with $\mathcal{K}(\mathcal{H})$-*symbolic structure*. Finally, we call an operator b *elliptic*, if its symbol $\tau_{\mathcal{B}}(b)$ is invertible in the C^*-algebra $\mathcal{Q}_{\mathcal{B}}$.

Theorem 1.3.7. *Let $\mathcal{V}$ be a finite system of densely defined, symmetric, closed operators on a Hilbert space $\mathcal{H}$. Furthermore, we assume that $\mathcal{H}_{\mathcal{V}}^\infty$ is dense in $\mathcal{H}$. Let $\mathcal{A}$ be a symmetric subalgebra with $\mathcal{K}(\mathcal{H})$-symbolic structure of $\mathscr{L}(\mathcal{H})$ and suppose $\mathcal{A} \subseteq \Psi_\infty^{\mathcal{V}}$. Let $\mathcal{A}_{\mathcal{V}}$ be the closure of $\mathcal{A}$ in $\Psi_\infty^{\mathcal{V}}$. Now, if $a \in \mathcal{A}_{\mathcal{V}}$ is elliptic and $u \in \mathcal{H}$ is arbitrary with $au =: f \in \mathcal{H}_{\mathcal{V}}^m$ for some $m \in \mathbb{N}_0 \cup \{\infty\}$, then $u \in \mathcal{H}_{\mathcal{V}}^m$.*

The proof of this theorem will use the following lemma (see e.g. [60]):

Lemma 1.3.8. *Let D be a dense subspace of a normed space H, and $p \in \mathscr{L}(H)$ be with $p(D) \subseteq D$ and $\dim \mathrm{r}\,(p) < \infty$. Then one has $\mathrm{r}\,(p) \subseteq D$.*

Proof of 1.3.7. Since a is elliptic, we know, that it is a Fredholm operator (see, for instance, [60, Theorem 2.5.8]). Thus $\mathrm{r}\,(a) \subseteq \mathcal{H}$ is closed and there exists $b \in \mathcal{A}_{\mathcal{V}} \subseteq \Psi_\infty^{\mathcal{V}}$ such that $p := id_{\mathcal{H}} - ba$ is the orthogonal projection onto $\ker(a)$. Moreover, $\ker(a)$ is finite dimensional, thus

$$u = bau + pu = bf + pu \in \mathcal{H}_{\mathcal{V}}^m$$

since $bf \in \mathcal{H}_{\mathcal{V}}^m$ and $pu \in \mathcal{H}_{\mathcal{V}}^\infty$, and we have finished the proof. □

1.4 Order shift algebras on compact Riemannian manifolds

Let us give a direct application of the previous sections: Denote by (M, g) a closed *Riemannian* manifold and let $\Delta := \Delta_g$ be the induced *Laplace-Beltrami*-operator on M.

We define $\mathcal{H}^0(M) := L^2(M)$ to be the space of square integrable functions on M with respect to the metric g.

The operator Λ given by $\Lambda := (1-\Delta)^{\frac{1}{2}}$ is then known to be a positive selfadjoint operator on M. If $s \in \mathbb{R}$ is arbitrary, we define the Sobolev space $\mathcal{H}^s(M)$ of order s to be the closure of $\mathcal{D}(\Lambda^s)$ with respect to the norm $||v||_s := ||\Lambda^s v||_0$.

This gives raise to continuous embeddings

$$\mathcal{H}^s(M) \hookrightarrow \mathcal{H}^t(M)$$

if $s > t$, and we get a scale of Sobolev-spaces $\{\mathcal{H}^s(M) : s \in \mathbb{R}\}$, i.e.

$$\mathcal{H}^{-s}(M) \supseteq \mathcal{H}^{-k}(M) \supseteq \ldots \supseteq \mathcal{H}^0(M) \supseteq \ldots \mathcal{H}^k(M) \supseteq \mathcal{H}^s(M)$$

if $s \geq k > 0$. Finally, we set $\mathcal{H}^\infty(M) := \bigcap_{s\in\mathbb{R}} \mathcal{H}^s(M)$ and $\mathcal{H}^{-\infty}(M) := \bigcup_{s\in\mathbb{R}} \mathcal{H}^s(M)$. To shorten notation we will write $\mathcal{H}^s$ etc. instead of $\mathcal{H}^s(M)$. It follows:

Proposition 1.4.1. *$\mathcal{H}^\infty$ is dense in $\mathcal{H}^s$ for all $s \in \mathbb{R}$ and $\mathcal{H}^\infty = \mathscr{C}^\infty(M)$ holds.*

Proof. See [23, Page 30; Page 181, Lemma 4.1] or [107, Corollary 7.4]. □

Moreover, we get a pairing

$$\langle u \,|\, v\rangle := \langle \Lambda^s u \,|\, \Lambda^{-s} v\rangle \;\; u \in \mathcal{H}^s,\, v \in \mathcal{H}^{-s},$$

where $\langle \cdot \,|\, \cdot\rangle$ denotes the inner product on $\mathcal{H}^0 = L^2(M)$ with respect to g. Now, an application of the *Riesz-Fréchet*-theorem shows, that we can identify $(\mathcal{H}^s)^*$ with $\mathcal{H}^{-s}$. In what follows, we fix $0 \leq \varepsilon \leq 1-\delta$ ($\delta \leq 1$).

To complete notation, let us denote by $[\mathcal{X}_0, \mathcal{X}_1]_\theta$ ($0 < \theta < 1$) the complex interpolation space of $\mathcal{X}_0$ and $\mathcal{X}_1$, where $(\mathcal{X}_0, \mathcal{X}_1)$ is an interpolation couple of complex Banach spaces (see [19, Section 1.5] for more details).

Proposition 1.4.2. *Define $\mathcal{A}_\varepsilon$ to be the following algebra:*

$$\begin{aligned}\mathcal{A}_\varepsilon := \big\{A \in \mathscr{L}(L^2(M)) \big| A(\mathcal{D}(\Lambda^\infty)) \subseteq \mathcal{D}(\Lambda^\infty),\\ \forall \nu \in \mathbb{N}\, \exists a_\nu \geq 0 : ||\mathrm{ad}\,[\Lambda^\varepsilon]^\nu(A)x|| \leq a_\nu ||x||\big\}.\end{aligned}$$

Then $\mathcal{A}_\varepsilon$ is a Ψ^*-algebra, and $0 \leq \varepsilon \leq \varepsilon'(\leq 1-\delta)$ implies $\mathcal{A}_{\varepsilon'} \subseteq \mathcal{A}_\varepsilon$.

Proof. Since Λ^ε is a positive selfadjoint operator, the first part of the proposition follows by 1.2.7. The proposition is then a consequence of [19, 2.3.11 resp. 2.3.12]. □

Theorem 1.4.3. *We have $\mathcal{A}_\varepsilon \subseteq \bigcap_{s\in\mathbb{R}} \mathscr{L}(\mathcal{H}^s)$.*

Proof. Let us first show, that $\mathcal{A}_\varepsilon \subseteq \bigcap_{s\geq 0} \mathscr{L}(\mathcal{H}^s)$ holds. Suppose that $A \in \mathcal{A}_\varepsilon$ is arbitrary, then $A^* \in \mathcal{A}_\varepsilon$, since $\mathcal{A}_\varepsilon$ is a Ψ^*-algebra. The proof of 1.2.4, shows, that $A \in \mathcal{A}_\varepsilon$ and $A(\mathcal{D}(\Lambda^{\varepsilon k})) \subseteq \mathcal{D}(\Lambda^{\varepsilon k})$ holds for all $k \in \mathbb{N}_0$. Therefore, we get $A, A^* \in \bigcap_{k\in\mathbb{N}_0} \mathscr{L}(\mathcal{H}^{\varepsilon k})$. Now, let $s > 0$ be arbitrary, then there are $k \in \mathbb{N}$ and $\theta \in [0,1]$ such that $s = \theta\varepsilon k$.

Moreover, $A, A^* \in \mathscr{L}(\mathcal{H}^0) \cap \mathscr{L}(\mathcal{H}^{\varepsilon k})$ and an application of interpolation theory gives $A, A^* \in \mathscr{L}([\mathcal{H}^0, \mathcal{H}^{\varepsilon k}]_\theta)$ and also

$$[\mathcal{H}^0, \mathcal{H}^{\varepsilon k}]_\theta = [\mathcal{D}(\Lambda^0), \mathcal{D}(\Lambda^{\varepsilon k})]_\theta = \mathcal{D}(\Lambda^s) = \mathcal{H}^s.$$

We get $A, A^* \in \mathscr{L}(\mathcal{H}^s)$ and thus $A, A^* \in \bigcap_{s \geq 0} \mathscr{L}(\mathcal{H}^s)$ follows.

Now, let $p \geq 0$ be arbitrary. Then $A^* \in \mathscr{L}(\mathcal{H}^p)$ induces a selfadjoint operator $(A^*)^*_{-p} \in \mathscr{L}(\mathcal{H}^{-p})$, cf. [23, Chapter 1, Proposition 6.4]. If $q \geq 0$, we get

$$\begin{aligned} \langle (A^*)^*_{-p} u \,|\, v \rangle - \langle (A^*)^*_{-q} u \,|\, v \rangle &= \langle ((A^*)^*_{-p} - (A^*)^*_{-q}) u \,|\, v \rangle \\ &= \langle u \,|\, (A_p - A_q) v \rangle \\ &= 0 \end{aligned}$$

where $v \in \mathcal{H}^\infty$ and $u \in \mathcal{H}^{-p} \cap \mathcal{H}^{-q} = \mathcal{H}^{\max\{-p,-q\}}$, since $\mathcal{H}^\infty$ is dense in $\mathcal{H}^r$ for all $r \in \mathbb{R}$. This shows that $(A^*)^*_{-p} = (A^*)^*_{-q}$ holds for $p, q \geq 0$. If we choose $q = 0$, we see that $(A^*)^*_{-p} = (A^*)^* = A$ and therefore A induces a continuous operator on $\mathcal{H}^{-p}$. This proves the theorem. □

Now let $\mathcal{M}$ be a family of functions $N \in \mathscr{C}^\infty(M, \mathbb{R})$ and $\mathcal{V}$ be a family of differential operators of order one on M. Then $N : \mathcal{H}^s(M) \longrightarrow \mathcal{H}^s(M)$ and $\mathcal{V} \ni X : \mathcal{H}^s(M) \longrightarrow \mathcal{H}^{s-1}(M)$ are continuous and we define:

Definition 1.4.4. Define $\widetilde{\Psi}^\varepsilon_{\rho,\delta}(M)$ to be the set

$$\begin{aligned} \widetilde{\Psi}^\varepsilon_{\rho,\delta}(M) := \{T \in A_\varepsilon \,:\, \mathrm{ad}\,[M]^\alpha \mathrm{ad}\,[X]^\beta(T) \in \mathscr{L}(\mathcal{H}^s, \mathcal{H}^{s+\rho|\alpha|-\delta|\beta|}) \\ \forall s \in \mathbb{R}\, \forall \alpha \in \mathbb{N}_0^n\, \forall \beta \in \mathbb{N}_0^m\}. \end{aligned}$$

Here we used the notation

$$\mathrm{ad}\,[X]^\beta(T) := \mathrm{ad}\,[X_1]^{\beta_1} \mathrm{ad}\,[X_2]^{\beta_2} \cdots \mathrm{ad}\,[X_n]^{\beta_n}(T)$$

for $X \in \mathcal{V}^n$ and an analogous notation for $M \in \mathcal{M}^m$.

Proposition 1.4.5. *We have:*

(i) $\widetilde{\Psi}^\varepsilon_{\rho,\delta}(M)$ *is an algebra.*

(ii) $\widetilde{\Psi}^\varepsilon_{\rho,\delta}(M) \cap (\mathscr{L}(L^2(M)))^{-1} = (\widetilde{\Psi}^\varepsilon_{\rho,\delta}(M))^{-1}$, *i.e.* $\widetilde{\Psi}^\varepsilon_{\rho,\delta}(M)$ *is spectrally invariant.*

(iii) *Moreover, if* $|\mathcal{V}| < \infty$ *and* $|\mathcal{M}| < \infty$ *hold,* $\widetilde{\Psi}^\varepsilon_{\rho,\delta}(M)$ *is a* Ψ_0*-algebra.*

(iv) *If in addition to* (iii) $X \in \mathcal{V}$ *also implies* $X^* \in \mathcal{V}$, *then* $\widetilde{\Psi}^\varepsilon_{\rho,\delta}(M)$ *gets to be a* Ψ^**-algebra.*

Finally, the mapping $\widetilde{\Psi}^\varepsilon_{\rho,\delta}(M) \times \mathcal{H}^\infty \longrightarrow \mathcal{H}^\infty$ $(T, u) \longmapsto T(u)$ *is continuous and bilinear.*

Let us sketch the proof:

Proof. (i) is clear.
(ii) Let $T \in \widetilde{\Psi}^{\varepsilon}_{\rho,\delta}(M) \cap \mathscr{L}(L^2(M))^{-1}$ be arbitrary. Then we have $T \in A_\varepsilon \subseteq \bigcap_{s\geq 0} \mathscr{L}(\mathcal{H}^s(M))$, and since this algebra is spectrally invariant $T^{-1} \in A_\varepsilon$ follows. Now, we are going to show that $T^{-1} \in \widetilde{\Psi}^{\varepsilon}_{\rho,\delta}(M)$ holds. Using an induction with respect to $|\alpha| + |\beta|$ we see, that the following formula holds:

$$\begin{aligned} &\mathrm{ad}\,[M]^\alpha \mathrm{ad}\,[X]^\beta (T^{-1}) \\ &\quad = \sum_{\substack{\alpha^1+\ldots+\alpha^n=\alpha \\ \beta^1+\ldots+\beta^n=\beta}} c_{\substack{\alpha^1,\ldots,\alpha^n \\ \beta^1,\ldots,\beta^n}} T^{-1} \left(\mathrm{ad}\,[M]^{\alpha^1} \mathrm{ad}\,[X]^{\beta^1}(T)\right) T^{-1} \cdot \ldots \\ &\qquad \cdot c_{\substack{\alpha^1,\ldots,\alpha^n \\ \beta^1,\ldots,\beta^n}} T^{-1} (\mathrm{ad}\,[M]^{\alpha^1} \mathrm{ad}\,[X]) \cdot \left(\mathrm{ad}\,[M]^{\alpha^n} (\mathrm{ad}\,[X]^{\beta^n}(T))\right) T^{-1}, \end{aligned} \tag{1.4.1}$$

where $c_{\substack{\alpha^1,\ldots,\alpha^n \\ \beta^1,\ldots,\beta^n}} \in \mathbb{R}$ have to be chosen suitably.

But then (1.4.1) implies

$$\mathrm{ad}\,[X_j]^{\beta_j}(T) \in \mathscr{L}(\mathcal{H}^s(M), \mathcal{H}^{s-\delta|\beta_j|}(M))$$

for $T \in \widetilde{\Psi}^{\varepsilon}_{\rho,\delta}(M)$ and all $0 \leq j \leq l$. Since $T^{-1} \in A_\varepsilon$, we have

$$T^{-1}(\mathrm{ad}\,[X_j](T))T^{-1} \in \mathscr{L}(\mathcal{H}^s(M), \mathcal{H}^{s-\delta|\beta_j|}(M)),$$

and therefore $\mathrm{ad}\,[X]^\beta(T) \in \mathscr{L}(\mathcal{H}^s(M), \mathcal{H}^{s-\delta|\beta|}(M))$ as desired. An analogous calculation gives the same result for multiplication operators, which completes (ii).

(iii) We define a topology $\widetilde{\tau}$ on $\widetilde{\Psi}^{\varepsilon}_{\rho,\delta}$ by

$$||P||_{s,l,l'} := \sup_{\substack{|\alpha|\leq l \\ |\beta|\leq l'}} ||\mathrm{ad}\,[M]^\alpha \mathrm{ad}\,[V]^\beta P||_{\mathscr{L}(\mathcal{H}^s(M), \mathcal{H}^{s+\rho|\alpha|-\delta|\beta|}(M))}$$

where $s \in \mathbb{R}$ and $P \in \widetilde{\Psi}^{\varepsilon}_{\rho,\delta}$. To given $s \in \mathbb{R}\setminus\mathbb{N}_0$ exits a $\theta \in]0,1[$, such that $s = \theta + k$, $k \in \mathbb{Z}$ holds. Using interpolation we get an equivalent norm $\widetilde{\tau}$ by restricting to $s \in \mathbb{Z}$. Now, a construction analogous to [46, Prop. 3.4, 3.5] gives a countable system of seminorms on $\widetilde{\Psi}^{\varepsilon}_{\rho,\delta}(M)$, such that the induced topology is finer than the operator topology.
(iii) This follows from

$$\mathrm{ad}\,[X](T^*) = XT^* - T^*X = (TX^* - X^*T)^* = -\left(\mathrm{ad}\,[X^*](T)\right)^*$$

and induction. $\square$

Chapter 2

Ψ^*-algebras on manifolds with corners

In this chapter, we aim to prove the existence of certain Ψ^*-completions for operator algebras of b-pseudodifferential operators on manifolds with corners. To achieve this, we will use the methods developed in chapter one, i.e. we will define appropriate Ψ^*-algebras using commutator methods. Note that in [60] resp. [65] this has been done for b-pseudodifferential operators resp. 0-pseudodifferential operators on manifolds with boundary.

2.1 Review of algebras of operators on manifolds with corners

Throughout this section $\mathcal{Z}$ stands for a connected, smooth, compact manifold with corners of dimension m in the sense of [87, Section 2.3]. Roughly speaking, this means that $\mathcal{Z}$ can be embedded into a smooth, closed manifold $\widetilde{\mathcal{Z}}$ of the same dimension, such that $\mathscr{C}^\infty(\mathcal{Z}) = \mathscr{C}^\infty(\widetilde{\mathcal{Z}})_{|\mathcal{Z}}$ and there exists a finite set $\varrho_i \in \mathscr{C}^\infty(\widetilde{\mathcal{Z}})$ $(i = 1, \ldots, K)$ of smooth functions - called *boundary defining functions* - with $\mathcal{Z} = \bigcup_{i=1}^{K} \left\{x \in \widetilde{\mathcal{Z}} \,:\, \varrho_i(x) \geq 0\right\}$, such that $(d\varrho_i(x))_{i\in I}$ are linear independent for all $x \in F$ and all $I \subseteq \{1, \ldots, K\}$, where

$$F := F_I := \bigcap_{i\in I} \{x \in \mathcal{Z} \,:\, \varrho_j(x) = 0\}$$

is a *boundary face* of codimension $k := |I|$ of $\mathcal{Z}$.

The set of *boundary faces* resp. *boundary faces of codimension* k of $\mathcal{Z}$ is denoted by $\mathcal{F}(\mathcal{Z})$ resp. $\mathcal{F}_k(\mathcal{Z})$. Sometimes we also use the notation $\mathcal{F}^{m-k}(\mathcal{Z}) := \mathcal{F}_k(\mathcal{Z})$ to denote the *boundary faces of dimension* $m-k$ of $\mathcal{Z}$. Clearly, $\mathcal{F}(\mathcal{Z}) = \bigcup_{k=1}^{K} \mathcal{F}_k(\mathcal{Z})$ holds. The elements of $\mathcal{F}_1(\mathcal{Z})$ are called *boundary hypersurfaces.* For $F \in \mathcal{F}_k(\mathcal{Z})$ let

$$\mathcal{E}(F) := \{H \in \mathcal{F}_1(\mathcal{Z}) \,:\, F \subseteq H\}$$

and set $M^{\mathcal{E}(F)} := \left\{(x_{F'})_{F'\in\mathcal{E}(F)} \,:\, x_{F'} \in M\right\}$, where M is an arbitrary set. Clearly, a fixed order on $\mathcal{F}_1(\mathcal{Z})$ gives bijections $\mathcal{E}(F) \longrightarrow \{1, \ldots, k\}$ and $M^{\mathcal{E}(F)} \longrightarrow M^k$, and we are going to use this identification in the sequel without further comments. Note that by definition a boundary face of codimension k itself is a manifold with corners of dimension $m - k$.

The space of all smooth vector fields on $\mathcal{Z}$ tangent to all boundary faces is denoted by $\mathcal{V}_b(\mathcal{Z})$. If $p \in \mathcal{Z}$ is an arbitrary point, then there is a face $F \in \mathcal{F}_k(\mathcal{Z})$ with k maximal such that $p \in \mathcal{F}$ (note that we allow k to be zero, i.e. p is then an element of the smooth interior of $\mathcal{Z}$). Then $\mathcal{V}_b(\mathcal{Z})$ is locally spanned (at p) by the vectorfields

$$x_1\partial_{x_1}, \ldots, x_k\partial_{x_k}, \partial_{x_{k+1}}, \ldots, \partial_{x_m},$$

i.e. with respect to local coordinates a vectorfield $V \in \mathcal{V}_b(\mathcal{Z})$ can be written as

$$V(p) = \sum_{j=1}^{k} a_j(p)x_j\partial_{x_j} + \sum_{j=k+1}^{n} a_j(p)\partial_{x_j},$$

with smooth functions a_j.

The associated smooth vector bundle ${}^bT\mathcal{Z} \longrightarrow \mathcal{Z}$, called the *b-tangent bundle*, comes together with a smooth map $j^b : {}^bT\mathcal{Z} \longrightarrow T\mathcal{Z}$ between vector bundles, such that $\mathcal{V}_b(\mathcal{Z}) = j^b(\mathscr{C}^\infty(\mathcal{Z}, {}^bT\mathcal{Z}))$; note that j^b is an isomorphism in the interior of $\mathcal{Z}$. Moreover, denote by ${}^b\Omega^{\frac{1}{2}} = {}^b\Omega^{\frac{1}{2}}(\mathcal{Z})$ the bundle of complex-valued half-densities associated to the dual bundle $p : {}^bT^*\mathcal{Z} \longrightarrow \mathcal{Z}$ of ${}^bT\mathcal{Z}$ and let $L^2_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ be the corresponding Hilbert space of square integrable b-half-densities.

Let $F \longrightarrow \mathcal{Z}$ be a smooth vector bundle. Then $\dot{\mathscr{C}}^\infty(\mathcal{Z}, F)$ denotes the space of all smooth section of F vanishing with all derivatives at the boundary faces of $\mathcal{Z}$. We then define the space of *extendible distributions* $\mathscr{C}^{-\infty}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ on $\mathcal{Z}$ to be the space

$$\mathscr{C}^{-\infty}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) := [\dot{\mathscr{C}}^\infty(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})]'.$$

Before we give the definition of (parameter-dependent) b-pseudodifferential operators, let us briefly discuss the used blow up spaces. Let $S := \bigcup_{H \in \mathcal{F}_1(\mathcal{Z})} H^2 \subseteq \mathcal{Z}^2$; then there exists a compact manifold $\mathcal{Z}^2_b$ with corners together with a smooth map

$$\beta^2_b : \mathcal{Z}^2_b := [\mathcal{Z}^2; S] \longrightarrow \mathcal{Z}^2$$

such that $\mathring{\mathcal{Z}}^2_b = \mathring{\mathcal{Z}}^2$, $\beta^2_{b|\mathring{\mathcal{Z}}^2_b} = id$ and that the *lifted diagonal* $\Xi_b := \overline{(\beta^2_b)^{-1}(S)} \subseteq \mathcal{Z}^2_b$ intersects the *b-front face* $\mathrm{ff}^b := \overline{(\beta^2_b)^{-1}(S)}$ transversally. Let $\chi : \mathcal{Z} \supseteq U \longrightarrow \overline{\mathbb{R}}^k_+ \times \mathbb{R}^{m-k}$ be a local chart near $p \in \mathcal{F}_k(\mathcal{Z})$ with $\chi(F \cap U) \subseteq \{0\} \times \mathbb{R}^{m-k}$ and $\chi(p) = (0, 0)$. Then the pull back of (x, y) resp. (x', y') of χ to the first resp. second factor of $U \times U$ yields the following local coordinates

$$(\tau, r, y, y') : \mathcal{Z}^2_b \supseteq (\beta^2_b)^{-1}(U \times U) \longrightarrow [-1, 1]^k \times \overline{\mathbb{R}}^k_+ \times \mathbb{R}^{m-k} \times \mathbb{R}^{m-k}$$

on $\mathcal{Z}^2_b$ near $(\beta^2_b)^{-1}(p, p)$, where we set

$$r_j := x_j + x'_j \text{ and } \tau_j := \frac{x_j - x'_j}{x_j + x'_j} \quad (j = 1, \ldots, k).$$

We now follow the definition given in [64, Definition 2.1] for a b-pseudodifferential operator depending on an additional parameter $\lambda \in \mathbb{R}^l$. Note, that this is exactly the definition of [87] in the case $l = 0$:

Definition 2.1.1. A family of continuous operators

$$a(\lambda) : \dot{\mathscr{C}}^\infty(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathscr{C}^{-\infty}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}), \ \ (\lambda \in \mathbb{R}^l),$$

belongs to the space $\Psi^j_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^l_\lambda)$ of *classical, parameter-dependent b-pseudodifferential operators* of order $j \in \mathbb{R} \cup \{-\infty\}$ if the following conditions hold:

(i) We have $\operatorname{singsupp} \kappa_{a(\lambda)} \subseteq \Xi_b$ for the lifted Schwartz kernel $\kappa_{a(\lambda)}$ of $a(\lambda)$, $\kappa_{a(\lambda)}$ vanishes with all derivatives at $\partial \mathcal{Z}^2_b \setminus \mathrm{ff}^b$ and for any $\omega \in \mathscr{C}^\infty(\mathcal{Z}^2_b)$ with $\omega \equiv 0$ in a neighbourhood of Ξ_b we have $\omega\kappa_{a(\cdot)} \in \mathcal{S}(\mathbb{R}^l_\lambda, \mathscr{C}^\infty(\mathcal{Z}^2_b, {}^b\Omega^{\frac{1}{2}}))$.

(ii) In a neighbourhood of $\Xi_b \setminus \mathrm{ff}_b$ the lifted Schwartz kernel $\kappa_{a(\lambda)}$ is given by the oscillatory integral

$$\kappa_{a(\lambda)}(z, z') = \int_{\mathbb{R}^n_\zeta} e^{i(z-z')\zeta} \sigma_a(z, \zeta, \lambda) đ\zeta |dz\, dz'|^{\frac{1}{2}}$$

with a classical symbol $\sigma_a \in \mathscr{S}^j_{cl}(\mathbb{R}^m_z, \mathbb{R}^m_\zeta \times \mathbb{R}^m_\lambda)$;

(iii) If (τ, r, y, y') are local coordinates near $(\beta^2_b)^{-1}(p,p)$ with $p \in \mathcal{F}_k(\mathcal{Z})$, then the lifted Schwartz-kernel $\kappa_{a(\lambda)}$ is given by the oscillatory integral

$$\kappa_{a(\lambda)}(\tau, r, y, y') = \int_{\mathbb{R}^{n-k}_\eta \times \mathbb{R}^k_\xi} \left(\frac{1+\tau}{1-\tau}\right)^{i\xi} e^{i(y-y')\eta} \beta_a(r, y, \xi, \lambda) đ\xi đ\eta |d\mu|^{\frac{1}{2}}$$

with a symbol $\beta_a \in S^j_{cl}(\mathbb{R}^k_r \times \mathbb{R}^{m-k}_y, \mathbb{R}^k_\xi \times \mathbb{R}^{m-k}_\eta \times \mathbb{R}^l_\lambda)$ and the density $|d\mu|^{\frac{1}{2}} := \left|\frac{d\tau}{1-\tau^2}\frac{dr}{r} dy\, dy'\right|^{\frac{1}{2}}$.

Note that we used the obvious abbreviation

$$\left(\frac{1+\tau}{1-\tau}\right)^{i\xi} = \left(\frac{1+\tau_1}{1-\tau_1}\right)^{i\xi_1} \cdot \ldots \cdot \left(\frac{1+\tau_k}{1-\tau_k}\right)^{i\xi_k}$$

for $\tau = (\tau_1, \ldots, \tau_k) \in [-1,1]^k$ and $\xi = (\xi_1, \ldots, \xi_k) \in \mathbb{R}^k_\xi$ in 2.1.1 (iii) (analogous for $d\tau/(1-\tau^2)$ and dr/r).

Exactly as in the case of b-pseudodifferential operators without parameters the local symbols σ_a resp. β_a fit together to a well defined *parameter-dependent homogeneous principal symbol*

$${}^b\tilde{\sigma}^{(j)}(a) : {}^bT^*\mathcal{Z} \times \mathbb{R}^l_\lambda \setminus \{0\} \longrightarrow \mathbb{C},$$

homogeneous of degree j in the fibres. The corresponding system of maps

$${}^b\tilde{\sigma}^{(j)} : \Psi^j_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^l_\lambda) \longrightarrow \mathscr{C}^\infty(S({}^bT^*\mathcal{Z} \times \mathbb{R}^l_\lambda)),$$

where $S({}^bT^*\mathcal{Z} \times \mathbb{R}^l_\lambda)$ denotes the sphere bundle of ${}^bT^*\mathcal{Z} \times \mathbb{R}^l_\lambda \longrightarrow \mathcal{Z}$, then has the usual multiplication property

$${}^b\tilde{\sigma}^{(j_1+j_2)}(a_1 a_2) = {}^b\tilde{\sigma}^{(j_1)}(a_1)\, {}^b\tilde{\sigma}^{(j_2)}(a_2)$$

for $a_1 \in \Psi^{j_1}_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^l_\lambda)$, $a_2 \in \Psi^{j_2}_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^l_\lambda)$, and we get the exact sequence

$$0 \to \Psi^{j-1}_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^l) \to \Psi^{j}_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^l) \to \mathscr{C}^\infty(S({}^bT^*\mathcal{Z} \times \mathbb{R}^l)) \to 0. \tag{2.1.1}$$

If $l = 0$, we will write ${}^b\sigma^{(j)}$ instead of ${}^b\tilde{\sigma}^{(j)}$ in the sequel. Recall that for given $F, G \in \mathcal{F}(\mathcal{Z})$ such that $G \subseteq F$, we get a defining function for G as a boundary face of the manifold with corners F via the family $\{\varrho_{H|F} : H \in \mathcal{E}(G) \setminus \mathcal{E}(F)\}$.

Let $a \in \Psi^j_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ be a b-pseudodifferential operator. Then the *indical family* $I_{F\mathcal{Z}}(a)$ of a at F is given by

$$I_{F\mathcal{Z}}(a)(z) = (\varrho^{-iz} a \varrho^{iz})_{|F} \in \Psi^j_{b,cl}(F, {}^b\Omega^{\frac{1}{2}}),\ z \in \mathbb{C}^{\mathcal{E}(F)},$$

where we used the notation $\varrho^z := \prod_{H \in \mathcal{E}(F)} \varrho_H^{z_H} : \mathcal{Z} \setminus \partial\mathcal{Z} \longrightarrow \mathbb{C}$ for $z = (z_H)_{H \in \mathcal{E}(F)} \in \mathbb{C}^{\mathcal{E}(F)}$.

To give a complete characterisation of the (joint-) symbol space, we need the following definition:

Definition 2.1.2. Let $\mathcal{M}^j_{b,\mathcal{O}}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}, \mathbb{C}^l)$ ($j \in \mathbb{R} \cup \{-\infty\}$) be the space of all entire maps $a : \mathbb{C}^l \longrightarrow \Psi^j_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$, such that $[a_\mu : \mathbb{R}^l \ni \lambda \longmapsto a(\lambda + i\mu)] \in \Psi^j_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^l_\lambda)$ holds uniformly for μ in compact subsets of $\mathbb{R}^l$.

This enables us to give a full characterisation of the symbol space by means of the joint symbol map

$$\tau_\Psi : \Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathscr{C}({}^bS^*\mathcal{Z}) \oplus \bigoplus_{F \in \mathcal{F}(\mathcal{Z})} \mathcal{M}^0_{b,\mathcal{O}}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{C}^{\mathcal{E}(F)})$$

(cf. [64, Proposition 2.28] and [85, Proposition 3]):

Proposition 2.1.3. *Let $Q_\Psi(\mathcal{Z})$ denote the algebra of all*

$$(f, (h_F)_{F \in \mathcal{F}(\mathcal{Z})}) \in \mathscr{C}^\infty({}^bS^*\mathcal{Z}) \oplus \bigoplus_{F \in \mathcal{F}(\mathcal{Z})} \mathcal{M}^0_{b,\mathcal{O}}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{C}^{\mathcal{E}(F)})_{|\mathbb{R}^{\mathcal{E}(F)}}$$

satisfying the following compatibility conditions:

$$\begin{aligned} {}^b\widetilde{\sigma}^{(0)}(h_F) &= f_{|{}^bS^*Z_{|F}} \text{ for all } F \in \mathcal{F}(\mathcal{Z}), \text{ and} && (2.1.2)\\ h_G(\lambda) &= I_{GF}(h_F((\lambda_H)_{H \in \mathcal{E}(F)}))((\lambda_H)_{H \in \mathcal{E}(G) \setminus \mathcal{E}(F)}) && (2.1.3) \end{aligned}$$

for all $\lambda \in \mathbb{R}^{\mathcal{E}(G)}$ and all boundary faces $F, G \in \mathcal{F}(\mathcal{Z})$ with $G \subseteq F$. Then the sequence

$$0 \to \Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \cap \mathcal{K}(L^2(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})) \to \Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \xrightarrow{\tau_\Psi} Q_\Psi(\mathcal{Z}) \to 0$$

is exact.

Let $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ denote the closure of $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ in the C^*-Algebra $\mathscr{L}(L^2_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}))$. Then the joint symbol map τ_Ψ extends to a homomorphism

$$\tau_{\mathcal{B}} : \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \to \mathscr{C}({}^bS^*\mathcal{Z}) \oplus \bigoplus_{F \in \mathcal{F}(\mathcal{Z})} \mathscr{C}_b\left(\mathbb{R}^{\mathcal{E}(F)}, \mathscr{L}\left(L^2_b(F, {}^b\Omega^{\frac{1}{2}})\right)\right)$$

and we will use the abbreviation $\mathcal{Q}_\tau$ for the right hand side of the above formula to shorten notation.

Since the space of operators with smooth kernel, compactly supported in the interior of $\mathcal{Z}^2$ belongs to $\Psi^{-\infty}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ and is dense in $\mathcal{K}(L^2_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}))$ we get $\mathcal{K}(L^2_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})) \subseteq \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. Thus the following sequence of C^*-algebras is exact:

$$0 \longrightarrow \mathcal{K}(L^2_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})) \longrightarrow \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \xrightarrow{\tau_{\mathcal{B}}} Q_{\mathcal{B}}(\mathcal{Z}) \longrightarrow 0, \tag{2.1.4}$$

where the symbol space $Q_{\mathcal{B}}(\mathcal{Z})$ will be completely characterised in 2.1.4. In what follows we will denote the closure of $\Psi^{-1}_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ in $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ by $\mathcal{B}^-(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. Additionally we have mappings

$$\begin{aligned} {}^b\sigma^{(0)}_{\mathcal{B}} &: \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathscr{C}({}^bS^*\mathcal{Z}), \quad \text{resp.} && (2.1.5)\\ I^{\mathcal{B}}_{FZ} &: \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathscr{C}_b(\mathbb{R}^{\mathcal{E}(F)}, \mathscr{L}(L^2_b(F, {}^b\Omega^{\frac{1}{2}}))) && (2.1.6) \end{aligned}$$

given by the composition of $\tau_{\mathcal{B}}$ with the projection of $Q_{\mathcal{B}}(\mathcal{Z})$ onto the components $\mathscr{C}({}^bS^*\mathcal{Z})$ resp. $\mathscr{C}_b(\mathbb{R}^{\mathcal{E}(F)}, \mathscr{L}(L^2_b(F, {}^b\Omega^{\frac{1}{2}})))$.

To treat the parameter-dependent case, first note that the inward pointing normal bundle $\overline{N^+F} \cong F \times [-1,1]^k$ ($F \in \mathcal{F}(\mathcal{Z})$) is a compact manifold with corners of dimension m, too, and that all above results also apply to the boundary faces of $\mathcal{Z}$. Moreover, we see that the (parameter-dependent) homogeneous principal symbol map

$${}^b\tilde{\sigma}^{(0)} : \mathcal{M}^0_{b,\mathcal{O}}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{C}^{\mathcal{E}(F)})_{|\mathbb{R}^{\mathcal{E}(F)}} \longrightarrow \mathscr{C}^\infty({}^bS^*\mathcal{Z}_{|F})$$

extends to a homeomorphism

$${}^b\tilde{\sigma}^{(0)}_{\mathcal{B}} : \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)}) \longrightarrow \mathscr{C}({}^bS^*\mathcal{Z}_{|F}).$$

Here $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$ denotes the closure of $\mathcal{M}^0_{b,\mathcal{O}}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{C}^{\mathcal{E}(F)})_{|\mathbb{R}^{\mathcal{E}(F)}}$ in the C^*-Algebra $\mathscr{C}_b(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}})) \hookrightarrow \mathscr{C}_b(\mathbb{R}^{\mathcal{E}(F)}, \mathscr{L}(L^2_b(F, {}^b\Omega^{\frac{1}{2}})))$. Again let $\mathcal{B}^-(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$ be the closure of $\Psi^{-1}_{b,cl}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$ in $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$. Then, by the fact that $S^{-\infty}$ is dense in S^{-1} with respect to the topology of S^0 (see [52, Proposition 1.1.11]), we get that $\Psi^{-\infty}_b(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$ is dense in $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$. Since we also have the dense inclusions

$$\mathscr{C}^\infty_c(\mathbb{R}^{\mathcal{E}(F)}) \otimes \Psi^{-\infty}_b(F, {}^b\Omega^{\frac{1}{2}}) \hookrightarrow \mathcal{B}^-(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$$

and

$$\mathscr{C}^\infty_c(\mathbb{R}^{\mathcal{E}(F)}) \otimes \Psi^{-\infty}_b(F, {}^b\Omega^{\frac{1}{2}}) \hookrightarrow \mathscr{C}_0(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{B}^-(F, {}^b\Omega^{\frac{1}{2}}))$$

it follows that

$$\mathcal{B}^-(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)}) \cong \mathscr{C}_0(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{B}^-(F, {}^b\Omega^{\frac{1}{2}}))$$

holds. This leads to the following exact sequence (cf. [86, Proposition 9, Corollary 3]):

$$0 \to \mathscr{C}_0(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{B}^-(F, {}^b\Omega^{\frac{1}{2}})) \to \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)}) \xrightarrow{{}^b\tilde{\sigma}_{\mathcal{B}}^{(0)}} \mathscr{C}({}^bS^*\mathcal{Z}_{|F}) \to 0, \tag{2.1.7}$$

and thus we get the following description of the symbol space associated to $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$:

Proposition 2.1.4. *The C^*-algebra $Q_{\mathcal{B}}(\mathcal{Z})$ consists of all*

$$(f, (h_F)_{F\in\mathcal{F}(\mathcal{Z})}) \in \mathscr{C}({}^bS^*\mathcal{Z}) \oplus \bigoplus_{F\in\mathcal{F}(\mathcal{Z})} \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$$

satisfying the following compatibility conditions:

$$\begin{aligned} {}^b\tilde{\sigma}_{\mathcal{B}}^{(0)}(h_F) &= f_{|{}^bS^*\mathcal{Z}_{|F}} \text{ for all } F \in \mathcal{F}(\mathcal{Z}), \text{ and} && (2.1.8)\\ h_G(\lambda) &= I_{GF}^{\mathcal{B}}(h_F((\lambda_H)_{H\in\mathcal{E}(F)}))((\lambda_H)_{H\in\mathcal{E}(G)\setminus\mathcal{E}(F)}) && (2.1.9)\end{aligned}$$

for all $\lambda \in \mathbb{R}^{\mathcal{E}(G)}$ and all boundary faces $F, G \in \mathcal{F}(\mathcal{Z})$ with $G \subseteq F$.

Proof. See [86, Proposition 11]. □

2.2 Ψ^*-completions of $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$

Now, that we have collected all relevant material and notation in section one of this chapter, we are finally able to prove the existence of certain Ψ^*-*completions of* $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ in the C^*-algebra $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. The construction itself will be done by induction with respect to the codimension of $\mathcal{Z}$; recall that the codimesion of $\mathcal{Z}$ is given by

$$k_{\mathcal{Z}} := \operatorname{codim} \mathcal{Z} := \max\{j \in \mathbb{N} : \mathcal{F}_j(\mathcal{Z}) \neq \emptyset\}.$$

So let us first discuss the case $k_{\mathcal{Z}} = 1$ in detail. Then all faces of $\mathcal{Z}$ are (embedded) hypersurfaces that do not intersect with each other, i.e.

$$\mathcal{F}(\mathcal{Z}) = \mathcal{F}_1(\mathcal{Z}) = \{M_1, \dots, M_p\}, \quad M_i \cap M_j = \emptyset \text{ if } i \neq j$$

and the M_j are closed manifolds themselves. We define $Q_{\mathcal{A}}(\mathcal{Z})$ to be the set of all

$$(f, (h_l)_{l=1}^p) \in \mathscr{C}^\infty({}^bS^*\mathcal{Z}) \oplus \bigoplus_{i=1}^p \Psi^0_{cl}(M_i, \Omega^{\frac{1}{2}}; \mathbb{R})$$

such that

$$f_{|{}^bS^*\mathcal{Z}_{|M_i}} = {}^b\widetilde{\sigma}_{\psi}^{(0)}(h_i).$$

Here $\Psi^0_{cl}(M_i, \Omega^{\frac{1}{2}}; \mathbb{R})$ denotes the algebra of all classical parameter-dependent pseudodifferential operators on the closed manifold M_i (see for instance [60, Section 4.2] for a definition). This gives

$$Q_\Psi(\mathcal{Z}) \subseteq Q_{\mathcal{A}}(\mathcal{Z}) \subseteq Q_{\mathcal{B}}(\mathcal{Z}), \tag{2.2.1}$$

where $Q_\Psi(\mathcal{Z}) = \tau_\Psi(\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}))$ is the range of the symbol map and $Q_\mathcal{A}(\mathcal{Z}) \hookrightarrow Q_\mathcal{B}(\mathcal{Z})$ is a dense Ψ^*-algebra by [60, Lemma 6.1.2]. Finally, we set

$$\mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) := \left\{ a \in \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \,:\, \tau_\mathcal{B}(a) \in Q_\mathcal{A}(\mathcal{Z}) \right\},$$

then $\mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ is a Ψ^*-algebra, cf. 1.1.6, and we have an exact sequence

$$0 \to \mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \cap \mathcal{K}(L^2(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})) \to \mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \xrightarrow{\tau_\mathcal{A}} Q_\mathcal{A}(\mathcal{Z}) \to 0, \tag{2.2.2}$$

where $\tau_\mathcal{A} := \tau_{\mathcal{B}_{|\mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})}} : \mathcal{A}(Z, {}^b\Omega^{\frac{1}{2}}) \longrightarrow Q_\mathcal{A}(\mathcal{Z})$ denotes the restriction of the joint symbol map. Moreover, we get the dense inclusions

$$\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \hookrightarrow \mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \hookrightarrow \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}). \tag{2.2.3}$$

Now, assume that we have already constructed a Ψ^*-algebra completion for $\Psi^0_{b,cl}(F, \Omega^{\frac{1}{2}})$ for all F with $1 \le \operatorname{codim} F \le n-1$ fulfilling (2.2.1), (2.2.2) and (2.2.3) (where $\mathcal{Z}$ has to be replaced by F). It is worth pointing out, that one can regard $\mathcal{M}^0_{b,\mathcal{O}}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{C}^{\mathcal{E}(F)})_{|\mathbb{R}^{\mathcal{E}(F)}}$ as a subset of $\mathscr{C}^\infty_b(\mathbb{R}^{\mathcal{E}(F)}, \Psi^0_{b,cl}(F, {}^b\Omega^{\frac{1}{2}}))$ for $F \in \mathcal{F}_k(\mathcal{Z})$ and that F is then a manifold with corners itself. Using the induction hypothesis yields a Ψ^*-algebra completion $\mathcal{A}(F, {}^b\Omega^{\frac{1}{2}})$ of $\Psi^0_{b,cl}(F, {}^b\Omega^{\frac{1}{2}})$ with

$$Q_\Psi(F) \hookrightarrow Q_\mathcal{A}(F) \hookrightarrow Q_\mathcal{B}(F) \tag{2.2.4}$$

dense, where $Q_\mathcal{A}(F) = \tau_\mathcal{B}(\mathcal{A}(F, {}^b\Omega^{\frac{1}{2}}))$ holds. The continuous inclusions

$$\begin{aligned} \mathscr{C}^\infty_b(\mathbb{R}^{\mathcal{E}(F)}, \Psi^0_{b,cl}(F, {}^b\Omega^{\frac{1}{2}})) &\hookrightarrow \mathscr{C}^\infty_b(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{A}(F, {}^b\Omega^{\frac{1}{2}})) \\ &\hookrightarrow \mathscr{C}_b(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}})) \end{aligned}$$

leads to the following definition

$$Q_F := \mathscr{C}^\infty_b(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{A}(F, {}^b\Omega^{\frac{1}{2}})) \cap \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)}).$$

Note, that by definition $\mathcal{M}^0_{b,\mathcal{O}}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{C}^{\mathcal{E}(F)})_{|\mathbb{R}^{\mathcal{E}(F)}} \subseteq Q_F$. Let $Q_\mathcal{A}(\mathcal{Z})$ be the set of all

$$(f, (h_F)_{F \in \mathcal{F}(\mathcal{Z})}) \in \mathscr{C}^\infty({}^bS^*\mathcal{Z}) \oplus \bigoplus_{F \in \mathcal{F}(\mathcal{Z})} Q_F,$$

such that

$$\begin{aligned} {}^b\tilde{\sigma}^{(0)}_\mathcal{B}(h_F) &= f_{|{}^bS^*\mathcal{Z}_{|F}} \text{ for all } F \in \mathcal{F}(\mathcal{Z}), \text{ and} && (2.2.5) \\ h_G(\lambda) &= I^\mathcal{B}_{GF}(h_F((\lambda_H)_{H \in \mathcal{E}(F)}))((\lambda_H)_{H \in \mathcal{E}(G) \setminus \mathcal{E}(F)}) && (2.2.6) \end{aligned}$$

for all $\lambda \in \mathbb{R}^{\mathcal{E}(G)}$ and all boundary faces $F, G \in \mathcal{F}(\mathcal{Z})$ with $G \subseteq F$.

Lemma 2.2.1. *We have:*

(i) $Q_\Psi(\mathcal{Z}) \subseteq Q_\mathcal{A}(\mathcal{Z}) \subseteq Q_\mathcal{B}(\mathcal{Z})$ *where* $Q_\Psi(\mathcal{Z}) = \tau_\Psi(\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}))$ *is the range of the symbol map.*

(ii) $Q_{\mathcal{A}}(\mathcal{Z}) \hookrightarrow Q_{\mathcal{B}}(\mathcal{Z})$ *is a dense* Ψ^**-algebra.*

Proof. (i) This is clear by definition.
(ii) Since $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ is dense in $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ by definition and since $\tau_{\mathcal{B}}(\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})) = Q_\Psi(\mathcal{Z})$ as well as $\tau(\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})) = Q_{\mathcal{B}}(\mathcal{Z})$, $Q_\Psi(\mathcal{Z})$ is dense in $Q_{\mathcal{B}}(\mathcal{Z})$. Moreover,

$$Q_\infty(\mathcal{Z}) := \mathscr{C}^\infty({}^bS^*\mathcal{Z}) \oplus \bigoplus_{F \in \mathcal{F}(\mathcal{Z})} Q_F$$

is a Ψ^*-algebra in

$$Q_b(\mathcal{Z}) := \mathscr{C}({}^bS^*\mathcal{Z}) \oplus \bigoplus_{F \in \mathcal{F}(\mathcal{Z})} \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$$

(cf. 1.1.5 (ii)), hence 1.1.7 applies to $Q_\infty(\mathcal{Z})$, $Q_b(\mathcal{Z})$ and the systems of C^*-algebra homomorphisms

$$\begin{aligned} Q_b(\mathcal{Z}) \longrightarrow \mathbb{C} \quad &: \quad (f, (h_F)_{F\in\mathcal{F}(\mathcal{Z})}) \longmapsto f_{|{}^bS^*\mathcal{Z}_{|H}}(\eta) \\ Q_b(\mathcal{Z}) \longrightarrow \mathbb{C} \quad &: \quad (f, (h_F)_{F\in\mathcal{F}(\mathcal{Z})}) \longmapsto {}^b\tilde{\sigma}^{(0)}_{\mathcal{B}}(h_H)(\eta), \end{aligned}$$

where $\eta \in {}^bS^*\mathcal{Z}_{|H}$ and $H \in \mathcal{F}(\mathcal{Z})$ resp.

$$\begin{aligned} Q_b(\mathcal{Z}) &\longrightarrow \mathcal{B}(G, {}^b\Omega^{\frac{1}{2}}) : \\ (f, (h_F)_{F\in\mathcal{F}(\mathcal{Z})}) &\longmapsto h_G(\lambda); \\ Q_b(\mathcal{Z}) &\longrightarrow \mathcal{B}(G, {}^b\Omega^{\frac{1}{2}}) : \\ (f, (h_F)_{F\in\mathcal{F}(\mathcal{Z})}) &\longmapsto I_{GF}(h_F((\lambda_H)_{H\in\mathcal{E}(F)}))((\lambda_H)_{H\in\mathcal{E}(G)\setminus\mathcal{E}(F)}), \end{aligned}$$

where $\lambda \in \mathcal{E}(G)$, $F, G \in \mathcal{F}(\mathcal{Z})$ such that $G \subseteq F$, i.e. $Q_{\mathcal{A}}(\mathcal{Z}) = Q_\infty(\mathcal{Z}) \cap Q_{\mathcal{B}}(\mathcal{Z})$ is a Ψ^*-algebra. □

Proposition 2.2.2. *The* Ψ^**-algebra defined by*

$$\mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) := \left\{ a \in \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \,:\, \tau_{\mathcal{B}}(a) \in Q_{\mathcal{A}}(\mathcal{Z}) \right\}$$

is a Ψ^**-completion of* $\Psi_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ *with* $\tau_{\mathcal{B}}(\mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})) = Q_{\mathcal{A}}(\mathcal{Z})$.

Proof. By definition $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \subseteq \mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. Since

$$\tau_{\mathcal{B}} : \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow Q_{\mathcal{B}}(\mathcal{Z})$$

is onto, we also get $\tau_{\mathcal{B}}(\mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})) = Q_{\mathcal{A}}(\mathcal{Z})$. □

Remark 2.2.3. Note that we have $\tau_{\mathcal{B}}(\mathcal{K}(L^2(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}))) = \{0\} \subseteq Q_{\mathcal{A}}(\mathcal{Z})$, which shows that the algebra $\mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ contains all compact operators on $L^2_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. Thus the algebra $\mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ is too large in order to obtain e.g. also mapping properties for a scale of Sobolev spaces and the corresponding result on ellipitic regularity. But the next section shows that it is possible to construct a refined version of the well-behaved completion $\mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ that still respects Sobolov mapping properties (see 2.3.3).

2.3 Refined Ψ^*-completions acting on Sobolev spaces generated by the b-Laplacian

First, let us recall some definitions and facts on b-differential operators; hereby we closely follow [82]. Recall that $\mathcal{V}_b(\mathcal{Z})$ denotes the space of all smooth vector fields $V \in \mathcal{V}(\mathcal{Z})$ on a manifold with corners $\mathcal{Z}$ of dimension m that are tangent to the boundary. Then $\mathcal{V}_b(\mathcal{Z})$ is a Lie-algebra and a $\mathscr{C}^\infty(\mathcal{Z})$-module, so it is quite natural to pass to its enveloping algebra, i.e. we consider the space $\mathrm{Diff}_b^j(\mathcal{Z})$ of j-th order differential operators $D : \mathscr{C}^\infty(\mathcal{Z}) \longrightarrow \mathscr{C}^\infty(\mathcal{Z})$, where

$$\mathrm{Diff}_b^j(\mathcal{Z}) := \mathrm{span}_{\mathscr{C}^\infty(\mathcal{Z})} \bigcup_{l=0}^{j} \{V_1 \circ \ldots \circ V_l \,:\, V_l \in \mathcal{V}_b(\mathcal{Z})\}\,.$$

Also we can define a second order differential operator $\Delta_b \in \mathrm{Diff}_b^2(\mathcal{Z})$ called the *b-Laplacian* fulfilling ${}^b\sigma^{(2)}(\Delta_b) = |\zeta|^2$, where $\zeta \in {}^bT^*\mathcal{Z}$ and ${}^b\sigma^{(2)}$ denotes the principal symbol map (see [82, Proposition 2.9.2]). Thus the operator $\Lambda_b := (1-\Delta_b)^{\frac{1}{2}}$ is (strictly) positive and selfadjoint and gives raise to a scale of Sobolev spaces $\left\{\mathcal{H}_b^s(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \,:\, s \in \mathbb{R}\right\}$ with $\mathcal{H}_b^s(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \subseteq \mathcal{H}_b^{s'}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ $(s < s')$ in the sense of Cordes [23]. Let $a \in \Psi_{b,cl}^0(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ be arbitrary, then since $\Lambda_b^2 \in \Psi_{b,cl}^2(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ holds we get $\Lambda_b^2 a$, $a\Lambda_b^2 \in \Psi_{b,cl}^2(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. The symbolic calculus now yields

$${}^b\sigma^{(2)}(a\Lambda_b) - {}^b\sigma^{(2)}(\Lambda_b a) = {}^b\sigma^{(0)}(a)^b\sigma^{(2)}(\Lambda_b) - {}^b\sigma^{(2)}(\Lambda_b)^b\sigma^{(0)}(a) = 0,$$

i.e. $\mathrm{ad}\,[\Lambda_b^2](a) \in \Psi_{b,cl}^1(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}))$ for all $a \in \Psi_{b,cl}^0(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. Using 1.3.6, we now get

$$\mathrm{ad}\,[\Lambda_b]^\nu(a) \in \mathscr{L}(\mathcal{H}_b^s(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}), \mathcal{H}_b^s(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})) \quad (\nu \in \mathbb{N},\, s \in \mathbb{R})$$

for all $a \in \Psi_{b,cl}^0(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. Consequently, if we set

$$\begin{aligned}\mathcal{A}_1 := \big\{a \in \mathscr{L}(L_b^2(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})) \,:\, a(D(\Lambda_b^\infty)) \subseteq D(\Lambda_b^\infty),\\ \forall\, \nu \in \mathbb{N}\, \exists\, c_\nu \geq 0 : ||ad[\Lambda_b]^\nu(a)x|| \leq c_\nu ||x||\big\},\end{aligned}$$

it follows:

Proposition 2.3.1. *We have* $\Psi_{b,cl}^0(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \subseteq \mathcal{A}_1$.

As a first step, let us now define a refined Ψ^*-algebra completion of $\Psi_{b,cl}^0(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$, which will "act" on the scale of Sobolev spaces and will be a Ψ^*-subalgebra of $\mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$.

Definition 2.3.2. The closure of $\Psi_{b,cl}^0(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ in the Ψ^*-algebra $\mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \cap \mathcal{A}_1$ will be denoted by $\mathcal{A}_1(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$.

Proposition 2.3.3. *For each* $s \in \mathbb{R}$ *the pairing*

$$\mathcal{A}_1(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \times \mathcal{H}_b^s(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{H}_b^s(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) :\, (a, u) \longmapsto a(u)$$

is well defined, bilinear and continuous. In particular, one has

$$\mathcal{A}_1(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \subseteq \bigcap_{s \in \mathbb{R}} \mathscr{L}(\mathcal{H}_b^s(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})).$$

Proof. Since $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ is a symmetric subalgebra of $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$, we get that $\mathcal{A}_1(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ is a Ψ^*-algebra in the C^*-algebra $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. Moreover, $\mathcal{A}_1(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ is a subalgebra of the Ψ^*-algebra $\mathcal{A}_1$, so we also get the the Sobolev mapping properties. □

Finally, let us show that one can even achieve more: It is possible to define a Ψ^*-completion of $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ such that the operators within this completion behave like ordinary pseudodifferential operators if one localises them to the interior of $\mathcal{Z}$. Namely, we have:

Theorem 2.3.4. *There exists a submultiplicative Ψ^*-algebra completion $\mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ of $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ in the C^*-algebra $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$, such that:*

(i) *$\mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ is a dense subalgebra of $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$.*

(ii) *Any $a \in \mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ extends to a bounded operator*

$$a : \mathcal{H}_b^s(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{H}_b^s(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \quad (s \in \mathbb{R}).$$

Moreover, the associated blinear map

$$\mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \times \mathcal{H}_b^s(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{H}_b^s(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}); (a, u) \longmapsto a(u)$$

is jointly continuous.

(iii) *Any $a \in \mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ has smooth symbols, i.e.*

$$\begin{aligned} &{}^b\sigma_{\mathcal{B}}^{(0)}(a) \in \mathscr{C}^\infty({}^bS^*\mathcal{Z}) \text{ and} \\ &I^{\mathcal{B}}_{F,\mathcal{Z}}(a) \in \mathscr{C}^\infty(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{A}(F, {}^b\Omega^{\frac{1}{2}})) \cap \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)}), \end{aligned}$$

where $\mathcal{A}(F, {}^b\Omega^{\frac{1}{2}})$ is a suitable submultiplicative Ψ^*-algebra on the manifold with corners F, having property (2.2.4).

(iv) *Let $\omega_1, \omega_2 \in \mathcal{C}_c^\infty(\mathring{\mathcal{Z}})$ and $a \in \mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. Then $\omega_1 a \omega_2$ is an ordinary, compactly supported pseudodifferential operator in the interior $\mathring{\mathcal{Z}}$ of $\mathcal{Z}$.*

Before we give the proof of this theorem, we have to introduce some more notations and make some preparations:

Notations 2.3.5. Let $p \in \mathcal{Z}$ be arbitrary. Then there exists a face $F \in \mathcal{F}_{k(p)}(\mathcal{Z})$ with $k(p)$ maximal, such that $p \in F$ (note, that $k(p) = 0$ is possible if p is in the interior of $\mathcal{Z}$). We assume, that $\varrho_1, \ldots, \varrho_{k(p)}$ are the boundary functions that vanish at p. Thus we get that

$$d\varrho_1(p), \ldots, d\varrho_{k(p)}(p)$$

are linear independent and we can use $\varrho_1, \ldots, \varrho_{k(p)}$ to get the following coordinate charts for a neighbourhood $\mathcal{W}_p$ of p:

$$\chi_p : \mathcal{W}_p \longrightarrow [0, r[^{k(p)} \times \mathbb{R}^{n-k}.$$

Here $[0, r[^{k(p)}$ means $[0, r[_{\varrho_1} \times \ldots \times [0, r[_{\varrho_k}$. Since $\mathcal{Z}$ is compact, we can choose r to be the same for all coordinate patches to get a finite atlas

$$\big\{(\mathcal{W}_{p_j}, \chi_{p_j}) \,:\, j = 1, \ldots, l\big\}$$

for $\mathcal{Z}$. Let $(\varphi_j)_{j\in I}$, $I := \{1, \ldots, l\}$, be a partition of unity that is subordinated to this (fixed) covering of coordinate neighbourhoods. Let J_r denote the subset of I such that $V_j \cap \partial\mathcal{Z} = \emptyset$ holds for all $j \in J_r$ and choose families of functions $(\psi_j)_{j\in I}$, $(\beta_j)_{j\in I}$, $(\gamma_j)_{j\in I} \subseteq \mathscr{C}_c^\infty(V_j)$ with $\varphi_j \prec \beta_j \prec \gamma_j \prec \psi_j$. We define systems $\widetilde{\mathcal{D}}_{int}$, $\widetilde{\mathcal{D}}_{\partial Z}$ and $\widetilde{\mathcal{M}}$ of b-differential operators as follows (always modulo the obvious chart diffeomorphisms):

- Let $\widetilde{\mathcal{D}}_{int} := \bigcup_{j\in J} \widetilde{\mathcal{D}}^j_{int}$ denote the union of differential operators $\widetilde{\mathcal{D}}^j_{int}$ that are given by $D_{j,l} := i\psi_j \partial_{z_l} \psi_j$, where $j \in J$ and $l = 1, \ldots, m$.
- Denote by $\widetilde{\mathcal{D}}_{\partial\mathcal{Z}} := \bigcup_{j\in I\setminus J} \widetilde{\mathcal{D}}^j_{\partial\mathcal{Z}}$ the set of differential operators that are given by $D^{\partial\mathcal{Z}}_{j,l} := i\psi_j x_l \partial_{x_l} \psi_j$ if $l = 1, \ldots, k_0$ and $D^{\partial\mathcal{Z}}_{j,l} := i\psi_j \partial_{x_l} \psi_j$ if $l = k_0 + 1, \ldots, n$.
- The set of multiplication operators $\widetilde{\mathcal{M}}$ is given by $\widetilde{\mathcal{M}} := \bigcup_{j\in J} \mathcal{M}_j$, where the operators in $\widetilde{\mathcal{M}}_j$ are given by $\psi_j z_l \psi_j$ for $l = 1, \ldots, m$.

Then these operators are densely defined and symmetric, hence closable operators and we denote by $\mathcal{D}_{int}$, $\mathcal{D}_{\partial\mathcal{Z}}$ and $\mathcal{M}$ the corresponding sets of minimal closed extension of them.

Finally, let us introduce the family $\widetilde{\mathcal{V}}$ of all operators given by

- $V_{j,l} := \beta_j D_{j,l} \beta_j$ and $W_{j,l} := (1 - \beta_j) D_{j,l} (1 - \beta_j)$ $(D_{j,l} \in \widetilde{\mathcal{D}}^j_{int})$;
- $V^{\partial\mathcal{Z}}_{j,l} := \beta_j D^{\partial\mathcal{Z}}_{j,l} \beta_j$ and $W_{j,l} := (1 - \beta_j) D^{\partial\mathcal{Z}}_{j,l} (1 - \beta_j)$ $(D^{\partial\mathcal{Z}}_{j,l} \in \widetilde{\mathcal{D}}^j_{\partial\mathcal{Z}})$.

Again, denote by $\mathcal{V}$ the corresponding set of minimal closed extension.

Definition 2.3.6. We define $\widetilde{\mathcal{A}}_{b,r}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ to be the set of all $a \in \mathcal{A}_1$, such that

$$\mathrm{ad}\,[M]^\alpha \mathrm{ad}\,[D]^\beta(a) \in \mathscr{L}(\mathcal{H}^s(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}), \mathcal{H}^{s-|\alpha|}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}))$$

holds for all $D \in (\mathcal{D}_{int} \cup \mathcal{D}_{\partial\mathcal{Z}} \cup \mathcal{V})^{|\beta|}$, for all $M \in \mathcal{M}^{|\alpha|}$ and for all $s \in \mathbb{R}$, $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0^m$. Here we used the definitions

$$\mathrm{ad}\,[D]^\beta(a) := \mathrm{ad}\,[D_1]^{\beta_1} \mathrm{ad}\,[D_2]^{\beta_2} \cdots \mathrm{ad}\,[D_n]^{\beta_n}(a)$$

resp.

$$\mathrm{ad}\,[M]^\alpha(a) := \mathrm{ad}\,[M_1]^{\alpha_1} \mathrm{ad}\,[M_2]^{\alpha_2} \cdots \mathrm{ad}\,[M_n]^{\alpha_m}(a).$$

We then get the following proposition:

Proposition 2.3.7. *$\widetilde{\mathcal{A}}_{b,r}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ is a Ψ^*-algebra.*

Proof. This follows from [110] (see also [46]). □

Remark 2.3.8. Recall that r denotes the "collar" parameter that was chosen for the coordinate charts that intersect with $\partial\mathcal{Z}$. In particular, setting $r := 1/k$ ($k \in \mathbb{N}$) yields systems of coordinate charts that "shrink" around the faces.

Let us now give the proof of theorem 2.3.4:

Proof of 2.3.4. In what follows, denote by $\mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ the closure of $\Psi^0_{b,cl}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ in the Ψ^*-algebra

$$\mathcal{A}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \cap \bigcap_{k\in\mathbb{N}} \widetilde{\mathcal{A}}_{b,\frac{1}{k}}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \hookrightarrow \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}).$$

Then property (i) is clear by definition. The Sobolev mapping property (ii) is due to the fact, that $\mathcal{A}_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ is a subalgebra of $\mathcal{A}_1$ by construction. To see (iii), we first decompose the operator $\omega_1 a \omega_2$ using a partition of unity according to 2.3.5:

$$\omega_1 a \omega_2 = \sum_{j\in I} \gamma_j \omega_1 a \omega_2 \varphi_j + \sum_{j\in I} (1-\gamma_j)\omega_1 a \omega_2 \varphi_j.$$

Thus it is enough to treat operators of the form $\gamma_j \omega_1 a \omega_2 \varphi_j$ resp. of the form $(1 - \gamma_j)\omega_1 a \omega_2 \varphi_j$. Now, let $k \in \mathbb{N}$ be such that

$$\operatorname{supp}\omega_i \cap \operatorname{supp}\varphi_j = \emptyset \quad (i = 1, 2) \tag{2.3.1}$$

for all $j \in I \setminus J_{1/k}$ (clearly such an k exists since $\omega_i \in \mathcal{C}^\infty_c(\mathring{\mathcal{Z}})$; note, that (2.3.1) then also holds for all $l \geq k$). Then $\omega_i \varphi_j = 0$ for all $j \in I \setminus J$ and thus we have to treat only operators of the form $\gamma_j \omega_1 a \omega_2 \varphi_j$ resp. of the form $(1 - \gamma_j)\omega_1 a \omega_2 \varphi_j$ with $j \in I \setminus J$. Using [60, Proposition 6.2.49 (c)] we see that $\gamma_j \omega_1 a \omega_2 \varphi_j$ (see also [60, Proposition 6.2.37 (c)]) is a pseudodifferential operator with local symbol in $S^0(\mathbb{R}^n, \mathbb{R}^n)$.

To treat $(1 - \gamma_j)\omega_1 a \omega_2 \varphi_j$, we first use the partition of unity again:

$$(1-\gamma_j)\omega_1 a \omega_2 \varphi_j = \sum_{l\in I} \varphi_l (1-\gamma_j)\omega_1 a \omega_2 \varphi_j$$

Thus we have to treat only operators $\varphi_l(1 - \gamma_j)\omega_1 a \omega_2 \varphi_j$ with $l, j \notin J$ again. So again by [60, Proposition 6.2.49 (b)] (see also the last appendix of this thesis) we get, that $\varphi_l(1 - \gamma_j)\omega_1 a \omega_2 \varphi_j$ is an operator with smooth kernel. □

Remark 2.3.9. Using a refined construction given by [27], we could also achieve that the operators in 2.3.4 (iv) can be represented by pseudodifferential operators with *classical symbols*.

Chapter 3

Localisation of C^*- and Ψ^*-chains

3.1 Representations of C^*- and Ψ^*-algebras

In this section we want to summarize the main definitions and results of representation theory of C^*-algebras. Most of the proofs are omitted, but the interested reader can find them easily in the literature (see [31] or [90] for example).

Definition 3.1.1. Let $\mathcal{B}$ be an (unital) algebra with involution and $\mathcal{H}$ be a Hilbert space.

(i) Let $\pi : \mathcal{B} \longrightarrow \mathscr{L}(\mathcal{H})$ be a $*$-algebra homomorphism. Then we call the pair $(\mathcal{H}, \pi)$ a *representation of* $\mathcal{B}$. $\mathcal{H}$ is called the *representation space* and the Hilbert dimension of $\mathcal{H}$ is called the *dimension* $(\dim \pi)$ of the representation.

(ii) Two representations $\pi_j : \mathcal{B} \longrightarrow \mathscr{L}(\mathcal{H}_j)$, $j \in \{0, 1\}$, of $\mathcal{B}$ are said to be (*unitarily*) *equivalent* if there exists a unitary operator $U : \mathcal{H}_0 \longrightarrow \mathcal{H}_1$, such that $U\pi_0(b) = \pi_1(b)U$ holds for all $b \in \mathcal{B}$.

(iii) A representation $(\mathcal{H}, \pi)$ is called *irreducible* if $\mathcal{H} \neq 0$ and $\{0\}$ as well as $\mathcal{H}$ are the only closed subspaces of $\mathcal{H}$ that are invariant under $\pi(b)$ for all $b \in \mathcal{B}$.

(iv) We say that $x \in \mathcal{H}$ is a *cyclic vector* for $(\mathcal{H}, \pi)$ if one has

$$\overline{LH}(\pi(b)x \,:\, b \in \mathcal{B}) = \mathcal{H}.$$

Here $LH(V)$ denotes the linear hull of a given set of vectors V.

Proposition 3.1.2. *Let $\mathcal{B}$ be an involutive algebra, and $(\mathcal{H}, \pi)$ a representation of $\mathcal{B}$. Then the following conditions are equivalent:*

(i) $(\mathcal{H}, \pi)$ *is irreducible.*

(ii) $\pi(\mathcal{B})' = \mathbb{C} \cdot id_{\mathcal{H}}$, *where*

$$\pi(\mathcal{B})' = \{a \in \mathscr{L}(\mathcal{H}) \,:\, a\pi(b) = \pi(b)a \text{ for all } b \in \mathcal{B}\}$$

denotes the commutant of $\pi(\mathcal{B})$.

(iii) *Either every* $0 \neq \xi \in \mathcal{H}$ *is cyclic for* π, *or* π *is the null representation of dimension* 1.

Proof. See [31, Proposition 2.3.1] or [90, Theorem 5.1.5]. □

Proposition 3.1.3. *Let $\mathcal{B}$ be a C^*-algebra and $a \in \mathcal{B}$. Then there is an irreducible representation $(\mathcal{H}, \pi)$ of $\mathcal{B}$, such that $||a||_{\mathcal{B}} = ||\pi(a)||_{\mathcal{H}}$.*

Proof. See [90, Theorem 5.1.12] □

Lemma 3.1.4. *Let $\mathcal{B}$ be a C^*-algebra. Then the following properties are equivalent:*

(i) *$\mathcal{B}$ is commutative.*

(ii) *All irreducible representations of $\mathcal{B}$ have dimension one.*

Proof. Suppose that $\mathcal{B}$ is commutative and let $(\mathcal{H}, \varphi)$ be an arbitrary irreducible representation of $\mathcal{B}$. Then we get $\varphi(\mathcal{B})' = \mathbb{C} \cdot id_{\mathcal{H}}$ by 3.1.2 (ii). Since $\mathcal{B}$ is commutative, we have $\varphi(\mathcal{B}) \subseteq \varphi(\mathcal{B})'$ and therefore $\varphi(\mathcal{B}) = \mathbb{C} \cdot id_{\mathcal{H}}$. This implies $\dim \mathcal{H} = 1$, since φ has no non-trival invariant closed subspaces.

Now suppose that all irreducible representations of $\mathcal{B}$ have dimension one. Let $a, b \in \mathcal{B}$ be arbitrary. Then there is a representation $(\mathcal{H}, \varphi)$ of $\mathcal{B}$ with

$$||ab - ba||_{\mathcal{B}} = ||\varphi(ab - ba)||_{\mathcal{H}}.$$

Since φ is one-dimensional, it vanishes on commutators, i.e. $\varphi(ab - ba) = 0$. This implies $||ab - ba||_{\mathcal{B}} = 0$ and therefore $\mathcal{B}$ is commutative. □

The Jacobson topology

Definition 3.1.5. Let $\mathcal{B}$ be a C^*-algebra. Then a closed, two-sided ideal $\mathcal{I} \subseteq \mathcal{B}$ is said to be a *primitive ideal* if it is the kernel of an irreducible representation.

Let $\widehat{\mathcal{B}}$ be the space of all equivalence classes of irreducible representations of a C^*-algebra $\mathcal{B}$. If $(\mathcal{H}, \varphi)$ is a non-zero representation of $\mathcal{B}$, we denote its equivalence class in $\widehat{\mathcal{B}}$ by $[\mathcal{H}, \varphi]$ and we set $\ker [\mathcal{H}, \varphi] = \ker \varphi$. Furthermore, let $\mathrm{Prim}(\mathcal{B})$ be the set of all primitive ideals of the C^*-algebra $\mathcal{B}$. If $\mathcal{R}$ is a subset of $\mathcal{B}$, we denote by $\mathrm{Hull}(\mathcal{R})$ the set of all primitive ideals of $\mathcal{B}$ containing $\mathcal{R}$. If $\mathcal{T} \subseteq \mathrm{Prim}(\mathcal{B})$ is non empty, we define $\mathrm{Ker}(\mathcal{T})$ to be the intersection of all elements of $\mathcal{T}$ and we set $\mathrm{Ker}(\emptyset) = \mathcal{B}$. Then there is a unique topology on $\mathrm{Prim}(\mathcal{B})$, such that for each subset $\mathcal{R} \subseteq \mathcal{B}$, the set $\mathrm{Hull}(\ker(\mathcal{R}))$ is the closure of $\mathcal{R}$ with respect to this topology. This topology is called the *Jacobson* or *hull kernel topology* on $\mathrm{Prim}(\mathcal{B})$. The weakest topology making the (surjective) map

$$\theta : \widehat{\mathcal{B}} \longrightarrow \mathrm{Prim}(\mathcal{B}) : \ [\mathcal{H}, \pi] \longmapsto \ker \pi$$

continuous is called the *spectrum of* $\mathcal{B}$. This topology on $\widehat{\mathcal{B}}$ is also called the *Jacobson topology*. If $\mathcal{R}$ is a subset of $\mathcal{B}$, let $\mathrm{Hull}'(\mathcal{R}) := \theta^{-1}(\mathrm{Hull}(\mathcal{R}))$; see [31, Chapter 3] or [90, Chapter 5.4] for more details.

Example 3.1.6. Let Ω be a locally compact Hausdorff space. Then the point evaluation

$$\delta_\omega : \mathscr{C}_0(\Omega) \longrightarrow \mathbb{C} : \ f \longmapsto f(\omega)$$

induces a homeomorphism $\Omega \xrightarrow{\cong} \widehat{\mathscr{C}_0(\Omega)} : \omega \longmapsto \delta_\omega$.

Suppose that $\mathcal{I} \subseteq \mathcal{B}$ is a two-sided closed ideal in $\mathcal{B}$. Then we define:

$$\widehat{\mathcal{B}}^{\mathcal{I}} := \left\{\pi \in \widehat{\mathcal{B}} \,:\, \pi(\mathcal{I}) \neq \{0\}\right\} \quad \text{and} \quad \widehat{\mathcal{B}}_{\mathcal{I}} := \left\{\pi \in \widehat{\mathcal{B}} \,:\, \pi(\mathcal{I}) = \{0\}\right\}.$$

Lemma 3.1.7. *Let $\mathcal{B}$ be a C^*-algebra. Suppose that $\mathcal{I} \subseteq \mathcal{B}$ is a two-sided closed ideal in $\mathcal{B}$. Moreover, we endow the spectrum $\widehat{\mathcal{B}}$ of $\mathcal{B}$ with the Jacobson-topology. Then we have:*

(i) *The mapping $\widehat{\mathcal{B}}^{\mathcal{I}} \longrightarrow \widehat{\mathcal{I}} \,:\, \pi \longmapsto \pi_{|\mathcal{I}}$ is a homeomorphism.*

(ii) *The mapping*

$$\widehat{\mathcal{B}}_{\mathcal{I}} \longrightarrow \widehat{\mathcal{B}/\mathcal{I}} \,:\, [\pi : \mathcal{B} \to \mathscr{L}(\mathcal{H})] \longmapsto [\widehat{\pi} : \mathcal{B}/\mathcal{I} \to \mathscr{L}(\mathcal{H}) : b + \mathcal{I} \mapsto \pi(b)]$$

is a homeomorphism.

(iii) *$\widehat{\mathcal{B}}^{\mathcal{I}}$ is open and $\widehat{\mathcal{B}}_{\mathcal{I}}$ is closed in $\widehat{\mathcal{B}}$.*

Proof. See [31, Proposition 2.11.2, Proposition 3.2.1]. □

Lemma 3.1.8. *Let $\mathcal{B}$ be a C^*-algebra and $\mathcal{I} \subseteq \mathcal{B}$ a two-sided closed ideal in $\mathcal{B}$. Moreover, let $(\mathcal{H}, \varphi)$ be a representation of $\mathcal{I}$. Then there is a unique representation $(\mathcal{H}, \pi)$ of $\mathcal{B}$, which extends the representation $(\mathcal{H}, \varphi)$.*

Proof. See [31, Proposition 2.10.4]. □

The following theorem is certainly well known (see [65, Proposition 7.4.3] for an application of it). But as we could not find any bibliographical reference, we also give a proof of it.

Theorem 3.1.9. *Let $\mathcal{B}$ be a C^*-algebra. Let $\mathcal{I} \subseteq \mathcal{B}$ be a two-sided closed ideal in $\mathcal{B}$. Then if $\mathcal{I}$ and $\mathcal{B}/\mathcal{I}$ are commutative, we get $\mathcal{B}$ to be commutative, too.*

Proof. In view of 3.1.4 we only have to show, that all irreducible representations of $\mathcal{B}$ have dimension one. We have $\widehat{\mathcal{B}} = \widehat{\mathcal{B}}^{\mathcal{I}} \uplus \widehat{\mathcal{B}}_{\mathcal{I}}$, where all representations of $\widehat{\mathcal{B}}_{\mathcal{I}} \cong \widehat{\mathcal{B}/\mathcal{I}}$ are one-dimensional. Thus we only have to consider representations $\pi \in \widehat{\mathcal{I}} \cong \widehat{\mathcal{B}}^{\mathcal{I}}$, which are all one-dimensional due to the assumptions of the theorem. By 3.1.8 all these representations extend uniquely to representations of $\mathcal{B}$. Therefore all representations of $\widehat{\mathcal{B}}^{\mathcal{I}}$ have dimension one. The theorem follows. □

Hereditary C^*-subalgebras

To the end of this section let $\mathcal{A}$ always be a non-zero C^*-subalgebra of a C^*-algebra $\mathcal{B}$. Let us denote by $\mathcal{B}^+$ the set of all positive elements of $\mathcal{B}$. Remember that an element $b \in \mathcal{B}$ is called positive if b is hermitian and $\sigma(b) \subseteq [0, \infty[$.

Definition 3.1.10. $\mathcal{A}$ is said to be *hereditary* in $\mathcal{B}$ if for $b \in \mathcal{B}^+$ and $a \in \mathcal{A}^+$ the inequality $b \leq a$ implies $b \in \mathcal{A}$.

Theorem 3.1.11. *$\mathcal{A}$ is hereditary in $\mathcal{B}$ if and only if $aba' \in \mathcal{A}$ for all $a, a' \in \mathcal{A}$ and $b \in \mathcal{B}$. Especially every closed ideal $\mathcal{I}$ of a C^*-algebra is hereditary.*

Proof. See [90, Theorem 3.2.2]. $\square$

Let $(\mathcal{H}, \varphi)$ be a representation of $\mathcal{B}$ and suppose that K is a closed vector subspace of $\mathcal{H}$ invariant for $\varphi(\mathcal{A})$. Then the map

$$\varphi : \mathcal{A} \longrightarrow \mathscr{L}(K); \quad b \longmapsto \varphi(b)_{|K}$$

is a $*$-homomorphism. Let $(\mathcal{H}, \varphi)_{|\mathcal{A},K}$ denote the representation (K, ψ). In the case that $K = \overline{LH}\,\{\varphi(a)h \,:\, a \in \mathcal{A},\ h \in \mathcal{H}\}$ we simply write $(\mathcal{H}, \varphi)_{|\mathcal{A}}$.

Theorem 3.1.12. *Let $\mathcal{A}$ be hereditary in $\mathcal{B}$ and suppose that (H, φ) is an irreducible representation of $\mathcal{B}$. Then $(\mathcal{H}, \varphi)_{|\mathcal{A}}$ is an irreducible representation of $\mathcal{A}$. Moreover, $\varphi(\mathcal{A})\mathcal{H}$ is closed.*

Proof. See [90, Theorem 5.5.2]. $\square$

We call the maps

$$\begin{array}{rclcrcl} \mathrm{Prim}(\mathcal{B}) \setminus \mathrm{Hull}(\mathcal{A}) & \longrightarrow & \mathrm{Prim}(\mathcal{A}), & & \mathcal{I} & \longmapsto & \mathcal{I} \cap \mathcal{A}, \\ \widehat{\mathcal{B}} \setminus \mathrm{Hull}'(\mathcal{A}) & \longrightarrow & \widehat{\mathcal{A}}, & & [\mathcal{H}, \varphi] & \longmapsto & [(\mathcal{H}, \varphi)_{|\mathcal{A}}], \end{array} \tag{3.1.1}$$

the *canonical* ones. We then have the following theorem:

Theorem 3.1.13. *Let $\mathcal{A}$ be hereditary in $\mathcal{B}$. The following diagram is commutative, where the maps are the canonical ones:*

$$\begin{tikzcd} \widehat{\mathcal{B}} \setminus \mathrm{Hull}'(\mathcal{A}) \arrow[r] \arrow[d] & \widehat{\mathcal{A}} \arrow[d] \\ \mathrm{Prim}(\mathcal{B}) \setminus \mathrm{Hull}(\mathcal{A}) \arrow[r] & \mathrm{Prim}(\mathcal{A}). \end{tikzcd}$$

Moreover, the horizontal maps are homeomorphisms.

Proof. See [90, Theorem 5.5.5] or appendix D.2.1. $\square$

Remark 3.1.14. Clearly, 3.1.7 is a special case of 3.1.13, since closed ideals are always hereditary.

The following can be found in [61]:

Theorem 3.1.15. *Let $\mathcal{A} \subseteq \mathcal{B}$ be a Ψ^*-algebra, which is dense in the C^*-algebra $\mathcal{B}$. Then the map*

$$\Phi : \widehat{\mathcal{B}} \longrightarrow \widehat{\mathcal{A}} : \ [\pi] \longmapsto [\pi_{|\mathcal{A}}]$$

is a bijection.

Proof. See [61, Theorem 2.10]. $\square$

Here $[\pi]$ resp. $[\pi_{|\mathcal{A}}]$ denotes the equivalence class of the irreducible representation π resp $\pi_{|\mathcal{A}}$.

Definition 3.1.16. Let $\mathcal{A}$ be a Ψ^*-subalgebra of a C^*-algebra $\mathcal{B}$. Then $\mathcal{A}$ is said to have the *property* E_0 in $\mathcal{B}$ provided for each equivalence class $[\pi_0] \in \widehat{\mathcal{B}}$ and each open set $U \in \tau(\widehat{\mathcal{B}})$ with $[\pi_0] \in U$ there exists an $a \in \mathcal{A}$ satisfying $\pi_0(a) \neq 0$ and $\pi(a) = 0$ for all $[\pi] \notin U$.

Theorem 3.1.17. *Let $\mathcal{A}$ be a dense Ψ^*-subalgebra of a C^*-algebra $\mathcal{B}$. Then the following conditions are equivalent.*

(i) *$\mathcal{A}$ has the property E_0 in $\mathcal{B}$.*

(ii) *The bijection $\Phi : (\widehat{\mathcal{B}}, \tau(\widehat{\mathcal{B}})) \longrightarrow (\widehat{\mathcal{A}}, \tau(\widehat{\mathcal{A}}))$ is a homeomorphism.*

(iii) *If $\mathcal{I} \trianglelefteq \mathcal{B}$ is a closed ideal in $\mathcal{B}$, then $\mathcal{I} \cap \mathcal{A}$ is dense in $\mathcal{I}$.*

Proof. See [61, Theorem 3.3] □

Example 3.1.18. Recall that we stated in 1.1.5, that $\Psi^0_{cl}(Y, \Omega^{\frac{1}{2}}, \mathbb{R}^l)$ is a Ψ^*-algebra in $\mathcal{C}_b(\mathbb{R}^l, \mathscr{L}(L^2(Y, \Omega^{\frac{1}{2}})))$. $\Psi^0_{cl}(Y, \Omega^{\frac{1}{2}}, \mathbb{R}^l)$ also has the property E_0 in $\mathcal{B}(Y, \Omega^{\frac{1}{2}}, \mathbb{R}^l)$ by [61, Theorem 4.9 (b)].

3.2 Solvable C^*- and Ψ^*-algebras

Before giving the definition of solvability for Ψ^*-algebras, let us first recall the definition of solvability for C^*-algebras and discuss some examples.

Definition 3.2.1 (Solvable C^*-algebras)**.** Let $\mathcal{B}$ be a C^*-algebra.

(i) A *composition series* for $\mathcal{B}$ is a family $(\mathcal{I}_\beta)_{\beta \leq \alpha}$ of closed ideals $\mathcal{I}_\beta$ of $\mathcal{A}$ indexed by the ordinals β less or equal to a fixed ordinal α, such that

  (a) $\mathcal{I}_0 = \{0\}$, $\mathcal{I}_\alpha = \mathcal{B}$,

  (b) $\mathcal{I}_\gamma \subset \mathcal{I}_\beta$, if $\gamma < \beta \leq \alpha$,

  (c) if β is a limit ordinal, $\beta \leq \alpha$, we have $\mathcal{I}_\beta = \left(\bigcup_{\gamma<\beta} \mathcal{I}_\gamma\right)^-$.

(ii) $\mathcal{B}$ is called *solvable* if we have

$$\mathcal{I}_{\beta+1}/\mathcal{I}_\beta \cong \mathscr{C}_0(T_\beta, \mathcal{K}(H_\beta)),$$

where T_β is a locally compact Hausdorff space and H_β is a separable Hilbert space.

In the case that the index set is finite, this reduces to:

Definition 3.2.2 (C^*-case, [32])**.** Let $\mathcal{B}$ be a C^*-algebra. Then $\mathcal{B}$ is said to be *solvable* if there exists a finite sequence

$$\mathcal{B} := \mathcal{J}_{l+2} \supseteq \mathcal{J}_{l+1} \supseteq \ldots \supseteq \mathcal{J}_1 \supseteq \mathcal{J}_0 := \{0\} \tag{3.2.1}$$

of closed ideals, such that $\mathcal{J}_{k+1}/\mathcal{J}_k \cong \mathcal{C}_0(T_k, \mathcal{K}(H_k))$ for some locally compact Hausdorff space T_k and some separable Hilbert space H_k. Moreover, the composition series (3.2.1) is said to be *solving of length* l, and the smallest length of such a series is called the *length of* $\mathcal{B}$, denoted by $\mathfrak{l}(\mathcal{B})$.

Example 3.2.3. Let $\mathcal{Z}$ be a manifold with corners of dimension m. Define $\mathcal{I}_{m+1} := \ker {}^b\sigma_{\mathcal{B}}^{(0)}$ and for $l = 1, \ldots, m$ set

$$\mathcal{I}_l := \left\{ a \in \mathcal{I}_{l+1} \,:\, I_{F\mathcal{Z}}^{\mathcal{B}}(a) = 0, \forall\, F \in \mathcal{F}_l(\mathcal{Z}) \right\}.$$

Then we get the nested sequence

$$\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \supseteq \mathcal{I}_{m+1} \supseteq \mathcal{I}_m \supseteq \ldots \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_0 := \{0\} \tag{3.2.2}$$

and this is a solving composition series for $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ (cf. [86, Theorem 2.2]), which is solving of minimal length (cf. [60]). The partial quotients are given by the isomorphisms ${}^b\sigma_{\mathcal{B}}^{(0)} : \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})/\mathcal{I}_{m+1} \xrightarrow{\cong} \mathscr{C}({}^bS^*\mathcal{Z})$ and

$$\mathcal{I}_{l+1}/\mathcal{I}_l \cong \bigoplus_{F \in \mathcal{F}_l(\mathcal{Z})} \mathscr{C}_0\left(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{K}(L_b^2(F))\right), \quad 0 \le l \le m. \tag{3.2.3}$$

In particular we have $\mathcal{I}_1 = \mathcal{K}(L_b^2(\mathcal{Z}))$. See also 3.4.11 where we discuss this result.

Remark 3.2.4. The notion of solvable C^*-algebras has been introduced by Dynin [32] in the context of pseudodifferential operators. Upmeier gave in [111] an example of a solvable *Toeplitz* C^*-algebra, namely he proved, that for a bounded symmetric domain $D \subseteq Z$ of rank r, the Toeplitz C^*-algebra $\mathscr{T}$ generated by all Toeplitz operators with continuous symbol functions on the *Shilov* boundary S of D is solvable of length r. In this case the partial quotients $\mathscr{I}_{k+1}/\mathscr{I}_k$ of the solving series

$$\mathscr{T} = \mathscr{I}_{r+1} \supseteq \mathscr{I}_r \supseteq \ldots \supseteq \mathscr{I}_1 \supseteq \mathscr{I}_0 := \{0\}$$

are given by

$$\mathscr{I}_{k+1}/\mathscr{I}_k \cong \mathscr{C}(S_k) \otimes \mathcal{K}(H_k)$$

where S_k denotes the manifold of all tripotents $p \in Z$ (i.e. $pp^*p = p$ holds) of rank k (see also [50] for the case of *Hardy-Toeplitz* C^*-algebras).

Remark 3.2.5. Note, that to $l \in \mathbb{N}$ arbitrary, there exists a manifold $\mathcal{Z}_l$ with corners (of dimension l), such that the associated (solvable) C^*-algebra $\mathcal{B}_l := \mathcal{B}(\mathcal{Z}_l, {}^b\Omega^{\frac{1}{2}})$ of b-pseudodifferential operators has length l (cf. [63]). A simple example is $\mathcal{Z}_l := \mathcal{I}^l$, where $\mathcal{I} := [0, 1]$ denotes the unit interval.

Definition 3.2.6 (Ψ^*-case)**.** Let $\mathcal{A}$ be a Ψ^*-subalgebra of a C^*-algebra $\mathcal{C}$. Then $\mathcal{A}$ is said to be *solvable* if there exists a finite sequence

$$\mathcal{A} = \mathcal{I}_{l+2} \supseteq \mathcal{I}_{l+1} \supseteq \ldots \supseteq \mathcal{I}_1 = \{0\} \tag{3.2.4}$$

of two-sided closed (with respect to the topology on $\mathcal{A}$) ideals $\mathcal{I}_k \trianglelefteq \mathcal{A}$ ($k = 1, \ldots, l + 2$), such that the quotient spaces $\mathcal{I}_{k+1}/\mathcal{I}_k$ are isomorphic to a dense Ψ^*-subalgebra of $\mathscr{C}_0(T_k, \mathcal{K}(H_k))$, where the T_k are locally compact Hausdorff spaces and the H_k are separable Hilbert spaces ($k = 1, \ldots, l + 2$).

Again, we take the same definition for the length of a solvable Ψ^*-algebra as in the C^*-case. Then we have the following theorem:

Theorem 3.2.7. *Let $\mathcal{B}$ be a solvable C^*-algebra and $\mathcal{A}$ a dense Ψ^*-subalgebra of $\mathcal{B}$ with property E_0. Then $\mathcal{A}$ is a solvable Ψ^*-algebra. Moreover, $\mathfrak{l}(\mathcal{B}) \geq \mathfrak{l}(\mathcal{A})$ holds.*

Proof. Let

$$\mathcal{B} = \mathcal{J}_{n+2} \supseteq \mathcal{J}_{n+1} \supseteq \ldots \supseteq \mathcal{J}_1 := \{0\}$$

be a solving composition series for $\mathcal{B}$ with $\mathcal{J}_{k+1}/\mathcal{J}_k \cong \mathscr{C}_0(T_k, \mathcal{K}(H_k))$ where T_k resp. H_k is a locally compact Hausdorff resp. separable Hilbert space. By 3.1.17 (iii) $\mathcal{I}_k := \mathcal{A} \cap \mathcal{J}_k$ ($k = 1, \ldots, n + 2$) is a dense Ψ^*-algebra in $\mathcal{J}_k$ and as $\mathcal{J}_k$ is a closed ideal in $\mathcal{B}$, it is a C^*-(sub)algebra of $\mathcal{B}$ itself (conf. [90, 3.1.3]). Clearly, we have the nested sequence of two-sided ideals

$$\mathcal{A} = \mathcal{I}_{n+2} \supseteq \mathcal{I}_{n+1} \supseteq \ldots \supseteq \mathcal{I}_1 = \{0\}. \tag{3.2.5}$$

Since $\mathcal{J}_{k+1} \cap \mathcal{J}_k = \mathcal{J}_k$ holds, we get

$$\mathcal{I}_{k+1} \cap \mathcal{J}_k = \mathcal{A} \cap \mathcal{J}_{k+1} \cap \mathcal{J}_k = \mathcal{A} \cap \mathcal{J}_k = \mathcal{I}_k,$$

and therefore $\mathcal{I}_{k+1} \cap \mathcal{J}_k$ is dense in $\mathcal{J}_k$. But then $j_{k+1}(\mathcal{I}_{k+1}/\mathcal{I}_k) = j_{k+1}(\mathcal{I}_{k+1}/(\mathcal{I}_{k+1} \cap \mathcal{J}_k))$ is a Ψ^*-algebra in the C^*-algebra $\mathcal{J}_{k+1}/\mathcal{J}_k$ by 1.1.9 (iv), where $j_k : \mathcal{I}_{k+1}/(\mathcal{I}_{k+1} \cap \mathcal{J}_k) \hookrightarrow \mathcal{J}_{k+1}/\mathcal{J}_k$ denotes the embedding 1.1.9 (ii). As $j_{k+1}(\mathcal{I}_{k+1}/\mathcal{I}_k)$ is also dense in $\mathcal{J}_{k+1}/\mathcal{J}_k$, we get the solvability of $\mathcal{A}$.

Obviously the construction leading to (3.2.5) gives $\mathfrak{l}(\mathcal{A}) \leq \mathfrak{l}(\mathcal{B})$. □

Remark 3.2.8. Let $\mathcal{C}_k := \mathcal{I}_{k+1}/\mathcal{I}_k$, with $\mathcal{I}_k$ defined as in (3.2.5). Then combining the isomorphisms in 3.1.15 and in 3.1.6 leads to $\widehat{\mathcal{C}_k} \cong T_k$.

3.3 The length of $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$

Fix $F \in \mathcal{F}_k(\mathcal{Z})$ and set $\dim F = m - k := n$. Then the following result by Melrose and Nistor (cf. [86, Theorem 3, Corollary 3]) holds:

Theorem 3.3.1. *The algebra $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$ has a composition series*

$$\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)}) \supseteq \mathfrak{I}_{n+1} \supseteq \mathfrak{I}_n \supseteq \ldots \supseteq \mathfrak{I}_1 \supseteq \mathfrak{I}_0 := \{0\},$$

where $n = \dim F$. *Here* $\mathfrak{I}_{n+1} := \ker {}^b\tilde{\sigma}^{(0)}_{\mathcal{B}}$ *and* $\mathfrak{I}_l$ *denotes the ideal*

$$\mathfrak{I}_l := \left\{ a \in \mathfrak{I}_{l+1} \,:\, I^{\mathcal{B}}_{F\mathcal{Z}}(a) = 0 \,\forall\, F' \subseteq F,\ F' \in \mathcal{F}_l(F) \right\}.$$

The partial quotients are given by the isomorphisms

$${}^b\tilde{\sigma}^{(0)}_{\mathcal{B}} : \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})/\mathfrak{I}_{n+1} \xrightarrow{\cong} \mathscr{C}({}^bS^*\mathcal{Z}_{|F})$$

and

$$\mathfrak{I}_{l+1}/\mathfrak{I}_l \cong \bigoplus_{F' \in \mathcal{F}_l(F)} \mathscr{C}_0 \left(\mathbb{R}^{\mathcal{E}(F')}, \mathcal{K}(L^2_b(F', {}^b\Omega^{\frac{1}{2}})) \right) \ 0 \le l \le n. \tag{3.3.1}$$

Proof. See [86, Theorem 3] (see also the proof of 3.4.11). □

This theorem enables us to calculate the length of $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$. But before we can do this, we have to present some additional results of Lauter (see [63] and [64]): Let $\widehat{Q_{\mathcal{B}}(\mathcal{Z})}$ be the spectrum of the C^*-algebra $Q_{\mathcal{B}}(\mathcal{Z})$, the algebra of symbols of $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. Then there is a canonical bijective mapping

$$\Phi : \widehat{Q_{\mathcal{B}}(\mathcal{Z})} \longrightarrow T := {}^bS^*\mathcal{Z} \uplus \biguplus_{F \in \mathcal{F}(\mathcal{Z})} \mathbb{R}^{\mathcal{E}(F)} : [\pi_t] \longmapsto t,$$

where the irreducible representations for $\eta \in {}^bS^*\mathcal{Z}$, $\lambda \in \mathbb{R}^{\mathcal{E}(F_0)}$ and F_0 a face in $\mathcal{Z}$ are given by

$$\begin{aligned} \pi_\eta : Q_{\mathcal{B}}(\mathcal{Z}) \longrightarrow \mathbb{C} \quad &; \quad (f, (h_F)_{F \in \mathcal{F}(\mathcal{Z})}) \longmapsto f(\eta) \\ \pi_{F_0,\lambda} : Q_{\mathcal{B}}(\mathcal{Z}) \longrightarrow \mathscr{L}(L^2_b(F_0, {}^b\Omega^{\frac{1}{2}})) \quad &; \quad (f, (h_F)_{F \in \mathcal{F}(\mathcal{Z})}) \longmapsto h_{F_0}(\lambda). \end{aligned}$$

Let $p : {}^bS^*\mathcal{Z} \longrightarrow \mathcal{Z}$ be the canonical projection. Set

$$\mathfrak{D}^\wedge_{\gamma_0}(D, \mathbb{R}^l) := \{\gamma d \,:\, \gamma > \gamma_0,\ d \in D\},$$

where $\emptyset \ne D \subseteq \mathbb{R}^l$ is a bounded set and $\gamma_0 \ge 0$ is arbitrary. We define the following topology $\mathcal{T}$ on T:

- Let $\zeta \in {}^bS^*\mathcal{Z}$ with $p(\zeta) \notin \partial Z$. We denote with $\mathfrak{U}(\zeta)$ the family of all open sets $U \subseteq {}^bS^*\mathcal{Z}$ with $\zeta \in U$ and $p(U) \cap \partial\mathcal{Z} = \emptyset$.
- Let $\zeta_0 \in {}^bS^*\mathcal{Z}$ with $p(\zeta_0) \in G \in \mathcal{F}_k(\mathcal{Z})$ and k maximal. Choose local coordinates $\zeta = (x, (x_H)_{H \in \mathcal{E}(G)}, y, (\xi_H)_{H \in \mathcal{E}(G)}, \eta)$ near ζ_0 with $\zeta_0 = (0, y^{(0)}, (\xi^{(0)}_H)_{H \in \mathcal{E}(G)}, \eta^{(0)})$. Then we denote by $\mathfrak{U}(\zeta_0)$ all sets of the form

  $$U \uplus \biguplus_{G \subseteq F \in \mathcal{F}(\mathcal{Z})} \mathfrak{D}^\wedge_{\gamma_F}(D_F, \mathbb{R}^{\mathcal{E}(F)}),$$

  where

  (i) $U \subseteq {}^bS^*\mathcal{Z}$ is open with $\zeta_0 \in U$,

(ii) $p(U) \cap H \neq \emptyset$, $H \in \mathcal{F}_1(\mathcal{Z})$ holds for all $H \in \mathcal{E}(F)$,

(iii) $\gamma_F \geq 0$ and $D_F \subseteq \mathbb{R}^{\mathcal{E}(F)}$ is open and bounded with $(\xi_H^{(0)})_{H\in\mathcal{E}(F)} \in D_F$.

- For $\lambda = (\lambda_H)_{H\in\mathcal{E}(G)} \in \mathbb{R}^{\mathcal{E}(G)}$ let

$$\mathfrak{U}(G,\lambda) := \left\{ \biguplus_{G\subseteq F\in\mathcal{F}(\mathcal{Z})} \prod_{H\in\mathcal{E}(F)} V_H \;:\; V_H \subseteq \mathbb{R} \text{ open with } \lambda_H \in V_H \right\} \tag{3.3.2}$$

We have the following theorem (cf. [64, Theorem 5.4, Theorem 5.5]):

Theorem 3.3.2. *Let $T_0 := T \uplus \{[id]\}$. Then the canonical bijections $\Phi : \widehat{Q_{\mathcal{B}}(\mathcal{Z})} \longrightarrow T$ resp. $\Phi_0 : \mathcal{B}(\widehat{\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}}) \longrightarrow T_0$ are homeomorphisms provided $\widehat{Q_{\mathcal{B}}(\mathcal{Z})}$ resp. $\mathcal{B}(\widehat{\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}})$ are endowed with the Jacobson topology, and T with the topology $\mathcal{T}$ above.*

The Jacobson-topology of $\mathcal{B}(\widehat{\mathcal{Z},{}^b\Omega^{\frac{1}{2}}})$ is determined by

$$\{\emptyset\} \cup \left\{ \{[id]\} \cup \{[\rho\circ\tau_{\mathcal{B}}] \;:\; [\rho] \in V\} \;:\; V \subseteq \widehat{Q_{\mathcal{B}}(\mathcal{Z})} \text{ open} \right\},$$

so we get all irreducible representations of $\mathcal{B}(\widehat{\mathcal{Z},{}^b\Omega^{\frac{1}{2}}})$:

$$id \;:\; \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathscr{L}(L^2_b(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})), \tag{3.3.3}$$

$$\pi_\eta \circ \tau_{\mathcal{B}} \;:\; \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathbb{C}, \quad (\eta \in {}^bS^*\mathcal{Z}), \tag{3.3.4}$$

$$\pi_{F_0,\lambda} \circ \tau_{\mathcal{B}} \;:\; \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathscr{L}(L^2_b(F_0, {}^b\Omega^{\frac{1}{2}})),\ \lambda \in \mathbb{R}^{\mathcal{E}(F_0)}. \tag{3.3.5}$$

Note, that we have

$$\begin{aligned} \pi_\eta \circ \tau_{\mathcal{B}}(a) &= {}^b\sigma^{(0)}_{\mathcal{B}}(a)(\eta) \quad \text{and} \\ \pi_{F_0,\lambda} \circ \tau_{\mathcal{B}}(a) &= I^{\mathcal{B}}_{F_0\mathcal{Z}}(a)(\lambda). \end{aligned}$$

From now on let $F \in \mathcal{F}_k(\mathcal{Z})$ be a fixed boundary surface of dimension $n = m - k$. We want to study the quotient $\mathcal{B}/\mathcal{J}_F$, where $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) =: \mathcal{B}$ and $\mathcal{J}_F$ denotes the kernel of the (extended) indical family

$$I^{\mathcal{B}}_{FZ} : \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)}).$$

Recall that the kernel of $I_{F\mathcal{Z}}$ is given by $\varrho_F \Psi^0_{b,cl}(\mathcal{Z}, {}^0\Omega^{\frac{1}{2}})$, where ϱ_F denotes the product of all defining functions ϱ_H with $H \in \mathcal{E}(F)$ (cf. [85, Lemma 1]) and that we have the following result (cf. [64, Lemma 3.1]):

Lemma 3.3.3. *For each $F \in \mathcal{F}(\mathcal{Z})$ the extended indical family*

$$I^{\mathcal{B}}_{F\mathcal{Z}} : \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$$

is onto.

Consequently, we get the following exact sequence

$$0 \longrightarrow \mathcal{J}_F \longrightarrow \mathcal{B} \xrightarrow{I^{\mathcal{B}}_{F\mathcal{Z}}} \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)}) \longrightarrow 0. \tag{3.3.6}$$

This in orchestra with 3.1.7 leads to

$$\Big(\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})\Big)^{\widehat{}} \cong \widehat{\mathcal{B}/\mathcal{J}_F} \cong \widehat{\mathcal{B}}_{\mathcal{J}_F}, \tag{3.3.7}$$

where again $\widehat{\mathcal{B}}_{\mathcal{J}_F} = \Big\{\pi \in \widehat{\mathcal{B}} : \pi(\mathcal{J}_F) = \{0\}\Big\}$. It follows that we can give an explicit description of all irreducible elements of the spectrum of $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$:

Theorem 3.3.4. *The following representations of $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$ are irreducible and pairwise inequivalent:*

$$\pi_\eta \circ \tau_{\mathcal{B}} \;:\; \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)}) \longrightarrow \mathbb{C}, \tag{3.3.8}$$
$$\pi_{F_0,\lambda} \circ \tau_{\mathcal{B}} \;:\; \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)}) \longrightarrow \mathscr{L}(L^2_b(F_0, {}^b\Omega^{\frac{1}{2}})), \tag{3.3.9}$$

where $F_0 \subseteq F$ is a face with $\operatorname{codim} F_0 \geq \operatorname{codim} F$ *and $\eta \in {}^bS^*\mathcal{Z}_{|F}$. Moreover, any irreducible representation of $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$ is equivalent to one of them.*

Proof. According to (3.3.7) we just have to check, which representations of $\mathcal{B}$ actually vanish on $\mathcal{J}_F$. First, let $\eta \in {}^bS^*\mathcal{Z}_{|F}$ be arbitrary. Then by proposition 2.1.4 we get, that

$${}^b\tilde{\sigma}^{(0)}_{\mathcal{B}}(I^{\mathcal{B}}_{F\mathcal{Z}}(a)) = {}^b\sigma^{(0)}_{\mathcal{B}}(a)_{|{}^bS^*\mathcal{Z}_{|F}},$$

i.e. for each $\eta \in S^*\mathcal{Z}_{|F}$ it holds $\pi_\eta \circ \tau_{\mathcal{B}}(a) = {}^b\sigma^{(0)}_{\mathcal{B}}(a)(\eta) = 0$ for all $a \in \mathcal{J}_F$. Now, let F_0 be a face with $\operatorname{codim} F_0 \geq \operatorname{codim} F$ and $F_0 \subseteq F$. Again the compatibility conditions in 2.1.4 yield

$$I^{\mathcal{B}}_{F_0\mathcal{Z}}(a)(\lambda) = I^{\mathcal{B}}_{F_0F}(I^{\mathcal{B}}_{F\mathcal{Z}}(a)((\lambda_H)_{H\in\mathcal{E}(F)}))((\lambda_H)_{H\in\mathcal{E}(F_0)\setminus\mathcal{E}(F)})$$

for all $\lambda \in \mathcal{E}(F_0)$. But this implies

$$\pi_{F_0,\lambda} \circ \tau_{\mathcal{B}}(a) = I^{\mathcal{B}}_{F_0\mathcal{Z}}(a)(\lambda) = 0$$

for all $a \in \mathcal{J}_F$ and finishes the proof. □

Lemma 3.3.5. *Let $k_0 := \max\{k : \mathcal{F}_k(F) \neq \emptyset\}$. Then we have:*

(i) $\mathfrak{l}\Big(\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})\Big) \leq n$, *if $k_0 = n$,*

(ii) $\mathfrak{l}\Big(\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})\Big) \leq k_0 + 1$ *if $k < n$.*

Proof. (i) We take a closer look at the partial quotients in (3.3.1): We have

$$\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})/\mathfrak{I}_{n+1} \cong \mathscr{C}_0({}^bS^*\mathcal{Z}_{|F}) \text{ and } \mathfrak{I}_{n+1}/\mathfrak{I}_n \cong \bigoplus_{F'\in\mathcal{F}^0(F)} \mathscr{C}_0(\mathbb{R}^n),$$

since $\mathcal{K}(L^2_b(F')) = \mathbb{C}$ if F' has dimension 0. Thus $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})/\mathfrak{I}_n$ is commutative by 3.1.9 and (3.3.1) is solvable with length n.

(ii) By the definition of k_0, we have $\mathcal{F}_l(F) = \emptyset$ if $l \geq k_0 + 1$. We get

$$\mathfrak{I}_{n+1} = \ldots = \mathfrak{I}_{k_0+1} \supseteq \mathfrak{I}_{k_0} \supseteq \ldots \supseteq \mathfrak{I}_1 \supseteq \{0\},$$

which is a solving composition series of length $k_0 + 1$. This proves (ii). □

For a C^*-algebra $\mathcal{Q}$ we define

$$(\widehat{\mathcal{Q}}^{\mathcal{I}_2})_{\mathcal{I}_1} := \widehat{\mathcal{Q}}^{\mathcal{I}_2} \cap \widehat{\mathcal{Q}}_{\mathcal{I}_1},$$

where $\mathcal{I}_1 \subseteq \mathcal{I}_2$ are both two-sided closed ideals in $\mathcal{Q}$. The following lemma can be found in [63, Lemma 3.4] (for the convenience of the reader, we will also present the proof).

Lemma 3.3.6. *Let $\mathcal{Q}$ be a C^*-algebra and $\mathcal{I}_1, \mathcal{I}_2$ two closed ideals of Q with $\mathcal{I}_1 \subseteq \mathcal{I}_2$. Then the map*

$$\begin{array}{rcl} \varphi : (\widehat{\mathcal{Q}}^{\mathcal{I}_2})_{\mathcal{I}_1} & \longrightarrow & \widehat{\mathcal{I}_2/\mathcal{I}_1} \\ [\pi] & \longmapsto & [x + \mathcal{I}_1 \longmapsto \pi(x)] \end{array}$$

induces a homeomorphism provided $(\widehat{\mathcal{Q}}^{\mathcal{I}_2})_{\mathcal{I}_1}$ is endowed with the topology induced by the inclusion $(\widehat{\mathcal{Q}}^{\mathcal{I}_2})_{\mathcal{I}_1} \hookrightarrow \widehat{\mathcal{Q}}$.

Proof. By 3.1.7 (i) the restriction map $\chi : \widehat{\mathcal{Q}}^{\mathcal{I}_2} \longrightarrow \widehat{\mathcal{I}}_2$ is a homeomorphism and we have $\chi((\widehat{\mathcal{Q}}^{\mathcal{I}_2})_{\mathcal{I}_1}) = \widehat{\mathcal{I}}_{2\mathcal{I}_1} \cong \widehat{\mathcal{I}_2/\mathcal{I}_1}$. Moreover, $\varphi = \chi_{|(\widehat{\mathcal{Q}}^{\mathcal{I}_2})_{\mathcal{I}_1}}$ holds; this completes the proof. □

Lemma 3.3.7. *Let $k_0 := \max\{k : \mathcal{F}_k(F) \neq \emptyset\}$. Then we have:*

(i) $\mathfrak{l}\Big(\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})\Big) \geq k_0$, *if $k_0 = n$,*

(ii) $\mathfrak{l}\Big(\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})\Big) \geq k_0 + 1$ *if $k < n$.*

The proof of this lemma follows closely the proof of [63, Lemma 3.5]. To shorten notation, we set $\mathcal{B}_F := \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})$.

Proof. Let $\mathcal{B}_F =: \mathcal{I}_{l+1} \supseteq \mathcal{I}_l \supseteq \cdots \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_0 = \{0\}$ be an arbitrary solving composition series of length l, where the subquotients are given by $\mathcal{I}_{j+1}/\mathcal{I}_j \cong \mathscr{C}_0(T_j, \mathcal{K}(H_j))$, $j = 0, \ldots, l$. Then we have a canonical bijective map $\psi : \widehat{\mathcal{B}_F} \longrightarrow \biguplus_{j=0}^{l} T_j$. By [31, Proposition 3.2.2] we get the following increasing sequence of open subsets of $\mathcal{B}_F$:

$$\emptyset = \widehat{\mathcal{B}_F}^{\mathcal{I}_0} \subseteq \widehat{\mathcal{B}_F}^{\mathcal{I}_1} \subseteq \cdots \subseteq \widehat{\mathcal{B}_F}^{\mathcal{I}_{l+1}} = \widehat{\mathcal{B}_F}.$$

Let $G \in \mathcal{F}_{k_0}(F)$ be arbitrary. We have $G \in \mathcal{F}_{k_0+k}(\mathcal{Z})$ and thus

$$\mathcal{E}(G) = \{H_1, \ldots, H_{k_0}, H_{k_0+1}, \ldots, H_{k_0+k}\} = \{1, \ldots, k_0, k_0+1, \ldots, k_0+k\}$$

is a set of $k_0 + k$ elements. Now, fix $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{k_0}) \in \mathbb{R}^{\mathcal{E}(G)}$, where $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k_0$ and $\lambda_0 \in \mathbb{R}^k$. By (3.3.2) we get $\mu_0 := \lambda_0 \in \widehat{\mathcal{B}_F}$, $\mu_1 := (\lambda_0, \lambda_1) \in \widehat{\mathcal{B}_F}$, $\mu_2 := (\lambda_0, \lambda_1, \lambda_2) \in \widehat{\mathcal{B}_F}$ and so on. Moreover, there exist $l_0, l_1 \in \{1, \ldots, l+1\}$,m such that $\mu_0 \in (\widehat{\mathcal{B}_F}^{\mathcal{I}_{l_0}})_{\mathcal{I}_{l_0-1}}$ and $\mu_2 \in (\widehat{\mathcal{B}_F}^{\mathcal{I}_{l_1}})_{\mathcal{I}_{l_1-1}}$.

Suppose that $l_1 < l_0$. We know that $\widehat{\mathcal{B}_F}^{\mathcal{I}_{l_1}}$ is open, thus there exist open sets $V_0 \subseteq \mathbb{R}^k$ and $V_1 \subseteq \mathbb{R}$ with $\lambda_i \in V_i$ $(i = 0, 1)$, such that

$$(V_0 \times V_1) \uplus V_0 \uplus V_1 \subseteq \widehat{\mathcal{B}_F}^{\mathcal{I}_{l_1}} \subseteq \widehat{\mathcal{B}_F}^{\mathcal{I}_{l_0-1}}$$

by (3.3.2) which contradicts our choice of l_0. Now, we assume that $l_0 = l_1$, i.e. $\mu_0, \mu_1 \in (\widehat{\mathcal{B}_F}^{\mathcal{I}_{l_0}})_{\mathcal{I}_{l_0-1}}$. By 3.3.6 the space $(\widehat{\mathcal{B}_F}^{\mathcal{I}_{l_0}})_{\mathcal{I}_{l_0-1}}$ is Hausdorff in the relative topology, which is a contradiction to the fact that we have $\mu_0 \in W_0 \cap W_1$ for any choice of open sets $W_i \subseteq \widehat{\mathcal{B}_F}$ with $\mu_i \in W_i$.

Thus we get an increasing sequence $1 \leq l_0 \leq \ldots \leq l_{k_0} \leq l + 1$, i.e. $l \geq k_0$ and this completes the proof in case of $k_0 = n$.

Now suppose that $k_0 < n$. Then again by 3.3.6 we have

$$\psi\left(\biguplus_{k=0}^{k_0}\left(\widehat{\mathcal{B}_F}^{\mathcal{I}_{j_k}}\right)_{\mathcal{I}_{j_k-1}}\right) = \biguplus_{k=0}^{k_0} T_{j_k-1} \subseteq \biguplus_{j=1}^{l+1} T_j,$$

where each T_{j_k} corresponds to an infinite-dimensional representation μ_k. Adding the one-dimensional representations (cf. (3.3.8)) of $\widehat{\mathcal{B}_F}$ we get $k_0 + 1 < l$. The lemma follows. □

Combining the two above lemmata yields the following theorem:

Theorem 3.3.8. *Let $F \in \mathcal{F}^n(\mathcal{Z})$ be arbitrary and*

$$k_0 := \max\{k \,:\, \mathcal{F}_k(F) \neq \emptyset\}.$$

Then we have

$$\mathfrak{l}(\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^{\mathcal{E}(F)})) = \begin{cases} n, & \text{if } k_0 = n \\ k_0 + 1, & \text{if } k_0 < n. \end{cases}$$

Remark 3.3.9. Let F be a manifold with corners of dimension n and $k \in \mathbb{N}$ be arbitrary. Then using an appropriate Cartesian product construction we can embed F into a manifold with corners $\mathcal{Z}$, such that F is a face of codimension $k = \mathcal{E}(F)$ in $\mathcal{Z}$. Thus we can also calculate the length of solving series for $\mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}, \mathbb{R}^k)$.

3.4 The local length of $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$

Preliminaries

We now want to give the notion of *local length* for certain classes of solvable C^*-algebras. To this end let $\mathcal{Z}$ be a Banach space with measure μ and $\mathcal{B}$ be a C^*-subalgebra of $\mathscr{L}(L^2(\mathcal{Z}, \mu))$ with $M_\varphi \in \mathcal{B}$ for all $\varphi \in \mathscr{C}_c^\infty(\mathcal{Z})$.

Definition 3.4.1. Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{B}$. Moreover, let $\mathcal{B}$ be solvable. Then every solving series of ideals $(\mathcal{I}_k)_k$ for $\mathcal{B}$ induces an ideal chain $(\mathfrak{I}_k)_k$, where $\mathfrak{I}_k := \mathcal{I}_k \cap \mathcal{A}$, for $\mathcal{A}$. If $(\mathfrak{I}_k)_k$ is not trivial we say that $(\mathfrak{I}_k)_k$ is the *associated ideal chain* to $(\mathcal{I}_k)_k$ with respect to $\mathcal{A}$.

Definition 3.4.2. Let $\varphi \in \mathscr{C}_c^\infty(\mathcal{Z})$ be arbitrary. Let $\mathcal{B}_\varphi$ be the C^*-closure of the algebra

$$\mathfrak{B} := \{b \in \mathcal{B} \,:\, b = M_\varphi a M_\varphi,\ a \in \mathcal{B}\} \subseteq \mathcal{B}$$

with respect to C^*-norm of $\mathcal{B}$.

Proposition 3.4.3. *Let $\varphi \in \mathscr{C}_c^\infty(\mathcal{Z})$. Then we have:*

(i) *$\mathcal{B}_\varphi$ is a C^*-subalgebra of $\mathcal{B}$.*

(ii) *$\mathcal{B}_\varphi$ is hereditary in $\mathcal{B}$.*

(iii) *Suppose that $\mathcal{I}$ is a closed two-sided ideal of $\mathcal{B}$, then $\mathfrak{I} := \mathcal{I} \cap \mathcal{B}_\varphi$ is a C^*-subalgebra of $\mathcal{B}$. Moreover, $\mathfrak{I}$ is a closed two-sided ideal in $\mathcal{B}_\varphi$ and therefore a C^*-algebra in $\mathcal{B}_\varphi$ itself.*

Proof. (i) This follows easily, since $\mathcal{B}$ is selfadjoint and $ab \in \mathcal{B}$ holds for all $a, b \in \mathcal{B}$.
(ii) In view of 3.1.11 we have to show that $b_1 b b_2 \in \mathcal{B}_\varphi$ holds for all $b_1, b_2 \in \mathcal{B}_\varphi$ and $b \in \mathcal{B}$. First of all let $c_1, c_2 \in \mathfrak{B}$, then $c_1 = M_\varphi a_1 M_\varphi$ resp. $c_2 = M_\varphi a_2 M_\varphi$ with $a_1, a_2 \in \mathcal{B}$. Since $M_\varphi, a_1, a_2, b \in \mathcal{B}$ we have $a_1 M_\varphi b M_\varphi a_2 \in \mathcal{B}$, which shows $c_1 b c_2 \in \mathfrak{B}$, so $\mathfrak{B}$ has the desired property. Now if $b_1, b_2 \in \mathcal{B}_\varphi$, there are c_1^n and c_2^n with $c_i^n \xrightarrow{n\to\infty} b_i$ $(i = 1, 2)$ with respect to the C^*-norm on $\mathcal{B}$. Moreover, we have

$$\begin{aligned} ||c_1^n b c_2^n - b_1 b b_2||_{\mathcal{B}} &\leq ||c_1^n b c_2^n - c_1^n b b_2||_{\mathcal{B}} + ||c_1^n b b_2 - b_1 b b_2||_{\mathcal{B}} \\ &\leq ||c_1^n||_{\mathcal{B}} ||b||_{\mathcal{B}} ||(c_2^n - b_2)||_{\mathcal{B}} \\ &\qquad + ||(c_1^n - b_1)||_{\mathcal{B}} ||b||_{\mathcal{B}} ||b_2||_{\mathcal{B}} \\ &\xrightarrow{n\to\infty} 0. \end{aligned}$$

This shows $b_1 b b_2 \in \mathcal{B}_\varphi$, since $c_1^n b c_2^n \in \mathfrak{B}$ for all $n \in \mathbb{N}$.
(iii) Clearly $\mathfrak{I}$ is selfadjoint and a C^*-algebra itself. To see that $\mathfrak{I}$ is closed with respect to $\mathcal{B}$, let $(a_k)_{k\in\mathbb{N}} \subseteq \mathfrak{I}$ be an arbitrary series, with $a_k \xrightarrow{k\to\infty} a \in \mathcal{B}$. Since $\mathcal{I}$ and $\mathcal{B}_\varphi$ are closed with respect to the C^*-norm on $\mathcal{B}$, we get $a \in \mathcal{I} \cap \mathcal{B}_\varphi$, and $\mathfrak{I}$ is closed, too. The rest of (iii) follows by a calculation analogous to (ii). □

Definition 3.4.4. Let $\mathcal{B}$ be solvable and $(\mathcal{I}_j)_{j=0}^n$ for $\mathcal{B}$ a solving series for $\mathcal{B}$ of minimal length. Then $\mathcal{B}$ has *local length* l_0 in $p \in \mathcal{Z}$ with respect to $(\mathcal{I}_j)_{j=0}^n$, if there exists an open neighbourhood $U \subseteq \mathcal{Z}$ of p, such that for all cut off functions $\varphi \in \mathscr{C}_0^\infty(\mathcal{Z})$, with $\operatorname{supp}\varphi \subset U$ and $\varphi \equiv 1$ in a neighbourhood $W \subset V$ of p, the length of the associated ideal-chain to $(\mathcal{I}_j)_{j=0}^n$ with respect to the C^*-subalgebra $\mathcal{B}_\varphi$, cf. 3.4.1, is l_0. We denote the local length of $\mathcal{B}$ in p with respect to $(\mathcal{I}_j)_{j=0}^n$ by $\mathfrak{l}_p(\mathcal{B}, (\mathcal{I}_j)_{j=0}^n)$.

The following theorem is certainly well-known and implicitly given in [63, Lemma 3.1].

Theorem 3.4.5. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B} \subseteq \mathscr{L}(\mathcal{H})$ be a C^*-algebra with $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}$. Suppose that $\mathcal{B}$ is solvable and $\mathcal{B} := \mathcal{I}_{n+1} \supseteq \mathcal{I}_n \supseteq \ldots \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_0 := \{0\}$ is an arbitrary solving series for $\mathcal{B}$, then $\mathcal{I}_n = \mathcal{K}(\mathcal{H})$ holds.*

Proof. Let $\mathcal{B} := \mathcal{I}_{n+1} \supseteq \mathcal{I}_n \supseteq \ldots \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_0 = \{0\}$ be an arbitrary solving series for $\mathcal{B}$. Since $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}$, the representation id of $\mathcal{B}$ on $\mathcal{H}$ is irreducible. By the definition of the Jacobson topology the set $\widehat{\mathcal{B}}^{\mathcal{K}(\mathcal{H})} = \{[id]\}$ is dense in $\mathcal{B}$. This gives $\widehat{\mathcal{B}}^{\mathcal{K}(\mathcal{H})} \subseteq \widehat{\mathcal{B}}^{\mathcal{I}_1} \cong \widehat{\mathcal{I}_1}$, since $\widehat{\mathcal{B}}^{\mathcal{I}_1}$ is open. Moreover, $\widehat{\mathcal{I}_1} = \widehat{\mathcal{I}_1/\mathcal{I}_0}$ is a locally compact Hausdorff space, because the compositions series is solving. Therefore $\widehat{\mathcal{B}}^{\mathcal{K}(\mathcal{H})}$ is dense and relatively closed in $\widehat{\mathcal{B}}^{\mathcal{I}_1}$, which implies $\widehat{\mathcal{B}}^{\mathcal{K}(\mathcal{H})} = \widehat{\mathcal{B}}^{\mathcal{I}_1}$. This gives $\mathcal{K}(\mathcal{H}) = \mathcal{I}_1$ by [31, Proposition 3.2.2]. $\square$

Remark 3.4.6. From 3.4.5 we see, that $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}$ implies that the associated ideal chain to a C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}$ is non-trivial: Remember that a closed ideal $\mathcal{I}$ in a C^*-algebra $\mathcal{B}$ is called *essential* if $a\mathcal{I} = 0 \Rightarrow a = 0$. It is well-known from general C^*-theory ([90, page 82]), that

$$\mathcal{I} \text{ is essential} \quad \Longleftrightarrow \quad \mathcal{I} \cap \mathcal{J} \neq 0 \text{ for all non-zero closed ideals } \mathcal{J} \text{ in } \mathcal{B}$$

holds. Especially one can show that the ideal of compact operators $\mathcal{K}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is an essential ideal in $\mathscr{L}(\mathcal{H})$ (see [90, Example 3.1.2] for instance).

Local length for manifolds with corners

Now let $\mathcal{Z}$ again be a manifold with corners of dimension m and $p \in \mathcal{Z}$ be arbitrary. Recall, that there is $F \in \mathcal{F}_{l_0}(\mathcal{Z})$ with l_0 maximal, such that $p \in F$. We want to calculate the local length of the C^*-algebra $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ in p with respect to a solving series given by Melrose and Nistor [86, Theorem 2].

To simplify notation, let $\mathcal{B}$ denote $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. We choose a neighbourhood U of $p \in F$, such that

$$\chi : \mathcal{Z} \supseteq U \longrightarrow \overline{\mathbb{R}}_+^{\mathcal{E}(F)} \times \mathbb{R}^{m-l_0}$$

is a diffeomorphism with $\chi(p) = (0,0)$.

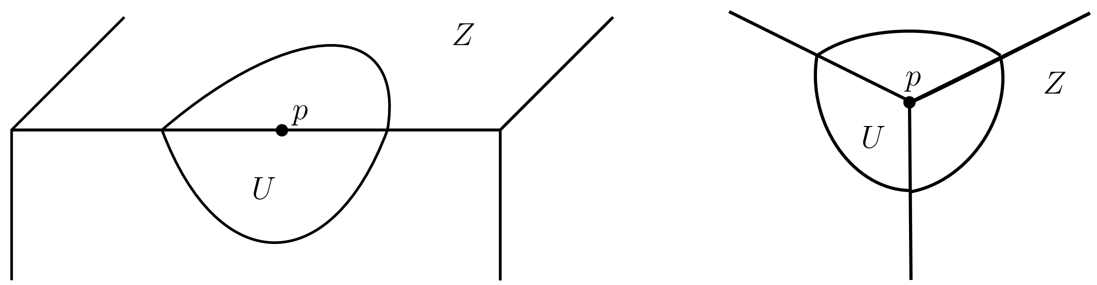

Figure 3.1: Localization at p with different codimensions.

First of all let us construct certain elements in $\mathcal{B}$ and in $\mathcal{B}^-(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$. To this end let $\varphi \in \mathscr{C}_0^\infty(\mathcal{Z})$ with $\operatorname{supp}\varphi \subseteq U$, $\varphi \equiv 1$ in an open neighbourhood $W \subseteq U$ of p. Fix $G \in \mathcal{F}_l(\mathcal{Z})$ with $\operatorname{supp}\varphi \cap G \neq \emptyset$. Note that this implies $F \subseteq G$ by the choice of l_0, i.e. $l_0 \leq l$. Set $V := \overset{\circ}{G} \cap W$, then $V \neq \emptyset$ and V is open.

Furthermore, let $\zeta_0 \in {}^bS^*\mathcal{Z}$ be with $\pi(\zeta_0) \in V$, then $\pi^{-1}(V)$ is an open neighbourhood of $\zeta_0 \in {}^bS^*\mathcal{Z}$. Let $\mathbb{R}_+^{\mathcal{E}(G)} \times \mathbb{R}^{m-l} \times \mathbb{R}^{\mathcal{E}(G)} \times \mathbb{R}^{m-l}$ be local coordinates of ${}^bT^*\mathcal{Z}$ near ζ_0 with $\zeta_0 = (0, y^{(0)}, (\xi_H^{(0)})_{H\in\mathcal{E}(G)}, \eta^{(0)})$. Choose $\mathcal{V} \subseteq \pi^{-1}(V)$ open, and for each $K \in \mathcal{F}(\mathcal{Z})$ with $G \subseteq K$ let $\gamma_K \geq 0$ and $D_K \subseteq \mathbb{R}^{\mathcal{E}(K)}$ be an open and bounded set with $(\xi_H^{(0)})_{H\in\mathcal{E}(K)} \in D_K$.

Proposition 3.4.7. *There exists $a \in \mathcal{B}$, such that*

(i) ${}^b\sigma_{\mathcal{B}}^{(0)}(\varphi a \varphi) \neq 0$ *and* $\operatorname{supp}{}^b\sigma_{\mathcal{B}}^{(0)}(a) \subseteq \mathcal{V}$,

(ii) $I_{K\mathcal{Z}}^{\mathcal{B}}(a) = 0$ *for all* $K \in \mathcal{F}(\mathcal{Z})$ *with* $G \cap K \neq G$ *and*

(iii) $I_{K\mathcal{Z}}^{\mathcal{B}}(a)(\lambda) = 0$ *for all* $K \in \mathcal{F}(\mathcal{Z})$ *with* $G \subseteq K$ *and all* $\lambda \notin \mathfrak{D}_{\gamma_K}^\wedge(D_K, \mathbb{R}^{\mathcal{E}(F)})$.

Proof. Assume that we have already constructed such an operator a and let $\zeta \in \mathcal{V}$ be arbitrary. Then we would get

$$ {}^b\sigma_{\mathcal{B}}^{(0)}(\varphi a \varphi)(\zeta) = {}^b\sigma_{\mathcal{B}}^{(0)}(a)(\zeta), $$

since $\pi(\zeta) \in V$ and $\varphi \equiv 1$ on V. Thus it is enough to construct an operator $a \in \mathcal{B}$ such that $\operatorname{supp}{}^b\sigma_{\mathcal{B}}^{(0)}(a) \subseteq \mathcal{V}$ and $|{}^b\sigma_{\mathcal{B}}^{(0)}(\zeta_0)| > 0$. But this follows using [64, Proposition 4.4]. □

Proposition 3.4.8. *Let $\lambda = (\lambda_H)_{H\in\mathcal{E}(G)} \in \mathbb{R}^{\mathcal{E}(G)}$ be arbitrary and $V_H \subseteq \mathbb{R}$ be open with $\lambda_H \in V_H$ for all $H \in \mathcal{E}(G)$. Then there exists $a \in \mathcal{B}^-(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$ with*

(i) ${}^b\sigma_{\mathcal{B}}^{(0)}(a) = 0$ *and* $0 \neq I_{GZ}^{\mathcal{B}}(M_\varphi a M_\varphi)$,

(ii) $I_{K\mathcal{Z}}^{\mathcal{B}}(a) = 0$ *for all* $K \in \mathcal{F}(Z)$ *with* $G \cap K \neq G$, *and*

(iii) $\operatorname{supp} I_{K\mathcal{Z}}^{\mathcal{B}}(a) \subseteq \prod_{H\in\mathcal{E}(F)} V_H$ *for all* $K \in \mathcal{F}(Z)$ *with* $G \subseteq K$.

Proof. This proposition follows from [64, Proposition 4.6] if we ensure, that (i) can be achieved by the construction given there. Let

$$ \widetilde{\chi} : \mathcal{Z} \supseteq V \longrightarrow \overline{\mathbb{R}}_+^{\mathcal{E}(G)} \times \mathbb{R}^{m-l} $$

be a local model for G. Choose $L \subseteq \mathbb{R}^{m-l}$ compact with $L \subset \operatorname{supp}(\varphi_{|G} \circ \widetilde{\chi}^{-1})$ and a function

$$ h_G \in \mathscr{C}_c^\infty(\mathbb{R}^{\mathcal{E}(G)}, \mathscr{C}^\infty(\mathbb{R}^{m-l} \times \mathbb{R}^{m-l})) $$

with $\operatorname{supp} h_G \subseteq \prod_{h\in\mathcal{E}(G)} V_H$ and $\operatorname{supp} h_G(\xi) \subseteq L \times L$ for all ξ. Now the construction in [64] of the corresponding kernel resp. the corresponding operator a yields the proposition. □

First let us note that

$$
\begin{aligned}
{}^b\sigma_{\mathcal{B}}^{(0)}(M_\varphi a M_\varphi) &= (\varphi\circ\pi)\,{}^b\sigma_{\mathcal{B}}^{(0)}(a)\,(\varphi\circ\pi) \text{ and} && (3.4.1)\\
I^{\mathcal{B}}_{GZ}(M_\varphi a M_\varphi) &= \varphi_{|G}\, I^{\mathcal{B}}_{G\mathcal{Z}}(a)\,\varphi_{|G}, \quad G\in\mathcal{F}(\mathcal{Z}) && (3.4.2)
\end{aligned}
$$

holds for all $a\in\Psi^0_{b,cl}(\mathcal{Z},{}^b\Omega^{\frac12})$ and all $\varphi\in\mathscr{C}^\infty_c(\mathcal{Z})$, where $\pi : {}^bS^*\mathcal{Z}\longrightarrow\mathcal{Z}$ denotes the canonical projection. This gives

$$\pi(\eta)\notin\operatorname{supp}\varphi \Rightarrow \pi_\eta\circ\tau_{\mathcal{B}}(M_\varphi a M_\varphi) = {}^b\sigma_{\mathcal{B}}^{(0)}(M_\varphi a M_\varphi)(\eta) = 0, \tag{3.4.3}$$

and

$$\varphi_{|G}\equiv 0 \Rightarrow \pi_{G,\lambda}\circ\tau_{\mathcal{B}}(M_\varphi a M_\varphi) = I^{\mathcal{B}}_{G\mathcal{Z}}(M_\varphi a M_\varphi)(\lambda) = 0, \tag{3.4.4}$$

for all $\lambda\in\mathbb{R}^{\mathcal{E}(G)}$ and all $a\in\Psi^0_{b,cl}(\mathcal{Z},{}^b\Omega^{\frac12})$. Especially, if we have $\operatorname{supp}\varphi\subseteq U$ with $\varphi\equiv 1$ in $W\subseteq V$ open and $G\in\mathcal{F}_l(\mathcal{Z})$ with $l_0+1\le l$, then (3.4.4) holds. Note that by the density of $\Psi^0_{b,cl}(\mathcal{Z},{}^b\Omega^{\frac12})$ in $\mathcal{B}$ (3.4.1) and (3.4.2) also hold for $\mathcal{B}$.

Lemma 3.4.9. *Let $\varphi\in\mathscr{C}^\infty_0(\mathcal{Z})$ and $G\in\mathcal{F}(\mathcal{Z})$ be with $G\cap\operatorname{supp}\varphi=\emptyset$. Then $I^{\mathcal{B}}_{G\mathcal{Z}}(a)=0$ holds for all $a\in\mathcal{B}_\varphi$.*

Proof. Let $a\in\mathcal{B}_\varphi$ be arbitrary. Then there exists $a_k\in\mathfrak{B}$ with $a_k\xrightarrow{k\to\infty}a$. By definition $a_k=M_\varphi b_k M_\varphi$ for a suitable $b_k\in\mathcal{B}$. Since $\mathcal{B}$ is the closure of $\Psi^0_{b,cl}(\mathcal{Z},{}^b\Omega^{\frac12})$ there exists $c^k_j\in\Psi^0_{b,cl}(\mathcal{Z},{}^b\Omega^{\frac12})$ with $c^k_j\xrightarrow{j\to\infty}b_k$ for all $k\in\mathbb{N}$. Thus

$$I^{\mathcal{B}}_{G\mathcal{Z}}(a) = \lim_{k\to\infty} I^{\mathcal{B}}_{G\mathcal{Z}}(\varphi b_k\varphi) = \lim_{k\to\infty}\lim_{j\to\infty} I_{G\mathcal{Z}}(\varphi c^k_j\varphi) = 0$$

holds by (3.4.4). The lemma follows. □

By 3.4.7 and 3.4.8 there are operators $a,b\in\mathcal{B}_\varphi$, such that $\pi_\eta\circ\tau_{\mathcal{B}}(a)\neq 0$ and $\pi_{\eta'}\circ\tau_{\mathcal{B}}(a)=0$ $(\eta\neq\eta')$ resp. $\pi_{G,\lambda}\circ\tau_{\mathcal{B}}(b)\neq 0$ and $\pi_{G',\lambda'}\circ\tau_{\mathcal{B}}(b)=0$ $(\lambda\neq\lambda'$ or $G\neq G')$. So we can give a description of all inequivalent irreducible representations of $\widehat{\mathcal{B}_\varphi}$ via the homeomorphism $\widehat{\mathcal{B}}\setminus\operatorname{Hull}'(\mathcal{B}_\varphi)\cong\widehat{\mathcal{B}_\varphi}$:

Proposition 3.4.10. *The representations $\pi_\eta\circ\tau_{\mathcal{B}}$, where $\eta\in{}^bS^*\mathcal{Z}$ with $\pi(\eta)\in\operatorname{supp}\varphi$, $\pi_{G,\lambda}\circ\tau_{\mathcal{B}}$, where $G\in\mathcal{F}(\mathcal{Z})$ with $\varphi_{|G}\not\equiv 0$, and id of $\widehat{\mathcal{B}_\varphi}$ are irreducible and pairwise inequivalent. Moreover, any irreducible representation of $\widehat{\mathcal{B}_\varphi}$ is equivalent to one of them.*

Now let $\mathcal{I}_{m+1} := \ker{}^b\sigma^{(0)}_{\mathcal{B}}$ and

$$\mathcal{I}_l := \left\{a\in\mathcal{I}_{l+1} \,:\, I^{\mathcal{B}}_{F,\mathcal{Z}}(a)=0,\ \forall\, F\in\mathcal{F}_l(\mathcal{Z})\right\} \quad (l=1,\ldots,m).$$

We will calculate the local length in p with respect to the following solving series:

Theorem 3.4.11. *The nested sequence*

$$\mathcal{B}(\mathcal{Z},{}^b\Omega^{\frac12})\supseteq\mathcal{I}_{m+1}\supseteq\mathcal{I}_m\supseteq\ldots\supseteq\mathcal{I}_1\supseteq\mathcal{I}_0:=\{0\}, \tag{3.4.5}$$

where $m = \dim \mathcal{Z}$, *is a solving composition series for* $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})$, *which is solving of minimal length. The partial quotients are given by the isomorphisms*

$$ {}^b\sigma_{\mathcal{B}}^{(0)} : \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})/\mathcal{I}_{m+1} \xrightarrow{\cong} \mathscr{C}({}^bS^*\mathcal{Z}) $$

and

$$ \mathcal{I}_{l+1}/\mathcal{I}_l \cong \bigoplus_{F\in\mathcal{F}_l(\mathcal{Z})} \mathscr{C}_0\left(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{K}(L^2_b(F, {}^b\Omega^{\frac{1}{2}}))\right) \ 0 \leq l \leq m. \tag{3.4.6} $$

In particular we have $\mathcal{I}_1 = \mathcal{K}(L^2_b(\mathcal{Z}))$.

The solving result has first been proven by Melrose and Nistor in [86, Theorem 2.2]. The fact, that the series is solving of minimal length has been proven by Lauter in [60].

Proof. That the principal symbol map ${}^b\sigma_{\mathcal{B}}^{(0)} : \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathscr{C}({}^bS^*\mathcal{Z})$ is onto is due to the facts, that $\mathscr{C}^\infty({}^bS^*\mathcal{Z}) \subseteq \mathrm{r}\,({}^b\sigma_{\mathcal{B}}^{(0)})$ by (2.1.1) and that $\mathrm{r}\,({}^b\sigma_{\mathcal{B}}^{(0)})$ is closed. This establishes the first isomorphism.

Now, let $a \in \mathcal{I}_l$ and $F \in \mathcal{F}_{n-l}(\mathcal{Z})$ be arbitrary. By (2.1.6) we have

$$ I^{\mathcal{B}}_{F\mathcal{Z}}(a) \in \mathscr{C}_b(\mathbb{R}^{\mathcal{E}(F)}, \mathscr{L}(L^2_b(F, {}^b\Omega^{\frac{1}{2}}))). $$

The compatibility condition (2.1.8) gives

$$ 0 = {}^b\sigma_{\mathcal{B}}^{(0)}(a)_{|{}^bS^*\mathcal{Z}_{|F}} = {}^b\tilde{\sigma}_{\mathcal{B}}^{(0)}(I^{\mathcal{B}}_{F\mathcal{Z}}(a)), $$

thus by (2.1.7) we conclude $I^{\mathcal{B}}_{F\mathcal{Z}}(a) \in \mathscr{C}_0(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{B}^-(F, {}^b\Omega^{\frac{1}{2}}))$. The definition of $\mathcal{I}_l$ together with the exact sequence (2.1.4) for the manifold with corners F

$$ 0 \longrightarrow \mathcal{K}(L^2_b(F, {}^b\Omega^{\frac{1}{2}})) \longrightarrow \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}}) \longrightarrow Q_{\mathcal{B}}(F) \longrightarrow 0 $$

finally yields $I^{\mathcal{B}}_{F\mathcal{Z}}(a) \in \mathscr{C}_0(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{K}(L^2_b(F, {}^b\Omega^{\frac{1}{2}})))$. Moreover, the map

$$ \mathcal{I}_l \longrightarrow \bigoplus_{F\in\mathcal{F}_{n-k}(\mathcal{Z})} \mathscr{C}_0(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{K}(L^2_b(F, {}^b\Omega^{\frac{1}{2}}))) $$

is onto with kernel $\mathcal{I}_{l+1}$ and we have finished the proof of the solving result.

The fact, that the series is solving of minimal length is a combination of the arguments in the proof of 3.3.5 and the proof of 3.3.7 hence we will omit it. □

Theorem 3.4.12. *Let* $p \in \mathcal{Z}$ *be arbitrary. Then we have:*

(i) *The local length in* p *with respect to (3.4.5) is given by*

$$ \mathfrak{l}_p(\mathcal{B}, (\mathcal{I}_l)_{l=0}^m) = \begin{cases} m, & \text{if } l_0 = m, \\ l_0 + 1, & \text{if } l_0 < m. \end{cases} $$

(ii) *The function* $\mathfrak{l}_{.}(\mathcal{B}, (\mathcal{I}_l)_{l=0}^m) : \mathcal{Z} \longrightarrow \mathbb{N}$ *is upper semi-continuous.*

Proof. (i) Let $\varphi \in \mathscr{C}_0^\infty(\mathcal{Z})$ with $\operatorname{supp}\varphi \subset U$, $\varphi \equiv 1$ in a neighbourhood $W \subseteq V$ of p be arbitrary. Set $\mathfrak{I}_k := \mathcal{I}_k \cap \mathcal{B}_\varphi$. Then we have $I^{\mathcal{B}}_{G\mathcal{Z}}(\varphi a \varphi) = 0$ for all $G \in \mathcal{F}^k(\mathcal{Z})$ where $k < m - l_0 - 1$ by 3.4.9. This gives

$$\mathfrak{I}_0 = \mathfrak{I}_1 = \cdots = \mathfrak{I}_{m-l_0} \supset \mathfrak{I}_{m-l_0+1} \supset \ldots \supset \mathfrak{I}_m,$$

which is a series of ideals of length $m - (m - l_0) + 1 = l_0 + 1$ (by 3.4.7 and 3.4.8 we have $\mathfrak{I}_{m-l_0} \supset \mathfrak{I}_{m-l_0+1}$ etc.), which proofs the assertion in the case $l_0 < m$. Now let $l_0 = m$. Then by the same argument as in the proof of 3.3.5, $\mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})/\mathcal{I}_1$ is commutative, so (3.4.5) is solving of length m. Thus the length of the associated series $(\mathfrak{I}_k)_k$ is m, too. Since φ has been arbitrary (i) follows.
(ii) Let $\alpha \in \mathbb{R}$ be arbitrary. We have to show, that $\mathfrak{L}_\alpha := \{p \in \mathcal{Z} : \mathfrak{l}_p(\mathcal{B}, (\mathcal{I}_l)_{l=0}^m) < \alpha\}$ is open in $\mathcal{Z}$. Since $\mathfrak{l}_p(\mathcal{B}, (\mathcal{I}_l)_{l=0}^m) \in \mathbb{N}$ for all $p \in \mathcal{Z}$, we can restrict ourself to the case $\alpha \in \mathbb{N}$.

(a) If $\alpha \geq m + 1$, then there is nothing to prove.

(b) Let $\alpha \leq m$. Now, if $l_0 \geq \alpha - 1$, then

$$\begin{array}{rcll} \mathfrak{l}_q(\mathcal{B}, (\mathcal{I}_l)_{l=0}^m) = m & \geq & \alpha & \text{if} \quad m = l_0 \text{ resp.} \\ \mathfrak{l}_q(\mathcal{B}, (\mathcal{I}_l)_{l=0}^m) = l_0 + 1 & \geq & \alpha & \text{if} \quad l_0 < m \end{array}$$

holds for all $q \in H$, where $H \in \mathcal{F}_{l_0}(\mathcal{Z})$. We conclude

$$\mathfrak{l}_q(\mathcal{B}, (\mathcal{I}_l)_{l=0}^m) \geq \alpha \quad \Longleftrightarrow \quad q \in H,\ H \in \mathcal{F}_l(\mathcal{Z}) \text{ with } l \geq \alpha - 1$$

and since every $H \in \mathcal{F}(\mathcal{Z})$ is closed with respect to the topology of $\mathcal{Z}$,

$$\mathfrak{D} := \bigcup_{\substack{H \in \mathcal{F}_l(\mathcal{Z}) \\ l \geq \alpha - 1}} H = \{q \in \mathcal{Z} : \mathfrak{l}_q(\mathcal{B}, (\mathcal{I}_l)_{l=0}^m \geq \alpha\}$$

is closed, too. Consequently $\mathfrak{L}_\alpha = \complement\mathfrak{D}$ is open. The combination of (a) and (b) shows, that $\mathfrak{l}_\cdot(\mathcal{B}, (\mathcal{I}_l)_{l=0}^m) : \mathcal{Z} \longrightarrow \mathbb{N}$ is upper semi-continuous. $\square$

Chapter 4

Infinite solving series

4.1 Constructing infinite ideal chains on direct sums of C^*-algebras

It is clear from our previous discussion, that to given $k \in \mathbb{N}$ there exists a manifold $\mathcal{Z}_k$ with corners (of dimension k), such that the associated (solvable) C^*-algebra $\mathcal{B}_k := \mathcal{B}(\mathcal{Z}_k, {}^b\Omega^{\frac{1}{2}})$ of b-pseudodifferential operators has length k. In the sequel let $(\mathcal{B}_k)_{k\in\mathbb{N}}$ be such a fixed family of operator algebras. We will use the following infinite product construction for C^*-algebras:

Proposition 4.1.1. *Let $(\mathcal{A}_\lambda)_{\lambda\in\Lambda}$ be a family of C^*-algebras.*

(i) *The direct sum $\mathcal{A} = \bigoplus_{\lambda\in\Lambda} \mathcal{A}_\lambda$ of all $(a_\lambda) \in \prod_{\lambda\in\Lambda} \mathcal{A}_\lambda$ such that $||(a_\lambda)|| := \sup_\lambda ||a_\lambda|| < \infty$ is a C^*-algebra under the pointwise-defined operations and involution, and the norm given by*

$$(a_\lambda) \longmapsto ||(a_\lambda)||.$$

(ii) *The restricted sum $\mathcal{A}_{c_0} := \bigoplus_{\lambda\in\Lambda}^{c_0} \mathcal{A}_\lambda$ of all ellements $(a_\lambda) \in \mathcal{A}$, such that for each $\varepsilon > 0$ there exits a finite subset Ω of Λ for which $||a_\lambda|| < \varepsilon$ if $\lambda \in \Lambda \setminus \Omega$, is a closed self adjoint ideal in $\mathcal{A}$.*

Definition 4.1.2. Set $\mathcal{B} := \bigoplus_{k\in\mathbb{N}} \mathcal{B}_k$ to be the direct sum of C^*-algebras defined as above. Then every $\mathcal{B}_k$ is a closed two-sided ideal in $\mathcal{B}$ (under the usual inclusion).

The following proposition shows, that one can construct a solving series for the infinite product in the case that all component algebras are solvable:

Proposition 4.1.3. *Let $(\mathcal{I}_i^k)_{i\in\mathbb{N}}$ be a solving series of length k for $\mathcal{B}_k$ ($k \in \mathbb{N}$). Then we define $(\mathcal{J}_j)_{j\in\mathbb{N}}$ as follows:*

(i) $\mathcal{J}_0 := \bigoplus_{k\in\mathbb{N}} \{0\}$.

(ii) *Let $j \in \mathbb{N}$, then there is $l \in \mathbb{N}$, such that $\sum_{i=1}^{l} i =: h \leq j < \sum_{i=1}^{l+1} i$. Set $m := j - h$ and define $\mathcal{J}_j := \bigoplus_{k\in\mathbb{N}} \mathcal{C}_k$, where*

$$\mathcal{C}_k := \begin{cases} \mathcal{B}_k & \text{if } k \leq l \\ \mathcal{I}_m^k & \text{if } k = l+1 \\ \{0\} & \text{if } k > l+1 \end{cases}.$$

Then $(\mathcal{J}_j)_{j\in\mathbb{N}}$ is a solving series for $\mathcal{B}$.

Proof. Let $j \in \mathbb{N}$ be arbitrary. Let h and m be chosen as above. Then $\mathcal{J}_j$ is given by

$$\mathcal{J}_j = \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \ldots \oplus \mathcal{B}_m \oplus \mathcal{I}_m^k \oplus \{0\},$$

where $k = j + 1 - h$ (and we suppress the infinite sum of $\{0\}$ at the end). This gives:

$$\mathcal{J}_{j+1}/\mathcal{J}_j \cong \mathcal{I}_k^{m+1}/\mathcal{I}_k^m \cong \mathscr{C}_0(T_m^k, \mathcal{K}(H_m^k)).$$

Note, that $\mathcal{J}_0 = \{0\}$. □

Before we prove that $\mathcal{B}$ can not be solvable of finite length, we state some general results on C^*-algebras (see for instance [90]).

Lemma 4.1.4. *Let $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{J}$ be closed two-sided ideals in a C^*-algebra $\mathcal{B}$ with $\mathcal{I}_2 \subseteq \mathcal{I}_1$. Moreover, let $\mathcal{I}$ be a closed ideal in $\mathcal{J}$.*

(i) *We have $\mathcal{I}_k \cap \mathcal{J} = \mathcal{I}_k\mathcal{J}$ $(k = 1, 2)$.*

(ii) *$(\mathcal{I}_1 \cap \mathcal{J})/(\mathcal{I}_2 \cap \mathcal{J})$ is a closed two sided ideal in $\mathcal{I}_1/\mathcal{I}_2$.*

(iii) *$\mathcal{I}$ is an ideal in $\mathcal{B}$.*

Proof. (i) We have to prove $\mathcal{I}_k \cap \mathcal{J} \subseteq \mathcal{I}_k\mathcal{J}$. Let a be a positive element in $\mathcal{I}_k \cap \mathcal{J}$. In particular, we have $a^{\frac{1}{2}} \in \mathcal{I}_k \cap \mathcal{J}$. Choose an approximate unit $(\mu_\lambda)_{\lambda\in\Lambda}$ for $\mathcal{I}_k$, then

$$a = \lim_\lambda (\mu_\lambda a^{\frac{1}{2}})a^{\frac{1}{2}}$$

holds. Since $\mu_\lambda a^{\frac{1}{2}} \in \mathcal{I}_k$ for all λ, we get $a \in \mathcal{I}_k\mathcal{J}$, which completes the proof.
(ii) Let us first note how to include $(\mathcal{I}_1 \cap \mathcal{J})/(\mathcal{I}_2 \cap \mathcal{J})$ injectively in $\mathcal{I}_1/\mathcal{I}_2$. To shorten notation we set $\mathcal{J}_k := \mathcal{I}_k \cap \mathcal{J}$ $(i = 1, 2)$. Then we have the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_2 \arrow[r] & \mathcal{I}_1 \arrow[r] & \mathcal{I}_1/\mathcal{I}_2 \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{J}_2 \arrow[r] \arrow[u, hook, "id"] & \mathcal{J}_1 \arrow[r] \arrow[u, hook, "id"] & \mathcal{J}_1/\mathcal{J}_2 \arrow[r] \arrow[u, hook, "\tau"] & 0,
\end{tikzcd}$$

where τ is defined by

$$\tau : \mathcal{J}_1/\mathcal{J}_2 \ni j_1 + \mathcal{J}_2 \longmapsto j_1 + \mathcal{I}_2 \in \mathcal{I}_1/\mathcal{I}_2$$

and all inclusions are given with respect to the same topology. Clearly τ is well defined and a $*$-algebra homomorphism between two C^*-algebras (note that τ is continuous since $||j_1 + I_2|| \leq ||j_1 + J_2||$), therefore $\tau(\mathcal{J}_1/\mathcal{J}_2)$ is a C^*-subalgebra of $\mathcal{I}_1/\mathcal{I}_2$. It remains to prove the injectivity of τ. Let $j_1 + \mathcal{J}_2 \in \mathcal{J}_1/\mathcal{J}_2$ be with $\tau(j_1 + \mathcal{J}_2) = 0$. Then $j_1 + \mathcal{I}_2 = 0$ by the definition of τ, and therefore $j_1 \in \mathcal{I}_2$ follows. Since $j_1 \in \mathcal{J}$, we get $j_1 \in \mathcal{I}_2 \cap \mathcal{J} = \mathcal{J}_2$. We conclude that τ is injective and suppress this mapping in the following to shorten notation. It is left to prove the ideal property. Let $i_1 + \mathcal{I}_2 \in \mathcal{I}_1/\mathcal{I}_2$ and $j_1 + \mathcal{J}_2 \in \mathcal{J}_1/\mathcal{J}_2$ be arbitrary. Then we have

$$(i_1 + \mathcal{I}_2)(j_1 + \mathcal{J}_2) = i_1 j_1 + i_1\mathcal{J}_2 + \mathcal{I}_2 j_1 + \mathcal{I}_2\mathcal{J}_2 \in \mathcal{J}_1/\mathcal{J}_2,$$

since $\mathcal{I}_2 j_1 \subseteq \mathcal{I}_2\mathcal{J}_1 = \mathcal{I}_2 \cap \mathcal{J}_1 = \mathcal{I}_2 \cap \mathcal{I}_1 \cap \mathcal{J} = \mathcal{J}_2$. Using this, the ideal property is now a straightforward calculation.
(iii) First of all note, that $\mathcal{I}$ is linearly spanned by its set of positive elements $\mathcal{I}^+$ and $\mathcal{J}$ has an approximate unit $(\mu_\lambda)_{\lambda\in\Lambda}$. Let $a \in \mathcal{I}$ be positive and $b \in \mathcal{B}$ be arbitrary. Then $a^{\frac{1}{2}} \in \mathcal{I}$ and thus

$$\lim_\lambda \mu_\lambda a^{\frac{1}{2}} = a^{\frac{1}{2}}.$$

This shows, that

$$ba = \lim_\lambda b\mu_\lambda a^{\frac{1}{2}} a^{\frac{1}{2}},$$

i.e. $ba \in \mathcal{I}$ since $\mathcal{I}$ is an ideal in $\mathcal{J}$. We conclude $b^*a \in \mathcal{I}$, so $ab \in \mathcal{I}$. □

The following theorem is certainly well-known, but since we could not find a bibliographical reference, we will prove it here.

Theorem 4.1.5. *Let $\mathcal{L}$ be a closed two-sided ideal in the C^*-algebra $\mathscr{C}_0(T, \mathcal{K}(H))$, where T is a locally compact Hausdorff space and H is a separable Hilbert space. Then $\mathcal{L} \cong \mathscr{C}_0(T', \mathcal{K}(H))$, where $T' \subseteq T$ is locally compact and Hausdorff.*

Proof. To shorten notation, let $\mathscr{C}_0 := \mathscr{C}_0(T, \mathcal{K}(H))$. First we observe that $T \cong \widehat{\mathscr{C}_0}$ via the point evaluation $w \longmapsto \delta_w$ (we will use this identification in the sequel without any comment). Since $\mathcal{L}$ is a two-sided ideal in $\mathscr{C}_0$, the set

$$\widehat{\mathscr{C}_0}_{\mathcal{L}} := \left\{\pi \in \widehat{\mathscr{C}_0} \,:\, \pi(\mathcal{L}) = \emptyset\right\}$$

is closed in $\widehat{\mathscr{C}_0}$ by [31, Proposition 2.11.2, Proposition 3.2.1]. By [90, Theorem 5.4.3] we know, that every (proper) closed two-sided ideal $\mathcal{I}$ in a C^*-algebra $\mathcal{A}$ is the intersection of all primitive ideals[1] that contain it. Thus we have

$$\mathcal{L} = \bigcap_{\mathcal{L}\subseteq\ker\delta_w} \ker\delta_w = \bigcap_{\mathcal{L}\subseteq\ker\delta_w} \{f \in \mathcal{C}_0 \,:\, 0 = \delta_\omega(f) = f(\omega),\, \omega \in T\}$$

and

$$\begin{aligned}
\widehat{\mathscr{C}_0}_{\mathcal{L}} &= \left\{\delta_w \in \widehat{\mathscr{C}_0} \,:\, \delta_w(\mathcal{L}) = \{0\}\right\} \\
&= \{w \in T \,:\, \delta_w(\mathcal{L}) = \{0\}\} \\
&= \{w \in T \,:\, \mathcal{L} \subseteq \ker\delta_w\}.
\end{aligned}$$

Now, we show that $\mathcal{L} \cong \mathscr{C}_0(T', \mathcal{K}(H))$, where $T' := T\backslash\widehat{\mathscr{C}_0}_{\mathcal{L}}$. Clearly, T' is a locally compact Hausdorff space. Let us first remark, that we can regard $\mathscr{C}_0(T', \mathcal{K}(H))$ as a closed two-sided ideal in $\mathscr{C}_0(T, \mathcal{K}(H))$ if we set $f(x) = 0$ for $x \in T \setminus T'$, where $f \in \mathscr{C}_0(T', \mathcal{K}(H))$.

[1] Recall, that an ideal $\mathcal{I}$ is called primitive, if it is the kernel of an irreducible representation.

Thus $f \in \mathscr{C}_0(T', \mathcal{K}(H))$ implies, that $f \in \mathscr{C}_0(T, \mathcal{K}(H))$ with $f(\omega) = 0$ for all $\omega \in \widehat{\mathscr{C}_{0\mathcal{L}}}$. But then $f \in \ker \delta_\omega$ for all $\omega \in \widehat{\mathscr{C}_{0\mathcal{L}}}$, which shows that $f \in \mathcal{L}$. Now, let

$$R : \mathscr{C}_0(T, \mathcal{K}(H)) \longrightarrow \mathscr{C}(T', \mathcal{K}(H)) : \quad f \longmapsto f_{|T'},$$

be the restriction operator to T'. Then $R : \mathcal{L} \longrightarrow \mathscr{C}_0(T', \mathcal{K}(H))$ is a linear $*$-isomorphism: Let us first show, that R maps $\mathcal{L}$ to $\mathscr{C}_0(T', \mathcal{K}(H))$. For this let $f \in \mathcal{L}$ be arbitrary. We have to prove that $R(f) = f_{|T'} \in \mathscr{C}_0(T', \mathcal{K}(H))$, i.e. that for all $\varepsilon > 0$ the set $\big\{\omega \in T' \,:\, ||f_{|T'}(\omega)|| \geq \varepsilon\big\}$ is compact. But

$$\big\{\omega \in T' \,:\, ||f_{|T'}(\omega)|| \geq \varepsilon\big\} = \{\omega \in T \,:\, ||f(\omega)|| \geq \varepsilon\}$$

since $f(\omega) = 0$ for all $\omega \in \widehat{\mathscr{C}_{0\mathcal{L}}} = \complement T'$ and this set is compact by the fact that $f \in \mathscr{C}_0(T, \mathcal{K}(H))$.

Now R is surjective by our previous remark, so it is left to show injectivity: Let $f \in \mathcal{L}$ be with $R(f) = 0$, i.e. $f_{|T'} = 0$. Again by

$$\begin{aligned} f \in \mathcal{L} &\iff f(\omega) = 0 \text{ for all } \omega \in T \text{ such that } \mathcal{L} \subseteq \ker \delta_\omega, \\ &\iff f(\omega) = 0 \text{ for all } \omega \in \widehat{\mathscr{C}_{0\mathcal{L}}}, \end{aligned}$$

and $T = T' \cup \widehat{\mathscr{C}_{0\mathcal{L}}}$, we get $f \equiv 0$ and R is injective. □

Again, the following proposition seems to be well-known:

Proposition 4.1.6. *Let $\mathcal{A}$ be a closed two-sided ideal in a C^*-algebra $\mathcal{C}$. If $\mathcal{C}$ is solvable then $\mathcal{A}$ is solvable, too. In particular, we have $\mathfrak{l}(\mathcal{C}) \geq \mathfrak{l}(\mathcal{A})$.*

Proof. Let

$$\mathcal{C} := \mathcal{I}_{n+1} \supseteq \mathcal{I}_n \supseteq \ldots \supseteq \mathcal{I}_1 \supseteq \mathcal{I}_0 = \{0\}$$

be a solving series for $\mathcal{C}$. Set $\mathcal{J}_k := \mathcal{I}_k \cap \mathcal{A}$. Then $(\mathcal{J}_k)_{k=1}^n$ is a nested sequence of two-sided closed ideals in $\mathcal{A}$ and by 4.1.4 (ii) $\mathcal{J}_k/\mathcal{J}_{k-1}$ is a two-sided closed ideal in $\mathcal{I}_k/\mathcal{I}_{k-1} \cong \mathscr{C}_0(T_k, \mathcal{K}(H_k))$. Therefore $\mathcal{J}_k/\mathcal{J}_{k-1} \cong \mathscr{C}_0(T_k', \mathcal{K}(H_k))$ by 4.1.5 and $(\mathcal{J}_k)_{k=1}^n$ is a solving series for $\mathcal{A}$. □

Theorem 4.1.7. *$\mathcal{B}$ is not solvable of finite length.*

Proof. By the definition of $\mathcal{B}$ we have that every $\mathcal{B}_k$ $(k \in \mathbb{N})$ is a closed two-sided ideal in $\mathcal{B}$. Since $\mathfrak{l}(\mathcal{B}_k) = k$ it follows by 4.1.6 that $\mathfrak{l}(\mathcal{B}) \geq \mathfrak{l}(\mathcal{B}_k) = k$ for all $k \in \mathbb{N}$, which shows that $\mathcal{B}$ can not be solvable of finite length. □

Remark 4.1.8. As pointed out by Prof. V. Nistor (January 2007) it is also possible to construct an algebra of pseudodifferential operators with an infinite long solving series without using a product construction that leads to an "infinite dimensional" manifold. However, the solving series of this example will not be minimal; in fact the algebra has a minimal solving series that has length two. Nevertheless, let us sketch the construction of such a manifold here, since it is instructional to see some "exotic" examples. Let $\mathcal{Z}_\infty := \mathcal{S}^1 \times [0, \infty[$ be the infinite cylinder over $\mathcal{S}^1$ and $\mathcal{Z} := \mathcal{S}^1 \times [0, 1]$ its finite analogue.

Using a gluing construction, we can attach infinitely many $\mathcal{Z}$'s to $\mathcal{Z}_\infty$, so that we get a manifold $\mathfrak{Z}$ with (infinitely many) multi-cylindrical ends. To fix notation, we will denote by $\mathcal{S}_0^1$ the boundary component of $\mathfrak{Z}$ given by the boundary of $\mathcal{Z}$ and with $\mathcal{S}_k^1$ the boundary component of $\mathfrak{Z}$ that stems from the k-th attached cylinder. On $\mathfrak{Z}$ we define an algebra of pseudodifferential operators $\Psi_{b,cy}^0(\mathfrak{Z})$ of order zero, such that the operators behave like b-type operators at the boundaries of the cylinders and we will denote by $\mathcal{B}(\mathfrak{Z})$ the corresponding C^*-closure of $\Psi_{b,cy}(\mathfrak{Z})$ in $L^2(\mathfrak{Z})$; consequently, each boundary gives rise to its own indical family

$$I_k := I_{\mathcal{S}_k^1} : \mathcal{B}(\mathfrak{Z}) \longrightarrow \mathscr{C}_b(\mathbb{R}, \mathscr{L}(L^2(\mathcal{S}^1))) \quad (k \in \mathbb{N}_0).$$

Now, we are able to define the following ideals:

$$\begin{aligned} \mathcal{J}_0 &:= \left\{a \in \mathcal{B}(\mathfrak{Z}) \,:\, a \in \ker(\sigma_{\mathcal{B}(\mathfrak{Z})}),\ I_k(a) = 0\, \forall k \in \mathbb{N}_0\right\}, \\ \mathcal{J}_l &:= \left\{a \in \mathcal{B}(\mathfrak{Z}) \,:\, a \in \ker(\sigma_{\mathcal{B}(\mathfrak{Z})}),\ I_k(a) = 0\, \forall k \in \mathbb{N},\ k \geq l\right\}, \end{aligned}$$

where $\sigma_{\mathcal{B}(\mathfrak{Z})}$ denotes the extension of the principal symbol map. Then by definition $\mathcal{J}_l \subseteq \mathcal{J}_{l+1}$ and

$$\mathcal{J}_{l+1}/\mathcal{J}_l \cong \mathscr{C}_0(\mathbb{R}, \mathcal{K}(L^2(\mathcal{S}^1))),$$

so this series is solving. But it is not solving of minimal length: Define $\mathcal{I}_1 := \ker(\sigma_{\mathcal{B}(\mathfrak{Z})})$, $\mathcal{I}_0 := \{a \in \mathcal{I}_0 \,:\, I_k(a) = 0\, \forall k \in \mathbb{N}_0\}$, then

$$\{0\} \subseteq \mathcal{I}_0 \subseteq \mathcal{I}_1 \subseteq \mathcal{B}(\mathfrak{Z})$$

is also solving.

Let us close this section with the following definition:

Definition 4.1.9 (Solving ideal chain)**.** Let $\mathcal{B}$ be a C^*-algebra. A finite sequence of closed ideals $(\mathcal{J}_k)_{k=1}^l$ is said to be a *solving ideal chain* for $\mathcal{B}$, if

$$\mathcal{B} \supset \mathcal{J}_l \supseteq \mathcal{J}_{l-1} \supseteq \ldots \supseteq \mathcal{J}_1 \supset \mathcal{J}_0 := \{0\}$$

and $\mathcal{J}_{k+1}/\mathcal{J}_k \cong \mathscr{C}_0(T_k, \mathcal{K}(H_k))$ $(1 < k \leq l-1)$ for some locally compact Hausdorff space T_k and some separable Hilbert space H_k.

4.2 Connecting the product algebra

Let us first recall again, that for a manifold with corners $\mathcal{Z}$ all irreducible representations of $\mathcal{B}(\widehat{\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}})$ are given by (cf. [61, Proposition 5.2]):

$$\begin{aligned} id &: \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathscr{L}(L_b^2(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}})), && (4.2.1) \\ \pi_\eta \circ \tau_{\mathcal{B}} &: \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathbb{C}, && (4.2.2) \\ \pi_{F_0,\lambda} \circ \tau_{\mathcal{B}} &: \mathcal{B}(\mathcal{Z}, {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathscr{L}(L_b^2(F_0, {}^b\Omega^{\frac{1}{2}})), && (4.2.3) \end{aligned}$$

where $\eta \in {}^bS^*\mathcal{Z}$, $\lambda \in \mathbb{R}^{\mathcal{E}(F_0)}$ and $\tau_{\mathcal{B}}$ denotes the joint symbol map.

Let $\mathcal{B}_n := \mathcal{B}(\mathcal{Z}_n, {}^b\Omega^{\frac{1}{2}})$ be the C^*-algebra of b-pseudodifferential operators on a manifold with corners $\mathcal{Z}_n$ with $\dim \mathcal{Z}_n = n$; moreover, we assume that $\mathcal{F}_n(\mathcal{Z}_n) \neq \emptyset$. Let $\Omega_n \subset \overline{\Omega}_n \subset \mathring{\mathcal{Z}}_n$ be open and choose a continuous section $\Sigma_n \in \mathscr{C}(\mathcal{Z}_n, {}^bS^*\mathcal{Z}_n)$ of the cosphere-bundle ${}^bS^*\mathcal{Z}_n$. We define

$$\mathcal{A}_n := \left\{ a \in \mathcal{B}_n \,:\, {}^b\sigma^{(0)}_{\mathcal{B}_n}(a)(\Sigma_n(x)) = 0 \,\forall x \in \Omega_n \right\}. \tag{4.2.4}$$

Proposition 4.2.1. *$\mathcal{A}_n$ is a selfadjoint two-sided closed ideal in the C^*-algebra $\mathcal{B}_n$. Therefore all irreducible representation of $\mathcal{A}_n$ are unitarily equivalent to one of the following pairwise inequivalent representations:*

$$id \;:\; \mathcal{A}_n \longrightarrow \mathscr{L}(L^2_b(\mathcal{Z}_n, {}^b\Omega^{\frac{1}{2}})), \tag{4.2.5}$$

$$\pi_\eta \circ \tau_{\mathcal{B}_n} \;:\; \mathcal{A}_n \longrightarrow \mathbb{C}, \quad \eta \in {}^bS^*\mathcal{Z}_n \setminus \{\Sigma_n(x) \,:\, x \in \Omega_n\}, \tag{4.2.6}$$

$$\pi_{F,\lambda} \circ \tau_{\mathcal{B}_n} \;:\; \mathcal{A}_n \longrightarrow \mathscr{L}(L^2_b(F, {}^b\Omega^{\frac{1}{2}})). \tag{4.2.7}$$

Proof. The set

$$\mathfrak{L} := \left\{ f \in \mathscr{C}({}^bS^*\mathcal{Z}_n) \,:\, f(\Sigma_n(x)) = 0 \text{ for all } x \in \Omega_n \right\}$$

is closed in $\mathscr{C}({}^bS^*\mathcal{Z}_n)$. Therefore $\mathcal{A}_n = ({}^b\sigma^{(0)}_{\mathcal{B}_n})^{-1}(\mathfrak{L})$ is also closed in $\mathcal{B}_n$, since ${}^b\sigma^{(0)}_{\mathcal{B}_n} : \mathcal{B}_n \longrightarrow \mathscr{C}({}^bS^*\mathcal{Z}_n)$ is a continuous $*$-homomorphism that is onto. Moreover the calculation

$${}^b\sigma^{(0)}_{\mathcal{B}_n}(abc)(\eta) = {}^b\sigma^{(0)}_{\mathcal{B}_n}(a)(\eta){}^b\sigma^{(0)}_{\mathcal{B}_n}(b)(\eta){}^b\sigma^{(0)}_{\mathcal{B}_n}(c)(\eta) = 0$$

where $\eta = \Sigma_n(x)$, $x \in \Omega_n$, $a, c \in \mathcal{B}_n$ and $b \in \mathcal{A}_n$ shows, that $\mathcal{A}_n$ is also an ideal in $\mathcal{B}_n$. Thus $\mathcal{A}_n$ is a closed two-sided $*$-ideal in $\mathcal{B}_n$.

For the second part of the proposition, we first note, that one has $\widehat{\mathcal{A}_n} \cong \widehat{\mathcal{B}_n}^{\mathcal{A}_n}$, since $\mathcal{A}_n$ is a two-sided closed ideal in $\mathcal{B}_n$. This shows that

$$\begin{aligned} \widehat{\mathcal{A}_n} &= \left\{ \pi \in \widehat{\mathcal{B}_n} \,:\, \pi(\mathcal{A}_n) \neq \{0\} \right\} \\ &= \widehat{\mathcal{B}_n} \setminus \{\pi_\eta \circ \tau_{\mathcal{B}_n} \,:\, \eta = \Sigma_n(x),\, x \in \Omega_n\}, \end{aligned}$$

which proves our claim. □

Theorem 4.2.2. *$\mathcal{A}_n$ is a solvable C^*-algebra and we have $\mathfrak{l}(\mathcal{A}_n) = n$.*

Proof. Since $\mathcal{A}_n$ is a selfadjoint closed two-sided ideal in $\mathcal{B}_n$, $\mathcal{A}_n$ is solvable and $n = \mathfrak{l}(\mathcal{B}_n) \geq \mathfrak{l}(\mathcal{A}_n)$ holds by 4.1.6. What remains is to prove the reversed inequality. Let

$$\mathcal{A}_n := \mathcal{J}_{l+1} \supseteq \mathcal{J}_l \supseteq \ldots \supseteq \mathcal{J}_1 \supseteq \mathcal{J}_0 = \{0\}$$

be an arbitrary solving series for $\mathcal{A}_n$. Since we have

$$\widehat{\mathcal{A}_n} = \widehat{\mathcal{B}_n} \setminus \{\pi_\eta \circ \tau_{\mathcal{B}_n} \,:\, \eta = \Sigma_n(x)\},$$

we get $\mathfrak{l}(\mathcal{A}_n) \geq n$ by the same argument as in the proof of [63, Lemma 3.5], which shows that $\mathfrak{l}(\mathcal{A}_n) = n$ holds. □

Definition 4.2.3. To $n \in \mathbb{N}$ let $\Omega_n \subset \overline{\Omega}_n \subset \mathring{\mathscr{Z}}_n$ be arbitrary. Choose continuous sections $\Sigma_n \in \mathscr{C}(\mathscr{Z}_n, {}^bS^*\mathscr{Z}_n)$ as above. Let $\mathcal{A} := \prod_{n\in\mathbb{N}} \mathcal{A}_n$ be the direct sum algebra cf. 4.1.1, where the $\mathcal{A}_n$ are given by (4.2.4) (with respect to Σ_n).

Since $\mathcal{A}_n$ is a solvable C^*-algebra of length n, we have the following proposition.

Proposition 4.2.4. *The algebra $\mathcal{A}$ is a solvable C^*-algebra. A solving series can be given analogously to 4.1.3.*

But again, we cannot give a solving series for $\mathcal{A}$ that has finite length:

Theorem 4.2.5. *$\mathcal{A}$ is not solvable with finite length.*

Proof. Since $\mathcal{A}_n$ is a two-sided closed ideal in $\mathcal{A}$, we have $\mathfrak{l}(\mathcal{A}) \geq \mathfrak{l}(\mathcal{A}_n) = n$ for all $n \in \mathbb{N}$. □

Notations 4.2.6. Now we additionally assume that there exist continuous mappings

$$\iota_n : \Omega_n \longrightarrow \mathring{\mathscr{Z}}_{n+1},$$

such that $\overline{\iota_n(\Omega_n)} \subseteq \mathring{\mathscr{Z}}_{n+1}$ and $\iota_n(\Omega_n) \cap \Omega_{n+1} = \emptyset$. Moreover, denote by Σ_n a continuous section of ${}^bS^*\mathscr{Z}_n$ and by $\iota_{n,*}(\Sigma_n)$ the push-forward of Σ_n under the differential $\iota_{n,*}$ of ι_n. Again we define $\mathcal{B}$ as in 4.1.2.

Example 4.2.7. A possible realisation of $\mathscr{Z}_n$ is the direct product of the unit interval $\mathscr{Z}_n := [0,1]^n$ with Ω_n localized in the interior $]0,1[^n$ of $\mathscr{Z}_n$.

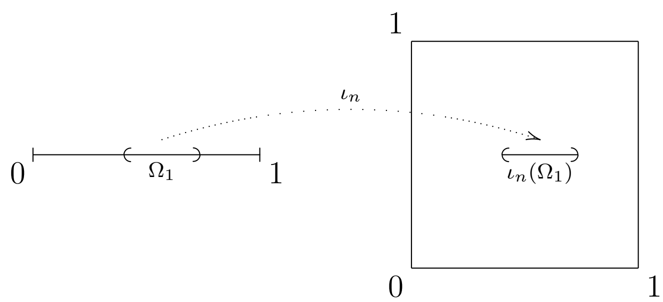

Figure 4.1: $\Omega_1 \subset \mathscr{Z}_1 := [0,1]$ mapped to $\mathscr{Z}_2 := [0,1]^2$.

Definition 4.2.8. Denote by $\mathcal{F}$ the space of all $(a_j)_{j\in\mathbb{N}} \in \mathcal{B}$, such that

$$\sigma_{\mathcal{B}_j}(a_j)(\Sigma_j(x)) - \sigma_{\mathcal{B}_{j+1}}(a_{j+1})((\iota_{j,*}\Sigma_j)(\iota_j(x))) = 0 \tag{4.2.8}$$

for all $x \in \Omega_j$ and for all $j \in \mathbb{N}$.

Proposition 4.2.9. *We have:*

(i) *$\mathcal{F}$ is a C^*-subalgebra of $\mathcal{A}$.*

(ii) *To $n \in \mathbb{N}$ exists a closed two sided ideal $\mathcal{F}_n \leq \mathcal{F}$, such that $\mathcal{F}_n$ is solvable with $\mathfrak{l}(\mathcal{F}_n) = n$.*

Proof. To simplify notation, we set $\widetilde{\Sigma}_j := \iota_{j,*}(\Sigma_j)$. For

$$\widetilde{x} := (x_j)_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}} \Omega_j =: \Omega$$

consider the map

$$\begin{aligned} \Phi_{\widetilde{x}} : \mathcal{B} &\longrightarrow \mathbb{C}^{\mathbb{N}}, \\ a = (a_j) &\longmapsto \left({}^b\sigma^{(0)}_{\mathcal{B}_j}(a_j)(\Sigma_j(x_j)) - {}^b\sigma^{(0)}_{\mathcal{B}_{j+1}}(a_{j+1})(\widetilde{\Sigma}_j(\iota_j(x_j)))\right)_{j\in\mathbb{N}}, \end{aligned}$$

where we use 4.1.1 for $\mathbb{C}^{\mathbb{N}}$.
(i) It is clear, that $\Phi_{\widetilde{x}}$ is a linear map; let us prove that $\Phi_{\widetilde{x}}$ is continuous. We have

$$\begin{aligned} &|{}^b\sigma^{(0)}_{\mathcal{B}_j}(a_j)(\Sigma_j(x_j)) - {}^b\sigma^{(0)}_{\mathcal{B}_{j+1}}(a_{j+1})(\widetilde{\Sigma}_j(\iota_j(x_j)))| \\ &\qquad\qquad \leq |{}^b\sigma^{(0)}_{\mathcal{B}_j}(a_j)(\Sigma_j(x_j))| + |{}^b\sigma^{(0)}_{\mathcal{B}_{j+1}}(a_{j+1})(\widetilde{\Sigma}_j(\iota_j(x_j)))| \\ &\qquad\qquad \leq C_1||a_j|| + C_2||a_{j+1}|| \\ &\qquad\qquad \leq D \sup_{j\in\mathbb{N}} ||a_j|| = D||a|| \; (< \infty), \end{aligned}$$

which gives $||\Phi_{\widetilde{x}}(a)|| \leq D||a||$ as desired. This implies, that $\ker \Phi_{\widetilde{x}}$ is closed and since

$$\mathcal{F} = \bigcap_{\widetilde{x}\in\Omega} \ker \Phi_{\widetilde{x}}$$

holds, $\mathcal{F}$ is also closed. What remains to be proven is, that $\mathcal{F}$ is actually an algebra. Let $a = (a_n)$, $b = (b_n) \in \mathcal{F}$ be arbitrary. To shorten notation set $\sigma_k := {}^b\sigma^{(0)}_{\mathcal{B}_k}$ and $\sigma_{k+1} := {}^b\sigma^{(0)}_{\mathcal{B}_{k+1}}$. Then we have

$$\begin{aligned} &\sigma_k(a_kb_k)(\Sigma_k(x_k)) - \sigma_{k+1}(a_{k+1}b_{k+1})(\widetilde{\Sigma}_k(\iota_k(x_k))) \\ &= \sigma_k(a_k)(\Sigma_k(x_k))\sigma_k(b_k)(\Sigma_k(x_k)) \\ &\qquad - \sigma_{k+1}(a_{k+1})(\widetilde{\Sigma}_k(\iota_k(x_k)))\sigma_{k+1}(b_{k+1})(\widetilde{\Sigma}_k(\iota_k(x_k))) \\ &= \left[\sigma_k(a_k)(\Sigma_k(x_k)) - \sigma_{k+1}(a_{k+1})(\widetilde{\Sigma}_k(\iota_k(x_k)))\right] \sigma_k(b_k)(\Sigma_k(x_k)) \\ &\quad + \sigma_{k+1}(a_{k+1})(\widetilde{\Sigma}_k(\iota_k(x_k))) \left[\sigma_k(b_k)(\Sigma_k(x_k)) - \sigma_{k+1}(b_{k+1})(\widetilde{\Sigma}_k(\iota_k(x_k)))\right] \\ &= 0, \end{aligned}$$

for all $x_k \in \Omega_k$, which shows that $ab \in \mathcal{F}$.
(ii) To $n \in \mathbb{N}$ we define $\mathcal{F}_n := \mathcal{A}_n$, where $\mathcal{A}_n$ is given by (4.2.4). Then $\mathcal{F}_n$ is a closed two-sided ideal in $\mathcal{F}$ using the usual embedding and $\mathfrak{l}(\mathcal{F}_n) = n$, cf. 4.2.2. □

Remark 4.2.10. It is worth pointing out, that the construction of $\mathcal{F}$ yields an algebra with an arcwise connected space of one-dimensional representations by condition (4.2.2).

4.3 Transmission algebras with infinitely long ideal chains

Let $(\mathcal{Z}_i)_{i\in\mathbb{N}}$ be a family of manifolds with corners such that $\dim \mathcal{Z}_i = i$. Again, we assume that each $\mathcal{Z}_i$ has a face of codimension i. Let $\Psi^0_{b,cl}(\mathcal{Z}_i, {}^b\Omega^{\frac{1}{2}})$ denote the corresponding algebras of b-pseudodifferential operators of order zero on $\mathcal{Z}_i$. Moreover, let $\mathcal{W}_i$ be a subset of the (smooth) interior $\mathring{\mathcal{Z}}_i$ of $\mathcal{Z}_i$, such that there exists a compact set $\mathcal{K}_i \subset \mathring{\mathcal{Z}}_i$ with $\mathcal{W}_i \subseteq \mathcal{K}_i$. We also assume that there exist smooth maps $\iota_i : \mathcal{Z}_i \longrightarrow \mathcal{Z}_{i+1}$, such that $\iota_i(\mathcal{W}_i) =: \widetilde{\mathcal{W}}_{i+1}$ is a subset of a compact set $\widetilde{\mathcal{K}} \subset \mathring{\mathcal{Z}}_{i+1}$ and that $\widetilde{\mathcal{W}}_i \cap \mathcal{W}_i = \emptyset$ $(i > 1)$.

Remark 4.3.1. To shorten notation, we will suppress all density bundles in what follows. Note also, that we do not ask $\mathcal{W}_i$ (resp. $\widetilde{\mathcal{W}}_i$) to be a submanifold; for example $\mathcal{W}_i$ (resp. $\widetilde{\mathcal{W}}_i$) could be partially a submanifold and partially a fractal set.

4.3.1 Transmission spaces

Since $\widetilde{\mathcal{W}_n}$ resp. $\mathcal{W}_{n-1}$ are subsets of the smooth interior of $\mathcal{Z}_n$ resp. $\mathcal{Z}_{n-1}$, there exists $d \in \mathbb{N}$ to fixed $n \in \mathbb{N}$, such that the point evaluation

$$ev_{p,i} : \mathcal{H}^m_b(\mathcal{Z}_i) \longrightarrow \mathbb{C} : \; f \longmapsto f(p)$$

is well defined and continuous for all $p \in \mathcal{W}_i \cup \widetilde{\mathcal{W}}_i$, if $m > d$ where $\mathcal{H}^d_b(\mathcal{Z}_i)$ denotes the b-Sobolev space of order d. In particular, this is true for $\mathcal{D}^\infty_i := \mathcal{H}^\infty_b(\mathcal{Z}_i)$. We denote by $\widetilde{\mathfrak{H}}$ resp. $\widetilde{\mathfrak{D}}$ the following Hilbert spaces

$$\widetilde{\mathfrak{H}} := \left\{ v = (v_i)_{i\in\mathbb{N}} \in \bigoplus_{i\in\mathbb{N}} L^2_b(\mathcal{Z}_i) \; : \; ||v||_{\widetilde{\mathfrak{H}}} < \infty \right\}$$

and

$$\widetilde{\mathfrak{D}} := \left\{ v = (v_i)_{i\in\mathbb{N}} \in \bigoplus_{i\in\mathbb{N}} \mathcal{D}^\infty_i \; : \; ||v||_{\widetilde{\mathfrak{D}}} < \infty \right\}$$

where

$$||f||_{\widetilde{\mathfrak{H}}} := \langle f \mid f \rangle^{1/2}_{\widetilde{\mathfrak{H}}} \text{ and } \langle f \mid g \rangle_{\widetilde{\mathfrak{H}}} := \sum_{i\in\mathbb{N}} \langle f_i \mid g_i \rangle_{L^2(\mathcal{Z}_i)}$$

(analogously for $\widetilde{\mathfrak{D}}$). Moreover, we define $\mathfrak{D}(\mathcal{Z})$ to be the following function space on $\mathcal{Z} := \bigcup_{i\in\mathbb{N}} \mathcal{Z}_i$:

$$\mathfrak{D}(\mathcal{Z}) := \left\{ f \in \widetilde{\mathfrak{D}} \; : \; f_i(p) = f_{i+1}(\iota_i(p)) \; \forall p \in \mathcal{W}_i, \; i \in \mathbb{N} \right\}.$$

Remark 4.3.2. $\mathfrak{D}(\mathcal{Z})$ is a closed subspace of $\widetilde{\mathfrak{D}}$.

Definition 4.3.3 (Transmission space)**.** The space $\mathfrak{D} := \mathfrak{D}(\mathcal{Z})$ is called *transmission space*. The closure of $\mathfrak{D}$ in $\widetilde{\mathfrak{H}}$ will be denoted by $\mathfrak{H}$.

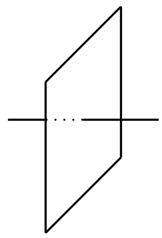

Figure 4.2: The unit interval intersects the unit square: Transmission at one point.

Definition 4.3.4. Let $\widetilde{\mathcal{A}}(\mathcal{Z}) \subseteq \mathscr{L}(\mathfrak{H})$ be the algebra of all continuous linear maps, such that $a(\mathfrak{H}) \subseteq \mathfrak{H}$ and $a^*(\mathfrak{H}) \subseteq \mathfrak{H}$ holds for all $a \in \widetilde{\mathcal{A}}(\mathcal{Z})$. Denote by $\overline{\mathcal{A}}(\mathcal{Z})$ the corresponding C^*-closure in $\mathscr{L}(\widetilde{\mathfrak{H}})$.

Notations 4.3.5.

(i) Denote by $\mathfrak{Y} \subseteq \overline{\mathcal{A}}(\mathcal{Z})$ the set of all operators X, such that for all $i \in \mathbb{N}$ there exist open neighbourhoods $\widetilde{\mathcal{V}}_i \subseteq \overline{\widetilde{\mathcal{V}}}_i \subseteq \mathcal{Z}_i$ with $\mathcal{W}_i \cup \widetilde{\mathcal{W}}_i \subseteq \widetilde{\mathcal{V}}_i$ and $X_{|\mathcal{Z}_i \setminus \widetilde{\mathcal{V}}_i}$ is an element of $\mathrm{Diff}_b^1(\mathcal{Z}_i) \cup \mathrm{Diff}_b^0(\mathcal{Z}_i)$.

(ii) Denote by $\mathfrak{N} \subseteq \overline{\mathcal{A}}(\mathcal{Z})$ the set of all operators M, such that for all $i \in \mathbb{N}$ there exist open neighbourhoods $\widetilde{\mathcal{O}}_i \subseteq \overline{\widetilde{\mathcal{O}}}_i \subseteq \mathcal{Z}_i$ with $\mathcal{W}_i \cup \widetilde{\mathcal{W}}_i \subseteq \widetilde{\mathcal{O}}_i$ and $M_{|\mathcal{Z}_i \setminus \widetilde{\mathcal{O}}_i}$ is given by a multiplication operator with a smooth function.

Remark 4.3.6. In particular $\mathfrak{Y}$ includes the following special cases:

(i) Vectorfields X possibly vanishing tangential to $\prod_{i \in \mathbb{N}} \mathcal{W}_i$ whenever this is a submanifold.

(ii) Vectorfields X possibly vanishing normal to $\prod_{i \in \mathbb{N}} \mathcal{W}_i$ whenever this is a submanifold.

Let us discuss two examples of operators that are elements of $\mathfrak{Y}$ resp. $\mathfrak{N}$; to this end, let $\mathscr{D}(\mathcal{Z}_i)$ denote the set of all $D_i \in \mathrm{Diff}_b^1(\mathcal{Z}_i) \cup \mathrm{Diff}_b^0(\mathcal{Z}_i)$, such that there exist closed subsets $\mathcal{V}_i \subseteq \mathcal{Z}_i$ with $\mathcal{W}_i \cup \widetilde{\mathcal{W}}_i \subseteq \mathcal{V}_i$ and $D_i = 0$ on $\mathcal{V}_i$. Note, that $\mathscr{D}(\mathcal{Z}_i)$ includes all operators D that are given by $D = M_\varphi X M_\varphi$ where X is a vector field and φ denotes a smooth function with $\operatorname{supp} \varphi \subseteq \mathring{\mathcal{Z}}_i \setminus (\mathcal{W}_i \cup \widetilde{\mathcal{W}}_i)$.

Example 4.3.7.

(i) Let $m \in \mathbb{N}$ and $D_i \in \mathscr{D}(\mathcal{Z}_i)$ $(i = 1, \dots, m)$ be given. Then $D := (D_i)_{i \in \mathbb{N}}$, where we set $D_i := 0$ for $i > m$, is an element of $\mathfrak{Y}$.

(ii) Let $m \in \mathbb{N}$ and $M_i := M_{\varphi_i}$ be the multiplication operator by the function φ_i, where $\varphi_i \in \mathcal{C}_c(\mathring{\mathcal{Z}}_i \setminus (\mathcal{W}_i \cup \widetilde{\mathcal{W}_i}))$. Then $M := (M_i)_{i\in\mathbb{N}}$, where we set $M_i := 0$ for $i > m$, is an element of $\mathfrak{N}$.

Notations 4.3.8. Let $\mathcal{V}_i$ be fixed closed subsets of $\mathcal{Z}_i$, such that $\mathcal{W}_i \cup \widetilde{\mathcal{W}_i} \subseteq \mathcal{V}_i$ and $(\varphi_j^{(i)})_{j=1}^l$ be a (finite) partition of unity for the (compact) manifold with corners $\mathcal{Z}_i$. If $\operatorname{supp}\varphi_j^{(i)} \cap \mathcal{V}_i = \emptyset$ is fulfilled, we can choose finitely many differential operators in $\operatorname{Diff}_1(\mathcal{Z}_{i|\operatorname{supp}\varphi_j^{(i)}}) \cup \operatorname{Diff}_0(\mathcal{Z}_{i|\operatorname{supp}\varphi_j^{(i)}})$ and also finitely many multiplication operators in $\operatorname{Diff}_0(\mathcal{Z}_{i|\operatorname{supp}\varphi_j^{(i)}})$, such that they enable us to give a *Beals*-type characterisation in the sense of 2.3.4 (iv). We fix these subsets $\mathcal{V}_i$ in what follows.

Definition 4.3.9. We define the following sets of operators inductively:

- Let $\mathfrak{X}^{(2)}$ denote a countable subset of $\mathfrak{Y}$, such that $D \in \mathfrak{X}^{(2)}$ can be represented by $D = (D_1, D_2, 0, \ldots)$ and $\mathfrak{X}^{(2)}$ includes at least all operators $(D_1, 0, \ldots)$, $(0, D_2, 0, \ldots)$ etc. that are necessary for a Beals-type-characterisation on $\mathcal{Z}_1$ resp. $\mathcal{Z}_2$ in the sense of 2.3.4 (iv)[2] and 4.3.8.

- Let $\mathfrak{M}^{(2)}$ denote a countable subset of $\mathfrak{N}$, such that $M \in \mathfrak{M}^{(2)}$ can be represented by $M = (M_1, M_2, 0, \ldots)$ and $\mathfrak{M}^{(2)}$ includes at least all operators $(M_1, 0, \ldots)$, $(0, M_2, 0, \ldots)$ etc. that are necessary for a Beals-type-characterisation on $\mathcal{Z}_1$ resp. $\mathcal{Z}_2$ in the sense of 2.3.4 (iv)[3] and 4.3.8.

So let $\mathfrak{X}^{(m)} \subseteq \mathfrak{Y}$ and $\mathfrak{M}^{(m)} \subseteq \mathfrak{N}$ be already chosen. Then:

- Let $\mathfrak{X}^{(m+1)}$ be a countable subset of $\mathfrak{Y}$, such that $\mathfrak{X}^{(m)} \subseteq \mathfrak{X}^{(m+1)}$ and $\mathfrak{X}^{(m+1)}$ includes (at least) the (differential-) operators that we have chosen according to 4.3.8 for the transmission space $\mathcal{Z}_{m+1}$.

- Let $\mathfrak{M}^{(m)} \subseteq \mathfrak{N}$ denote a countable subset, such that $\mathfrak{M}^{(m)}$ includes (at least) the (multiplication-) operators that we have chosen according to 4.3.8 for the transmission space $\mathcal{Z}_{m+1}$.

Definition 4.3.10. We define the following Ψ^*-algebras

$$\mathcal{A}^{\mathfrak{M}}(\mathcal{Z}) := \bigcap_{k\in\mathbb{N}} \mathcal{A}^{\mathfrak{M}^{(k)}}(\mathcal{Z}) \text{ and } \mathcal{A}^{\mathfrak{X}}(\mathcal{Z}) := \bigcap_{k\in\mathbb{N}} \mathcal{A}^{\mathfrak{X}^{(k)}}(\mathcal{Z})$$

according to 1.3.5 where the construction is done within the C^*-algebra $\overline{\mathcal{A}}(\mathcal{Z})$. Finally, we set $\mathcal{A}(\mathcal{Z}) := \mathcal{A}^{\mathfrak{M}}(\mathcal{Z}) \cap \mathcal{A}^{\mathfrak{X}}(\mathcal{Z})$ and denote by $\mathcal{B}(\mathcal{Z})$ the C^*-closure of $\mathcal{A}(\mathcal{Z})$ in $\mathscr{L}(\mathfrak{H})$.

Definition 4.3.11. Define the following (two-sided) ideals $\widetilde{\mathcal{J}_i}$ in $\mathcal{A}(\mathcal{Z})$:

$$\begin{aligned}\widetilde{\mathcal{J}_i} := \Big\{ a \in \mathcal{A}(\mathcal{Z}) \,:\, a = &\sum_{\nu \text{ finite}} b_\nu M_{\phi_\nu} c_\nu, \\ & b_\nu, c_\nu \in \mathcal{A}(\mathcal{Z}),\, \phi_\nu \in \mathfrak{D}(\mathcal{Z}) \,, \operatorname{supp}\phi_\nu \subset \mathring{\mathcal{Z}}_i \Big\}.\end{aligned} \tag{4.3.1}$$

[2]See also the definition of the differential operators in 2.3.5.
[3]See also the definition of the multiplication operators in 2.3.5.

Proposition 4.3.12.

(i) *The closure* $\mathcal{J}_i$ *of* $\widetilde{\mathcal{J}}_i$ *in* $\mathcal{A}(\mathcal{Z})$ *is a proper two-sided closed ideal in* $\mathcal{A}(\mathcal{Z})$.

(ii) *The closure of* $\widetilde{\mathcal{J}}_i$ *in* $\mathcal{B}(\mathcal{Z})$*, also denoted by* $\mathcal{J}_i$*, is a proper two-sided closed ideal in* $\mathcal{B}(\mathcal{Z})$.

Proof. (i) Clearly, the closure $\mathcal{J}_i$ of $\widetilde{\mathcal{J}}_i$ is a closed two-sided ideal in $\mathcal{A}(\mathcal{Z})$. So it remains to prove, that $\mathcal{J}_i \neq \mathcal{A}(\mathcal{Z})$: Since $\mathcal{A}(\mathcal{Z})$ is a Ψ^*-algebra there exists a neighbourhood V of id, such that $a^{-1} \in \mathcal{A}(\mathcal{Z})$ for all $a \in V$. Suppose, that we can approximate the identity operator id in $\mathcal{A}(\mathcal{Z})$ by elements of $\widetilde{\mathcal{J}}_i$, i.e. suppose that there exists an element $c \in \widetilde{\mathcal{J}}_i$, such that $c \in V$ holds. It follows that $c^{-1} \in \mathcal{A}(\mathcal{Z})$, thus

$$id = c^{-1}c = \sum_{\nu \text{ finite}} c^{-1} a_\nu M_{\phi_\nu} b_\nu, \tag{4.3.2}$$

with suitable $b_\nu, a_\nu \in \mathcal{A}(Z)$ and $\phi_\nu \in \mathfrak{D}(\mathcal{Z})$ with $\operatorname{supp}\phi_\nu \cap \partial\mathcal{Z}_i = \emptyset$. Let $u \in L^2(\mathcal{Z})$ be such that $u \equiv 1$ on a neighbourhood O_i where $O_i \subseteq \mathcal{Z}_i \setminus (\operatorname{supp}\phi_{\nu|\mathcal{Z}_i} \cup \partial\mathcal{Z}_i)$ and $u \equiv 0$ otherwise. Moreover, it is possible to choose O_i, such that there exists $\omega \in \mathscr{C}_c^\infty(\mathcal{Z})$ with $\operatorname{supp}\omega \cap \operatorname{supp}\phi_{\nu|\mathcal{Z}_i} = \emptyset$ and $\omega \equiv 1$ in a neighbourhood of O_i for all ν. Then $M_\omega c^{-1} M_{\phi_\nu} b_j M_\omega$ is a smoothing operator:

We have $\operatorname{supp}\omega \cap \operatorname{supp}\phi_{\nu|\mathcal{Z}_i} = \emptyset$ and we choose a function $\widetilde{\phi}_\nu \in \mathscr{C}_c(\mathcal{Z})$, such that $\operatorname{supp}\widetilde{\phi}_\nu \subseteq \mathcal{Z} \setminus (\mathcal{W}_i \cup \widetilde{\mathcal{W}}_i)$ and $\widetilde{\phi}_\nu \equiv \phi_\nu$ outside a suitable neighbourhood of $\mathcal{W}_i \cup \widetilde{\mathcal{W}}_i$. Let X be an element of $\mathfrak{X}$; then

$$\operatorname{ad}(\widetilde{\phi}_\nu X)(M_{\phi_\nu} b_i M_\omega) = \widetilde{\phi}_\nu X M_{\phi_\nu} b_i M_\omega,$$

holds and therefore $\operatorname{ad}(\widetilde{\phi}_\nu X)^k (M_{\phi_\nu} b_i M_\omega) = (\widetilde{\phi}_\nu X)^k M_{\phi_\nu} b_i M_\omega \in L^2(\mathcal{Z})$ for all $X \in \mathfrak{X}$ by the definition of the algebra $\mathcal{A}(\mathcal{Z})$.

This is a contradiction to the corresponding right hand side of (4.3.2).

(ii) The prove is literally the same as the one of (i), if we use elements of $\mathcal{A}(\mathcal{Z})$ to approximate the ones in $\mathcal{B}(\mathcal{Z})$, hence we will omit it. □

Notations 4.3.13. In what follows, we assume, that $\mathcal{Z}_i$ has a face H_i of codimension i and we choose functions ϕ_i, $\theta_i \in \mathscr{C}_c(\mathcal{Z}_i)$ $(i \in \mathbb{N})$, such that $\operatorname{supp}\phi_i, \operatorname{supp}\theta_i \subset \mathring{\mathcal{Z}}_i$, $\phi_i \equiv 1$ on a neighbourhood of $\mathcal{W}_i \cup \widetilde{\mathcal{W}}_i$ and $\phi_i \prec \theta_i$. Then we have for all $i \in \mathbb{N}$ the following decomposition for $a \in \mathcal{A}(\mathcal{Z})$:

$$a = (1-\phi_i)a(1-\theta_i) + (1-\phi_i)a\theta_i + \phi_i a(1-\theta_i) + \phi_i a\theta_i. \tag{4.3.3}$$

Definition 4.3.14.

(i) Denote by $\widetilde{\Psi}^0(\mathcal{Z})$ (resp. $\widetilde{\Psi}^-(\mathcal{Z})$) the set of all operators $a \in \mathcal{A}(\mathcal{Z})$ such that

(a) $(1-\phi_i)a(1-\theta_i)$ and $(1-\theta_i)a^*(1-\phi_i)$ are elements of $\Psi^0_{b,cl}(\mathcal{Z}_i)$ (resp. $\Psi^{-1}_{b,cl}(\mathcal{Z}_i)$) for all $i \in \mathbb{N}$ and for all decompositions like 4.3.13 resp. (4.3.3).

(b) $I^{\mathcal{B}}_{F\mathcal{Z}_i}((1-\phi_i)a(1-\theta_i)(1-\phi_i)b(1-\theta_i)) = I^{\mathcal{B}}_{F\mathcal{Z}_i}((1-\phi_i)ab(1-\theta_i))$ holds for all $i \in \mathbb{N}$ and for all decompositions like 4.3.13 resp. (4.3.3).

(ii) Denote by $\Psi^0(\mathcal{Z})$ the closure of the algebraic span of $\widetilde{\Psi}^0(\mathcal{Z})$ in $\mathcal{A}(\mathcal{Z})$ and by $\Psi^-(\mathcal{Z})$ the closure of the algebraic span of $\widetilde{\Psi}^-(\mathcal{Z})$ in $\Psi^0(\mathcal{Z})$.

(iii) Denote by $\mathcal{B}_b(\mathcal{Z})$ the C^*-closure of the algebraic span of $\widetilde{\Psi}^0(\mathcal{Z})$ in the C^*-algebra $\mathcal{B}(\mathcal{Z})$ and by $\mathcal{B}_b^-(\mathcal{Z})$ the closure of the algebraic span of $\widetilde{\Psi}^-(\mathcal{Z})$ in $\mathcal{B}_b(\mathcal{Z})$.

(iv) Denote by $\mathcal{J}_{b,i}$ the ideal that corresponds to $\mathcal{J}_i$ using (ii) and (iii).

Theorem 4.3.15. *Let $a \in \Psi^0(\mathcal{Z})$ be arbitrary.*

(i) *If $\omega_1 \in \mathscr{C}_c^\infty(\mathcal{Z}_i)$ and $\omega_2 \in \mathscr{C}_c^\infty(\mathcal{Z}_j)$ are given such that*

$$\begin{aligned}&\operatorname{supp}\omega_1 \cap \mathcal{V}_i = \emptyset = \operatorname{supp}\omega_2 \cap \mathcal{V}_j \text{ and}\\ &\operatorname{supp}\omega_1 \cap \operatorname{supp}\omega_2 = \emptyset \quad (\text{if } i = j),\end{aligned}$$

then $\omega_1 a \omega_2$ is a pseudolocal operator.

(ii) *If $\omega_1, \omega_2 \in \mathscr{C}_c^\infty(\mathcal{Z}_i)$ are given such that*

$$\operatorname{supp}\omega_j \cap \mathcal{V}_i = \emptyset \quad (j = 1, 2)$$

and $\operatorname{supp}\omega_j$ *are both contained in a chart compatible with the* Beals-Type-*characterisation, cf. 4.3.8, then $\omega_1 a \omega_2$ is an ordinary, compactly supported pseudodifferential operator in the interior $\mathring{\mathcal{Z}}_i$ of $\mathcal{Z}_i$.*

Proof. The proof of (i) is exactly the same as part of the proof of 4.3.12, where we show, that one gets a smoothing operator. (ii) follows by 2.3.4 (iv). □

4.3.2 Ideal chains

Definition 4.3.16. Let $F \in \mathcal{F}(\mathcal{Z}_i)$ be a face in $\mathcal{Z}_i$. Then we define a linear map

$$I_{F,\mathcal{Z}_i} : \mathcal{B}_b(\mathcal{Z}) \longrightarrow \mathcal{B}(F, {}^b\Omega^{\frac{1}{2}})$$

by $I_{F,\mathcal{Z}_i}(a) = 0$ if $a \in \mathcal{J}_{b,i}$ and $I_{F,\mathcal{Z}_i}(a) = I^{\mathcal{B}}_{F\mathcal{Z}_i}(M_{(1-\phi_i)} a M_{(1-\theta_i)})$ else.

Proposition 4.3.17. *$I_{F,\mathcal{Z}_i}$ is a well-defined continuous $*$-homomorphism that is independent of the special choice of the decomposition (4.3.3).*

Proof. Let $(1 - \widetilde{\phi}_i)a(1 - \widetilde{\theta}_i)$ be another decomposition (4.3.3). Then

$$\begin{aligned} I^{\mathcal{B}}_{F\mathcal{Z}_i}(M_{(1-\phi_i)} a M_{(1-\theta_i)}) &= I^{\mathcal{B}}_{F\mathcal{Z}_i}(M_{(1-\phi_i)} a M_{(1-\theta_i)}) \\ &= I^{\mathcal{B}}_{F\mathcal{Z}_i}(M_{(1-\widetilde{\phi}_i)} M_{(1-\phi_i)} a M_{(1-\theta_i)} M_{(1-\widetilde{\theta}_i)}) \\ &= I^{\mathcal{B}}_{F\mathcal{Z}_i}(M_{(1-\widetilde{\phi}_i)} a M_{(1-\widetilde{\theta}_i)}), \end{aligned}$$

thus the $*$-homomorphism $I_{F,\mathcal{Z}_i}$ is well-defined and independent of the decomposition. To see continuity, we first note that

$$||I_{F,\mathcal{Z}_i}(M_{(1-\phi_i)} a M_{(1-\theta_i)})|| \le ||M_{(1-\phi_i)} a M_{(1-\theta_i)}||_{\mathscr{L}(L^2(\mathcal{Z}_i, {}^b\Omega^{\frac{1}{2}}))}$$

holds, since $I^{\mathcal{B}}_{F,\mathcal{Z}_i}$ is continuous. Now $M_{(1-\phi_i)}aM_{(1-\theta_i)}$ can be regarded as an element in $\mathscr{L}(\mathfrak{H})$ if we extend it by zero to act on $\mathfrak{H}$. Then by definition

$$||M_{(1-\phi_i)}aM_{(1-\theta_i)}||_{\mathscr{L}(L^2(\mathcal{Z}_i,{}^b\Omega^{\frac{1}{2}}))} \leq ||M_{(1-\phi_i)}aM_{(1-\theta_i)}||_{\mathscr{L}(\mathfrak{H})}$$

and $||M_{\varphi_i}||_{\mathscr{L}(\mathfrak{H})} \leq C$. Thus the continuity follows. □

Definition 4.3.18. Let $2 \leq l \leq i$. Denote by $\mathcal{J}^{(i)}_l$ the closure in $\mathcal{B}_b(\mathcal{Z})$ of the ideal of all operators in $\widetilde{\Psi}^-(\mathcal{Z})$ such that $I_{F,\mathcal{Z}_i}$ vanishes for all $F \in \mathcal{F}_l(\mathcal{Z}_i)$.

Proposition 4.3.19. *We get the nested sequence*

$$\mathcal{B}_b(\mathcal{Z}) \supset \mathcal{J}^{(i)}_i \supset \mathcal{J}^{(i)}_{i-1} \supset \ldots \supset \mathcal{J}^{(i)}_2 \supseteq \mathcal{J}_1 := \{0\}$$

and $(\mathcal{J}^{(i)}_l)^i_{l=2}$ *is a solving ideal chain in the sense of 4.1.9.*

Proof. The only thing that we have to prove is the special structure of the quotients $\mathcal{J}^{(i)}_{l+1}/\mathcal{J}^{(i)}_l$. To see that

$$\mathcal{J}^{(i)}_{l+1}/\mathcal{J}^{(i)}_l \cong \bigoplus_{F\in\mathcal{F}_k(\mathcal{Z}_i)} \mathscr{C}_0(\mathbb{R}^{\mathcal{E}(F)}, \mathcal{K}(L^2_b(F,{}^b\Omega^{\frac{1}{2}})))$$

holds for $l = 1, \ldots, i-1$, let us first note, that if $\phi \in \mathcal{C}^\infty_c(\mathcal{Z})$ is a function with $\operatorname{supp}\phi \subseteq \mathring{\mathcal{Z}}_i$ and $\phi \equiv 1$ on $\mathcal{W}_i \cup \widetilde{\mathcal{W}_i}$, we get that $M_\psi a M_\psi \in \mathcal{B}_b(\mathcal{Z})$ for all $a \in \mathcal{B}(\mathcal{Z}_i, {}^b\Omega^{\frac{1}{2}})$ where $\psi := 1-\phi$. Then we have

$$I_{F,\mathcal{Z}_i}(a) = I^{\mathcal{B}}_{F\mathcal{Z}_i}(M_\psi a M_\psi) = I^{\mathcal{B}}_{F\mathcal{Z}_i}(M_\psi) I^{\mathcal{B}}_{F\mathcal{Z}_i}(a) I^{\mathcal{B}}_{F\mathcal{Z}_i}(M_\psi) = I^{\mathcal{B}}_{F\mathcal{Z}_i}(a),$$

since the restriction of ψ to any boundary face F is one. Hence the usual proof (see for instance [85, Theorem 2]) of the isomorphism is applicable and we have finished the proof. □

Corollary 4.3.20. *To each* $i \in \mathbb{N}$*, there exists a solving ideal chain in the sense of 4.1.9 for* $\mathcal{B}_b(\mathcal{Z})$*.*

Defining $\mathcal{I}^{(i)}_l := \Psi^0(\mathcal{Z}) \cap \mathcal{J}^{(i)}_l$, an application of 1.1.9 yields the following theorem for the Ψ^*-algebra $\widetilde{\Psi}^0(\mathcal{Z})$:

Theorem 4.3.21. *For each* $i \in \mathbb{N}$ *the ideals* $(\mathcal{I}^{(i)}_l)^i_{l=2}$ *yield a nested sequence of closed two-sided ideals in the* Ψ^**-algebra* $\Psi^0(\mathcal{Z})$ *of length* $i-1$*. Moreover, the closures of* $\mathcal{I}^{(i)}_l$*,* $l = 2, \ldots, i$*, in the* C^**-algebra* $\mathcal{B}_b(\mathcal{Z})$ *give a solving ideal chain for* $\mathcal{B}_b(\mathcal{Z})$*.*

4.3.3 A refined transmisson model

We will denote by $\Delta_{b,i}$ the b-Laplacian acting on the function space $\mathscr{C}^\infty(\mathcal{Z}_i)$ in what follows. Moreover, we set $\Lambda_{b,i} := (1-\Delta_{b,i})^{1/2}$.

Before we give a construction of a transmission algebra acting on a scale of Sobolev spaces let us recall the following well-known facts about the *Friedrichs extension*.

To this end we denote by $\mathcal{H}$ a Hilbert space and $q : \mathcal{D}(q)\times\mathcal{D}(q) \longrightarrow \mathbb{C}$ a quadratic form, where $\mathcal{D}(q) \hookrightarrow \mathcal{H}$ should be dense. Recall that a quadratic form is called *semibounded* with bound $\lambda \in \mathbb{R}$, if q is symmetric and $q(x,x) \geq \lambda ||x||^2_{\mathcal{H}}$ holds for all $x \in \mathcal{D}(q)$. It is also well-known, that one can extend a closable semibounded quadratic form q to a closed quadratic form $\overline{q} : \mathcal{D}(\overline{q}) \times \mathcal{D}(\overline{q}) \longrightarrow \mathbb{C}$. Then we have:

Proposition 4.3.22. *Let $q : \mathcal{D}(q) \times \mathcal{D}(q) \longrightarrow \mathbb{C}$ be a closed semibounded quadratic form. Then there exists an unique selfadjoint linear operator $Q : \mathcal{D}(Q) \longrightarrow \mathcal{H}$ with $\mathcal{D}(Q) \subseteq \mathcal{D}(q)$ and $q(x,y) = \langle Qx \,|\, y\rangle$ for $x \in \mathcal{D}(Q)$, $y \in \mathcal{D}(q)$. Q is called the selfadjoint operator induced by q.*

Proof. See [19, Proposition 1.2.9] for instance. □

This leads to the following theorem:

Theorem 4.3.23. *Let $A : \mathcal{D}(A) \longrightarrow \mathcal{H}$ be symmetric operator with dense domain $\mathcal{D}(A) \subseteq \mathcal{H}$. Suppose, that there exists $\lambda > 0$, such that $\langle Ax \,|\, x\rangle_{\mathcal{H}} \geq \lambda ||x||_{\mathcal{H}}$ holds for all $x \in \mathcal{D}(A)$. Then $q(x,y) := \langle Ax \,|\, y\rangle_{\mathcal{H}}$ defines a closable semibounded quadratic form. The operator Q induced by the closure $\overline{q}$ of q is a selfadjoint extension of A with $\langle Qx \,|\, x\rangle_{\mathcal{H}} \geq \lambda ||x||_{\mathcal{H}}$ for all $x \in \mathcal{D}(Q)$. Q is called the* Friedrichs *extension of A.*

Proof. See again [19, Proposition 1.2.10] for instance. □

Now, we want to define an interaction operator, that will generate an appropriate scale of Sobolev spaces:

Definition 4.3.24 (Transmission form)**.** We define $q_{\Lambda^2_{\mathfrak{D}}} : \mathfrak{D}\times\mathfrak{D} \longrightarrow \mathbb{R}$ to be the following quadratic form

$$q_{\Lambda^2_{\mathfrak{D}}}(f,g) := \sum_{i\in\mathbb{N}} \langle \Lambda^2_{b,i} f_i \,|\, g_i\rangle_{L^2(\mathcal{Z}_i)}$$

and call it the *transmission form* corresponding to the transmission space $\mathfrak{D}$.

Lemma 4.3.25. *The quadratic form $q_{\Lambda^2_{\mathfrak{D}}}$ is continuous and semibounded, i.e. there exists $\lambda > 0$, such that*

$$q_{\Lambda^2_{\mathfrak{D}}}(f,f) \geq \lambda ||f||^2_{\mathfrak{D}}$$

holds for all $f \in \mathfrak{D}$. Denote by $\widetilde{\Lambda^2_{\mathfrak{D}}} : \mathcal{D}(\widetilde{\Lambda^2_{\mathfrak{D}}}) \longrightarrow \mathfrak{H}$ the corresponding (positive) selfadjoint operator cf. 4.3.22.

Proof. Since $\Lambda^2_{b,i}$ is a (strictly) positive selfadjoint operator that is bounded from below (for all $i \in \mathbb{N}$) the claim follows. □

Definition 4.3.26. Denote by $\mathcal{H}^s(\mathcal{Z})$ the scale of Sobolev spaces generated by $\widetilde{\Lambda^2_{\mathfrak{D}}}$. In particular, we have $\mathcal{H}^0(\mathcal{Z}) \subseteq \mathfrak{H} \subseteq \widetilde{\mathfrak{H}}$.

Definition 4.3.27 (Transmission algebra)**.** Let $0 < \epsilon \leq 1$. We denote by $\mathcal{A}_{\epsilon}^{\Lambda_{\mathfrak{D}}}$ the following Ψ^*-algebra:

$$\mathcal{A}_{\epsilon}^{\Lambda_{\mathfrak{D}}} := \Big\{ a \in \mathscr{L}(\mathfrak{H}) : a(D(\Lambda_{\mathfrak{D}}^{\infty})) \subseteq D(\Lambda_{\mathfrak{D}}^{\infty}), \\ \forall\, \nu \in \mathbb{N}\, \exists\, c_\nu \geq 0 : ||ad(\Lambda_{\mathfrak{D}}^{\epsilon})^{\nu}(a)x|| \leq c_\nu ||x|| \Big\}.$$

In what follows, we set $\epsilon = 1$. Let us now refine this Ψ^*-algebra by using iterated commutators of suitable first order differential operators. These operator families will be a slight modification of 4.3.5:

Notations 4.3.28.

(i) Denote by $\mathfrak{Y} \subseteq \bigcap_{s\in\mathbb{R}} \mathscr{L}(\mathcal{H}^s(\mathcal{Z}), \mathcal{H}^{s-1}(\mathcal{Z}))$ the set of all operators X, such that there exist open neighbourhoods $\widetilde{\mathcal{V}}_i \subseteq \mathcal{Z}_i$ with $\mathcal{W}_i \cup \widetilde{\mathcal{W}_i} \subseteq \widetilde{\mathcal{V}}_i$ and $X_{|\mathcal{Z}_i\setminus\widetilde{\mathcal{V}}_i}$ is an element of $\mathrm{Diff}_b^1(\mathcal{Z}_i) \cup \mathrm{Diff}_b^0(\mathcal{Z}_i)$ $(i \in \mathbb{N})$.

(ii) Denote by $\mathfrak{N} \subseteq \bigcap_{s\in\mathbb{R}} \mathscr{L}(\mathcal{H}^s(\mathcal{Z}), \mathcal{H}^{s}(\mathcal{Z}))$ the set of all operators M, such that there exist open neighbourhoods $\widetilde{\mathcal{O}}_i \subseteq \mathcal{Z}_i$ with $\mathcal{W}_i \cup \widetilde{\mathcal{W}_i} \subseteq \widetilde{\mathcal{O}}_i$ and $M_{|\mathcal{Z}_i\setminus\widetilde{\mathcal{O}}_i}$ is given by a multiplication operator with a smooth function.

Again, the operator-families given in 4.3.7 are examples of such possible operators, more precisely:

Example 4.3.29.

(i) Let $m \in \mathbb{N}$ and $D_i \in \mathscr{D}(\mathcal{Z}_i)$ $(i = 1, \dots, m)$ be given. Then $D := (D_i)_{i\in\mathbb{N}}$, where we set $D_i := 0$ for $i > m$, acts on the scale of Sobolev spaces $\mathcal{H}^s(\mathcal{Z})$ as an operator of order one, i.e. we have the mapping property

$$D : \mathcal{H}^s(\mathcal{Z}) \longrightarrow \mathcal{H}^{s-1}(\mathcal{Z})$$

for all $s \in \mathbb{R}$. In particular D is an element of $\mathfrak{Y}$.

(ii) Let $m \in \mathbb{N}$ be fixed. We choose a family $(\varphi_i)_{i=1}^m$ of functions $\varphi \in \mathscr{C}^\infty(\mathcal{Z}_i)$ with $\operatorname{supp}\varphi_i \subseteq \mathcal{O}_i$, where $\mathcal{O}_i$ is a closed subset of $\mathcal{Z}_i$ with $\mathcal{W}_i \cup \widetilde{\mathcal{W}_i} \subset \mathcal{O}_i$. We define $M_i := M_{\varphi_i}$ if $i \leq m$ and $M_i = 0$ if $i > 0$, then $M := (M_i)_{i\in\mathbb{N}}$ acts on the scale of Sobolev spaces $\mathcal{H}^s(\mathcal{Z})$ as an operator of order zero, i.e. we have the mapping property

$$D : \mathcal{H}^s(\mathcal{Z}) \longrightarrow \mathcal{H}^s(\mathcal{Z})$$

for all $s \in \mathbb{R}$. In particular M is an element of $\mathfrak{N}$.

Now, using an analogous definition to 4.3.9 we get families $\mathfrak{M}_{int}^{(k)}$ and $\mathfrak{X}_{int}^{(k)}$ of operators that are adapted to the scale of Sobolev spaces generated by $\Lambda_{\mathfrak{D}}$.

Definition 4.3.30 (Refined interaction algebra)**.** We define the *refined interaction algebra* $\Psi^0_{int}(\mathcal{Z})$ to be the Ψ^*-algebra given by

$$\Psi^0_{int}(\mathcal{Z}) := \bigcap_{k\in\mathbb{N}} \Psi^0_{int,k}(\mathcal{Z}),$$

where $\Psi^0_{int,k}(\mathcal{Z})$ consists of all $a \in \mathcal{A}_1^{\Lambda_{\mathfrak{D}}}$, such that

$$\mathrm{ad}\,[M]^\alpha \mathrm{ad}\,[V]^\beta(a) \in \bigcap_{s\in\mathbb{R}} \mathscr{L}(\mathcal{H}^s(\mathcal{Z}), \mathcal{H}^{s-|\alpha|}(\mathcal{Z}))$$

holds for all $V \in (\mathfrak{X}^{(k)}_{int})^{|\beta|}$, for all $M \in (\mathfrak{M}^{(k)}_{int})^{|\alpha|}$ and for all $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0^m$.

Exactly as in 4.3.15 we get:

Theorem 4.3.31. *Let $a \in \Psi^0_{int}(\mathcal{Z})$ be arbitrary.*

(i) *If $\omega_1 \in \mathcal{C}_c^\infty(\mathcal{Z}_i)$ and $\omega_2 \in \mathcal{C}_c^\infty(\mathcal{Z}_j)$ are given, such that*

$$\begin{aligned} &\mathrm{supp}\,\omega_1 \cap \mathcal{V}_i = \emptyset = \mathrm{supp}\,\omega_2 \cap \mathcal{V}_j \textit{ and} \\ &\mathrm{supp}\,\omega_1 \cap \mathrm{supp}\,\omega_2 = \emptyset \quad (\textit{if } i = j), \end{aligned}$$

then $\omega_1 a \omega_2$ is a pseudo-local operator.

(ii) *If $\omega_1, \omega_2 \in \mathcal{C}_c^\infty(\mathcal{Z}_i)$ are given, such that*

$$\mathrm{supp}\,\omega_j \cap \mathcal{V}_i = \emptyset, \quad (j = 1, 2),$$

and $\mathrm{supp}\,\omega_j$ *are both contained in a chart compatible with the* Beals-Type-*characterisation, cf. 4.3.8, then $\omega_1 a \omega_2$ is an ordinary, compactly supported pseudodifferential operator in the interior $\mathring{\mathcal{Z}}_i$ of $\mathcal{Z}_i$.*

Chapter 5

K-theory for conformally compact spaces

An open Riemannian manifold (X_0, g_0) is called *conformally compact space* if it is isometric to the interior of a compact manifold X with boundary ∂X, where X is endowed with the metric $g := \varrho^{-2}h$. Thereby h is a (given) smooth metric on X and $\varrho : X \longrightarrow \overline{\mathbb{R}}_+$ is a boundary defining function.

5.1 Review of algebras of operators on conformally compact spaces

Throughout this chapter, X denotes a smooth, compact, n-dimensional manifold with boundary ∂X. Moreover, we assume that X and ∂X are connected. Let us give a brief overview of the basic objects in 0-calculus. Most of the proofs will be omitted, we refer to [65] for a detailed treatment of the subject.

Let us first introduce the main structure objects: We denote by $\mathcal{V}_0(X)$ the space of all vector fields on X, that vanish at the boundary. Elements of $\mathcal{V}_0(X)$ are called 0-*vector fields* and $\mathcal{V}_0(X)$ is a Lie subalgebra of the Lie algebra of all smooth vector fields on X. If $q \in \partial X$ is arbitrary and

$$(x, y) : X \supseteq V \longrightarrow \overline{\mathbb{R}}_+ \times \mathbb{R}^{n-1}_y, \quad \text{where } \overline{\mathbb{R}}_+ := [0, \infty[, \tag{5.1.1}$$

are local coordinates near q, then $\mathcal{V}_0(X)_{|V}$ is spanned over $\mathscr{C}^\infty(V)$ by the vector fields $x\partial_x$ and $x\partial_{y_j}$ $(j = 1, \ldots, n-1)$. And there is a vector bundle of rank $n = \dim X$ over X together with a natural map $j^0 : {}^0TX \longrightarrow TX$, such that $\mathcal{V}_0(X) = j^0(\mathscr{C}^\infty(X, {}^0TX))$; ${}^0TX$ is called the 0-*tangent bundle*. Note, that the fibres of ${}^0TX$ are given by

$${}^0T_qX = \mathcal{V}_0(X)/\mathcal{I}_q\mathcal{V}_0(X)$$

for $q \in \partial X$, where $\mathcal{I}_q\mathcal{V}_0(X) := \{f \in \mathscr{C}^\infty(X) \,:\, f(q) = 0\}$. Thus $\mathcal{V}_0(X)$ is a finitely generated, projective $\mathscr{C}^\infty(X)$-module (cf. [114, Theorem 1.13]). And it is worth pointing out, that j^0 is an isomorphism over the interior $\mathring{X}$ of X.

Since $\mathcal{I}_q\mathcal{V}_0(X)$ is an ideal in $\mathcal{V}_0(X)$, ${}^0T_qX$ has a Lie algebra structure as well. Moreover, there is a natural vector bundle map ${}^0TX \longrightarrow {}^bTX$ and we denote the restriction of the kernel of this map to ∂X by ${}^0T\partial X$. Note, that the choice of a boundary defining function $\varrho : X \longrightarrow \overline{\mathbb{R}}_+$ (i.e. $\varrho \geq 0$, $\partial X = \{\varrho = 0\}$ and $d\varrho_{|\partial X} \neq 0$) and the exact sequence of Lie-algebras

$$0 \longrightarrow {}^0T\partial X \longrightarrow {}^0TX_{|\partial X} \longrightarrow {}^0TX_{|\partial X}/{}^0T\partial X \longrightarrow 0 \tag{5.1.2}$$

give rise to the identifications

$$^0T\partial X \longrightarrow T\partial X: \ [V] \longmapsto [(1/\varrho_N V)_{|\partial X}]$$

and

$$^0TX_{|\partial X}/^0T\partial X \longrightarrow \partial X \times \mathbb{R}: \ [V] \longmapsto (\pi[V], (1/\varrho_N V \varrho_N)_{|\pi[V]})$$

(see [65, page 4]). Moreover, a normal fibration ν to the boundary together with the boundary defining function ϱ yield a collar neighbourhood of the boundary, such that we get the splitting

$$^0TX_{|\partial X} \cong {}^0T\partial X \oplus {}^0TX_{|\partial X}/^0T\partial X. \tag{5.1.3}$$

Let

$$\begin{aligned} j_\varrho : T^*\partial X \oplus (\partial X \times \mathbb{R}) \ \longrightarrow \ & {}^0T^*\partial X \oplus ({}^0TX_{|\partial X}/^0T\partial X)^* \\ & \cong {}^0T^*X_{|\partial X} \end{aligned} \tag{5.1.4}$$

denote the identification of the corresponding covector bundles over ∂X.

Before we can give the definition of pseudodifferential operators in this setting, let us briefly discuss the used blow up spaces. For this, let

$$\beta_b^2 : X_b^2 := [X^2; (\partial X)^2] \longrightarrow X^2$$

be the *b-blow up* (cf. [83]) and $B := (\beta_b^2)^{-1}(\partial\Delta)$ the preimage of the boundary of the diagonal $\Delta \subseteq X^2$. Let $\Delta_b := \overline{(\beta_b^2)^{-1}(\Delta \setminus (\partial X)^2)}^{X_b^2}$ denote the b-diagonal of the blow up space[1]. Then we define the *extended* 0-*double space* $X_{0,e}^2$ to be the manifold given by the *extended* 0-*blow up*

$$\beta_{0,e}^2 : X_{0,e}^2 := [X_b^2; B] \xrightarrow{\beta} X_b^2 \xrightarrow{\beta_b^2} X^2.$$

We call $\Delta_{0,e} := \overline{\beta^{-1}(\Delta_b \setminus \partial X_b^2)}^{X_{0,e}^2}$ the *extended* 0-*diagonal* and the new boundary hypersurface $\mathrm{ff}^{0,e}$ produced by the second blow up the *extended* 0-*front face.*

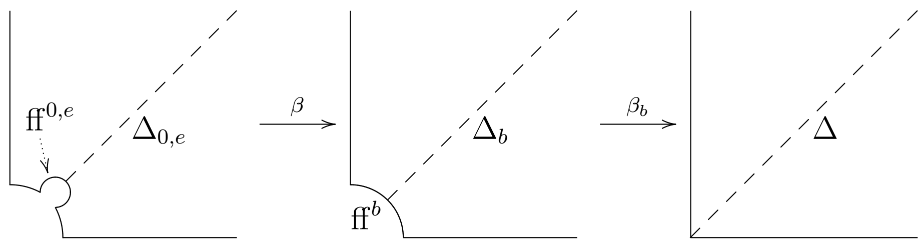

Figure 5.1: The *extended* 0-*blow up* $X_{0,e}^2$ of X.

[1]Note, that there is a canonical diffeomorphism $B \cong [-1, 1] \times \partial X$.

Let us denote by (x,y) resp. (x',y') the lift of local coordinates given by (5.1.1) through the projection onto the left resp. right factor of X^2. Then we get a system of local (projective) coordinates $(\tau,U,r,y') \in [-1,1]\times \mathbb{R}^{n-1}_U \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{n-1}_{y'}$, where

$$\tau = \frac{x-x'}{x+x'},\ r = x+x' \text{ and } U = \frac{y-y'}{x+x'}.$$

With respect to this coordinates we have

$$\Delta_{0,e} = \{\tau = 0,\ U = 0\} \text{ and } \mathrm{ff}^{0,e} = \{r = 0\}.$$

Note, that $\{\tau=-1\}$ resp. $\{\tau = 1\}$ correspond to the lift of the left resp. right boundary of the b-double space X^2_b to $X^2_{0,e}$. The lift of the b-front face to $X^2_{0,e}$ would be given by $\{|U| = \infty\}$, so the projective coordinates are only valid apart from that face.

To complete notation, let us recall the densities adapted to this calculus. If we apply the smooth density functor Ω^α of α-densities to the 0-tangent bundle ${}^0TX$ we obtain the smooth bundle of 0-α-*densities* ${}^0\Omega^\alpha(X)$. A non vanishing section is given by $|\frac{dx}{x^n}dy|^\alpha$ and we get a well-defined integral

$$\int\limits_X : \dot{\mathscr{C}}^\infty(X, {}^0\Omega^1(X)) \longrightarrow \mathbb{C}.$$

Thus we can define a natural scalar product on $\dot{\mathscr{C}}^\infty(X, {}^0\Omega^{\frac12}(X))$ given by

$$\langle f,g\rangle_{L^2(X,{}^0\Omega^{\frac12})} := \int\limits_X f\overline{g},$$

where $f,g \in \dot{\mathscr{C}}^\infty(X,{}^0\Omega^{\frac12}(X))$. We denote by $L^2(X,{}^0\Omega^{\frac12})$ the closure of $\dot{\mathscr{C}}^\infty(X,{}^0\Omega^{\frac12}(X))$ with respect to this inner product. Finally let us denote by $KD^{\frac12}_{0,e}$ the *extended* 0-*kernel half-density bundle* $\varrho_{\mathrm{ff}^{0,e}}^{-\frac{n}{2}}\Omega^{\frac12}(X^2_{0,e})$, where $\varrho_{\mathrm{ff}^{0,e}}$ denotes a defining function for $\mathrm{ff}^{0,e}$. Now we are able to give the definition of pseudodifferential operators:

Definition 5.1.1. A bounded linear operator

$$A : \dot{\mathscr{C}}^\infty(X,{}^0\Omega^{\frac12}) \longrightarrow \mathscr{C}^{-\infty}(X,{}^0\Omega^{\frac12})$$

is said to be a *classical* 0-*pseudodifferential operator of order* $m\in\mathbb{C}$ in the *small calculus* provided

$$\kappa_A \in \left\{\kappa \in I^m_{cl}(X^2_{0,e},\Delta_{0,e};KD^{\frac12}_{0,e}) \,:\, \kappa\equiv 0 \text{ at } \partial X^2_{0,e}\setminus \mathrm{ff}^{0,e}\right\},$$

where $\equiv$ means vanishing to infinite order. The space of all classical 0-pseudodifferential operators of order m is denoted by $\Psi^m_0(X,{}^0\Omega^{\frac12})$.

Following this definition, one can prove, that there is a well defined *homogeneous principal symbol map*

$${}^0\sigma^{(m)} : \Psi^m_0(X,{}^0\Omega^{\frac12}) \longrightarrow S^{[m]}({}^0T^*X),$$

which has the usual multiplication property

$$^0\sigma^{(m_1+m_2)}(A_1A_2) = {}^0\sigma^{(m_1)}(A_1)^0\sigma^{(m_2)}(A_2) \in S^{[m_1+m_2]}({}^0T^*X),$$

for $A_i \in \Psi_0^{m_i}(X, {}^0\Omega^{\frac{1}{2}})$ ($m_i \in \mathbb{R}$). Moreover, one has to define two more classes of 0-operators, namely

- $\Psi_0^{m,k}(X, {}^0\Omega^{\frac{1}{2}})$ ($k, m \in \mathbb{C}$) (k-type operators), which are given by

$$\Psi_0^{m,k}(X, {}^0\Omega^{\frac{1}{2}}) := \varrho^{-k}\Psi_0^m(X, {}^0\Omega^{\frac{1}{2}}) \tag{5.1.5}$$

  (see [65, Definition 2.3.4]), and

- $\Psi_0^{m,k,\gamma}(X, {}^0\Omega^{\frac{1}{2}})$, which is the space of all m^{th}*-order* 0*-pseudodifferential operators of type* k *with bounds* γ (see [65, Definition 2.4.3]).

Remark 5.1.2. Note, that (5.1.5) makes sense since $\Psi_0^m(X, {}^0\Omega^{\frac{1}{2}})$ is invariant under conjugations by powers of the boundary defining function ϱ, i.e. if $z \in \mathbb{C}$ is arbitrary then

$$\varrho^{-z}\Psi_0^m(X, {}^0\Omega^{\frac{1}{2}})\varrho^z = \Psi_0^m(X, {}^0\Omega^{\frac{1}{2}}) \tag{5.1.6}$$

holds, cf. [65, Lemma 2.3.3].

The reduced normal operator

In addition to the principal symbol one has to introduce an operator valued boundary symbol to give a full description of the Fredholm conditions for pseudodifferential operators on conformally compact spaces. Since we need this so-called *reduced normal operator* for the K-theoretic calculation, we give a short overview of its definition.

To this end let $\varrho : X \longrightarrow \overline{\mathbb{R}}_+$ be a (fixed) boundary defining function, then it uniquely determines a trivialization of the positive normal bundle

$$\begin{array}{rccc} \Phi : N^+\partial X := T^+X_{|\partial X}/T\partial X & \xrightarrow{\cong} & \overline{\mathbb{R}}_+ \times \partial X; \\ [V_q] & \longmapsto & (d_{\varrho_N}(q)(V_q), q) \end{array}$$

of the boundary. Now, let $\nu : \Gamma \xrightarrow{\cong} \nu(\Gamma) \subseteq X$ be a *normal fibration* to the boundary ∂X; here $\partial X \subseteq \Gamma \subseteq N^+\partial X$ is an open neighbourhood of the zero-section ∂X in $N^+\partial X$. Thus local coordinates $\chi : \partial X \supseteq V \longrightarrow \mathbb{R}_Y^{n-1}$ on the boundary give raise to local coordinates

$$(x, y) : X \supseteq \mathcal{V} := \nu(\Phi^{-1}(\overline{\mathbb{R}}_+ \times \mathcal{W})) \longrightarrow \overline{\mathbb{R}}_+ \times \mathbb{R}_y^{n-1}$$

for $q \in \mathcal{W} \subseteq \partial X$. We then define the *reduced normal operator* $\mathcal{N}_\varrho^{\nu,\chi}(A)$ of an element $A \in \Psi_0^{m,k,\gamma}(X, {}^0\Omega^{\frac{1}{2}})$ at $(0, y) = q \in \partial X$ to be the distributional density

$$\begin{aligned} \mathcal{N}_\varrho^\nu(A)(y, \eta, \tau, r) \;&:=\; \mathcal{F}_\varrho^\nu(A)(y, r\eta, \tau) \left|\frac{dr}{r} d\tau\right|^{\frac{1}{2}} \\ &\in \mathscr{C}^{-\infty}([-1, 1], \Omega^{\frac{1}{2}})\widehat{\otimes}_\pi \mathscr{C}^{-\infty}(\overline{\mathbb{R}}_+, {}^b\Omega^{\frac{1}{2}}), \\ \mathcal{F}_\varrho^\nu(A)(y, \eta, \tau) \;&:=\; \int_{\mathbb{R}_U^{n-1}} e^{-iU\eta}\widehat{\kappa}_A(\tau, U, 0, y)\, dU \in \mathscr{C}^{-\infty}([-1, 1]). \end{aligned}$$

Here κ_A denotes the lifted Schwartz kernel of the operator A written in coordinates near the extended 0-front face $\mathrm{ff}^{0,e}$ as chosen above, i.e.

$$\kappa_A = \widehat{\kappa}_A(\tau, U, r, y')|\, d\tau\, dU\, \frac{dr}{r^n}\, dy'|^{\frac{1}{2}}$$

with

$$\widehat{\kappa}_A \in \mathscr{C}^{-\infty}([-1,1])\widehat{\otimes}_\pi \mathscr{S}'(\mathbb{R}^{n-1}_U)\widehat{\otimes}_\pi \mathscr{C}^{-\infty}(\overline{\mathbb{R}}_+)\widehat{\otimes}_\pi \mathscr{C}^{-\infty}(\mathbb{R}^{n-1}_{y'}).$$

For a detailed treatment of the *reduced normal operator* we again refer to [65, Chapter 4].

5.2 Review of C^*-algebras of b-c-operators

Throughout this section let M be the closed interval $[0,1]$. The reduced normal operator of a pseudodifferential operator in $\Psi_0(X, {}^0\Omega^{\frac{1}{2}})$ associates to a given operator a family of b-c-pseudodifferential operators on M. Roughly speaking the algebra of b-c-operators consists of pseudodifferential operators with different behaviours at the two endpoints of M: they are of b-type at $\{0\}$ and of *cusp*-type at $\{1\}$. For a detailed treatment of this calculus see again [65, Chapter 3].

Preliminaries

Let $\mathcal{B} \subseteq \mathscr{C}(M \times \{\pm 1\}) \oplus \mathscr{C}([-1,1]) \oplus \mathscr{C}([-1,1])$ be the C^*-subalgebra consisting of all triples (f_1, f_2, f_3) satisfying

$$f_1(0, \pm 1) = f_2(\pm 1) \quad \text{and} \quad f_1(1, \pm 1) = f_3(\pm 1). \tag{5.2.1}$$

Lemma 5.2.1. *We have $\mathcal{B} \cong \mathscr{C}(\mathcal{S}^1)$.*

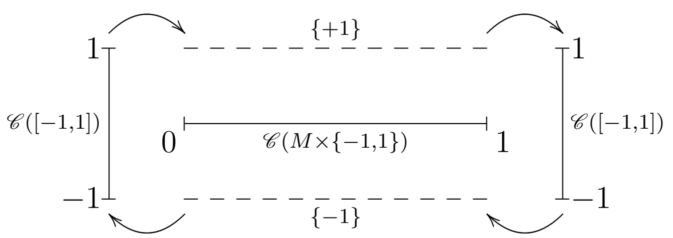

Figure 5.2: Gluing $\mathscr{C}(\mathcal{S}^1)$ out of $\mathcal{B}$.

Proof. We define two mappings $\mathfrak{K} : \mathcal{B} \to \mathscr{C}(S^1)$ and $\mathfrak{M} : \mathscr{C}(S^1) \to \mathcal{B}$ via

$$\mathfrak{K} : (f_1, f_2, f_3) \mapsto f, \text{ where } f(\varphi) := \begin{cases} f_1(\frac{2\varphi}{\pi}, 1), & 0 \le \varphi \le \frac{\pi}{2}, \\ f_3(-\frac{4\varphi}{\pi} + 3), & \frac{\pi}{2} < \varphi \le \pi, \\ f_1(-\frac{2\varphi}{\pi} + 3, -1), & \pi < \varphi \le \frac{3\pi}{2}, \\ f_2(\frac{4\varphi}{\pi} - 7), & \frac{3\pi}{2} < \varphi \le 2\pi, \end{cases}$$

$$\mathfrak{M}: f \mapsto (f_1, f_2, f_3), \text{ where } \begin{array}{ll} f_1(x,1) := f(\frac{\pi}{2}x), & x \in M, \\ f_1(x,-1) := f(-\frac{\pi}{2}x + \frac{3}{2}\pi) & x \in M, \\ f_2(t) := f(\frac{\pi}{4}t + \frac{7}{4}\pi), & t \in [-1,1], \\ f_3(r) := f(-\frac{\pi}{4}r + \frac{3}{4}\pi), & r \in [-1,1]. \end{array}$$

It is easy to check, that $\mathfrak{K}$ and $\mathfrak{M}$ are inverse maps, so we get $\mathcal{B} \cong \mathscr{C}(S^1)$ as desired. □

Remark 5.2.2. Let $\widehat{\mathcal{B}}$ denote all elements in $\mathcal{B}$, such that (in addition to (5.2.1)) $f_1(z,1) = c$ and $f_2(z,-1) = d$ holds for all $z \in M$. Then we have $\widehat{\mathcal{B}} \cong \mathscr{C}(\mathcal{S}^1)$.

b-c-operatoralgebras

Let $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{R}$ be fixed real numbers. Furthermore we denote by $\varrho_0 : x \longmapsto x$ resp. $\varrho_1 : x \longmapsto (1-x)$ defining functions for the components $\{0\}$ resp. $\{1\}$ of ∂M. Again let us give a short overview on the main structure elements and blow up spaces.

Definition 5.2.3. A *b-c-vector field* on M is a smooth vector field V, such that $Vf \in \varrho_0\varrho_1^2\mathscr{C}^\infty(M)$ for all $f \in \mathscr{C}^\infty(M)$.

The vector bundle corresponding to the b-c-vector fields will be denoted by ${}^{b,c}TM$. Moreover, let

$$\beta_b^2 : M_b^2 := [M^2; \{(0,0)\} \cap \{(1,1)\}] \longrightarrow M^2$$

be the b-blow up of M^2 with b-diagonal $\Delta_b := \overline{(\beta_b^2)^{-1}(\Delta \setminus \partial M^2)}^{M_b^1}$. The new boundary faces obtained by blowing up (j,j) $(j = 0,1)$ are denoted by $\mathrm{ff}^b(j)$ and we set $B := \Delta_b \cap \mathrm{ff}^b(1)$. Then we call the map

$$\beta_{b,c}^2 : M_{b,c}^2 := [M_b^2; B] \longrightarrow M_b^2 \xrightarrow{\beta_b^2} M^2$$

the *b-c-blow up* and $M_{b,c}^2$ the *b-c-double space*. The new boundary surface produced by this last blow up will be denoted by ff^c and is called the *cusp front face*.

Moreover, if we apply the smooth density functor Ω^α of α-densities to the b-c-tangent bundle ${}^{b,c}TM$ we obtain the smooth bundle of *b-c-α-densities* ${}^{b,c}\Omega^\alpha(X)$. In this way we also obtain a natural scalar product on $\dot{\mathscr{C}}^\infty(X, {}^{b,c}\Omega^{\frac{1}{2}}(X))$ given by

$$\langle f, g\rangle_{L^2(M,{}^{b,c}\Omega^{\frac{1}{2}})} := \int_M f_1\overline{f_2} = \int_0^1 \widehat{f}_1(z)\overline{\widehat{f}_2(z)}\,\frac{dz}{z(1-z)^2},$$

where $f_j = \widehat{f}_j \left|\frac{dz}{z(1-z)^2}\right|^{\frac{1}{2}} \in \dot{\mathscr{C}}^\infty(M, {}^{b,c}\Omega^{\frac{1}{2}})$. Let $L^2(M, {}^{b,c}\Omega^{\frac{1}{2}})$ denote the *Hilbert* space completion of $\dot{\mathscr{C}}^\infty(X, {}^{b,c}\Omega^{\frac{1}{2}}(X))$ with respect to this inner product.

Definition 5.2.4. A bounded linear operator

$$A : \dot{\mathscr{C}}^\infty(M, {}^{b,c}\Omega^{\frac{1}{2}}) \longrightarrow \mathscr{C}^{-\infty}(M, {}^{b,c}\Omega^{\frac{1}{2}})$$

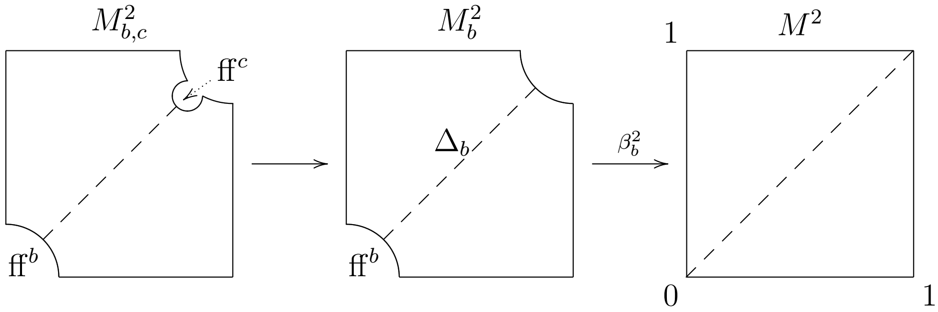

Figure 5.3: The b-c-blow up of M^2.

is said to be a *classical b-c-pseudodifferential operator of order* $m \in \mathbb{C}$ in the *small calculus* provided

$$\kappa_A \in \left\{ \kappa \in I^m_{cl}(M^2_{b,c}, \Delta_{b,c}; KD^{\frac{1}{2}}_{b,c}) \,:\, \kappa \equiv 0 \text{ at } \partial M^2_{b,c} \setminus (\mathrm{ff}^b \cup \mathrm{ff}^c) \right\},$$

where $\equiv$ means vanishing to infinite order. The space of all classical b-c-pseudodifferential operators of order m is denoted by $\Psi^m_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$.

Here, $KD^{\frac{1}{2}}_{b,c}$ denotes the *b-c-kernel half density bundle* $\varrho_{\mathrm{ff}^b}^{-\frac{1}{2}} \varrho_{\mathrm{ff}^c}^{-1} \Omega^{\frac{1}{2}}(M^2_{b,c})$, where ϱ_{ff^b} resp. ϱ_{ff^c} are defining functions for ff^b resp. ff^c.

The algebra of all b-c-pseudodifferential operators $\Psi^0_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$ can be realized as a symmetric subalgebra of $\mathscr{L}(\varrho_0^{\mathfrak{a}_0} \varrho_1^{\mathfrak{a}_1} L^2(M, {}^{b,c}\Omega^{\frac{1}{2}}))$. We have the following symbol morphism:

$$\begin{aligned} \tau_{b,c} : \Psi^0_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}}) &\longrightarrow \mathscr{C}({}^{b,c}S^*M) \oplus \mathcal{B}(\Gamma_{\mathfrak{a}_0}) \oplus \mathcal{B}(\mathbb{R}) : \\ A &\longmapsto \left({}^{b,c}\sigma^{(0,0)}(A), I_b(A)_{|\Gamma_{\mathfrak{a}_0}}, I_c(A)\right), \end{aligned}$$

where $\Gamma_{\mathfrak{a}_0} = \{z \in \mathbb{C} \,:\, \operatorname{Im} z = -\mathfrak{a}_0\} \cong \mathbb{R}$. Here $\mathcal{B}(\mathbb{R})$ denotes the C^*-closure of $S^0_{cl}(\mathbb{R})$ in $\mathscr{C}_b(\mathbb{R})$ and $\mathcal{B}(\Gamma_{\mathfrak{a}_0})$ the closure of $\mathcal{M}^0_{|\Gamma_{\mathfrak{a}_0}}$ [2] in $\mathscr{C}_b(\Gamma_{\mathfrak{a}_0})$. Note, that $\mathcal{B}(\Gamma_{\mathfrak{a}_0})$ and $\mathcal{B}(\mathbb{R})$ are both isomorphic to $\mathscr{C}[-1,1]$ by [75], but we keep the notations $\mathcal{B}(\mathbb{R})$ resp. $\mathcal{B}(\Gamma_{\mathfrak{a}_0})$ to enable comparison to [65]. Especially, we refer to [65, chapter 3] for the exact definition of the single symbol maps ${}^{b,c}\sigma^{(0,0)}(A)$ (the principal symbol), $I_b(A)_{|\Gamma_{\mathfrak{a}_0}}$ (the *b-indical family*) and $I_c(A)$ (the *c-indical family*) for an operator $A \in \Psi^0_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$.

Moreover, let $\mathcal{B}^{(\mathfrak{a}_0,\mathfrak{a}_1)}_{b,c}$ be the C^*-algebra generated by all operators of $\Psi^0_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$ in the C^*-algebra $\mathscr{L}(\varrho_0^{\mathfrak{a}_1} \varrho_1^{\mathfrak{a}_1} L^2(M, {}^{b,c}\Omega^{\frac{1}{2}}))$ and $\mathcal{K}^{(\mathfrak{a}_0,\mathfrak{a}_1)}_{b,c}$ denote the ideal of compact operators in this algebra. Then we get the following exact sequence of C^*-algebras:

$$0 \longrightarrow \mathcal{K}^{(\mathfrak{a}_0,\mathfrak{a}_1)}_{b,c} \longrightarrow \mathcal{B}^{(\mathfrak{a}_0,\mathfrak{a}_1)}_{b,c} \longrightarrow Q^{(\mathfrak{a}_0,\mathfrak{a}_1)}_{b,c} \longrightarrow 0, \tag{5.2.2}$$

[2] $\mathcal{M}^0$ denotes the space of all entire maps h, such that $[\xi \longmapsto h(\xi + i\mu)] \in S^0_{cl}(\mathbb{R}_\xi)$ with uniform estimates for μ in compact subsets of $\mathbb{R}$.

where the C^*-algebra of joint b-c-symbols $Q_{b,c}^{(\mathfrak{a}_0,\mathfrak{a}_1)} := R(\tau_{b,c})$ (see also [65, (3.14)] consists of all triples

$$(f_0, f_1, f_2) \in \mathscr{C}({}^{b,c}S^*M) \oplus \mathcal{B}(\Gamma_{\mathfrak{a}_0}) \oplus \mathcal{B}(\mathbb{R})$$

satisfying

$$\sigma_{\mathcal{B}}^{(0)}(f_1)(\pm 1) = f_0(0, \pm 1) \quad \text{and} \quad \sigma_{\mathcal{B}}^{(0)}(f_2)(\pm 1) = f_0(1, \pm 1). \tag{5.2.3}$$

Here $\sigma_{\mathcal{B}}^{(0)} : \mathcal{B}(R) \longrightarrow \mathscr{C}(\{\pm 1\})$ denotes the map given by the extension of the homogeneous principal part $\sigma^{(0)} : S^0_{cl}(\mathbb{R}) \to \mathscr{C}^\infty(\{\pm 1\}) = \mathbb{C} \oplus \mathbb{C}$ (cf. [75]); note, that we have the sequence

$$0 \longrightarrow \mathscr{C}_0(\mathbb{R}) \longrightarrow \mathcal{B}(\mathbb{R}) \longrightarrow \mathbb{C} \oplus \mathbb{C} \longrightarrow 0.$$

By (5.2.1) and lemma 5.2.1, we conclude $Q_{b,c}^{(\mathfrak{a}_0,\mathfrak{a}_1)} \cong \mathscr{C}(S^1)$.

Remark 5.2.5. By [65, Formula (3.5)]

$$\varrho_0^{z_0} \varrho_1^{z_1} \Psi^0_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}}) \varrho_0^{-z_0} \varrho_1^{-z_1} = \Psi^0_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$$

holds for all z_0, $z_1 \in \mathbb{C}$. So C^*-closures of $\Psi^0_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$ with respect to different weighted $L^2(M, {}^{b,c}\Omega^{\frac{1}{2}})$ spaces differ only by a Hilbert space isometry. This shows that - for K-theory computation - we can restrict ourself to the case $\mathfrak{a}_0 = 0 = \mathfrak{a}_1$. Using this fact, we also avoid heavy notation in the sequel of this section: from now on we drop the weight indices, i.e. we write $\mathcal{B}_{b,c}$ instead of $\mathcal{B}_{b,c}^{(\mathfrak{a}_0,\mathfrak{a}_1)}$ and so on.

Lemma 5.2.6. *There exists $A \in \mathcal{B}_{b,c}$, such that* ind $(A) = 1$. *Especially the index map is surjective.*

We owe the outline of the proof S. Moroianu.

Proof. First of all note, that we can regard a b-c-operator as a c-operator at both ends if it behaves like a b-differential operator at the b-end, using the transcendental blow-up

$$[0, \infty[\ni x \longmapsto t = e^{-\frac{1}{x}} \in [0, \infty[$$

at the b-end $\{0\}$ of M (see [85, Appendix C]). So we can use [68, Formula (27)]

$$\begin{aligned} \operatorname{ind}(A) &= \overline{\mathrm{AS}}(A) - i\frac{1}{2\pi}\Big(\overline{\mathrm{Tr}}_{Q_{\{0\}}}\left(I_c^{\{0\}}(A)\partial_\xi\left[I_c^{\{0\}}(A)^{-1}\right]\right) & (5.2.4) \\ &\qquad + \overline{\mathrm{Tr}}_{Q_{\{1\}}}\left(I_c^{\{1\}}(A)\partial_\xi\left[I_c^{\{1\}}(A)^{-1}\right]\right)\Big) \\ &= \overline{\mathrm{AS}}'(A) + \frac{1}{2}(\eta_{\{0\}}(A) + \eta_{\{1\}}(A)) & (5.2.5) \end{aligned}$$

to find the desired operator within this special class of b-c-operators. Here (5.2.5) uses the η invariant defined in [84] (see also [68, Page 15] for details). Note, that the $1/2\pi$-factor arises by the slightly different definitions for the η-invariant in the papers [84] and [68]. Choose an operator $R \in \Psi^{-\infty}_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$, such that:

- The smooth Schwartz-kernel κ_R of R vanishes in a neighbourhood of the b-front face ff^b.

- If we define $A := Id + R$, then $I_c(A)$ should extend to an invertible symbol in $S^0_{cl}(\mathbb{R}_\xi)$. [3]

Then A is a fully elliptic operator in $\mathcal{B}_{b,c}$ with ${}^{b,c}\sigma^{(0,0)}(A) \equiv 1$ and $I_b(A)_{|\Gamma_{\mathfrak{a}_0}} \equiv 1$. Note, that this forces $\sigma^{(0)}_{\mathcal{B}}(I_c(A))(\pm 1) = 1$ by (5.2.3) and we can think of $I_c(A)$ as a function from $\mathcal{S}^1$ to itself, so our last requirement on R is:

- $I_c(A)$, regarded as a function from $\mathcal{S}^1$ to itself, should have winding number 1.

Finally let us note, that for any operator B which inverts A up to trace class remainders we have $I_c^{\{0\}}(B) = I_c^{\{0\}}(A)^{-1}$ and $I_c^{\{1\}}(B) = I_c^{\{1\}}(A)^{-1}$. Now we apply the index formula (5.2.5) and get

$$\begin{aligned} \operatorname{ind}(A) &= \overline{\mathrm{AS}}(A) + \frac{1}{2\pi i}\,\overline{\mathrm{Tr}}_Q\left(I_c^{(1)}(A)\partial_\xi\left[I_c^{(1)}(A)^{-1}\right]\right) \\ &= \overline{\mathrm{AS}}'(A) + \frac{1}{2}\,\eta_{\{1\}}(A), \end{aligned}$$

since the c-indical contribution at $\{0\}$ vanishes. Moreover, we have $\overline{\mathrm{AS}}'(A) = \overline{\mathrm{AS}}'(1) = 0$, because $\overline{\mathrm{AS}}'$ is invariant under perturbations of high enough negative order (remember, that R is of order $-\infty$). Now, by [84, Formula (5.3)] $\frac{1}{2}\eta_{\{1\}}(A)$ computes the winding number of $I_c(A)$, so $\operatorname{ind}(A) = 1$ and we have found our desired operator. $\square$

Theorem 5.2.7. *We have $K_0(\mathcal{B}_{b,c}) = \mathbb{Z}$ and $K_1(\mathcal{B}_{b,c}) = 0$.*

Proof. (5.2.2) induces the following six term exact sequence

$$\begin{tikzcd} \mathbb{Z} \arrow[r] & K_0(\mathcal{B}_{b,c}) \arrow[r] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[u, "\operatorname{ind}"] & K_1(\mathcal{B}_{b,c}) \arrow[l] & 0 \arrow[l] \end{tikzcd}.$$

Since the index-mapping ind is surjective, cf. lemma 5.2.6, we conclude $K_0(\mathcal{B}_{b,c}) = \mathbb{Z}$ and $K_1(\mathcal{B}_{b,c}) = 0$. $\square$

Now, let $\mathcal{D}$ be the set of all $N \in \mathcal{B}_{b,c}$ satisfying the additional condition

$$ {}^{b,c}\sigma^{(0,0)}(N)(z,1) = c, \quad {}^{b,c}\sigma^{(0,0)}(N)(z,-1) = d \tag{5.2.6}$$

for all $z \in [0,1]$ ($c, d \in \mathbb{R}$), then we have:

Proposition 5.2.8. *$\mathcal{D}$ is a C^*-subalgebra of $\mathcal{B}_{b,c}$. Moreover, we have $K_0(\mathcal{D}) = \mathbb{Z}$, $K_1(\mathcal{D}) = 0$ and the map $\pi : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{K}_{b,c}$ induces an isomorphism $K_0(\mathcal{D}) \cong K_0(\mathscr{C}(\mathcal{S}^1))$.*

---

[3] Note, that if $\widehat{\kappa}_R(S,x')\left|dS\frac{dx'}{x'}\right|^{1/2}$ is a local representation of κ_R in coordinates $S = \frac{1}{x'} - \frac{1}{x}$ and x' near the c-front face ff^c then the c-indical family $I_c(R)(\xi) = \int_{-\infty}^{\infty} e^{-iS\xi}\widehat{\kappa}(S,0)\,dS$ is a Schwartz-function. Moreover, the c-indical family $I_c(B)$ of an operator $B \in \Psi^m_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$ is an element of $S^m_{cl}(\mathbb{R}^n)$.

Proof. To shorten notation, we set ${}^{b,c}\sigma^{(0,0)} =: \sigma$. Let us first prove, that $\mathcal{D}$ is an algebra. Thus let $N_1,\ N_2 \in \mathcal{D}$ be arbitrary. Then

$$\sigma(N_1N_2)(z,1) = \sigma(N_1)(z,1)\sigma(N_2)(z,1) = c_1c_2,$$

which shows that $\mathcal{D}$ is an algebra. $\mathcal{D}$ includes all adjoints, since σ is a $*$-homomorphism. So what is left, is to prove, that $\mathcal{D}$ is closed: To this end, let $(N_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ be a sequence with

$$N_n \xrightarrow{n\to\infty} N \in \mathcal{B}_{b,c}.$$

Then it follows that $(\sigma(N_n)(z,\pm1))_{n\in\mathbb{N}}$ are *Cauchy*-sequences in $\mathbb{C}$ for all $z \in [0,1]$, since σ is continuous. Hence there exist $c^{\pm} \in \mathbb{C}$, such that $\sigma(N_n)(z,\pm1) \xrightarrow{n\to\infty} c^{\pm}$ (for all $z \in [0,1]$). But this implies

$$\begin{aligned} |\sigma(N)(z,\pm1) - c^{\pm}| &\leq |\sigma(N-N_n)(z,\pm1)| + |\sigma(N_n)(z,\pm1) - c^{\pm}| \\ &\leq C||N-N_n|| + |\sigma(N_n)(z,\pm1) - c^{\pm}| \\ &\xrightarrow{n\to\infty} 0, \end{aligned}$$

which shows that $N \in \mathcal{D}$ and we have proved that $\mathcal{D}$ is a C^*-subalgebra of $\mathcal{B}_{b,c}$. Moreover, by 5.2.2 we have

$$K_*(\mathcal{D}/\mathcal{K}_{b,c}) \cong K_*(\mathscr{C}(\mathcal{S}^1)),$$

so we can use the same argument as in the proof of the previous theorem to compute the K-theory of $\mathcal{D}$. Note, that 5.2.6 actually proves that one can find an index one operator in $\mathcal{D}$. □

Let $\widehat{\mathcal{D}}$ denote the set

$$\widehat{\mathcal{D}} := \left\{N \in \mathcal{D} \,:\, I_b(N)_{|\Gamma_{\mathfrak{a}_0}} = 0\right\}. \tag{5.2.7}$$

Then $\widehat{\mathcal{D}}$ is a closed two sided ideal in $\mathcal{D}$, since $\widehat{\mathcal{D}}$ is given as the kernel of a linear $*$-homomorphism.

Theorem 5.2.9. *We have $K_i(\widehat{\mathcal{D}}) = 0$ $(i = 0,1)$.*

Proof. First note, that the map $A \mapsto I_b(A)_{|\Gamma_{\mathfrak{a}_0}}$ induces an isomorphism

$$\mathcal{D}/\widehat{\mathcal{D}} \cong \mathscr{C}([-1,1]).$$

And we have the split exact sequence

$$0 \longrightarrow \mathscr{C}_0(\mathbb{R}) \longrightarrow \mathscr{C}(\mathcal{S}^1) \xrightarrow[ev]{} \mathbb{C} \longrightarrow 0,$$

where ev denotes the point evaluation. Consequently ev_* induces an isomorphism

$$K_0(\mathscr{C}(\mathcal{S}^1)) \cong \mathbb{Z}$$

in K_0. Now we consider the commutative diagram and its image through the K_0-functor

$$\begin{array}{ccccccc} \mathcal{D} & \xrightarrow{\varphi} & \mathscr{C}([-1,1]) & & K_0(\mathcal{D}) & \xrightarrow{K_0(\varphi)} & K_0(\mathscr{C}([-1,1])) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle ev} & \xmapsto{K_0} & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \mathscr{C}(\mathcal{S}^1) & \xrightarrow{ev} & \mathbb{C} & & K_0(\mathscr{C}(\mathcal{S}^1)) & \xrightarrow{\cong} & \mathbb{Z}, \end{array}$$

where π is the map from 5.2.8 and $\varphi := I_b(\cdot)_{|\Gamma_{\mathfrak{a}_0}}$. We conclude, that $K_0(\varphi)$ is also an isomorphism. Then the short exact sequence

$$0 \longrightarrow \widehat{\mathcal{D}} \longrightarrow \mathcal{D} \longrightarrow \mathscr{C}([-1,1]) \longrightarrow 0$$

induces the following six term exact sequence

$$\begin{tikzcd} K_0(\widehat{\mathcal{D}}) \arrow[r] & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \arrow[d] \\ 0 \arrow[u] & 0 \arrow[l] & K_1(\widehat{\mathcal{D}}) \arrow[l] \end{tikzcd} ,$$

in K-theory. This proves our claim. $\square$

Corollary 5.2.10. *Let X be a compact manifold with boundary ∂X. Then we have*

$$K_i(\mathscr{C}(S^*\partial X) \otimes \widehat{\mathcal{D}}) = 0.$$

Proof. $S^*\partial X$ is a compact Hausdorff space, thus the *Künneth* theorem for tensor products [16, Theorem 23.1.3] (see also [16, 22.3.5 (d)]) yields the assertion. $\square$

5.3 K-theory for operators on conformally compact spaces

By [65, Lemma 5.1.4] $\Psi_0^0(X, {}^0\Omega^{\frac{1}{2}})$ can be realized as a symmetric subalgebra of the C^*-algebra $\mathscr{L}(\varrho^{\mathfrak{a}}L^2(X, {}^0\Omega^{\frac{1}{2}}))$. To shorten notation, let us denote the compact operators in $\mathscr{L}(\varrho^{\mathfrak{a}}L^2(X, {}^0\Omega^{\frac{1}{2}}))$ by $\mathcal{K}_0^{(\mathfrak{a})}$. Then we have a joint-symbol map given by

$$\tau_0 := ({}^0\sigma^{(0)}, \mathcal{N}_\varrho^\nu) : \Psi_0^0(X, {}^0\Omega^{\frac{1}{2}}) \longrightarrow \mathscr{C}({}^0S^*X) \oplus \mathscr{C}(S^*\partial X, \mathcal{B}_{b,c}^{(a)}) =: Q^{(\mathfrak{a})},$$

where $\mathcal{B}_{b,c}^{(\mathfrak{a})} := \mathcal{B}_{b,c}^{(\mathfrak{a},0)}$. Moreover, if we denote by $\mathcal{B}_0^{(\mathfrak{a})}$ the C^*-closure of $\Psi_0^0(X, {}^0\Omega^{\frac{1}{2}})$ in $\mathscr{L}(\varrho^{\mathfrak{a}}L^2(X, {}^0\Omega^{\frac{1}{2}}))$, then we get the following exact sequence

$$0 \longrightarrow \mathcal{K}^{(\mathfrak{a})} \longrightarrow \mathcal{B}_0^{(\mathfrak{a})} \longrightarrow \mathcal{Q}_0^{(\mathfrak{a})} \longrightarrow 0,$$

where $\mathcal{Q}_0^{(\mathfrak{a})}$ is given as the C^*-closure of $\tau_0(\Psi_0^0(X, {}^0\Omega^{\frac{1}{2}}))$ in the C^*-algebra $Q^{(\mathfrak{a})}$. Finally let us denote by ${}^0\sigma^{(0)}$ resp. by $\mathcal{N}_\varrho^\nu$ the composition of τ_0 with the projection onto the first resp. the second component of $Q^{(\mathfrak{a})}$.

Remark 5.3.1. To avoid clumsy notation and since two closures of $\Psi_0^0(X, {}^0\Omega^{\frac{1}{2}})$ with respect to different weighted $L^2(X, {}^0\Omega^{\frac{1}{2}})$ spaces only differ by a Hilbert space isometry (cf. (5.1.6)), we will suppress all indices dealing with $\mathfrak{a}$, i.e. we will write $\mathcal{K}_0$, $\mathcal{B}_0$ etc. instead of $\mathcal{K}_0^{(\mathfrak{a})}$ and $\mathcal{B}_0^{(\mathfrak{a})}$.

Let us give a brief overview of some results concerning the range of the reduced normal operator. To this end we define

$$\mathcal{B}_{b,c}(S^*\partial X) := R\left(\mathcal{N}_\varrho^\nu : \mathcal{B}_0 \longrightarrow \mathscr{C}(S^*\partial X, \mathcal{B}_{b,c})\right),$$

i.e. $\mathcal{B}_{b,c}(S^*\partial X)$ denotes the range of the reduced normal operator. Note that $\mathcal{B}_{b,c}(S^*\partial X)$ is a C^*-subalgebra of the C^*-algebra $\mathscr{C}(S^*\partial X, \mathcal{B}_{b,c})$, since the range of a $*$-morphism between C^*-algebras is always closed. Moreover, let

$$\widehat{\cdot} : T^*\partial X \setminus \{0\} \longrightarrow S^*\partial X : \eta \longmapsto \widehat{\eta}$$

denote the natural projection. Again, the following results can be found in [65]:

Lemma 5.3.2. *Let $N \in \mathcal{B}_{b,c}(S^*\partial X)$ be arbitrary.*

(i) ${}^{b,c}\sigma^{(0,0)}(N(\widehat{\eta}))(z,\xi)$ *depends only on $\pi(\widehat{\eta}) \in \partial X$ and the sign of $\xi \in \mathbb{R} \setminus \{0\}$.*

(ii) *The equation*

$$\begin{aligned}\sigma_{\mathcal{B}}^{(0)}(I_c(N(\widehat{\eta})))(\pm 1) &= {}^{b,c}\sigma^{(0,0)}(N(\widehat{\eta}))(1,\pm 1)\\ &= {}^{b,c}\sigma^{(0,0)}(N(\widehat{\eta}))(0,\pm 1) = \sigma_{\mathcal{B}}^{(0)}(I_b(N(\widehat{\eta})))(\pm 1)\end{aligned}$$

holds for all arbitrary $\eta \in T^*\partial X \setminus \{0\}$.

(iii) *The function $I_b(N(\widehat{\eta})) \in \mathcal{B}(\Gamma_{\mathfrak{a}})$ depends only on $\pi(\widehat{\eta}) \in \partial X$.*

Proof. See [65, Lemma 7.4.1]. □

This last lemma leads to the following characterization of $\mathcal{Q}_0$:

Proposition 5.3.3. *The C^*-algebra $\mathcal{Q}_0$ of joint 0-symbols consists of all pairs*

$$(f, N) \in \mathscr{C}({}^0S^*X) \oplus \mathcal{B}_{b,c}(S^*\partial X)$$

satisfying for each $\eta \in T^*\partial X \setminus \{0\}$ the following compatibility conditions:

$$\begin{aligned}{}^{b,c}\sigma^{(0,0)}(N(\widehat{\eta}))(z,\xi) &= f_0(j_\varrho(0,(\pi(\eta),\xi))) && (5.3.1)\\ I_c(N(\widehat{\eta}))(\xi/|\eta|_s) &= f_0(j_\varrho(\eta,(\pi(\eta),\xi))), && (5.3.2)\end{aligned}$$

where $(z,\xi) \in {}^{b,c}T^*M \setminus \{0\}$ and $\xi \in \mathbb{R}$. Here $f_0 \in \mathscr{C}({}^0T^*X \setminus \{0\})$ is the function homogeneous of degree 0 in the fibres, that corresponds naturally to $f \in \mathscr{C}({}^0S^*X)$, i.e. $f_0(\zeta) = f(\widehat{\zeta})$ for all $\zeta \in {}^0T^*X \setminus \{0\}$.

Proof. See [65, Proposition 7.5.1]. □

Lemma 5.3.4. *Let $\mathcal{J}$ denote the set of all $N \in \mathscr{C}(S^*\partial X, \mathcal{B}_{b,c})$ satisfying $I_b(N(\widehat{\eta})) = 0$ for all $\widehat{\eta} \in S^*\partial X$, ${}^{b,c}\sigma^{(0,0)}(N(\widehat{\eta}))(z,1) = c_1$ and ${}^{b,c}\sigma^{(0,0)}(N(\widehat{\eta}))(z,-1) = c_2$ for all $z \in [0,1]$. Then $\mathcal{J}$ is a closed, two-sided ideal in $\mathcal{B}_{b,c}(S^*\partial X)$.*

Proof. Let $N \in \mathscr{J}$ be arbitrary. Then by (5.2.3)

$$\begin{aligned} c_1 &= {}^{b,c}\sigma^{(0,0)}(N(\widehat{\eta}))(z,1) = {}^{b,c}\sigma^{(0,0)}(N(\widehat{\eta}))(0,1) \\ &= \sigma^{(0)}_{\mathcal{B}}(I_b(N(\widehat{\eta})))(1) = 0, \end{aligned}$$

which shows that $c_1 = 0 = c_2$. Therefore $\mathscr{J}$ is a closed selfadjoint ideal and what remains, is to show that $\mathscr{J} \subseteq \mathcal{B}_{b,c}(S^*\partial X)$ holds. First note, that by our previous discussion

$$\sigma^{(0)}_{\mathcal{B}}\left(I_c\left(N(\widehat{\eta})\right)\right)(\pm 1) = 0 \ \forall \widehat{\eta} \in S^*\partial X,$$

which shows that $I_c\left(N(\widehat{\eta})\right) \in \mathscr{C}_0(\mathbb{R}_\xi)$. By [65, Lemma 7.4.2 (b)] the $*$-morphism $A \longmapsto \left[(\widehat{\eta},\xi) \mapsto I_c\left(\mathcal{N}^\nu_\varrho(A)(\widehat{\eta})\right)(\xi)\right]$ induces a surjective map

$$I_c \circ \mathcal{N}^\nu_\varrho : \ker({}^{b,c}\sigma^{(0,0)} \circ \mathcal{N}^\nu_\varrho) \cap \ker(I_b \circ \mathcal{N}^{\nu,\mathfrak{a}}_\varrho) \longrightarrow \mathscr{C}_0(S^*\partial X \times \mathbb{R}_\xi);$$

therefore we can find $A \in \ker({}^{b,c}\sigma^{(0,0)} \circ \mathcal{N}^\nu_\varrho) \cap \ker(I_b \circ \mathcal{N}^\nu_\varrho) \subseteq \mathcal{B}^{(\mathfrak{a})}_0(X, {}^0\Omega^{\frac{1}{2}})$, such that $I_c(\mathcal{N}^\nu_\varrho(A)) = I_c(N)$. This implies

$$N - \mathcal{N}^\nu_\varrho(A) \in \mathscr{C}(S^*\partial X, \mathcal{K}_{b,c}),$$

where $\mathcal{K}_{b,c} := \mathcal{K}^{(\mathfrak{a},0)}_{b,c}$ and we get $N \in \mathcal{B}_{b,c}(S^*\partial X)$, since $\mathscr{C}(S^*\partial X, \mathcal{K}_{b,c}) \subseteq \mathcal{B}_{b,c}(S^*\partial X)$ by [65, Lemma 7.4.2 (a)]. $\square$

The following lemma is a reformulation of [65, Proposition 7.4.3]:

Lemma 5.3.5. *The map* ${}^{b,c}\sigma^{(0,0)} \circ \mathcal{N}^\nu_\varrho : \mathcal{B}_0 \longrightarrow \mathscr{C}(\partial X) \oplus \mathscr{C}(\partial X)$ *is onto.*

Proof. The lemma follows from [65, Proposition 7.4.3]; notice, that

$${}^{b,c}\sigma^{(0,0)} \circ \mathcal{N}^\nu_\varrho = F_1 \circ \mathcal{N}^\nu_\varrho$$

in the proof of [65, Proposition 7.4.3] and that the desired operator can be found in $\Psi^0_0(X, {}^0\Omega^{\frac{1}{2}})$. $\square$

Proposition 5.3.6.

(i) *We have* $\mathcal{B}_{b,c}(S^*\partial X)/\mathscr{J} \cong \mathscr{C}(\partial X \times [-1,1])$.

(ii) *For the kernel of* $\mathcal{N}^\nu_\varrho$ *we have:* $\ker\left(\mathcal{N}^\nu_\varrho\right)/\mathcal{K}_0 \cong \mathscr{C}_0(S^*\mathring{X})$.

Proof. Consider the mappings

$$\begin{aligned} F : \mathcal{B}_{b,c}(S^*\partial X) &\longrightarrow \mathscr{C}(\partial X \times [-1,1]), \\ N &\longmapsto I_b(N); \\ G : \ker\left(\mathcal{N}^\nu_\varrho\right) &\longrightarrow \mathscr{C}_0(S^*\mathring{X}), \\ A &\longmapsto {}^0\sigma^{(0)}(A). \end{aligned}$$

(i) The first part of the proposition follows, if we show that F is onto, since it is clear, that $\mathscr{J}$ is the kernel of F. By [65, Proposition 7.3.2] we see, that

$$\forall\, A \in \mathcal{B}_0,\ \forall\, \widehat{\eta} \in S^*\partial X:\ I_b(\mathcal{N}_\varrho^\nu(A)(\widehat{\eta})) \in \mathcal{B}(\Gamma_{\mathfrak{a}}) \cong \mathscr{C}([-1,1]) \text{ and}$$
$$\forall\, N \in \mathcal{B}_{b,c}(S^*\partial X),\ \forall \widehat{\eta}_1,\ \widehat{\eta}_2 \in S^*\partial X:\ I_b(N(\widehat{\eta}_1)) = I_b(N(\widehat{\eta}_2)),$$

which shows, that the map F is well-defined, and it remains to check surjectivity. So let $h \in \mathscr{C}(\partial X \times [-1,1])$ be arbitrary. We define

$$g_1(y) := h(y,1) \text{ and } g_2(y) := h(y,-1) \quad (y \in \partial X),$$

then $g := g_1 \oplus g_2 \in \mathscr{C}(\partial X) \oplus \mathscr{C}(\partial X)$ and we can find $A \in \mathcal{B}_0$ with ${}^{b,c}\sigma^{(0,0)}(\mathcal{N}_\varrho^\nu(A)) = g$ using 5.3.5. Now

$${}^{b,c}\sigma^{(0,0)}(\mathcal{N}_\varrho^\nu(A)(\widehat{\eta}))(0,\pm 1) = \sigma_{\mathcal{B}}^{(0)}(I_b(\mathcal{N}_\varrho^\nu(A)(\widehat{\eta})))(\pm 1),$$

thus $h_0 := I_b(\mathcal{N}_\varrho^\nu(A)(\widehat{\eta})) - h \in \mathscr{C}_0(\partial X \times \Gamma_{\mathfrak{a}})$, where we used the identification of $[-1,1]$ with the two point compactification of the weightline $\Gamma_{\mathfrak{a}}$. Now, by [65, Lemma 7.4.2 (c)] the map

$$I_b \circ \mathcal{N}_\varrho^\nu : \ker({}^0\sigma^{(0)}) \longrightarrow \mathscr{C}_0(\partial X \times \Gamma_{\mathfrak{a}})$$

is onto. Consequently, we can find an operator $B \in \ker({}^0\sigma^{(0)}) \subseteq \mathcal{B}_0$ with $I_b(\mathcal{N}_\varrho^\nu(B)) = h_0$. But then the operator $C := A - B$ fulfils

$$I_b(\mathcal{N}_\varrho^\nu(C)) = I_b(\mathcal{N}_\varrho^\nu(A)) - I_b(\mathcal{N}_\varrho^\nu(B)) = h$$

and we have finished the proof of (i).
(ii) Let $A \in \mathcal{B}_0$ be with $\mathcal{N}_\varrho^\nu(A) = 0$ and $\zeta \in {}^0T^*X_{|\partial X} \setminus \{0\}$ be arbitrary. By (5.1.3) and (5.1.4) we find $\eta \oplus (y,\xi) \in T^*\partial X \oplus (\partial X \times \mathbb{R})$, such that $j_\varrho(\eta,(y,\xi)) = \zeta$ (note, that $\pi(\eta) = y$). Now, if $\eta \neq 0$, we conclude by (5.3.2)

$$\begin{aligned}
{}^0\widetilde{\sigma}^{(0)}(A)(\zeta) &= {}^0\widetilde{\sigma}^{(0)}(A)(j_\varrho(\eta,(y,\xi))) \\
&= {}^0\widetilde{\sigma}^{(0)}(A)(j_\varrho(\eta,(\pi(\eta),\xi))) \\
&= I_c(\mathcal{N}_\varrho^\nu(A)(\widehat{\eta}))(\xi/|\eta|_s) = 0,
\end{aligned}$$

where ${}^0\widetilde{\sigma}^{(0)}$ is the function corresponding to ${}^0\sigma^{(0)}$ cf. 5.3.3. Conversely, if $\eta = 0$ (note that again $\pi(\eta) = y$ holds) we get

$$\begin{aligned}
{}^0\widetilde{\sigma}^{(0)}(A)(\zeta) &= {}^0\widetilde{\sigma}^{(0)}(A)(j_\varrho(0,(y,\xi))) \\
&= {}^0\widetilde{\sigma}^{(0)}(A)(j_\varrho(0,(\pi(\eta),\xi))) \\
&= {}^{b,c}\sigma^{(0,0)}(\mathcal{N}_\varrho^\nu(A))(\widehat{\eta})(z,\xi) = 0.
\end{aligned}$$

This shows, that ${}^0\sigma^{(0)}(A)$ vanishes at ${}^0S^*X_{|\partial X}$, i.e.

$${}^0\widetilde{\sigma}^{(0)}(A) \in \mathscr{C}_0({}^0S^*\mathring{X}) \cong \mathscr{C}_0(S^*\mathring{X})$$

(the last isomorphism holds, since $T\mathring{X} \cong {}^0T\mathring{X}$) and G is well-defined. It remains to show, that G is surjective. Since the range of a $*$-homomorphism is always closed and

$\mathscr{C}_c(S^*\mathring{X})$ is dense in $\mathscr{C}_0(S^*\mathring{X})$ it is enough to show, that for all $f \in \mathscr{C}_c(S^*\mathring{X})$ there exists an operator $A \in \ker(\mathcal{N}_\varrho^\nu)$, such that ${}^0\sigma^{(0)}(A)^{(0)} = f$. So let $f \in \mathscr{C}_c(S^*\mathring{X})$ be arbitrary. Since f is only supported in a compact subset K of the interior $\mathring{X}$ of X, we can find an operator $A \in \mathcal{B}_0^{(\mathfrak{a})}$ which is zero outside an open neighbourhood V of K with $V \cap \partial X = \emptyset$ and ${}^0\sigma^{(0)}(A) = f$. Clearly $A \in \ker(\mathcal{N}_\varrho^\nu)$, since A is zero at the boundary. □

Proposition 5.3.7.

(i) $K_i(\mathcal{B}_{b,c}(S^*\partial X)/\mathcal{J}) \cong K_i(\mathscr{C}(\partial X))$ *holds for* $i = 0, 1$.

(ii) *We have* $K_i(\mathcal{J}) = 0$ *for* $i = 0, 1$.

(iii) $K_i(\mathcal{B}_{b,c}(S^*\partial X))$ *is isomorphic to* $K_i(\mathscr{C}(\partial X))$ $(i = 0, 1)$.

Proof. (i) This is clear since $\mathcal{B}_{b,c}(S^*\partial X)/\mathcal{J} \cong \mathscr{C}(\partial X \times [-1,1])$ and $\partial X \times [-1,1] \simeq_h \partial X$.
(ii) By definition $\mathcal{J} = \mathscr{C}(S^*\partial X, \widehat{\mathcal{D}})$, where $\widehat{\mathcal{D}}$ is given by (5.2.7). Now, $K_i(\mathcal{J}) = 0$ follows by 5.2.10.
(iii) This follows from (ii) and the six term exact sequence in K-theory associated to the exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{B}_{b,c}(S^*\partial X) \longrightarrow \mathcal{B}_{b,c}(S^*\partial X)/\mathcal{J} \longrightarrow 0. \qquad \square$$

The following proposition is certainly well-known and can be found in [109, (8.3) Lemma] (without proof):

Proposition 5.3.8. *Let the following diagram of abelian groups and homomorphisms be commutative with exact rows:*

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{A}_i' \arrow[r, "f_i'"] & \mathcal{B}_i' \arrow[r, "g_i'"] & \mathcal{C}_i' \arrow[r, "h_i'"] & \mathcal{A}_{i+1}' \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathcal{A}_i \arrow[r, "f_i"] \arrow[u, "a_i"] & \mathcal{B}_i \arrow[r, "g_i"] \arrow[u, "b_i"] & \mathcal{C}_i \arrow[r, "h_i"] \arrow[u, "c_i"] & \mathcal{A}_{i+1} \arrow[r] \arrow[u, "a_{i+1}"] & \cdots
\end{tikzcd}$$

Furthermore suppose, that $c_i : \mathcal{C}_i \longrightarrow \mathcal{C}_i'$ *is an isomorphism for all* $i \in \mathbb{N}$. *Then the following sequence is exact:*

$$\cdots \longrightarrow \mathcal{A}_i \xrightarrow{(a_i, -f_i)} \mathcal{A}_i' \oplus \mathcal{B}_i \xrightarrow{\langle f_i', b_i \rangle} \mathcal{B}_i' \xrightarrow{h_i c_i^{-1} g_i'} \mathcal{A}_{i+1} \longrightarrow \cdots$$

Here, the map $\langle f_i', b_i \rangle$ *is given by* $f_i'(\alpha_i) + b_i(\beta_i)$.

Proof. (i) Let $(\alpha_i', \beta_i) \in \ker\langle f_i', b_i \rangle$ be arbitrary, i.e. $f_i'(\alpha_i') + b_i(\beta_i) = 0$. Applying g_i' to this, we get $g_i'(f_i'(\alpha_i') + b_i(\beta_i)) = 0$ which implies $g_i'(b_i(\beta_i)) = 0$, since $g_i' \circ f_i' \equiv 0$ by the exactness of the upper row. This gives $c_i^{-1}(g_i'(b_i(\beta_i))) = 0$ and we conclude $g_i(\beta_i) = 0$ by the commutativity of the diagram. Therefore we can find $\alpha_i \in \mathcal{A}_i$, such that $f_i(\alpha_i) = \beta_i$. Note, that we have $f_i(\alpha_i + h_{i-1}(\gamma_{i-1})) = \beta_i$ for all $\gamma_{i-1} \in \mathcal{C}_{i-1}$ by the exactness of the lower row. By commutativity we get $b_i(f_i(\alpha_i)) = f_i'(a_i(\alpha_i))$, which implies $f_i'(a_i(\alpha_i) + \alpha_i') = 0$. So again, we can find $\gamma_{i-1}' \in \mathcal{C}_{i-1}'$, such that $h_{i-1}'(\gamma_{i-1}') = a_i(\alpha_i) + \alpha_i'$. This gives

$$a_i(h_{i-1}(c_{i-1}^{-1}(\gamma_{i-1}'))) = a_i(\alpha_i) + \alpha_i',$$

since the diagram commutes and c_{i-1} is an isomorphism. But then $a_i(h_{i-1}(c_{i-1}^{-1}(\gamma'_{i-1})) - \alpha_i) = \alpha'_i$ and $\ker\langle f'_i, b_i\rangle \subseteq \mathrm{r}\,(a_i, -f_i)$. Clearly, $\mathrm{r}\,(a_i, -f_i) \subseteq \ker\langle f'_i, b_i\rangle$ since the diagram commutes, which gives exactness at $\mathcal{A}'_i \oplus \mathcal{B}_i$.
(ii) Exactness at $\mathcal{B}'_i$: Let $\beta'_i \in \ker(h_i c_i^{-1} g'_i)$ be arbitrary. Then we have $c_i^{-1} g'_i(\beta'_i) \in \ker h_i$ and we find $\beta_i \in \mathcal{B}_i$, such that $g(\beta_i) = c_i^{-1} g'_i(\beta'_i)$ by the exactness of the lower row. We get

$$g'_i(\beta'_i - b_i(\beta_i)) = g'_i(\beta'_i) - g'_i b_i(\beta_i) = g'_i(\beta_i) - c_i g_i(\beta_i) = 0$$

by commutativity and the fact, that c_i is an isomorphism. Again we can find $\alpha_i \in \mathcal{A}_i$, such that $f'_i(\alpha'_i) = \beta'_i - b_i(\beta_i)$ by the exactness of the upper row. But this implies $\langle f'_i(\alpha'_i), bi(\beta_i)\rangle = f'_i(\alpha'_i) + b_i(\beta_i) = \beta'_i$, i.e. $\ker(h_i c_i^{-1} g'_i) \subseteq \mathrm{r}\,\langle f'_i, bi\rangle$. On the contrary, if $\beta_i \in \mathrm{r}\,\langle f'_i, bi\rangle$ is given, we find $\alpha'_i \in \mathcal{A}'_i$ and $\beta_i \in \mathcal{B}_i$, such that $\beta_i = f'_i(\alpha'_i) + b_i(\beta_i)$. We deduce

$$h_i c_i^{-1} g'_i(\beta_i) = h_i c_i^{-1} \underbrace{g'_i f'_i}_{=0}(\alpha'_i) + h_i \underbrace{c_i^{-1} g'_i b_i}_{=g_i}(\beta_i) = 0$$

by exactness and commutativity. The reversed inclusion $\mathrm{r}\,\langle f'_i, bi\rangle \subseteq \ker(h_i c_i^{-1} g'_i)$ follows.
(iii) Exactness at $\mathcal{A}_{i+1}$: Let $\alpha_{i+1} \in \ker(a_{i+1}, -f_{i+1})$, i.e. $a_{i+1}(\alpha_i) = 0 = -f_{i+1}(\alpha_{i+1})$. By exactness of the lower row, we find $\gamma_i \in \mathcal{C}_i$ such that $h_i(\gamma_i) = \alpha_{i+1}$. This implies

$$h'_i c_i(\gamma_i) = a_{i+1} h_i(\gamma_i) = a_{i+1}(\alpha_{i+1}) = 0$$

i.e. there exists an element $b'_i \in \mathcal{B}'_i$, such that $g'_i(b'_i) = c_i(\gamma_i)$ by the exactness of the upper row. Therefore

$$h_i c_i'^{-1} g'_i(\beta'_i) = h_i c_i^{-1} c_i(\gamma_i) = h_i(\gamma_i) = \alpha_{i+1},$$

and we have proven $\ker(a_{i+1}, -f_{i+1}) \subseteq \mathrm{r}\,(h_i c_i'^{-1} g'_i)$. Finally, let $\alpha_{i+1} \in \mathrm{r}\,(h_i c_i^{-1} g'_i)$ be arbitrary; choose $\beta'_i \in \mathcal{B}'_i$, such that $h_i c_i^{-1} g'_i(\beta'_i) = \alpha_{i+1}$. Since $a_{i+1} h_i c_i^{-1} = h'_i$ we conclude

$$a_{i+1} h_i c_i^{-1} g'_i(\beta'_i) = h'_i g'_i(\beta'_i) = 0$$

by exactness of the upper row. Moreover, $f_{i+1} h_i = 0$ which gives $-f_{i+1} h_i c_i^{-1} g'_i(\beta'_i) = 0$ and shows $\mathrm{r}\,(h_i c_i'^{-1} g'_i) \subseteq \ker(a_{i+1}, -f_{i+1})$. □

Lemma 5.3.9. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian groups, $\alpha : \mathcal{A} \to \mathcal{A} \oplus \mathcal{B}$ the canonical injection $a \mapsto (a, 0)$ and $\beta : \mathcal{A} \to \mathcal{C}$ be a group homomorphism. Then*

$$(\mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C})\,/m(\mathcal{A}) \cong \mathcal{B} \oplus \mathcal{C}$$

holds, where $m : \mathcal{A} \to \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}$ is given by $m(a) := (\alpha(a), -\beta(a))$.

Proof. Let $\varphi : \mathcal{B} \oplus \mathcal{C} \longrightarrow (\mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C})\,/m(\mathcal{A})$ be the map given by

$$\varphi : \mathcal{B} \oplus \mathcal{C} \ni (b, c) \longmapsto [0, b, c] \in (\mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C})\,/m(\mathcal{A}).$$

Then φ is injective, since $\varphi(b, c) = 0$ implies

$$(a, b, c - \beta(a)) = (\tilde{a}, 0, -\beta(\tilde{a})).$$

Thus $b = 0$, $a = \tilde{a}$ and therefore $c = 0$.

To prove surjectivity, let $[a, b, c] \in (\mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C})\,/m(\mathcal{A})$ be arbitrary. Then

$$\begin{aligned}[a,b,c] &= (a + a_1, b, c - \beta(a_1)) \\ &= (a + a_1, 0, -\beta(a + a_1)) + (0, b, c + \beta(a)),\end{aligned}$$

i.e. $\varphi(b, c + \beta(a)) = [a, b, c]$. This shows that φ is a group isomorphism and we have proved the lemma. □

Remark 5.3.10. Let us note, that (5.3.3) enables us to give the inverse of φ used in the last proof, namely

$$\varphi^{-1} : [a, b, c] \longmapsto (b, c + \beta(a)). \tag{5.3.3}$$

Proposition 5.3.11. *The map $\varphi : \mathscr{C}(\partial X) \longrightarrow \mathcal{B}_{b,c}(S^*\partial X)$ defined by*

$$\varphi : f \longmapsto \mathcal{N}(M_g), \quad g \in \mathscr{C}(X) \text{ such that } g_{|\partial X} = f,$$

induces isomorphisms $\varphi_* : K_*(\partial X) \longrightarrow K_*(\mathcal{B}_{b,c}(S^*\partial X))$. Here $M_g \in \mathcal{B}_0$ denotes the multiplication operator given by the multiplication with g.

Proof. By 5.3.7 the canonical projection

$$\pi : \mathcal{B}_{b,c}(S^*\partial X) \longrightarrow \mathcal{B}_{b,c}(S^*\partial X)/\mathcal{J}$$

induces isomorphisms

$$\pi_* : K_*(\mathcal{B}_{b,c}(S^*\partial X)) \longrightarrow K_*(\mathscr{C}(\partial X))$$

in K-theory. Since φ injects, we get

$$\varphi : \mathscr{C}(\partial X) \longrightarrow r(\varphi) \subseteq \mathcal{B}_{b,c}(S^*\partial X)$$

to be a C^*-algebra isomorphism. Let $f \in \mathscr{C}(\partial X)$ be arbitrary and $g \in \mathscr{C}(X)$ be with $g_{|\partial X} = f$. We choose a local coordinate patch V given by (5.1.2), which yields a trivialization $T^*\partial X_{|\partial X \cap V} \cong \mathbb{R}^{n-1}_y \times \mathbb{R}^{n-1}_\eta$ and get

$$[\mathcal{N}^\nu_\varrho(M_g)(y,\eta)]h(x) = g(0,y)h(x) = f(y)h(x) \quad (h \in \mathscr{C}^\infty(M, {}^{b,c}\Omega^{\frac{1}{2}}))$$

according to [65, Proposition 4.6.1]. This shows $r(\varphi) \cong \mathscr{C}(\partial X)$ and we conclude

$$\varphi_* : K_*(\mathscr{C}(\partial X)) \xrightarrow{\cong} K_*(\mathscr{C}(\partial X)) \cong K_*(\mathcal{B}_{b,c}(S^*\partial X)). \qquad \square$$

Let m denote the natural identification of elements $f \in \mathscr{C}(X)$ with their induced multiplication operator M_f in $\mathcal{B}_0(X, {}^0\Omega^{\frac{1}{2}})$. To shorten notation and for systematic reasons (see 5.3.12) let $\mathcal{B}_0 =: \mathcal{B}$, $\mathcal{K}_0 =: \mathcal{K}$ and $\ker(\mathcal{N}^\nu_\varrho) =: \mathcal{I}$.

Remark 5.3.12. A careful comparison of the results for this 0-setting and the results in [80] and [79] shows, that one could use the same arguments as in the case of *Boutet de Monvel* calculus to prove the following theorem. Anyway, the proof provided here is very short and works also for *Boutet de Monvel's* algebra.

Theorem 5.3.13. *The commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}/\mathcal{K} \arrow[r] & \mathcal{B}/\mathcal{K} \arrow[r] & \mathcal{B}/\mathcal{I} \arrow[r] & 0 \\ 0 \arrow[r] & \mathscr{C}_0(\mathring{X}) \arrow[r, "l"] \arrow[u, "m"] & \mathscr{C}(X) \arrow[r] \arrow[u, "m"] & \mathscr{C}(\partial X) \arrow[r] \arrow[u, "\varphi"] & 0 \end{tikzcd}$$

with exact rows induces an isomorphism

$$K_*(\mathcal{B}/\mathcal{K}) \cong K_*(\mathscr{C}(X)) \oplus K_{*-1}(\mathscr{C}_0(T^*\mathring{X})). \tag{5.3.4}$$

Before we give the proof of this theorem let us fix some notation first: Let $p : \mathscr{C}_0(\mathring{X}) \longrightarrow \mathscr{C}_0(S^*\mathring{X})$ denote the pull-back of functions under the bundle projection $S^*\mathring{X} \longrightarrow \mathring{X}$. Moreover, we denote by $B^*\mathring{X}$ the unit ball bundle of $\mathring{X}$; note, that $\mathscr{C}_0(\mathring{X})$ is homotopy equivalent to $\mathscr{C}_0(B^*\mathring{X})$. Finally notice, that the composition of the isomorphism $\mathcal{I}/\mathcal{K} \cong \mathscr{C}_0(S^*\mathring{X})$ with m corresponds to p and we will use this fact in the following proof without any comment.

Proof. We have $\mathcal{I}/\mathcal{K} \cong \mathscr{C}_0(S^*\mathring{X})$ and

$$K_*(\mathscr{C}_0(S^*\mathring{X})) \cong K_*(\mathscr{C}_0(\mathring{X})) \oplus K_{*-1}(\mathscr{C}_0(T^*\mathring{X})).$$

Moreover, $p_* : K_*(\mathscr{C}_0(\mathring{X})) \longrightarrow K_*(\mathscr{C}_0(S^*\mathring{X}))$ corresponds to the canonical injection

$$\iota_* : K_*(\mathscr{C}_0(\mathring{X})) \hookrightarrow K_*(\mathscr{C}_0(\mathring{X})) \oplus K_{*-1}(\mathscr{C}_0(T^*\mathring{X}))$$

under this identification [79, Proposition 11] (note, that this also implies $K_*(\mathscr{C}(X)) \hookrightarrow K_*(\mathcal{B}/\mathcal{K})$ by the five lemma). Using this and our previous observations, we get the following commutative diagram with exact rows:

$$\begin{tikzcd} \cdots \arrow[r] & K_*(\mathscr{C}_0(S^*\mathring{X})) \arrow[r] & K_*(\mathcal{B}/\mathcal{K}) \arrow[r] & K_*(\mathscr{C}(\partial X)) \arrow[r] & \cdots \\ \cdots \arrow[r] & K_*(\mathscr{C}_0(\mathring{X})) \arrow[r, "l_*"] \arrow[u, hook, "p_*"] & K_*(\mathscr{C}(X)) \arrow[r] \arrow[u] & K_*(\mathscr{C}(\partial X)) \arrow[r] \arrow[u, equal] & \cdots \end{tikzcd}$$

which fits into 5.3.8. Consequently, the following sequence

$$\longrightarrow K_*(\mathscr{C}_0(\mathring{X})) \xrightarrow{(p_*,-l_*)} K_*(\mathscr{C}_0(S^*\mathring{X})) \oplus K_*(\mathscr{C}(X)) \longrightarrow K_*(\mathcal{B}/\mathcal{K}) \longrightarrow$$

is exact. Since p_* is an injection, it follows that

$$K_*(\mathscr{C}_0(\mathring{X})) \hookrightarrow K_*(\mathscr{C}_0(S^*\mathring{X})) \oplus K_*(\mathscr{C}(X)).$$

This leads to the short exact sequence

$$0 \longrightarrow K_*(\mathscr{C}_0(\mathring{X})) \longrightarrow K_*(\mathscr{C}_0(S^*\mathring{X})) \oplus K_*(\mathscr{C}(X)) \longrightarrow K_*(\mathcal{B}/\mathcal{K}) \longrightarrow 0.$$

If we denote $h_* := (p_*, -l_*)$, we conclude (using the identifications stated above)

$$\begin{aligned} K_*(\mathcal{B}/\mathcal{K}) &\cong \left(K_*(\mathscr{C}_0(S^*\mathring{X})) \oplus K_*(\mathscr{C}(X))\right)/h_*\left(K_*(\mathscr{C}_0(\mathring{X}))\right) \\ &\cong \frac{\left(K_*(\mathscr{C}_0(\mathring{X})) \oplus K_{*-1}(\mathscr{C}_0(T^*\mathring{X})) \oplus K_*(\mathscr{C}(X))\right)}{h_*\left(K_*(\mathscr{C}_0(\mathring{X}))\right)}. \end{aligned}$$

By 5.3.9 $K_*(\mathcal{B}/\mathcal{K}) \cong K_{*-1}(\mathscr{C}_0(T^*\mathring{X})) \oplus K_*(\mathscr{C}(X))$ follows. □

Proposition 5.3.14. *We have*

$$\begin{aligned} K_1(\mathcal{B}_0) &\cong \ker\left(\mathrm{ind} : K_1\left(\mathcal{B}_0/\mathcal{K}_0\right) \longrightarrow \mathbb{Z}\right) \text{ and} \\ K_0(\mathcal{B}_0) &\cong K_0(\mathscr{C}(X)) \oplus K_1(\mathscr{C}_0(T^*\mathring{X})), \end{aligned}$$

where ind *denotes the Fredholm index map, which is surjective.*

Proof. It suffices to prove that that there exists a $\mathcal{B}_0$ valued k-by-k-matrix, such that this matrix has index one. But this follows similar to the proof in [67, Theorem 5.18]. □

5.4 The index of a fully elliptic operator on a conformally compact space

To calculate the index, we have to introduce 0-operators acting between sections of finite dimensional vector bundles $\mathcal{E}_1, \mathcal{E}_2 \longrightarrow X$. First of all note, that we can assume that our operators act between the same bundles: Since X has a non empty boundary, there exists a section of ${}^0S^*X$. Thus the symbol of an elliptic operator gives an isomorphism between $\mathcal{E}_1$ and $\mathcal{E}_2$ and we restrict ourselves to the case $\mathcal{E}_1 = \mathcal{E}_2 =: \mathcal{E}$. Now, choose a complement $\mathcal{F}$ of $\mathcal{E}$, such that $\mathcal{E} \oplus \mathcal{F} = \mathbb{C}^n$ for a suitable $n \in \mathbb{N}$, where $\mathbb{C}^n$ denotes the n-dimensional trivial bundle. Thus, we consider the algebra $\Psi_0^m(X;\mathbb{C}^n) := \Psi_0^m(X;\mathbb{C}^n,\mathbb{C}^n)$ of all operators

$$\widetilde{A} = \begin{pmatrix} id & 0 \\ 0 & A \end{pmatrix} : \begin{matrix} \mathscr{C}^\infty(X,\mathcal{F}) \\ \oplus \\ \mathscr{C}^\infty(X,\mathcal{E}) \end{matrix} \longrightarrow \begin{matrix} \mathscr{C}^\infty(X,\mathcal{F}) \\ \oplus \\ \mathscr{C}^\infty(X,\mathcal{E}) \end{matrix}.$$

Note, that $\mathrm{ind}\,\widetilde{A} = \mathrm{ind}\,A$ and we will write A instead of $\widetilde{A}$ in what follows to simplify notation. Let us recall the characterization of Fredholm operators in $\Psi_0^m(X;\mathbb{C}^n)$ given in [65]:

Proposition 5.4.1. *Let $\mathfrak{a} \in \mathbb{R}$ be arbitrary and $A \in \Psi_0^m(X;\mathbb{C}^n)$ be with ${}^0\sigma(A)(\zeta) \neq 0$ for all $\zeta \in {}^0S^*X$. Then the operator*

$$A : \varrho_N^{\mathfrak{a}}\mathcal{H}_0^s(X;\mathbb{C}^n) \longrightarrow \varrho_N^{\mathfrak{a}}\mathcal{H}_0^{s-m}(X;\mathbb{C}^n)$$

is Fredholm provided the reduced normal operator

$$\mathcal{N}_{\varrho_N}(A)(\eta) : \varrho_0^{\mathfrak{a}}\varrho_1^{\mathfrak{a}_1}\mathcal{H}_{b,c}^m(M;\mathbb{C}^n_{\pi(\eta)}) \longrightarrow \varrho_0^{\mathfrak{a}}\varrho_1^{\mathfrak{a}_1-m}L_{b,c}^2(M;\mathbb{C}^n_{\pi(\eta)})$$

is invertible for all $\eta \in T^*\partial X \setminus \{0\}$ and some, hence any $\mathfrak{a}_1 \in \mathbb{R}$.

Therefore an operator is Fredholm, if it is *fully elliptic*. Before we state an index formula for Fredholm operators of order 0 let us make some remarks. First of all note, that by general group theory (5.3.4) induces a short exact split sequence

$$\begin{tikzcd} 0 \arrow[r] & K_1(\mathscr{C}(X)) \arrow[r] & K_1(\mathcal{B}_0/\mathcal{K}_0) \arrow[r, "\pi"'] & K_0(\mathscr{C}_0(T^*\mathring{X})) \arrow[l, bend right, dotted] \arrow[r] & 0 \end{tikzcd}$$

of abelian groups. And we see:

Lemma 5.4.2. *Every fully elliptic operator in $\mathcal{B}_0$ can be (stably) deformed to a direct sum of an operator which is the identity near the boundary, and a multiplication operator by a bundle endomorphism on X.*

Let $2X$ denote the *double of* X, i.e. the manifold that is obtained by gluing X along the boundary. We will write X_1 resp. X_2 for X, if we need to distinguish the "right" and the "left" component of $2X$. On $K^0(T^*2X)$ we have the *topological index map*, i.e. the push forward to a point

$$\mathrm{ind}_t : K^0(T^*2X) \longrightarrow K^0(\{\bullet\}) \cong \mathbb{Z},$$

which gives rise to a topological index for $K_0(\mathcal{C}_0(T^*\mathring{X}))$:

$$\mathrm{ind}_t : K_0(\mathscr{C}_0(T^*\mathring{X})) \cong K^0(T^*\mathring{X}) \hookrightarrow K^0(T^*2X) \longrightarrow \mathbb{Z}. \tag{5.4.1}$$

If $A \in \mathcal{B}_0$ is an operator that is the identity in a collar neighbourhood near the boundary, there exists an elliptic pseudodifferential operator $\widehat{A} \in \Psi^0(2X, \mathbb{C}^n)$, such that $\widehat{A}f = Af$ whenever f is supported in X, namely $\widehat{A}f := Af_{|X_1} + A\alpha(f_{|\mathring{X}_2})$. Here $\alpha : 2X \longrightarrow 2X$ denotes the *flip*, i.e. the diffeomorphism that interchanges X_0 and X_1 in $2X$. Since $\widehat{A}$ is the identity near the boundary and coincides with A on X, we have $\widehat{A}f = 0$ if and only if $A(f_{|X_1}) = 0$ and $A(\alpha(f_{|X_2})) = 0$, which gives

$$\mathrm{ind}\,\widehat{A} = 2\mathrm{ind}\,A.$$

$2X$ is a compact manifold without boundary, thus the *Atiyah-Singer* index theorem yields

$$\mathrm{ind}\,\widehat{A} = \mathrm{ind}_t \circ \delta\,[\sigma(\widehat{A})], \tag{5.4.2}$$

where σ denotes the homogeneous principal symbol map on $\Psi^0(2X, \mathbb{C}^n)$ (the closure of the algebra of 0-order pseudodifferential operators with respect to $L^2(2X, \mathbb{C}^n)$), $[\sigma(\widehat{a})]$ the K-class of $\sigma(\widehat{a})$ in $K_1(\mathscr{C}(S^*2X))$ and δ is the index map associated to the six term exact sequence in K-theory of the short exact sequence

$$0 \longrightarrow \mathscr{C}_0(T^*2X) \longrightarrow \mathscr{C}(B^*2X) \longrightarrow \mathscr{C}(S^*2X) \longrightarrow 0.$$

Here B^*2X resp. S^*2X is the cobundle of the *ball*- resp. *sphere*-bundle of $T2X \longrightarrow X$.

Let $\gamma : \mathscr{C}_0(T^*\mathring{X}) \oplus \mathscr{C}_0(T^*\mathring{X}) \longrightarrow \mathscr{C}(T^*2X)$ be the map given by

$$\gamma : f_1 \oplus f_2 \longmapsto \begin{cases} f_1(\zeta), & \zeta \in T^*2X_{|\mathring{X}_1}, \\ f_2(\zeta), & \zeta \in T^*2X_{|\mathring{X}_2}, \\ 0, & \zeta \in T^*2X_{|\partial X}. \end{cases}$$

Then $\gamma_1(f) := \gamma(f \oplus 0)$ resp. $\gamma_2(f) := \gamma(0 \oplus f)$ corresponds to the inclusion of $\mathscr{C}_0(T^*\mathring{X})$ in $\mathscr{C}(T^*2X)$ at the "right" resp. "left" side of $2X$. Note that we have

$$\alpha_*\gamma_*(m \oplus 0) = \gamma_*(0 \oplus m) \quad (m \in K_0(\mathscr{C}_0(T^*\mathring{X})))$$

by the naturality of the induced maps.

Theorem 5.4.3. *Let $A \in \mathcal{B}_0$ be a Fredholm operator. Then the index of A is given by:*

$$\operatorname{ind} A = \operatorname{ind}_t(\pi([\tau_0(A)])),$$

where $[\tau_0(A)]$ denotes the K_1 class of the joint symbol of A in $\mathcal{B}_0/\mathcal{K}_0$.

Proof. We have the following diagram, where the horizontal row is exact:

$$\begin{tikzcd}
0 \arrow[r] & K_1(\mathscr{C}(X)) \arrow[r] & K_1(\mathcal{B}_0/\mathcal{K}_0) \arrow[r, "\pi"] \arrow[d, "\operatorname{ind}"'] & K_0(\mathscr{C}_0(T^*\mathring{X})) \arrow[r] \arrow[d, "\operatorname{ind}_t"'] & 0 \\
& & \mathbb{Z} & \mathbb{Z} &
\end{tikzcd}$$

Clearly $\operatorname{ind} = \operatorname{ind}_t$ on $K_1(\mathscr{C}(X))$ since $K_1(\mathscr{C}(X))$ is on the one hand given by classes of (invertible) multiplication operators, therefore their index vanishes, and on the other hand it is mapped to zero by the exactness of the upper row.

So, by 5.4.2 we can assume, that A is given by an operator that is a bundle isomorphism in a collar neighbourhood of the boundary. Using the map $\gamma_{1,*}$ we get

$$2\gamma_{1,*}(\pi[\tau_0(A)]) = \gamma_*\left(\pi[\tau_0(A)] \oplus 0\right) + \alpha_*\gamma_*\left(0 \oplus \pi[\tau_0(A)]\right).$$

Since the topological index is independent of α, we conclude

$$2\operatorname{ind}_t\left(\gamma_{1,*}(\pi[\tau_0(A)])\right) = \operatorname{ind}_t\gamma_*\left(\pi[\tau_0(A)] \oplus \pi[\tau_0(A)]\right).$$

Note, that $\operatorname{ind}_t \circ \gamma_{1,*}$ is the definition of the topological index for $\mathscr{C}_0(T^*\mathring{X})$ cf. (5.4.1) and $\gamma_*(\pi[\tau_0(A)] \oplus \pi[\tau_0(A)]) = \delta[\sigma(\widehat{A})]$ by the very definition of the map γ and that of the K-classes $\pi[\tau_0(A)]$ and $\delta[\sigma(\widehat{A})]$. This in orchestra with (5.4.2) gives

$$\operatorname{ind}_t(\pi[\tau_0(A)]) = \operatorname{ind}_t\left(\gamma_{1,*}(\pi[\tau_0(A)])\right) = \frac{\operatorname{ind}_t \circ \delta\,[\sigma(\widehat{A})]}{2} = \operatorname{ind} A,$$

which proves the index theorem. □

5.5 The Dirac operator

In the sequel we assume, that X has an additional spin structure and that $\dim X = n \geq 2$. Then this spin structure on X canonically induces a spin structure on ∂X. If $\dim X = n$ is odd, then the restriction to ∂X of the spinor bundle ΣX of X is precisely the spinor bundle on ∂X, i.e. $\Sigma M_{|N} = \Sigma\partial X$. If n is even, then $\Sigma X_{|N} \cong \Sigma\partial X \oplus \Sigma\partial X$ holds. Note, that we will suppress the density bundles in what follows to shorten notation.

Let us denote by D^X the Dirac operator on X (with respect to the smooth metric h), which is given by $D^X = \sum_{i=1}^{n} c(X^i)\nabla^{\Sigma X}_{X_i}$, where

$$\nabla^{\Sigma X}_{X_j}\varphi = [b, X_j(\sigma)] + \frac{1}{2}\sum_{k<l}\Gamma^l_{jk}c(X^k)c(X^l)\varphi$$

for a spinor field $\varphi = [b,\sigma]$ and the Γ^l_{jk} are the *Christoffel* symbols. If ν denotes a unit normal field with respect to ∂X we get the following decomposition formula for the Dirac operator (cf. for example [11, formula (3.6)] and the references given there)

$$D^X = -c(\nu)D^{\partial X} - \frac{1}{2}tr(W)c(\nu) + c(\nu)\nabla^{\Sigma X}_{\nu}, \tag{5.5.1}$$

for sections of ΣX defined in a neighbourhood of ∂X. If we denote by D^0 the Dirac operator with respect to the 0-metric g, then

$$D^0 = \varrho_N \varrho_N^{\frac{n-1}{2}} D^X \varrho_N^{-\frac{n-1}{2}} = \varrho_N^{\frac{n+1}{2}} D^X \varrho_N^{-\frac{n-1}{2}}$$

holds by the usual formula for conformal changes (cf. [69, Chapter 2, §5 Theorem 5.24]). This in connection with (5.5.1) and [69, Chapter 2, §5 Lemma 5.5] gives

$$\begin{aligned}
D^0 &= \varrho_N^{\frac{n+1}{2}}\left(\varrho_N^{-\frac{n-1}{2}} D^X + c(d(\varrho_N^{-\frac{n-1}{2}}))\right)\\
&= \varrho_N D^X - \frac{n-1}{2}\sum_{i=1}^{n} X_i(\varrho_N)c(X^i)\\
&= \varrho_N\left(-c(\nu)D^{\partial X} - \frac{1}{2}tr(W)c(\nu) + c(\nu)\nabla^{\Sigma X}_{\nu}\right)\\
&\qquad\qquad - \frac{n-1}{2}\sum_{i=1}^{n} X_i(\varrho_N)c(X^i).
\end{aligned}$$

Here d denotes the total differential. Before we calculate the reduced normal operator of the above terms separately, let us fix some notation: To be consistent with the formulas given by [65, Proposition 4.6.1] let $(x,y) : X \supseteq V \longrightarrow \overline{\mathbb{R}}_+ \times \mathbb{R}^{n-1}_y$ coordinate chart in a neighbourhood of the boundary of X. Using this chart, we will identify $\nu = X_n$ with ∂_x and X_i with ∂_{y_i} $(i = 1, \ldots n-1)$ in the following calculations. Moreover note, that by a *Taylor* expansion $\varrho_N(x,y) = xa(x,y)$ where $a(0,y) > 0$ for all $y \in \partial X$ holds. Let us first treat the term involving the connection on the spinor bundle:

$$\begin{aligned}
\varrho_N c(\nu) D^{\partial X}\varphi &= \varrho_N c(\nu) \sum_{i=0}^{n-1} c(X^i)\nabla^{\Sigma\partial X}_{X_i}\varphi \\
&= \varrho_N c(\nu) \sum_{i=0}^{n-1} c(X^i)\nabla^{\Sigma\partial X}_{X_i}\varphi \\
&= \varrho_N c(\nu) \sum_{i=0}^{n-1} c(X^i)[b, X_i(\sigma)] \\
&\quad + \underbrace{\frac{1}{2}\varrho_N c(\nu) \sum_{i=0}^{n-1} c(X^i) \sum_{k<l} \Gamma^l_{jk} c(X^k) c(X^l)\,\varphi}_{(*)}
\end{aligned}$$

and

$$\varrho_N c(\nu)\nabla^{\Sigma X}_{\nu}\varphi = \varrho_N c(\nu) \left([b, X_n(\sigma)] + \frac{1}{2}\sum_{k<l}\Gamma^l_{nk} c(X^k)c(X^l)\varphi \right).$$

The term $(*)$ vanishes for $x = 0$ and has order zero, therefore its reduced normal operator vanishes. This gives:

$$\begin{aligned}
&\mathcal{N}\left(\varrho_N c(\nu)\left(-D^{\partial X} + \nabla^{\Sigma X}_{\nu}\right)\right)\psi \\
&\qquad = a(0,y)c(\nu)\left(\sum_{j=0}^{n-1} c(X^j)[e, -i\eta_j x\omega] + [e, x\partial_x\omega]\right).
\end{aligned}$$

For a spinor field $\psi = [e,\omega]$. The term $-\varrho_N tr(W)c(\nu)$ has order zero and vanishes for $x = 0$, so again the reduced normal operator also vanishes. Finally we get:

$$\sum_{i=1}^{n} X_i(\varrho_N)c(X^i) = \left(a(x,y)c(X^n) + \sum_{i=1}^{n} xX_i(a(x,y))c(X^i) \right).$$

Here, the reduced normal operator of the second term vanishes, too, while the first term contributes as a non-vanishing zero order term. Summarized we get:

Proposition 5.5.1. *The reduced normal operator of D^0 is given by:*

$$\begin{aligned}
&\mathcal{N}(D^0)(y,\eta)\psi \\
&\qquad = a(0,y)c(\nu)\left(\sum_{j=0}^{n-1} c(X^j)[e, -i\eta_j x\omega] + [e, (x\partial_x - \frac{n-1}{2})\omega]\right)
\end{aligned}$$

for $\psi = [e,\omega]$.

Since D^0 is (essentially) self adjoint (cf. [69, §5 Theorem 5.7]), $\mathcal{N}(D^0)(y,\eta)$ has this property, too, where $(y,\eta) \in \mathbb{R}^{n-1}_y \times \mathbb{R}^{n-1}_\eta$ cf. [65, Proposition 4.5.2]. And we have

$$\begin{aligned} {}^{b,c}\sigma(\mathcal{N}(D^0)(y,\eta))(z,\xi) &= ia(0,y)c(\nu)\xi, \\ I_b(\mathcal{N}(D^0)(y,\eta))(w) &= a(0,y)c(\nu)\left(-\frac{n-1}{2} + iw\right), \\ I_c^{(1)}(\mathcal{N}(D^0)(y,\eta))(\xi) &= ia(0,y)c(\nu)\left(\xi + \sum_{j=1}^{n} c(X^j)\eta_j\right), \end{aligned} \tag{5.5.2}$$

where ${}^{b,c}\sigma := {}^{b,c}\sigma^{(1,1)}$. Note that we again used the formulas and coordinates according to [65, Proposition 4.6.1]. Let $\mathfrak{a}_0 \in \mathbb{R}$ be arbitrary, then we deduce from (5.5.2):

$$I_b(\mathcal{N}(D^0)(y,\eta))(w - i\mathfrak{a}_0) = a(0,y)c(\nu)\left(-\frac{n-1}{2} + iw + \mathfrak{a}_0\right) \tag{5.5.3}$$

and get (cf. [65, Proposition 4.5.4]):

Proposition 5.5.2. *The reduced normal operator of the Dirac operator D^0 is a smooth family*

$$\mathcal{N}(D^0) : T^*\partial X \setminus \{0\} \to \mathscr{L}(\varrho_0^{\mathfrak{a}_0}\varrho_1^{\mathfrak{a}_1}\mathcal{H}^s_{b,c}(\Sigma M), \varrho_0^{\mathfrak{a}_0}\varrho_1^{\mathfrak{a}_1-1}\mathcal{H}^{s-1}_{b,c}(\Sigma M))$$

of Fredholm operators for any $\mathfrak{a}_1 \in \mathbb{R}$, provided $\mathfrak{a}_0 \neq \frac{n-1}{2}$.

Proof. We have $-\frac{n-1}{2}+iw+\mathfrak{a}_0 = 0$ if and only if $w = 0$ and $\mathfrak{a}_0 = \frac{n-1}{2}$. Moreover, $a(0,y) \neq 0$ holds for all $y \in \partial X$ and $c(\nu)$ is invertible. Now, (5.5.3) implies $I_b(\mathcal{N}(D^0)(y,\eta))(w - i\mathfrak{a}_0) \neq 0$ for all $y \in \partial X$ and all $w \in \mathbb{R}$ if $\mathfrak{a}_0 = \frac{n-1}{2}$ and we get the desired property by [65, Proposition 4.5.4]. □

Remark 5.5.3. Let us make a short remark about the notation ΣM used here: Since $M = [0,1]$ is an one dimensional manifold, the corresponding Spin(1)-structure is just two copies of the interval itself (the double cover of the orthonormal frame bundle), thus the associated spinor bundle ΣM is the trivial (complex) one line-bundle over M. We kept the spin notation for systematic reasons here.

But the Dirac operator is not Fredholm for unweighted spaces:

Theorem 5.5.4. *There exists $0 \neq \zeta \in T^*\partial X \setminus \{0\}$, such that the reduced normal operator*

$$\mathcal{N}(D^0)(\zeta) : \mathcal{H}^1_{b,c}(\Sigma M) \longrightarrow \varrho_1^{-1}L^2_{b,c}(\Sigma M)$$

of the Dirac operator is not injective. Especially,

$$D^0 : \mathcal{H}^s_0(\Sigma X) \longrightarrow \mathcal{H}^{s-1}_0(\Sigma X)$$

is not a Fredholm operator.

Proof. Let $0 \neq \varphi = [b, \sigma] \in \ker(\mathcal{N}(D^0)(y,\eta))$ be arbitrary, i.e.

$$a(0,y)c(\nu)\left[b, \left(-i\sum_{j=0}^{n-1} c(X^j)\eta_j x + x\partial_x - \frac{n-1}{2}\right)\sigma\right] = 0.$$

We have the identity

$$\begin{aligned} x^{\frac{n-1}{2}} x\partial_x \left(x^{-\frac{n-1}{2}}\sigma\right) &= x^{\frac{n-1}{2}} x \left(\partial_x (x^{-\frac{n-1}{2}})\sigma + x^{-\frac{n-1}{2}}\partial_x\sigma\right) \\ &= \left(x\partial_x - \frac{n-1}{2}\right)\sigma, \end{aligned}$$

which gives

$$\begin{aligned} &\left(-i\sum_{j=0}^{n-1} c(X^j)\eta_j x + x\partial_x - \frac{n-1}{2}\right)\sigma \\ &\qquad = -i\sum_{j=0}^{n-1} c(X^j)\eta_j x\sigma + x^{\frac{n+1}{2}}\partial_x\left(x^{-\frac{n-1}{2}}\sigma\right) \\ &\qquad = x^{\frac{n+1}{2}}\left(-i\sum_{j=0}^{n-1} c(X^j)\eta_j + \partial_x\right) x^{-\frac{n-1}{2}}\sigma \end{aligned}$$

Thus, it is enough to solve the differential equation

$$\left(-i\sum_{j=0}^{n-1} c(X^j)\eta_j x + \partial_x\right)\widetilde{\sigma} = 0,$$

where $\widetilde{\sigma} := x^{-\frac{n-1}{2}}\sigma$. We conclude

$$\widetilde{\sigma}(x) = e^{-i\sum\limits_{j=0}^{n-1} c(X^j)\eta_j x} v \Longleftrightarrow \sigma(x) = x^{\frac{n-1}{2}} e^{-i\sum\limits_{j=0}^{n-1} c(X^j)\eta_j x} v \tag{5.5.4}$$

($v \in \mathbb{R}^n$) and we have found an explicit solution. Now let $\eta = (1, 0, \ldots, 0)$, i.e. $\eta_j = 0$ for all $j \neq 1$. Since $c(X^1)^2 = -1$ all eigenvalues of this matrix are $\pm i$ and we choose v in the -1 eigenspace of $c(X_i)$. Then we get

$$\sigma(x) = x^{\frac{n-1}{2}} e^{-ic(X^1)x} v = x^{\frac{n-1}{2}} e^{-Ex} v = x^{\frac{n-1}{2}}\left(e^{-x}v_1, \ldots, e^{-x}v_n\right),$$

so we can calculate:

$$\begin{aligned} \int_0^{\epsilon_1} ||\sigma(x)||^2 \frac{dx}{x} &= \int_0^{\epsilon_1} x^{n-2}\sum_{i=0}^{n} e^{-2x} v_i^2\, dx \\ &= \sum_{i=0}^{n} v_i^2 \int_0^{\epsilon_1} x^{n-2} e^{-2x}\, dx < \infty \end{aligned} \tag{5.5.5}$$

($\epsilon_1 > 0$) and

$$\int_{\epsilon_2}^{\infty} ||\sigma(x)||^2 \, dx = \sum_{i=0}^{n} v_i^2 \int_{\epsilon_2}^{\infty} x^{n-1} e^{-2x} \, dx < \infty \tag{5.5.6}$$

($\epsilon_2 > 1$) which shows, that φ is an element in $L^2_{b,c}(\Sigma M, {}^{b,c}\Omega^{\frac{1}{2}})$ (see also Appendix C.1). Moreover, we have

$$\partial_x \sigma(x) = \frac{n-1}{2} x^{\frac{n-3}{2}} e^{-Ex} v - x^{\frac{n-1}{2}} e^{-Ex} v$$

for this special choice of η. Thus $x\partial_x\varphi$ is integrable in a neighbourhood of 0 with respect to dx/x and $\partial_x\varphi$ is integrable outside a neighbourhood of zero up to infinity with respect to dx using an analogous calculation. Therefore $\varphi \in \mathcal{H}^1_{b,c}(\Sigma M, {}^{b,c}\Omega^{\frac{1}{2}})$ (see again Appendix C.1). □

Remark 5.5.5.

(i) Lott proved in [73, Theorem 1] under somewhat similar conditions, that the Dirac operator D^0 has no L^2 kernel. However, he needed the condition $\dim X = 4k$ to prove that zero is an element of the essential spectrum of D^0 (see [73, Corollary 1]). Lott also proved that

$$\dim \ker D^{0,+} = \dim \ker D^{0,-}$$

(see [73, Corollary 5]) holds, where (cf. [15, Definition 3.4.6])

$$D^0 = \begin{pmatrix} 0 & D^{0,+} \\ D^{0,-} & 0 \end{pmatrix}.$$

(ii) In [89] A. Moroianu and S. Moroianu showed that the Dirac operator (D, g), with the metric g given by

$$g := \frac{h}{f} := \frac{dx^2 + h_M}{f(x)^2}, \tag{5.5.7}$$

does not have any distributional eigenspinors of real eigenvalue inside L^2 if X has a boundary component M (not necessarily compact) that is at infinite distance with respect to the metric g ([89, Theorem 2.1]). Note, that hereby the function f depends only on the variable x in a neighbourhood of M and the proof of their result relies on that fact; our boundary defining function ϱ is not asked to fulfil this. This assumption on f implies geometrically the existence of a non complete vector field on X, which is *gradient conformal* on an open subset of X. They also prove that the existence of a vector field on a complete manifold X, which is gradient conformal and non-vanishing outside a compact subset of X implies that there is an open subset of X, which is isometric to a warped product of an open cylinder and a complete Riemannian manifold M, such that the metric decomposes like (5.5.7) (see [89, Theorem 4.1, Proposition 4.2]). Note, that a general 0-metric needs not to fulfil this (see also the discussion in appendix C.2); however, their function f needs not to behave like x^2 as in the case of a 0-metric.

It is also worth pointing out, that their proof makes use of a system of first order differential equations induced directly by the Dirac operator - not for its reduced normal operator as in our case.

5.6 K-theory for Ψ^*-algebras on conformally compact spaces

In this section we want to discuss the K-theory for certain Ψ^*-completions of $\Psi^0_0(X, {}^0\Omega^{\frac{1}{2}})$. Lauter proved in [65, Proposition 8.2.1] that the algebra of operators of order 0 in the 0- and the b-c-calculus do not admit topologies making them topological algebras with an open group of invertible elements (see also [60, Theorem 4.7.2] for a similar result on the b-calculus on a manifold with boundary):

Proposition 5.6.1. *Neither on $\Psi^0_0(X, {}^0\Omega^{\frac{1}{2}})$ nor on $\Psi^0_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$ there exists a topology giving them the structure of a topological algebra with an open group of invertible elements. In particular, neither $\Psi^0_0(X, {}^0\Omega^{\frac{1}{2}})$ nor $\Psi^0_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$ can be realized as Ψ^*-algebras.*

However, it is possible to construct "*Ψ^*-completions*" of $\Psi^0_0(X, {}^0\Omega^{\frac{1}{2}})$ and $\Psi^0_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$, that own much finer properties than the C^*- completions, using methods developed in [46]:

Proposition 5.6.2. *For any $\mathfrak{a}_0, \mathfrak{a}_1 \in \mathbb{R}$ there exists a submultiplicative Ψ^*-algebra $\mathcal{A}^{\mathfrak{a}_0,\mathfrak{a}_1}_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$ in the C^*-algebra $\mathcal{B}^{\mathfrak{a}_0,\mathfrak{a}_1}_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$, such that*

(i) $\Psi^0_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$ *is a dense subalgebra of* $\mathcal{A}^{\mathfrak{a}_0,\mathfrak{a}_1}_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$.

(ii) $A \in \mathcal{A}^{\mathfrak{a}_0,\mathfrak{a}_1}_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$ *extends from* $\dot{\mathscr{C}}(M, {}^{b,c}\Omega^{\frac{1}{2}})$ *to a bounded operator*

$$A : \varrho_0^{\mathfrak{a}_0} \varrho_1^{\mathfrak{a}_1} H^s_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}}) \longrightarrow \varrho_0^{\mathfrak{a}_0} \varrho_1^{\mathfrak{a}_1} H^s_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$$

for all $s \in \mathbb{N}_0$.

(iii) *Let* $\omega_1, \omega_2 \in \mathscr{C}^\infty_c(]0,1[)$ *and* $a \in \mathcal{A}^{\mathfrak{a}_0,\mathfrak{a}_1}_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$*: Then* $\omega_1 A \omega_2$ *is an ordinary compactly supported pseudodifferential operator on the open manifold* $]0,1[$.

Proposition 5.6.3. *For any $\mathfrak{a} \in \mathbb{R}$ there exists a submultiplicative Ψ^*-algebra $\mathcal{A}^{\mathfrak{a}}_0(X, {}^0\Omega^{\frac{1}{2}})$ in the C^*-algebra $\mathcal{B}^{\mathfrak{a}}_0(X, {}^{b,c}\Omega^{\frac{1}{2}})$, such that*

(i) $\Psi^0_0(M, {}^0\Omega^{\frac{1}{2}})$ *is a dense subalgebra of* $\mathcal{A}^{\mathfrak{a}}_0(X, {}^0\Omega^{\frac{1}{2}})$.

(ii) *Any* $A \in \mathcal{A}^{\mathfrak{a}}_0(X, {}^0\Omega^{\frac{1}{2}})$ *extends from* $\dot{\mathscr{C}}(X, {}^0\Omega^{\frac{1}{2}})$ *to a bounded operator*

$$A : \varrho_N^{\mathfrak{a}} H^s_0(X, {}^0\Omega^{\frac{1}{2}}) \longrightarrow \varrho_N^{\mathfrak{a}} H^s_0(X, {}^0\Omega^{\frac{1}{2}})$$

for all $s \in \mathbb{N}_0$.

(iii) *Let* $\omega_1, \omega_2 \in \mathscr{C}^\infty_c(\mathring{X})$ *and* $A \in \mathcal{A}^{\mathfrak{a}}_0(X, {}^0\Omega^{\frac{1}{2}})$*: Then* $\omega_1 A \omega_2$ *is an ordinary compactly supported pseudodifferential operator on the interior* $\mathring{X}$ *of* X.

Proof. See [65, Theorem 8.2.3 and Theorem 8.2.4]. □

Theorem 5.6.4. *The continuous and dense injection of* $\mathcal{A}_0^{\mathfrak{a}}(X, {}^0\Omega^{\frac{1}{2}})$ *into* $\mathcal{B}_0^{\mathfrak{a}}(X, {}^0\Omega^{\frac{1}{2}})$ *induces an isomorphism*

$$K_*(\mathcal{A}_0^{\mathfrak{a}}(X, {}^0\Omega^{\frac{1}{2}})) \cong K_*(\mathcal{B}_0^{\mathfrak{a}}(X, {}^0\Omega^{\frac{1}{2}})).$$

Proof. Since $\mathcal{A}_0^{\mathfrak{a}}(X, {}^0\Omega^{\frac{1}{2}})$ is a spectrally invariant *Fréchet* algebra and is also dense in $\mathcal{B}_0^{\mathfrak{a}}(X, {}^0\Omega^{\frac{1}{2}})$, this follows using [17, Section A.2.2]. □

Remark 5.6.5. Note, that $\mathcal{A}_0^{\mathfrak{a}}(X, {}^0\Omega^{\frac{1}{2}})$ depends on various choices (for example, it depends on the choice of sequences of 0-vector fields). Since Ψ^*-algebras are closed under intersections, we can consider the algebra $\widetilde{\mathcal{A}}_0(X, {}^0\Omega^{\frac{1}{2}})$ given by the intersection over all possible choices. We obtain a complete topological algebra with jointly continuous multiplication and continuous inversion having properties 5.6.3 (a)-(c) and property 1.1.1 (ii). In particular $\widetilde{\mathcal{A}}_0(X, {}^0\Omega^{\frac{1}{2}})$ is closed under functional calculus. In general $\widetilde{\mathcal{A}}_0(X, {}^0\Omega^{\frac{1}{2}})$ is not expected to have a *Fréchet* topology.

Theorem 5.6.6. *The continuous and dense injection of* $\widetilde{\mathcal{A}}_0(X, {}^0\Omega^{\frac{1}{2}})$ *into* $\mathcal{B}_0^{\mathfrak{a}}(X, {}^0\Omega^{\frac{1}{2}})$ *induces an isomorphism*

$$K_*(\widetilde{\mathcal{A}}_0(X, {}^0\Omega^{\frac{1}{2}})) \cong K_*(\mathcal{B}_0^{\mathfrak{a}}(X, {}^0\Omega^{\frac{1}{2}})).$$

Proof. As noted before, $\widetilde{\mathcal{A}}_0(X, {}^0\Omega^{\frac{1}{2}})$ is a locally convex, complete topological algebra, with continuous inversion, jointly continuous multiplication and with a continuous and dense embedding into $\mathcal{B}_0$. Then using A.1.8 the claim follows. □

Appendix A

K-Theory

A.1 K-theory for certain spectrally invariant algebras

Let us present a sharpening and generalization of Karoubi's density theorem in K-theory:

$$K_*(\mathcal{A}) \cong K_*(\mathcal{B}) \tag{A.1.1}$$

for certain algebras $\mathcal{B}$ with suitable (dense) subalgebras $\mathcal{A}$. The result presented here is also a generalization and simplification of the appendix in [17].

Theorem A.1.1 (Karoubi 1978/79)**.** *If $\mathcal{B}$ is an unital* Banach *algebra over $\mathbb{C}$ and $\mathcal{A}$ a continuously and dense embedded* Banach-*subalgebra, such that*

$$\begin{aligned} \mathcal{A} \cap \mathcal{B}^{-1} &= \mathcal{A}^{-1} \quad \textit{and} & \text{(A.1.2)} \\ M_n(\mathcal{A}) \cap M_n(\mathcal{B})^{-1} &= M_n(\mathcal{A})^{-1},\ n \geq 2 & \text{(A.1.3)} \end{aligned}$$

is fulfilled, then (A.1.1) holds.

First of all, let us generalize the fact that (A.1.3) is a consequence of the other assumptions (cf. [59], [105], [17] and [39]):

Lemma A.1.2. *Let $\mathcal{B}$ be an unital algebra over $\mathbb{C}$ with a topology $\tau_{\mathcal{B}}$ on $\mathcal{B}$, which makes $\mathcal{B}$ into a separated topological algebra with jointly continuous multiplication, continuous inversion and an open group of invertible elements. Furthermore let $\mathcal{A}$ be an unital dense subalgebra of $\mathcal{B}$, such that*

(i) *there is a topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$, which makes $(\mathcal{A}, \tau_{\mathcal{A}})$ into a separated topological algebra with jointly continuous multiplication,*

(ii) *the inversion in $\mathcal{A}^{-1}$ is continuous with respect to $\tau_{\mathcal{A}}$,*

(iii) *the natural inclusion $(\mathcal{A}, \tau_{\mathcal{A}}) \hookrightarrow (\mathcal{B}, \tau_{\mathcal{B}})$ is continuous,*

(iv) *$\mathcal{A}$ is spectrally invariant in $\mathcal{B}$, i.e. $\mathcal{A} \cap \mathcal{B}^{-1} = \mathcal{A}^{-1}$,*

Then (A.1.3) holds.

Remark A.1.3. For sequentially complete, locally convex algebras $\mathcal{A}$ the assumptions (ii) of A.1.2 is superfluous if $\mathcal{A}$ is metrizable or submultiplicative.

Proof of A.1.2. Let us first note, that the following *Gauss* decomposition

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_{21}b_{11}^{-1} & 1 \end{pmatrix} \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} - b_{21}b_{11}^{-1}b_{12} \end{pmatrix} \begin{pmatrix} 1 & b_{11}^{-1}b_{12} \\ 0 & 1 \end{pmatrix}, \qquad (A.1.4)$$

holds if b_{11} is invertible, where the b_{ij} are matrix elements (or even matrices themselves). To finish preparation, let us mention that we denote by id_n the identity of $M_n(\mathcal{B})$ (resp. $M_n(\mathcal{A})$) in what follows.

Now we prove, that

$$M_n(\mathcal{A}) \cap \mathcal{U}_{\mathcal{B}}(\mathrm{id}_n) \subseteq M_n(\mathcal{A})^{-1}, \qquad (A.1.5)$$

where $\mathcal{U}_{\mathcal{B}}(\mathrm{id}_n)$ denotes an open neighbourhood of id_n, such that $b \in \mathcal{B}^{-1}$ for all $b \in \mathcal{U}_{\mathcal{B}}(\mathrm{id}_n)$. This will be done by induction with respect to n:

If $n = 1$ this is true, since $\mathcal{A}$ is spectrally invariant in $\mathcal{B}$. So let $n = 2$ and choose neighbourhoods $\mathcal{U}(0), \mathcal{V}(0)$ and $\mathcal{W}(0)$ of zero in $\mathcal{B}$, such that

$$\mathcal{B} \ni a \in e + \mathcal{U}(0),\ \mathcal{B} \ni b \in \mathcal{W}(0) \Longrightarrow\ a + b \in e + \mathcal{V}(0) \subseteq \mathcal{B}^{-1}.$$

The map $\Phi : \mathcal{B} \times \mathcal{B}^{-1} \times \mathcal{B} \longrightarrow \mathcal{B}$ given by

$$(b_1, b_2, b_3) \longmapsto b_1 b_2^{-1} b_3$$

is continuous, since the inversion and (joint) multiplication are continuous. In particular Φ is continuous in $(0, e, 0)$ and $\Phi(0, e, 0) = 0$. Thus, $\Phi^{-1}(\mathcal{W}(0))$ is open and we can find open subsets $\mathcal{O}_1(0)$, $\mathcal{O}_2(0)$ and $\mathcal{O}_3(0)$, such that

$$(b_1, b_2, b_3) \in \mathcal{O}_1(0) \times (e + \mathcal{O}_2(0)) \times \mathcal{O}_3(0) \Longrightarrow \Phi(b_1, b_2, b_3) \in \mathcal{W}(0).$$

We define $\mathcal{O}(0) := \bigcap_{i=1}^{3} \mathcal{O}_i(0)$ and denote by $\mathfrak{U}(0)$ the open neighbourhood of zero in $M_2(\mathcal{B})$ that corresponds to $\mathcal{U}(0) \times \mathcal{O}(0) \times \mathcal{O}(0) \times \mathcal{U}(0)$ with respect to the identification

$$(a_{11}, a_{12}, a_{21}, a_{22}) \longmapsto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Let $\mathfrak{a} = (a_{ij}) \in M_2(\mathcal{A}) \cap (\mathrm{id}_2 + \mathfrak{U}(0))$ be arbitrary. Then $a_{11} = a_{11} + 0 \in \mathcal{A} \cap (e + \mathcal{V}(0))$, i.e. $a_{11} \in \mathcal{B}^{-1}$ which implies $a_{11} \in \mathcal{A}^{-1}$ since $\mathcal{A}$ is spectrally invariant in $\mathcal{B}$. Thus we can use (A.1.4); note that the outer left and right matrices on the right hand side of (A.1.4) are always invertible in $\mathcal{A}$. By the choice of $\mathfrak{U}(0)$, we get $a_{22} \in \mathcal{U}(0)$ and $a_{21}a_{11}^{-1}a_{12} \in \mathcal{W}(0)$. This implies $a_{22} - a_{21}a_{11}^{-1}a_{12} \in e + \mathcal{V}(0)$ and we have $a_{22} - a_{21}a_{11}^{-1}a_{12} \in \mathcal{A}^{-1}$. We conclude $M_2(\mathcal{A}) \cap (\mathrm{id}_2 + \mathfrak{U}(0)) \subseteq M_2(\mathcal{A})^{-1}$, which proves (A.1.5) in the case $n = 2$.

Now, suppose that we had already proven (A.1.5) for $0 \leq k \leq n-1$: If $a = (a_{ij})_{ij} \in M_n(\mathcal{A})$ we set

$$b_{11} := a_{11},\ b_{12} := (a_{12} \cdots a_{1n}),\ b_{21} := \begin{pmatrix} a_{21} \\ \vdots \\ a_{n1} \end{pmatrix},\ b_{22} := \begin{pmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

and get $b_{12}, b_{21}^t \in \mathcal{A}^{n-1}$ and $b_{22} \in M_{n-1}(\mathcal{A})$. Thus we can use the same argument as in the case $n = 2$ (after the obvious changes). Note, that $b_{21}b_{11}^{-1}b_{12}$ defines an element in $M_{n-1}(\mathcal{A})$ (by matrix multiplication).

Now, $M_n(\mathcal{A}) \cap \mathcal{U}_{\mathcal{B}}(\mathrm{id}_n) \subseteq M_n(\mathcal{A})^{-1}$ implies $M_n(\mathcal{A}) \cap M_n(\mathcal{B})^{-1} = M_n(\mathcal{A})^{-1}$, since $M_n(\mathcal{A})$ is dense in $M_n(\mathcal{B})$ using [39, Lemma 5.3]. □

Lemma A.1.4. *Let $\mathcal{B}$ and $\mathcal{A}$ be given as in A.1.2. Then:*

(i) *The spectrum $\sigma_{\mathcal{B}}(b)$ of $b \in \mathcal{B}$ is compact and upper semi continuous with respect to b, i.e. for all open neighbourhoods $\mathcal{U}$ of $\sigma_{\mathcal{B}}(b)$ exists an open neighbourhood $\mathcal{W}(b) \subseteq \mathcal{B}$, such that $\sigma_{\mathcal{B}}(a) \subseteq \mathcal{U}$ holds for all $a \in \mathcal{W}(b)$.*

(ii) *For fixed $b \in \mathcal{B}$ and every closed set $\mathcal{K} \subseteq \mathbb{C}$ with $\mathcal{K} \cap \sigma(b) = \emptyset$ and for each neighbourhood $\mathcal{V}(0)$ of $0 \in \mathcal{B}$ there exists a neighbourhood $\mathcal{W}(b)$ of b, such that:*

  (a) $\mathcal{W}(b) \cap \mathcal{A} \neq \emptyset$,

  (b) $(\lambda e - a)^{-1} \in \mathcal{A}$ *for all* $a \in \mathcal{W}(b) \cap \mathcal{A}$ *and all* $\lambda \in \mathcal{K}$,

  (c) $(\lambda e - b)^{-1} - (\lambda e - c)^{-1} \in \mathcal{V}(0)$ *for all* $\lambda \in \mathcal{K}$ *and all* $c \in \mathcal{W}(b)$.

Proof. (i) Since $\mathcal{B}^{-1}$ is open, there is a neighbourhood $\mathcal{V}(0)$, such that $e - b$ is invertible for all $b \in \mathcal{V}(0)$. The scalar multiplication with $\lambda \in \mathbb{C}$ is continuous, which implies $\lambda b \xrightarrow{\lambda \to 0} 0$. This shows, that we can find $N \in \mathbb{N}$, such that $b/N \in \mathcal{V}(0)$. But then $(e - b/\lambda) \in \mathcal{B}^{-1}$ for all $\lambda \in \mathbb{C}$ fulfilling $|\lambda| \geq N$, and $(\lambda e - b)^{-1} = \lambda^{-1}(e - b/\lambda)^{-1}$ exists in $\mathcal{B}$. Thus $\sigma_{\mathcal{B}}(b)$ is bounded. Now, let $\lambda_0 \notin \sigma(b)$, i.e. $(\lambda_0 e - b)^{-1} \in \mathcal{B}$. Since $\mathcal{B}^{-1}$ is open, we can find $\mathcal{U}_\delta(\lambda_0) \subseteq \mathbb{C}$, such that $(\lambda e - b)^{-1} \in \mathcal{B}^{-1}$ for all $\lambda \in \mathcal{U}_\delta(\lambda_0)$. Therefore $\sigma_{\mathcal{B}}(b)$ is also closed, hence compact.

Now, let $\mathcal{U}$ be an open neighbourhood of $\sigma_{\mathcal{B}}(b)$. Since the inversion is continuous, we can choose a neighbourhood $\mathcal{V}(0)$ of zero, such that $(e - a) \in \mathcal{B}^{-1}$ holds for all $a \in \mathcal{V}(0)$. Let $r \gg 0$ be given such that

$$\sigma(b) \subseteq \mathcal{U} \subseteq K_r \text{ and } b/\lambda \in \mathcal{V}(0) \text{ if } |\lambda| > r \ (\lambda \in \mathbb{C}),$$

where $K_r := \{z \in \mathbb{C} : |z| \leq r\}$. Again, the continuity of the scalar multiplication yields, that we can find $\mathcal{W}_0(b) \subseteq \mathcal{B}$, such that $a/\lambda \in \mathcal{V}(0)$ holds for all $a \in \mathcal{W}_0(b)$ and for all $\lambda \in \mathbb{C}$ with $|\lambda| > r$. But this implies

$$(\lambda e - a)^{-1} = \lambda^{-1}\left(e - \frac{a}{\lambda}\right)^{-1} \in \mathcal{B}$$

for all $a \in \mathcal{W}_0(b)$ and for all $\lambda \notin K_r$.

Now, set $R_\epsilon := \{\lambda \in \mathbb{C} : \mathrm{dist}\,(\lambda, \sigma(b) < \epsilon\}$, where $\epsilon > 0$ is chosen, such that $\sigma_{\mathcal{B}}(b) \subseteq R_\epsilon \subseteq \mathcal{U}$. Then $\mathcal{D} := K_r \setminus R_\epsilon$ is compact in $\mathbb{C}$. Let $\lambda_0 \in K_r \setminus R_\epsilon (\subseteq \complement\sigma_{\mathcal{B}}(b))$ be arbitrary.

The map

$$\mathcal{D} \times \mathcal{B} \ni [(\lambda, a) \longmapsto (\lambda e - a)] \longmapsto (\lambda e - a)^{-1}$$

is continuous in (λ_0, b) and $\mathcal{B}^{-1}$ is open, thus we can find a neighbourhood $\mathcal{U}(\lambda_0)$ of λ_0 and a neighbourhood $\mathcal{W}_{\lambda_0}(b)$ of b, such that

$$\lambda \in \mathcal{U}(\lambda_0) \text{ and } a \in \mathcal{W}_{\lambda_0}(b) \Longrightarrow (\lambda e - a)^{-1} \in \mathcal{B}.$$

Clearly $\bigcup\limits_{\lambda_0 \in \mathcal{D}} \mathcal{U}(\lambda_0) \supseteq K_r \setminus R_\epsilon$ and we can use *Heine-Borel* to get $\bigcup\limits_{j=1}^{N} \mathcal{U}_j(\lambda_j) \supseteq K_r \setminus R_\epsilon$. Now, set $\mathcal{W}(b) := \bigcap\limits_{j=1}^{N} \mathcal{W}_{\lambda_j}(b) \cap \mathcal{W}_0(b)$, then:

$$a \in \mathcal{W}(b) \text{ and } \lambda \in \complement R_\epsilon \Longrightarrow (\lambda e - a)^{-1} \in \mathcal{B},$$

and we have finished the proof of (i).
(ii) Let $b \in \mathcal{B}$ be fixed, $\mathcal{K} \subseteq \complement\sigma(b)$ be closed and $\mathcal{V}(0)$ be an arbitrary open neighbourhood of zero in $\mathcal{B}$. Since $\mathcal{K}$ and $\sigma(b)$ are closed, we can find an open neighbourhood $\mathcal{U}$ of $\sigma(b)$, such that $\mathcal{K} \cap \mathcal{U} = \emptyset$. But then we can apply (i) and find an open neighbourhood $\mathcal{W}_1(b)$, such that $(\lambda e - c)^{-1} \in \mathcal{B}$ exists for all $c \in \mathcal{W}_1(b)$ and all $\lambda \in \mathcal{K}$. The function

$$r : \mathcal{K} \times \mathcal{W}_1(b) \ni (\lambda, c) \longmapsto (\lambda e - b)^{-1} - (\lambda e - a)^{-1} \in \mathcal{B}$$

is well-defined and continuous with $r(\lambda, b) = 0 \in \mathcal{V}(0)$ for all $\lambda \in \mathcal{K}$. Thus we can find a neighbourhood $\mathcal{W}(b) \subseteq \mathcal{W}_1(b)$, such that $(\lambda e - c)^{-1} \in \mathcal{B}$ for all $c \in \mathcal{W}(b)$ and

$$c \in \mathcal{W}(b) \text{ and } \lambda \in \mathcal{K} \Longrightarrow r(\lambda, c) \in \mathcal{V}(0),$$

which gives (c). Since $\mathcal{A} \cap \mathcal{B}^{-1} = \mathcal{A}^{-1}$ and $\mathcal{A}$ is dense in $\mathcal{B}$ we get also $\mathcal{A} \cap \mathcal{W}(b) \neq \emptyset$ and $(\lambda e - a)^{-1} \in \mathcal{A}$ for all $a \in \mathcal{W}(b) \cap \mathcal{A}$. □

In the following step for the proof of (A.1.1) we apply the functional calculus of *Waelbroeck* (see [112], [113]) together with lemma A.1.4.

Lemma A.1.5. *Assume that $\mathcal{A}$ and $\mathcal{B}$ have the same properties as in A.1.2 and are also locally convex and sequentially complete. Then:*

- *$P(\mathcal{A})$ is dense in $P(\mathcal{B})$ and*
- *$\pi_n(P(\mathcal{A})) \cong \pi_n(P(\mathcal{B}))$ holds for all $n \in \mathbb{N}_0$.*

Remark A.1.6. It is worth pointing out, that under the assumptions of A.1.5 the spectrum of a given element $b \in \mathcal{B}$ resp. $a \in \mathcal{A}$ is not empty.

Proof of A.1.5. First of all let us show, that $P(\mathcal{A})$ is dense in $P(\mathcal{B})$: Let $\varepsilon > 0$, $||\cdot||_\eta$ be a (fixed) seminorm on $\mathcal{B}$ and $q \in P(\mathcal{A})$ be arbitrary. We have

$$\lambda e - q = \lambda(e - q) + (\lambda - 1)q,$$

which gives $(\lambda e - q)^{-1} = \frac{e-q}{\lambda} + \frac{q}{\lambda-1}$, since $(e - q)q = 0$. Thus $\sigma(q) \subset \{0, 1\}$ and we get

$$q = \frac{1}{2\pi i} \int\limits_{|z-1|=\frac{1}{2}} (\lambda e - q)^{-1}\, d\lambda$$

using the functional calculus of Waelbroeck [112]. Moreover, $\mathcal{U}$ given by

$$\mathcal{U} := \left\{ z \in \mathbb{C} \,:\, |z| < \frac{1}{4} \right\} \cup \left\{ z \in \mathbb{C} \,:\, |z - 1| < \frac{1}{4} \right\}$$

is an open neighbourhood of $\sigma_{\mathcal{B}}(q)$ and we have $\Gamma \cap \mathcal{U} = \emptyset$ where

$$\Gamma := \left\{ z \in \mathbb{C} \,:\, |z-1| = \frac{1}{4} \right\}.$$

Set $\mathcal{V}(0) := \left\{ b \in \mathcal{B} \,:\, ||b||_\eta < \frac{\varepsilon}{C} \right\}$, where $C := L(\Gamma)/2\pi$ and $L(\Gamma)$ denotes the length of the compact contour $\Gamma \subseteq \mathbb{C}$. Then the last lemma together with the density of $\mathcal{A}$ in $\mathcal{B}$ implies, that we can find a (small) open neighbourhood $\mathcal{W}(q)$ of q such that $\mathcal{W}(q) \cap \mathcal{A} \neq \emptyset$ and

$$a \in \mathcal{A},\ a \in \mathcal{W}(q), \lambda \in \Gamma \implies \begin{array}{l} \text{(i) } (\lambda e - a)^{-1} \in \mathcal{A} \text{ and} \\ \text{(ii) } (\lambda e - b)^{-1} - (\lambda e - a)^{-1} \in \mathcal{V}(0). \end{array}$$

Note that $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$ holds since $\mathcal{A} \cap \mathcal{B}^{-1} = \mathcal{A}^{-1}$. Let $a \in \mathcal{A} \cap \mathcal{W}(q)$ and define

$$h(z) := \begin{cases} 1 & \text{if} \quad |1-z| < \frac{1}{2}, \\ 0 & \text{else.} \end{cases}$$

Then $h(z)^2 = h(z)$ holds if $|z-1| = 1/4$, thus we get

$$\frac{1}{2\pi i} \int\limits_{|z-1|=\frac{1}{4}} (\lambda e - a)^{-1}\, d\lambda = h(a) \in P(\mathcal{A}).$$

We conclude:

$$\begin{aligned} ||h(a) - q||_\eta &= \frac{1}{2\pi} \Bigg|\Bigg| \int\limits_{|z-1|=\frac{1}{4}} \left((\lambda e - a)^{-1} - (\lambda e - q)^{-1} \right) \, d\lambda \Bigg|\Bigg|_\eta \\ &\leq \frac{1}{2\pi} L(\Gamma) \sup_{\lambda \in \Gamma} \left(||(\lambda e - a)^{-1} - (\lambda e - q)^{-1}||_\eta \right). \end{aligned}$$

By the choice of $\mathcal{W}(q)$, we have $(\lambda e - a)^{-1} - (\lambda e - q)^{-1} \in \mathcal{V}(0)$ for all $\lambda \in \Gamma$ and it follows

$$||h(a) - q||_\eta \leq \frac{1}{2\pi} L(\Gamma) \sup_{\lambda \in \Gamma} ||(\lambda e - a)^{-1} - (\lambda e - q)^{-1}||_\eta < \varepsilon$$

and we have proved the density of $P(\mathcal{A})$ in $P(\mathcal{B})$.

Now, let $\mathcal{M}$ be a connected component of $P(\mathcal{B})$. Then $\mathcal{A} \cap \mathcal{M}$ is a dense connected component of $P(\mathcal{A})$: The previous discussion shows, that $h(a)$ is also in $\mathcal{M}$ if $q \in \mathcal{M}$. Let $p : [0,1] \longrightarrow P(\mathcal{B})$ be a continuous path in $P(\mathcal{B})$ with $p(0) \in \mathcal{A}$ and $p(1) \in \mathcal{A}$. Then there exists $g \in \mathcal{B}_e^{-1}$, such that $p(1) = gp(0)g^{-1}$, cf. [39, Lemma 2.2], where $\mathcal{B}_e^{-1}$ denotes the connected component of $e \in \mathcal{B}^{-1}$. Since $\mathcal{A}^{-1}$ is dense in $\mathcal{B}^{-1}$, we can choose $\widetilde{a} \in \mathcal{A}^{-1}$ fulfilling $\widetilde{a} - g \in \mathcal{U}(0)$ where $\mathcal{U}(0)$ is a suitable neighbourhood of zero in $\mathcal{B}$ and $\widetilde{a} \in \mathcal{A}_e^{-1}$ (see the proof of [39, Lemma 5.3]). This gives a continuous path from $p(0)$ to $\widehat{a} := \widetilde{a}p(0)\widetilde{a}^{-1}$ in $P(\mathcal{A})$. The function

$$\Phi : \mathcal{A} \ni c \longmapsto -p(1)(e-c) - c(e - p(1)) \in \mathcal{A}$$

is continuous in $\mathcal{A}$ and $\Phi(p(1)) = 0$. Let $\mathcal{V}(0)$ be a neighbourhood of zero (in $\mathcal{A}$), such that $e - r \in \mathcal{A}_e^{-1}$ for all $r \in \mathcal{V}(0)$. Then we find a neighbourhood $\mathcal{U}(p(1))$ of $p(1)$ (in $\mathcal{A}$) with $\Phi(p) \in \mathcal{V}(0)$ for all $p \in \mathcal{V}(0)$. We have

$$p(1) - \widehat{a} = gp(0)g^{-1} - \widetilde{a}p(0)\widetilde{a}^{-1} = (g - \widetilde{a})p(0)g^{-1} + \widetilde{a}p(0)(g^{-1} - \widetilde{a}^{-1})$$

and we choose $\mathcal{U}(0)$ so small, that

$$\widetilde{a} - g \in \mathcal{U}(0) \Rightarrow \widehat{a} \in \mathcal{U}(p(1)).$$

Then $-p(1)(e - \widehat{a}) - \widehat{a}(e - p(1)) = \Phi(\widehat{a}) \in \mathcal{V}(0)$, i.e.

$$c := (e - p(1))(e - \widehat{a}) + p(1)\widehat{a} \in \mathcal{A}_e^{-1}$$

and $c\widehat{a}c^{-1} = p(1)$. Thus, we can again find a path that connects $p(1)$ and $\widehat{a}$ in $P(\mathcal{A})$. This finishes the proof for π_0. If $n \geq 1$, note that

$$\mathscr{C}(\Omega, \mathcal{A}) \subseteq \mathscr{C}(\Omega, \mathcal{B}) \;\; (\Omega \text{ compact})$$

is a dense spectrally invariant algebra having the same properties as $\mathcal{A}$ and $\mathcal{B}$ (as stated in the assumption of A.1.5), hence we can apply π_0 to this algebra and get the result for $n \geq 1$. This finishes the proof. □

Corollary A.1.7. *Combining A.1.2 and A.1.5 (under the assumptions of A.1.5) we get*

$$\pi_k\left(P(M_n(\mathcal{A}))\right) \cong \pi_k\left(P(M_n(\mathcal{B}))\right)$$

for all $n \in \mathbb{N}$ and all $k \in \mathbb{N}_0$.

We have proven:

Theorem A.1.8. *Let $\mathcal{B}$ be unital algebra over $\mathbb{C}$ with a topology $\tau_{\mathcal{B}}$ on $\mathcal{B}$, which makes $\mathcal{B}$ into a locally convex algebra with jointly continuous multiplication, continuous inversion and an open group $\mathcal{B}^{-1}$ of invertible elements and let $\mathcal{B}$ be sequentially complete. Additionally let $\mathcal{A}$ be a dense sequentially complete subalgebra of $\mathcal{B}$, such that*

(i) *$e \in \mathcal{A}$,*

(ii) *there is a topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$, which makes $(\mathcal{A}, \tau_{\mathcal{A}})$ into a locally convex algebra, with jointly continuous multiplication, an open group $\mathcal{A}^{-1}$ of invertible elements and continuous inversion,*

(iii) *the natural inclusion $(\mathcal{A}, \tau_{\mathcal{A}}) \hookrightarrow (\mathcal{B}, \tau_{\mathcal{B}})$ is continuous,*

(iv) *$\mathcal{A}$ is spectrally invariant in $\mathcal{B}$, i.e. $\mathcal{A} \cap \mathcal{B}^{-1} = \mathcal{A}^{-1}$.*

Then

$$K_*(\mathcal{A}) \cong K_*(\mathcal{B})$$

holds.

Appendix B

Boutet de Monvel's algebra

B.1 Preliminaries

Let X be a n-dimensional compact manifold with boundary ∂X. Moreover, let Y be a closed manifold of dimension n where $X \hookrightarrow Y$. Besides some change in the order of variables, we closely follow [79].

Definition B.1.1.

(i) Let $u \in \mathscr{C}^\infty(X)$ and P be a (*classical*) pseudodifferential operator on Y. Then we get a map $P_+ : \mathscr{C}^\infty(X) \longrightarrow \mathscr{C}^\infty(\overset{\circ}{X})$, $u \longmapsto P_+ u$, which is given by the restriction of P to $\overset{\circ}{X}$ applied to the extension by zero of u to Y.

(ii) P has the *transmission property* if the image of P_+ is contained in $\mathscr{C}^\infty(X)$.

Let us note, that the transmission property of a classical polyhomogeneous pseudodifferential operator can be completely characterized by some sort of symmetry condition on its symbol:

Proposition B.1.2. *A symbol $p(x, y, \xi, \eta) \in S^0_{cl}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{n-1}_y, \mathbb{R}_\xi \times \mathbb{R}^{n+1}_\eta)$ has the transmission property at $x = 0$, if and only if*

$$\partial_x^\beta \partial_\xi^\alpha p_j(0, y, 1, 0) = (-1)^{j+|\alpha|} \partial_x^\beta \partial_\xi^\alpha p_j(0, y, -1, 0)$$

for all $\alpha, \beta \geq 0$ and $j \leq 0$.

Remark B.1.3. Some words about the notation used here. We have changed the notation of variables used in [49] resp. [79] and [80] slightly. The notation used here is given in a way, that it fits into the one used in [65] for the 0-calculus.

An operator in Boutet de Monvel's calculus is a matrix:

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} \mathscr{C}^\infty(X) \\ \oplus \\ \mathscr{C}^\infty(\partial X) \end{matrix} \longrightarrow \begin{matrix} \mathscr{C}^\infty(X) \\ \oplus \\ \mathscr{C}^\infty(\partial X), \end{matrix} \tag{B.1.1}$$

where P is a pseudodifferential operator satisfying the transmission condition, G is a singular *Green* operator, T is a *trace operator*, K is a *potential operator* and S is a pseudodifferential operator on ∂X.

If A is such an operator, then one can define the following two symbol maps:

(i) The *principal symbol* $\sigma(A)$, which is given by the principal symbol of P restricted to S^*X.

(ii) The *boundary symbol* $\gamma(A)$, which is an element of $\mathscr{C}^\infty(S^*\partial X, \mathscr{L}(L^2(\overline{\mathbb{R}}_+) \oplus \mathbb{C}))$.

Moreover, let $\mathcal{A}$ be the algebra of all operators of the form (B.1.1); denote by $\mathfrak{A}$ the closure of $\mathcal{A}$ in $\mathfrak{H} := L^2(X) \oplus L^2(\partial X)$ and by $\overline{\sigma}$ and $\overline{\gamma}$ the continuous extensions of σ and γ to $\mathfrak{A}$. Finally denote by $\mathfrak{K}$ the ideal of compact operators in $\mathfrak{H}$.

Let us make the notion of the boundary symbol map more explicit. To this end choose local coordinates

$$(x, y) : X \supseteq V \longrightarrow \overline{\mathbb{R}}_+ \times \mathbb{R}^{n-1}_y$$

near $q \in \partial X$, such that the boundary elements $V \cap \partial X$ are given by $\{x = 0\}$. Moreover, let $(\xi, \eta) \in \mathbb{R}_\xi \times \mathbb{R}^{n-1}_\eta$ be covariables to (x, y), such that we get a coordinate neighbourhood (x, y, ξ, η) of an appropriate lift of V to S^*X. Let $p(x, y, \xi, \eta)$, $g(x, y, \xi, \eta)$, $k(y, \xi, \eta)$, $t(y, \xi, \eta)$ and $s(y, \eta)$ be the symbols of P, G, K, T and S with respect to the above coordinates and denote by p_0, g_0 etc. the leading terms of the asymptotic expansions of p, g etc.

Then for fixed $(y, \eta) \in S^*\partial X$ we get

$$\gamma(A)(y, \eta) = \begin{pmatrix} p_0(0, y, D_\xi, \eta)_+ + g_0(y, D_\xi, \eta) & k_0(y, D_\xi, \eta) \\ t_0(y, D_\xi, \eta) & s_0(y, \eta) \end{pmatrix}. \tag{B.1.2}$$

If p is a classical pseudodifferential symbol of order zero on $\mathbb{R}^n$, then $p_0(0, y, \cdot, \eta)$ is a classical symbol on $\mathbb{R}_\xi$ for fixed (y, η). If in addition p has the transmission property, then the values of p in $\xi = \pm\infty$ coincide.

Denote by $U : L^2(\mathcal{S}^1) \longrightarrow L^2(\mathbb{R})$ the unitary mapping

$$(Ug)(t) := \frac{\sqrt{2}}{1 + it} g\left(\frac{1 - it}{1 + it}\right),$$

and by $\mathcal{H}_{-1}$ the image of $\mathscr{C}^\infty(\mathcal{S}^1)$ under U. Set $\mathcal{H}_0 := \mathcal{H}_{-1} \oplus \mathbb{C}$, then $p \in \mathcal{H}_0$ (if p has the transmission property). Let $\mathfrak{T}$ be the C^*-algebra of all bounded operators on $L^2(\overline{R}_+)$ generated by

$$\{p(D)_+ \,:\, p \in \mathcal{H}_0\}.$$

Then we have (cf. [79, page 149, (8)]):

$$\overline{\gamma}(\mathcal{A}) \subseteq \mathscr{C}(S^*\partial X) \otimes \begin{pmatrix} \mathfrak{T} & L^2(\overline{\mathbb{R}}_+) \\ L^2(\overline{\mathbb{R}}_+)^* & \mathbb{C} \end{pmatrix}. \tag{$B.1.3$}$$

But not every element of the above right hand side gives an element in the image of $\overline{\gamma}$. For this, note, that the map $p(D)_+ \mapsto p(\infty)$ extends to a $*$-homomorphism $\lambda : \mathfrak{T} \longrightarrow \mathbb{C}$. The following proposition is due to Melo, Nest and Schrohe [79]:

Proposition B.1.4.

(i) $\overline{\gamma}(\mathfrak{A})$ *is isomorphic to*

$$\begin{pmatrix} \mathscr{C}(\partial X) & 0 \\ 0 & 0 \end{pmatrix} \oplus \left(\mathscr{C}(S^*\partial X) \otimes \begin{pmatrix} \mathfrak{T}_0 & L^2(\overline{\mathbb{R}}_+) \\ L^2(\overline{\mathbb{R}}_+)^* & \mathbb{C} \end{pmatrix} \right),$$

where $\mathfrak{T}_0$ *denotes the kernel of* $\lambda : \mathfrak{T} \longrightarrow \mathbb{C}$.

(ii) $\ker(\overline{\gamma})/\mathfrak{K}$ *is isomorphic to* $\mathscr{C}_0(S^*\mathring{X})$.

B.2 K-theory for Boutet de Monvels algebra

The K-groups of Boutet de Monvels algebra have been calculated by Melo, Nest and Schrohe in [79] (imposing a torsion condition on $K_*(\mathscr{C}(\partial X))$) and by Melo, Schick and Schrohe in [80] (without the torsion condition). Let us shortly point out, that it is possible to use 5.3.8 to calculate the K-theory of Boutet de Monvel's algebra.

First we also introduce the maps defined in [79]:

- Let $b : \mathscr{C}(\partial X) \longrightarrow \mathrm{r}\,(\overline{\gamma})$ be given by $g \longmapsto \overline{\gamma}\left(\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}\right)$, where f is a function with $f_{|\partial X} = g$.
- Denote by $m : \mathscr{C}(X) \longrightarrow \mathfrak{A}/\mathfrak{K}$ the $*$-homomorphism that maps $f \in \mathscr{C}(X)$ to the class of $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$, where g denotes the restriction of f to ∂X.

Then Boutet de Monvels algebra also fits into the setting of 5.3.8 and we get the following result (cf. [80, Theorem 1]; the proof is exactly the same as in 5.3.13, hence we will omit it).

Theorem B.2.1. *Let* $\mathfrak{J} := \ker(\overline{\gamma})$. *Then the commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{J}/\mathfrak{K} \arrow[r] & \mathfrak{A}/\mathfrak{K} \arrow[r] & \mathfrak{A}/\mathfrak{J} \arrow[r] & 0 \\ 0 \arrow[r] & \mathscr{C}_0(\mathring{X}) \arrow[r, "\iota"] \arrow[u, "m"] & \mathscr{C}(X) \arrow[r] \arrow[u, "m"] & \mathscr{C}(\partial X) \arrow[r] \arrow[u, "b"] & 0 \end{tikzcd}$$

with exact rows induces an isomorphism

$$K_*(\mathfrak{A}/\mathfrak{K}) \cong K_*(\mathscr{C}(X)) \oplus K_{*-1}(\mathscr{C}_0(T^*\mathring{X})).$$

B.3 The structure elements of the 0-calculus revisited

In 5.3.14 we implicitly proved the following theorem:

Theorem B.3.1. *The C^*-algebra $\mathcal{B}(X, {}^0\Omega^{\frac{1}{2}})$ of 0-operators and the C^*-closure $\mathfrak{A}$ of Boutet de Monvel's algebra have isomorphic K-groups, i.e. we have*

$$K_*(\mathcal{B}(X, {}^0\Omega^{\frac{1}{2}})) \cong K_*(\mathfrak{A}).$$

However, we used a homotopy argument to reduce the joint symbol map in the b-c-calculus to a symbol that depends somehow only on the c-indical family of the reduced normal operator. Now, we shortly want to prove that there is a subalgebra in the zero calculus where we do not need this homotopy argument. Again, we use the definitions and notations introduced in chapter 5.

Definition B.3.2. Let $\mathcal{T}$ be the set of all $N \in \mathcal{B}_{b,c}$, such that there exists $c \in \mathbb{R}$ with

$$\begin{aligned} {}^{b,c}\sigma^{(0,0)}(N)(z, \pm 1) &= c \quad \forall\, z \in [0,1], \\ I_b(N)(\xi) &= c \quad \forall\, \xi \in \mathbb{R}. \end{aligned}$$

Then we have:

Proposition B.3.3. *$\mathcal{T}$ is a C^*-subalgebra of $\mathcal{B}_{b,c}$ with $\mathcal{K}_{b,c} \subseteq \mathcal{T}$. Moreover, we have $\mathcal{T}/\mathcal{K}_{b,c} \cong \mathscr{C}(\mathcal{S}^1)$.*

Proof. We only have to prove the second part of the proposition. Consider the linear map

$$\mathcal{T} \ni A \longmapsto I_c(A) \in \mathscr{C}([-1,1]),$$

with kernel $\mathcal{K}_{b,c}$. By the compatibility conditions we then get $\mathcal{T}/\mathcal{K}_{b,c} \cong R(I_{c|\mathcal{T}}) \cong \mathscr{C}(\mathcal{S}^1)$, if we glue -1 and 1 to -1 on $\mathcal{S}^1$. □

Using the identification stated in the last proposition and the joint symbol map of $\mathcal{B}_{b,c}$ we get the following $*$-homomorphism

$$\sigma_{b,c} : \mathcal{T} \xrightarrow{\tau_{b,c}} \mathscr{C}(\mathcal{S}^1) \xrightarrow{\mathrm{ev}_1} \mathbb{C},$$

where ev_1 denotes the point evaluation at $1 \in \mathcal{S}^1$. Note also, that the joint symbol $\tau_{c,b}(A)$ of an operator $A \in \mathcal{T}$ is completely determined by the c-indical family $I_c(A)$ of A.

Denote by $\mathcal{T}_0$ the kernel of $\sigma_{b,c}$. Then we get the exact sequence

$$0 \longrightarrow \mathcal{T}_0 \longrightarrow \mathcal{T} \longrightarrow \mathbb{C} \longrightarrow 0.$$

Proposition B.3.4. *Let $\mathcal{I}$ denote the set of all $N \in \mathcal{B}^{(\mathfrak{a})}_{b,c}(S^*\partial X)$ satisfying the conditions $I^{(\mathfrak{a})}_b(N(\widehat{\eta})) = d_1$, ${}^{b,c}\sigma^{(0,0)}(N(\widehat{\eta}))(z,1) = d_2$ and ${}^{b,c}\sigma^{(0,0)}(N(\widehat{\eta}))(z,-1) = d_3$ for all $z \in [0,1]$ and $\widehat{\eta} \in S^*\partial X$. Then $\mathcal{I}$ is a C^*-subalgebra of $\mathcal{B}^{(\mathfrak{a})}_{b,c}(S^*\partial X)$.*

Proof. First of all note, that we have $d_1 = d_2 = d_3 =: d$ due to the compatibility conditions. Let $N_1, N_2 \in \mathcal{I}$ be arbitrary and $\widehat{\eta} \in S^*\partial X$ fixed. Then

$$\begin{aligned} I^{(\mathfrak{a})}_b(N_1(\widehat{\eta})N_2(\widehat{\eta})) &= I^{(\mathfrak{a})}_b(N_1(\widehat{\eta}))I^{(\mathfrak{a})}_b(N_2(\widehat{\eta})) = c_1 d_1 =: e_1 \text{ and} \\ I^{(\mathfrak{a})}_b(N(\widehat{\eta})^*) &= I^{(\mathfrak{a})}_b(N(\widehat{\eta}))^* = c_1^*. \end{aligned}$$

The multiplication property of ${}^{b,c}\sigma^{(0,0)}$ gives a similar result for the other two symbol families, therefore we see, that $\mathcal{I}$ is closed by multiplication and by taking adjoints.

Let $(N_n)_{\in \mathbb{N}} \subseteq \mathcal{I}$ be with $N_n \xrightarrow{n\to\infty} N$. Then it remains to show, that $N \in \mathcal{I}$. We have

$$||I_b^{(\mathfrak{a})}(N_n(\widehat{\eta})) - I_b^{(\mathfrak{a})}(N(\widehat{\eta}))|| \leq C||N_n - N|| \xrightarrow{n\to\infty} 0,$$

by the continuity of the symbol map, which gives $I_b^{(\mathfrak{a})}(N(\widehat{\eta})) = c_1$ for $\widehat{\eta} \in S^*\partial X$. With a similar calculation for ${}^{b,c}\sigma^{(0,0)}$ we conclude, that $\mathcal{I}$ is closed, which finishes the proof. □

Remark B.3.5.

(i) Note, that we have $\mathscr{C}(S^*\partial X, \mathcal{K}_0) \subseteq \mathcal{I} \subseteq \mathscr{C}(S^*\partial X, \mathcal{T})$.

(ii) If $N \in \mathcal{I}$ is arbitrary, then $N(\widehat{\eta}) \in \mathcal{T}$ for all $\widehat{\eta} \in S^*\partial X$.

Proposition B.3.6. *Let $\mathcal{I}_0$ denote the set of all $N \in \mathcal{I}$ satisfying*

$$\sigma_{b,c}(I_b^{(\mathfrak{a})}(N(\widehat{\eta})))(-1) = 0$$

for all $\widehat{\eta} \in S^*\partial X$. Then $\mathcal{I}_0$ is a closed, two-sided ideal in $\mathcal{I}$.

Proof. Since $\mathcal{I}_0$ is given by the kernel of a $*$-homomorphism, this is clear. □

Lemma B.3.7.

(i) *We have $\mathcal{I}/\mathcal{I}_0 \cong \mathscr{C}(\partial X)$.*

(ii) *We have $\mathcal{I}_0 \cong \mathscr{C}(S^*\partial X, \mathcal{T}_0)$.*

Proof. (i) We consider the mapping $F : \mathcal{I} \longrightarrow \mathscr{C}(\partial X)$ given by

$$F : \mathcal{I} \ni N \longmapsto I_b(N) \in \mathscr{C}(\partial X).$$

Clearly, $\mathcal{I}_0 = \ker(F)$. Let $f \in \mathscr{C}(\partial X)$ be arbitrary. We choose $g \in \mathscr{C}_c(X)$ with $g_{|\partial X} = f$, then the reduced normal operator of M_g, where M_g denotes the multiplication operator with g, is given by:

$$[\mathcal{N}_\varrho(M_g)(y,\eta)]h(x) = g(0,y)\widehat{h}(x)\left|\frac{dx}{x}\right|^{\frac{1}{2}} \quad (h \in \mathscr{C}(M, {}^{b,c}\Omega^{\frac{1}{2}})).$$

Here we used coordinates given by the compactification $M = [0,1]$ of $\overline{\mathbb{R}}_+$. So $\mathcal{N}_\varrho(M_g) = M_f$ independent of the choice of g and

$$\begin{aligned} I_b(\mathcal{N}_\varrho(M_g)(y,\eta))(w) &= f(y), \\ {}^{b,c}\sigma^{(0,0)}(\mathcal{N}_\varrho(M_g)(y,\eta))(z,\xi) &= f(y) \text{ and} \\ I_c(\mathcal{N}_\varrho(M_g)(y,\eta))(\xi) &= f(y). \end{aligned} \tag{B.3.1}$$

We conclude $\mathcal{N}_\varrho(M_g) \in \mathcal{I}$ and the map F is surjective (by (B.3.1)), which gives (i). Note that the mapping $G : \mathscr{C}(\partial X) \longrightarrow \mathcal{I}$ given by

$$G : \mathscr{C}(\partial X) \ni f \longmapsto \mathcal{N}_g \in \mathcal{I}$$

is a well defined injective $*$-homomorphism with $F \circ G = id_{\mathscr{C}(\partial X)}$.
(ii) This is clear by definition. □

The last lemma gives rise to the following split exact sequence:

$$0 \longrightarrow \mathscr{C}(S^*\partial X, \mathcal{T}_0) \longrightarrow \mathcal{I} \underset{F}{\overset{G}{\rightleftarrows}} \mathscr{C}(\partial X) \longrightarrow 0. \tag{B.3.2}$$

Denote by $\mathcal{B}_{\mathrm{BM}}$ the following set:

$$\mathcal{B}_{\mathrm{BM}} := \{A \in \mathcal{B}_0 \, : \, \mathcal{N}_\varrho(A) \in \mathcal{I}\}\,.$$

Proposition B.3.8. *$\mathcal{B}_{\mathrm{BM}}$ is a C^*-subalgebra of $\mathcal{B}_0$ and $\mathcal{K}_0 \subset \mathcal{B}_{\mathrm{BM}}$.*

Proof. That $\mathcal{B}_{\mathrm{BM}}$ is closed under multiplication and taking adjoints is clear by the properties of the reduced normal operator. Moreover, if K is an element of $\mathcal{K}_0$ then $\mathcal{N}_\varrho^\nu$ vanishes identically, which gives $\mathcal{N}_\varrho^\nu(K) \in \mathcal{I}$.

So what remains to be checked is that $\mathcal{B}_{\mathrm{BM}}$ is closed: For this, let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_{\mathrm{BM}}$ be arbitrary with $A_n \xrightarrow{n\to\infty} A \in \mathcal{B}_0$. Then we have to prove that $A \in \mathcal{B}_{\mathrm{BM}}$ holds.

By definition, we have $\mathcal{N}_\varrho^\nu(A_n) \in \mathcal{I}$ for all $n \in \mathbb{N}$. Since $\mathcal{I}$ is a C^*-algebra itself, we know that the limit of the sequence $(\mathcal{N}_\varrho^\nu(A_n))_{n\in\mathbb{N}}$ (if it exists) will be also an element of $\mathcal{I}$. By continuity we get

$$||\mathcal{N}_\varrho^\nu(A_n) - \mathcal{N}_\varrho^\nu(A)|| \leq C||A_n - A|| \xrightarrow{n\to\infty} 0,$$

which shows, that $A \in \mathcal{B}_{\mathrm{BM}}$ using our previous remark. □

We denote by $\mathfrak{N}_\varrho^\nu$ the restriction of $\mathcal{N}_\varrho^\nu$ to $\mathcal{B}_{\mathrm{BM}}$ and by $\mathcal{Q}_{\mathrm{BM}}$ the quotient $\mathcal{B}_{\mathrm{BM}}/\mathcal{K}_0$.

Proposition B.3.9.

(i) *We have* $\ker(\mathfrak{N}_\varrho^\nu)/\mathcal{K}_0 \cong \mathscr{C}_0(S^*\mathring{X})$.

(ii) $\mathrm{Im}\,(\mathfrak{N}_\varrho^\nu) \cong \mathcal{I}$ *and the sequence*

$$0 \longrightarrow \mathscr{C}(S^*\partial X, \mathcal{T}_0) \longrightarrow \mathcal{I} \longrightarrow \mathscr{C}(\partial X) \longrightarrow 0$$

is split exact. Especially, we have

$$\mathrm{Im}\,(\mathfrak{N}_\varrho^\nu) \cong \mathscr{C}(\partial X) \oplus \mathscr{C}(S^*\partial X, \mathcal{T}_0).$$

Proof. (i) This part of the theorem will follow if we prove that

$$\ker(\mathcal{N}_\varrho^\nu) = \ker(\mathfrak{N}_\varrho^\nu)$$

holds: By [29, Proposition 4.5 8(ii)] we then conclude

$$\ker(\mathfrak{N}_\varrho^\nu)/\mathcal{K}_0 = \ker(\mathcal{N}_\varrho^\nu)/\mathcal{K}_0 \cong \mathscr{C}_0(S^*\mathring{X}).$$

Clearly, $\ker(\mathfrak{N}_\varrho^\nu) \subseteq \ker(\mathcal{N}_\varrho^\nu)$, so let $A \in \ker(\mathcal{N}_\varrho^\nu)$ be arbitrary, i.e. $\mathcal{N}_\varrho^\nu(A) = 0$. But this implies $I_c(\mathcal{N}_\varrho^\nu(A)) \equiv 0$, $I_b(\mathcal{N}_\varrho^\nu(A)) \equiv 0$ and ${}^{(0,0)}\sigma^{b,c}(\mathcal{N}_\varrho^\nu(A)) \equiv 0$, so $\mathcal{N}_\varrho^\nu(A) \in \mathcal{I}$, i.e. $A \in \mathcal{B}_{\mathrm{BM}}$.

(ii) The first part is clear by definition. The second part is a reformulation of B.3.7 (i) and (ii). Note, that the end of B.3.7 (i) actually proves that there is a splitting map, see (B.3.2). □

Now let us give a description of $\mathcal{Q}_{\mathrm{BM}}$:

Theorem B.3.10. *The C^*-algebra $\mathcal{Q}_{\mathrm{BM}}$ consists of all*

$$(f, N) \in \mathcal{Q}_0 \text{ such that } N \in \mathcal{C}(S^*\partial X, \mathcal{T}).$$

Proof. Clearly $\mathcal{Q}_{\mathrm{BM}} \subseteq \mathcal{Q}_0$. Moreover, we have

$$\mathrm{r}\,(\mathfrak{N}_\varrho) = \mathscr{C}(\partial X) \oplus \mathscr{C}(S^*\partial X, \mathcal{T}_0) \subseteq \mathscr{C}(S^*\partial X, \mathcal{T}),$$

which gives $N \in \mathscr{C}(S^*\partial X, \mathcal{T})$. Conversely, let $(f, N) \in \mathcal{Q}_0$ be with $N \in \mathscr{C}(S^*\partial X, \mathcal{T})$. Then we find $A \in \mathcal{B}_0$, such that

$$\tau_0(A) = ({}^0\sigma^{(0)}(A), \mathcal{N}_\varrho(A)) = (f, N).$$

Since $N \in \mathcal{B}_{b,c}(S^*\partial X)$, we conclude $\mathcal{N}_\varrho \in \mathcal{I}$, i.e. $A \in \mathcal{B}_{\mathrm{BM}}$. □

Corollary B.3.11. *Let $(f, N) \in \mathcal{Q}_{\mathrm{BM}}$. Then*

$$\begin{aligned} f_0(j_\varrho(0, (\pi(\eta), 1))) &= f_0(j_\varrho(0, (\pi(\eta), -1))) \text{ and} && \text{(B.3.3)} \\ I_c(N(\eta))(\xi) &= f_0(j_\varrho(\eta, (\pi(\eta), \xi))) && \text{(B.3.4)} \end{aligned}$$

for all $\xi \in \mathbb{R}$.

Having collected all this information, we are now able to use the computation of chapter 5 to get:

Proposition B.3.12. *We have the following isomorphism in K-theory:*

$$K_*(\mathcal{B}_{\mathrm{BM}}/\mathcal{K}_0) \cong K_*(\mathscr{C}(X)) \oplus K_{*-1}(\mathscr{C}_0(T^*\mathring{X})).$$

In particular this induces again the isomorphism:

$$K_*(\mathcal{B}_{\mathrm{BM}}) \cong K_*(\mathfrak{A}).$$

Appendix C

Some remarks on the 0-calculus

C.1 A remark on $L^2(M, {}^{b,c}\Omega^{\frac{1}{2}})$ and $\mathcal{H}^1_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$

Recall that the inner product on $L^2(M, {}^{b,c}\Omega^{\frac{1}{2}})$ is given by

$$\langle f_1, f_2\rangle_{L^2(M,{}^{b,c}\Omega^{\frac{1}{2}})} := \int\limits_M f_1\overline{f_2} = \int\limits_0^1 \widehat{f}_1(z)\overline{\widehat{f}_2(z)}\frac{dz}{z(1-z)^2},$$

where $f_j = \widehat{f}_j \left|\frac{dz}{z(1-z)^2}\right|^{\frac{1}{2}} \in L^2(M, {}^{b,c}\Omega^{\frac{1}{2}})$.

The map

$$RC : \overline{\mathbb{R}}_+ \longrightarrow M : \ x \longmapsto \frac{2}{\pi}\arctan(x) =: z$$

identifies the point at infinity with $z = 1$. If V is a smooth vector field on M, we have the representation $V = a(z)z(1-z)^2\partial_z$, where $a \in \mathscr{C}^\infty(M)$. The transformation of $z(1-z)^2\partial_z$ under RC is then given by

$$\begin{aligned} z(1-z)^2\partial_z &= z(1-z)^2\frac{\partial x}{\partial z}\partial_x \\ &= \frac{2}{\pi}\arctan(x)\,(1-\frac{2}{\pi}\arctan(x))^2\frac{2}{\pi}(1+x^2)\partial_x \\ &= f(x)\partial_x, \end{aligned}$$

where $f(x) := \frac{2}{\pi}\arctan(x)\,(1-\frac{2}{\pi}\arctan(x))^2\frac{2}{\pi}(1+x^2)$. Since $(1+x^2) \xrightarrow{x\to 0} 1$ and $(1-\frac{2}{\pi}\arctan(x)) \xrightarrow{x\to 0} 1$ we have $f(x) = xg(x)$ with a smooth function g fulfiling $g(0) \neq 0$ if $|x| < 1$ using the Taylor expansion of arctan. If $|x| > 1$ we have

$$\arctan(x) = \left(\frac{\pi}{2} + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)x^{2n+1}}\right),$$

thus it follows that

$$(1-\frac{2}{\pi}\arctan(x))^2\frac{2}{\pi}(1-x^2) = C + g(x),$$

where $C > 0$ is constant and $g(x)$ vanishes for $x \to \infty$. Consequently the identification of M with $[0,\infty[$ using RC yields the condition

$$\int\limits_0^\infty |\widehat{f}(x)|^2\,\rho(x)dx < \infty$$

for the space $L^2(M, {}^{b,c}\Omega^{\frac{1}{2}})$; here $f = \widehat{f}|\rho\, dx|^{\frac{1}{2}}$ and ρ is a smooth positive function, such that $\rho(x) = 1/x$ in a neighbourhood $\mathcal{U} \subseteq [0, 1/2[$ of $x = 0$ and $\rho(x) = 1$ for $x \geq 1$.

An analogous calculation shows, that $RC : \overline{\mathbb{R}}_+ \longrightarrow M$ transforms the *Sobolev* space $\mathcal{H}^1_{b,c}(M, {}^{b,c}\Omega^{\frac{1}{2}})$ given by all $f \in L^2(M, {}^{b,c}\Omega^{\frac{1}{2}})$, such that

$$\int_0^1 z(1-z)^2 |\partial_z \widehat{f}(z)|^2\, dz = \int_0^1 |z(1-z)^2 \partial_z \widehat{f}(z)|^2 \frac{dz}{z(1-z)^2} < \infty$$

to the conditions ($c < d$)

$$\int_0^c |x\partial_x \widehat{f}(x)|^2 \frac{dx}{x} < \infty \text{ and } \int_d^\infty |\partial_x \widehat{f}(x)|^2 dx < \infty.$$

C.2 A conformally compact manifold with 0-metrics that are not isometric

This appendix treats the question, if a general conformally compact metric can be isometrically changed, such that the metric decomposes like a warped product near the boundary. For this we will calculate the *sectional curvature* for a certain manifold $\mathcal{M}$, namely let $\mathcal{M}$ denote the manifold $]0, 1[\times \mathcal{S}^1$. Moreover, let $f(x, \vartheta) : \mathcal{M} \longrightarrow]0, \infty[$ be an arbitrary strictly positive function. We will work with the metric

$$g := x^{-2} \left(dx^2 + f(x, \vartheta)^{-2} d\vartheta^2 \right)$$

in what follows. Note, that in the case $f \equiv 1$ we have

$$g_0 := g = x^{-2}(dx^2 + d\vartheta^2).$$

Finally, we set $X := x\partial_x$ and $T := xf\partial_\vartheta$; then $\{X, T\}$ is an orthonormal frame with respect to the metric g. Recall, that g gives rise to an unique connection ∇ that is torsion free, i.e.

$$\nabla_T X - \nabla_X T - [T, X] = 0, \tag{C.2.1}$$

and fulfils

$$Ug(V, W) - g(\nabla_U V, W) - g(V, \nabla_U W) = 0$$

for all vectorfields U, V and W. It is also worth pointing out, that the action of $\nabla_U V$ on a vectorfield W with respect to g is given by

$$\begin{aligned} g(\nabla_U V, W) \;\; = \;\; & \frac{1}{2}\big(g([W, U], V) + g(U, [W, V]) - g([V, U], W) \\ & + Ug(V, W) + Vg(U, W) - Wg(U, V) \big) \end{aligned}$$

and we will use this in what follows. First of all, a direct computation gives

$$[T, X] = -\frac{\partial_x[xf]}{f}T.$$

Let V be an arbitrary vectorfield given with respect to the basis $\{T, X\}$, i.e. $V = a_1X + a_2T$ where $a_i(x, \vartheta) : \mathcal{M} \longrightarrow \mathbb{R}$ are suitable functions. Note, that

$$[V, X] = a_2[T, X] - (Xa_1)X - (Xa_2)T$$

and

$$[V, T] = a_1[X, T] - (Ta_1)X - (Ta_2)T$$

holds. Let us now compute the covariant derivatives in the direction of X resp. T: We have

$$\begin{aligned} g(\nabla_X X, V) &= \frac{1}{2}\big(g([V, X], X) + g(X, [V, X]) - g([X, X], V) \\ &\quad +Xg(X, V) + Xg(X, V) - Vg(X, X)\big), \end{aligned}$$

where

$$\begin{aligned} g([V, X], X) &= -(Xa_1) \quad \text{and} \\ Xg(X, V) &= (Xa_1), \end{aligned}$$

thus $\nabla_X X = 0$. For $\nabla_X T$ we get

$$\begin{aligned} g(\nabla_X T, V) &= \frac{1}{2}\big(g([V, X], T) + g(X, [V, T]) - g([T, X], V) \\ &\quad +Xg(T, V) + Tg(X, V) - Vg(X, T)\big), \end{aligned}$$

where

$$\begin{aligned} g([V, X], X) &= -a_2\frac{\partial_x[xf]}{f} - (Xa_2), \\ g(X, [V, T]) &= -(Ta_1) \quad \text{and} \\ g([T, X], V) &= -a_2\frac{\partial_x[xf]}{f}. \end{aligned}$$

Since $Xg(T, V) = (Xa_2)$ and $Tg(X, V) = (Ta_1)$ we again get $\nabla_X T = 0$. Now using (C.2.1) it follows $\nabla_T X = -\frac{\partial_x[xf]}{f}T$. For the sake of completeness, let us also compute $\nabla_T T$, although we will not need this expression for our purpose to compute the sectional curvature of g: We have

$$\begin{aligned} g(\nabla_T T, V) &= \frac{1}{2}\big(g([V, T], T) + g(T, [V, T]) - g([T, T], V) \\ &\quad +Tg(T, V) + Tg(T, V) - Vg(T, T)\big), \end{aligned}$$

where the addends are given by

$$\begin{aligned} g([V,T],T) &= a_1\frac{\partial_x[xf]}{f} - (Ta_2) \quad \text{and} \\ Tg(T,V) &= (Ta_2), \end{aligned}$$

hence

$$\nabla_T T = \frac{\partial_x[xf]}{f}X.$$

Now, we want to calculate the sectional curvature of g. Recall, that the sectional curvature Sc_g with respect to a 2-plane Π of the tangent space spanned by orthonormal vectors $\{X_i, X_j\}$ is given by

$$Sc_{g,X_i,X_j} = g(R_{X_iX_j}X_j, X_i),$$

where R denotes the *curvature endomorphism*

$$R_{VW}Z = \nabla_V\nabla_W Z - \nabla_W\nabla_V Z - \nabla_{[V,W]}Z.$$

With respect to the plane spanned by the orthonormal frame $\{X, T\}$ this becomes

$$Sc_{g,T,X} = g(R_{TX}X, T).$$

Since

$$R_{TX}X = \nabla_T\nabla_X X - \nabla_X\nabla_T X - \nabla_{[T,X]}X = -\nabla_X\nabla_T X - \nabla_{[T,X]}X$$

and

$$\begin{aligned} \nabla_X\nabla_T X &= -\partial_x\left(\frac{\partial_x[xf]}{f}\right)T, \\ \nabla_{[T,X]}X &= \left(\frac{\partial_x[xf]}{f}\right)^2 T \end{aligned}$$

we get $g(R_{TX}X,T) = \partial_x\left(\frac{\partial_x[xf]}{f}\right) - \left(\frac{\partial_x[xf]}{f}\right)^2$. Thus we finally have proven:

$$Sc_{g,T,X} = g(R_{XT}T, X) = \partial_x\left(\frac{\partial_x[xf]}{f}\right) - \left(\frac{\partial_x[xf]}{f}\right)^2.$$

Now, if $f \equiv 1$, then $Sc_{g,T,X} = -1$; however, if f is a positive function that depends non-trivial on x and ϑ, we get the sectional curvature also to be a non-trivial function in x and ϑ. Since the sectional curvature is an isometric invariant (see for instance [70, Chapter 7]), there cannot exist an isometric isomorphism between the manifold $\mathcal{M}$ equipped with $g_0 = x^{-2}(dx^2 + d\vartheta^2)$ and $g = x^{-2}(dx^2 + f(x,\vartheta)^{-2}d\vartheta^2)$ in such a non-trivial case. In particular it is not possible to isometrically change the metric g to behave like a warped product near the boundary.

Appendix D

Representations of hereditary C^*-algebras

This appendix deals with the proof of 3.1.13. In what follows, $\mathcal{B}$ denotes a C^*-algebra. Recall, that $\mathcal{A}$ is a hereditary subalgebra of $\mathcal{B}$ if and only if $aba' \in \mathcal{A}$ holds for all $a, a' \in \mathcal{A}$ and $b \in \mathcal{B}$ by 3.1.11.

D.1 Preliminaries

A linear map $\varphi : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ between C^*-algebras is said to be *positive* if $\varphi(\mathcal{B}_1^+) \subseteq \mathcal{B}_2^+$. Let $\tau : \mathcal{B} \longrightarrow \mathbb{C}$ be a positive linear functional. If $||\tau|| = 1$, we say that τ is a *state*. Moreover, τ is called a *pure state*, if for each positive linear functional ϱ with $\varrho \leq t\tau$ there exists a $t \in [0, 1]$ such that $\varrho = t\tau$.

Proposition D.1.1.

(i) *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{B}$ and let τ be a positive linear functional on $\mathcal{A}$. Then there is a positive linear functional τ' on $\mathcal{B}$ extending τ, such that $||\tau|| = ||\tau'||$.*

(ii) *If in addition to* (i) *$\mathcal{A}$ is hereditary in $\mathcal{B}$, the positive functional τ' on $\mathcal{B}$ extending τ is unique. Moreover, if $(\mu_\lambda)_{\lambda \in \Lambda}$ is an approximate unit for $\mathcal{B}$, then*

$$\tau'(b) = \lim_\lambda \tau(\mu_\lambda b \mu_\lambda)$$

holds for all $b \in \mathcal{B}$.

Proof. See [90, Theorem 3.3.8, Theorem 3.3.9]. □

Proposition D.1.2. *Let $(\mathcal{H}, \varphi)$ be a representation of $\mathcal{B}$ with cyclic vector v. Then the function*

$$\tau : \mathcal{B} \longrightarrow \mathbb{C}, \quad b \longmapsto \langle \varphi(b)v \,|\, v \rangle$$

is a state of $\mathcal{B}$ and $(\mathcal{H}_\tau, \varphi_\tau)$ is unitarily equivalent to $(\mathcal{H}, \varphi)$. If in addition $(\mathcal{H}, \varphi)$ is irreducible, then τ is a pure state.

Proof. See [90, Theorem 5.1.7]. □

The following is a generalisation of 3.1.8:

Proposition D.1.3. *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{B}$. Suppose that $(\mathcal{K}, \psi)$ is a (non-degenerate) representation of $\mathcal{A}$. Then there is a (non-degenerate) representation $(\mathcal{H}, \varphi)$ of $\mathcal{B}$ and a closed vector subspace $\mathcal{K}'$ of $\mathcal{H}$ invariant for $\varphi(\mathcal{B})$, such that $(\mathcal{K}, \psi)$ is unitarily equivalent to $(\mathcal{H}, \varphi)_{\mathcal{A},\mathcal{K}'}$. If $(\mathcal{K}, \psi)$ is cyclic resp. irreducible, we can choose $(\mathcal{H}, \varphi)$ cyclic resp. irreducible, too.*

Proof. See [90, Theorem 5.5.1]. □

Lemma D.1.4. *Suppose, that $\mathcal{A}$ is a non-zero hereditary C^*-subalgebra of a C^*-algebra $\mathcal{B}$, and that $\mathcal{I}$ is a primitive ideal of $\mathcal{B}$ not containing $\mathcal{A}$. Then $\mathcal{I} \cap \mathcal{A}$ is a primitive ideal in $\mathcal{A}$. Moreover, if $\mathcal{J}$ is a closed ideal of $\mathcal{B}$, such that $\mathcal{J} \cap \mathcal{B} \subseteq \mathcal{I}$, then $\mathcal{J} \subseteq \mathcal{I}$.*

Proof. See [90, Lemma 5.5.4]. □

Finally, let us note that an irreducible representation $(\mathcal{H}, \varphi)$ of a C^*-algebra $\mathcal{B}$ is *algebraically irreducible*, i.e. 0 and $\mathcal{H}$ are the only vector subspaces of $\mathcal{H}$ that are invariant for $\varphi(\mathcal{B})$ (cf. [90, Theorem 5.2.3]).

D.2 Hereditary subalgebras and their spectrum

Theorem D.2.1. *Let $\mathcal{A}$ be hereditary in $\mathcal{B}$. Then the following diagram is commutative, where the maps are the canonical ones:*

$$\begin{tikzcd}
\widehat{\mathcal{B}} \setminus \mathrm{Hull}'(\mathcal{A}) \arrow[r] \arrow[d] & \widehat{\mathcal{A}} \arrow[d] \\
\mathrm{Prim}(\mathcal{B}) \setminus \mathrm{Hull}(\mathcal{A}) \arrow[r] & \mathrm{Prim}(\mathcal{A}).
\end{tikzcd}$$

Moreover, the horizontal maps are homeomorphisms.

For the definition of the *canonical maps* used in the theorem see (3.1.1). We follow closely [90, Theorem 5.5.5].

Proof. Writing down the definitions, one immediately sees that the diagram is commutative. Let us denote by Φ resp. Φ' the upper resp. lower horizontal map in the diagram. First, we prove that Φ is injective: Suppose that $\Phi[\mathcal{H}_1, \varphi_1] = \Phi[\mathcal{H}_2, \varphi_2]$ holds. Then $(\mathcal{K}_j, \psi_j) = (\mathcal{H}_j, \varphi_j)_{|\mathcal{A}}$ $(j = 1, 2)$ are non-zero unitarily equivalent irreducible representations of $\mathcal{A}$. Thus there is a unitary $u : \mathcal{K}_1 \longrightarrow \mathcal{K}_2$, such that $\psi_2(a) = u\psi_1(a)u^*$ for $a \in \mathcal{A}$. Moreover, we choose unit vectors $v_1 \in \mathcal{K}_1$ and $v_2 \in \mathcal{K}_2$ with $u(v_1) = v_2$. Then we get $\varphi_1(\mathcal{B})v_1 = \mathcal{H}_1$ and $\varphi_2(\mathcal{B})v_2 = \mathcal{H}_2$ (since the representations are algebraically irreducible).

Moreover, the function

$$\varrho_j : \mathcal{B} \longrightarrow \mathbb{C}, \quad b \longmapsto \langle \varphi_j(b)v_j \,|\, v_j \rangle$$

is a pure state of $\mathcal{B}$ and $(\mathcal{H}_j, \varphi_j)$ is unitarily equivalent to $(\mathcal{H}_{\varrho_j}, \varphi_{\varrho_j})$ $(j = 1, 2)$ by D.1.2. Clearly, ϱ_j is an extension of

$$\varrho_j' : \mathcal{A} \longrightarrow \mathbb{C}, \quad a \longmapsto \langle \psi_j(a)v_j \,|\, v_j \rangle,$$

which is a pure state of $\mathcal{A}$, since $(\mathcal{K}_j, \psi_j)$ is irreducible. Moreover, for each $a \in \mathcal{A}$

$$\varrho_2(a) = \langle \Psi_2(a)u(v_1) \,|\, u(v_2) \rangle = \langle \psi_1(a)(v_1) \,|\, v_2 \rangle = \varrho_1'(a)$$

holds, thus $\varrho_2 = \varrho_1$ using D.1.1. But this proves, that $(\mathcal{H}_1, \varphi_1)$ and $(\mathcal{H}_2, \varphi_2)$ are unitarily equivalent, i.e. Φ is injective.

Now, we aim to prove surjectivity of Φ. Let $[\mathcal{K}, \psi] \in \widehat{\mathcal{A}}$ be arbitrary. Then by D.1.3 there exists an irreducible representation $(\mathcal{H}, \varphi)$ of $\mathcal{B}$ and a closed subspace $\mathcal{K}'$ of $\mathcal{H}$, such that $\mathcal{K}'$ is invariant with respect to the action of $\varphi(\mathcal{A})$ and that $(\mathcal{K}, \psi)$ and $(\mathcal{H}, \varphi)_{\mathcal{A},\mathcal{K}'}$ are unitarily equivalent. Clearly, $[\mathcal{H}, \varphi] \in \widehat{\mathcal{A}}$ since $(\mathcal{H}, \varphi)$ is non-zero. Moreover, we can choose $v \in \mathcal{K}'$ fulfilling $\varphi(\mathcal{A})v = \mathcal{K}'$ using the irreducibility of $(\mathcal{H}, \varphi)_{\mathcal{A},\mathcal{K}'}$. Let $a_0 \in \mathcal{A}$ be with $v = \varphi(a_0)(v)$, then $\varphi(\mathcal{B})v = \mathcal{H}$ implies that for each $w \in \mathcal{H}$ and each $a \in \mathcal{B}$ there exists $b \in \mathcal{B}$, such that

$$\varphi(a)w = \varphi(a)\varphi(b)v = \varphi(aba_0)v \in \varphi(\mathcal{A})v,$$

and we get that $\varphi(\mathcal{A})\mathcal{H} = \mathcal{K}'$. Therefore $(\mathcal{H}, \varphi)_{\mathcal{A},\mathcal{K}'} = (\mathcal{H}, \varphi)_{\mathcal{B}}$ holds and it follows $[\mathcal{H}, \varphi] \notin \mathrm{Hull}'(\mathcal{B})$ and $\Phi[\mathcal{H}, \varphi] = [\mathcal{K}, \psi]$, i.e. Φ is a bijection.

That Φ' is surjective now follows directly from the commutativity of the diagram. Let $\mathcal{I}_1, \mathcal{I}_2$ be with $\mathcal{I}_1 \cap \mathcal{A} = \mathcal{I}_2 \cap \mathcal{A}$. Then by D.1.4 we get that $\mathcal{I}_1 = \mathcal{I}_2$, thus Φ' is injective.

Now, we will prove, that Φ' is a homeomorphism (then it follows that Φ is also a homeomorphism): Let $\emptyset \neq C$ be a closed set in $\mathrm{Prim}(\mathcal{A})$, i.e. $C = \mathrm{Hull}_{\mathcal{A}}(\mathcal{I})$ for some closed ideal $\mathcal{I}$ of $\mathcal{A}$. We get

$$\begin{aligned} (\Phi')^{-1}(C) &= \{\mathcal{J} \in \mathrm{Prim}(\mathcal{B}) \,:\, \mathcal{I} \subseteq \mathcal{J} \cap \mathcal{A} \text{ and } \mathcal{A} \nsubseteq \mathcal{J}\} \\ &= \mathrm{Hull}_{\mathcal{B}}(\mathcal{I}) \cap (\mathrm{Prim}(\mathcal{B}) \setminus \ \mathrm{Hull}_{\mathcal{B}}(\mathcal{A}))\,. \end{aligned}$$

Thus $(\Phi')^{-1}(C)$ is closed in $\mathrm{Prim}(\mathcal{B}) \setminus \mathrm{Hull}_{\mathcal{B}}(\mathcal{A})$ and Φ' is continuous.

To prove that $(\Phi')^{-1}$ is also continuous, we prove that Φ' is a closed map. To this end, let $\emptyset \neq C$ be a closed set in $\mathrm{Prim}(\mathcal{B}) \setminus \mathrm{Hull}_{\mathcal{B}}(\mathcal{A})$. Then $C = \mathrm{Hull}_{\mathcal{B}}(\mathcal{I}) \setminus \mathrm{Hull}_{\mathcal{B}}(\mathcal{A})$ for a proper closed ideal $\mathcal{I}$ of $\mathcal{B}$. Let $\mathcal{J} \in \mathrm{Hull}_{\mathcal{B}}(\mathcal{I} \cap \mathcal{B})$, then $\mathcal{J} = \mathcal{J}' \cap \mathcal{B}$ holds for a suitable $\mathcal{J}' \in \mathrm{Prim}(\mathcal{B}) \setminus \mathrm{Hull}_{\mathcal{B}}(\mathcal{A})$ since Φ' is surjective. It follows $\mathcal{J} = \Phi'(\mathcal{J}') \in \Phi'(C)$ and thus $\mathrm{Hull}_{\mathcal{A}}(\mathcal{I} \cap \mathcal{A}) \subseteq \Phi'(\mathrm{C})$. The reverse inclusion is trivial, so we get $\mathrm{Hull}_{\mathcal{B}}(\mathrm{I} \cap \mathcal{B}) = \Phi'(\mathrm{C})$, and Φ' is a closed map. $\square$

Appendix E

The Theorem of Beals, Coifman-Meyer revisited

In what follows, we present a proof of the following theorem given in [21] (see [14] also) in the context of Ψ^*-algebras and commutator methods:

Theorem (Beals, Coifman-Meyer). *Let $A : \mathscr{C}^\infty(\mathcal{M}) \longrightarrow \mathscr{C}^\infty(\mathcal{M})$ be a continuously linear operator on a compact smooth manifold $\mathcal{M}$ without boundary. Then A is a pseudodifferential operator of order 0, if and only if for each sequence $V_1, V_2, \ldots$ of smooth vector fields on $\mathcal{M}$, each of the operators $A_1 := [V_1, A]$ and $A_{j+1} := [V_{j+1}, A_j]$ $(j \in \mathbb{N})$ has an extension to a bounded operator on $L^2(\mathcal{M})$.*

It is worth pointing out, that Lauter gave a similar description for manifolds with boundary in the setting of b-pseudodifferential operators in [60].

E.1 Notations and prerequisites

Before we give the proof in the case of compact manifold, let us fix some notation first.

Notations E.1.1. Let $\mathcal{M}$ be a closed smooth manifold of dimension n. Moreover, we assume $\mathcal{M}$ to be oriented and we choose a fixed open covering of $\mathcal{M}$ by coordinate charts $\mathcal{M} = \bigcup_{j=1}^{N} U'_j$ with coordinate maps $\chi_j : \mathcal{M} \supseteq U'_j \longrightarrow U_j \subseteq \mathbb{R}^n$. Let $(\varphi'_j)_{j=1,\ldots,N}$ be a smooth partition of unity subordinate to the covering $(U'_j)_j$ and choose $\vartheta'_j \in \mathscr{C}_0^\infty(U'_j)$, such that $0 \le \vartheta'_j \le 1$ and $\varphi'_j \prec \vartheta'_j$ holds.

Definition E.1.2. A continuous linear operator $A : \mathscr{C}^\infty(\mathcal{M}) \longrightarrow \mathscr{C}^\infty(\mathcal{M})$ is said to be a pseudodifferential operator in $\Psi^m_{1,0}(\mathcal{M})$, provided that for $j = 1, \ldots, N$ there exist symbols $\sigma_{a_j} \in S^m_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and $k_{j,l} \in \mathscr{C}^\infty(U_j \times U_l)$, such that

(i) $(\vartheta'_j A \varphi'_j) f(\chi_j^{-1}(x)) = \vartheta'_j \sigma_{a_j}(X, D_x)(\varphi'_j f)\left(\chi_j^{-1}(x)\right)$ holds, i.e. an operator is locally given by

$$\begin{aligned}(\vartheta'_j A \varphi'_j f)(\chi_j^{-1}(x)) \\ &= (\vartheta'_j)(\chi_j^{-1}(x)) \int\limits_{\mathbb{R}^n} e^{i\langle x \,|\, \xi\rangle} \sigma_{a_j}(x,\xi) \mathcal{F}\left((\varphi'_j f)(\chi^{-1}(\xi_j))\right) đ\xi,\end{aligned}$$

where $\mathcal{F}$ denotes the *Fourier*-transformation on $\mathbb{R}^n$ and $f \in \mathscr{C}^\infty(\mathcal{M})$,

(ii) the operator $[(1-\vartheta_j')A\varphi_j']$ has a $\mathscr{C}^\infty$-kernel representation, i.e. for each $1 \le l \le N$ we have for all $x \in U_l$

$$[(1-\vartheta_j')A\varphi_j' f](\chi_l^{-1}(x)) = \int\limits_{U_j} k_{j,l}(x,z) f(\chi_l^{-1}(z))\, dz,$$

where $k_{j,l} \in \mathscr{C}^\infty(U_l \times U_j)$.

Remark E.1.3. It is well-known, that this definition is independent of all choices we made; see for instance [58, Theorem 7.7.3] for a detailed prove of coordinate invariance.

The local definition of pseudodifferential operators on $\mathcal{M}$ shows, that it might be needed to have a local characterisation that classifies such operators. The following notion turns out to be helpful.

Definition E.1.4. Let $U \subseteq \mathbb{R}^n$ be open and $\emptyset \neq K \subseteq U$ compact. Denote by $\mathscr{L}_K(U)$ the algebra of operators $A \in \mathscr{L}(L^2(U))$ such that

(i) $A(\mathscr{C}_0^\infty(U)) \subseteq \mathscr{C}_0^\infty(U)$,

(ii) $\operatorname{supp}(A\varphi) \subseteq K$ for all $\varphi \in \mathscr{C}^\infty(U) \cap L^2(U)$,

(iii) $A\varphi = 0$ for all $\varphi \in \mathscr{C}^\infty(U) \cap L^2(U)$ fulfilling $\operatorname{supp}\varphi \cap K = \emptyset$.

We call $A \in \mathscr{L}(L^2(U))$ *ported* by K if A has the properties E.1.4 (ii) and (iii).

Remark E.1.5. Let A be an element of $\mathscr{L}_K(U)$, then A induces a bounded linear map

$$\widetilde{A} : \mathscr{C}^\infty(\mathbb{R}^n) \longrightarrow \mathscr{C}^\infty(\mathbb{R}^n) :\ f \longmapsto A(\upsilon f),$$

where $\upsilon \in \mathscr{C}_0^\infty(U)$ with $\upsilon \equiv 1$ on K. Moreover, $\widetilde{A}$ is independent of the choice of υ.

Proposition E.1.6. *Let $\vartheta, \varphi \in \mathscr{C}_0^\infty(U)$ be with $\varphi \prec \vartheta$. Denote by $\widetilde{A}$ the operator induced by $A \in \mathscr{L}_K(U)$, cf. E.1.5. We define $\sigma_A(x,\xi) := e^{-i\langle x\,|\,\xi\rangle}\widetilde{A}[x' \longmapsto e^{i\langle x'\,|\,\xi\rangle}](x)$, where $x \in \mathbb{R}^n_x$ and $\xi \in \mathbb{R}^n_\xi$. Then we get:*

(i) $\sigma_A \in \mathscr{C}^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ *and* $\sigma_A(x,\xi) = 0$ *for all* $\xi \in \mathbb{R}^n_\xi$*, if* $x \notin K$*,*

(ii) $\int\limits_{\mathbb{R}^n} |\sigma_A(x,\xi)|^2\, dx \le ||\upsilon||^2_{L^2(U)} ||A||^2_{\mathscr{L}(L^2(U))}$ *for all* $\xi \in \mathbb{R}^n_\xi$ *and for all* $\upsilon \in \mathscr{C}_0^\infty(U)$*, such that* $\upsilon \equiv 1$ *holds in a neighbourhood of* K*,*

(iii) $[\vartheta A\varphi f](x) = \vartheta(x)\int\limits_{\mathbb{R}^n} e^{i\langle x\,|\,\xi\rangle}\sigma_A(x,\xi)\mathcal{F}(\varphi f)(\xi)\, đ\xi$ *for all* $x \in \mathbb{R}^n$ *and for all* $f \in \mathscr{C}^\infty(\mathbb{R}^n)$.

Proof. (i) We use the tensor decomposition

$$\mathscr{C}^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) = \mathscr{C}^\infty(\mathbb{R}^n_x)\widehat{\bigotimes}_\pi \mathscr{C}^\infty(\mathbb{R}^n_\xi)$$

and get:

$$\sigma_A = e^{-i\langle x\,|\,\xi\rangle}\widetilde{A}\widehat{\otimes}_\pi id[(x',\xi) \longmapsto e^{i\langle x'\,|\,\xi\rangle}] \in \mathscr{C}^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi).$$

The second claim is a consequence of E.1.4 (ii).
(ii) We have

$$\begin{aligned}\int\limits_{\mathbb{R}^n} |\sigma_A(x,\xi)|^2\,dx &= ||A[x' \longmapsto \upsilon(x')e^{i\langle x'\,|\,\xi\rangle}]||^2_{L^2(U)} \\ &\leq ||A^2||^2_{\mathscr{L}(L^2(U))}||\upsilon||^2_{L^2(U)}.\end{aligned}$$

(iii) Choose $\upsilon \in \mathscr{C}_0^\infty(U)$, such that $\vartheta \prec \upsilon$ and $\upsilon = 1$ in a neighbourhood of K. To given $h \in \mathscr{S}(\mathbb{R}^n)$ we consider the following map

$$I : \mathbb{R}^n_\xi \ni \xi \longmapsto h(\xi)[x \longmapsto \upsilon(x)e^{i\langle x\,|\,\xi\rangle}].$$

Here the right hand side is an element of $\mathscr{C}^\infty(\mathbb{R}^n_x)$, since

$$\mathscr{C}^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) \cong \mathscr{C}^\infty(\mathbb{R}^n_\xi, \mathscr{C}^\infty(\mathbb{R}^n_x)).$$

Moreover, I is integrable for $h \in \mathscr{S}(\mathbb{R}^n_\xi)$. This gives

$$\int \widetilde{A}(I(\xi))\,đ\xi = \widetilde{A}\left(\int I(\xi)\,đ\xi\right)$$

by the continuity of $\widetilde{A}$. Especially, if we set $h := \mathscr{F}(\varphi f) \in \mathscr{S}(\mathbb{R}^n_\xi)$, we get:

$$\begin{aligned}&\vartheta(x)\int\limits_{\mathbb{R}^n} e^{i\langle x\,|\,\xi\rangle}\sigma_A(x,\xi)\mathscr{F}(\varphi f)(\xi)\,đ\xi \\ = \ &\vartheta(x)\int\limits_{\mathbb{R}^n} A[x' \longmapsto \upsilon(x')e^{\langle x'\,|\,\xi\rangle}](x)\mathscr{F}(\varphi f)(\xi)\,đ\xi \\ = \ &\vartheta(x)A[x' \longmapsto \upsilon(x')\varphi(x')f(x')] \\ = \ &[\vartheta A\varphi f](x).\end{aligned}$$

This proves (iii). □

E.2 Local classification

We have seen, that each operator in $\mathscr{L}_K(U)$ has a representation according to E.1.2 (i). Now, we want to classify all operators in $\mathscr{L}_K(U)$, whose local symbols are given by elements in $S^m_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$. Let us fix some notation first:

Notations E.2.1. Let $U \subseteq \mathbb{R}^n$ be open and $\varphi, \theta, \omega, \psi, \rho \in \mathscr{C}_0^\infty(U)$, where $\varphi \prec \theta \prec \omega \prec \psi \prec \rho$. We consider the following unbounded operators $L^2(U) \supseteq \mathscr{C}_0^\infty(U) \longrightarrow L^2(U)$:

$$\widetilde{V}_l := i\psi \frac{\partial}{\partial x_l}\psi,\ \widetilde{V}^k := \psi x_k \psi,$$
$$\widetilde{V}_l^k := i\psi x_k \frac{\partial}{\partial x_l}\psi \text{ if } l \neq k \text{ and } \widetilde{V}_k^k := i\psi x_k \frac{\partial}{\partial x_k}\psi + i\frac{\psi^2}{2} \quad \text{otherwise.}$$

Denote this family of operators by $\widetilde{\mathcal{V}}$. Since these operators are symmetric and densely defined, they are closable. We denote the minimal closed (symmetric) extensions of these operators by V_l, V^k resp. V_l^k and the corresponding family by $\mathcal{V}$.

Lemma E.2.2. *Let $U_\psi := \psi^{-1}(\mathbb{C} \setminus 0) \subseteq U$ and $\omega_0 \in \mathscr{C}_0^\infty(U_\psi) \subseteq \mathscr{C}_0^\infty(U)$. Then*

$$\omega_0 \mathcal{H}_{\mathcal{V}}^m = \omega_0 \mathcal{H}^m(\mathbb{R}^n)$$

holds for all $m \in \mathbb{N}_0$. In particular, we get $\mathscr{C}_0^\infty(U_\psi) \subseteq \mathcal{H}_{\mathcal{V}\,|U_\psi}^\infty \subseteq \mathscr{C}^\infty(U_\psi)$.

Proof. We divide the proof in two steps:
(i) First, we will show, that $\omega_1 \mathcal{H}_{\mathcal{V}}^m \subseteq \mathcal{H}^m(\mathbb{R}^n)$ and $\omega_1 \mathcal{H}^m(\mathbb{R}^n) \subseteq \mathcal{H}_{\mathcal{V}}^m$ holds for $\omega_1 \in \mathscr{C}_0^\infty(U_\psi)$ using induction:
The case $m = 0$ is obvious. Now, let $f \in \mathcal{H}_{\mathcal{V}}^m$ be with $m \geq 1$. The definition implies, that $f_{j,l} \in \mathscr{C}_0^\infty(U)$ exists, such that $f_{j,l} \xrightarrow{j\to\infty} f$ and $\widetilde{V}_l(f_{j,l}) \xrightarrow{j\to\infty} V_l(f) \in \mathcal{H}_{\mathcal{V}}^{m-1}$ with respect to $L^2(U)$ $(l = 1, \ldots, n)$. This gives

$$\begin{aligned}
\widetilde{V}_l(\omega_1 f_{j,l}) &= i\psi \frac{\partial}{\partial x_l}(\psi \omega_1 f_{j,l}) \\
&= \omega_1 \widetilde{V}_l(f_{j,l}) + i\psi^2 \left(\frac{\partial}{\partial x_l}\omega_1\right) f_{j,l} \\
&\xrightarrow{j\to\infty} \omega_1 V_l(f) + i\psi^2 \left(\frac{\partial}{\partial x_l}\omega_1\right) f
\end{aligned} \tag{E.2.1}$$

in $L^2(\mathbb{R}^n)$. In addition we also have

$$\widetilde{V}_l(\omega_1 f_{j,l}) = i\psi^2 \frac{\partial}{\partial x_l}(\omega_1 f_{j,l}) + i\psi\omega_1 f_{j,l} \frac{\partial}{\partial x_l}\psi. \tag{$E.2.2$}$$

Let Ω_1 be a relative compact neighbourhood of $\operatorname{supp}\omega_1$, such that $\overline{\Omega}_1 \subseteq U_\psi$. Denote by

$$h : x \longmapsto \begin{cases} \frac{1}{\psi^2(x)}, & \text{if } x \in \overline{\Omega}_1, \\ 0, & \text{else,} \end{cases}$$

the function that is bounded and continuous in a neighbourhood of $\operatorname{supp}\omega_1$. Since $\omega_1 f_{j,l} \xrightarrow{j\to\infty} \omega_1 f$ with respect to the topology in $L^2(\mathbb{R}^n)$, a combination of (E.2.1), (E.2.2) and the induction hypothesis yields

$$i\frac{\partial}{\partial x_l}(\omega_1 f_{j,l}) \xrightarrow{j\to\infty} h\left[i\psi^2 \left(\frac{\partial}{\partial x_l}\omega_1\right) f + \omega_1(V_l f) - i\psi\omega_1 f \left(\frac{\partial}{\partial x_l}\psi\right)\right],$$

where the right hand side is an element of $\mathcal{H}^{m-1}(\mathbb{R}^n)$. Consequently, we get $i\overline{\partial_l}(\omega_1 f) \in \mathcal{H}^{m-1}(\mathbb{R}^n)$ $(l = 1, \dots, n)$. The induction hypothesis $\omega_1 f \in \mathcal{H}^{m-1}(\mathbb{R}^n)$ implies $\omega_1 f \in \mathcal{H}^m(\mathbb{R}^n)$. The second part of the claim follows by a similar argument.
(ii) We choose $\omega_1 \in \mathscr{C}_0^\infty(U_\psi)$, such that $\omega_0 \prec \omega_1$. (i) then implies

$$\omega_0 \mathcal{H}_\mathcal{V}^m = \omega_0\omega_1\mathcal{H}_\mathcal{V}^m \subseteq \omega_0\mathcal{H}^m(\mathbb{R}^n) = \omega_0\omega_1\mathcal{H}^m(R^n) \subseteq \omega_0\mathcal{H}_\mathcal{V}^n,$$

which finishes the proof. □

Lemma E.2.3.

(i) *Let $f \in \mathscr{C}^\infty(U) \cap L^\infty(U)$ be given. Then $M_f \in \Psi_\infty^\mathcal{V}$.*

(ii) *Let $\varphi, \psi \in \mathscr{C}_0^\infty(U)$ and $A \in \Psi_\infty^\mathcal{V}$ be arbitrary. Then $M_\varphi A M_\psi \in \Psi_\infty^\mathcal{V}$ holds, i.e. the algebra $\Psi_\infty^\mathcal{V}$ is localizable.*

Proof. Let $\widetilde{V}$ in $\widetilde{\mathcal{V}}$ be arbitrary, i.e.

$$\widetilde{V} = \sum_{l=1}^n a_l \frac{\partial}{\partial x_l} + b : L^2(U) \supseteq \mathscr{C}_0^\infty(U) \longrightarrow L^2(U),$$

where $b, a_l \in \mathscr{C}_0^\infty(U)$. Since $M_f(\mathscr{C}_0^\infty(U)) \subseteq \mathscr{C}_0^\infty(U)$ and $[\widetilde{V}, M_f] = M_g$ with $g := \sum_{l=1}^n a_l \frac{\partial}{\partial x_l} f \in \mathscr{C}_0^\infty(U)$ holds, the operator $[\widetilde{V}, M_f]$ has a continuous extension. This implies $M_f \in \Psi_1^\mathcal{V}$, since $\widetilde{V} \in \mathcal{V}$ was arbitrary. Now, by $\delta_V M_f = M_g$ and $g \in \mathscr{C}_0^\infty(U) \subseteq \mathscr{C}^\infty(U) \cap L^\infty(U)$, we can use the same argument again and get (i) by iteration. (ii) follows by of (i), since $\Psi_\infty^\mathcal{V}$ is an algebra by construction. □

Now, we aim to show that operators in $\Psi_\infty^\mathcal{V}$ are pseudodifferential operators after some sort of localizing process. In the sequel let $K := \operatorname{supp}\omega \subseteq U$, where ω is the function given in E.2.1, and $\Psi_{\infty,K}^\mathcal{V}$ denotes the space of all $A \in \Psi_{\infty,K}^\mathcal{V}$ fulfilling E.1.4 (ii) and (iii).

Lemma E.2.4.

(i) *We have $\Psi_{\infty,K}^\mathcal{V} \subseteq \mathscr{L}_K(U)$.*

(ii) *For $A \in \Psi_{\infty,K}^\mathcal{V}$ and $V \in \mathcal{V}$ we get $\delta_V A \in \Psi_{\infty,K}^\mathcal{V}$.*

Proof. (i) It is enough to show $A(\mathscr{C}_0^\infty(U)) \subseteq \mathscr{C}_0^\infty(U)$ for arbitrary $A \in \Psi_{\infty,K}^\mathcal{V}$. Let $f \in \mathscr{C}_0^\infty(U)$ and $\omega_1 \in \mathscr{C}_0^\infty(U)$ be chosen, such that $\omega \prec \omega_1 \prec \psi$ holds. Then we have $\operatorname{supp} Af \subseteq K$ and $\omega_1 = 1$ in a neighbourhood of K. Properties E.1.4 (ii) and (iii) then give:

$$\begin{aligned} Af &= \omega_1 A(\omega_1 f) + \underbrace{(1-\omega_1)Af}_{=0} + \omega_1 A(\underbrace{(1-\omega_1)f}_{=0 \text{ near } K}) \\ &= \omega_1 A(\omega_1 f) \in \omega_1 A(\mathscr{C}_0^\infty(U)) \subseteq \omega_1 A(\mathcal{H}_\mathcal{V}^\infty) \subseteq \omega_1 \mathcal{H}_\mathcal{V}^\infty \subseteq \mathscr{C}_0^\infty(U), \end{aligned}$$

where we used 1.3.5 (iv) and E.2.2.
(ii) By 1.3.5 (v) we already have $\delta_V(A) \in \Psi_\infty^\mathcal{V}$, so what is left is to prove (ii) and (iii) of

E.1.4. Each $V \in \mathcal{V}$ is the minimal closed extension of an operator $\widetilde{V} = \psi\widetilde{V}_1\psi : L^2(U) \supseteq \mathscr{C}_0^\infty(U) \longrightarrow L^2(U)$, where $\widetilde{V}_1$ is differential operator of order one. Let $f \in \mathscr{C}^\infty(U) \cap L^2(U)$ be arbitrary. Then there exists a sequence $(f_k)_{k\in\mathbb{N}}$, such that $\mathscr{C}_0^\infty(U) \ni f_k \xrightarrow{k\to\infty} f$ with respect to $L^2(U)$; this implies $\delta_V(A)f_k \xrightarrow{k\to\infty} \delta_V(A)f$ since $\delta_V(A) \in \Psi_\infty^{\mathcal{V}} \subseteq \mathscr{L}(L^2(U))$ is bounded. The same argument gives $\chi_K\delta_V(A)f_k \xrightarrow{k\to\infty} \chi_K\delta_V(A)f$, where χ_K is the characteristic function on K. Moreover,

$$\chi_K\delta_V(A)f_k = \chi_K[\underbrace{\psi\widetilde{V}_1\psi Af_k}_{\mathrm{supp}\subseteq K} - \underbrace{A(\psi\widetilde{V}_1\psi f_k)}_{\mathrm{supp}\subseteq K}] = \delta_V(A)f_k,$$

which gives $\chi_K\delta_V(A)f = \delta_V(A)f$, i.e. $\mathrm{supp}\,\delta_V(A)f \subseteq K$ and (ii) of E.1.4 is fulfilled.

To show (ii), let $f \in \mathscr{C}^\infty(U) \cap L^2(U)$ be with $\mathrm{supp}\, f \cap K = \emptyset$ and $\omega_2 \in \mathscr{C}_0^\infty(U)$ be with $\omega \prec \omega_2$ and $\mathrm{supp}\,\omega_2 \cap \mathrm{supp}\, f = \emptyset$. Again, there exists $f_k \in \mathscr{C}_0^\infty(U)$ where $f_k \xrightarrow{k\to\infty} f$ with respect to $L^2(U)$; this gives $(1-\omega_2)f_k \xrightarrow{k\to\infty} (1-\omega_2)f = f$, since $\omega_2 = 0$ on $\mathrm{supp}\, f$ holds. Consequently $\delta_V(A)((1-\omega_2)f_k) \xrightarrow{k\to\infty} \delta_V(A)f$ by the boundedness of $\delta_V(A)$. Since $A \in \Psi_{\infty,K}^{\mathcal{V}}$ we also get

$$\delta_V(A)((1-\omega_2)f_k) = \psi\widetilde{V}_1\psi A((1-\omega_2)f_k) - A(\psi\widetilde{V}_1\psi(1-\omega_2)f_k) = 0,$$

and therefore $\delta_V(A)f = 0$. □

To show that the local symbols of $\Psi_{\infty,K}^{\mathcal{V}} \subseteq \mathscr{L}_K(U)$ are elements of $S_{1,0}^m(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$, we define the space

$$\Sigma_K := \left\{\sigma_A \,:\, A \in \Psi_{\infty,K}^{\mathcal{V}}\right\}.$$

Proposition E.2.5. *Let $\sigma \in \Sigma_K$ be a symbol and $\gamma \in \mathscr{C}^\infty(\mathbb{R}_x^n)$. Then $\gamma\cdot\sigma$, $\frac{\partial}{\partial x_l}\sigma$, $\frac{\partial}{\partial \xi_k}\sigma$ and $\xi_l\frac{\partial}{\partial \xi_k}\sigma$ are also symbols in Σ_K. Here we used the notation $\gamma\cdot\sigma$ for the mapping $(x,\xi) \longmapsto \gamma(x)\sigma(x,\xi)$.*

The proof of this lemma will use the following tensor product decomposition:

$$\mathscr{C}^\infty(\mathbb{R}_x^n \times \mathbb{R}_\xi^n) = \mathscr{C}^\infty(\mathbb{R}_x^n)\widehat{\bigotimes}_\pi \mathscr{C}^\infty(\mathbb{R}_\xi^n).$$

For $f \in \mathscr{C}^\infty(\mathbb{R}_x^n)\widehat{\bigotimes}_\pi\mathscr{C}^\infty(\mathbb{R}_\xi^n)$ and $g : \mathscr{C}^\infty(\mathbb{R}_x^n) \longrightarrow \mathscr{C}^\infty(\mathbb{R}_x^n)$ we get

$$\begin{aligned}(g\widehat{\otimes}_\pi id)\left(\frac{\partial}{\partial \xi_j}f\right) &= (g\widehat{\otimes}_\pi id)\left(id\widehat{\otimes}_\pi\frac{\partial}{\partial \xi_j}\right)f \\ &= \left(id\widehat{\otimes}_\pi\frac{\partial}{\partial \xi_j}\right)(g\widehat{\otimes}_\pi id)f \\ &= \frac{\partial}{\partial \xi_k}(g\widehat{\otimes}_\pi id)f,\end{aligned}$$

which shows

$$g\widehat{\otimes}_\pi id[(x',\xi) \mapsto ix_k' e^{\langle x' \,|\, \xi\rangle}] = \frac{\partial}{\partial \xi_k}(g\widehat{\otimes}_\pi id[(x',\xi) \mapsto e^{i\langle x' \,|\, \xi\rangle}]). \qquad (E.2.3)$$

Proof. Let $A \in \Psi^{\mathcal{V}}_{\infty,K}$ be with $\sigma = \sigma_A$. Then $\gamma \cdot \sigma = \sigma_{\gamma_{|U}A}$ and E.2.3 (i) implies $\gamma_{|U}A \in \Psi^{\mathcal{V}}_{\infty,K}$. Before we give the proof of the last three implications, it is worth pointing out, that $V \in \mathcal{V}$ implies $\delta_V(A) \in \Psi^{\mathcal{V}}_{\infty,K}$ (using the last lemma and $\omega \prec \psi$). We will show:

(i) $\sigma_{\delta_{V_l}A}(x,\xi) = i\frac{\partial}{\partial x_l}\sigma_A(x,\xi)$, where $l = 1,\dots,n$,

(ii) $\sigma_{\delta_{V^k}A}(x,\xi) = -i\frac{\partial}{\partial \xi_k}\sigma_A(x,\xi)$, where $k = 1,\dots,n$,

(iii) $\sigma_{\delta_{V_l^k}A}(x,\xi) = -ix_k\frac{\partial}{\partial x_l}\sigma_A(x,\xi) + i\xi_l\frac{\partial}{\partial \xi_k}\sigma_A(x,\xi)$, where $k,l = 1,\dots,n$.

Ad (i): We have

$$
\begin{aligned}
\sigma_{[i\frac{\partial}{\partial x_l},A]}(x,\xi) &= e^{-i\langle\xi\,|\,x\rangle}\left(\left[i\frac{\partial}{\partial x_l},A\right]e^{i\langle\xi\,|\,\cdot\rangle}\right)(x)\\
&= i^2\xi_l e^{-i\langle\xi\,|\,x\rangle}(Ae^{i\langle\xi\,|\,\cdot\rangle})(x)\\
&\qquad - ie^{-i\langle\xi\,|\,x\rangle}\frac{\partial}{\partial x_l}\left(Ae^{i\langle\xi\,|\,\cdot\rangle}\right)(x)\\
&= i\frac{\partial}{\partial x_l}\left(e^{-i\langle\xi\,|\,x\rangle}(Ae^{i\langle\xi\,|\,\cdot\rangle})\right)(x) = i\frac{\partial}{\partial x_l}\sigma(x,\xi).
\end{aligned}
$$

Ad (ii): We have

$$
\begin{aligned}
\sigma_{[A,M_{x_k}]} &= e^{-i\langle\xi\,|\,x\rangle}\left([A,M_{x_k}]e^{i\langle\xi\,|\,\cdot\rangle}\right)(x)\\
&= e^{-i\langle\xi\,|\,x\rangle}(-i)\left(A(iM_{x_k}e^{i\langle\xi\,|\,\cdot\rangle})\right)(x)\\
&\qquad - i\left(\left(\frac{\partial}{\partial \xi_k}e^{-i\langle\xi\,|\,x\rangle}Ae^{i\langle\xi\,|\,\cdot\rangle}\right)(x)\right)\\
&\overset{(*)}{=} -ie^{-i\langle\xi\,|\,x\rangle}A\left(\frac{\partial}{\partial \xi_k}e^{i\langle\xi\,|\,\cdot\rangle}\right)(x)\\
&\qquad - i\left(\frac{\partial}{\partial \xi_k}\sigma(x,\xi) - e^{-i\langle\xi\,|\,x\rangle}\frac{\partial}{\partial \xi_k}Ae^{i\langle\xi\,|\,\cdot\rangle}\right)(x)\\
&= -i\frac{\partial}{\partial \xi_k}\sigma(x,\xi).
\end{aligned}
$$

Here $(*)$ holds by (E.2.3).
Ad (iii): We have

$$
\begin{aligned}
\sigma_{[A,ix_k\frac{\partial}{\partial x_l}]}(x,\xi) &= e^{-i\langle\xi\,|\,x\rangle}\left(\left[A,ix_k\frac{\partial}{\partial x_l}\right]e^{i\langle\xi\,|\,\cdot\rangle}\right)(x)\\
&= i\Big(e^{-i\langle\xi\,|\,x\rangle}\left(Ax_k\frac{\partial}{\partial x_l}e^{i\langle\xi\,|\,\cdot\rangle}\right)(x)\\
&\qquad - e^{-i\langle\xi\,|\,x\rangle}\left(x_k\frac{\partial}{\partial x_l}Ae^{i\langle\xi\,|\,\cdot\rangle}\right)(x)\Big).
\end{aligned}
$$

We treat the summands separately and get for the first summand:

$$\begin{aligned}
& e^{-i\langle \xi \,|\, x\rangle} \left(A x_k \frac{\partial}{\partial x_l} e^{i\langle \xi \,|\, \cdot\rangle} \right)(x) \\
= \;& i\xi_l e^{-i\langle \xi \,|\, x\rangle} (A x_k e^{i\langle \xi \,|\, \cdot\rangle})(x) \\
= \;& \xi_l e^{-i\langle \xi \,|\, x\rangle} \left(A \left(\frac{\partial}{\partial \xi_k} e^{i\langle \xi \,|\, \cdot\rangle} \right) \right)(x) \\
= \;& \xi_l e^{-i\langle \xi \,|\, x\rangle} \frac{\partial}{\partial \xi_k} \left(A(e^{i\langle \xi \,|\, \cdot\rangle}) \right)(x) \\
= \;& \xi_l \frac{\partial}{\partial \xi_k} \left(e^{-i\langle \xi \,|\, x\rangle} A(e^{i\langle \xi \,|\, \cdot\rangle}) \right)(x) \\
& \qquad - \xi_l \left(\frac{\partial}{\partial \xi_k} e^{-i\langle \xi \,|\, x\rangle} \right) (A(e^{i\langle \xi \,|\, \cdot\rangle}))(x) \\
= \;& \xi_l \frac{\partial}{\partial \xi_k} \sigma(x,\xi) + \xi_l x_k e^{-i\langle \xi \,|\, x\rangle} \left(A(e^{i\langle \xi \,|\, \cdot\rangle}) \right)(x) \\
= \;& \xi_l \frac{\partial}{\partial \xi_k} \sigma(x,\xi) - q(x,\xi),
\end{aligned}$$

where $q(x,\xi) := -ix_k\xi_l\sigma(x,\xi)$. The second addend gives:

$$\begin{aligned}
& e^{-i\langle \xi \,|\, x\rangle} \left(x_k \frac{\partial}{\partial x_l} A e^{i\langle \xi \,|\, \cdot\rangle} \right)(x) \\
= \;& x_k \left(\frac{\partial}{\partial x_l} \left(e^{-i\langle \xi \,|\, x\rangle} (A e^{i\langle \xi \,|\, \cdot\rangle}) \right) \right)(x) \\
& \qquad - \left(\frac{\partial}{\partial x_l} \left(e^{-i\langle \xi \,|\, x\rangle} \right) \right) (A e^{i\langle \xi \,|\, \cdot\rangle})(x) \\
= \;& x_k \left(\frac{\partial}{\partial x_l} \sigma(x,\xi) + i\xi_j e^{-i\langle \xi \,|\, x\rangle} A(e^{i\langle \xi \,|\, \cdot\rangle})(x) \right) \\
= \;& x_k \frac{\partial}{\partial x_l} \sigma(x,\xi) + i x_k \xi_l \sigma(x,\xi) \\
= \;& x_k \frac{\partial}{\partial x_l} \sigma(x,\xi) - q(x,\xi).
\end{aligned}$$

The above calculations show, that

$$\sigma_{[A, ix_k \frac{\partial}{\partial x_l}]}(x,\xi) = -ix_k \frac{\partial}{\partial x_l} \sigma(x,\xi) + i\xi_l \frac{\partial}{\partial \xi_k} \sigma(x,\xi)$$

and we have proven (iii). $\square$

Corollary E.2.6. *Let $\sigma \in \Sigma_K$ be arbitrary. Then we have:*

(i) $\frac{\partial^\beta}{\partial x^\beta}\sigma \in \Sigma_K$ *for all* $\beta \in \mathbb{N}_0^n$.

(ii) $\frac{\partial^\alpha}{\partial \xi^\alpha}\sigma \in \Sigma_K$ *for all* $\alpha \in \mathbb{N}_0^n$.

Proof. This follows by induction using E.2.5. □

Corollary E.2.7. *Let $\alpha, \beta \in \mathbb{N}_0^n$ be multiindices, $p = p(\xi_1, \ldots, \xi_n)$ a polynomial of degree equal or less than $|\alpha|$ and $\sigma \in \Sigma_K$. Then the mapping $(x, \xi) \longmapsto p(\xi)[\partial_x^\beta \partial_\xi^\alpha \sigma(x, \xi)]$ is also an element of Σ_K.*

Proof. (i) Let $\sigma \in \Sigma$, $\alpha \in \mathbb{N}_0^n$ and p be a polynomial with $\operatorname{grad} p \leq |\alpha|$. Then we want to show, that $p\partial_\xi^\alpha \sigma \in \Sigma$ holds. Since $p = \sum_{|\gamma| \leq |\alpha|} a_\gamma \xi^\gamma$, it is enough to show that $\xi^\gamma \partial_\xi^\alpha \sigma \in \Sigma$ holds. We will do this by induction with respect to $|\alpha|$:

If $|\alpha| = 1$, then the claim follows by E.2.5. Now suppose the induction hypothesis holds for all $\beta \in \mathbb{N}_0^n$, such that $|\beta| < n$. Let $\alpha \in \mathbb{N}_0^n$ be with $|\alpha| - 1 \leq n$ and $\gamma \in \mathbb{N}_0^n$, such that $|\gamma| \leq |\alpha|$.

In the case that $\gamma = (0, \ldots, 0)$ we again get $\partial_\xi^\alpha \sigma \in \Sigma$. So let $\gamma \neq (0, \ldots, 0)$. Then there exists a minimal $j_1, j_2 \in \{1, \ldots, n\}$, such that $\gamma_{j_1} \neq 0$ and $\alpha_{j_2} \neq 0$. We conclude

$$
\begin{aligned}
\xi^\gamma \partial_\xi^\alpha \sigma(x, \xi) &= \xi_{j_1}^{\gamma_{j_1}} \cdots \xi_n^{\gamma_n} \partial_{\xi_{j_2}}^{\alpha_{j_2}} \cdots \partial_{\xi_n}^{\alpha_n} \sigma(x, \xi) \\
&= \xi_{j_1} \left(\xi_{j_1}^{\gamma_{j_1}-1} \xi_{j_1+1}^{\gamma_{j_1+1}} \cdots \xi_n^{\gamma_n} \partial_{\xi_{j_2}}^{\alpha_{j_2}} \cdots \partial_{\xi_n}^{\alpha_n} \sigma(x, \xi) \right) \\
&= \xi_{j_1} \frac{\partial}{\partial \xi_{j_2}} \left(\xi_{j_1}^{\gamma_{j_1}-1} \xi_{j_1+1}^{\gamma_{j_1+1}} \cdots \xi_n^{\gamma_n} \partial_{\xi_{j_2}}^{\alpha_{j_2}-1} \partial_{\xi_{j_2+1}}^{\alpha_{j_2+1}} \cdots \partial_{\xi_n}^{\alpha_n} \sigma(x, \xi) \right) \\
&\quad - \xi_{j_1} \frac{\partial}{\partial \xi_{j_2}} \left(\xi_{j_1}^{\gamma_{j_1}-1} \xi_{j_1+1}^{\gamma_{j_1+1}} \cdots \xi_n^{\gamma_n} \right) \partial_{\xi_{j_2}}^{\alpha_{j_2}-1} \partial_{\xi_{j_2+1}}^{\alpha_{j_2+1}} \cdots \partial_{\xi_n}^{\alpha_n} \sigma(x, \xi).
\end{aligned}
$$

Using the induction hypothesis we see, that

$$
\xi_{j_1}^{\gamma_{j_1}-1} \xi_{j_1+1}^{\gamma_{j_1+1}} \cdots \xi_n^{\gamma_n} \partial_{\xi_{j_2}}^{\alpha_{j_2}-1} \partial_{\xi_{j_2+1}}^{\alpha_{j_2+1}} \cdots \partial_{\xi_n}^{\alpha_n} \sigma(x, \xi) \in \Sigma,
$$

which gives

$$
\xi_{j_1} \frac{\partial}{\partial \xi_{j_2}} \left(\xi_{j_1}^{\gamma_{j_1}-1} \xi_{j_1+1}^{\gamma_{j_1+1}} \cdots \xi_n^{\gamma_n} \partial_{\xi_{j_2}}^{\alpha_{j_2}-1} \partial_{\xi_{j_2+1}}^{\alpha_{j_2+1}} \cdots \partial_{\xi_n}^{\alpha_n} \sigma(x, \xi) \right) \in \Sigma
$$

by E.2.6 and induction. The second summand s has to be treated in separated cases:

(I) $j_1 < j_2$: $\frac{\partial}{\partial \xi_{j_2}} (\xi_{j_1}^{\gamma_{j_1}-1} \xi_{j_1+1}^{\gamma_{j_1+1}} \cdots \xi_n^{\gamma_n}) = 0$, which gives

$$
s = \xi_{j_1} \frac{\partial}{\partial \xi_{j_2}} (\xi_{j_1}^{\gamma_{j_1}-1} \xi_{j_1+1}^{\gamma_{j_1+1}} \cdots \xi_n^{\gamma_n}) \partial_{\xi_{j_2}}^{\alpha_{j_2}-1} \partial_{\xi_{j_2+1}}^{\alpha_{j_2+1}} \cdots \partial_{\xi_n}^{\alpha_n} \sigma(x, \xi) = 0
$$

and $s \in \Sigma$.

(II) $j_1 = j_2$ und $\gamma_{j_1} = \gamma_{j_2} = 1$: Then again $s = 0 \in \Sigma$.

(III) $j_1 = j_2$ und $\gamma_{j_1} = \gamma_{j_2} \geq 2$: We get

$$
s(x, \xi) = (\gamma_{j_1} - 1) \xi_{j_1}^{\gamma_{j_1}-1} \xi_{j_1+1}^{\gamma_{j_1+1}} \cdots \xi_n^{\gamma_n} \partial_{\xi_{j_1}}^{\alpha_{j_1}-1} \partial_{\xi_{j_1+1}}^{\alpha_{j_1+1}} \cdots \partial_{\xi_n}^{\alpha_n} \sigma(x, \xi) \in \Sigma
$$

by induction.

(IV) $j_1 > j_2$: Again we have

$$s(x,\xi) = \gamma_{j_2}\left(\prod_{k\neq j_2} \xi_k^{i_k}\right) \xi_{j_2}^{\gamma_{j_2}-1} \partial_{\xi_{j_2}}^{\alpha_{j_2-1}} \cdots \partial_{\xi_n}^{\alpha_n} \sigma(x,\xi) \in \Sigma$$

by induction.

(ii) Now, we finally want to prove the lemma. Let $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a polynomial with degree $p \leq |\alpha|$, i.e. $p = \sum_{|\gamma|\leq|\alpha|} a_\gamma \xi^\gamma$. Then

$$p(\xi)\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi) = \sum_{|\gamma|\leq|\alpha|} a_\gamma \xi^\gamma \partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)$$

follows. By E.2.6 we get $\partial_x^\beta \sigma \in \Sigma$, which shows $\xi^\gamma \partial_\xi^\alpha \partial_x^\beta \sigma \in \Sigma$ by (i). This proves the claim. □

Satz E.2.8. *We have:* $\Sigma_K \in S_{1,0}^m(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$.

Proof. Let $\sigma \in \Sigma_K$ be arbitrary. Then $p \cdot \partial_x^\beta \partial \alpha_\xi \sigma \in \Sigma_K$ holds for all polynomials p in ξ of degree equal or less $|\alpha|$. Thus by E.1.6 (ii)

$$\int_{R^n} |p(\xi)\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)|^2 \, dx \leq C_{\alpha,\beta}$$

follows for all $\xi \in \mathbb{R}_\xi^n$. The *Sobolev* embedding theorem now implies the existence of a constant $\widetilde{C}_{\alpha,\beta} < \infty$, such that

$$\sup_{\xi\in\mathbb{R}_\xi^n} \sup_{x\in\mathbb{R}_x^n} |p(\xi)\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)| \leq \widetilde{C}_{\alpha,\beta}$$

holds for all $\alpha, \beta \in \mathbb{N}_0^n$ and all polynomial p of degree equal or less $|\alpha|$, i.e. $\sigma \in S_{1,0}^m(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$. □

E.3 Global classification

We have proven that pseudodifferential operators can locally be characterized by means of commutators of finitely many symmetric operators. Now, we have to globalize this approach to a criteria on a closed manifold. In particular we have to show that all operators satisfying the commutator conditions admit a $\mathscr{C}^\infty$-kernel representation.

First of all, we will describe the relationship between operators on $\mathcal{M}$ and their local representation defined on a coordinate chart in $\mathbb{R}^n$.

Notations E.3.1. Let $\chi : \mathcal{M} \supseteq U' \longrightarrow U \subseteq \mathbb{R}^n$ be a chart and $\varphi, \vartheta, \psi, \upsilon \in \mathscr{C}_0^\infty(U)$, where $\varphi \prec \vartheta \prec \psi \prec \upsilon$. We define $\varphi', \vartheta', \psi', \upsilon' \in \mathscr{C}_0^\infty(U')$ by $\varphi' := \varphi \circ \chi$ etc. Moreover, let $\widetilde{V} := \psi V_1 \psi : \mathscr{C}_0^\infty(U) \longrightarrow L^2(U)$ be a densely defined, symmetric operator, where V_1 is a partial differential operator of order 1. Let $\widetilde{V}' : \mathscr{C}^\infty(\mathcal{M}) \longrightarrow L^2(\mathcal{M})$ be the differential operator on $\mathscr{C}^\infty(\mathcal{M})$ that is locally given by $\widetilde{V}$, i.e. we have

$$(\widetilde{V}'f)(x') = \begin{cases} [\widetilde{V}(f \circ \chi^{-1})](\chi(x')), & x' \in U', \\ 0, & \text{else}, \end{cases}$$

for $f \in \mathscr{C}^\infty(\mathcal{M})$. Then $\widetilde{V}'$ is also densely defined and symmetric. We denote by $V : L^2(U) \supseteq \mathcal{D}(V) \longrightarrow L^2(U)$ and $V' : L^2(\mathcal{M}) \supseteq \mathcal{D}(V') \longrightarrow L^2(\mathcal{M})$ the corresponding minimal closed extensions of $\widetilde{V}$ resp. $\widetilde{V}'$.

A given operator $A \in \mathscr{L}(L^2(\mathcal{M}))$ induces a bounded operator $A_{\upsilon,\chi} \in \mathscr{L}(L^2(U))$

$$\begin{tikzcd} L^2(\mathcal{M}) \arrow[r, "\upsilon' A"] & L^2(U') \arrow[d, "(\cdot)\circ\chi^{-1}"] \\ L^2(U) \arrow[u, "(\cdot)\circ\chi"] \arrow[r, dashed, "A_{\upsilon,\chi}"'] & L^2(U) \end{tikzcd}$$

by $[A_{\upsilon,\chi}f](x) = (\upsilon' A(f \circ \chi))(\chi^{-1}(x))$.

Lemma E.3.2. *We have:*

(i) $f \in \mathcal{D}(V)$ *implies* $f \circ \chi \in \mathcal{D}(V')$ *and* $V'(f \circ \chi) = (Vf) \circ \chi$.

(ii) $g' \in \mathcal{D}(V')$ *implies* $h \in \mathcal{D}(V)$ *and* $Vh = (V'g') \circ \chi^{-1}$, *where* $h := (\upsilon' g') \circ \chi^{-1}$.

(iii) $A \in \mathcal{D}(\delta_{V'})$ *implies* $A_{\upsilon,\chi} \in \mathcal{D}(\delta_V)$ *and* $\delta_V(A_{\upsilon,\chi}) = (\delta_{V'}(A))_{\upsilon,\chi}$.

(iv) $f \in L^2(\mathcal{M})$ *implies*

$$(\vartheta' A \varphi' f)(\chi^{-1}(x)) = (\vartheta A_{\upsilon,\chi}(\varphi(f \circ \chi^{-1})))(x)$$

for all $x \in U$.

Proof. (i) Let $f \in \mathcal{D}(V)$ be arbitrary. Then we can choose a sequence $(f_k)_k \subseteq \mathscr{C}_0^\infty(U)$, such that $f_k \xrightarrow{k\to\infty} f$ and $\widetilde{V} f_k \xrightarrow{k\to\infty} Vf'$ with respect to $L^2(U)$. This gives $f_k \circ \chi \xrightarrow{k\to\infty} f \circ \chi$ and

$$\widetilde{V}'(f_k \circ \chi) = (\widetilde{V}(f_k \circ \chi \circ \chi^{-1})) \circ \chi = (\widetilde{V}(f_k)) \circ \chi \xrightarrow{k\to\infty} Vf \circ \chi$$

with respect to $L^2(\mathcal{M})$, i.e. $f \circ \chi \in D(V')$ and $V'(f \circ \chi) = (Vf) \circ \chi$.

(ii) Let $g' \in \mathcal{D}(V')$ be arbitrary. Then there exists $g'_k \in \mathscr{C}^\infty(\mathcal{M})$ such that $g'_k \xrightarrow{k\to\infty} g$ and $\widetilde{V}' g_k \xrightarrow{k\to\infty} V'g'$ with respect to $L^2(\mathcal{M})$. This implies $\upsilon' g_k \xrightarrow{k\to\infty} \upsilon' g$ and

$$h_k := (\upsilon' g'_k) \circ \chi^{-1} \xrightarrow{k\to\infty} (\upsilon' g') \circ \chi^{-1} = h.$$

Moreover, an application of (i) shows

$$\widetilde{V} h_k = \widetilde{V}((\upsilon' g'_k) \circ \chi^{-1}) = (\widetilde{V}' g'_k) \circ \chi^{-1} \xrightarrow{k\to\infty} (V'g') \circ \chi^{-1},$$

i.e. $h \in \mathcal{D}(V)$ and $Vh = (V'g') \circ \chi^{-1}$.

(iii) Let $f \in \mathcal{D}(V)$ be arbitrary, then (i) gives $f \circ \chi \in \mathcal{D}(V')$. We conclude $A(f \circ \chi) \in \mathcal{D}(V')$, since $A \in \mathcal{D}(\delta_{V'})$ and therefore by (ii): $A_{\upsilon,\chi} f = (\upsilon' A(f \circ \chi) \circ \chi^{-1}) \in \mathcal{D}(V)$. We calculate:

$$\begin{aligned} VA_{\upsilon,\chi} f - A_{\upsilon,\chi} Vf &= V\left[(\upsilon' A\,(f \circ \chi)) \circ \chi^{-1}\right] - [\upsilon' A\,((Vf) \circ \chi)] \circ \chi^{-1} \\ &= [\upsilon'(V'A - AV')(f \circ \chi)] \circ \chi^{-1} \\ &= [\upsilon' \delta_{V'}(A)(f \circ \chi)] \circ \chi^{-1}. \end{aligned}$$

This shows that $VA_{\upsilon,\chi} - A_{\upsilon,\chi} V : \mathcal{D}(V) \longrightarrow L^2(U)$ has the bounded extension $(\delta_{V'}(A))_{\upsilon,\chi}$, and we have proven (iii).

(iv) Let $f \in L^2(\mathcal{M})$. Then

$$\begin{aligned} \left[\vartheta A_{\upsilon,\chi}\left(\varphi(f \circ \chi^{-1})\right)\right](x) &= \vartheta(y)\left[\upsilon' A\left(\varphi(f \circ \chi^{-1}) \circ \chi\right)\right]\left(\chi^{-1}(x)\right) \\ &= \vartheta A\left(\varphi' f\right)\left(\chi^{-1}(x)\right) \end{aligned}$$

follows, which shows (iv). □

Notations E.3.3. Let $\mathsf{A} := (U'_j, \chi_j)_{j=1}^N$ be a (fixed) covering of $\mathcal{M}$ by coordinate charts and $(\varphi_j)_{j=1,\dots,N}$ a partition of unity subordinated to A. Moreover, we choose $\vartheta'_j, \eta'_j, \psi'_j \in \mathscr{C}_0^\infty(U'_j)$ to given $j = 1,\dots,N$, such that $0 \le \vartheta'_j, \eta'_j, \psi'_j \le 1$ and $\varphi'_j \prec \eta'_j \prec \vartheta'_j \prec \psi'_j$. We then define $\varphi_j, \eta_j, \vartheta_j, \psi_j \in \mathscr{C}_0^\infty(U_j)$ by $\varphi_j := \varphi'_j \circ \chi_j^{-1}$ etc.

Denote by $\widetilde{\mathcal{V}}_j$ resp. $\mathcal{V}_j$ the family of densely defined symmetric, hence closable operators defined, cf. E.2.1, where we replace U by U_j and ψ by ψ_j. Let $\widetilde{\mathcal{V}}'_j$ resp. $\mathcal{V}'_j$ be the families of densely defined symmetric closable resp. closed operators given by E.3.1 with respect to the charts χ_j and the local family $\widetilde{\mathcal{V}}_j$ etc.

Finally, we consider the following set of operators

$$\widetilde{V}_j^l := i\eta_j \frac{\partial}{\partial x_l} \eta_j : L^2(U_j) \supseteq \mathscr{C}_0^\infty(U_j) \longrightarrow L^2(U_j),$$

where $j = 1,\dots,N$, $l = 1,\dots,n$. Then again $\widetilde{V}_j^l$ is densely defined and symmetric, hence closable, and we denote the minimal closed extension by $V_j^l : D(V_j^l) \longrightarrow L^2(U_j)$ and the corresponding global operator by $V_j'^l : D(V_j'^l) \longrightarrow L^2(\mathcal{M})$.

Let $\widetilde{V}'^l_{D,j} \in \widetilde{\mathcal{V}}'_{D,j}$ be the operator that is locally given by $i\psi_j \frac{\partial}{\partial x_l} \psi_j$, and set

$$\widetilde{V}_j'^{l,m} := (1 - \eta'_m)\widetilde{V}'^l_{D,j}(1 - \eta'_m) : L^2(M) \supseteq \mathscr{C}_0^\infty(M) \longrightarrow L^2(M).$$

Then this operator is again symmetric and densely defined and therefore closable. We denote by $V_j'^{l,m} : \mathscr{C}_0^\infty(\mathcal{M}) \longrightarrow L^2(\mathcal{M})$ the corresponding minimal closed extension of this operator. Moreover, we define the following family of closable operators

$$\begin{aligned} \widetilde{\mathcal{W}} := \bigcup_{j=1}^N \widetilde{\mathcal{V}}'_j \cup \left\{\widetilde{\mathcal{V}}_j'^l \,:\, j = 1,\dots,N,\, l = 1,\dots,n\right\} \\ \cup \left\{\widetilde{\mathcal{V}}_j'^{l,m} \,:\, j, m = 1\dots,N,\, l = 1,\dots,n\right\}, \end{aligned}$$

resp. $\mathcal{W}$ the corresponding family of densely defined symmetric minimal extensions of elements from $\widetilde{\mathcal{W}}$. It is worth pointing out, that all of these operators are local operators.

Lemma E.3.4.

(i) $f \in \mathscr{C}^\infty(\mathcal{M})$ *implies* $M_f \in \Psi_\infty^{\mathcal{W}}$.

(ii) *We have* $\mathcal{H}_{\mathcal{W}}^\infty = \mathscr{C}^\infty(\mathcal{M})$.

Proof. Using E.3.1, the prove of (i) is given by an analogous calculation as in E.2.3, hence we omit it.

Let us treat (ii): Since $\mathscr{C}^\infty(\mathcal{M}) \subseteq \mathcal{D}(W')$ and

$$W'(\mathscr{C}^\infty(\mathcal{M})) = \widetilde{W}'(\mathscr{C}^\infty(\mathcal{M}))) \subseteq \mathscr{C}^\infty(\mathcal{M})$$

for all $W' \in \mathcal{W}$, we conclude $\mathscr{C}^\infty(\mathcal{M}) \subseteq \mathcal{H}_{\mathcal{W}}^\infty$. It is left to show the reversed implication. Let $f \in \mathcal{H}_{\mathcal{W}}^\infty$ be arbitrary, then an induction and E.3.2 (ii) shows, that $(\varphi_j' f) \circ \chi_j^{-1} \in \mathcal{H}_{\mathcal{V}_j}^\infty$ and therefore by E.2.2 $(\varphi_j' f) \circ \chi_j^{-1} \in \mathcal{C}_0^\infty(U_j)$ holds. Summing up, we see that $f = \sum\limits_{j=1}^{N} \varphi_j' f \in \mathscr{C}^\infty(\mathcal{M})$ as desired. $\square$

To treat operators given of the form $(1-\vartheta_j')A\varphi_j'$, we use the given partition of unity to localize, i.e.

$$(1-\vartheta_j')A\varphi_j' = \sum_{m=1}^{N} \varphi_m'(1-\vartheta_j')A\varphi_j'. \qquad (E.3.1)$$

We define:

Definition E.3.5.

(i) Denote by $\Psi_{j,m}$ the space of all $A \in \widetilde{\Psi}_\infty^{\mathcal{W}}(\mathcal{M})$, such that
   (a) $\operatorname{supp} Af \subseteq \operatorname{supp} \varphi_m' \cap \operatorname{supp} \left(1-\vartheta_j'\right)$ and
   (b) $\operatorname{supp} f \cap \operatorname{supp} \varphi_j' = \emptyset \Rightarrow Af = 0$

   holds for all $f \in \mathscr{C}^\infty(\mathcal{M})$.

(ii) Denote by $\Psi_{m,j}^*$ the space of all $A \in \widetilde{\Psi}_\infty^{\mathcal{W}}(\mathcal{M})$, such that
   (a) $\operatorname{supp} Af \subseteq \operatorname{supp} \varphi_j'$ and
   (b) $\operatorname{supp} f \cap \left(\operatorname{supp} \varphi_m' \cap \operatorname{supp} (1-\vartheta_j')\right) = \emptyset \Rightarrow Af = 0$

   holds for all $f \in \mathscr{C}^\infty(\mathcal{M})$.

Note, that $A \in \Psi_{j,m}$ induces a bounded operator $A_{j,m} : L^2(U_j) \longrightarrow L^2(U_m)$ by $A_{j,m} f = A(f \circ \chi_j) \circ \chi_m^{-1}$. Analogously each operator $A^* \in \Psi_{m,j}^*$ defines a bounded operator $A_{m,j}^*$:

$L^2(U_m) \longrightarrow L^2(U_j)$ via $A^*_{m,j}f = A(f \circ \chi_m) \circ \chi_j^{-1}$.

$$\begin{tikzcd}
L^2(M) \arrow[r, "A_{|U'_m}"] & L^2(U'_m) \arrow[d, "\circ\chi_m^{-1}"] \\
L^2(U_j) \arrow[u, "\circ\chi_j"] \arrow[r, shift left, "A_{j,m}"] & L^2(U_m) \arrow[l, shift left, "A^*_{m,j}"] \arrow[d, "\circ\chi_m"] \\
L^2(U'_j) \arrow[u, "\circ\chi_j^{-1}"] & L^2(M) \arrow[l, "A^*_{|U'_j}"]
\end{tikzcd}$$

The spaces of all local representations of operators from $\Psi_{j,m}$ resp. $\Psi^*_{m,j}$ will be denoted by $\mathcal{L}_{j,m}$ resp. $\mathcal{L}^*_{m,j}$. If $\widetilde{B} \in \mathcal{L}_{j,m} \cup \mathcal{L}^*_{m,j}$ is a local representation of $B \in \Psi_{j,m} \cup \Psi^*_{j,m}$, $\widetilde{B}$ induces an element of $\mathscr{L}(L^2(\mathbb{R}^n))$ resp. $\mathscr{L}(\mathscr{C}^\infty(\mathbb{R}^n))$, and we will use this in the sequel without any comment.

Remark E.3.6. Let $A \in \widetilde{\Psi}_{1,0}(\mathcal{M})$ be arbitrary. Then we have:

(i) $\varphi'_m(1-\vartheta'_j)A\varphi'_j \in \Psi_{j,m}$.

(ii) $\varphi'_j A^*(1-\vartheta'_j)\varphi'_m \in \Psi^*_{m,j}$.

(iii) Let $B_1 : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ be the local representation of $\varphi'_m(1-\vartheta'_j)A\varphi'_j$ and $B_2 : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ the representation of $\varphi'_j A^*(1-\vartheta'_j)\varphi'_m \in \Psi^*_{j,m}$. Then $B_1^* = B_2$.

Lemma E.3.7. *We have* $\delta_{W'}(\Psi_{j,m}) \subseteq \Psi_{j,m}$ *and* $\delta_{W'}(\Psi^*_{j,m}) \subseteq \Psi^*_{j,m}$.

Proof. By 1.3.5 we have $\delta_{W'}(\Psi_{j,m}) \subseteq \Psi^{\mathcal{W}}_\infty$ and $\delta_{W'}(\Psi^*_{m,j}) \subseteq \Psi^{\mathcal{W}}_\infty$. So what is left is to prove (i) and (ii) of E.3.6. For given $A \in \Psi^{\mathcal{W}}_\infty$ and $f \in \mathscr{C}^\infty(\mathcal{M})$ we get

$$\delta_{W'}(A)f = (W'A - AW')f = \widetilde{W'}Af - A\widetilde{W'}f.$$

Moreover, $\widetilde{W'} \in \mathcal{W}$ is a local operator, and (i) and (ii) is due to E.3.6. □

Lemma E.3.8. *We have:*

(i) $B \in \mathcal{L}_{j,m} \Rightarrow B : L^2(\mathbb{R}^n) \longrightarrow \mathcal{H}^k(\mathbb{R}^n)$ *is bounded for all* $k \in \mathbb{N}_0$,

(ii) $B \in \mathcal{L}^*_{m,j} \Rightarrow B : L^2(\mathbb{R}^n) \longrightarrow \mathcal{H}^k(\mathbb{R}^n)$ *is bounded for all* $k \in \mathbb{N}_0$.

Proof. We will prove the lemma by induction with respect to $k \in \mathbb{N}$.

(i) Let $k = 1$ and $A \in \Psi_{j,m}$, where $B = A_{j,m} \in \mathcal{L}_{j,m}$ and $f \in L^2(\mathbb{R}^n)$ are arbitrary. Then we have to show that $Bf \in \mathcal{H}^1(\mathbb{R}^n)$. Since $\mathscr{C}^\infty_0(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, there exists a sequence $(f_k)_k \subseteq \mathscr{C}^\infty_0(\mathbb{R}^n)$ with $f_k \xrightarrow{k\to\infty} f$ with respect to the topology $L^2(\mathbb{R}^n)$. Because of $B \in \mathcal{L}_{j,m}$ we get $Bf_k \xrightarrow{k\to\infty} Bf \in L^2(\mathbb{R}^n)$ and $Bf_k \in \mathscr{C}^\infty_0(\mathbb{R}^n)$. Moreover,

$$A(1-\eta'_j)(\widetilde{V}'^{l}_{D,m})(1-\eta'_j)(f_k \circ \chi_j) = 0, \qquad (E.3.2)$$

holds by

$$\operatorname{supp}\varphi_j' \cap \operatorname{supp}\left((1-\eta_j')(\widetilde{V}_{D,j}'^{,l})(1-\eta_j')(f\circ\chi_j)\right)=\emptyset$$

and E.3.5 (i) (b). If $\delta_{V_m'^{l,j}}(A)(f_k\circ\chi_j)\circ\chi_m^{-1}\in L^2(\mathbb{R}^n)$ is arbitrary we get:

$$\begin{aligned}
&\delta_{V_m'^{l,j}}(A)(f_k\circ\chi_j)\circ\chi_m^{-1}\\
&\quad=\Big((1-\eta_j')(\widetilde{V}_{D,m}'^{l})(1-\eta_j')A(f_k\circ\chi_j)\\
&\qquad\quad-\underbrace{A(1-\eta_j')(\widetilde{V}_{D,m}'^{l})(1-\eta_j')(f_k\circ\chi_j)}_{=0 \text{ cf. } (E.3.2)}\Big)\circ\chi_m^{-1}\\
&\quad=i\psi_m\frac{\partial}{\partial x_l}\psi_m(A(f_k\circ\chi_j))\circ\chi_m^{-1}=i\frac{\partial}{\partial x_l}A_{j,m}f_k\\
&\quad=i\frac{\partial}{\partial x_l}Bf_k,
\end{aligned}$$

since in E.3.5 (i) (a) $\operatorname{supp}(A(f_k\circ\chi_j))\subseteq\operatorname{supp}(1-\vartheta_j')\cap\operatorname{supp}\varphi_m'$ holds, where $\psi_m\equiv 1$. We get $i\overline{\frac{\partial}{\partial x_l}}Bf=\operatorname{ad}[V_m'^{l,j}](A)(f\circ\chi_j)\circ\chi_m^{-1}$ and $i\overline{\frac{\partial}{\partial x_l}}Bf\in L^2(\mathbb{R}^n)$ for all $l\in\{1,\dots,n\}$. Since $Bf\in L^2(\mathbb{R}^n)$ holds, $Bf\in\mathcal{H}^1(\mathbb{R}^n)$ follows[1]. If $\delta_{V_m'^{l,j}}(A)$ denotes the operator given by

$$\delta_{V_m'^{l,j}}(A)g:=\operatorname{ad}[V_m'^{l,j}](A)(g\circ\chi_j)\circ\chi_m^{-1},\qquad (g\in L^2(\mathbb{R}^n))$$

then it is the local representation of an operator in $\Psi_{j,m}$ by E.3.7, i.e. $\delta_{V_m'^{l,j}}(A)\in\mathcal{L}_{j,m}$. Since $i\overline{\frac{\partial}{\partial x_l}}Bf=\delta_{V_m'^{l,j}}(A)f$, where $\delta_{V_m'^{l,j}}(A)\in\mathcal{L}_{j,m}$ the inductional step becomes trivial.

(ii) Again, let $k=1$ and $A\in\widetilde{\Psi}_{j,m}^{*}$ be given with $B=A_{m,j}^{*}\in\mathscr{L}(L^2(\mathbb{R}^n))$ and $f\in L^2(\mathbb{R}^n)$ arbitrary. Then there is $(f_k)_k\subseteq\mathscr{C}_0^\infty(\mathbb{R}^n)$, such that $f_k\xrightarrow{k\to\infty}f$ with respect to $L^2(\mathbb{R}^n)$. But this again gives $Bf_k\xrightarrow{k\to\infty}Bf$ with respect to the topology of $L^2(\mathbb{R}^n)$, where $Bf_k\in\mathscr{C}_0^\infty(\mathbb{R}^n)$. In addition we get

$$A(\widetilde{V}_j'^{l})(f_k\circ\chi_m)=0, \tag{E.3.3}$$

because $\widetilde{V}_j^{l'}(f_k\circ\chi_m)=i\eta_j\frac{\partial}{\partial x_l}\eta_j(f_k\circ\chi_m)$ and thus

$$\operatorname{supp}(\widetilde{V}_j'^{l})(f_k\circ\chi_m)\cap\left(\operatorname{supp}\vartheta_m'\cap(1-\vartheta_j')\right)=\emptyset$$

holds. Consequently, we get the following equation:

$$\begin{aligned}
L^2(\mathbb{R}^n)\ni{}&\operatorname{ad}[V_j'^{l}](A)(f_k\circ\chi_m)\circ\chi_j^{-1}\\
&=\Big((\widetilde{V}_j'^{l})A(f_k\circ\chi_m)-\underbrace{A(\widetilde{V}_j'^{l})(f_k\circ\chi_m)}_{=0\text{ by }(E.3.3)}\Big)\circ\chi_j^{-1}\\
&=\Big((\widetilde{V}_j'^{l})A(f_k\circ\chi_m)\Big)\circ\chi_j^{-1}=\eta_j\frac{\partial}{\partial x_l}\eta_j\left(A(f_k\circ\chi_m)\circ\chi_j^{-1}\right)\\
&=\frac{\partial}{\partial x_l}Bf_k.
\end{aligned}$$

[1] see also [98, Proposition 1.14]

This implies $Bf \in L^2(\mathbb{R}^n)$, and the inductional step is again trivial. $\square$

Theorem E.3.9. *We have $\Psi^0_{1,0}(\mathcal{M}) = \Psi^{\mathcal{W}}_\infty$. In particular $\Psi^0_{1,0}(\mathcal{M})$ is a Ψ^*-algebra.*

Proof. On the one hand, we know that $\Psi^0_{1,0}(\mathcal{M})^* = \Psi^0_{1,0}(\mathcal{M})$ and

$$[\Psi^0_{1,0}(\mathcal{M}), \Psi^0_{0,1}(\mathcal{M})] \subseteq \Psi^0_{1,0}(\mathcal{M}) \subseteq \mathscr{L}(L^2(\mathcal{M}))$$

holds using $\widetilde{\mathcal{W}} \subseteq \Psi^0_{1,0}(\mathcal{M})$ and by the well-known symbolic calculus for pseudodifferential operators we get that $A \in \Psi^0_{1,0}(\mathcal{M})$ and $W' \in \mathcal{W}$ give $\delta_{W'}(A) \in \Psi^0_{1,0}(\mathcal{M})$; thus an iteration process finally yields $\Psi^0_{1,0}(\mathcal{M}) \subseteq \Psi^{\mathcal{W}}_\infty$.

On the other hand, if $A \in \Psi^{\mathcal{W}}_\infty$ is given, we first use the decomposition

$$A = \sum_{j=1}^{N} \vartheta'_j A \varphi'_j + \sum_{j=1}^{N} (1 - \vartheta'_j) A \varphi'_j.$$

Then E.1.6, E.2.8 and E.3.2 imply that E.1.2 (i) is fulfilled.

What is left to show, is that $(1 - \vartheta'_j) A \varphi'_j$ is an integral operator for all $j \in \{1, \dots, N\}$ having a $\mathscr{C}^\infty$-kernel representation. For this we have to treat operators B given by

$$B = \varphi'_m (1 - \vartheta'_j) A \varphi'_j \in \Psi_{j,m}$$

according to (E.3.1). Let $B_1 \in \mathscr{L}_{j,m}$ be the local representation of B and $B_1^* \in \mathscr{L}^*_{j,m}$ be the local representation of $\varphi'_j A^* (1 - \vartheta'_j) \varphi_m \in \Psi^*_{j,m}$. Then $(B_1^*)^* = B_1$ follows, cf. E.3.6. This implies $B_1, B_1^* : L^2(\mathbb{R}^n) \longrightarrow \mathcal{H}^k(\mathbb{R}^n)$ for all $k \in \mathbb{N}_0$ by E.3.8. Since both operators are adjoint to each other, we conclude that B_1 extends to a bounded operator $\mathcal{H}^{-k}(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ for all $k \in \mathbb{N}_0$. By interpolation theory (see, for instance, [19, Theorem 1.5.10])

$$[L^2(\mathbb{R}^n), \mathcal{H}^{2k}(\mathbb{R}^n)]_{\frac{1}{2}} = [D(\Lambda^0), D(\Lambda^{2k})]_{\frac{1}{2}} = D(\Lambda^k) = \mathcal{H}^k(\mathbb{R}^n),$$

resp.

$$[\mathcal{H}^{-2k}(\mathbb{R}^n), L^2(\mathbb{R}^n)]_{\frac{1}{2}} = \mathcal{H}^{-k}(\mathbb{R}^n)$$

holds, which implies that B_1 can be extended to a bounded operator

$$B_1 : \mathcal{H}^{-k}(\mathbb{R}^n) \longrightarrow \mathcal{H}^k(\mathbb{R}^n)$$

for all $k \in \mathbb{N}_0$ by [19, Theorem 1.5.5]. This shows that B_1 has a $\mathscr{C}^\infty$-kernel representation using the classical result by Seeley [106]. Therefore E.1.2 (ii) holds and we have finished the proof. $\square$

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List of Notations

Index

Curriculum vitae

Personalien

Name:	Jochen Ditsche
Geboren am:	16. Dezember 1977
Geburtsort:	Düsseldorf

Schule & Studium

1994 - 1997	Willigis Gymnasium Mainz
1997 - 2003	Studium der Mathematik mit Nebenfach Informatik an der Johannes Gutenberg-Universität Mainz
09/2000	Vordiplom in Mathematik
09/2003	Diplom in Mathematik

Promotion

11/2003 - 03/2008	Landesstelle als wissenschaftl. Mitarbeiter an der Johannes Gutenberg-Universität Mainz
11/2005 - 03/2006	Forschungsaufenthalt an der Pennsylvania State University, State College, PA (USA)

Studienbegleitende Tätigkeiten

1999 - 2003	Werkstudent bei der Schott Glas AG
2000 - 2003	Wissenschaftl. Hilfskraft am Institut für Mathematik der Johannes Gutenberg-Univerität Mainz, Übungsgruppenleiter
2003 - 2008	Wissenschaftl. Mitarbeiter in der Arbeitsgruppe Funktionalanalysis am Institut für Mathematik der Johannes Gutenberg-Universität Mainz

Stipendien

03/2003 - 07/2003	Förderstipendium der Johannes Gutenberg-Universität Mainz
11/2005 - 03/2006	DAAD-Stipendium für Doktoranden

Ersatzdienst

07/1997 - 07/1998	Ziviler Ersatzdienst beim Amt für Grünanlagen und Naturschutz in Mainz

Mainz, 30. März 2008