Integrable systems and a moduli space for (1,6)-polarised abelian surfaces

Dissertation

zur Erlangung des Grades

"Doktor der Naturwissenschaften"

am Fachbereich Physik, Mathematik und Informatik

der Johannes Gutenberg-Universität

in Mainz

Laura Biroth
geb. in Bad Homburg

Mainz, den 12.09.2019

Prüfungsdatum: 28. November 2019

Abstract

A Hamiltonian system is a type of differential equation used in physics to describe the evolution of a mechanical system like a particle in a potential. Certain particularly well-behaved Hamiltonian systems are called *integrable.* For us an integrable system on $\mathbb{C}^{2n}$ is simply a set of n independent Poisson-commuting polynomials in $2n$ variables. In case the system is *algebraically completely integrable* the fibres of the induced map are affine parts of abelian varieties.
In this thesis we study a projective model for the moduli-space of embedded (1,6)-polarised abelian surfaces first described by Gross and Popescu. We analyse its discriminant locus, the degenerations occurring, the form of the equations describing each surface and the automorphisms of this moduli space.
In the last chapter we compute the cohomology of some quasi-homogeneous integrable systems on $\mathbb{C}^4$.

Zusammenfassung

Ein Hamiltonsches System ist ein Typ von Differentialgleichung, der in der Physik benutzt wird um mechanische Systeme, wie zum Beispiel eine Punktmasse in einem Potential, zu beschreiben. Eine bestimmte Klasse Hamiltonscher Systeme, die sich besonders gut verhält, heißt *integrabel*. Für uns ist ein integrables System auf $\mathbb{C}^{2n}$ einfach eine Menge von n unabhängigen Poisson-kommutierenden Polynomen in $2n$ Variablen. Im Fall dass das System *algebraisch vollständig integrabel* ist, sind die Fasern der induzierten Abbildung affine Teile von abelschen Varietäten.
In dieser Arbeit untersuchen wir ein projektives Model für den Modulraum von eingebetteten (1,6)-polarisierten abelschen Flächen, der erstmals von Gross und Popescu beschrieben wurde. Wir analysieren seine Diskriminante, die auftretenden Entartungen, die Form der Gleichungen, die jede Fläche beschreiben, und die Automorphismen dieses Modulraums.
Im letzten Kapitel berechnen wir die Kohomologie einiger quasi-homogener integrabler Systeme auf $\mathbb{C}^4$.

Contents

Introduction

Hamiltonian systems are differential equations that describe the movement of a mechanical system with conservative forces. Written in Hamilton's canonical form, they take the form of a first order system determined by a Hamiltonian function H, that describes the total energy of the system. Already of great interest are Hamiltonians of the form

$$H(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{2} \sum p_i^2 + V(\boldsymbol{q}),$$

that consist of the sum of kinetic and potential energy, describing a point particle in a potential field V. In the context of differential equations, to integrate an equation means to solve it from given initial conditions. Liouville's insight was that this can be done explicitly, if one can find enough constants of motion (c.f. [Lio55] or [Arn89] p.271 ff). In such a case we call the system *integrable (in the sense of Liouville).* In the real domain, the level sets of these constants of motion are, if they are compact, diffeomorphic to real tori and the motion on it is quasi-periodic given by a linear vector field. This is also known as the Arnold-Liouville theorem.

In the complex domain, the best one can hope for is that the level sets are generically affine parts of complex tori in projective space, i.e. abelian varieties. A Hamiltonian system is called *algebraically completely integrable* or *a.c.i.*, if this is the case and furthermore the Hamiltonian vector fields extend and are translation invariant when restricted to these tori. The geodesic flow on a general ellipsoid was recognised to be a.c.i. by Jacobi, which leads to a description of the motions in terms of theta-functions. Sofia Kowalewskaya discovered a powerful method to find a.c.i systems, that was developed further and put in a modern perspective by Adler and van Moerbeke. Although not all integrable systems are a.c.i. (c.f. Section 1.5), the geometry of abelian varieties and the geometry of integrable systems are closely interrelated ([Mum84]).

That integrability is a highly non-generic property of a Hamiltonian H was already known to Liouville. Maria Przybylska has shown in several papers (for example [Prz07]) that in a generic class of potentials V of degree $k \geq 3$ in n variables only a *finite number* (up to change of coordinates) is integrable and listed these generic integrable potentials for small values of n and k.

One of her integrable systems with a potential of degree 3 was originally described by Dorizzi, Grammaticos and Ramani in [DGR82]. Semmel and van Straten have shown in [SvS13] that this so-called *DGR-system* is a.c.i. and its fibres complete to $(1, 6)$-polarised abelian surfaces.

Abelian varieties are complex tori which can be embedded into projective space and have a long history. Their first instances studied were elliptic curves. In the first half of the nineteenth century this was extended by Abel, Jacobi and others to dimension two, i.e. abelian surfaces. In 1857 Riemann laid the foundations for further work on abelian varieties in dimension > 1, introducing the Riemann bilinear relations and Riemann theta functions (c.f. [Rie57]). By the end of the 19th century, mathematicians had begun to use geometric methods in the study of abelian functions. Eventually, in the 1920s, Lefschetz laid the basis for the study of abelian functions in terms of complex tori. He also appears to be the first to use the name "abelian variety". It was André Weil in the 1940s who gave the subject its modern foundations in the language of algebraic geometry. Around 1967 David Mumford developed a new theory of the equations defining abelian varieties (see for example [Mum83], [Mum84], [Mum91]).

There is a coarse moduli space of abelian abelian varieties of given type and dimension with or without different additional structures (c.f. Chapter 3 and Section 7.4). Usually these are described as quotients of Siegel upper half space by certain subgroups of the paramodular group. Igusa has shown that all these moduli spaces are quasi-projective varieties, which in general posses quotient singularities. The question of (uni) rationality of these moduli was answered positively for abelian surfaces of type $(1, d)$ for some small values of d, while Gritsenko has shown in 1994 ([Gri94]) that the moduli space $\mathcal{A}_d$ of $(1, d)$-polarised abelian surfaces is not unirational for $d \geq 13$ and $d \neq 14, 15, 16, 20, 24, 30, 36$. At that time it was not known if $\mathcal{A}_6$ was unirational (or even rational) or not. In their 2001 paper [GP01] Gross and Popescu give a nice projective model for the moduli space $\mathcal{A}_6^{emb}$ of embedded $(1, 6)$-polarised abelian surfaces which is rational; in fact it can be identified with a quadric.

The central question around which this thesis arose is the following:

Is it possible to identify the system of Dorizzi, Grammaticos and Ramani inside the moduli space of (1,6)-polarised surfaces described by Gross and Popescu?

To achieve such a goal, a deeper study of the projective geometric properties of such abelian surfaces is necessary. An explicit form of the coordinate transformation would lead to the solution of the Hamilton equations for the DGR-potential in terms of theta-functions belonging to the $(1, 6)$-polarised abelian surface.
In this thesis we give at least a partial answer to the above question in Section 5.7.

In Chapter 1 I give an introduction to Hamiltonian systems, integrability in the sense of Liouville, algebraic complete integrability and some finiteness results about polynomial integrable systems proven by Przybylska et al.

In Chapter 2 I give an introduction to abelian varieties over $\mathbb{C}$, their line bundles and theta functions. In Section 2.7.3 I introduce the Heisenberg group acting on each abelian variety, which plays a crucial role for the understanding of the embedding of abelian varieties.

In Chapter 3 I describe the algebraic information encoding a Heisenberg invariant embedding of an abelian variety and describe the moduli space of Heisenberg invariantly embedded abelian varieties of a given type as a quotient of Siegel upper half space. Many authors are quite imprecise about this point.

In Chapter 4 I give a short overview of what is known about $(1, d)$-polarised abelian varieties and their moduli spaces for $d = 1, \ldots, 5$.

Chapter 5 is the central part of this thesis. Here I examine in detail the projective model of the moduli space of Heisenberg invariantly embedded abelian surfaces of type $(1, 6)$ introduced by Gross and Popescu in [GP01]. I analyse its discriminant locus, its stratification and the degenerations occurring, the form of the equations describing each surface and the automorphisms of this moduli space. Furthermore I identify two subfamilies described by Hulek and Ranestad in [HR00] and the family described by Semmel and van Straten in [SvS13], related to the Dorizzi-Grammaticos-Ramani integrable system.

Chapter 6 is a digression to understand the topology of some particular integrable systems. Here I use a complex introduced by Garay and van Straten in [GvS10] to compute the cohomology of the smooth fibres of four polynomial integrable systems. Because we have no criterion when calculations are finished, this only leads us to some conjectures about the cohomology modules. This chapter is only loosely related to the others.

The appendix (Chapter 7) contains some further details about complex tori and abelian varieties and is included for the convenience of the reader. In particular it contains a more detailed version of the Appell-Humbert-Theorem that describes their line bundles. Section 7.3 is devoted to the (rational) map induced by a line bundle, and contains some criteria when it is base-point free or gives an embedding, and make some statements about the equations describing its image. In Section 7.4 contains known results about the moduli spaces of abelian varieties with several additional structures.

1 Introduction to integrable systems

1.1 Classical mechanics

According to Newton, the equation of motion of a point particle of mass m, moving in n-dimensional space $\mathbb{R}^n$ is given by

$$\boldsymbol{F} = m \cdot \boldsymbol{a},$$

where $\boldsymbol{F}$ denotes the force acting on the particle, and $\boldsymbol{a} = \ddot{\boldsymbol{q}}$ denotes its acceleration. If the particle moves in a conservative force field given by a potential function $V(\boldsymbol{q})$, then the force is given by

$$\boldsymbol{F} = -\frac{\partial V}{\partial \boldsymbol{q}}.$$

When we introduce the momentum $\boldsymbol{p} = m\dot{\boldsymbol{q}}$ and the Hamiltonian

$$H(\boldsymbol{p}, \boldsymbol{q}) = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{q}),$$

the equations of motion can be rewritten as

$$\begin{aligned} \dot{\boldsymbol{q}} &= \frac{\partial H}{\partial \boldsymbol{p}}, \\ \dot{\boldsymbol{p}} &= -\frac{\partial H}{\partial \boldsymbol{q}}. \end{aligned} \tag{1.1}$$

Now for any other quantity $K(\boldsymbol{p}, \boldsymbol{q})$ depending on positions and momenta, we have

$$\begin{aligned} \frac{\partial K}{\partial t} &= \sum_{i=1}^{n} \frac{\partial K}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial K}{\partial p_i}\frac{\partial p_i}{\partial t} \\ &\overset{(1.1)}{=} \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial K}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial K}{\partial p_i} =: \{H, K\}. \end{aligned}$$

With this notion, called the *Poisson bracket*, we can rewrite (1.1) as

$$\begin{aligned} \dot{\boldsymbol{q}} &= \{H, \boldsymbol{q}\} \\ \dot{\boldsymbol{p}} &= \{H, \boldsymbol{p}\}. \end{aligned} \tag{1.2}$$

These are called *Hamilton's equations of motion.*

As an example, consider in a radial symmetric potential in the plane, i.e. a potential depending only on $r^2 = q_1^2 + q_2^2$. In this case the angular momentum $L = p_1 q_2 - p_2 q_1$ of a particle following (1.2) is constant:

$$\begin{aligned}\frac{\partial L}{\partial t} &= \{H, L\} \\ &= -p_1 p_2 - \frac{\partial H}{\partial q_1} q_2 + p_2 p_1 + \frac{\partial H}{\partial q_2} q_1 \\ &= -p_1 p_2 - 2q_1 \frac{\partial H}{\partial r^2} q_2 + p_2 p_1 + 2q_2 \frac{\partial H}{\partial r^2} q_1 \\ &= 0.\end{aligned}$$

We call L an *constant of motion*, *(first) integral* or *conserved quantity.*
In the case that we have *as many first integrals as possible* (see Section 1.2) we speak of an *integrable system* because Equation (1.2) can be solved by using only algebraic operations, the implicit function theorem and integration.

1.2 Poisson rings

Now we want to establish a more abstract notion of an integrable system. For this we need the notion of a *Poisson ring.*

Definition 1.1. A *Poisson ring* is a commutative ring with an additional binary operation $\{\cdot,\cdot\}\colon R \times R \longrightarrow R$ (called *Poisson bracket* or *Poisson structure*) satisfying the following axioms:

- $\{\cdot,\cdot\}$ is skew-symmetric, i.e.

  $$\{f, g\} = -\{g, f\}.$$

- $\{\cdot,\cdot\}$ is a derivation in both variables, i.e. it is bilinear and satisfies the Leibniz identity

  $$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

- $\{\cdot,\cdot\}$ satisfies the Jacobi identity

  $$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

  So the Poisson bracket defines a Lie algebra structure on R.

Definition 1.2. For each $f \in R$, we denote by X_f the derivation of R defined by $\{f, \cdot\}$, called *hamiltonian vector field* associated to f.
Functions f for which $X_f = 0$ are called *Casimir elements.* Two elements $f, g \in R$ with $\{f, g\} = 0$ are said to *Poisson commute* or be *in involution.*

The map

$$\begin{aligned} R &\longrightarrow Der(R) \\ f &\longmapsto X_f \end{aligned}$$

is a morphism from the Lie algebra R (equipped with the Poisson bracket) to the Lie algebra of derivations of R (w.r.t. the commutator).
Usually a Poisson ring consists of some kind of functions on some underlying space.

Definition 1.3. A *Poisson manifold* is a smooth manifold M together with Poisson structure on $\mathcal{C}^\infty(M)$.

We can make this definition also in the category of complex manifolds (replacing $\mathcal{C}^\infty(M)$ by the algebra of holomorphic or meromorphic functions on M), or complex affine varieties, that are possibly singular (replacing $\mathcal{C}^\infty(M)$ by the algebra of regular functions $\mathcal{O}_M(M)$).

Definition 1.4. For $m \in M$, $\{X_H(m) \mid H \in \mathcal{C}^\infty(M)\}$ is an (even-dimensional) vector space. We denote its dimension by $\mathrm{rk}_m\{\cdot,\cdot\}$ and call it the *rank of* $\{\cdot,\cdot\}$ *at* m. The *rank of* $\{\cdot,\cdot\}$, denoted by $\mathrm{rk}\{\cdot,\cdot\}$, is the maximum of all ranks $\mathrm{rk}_m\{\cdot,\cdot\}$ with $m \in M$. For $s \in \mathbb{N}$, we denote by $M_{(s)}$ the subset $\{m \in M \mid \mathrm{rk}_m\{\cdot,\cdot\} \geq 2s\}$. A Poisson structure of constant rank is called a *regular* Poisson structure.
In this case the vector field X_H for $H \in \mathcal{C}^\infty(M)$ is defined by $X_H := \{H,\cdot\}$ and an integrable system on a Poisson manifold of rank $2r$ is defined to be a set of $s = \dim(M) - r$ independent functions in involution.

Theorem 1.5 (Darboux)**.** *If the rank of $\{\cdot,\cdot\}$ is constant with value $2r$ in the neighbourhood of a point $m \in M$, then there exist local coordinates $(q_1,\dots,q_r,p_1,\dots,p_r, z_1,\dots,z_s)$ around m such that the Poisson bracket takes the following canonical form:*

$$\{q_i,q_j\} = \{p_i,p_j\} = \{q_i,z_k\} = \{p_i,z_k\} = \{z_k,z_l\} = 0 \quad \{q_i,p_j\} = \delta_{ij}$$

for all $1 \leq i,j \leq r$ and $1 \leq k,l \leq s$.

For us the key case is the symplectic case, i.e. a regular Poisson structure of rank r where $\dim(M) = 2r$. This means $s = 0$, i.e. there are no Casimir elements z_i. In this case the Poisson bracket can and is often described using a closed, non-degenerate 2-form ω on M. Then the vector field associated to a smooth function $H\colon M \longrightarrow \mathbb{R}$ is given by

$$\omega_x(Y, X_H(x)) = (dH)_x(Y) \text{ for all } Y \in T_xM$$

and the Poisson-bracket can be described as

$$\{f,g\} := X_f \cdot g = \omega(X_f, X_g).$$

Example 1.6. The Poisson bracket induced by the symplectic form $\omega = \sum dp_i \wedge dq_i$ on $\mathbb{R}^n \times \mathbb{R}^n$ is just the standard Poisson bracket

$$\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}. \tag{1.3}$$

A general reference for Poisson varieties is for example [Van96].

1.3 Integrable systems

Theorem 1.7 (Liouville, [Van08]). *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold of rank $2r$, $s = \dim(M) - r$. Suppose we are given a point $m \in M$ and s functions $f_1, \ldots, f_s$ in involution with differentials $df_1, \ldots, df_s$ independent at m. Then the integral curve, starting at m, of each of the Hamiltonian vector fields X_{f_i} can be obtained locally by using only algebraic operations, the implicit function theorem and integration.*

This leads us to the following definition:

Definition 1.8. Let us call an s-tuple of functions $F = (f_1, \ldots, f_s)$ on M *independent* when the open subset $\mathcal{U}_F := \{m \in M \mid df_1(m) \wedge \cdots \wedge df_s(m) \neq 0\}$ is dense in M. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold of rank $2r$, $s = \dim(M) - r$. Then an s-tuple $F = (f_1, \ldots, f_s)$ of involutive and independent functions on M is called *(Liouville) integrable* and $(M, \{\cdot,\cdot\}, F)$ is a *(Liouville) integrable system*. The vector fields X_{f_i} are then called *integrable* vector fields and the map F is called the *momentum map*.

Theorem 1.9 (Liouville theorem: Liouville, Mineur, Arnold, [Van08]). *Let $(M, \{\cdot,\cdot\}, F)$ be a real integrable system, where $F = (f_1, \ldots, f_s)$, and consider a point $m \in \mathcal{U}_F \cap M_{(r)}$, where $2r$ denotes the rank of $\{\cdot,\cdot\}$. Denote by F_m the connected component of $F^{-1}(F(m)) \cap \mathcal{U}_F \cap M_{(r)}$ containing m. Then:*

1. *If F_m is compact, then it is diffeomorphic to a torus $T^r = (\mathbb{R}/\mathbb{Z})^r$.*
2. *If F_m is not compact, but the flow of each of the vector fields X_{f_i} is complete on F_m, then F_m is diffeomorphic to a cylinder $\mathbb{R}^{r-q} \times T^q$, $(0 \leq q < r)$.*

In both cases the diffeomorphism can be chosen in such a way that the vector fields $X_{f_1}, \ldots, X_{f_s}$ are mapped to linear (i.e. translation invariant) vector fields.

A proof of this can be found for example in [Arn89].

1.4 First examples

Here are three examples of integrable systems on a symplectic manifold:

Example 1.10. 1. The Hamiltonian

$$H = \frac{1}{2}\boldsymbol{p}^2 + \frac{1}{2}\boldsymbol{q}^2$$

gives the differential system

$$\dot{\boldsymbol{q}} = \boldsymbol{p}, \quad \dot{\boldsymbol{p}} = -\boldsymbol{q}$$

describing n *independent harmonic oscillators*. It has n independent Poisson commuting first integrals, namely

$$f_i = \frac{1}{2}p_i^2 + \frac{1}{2}q_i^2, \qquad 1 \leq i \leq n,$$

the energy of the ith oscillator, hence it is completely integrable.

2. *The spherical pendulum:*

   The restriction of the symplectic form on $\mathbb{R}^3 \times \mathbb{R}^3$ to

   $$TS^2 = \{(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|\boldsymbol{q}\|^2 = 1 \text{ and } \boldsymbol{q} \cdot \boldsymbol{p} = 0\}$$

   continues to be a symplectic form.

   The Hamiltonian

   $$H = \frac{1}{2}\|\boldsymbol{p}\|^2 - \boldsymbol{\Gamma} \cdot \boldsymbol{q}$$

   then defines the differential system

   $$\dot{\boldsymbol{q}} = \boldsymbol{p}, \quad \dot{\boldsymbol{p}} = \boldsymbol{\Gamma} - (\boldsymbol{q} \cdot \boldsymbol{\Gamma} + \|\boldsymbol{p}\|^2)\boldsymbol{q}.$$

   It describes the motion of a mass moving on the surface of a sphere under the gravitational force $\boldsymbol{\Gamma}$. Besides the Hamiltonian itself, it has the first integral

   $$K = (\boldsymbol{p} \times \boldsymbol{q}) \cdot \boldsymbol{\Gamma},$$

   the angular momentum with respect to the axis $\boldsymbol{\Gamma}$. Thus, the spherical pendulum also is an integrable system.

   More details about this example can be found in [Knö86].

3. A rigid body with mass 1, centre of gravity $\boldsymbol{G}$ fixed at point $\boldsymbol{O}$ in a constant gravitational field, can be described as follows:

   Use a frame attached to the rigid body. Then the constant gravitational field becomes a vector $\boldsymbol{\Gamma}(t)$ depending on time. Denote by $\boldsymbol{\Omega}$ the instantaneous rotation, and by $\boldsymbol{M}$ the angular momentum of the solid. They are connected via the relation $\boldsymbol{M} = J \cdot \boldsymbol{\Omega}$, where J is the matrix of inertia of the solid, a real, constant, symmetric matrix.

   Then the total energy of the solid is

   $$H(\boldsymbol{\Gamma}, \boldsymbol{M}) = \frac{1}{2}\boldsymbol{M} \cdot \boldsymbol{\Omega} + \boldsymbol{\Gamma} \cdot \boldsymbol{L}$$

   (with $\boldsymbol{L} := \boldsymbol{O} - \boldsymbol{G}$) leading to the equations of motion

   $$\begin{aligned} \dot{\boldsymbol{\Gamma}} &= \boldsymbol{\Gamma} \times \boldsymbol{\Omega} \\ \dot{\boldsymbol{M}} &= \boldsymbol{M} \times \boldsymbol{\Omega} + \boldsymbol{\Gamma} \times \boldsymbol{L}. \end{aligned}$$

   Such a rigid body is called *Kovalevskaya top* if the matrix of inertia is given by $J = \left(\begin{smallmatrix} 2 & & \\ & 2 & \\ & & 1 \end{smallmatrix}\right)$ w.r.t. an orthonormal basis whose first vector is collinear with $\boldsymbol{L}$.

   If we fix $\boldsymbol{L} = \left(\begin{smallmatrix} -1 \\ 0 \\ 0 \end{smallmatrix}\right)$ and write in the same basis

   $$\boldsymbol{\Omega} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad \boldsymbol{\Gamma} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix},$$

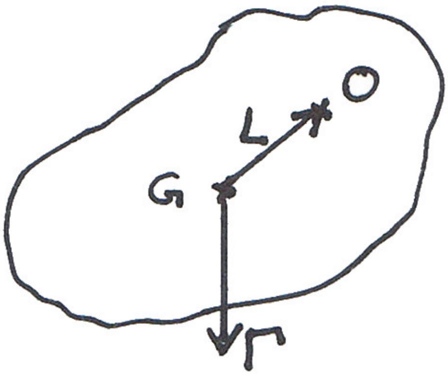


Figure 1.1: The rigid body

then

$$K = |(p + iq)^2 + (\gamma_1 + i\gamma_2)|^2$$

is a first integral of the system, called the *Kovalevskaya integral*.

Some more details on this example can be found in [Aud08], an extensive treatise in [Aud96] or [Kov89].

1.5 Algebraic integrability

We now discuss the notion of algebraic integrability. The idea is to consider complex integrable systems, whose (complex) geometry is the best possible analogue of the (real) geometry that appears in the Arnold-Liouville theorem. This idea goes back to Kovalevskaya ([Kov89]) and was revived by Adler and van Moerbeke ([AvMV04]).

Definition 1.11. Let $(M, \{\cdot,\cdot\}, F)$ be an integrable system, where M is a non-singular affine variety and $F = (f_1, \ldots, f_s)$. We say that $(M, \{\cdot,\cdot\}, F)$ is an *algebraic completely integrable* system or an *a.c.i.* system if for generic $m \in M$ the invariant manifold F_m is an affine part of an abelian variety and the Hamiltonian vector fields X_{f_i} are translation invariant, when restricted to these tori.

In our examples from the last section we can see that not every real integrable system is a.c.i. after complexification:
For a single harmonic oscillator the generic invariant manifold looks like

$$\{(p, q) \in \mathbb{C}^2 \mid p^2 + q^2 = m\}, \quad m \neq 0.$$

While this describes a circle over $\mathbb{R}$, over $\mathbb{C}$ it is isomorphic to

$$\{(p', q') \in \mathbb{C}^2 \mid p' \cdot q' = m\}, \quad m \neq 0 \quad (\text{set } p' = p + iq \text{ and } q' = p - iq),$$

i.e. isomorphic to $\mathbb{C}^*$ (via projection to one variable) which is a complex abelian Lie group but not an open subset of an abelian variety.
In the case of the spherical pendulum the generic fibre is isomorphic to $\mathbb{C}^* \times A$ with A an affine part of an elliptic curve, while the Kovalevskaya top is indeed a.c.i..
So it might be the "better" notion to talk about *semi-abelian varieties* (i.e. extensions of abelian varieties by some $(\mathbb{C}^*)^r$), and *generalised algebraic complete integrability*, where the invariant manifold F_m only has to be affine part of a semi-abelian variety (c.f. the PhD thesis of Michael Semmel [Sem12] for more details about this topic).

The example of the Kovalevskaya top was found by a method developed by Kovalevskaya ([Kov89]) to show that a given Liouville integrable system $\boldsymbol{F} = (H = f_1(x_1, \ldots, x_{2n}), \ldots,$
$f_n(x_1, \ldots, x_{2n}))$ is a.c.i.. A modern treatment can be found in [AvMV04].
Reproducing the proof of the Liouville theorem over $\mathbb{C}$ one arrives (in the symplectic case) at

Theorem 1.12 (Complex Liouville theorem, [Van08])**.** *Let $A \in \mathbb{C}^{2n}$ be a non-singular affine variety of dimension n which supports n holomorphic vector fields $V_1, \ldots, V_n$ and let $\varphi\colon A \longrightarrow \mathbb{C}^N \subset \mathbb{P}^N$ be an embedding. We define $\Delta = \overline{\varphi(A)} \setminus \varphi(A)$ and denote the union of all irreducible components of Δ of dimension $r-1$ by Δ'. Suppose the following:*

1. *The vector fields commute pairwise, $[V_i, V_j] = 0$, for $1 \leq i, j \leq n$.*
2. *At every point $m \in A$ the vector fields $V_1, \ldots, V_n$ are independent.*
3. *The vector field $\varphi_* V_1$ extends to a vector field $\overline{V_1}$ which is holomorphic on a neighbourhood of Δ' in $\mathbb{P}^N$.*
4. *The integral curves of $\overline{V_1}$ that start at points $m \in \Delta'$ go immediately into $\varphi(A)$.*

Then $\overline{\varphi(A)}$ is an abelian variety of dimension r and the vector fields $\varphi_* V_1, \ldots, \varphi_* V_r$ extend to holomorphic (hence linear) vector fields on $\overline{\varphi(A)}$. Moreover, $\Delta' = \Delta$.

This theorem can be used to show algebraic complete integrability of a given weighted homogeneous Liouville integrable system $\boldsymbol{F} = (H = f_1(x_1, \ldots, x_{2n}), \ldots,$ $f_n(x_1, \ldots, x_{2n}))$ in the following steps:
Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_{2n})$ be a weight-vector such that $f_1, \ldots, f_n$ are quasi-homogeneous with respect to these weights.

1. Try and find Laurent series solutions

$$x_i(t) = \sum_{k=0}^{\infty} x_i^{(k)} t^{k-\nu_i}, \quad i = 1, \ldots, 2n$$

solving the differential equation

$$\dot{\boldsymbol{x}} = \{H, \boldsymbol{x}\}.$$

This gives polynomial equations for the $x_i^{(0)}$, namely

$$\nu_i x_i^{(0)} + g_i(x_1^{(0)}, \ldots, x_{2n}^{(0)}) = 0, \quad i = 1, \ldots, 2n \tag{1.4}$$

where $g_i = \{H, x_i\}$. The zero locus $\mathcal{I}$ of (1.4) is called the *indicial locus* of $\boldsymbol{F}$. Decompose $\mathcal{I}$ into irreducible components (for $n = 2$ these will be just points). Now for each $m \in \mathcal{I}$ compute the *Kovalevskaya matrix*

$$K(m)_{ij} = \frac{\partial g_i}{\partial x_j}(m) + \nu_i \delta_{ij}, \quad 1 \leq i, j \leq 2n.$$

We are only interested in the *principal balances* of $\boldsymbol{F}$, i.e. the solutions corresponding to the irreducible components $\mathcal{I}^{(1)}, \ldots, \mathcal{I}^{(d)}$ of $\mathcal{I}$ for which:

- $K(m)$ has $2n-1$ non-negative integer eigenvalues (counted with algebraic multiplicities).
- $K(m)$ is diagonalisable for all $m \in \mathcal{I}^{(j)}$ and
- $\mathcal{I}$ is non-singular at all point of $\mathcal{I}^{(j)}$.

For each principal balance $\mathcal{I}^{(j)}$ compute the first $k_p + 1$ terms (where k_p is the maximal eigenvalue of $K(m)$, $m \in \mathcal{I}^{(j)}$) of the series $\boldsymbol{x}(t; \mathcal{I}^{(j)})$ starting at $\mathcal{I}^{(j)}$ by substituting the Laurent polynomials (up to the term $x^{(k_p)}$) in the differential equation $\dot{\boldsymbol{x}} = \{H, \boldsymbol{x}\}$. There will be $2n-1$ free parameters showing up in the series $\boldsymbol{x}(t; \mathcal{I}^{(j)})$. Each (formal) Laurent-series obtained this way will automatically be convergent.

2. To embed the fibre $\boldsymbol{F}_{\boldsymbol{c}} := \boldsymbol{F}^{-1}(\boldsymbol{c})$ into projective space, choose a pole vector $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_d)$ with ρ_i non-negative integers ($\boldsymbol{\rho} = (3, 0, \ldots, 0)$ will always be enough). Compute a $\mathbb{C}$-basis $z_0 = 1, \ldots, z_N$ of the vector space

$$\mathcal{Z}_\rho := \{z \in \mathbb{C}[x_1, \ldots, x_{2n}] \mid \operatorname{ord}_{t=0} z(\boldsymbol{x}(t; \mathcal{I}^{(j)})) \geq -\rho_i \text{ for } 1 \leq j \leq d\}$$

organising computations by weight. For sufficiently big $\boldsymbol{\rho}$ we obtain an isomorphic embedding

$$\begin{aligned} \varphi_{\boldsymbol{c}} \colon \boldsymbol{F}_{\boldsymbol{c}} &\longrightarrow \mathbb{P}^N \\ \boldsymbol{x} &\longmapsto (1 : z_1(\boldsymbol{x}) : \cdots : z_N(\boldsymbol{x})). \end{aligned}$$

Define $\mathcal{A}_{\boldsymbol{c}} := \overline{\varphi_{\boldsymbol{c}}(\boldsymbol{F}_{\boldsymbol{c}})}$, $\Delta_{\boldsymbol{c}} := \overline{\varphi_{\boldsymbol{c}}(\boldsymbol{F}_{\boldsymbol{c}})} \setminus \varphi_{\boldsymbol{c}}(\boldsymbol{F}_{\boldsymbol{c}})$ and $\Delta'_{\boldsymbol{c}}$ as the union of all irreducible components of $\Delta_{\boldsymbol{c}}$ of dimension $n-1$.

3. Prove that the vector field $\varphi_* X_H$ extends to a holomorphic vector field on $\mathbb{P}^N$ by showing that the *Wronskians* $W(z_i, z_j) := \dot{z}_i z_j - z_i \dot{z}_j$ are expressible as a quadratic polynomial in the z_k

$$W(z_i, z_j) = \sum_{k,l=0}^{N} \alpha_{ij}^{kl} z_k z_l$$

(where each α_{ij}^{kl} depends on the values of $\boldsymbol{c}$ only) either by explicit computation or by more abstract arguments if possible. It is enough to do this in two affine charts $z_i \neq 0$.

4. Show that the integral curves of X_H starting at points $m \in \Delta'$ do immediately go into $\varphi_c(\boldsymbol{F}_c)$. More details on this step can be found in [AvMV04], Chapter 7.

If one succeeds doing all of these steps, then $\mathcal{A}_{\boldsymbol{c}}$ is an abelian variety and all the the vector fields $\varphi_* X_{f_i}$ extend to linear vector fields on $\mathcal{A}_{\boldsymbol{c}}$ i.e. the system is algebraic completely integrable by the complex Liouville theorem.

This is the method Kovalevskaya essentially used to construct her top (c.f. Section 1.4).

1.6 About (non-)integrability

One question arising naturally is the following: Given a Hamiltonian H on M , can it be completed to an integrable system $f_1 = H, f_2, \dots, f_n$? In this case, H is called *integrable*, and *non-integrable* otherwise.
While it is in principle clear how to prove integrability (just write down the necessary first integrals), it is not obvious how to prove non-integrability.
There are powerful methods of Ziglin ([Zig82], [Zig83]) considering the monodromy group of variational equations along a particular solution and generalisations by Morales-Ruiz and Ramis ([MR99], [MRR01]) using differential Galois theory that can be used effectively to prove non-integrability in many cases.

We are especially interested in polynomial integrable systems on $\mathbb{C}^{2n}$. Such polynomial integrable systems are however very rare.
Maria Przybylska studied (for example in [Prz07]) Hamiltonians describing a particle in a potential

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}). \tag{1.5}$$

A *Darboux point* is a point $\boldsymbol{d} \in \mathbb{P}_{\mathbb{C}}^{n-1}$, such that $V'(\boldsymbol{d}) = \boldsymbol{d}$ in $\mathbb{P}^{n-1}$. For each of these Darboux points we choose a representative in $\boldsymbol{d} \in \mathbb{C}^n$ such that $V'(\boldsymbol{d}) = \boldsymbol{d}$ in $\mathbb{C}^n$. In the case of a homogeneous potential the Kovalevskaya matrix is given by

$$K(\boldsymbol{d}) = V''(\boldsymbol{d}) - \mathrm{id}\,.$$

Denote its $n-1$ nontrivial eigenvalues by $\Lambda_1(\boldsymbol{d}), \dots, \Lambda_{n-1}(\boldsymbol{d})$ (the nth so-called trivial eigenvalue is always $\Lambda_n(\boldsymbol{d}) = k - 1$). Let $\lambda_i(d) := \Lambda_i(\boldsymbol{d}) + 1$ be the corresponding eigenvalues of the Hessian $V''(\boldsymbol{d})$. We call Λ_i the *Kovalevskaya exponents* of V.

She proves that potentials with the maximal number of exactly $\frac{(k-1)^n-1}{k-2}$ Darboux points and all the Kovalevskaya exponents different from zero form a non-empty open set in the space of all homogeneous polynomials of degree k.
In the following she identifies potentials that only differ by an orthogonal linear transformation of coordinates and states the following theorem:

Theorem 1.13 (Theorem 4 in [Prz07]). *Among Hamiltonian systems given by (1.5) with homogeneous potentials of fixed degree $k > 2$ admitting the maximal number of Darboux points only a finite number (of equivalence classes) is integrable.*

In her proof the following theorem of Morales-Ruiz and Ramis is needed:

Theorem 1.14 ([MRR01]). *If the Hamiltonian system given by (1.5) with the polynomial homogeneous potential $V(q)$ of degree $k > 2$ is meromorphically integrable in the Liouville sense, then values of (k, λ_i) for $i = 1, \dots, n$ belong to the following list*

1. $(k, \frac{k}{2}p(p-1)+p)$
2. $(k, \frac{k-1}{2k}+p(p+1)\frac{k}{2})$
3. $(3, \frac{1}{6}(1+3p)^2 - \frac{1}{24})$
4. $(3, \frac{3}{32}(1+4p)^2 - \frac{1}{24})$
5. $(3, \frac{3}{50}(1+5p)^2 - \frac{1}{24})$
6. $(3, \frac{3}{50}(2+5p)^2 - \frac{1}{24})$
7. $(4, \frac{2}{9}(1+3p)^2 - \frac{1}{8})$
8. $(5, \frac{5}{18}(1+3p)^2 - \frac{9}{40})$
9. $(5, \frac{1}{10}(2+5p)^2 - \frac{9}{40})$

where p is an integer.

She derives several relations among the $\boldsymbol{\Lambda}(\boldsymbol{d})$ for all Darboux points $\boldsymbol{d}$ of the potential, in particular

$$\sum_{\boldsymbol{d}\in\mathcal{D}_V}\sum_{i=1}^{n-1}\frac{1}{\Lambda_i(\boldsymbol{d})} = -\frac{(k-1)^n - n(k-2) - 1}{(k-2)^2} \tag{1.6}$$

where $\mathcal{D}_V$ is the set of all Darboux points of V and concludes that there are only finitely many possibilities to solve equation (1.6) with values from Theorem 1.14.
Now she claims that these finitely many solutions correspond only to a finite number of (equivalence classes of) potentials, although she only explains how to reconstruct the potential from the spectra of $K(\boldsymbol{d})$, $\boldsymbol{d} \in \mathcal{D}_V$ in the cases $n = k = 3$ in [Prz09a] and for $n = 2$ in [MP05].

The usefulness of this result is stressed by the following fact: A polynomial potential $V(\boldsymbol{q})$ can be written as a sum of homogeneous components $V(\boldsymbol{q}) = V_{min}(\boldsymbol{q}) + \cdots + V_{max}(\boldsymbol{q})$ where $V_{min}(\boldsymbol{q})$ and $V_{max}(\boldsymbol{q})$ are the homogeneous components of the lowest and the highest order, respectively. By scaling the coordinates (i.e. replacing $\boldsymbol{q}$ by $\lambda\boldsymbol{q}$) and looking at the limits $\lambda \to 0$ and $\lambda \to \infty$, one sees: If the Hamiltonian (1.5) with polynomial potential $V(\boldsymbol{q})$ is integrable, then so are the homogeneous polynomial potentials $V_{min}(\boldsymbol{q})$ and $V_{max}(\boldsymbol{q})$.

In the following papers she gives lists of all integrable potentials for certain values of n and k:
In [MP04] Przybylska and Maciejewski describe all integrable Hamiltonians of this type for $n = 2$, $k = 3$, which are up to coordinate transformation the following five potentials:

$$\begin{aligned} V_1 &= q_1^3 \\ V_2 &= \frac{1}{3}q_1^3 + \frac{1}{3}q_2^3 \\ V_3 &= \frac{1}{2}q_1^2 q_2 + q_2^3 \\ V_4 &= \frac{1}{2}q_1^2 q_2 + \frac{8}{3}q_2^3 \\ V_5 &= \frac{\sqrt{-3}}{18}q_1^3 + \frac{1}{2}q_1^2 q_2 + q_2^3 \end{aligned}$$

which actually all had been known for a long time. Potential V_1 depends on one variable, potentials V_2 to V_4 are of Hénon-Heiles type. The fifth one was discovered over thirty years ago by Dorizzi, Grammaticos and Ramani in 1982 ([DGR82]). Let us call this one the DGR-system in the following. We will revisit potentials V_3 to V_5 in Chapter 6 after a slight change of coordinates.
In [Prz09a] she found all integrable potentials for $n = k = 3$ with the maximal number of Darboux points.
In [Prz09b] she classifies the non-generic potentials for these values of n and k.
In [MP05] Przybylska and Maciejewski analyse the case $n = 2$, $k = 4$.

In their paper [SvS13] Michael Semmel and Duco van Straten examined the DGR-system and an integrable (but non-homogeneous) deformation of it using Kovalevskayas method described in Section 1.5. They showed

Theorem 1.15. *The system of Dorizzi-Grammaticos-Ramani is algebraic completely integrable. Its general fibre is isomorphic to $\mathcal{A}\setminus\mathcal{D}$, where $\mathcal{A}$ is an abelian surface and $\mathcal{D}$ a curve of geometric genus 4 having a singularity of type D_4. The Hamiltonian vector fields extend to linear vector fields on $\mathcal{A}$ and $\mathcal{D}$ puts a (1,6)-polarisation on $\mathcal{A}$. If the deformation parameter $a = 0$ (i.e. for the potential V_5 given above), $\mathcal{A}$ is isomorphic to the self-product of the elliptic curve with an automorphism of order 6.*

Since the fibres of algebraically completely integrable systems complete to abelian varieties, one is interested in studying these in the following chapters. The DGR-system gives a particular interest in (1,6)-polarised abelian surfaces. In Chapter 5 we describe their equations and their moduli space and we will find a codimension one subset which contains the DGR-system.

2 Preliminaries about abelian varieties

In this chapter we give some basic information and notation about abelian varieties. More details can be found in the appendix and in the book [BL04], which serves as our standard reference and where proofs of most statements can be found.

2.1 Complex tori

The quotient of a complex vector space $V \cong \mathbb{C}^g$ by a lattice $\mathbb{Z}^{2g} \cong \Lambda \subset V$ is an abelian compact complex Lie group (with respect to the operation $+$ induced from the vector space), called a *complex torus* $X = V/\Lambda$. We denote the canonical projection by $\pi \colon V \longrightarrow X$, which also is the universal covering of X as a topological space.

Proposition 2.1. *The n-th singular cohomology group $H^n(X,\mathbb{Z})$ of a complex torus X is isomorphic to the group $Alt^n(\Lambda,\mathbb{Z})$ of $\mathbb{Z}$-valued alternating linear n-forms on its lattice Λ.*

Proof. For any sufficiently nice topological space X (in particular a complex torus homeomorphic to $(S^1)^{2g}$) we have

$$H^1(X,\mathbb{Z}) \cong Hom(\pi_1(X),\mathbb{Z}).$$

Since for a complex torus $\pi_1(X) \cong \Lambda$, we get that $H^1(X,\mathbb{Z}) \cong Hom(\Lambda,\mathbb{Z}) \cong Alt^1(\Lambda,\mathbb{Z})$.
The rest follows by induction on the dimension using the Künneth formula. □

2.2 Line bundles on complex tori and the Appell-Humbert-Theorem

An abelian variety is a complex torus which can be embedded into projective space, i.e. which carries an ample line bundle. For this reason we will have a closer look at line bundles on complex tori.

Recall that a *hermitian form* on a vector space V is a map $H \colon V \times V \longrightarrow \mathbb{C}$ that is $\mathbb{C}$-linear in the first argument and satisfies $H(v,w) = \overline{H(w,v)}$ for all $v, w \in V$.

Define the *Néron-Severi group* $NS(\Lambda)$ to be the (additive) group of hermitian forms $H\colon V\times V \longrightarrow \mathbb{C}$ with $\operatorname{Im} H(\Lambda,\Lambda)\subseteq \mathbb{Z}$. It is isomorphic to the group of $\mathbb{R}$-valued alternating forms E on V satisfying $E(\Lambda,\Lambda)\subseteq\mathbb{Z}$ and $E(iv,iw)=E(v,w)$ via

$$H \mapsto E = \operatorname{Im}(H).$$

By restriction to Λ, one can see $NS(X)$ as a subgroup of $H^2(X,\mathbb{Z})\cong Alt^2(\Lambda,\mathbb{Z})$. To each line bundle L on X we can associate an element of $NS(\Lambda)\subseteq H^2(X,\mathbb{Z})$, called its *first Chern class.*

A *semicharacter* for a hermitian form $H\in NS(\Lambda)$ is a map $\chi\colon\Lambda\longrightarrow S^1\subset\mathbb{C}^*$ satisfying

$$\chi(\lambda+\mu)=\chi(\lambda)\chi(\mu)\exp(\pi i \operatorname{Im} H(\lambda,\mu)) \text{ for all } \lambda,\mu\in\Lambda.$$

We define

$$\mathcal{P}(\Lambda):=\{(H,\chi)\mid H\in NS(\Lambda),\chi \text{ semicharacter for } H\}.$$

Obviously $\mathcal{P}(\Lambda)$ is a group with respect to the composition

$$(H_1,\chi_1)\circ(H_2,\chi_2)=(H_1+H_2,\chi_1\chi_2).$$

Theorem 2.2 (Appell-Humbert-Theorem)**.** *There is an isomorphism*

$$\mathcal{P}(\Lambda)\longrightarrow Pic(X).$$

We will denote the line bundle associated to a pair (H,χ) by $L(H,\chi)$.

Recall that the first Chern class H of a line bundle L on X is a hermitian form on V, whose alternating form $E=\operatorname{Im} H$ takes integer values on the lattice Λ. According to the elementary divisor theorem, there is a basis $\lambda_1,\dots,\lambda_g,\mu_1,\dots,\mu_g$ – called *symplectic basis* – with respect to which E is given by the matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where $D=diag(d_1,\dots,d_g)$ with non-negative integers d_j satisfying $d_j|d_{j+1}$ for $j=1,\dots,g-1$. The elementary divisors $d_1,\dots,d_g$ are uniquely determined by L. We call D or the tuple $(d_1,\dots,d_g)$ the *type* of L.

Proposition 2.3. *L is ample if and only if $d_i>0$ for all i.*

2.3 Certain Subgroups

For a line bundle L on X we define

$$K(L):=\{x\in X\mid t_x^*L\simeq L\},$$

where

$$t_x \colon X \longrightarrow X, y \longmapsto y + x$$

is the translation by x. Clearly $K(L)$ is a subgroup of X.
If $L = L(H, \chi)$, we set

$$\Lambda(L) := \{v \in V \mid \operatorname{Im} H(v, \Lambda) \subseteq \mathbb{Z}\}.$$

Then $K(L)$ can be described as

$$K(L) = \Lambda(L)/\Lambda.$$

Since $K(L)$ only depends on $H = c_1(L)$, we sometimes write $K(H)$ instead of $K(L)$.

2.4 Decompositions and characteristics

Let L be a ample line bundle on X. A *decomposition for* L (or H or E) is a direct sum decomposition

$$\Lambda = \Lambda_1 \oplus \Lambda_2,$$

where Λ_1 and Λ_2 are isotropic with respect to E, i.e. $E|_{\Lambda_1 \times \Lambda_1} = 0$ and $E|_{\Lambda_2 \times \Lambda_2} = 0$. Such a decomposition always exists, because $\Lambda = \langle \lambda_1, \ldots, \lambda_g \rangle \oplus \langle \mu_1, \ldots, \mu_g \rangle$ is a decomposition for L whenever $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ is a symplectic basis. The other way around, each decomposition is of this form for some choice of symplectic basis.
A decomposition

$$V = V_1 \oplus V_2$$

with real subvector spaces V_1 and V_2 is called a *decomposition for* L if $(V_1 \cap \Lambda) \oplus (V_2 \cap \Lambda)$ is a decomposition of Λ for L. Clearly a decomposition of V for L is a decomposition into maximal isotropic subvector spaces, but not every decomposition of V into maximal isotropic subvector spaces is a decomposition for L.

Lemma 2.4. *Let L be an ample line bundle on X and $\Lambda = \Lambda_1 \oplus \Lambda_2$ a decomposition for L with induced decomposition $V = V_1 \oplus V_2$. Then:*

1. $\Lambda(L) = \Lambda(L)_1 \oplus \Lambda(L)_2$ *with* $\Lambda(L)_i = V_i \cap \Lambda(L)$ *for* $i = 1, 2$.
2. $K(L) = K_1 \oplus K_2$ *with* $K_i = \Lambda(L)_i/\Lambda_i$ *for* $i = 1, 2$.
3. $K_i \simeq \mathbb{Z}^g/D\mathbb{Z}^g = \bigoplus_{j=1}^g \mathbb{Z}/d_j\mathbb{Z}$ *for* $i = 1, 2$ *if the line bundle L is of type* $D = diag(d_1, \ldots, d_g)$

The decompositions of Lemma 2.4 are also called *decompositions for L.*

For each non-degenerate $H \in NS(X)$ and each decomposition $V = V_1 \oplus V_2$ for H we can define a map $\chi_0 \colon V \longrightarrow S^1$ by

$$\chi_0(v) = \exp(\pi i E(v_1, v_2)),$$

where $v = v_1 + v_2$ with $v_i \in V_i$. $\chi_0|_\Lambda$ is a semicharacter for H. Let $L_0 = L(H, \chi_0)$ denote the corresponding line bundle. With this notation we have:

Lemma 2.5. *Suppose H is a non-degenerate hermitian form on V and $V = V_1 \oplus V_2$ a decomposition for H. Then:*

1. *L_0 is the unique line bundle in $Pic^H(X)$ whose semicharacter is trivial on $\Lambda_i = V_i \cap \Lambda$ for $i = 1, 2$.*

2. *For every $L = L(H, \chi)$ there is a point $c \in V$, uniquely determined up to translation by elements of $\Lambda(L)$, such that $L \simeq t_{\bar{c}}^* L_0$ or equivalently $\chi = \chi_0 \exp(2\pi i E(c, \cdot))$.*

We call c a *characteristic of the line bundle L with respect to the chosen decomposition.* If a decomposition for L is fixed, we speak only of a characteristic c of L.

2.5 Theta functions

If $L = L(H, \chi)$ is a line bundle on X, then its space of global sections $H^0(L)$ can be identified with the set of holomorphic functions $\vartheta \colon V \longrightarrow \mathbb{C}$ satisfying

$$\vartheta(v + \lambda) = a_L(\lambda, v)\vartheta(v) \text{ for all } v \in V, \lambda \in \Lambda,$$

where $a_L \colon \Lambda \times V \longrightarrow \mathbb{C}^*$ is a factor of automorphy for $L(H, \chi)$ given by

$$a_L(\lambda, v) = \chi(\lambda) \exp(\pi H(v, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)).$$

Such a function is called a *theta function* for a_L.
For a characteristic $c \in V$ of L with respect to the decomposition $V = V_1 \oplus V_2$ we define a function

$$\vartheta^c(v) = \exp\Big(-\pi H(v, c) - \frac{\pi}{2} H(c, c) + \frac{\pi}{2} B(v + c, v + c)\Big) \cdot \sum_{\lambda \in \Lambda_1} \exp\Big(\pi (H - B)(v + c, \lambda) - \frac{\pi}{2} (H - B)(\lambda, \lambda)\Big)$$

where B is the $\mathbb{C}$-linear extension of the symmetric form $H|_{V_2 \times V_2}$ to $V \times V$.
We can define the so called *canonical theta functions* by

$$\vartheta^c_{\bar{w}}(v) = a_L(w, v)^{-1} \vartheta^c(v + w).$$

Theorem 2.6. *Suppose $L = L(H, \chi)$ is an ample line bundle on X and let c be a characteristic with respect to a decomposition $V = V_1 \oplus V_2$ for L. Then the set $\{\vartheta^c_{\bar{w}} \mid \bar{w} \in K(L)_1\}$ is a basis of the vector space $H^0(L)$.*

2.6 Polarised abelian varieties

A *polarisation* on a complex torus $X = V/\Lambda$ is by definition the first Chern class $H = c_1(L)$ of an ample line bundle L on X. By abuse of notation sometimes the line bundle L itself is considered as a polarisation. The type of L is called the *type of the polarisation*. A polarisation of type $(1, \dots, 1)$ is called a *principal polarisation*.
An *abelian variety* is defined to be a complex torus X admitting a polarisation $H = c_1(L)$. The pair (X, H) or (X, L) is called a *polarised abelian variety*.
A *homomorphism of polarised abelian varieties* $f\colon (Y, M) \longrightarrow (X, L)$ is a homomorphism of complex tori such that $f^*c_1(L) = c_1(M)$. This does not mean that $f^*L = M$ but only that f^*L and M are analytically equivalent.

A line bundle L on a complex torus $X = V/\Lambda$ induces in the usual way a meromorphic map $X \dashrightarrow \mathbb{P}(H^0(L)^*)$. After a choice of basis $\vartheta_0, \dots, \vartheta_N$ of $H^0(L)$, φ_L can be described as a map

$$\varphi_L\colon X \dashrightarrow \mathbb{P}^N,$$

given by

$$\varphi_L(\bar{v}) = (\vartheta_0(v) : \dots : \vartheta_N(v)),$$

whenever not all ϑ_j vanish simultaneously at v. This version of φ_L depends on the choice of the basis of $H^0(L)$ and a change of this basis means composing φ_L with a projective transformation of $\mathbb{P}^N$.
There are several useful theorems which imply that φ_L, under certain conditions, is base point free or an embedding. For more details see Section 7.3.

2.7 Theta- and Heisenberg-groups

Let L be a (very) ample line bundle on an abelian variety $X = V/\Lambda$ and $\varphi_L\colon X \dashrightarrow \mathbb{P}^N$ the corresponding rational map (embedding). Recall the group $K(L)$ consisting of all $x \in X$ with $t_x^*L \simeq L$. We will see that the translations of X by elements of $K(L)$ extend to linear automorphisms of $\mathbb{P}^N$. This leads to a projective representation $\rho\colon K(L) \to PGL_N(\mathbb{C})$, with respect to which the embedding φ_L is equivariant. It will be an important tool in the investigation of equations and geometric properties of the embedded abelian variety $\varphi_L(X)$ in $\mathbb{P}^N$. We will define an extension group $\mathcal{G}(L)$, the so called theta group of L, for which ρ lifts to an ordinary representation $\tilde{\rho}\colon \mathcal{G}(L) \to GL_N(\mathbb{C})$. We will also introduce the Heisenberg group $\mathcal{H}(L)$ which is an abstract version of the theta group depending only on the type of L, plus a corresponding representation.

2.7.1 Theta-group

Definition 2.7. The *theta-group* $\mathcal{G}(L)$ of a line bundle L is the set of all pairs (φ, x) where $x \in X$ and $\varphi\colon X \to X$ is a linear automorphism of L over x, i.e. such that the diagram

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[d] & L \arrow[d] \\ X \arrow[r, "t_x"'] & X \end{tikzcd}$$

commutes. It is a group with respect to the composition $(\varphi_1, x_1)(\varphi_2, x_2) = (\varphi_1\varphi_2, x_1 + x_2)$.

We write (φ, x) for convenience, although the automorphism φ determines the point x uniquely.

Proposition 2.8. *The sequence*

$$1 \longrightarrow \mathbb{C}^* \xrightarrow{i} \mathcal{G}(L) \xrightarrow{p} K(L) \longrightarrow 0$$

with $i(\alpha) = (\alpha, 0)$ and $p((\varphi, x)) = x$ is exact and $\mathcal{G}(L)$ is a central extension of $K(L)$ by $\mathbb{C}^*$.

Since $\mathcal{G}(L)$ is a central extension of abelian groups, its commutator induces a map

$$e^L\colon K(L) \times K(L) \longrightarrow \mathbb{C}^*.$$

The map e^L can be expressed in terms of the first Chern class H of L as follows:

Proposition 2.9. *For all $w_1, w_2 \in \Lambda(L)$*

$$e^L(\bar{w}_1, \bar{w}_2) = exp(-2\pi i \operatorname{Im} H(w_1, w_2)).$$

The map e^L is a multiplicative alternating form with values in $\mathbb{C}^*$. We will come back to this in Section 7.4.3.

2.7.2 Canonical representation of the theta group

Assume s is a section of the line bundle L and $(\varphi, x) \in \mathcal{G}(L)$. As the following commutative diagram shows

$$\begin{tikzcd} L \arrow[r, "\varphi"] & L \\ X \arrow[u, "s"] \arrow[r, "t_x"'] & X \arrow[u, "\varphi s t_{-x}"'] \end{tikzcd}$$

$\varphi s t_{-x}$ is also a section of L. The assignment $((\varphi, x), s) \mapsto \varphi s t_{-x}$ defines an action of $\mathcal{G}(L)$ on $H^0(L)$. The corresponding map

$$\tilde{\rho}\colon \mathcal{G}(L) \longrightarrow GL(H^0(L))$$

is called the *canonical representation of the theta group $\mathcal{G}(L)$*.
Since $\mathbb{C}^*$ acts by multiplication on $H^0(L)$, $\tilde{\rho}$ induces a projective representation

$$\rho\colon K(L) \longrightarrow PGL(H^0(L)).$$

2.7.3 The Heisenberg group

For an ample line bundle L and a basis $\vartheta_0, \dots, \vartheta_N$ of $H^0(L)$ the associated rational map is given by $\varphi_L\colon X \dashrightarrow \mathbb{P}^N$, $\bar{v} \longmapsto (\vartheta_0(v) : \dots : \vartheta_N(v))$. The group $K(L)$ acts on both sides, by translation on X and via the representation ρ on $\mathbb{P}^N$. The map φ_L is equivariant with respect to these actions. In particular the image $\varphi_L(X)$ is invariant under the action of $K(L)$ which can be described explicitly as a matrix with respect to a basis of $H^0(L)$ ([BL04] Proposition 6.4.2). This can be used for example to derive information on the equations for $\varphi_L(X)$.

However, the projective variety $\varphi_L(X)$ does not depend on the particular choice of L within its algebraic equivalence class, whereas the formula describing the action of $K(L)$ on $H^0(L)$ does. Thus it would be desirable to have a description of the theory of theta functions depending only on the polarisation. This leads to the theory of Heisenberg groups which we will discuss now.

Let $H \in NS(X)$ be a polarisation of type $D = diag(d_1, .., d_g)$. Then define the group

$$K(D) := \mathbb{Z}^g/D\mathbb{Z}^g \oplus \mathbb{Z}^g/D\mathbb{Z}^g$$

and as a set

$$\mathcal{H}(D) := \mathbb{C}^* \times K(D).$$

The *Heisenberg group of* D is the set $\mathcal{H}(D)$ together with the following group structure: Let $f_1, \dots, f_{2g}$ be the standard basis of $K(D)$. Define the alternating, $\mathbb{Z}$-linear map $e^D\colon K(D) \times K(D) \longrightarrow \mathbb{C}^*$ via

$$e^D(f_\nu, f_\mu) = \begin{cases} \exp(-\frac{2\pi i}{d_\nu}) & \mu = g + \nu \\ \exp(\frac{2\pi i}{d_\nu}) & \nu = g + \mu \\ 1 & \text{else} \end{cases}$$

and set for any $(\alpha, x_1, x_2), (\beta, y_1, y_2) \in \mathcal{H}(D)$:

$$(\alpha, x_1, x_2) \cdot (\beta, y_1, y_2) = \big(\alpha\beta e^D\big((x_1, 0), (0, y_2)\big), x_1 + y_1, x_2 + y_2\big).$$

Lemma 2.10. *With this composition* $\mathcal{H}(D)$ *is a group and the following sequence is exact*

$$1 \longrightarrow \mathbb{C}^* \xrightarrow{i} \mathcal{H}(D) \xrightarrow{p} K(D) \longrightarrow 0$$

and $\mathcal{H}(D)$ *is a central extension.*
Here, $i(\alpha) = (\alpha, 0, 0)$ *and* $p(\alpha, x_1, x_2) = (x_1, x_2)$. *The neutral element of* $\mathcal{H}(D)$ *is* $(1, 0, 0)$ *and* $(\alpha, x_1, x_2)^{-1} = (\alpha^{-1} e^D(-(x_1, 0), (0, x_2)), -x_1, -x_2)$.

An isomorphism $b\colon \mathcal{G}(L) \longrightarrow \mathcal{H}(D)$ that restricts to the identity on $\mathbb{C}^*$ is called a *theta structure*. Such a theta structure induces an isomorphism $\bar{b}\colon K(L) \longrightarrow K(D)$ such that the following diagram commutes:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d, equal] & \mathcal{G}(L) \arrow[r] \arrow[d, "b"] & K(L) \arrow[r] \arrow[d, "\bar{b}"] & 0 \\ 1 \arrow[r] & \mathbb{C}^* \arrow[r] & \mathcal{H}(D) \arrow[r] & K(D) \arrow[r] & 0 \end{tikzcd}$$

Lemma 2.11. *For any theta structure $b\colon \mathcal{G}(L) \longrightarrow \mathcal{H}(D)$, the induced map*

$$\bar{b}\colon K(L) \longrightarrow K(D)$$

is a symplectic isomorphism with respect to the forms e^L and e^D, i.e.

$$\bar{b}^* e^D = e^L.$$

We will come back to this topic in Section 7.4.3.

Theorem 2.12. *Every ample line bundle L on X of type D admits exactly*

$$\# Sp(D) \cdot h^0(L)^2$$

theta structures, where $Sp(D)$ is the group of all automorphisms of $K(D)$ which preserve the alternating form e^D.
In particular, $\mathcal{G}(L)$ is isomorphic to $\mathcal{H}(D)$.

2.7.4 Schrödinger representation

Now we build a substitute for the canonical representation $\tilde{\rho}\colon \mathcal{G}(L) \longrightarrow GL(H^0(L))$ for the Heisenberg group.

Let $\mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g)$ be the $\mathbb{C}$-vector space of complex valued functions on $\mathbb{Z}^g/D\mathbb{Z}^g$. A basis of this vector space is given by $\{\delta_x \mid x \in \mathbb{Z}^g/D\mathbb{Z}^g\}$ where

$$\delta_x(y) = \begin{cases} 1 & x = y \\ 0 & \text{else.} \end{cases}$$

The group $\mathcal{H}(D)$ acts on this vector space via

$$(\alpha, x_1, x_2) \cdot \gamma = \alpha e^D((\cdot, 0), (0, x_2))\gamma(\cdot + x_1).$$

The induced representation

$$\tilde{\rho}\colon \mathcal{H}(D) \longrightarrow GL(\mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g))$$

is called the *Schrödinger representation* of $\mathcal{H}(D)$.
Let $b\colon \mathcal{G}(L) \longrightarrow \mathcal{H}(D)$ be a theta structure, $\bar{b}\colon K(L) \longrightarrow K(D) = (\mathbb{Z}^g/D\mathbb{Z}^g)_1 \oplus (\mathbb{Z}^g/D\mathbb{Z}^g)_2$ the induced symplectic isomorphism. Define a decomposition $K(L) = K_1 \oplus K_2$ by $K_j := \bar{b}^{-1}((\mathbb{Z}^g/D\mathbb{Z}^g)_j)$. Let c be a characteristic for L with respect to this decomposition. According to Theorem 2.6 the vector space $H^0(L)$ has a basis indexed by K_1. Then $\bar{b}$ induces an isomorphism $\beta\colon H^0(L) \longrightarrow \mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g)$ via $\beta(\vartheta_x^c) = \delta_{\bar{b}(x)}$.

Proposition 2.13. *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{G}(L) \times H^0(L) \arrow[r, "\tilde{\rho}"] \arrow[d, "{(b,\beta)}"'] & H^0(L) \arrow[d, "\beta"] \\
\mathcal{H}(D) \times \mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g) \arrow[r, "\tilde{\rho}"'] & \mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g)
\end{tikzcd}$$

This exactly means that there is an isomorphism between the Schrödinger representation $\tilde{\rho}\colon \mathcal{H}(D) \longrightarrow GL(\mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g))$ and the canonical representation $\tilde{\rho}\colon \mathcal{G}(L) \longrightarrow GL(H^0(L))$ in the category of representations. Because the centre $\mathbb{C}^*$ of $\mathcal{H}(D)$ acts by multiplication by a scalar, $\tilde{\rho}$ descends to a projective representation $\rho\colon K(D) \longrightarrow PGL_N(\mathbb{C})$ which is isomorphic to the representation $K(L) \longrightarrow PGL_N(\mathbb{C})$ of Section 2.7.2.

Example 2.14. Let X be an abelian surface, i.e. $g = 2$ and $D = \left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$. Then

$$\mathcal{H}(D) = \mathbb{C}^* \times (\mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z}) \times (\mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z})$$

and

$$e^D((\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_1', \bar{\nu}_2'), (\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_1', \bar{\mu}_2')) = \exp\left(\frac{2\pi i}{d_1}(\nu_1'\mu_1 - \nu_1\mu_1') + \frac{2\pi i}{d_2}(\nu_2'\mu_2 - \nu_2\mu_2')\right)$$

where $\nu_j, \nu_j', \mu_j, \mu_j'$ are representatives in $\mathbb{Z}$ of $\bar{\nu}_j, \bar{\nu}_j', \bar{\mu}_j, \bar{\mu}_j'$ in $\mathbb{Z}/d_j\mathbb{Z}$.
The generators $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$ of $K(D)$ are represented by $\sigma_1 = (1,1,0,0,0), \sigma_2 = (1,0,1,0,0), \tau_1 = (1,0,0,1,0), \tau_2 = (1,0,0,0,1)$ in $\mathcal{H}(D)$. Their images under $\tilde{\rho}$ act on $\{\delta_{(\nu_1,\nu_2)} \mid (\nu_1,\nu_2) \in \mathbb{Z}^2/D\mathbb{Z}^2\}$ as follows:

$$\begin{aligned}
\sigma_1\colon \delta_{(\nu_1,\nu_2)} &\longmapsto \delta_{(\nu_1-1,\nu_2)} \\
\sigma_2\colon \delta_{(\nu_1,\nu_2)} &\longmapsto \delta_{(\nu_1,\nu_2-1)} \\
\tau_1\colon \delta_{(\nu_1,\nu_2)} &\longmapsto \xi_1^{-\nu_1}\delta_{(\nu_1,\nu_2)} \\
\tau_2\colon \delta_{(\nu_1,\nu_2)} &\longmapsto \xi_2^{-\nu_2}\delta_{(\nu_1,\nu_2)}
\end{aligned}$$

where $\xi_j = \exp(\frac{2\pi i}{d_j})$.

For $(d_1, d_2) = (1, d)$ the elements $\sigma_1, \tau_1 \in GL(\mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g))$ are the identity. So, with $\xi = \exp(\frac{2\pi i}{d})$ there are only two generators left, namely

$$\begin{aligned}
\sigma\colon \delta_\nu &\longmapsto \delta_{\nu-1} \\
\tau\colon \delta_\nu &\longmapsto \xi^{-\nu}\delta_\nu.
\end{aligned}$$

We will write $\mathbf{H}_d$ for the subgroup of $\mathcal{H}(1,d)$ generated by σ and τ in the sequel. $\mathbf{H}_d$ is a finite group of order d^3.

2.7.5 Symmetric theta structures

If the line bundle L is symmetric, i.e. $(-1)^*L \cong L$, we can get an even bigger group acting on $\varphi_L(X)$.
Think of $K(L)$ as a group of translations of X, i.e. $K(L) \subseteq Aut(X)$. Since $(-1)_X t_x (-1)_X = t_{-x}$, conjugation by elements of $\langle (-1)_X \rangle$ leaves $K(L)$ fixed. Hence we can consider the semidirect product $K^e(L) := K(L) \rtimes \langle (-1)_X \rangle$.
Similarly, we can construct the *extended theta group* $\mathcal{G}^e(L)$, $K^e(D)$ and the *extended Heisenberg group* $\mathcal{H}^e(D)$. They all sit in similar exact sequences and form central extensions as their unextended analogues that we will not spell out, but in particular $\mathcal{H}^e(D)$ is isomorphic to $\mathcal{G}^e(L)$.

Definition 2.15. An *extended theta structure* is an isomorphism

$$b^e \colon \mathcal{G}^e(L) \longrightarrow \mathcal{H}^e(D),$$

which restricts to the identity on $\mathbb{C}^*$.

Any extended theta structure can be restricted to a ordinary one, but not every ordinary theta structure extends; it extends to an extended theta structure if and only if it is *symmetric*, i.e. the following diagram commutes

$$\begin{tikzcd}
\mathcal{G}(L) \arrow[r, "(-1)_L"] \arrow[d, "b"'] & \mathcal{G}(L) \arrow[d, "b"] \\
\mathcal{H}(D) \arrow[r, "\iota"'] & \mathcal{H}(D)
\end{tikzcd}$$

where $\iota((\alpha, x, y)) = (\alpha, -x, -y)$.
The Schrödinger representation can also be extended to $\mathcal{H}^e(D)$ via

$$\tilde{\rho}(\iota) \colon \delta_x \longmapsto \delta_{-x}.$$

2.8 Moduli spaces

Define a *polarised abelian variety of type D with symplectic basis* to be a triplet

$$(X, H, \{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\})$$

where $X = V/\Lambda$ is an abelian variety, H a polarisation of type D on X, and $\{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\}$ a basis of Λ for H such that H is of the form $\left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)$ with respect to this basis.

The set

$$\mathcal{H}_g := \{Z \in \mathbb{C}^{g \times g} \mid Z^\top = Z, \mathrm{Im}(Z) > 0\}$$

is called the *Siegel upper half space*. It is a $\frac{1}{2}g(g+1)$-dimensional open submanifold of the vector space $\mathbb{C}^{g \times g}$.

The assignment

$$\Phi\colon Z \longmapsto (X_Z, H_Z, \{\text{columns of } (Z,D)\})$$

with $\Lambda_Z := (Z,D)\mathbb{Z}^{2g}$, $X_Z := \mathbb{C}^g/\Lambda_Z$ and H_Z the hermitian form described by $(\mathrm{Im}(Z))^{-1}$ with respect to the standard basis of $\mathbb{C}^g$, associates a polarised abelian variety with symplectic basis to any point in $\mathcal{H}_g$. The other way around, Riemann's bilinear relations tell us that any polarised abelian variety with symplectic basis is isomorphic to one in the image of Φ. Since for $Z \neq Z' \in \mathcal{H}_g$ the associated p.a.v. with symplectic basis are never isomorphic, we have:

Proposition 2.16. *Given a type D, the Siegel upper half space $\mathcal{H}_g$ is a coarse moduli space for polarised abelian varieties of type D with choice of a symplectic basis.*

If we do not include a symplectic basis in the datum but consider simply polarised abelian varieties of a given type or abelian varieties with a level D-structure, we obtain certain quotients of Siegel upper half space as moduli spaces.

Theorem 2.17. *The normal complex analytic space $\mathcal{A}_D := \mathcal{H}_g/\Gamma_D$ is a moduli space for polarised abelian varieties of type D. Here*

$$\Gamma_D = \{M \in \mathbb{R}^{2g\times 2g} \mid M \left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right) M^\top = \left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)\}$$

is the paramodular group *acting on $\mathcal{H}_g$ via*

$$M\langle Z\rangle = (aZ + bD)(D^{-1}cZ + D^{-1}dD)^{-1} \text{ for all } M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

A symplectic basis cannot be given in algebraic terms, but a *level D-structure* is kind of the closest replacement for this notion.

Let $(X = V/\Lambda, H)$ be a polarised abelian variety of type $D = diag(d_1, \dots, d_g)$. Recall the (multiplicative) alternating form $e^H\colon K(H) \times K(H) \longrightarrow \mathbb{C}^*$, $e^H(\bar v, \bar w) = \exp(-2\pi i \,\mathrm{Im}\, H(v,w))$. In Section 2.7.3 we introduced the group $K(D) = (\mathbb{Z}^g/D\mathbb{Z}^g)^2$ and the (multiplicative) alternating form $e^D\colon K(D) \times K(D) \longrightarrow \mathbb{C}^*$. A *level D-structure* on (X,H) is by definition a symplectic isomorphism $\bar b\colon K(H) \longrightarrow K(D)$. The symplectic isomorphism $\bar b\colon K(H) \longrightarrow K(D)$ can be identified with the ordered set $\{\bar b^{-1}(f_1), \dots, \bar b^{-1}(f_{2g})\}$ where $f_1, \dots, f_{2g}$ denotes the standard generators of $K(D)$. This is a basis of $K(L)$.

Theorem 2.18. *The normal complex analytic space $\mathcal{A}_D(D) := \mathcal{H}_g/\Gamma_D(D)$ with*

$$\Gamma_D(D) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_D \mid a - 1_g \equiv b \equiv c \equiv d - 1_g \equiv 0 \mod D\}$$

where we write $a \equiv 0 \mod D$ for $a \in D \cdot \mathbb{Z}^{g\times g}$ is a moduli space for polarised abelian varieties of type $D = diag(d_1, \dots, d_g)$ with level D-structure. The embedding $\Gamma_D(D) \hookrightarrow \Gamma_D$ induces a holomorphic map $\mathcal{A}_D(D) \to \mathcal{A}_D$ of finite degree.

3 The moduli space of Heisenberg invariantly embedded abelian varieties

Given a polarised abelian variety A with a very ample line bundle L of type D, the embedding

$$\varphi_L\colon A \longrightarrow \mathbb{P}^N$$

depends of course on the choice of a basis of $H^0(L)$. Two different choices of a basis lead to maps that differ by a projective linear transformation.
In this section we want to find an algebraic description of the datum necessary to describe a particular Heisenberg invariant embedding and derive a description of the moduli space of Heisenberg invariantly embedded abelian surfaces of a given type D as a quotient of Siegel upper half space (cf. Section 7.4).

3.1 The algebraic datum encoding an embedding

Lemma 3.1. *The centraliser of $\mathcal{H}(D)$ in $GL_N(\mathbb{C})$ is $\mathbb{C}^*$.*

Proof. For $D = diag(1, d)$ this can be seen as follows (a similar argument works in general): Suppose $\beta(e_i) = \sum_j b_{ij}e_j$. Then commutativity with σ, which sends e_i to e_{i-1}, means that $b_{i-1,j} = b_{i,j+1}$ for all i and j, i.e. the entries of the matrix (b_{ij}) for β are constant on diagonals. Commutativity with τ, where τ sends e_i to $\zeta_d^{-i}e_i$ ($\zeta_d = \exp(\frac{2\pi i}{d})$), translates to the condition that $b_{ij} = 0$ for $i \neq j$. So (b_{ij}) has to be of the form $c \cdot \mathrm{id}$ for $c \in \mathbb{C}^*$. □

Proposition 3.2. *There is a 1-to-1-correspondence between Heisenberg invariant embeddings $\varphi\colon X \longrightarrow \mathbb{P}^N$ of an abelian variety X of type D and injective group homomorphisms $\mathbb{Z}^g/D\mathbb{Z}^g \longrightarrow \mathbb{Z}^g/D\mathbb{Z}^g \oplus \mathbb{Z}^g/D\mathbb{Z}^g$ which can be extended to a symplectic automorphism of $\mathbb{Z}^g/D\mathbb{Z}^g \oplus \mathbb{Z}^g/D\mathbb{Z}^g$.*

Proof. Under which condition is an embedding Heisenberg invariant? In Section 2.7 we defined the groups $\mathcal{G}(L)$ and $\mathcal{H}(D)$ to be central extensions of $K(L)$ resp. $K(D) = (\mathbb{Z}^g/D\mathbb{Z}^g)^2$ by $\mathbb{C}^*$. We defined a theta-structure to be a group isomorphism $b\colon \mathcal{G}(L) \longrightarrow \mathcal{H}(D)$ that restricts to the identity on $\mathbb{C}^*$ and saw that such a theta structure always exists. A theta structure induces a symplectic isomorphism $\bar{b}\colon K(L) \longrightarrow K(D)$ (and any such symplectic isomorphism is induced by a theta structure) and an isomorphism of vector spaces $\beta\colon H^0(L) \longrightarrow \mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g)$. These isomorphisms make the following diagram commute

$$\begin{tikzcd}
\mathcal{G}(L) \times H^0(L) \arrow[r] \arrow[d, "{(b,\beta)}"'] & H^0(L) \arrow[d, "\beta"] \\
\mathcal{H}(D) \times \mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g) \arrow[r] & \mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g).
\end{tikzcd}$$

where the horizontal arrows indicate the action of $\mathcal{G}(L)$ on $H^0(L)$ resp of $\mathcal{H}(D)$ on $\mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g)$ as described in Section 2.7.
Given b, we can show that β is (up to multiplication by a constant) the unique vector space isomorphism making the above diagram commute, by the following argument: Suppose there are two different vector space isomorphisms $\beta_1, \beta_2\colon H^0(L) \longrightarrow \mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g)$ making the above diagram commute. Then $\tilde{\beta} := \beta_1 \circ \beta_2^{-1}$ is an isomorphism of $\mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g)$ fitting into the following commutative diagram:

$$\begin{tikzcd}
\mathcal{H}(D) \times \mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g) \arrow[r] \arrow[d, "{(\mathrm{id},\tilde{\beta})}"'] & \mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g) \arrow[d, "\tilde{\beta}"] \\
\mathcal{H}(D) \times \mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g) \arrow[r] & \mathbb{C}(\mathbb{Z}^g/D\mathbb{Z}^g).
\end{tikzcd}$$

This means that $\tilde{\beta}$ commutes with any element from $\mathcal{H}(D)$. By Lemma 3.1 this means that $\tilde{\beta}$ is only multiplication by a constant.
From this we can see that the embedding is Heisenberg-invariant if and only if the chosen basis of $H^0(L)$ is of the form $c \cdot \beta^{-1}(\delta_x)$, $x \in \mathbb{Z}^g/D\mathbb{Z}^g, c \in \mathbb{C}^*$, β and δ_x as in Section 2.7. So for each β there is exactly one Heisenberg invariant projective embedding $\varphi\colon X \longrightarrow \mathbb{P}^N$.

Now try and count such bases: The map β and thus the basis of $H^0(L)$ is completely determined by $\bar{b}$. Suppose we are given two symplectic isomorphisms $\bar{b}_1, \bar{b}_2\colon K(L) \longrightarrow K(D)$. Then their difference $\bar{b}_1 \circ \bar{b}_2^{-1}$ is a symplectic automorphism of $K(D)$.
But not all $\bar{b}$ give different bases because β only depends on $\bar{b}|_{K_1}$. More precisely: $\bar{b}_1$ and $\bar{b}_2$ give the same basis if and only if $\bar{b}_1^{-1}|_{(\mathbb{Z}^g/D\mathbb{Z}^g)_1} = \bar{b}_2^{-1}|_{(\mathbb{Z}^g/D\mathbb{Z}^g)_1}$ where $(\mathbb{Z}^g/D\mathbb{Z}^g)_1$ is the first summand of $K(D)$. These restricted maps live in $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^g/D\mathbb{Z}^g, K(L))$ and are injective. By composition with one fixed symplectic isomorphism $\bar{b}_0\colon K(L) \longrightarrow K(D) = (\mathbb{Z}^g/d\mathbb{Z}^g)^2$ we obtain the desired result. □

Remark 3.3. *1. For $D = diag(1, d)$, the group $K(D)$ is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^2$ and any automorphism of this group is symplectic, so the group of symplectic automorphisms of $K(D)$ is isomorphic to $GL_2(\mathbb{Z}/d\mathbb{Z})$.*

2. In the case $d = 6$ the group $GL_2(\mathbb{Z}/d\mathbb{Z})$ has 288 elements.

Since we will later look at abelian surfaces of type $(1, d)$ (in particular of type (1,6) in Chapter 5), the following fact is of interest:

Lemma 3.4. *Any injective group homomorphism from $\mathbb{Z}/d\mathbb{Z}$ to $(\mathbb{Z}/d\mathbb{Z})^2$ can be extended to an automorphism of $(\mathbb{Z}/d\mathbb{Z})^2$.*

Proof. For any group monomorphism $\alpha\colon \mathbb{Z}/d\mathbb{Z} \to (\mathbb{Z}/d\mathbb{Z})^2$ the image $\alpha(1)$ is an element of order d in $(\mathbb{Z}/d\mathbb{Z})^2$. Such an element is of the form (a,b) with $a,b \in \mathbb{Z}$ and $\gcd(a,b,d) = 1$. This means there are coefficients $s,t,u \in \mathbb{Z}$ such that

$$sa + tb + ud = 1.$$

We claim that (a,b) and $(-t,s)$ together generate $(\mathbb{Z}/d\mathbb{Z})^2$, so

$$\tilde{\alpha}\colon (\mathbb{Z}/d\mathbb{Z})^2 \to (\mathbb{Z}/d\mathbb{Z})^2, \quad (1,0) \mapsto (a,b), \quad (0,1) \mapsto (-t,s)$$

is an automorphism extending α. For this, we look at the matrix

$$M = \begin{pmatrix} a & -t \\ b & s \end{pmatrix}.$$

It has determinant equal to $1 \in \mathbb{Z}/d\mathbb{Z}$, thus is invertible and the columns of any invertible matrix in $R^{2\times 2}$ (R any ring) generate the free module R^2. □

Example 3.5. For general diagonal matrix D not every injective group homomorphism $\mathbb{Z}^g/D\mathbb{Z}^g \longrightarrow \mathbb{Z}^g/D\mathbb{Z}^g \oplus \mathbb{Z}^g/D\mathbb{Z}^g$ can be extended to an automorphism of $\mathbb{Z}^g/D\mathbb{Z}^g \oplus \mathbb{Z}^g/D\mathbb{Z}^g$:
Consider the case $D = diag(2,4)$. We write $G := \mathbb{Z}^2/D\mathbb{Z}^2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and consider the monomorphism

$$\alpha\colon G \longrightarrow G \oplus G$$

given by $\alpha(e_1) = 2f_2$ and $\alpha(e_2) = f_4$, where e_1, e_2 and f_1, f_2, f_3, f_4 are generators for the cyclic factors on the left- resp. right-hand side in the given order. This monomorphism can not be extended to an automorphism of the right-hand group because

$$(G \oplus G)/\alpha(G) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle f_1 \rangle \times \langle \bar{f_2} \rangle \times \langle f_3 \rangle.$$

But if there was an automorphism $\tilde{\alpha}$ extending α, then $\tilde{\alpha}(e_4)$ would need to have order four in $(G \oplus G)/\alpha(G)$ and there is no such element.

Corollary 3.6. *So for $D = diag(1,d)$ the number of embedding maps is exactly the number of elements of order d in $K(L) \cong (\mathbb{Z}/d\mathbb{Z})^2$.*

If we only want to count images of such embeddings, there will be at most half as many. To see this, consider any (symplectic) isomorphism $\bar{b}\colon K(L) \longrightarrow (\mathbb{Z}^g/D\mathbb{Z}^g)^2$. Then $-\bar{b}$ is another isomorphism. Let by $e_0, e_1, \dots, e_N$ be all elements of $\mathbb{Z}^g/D\mathbb{Z}^g$ in a fixed ordering and let $\sigma_i := \bar{b}^{-1}(e_i)$. Then $\bar{b}$ gives us the basis $\vartheta_{\sigma_0}, \vartheta_{\sigma_1}, \dots, \vartheta_{\sigma_N}$, whereas $-\bar{b}$ gives us the same basis but in a different order $\vartheta_{-\sigma_0}, \vartheta_{-\sigma_1}, \dots, \vartheta_{-\sigma_N}$. If we denote the corresponding coordinates of $\mathbb{P}^N$ by $x_{\sigma_0}, \dots, x_{\sigma_N}$, then the linear projective transformation converting one embedding into the other is given by $\iota\colon x_i \longmapsto x_{-i}$ (indices read as elements of $\mathbb{Z}^g/D\mathbb{Z}^g$).

Proposition 3.7. *Let X be any abelian variety, $\varphi_L\colon X \longrightarrow \mathbb{P}^N$ a $\mathcal{H}(D)$-invariant embedding. Then $\overline{X} := \varphi_L(X)$ is also invariant under ι.*

Proof. Lemma 4.6.1 in [BL04] tells us that

$$\begin{tikzcd} X \arrow[rr, "t_x"] \arrow[dr, "\varphi_{t_x^*L}"'] & & X \arrow[dl, "\varphi_L"] \\ & \mathbb{P}^N & \end{tikzcd}$$

commutes for the right choice of bases of $H^0(L)$ and $H^0(t_x^*L)$. Since two line bundles L and L' are algebraically equivalent if and only if $L' = t_x^*L$ for some $x \in X$ this implies that the image $\overline{X}$ of φ_L in $\mathbb{P}^N$ does not depend on L itself but only on its algebraic equivalence class. So instead of the given line bundle $L = L(H,\chi)$ we can consider the line bundle $L_0 = L(H,\chi_0)$ as described in Section 2.4 giving the same embedding. The advantage of L_0 is that it is symmetric and admits an extended theta structure (proof of Theorem 6.9.5 in [BL04]). This means that the extended Heisenberg group $\mathcal{H}^e(D) = \langle \mathcal{H}(D), \iota\rangle$ acts on $H^0(L_0)$ and $\overline{X} := \varphi_{L_0}(X)$ is $\mathcal{H}^e(D)$-invariant. □

3.2 The moduli space of Heisenberg invariantly embedded abelian varieties of type D

In Proposition 3.2 we have seen that there is a 1-to-1-correspondence between Heisenberg equivariant embeddings $\varphi_L\colon X \longrightarrow \mathbb{P}^N$ of a polarised abelian variety (X, L) of type D and injective group homomorphisms $\mathbb{Z}^g/D\mathbb{Z}^g \longrightarrow K(L)$ which can be extended to a symplectic automorphism of $K(L)$.
Such a monomorphism $\lambda\colon \mathbb{Z}^g/D\mathbb{Z}^g \longrightarrow K(D)$ can be identified with the ordered set $\{\lambda(f_1), \dots, \lambda(f_g)\}$ where $f_1, \dots, f_g$ denotes the standard generators of $\mathbb{Z}^g/D\mathbb{Z}^g$. Since λ can be extended to a symplectic automorphism, this is (the first) half of a symplectic basis of $K(D)$.
We define an *isomorphism of polarised abelian varieties with a half-basis* similarly to the definition made in Section 7.4.3 as an isomorphism of polarised abelian varieties that maps the j-th element of the given half-basis of $K(L)$ to the corresponding element of the given half-basis of $K(L')$.
Given a symplectic isomorphism $\bar{b} : K(L) \to K(D)$, there is a symplectic basis $\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g$ of Λ for H such that $\bar{b}(\overline{\frac{1}{d_i}\lambda_i}) = f_i$ for $1 \le i \le g$.

Now will use an argument similar to those we sketched in Section 7.4 (and which can be found in more detail in [BL04], Chapter 8) to describe the moduli space of Heisenberg invariantly embedded abelian varieties:
Every $Z \in \mathcal{H}_g$ determines a polarised abelian variety of type D with half-basis of $K(D)$:

$$Z \longmapsto \Big(X_Z, H_Z, \{\overline{\tfrac{1}{d_1}\lambda_1}, \dots, \overline{\tfrac{1}{d_g}\lambda_g}\}\Big)$$

where $(X_Z, H_Z, \{\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g\})$ is the polarised abelian variety of type D with symplectic basis of Proposition 7.26.
By what we said above it is clear that every polarised abelian variety with level D-structure is isomorphic to one of these. Suppose that

$$\varphi\colon \Big(X_Z, H_Z, \{\overline{\tfrac{1}{d_1}\lambda_1}, \dots, \overline{\tfrac{1}{d_g}\lambda_g}\}\Big) \longrightarrow \Big(X_{Z'}, H_{Z'}, \{\overline{\tfrac{1}{d_1}\lambda'_1}, \dots, \overline{\tfrac{1}{d_g}\lambda'_g}\}\Big)$$

is an isomorphism of polarised abelian varieties with a half-basis. Let $A \in \mathbb{C}^{g\times g}$ be the analytic and $R^T \in \mathbb{Q}^{2g\times 2g}$ the rational representation of φ. Then φ being an isomorphism of polarised abelian varieties with half-basis translates to

1. $A(Z', D) = (Z, D)R^T$ (cf. Section 2.1) and
2. $AZ'D^{-1} \equiv ZD^{-1} \mod \Lambda_Z = (Z, D)\mathbb{Z}^{2g}$.

Condition 2 is equivalent to

$$AZ'D^{-1} - ZD^{-1} \in (Z, D)\mathbb{Z}^{2g\times g}$$

or (after multiplying with D) to

$$AZ' - Z \in (Z, D)\mathbb{Z}^{2g\times g}D. \tag{3.1}$$

Now we will reformulate condition 1 writing $R^T = \left(\begin{smallmatrix} R_{11} & R_{12} \\ R_{21} & R_{22}\end{smallmatrix}\right)$ with $g\times g$-blocks R_{ij}. Looking only at the right block of condition 1 states

$$AZ' = ZR_{11} + DR_{21}.$$

Plugging this in shows that (3.1) is equivalent to

$$(Z, D)\left(\begin{smallmatrix} R_{11}-1 \\ R_{21}\end{smallmatrix}\right) = Z(R_{11} - 1) + DR_{21} \in (Z, D)\mathbb{Z}^{2g\times g}D.$$

Since (Z, D) encodes an isomorphism of $\mathbb{R}$-vector spaces $\mathbb{R}^{2g} \longrightarrow \mathbb{C}^g$ it is invertible and we arrive at

$$\left(\begin{smallmatrix} R_{11}-1 \\ R_{21}\end{smallmatrix}\right) \in \mathbb{Z}^{2g\times g}D.$$

Thus, the matrix R is an element of the group

$$\Gamma_D^{emb} = \{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_D \mid a - 1_g \equiv b \equiv 0 \mod D\}$$

where we write $a \equiv 0 \mod D$ for $a \in D\cdot\mathbb{Z}^{g\times g}$.
Similarly to $\Gamma_D(D)$ in Chapter 7, Γ_D^{emb} is a subgroup of finite index in Γ_D and acts properly and discontinuously on $\mathcal{H}_g$. So we obtain

Theorem 3.8. *The normal complex analytic space $\mathcal{A}_D^{emb} := \mathcal{H}_g/\Gamma_D^{emb}$ is a moduli space for polarised abelian varieties of type $D = diag(d_1, \dots, d_g)$ with a Heisenberg equivariant embedding map (i.e. a half-basis of $K(D)$). The embedding $\Gamma_D^{emb} \hookrightarrow \Gamma_D$ induces a holomorphic map $\mathcal{A}_D^{emb} \to \mathcal{A}_D$ of finite degree.*

4 Abelian surfaces

Given an embedding of an abelian variety $\varphi_L : A \to \mathbb{P}^N$, it is natural to ask for the geometrical properties of image. In particular, one would like to understand the defining ideal of $\varphi_L(A)$ and write down explicit equations. If we vary the abelian variety A in its moduli space, the coefficients of these equations will depend on the moduli point.
Igusa has shown in [Igu72] that the moduli spaces $\mathcal{A}_D$ and $\mathcal{A}_D(D)$ etc. discussed in Section 7.4 all are (possibly singular) quasi-projective varieties. These can be resolved and compactified in various ways to smooth projective varieties and the boundary strata will correspond to certain singular degenerations of A. We refer to Hulek, Kahn and Weintraub who in [HKW93] described certain non-singular compactifications $\widetilde{\mathcal{A}}_D$ resp. $\widetilde{\mathcal{A}}_D(D)$ in the case of abelian surfaces.
It is of great interest to know if these moduli spaces are rational (i.e. birational to $\mathbb{P}^n$ for some n) or unirational (X is called unirational, if there exists a dominant rational map from $\mathbb{P}^n$ to X). If this is the case, one may hope to be able to write down explicit equations for the image $\varphi_L(A) \subset \mathbb{P}^N$. Gritsenko has proven in [Gri94] that $\widetilde{\mathcal{A}}_d := \widetilde{\mathcal{A}}_{(1,d)}$ is *not* unirational for $d \geq 13$ and $d \neq 14, 15, 16, 18, 20, 24, 30, 36$, by constructing cusp forms of weight 3 with respect to the paramodular group $\Gamma_D(D)$, while it was known that $\widetilde{\mathcal{A}}_d$ is rational or unirational for $d = 1, 2, 3, 4, 5, 7, 9$.
In this chapter we want to review embeddings of abelian surface A of type $(1, d)$ for $d = 1, 2, 3, 4, 5$. One always starts with a basis of $H^0(L)$ such that $\varphi_L(A)$ is Heisenberg-invariant in the sense of Section 2.7.4. We note that for a very ample line bundle L of type (d_1, d_2) we have $\dim(H^0(L)) = d_1 d_2$ so that the image

$$\varphi_L(A) \subset \mathbb{P}^{d_1 d_2 - 1}$$

is a surface of degree $2d_1 d_2$.
Caveat: Many authors write *moduli space of abelian varieties with level structure*, but actually mean different things. We adapted the notation to our definitions in Section 7.4 and 3.1 as far as possible.

4.1 Polarisation of type (1,1)

A treatise of the principally polarised case can be found for example in [BL04] (Chapter 10.2), but the analyses of the Kummer surface appearing goes back to the nineteenth century.
For an irreducible principal polarisation L, the map

$$\varphi = \varphi_{L^2} \colon A \longrightarrow \mathbb{P}^3$$

factors via an embedding

$$\psi\colon K := A/\langle(-1)_X\rangle \longrightarrow \mathbb{P}^3$$

(cf. Theorem 7.15).

We will identify K with its image under ψ and try to find equations describing it.

Starting with the standard coordinates x_{ij}, $i, j \in \mathbb{Z}/2\mathbb{Z}$, on which $\mathcal{H}(2,2)$ acts in the standard way:

$$\begin{aligned} \sigma_1\colon x_{ij} &\mapsto x_{i+1,j} & \qquad \tau_1\colon x_{ij} &\mapsto (-1)^i x_{ij} \\ \sigma_2\colon x_{ij} &\mapsto x_{i,j+1} & \qquad \tau_2\colon x_{ij} &\mapsto (-1)^j x_{ij} \end{aligned}$$

It is useful to introduce new coordinates

$$\begin{aligned} y_0 &= x_{01} + x_{10} & \qquad y_2 &= x_{01} - x_{10} \\ y_1 &= x_{11} + x_{00} & \qquad y_3 &= x_{11} - x_{00}. \end{aligned}$$

In these coordinates one can show:

1. K is singular in the coordinate points.
2. The coordinate planes touch K in smooth conics.

These properties can be used to find an equation for K. The map

$$\varphi\colon A \longrightarrow K \subset \mathbb{P}^3$$

is $\mathcal{H}(2,2)$-equivariant. In the new coordinates $y_0, \ldots, y_3$ the elements σ_1 and τ_1 act as follows:

$$\sigma\colon \begin{cases} y_0 \mapsto y_2 \\ y_1 \mapsto y_3 \\ y_2 \mapsto y_0 \\ y_3 \mapsto y_1 \end{cases} \qquad \tau\colon \begin{cases} y_0 \mapsto y_1 \\ y_1 \mapsto y_0 \\ y_2 \mapsto -y_3 \\ y_3 \mapsto -y_2 \end{cases} \tag{4.1}$$

Let Q be the quartic defining the Kummer surface K, i.e.

$$K = \{y \in \mathbb{P}^3 \mid Q(y) = 0\}.$$

The Kummer surface K is invariant under the action of $K(L_0^2)$, i.e. there is a character $\chi\colon K(L_0^2) \longrightarrow \mathbb{C}^*$ such that

$$\alpha^* Q = \chi(\alpha) Q \text{ for all } \alpha \in K(L_0^2). \tag{4.2}$$

From Property 2 it follows that

$$Q(y_0, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_3) = F_i^2(y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_3)$$

for some quadric polynomials F_i, $i = 0, \dots, 3$.
Since K contains all four coordinate points (Property 1), it follows that F_0 is of the shape

$$F_0(y_1, y_2, y_3) = \lambda_1 y_2 y_3 + \lambda_2 y_1 y_3 + \lambda_3 y_1 y_2$$

for some $\lambda_i \in \mathbb{C}$.
Now by using Equation (4.2), and the fact that σ and τ act on the y_i as described in (4.1) one obtains

$$\begin{aligned} F_1^2(y_0, y_2, y_3) &= \chi(\tau) F_0^2(y_0, -y_3, -y_2), \\ F_2^2(y_0, y_1, y_3) &= \chi(\sigma) F_0^2(y_3, y_0, y_1), \\ F_3^2(y_0, y_1, y_2) &= \chi(\sigma\tau) F_0^2(y_2, -y_1, -y_0). \end{aligned}$$

Under the assumption that $\chi \equiv 1$ we obtain

$$\begin{aligned} F_0^2 + F_1^2 + F_2^2 + F_3^2 =& 2\lambda_1^2(y_2^2 y_3^2 + y_0^2 y_1^2) + 2\lambda_2^2(y_1^2 y_3^2 + y_0^2 y_2^2) + 2\lambda_3^2(y_1^2 y_2^2 + y_0^2 y_3^2) \\ &+ 2\lambda_1\lambda_2(y_0 y_1 + y_2 y_3)(y_1 y_3 - y_0 y_2) \\ &+ 2\lambda_1\lambda_3(y_0 y_3 - y_1 y_2)(y_0 y_1 - y_2 y_3) \\ &+ 2\lambda_2\lambda_3(y_1 y_2 + y_0 y_3)(y_1 y_3 + y_0 y_2). \end{aligned}$$

Now for $p = Q - F_0^2 - F_1^2 - F_2^2 - F_3^2$ we have

$$p|_{y_0=0} = -F_1^2|_{y_0=0} - F_2^2|_{y_0=0} - F_3^2|_{y_0=0}, \text{ hence}$$

$$p \equiv -\lambda_1^2 y_2^2 y_3^2 - \lambda_2^2 y_1^2 y_3^2 - \lambda_3^2 y_1^2 y_2^2 \mod y_0.$$

This congruence together with the three other ones obtained by setting $y_1, \dots, y_3 = 0$ have a solution in degree four unique modulo $y_0 y_1 y_2 y_3$, hence

$$p = -\lambda_1^2(y_2^2 y_3^2 + y_0^2 y_1^2) - \lambda_2^2(y_1^2 y_3^2 + y_0^2 y_2^2) - \lambda_3^2(y_1^2 y_2^2 + y_0^2 y_3^2) + \lambda_0^2 y_0 y_1 y_2 y_3.$$

If $\chi \not\equiv 1$, the congruences determining p have no solution in degree four, so these cases can not occur.
This means:

Proposition 4.1. *The coordinates of $\mathbb{P}^3$ can be chosen in such a way that the Kummer surface K associated to the abelian surface A with irreducible principle polarisation is given by an equation of the form*

$$\begin{aligned} Q =& \lambda_1^2(y_2^2 y_3^2 + y_0^2 y_1^2) + \lambda_2^2(y_1^2 y_3^2 + y_0^2 y_2^2) + \lambda_3^2(y_1^2 y_2^2 + y_0^2 y_3^2) \\ &+ 2\lambda_1\lambda_2(y_0 y_1 + y_2 y_3)(y_1 y_3 - y_0 y_2) \\ &+ 2\lambda_1\lambda_3(y_0 y_3 - y_1 y_2)(y_0 y_1 - y_2 y_3) \\ &+ 2\lambda_2\lambda_3(y_1 y_2 + y_0 y_3)(y_1 y_3 + y_0 y_2) \\ &+ \lambda_0^2 y_0 y_1 y_2 y_3 \end{aligned} \tag{4.3}$$

for some $\lambda = (\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^3$.

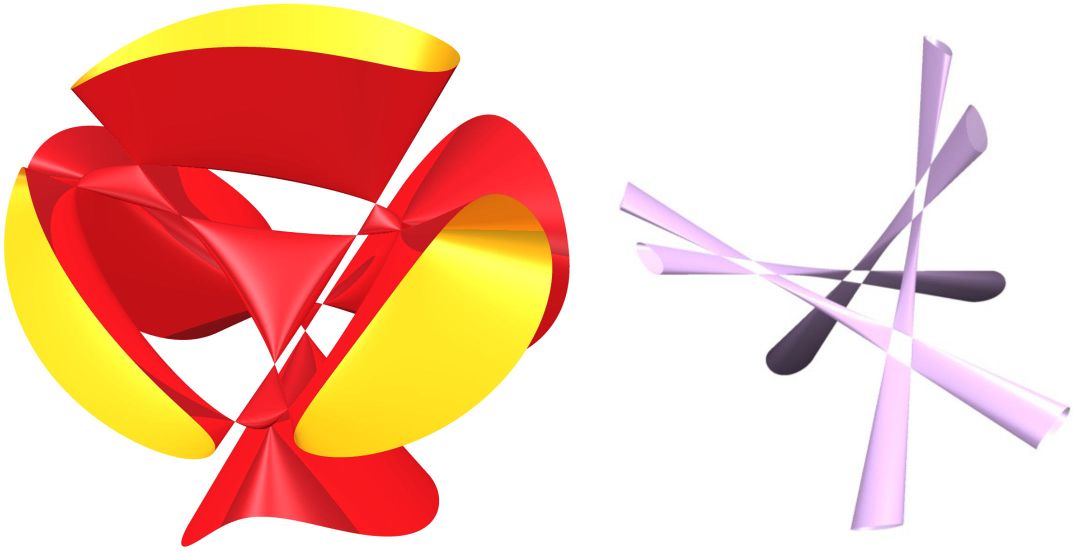


Figure 4.1: The Kummer surface in the affine chart $y_0 = -x + y + z + 1$, $y_1 = x - y + z + 1$, $y_2 = x + y - z + 1$, $y_3 = x + y + z - 1$ for different values of the parameters

The Kummer surface has many interesting geometrical properties, that can be found in classical literature, like for example [Hud90]:

- K has 16 singular points which are exactly the images of the 16 two-torsion points $z \in A_2 \subset A$.
- For any $z \in A_2$ we will denote the unique divisor in the linear system $|t_z^* L|$ by D_z. The curve $C_z = \varphi(D_z)$ is a conic and $2C_z$ is a complete intersection of K with a plane in $\mathbb{P}^3$. We denote this plane by P_z. Geometrically this means that P_z touches K along C_z. The 16 planes P_z, $z \in A_2$, are called *singular planes* of K in $\mathbb{P}^3$.

Now we can analyse the configuration of these points and planes explicitly. Summarizing we get:

1. The 16 singular planes and the 16 singular points of K form a 16_6 configuration, i.e.
   - any singular plane contains exactly 6 singular points.
   - any singular point is contained in exactly 6 singular planes.
2. Any two different singular planes have exactly two singular points in common.
3. Three pairwise different singular planes P_{z_1}, P_{z_2} and P_{z_3} always intersect in one point p. The point p is singular if and only if the singular points z_1, z_2 and z_3 span the singular plane P_p. This reflects some kind of self-duality of K (see [Hud90]).

We can consider two different kinds of tetrahedra build by these singular planes:

1. A *Rosenhain tetrahedron* for K is a tetrahedron in $\mathbb{P}^3$ with singular planes of K as faces and singular points of K as vertices.

2. A *Göpel tetrahedron* for K is a tetrahedron in $\mathbb{P}^3$ with singular planes of K as faces such that the vertices are not singular points.

The following relations between points $z_1, \ldots, z_4$ determine if their corresponding singular planes form a Göpel or a Rosenhain tetrahedron:

1. Singular planes $P_{z_1}, P_{z_2}, P_{z_3}$ and P_{z_4} form a Rosenhain tetrahedron if and only if z_1, z_2 and z_3 span a singular plane and $z_4 = z_1 + z_2 + z_3$.

2. There are exactly 80 Rosenhain tetrahedra for K.

3. Singular planes $P_{z_1}, P_{z_2}, P_{z_3}$ and P_{z_4} form a Göpel tetrahedron if and only if z_1, z_2 and z_3 do not span a singular plane and $z_4 = z_1 + z_2 + z_3$.

4. There are exactly 60 Göpel tetrahedra for K.

The coordinates used in the equation given above correspond to a Rosenhain tetrahedron.

The equation given above allows the following statement about moduli spaces:

Remark 4.2. *The moduli space $\mathcal{A}_{(1,1)}$ of principally polarised abelian varieties is of dimension three. Hence the family of Kummer surfaces is also of dimension three. Since the family of quartics given above is parametrised by $\mathbb{P}^3$, this implies that for a general $\lambda \in \mathbb{P}^3$ Equation (4.3) defines a Kummer surface.*
In other words: The moduli space of Kummer surfaces is birational to $\mathbb{P}^3$.
According to [BLvS89] the exceptional locus is $\Delta = \{\lambda_1\lambda_2\lambda_3 = 0\}$.

4.2 Polarisation of type (1,2)

The case of the (1,2)-polarisation was analysed by Wolf Barth in his paper *Abelian surfaces with (1,2)-polarisation* ([Bar87]). His motivation was the work of Adler-van Moerbeke ([AvM82]) and Haine ([Hai83]) on certain cases of geodesic flow on $SO(4)$ leading to integrable Hamiltonian systems.
In this case φ_L is a rational map from the abelian surface to $\mathbb{P}^1$, so even for dimensional reasons it can not be an embedding.
But the line bundle L^2 of type $(2,4)$ is very ample, hence induces an embedding

$$\varphi_{L^2} \colon A \longrightarrow \mathbb{P}^7.$$

In this case let $\{x_{jk} \mid 0 \le j \le 1, 0 \le k \le 3\}$ be the standard coordinates of $\mathbb{P}^7$ (in the sense of Section 2.7.4) on which $\mathcal{H}^e(2,4)$ acts as follows:

$$\begin{aligned}
\sigma_1 &\colon x_{jk} \mapsto x_{j+1,k}\\
\sigma_2 &\colon x_{jk} \mapsto x_{j,k+1}\\
\tau_1 &\colon x_{jk} \mapsto (-1)^j x_{jk}\\
\tau_2 &\colon x_{jk} \mapsto i^k x_{jk}\\
\iota &\colon x_{jk} \mapsto x_{-j,-k}
\end{aligned}$$

He introduces new coordinates on $\mathbb{P}^7$ by

$$\begin{array}{llll}
y_1 = x_{00} + x_{02} & y_3 = x_{01} + x_{03} & y_5 = x_{00} - x_{02} & y_7 = x_{01} - x_{03}\\
y_2 = x_{10} + x_{12} & y_4 = x_{11} + x_{13} & y_6 = x_{10} - x_{12} & y_8 = x_{11} - x_{13}.
\end{array}$$

Then in the new coordinates $\varphi_{L^2}(A)$ is cut out by the six quadrics

$$\begin{aligned}
q_1 &= \mu_1(y_1^2 + y_2^2) - \lambda_1(y_3^2 + y_4^2) + \mu_1(y_5^2 + y_6^2) + \lambda_1(y_7^2 + y_8^2)\\
q_2 &= -\lambda_1(y_1^2 + y_2^2) + \mu_1(y_3^2 + y_4^2) + \lambda_1(y_5^2 + y_6^2) + \mu_1(y_7^2 + y_8^2)\\
q_3 &= \mu_1(y_1^2 - y_2^2) - \lambda_1(y_3^2 - y_4^2) + \mu_1(y_5^2 - y_6^2) + \lambda_1(y_7^2 - y_8^2)\\
q_4 &= -\lambda_1(y_1^2 - y_2^2) + \mu_1(y_3^2 - y_4^2) + \lambda_1(y_5^2 - y_6^2) + \mu_1(y_7^2 - y_8^2)\\
q_5 &= \mu_1(y_1y_2) - \lambda_1(y_3y_4) + \mu_1(y_5y_6) + \lambda_1(y_7y_8)\\
q_6 &= -\lambda_1(y_1y_2) + \mu_1(y_3y_4) + \lambda_1(y_5y_6) + \mu_1(y_7y_8)
\end{aligned} \tag{4.4}$$

for some projective parameters $(\lambda_j : \mu_j) \in \mathbb{P}^1$, $j = 1, 2, 3$.
The other way around, one may ask, under which condition a point $(\lambda_1 : \mu_1), (\lambda_2 : \mu_2), (\lambda_3 : \mu_3)$ in $(\mathbb{P}^1)^3$ actually describes an abelian surface. For this purpose Barth defines

$$r_{jk} = r_{jk}(\lambda, \mu) := (\lambda_j^2\mu_k^2 - \lambda_k^2\mu_j^2)(\lambda_j^2\lambda_k^2 - \mu_j^2\mu_k^2)$$
$$\text{and } r = r_{12} \cdot r_{23} \cdot r_{31}.$$

The equation $r = 0$ is equivalent to the fact there exists some $k \neq l$ such that $\frac{\mu_j}{\lambda_j} = \pm(\frac{\mu_k}{\lambda_k})^{\pm 1}$.

Theorem 4.3. *For $(\lambda_1 : \mu_1), (\lambda_2 : \mu_2), (\lambda_3 : \mu_3) \in (\mathbb{P}^1)^3$ the following properties are equivalent:*

1. *$r \neq 0$.*

2. *The quadrics $q_1, \dots, q_6$ generate the ideal sheaf of a smooth abelian surface (of degree 16 with a (2,4)-polarisation) in $\mathbb{P}^7$.*

In other words: The moduli space $\mathcal{A}_{(2,4)}^{emb}$ for Heisenberg invariantly embedded abelian surfaces with a (2,4)-polarisation is $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$ where $\Delta = \{((\lambda_1 : \mu_1), (\lambda_2 : \mu_2), (\lambda_3 : \mu_3)) \in (\mathbb{P}^1)^3 \mid r = 0\}$. In particular, this moduli space is rational.

4.3 Polarisation of type (1,3)

In their paper [BL95] Christina Birkenhake and Herbert Lange study the moduli spaces $\mathcal{A}^{iso}_{(1,d)}$ of abelian surfaces of type $(1,d)$ with an isogeny to a principally polarised abelian surface, especially for the cases $d = 2$ and $d = 3$.

In a first step, they identify $\mathcal{A}^{iso}_{(1,d)}$ with the moduli space $\mathcal{C}^d_2$ of cyclic étale d-fold coverings of curves of genus 2 in the following way:
Any isogeny π from a $(1,d)$-polarised abelian surface (X, L) to a principally polarised abelian surface (Y, P) is restricted to $\pi|_C : C \longrightarrow H$ where H is a curve of genus two such that Y is the Jacobian $J(H)$ and $C = \pi^{-1}(H)$.
The other way around any cyclic étale d-fold covering $f : C \longrightarrow H$ extends to a d-fold covering (i.e. isogeny of degree d) $\pi : X \longrightarrow Y = J(H)$ and $L = \pi^*\mathcal{O}_Y(H)$ defines a polarisation of type $(1,d)$ on X.

Let $\mathcal{A}^0_{(1,3)} \subset \mathcal{A}^{iso}_{(1,3)}$ be the subset of abelian surfaces of type (1,3) with an isogeny onto a Jacobian of a smooth curve of genus 2.
Let us construct a map as follows: Let (X, L, π) be an element of $\mathcal{A}^0_{(1,3)}$. By our considerations at the beginning of this section it corresponds to an étale 3-fold covering $f : C \longrightarrow H$ in $\mathcal{C}^3_2$. The hyperelliptic covering lifts to a covering $C \longrightarrow E$, where E is an elliptic curve uniquely determined by f. Let $\hat{\Xi}$ denote the polarisation on $E \times E$ defined by the divisor $E \times \{0\} + \{0\} \times E + A$. Then one can show that there exists an embedding $C \hookrightarrow E \times E$ whose image is contained in the linear system $|E \times \{0\} + \{0\} \times E + A|$ and it is uniquely determined modulo translation by elements of $K(\hat{\Xi})$. By looking at the action of $T = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$ on $E \times E$ they prove that C does not contain any point from $K(\hat{\Xi})$.
Denote by $\mathcal{M}$ the (coarse) moduli space of pairs (E, C) where E is an elliptic curve and C a smooth curve in the linear system $|E \times \{0\} + \{0\} \times E + A|$ such that $K(\hat{\Xi}) \cap C = \emptyset$ modulo translation by elements in $K(\hat{\Xi})$.
Now we have constructed a map $\psi : \mathcal{A}^{iso}_{(1,3)} \longrightarrow \mathcal{M}$.

Theorem 4.4. *$\psi : \mathcal{A}^{iso}_{(1,3)} \longrightarrow \mathcal{M}$ is an isomorphism of algebraic varieties.*

To show this, they construct the inverse map as follows: For $(E, C) \in \mathcal{M}$ the automorphism T of $E \times E$ given above acts on every curve of the linear system. In particular, T restricts to an automorphism τ of C which is of order 3. Moreover, τ is fixed point free, so it induces an étale 3-fold covering $C \longrightarrow H = C/\tau$, which then corresponds to an element $(X, L, \pi) \in \mathcal{A}^0_{(1,3)}$.

Corollary 4.5. *$\mathcal{A}^{iso}_{(1,3)}$ is rational.*

The idea of the proof here is to show that $\mathcal{M}^0 := \{(E, C) \in \mathcal{M} \mid E$ admits no nontrivial automorphisms$\}$ is rational. To show that, they consider the open set $U = \mathbb{C} \setminus \{0, 1728\}$ parametrising elliptic curves without nontrivial automorphisms and the universal family $\mathcal{E} \longrightarrow U$. They construct a vector bundle of rank 3 over U whose projectivisation $\mathbb{P}_U$ parametrises the linear system $|E \times \{0\} + \{0\} \times E + A|$. By construction $\mathcal{M}^0$ is an open subset of the quotient $P_U/K(\hat{\Xi})$. Since every vector bundle on U is trivial, $\mathbb{P}_U \cong \mathbb{P}^2 \times U$ and $\mathbb{P}_U/K(\hat{\Xi}) \cong \mathbb{P}^2/(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \times U$, which is rational by Lüroth's theorem.

Since $\mathcal{A}^{iso}_{(1,3)}$ is a finite covering of $\mathcal{A}_{(1,3)}$, this shows:

Corollary 4.6. *The moduli space $\mathcal{A}_{(1,3)}$ of (1,3)-polarised abelian surfaces without extra structure is at least unirational.*

4.4 Polarisation of type (1,4)

The case of the (1,4)-polarisation is treated by Birkenhake, Lange and van Straten in their paper *Abelian surfaces of type (1,4)* ([BLvS89]).

They consider A to be a complex abelian surface with an ample line bundle L of type (1,4) on it. One can assume L to be symmetric without changing the corresponding map

$$\varphi_L \colon A \longrightarrow \mathbb{P}^3.$$

First of all they exclude the case that (A, L) is isomorphic to a product of elliptic curves as polarised abelian variety, i.e. that there are elliptic curves E_1 and E_2 on A and line bundles L_1 on E_1 of degree 4 and L_2 on E_2 of degree 1, s.t. $(A, L) \cong (E_1 \times E_1, p_1^* L_1 \otimes p_2^* L_2)$, in which the complete linear system $|L|$ has a fixed component. From now on we will assume that A is not a product of elliptic curves. Then $|L|$ is base point free (Lemma 7.23).
So φ_L is a well-defined map on the whole of A whose image $\bar{A} \subset \mathbb{P}^3$ is invariant under the action of $\mathbf{H}_4^e = \langle \sigma, \tau, \iota \rangle$ acting on $\mathbb{P}^3$ with coordinates $x_0, \ldots, x_3$ via

$$\sigma \colon x_j \longmapsto x_{j-1} \qquad \tau \colon x_j \longmapsto i^{-j} x_j \qquad \iota \colon x_j \longmapsto x_{-j}.$$

Now they introduce other coordinates similar to Barth in Section 4.2

$$y_0 = x_0 + x_2 \quad y_1 = x_0 - x_2 \quad y_2 = x_1 + x_3 \quad y_3 = x_1 - x_3$$

and show:

Lemma 4.7. 1. *$\bar{A}$ is a surface of degree $d = 8$ or 4 in $\mathbb{P}^3$.*

2. *The defining polynomial Q of $\bar{A}$ is actually a polynomial in the squares $y_0^2, y_1^2, y_2^2, y_3^2$, i.e. there is a $\widetilde{Q} \in \mathbb{C}[z_0, \ldots, z_3]$ such that*

$$Q(y_0, y_1, y_2, y_3) = \widetilde{Q}(y_0^2, y_1^2, y_2^2, y_3^2).$$

Denote by $\overline{C}$ the subset of $\mathbb{P}^3 = \mathbb{P}^3(z_0, \ldots, z_3)$ defined by $\widetilde{Q}(z_0, z_1, z_2, z_3) = 0$. Then our reasoning above means geometrically:

Lemma 4.8. *The map $\mathbb{P}^3(y_0, \ldots, y_3) \longrightarrow \mathbb{P}^3(z_1, \ldots, z_3)$, $z_i = y_i^2$ induces a covering $\bar{p}\colon \overline{A} \longrightarrow \overline{C}$ that is $8:1$ outside the coordinate planes.*

In the case $\overline{A}$ is of degree eight, one finds with the same arguments than in Section 4.1 that $\widetilde{Q}$ is of the form

$$\begin{aligned}\widetilde{Q} =&\lambda_1^2(z_2^2z_3^2 + z_0^2z_1^2) + \lambda_2^2(z_1^2z_3^2 + z_0^2z_2^2) + \lambda_3^2(z_1^2z_2^2 + z_0^2z_3^2)\\ &+ 2\lambda_1\lambda_2(z_0z_1 + z_2z_3)(z_1z_3 - z_0z_2)\\ &+ 2\lambda_1\lambda_3(z_0z_3 - z_1z_2)(z_0z_1 - z_2z_3)\\ &+ 2\lambda_2\lambda_3(z_1z_2 + z_0z_3)(z_1z_3 + z_0z_2)\\ &+ \mu_0z_0z_1z_2z_3\end{aligned} \tag{4.5}$$

for some $(\lambda_1, \lambda_2, \lambda_3, \mu_0) \in \mathbb{C}^4 \setminus \{0\}$.

If $\overline{A}$ and hence Q is a quartic, then $Q(y_0, y_1, y_2, y_3) = \widetilde{Q}(y_0^2, y_1^2, y_2^2, y_3^2)$ for some quadratic polynomial $\widetilde{Q} \in \mathbb{C}[z_0, \ldots, z_4]$ and up to a projective transformation $\widetilde{Q}$ is of the form

$$\widetilde{Q} = \lambda_1(z_0z_1 + z_2z_3) + \lambda_2(z_1z_3 - z_0z_2) \tag{4.6}$$

for some $(\lambda_1 : \lambda_2) \in \mathbb{P}^2$.

An alternative description of the two cases occurring is as follows: Consider $K(L) = K_1 \oplus K_2$ a decomposition into maximal isotropic subspaces ($K_1 \cong K_2 \cong \mathbb{Z}/4\mathbb{Z}$) and denote by $\pi\colon A \longrightarrow B = A/K_2$ the natural projection. There is a line bundle M on B with $L = \pi^*M$. Let X be the unique divisor of $|M|$ and $Y = \pi^{-1}(X)$.
With this set-up they prove the following two characterisations:

Theorem 4.9. *Suppose X and Y do not admit elliptic involutions compatible with the action of K_2. Then $\overline{A}$ is an octic and $\varphi_L\colon A \longrightarrow \overline{A} \subseteq \mathbb{P}^3$ is birational.*

Theorem 4.10. *Assume X and Y admit elliptic involutions compatible with the action of K_2. Then $\overline{A}$ is an quartic and $\varphi_L\colon A \longrightarrow \overline{A} \subseteq \mathbb{P}^3$ is of degree 2 onto its image.*

In both cases $\overline{A}$ is cut out set-theoretically by the equation $Q = 0$ with

$$\begin{aligned}Q(y_0, y_1, y_2, y_3) =&\lambda_1^2(y_2^4y_3^4 + y_0^4y_1^4) + \lambda_2^2(y_1^4y_3^4 + y_0^4y_2^4) + \lambda_3^2(y_1^4y_2^4 + y_0^4y_3^4)\\ &+ 2\lambda_1\lambda_2(y_0^2y_1^2 + y_2^2y_3^2)(y_1^2y_3^2 - y_0^2y_2^2)\\ &+ 2\lambda_1\lambda_3(y_0^2y_3^2 - y_1^2y_2^2)(y_0^2y_1^2 - y_2^2y_3^2)\\ &+ 2\lambda_2\lambda_3(y_1^2y_2^2 + y_0^2y_3^2)(y_1^2y_3^2 + y_0^2y_2^2)\\ &+ \lambda_0^2y_0^2y_1^2y_2^2y_3^2\end{aligned} \tag{4.7}$$

for some $\lambda = (\lambda_0 : \cdots : \lambda_3)$. In the octic case this is achieved by just setting λ_0 to be a square-root of μ_0. In the quartic case one has to square Equation (4.6) to obtain

(4.7) with $\lambda_3 = 0$ and $\lambda_0^2 = 2(\lambda_1^2 - \lambda_2^2)$ (or $\lambda_2 = 0$ and $\lambda_0^2 = -2(\lambda_1^2 + \lambda_3^2)$ resp. $\lambda_1 = 0$ and $\lambda_0^2 = 2(\lambda_2^2 - \lambda_3^2)$ in the other two cases).
Finally they show that in the octic case a point $\lambda \in \mathbb{P}^3$ determines an abelian surface via Equation (4.7) or more precisely:

Theorem 4.11. *Let Δ be the set of all $\lambda \in \mathbb{P}^3$ such that (4.5) does not describe a Kummer surface, i.e. $\Delta = \{\lambda_1\lambda_2\lambda_3 = 0\}$. Then*

$$(\mathbb{P}^3 \setminus \Delta)/(\lambda_0 \mapsto -\lambda_0)$$

is the moduli space of embedded abelian surfaces of type $(1, 4)$.

This means that this moduli space is unirational.
They also describe some degenerations occurring for $\lambda \in \Delta$.

4.5 Polarisation of type (1,5)

The case of the (1,5)-polarisation was treated by G. Horrocks and D. Mumford taking another approach than in the previous cases. In [HM73] they construct a rank 2 vector bundle $\mathcal{F}$ on $\mathbb{P}^4$ from the Koszul complex of $\mathcal{O}(1) \otimes_{\mathbb{C}} V$ where $V = \mathbb{C}(\mathbb{Z}/5\mathbb{Z})$ is the Schrödinger representation of $\mathbf{H}_5$. This bundle turns out to be irreducible. Before, it was not known if any indecomposable vector bundles of rank 2 on $\mathbb{P}^4$ exist.
Since $\bigwedge^2 \mathcal{F} \simeq \mathcal{O}(5)$ and all global sections of $\mathcal{F}$ are $\mathbf{H}_5$-invariant, $\bigwedge^2 \Gamma(\mathcal{F}) \simeq \Gamma(\mathcal{O}(5))^{\mathbf{H}_5}$. The dimension of $\Gamma(\mathcal{O}(5))^{\mathbf{H}_5}$ is computed to be 6 using character theory and an explicit $\mathbb{C}$-basis is given by

$$\begin{aligned}
S &= \sum_i x_i^5 & Q &= \sum_i x_{i-1}x_i^3x_{i+1} & Q' &= \sum_i x_{i-2}x_i^3x_{i+2} \\
Y &= 5\prod_i x_i & R &= \sum_i x_{i-1}^2x_ix_{i+1}^2 & R' &= \sum_i x_{i-2}x_i^2x_{i+1}^2.
\end{aligned}$$

The simultaneous vanishing locus L of all elements of $\Gamma_{\mathbf{H}_5}(\mathcal{O}(5))$ consists of 25 skew lines.
They analyse the normalizer N of the Heisenberg group $\mathbf{H}_5$ which turns out to be an semidirect product $N = \mathbf{H}_5 \rtimes SL_2(\mathbb{Z}/5\mathbb{Z})$ with 15000 elements all given in explicit matrix form.
Using this they proof the two main theorems

Theorem 4.12. *For almost all $s \in \Gamma(\mathcal{F})$, $V(s)$ is a non-singular surface $X_s \subset \mathbb{P}^4$ of degree 10. Whenever X_s is not singular, it is an abelian surface.*

Singularities may only occur where X_s intersects L.

Theorem 4.13. *Every abelian surface $Z \subset \mathbb{P}^4$ is projectively equivalent to the zero set of some section s of $\mathcal{F}$.*

Now denote by $\mathcal{A}_{(1,5)}(1,5)^*$ the subset of all $(X, \lambda, \alpha) \in \mathcal{A}_{(1,5)}(1,5)$ for which L_λ is very ample and by $\mathbb{P}(\Gamma(\mathcal{F}))^*$ the subset of all $\mathbb{C}s \in \mathbb{P}(\Gamma(\mathcal{F}))$ for which $V(s)$ is smooth. Then $\Gamma_D/\Gamma_D(D) \cong SL_2(\mathbb{Z}/5\mathbb{Z})$ acts on $\mathcal{A}_{(1,5)}(1,5)$ (c.f. Section 7.4) and $N/H \cong SL_2(\mathbb{Z}/5\mathbb{Z})$ acts on $\mathbb{P}(\Gamma(\mathcal{F}))$.
Finally, they state the following result

Theorem 4.14. $\mathcal{A}_{(1,5)}(1,5)^* \cong \mathbb{P}(\Gamma(\mathcal{F}))^*$ *as quasi-projective varieties and the action of* $\Gamma_D/\Gamma_D(D)$ *on* $\mathcal{A}_{(1,5)}(1,5)$ *corresponds to the action of* N/H *on* $\mathbb{P}(\Gamma(\mathcal{F}))$.

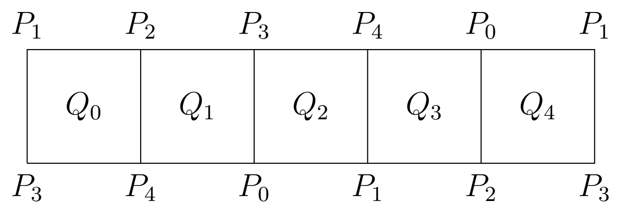
This shows that $\mathcal{A}_{(1,5)}(1,5)$ is rational.
They also describe some degenerations occurring outside the open sets *. A complete analysis of all degenerations was given by Barth, Hulek and Moore in [BHM87]: Despite being an smooth abelian surface, X_s can also take the following forms:

1. a translation scroll associated to a normal elliptic quintic curve,
2. the tangent scroll of a normal elliptic quintic curve,
3. a quintic elliptic scroll of multiplicity 2,
4. a union of five smooth quadric surfaces,
5. a union of five planes, each of multiplicity 2,

and this list is complete, i.e. these are the only degenerations that can occur.
The hierarchy of this degenerations is as follows:

$$\begin{tikzcd}[column sep=small]
 & \text{abelian surface} \arrow[d, dash] & & \text{dim } 3 \\
 & \text{translation scroll} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & \text{dim } 2 \\
\text{tangent scroll} \arrow[dr, dash] & \text{quintic scroll} \arrow[d, dash] & \text{5 quadrics} \arrow[dl, dash] & \text{dim } 1 \\
 & \text{5 double planes} & & \text{dim } 0
\end{tikzcd}$$

Let us have a closer look at the degeneration in case 4: In this case, the five smooth quadric surfaces Q_i intersect in the following manner building combinatorially a kind of twisted torus:



In his paper [Man88] Manolache shoes that the ideal of a general such surface is generated by 3 (Heisenberg invariant) quintics and 15 sextics.

In the next chapter we will take a closer look at $(1, 6)$ polarised surfaces and we will see in Section 5.4.3 that the diagram of degenerations in that case shares some common features with the above.

5 Moduli and geometry of (1,6)-polarised abelian surfaces

At the time Gritsenko wrote his paper [Gri94], it was not known, if $\mathcal{A}_6$ was rational resp. unirational. To our knowledge this question was first answered by Gross and Popescu in their paper [GP01].
In this chapter we will study the projective model Q of the moduli space $\mathcal{A}_6^{emb}$ they describe in some detail, and find for example its discriminant, that is the locus of points of Q, which do not describe a smooth abelian surface. We describe the degenerations of the surfaces occurring and the automorphisms of Q as a moduli space.
In Section 5.6 and 5.7 we will identify certain subfamilies of $(1,6)$-polarised abelian surfaces inside Q. One of these is the family of abelian surfaces that appear in the DGR-integrable system.

5.1 Introduction

If L is a line bundle of type $(1,6)$ on an abelian surface A, then $\dim(H^0(L)) = 6$, so L induces a rational map $\varphi_L\colon A \dashrightarrow \mathbb{P}^5$. It follows from Lemma 7.23 that φ_L is in fact basepoint free. If A does not contain an elliptic curve, then a result of Ramanan (Lemma 7.24 in this thesis) implies that L is very ample, so we get an embedding

$$\varphi_L : A \hookrightarrow \mathbb{P}^5.$$

The image then is a smooth surface of degree $2 \cdot 1 \cdot 6 = 12$.

In the following we want to study the image of this embedding. From examples e.g. from [SvS13], we expect the general surface to be cut out by four cubics and six quartics. From the exact sequence for the ideal sheaf $\mathcal{I}_A$ of A, twisted by $\mathcal{O}(3)$,

$$0 \longrightarrow \mathcal{I}_A(3) \longrightarrow \mathcal{O}_{\mathbb{P}^5}(3) \longrightarrow \mathcal{O}_A(3) = L^3 \longrightarrow 0,$$

we learn that

$$\dim H^0(\mathcal{I}_A(3)) \geq \dim H^0(\mathcal{O}_{\mathbb{P}^5}(3)) - \dim H^0(L^3) = 56 - 54 = 2,$$

so that there are at least two cubics in the ideal of A. To see that there are in fact at least four cubics in the ideal, we need to take the Heisenberg group into account.

5.1.1 A representation of the Heisenberg group

First we look at the finite Heisenberg group $\mathbf{H}_6$ generated by σ and τ. $\mathbf{H}_6$ acts on the polynomial ring $\mathbb{C}[x_0, \ldots, x_5]$ by

$$\sigma(x_i) = x_{i-1}, \quad \tau(x_i) = \xi^{-i} x_i,$$

where ξ is a primitive sixth root of unity. As the image of A is $\mathbf{H}_6$-invariant, also the ideal generated by the cubics in the ideal has to be invariant under $\mathbf{H}_6$. One might hope for invariant cubics, but one has:

Lemma 5.1. $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}_6} = 0.$

The proof is an argument similar to the one of Lemma 5.3.
Now a crucial idea of Gross and Popescu is to look at cubic polynomials that are invariant under a certain *subgroup* of $\mathbf{H}_6$.

Definition 5.2. We define $\mathbf{H}'$ as the subgroup of $\mathbf{H}_6$ generated by the elements σ^2 and τ^2.

$\mathbf{H}'$ is isomorphic to the Heisenberg group $\mathbf{H}_3$ of a (1,3)-polarised abelian surface. σ^2 and τ^2 both have order 3 and commute up to a constant from the subgroup $\mu_3 \subset \mathbb{C}^*$ of third roots of unity. So $\mathbf{H}'$ has 27 elements.

Lemma 5.3. *The space $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$ of $\mathbf{H}'$ invariant cubic forms on $\mathbb{P}^5$ has a basis $f_0, \ldots, f_3, g_0 = \sigma f_0, \ldots, g_3 = \sigma f_3$ where*

$$\begin{aligned}
f_0 &= x_0^3 + x_2^3 + x_4^3 & g_0 &= x_1^3 + x_3^3 + x_5^3 \\
f_1 &= x_1^2 x_4 + x_3^2 x_0 + x_5^2 x_2 & g_1 &= x_2^2 x_5 + x_4^2 x_1 + x_0^2 x_3 \\
f_2 &= x_1 x_2 x_3 + x_3 x_4 x_5 + x_5 x_0 x_1 & g_2 &= x_2 x_3 x_4 + x_4 x_5 x_0 + x_0 x_1 x_2 \\
f_3 &= x_0 x_2 x_4 & g_3 &= x_1 x_3 x_5.
\end{aligned}$$

Proof. All monomials are eigenvectors with respect to the action of τ and there are exactly 20 monomials of degree 3 invariant under τ^2, namely those belonging to the f_i and g_i in the statement of this lemma. Half of them (those belonging to the f_i) have τ-eigenvalue 1, the others have τ-eigenvalue -1.
The whole group $\mathbf{H}_6$ acts on $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$. We will show now that we can choose a basis of $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$ in such a way that it consists only of eigenvectors of τ. Suppose $f \in H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$ with $f = v + w$ such that $\tau v = v$, $\tau w = -w$ and $v \neq 0 \neq w$. Then $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$ also contains $\tau f = v - w$. Thus $\langle f, \tau f \rangle_{\mathbb{C}} = \langle v, w \rangle_{\mathbb{C}}$.
To obtain such a basis, we may only combine monomials from the same eigenspace. A polynomial in $\ker(\tau \pm \mathrm{id})$ containing a certain monomial m is invariant under σ^2 if and only if it contains $\sigma^2 m$ and $\sigma^4 m$ with the same coefficient. So it has to be a linear combination of the f_i and g_i given above. Conversely, it is easy to check that these are all $\mathbf{H}'$-invariant. $\square$

$\mathbf{H}_6$ acts on this space (σ by interchanging f_i and g_i, τ by leaving the f_i fixed and multiplying all g_i by -1). Thus $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$ as a representation of $\mathbf{H}_6$ splits up into four isomorphic irreducible subrepresentations

$$H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'} \cong \bigoplus_{i=0}^{3} \langle f_i, g_i \rangle.$$

Definition 5.4. We will identify $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$ with $V_0 \otimes W$ where V_0 is an irreducible two-dimensional representation of $\mathbf{H}_6$ (σ acting as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, τ as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$) and W is a four-dimensional complex vector space with basis $e_0, \ldots, e_3$ such that $V_0 \otimes e_i = \langle f_i, g_i \rangle$.

Then any $\mathbf{H}_6$-subrepresentation of $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$ is of the form $V_0 \otimes T$ for some subspace $T \subseteq W$.

Lemma 5.5. *Let A be an abelian surface of type $(1, 6)$ embedded $\mathbf{H}_6^e$-invariantly into $\mathbb{P}^5$ and let $\mathcal{I}_A$ be the homogeneous ideal defining A. Then*

$$\dim(H^0(\mathcal{I}_A(3))^{\mathbf{H}'}) \geq 4.$$

The idea of the proof is to look at the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'} \longrightarrow H^0(L^3)^{\mathbf{H}'}$$

and the action of ι on both sides. While ι fixes the elements of $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$, Gross and Popescu argue that $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$ is as a $\mathbf{H}_6$-representaion isomorphic to $H^0(M)$ for a (1,6)-polarised line bundle M. So $H^0(L^3)^{\mathbf{H}'} \cong \mathbb{C}^6$ decomposes into a two- and a four-dimensional eigenspace for ι. So $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$ must map to one of these two eigenspaces. Hence the kernel $H^0(\mathcal{I}_A(3))^{\mathbf{H}'}$ of the above restriction map is at least four dimensional.

5.1.2 Points, lines and planes

There are several important $\mathbf{H}_6$-invariant loci in $\mathbb{P}^5$. The following facts can be verified by a simple calculation, preferably using a computer algebra system:

Proposition 5.6.

1. *The simultaneous vanishing locus of all f_i and g_i are exactly the nine lines l_{ij}, $i, j \in \mathbb{Z}/3\mathbb{Z}$, given by $\{x_i = x_{i+3} = x_{i-1} + \omega^j x_{i+1} = x_{i+2} + \omega^j x_{i+4} = 0\}$ with ω a third root of unity.*

2. *The radical ideal I_{lines} corresponding to the union L of these nine lines is generated by the eight cubics f_i, g_i and the three determinantal conics*

$$x_1x_2 - x_4x_5, \quad x_0x_1 - x_3x_4, \quad x_2x_3 - x_0x_5.$$

3. *$\mathbf{H}_6$ acts on the nine lines as follows: σ sends l_{ij} to $l_{i-1,j}$ and τ sends l_{ij} to $l_{i,j-1}$.*

Proposition 5.7. *1. The four polynomials $f_0, \dots, f_3$ define a scheme B of dimension 2 and degree 12, decomposing into ten planes*

$$P = \{x_0 = x_2 = x_4 = 0\} \text{ (counted three times) and}$$

$$P_{ij} = \{x_{2i} = x_{2i-1} - \omega^j x_{2i+1} = x_{2i-2} + \omega^j x_{2i+2} = 0\}, i, j \in \mathbb{Z}/3\mathbb{Z},$$

(each counted once) and the nine lines described in Proposition 5.6.

2. τ fixes the plane P while it permutes the P_{ij} in 3-cycles: $P_{ij} \mapsto P_{i,j-1}$. σ^2 fixes P and maps P_{ij} to $P_{i-1,j}$.

3. The scheme C defined by $g_0, \dots, g_3$ is isomorphic to B via the action of σ. We will write P' for σP and P'_{ij} for σP_{ij} in the sequel.

The following locus will also play an important role.

Definition 5.8. The scheme $Z \subset \mathbb{P}^5$ is defined by the ideal generated by the six 2×2 minors of the matrix

$$\begin{pmatrix} f_0 & f_1 & f_2 & f_3 \\ g_0 & g_1 & g_2 & g_3 \end{pmatrix}.$$

Clearly, Z is the locus where the vectors

$$f(y) := (f_0(y), f_1(y), f_2(y), f_3(y)) \text{ and}$$
$$g(y) := (g_0(y), g_1(y), g_2(y), g_3(y))$$

become linearly dependent.

Proposition 5.9. *Z is a three-dimensional scheme of degree 18. Its reduction is also of dimension three and of degree 18.*
The ideal of minors of the Jacobi-matrix of the given ideal define a one-dimensional subscheme of degree 45 of Z. It decomposes into the nine lines described in Proposition 5.6 (each with multiplicity 5) and 144 isolated points.
72 of these points lie on the nine lines (8 on each line) and form two orbits of length 36 under the $\mathbf{H}_6$-action. The other 72 points lie not on the lines and form 12 orbits of length 6.

So we have Heisenberg-invariant configurations of objects in different dimensions:

Dimension	Objects
3	the locus Z
3	the scroll $S = \{x_1x_2 - x_4x_5 = x_0x_1 - x_3x_4 = x_2x_3 - x_0x_5 = 0\}$
2	20 planes P, P', P_{ij}, P'_{ij}
1	9 lines l_{ij}
0	144 points: 72 points on the lines
	and 72 points off the lines

One can easily check the following facts about this configuration:

Proposition 5.10. *1. The intersection of Z with the scroll S consists of certain lines and 6 planes $P^{(k)}$, including P and P' described above.*

2. Each plane P_{ij} resp. P'_{ij} intersects each $P^{(k)}$ in a line. Write $\tilde{l}_{ij} = P \cap P_{ij}$. Caveat: $\tilde{l}_{ij}$ is different from l_{ij} given above.

3. Each of the planes $P^{(k)}$ contains 12 of the off-line points from Proposition 5.9.

4. Each of these points is contained in three lines and each of these lines contains four points. This means, the 12 points and 9 lines form a dual Hesse configuration inside $P^{(K)}$.

5. Each line l_{ij} intersects each of the planes $P^{(k)}$ in one point not belonging to the designated points from Proposition 5.9.

These facts give an indication of the great richness and complexity of the $\mathbf{H}_6$-geometry of $\mathbb{P}^5$.

5.1.3 The morphism ϕ

Gross and Popescu define a rational map that plays a fundamental role in the geometry of $(1,6)$-polarised abelian surfaces. The simplest way to describe it as the map

$$\psi \colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^5, \quad y \mapsto (p_{01}(y) : p_{02}(y) : \ldots : p_{23}(y))$$

where $p_{ij}(y)$ is the (i,j)-minor of the matrix

$$\begin{pmatrix} f_0 & f_1 & f_2 & f_3 \\ g_0 & g_1 & g_2 & g_3 \end{pmatrix}.$$

Clearly, by the definition of Z, the domain of definition of the map ψ is precisely $\mathbb{P}^5 \setminus Z$.
However, the two copies of $\mathbb{P}^5$ in the above definition of ψ play a very different role.

In more intrinsic terms, the map is the composition of the map

$$\phi \colon \mathbb{P}^5 \dashrightarrow Grass(2, W^\vee), \quad y \mapsto V_y$$

and the Plücker-embedding

$$Grass(2, W^\vee) \hookrightarrow \mathbb{P}(\wedge^2 W^\vee) = \mathbb{P}^5.$$

Here we denote, for $y \in \mathbb{P}^5 \setminus Z$, by

$$V_y := \langle (f_0(y), \ldots, f_3(y)), (g_0(y), \ldots, g_3(y)) \rangle_{\mathbb{C}} \subseteq W^\vee$$

the plane spanned by $f(y)$ and $g(y)$. The Plücker-embedding $Grass(2, W^\vee) \hookrightarrow \mathbb{P}^5$ is realised using Plücker coordinates. The coordinates for a 2-plane spanned by $(x_0, \ldots, x_3)$ and $(y_0, \ldots, y_3)$ are

$$p_{ij} = x_i y_j - x_j y_i.$$

These satisfy the *Plücker relation*

$$p_{03}p_{12} - p_{02}p_{13} + p_{01}p_{23} = 0,$$

which gives the equation for $Grass(2, W^\vee) \subseteq \mathbb{P}^5$.

Definition 5.11. We put

$$\begin{aligned} G &:= \{p_{03}p_{12} - p_{02}p_{13} + p_{01}p_{23} = 0\}, \\ H &:= \{p_{03} + p_{12} = 0\}, \\ Q &:= H \cap G. \end{aligned}$$

We note that Q, the intersection of the hyperplane H with the Plücker quadric G, is a smooth quadric.

Proposition 5.12. *The image of ϕ is contained in the quadric Q.*

Proof. Check that

$$(f_0 g_3 - f_3 g_0) + (f_1 g_2 - f_2 g_1) = 0,$$

so the image of ϕ actually lies in Q. □

So we will consider now ϕ as a rational map

$$\phi : \mathbb{P}^5 \dashrightarrow Q,$$

defined outside the locus Z.

There is a slightly different point of view on the map ϕ. First remark that the map which maps a subspace $V \subseteq W^\vee$ to its annihilator $V^0 \subseteq W$ defines an isomorphism

$$Grass(2, W^\vee) \longrightarrow Grass(2, W).$$

The composition of this isomorphism with the map $\phi \colon \mathbb{P}^5 \dashrightarrow Grass(2, W^\vee)$ maps any point $y \in \mathbb{P}^5$ to the subspace $V_y^0 \subseteq W$ which is characterised by the property that

$$V_0 \otimes V_y^0 \subset V_0 \otimes W$$

is the largest $\mathbf{H}_6$-subrepresentation of $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$ consisting entirely of polynomials vanishing at the point $y \in \mathbb{P}^5 \setminus Z$.

5.1.4 The fibres of ϕ

In order to speak sensibly about the fibres of ϕ, we have to close them up. So we consider the following diagram

$$\begin{tikzcd}
 & \mathcal{X} \subseteq \mathbb{P}^5 \times Q \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{P}^5 \arrow[rr, dashed, "\phi"'] & & Q
\end{tikzcd}$$

where $\mathcal{X}$ is the closure of the graph of ϕ and π_1 and π_2 are the projections to $\mathbb{P}^5$ respectively Q. Then $\pi_1 \colon \mathcal{X} \longrightarrow \mathbb{P}^5$ is birational and $\pi_2 = \phi \circ \pi_1$ wherever all maps are defined. The fibres of the resulting map

$$\pi_2 \colon \mathcal{X} \longrightarrow Q$$

are birational to the (closures of) the fibres of ϕ, hence have the same dimension. We note that as $\dim Q = 3$ and $\dim \mathcal{X} = 5$, all non-empty fibres $\mathcal{X}_V$ of π_2 have dimension ≥ 2. It is useful to introduce another space $\mathcal{C}$ that sits in a diagram:

$$\begin{tikzcd}
\mathcal{X} \arrow[r, hook] \arrow[dr] & \mathcal{C} \arrow[r, hook] \arrow[d] & \mathbb{P}^5 \times Q \arrow[dl] \\
 & Q &
\end{tikzcd}$$

Definition 5.13. For a point $V \in Q$, considered as a two-dimensional subspace $V \subset W^\vee$, let

$$I_V \subset H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$$

be the ideal generated by the cubics in $V_0 \otimes V^0$.

We let $\mathcal{C}_V$ be the subscheme of $\mathbb{P}^5$ defined by the ideal I_V and write $\mathcal{C}$ for the smallest subscheme of $\mathbb{P}^5 \times Q$ containing all pairs $(\mathcal{C}_V, V)$, $V \in Q$.

Proposition 5.14. *The fibre of $\mathcal{C} \longrightarrow Q$ over*

$$V = \langle (1,0,0,0), (0,1,1,0) \rangle_{\mathbb{C}} \in Gr(2, W^\vee)$$

is of dimension 2 *and degree* 12.

Proof. For V given above, $V^0 = \langle (0,1,-1,0), (0,0,0,1) \rangle_{\mathbb{C}}$, so the ideal I_V is generated by

$$f_1 - f_2, g_1 - g_2, f_3, g_3.$$

Explicitly, $\mathcal{C}_V$ is described by the equations

$$\begin{aligned} x_0x_2x_4 = x_1x_3x_5 = x_1^2x_4 + x_3^2x_0 + x_5^2x_2 - (x_1x_2x_3 + x_3x_4x_5 + x_5x_0x_1) = \\ x_1x_4^2 + x_3x_0^2 + x_5x_2^2 - (x_2x_3x_4 + x_4x_5x_0 + x_0x_1x_2) = 0. \end{aligned}$$

By factorising the first two equations, it is clear that $\mathcal{C}_V$ is contained in the union of the nine 3-dimensional linear subspaces $L_{ij} = \{x_i = x_j = 0\}$ for $i \in \{0, 2, 4\}$ and $j \in \{1, 3, 5\}$.
If $i = j + 3$ (in $\mathbb{Z}/6\mathbb{Z}$), then $L_{ij} \cap \mathcal{C}_V$ is given by

$$x_i = x_j = \sum_{\substack{k \in 0,2,4 \\ k \neq i}} x_k^2 x_{k+3} = \sum_{\substack{k \in 0,2,4 \\ k \neq i}} x_k x_{k+3}^2 = 0,$$

which can easily seen to be a curve.
If $i \neq j + 3$, for example $i = 0, j = 1$, then $L_{ij} \cap \mathcal{C}_V$ is described by the equations

$$x_0 = x_1 = x_5^2 x_2 - x_3 x_4 x_5 = x_5 x_2^2 - x_2 x_3 x_4 = 0,$$

which are satisfied if either $x_2 = x_5 = 0$ or $x_5 x_2 - x_3 x_4 = 0$. Thus, in this case $L_{ij} \cap \mathcal{C}_V$ is the union of a line and a quadric surface.
So $\mathcal{C}_V$ is a union of 6 quadric surfaces and a number of (possibly embedded) curves, so it is two-dimensional and of degree 12. □

As $\mathcal{X}_V \subset \mathcal{C}_V$ and $\dim \mathcal{X}_V \geq 2$ by semicontinuity we may conclude:

Corollary 5.15. *The map ϕ is dominant and the closure of the generic fibre of ϕ has dimension two and degree 12.*

5.1.5 Moduli-interpretation of ϕ

Lemma 5.16. *For a general abelian surface A of type (1,6) embedded $\mathbf{H}_6$-invariantly into $\mathbb{P}^5$ with ideal $\mathcal{I}_A$*

$$\dim(H^0(\mathcal{I}_A(3))^{\mathbf{H}'}) = 4$$

and the cubics in $H^0(\mathcal{I}_A(3)))^{\mathbf{H}'}$ cut out a scheme of dimension ≤ 2.

Proof. To prove the statement about the dimension of $H^0(\mathcal{I}_A(3))^{\mathbf{H}'}$, let $y \in A$ be a general point (at which ϕ is defined), $V = \phi(y)$, V^0 its annihilator. Then $V_0 \otimes V^0$ is the largest $\mathbf{H}_6$-subrepresentation of $H^0(\mathcal{I}_A(3))^{\mathbf{H}'}$ consisting entirely of polynomials vanishing at y. In other words $V_0 \otimes V^0$ is the largest subspace of $H^0(\mathcal{I}_A(3))^{\mathbf{H}'}$ consisting of polynomials vanishing at the whole orbit $\mathbf{H}_6 y$. Thus,

$$H^0(\mathcal{I}_A(3))^{\mathbf{H}'} \subseteq H^0(\mathcal{I}_{\mathbf{H}_6 y}(3))^{\mathbf{H}'} \subseteq V^0 \otimes V_0.$$

But the latter is four-dimensional and the first has dimension ≥ 4 by Lemma 5.5, so equality holds.
That the dimension of the scheme defined by $H^0(\mathcal{I}_A(3)))^{\mathbf{H}'}$ is ≤ 2 is shown again by a degeneration argument. □

This leads to the key observation that relates the map ϕ to abelian surfaces:

Proposition 5.17. *For a general* $(1,6)$*-polarised abelian surface* $A \subset \mathbb{P}^5$ *the set*

$$\phi(A \cap (\mathbb{P}^5 \setminus Z))$$

consists of a single point.

Corollary 5.18. *The closure of the general fibre of* ϕ *contains exactly one* $\mathbf{H}_6$*-invariant abelian surface.*

Proof. By Corollary 5.15 the fibres are non-empty and there cannot be two distinct abelian surfaces A, A' in one fibre. □

Corollary 5.19. *Proposition 5.17 means that* ϕ *induces a rational map*

$$\Theta_6 \colon \mathcal{A}_6^{emb} \dashrightarrow Q \subset Grass(2, W^{\vee})$$

by sending the general Heisenberg invariant (1,6)-polarized abelian surface $A \subseteq \mathbb{P}^5$ *to* $\phi(A \cap \mathbb{P}^5 \setminus Z)$*. By 5.18,* Θ_6 *must be generically 1 to 1.*

5.2 Finding the discriminant

According to Corollary 5.15 for general $V \in Q$ the scheme $\mathcal{C}_V$ is two dimensional, but it obviously it also contain the union L of the nine lines l_{ij} (c.f. Proposition 5.6).

Definition 5.20. We write

$$A_V := \overline{\mathcal{C}_V \setminus L}.$$

It turns out that A_V is in general smooth of pure dimension 2 and each line l_{ij} intersects the abelian surface in four points, making in total 36 (different) singular points of $\mathcal{C}_V$. To give A_V a scheme structure, we use

$$(I_V : I_{lines}^{\infty})$$

as an defining ideal. It seems to be always generated by the four cubics we started with and six additional quartics, as expected. For further details about this see Section 5.5.

In the following we want to study the fibres A_V in more detail. Q is only birational to the moduli space of embedded abelian surfaces, so there are points in Q which actually do not encode smooth abelian surfaces, but also a singular degeneration thereof. We denote the set of all points where this is the case by

$$\widetilde{D} := \{V \in Q \mid A_V \text{ is not smooth}\}$$

and call it the *discriminant.*

It is easy to see that $V \in \widetilde{D}$ whenever $p_{12} = p_{03} = 0$ in the corresponding Plücker-coordinates (c.f. Section 5.4.3). Thus, we know that $\widetilde{D}$ is non-empty and has codimension 1.

Computing equations for $\widetilde{D}$ using for example SINGULAR is very time consuming respectively does not terminate at all on my computer if it is done in the straight forward manner. We need two tricks to approach the problem nevertheless:

1. Work in affine charts.
2. Look at as many 1-parameter families in one chart as necessary to find at least the codimension one part of $\widetilde{D}$ in this chart.

The projective variety $Q \subset \mathbb{P}^5$ is covered by the four affine charts $\{p_{01} \neq 0\}$, $\{p_{02} \neq 0\}$, $\{p_{13} \neq 0\}$ and $\{p_{23} \neq 0\}$. The two other charts $\{p_{03} \neq 0\}$ and $\{p_{12} \neq 0\}$ belonging to the standard affine cover of $\mathbb{P}^5$ are not necessary to cover Q because

$$\begin{aligned} &Q \cap \{p_{01} = p_{02} = p_{13} = p_{23} = 0\} \\ =&\{p_{01} = p_{02} = p_{13} = p_{23} = p_{03} + p_{12} = p_{03}p_{12} = 0\} = \emptyset \subset \mathbb{P}^5. \end{aligned}$$

The following table shows how the vector spaces and their annihilators belonging to a point in each chart look like:

Chart	$V \subset W$	$V^0 \subset W^\vee$	p-coordinates
$p_{01} = 1$	$\langle \begin{pmatrix} 1 \\ 0 \\ a \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ c \\ a \end{pmatrix} \rangle$	$\langle \begin{pmatrix} -a \\ -c \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -b \\ -a \\ 0 \\ 1 \end{pmatrix} \rangle$	$c = p_{02},\ a = p_{03} = -p_{12},\ b = -p_{13},\ p_{23} = a^2 - bc$
$p_{02} = 1$	$\langle \begin{pmatrix} 1 \\ -a \\ 0 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ c \\ 1 \\ a \end{pmatrix} \rangle$	$\langle \begin{pmatrix} a \\ 1 \\ -c \\ 0 \end{pmatrix}, \begin{pmatrix} -b \\ 0 \\ -a \\ 1 \end{pmatrix} \rangle$	$c = p_{01},\ a = p_{03} = -p_{12},\ b = -p_{23},\ p_{13} = -a^2 - bc$
$p_{13} = 1$	$\langle \begin{pmatrix} a \\ 1 \\ c \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \\ -a \\ 1 \end{pmatrix} \rangle$	$\langle \begin{pmatrix} 1 \\ -a \\ 0 \\ -b \end{pmatrix}, \begin{pmatrix} 0 \\ -c \\ 1 \\ a \end{pmatrix} \rangle$	$b = -p_{01},\ a = p_{03} = -p_{12},\ c = p_{23},\ p_{02} = -a^2 - bc$
$p_{23} = 1$	$\langle \begin{pmatrix} a \\ c \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ a \\ 0 \\ 1 \end{pmatrix} \rangle$	$\langle \begin{pmatrix} 1 \\ 0 \\ -a \\ -b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -c \\ -a \end{pmatrix} \rangle$	$p_{01} = a^2 - bc,\ a = p_{03} = -p_{12},\ b = -p_{02},\ c = p_{13}$

In general, the annihilator V^0 of the vector space V belonging to a point p is generated by the four linearly dependent vectors

$$\begin{pmatrix} p_{12} \\ -p_{02} \\ p_{01} \\ 0 \end{pmatrix}, \begin{pmatrix} p_{13} \\ p_{12} \\ 0 \\ p_{01} \end{pmatrix}, \begin{pmatrix} p_{23} \\ 0 \\ -p_{03} \\ p_{02} \end{pmatrix}, \begin{pmatrix} 0 \\ -p_{02} \\ p_{13} \\ -p_{12} \end{pmatrix}.$$

Remember that the equations of $\mathcal{C}_V$ are exactly the cubics in $V^0 \otimes V_0 \subset H^0(\mathcal{O}_{\mathbb{P}^5}(3))$. To reduce the number of parameters and so the computational complexity, calculations should be done in these charts whenever necessary.

Finding the discriminant locus in the first chart, i.e. the set of all $V \in Q$ where $\widetilde{\mathcal{C}}_V$ is not smooth, using SINGULAR could be done in the following manner:

```
ring R = 0, (x(0..5),a,b,c), dp;

poly f0 = x(0)^3+x(2)^3+x(4)^3;
poly g0 = x(1)^3+x(3)^3+x(5)^3;
poly f1 = x(1)^2*x(4)+x(3)^2*x(0)+x(5)^2*x(2);
poly g1 = x(2)^2*x(5)+x(4)^2*x(1)+x(0)^2*x(3);
poly f2 = x(1)*x(2)*x(3)+x(3)*x(4)*x(5)+x(5)*x(0)*x(1);
poly g2 = x(2)*x(3)*x(4)+x(4)*x(5)*x(0)+x(0)*x(1)*x(2);
poly f3 = x(0)*x(2)*x(4);
poly g3 = x(1)*x(3)*x(5);

LIB "primdec.lib";

ideal lines = x(2)*x(3)-x(0)*x(5), x(1)*x(2)-x(4)*x(5), x(0)*x(1)-x(3)*x(4),
f0, g0, f1, g1, f2, g2, f3, g3;

ideal I = -a*f0-c*f1+f2, -a*g0-c*g1+g2, -b*f0-a*f1+f3, -b*g0-a*g1+g3;
ideal re = quotient(I,lines);
re = mres(re,1)[1];    //reduces number of generators to 22

matrix m = jacob(re);
matrix B = submat(m,1..22,1..6);

ideal sing = I+wedge(B,3);

ideal sings = sat(sing, maxid)[1];

ideal disc = eliminate(sings,x(0)*x(1)*x(2)*x(3)*x(4)*x(5));
```

Unfortunately, the calculation of `sat(sing, maxid)` does not seem to terminate and it does not seem to get better using only a subset of the Jacobi minors.

The best thing we could do is to look at linear 1-parameter-families like $a = b = 0$ or $a = 1, b = 1$ and compute their discriminant. This method actually can be used to determine the codimension one part of the discriminant.

Definition 5.21. In the following we will write D for the codimension one part of the discriminant $\widetilde{D}$.

With this method nothing can be said about potential lower dimensional parts. D has to be non-empty and determined by one single polynomial $f \in \mathbb{C}[p_{ij}]$. Of course, f is only unique modulo $I_Q = \langle p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}, p_{03} + p_{12} \rangle$.

5.2.1 The degree of D

To find the degree of the discriminant D, we count the number of intersection points of D with a general line in $\mathbb{P}^5$ that is contained in Q. We use the line

$$C = \{2p_{23} = p_{12}, 3p_{23} = p_{02}, 4p_{23} + 3p_{13} = p_{01}, p_{03} + p_{12} = 0\}.$$

In the chart $p_{23} = 1$, C corresponds to the line given by $a = -2, b = -3$; we could have chosen almost any other values for a and b and any other chart here.
The following table shows how C is described in the different charts and for which values it intersects the discriminant (the polynomials g_{ij} is obtained by the computation shown above after replacing a and b with -2 and -3):

Chart

$p_{01} \neq 0$: General point: $(2 : 3c : -2c : 2c : \frac{2-4c}{3} : c)$

Vector spaces: $V = \langle \begin{pmatrix} 6 \\ 0 \\ -6c \\ 2(2c-1) \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 9c \\ -6c \end{pmatrix} \rangle$, $V^0 = \langle \begin{pmatrix} 6c \\ -9c \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 2(1-2c) \\ 6c \\ 0 \\ 9 \end{pmatrix} \rangle$

Discriminant: $g_{01} = 138975c^9 - 80030c^8 - 39034c^7 + 26514c^6 - 2226c^5 + 3552c^4 - 2832c^3 + 608c^2 - 32c$

$p_{02} \neq 0$: General point: $(4 - 3c : 3 : -2 : 2 : -c : 1)$

Vector spaces: $V = \langle \begin{pmatrix} 3 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4-3c \\ 3 \\ -2 \end{pmatrix} \rangle$, $V^0 = \langle \begin{pmatrix} -2 \\ 3 \\ 3c-4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix} \rangle$

Discriminant: $g_{02} = 243c^8 + 486c^7 - 7074c^6 + 14184c^5 + 3339c^4 - 19198c^3 - 14892c^2 - 4088c - 392$

$p_{13} \neq 0$: General point: $(3 - 4c : -3c : 2c : -2c : 1 : -c)$

Vector spaces: $V = \langle \begin{pmatrix} 2c \\ 1 \\ -c \\ 0 \end{pmatrix}, \begin{pmatrix} 4c-3 \\ 0 \\ -2c \\ 1 \end{pmatrix} \rangle$, $V^0 = \langle \begin{pmatrix} 1 \\ -2c \\ 0 \\ 3-4c \end{pmatrix}, \begin{pmatrix} 0 \\ c \\ 1 \\ 2c \end{pmatrix} \rangle$

Discriminant: $g_{13} = 392c^9 + 4088c^8 + 14892c^7 + 19198c^6 - 3339c^5 - 14184c^4 + 7074c^3 - 486c^2 - 243c$

$p_{23} \neq 0$: General point: $(4 - 3c : 3 : -2 : 2 : -c : 1)$

Vector spaces: $V = \langle \begin{pmatrix} -2 \\ -c \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \rangle$, $V^0 = \langle \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ c \\ 2 \end{pmatrix} \rangle$

Discriminant: $g_{23} = 243c^8 + 486c^7 - 7074c^6 + 14184c^5 + 3339c^4 - 19198c^3 - 14892c^2 - 4088c - 392$

Since neither 0 nor $\frac{4}{3}$ is a zero of $g_{02} = g_{23}$, all 8 intersection points of C with the discriminant in these charts lie in all four charts simultaneously. The ninth point in $p_{01} \neq 0$ corresponding to $c = 0$ lies in $\{p_{01} \neq 0\} \cap \{p_{13} \neq 0\}$ but not in the other two charts and is identical with the point corresponding to $c = 0$ in $\{p_{13} \neq 0\}$. These are in total 9 intersection points, so D is of degree 9.

5.2.2 Determination of the polynomial f

Now we make the ansatz

$$f(p_{01}, \dots, p_{23}) = \sum_{i_{01}+\cdots+i_{23}=9} f_{i_{01}\dots i_{23}} p_{01}^{i_{01}} \cdots p_{23}^{i_{23}}, \quad f_{i_{01}\dots i_{23}} \in \mathbb{Q}.$$

Since f is only given modulo I_Q, we can force $i_{03} = 0$ and $i_{12} \in \{0, 1\}$. Hence we have to find $\binom{12}{3} + \binom{11}{3} = 385$ unknown coefficients.
Now we go back to the chart $p_{01} \neq 0$. Here we have

$$p_{01} = 1, \;\; p_{02} = c, \;\; p_{03} = -p_{12} = a, \;\; p_{13} = -b \text{ and } p_{23} = a^2 - bc.$$

Denote the corresponding map $\mathbb{C}[p_{01}, \dots, p_{23}] \longrightarrow \mathbb{C}[a, b, c]$ by φ. So $\varphi(f) = \tilde{f}(a, b, c)$ is a unique (inhomogeneous) polynomial $\tilde{f}$ of degree at most 18. $\tilde{f}$ lies in the linear subspace of $\mathbb{C}[a, b, c]$ generated by

$$\varphi(p_{01}^{i_{01}} p_{02}^{i_{02}} p_{12}^{i_{12}} p_{13}^{i_{13}} p_{23}^{i_{23}}), \quad 0 \leq i_{kl} \leq 9, \quad 0 \leq i_{12} \leq 1, \quad \sum i_{kl} = 9. \tag{5.1}$$

To determine the unknown coefficients we repeat the computations given in the code above after replacing a and b by some constants $A, B \in \mathbb{Q}$. This results in a polynomial in c,

$$g_{A,B}(c) = \sum g_n^{A,B} c^n \in \mathbb{C}[c],$$

which should have the same zero locus as $\tilde{f}(A, B, c)$.
This means that

$$\sqrt{\langle g_{A,B}(c) \rangle} = \sqrt{\langle \tilde{f}(A, B, c) \rangle} \subset \mathbb{C}[c].$$

From this we get the weaker, but easier to evaluate condition that

$$g_{A,B}^{sf}(c) \mid \tilde{f}(A, B, c),$$

where $g_{A,B}^{sf} = \frac{g_{A,B}}{\gcd(g_{A,B}, g'_{A,B})}$ denotes the square-free part of $g_{A,B}$.
This can be rewritten as

$$\tilde{f}^{\langle a-A, b-B, g_{A,B}^{sf}(c) \rangle} = 0.$$

Here g^I denotes the reduction of g modulo the ideal I. Note that this is easy to compute, because $a - A, b - B, g_{A,B}^{sf}(c)$ is already a Groebner basis (in any monomial ordering), since the three polynomials consist of disjoint variables.

This condition can be translated into a system of linear equations in the following way:

1. Reduce all basis elements m_i given by (5.1) modulo $I_{A,B} = \langle a-A, b-B, g_{A,B}^{sf}(c) \rangle$.

2. You will receive a polynomial in c of degree ≤ 8 (in fact ≤ 7), say $\sum_{j=0}^{8} a_{ij} c^j$.
3. If $f = \sum_i f_i m_i$, then, since reduction modulo I is $\mathbb{C}$-linear, $\sum_i f_i a_{ij} = 0$ for all $0 \leq j \leq 8$.
4. Repeat this for different A and B until you have enough equations.

Doing this for all $(B, C) \in \{-4, -\frac{7}{2}, -3, -\frac{5}{2}, \dots, \frac{7}{2}, 4\}^2$, yields a linear system of equations which has a one-dimensional kernel, generated by the following polynomial:

Proposition 5.22. *The polynomial f describing the codimension one part D of the discriminant is*

$$f = p_{12} \cdot (-p_{02}^2 - 10p_{02}p_{13} - 9p_{13}^2 + 4p_{01}p_{23}) \cdot g \text{ with}$$

$$\begin{aligned} g = &- p_{01}^3 p_{02}^2 p_{13} - 2p_{01}^3 p_{02} p_{13}^2 - 27p_{02}^4 p_{13}^2 - p_{01}^3 p_{13}^3 - 108p_{02}^3 p_{13}^3 - 162p_{02}^2 p_{13}^4 - 108p_{02} p_{13}^5 \\ &- 27p_{13}^6 + 4p_{01}^4 p_{13} p_{23} + 18p_{01} p_{02}^3 p_{13} p_{23} + 198p_{01} p_{02}^2 p_{13}^2 p_{23} + 342p_{01} p_{02} p_{13}^3 p_{23} \\ &+ 162p_{01} p_{13}^4 p_{23} + p_{01}^2 p_{02}^2 p_{23}^2 - 78p_{01}^2 p_{02} p_{13} p_{23}^2 - 207p_{01}^2 p_{13}^2 p_{23}^2 - 4p_{01}^3 p_{23}^3 - 16p_{02}^3 p_{23}^3 \\ &- 72p_{02}^2 p_{13} p_{23}^3 - 864p_{02} p_{13}^2 p_{23}^3 + 216p_{13}^3 p_{23}^3 + 72p_{01} p_{02} p_{23}^4 + 648p_{01} p_{13} p_{23}^4 - 432p_{23}^6. \end{aligned}$$

5.3 Geometry of Q and its discriminant

5.3.1 Geometry of the discriminant

All factors of f given above are irreducible over $\mathbb{Q}$, but the ideal $I_D = I_Q + \langle f \rangle$ describing the codimension one part of the discriminant D decomposes into four primary factors.

Theorem 5.23. *D decomposes over $\mathbb{Q}$ into four irreducible components: E_1, E_2, E_3 and Δ described by the following ideals*

$$\begin{aligned} I_{E_1} &= I_Q + \langle p_{12} \rangle \\ I_{E_2} &= I_Q + \langle p_{02} - 2p_{03} + 3p_{13} \rangle \\ I_{E_3} &= I_Q + \langle p_{02} + 2p_{03} + 3p_{13} \rangle \\ I_{\Delta} &= I_Q + \langle g \rangle \end{aligned}$$

where g is the factor of degree six in the decomposition of f given in Proposition 5.22.

The singular locus and the intersection behaviour of the components of D can be described as follows:

Proposition 5.24. *1. The hyperplane sections E_i are smooth quadric surfaces.*

2. They all intersect in one smooth conic:

$$E_1 \cap E_2 = E_1 \cap E_3 = E_2 \cap E_3 = E_1 \cap E_2 \cap E_3 = \{p_{12} = p_{02} + 3p_{13} = 0\} \subset Q$$

Proposition 5.25. *The intersection of Δ with each of the E_i is a curve of degree 12. It consists of the following four irreducible components over $\mathbb{Q}$:*

- *Two lines,*
- *a curve of degree two that decomposes into two lines over $\mathbb{Q}[\omega]$ (ω third root of unity),*
- *a curve of degree four counted twice. This is the curve C_i described in detail in the next proposition.*

The four lines are skew.

Proposition 5.26. *1. The singular locus of Δ is a curve of degree 24.*

2. $Sing(\Delta)$ decomposes into four irreducible components over $\mathbb{Q}$:

- *Three curves C_i of degree 4 and genus 0, each of them lying in the intersection of Δ with one of the E_i.*
  The curves $C_i \subset Q$ are described by the following equations:

$$\begin{aligned}
C_1 : \; & p_{13}^3 - p_{01}p_{13}p_{23} - 4p_{23}^3 = 0 \\
& p_{01}p_{13}^2 - p_{01}^2p_{23} - 4p_{02}p_{23}^2 = 0 \\
& p_{01}^2p_{02} - p_{01}^2p_{13} + 4p_{02}^2p_{23} = 0 \\
C_2 : \; & 4p_{12}p_{13}^2 + 4p_{13}^3 + p_{01}p_{13}p_{23} - p_{23}^3 = 0 \\
& p_{01}p_{12}p_{13} + p_{01}p_{13}^2 - p_{12}p_{23}^2 + 3p_{13}p_{23}^2 = 0 \\
& p_{01}^2p_{13} + 4p_{12}p_{13}p_{23} - 12p_{13}^2p_{23} - p_{01}p_{23}^2 = 0 \\
C_3 : \; & 4p_{12}p_{13}^2 - 4p_{13}^3 - p_{01}p_{13}p_{23} + p_{23}^3 = 0 \\
& p_{01}p_{12}p_{13} - p_{01}p_{13}^2 - p_{12}p_{23}^2 - 3p_{13}p_{23}^2 = 0 \\
& p_{01}^2p_{13} - 4p_{12}p_{13}p_{23} - 12p_{13}^2p_{23} - p_{01}p_{23}^2 = 0.
\end{aligned}$$

- *One curve R of degree 12 and genus 1. $R \subset Q$ is described by the following equations:*

$$\begin{aligned}
& p_{01}^2 - 12p_{02}p_{23} + 36p_{13}p_{23} = 0 \\
& 9p_{02}^2p_{13} + 18p_{02}p_{13}^2 + 9p_{13}^3 - p_{01}p_{02}p_{23} - 33p_{01}p_{13}p_{23} = 0
\end{aligned}$$

3. The curve R has twelve singular points, four on each of the E_i.

4. Each C_i has eight singular points:

- *Four of them lie on all E_i and C_i simultaneously. Explicitly these are:*

$$\begin{aligned}
&(-3 : -3 : 0 : 0 : 1 : 1) \\
&(-3\omega^2 : -3\omega : 0 : 0 : \omega : 1) \\
&(-3\omega : -3\omega^2 : 0 : 0 : \omega^2 : 1) \\
&(1 : 0 : 0 : 0 : 0 : 0)
\end{aligned}$$

- *The other four coincide with the singularities of R lying on E_i.*

5.3.2 Automorphisms of Q

Given a polarised abelian variety A with a line bundle L of type D, the rational map $\varphi_L\colon A \longrightarrow \mathbb{P}^N$ depends on the choice of a basis of $H^0(L)$. Two different choices of a basis lead to maps that differ by a projective linear transformation.
Gross and Popescu classified Heisenberg invariantly *embedded* abelian varieties. Thus it has to be expected that they distinguish several embeddings of the same variety that only differ by a projective linear transformation.

From Corollary 3.6 we know that for a (1,6)-polarised abelian surface there is one Heisenberg equivariant embedding map for each element of order 6 in $(\mathbb{Z}/6\mathbb{Z})^2$.
These are exactly

$$\#(\mathbb{Z}/6\mathbb{Z})^2 - \#3\text{-torsion of } (\mathbb{Z}/6\mathbb{Z})^2 - \#2\text{-torsion of } (\mathbb{Z}/6\mathbb{Z})^2 + 1 = 36-9-4+1 = 24.$$

So there are exactly 24 Heisenberg invariant embedding maps from A to $\mathbb{P}^5$ which yield at most 12 different images.
In the sequel we will for a general such surface indeed find 12 different embeddings, so this estimate is sharp now.

We hope that the projective transformations mapping one embedding of A to another do not only act set-theoretically on the points of Q, but induce global automorphisms of the projective variety Q.
In the following we will also write I_p or $\mathcal{C}_p$ instead of I_V or $\mathcal{C}_V$, if $V \subset W^\vee$ is encoded by $p \in G \subset \mathbb{P}^5$.

Example 5.27. If $p = (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23})$ is a point in Q, then $p' = (p_{01} : p_{02} : -p_{03} : -p_{12} : p_{13} : p_{23})$ is also in Q and the corresponding ideals I_p and $I_{p'}$ are isomorphic via

$$\begin{aligned} \varphi\colon x_0 &\mapsto \zeta x_0 & x_1 &\mapsto x_1 & x_2 &\mapsto \zeta^5 x_2 \\ x_3 &\mapsto \zeta^4 x_3 & x_4 &\mapsto \zeta^9 x_4 & x_5 &\mapsto \zeta^8 x_5, \end{aligned}$$

where ζ is a primitive twelfth root of unity.
This means the maps φ and

$$\psi\colon (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) \longmapsto (p_{01} : p_{02} : -p_{03} : -p_{12} : p_{13} : p_{23})$$

make the following diagram commute:

$$\begin{tikzcd} \mathbb{P}^5 \arrow[r, "\phi"] \arrow[d, "\varphi"'] & H \arrow[d, "\psi"] \\ \mathbb{P}^5 \arrow[r, "\phi"'] & H \end{tikzcd}$$

Here H is hyperplane $p_{03}+p_{12} = 0$ introduced in Section 5.1.3 and $\phi\colon \mathbb{P}^5 \longrightarrow Q \subset H$.

In this section we will study the group $\mathrm{Aut}^{mod}(Q)$ of all such automorphisms of Q.

Definition 5.28. An *automorphism of Q as a moduli space* is a linear isomorphism $\psi\colon H \longrightarrow H$ which restricts to an automorphism of Q and which is induced by a linear isomorphism $\varphi\colon \mathbb{P}^5 \longrightarrow \mathbb{P}^5$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{P}^5 \arrow[r, "\phi"] \arrow[d, "\varphi"'] & H \arrow[d, "\psi"] \\ \mathbb{P}^5 \arrow[r, "\phi"'] & H \end{tikzcd}$$

To see if any given $\varphi\colon \mathbb{P}^5 \longrightarrow \mathbb{P}^5$ induces such a map $\psi\colon H \longrightarrow H$, one can look at the corresponding map of rings $\varphi^*\colon \mathbb{C}[x_0,\dots x_5] \longrightarrow \mathbb{C}[x_0,\dots x_5]$ write $p_{ij} = f_i g_j - f_j g_i$ as a polynomial in the x_i and then check if $\varphi^*(p_{ij})$ can be expressed as a linear combination of all p_{ij} for all $0 \le i < j \le 3$. Here one can see that ψ being linear and hence an automorphism of H and not only of Q is a direct consequence of being induced by a linear map on the x_i and not a restriction for ψ.
Since I_p describes the preimage of p under ϕ, saying that ψ is induced by φ in the above sense is equivalent to $I_{\psi(p)} = \varphi^*(I_p)$ for all $p \in H$. Recall that we denote by I_p the ideal $I_V = \langle V^0 \otimes W\rangle$ where $V \subseteq W^\vee$ is the subspace corresponding to $p \in Q \subset Gr(2, W^\vee) \subset \mathbb{P}^5$.

Theorem 5.29. *The automorphism group $\mathrm{Aut}^{mod}(Q)$ of Q as a moduli space is generated by the three projective transformation ψ_0, ψ_1 and ψ_2 described in the proof, permutes the hyperplanes E_i arbitrarily and is isomorphic to*

$$\mathrm{Aut}^{mod}(Q) \cong \mathbb{Z}/2\mathbb{Z} \times S_3 \cong D_6.$$

The subgroup $\mathrm{Aut}^0(Q)$ of automorphisms fixing all three hyperplanes E_i is generated by ψ_0 and is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Proof. Let us describe two automorphisms of Q explicitly: There are linear isomorphisms

$$\varphi_1^*, \varphi_2^*\colon \mathbb{C}[x_0,\dots,x_5] \longrightarrow \mathbb{C}[x_0,\dots,x_5]$$

given by

$$\begin{aligned} \varphi_1^*\colon x_0 &\mapsto \zeta x_0 & \qquad \varphi_2^*\colon x_0 &\mapsto x_0 + x_3 \\ x_1 &\mapsto x_1 & x_1 &\mapsto x_5 - x_2 \\ x_2 &\mapsto \zeta^5 x_2 & x_2 &\mapsto x_1 + x_4 \\ x_3 &\mapsto \zeta^4 x_3 & x_3 &\mapsto x_3 - x_0 \\ x_4 &\mapsto \zeta^9 x_4 & x_4 &\mapsto x_2 + x_5 \\ x_5 &\mapsto \zeta^8 x_5 & x_5 &\mapsto x_1 - x_4 \end{aligned}$$

where ζ is a primitive twelfth root of unity.

φ_1^* acts on $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$ as follows:

$$\begin{array}{llll} f_0 \mapsto if_0 & f_1 \mapsto -if_1 & f_2 \mapsto -if_2 & f_3 \mapsto if_3 \\ g_0 \mapsto g_0 & g_1 \mapsto -g_1 & g_2 \mapsto -g_2 & g_3 \mapsto g_3 \end{array}$$

whereas

$$\begin{array}{ll} \varphi_2^*\colon f_0 \mapsto f_0 + g_0 + 3f_1 + 3g_1 & g_0 \mapsto -f_0 + g_0 - 3f_1 + 3g_1 \\ \quad f_1 \mapsto f_0 + g_0 - f_1 - g_1 & g_1 \mapsto -f_0 + g_0 + f_1 - g_1 \\ \quad f_2 \mapsto -f_2 - g_2 + 3f_3 + 3g_3 & g_2 \mapsto f_2 - g_2 - 3f_3 + 3g_3 \\ \quad f_3 \mapsto f_2 + g_2 + f_3 + g_3 & g_3 \mapsto -f_2 + g_2 - f_3 + g_3. \end{array}$$

Hence both induce maps

$$\psi_1^*, \psi_2^*\colon \mathbb{C}[p_{01}, \ldots, p_{23}]/(p_{03} + p_{12}) \longrightarrow \mathbb{C}[p_{01}, \ldots, p_{23}]/(p_{03} + p_{12})$$

given by $p_{ij} = f_i g_j - f_j g_i$ which can (after rescaling, because we are only interested in projective properties) be described as follows:

$$\begin{array}{ll} \psi_1^*\colon p_{01} \mapsto p_{01} & \psi_2^*\colon p_{01} \mapsto 4p_{01} \\ \quad p_{02} \mapsto p_{02} & \quad p_{02} \mapsto p_{02} - 6p_{03} - 9p_{13} \\ \quad p_{03} \mapsto -p_{03} & \quad p_{03} \mapsto -p_{02} + 2p_{03} - 3p_{13} \\ \quad p_{13} \mapsto p_{13} & \quad p_{13} \mapsto -p_{02} - 2p_{03} + p_{13} \\ \quad p_{23} \mapsto p_{23} & \quad p_{23} \mapsto 4p_{23}. \end{array}$$

Let $\psi_i\colon H \to H$ be the corresponding projective isomorphisms.
The maps ψ_i restrict to isomorphisms $Q \to Q$ and $D \to D$ where ψ_2 maps E_1 to E_2 and vice versa fixing E_3, while ψ_1 exchanges E_2 and E_3 and leaves E_1 fixed.
As projective automorphisms of H both ψ_i are diagonalisable and have four times the eigenvalue 1 and one time the eigenvalue -1, so both are of order two, whereas their product $\psi_1 \circ \psi_2$ has order three. So they induce a faithful action of $D_3 = S_3 = \langle s, t \mid s^2, t^2, (st)^3 \rangle$ on H resp. Q resp. D which permutes the three hyperplanes E_1, E_2, E_3 arbitrarily and changes the equations of the corresponding varieties $\widetilde{\mathcal{C}}_p$ only by a linear change of coordinates.

Since the existence of ψ_1 and ψ_2 is everything we need in Section 5.4, we can use the results derived in this section from now on.
Any element of $\mathrm{Aut}^{mod}(Q)$ has to map the discriminant D to itself and hence permutes E_1, E_2 and E_3. Since the group $S_3 = \langle \psi_1, \psi_2 \rangle$ already allows arbitrary permutations of the E_i, any element $\alpha \in \mathrm{Aut}^{mod}(Q)$ can be written as $\alpha = \psi\beta$ with $\psi \in S_3$ and β fixing all the E_i.
Denote the group of all such automorphisms by $\mathrm{Aut}^0(Q)$.
Following our considerations above we have an exact sequence

$$0 \longrightarrow \mathrm{Aut}^0(Q) \longrightarrow \mathrm{Aut}^{mod}(Q) \longrightarrow S_3 \longrightarrow 0$$

which right-splits, because S_3 is realised as the subgroup generated by ψ_1 and ψ_2 in $\mathrm{Aut}^{mod}(Q)$. Thus, $\mathrm{Aut}^{mod}(Q) = \mathrm{Aut}^0(Q) \rtimes S_3$.

A computation along the following lines will find all elements of $\mathrm{Aut}^0(Q)$:
From Section 5.4 we know that for a generic point $p \in E_1$, the surface A_p is singular along the curves

$$\begin{aligned} C_1 &= \{x_1 = x_3 = x_5 = f_0 - af_3 = 0\} \\ \text{and } C_2 &= \{x_0 = x_2 = x_4 = g_0 - ag_3 = 0\}. \end{aligned}$$

Similarly for $p \in E_2$ we have A_p is singular along

$$\begin{aligned} C_1' &= \{x_0 + x_3 = x_1 + x_4 = x_2 + x_5 = f_0 - bf_3 = 0\} \\ \text{and } C_2' &= \{x_0 - x_3 = x_1 - x_4 = x_2 - x_5 = g_0 - bg_3 = 0\}, \end{aligned}$$

and for $p \in E_3$ along the curves

$$\begin{aligned} C_1'' &= \{x_0 + ix_3 = x_1 + ix_4 = x_2 + ix_5 = f_0 - cf_3 = 0\} \\ \text{and } C_2'' &= \{x_0 - ix_3 = x_1 - ix_4 = x_2 - ix_5 = g_0 - cg_3 = 0\}. \end{aligned}$$

Thus any element of $\mathrm{Aut}^0(Q)$ will have to either fix or exchange the vector spaces

$$\langle x_0, x_2, x_4 \rangle_{\mathbb{C}} \text{ and } \langle x_1, x_3, x_5 \rangle_{\mathbb{C}}$$

and the same for

$$\langle x_0 + x_3, x_1 + x_4, x_2 + x_5 \rangle_{\mathbb{C}} \text{ and } \langle x_0 - x_3, x_1 - x_4, x_2 - x_5 \rangle_{\mathbb{C}}$$

(C_1'' and C_2'' give no further restrictions). These are four cases which can be treated separately.
Let us work out the case that each of these four spaces is mapped to itself. In this case the map $\varphi^*\colon \mathbb{C}[x_0, \ldots, x_5] \longrightarrow \mathbb{C}[x_0, \ldots, x_5]$ can be represented by the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 \\ a_4 & a_5 & a_6 & 0 & 0 & 0 \\ a_7 & a_8 & a_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_9 & a_7 & a_8 \\ 0 & 0 & 0 & a_3 & a_1 & a_2 \\ 0 & 0 & 0 & a_6 & a_4 & a_5 \end{pmatrix}$$

in the basis $x_0, x_2, x_4, x_1, x_3, x_5$ for any invertible matrix $A = (a_i)$.
Further we need that $\varphi^*(f_0), \varphi^*(f_3) \in \langle f_0, f_3 \rangle$. From each of these conditions we can derive eight equations in the a_i by reducing $\varphi^*(f_0)$ resp. $\varphi^*(f_3)$ (with the parameters a_i) modulo $\langle f_0, f_3 \rangle$ with the division algorithm, interpret the remainder as a polynomial in $\mathbb{C}[a_1, \ldots a_9][x_0, \ldots, x_5]$, and set all its coefficients to zero. Together with the normalisation $\det(A) = 1$ this defines a zero-dimensional ideal in $\mathbb{C}[a_1, \ldots, a_9]$. It

decomposes into 648 distinct points over $\mathbb{Q}(\zeta_{36})$ (ζ_{36} a primitive 36th root of unity). A direct primary decomposition might not be successful, but for example SINGULAR can find a primary decomposition over $\mathbb{Q}$ and then decompose the resulting factors further over $\mathbb{Q}(\zeta_{36})$.
One can easily check that each of these φ^* induces a map $\psi^*\colon \mathbb{C}[p_{01},\dots,p_{23}] \longrightarrow \mathbb{C}[p_{01},\dots,p_{23}]$ but only 108 of them map Q to itself. In fact, these 108 φ induce only two different ψ, the identity and

$$\psi_0^* = \begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{9} & \frac{1}{9} \\ -\frac{2}{3} & \frac{1}{3} & 0 & \frac{2}{9} & -\frac{1}{9} \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & \frac{1}{3} & \frac{1}{3} \\ 4 & -2 & 0 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

described with respect to the basis $p_{01}, p_{02}, p_{03}, p_{13}, p_{23}$. ψ_0^* is induced for example by the following map on the x_i:

$$\begin{aligned} \varphi_0^*\colon x_0 &\mapsto x_0 + x_2 + x_4 \\ x_1 &\mapsto \omega^2 x_1 + x_3 + \omega x_5 \\ x_2 &\mapsto x_0 + \omega^2 x_2 + \omega x_4 \\ x_3 &\mapsto x_1 + x_3 + x_5 \\ x_4 &\mapsto x_0 + \omega x_2 + \omega^2 x_4 \\ x_5 &\mapsto \omega x_1 + x_3 + \omega^2 x_5 \end{aligned}$$

where ω is a primitive third root of unity.
In the other three cases, we only find the same maps id and ψ_0^*.
This shows that $\mathrm{Aut}^{mod}(Q) = \langle \psi_0, \psi_1, \psi_2 \rangle$.

The observation that $\psi_0^2 = \mathrm{id}$ and that ψ_0 commutes with both ψ_1 and ψ_2, proves

$$\mathrm{Aut}^{mod}(Q) \cong \mathbb{Z}/2\mathbb{Z} \times S_3,$$

whereas the last isomorphism $\mathbb{Z}/2\mathbb{Z} \times S_3 \cong D_6$ is a known result from the classification of small groups. □

The reason that we find $4 \cdot 54 = 216 = 3 \cdot 72$ maps $\varphi^*\colon \mathbb{C}[x_0,\dots,x_5] \longrightarrow \mathbb{C}[x_0,\dots,x_5]$ for each automorphism $\psi\colon Q \longrightarrow Q$ is the fact that our abelian surfaces parametrised by Q are invariant under the 72 elements of the extended Heisenberg group $\mathbf{H}_6^e$ and our normalisation $\det(A) = 1$ fixes the matrix belonging to each element of $\mathbf{H}_6^e$ only up to multiplication with ω.

Small orbits

Under the action of $\mathrm{Aut}^{mod}(Q)$ the general point ought to have an orbit of length twelve. Finding points with smaller orbits (which probably correspond to more symmetric embedded surfaces) can be done as follows:

Any point with a small orbit (i.e. less than twelve elements) has to be fixed by a nontrivial element of $\mathrm{Aut}^{mod}(Q)$. The points in H fixed by $\psi \in \mathrm{Aut}^{mod}(Q)$ can easily be computed using linear algebra as the eigenspaces of the matrix describing ψ. There are in total 18 different subspaces of $\mathbb{P}^4 = H$ occurring as eigenspaces of the non-trivial elements of $\mathrm{Aut}^{mod}(Q)$. Luckily their intersection behaviour is quite simple, which means no new subspaces occur as their intersections.
Looking at them carefully, we arrive at the following results:

- There are six isolated points in H which have orbits as follows:
  - The point $(6 : -3 : 0 : 0 : 1 : 1)$ is fixed by all elements of $\mathrm{Aut}^{mod}(Q)$ (orbit of length 1). It does not lie in Q and has no obvious geometric meaning.
  - The points $(0 : 3 : \pm\sqrt{3} : \mp\sqrt{3} : 1 : 0)$ form together an orbit of length 2. They both lie in $Q \setminus D$, so they encode two very special smooth abelian surfaces.
  - The three points $(0 : 0 : 1 : -1 : 0 : 0)$, $(0 : 3 : 1 : -1 : 1 : 0)$, $(0 : 3 : -1 : 1 : 1 : 0)$ form together an orbit of length 3. They do not lie in Q, so they do not define abelian surfaces. But they will show up as projection points in Section 5.3.3 again.
- There are four lines in H with small orbits. They each intersect Q in two points.
  - The two points $(-12 \mp 6\sqrt{3} : -3 \mp 3\sqrt{3} : 0 : 0 : 1 \pm \sqrt{3} : 1)$ are fixed by all the automorphisms (i.e. their orbit consists of one point each). They are generic within the the intersection of the three hyperplanes $E_1 \cap E_2 \cap E_3$.
  - The six points $(6 : -3 \pm 3\sqrt{3} : 0 : 0 : 1 \pm \sqrt{3} : 1)$, $(6 : -3 \pm \frac{3}{2}\sqrt{3} : \mp\frac{3}{2}\sqrt{3} : \pm\frac{3}{2}\sqrt{3} : 1 \pm \frac{\sqrt{3}}{2} : 1)$ and $(6 : -3 \pm \frac{3}{2}\sqrt{3} : \pm\frac{3}{2}\sqrt{3} : \mp\frac{3}{2}\sqrt{3} : 1 \pm \frac{\sqrt{3}}{2} : 1)$ form two orbits of length 3. They all lie in the degree four component of the intersection of Δ with one of the hyperplanes E_i. The corresponding surfaces are described in Section 5.4.3.
- The five two-dimensional subspaces of H with small orbits look as follows:
  - All points of $E_1 \cap E_2 \cap E_3$, except the two fix-points described above, build orbits of length 2.
  - The subspace defined by the equations $p_{01} + p_{02} - 3p_{13} = p_{01} - 6p_{23} = 0$ is fixed pointwise by φ_0, while φ_1 and φ_2 act non-trivially on that space. Thus, its points form orbits of length 6.
    It intersects Q in a smooth curve of degree two with no obvious geometric meaning.
  - The three subspaces defined by the equations $p_{01} + 6p_{13} + 6p_{23} = p_{02} + 3p_{13} = 0$ resp. $p_{01} - 6p_{03} + 6p_{13} + 6p_{23} = p_{02} - 6p_{03} + 3p_{13} = 0$ resp. $p_{01} + 6p_{03} + 6p_{13} + 6p_{23} = p_{02} + 6p_{03} + 3p_{13} = 0$ each intersect Q in a smooth curve of degree two with no obvious geometric meaning.
    Their points also form orbits of length 6 spread over all three planes.

- The points on each of the hyperplanes E_1, E_2 and E_3 have orbits of length 6 if they do not belong to one of the special cases described above.

5.3.3 The three double covers

The Plücker quadric $q = p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}$ gives a correspondence between linear subspaces of complementary dimension in $\mathbb{P}^5$ via

$$L \mapsto \{p' \in \mathbb{P}^5 \mid \sum_{0 \le i < j \le 3} \partial_{ij} q(p) \cdot p'_{ij} = 0 \text{ for all } p \in L\}.$$

We call the image of L under this map the subspace *polar* to L.

Each of the components E_i of the discriminant is the intersection of Q with a hyperplane in $\mathbb{P}^5$, denote this hyperplane by A_i. We observe that for any of the hyperplanes A_i, q_i the point polar to A_i, the projection $\pi_i \colon \mathbb{P}^5 \dashrightarrow A_i$ from $q_i \in H$ onto A_i induces a double cover $Q \longrightarrow \mathbb{P}^3 = A_i \cap H$ branched along a smooth quadric.
The points polar to each hyperplane are

$$\begin{aligned} A_1: &\quad q_1 = (0:0:1:-1:0:0) \\ A_2: &\quad q_2 = (0:3:1:-1:1:0) \\ A_3: &\quad q_3 = (0:-3:1:-1:-1:0). \end{aligned}$$

We take a closer look at the projection from q_1 onto A_1. The other two look the same via the automorphisms given in Section 5.3.2.
In coordinates this reads as

$$\begin{aligned} \pi_1 \colon \mathbb{P}^5 &\dashrightarrow \mathbb{P}^3 \\ (p_{01}:p_{02}:p_{03}:p_{12}:p_{13}:p_{23}) &\longmapsto (p_{01}:p_{02}:p_{13}:p_{23}) =: (v:w:x:y). \end{aligned}$$

In this case the branching locus is described by $vy - wx = 0$.
The isomorphism ψ_1 exchanges exactly the two preimages of a point in $\mathbb{P}^3$. This means it induces the identity on $\mathbb{P}^3$. ψ_0 induces an automorphism of $\mathbb{P}^3$ of order 2, while ψ_2 and all products involving ψ_2 are not compatible with the projection. So only the $\mathbb{Z}/2\mathbb{Z}$-action by ψ_0 is left on $\mathbb{P}^3$.

π_1 maps the components of the discriminant to the following subvarieties of $\mathbb{P}^3$: E_1 (which is fixed by ψ_1) is mapped to the branching locus of π_1

$$\varepsilon_1 = \{vy - wx = 0\}.$$

E_2 and E_3 (which are exchanged by ψ_1) are mapped both to the same quadric

$$\varepsilon_2 = \{w^2 + 10wx + 9x^2 - 4vy = 0\}.$$

The image δ of Δ is again described by the same polynomial g. Note that neither p_{03} nor p_{12} occur in g, c.f. page 53!

The singular curve R of Δ is mapped to the curve C_6 described below.
This means that $\Delta_0 := \{x \in H \mid g(x) = 0\}$ is the cone over δ with apex q_1, i.e. for any point $p \in \delta$ the whole line spanned by p and q_1 lies in Δ_0 and Δ_0 is exactly the union of all these lines, and $\Delta = Q \cap \Delta_0$.

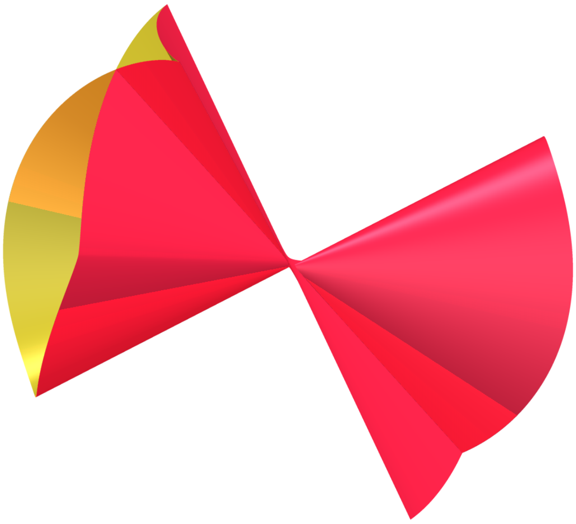


Figure 5.1: The component δ of D

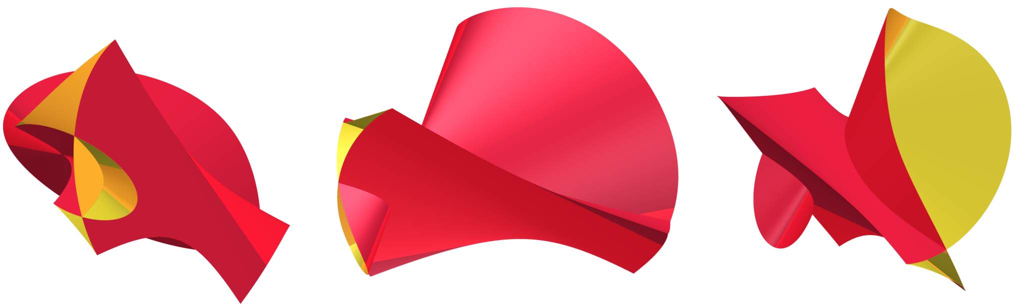


Figure 5.2: A closer look at δ from different perspectives

δ is a surface of degree 6 in $\mathbb{P}^3$, singular along a curve of degree 16, that decomposes into a smooth curve C_4 of degree four and a curve C_6 of degree 6 (counting twice) which has four cusps. C_6 is given by the equations:

$$g_1 := v^2 - 12wy + 36xy = 0$$
$$g_2 := 9w^2x + 18wx^2 + 9x^3 - vwy - 33vxy + 36y^3 = 0$$

Both of these curves have genus zero. C_4 intersects C_6 exactly in its singular points

$$P_0 = (0 : 1 : -1 : 0)$$

$$
\begin{aligned}
P_1 =&(24 : 15 : 1 : 4)\\
P_2 =&(24\omega^2 : 15 : 1 : 4\omega)\\
P_3 =&(24\omega : 15 : 1 : 4\omega^2).
\end{aligned}
$$

Here ω is a primitive third root of unity. In these intersection points C_4 has the same tangent direction as C_6.

Proposition 5.30. *The tangent scroll* $\operatorname{Tan} C_6 := \bigcup_{\substack{p \in C_6\\ smooth}} T_p C_6$ *is exactly the surface* δ.

Proof. Its a simple computation using a computer algebra system to find equations for

$$\bigcup_{p \in X} T_p X = \{x \in \mathbb{P}^3 \mid \exists y \in \mathbb{P}^3 \text{ s.t. } \sum_{i=0}^{3} \partial_i g_j(y) x_i = 0 \text{ for } j = 1, 2\}.$$

One can check that this variety decomposes into the four tangent planes at the cusps and the surface δ. □

As already stated above, C_6 is a rational curve. A parametrisation can be found as follows:
Project C_6 to $\mathbb{P}^2$ by eliminating v. We denote the image of this projection by $\overline{C}_6$. Now consider the radical ideal J_6 describing $Sing(\overline{C}_6)$. Its degree three part is two-dimensional while it contains no elements of degree less or equal than two. So there is a pencil of cubics through the singular points of $\overline{C}_6$ parametrised by, say t. Each of its elements intersects $\overline{C}_6$ in its singularities and one more point depending on t. Writing the pencil in the form

$$(495x^2y - 18xyz - yz^2) + t(63x^3 - 12y^3 + 62x^2z - xz^2)$$

the moving point is described by

$$
\begin{aligned}
16yt^6 - 108wt^4 - 756yt^3 + 729y = 0\\
-\frac{128}{729}yt^5 + \frac{32}{27}wt^3 + \frac{220}{27}yt^2 + x + w = 0
\end{aligned}
$$

which yields the parametrisation

$$t \longmapsto (16t^6 - 756t^3 + 729 : -108t^3 - 729 : 108t^4)$$

for $\overline{C}_6$.
Now plugging this into the equations describing the original curve C_6 and computing a Groebner basis w.r.t. the lexicographical ordering with $v > t$ gives an equation linear in v which can easily be solved for v.
This all together yields the following parametrisation:

Proposition 5.31. *The map* $\mu \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ *given by*

$$t \longmapsto (-144t^5 + 1944t^2 : 16t^6 - 756t^3 + 729 : -108t^3 - 729 : 108t^4)$$

is a parametrisation of C_6.

Under this parametrisation the parameter values for the four cusps are

$$t_0 = 0 \qquad t_1 = -3 \qquad t_2 = -3\omega \qquad t_3 = -3\omega^2$$

which yields a cross-ratio

$$\lambda = \frac{t_1 - t_3}{t_2 - t_1} : \frac{t_0 - t_3}{t_2 - t_0} = -\omega \qquad\qquad (\omega^3 = 1)$$

and a j-invariant of

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1-\lambda)^2} = 0.$$

From this parametrisation we obtain a parametrisation of the tangent variety δ as follows:
Compute the Jacobi matrix of C_6 at the point $\mu(t)$. It has entries in the polynomial ring $\mathbb{C}[t]$. A computer algebra system tells us, that its kernel is generated by

$$v_1(t) = \begin{pmatrix} 27 \\ 0 \\ 8t^3 - 27 \\ -36t^2 \end{pmatrix} \text{ and } v_2(t) = \begin{pmatrix} 9t \\ -6t^2 \\ 27t \\ 4t^3 - 54 \end{pmatrix},$$

which span T_pC_6 for any smooth point $p = \mu(t)$. Then $(s,t) \mapsto v_1(t) + sv_2(t)$ is a parametrisation of its tangent variety.

Proposition 5.32. *The map* $\nu\colon \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ *given by*

$$(s,t) \longmapsto (27 + 9st : -6st^2 : 8t^3 + 27st - 27 : 4st^3 - 36t^2 - 54s)$$

is a parametrisation of δ*.*

The intersection behaviour of δ, ε_1 and ε_2 can be described as follows:

- ε_1 and ε_2 intersect in a smooth conic of genus 0, counted twice.
- δ intersects each ε_i in a smooth genus zero quartic and four skew lines. The quartic intersects each of the lines in two different points (with multiplicity two in one of them).
- Each line from $\varepsilon_1 \cap \delta$ intersects exactly one of the lines from $\varepsilon_2 \cap \delta$ in the same point where the quartic intersects.

Compare with the picture on page 68, which shows the intersection behaviour on E_1 which is projected to ε_1.

5.4 Degenerations on the discriminant

5.4.1 Translation scrolls

A line bundle of degree $d \geq 3$ on an elliptic curve E is very ample so defines an embedding

$$\varphi_L \colon E \longrightarrow \mathbb{P}^{d-1}$$

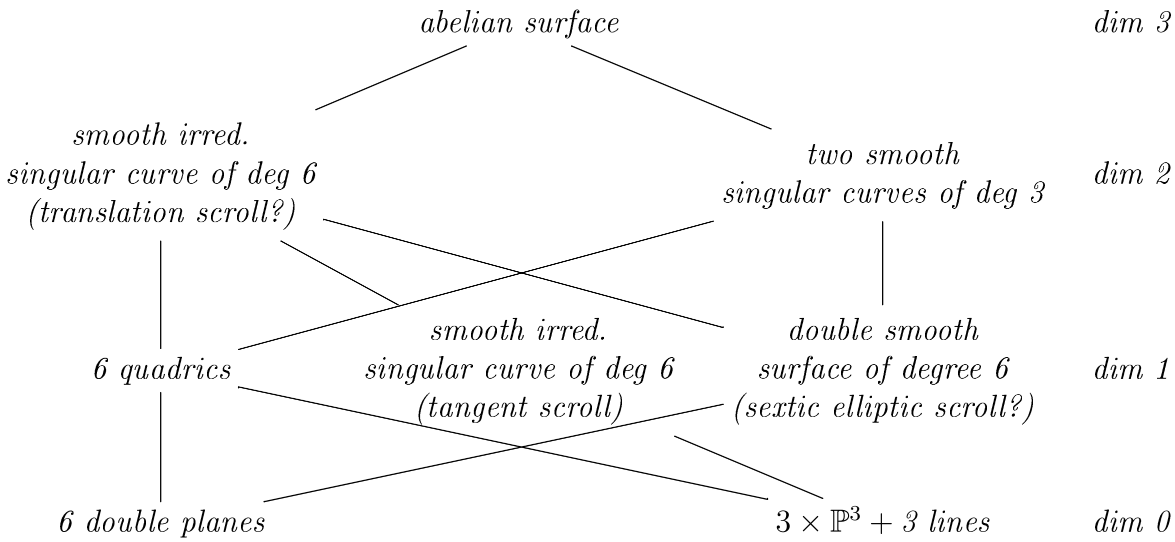
and the image is called *elliptic normal curve of degree d in $\mathbb{P}^{d-1}$*. For $d = 3$ we have the usual plane cubic, for $d = 4$ the intersection of two quadrics in $\mathbb{P}^3$, etc.
The union of all lines $L_{e,e+a}$ connecting a point $e \in E$ and the point $e + a \in E$, obtained by translating e by a fixed $a \in E$ form a surface, called a *translation scroll*

$$S = S(E, a) = \bigcup_{e \in E} L_{e,e+a}.$$

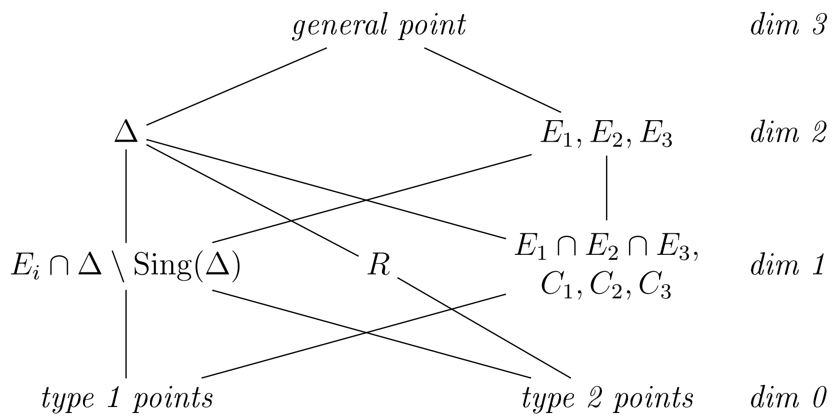
The two lines $L_{e,e+a}$ and $L_{e-a,e}$ pass through e and hence in general S is a surface of degree $2d$, singular along E. There are two notable exceptional cases:
First, if $a = 0$, these two lines degenerate to the tangent line T_eE, counted twice and the surface degenerates into the *tangent scroll* of E, which has a cuspidal ridge along E.
If a is a non-trivial 2-torsion point, then $e+a = e-a$, so these two lines also coincide. The union of all the 2-torsion secants forms a smooth surface of degree d, called *degree d elliptic scroll*, and the scroll S degenerates to this surface, counted twice.
These scrolls appear naturally as degenerations of abelian surfaces of type $(1, d)$ in $\mathbb{P}^{d-1}$.

5.4.2 Statement

Theorem 5.33. *An overview about all degenerations occurring is given in the following diagram:*



These are the subsets of Q on which each type of degeneration occurs (the corresponding equations can be found in Section 5.3.1):



Points of type 1 are the points in the intersection of all four components of the discriminant, i.e. $\Delta \cap E_1 \cap E_2 \cap E_3$. They are singular points of all curves C_i but of R.
Points of type 2 can be characterised as the singular points of R. They automatically lie on one of the E_i.

A list of all points of type 1 resp. type 2 can be found below:

type 1	type 2
$(-3 : -3 : 0 : 0 : 1 : 1)$	$(0 : 1 : 0 : 0 : 0 : 0)$
$(-3\omega^2 : -3\omega : 0 : 0 : \omega : 1)$	$(6 : -3 : 0 : 0 : -2 : 1)$
$(-3\omega : -3\omega^2 : 0 : 0 : \omega^2 : 1)$	$(6\omega^2 : -3\omega : 0 : 0 : -2\omega : 1)$
$(1 : 0 : 0 : 0 : 0 : 0)$	$(6\omega : -3\omega^2 : 0 : 0 : -2\omega^2 : 1)$
	$(0 : -1 : 1 : -1 : 1 : 0)$
	$(24 : 15 : 9 : -9 : 1 : 4)$
	$(24\omega^2 : 15\omega : 9\omega : -9\omega : \omega : 4)$
	$(24\omega : 15\omega^2 : 9\omega^2 : -9\omega^2 : \omega^2 : 4)$
	$(0 : 1 : 1 : -1 : -1 : 0)$
	$(24 : 15 : -9 : 9 : 1 : 4)$
	$(24\omega^2 : 15\omega : -9\omega : 9\omega : \omega : 4)$
	$(24\omega : 15\omega^2 : -9\omega^2 : 9\omega^2 : \omega^2 : 4)$

5.4.3 Proof

The hyperplanes

In Section 5.3.2 we have seen that there are isomorphisms ψ_1 and ψ_2 of Q as a moduli space that act transitively on the three hyperplanes E_1, E_2 and E_3.
Thus it is enough to study one hyperplane. We choose E_1 because here the equations are simpler. All results carry over to the other two hyperplanes via those isomorphisms.

The hyperplane E_1

For each point in E_1, i.e. 2-dimensional subspace $V = \langle x, y\rangle \subseteq W^\vee$ with $p_{03} = p_{12} = 0$ the projections of x and y to the first and last respectively the middle two components get linear dependent. So the annihilator is of the form

$$V^0 = \langle \begin{pmatrix} a_1 \\ 0 \\ 0 \\ -a_2 \end{pmatrix}, \begin{pmatrix} 0 \\ b_1 \\ -b_2 \\ 0 \end{pmatrix} \rangle$$

with $(a_1, a_2), (b_1, b_2) \in \mathbb{P}^1$, i.e. the ideal defining $\mathcal{C}_V$ is generated by

$$a_1 f_0 - a_2 f_3, \quad a_1 g_0 - a_2 g_3, \quad b_1 f_1 - b_2 f_2, \quad b_1 g_1 - b_2 g_2.$$

By computing the singular locus using these parameters one can show that $\mathcal{C}_V$ (with V as above) is always singular along the two curves

$$\begin{aligned} C_1 &= \{x_1 = x_3 = x_5 = a_1 f_0 - a_2 f_3 = 0\} \\ \text{and } C_2 &= \{x_0 = x_2 = x_4 = a_1 g_0 - a_2 g_3 = 0\}. \end{aligned}$$

To simplify computations we choose the chart $p_{23} \neq 0$ (for V!) so we can normalize our generators to

$$f_0 - a f_3, \quad g_0 - a g_3, \quad f_1 - b f_2, \quad g_1 - b g_2$$

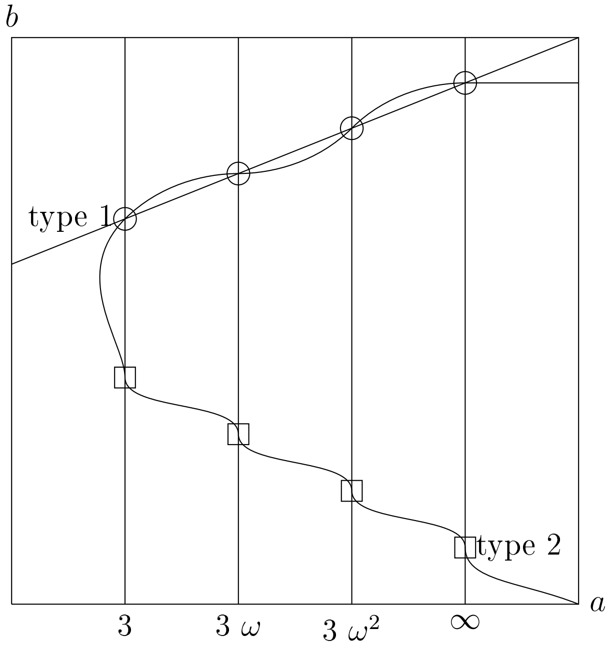
with $a = -p_{02}$ and $b = p_{13}$.

Now we can again compute the singular locus, divide out C_1 and C_2 and project the remaining part to the a-b-plane, and find out that there are worse singularities exactly above

$$a^3 = 27, \quad a = 3b \quad \text{and} \quad ab^2 + b^3 = 4.$$

An examination of the special values $a = \infty$ or $b = \infty$ shows that there are no further badly singular fibres over $b = \infty$, while the whole line $a = \infty$ has more singularities.

The line $a = 3b$ (resp. $p_{02} + 3p_{13} = 0$) is exactly the intersection locus of all the hyperplanes E_i, the vertical lines $a^3 = 27$, $a = \infty$, and the curve $ab^2 + b^3 = 4$ are the components of the intersection of E_1 with Δ.



Bad a-values

For $a^3 = 27$ or $a = \infty$ the polynomials $f_0 - af_3 = x_0^3 + x_2^3 + x_4^3 - ax_0x_2x_4$ resp. $g_0 - ag_3 = x_1^3 + x_3^3 + x_5^3 - ax_1x_3x_5$ become reducible, so also $\mathcal{C}_V$ decomposes into several components.

Let us study the case $a = \infty$ in more detail: In this case $\mathcal{C}_V$ is given by the equations

$$x_0x_2x_4 = x_1x_3x_5 = f_1 - bf_2 = g_1 - bg_2 = 0.$$

Thus, $\mathcal{C}_V$ is contained in the union of the subspaces $L_{ij} = \{x_i = x_j = 0\}$, $i = 0, 2, 4$, $j = 1, 3, 5$.
Now $\mathcal{C}_V \cap L_{i,i+1}$ is given by the equations

$$x_i = x_{i+1} = x_{i-1}(x_{i+2}x_{i-1} - bx_{i+3}x_{i-2}) = x_{i+2}(x_{i+2}x_{i-1} - bx_{i+3}x_{i-2}) = 0.$$

Here all indices are meant to be read modulo 6. So it is the union of the quartic surface $S_i = \{x_i = x_{i+1} = x_{i+2}x_{i-1} - bx_{i+3}x_{i-2} = 0\}$ and the line $l'_{i+3,i+4} = \{x_{i-1} = x_i = x_{i+1} = x_{i+2} = 0\}$ where the indices of l' stand for the free coordinates.
Similarly, $\mathcal{C}_V \cap L_{i,i+3}$ is given by the equations

$$x_i = x_{i+3} = x_{i+1}^2x_{i-2} + x_{i-1}^2x_{i+2} = x_{i+1}x_{i-2}^2 + x_{i-1}x_{i+2}^2 = 0$$

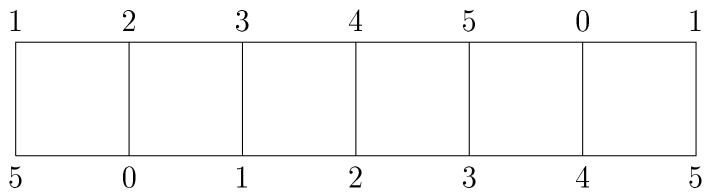
which turns out to be the union of the four lines $l'_{i+1,i+2}, l'_{i+2,i-2}, l'_{i-1,i+1}, l'_{i-2,i-1}$ and the three lines l_{ij}, $j = 0, 1, 2$, described in Proposition 5.6. Since the lines l'_{ij} are all contained in some of the surfaces S_i, $\mathcal{C}_V$ actually consists of six quadric surfaces and the nine lines l_{ij}.

Thus, A_V is the union of the six quadrics S_i. The components intersect as follows:

$$\begin{aligned} S_i \cap S_{i+1} &= l'_{i-1,i+1} \cup p_{i-2} \\ S_i \cap S_{i+2} &= l'_{i-2,i-1} \\ S_i \cap S_{i+3} &= p_{i-1} \cup p_{i+2} \end{aligned}$$

where $\mathbb{P}^5 \ni p_i = (0, \dots, 1, \dots, 0)$ with all but the ith component equal to zero.
Now we have:

- Each quadric intersects all but one of the others in a line. In total there are twelve lines, four on each quadric.
- The lines do not depend on the value of b.
- In each of the six points p_i intersect exactly four lines.
- The four lines on each quadric cut out a square. The six squares build a (degenerate) torus glued with a twist of two in one direction.



For the other special values of a the exact formulas for the six quadrics S_i, the twelve lines l'_{ij} and the six points p_i differ, but qualitatively exactly the same happens. We always obtain a degenerate torus with a twist of two in one direction.

The line $a = 3b$

If $a = 3b$, our degree 12 surface A_V degenerates to a double smooth surface of degree 6. The reduced variety associated to A_V corresponds to the ideal minimally generated by the four given cubics and the three determinantal conics $x_1x_2 - x_4x_5, x_0x_1 - x_3x_4, x_2x_3 - x_0x_5$.

The curve $ab^2 + b^3 = 4$

For a, b on the curve $ab^2 + b^3 = 4$ we also get to a double smooth surface of degree 6. But this time the ideal of the reduced variety corresponding to A_V is minimally generated by only the two cubics $f_0 - af_3$ and $g_0 - ag_3$ and the three conics

$$x_{i+1}x_{i+2} + 2x_ix_{i+3} + x_{i-1}x_{i-2}, \quad i = 0, 1, 2.$$

Intersection points

The lines and curves given above have less intersection points then one would expect. Wherever the line $b = 3a$ intersects the vertical lines $a = const$ this is also an intersection point with the curve $ab^2 + b^3 = 4$. Let us call these intersection points of type 1.
Despite this the curve $ab^2 + b^3 = 4$ has only one more intersection point with each of the vertical lines, but with multiplicity two. Call these intersection points of type 2.

In each intersection point of type 1 the fibre $\mathcal{C}_V$ becomes three-dimensional of degree three. For the case $a = b = \infty$ a direct primary decomposition shows that it is the union of the three linear subspaces $\{x_0 = x_3 = 0\}$, $\{x_1 = x_4 = 0\}$, $\{x_2 = x_5 = 0\}$ and the three lines $l'_{i,i+3}$ with some multiplicities.
In the other cases a direct primary decomposition of A_V is not successful, but one can decompose the singular locus into three fivefold lines, combine each two of them to get a three-dimensional linear subspace of $\mathbb{P}^5$, and in fact A_V turns out to be the union of these three subspaces and these three lines.

In each intersection point of type 2 A_V degenerates into six plains (each with multiplicity two). Each plane intersects exactly two of the others in a line. The six lines form a hexagon.

The component Δ

For the component Δ of D described by g, we have the problem that there is no obvious rational point lying on Δ but on none of the hyperplanes. This problem can be approached in positive characteristic. For example in characteristic 601 the point $p = (1 : 7 : 5 : -5 : -6 : -17)$ belongs to Δ but to none of the hyperplanes. This and similar experiments show that the corresponding variety A_p is a surface of degree twelve singular along a smooth irreducible curve of degree 6, whose ideal is generated by nine quadrics.

The curve $R \subset Sing(\Delta)$

A point on R can be found using the parametrisation μ from Proposition 5.31. For $t = 1$ we obtain the point $(1800 : -11 : -837 : 108) \in C_6$. Choosing a characteristic such that $1800 \cdot 108 - 11 \cdot 837$ has a square root x, we find a point $(1800 : -11 : x : -x : -837 : 108)$ on R.
In characteristic 73, this yields the point $p = (-25 : -11 : 24 : -24 : -34 : 35)$ lying on R. The corresponding variety A_p is a reduced surface of degree 12 singular along a smooth curve of degree 6, which turns out to be an elliptic normal curve defined by nine quadrics.
The original surface $A_p$is in fact the tangent scroll of this curve. This is what we expected by comparison with the $(1, 5)$-polarised case (Section 4.5).

5.5 The Quartics

In this section we want to describe the generators of the ideal corresponding to A_V i.e. the saturation $\tilde{I}_V = (I_V : I_{lines}^{\infty})$.
In all examples we studied so far $\tilde{I}_V$ is minimally generated by the four original cubics and a number of (in general and at most six) quartics which can be chosen in such a way that σ acts on them as a permutation in three-cycles.

In Section 5.5 we gave a basis $f_0, \ldots, f_3, g_0, \ldots, g_3$ of $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$, where $\mathbf{H}'$ is the subgroup of the Heisenberg group generated by σ^2 and τ^2, and decomposed it as $V_0 \otimes W$ where W is a four dimensional complex vector space with basis $e_0, \ldots, e_3$ such that $V_0 \otimes e_i = \langle f_i, g_i \rangle$.
Now we will do similarly for quartics invariant under σ^3 and τ^3.
A basis of $H^0(\mathcal{O}_{\mathbb{P}^5}(4))^{\langle \sigma^3, \tau^3 \rangle}$ is given by $s_i, \sigma s_i, \sigma^2 s_i$, $i = 1, \ldots, 12$ where s_i are the following polynomials:

$$
\begin{aligned}
s_1 &= x_0^4 + x_3^4 & s_7 &= x_0^2 x_2 x_4 + x_3^2 x_5 x_1 \\
s_2 &= x_0^2 x_3^2 & s_8 &= x_5^2 x_0 x_2 + x_2^2 x_3 x_5 \\
s_3 &= x_1^2 x_2^2 + x_3^2 x_4^2 & s_9 &= x_1^2 x_4 x_0 + x_4^2 x_1 x_3 \\
s_4 &= x_2^2 x_4^2 + x_5^2 x_1^2 & s_{10} &= x_0^2 x_1 x_5 + x_3^2 x_4 x_2 \\
s_5 &= x_1^3 x_3 + x_4^3 x_0 & s_{11} &= x_0 x_1 x_2 x_3 + x_3 x_4 x_5 x_0 \\
s_6 &= x_2^3 x_0 + x_5^3 x_3 & s_{12} &= x_1 x_2 x_4 x_5.
\end{aligned}
$$

$\mathbf{H}_6$ acts on this space: σ as a permutation of order three on this basis interchanging s_i, σs_i and $\sigma^2 s_i$, τ by multiplying $\sigma^j s_i$ by ω^j, ω being a third root of unity. Thus $H^0(\mathcal{O}_{\mathbb{P}^5}(4))^{\langle \sigma^3, \tau^3 \rangle}$ as a representation of $\mathbf{H}_6$ splits up into twelve isomorphic subrepresentations

$$H^0(\mathcal{O}_{\mathbb{P}^5}(4))^{\langle \sigma^3, \tau^3 \rangle} \cong \bigoplus_{i=1}^{12} \langle s_i, \sigma s_i, \sigma^2 s_i \rangle.$$

We will identify $H^0(\mathcal{O}_{\mathbb{P}^5}(4))^{\langle \sigma^3, \tau^3 \rangle}$ with $T_0 \otimes S$ where T_0 is a three dimensional representation of $\mathbf{H}_6$ and S is a twelve dimensional complex vector space with basis $e_1, \ldots, e_{12}$ such that $T_0 \otimes e_i = \langle s_i, \sigma s_i, \sigma^2 s_i \rangle$.
By abuse of notation we will identify W resp. S with the vector spaces generated by $f_0, \ldots, f_3$ and $s_1, \ldots, s_{12}$ i.e. the eigenspace of τ with eigenvalue 1 in $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$ resp. $H^0(\mathcal{O}_{\mathbb{P}^5}(4))^{\langle \sigma^3, \tau^3 \rangle}$.
In this sense tensoring with V_0 resp. T_0 and intersecting with W resp. S are operations inverse to each other.

Thus the map

$$\phi\colon I_V \mapsto (I_V : I_{lines}^{\infty})$$

can apparently be interpreted as a map from the set $\mathcal{I}_3$ of all ideals in $\mathbb{C}[x_0, \ldots, x_5]$ generated by a four-dimensional subrepresentation of $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$ to the set $\mathcal{I}_4$

of all ideals in $\mathbb{C}[x_0, \ldots, x_5]$ generated by a four-dimensional subrepresentations of $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$ and an at most six-dimensional subrepresentation of $H^0(\mathcal{O}_{\mathbb{P}^5}(4))^{\langle\sigma^3,\tau^3\rangle}$.

According to our consideration above any 2-dimensional subspace $V \subseteq W$ induces an ideal I_V in $\mathcal{I}_3$ generated by the subrepresentation $V^0 \otimes V_0$ of $H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$. Accordingly, any pair (V, T) of a 2-dimensional subspace $V \subseteq W$ and an at most 2-dimensional subspace $T \subseteq S$ induce an ideal $I_{V,T}$ in $\mathcal{I}_4$ generated by $V^0 \otimes V_0 \oplus T \otimes T_0 \subseteq H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'} \oplus H^0(\mathcal{O}_{\mathbb{P}^5}(4))^{\langle\sigma^3,\tau^3\rangle} \subseteq \mathbb{C}[x_0, \ldots x_5]$. Let us call these maps p_3 and p_4.

Example 5.34. Consider the ideal I_V given by $\langle f_0, g_0, f_1, g_1\rangle$ corresponding to the vector space $V = \langle e_2, e_3\rangle$.
Then

$$\tilde{I}_V = (I_V : I_{lines}^{\infty}) = \langle f_0, g_0, f_1, g_1, h_1, \sigma(h_1), \sigma^2(h_1), h_2, \sigma(h_2), \sigma^2(h_2)\rangle$$

with $h_1 = s_5 - 2s_{12}$ and $h_2 = -s_{10} + s_{11}$.
But we can find other generators of this form, for example

$$\begin{aligned} h_1' &= h_1 + x_0 f_0 + x_3\sigma(f_0) = s_1 + 2s_5 + s_6 - 2s_{12} \\ h_2' &= h_2 + x_0 f_1 + x_3\sigma(f_1) = 2s_4 + s_7 + s_9 - s_{10} - s_{11}. \end{aligned}$$

So (V, T) and (V, T') with

$$T = \langle e_5 - 2e_{12}, -e_{10} + e_{11}\rangle \text{ and}$$

$$T' = \langle e_1 + 2e_5 + e_6 - 2e_{12}, 2e_4 + e_7 + e_9 - e_{10} - e_{11}\rangle$$

are two pairs of subspaces of W resp. S with $p_4(V, T) = p_4(V, T') = \tilde{I}_V$.
More generally, we can add any linear combination of $x_0 f_0 + x_3\sigma(f_0)$ and $x_0 f_1 + x_3\sigma(f_1)$ to h_1 and h_2 without changing the ideal $\tilde{I}_V$ nor the way $\mathbf{H}$ acts on its generators.

Now let us say a few words about how uniquely V and T can be reconstructed from their images in $\mathcal{I}_3$ resp $\mathcal{I}_4$ in general:
For $I_3 \in \mathcal{I}_3$ the subspace $V \subseteq W$ such that $p_3(V) = I_3$ is uniquely given by the annihilator of the intersection of I_3 with W.
For $I_4 \in \mathcal{I}_4$ we have $V = (I_4^{(3)} \cap W)^0$ where $I^{(k)}$ denotes the degree k part of an homogeneous ideal I. The subspace T representing the degree 4 generators is not unique. Let J be a complement of $\mathfrak{m}I_4^{(3)}$ in $I_4^{(4)}$, i.e. $I_4^{(4)} = \mathfrak{m}I_4^{(3)} \oplus J$ as $\mathbb{C}$-vector spaces. After Maschke's theorem J can be chosen to be subrepresentation of $H^0(\mathcal{O}_{\mathbb{P}^5}(4))^{\langle\sigma^3,\tau^3\rangle}$. If the ideal is of the general form described above, J has dimension 6 (in special cases less). So $T = J \cap S$ is a two-dimensional subspace of S and $p_4(V, T) = I_4$.
The other way around, if $p_4(V, T) = I_4$, then $J = T_0 \otimes T$ always is a subrepresentation of $\mathbf{H}$ in $I_4^{(4)}$, complementary to $\mathfrak{m}I_4^{(3)}$ where $I_4^{(3)} = V_0 \otimes V$. So the only choice we have to make is those of the complement J.
If $I_4^{(3)} = \langle q_1, \ldots, q_4\rangle$ (with $\sigma(q_i) = q_{i+1 \mod 2}, \tau(q_i) = (-1)^{i-1} q_i$) and J and J' are two such complements, $J = \langle h_1, \ldots, h_6\rangle$, $\sigma(h_i) = h_{i+1 \mod 3}, \tau(h_i) = \omega^{i-1} h_i$, then J' can

be generated by $h'_1, \dots, h'_6$ where $h'_i = h_i + b_i$, $b_i \in \mathfrak{m} I_4^{(3)}$ and σ and τ act in the same way. The condition that $\sigma^3(h'_i) = h'_i$ implies that $b_i \in \langle q_1 x_0 + q_2 x_3, q_3 x_0 + q_4 x_3 \rangle$.
So T is only defined up to addition of these polynomials to its generators. This ambiguity can be used to modify the map f such that the diagramm below commutes. To make this more visible we choose another basis for $H^0(\mathcal{O}_{\mathbb{P}^5}(4))^{\langle \sigma^3, \tau^3 \rangle}$, namely $t_i, \sigma t_i, \sigma^2 t_i$ for:

$$\begin{aligned}
t_1 &= x_0 f_0 + x_3 g_0 & t_7 &= s_4 \\
t_2 &= x_0 f_1 + x_3 g_1 & t_8 &= s_5 \\
t_3 &= x_0 f_2 + x_3 g_2 & t_9 &= s_6 \\
t_4 &= x_0 f_3 + x_3 g_3 & t_{10} &= s_7 \\
t_5 &= s_2 & t_{11} &= s_{11} \\
t_6 &= s_3 & t_{12} &= s_{12}.
\end{aligned} \tag{5.2}$$

In this basis the subspace T belonging to a given subspace V may only vary by adding elements of the form $\iota(v), v \in V$, to its generators. Here $\iota : W \longrightarrow S, e_i \longmapsto t_i$ means the embedding of W into the first four components of S.

For any element $x \wedge y$ of $V \wedge V$ the "minors" $p_{ij} = x_i y_j - x_j y_i$ are well defined (independent of the choice of a representative). Let Q be the 5-dimensional subspace of $V \wedge V$ given by $p_{03} + p_{12} = 0$. By Q_0 we denote the set of non-zero "elementary wedges" in Q, i.e. $Q_0 = \{x \wedge y \mid x, y \in V, x_0 y_3 - x_3 y_0 + x_1 y_2 - x_2 y_1 = 0\}$.
A basis of Q lying in Q_0 is given by

$$\begin{aligned}
&b_1 = e_0 \wedge e_1 \qquad b_2 = e_0 \wedge e_2 \qquad b_3 = e_1 \wedge e_3 \\
&b_4 = (e_1 + e_2) \wedge e_3 \qquad b_5 = (e_1 - e_3) \wedge (e_2 - e_4).
\end{aligned} \tag{5.3}$$

Let $\pi_3 \colon Q_0 \longrightarrow \mathcal{I}_3$ be the map sending an element $v_1 \wedge v_2$ of Q_0 to the ideal generated by $V_0 \otimes \langle v_1, v_2 \rangle \subset H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'}$, i.e. $p_3(\langle v_1, v_2 \rangle)$.
Let $\pi_4 \colon Q_0 \times (S \times S) \longrightarrow \mathcal{I}_4$ be the map sending $(v_1 \wedge v_2, s_1, s_2)$ to the ideal generated by $V_0 \otimes \langle v_1, v_2 \rangle \oplus T_0 \otimes \langle s_1, s_2 \rangle \subset H^0(\mathcal{O}_{\mathbb{P}^5}(3))^{\mathbf{H}'} \oplus H^0(\mathcal{O}_{\mathbb{P}^5}(4))^{\langle \sigma^3, \tau^3 \rangle} \subset \mathbb{C}[x_0, \dots x_5]$, i.e. $p_4(\langle v_1, v_2 \rangle, \langle s_1, s_2 \rangle)$.
In the following we will construct a map $f \colon Q \longmapsto S \times S$ such that we have the following diagram

$$\begin{tikzcd}
Q_0 \arrow[r, "id \times f|_{Q_0}"] \arrow[d, twoheadrightarrow, "\pi_3"'] & Q_0 \times (S \times S) \arrow[d, twoheadrightarrow, "\pi_4"] \\
\mathcal{I}_3 \arrow[r, dashed, "\phi", "I_V \mapsto (I_V : I_{lines}^{\infty})"'] & \mathcal{I}_4
\end{tikzcd} \tag{5.4}$$

We discussed above that $\langle v_1, v_2 \rangle$ can be uniquely reconstructed from its corresponding ideal whereas $\langle s_1, s_2 \rangle$ can not. This means in particular that the preimage of some ideal $I \in \mathcal{I}_3$ under π_3 is unique up to a scalar.

Proposition 5.35. *There is a map of* $\mathbb{C}$*-vector spaces* $f\colon Q \longmapsto S \times S$ *such that the following holds:*

- *If* $I_V = \pi_3(v_1 \wedge v_2) \in \mathcal{I}_3$*, then* A_V *is cut out set-theoretically by the ideal* $\pi_4(id \times f|_{Q_0}(v_1 \wedge v_2))$.
- *The ideal* $(I_V : I^\infty_{lines})$ *actually contains* $\pi_4(id \times f|_{Q_0}(v_1 \wedge v_2))$.

Proof. 1. According to our considerations above, if $f(b_i) = (v_i, w_i)$, then (v'_i, w'_i) with $v'_i = v_i + r_i\iota(b_i^{(1)}) + s_i\iota(b_i^{(2)})$, $w'_i = w_i + t_i\iota(b_i^{(1)}) + u_i\iota(b_i^{(2)})$ will generate the same ideal. So the fact that the diagram commutes for the five base vectors b_i leaves us with twenty free parameters. But considering the images of further linear combinations of the base vectors, results in linear relations between these parameters, so that only eight of them survive. For the same reason, the set of all admissible parameters has to be a linear subspace of $\mathbb{C}^8$.

So if there is an f that makes the diagram above commute, it has to be one of the maps $f_{p_1,p_2,q_1,q_2,r_1,r_2,s_1,s_2}$ given by

$$\begin{aligned}
b_1 \mapsto &(-s_1 - \frac{1}{3}, p_1 - r_1, -\frac{2}{3}, 0, 0, 1, 0, 0, 0, 0, 3, 0),\\
&(-s_2 + \frac{5}{3}, p_2 - r_2 - \frac{1}{3}, -\frac{5}{3}, \frac{1}{3}, 2, 0, 0, -2, -2, 0, 2, 4)\\
b_2 \mapsto &(p_1 + r_1, \frac{1}{3}, p_1 - r_1 - 1, 0, 0, 0, 0, 0, 0, 0, 0, 3),\\
&(p_2 + r_2 + \frac{1}{3}, -\frac{1}{3}, p_2 - r_2, 0, 2, 0, 0, 0, 0, 0, 0, -4)\\
b_3 \mapsto &(-\frac{1}{3}, q_1 - s_1, \frac{1}{3}, s_1, 1, 0, 0, 0, 0, 0, 0, 1),\\
&(0, q_2 - s_2 - 2, \frac{1}{3}, s_2, 0, 0, 6, 0, 0, 0, 0, 0)\\
b_4 \mapsto &(0, q_1 - s_1 + \frac{1}{3}, q_1 - s_1, s_1 - p_1 - r_1, 0, 0, 0, 0, 0, 0, 0, 1),\\
&(0, q_2 - s_2 - \frac{4}{3}, q_2 - s_2, s_2 - p_2 - r_2 - \frac{1}{3}, 0, 0, 2, 0, 0, 0, 0, 0)\\
b_5 \mapsto &(-q_1, 2p_1 + \frac{1}{3}, q_1 + \frac{1}{3}, -2p_1 + \frac{1}{3}, 0, 0, 0, 0, 0, 0, 0, 1),\\
&(-q_2, 2p_2 - \frac{1}{3}, q_2 - \frac{1}{3}, -2p_2 - \frac{1}{3}, 0, 0, 2, 0, 0, 0, 2, 0)
\end{aligned}$$

in the basis t_i for some $p_i, q_i, r_i, s_i \in \mathbb{C}, i = 1, 2$.

2. For $p_i = q_i = r_i = s_i = 0$ for example we can show the following:

Take a generic point from each chart of Q as described in the table on page 49 e.g. $V = \langle \begin{pmatrix} 1 \\ 0 \\ a \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ c \\ a \end{pmatrix} \rangle$. Calculate the ideal $\tilde{I}_V = (I_V : I^\infty_{lines})$ in the ring $\mathbb{C}[x_0, \ldots, x_5, a, b, c]$.

On the other hand lift V to an element of $v_1 \wedge v_2 \in Q_0$ e.g. $\begin{pmatrix} 1 \\ 0 \\ a \\ b \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ c \\ a \end{pmatrix}$. This is unique up to a scalar, thus $(V, T) := \pi_4 \circ (id \times f)(v_1 \wedge v_2)$ is well-defined. Find generators of the ideal $p_4(V, T) \subseteq \mathbb{C}[x_0, \ldots, x_5, a, b, c]$. Now we can check using for example SINGULAR that these ideals are the same for all four charts.

3. In the same manner we can show that the ideal $\pi_4(v_1 \wedge v_2, f(v_1, \wedge v_2))$ is the same for $f_{1,0,\ldots,0}, \ldots, f_{0,\ldots,0,1}$ and $f_{0,\ldots,0}$. Since the space of all admissible parameter values is a linear, it has to be empty or all of $\mathbb{C}^8$. So the choice of parameters does not matter.

4. Since lifting along π_3 is unique up to a scalar, the map along the upper way of diagram (5.4)

$$g\colon \mathcal{I}_3 \longrightarrow \mathcal{I}_4, I \longmapsto \pi_4 \circ (id \times f)(v_1 \wedge v_2) \qquad \text{where } I = \pi_3(v_1 \wedge v_2)$$

is well-defined and commutativity of (5.4) is equivalent to the equality of g and ϕ.

Any ideal $I \in \mathcal{I}_3$ corresponds to a 2-dimensional subspace V of W. In the following we only treat subspaces of the form $\langle \begin{pmatrix} 1 \\ 0 \\ z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ z_3 \\ z_1 \end{pmatrix} \rangle$, $z = (z_1, z_2, z_3) \in \mathbb{C}^3$, and denote the corresponding ideal by I_z. In the other charts the argument is completely equivalent.

In point 2 we considered the parameters a, b and c as variables which is equivalent to tensoring the whole diagram with $\mathbb{C}[a, b, c]$. Denote the corresponding maps in this diagram by g_{abc} and ϕ_{abc}.

Denote by I_{abc} the ideal in $\mathbb{C}[x_0, ..., x_5, a, b, c]$ corresponding to $\langle \begin{pmatrix} 1 \\ 0 \\ a \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ c \\ a \end{pmatrix} \rangle$.

Then we have already shown that

$$g_{abc}(I_{abc}) = \phi_{abc}(I_{abc}). \tag{$*$}$$

Let $\varphi_z\colon \mathbb{C}[x_0, ..., x_5, a, b, c] \longrightarrow \mathbb{C}[x_0, ..., x_5]$ be the map that inserts the complex numbers z_1, z_2, z_3 for a, b resp. c. For simplicity we denote the induced maps on any of the sets occurring above also by φ_z. Then $I_z = \varphi_z(I_{abc})$ and $g \circ \varphi_z = \varphi_z \circ g_{abc}$ because the construction of g involves only linear algebra.

We would like to show that

$$g(I_z) = \phi(I_z) \text{ for all } z \in \mathbb{C}^3. \tag{$**$}$$

This does in fact not follow from $(*)$. To get a taste of the problem look at Example 5.37 given below.

But we can conclude that:

(i) $g(I)$ and $\phi(I)$ have the same zero set for all $I \in \mathcal{I}_3$.

(ii) $g(I) \subseteq (I : I_{lines}^{\infty})$ for all $I \in \mathcal{I}_3$.

For (i): Denote by I_{lines}^{abc} resp. I_{lines} the ideals generated by

$$f_i, \quad g_i, \quad i = 0, \dots, 3, \quad x_1x_2 - x_4x_5, \quad x_0x_1 - x_3x_4, \quad x_2x_3 - x_0x_5$$

in $\mathbb{C}[x_0, ..., x_5, a, b, c]$ resp. $\mathbb{C}[x_0, ..., x_5]$ as discussed in Proposition 5.6.

We have $g(I_z) = g(\varphi_z(I_{abc})) = \varphi_z(g_{abc}(I_{abc})) = \varphi_z(I_{abc} : {I_{lines}^{abc}}^{\infty})$.

Let us describe the zero set of this ideal: Denote by $\mathcal{C}_z$ the zero-set in $\mathbb{P}^5$ corresponding to I_z.

Then the zero set of I_{abc} is

$$\mathcal{C} := \{(z, \mathcal{C}_z) \mid z = (z_1, z_2, z_3) \in \mathbb{C}^3 \cong \mathbb{A}^3\} \subset \mathbb{A}^3 \times \mathbb{P}^5$$

and I_{lines}^{abc} corresponds to $\mathbb{A}^3 \times L \in \mathbb{A}^3 \times \mathbb{P}^5$.

So $J_{abc} := (I_{abc} : {I_{lines}^{abc}}^{\infty})$ describes the set $\mathcal{C} \setminus (\mathbb{A}^3 \times L)$.

Let a, b, c be the coordinates of $\mathbb{A}^3$. Then $\varphi_z(J_{abc})$ corresponds to the set

$$(\mathcal{C} \setminus (\mathbb{A}^3 \times L)) \cap \{a = z_1, b = z_2, c = z_3\} = \mathcal{C}_z \setminus L$$

which is exactly the set described by $(I_z : I_{lines}^{\infty}) = \phi(I_z)$.

For (ii): Let I be any ideal from $\mathcal{I}_3$ such that $I = \varphi_z(I_{abc})$ for $z \in \mathbb{C}^3$. Then we have

$$\begin{aligned}(I : I_{lines}^{\infty}) &= (\varphi_z(I_{abc}) : \varphi_z(I_{lines}^{abc})^{\infty}) \supseteq \varphi_z(I_{abc} : {I_{lines}^{abc}}^{\infty}) \\ &= \varphi_z(\phi_{abc}(I_{abc})) = \varphi_z(g_{abc}(I_{abc})) = g(\varphi_z(I_{abc})) = g(I)\end{aligned}$$

using the following lemma and corollary plus the fact that $\varphi_{z_1,z_2,z_3}(I_{lines}^{abc}) = I_{lines}$. ∎

Lemma 5.36. *Let $\iota \colon R \longrightarrow S$ be a ring extension and $\varphi \colon S \longrightarrow R$ a ring homomorphism with $\varphi \circ \iota = \mathrm{id}_R$. Let I, J be two ideals in S. Then*

$$\varphi(I : J) \subseteq (\varphi(I) : \varphi(J)).$$

The other inclusion in in general not true.

Proof. Let f be any element of $\varphi(I : J)$. This means that there existst an $h \in S$ such that $f = \varphi(h)$ and $h \cdot j \in I$ for all $j \in J$. But then

$$f \cdot \varphi(j) = \varphi(h) \cdot \varphi(j) = \varphi(h \cdot j) \in \varphi(I)$$

for all $j \in J$, so $f \in (\varphi(I) : \varphi(J))$. ∎

Example 5.37. For the other inclusion consider $R = \mathbb{C}[x]$, $S = \mathbb{C}[x, a]$, $I = (x^2 - a)$, $J = (x)$ and $\varphi\colon S \longrightarrow R$ the $\mathbb{C}$-algebra homomorphism given by $x \mapsto x$ and $a \mapsto 0$. Then $(I : J) = (x^2 - a)$, so $\varphi(I : J) = (x^2)$, but $\varphi(I) = (x^2)$ and $\varphi(J) = (x)$, so $(\varphi(I) : \varphi(J)) = (x)$ which is actually the larger ideal.

Corollary 5.38. *The same inclusion holds for the saturation* $(I : J^\infty) = \bigcup_{k=0}^{\infty}(I : J^k)$ *i.e.*

$$\varphi(I : J^\infty) \subseteq (\varphi(I) : \varphi(J)^\infty).$$

In other words, this means that given an ideal I_V to determine the generators of $\tilde{I}_V$ one can go along the following steps:

- Choose any two generators v_1 and v_2 of V^0.
- Decompose $v_1 \wedge v_2$ in basis (5.3). This can be done by looking at the minors.
- Use the description of f given above to determine $f(v_1 \wedge v_2) = (s_1, s_2) \in S \times S$.
- Then $\tilde{I}_V$ is generated by the σ-orbits of the linear combinations of the f_i corresponding to v_1, v_2 and the linear combinations of the t_i corresponding to s_1 and s_2.

5.6 A subfamily with two fibrations

In [HR00] Klaus Hulek and Kristian Ranestad describe the family $\mathcal{A}$ of all abelian surfaces of type (1,6) embedded $\mathbf{H}_6^e$-invariantly into $\mathbb{P}^5$ with two plane cubic fibrations.
This subfamily can be characterised as follows: There are four special points in Q corresponding to $\mathbf{H}_6^e$-orbits of three lines in $\mathbb{P}^5$. These are described in Lemma 7.4, Proposition 7.7 and Remark 7.8 in [HR00]. They can be given explicitly by

$$\begin{aligned}
p_1 &= (1 : 0 : 0 : 0 : 0 : 0) & V_1 &= \langle \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \rangle \\
p_2 &= (3 : 3 : 0 : 0 : -1 : -1) & V_2 &= \langle \begin{pmatrix} 1\\0\\0\\-3 \end{pmatrix}, \begin{pmatrix} 0\\1\\-1\\0 \end{pmatrix} \rangle \\
p_3 &= (3 : 3\omega : 0 : 0 : -\omega : -\omega^2) & V_3 &= \langle \begin{pmatrix} \omega\\0\\0\\-3 \end{pmatrix}, \begin{pmatrix} 0\\\omega\\-1\\0 \end{pmatrix} \rangle \\
p_4 &= (3 : 3\omega^2 : 0 : 0 : -\omega^2 : -\omega) & V_4 &= \langle \begin{pmatrix} \omega^2\\0\\0\\-3 \end{pmatrix}, \begin{pmatrix} 0\\\omega^2\\-1\\0 \end{pmatrix} \rangle
\end{aligned}$$

with ω a primitive third root of unity. These are exactly the intersection points of type 1 from Section 5.4.
Now recall that points in $Q \subseteq Gr(2, W^\vee)$ correspond to lines in $\mathbb{P}^3$. The proofs of Corollary 7.12 and 7.13 tell us that an abelian surfaces has two plane cubic fibrations

if and only if it corresponds to a line that intersects two of the special lines given above.

Equations for the component $\mathcal{A}_{ij}$ of $\mathcal{A}$ consisting of all lines in $\mathbb{P}^3$ intersecting the lines V_i and V_j can be found as follows:

If $L \subset W$ two-dimensional intersects for example V_1 and V_2 non-trivially, then there has to be an $0 \neq x \in L \cap V_1$ and an $0 \neq y \in L \cap V_2$. Since $V_1 \cap V_2 = 0$, it follows that $L = \langle x, y \rangle$. This means that $L = \langle \begin{pmatrix} 0 \\ 0 \\ a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \\ -d \\ -3c \end{pmatrix} \rangle$ for some $(a : b), (c : d) \in \mathbb{P}^1$. Thus, L corresponds to the point $(-3ac + bd : bd : -ad : -bc : ac : 0) \in Gr(2, W^\vee)$. Equations describing all points of this form can be computed by a simple elimination argument and we obtain

$$\begin{aligned}
\mathcal{A}_{12} &= \{p_{23} = p_{01} - p_{02} + 3p_{13} = 0\} \\
\mathcal{A}_{13} &= \{p_{23} = p_{01} - \omega^2 p_{02} + 3\omega^2 p_{13} = 0\} \\
\mathcal{A}_{14} &= \{p_{23} = p_{01} - \omega p_{02} + 3\omega p_{13} = 0\} \\
\mathcal{A}_{23} &= \{p_{02} - 3p_{13} - 3\omega p_{23} = p_{01} + 3\omega^2 p_{23} = 0\} \\
\mathcal{A}_{24} &= \{p_{02} - 3p_{13} - 3\omega^2 p_{23} = p_{01} + 3\omega p_{23} = 0\} \\
\mathcal{A}_{34} &= \{p_{02} - 3p_{13} - 3p_{23} = p_{01} + 3p_{23} = 0\},
\end{aligned}$$

all described in Q.

5.7 Finding the integrable systems

As we have already seen in Chapter 1, Theorem 1.15, the DGR system is algebraically completely integrable, and its fibres embed as (1,6)-polarized abelian surfaces in $\mathbb{P}^5$. In [SvS13] Semmel and van Straten give explicit equations (four cubics and six quartics) which describe these surfaces. We want to determine where these surfaces sit in the moduli space of (1,6)-polarized abelian surfaces studied here.

This is not possible by just looking at the cubic equations, because they are not Heisenberg invariant in our sense of the word, respectively the Heisenberg group acting on these surfaces is not in standard form.

One approach might be to change this by looking at the lines that are cut out by the cubics in addition the surface and find a change of coordinates that converts them in the standard form described in Proposition 5.6. But this seems to be quite challenging, especially because the lines in the form given in that paper can only be distinguished over a quite complicated field extension.

A simpler first approach is given by the observation that one of the nine lines in the DGR-system is defined over $\mathbb{Q}$. It has the remarkable property that it intersects each surface in four points, whose j-invariant is zero.

In the coordinates described here any of the lines l_{ij} described in Proposition 5.6 intersects the general surface in four points and the j-invariant of these points is

independent of the choice of the line l_{ij}, because $\mathbf{H}_6$ acts linearly and transitively on the nine lines, the surface itself is $\mathbf{H}_6$-invariant and the j-invariant does not change under linear changes of coordinates.
But the j-invariant in fact depends on the choice of the particular surface. So only those surfaces whose j-invariant with respect to these lines is 0 can come from the DGR-system.
Let us look at surfaces encoded by a point in the chart $p_{23} \neq 0$ and the line

$$l_{00} = \{x_0 = x_3 = x_1 + x_5 = x_2 + x_4 = 0\}.$$

Then the surface $\mathcal{C}_p$ is described by the ideal

$$\langle \begin{pmatrix} 1 \\ 0 \\ a \\ c \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ b \\ a \end{pmatrix} \rangle \otimes W = \langle f_0 + af_2 + cf_3, g_0 + ag_2 + cg_3, f_1 + bf_2 + af_3, g_1 + bg_2 + ag_3 \rangle$$

for some $a, b, c \in \mathbb{C}$ and A_p intersects l_{00} in the points

$$(0 : -1 : -x_4 : 0 : x_4 : 1) \text{ with } ax_4^4 + (c - 3b)x_4^2 + a = 0. \qquad (*)$$

Equation $(*)$ has less than four solutions if and only if either $a = 0$ or $y^2 + \frac{c-3b}{a}y + 1 = 0$ has only one solution y. The latter condition is equivalent to $(c - 3b)^2 = (2a)^2$, i.e. $\pm 2a - 3b + c = 0$. These three cases read like

$$p_{12} = 0 \text{ and } p_{02} \pm 2p_{12} + 3p_{13} = 0$$

which corresponds exactly exactly to the three hyperplane components E_1, E_2, E_3 of the discriminant. In these cases the j-invariant is said to be infinite.

Now assume the point p lies on none of these hyperplanes and abbreviate $\alpha = \frac{3b-c}{a}$.
Let u and v be the solutions of $y^2 - \alpha y + 1 = 0$, i.e. $v + w = \alpha$ and $v \cdot w = 1$. Now let $\tilde{u}, \tilde{v} \in \mathbb{C}$ such that $\tilde{u}^2 = u$ and $\tilde{v}^2 = v$. This means that $\tilde{u}, -\tilde{u}, \tilde{v}, -\tilde{v}$ are the solutions of $(*)$.
Then the (or better: a possible) cross ratio of these four values is:

$$\lambda := \frac{\tilde{u} - (-\tilde{u})}{(-\tilde{v}) - \tilde{u}} : \frac{\tilde{v} - (-\tilde{u})}{(-\tilde{v}) - \tilde{v}} = \frac{2\tilde{u}}{-(\tilde{u} + \tilde{v})} : \frac{\tilde{u} + \tilde{v}}{-2\tilde{v}} = \frac{4\tilde{u}\tilde{v}}{(\tilde{u} + \tilde{v})^2}.$$

Now $\tilde{u} \cdot \tilde{v} = \pm 1$ and without loss of generality we can choose them in such a way that $\tilde{u} \cdot \tilde{v} = 1$. Furthermore

$$(\tilde{u} + \tilde{v})^2 = \tilde{u}^2 + 2\tilde{u}\tilde{v} + \tilde{v}^2 = u + 2 + v = \alpha + 2.$$

So $\lambda = \frac{4}{\alpha+2}$ and the j-invariant can be simplified to

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 \cdot (\lambda - 1)^2} = \frac{16(\alpha^2 + 12)^3}{(\alpha^2 - 4)^2}.$$

Hence $j = 0$ if and only if $\alpha^2 = -12$ i.e. $\alpha = \pm(4\omega + 2)$.
Translating this back to the global p_{ij}-coordinates we arrive at the equation $(p_{02} + 3p_{13})^2 + 12p_{12}^2 = 0$.

The closure of all other fibres of the function $j\colon Q \setminus (E_1 \cup E_2 \cup E_3) \longrightarrow \mathbb{C}$ is defined by the equation

$$j \cdot \left((3p_{13} + p_{02})^2 p_{03} - 4p_{03}^3\right)^2 - 16 \cdot \left((3p_{13} + p_{02})^2 + 12p_{03}^2\right)^3 = 0.$$

For $j \neq 0, \infty$ this is a surface of degree twelve which is singular exactly along the curve given by

$$p_{12} = p_{02} + 3p_{13} = 3p_{13}^2 + p_{01}p_{23} = p_{03} + p_{12} = 0$$

which is exactly the intersection of the hyperplanes $E_1 \cap E_2 \cap E_3$ in Q.

A similar calculation in the other charts gives the same result modulo I_Q if one pays attention to the fact that homogenising a given set of generators is not the same as homogenising an ideal. For example in the chart $p_{02} \neq 0$ by doing this one arrives at the ideal

$$I_Q + \langle (p_{02}^2 - 3p_{03}^2 + 3p_{01}p_{23})^2 + 12(p_{02}p_{03})^2 \rangle = (I_Q + \langle (p_{02} + 3p_{13})^2 + 12p_{12}^2 \rangle) \cap (I_Q + \langle p_{02}^2 \rangle)$$

where the latter ideal is obviously an artefact from bad homogenisation.

Theorem 5.39. *The subset of all (1,6)-polarised abelian surfaces which intersect the lines cut out by their cubic equations with a j-invariant of 0 is described by the ideal*

$$I_{j=0} = I_Q + \langle (p_{02} + 3p_{13})^2 + 12p_{12}^2 \rangle.$$

Since this is a subvariety of Q of dimension 2 and the integrable system in [SvS13] depends on two parameters, we already have found a subvariety of the right dimension. But the subvariety of Q describing the abelian surfaces with $j = 0$ is obviously reducible over $\mathbb{C}$. So it is not clear yet if all or only one (and in this case which) component really contains this integrable system.

6 Cohomology calculations

In this chapter we want to calculate the cohomology of the smooth fibres of the four integrable systems introduced in Section 6.3. To do this, we use a complex associated to each system introduced by Garay and van Straten.

6.1 The complex

In their paper [GvS10] M. Garay and D. van Straten associate to each involutive system $(f_1, \dots, f_n)$ in a Poisson ring R a complex in the following way: Set $T = \mathbb{C}[t_1, \dots, t_n]$. Then R becomes a T-module via the rule $t_i \cdot g := f_i \cdot g$. They define the complex

$$C_f^\bullet \colon 0 \longrightarrow C^0 \xrightarrow{\partial_0} C^1 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{n-1}} C^n \longrightarrow 0$$

with $C^k = R \otimes_T \bigwedge^k T^n$ and $\partial_k(g \otimes v) = \sum_{i=1}^n \{f_i, g\} e_i \wedge v$ where $e_1, \dots, e_n$ is the standard basis of T^n.
Because of the Leibniz identity for the Poisson bracket and because all f_k Poisson-commute, the morphisms ∂_k are morphisms of T-modules (but not of R-modules!) and $\partial^2 = 0$. So also the cohomology groups

$$H^i(f) := H^i(C_f^\bullet)$$

are T-modules, but in general not R-modules.
In the special case $R = \mathbb{C}[p_1, \dots, p_n, q_1, \dots, q_n]$ there is a relation between the complex $C_f^\bullet$ and the relative de Rham-complex $\Omega^\bullet_{R/T}$ defined by

$$\Omega^0_{R/T} = R, \quad \Omega^k_{R/T} = \Omega^k_R / f^*\Omega^1_T \wedge \Omega^{k-1}_r \text{ for } k \geq 0.$$

We define a morphism of graded complexes

$$\varphi^\bullet \colon (\Omega^\bullet_{R/T}, d) \longrightarrow (C_f^\bullet, \delta)$$

as follows: Denote by $v_1, \dots, v_n$ the hamiltonian vector fields of the functions $f_1, \cdots, f_n$. Then the mapping

$$\begin{aligned} \varphi^1 \colon \Omega^1_{R/T} &\longrightarrow C_f^1 \simeq R^n \\ \alpha &\longmapsto (i_{v_1}\alpha, \dots, i_{v_n}\alpha) \end{aligned}$$

induces morphisms

$$\varphi^k \colon \Omega^k_{R/T} = \bigwedge^k \Omega^1_{R/T} \longrightarrow \bigwedge^k C_f^1 = C_f^k,$$

which form a morphism of complexes

$$\varphi^\bullet\colon (\Omega^\bullet_{R/T}, d) \longrightarrow (C^\bullet_f, \delta),$$

i.e. commute with differentials.
Both complexes $\Omega^\bullet_{R/T}$ and $C^\bullet_f$ can be sheafified to complexes of sheaves $(\Omega^\bullet_f, d)$ and $(\mathcal{C}^\bullet_f, \delta)$ on the affine space $\mathbb{C}^{2n} = \mathrm{Spec}(R)$ with analytic topology.

Proposition 6.1. *If the morphism $f\colon \mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(T)$ is smooth at a point, then the map $\varphi^\bullet\colon (\Omega^\bullet_f, d) \longrightarrow (\mathcal{C}^\bullet_f, \delta)$ is an isomorphism of differential graded algebras at this point.*

This implies in particular that the cohomology modules $H^i(f)$ are isomorphic to the de Rham cohomology of the smooth fibres of f.
Garay and van Straten also state the following theorem, which we will not use here, but which inspires my conjecture in Section 6.9:

Theorem 6.2. *If f is a pyramidal (cf. Section 6.9) holomorphic integrable system, then the direct image sheaves of the complex $\mathcal{C}^\bullet_f$ are coherent and $(R^i f_* \mathcal{C}^\bullet_f)_0$ is isomorphic to $H^i(f)$.*

In our non-sheaf language, this means that $H^i(f)$ is finitely generated for pyramidal systems f.

6.2 The quasihomogeneous case $n = 2$

We use this to compute the cohomology of smooth fibres for some two-dimensional integrable systems (H, G) in $R := \mathbb{C}[p, P, q, Q]$, i.e. we compute the cohomology of the complex

$$0 \longrightarrow C^0 = R \xrightarrow{\partial_0} C^1 = R^2 \xrightarrow{\partial_1} C^2 = R \longrightarrow 0 \tag{6.1}$$

with

$$\partial_0 : f \longmapsto (\{H, f\}, \{G, f\})$$
$$\partial_1 : (f, g) \longmapsto \{G, f\} - \{H, g\}.$$

This is not a straight-forward-calculation with a computer algebra system, as the terms in in the complex $C^\bullet$ are not finitely generated as T-modules.
But we can consider R as a graded ring $R = \bigoplus_{d=0}^\infty R_d$. Then all our four examples are quasi-homogeneous with $\mathrm{wt}(p) = \mathrm{wt}(P) = w_1$ and $\mathrm{wt}(q) = \mathrm{wt}(Q) = w_2$. In this case all arrows above are homomorphisms of graded algebras, using the gradings (with the abbreviations $k_1 := \deg(H) - w_1 - w_2$ and $k_2 := \deg(G) - w_1 - w_2$)

$$\begin{aligned} C^0_d &= R_d, \\ C^1_d &= R_{d+k_1} \oplus R_{d+k_2}, \\ C^2_d &= R_{d+k_1+k_2}, \end{aligned}$$

i.e. $C^0 = R,\ C^1 = R(k_1) \oplus R(k_2),\ C^2 = R(k_1 + k_2)$ as graded R-modules.

In the following we will try to find generators (and relations) of the cohomology modules, by going through C^i degree by degree and compute the kernel and image of the above maps in each degree via (finite dimensional) linear algebra.
We will explain this at the example of $H^1(f)$. For $H^0(f)$ and $H^2(f)$ only part of these calculation is needed, because there is no image, hence no relations in $H^0(f) = \ker(\partial_0)$, and we don't have to specify a kernel for $H^2(f) = C^2/\operatorname{im}(\partial_1)$.

I used MAPLE for these computations, but any other program able to handle polynomials and vector spaces will do.

Algorithm for the generators of $H^1(f)$

For all degrees d from 0 to some upper bound d_{max} do:

- Determine a $\mathbb{C}$-basis of the kernel Z_d of $\partial_1|_{C^1_d} : C^1_d \longrightarrow C^2_d$.
- Determine a $\mathbb{C}$-basis of the image B_d of $\partial_0|_{C^0_d} : C^0_d \longrightarrow C^1_d$.
- Determine a basis of those elements of Z_d which already are multiples of generators found in lower degrees, i.e. $H \cdot Z_{d-\deg(H)}$ and $G \cdot Z_{d-\deg(G)}$.
- Complete the joint bases of B_d , $H \cdot Z_{d-\deg(H)}$ and $G \cdot Z_{d-\deg(G)}$ to a basis of Z_d. The elements we have to add are (minimal) generators of H^1.

Algorithm for the relations of $H^1(f)$

Given generators $e_1, \ldots, e_n$ with degree $d_1, \ldots, d_n$, first search for (a $\mathbb{C}$-basis of) all relations in degree d:

- Write down generic coefficients $c_1, \ldots, c_n$ with $\deg(c_i) = d - d_i$.
- Use the basis $g_1^{(d)}, \ldots, g_m^{(d)}$ of B^d we computed while searching for generators.
- Solve the system of linear equations given by $\sum_{i=1}^n c_i e_i = \sum_{j=1}^m a_j g_j^{(d)}$.
- Forget about the a_j and keep a basis of all c_i solving the system.
  These form a $\mathbb{C}$-basis of all relations Rel_d in degree d.

Now to find minimal generators of the $\mathbb{C}[t_1, t_2]$-module of all relations Rel up to degree $\tilde{d}_{max}$ (reasonably we choose $\tilde{d}_{max} \geq d_{max}$) we complete a basis of $H \cdot Rel_{d-\deg(H)} \oplus G \cdot Rel_{d-\deg(G)}$ to a basis of Rel_d for each $d \leq \tilde{d}_{max}$.
The elements we have to add are (minimal) generators of Rel (up to the given degree $\tilde{d}_{max}$).

6.3 The Examples

In this chapter we will study the cohomology of several examples of the form (1.5), all four mentioned by Michael Semmel in his thesis [Sem12]. Three of them come from Przybylskas and Maciejewskis paper [MP04] and have potentials of degree 3. The fourth one is another example of Grammaticos, this time with a potential of degree 4.
Here we give a first overview over these examples and first elementary algebraic geometric properties. Note that every integrable system $F = (f_1, \ldots, f_n)$ in n degrees of freedom induces a map $\mathbb{C}^{2n} \longrightarrow \mathbb{C}^n$ by $(\boldsymbol{p}, \boldsymbol{q}) \longmapsto (f_1(\boldsymbol{p}, \boldsymbol{q}), \ldots, f_n(\boldsymbol{p}, \boldsymbol{q}))$. We will give some description for the fibres of these maps in the following.

Our systems will all be two-dimensional. For simplicity of notation we will write q, Q, p, P instead of q_1, q_2, p_1, p_2 and H, G instead of f_1, f_2. We will denote the fibre $\{H = h, G = g\}$ by $X_{g,h}$.

6.3.1 The Hénon-Heiles potential

The Hénon-Heiles potential is given by

$$V(q, Q) = \frac{1}{2}(q^2 + Q^2) + \frac{\epsilon}{3}q^3 + qQ^2.$$

Often one also finds

$$V(q, Q) = \frac{\epsilon}{3}q^3 + qQ^2 \tag{6.2}$$

where integrability of the latter implies integrability of the first variant because its the homogeneous component of maximal degree (c.f. Section 1.6).
It was considered by Michel Hénon and Carl Heiles in the 1960s ([HH64]) to describe a star moving around a galactic centre in a slightly perturbed axial symmetric potential. It turned out that (6.2) is integrable only in some special cases, namely if and only if $\epsilon \in \{1, 6, 16\}$. For $\epsilon = 1$, the system decomposes as a product of two one-dimensional integrable systems, thus is not interesting to us.

Parameter 6

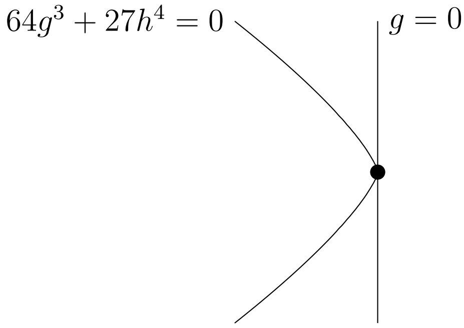
The polynomials

$$H = \frac{1}{2}(p^2 + P^2) + 2q^3 + qQ^2,$$

$$G = pPQ - P^2q + q^2Q^2 + \frac{1}{4}Q^4$$

are quasi-homogeneous of degree 6 resp. 8 with $\mathrm{wt}(p) = \mathrm{wt}(P) = 3$ and $\mathrm{wt}(q) = \mathrm{wt}(Q) = 2$.
The fibre $X_{g,h}$ is always a surface of degree 6. It is smooth except for

$$g(64g^3 + 27h^4) = 0.$$



In the case $g = 0$, $h \neq 0$, $X_{0,h}$ becomes singular along the smooth curve $Q = P = p^2 + 4q^3 - 2h = 0$ (degree 2), while for $64g^3 + 27h^4 = 0$ the singular locus is a smooth curve of degree 4. In both cases $X_{g,h}$ has a singularity of type A_1 transverse to the singular curve.
The zero fibre $X_{0,0}$ of this system is reduced and irreducible. The singular loci of the two cases described above coincide, the singular curve is given by $Q = P = p^2 + 4q^3 = 0$ and has a cusp in the origin.

Parameter 16

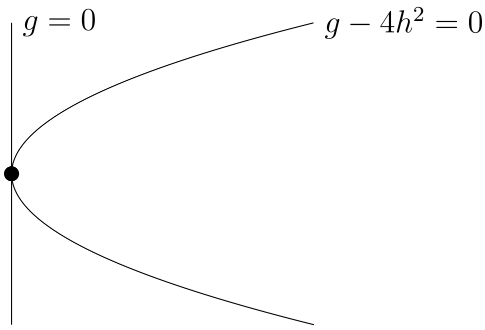
The polynomials

$$H = \frac{1}{2}(p^2 + P^2) + \frac{16}{3}q^3 + qQ^2,$$

$$G = P^4 + 4qQ^2P^2 - \frac{4}{3}Q^3pP - \frac{4}{3}q^2Q^4 - \frac{2}{9}Q^6$$

are quasi-homogeneous of degree 6 resp. 12 with $\mathrm{wt}(p) = \mathrm{wt}(P) = 3$ and $\mathrm{wt}(q) = \mathrm{wt}(Q) = 2$.
The fibre over (g, h) is always a surface of degree 8. It is smooth except for

$$g(g - 4h^2) = 0.$$



Over the line $g = 0$, it becomes singular with the smooth curve given by $Q = P = 3p^2 + 32q^3 - 6h = 0$ as singular locus, transverse to this curve the fibre has an $\widetilde{E}_7$-singularity (i.e. a fourfold point with four different tangents), while for $g - 4h^2 = 0$

the singular locus is a smooth curve of degree 8 with an A_2-singularity transverse to it.
The fibre over 0 of this system is reduced and irreducible, singular along a the curve given by $Q = P = 3p^2 + 32q^3 = 0$ with a cusp in the origin.

6.3.2 Grammaticos example of degree 3

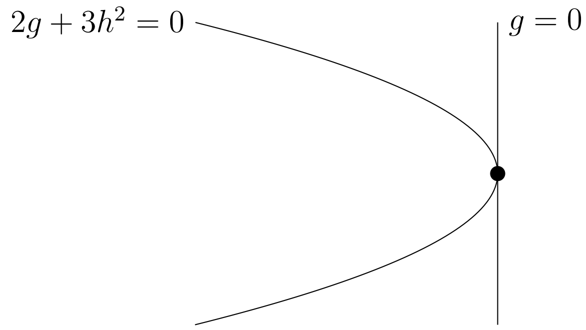
The polynomials

$$H = \frac{1}{2}p^2 - \frac{1}{6}P^2 + q^3 - \frac{3}{2}qQ^2 + \frac{1}{2}Q^3,$$

$$G = \frac{1}{9}(p - \frac{1}{2}P)P^3 - \frac{3}{2}Q^3p^2 - \frac{3}{2}qQ^2pP - \frac{3}{2}Q^3pP + (\frac{1}{2}Q^3 - qQ^2 - q^2Q)P^2 \\ - \frac{3}{2}q^3Q^3 + \frac{9}{8}q^2Q^4 + \frac{9}{4}qQ^5 - \frac{15}{8}Q^6$$

are quasi-homogeneous of degree 6 resp. 12 with $\mathrm{wt}(p) = \mathrm{wt}(P) = 3$ and $\mathrm{wt}(q) = \mathrm{wt}(Q) = 2$.
Here $X_{g,h}$ is always a surface of degree 8. It is smooth except for

$$g(2g + 3h^2) = 0.$$



In both cases $g = 0$ resp. $2g + 3h^2 = 0$ the fibre becomes singular along a smooth curve of degree 2 with a D_4-singularity transverse to this curve.
For this system $X_{0,0}$ decomposes into two components, both of degree 4, one of which is reduced, while the other corresponds to a reduced surface of degree 2, counted twice. The singular locus of the degree 4 surface is a line with an E_8-singularity transverse to it, while the degree 2 component is singular only in the origin. The two components intersect each other in two cuspidal curves.

6.3.3 Grammaticos example of degree 4

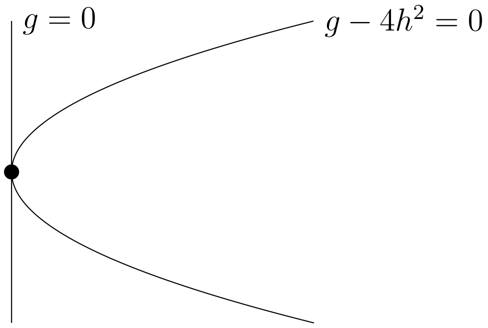
The polynomials

$$H = \frac{1}{2}(p^2 + P^2) + q^4 + \frac{3}{4}q^2Q^2 + \frac{1}{8}Q^4,$$

$$G = P^4 + \frac{1}{2}Q^4p^2 - 2qQ^3pP + (3q^2Q^2 + \frac{1}{2}Q^4)P^2 + \frac{1}{4}q^4Q^4 + \frac{1}{4}q^2Q^6 + \frac{1}{16}Q^8$$

are quasi-homogeneous of degree 4 resp. 8 with $\mathrm{wt}(p) = \mathrm{wt}(P) = 2$ and $\mathrm{wt}(q) = \mathrm{wt}(Q) = 1$.
The fibre over (g, h) is always a surface of degree 8. It is smooth except for

$$g(g - 4h^2) = 0.$$



In the case $g = 0$ it has a singularity of type $\widetilde{E}_7$ along a smooth curve plane curve of degree 2, while for $g - 4h^2 = 0$ the singular locus also is smooth curve of degree 2 with an A_2-singularity transverse to it.
The fibre $X_{0,0}$ of this system decomposes into two components one of degree 4, one of which is reduced, while the other corresponds to a reduced surface of degree 2, counted twice. The singular locus of the degree 4 surface is a curve of degree 3 (decomposing into a line and a singular conic), while the degree 2 component is singular only in the origin. The two components intersect each other in an irreducible curve of degree 2 with multiplicity 6 with an A_3-singularity in the origin.

6.4 Zeroth Cohomology

In all four examples a search (up to degree 36 in the first three, and up to degree 24 in the last example) suggests $\ker(\{H, \cdot\})$ and $\ker(\{G, \cdot\})$ are both generated as T-module by the constant polynomial 1, i.e. $\ker(\{H, \cdot\}) = \ker(\{G, \cdot\}) = \mathbb{C}[H, G]$. Hence $H^0(f) = \ker(\{H, \cdot\}) \cap \ker(\{G, \cdot\}) = \langle 1 \rangle_T$ as expected.

6.5 First Cohomology

One consideration about $\ker(\partial_1)$:
Since $0 \in R_d$ for every d, C_d^1 consists not only of the pairs (f, g) with $\deg(f) = d + k_1$ and $\deg(g) = d + k_2$, but also of the 2-tuples $(f, 0)$ and $(0, g)$ with f resp. g of the right degree. The latter are mapped by ∂_1 to $\{G, f\}$ resp. $-\{H, g\}$. So we already found all elements of $\ker(\partial_1)$ that are of this form by our calculations in Section 6.4. In all our four examples these elements are generated by the two generators $(1, 0)$, $(0, 1)$ of $\ker(\partial_1)$.

Thus in the following we only have to look for pairs of the form (f, g) with $\deg(f) = d + k_1$ and $\deg(g) = d + k_2$ $(f, g \neq 0)$. I ran the algorithm described above until either the runtime exceeded some hours or Maple started crashing because of overflow problems. The following table gives the bound d_{max} reached in each example and all generators of $H^1(f)$ found up to this bound. The name e_d for a generator indicates that it lives in C^1_d.

Example	Search in degrees	Generators found
Hénon-Heiles 6	$d_{max} = 31$	e_1, e_3
Hénon-Heiles 16	$d_{max} = 29$	e'_1, e_7
Grammaticos 3	$d_{max} = 29$	e''_1, e'_7
Grammaticos 4	$d_{max} = 15$	e_0, e'''_1, e_5

There were no relations found in any of the examples (despite a search up to degree 41 in the first three and up to degree 29 in the last example).

Conjecture 6.3. *In any of the examples $H^1(f)$ is a free module with either four or five generators. Their degrees can be read off from the table above.*

6.6 Second Cohomology

The second cohomology modules are the most complicated ones. A search up to degree d_{max} found the following number of generators in each degree:

Hénon-Heiles 6 ($d_{max} = 33$)

degree	-3	-1	1	3	5	6	7	$\cdots$
# generators	1	2	1	2	1	0	0	$\cdots$

Hénon-Heiles 16 ($d_{max} = 23$)

degree	-7	-5	-3	-2	-1	1	3	4	5	7	9	10	11	$\cdots$
# generators	1	2	1	1	2	3	2	1	1	2	1	0	0	$\cdots$

Grammaticos degree 3 ($d_{max} = 16$)

degree	-8	-6	-4	-3	-2	0	2	3	4	6	8	10	12	14	16
# generators	1	2	1	1	2	3	2	1	1	2	1	1	1	1	1

Grammaticos degree 4 ($d_{max} = 11$)

degree	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
# generators	1	2	3	3	5	5	6	5	5	3	3	2	2	2	2	2	2

Both lists for the Hénon-Heiles examples seem to be complete, while in the examples by Grammaticos we seem to see an infinite sequence of one generator in every even degree resp. two generators in every degree for large enough degrees.

We find no relations in the Hénon-Heiles examples.

In Grammaticos example of degree 3 looking for relations among these generators we find one nontrivial relation (i.e. which can not be used to eliminate a generator) in every even degree between 10 and 28, and we may suspect that it will go further on like this. If so, these infinitely many generators and relations generate a module of exactly the dimension in any degree as the Poincaré series calculated in Section 6.7.3 suggests.
In Grammaticos example of degree 4 we find one relation in degree 7 (making one of the degree -1 generators e_{-1} torsion, i.e. $(G-4H^2)\cdot e_{-1} \in \mathrm{im}(\partial_1)$) and two in each higher degree (up to 15, where I stopped searching). Together this gives the right dimensions w.r.t. the calculations in Section 6.7.4.

Conjecture 6.4.
1. *For the Hénon-Heiles examples $H^2(f)$ is a finitely generated free T-module with 7 resp. 17 generators in the degrees given above.*
2. *In the examples by Grammaticos $H^2(f)$ is an infinitely generated T-modules with infinitely many relations.*

6.7 Poincaré series

We use the theory of Poincaré series to check the consistency of our previous calculations.

Definition 6.5. Let $M = \bigoplus_{j\in\mathbb{Z}} M_j$ be a graded k-module with all M_k finite dimensional k vector spaces. Then the *Poincaré series* of M is given by $P_M(t) := \sum_{j\in\mathbb{Z}} \dim_k(M_j)t^j$

Lemma 6.6. *For the polynomial ring $R = k[x_1, \dots, x_n]$ with $\deg(x_i) = w_i$ we have*

$$P_R(t) = \frac{1}{(1-t^{w_1})\cdots(1-t^{w_n})}.$$

Lemma 6.7. *If M is a graded k-module with Poincaré series P_M and $M(d)$ is its shift by d, i.e. $M(d)_k = M_{k+d}$, then $P_{M(d)} = t^{-d}P_M$.*

Theorem 6.8. *Let $C_\bullet\colon 0 \longrightarrow C_n \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$ be a finite chain complex of graded k-algebras with $(C_j)_k$ finite dimensional for all j and k. Then the* Euler characteristic *of this complex*

$$\chi_C := P_{C_0} - P_{C_1} + P_{C_2} - \cdots \pm P_{C_n}$$

is equal to the Euler characteristic of its homology

$$\chi_H := P_{H_0(C)} - P_{H_1(C)} + P_{H_2(C)} - \cdots \pm P_{H_n(C)}.$$

The proofs are just elementary combinatorics and linear algebra.

With respect to our gradings given in Section 6.1, the maps ∂_0, ∂_1 of our complex are homomorphisms of graded algebras. With respect to this grading we have the following Poincaré series:

$$\begin{aligned}
P_{C^0} &= \frac{1}{(1-t^{w_1})^2(1-t^{w_2})^2} \\
P_{C^1} &= \frac{t^{-k_1}+t^{-k_2}}{(1-t^{w_1})^2(1-t^{w_2})^2} \\
P_{C^2} &= \frac{t^{-k_1-k_2}}{(1-t^{w_1})^2(1-t^{w_2})^2}.
\end{aligned}$$

So the Euler characteristic of our complex (for $n=2$) is:

$$\begin{aligned}
\chi_C &:= P_{C^0} - P_{C^1} + P_{C^2} \\
&= \frac{1-t^{-k_1}-t^{-k_2}+t^{-k_1-k_2}}{(1-t^{w_1})^2(1-t^{w_2})^2}.
\end{aligned}$$

A check with Singular gives in all four cases, that $\mathbb{C}[t_1,t_2] \longrightarrow \mathbb{C}[p,P,q,Q], t_1 \mapsto G, t_2 \mapsto H$ is injective, i.e. $\mathbb{C}[G,H] \cong \mathbb{C}[t_1,t_2]$.

6.7.1 Hénon-Heiles 6

In this example we have $w_1=3, w_2=2, \deg(H)=6, \deg(G)=8$, so $k_1=1$ and $k_2=3$, hence

$$\chi_C = \frac{1-t^{-1}-t^{-3}+t^{-4}}{(1-t^2)^2(1-t^3)^2}.$$

Following the calculations in Section 6.4 and 6.5 we assume $H^0=\mathbb{C}[G,H]$, so

$$P_{H^0} = \frac{1}{(1-t^6)(1-t^8)}$$

and $H^1_{d+1} = \mathbb{C}[G,H]_d(1,0) \oplus \mathbb{C}[G,H]_{d+2}(0,1) \oplus \mathbb{C}[G,H]_{d-2}e_2 \oplus \mathbb{C}[G,H]_{d-4}e_4^{(2)}$, so

$$P_{H^1} = \frac{t^3+t+t^{-1}+t^{-3}}{(1-t^6)(1-t^8)}.$$

So we expect the following Poincaré series for H^2:

$$P_{H^2} = \frac{t^4+2t^2+1+2t^{-2}+t^{-4}}{(1-t^6)(1-t^8)}.$$

This matches our calculations in Section 6.6.

6.7.2 Hénon-Heiles 16

In this example we have $w_1 = 3, w_2 = 2, \deg(H) = 6, \deg(G) = 12$, so $k_1 = 1$ and $k_2 = 7$, hence

$$\chi_C = \frac{1 - t^{-1} - t^{-7} + t^{-8}}{(1-t^2)^2(1-t^3)^2}.$$

Following the calculations in Section 6.4 and 6.5 we assume $H^0 = \mathbb{C}[G,H]$, so

$$P_{H^0} = \frac{1}{(1-t^6)(1-t^{12})}$$

and $H^1_{d+1} = \mathbb{C}[G,H]_d(1,0) \oplus \mathbb{C}[G,H]_{d+6}(0,1) \oplus \mathbb{C}[G,H]_{d-2}e_2 \oplus \mathbb{C}[G,H]_{d-8}e_8$ so

$$P_{H^1} = \frac{t^{-7} + t^{-1} + t + t^7}{(1-t^6)(1-t^{12})}.$$

So we expect the following Poincaré series for H^2:

$$P_{H^2} = \frac{t^8 + 2t^6 + t^4 + t^3 + 2t^2 + 3 + 2t^{-2} + t^{-3} + t^{-4} + 2t^{-6} + t^{-8}}{(1-t^6)(1-t^{12})}.$$

This also matches our calculations in Section 6.6.

Note, that this calculation as well as the degrees of all generators coincide exactly with those in Grammaticos example of degree 3. But a check in Singular tells us, that the two are not isomorphic in the sense of [Prz07] (i.e. there is no symplectic matrix A s.t. $H_{HH16}(\boldsymbol{p},\boldsymbol{q}) = H_{G3}(A\cdot(\boldsymbol{p},\boldsymbol{q}))$). This is also obvious from the fact, that the Hénon-Heiles examples have reduced fibre over 0 while those by Grammaticos have not. Additionally, they are also listed as two different integrable potentials in [MP04].

6.7.3 Grammaticos 3

In this example we have $w_1 = 3, w_2 = 2, \deg(H) = 6, \deg(G) = 12$, so $k_1 = 1$ und $k_2 = 7$, hence

$$\chi_C = \frac{1 - t^{-1} - t^{-7} + t^{-8}}{(1-t^2)^2(1-t^3)^2}.$$

The calculations in Section 6.4 and 6.5 suggest $H^0 = \mathbb{C}[G,H]$, so

$$P_{H^0} = \frac{1}{(1-t^6)(1-t^{12})}$$

and $H^1_{d+1} = \mathbb{C}[G,H]_d(1,0) \oplus \mathbb{C}[G,H]_{d+6}(0,1) \oplus \mathbb{C}[G,H]_{d-2}e_2 \oplus \mathbb{C}[G,H]_{d-8}e_8$ so

$$P_{H^1} = \frac{t^7 + t + t^{-1} + t^{-7}}{(1-t^6)(1-t^{12})}.$$

So if our calculations are correct, we expect the following Poincaré series for H^2:

$$\begin{aligned} P_{H^2} &= \chi_{\mathcal{C}} - P_{H^0} + P_{H^1} \\ &= \frac{t^8 + 2t^6 + t^4 + t^3 + 2t^2 + 3 + 2t^{-2} + t^{-3} + t^{-4} + 2t^{-6} + t^{-8}}{(1-t^6)(1-t^{12})}. \end{aligned}$$

Also this looks like the Poincaré series of a free module with the generators in Section 6.6 up to degree 8, it also matches the module with infinitely many generators and relations described there.

To see this, one can argue as follows: Suppose the module H^2 has a free resolution

$$0 \longrightarrow R \longrightarrow F \longrightarrow H^2 \longrightarrow 0$$

where F is the free module generated by the generators given in Section 6.6, and R is the free module generated by the relations described there.
Note: We do not know a priori, that a free resolution really ends after R, but we know that a minimal free resolution (a beginning of which we have computed with F and R) could have at most one additional term because of Hilberts syzygy theorem. (The ring $T = \mathbb{C}[t_1, t_2]$ has projective dimension two.) Since $P_{H^2} = P_F - P_R$ if our conjectures are right, the potential third term of a minimal free resolution is zero. Then F had a Poincaré series given by

$$P_F = \frac{1 + 2t^2 + t^4 + t^5 + 2t^6 + 3t^8 + 2t^{10} + t^{11} + t^{12} + 2t^{14} + \sum_{k=8}^{\infty} t^{2k}}{t^8(1-t^6)(1-t^{12})}$$

and the Poincaré series of R would look like

$$P_R = \frac{\sum_{k=9}^{\infty} t^{2k}}{t^8(1-t^6)(1-t^{12})}.$$

From this we get that the Poincaré series $P_{H^2} = P_F - P_R$ is like stated above.
So the sheer fact that we have the same number of generators as of free relations in any degree up from a certain point, hides the generators and relations from the Poincaré series.

6.7.4 Grammaticos 4

In this example we have $w_1 = 2, w_2 = 1, \deg(H) = 4, \deg(G) = 8$, so $k_1 = 1$ and $k_2 = 5$, hence

$$\chi_C = \frac{1 - t^{-1} - t^{-5} + t^{-6}}{(1-t)^2(1-t^2)^2}.$$

Following the calculations in Section 6.4 and 6.5 we assume $H^0 = \mathbb{C}[G, H]$, so

$$P_{H^0} = \frac{1}{(1-t^4)(1-t^8)}$$

and

$$H^1_{d+1} = \mathbb{C}[G,H]_{d+4}(0,1) \oplus \mathbb{C}[G,H]_d(1,0) \oplus \mathbb{C}[G,H]_{d-1}e_1 \oplus \mathbb{C}[G,H]_{d-2}e_2 \oplus \mathbb{C}[G,H]_{d-6}e_6$$

so

$$P_{H^1} = \frac{t^{-5} + t^{-1} + 1 + t + t^5}{(1-t^4)(1-t^8)}.$$

So we expect the following Poincaré series for H^2:

$$\begin{aligned} P_{H^2} &= \frac{t^6 + 2t^5 + 2t^4 + t^3 + 2t^2 + 2t + 1}{t^6(1-t^2)^2(1+t^2)} \\ &= \frac{t^6 + 2t^5 + 3t^4 + 3t^3 + 5t^2 + 5t + 6 + 5t^{-1} + 5t^{-2} + 3t^{-3} + 3t^{-4} + 2t^{-5} + t^{-6}}{(1-t^4)(1-t^8).} \end{aligned}$$

This looks like the Poincaré series of a free module with the generators in Section 6.6 up to degree 6 and one of the generators in degree 7. But our presumably infinite series of generators and relations also produces the same Poincaré series with the same argument than above.

6.8 An observation

In all four examples the Poincaré series for all cohomology modules $H^i(f)$ has a numerator, that is symmetric in the sense, that there is always the same coefficient in front of t^k as in front of t^{-k}. This means, it could be written as a polynomial in $t + t^{-1}$. This fact might reflect some kind of self-duality of the cohomology modules.

6.9 Attempt of an explanation

In both of our Hénon-Heiles examples all cohomology modules are free modules, whereas both examples by Grammaticos have nontrivial relations in the H^2-module. There are several aspects that differ between these two pairs of examples each of which could be an explanation for this difference:

- As already mentioned in Section 6.3, the Hénon-Heiles examples have reduced zero-fibre while the examples by Grammaticos have not.
- In the Hénon-Heiles examples H has isolated singularity, while in the examples by Grammaticos, it has not.
- In their paper [GvS10] Garay and van Straten define a similar but stronger criterion of being *pyramidal* from which they deduce that all $H^k(f)$ are coherent. They denote by $v_1, \ldots, v_n$ the hamiltonian vector fields of $f_1, \ldots, f_n$ and put

$$Mk(f) = \{x \in M \mid \dim \mathrm{Span}\{v_1(x), ..., v_n(x)\} = k\}.$$

They call an integrable system pyramidal if

$$\dim M_k(f) \leq k.$$

Again, both Hénon-Heiles examples are pyramidal, those by Grammaticos are not.

I suspect pyramidality is the right criterion to look at. It would be interesting to look at examples that differ only in one or two of the aspects above, but this may be difficult, since, following Przybylska, there are only finitely many polynomial integrable potentials in each given degree and number of variables (in particular, we analysed all interesting examples in degree 3 in 2 variables), and raising the degree and/or the number of variables will probably increase computational complexity quite quickly.

7 Appendix: More about abelian varieties

In this chapter we give some more details about abelian varieties, their embeddings and their moduli. Proofs and further details can be found in [BL04]. As in Chapter 2 $X = V/\Lambda$ is a complex torus.

7.1 Homomorphisms of complex tori

With the group structure on X we can define the *translation* maps given by $t_x \colon X \longrightarrow X$, $y \longmapsto x + y$.

Proposition 7.1. *Let $X = V/\Lambda$ and $X' = V'/\Lambda'$ be two complex tori and $h \colon X \longrightarrow X'$ a holomorphic map. Then:*

1. *There exists a unique homomorphism $f \colon X \longrightarrow X'$ s.t. $h = t_{h(0)} \circ f$.*
2. *There exists a unique $\mathbb{C}$-linear map $F \colon V \longrightarrow V'$ inducing f. F maps the lattice Λ to Λ'.*

Both maps $F \colon V \longrightarrow V'$ and $F|_\Lambda \colon \Lambda \longrightarrow \Lambda'$ determine f uniquely, i.e. there are injective homomorphisms of abelian groups

$$\rho_a \colon Hom(X, X') \longrightarrow Hom(V, V'), f \longmapsto F$$

and

$$\rho_r \colon Hom(X, X') \longrightarrow Hom(\Lambda, \Lambda'), f \longmapsto F|_\Lambda,$$

called the *analytic* resp. *rational representation* of $Hom(X, X')$. The maps $\rho_a(f)$ and $\rho_r(f)$ are also called analytic resp. rational representation of f.

Analytic and rational representation of a homomorphism f can be seen as matrices, which are connected in the following way: Let $e_1, \dots e_g$ be a $\mathbb{C}$-basis of V and $\lambda_1, \dots, \lambda_{2g}$ a $\mathbb{Z}$-basis of the lattice Λ. Write λ_i in terms of the basis $e_1, \dots, e_g$: $\lambda_i = \sum_{j=1}^{g} \lambda_{ij} e_j$. The matrix

$$\Pi = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1,2g} \\ \vdots & & \vdots \\ \lambda_{g1} & \cdots & \lambda_{g,2g} \end{pmatrix} \in \mathbb{C}^{g \times 2g}$$

is called a *period matrix* for X.
Let Π' be the period matrix for $\Lambda' \subset V'$ w.r.t. the bases $e'_1, \dots, e'_{g'}$ and $\lambda'_1, \dots, \lambda'_{2g'}$.

Let $A \in \mathbb{C}^{g \times g'}$ be the matrix of $\rho_a(f) \colon V \longrightarrow V'$ w.r.t. the bases $e_1, \dots, e_g$ resp. $e'_1, \dots, e'_{g'}$ and $R \in \mathbb{Z}^{2g \times 2g'}$ the matrix of $\rho_r(f)$ with respect to the bases $\lambda_1, \dots, \lambda_{2g}$ and $\lambda'_1, \dots, \lambda'_{2g'}$. Then V resp. V' can be identified with $\mathbb{C}^g$ resp. $\mathbb{C}^{g'}$, Λ and Λ' with $\mathbb{Z}^{2g}$ and $\mathbb{Z}^{2g'}$, Π and Π' represent the embeddings of Λ in V resp. Λ' in V' and the fact that R and A represent the same map translates to the commutativity of the following diagram

$$\begin{tikzcd} \mathbb{Z}^{2g} \arrow[r, "R"] \arrow[d, "\Pi"'] & \mathbb{Z}^{2g'} \arrow[d, "\Pi'"] \\ \mathbb{C}^g \arrow[r, "A"'] & \mathbb{C}^{g'}, \end{tikzcd}$$

i.e. $A \cdot \Pi = \Pi' \cdot R$.

Definition 7.2. A homomorphism $f \colon X \longrightarrow X'$ of complex tori is called an *isogeny* if it is surjective and has finite kernel.

A special case of isogenies are the maps $n_X \colon X \longrightarrow X$ $(n \in \mathbb{Z})$ which send a point x to its n-fold sum nx. For $n \neq 0$ this is an isogeny whose kernel are the n-torsion points of X, denoted by $X_n \cong (\mathbb{Z}/n\mathbb{Z})^g$.

7.2 Line bundles on complex tori and the Appell-Humbert-Theorem

Here we give some more details about the description of line bundles in terms of a factor of automorphy and the Appel-Humbert-Theorem.

For the next paragraphs let X be a arbitrary complex manifold and $\pi \colon \widetilde{X} \longrightarrow X$ the universal covering, $\pi_1(X)$ the fundamental group. We will describe all line bundles on X with trivial pullback to $\widetilde{X}$.
A *factor of automorphy* is a holomorphic function $f \colon \pi_1(X) \times \widetilde{X} \longrightarrow \mathbb{C}^*$ such that $f(g_1 g_2, \tilde{x}) = f(g_2, g_1 \tilde{x}) \cdot f(g_1, \tilde{x})$.
Such functions can be identified with 1-cocycles in the sense of group cohomology, i.e. with elements of $Z^1(\pi_1(X), H^0(\mathcal{O}^*_{\widetilde{X}}))$.

Theorem 7.3. *There is a canonical isomorphism*

$$\phi \colon H^1(\pi_1(X), H^0(\mathcal{O}^*_{\widetilde{X}})) \longrightarrow \ker(H^1(X, \mathcal{O}^*_X) \xrightarrow{\pi^*} H^1(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}})).$$

In other words: Every line bundle on X which has trivial pull-back to $\widetilde{X}$ can be described by a factor of automorphy which is unique up to an element of $B^1(\pi_1(X), H^0(\mathcal{O}^*_{\widetilde{X}}))$.

One possible description of ϕ is as follows: Any factor of automorphy f, seen as a holomorphic function $f\colon \pi_1(X) \times \widetilde{X} \longrightarrow \mathbb{C}^*$ as above, describes an action of $\pi_1(X)$ on the trivial line bundle $\widetilde{X} \times \mathbb{C} \longrightarrow \widetilde{X}$ by

$$\lambda \cdot (\tilde{x}, t) := (\lambda\tilde{x}, f(\lambda, \tilde{x})t).$$

Note that the cocycle condition for f translates to the fact that $\lambda_1 \cdot \lambda_2 \cdot (\tilde{x}, t) = (\lambda_1\lambda_2) \cdot (\tilde{x}, t)$. Then $\phi(f) = (\widetilde{X} \times \mathbb{C})/\pi_1(X)$.
For another description see [BL04], Appendix B.

Now for a complex torus X the universal cover is $\pi\colon V \longrightarrow X$, $\pi_1(X) \cong \Lambda$. Because every vector bundle on a complex vector space is trivial, we obtain:

Corollary 7.4.

$$H^1(X, \mathcal{O}_X^*) \cong H^1(\Lambda, H^0(\mathcal{O}_V^*)).$$

But $Pic(X) \cong H^1(X, \mathcal{O}_X^*)$, which can be seen best by thinking about line bundles in the topological sense i.e. as maps $p\colon L \longrightarrow X$ whose fibres are one-dimensional vector spaces and which can be trivialised over certain subsets forming an open cover $(U_i)_{i\in I}$. Then the collection of all transition functions $g_{ij}\colon U_i \cap U_j \longrightarrow \mathbb{C}^*$ is an element of the Čech cohomology $H^1(\mathcal{O}_X^*)$.
So every holomorphic line bundle on X can be described by a factor of automorphy.

Now we are looking for a simpler description for the factors of automorphy on a complex torus X.
For this consider the exponential exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

and its long exact cohomology sequence

$$\cdots \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow \cdots .$$

As $H^1(X, \mathcal{O}_X^*) \cong Pic(X)$ and $H^2(X, \mathbb{Z}) \cong Alt^2(\Lambda, \mathbb{Z})$ by Proposition 2.1, we can see c_1 as a map associating to each line bundle L a $\mathbb{Z}$-valued alternating form $c_1(L)$ on the lattice Λ. We will call $c_1(L)$ the *first Chern-class* of the line bundle L. $c_1(L)$ can be described explicitly in terms of the factor of automorphy of L.

Proposition 7.5. *For an alternating form $E\colon V \times V \longrightarrow \mathbb{R}$ the following conditions are equivalent:*

1. $E|_\Lambda = c_1(L)$ *for a holomorphic line bundle L on X.*
2. $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$ *and* $E(iv, iw) = E(v, w)$.

Recall that a *hermitian form on V* is a map $H\colon V \times V \longrightarrow \mathbb{C}$ that is $\mathbb{C}$-linear in the first argument and satisfies $H(v, w) = \overline{H(w, v)}$ for all $v, w \in V$.

Lemma 7.6. *There is a one-to-one-correspondence between the set of hermitian forms H on V and the set of real valued alternating forms E on V satisfying $E(iv, iw) = E(v, w)$ given by*

$$E(v, w) = \operatorname{Im} H(v, w) \quad \textit{and} \quad H(v, w) = E(iv, w) + iE(v, w)$$

for all $v, w \in V$.

In the sequel we will consider the first Chern class of a line bundle L on X either as an alternating or a hermitian form on V.
Define the *Néron-Severi group* $NS(X)$ to be the image of the homomorphism

$$c_1 \colon H^1(\mathcal{O}_X^*) \longrightarrow H^2(X; \mathbb{Z}).$$

According to Proposition 7.5 and Lemma 7.6, $NS(X)$ can be identified either with the group of hermitian forms $H \colon V \times V \longrightarrow \mathbb{C}$ with $\operatorname{Im} H(\Lambda, \Lambda) \subseteq \mathbb{Z}$ or with the group of $\mathbb{R}$-valued alternating forms E on V satisfying $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$ and $E(iv, iw) = E(v, w)$.

A *semicharacter* for a hermitian form H is a map $\chi \colon \Lambda \longrightarrow S^1 \subset \mathbb{C}^*$ satisfying

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)\exp(\pi i \operatorname{Im} H(\lambda, \mu)) \text{ for all } \lambda, \mu \in \Lambda.$$

The semicharacters for $0 \in NS(X)$ are exactly the group homomorphisms from Λ to S^1.
We define

$$\mathcal{P}(\Lambda) = \{(H, \chi) \mid H \in NS(\Lambda), \chi \text{ semicharacter for } H\}.$$

Obviously, $\mathcal{P}(\Lambda)$ is a group with respect to the composition

$$(H_1, \chi_1) \circ (H_2, \chi_2) = (H_1 + H_2, \chi_1\chi_2)$$

and the following sequence is exact

$$0 \longrightarrow Hom(\Lambda, S^1) \xrightarrow{\iota} \mathcal{P}(\Lambda) \xrightarrow{p} NS(X)$$

where $\iota(\chi) = (0, \chi)$ and $p(H, \chi) = H$.
In fact, $p \colon \mathcal{P}(\Lambda) \longrightarrow NS(X)$ is surjective and we have the following theorem:

Theorem 7.7 (Appell-Humbert-Theorem)**.** *There is a canonical isomorphism of exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & Hom(\Lambda, S^1) \arrow[r] \arrow[d, "\simeq"] & \mathcal{P}(\Lambda) \arrow[r] \arrow[d, "\simeq"] & NS(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Pic^0(X) \arrow[r] & Pic(X) \arrow[r] & NS(X) \arrow[r] & 0
\end{tikzcd}$$

where the isomorphism $\mathcal{P}(\Lambda) \longrightarrow Pic(X)$ is given by associating to each pair $(H, \chi) \in \mathcal{P}(\Lambda)$ the line bundle L on X that is described by the factor of automorphy

$$a_{(H,\chi)}(\lambda, v) = \chi(\lambda)\exp\left(\pi H(v, \lambda) + \frac{\pi}{2}H(\lambda, \lambda)\right).$$

We will denote the line bundle associated to a pair (H, χ) by $L(H, \chi)$.
Using this theory one can see for example:

Lemma 7.8. *For any $L = L(H, \chi) \in Pic(X)$ and $\bar{v} \in X$ with representative $v \in V$*

$$t_{\bar{v}}^* L(H, \chi) = L\big(H, \chi \exp(2\pi i \operatorname{Im} H(v, \cdot))\big).$$

Lemma 7.9. *Let $f\colon X' \longrightarrow X$ be a homomorphism with analytic representation $F\colon V' \longrightarrow V$ and rational representation $F_\Lambda\colon \Lambda' \longrightarrow \Lambda$. Then*

$$f^* L(H, \chi) = L(F^* H, F_\Lambda^* \chi).$$

7.3 Projective embeddings and equations

In this section we will study the map φ_L and see for example in which cases φ_L is an embedding. We will also make some statement about the equations describing the image $\varphi_L(X)$.

Proposition 7.10. *If L is a positive definite line bundle on X of type $(d_1, \dots, d_g)$ and $d_1 \geq 2$, then φ_L has no base point, i.e. is a holomorphic map.*

Theorem 7.11 (Lefschetz)**.** *If L is a positive definite line bundle on X of type $(d_1, \dots, d_g)$ and $d_1 \geq 3$, then φ_L is an embedding.*

Note that the type of L^n is n times the type of L.

Corollary 7.12. *If L is an ample line bundle on X, then L^n is very ample for every $n \geq 3$.*

Now we want to study the case $d_1 = 2$. One can show that a line bundle of type $(2, d_2, \dots)$ is of the form L^2 for an ample line bundle L on X.

Theorem 7.13 (Decomposition Theorem)**.** *Let L be a line bundle on X. Then as polarised abelian varieties*

$$(X, L) \xrightarrow{\cong} (X_1 \times X_2 \times \cdots \times X_r, q_1^* M \otimes q_2^* N_2 \otimes \cdots \otimes q_r^* N_r)$$

where M is a line bundle on X_1 without fixed components (in the corresponding linear system of divisors), the N_i are irreducible principal polarisations on the X_i and the q_i are the projections from $X_1 \times X_2 \times \cdots \times X_r$ onto its factors.

To study the map φ_{L^2}, decompose L as in the theorem above. Let $\varphi_{M^2}\colon X_M \to \mathbb{P}^{n_1}$ and $\varphi_{N_i^2}\colon X_i \to \mathbb{P}^{n_i}$, $i = 2, \dots, r$, be the corresponding holomorphic maps and denote by $\psi\colon \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \to \mathbb{P}^N$ the Segre embedding. Then the holomorphic map $\varphi_{L^2}\colon X \to \mathbb{P}^N$ decomposes as

$$\varphi_{L^2} = \psi \circ (\varphi_{M^2} \times \varphi_{N_2^2} \times \cdots \times \varphi_{N_r^2}).$$

Thus it is enough to consider the cases

1. $L = M$ a polarisation without fixed components,
2. $L = N_i$ an irreducible principal polarisation.

In this two cases we have the following results:

Theorem 7.14. *If L is an ample line bundle without fixed components, then L^2 is very ample.*

Theorem 7.15. *If L is a symmetric line bundle defining an irreducible principal polarisation on X, then φ_{L^2} induces an embedding of the Kummer variety $K_X := X/\langle(-1)_X\rangle$ to $\mathbb{P}^N$.*

It is no restriction to assume that L is symmetric because of the following facts:

Lemma 7.16. *Let L be an ample line bundle on X and $x \in X$ a point. Let $\vartheta_0, \ldots, \vartheta_N$ be an basis of $H^0(L)$. Then $t_x^*\vartheta_0, \ldots, t_x^*\vartheta_N$ is a basis of $H^0(t_x^*L)$ and with $\varphi_L, \varphi_{t_x^*L}$ the corresponding maps to $\mathbb{P}^N$ the following diagram commutes:*

$$\begin{tikzcd} L \arrow[rr, "t_x"] \arrow[dr, "\varphi_{t_x^*L}"'] & & L \arrow[dl, "\varphi_L"] \\ & \mathbb{P}^N & \end{tikzcd}$$

Proposition 7.17. *For two line bundles L and L' on X the following statements are equivalent:*

1. *L and L' are analytically equivalent.*
2. *$L' = t_x^*L$ for some $x \in X$.*
3. *$c_1(L) = c_1(L')$.*

So the image $\overline{X}$ of φ_L in $\mathbb{P}^N$ does not depend on L itself but only on its analytic equivalence class resp. its first Chern class.

Definition 7.18. A line bundle L is called *symmetric* if $(-1)_X^*L \cong L$.

The following argument shows that for each $H \in NS(X)$ there is a symmetric line bundle L on X such that $c_1(L) = H$: Lemma 7.9 shows that $L(H, \chi)$ is symmetric if and only if χ has values in $\{\pm 1\}$. But since $NS(X)$ consists only of those hermitian forms whose imaginary part takes integral values on $\Lambda \times \Lambda$, the semicharacter $\chi_0 \colon V \longrightarrow S^1$ defined by $\chi_0(v) = \exp(\pi i E(v_1, v_2))$ where $v = v_1 + v_2$ with $v_i \in V_i$ ($V = V_1 \oplus V_2$ a decomposition for H) only takes values ± 1. So for each H, $L(H, \chi_0)$ is symmetric.

Now we want to make some statements about the equations describing the image of φ_L:

Definition 7.19. A projective variety $Y \subseteq \mathbb{P}^N$ is called *projectively normal* in $\mathbb{P}^N$ if its homogeneous coordinate ring is an integrally closed domain.
We call a line bundle M on Y *normally generated* if it is very ample and Y is projectively normal under the associated projective embedding.

Theorem 7.20. *Let L be an ample line bundle on an abelian variety X. Then:*

1. *L^n is normally generated for any $n \geq 3$.*
2. *If L is of characteristic c, then L^2 is normally generated if and only if no point of $t_{\bar{c}}^*$ is a base point of L.*

A very ample line bundle on M on an abelian variety X gives an associated embedding $\varphi_M \colon X \hookrightarrow \mathbb{P}^N$. For the degree of the generators of the homogeneous ideal $I(M)$ of all polynomials vanishing on $\varphi_M(X) \subseteq \mathbb{P}^N$ we have the following theorem:

Theorem 7.21. *Suppose L is an ample line bundle on X and L^n is normally generated, then*

1. *the ideal $I(L^n)$ is generated by forms of degree 2 whenever $n \geq 4$,*
2. *the ideal $I(L^3)$ is generated by forms of degree 2 and 3,*
3. *the ideal $I(L^2)$ is generated by forms of degree 2, 3 and 4.*

For the special case of abelian surfaces the Decomposition Theorem reads like:

Lemma 7.22. *L has a fixed component if and only if there are elliptic curves E_1 and E_2 with line bundles L_1 of type (1) on E_1 and L_2 of type (d) on L_2 such that $(X, L) \cong (E_1 \times E_2, p_1^* L_1 \otimes p_2^* L_2)$.*

Assuming L has no fixed components, we have:

Lemma 7.23. *Let L be a line bundle of type $(1, d)$.*

1. *If $d \geq 3$, L has no base point.*
2. *If $d = 2$, L has exactly four base points.*

Some more results about ampleness in dimension two can be found in the paper [Ram85] by Ramanan.

Theorem 7.24. *Let A be an abelian surface not containing elliptic curves and let L be an ample line bundle on A with $c_1(L)$ of type (d_1, d_2). Then L is very ample in either of the following cases*

1. *$d_1 = 1$ and $d_2 \geq 5$,*
2. *$d_1 = 2$ and $d_2 \geq 4$, or*
3. *$d_1 \geq 3$.*

7.4 Moduli spaces

7.4.1 Siegel upper half space

Suppose $X = V/\Lambda$ is an abelian variety of dimension g and $H \in NS(X)$ a hermitian form on V defining a polarisation of type $D = diag(d_1, \ldots, d_g)$. Then by definition there is a symplectic basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ of Λ for H such that the alternating form $\mathrm{Im}(H)$ is given by the matrix $\left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)$ with respect to this basis.
Define $e_\nu = \frac{1}{d_\nu}\mu_\nu$ for $\nu = 1, \ldots, g$. The vectors $e_1, \ldots, e_g$ form a $\mathbb{C}$-basis for V. With respect to these bases the period matrix is of the form

$$\Pi = (Z, D)$$

for some $Z \in \mathbb{C}^{g\times g}$.

Proposition 7.25 (Riemann Bilinear Relations)**.**

1. *$Z^\top = Z$ and $\mathrm{Im}(Z)$ is positive definite.*
2. *$(\mathrm{Im}(Z))^{-1}$ is the matrix of the hermitian form H with respect to the basis $e_1, \ldots, e_g$.*

Define a *polarised abelian variety of type D with symplectic basis* to be a triplet

$$(X, H, \{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\})$$

where $X = V/\Lambda$ is an abelian variety, H a polarisation of type D on X, and $\{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\}$ a basis of Λ for H such that H is of the form $\left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)$ with respect to this basis.
Two polarised abelian varieties $(X = V/\Lambda, H, \{\lambda_1, \ldots, \mu_g\})$ and $(X' = V'/\Lambda', H',$ $\{\lambda_1', \ldots, \mu_g'\})$ of type D with symplectic basis are said to be *isomorphic* if there is a linear isomorphism $\varphi\colon V \longrightarrow V'$ which maps λ_j to λ_j' and μ_j to μ_j' for all $j = 1, \ldots, g$. In this case φ automatically maps Λ to Λ' and pulls back H' to H i.e. it is an isomorphism of polarised abelian varieties.

The set
$$\mathcal{H}_g := \{Z \in \mathbb{C}^{g\times g} \mid Z^\top = Z, \mathrm{Im}(Z) > 0\}$$
is called the *Siegel upper half space*. It is a $\frac{1}{2}g(g+1)$-dimensional open submanifold of the vector space $\mathbb{C}^{g\times g}$.
We have seen that a polarised abelian variety of type D with symplectic basis determines a point Z in $\mathcal{H}_g$. Conversely, given a type D, the assignment

$$\Phi\colon Z \longmapsto (X_Z, H_Z, \{\text{columns of } (Z, D)\})$$

with $\Lambda_Z := (Z, D)\mathbb{Z}^{2g}$, $X_Z := \mathbb{C}^g/\Lambda_Z$ and H_Z the hermitian form described by $(\mathrm{Im}(Z))^{-1}$ with respect to the standard basis of $\mathbb{C}^g$, associates a polarised abelian variety with symplectic basis to any point in $\mathcal{H}_g$. As we argued above, any polarised abelian variety of type D with symplectic basis is isomorphic to one in the image of Φ. By definition for $Z \neq Z' \in \mathcal{H}_g$ the associated p.a.v. with symplectic basis are never isomorphic. Hence we have:

Proposition 7.26. *Given a type D the Siegel upper half space $\mathcal{H}_g$ is a moduli space for polarised abelian varieties of type D with choice of a symplectic basis.*

7.4.2 The analytic moduli space

To construct a moduli space for polarised abelian varieties of type D we have to analyse which points of $\mathcal{H}_g$ determine isomorphic polarised abelian varieties.
By analysing the action of the matrices of $\rho_a(f)$ and $\rho_r(f)$ of the analytic and the rational representation of an possible isomorphism f on Z one obtains:

Proposition 7.27. *For a given type D and $Z, Z' \in \mathcal{H}_g$ the following statements are equivalent:*

1. *The polarised abelian varieties (X_Z, H_Z) and $(X_{Z'}, H_{Z'})$ are isomorphic.*
2. *$Z' = M(Z)$ for some $M \in G_D$.*

Here

$$G_D := \{M \in Sp_{2g}(\mathbb{Q}) \mid M^\top \Lambda_D \subseteq \Lambda_D\}$$

where $\Lambda_D := \left(\begin{smallmatrix} 1_g & \\ & D \end{smallmatrix}\right) \mathbb{Z}^{2g}$ and

$$Sp_{2g}(R) = \{M \in R^{2g\times 2g} \mid N^\top \begin{pmatrix} & 1_g \\ -1_g & \end{pmatrix} N = \begin{pmatrix} & 1_g \\ -1_g & \end{pmatrix}\}$$

is called the *symplectic group* for any commutative ring R with 1. The action of G_D on $\mathcal{H}_g$ is given by $M(Z) := (\alpha Z + \beta)(\gamma Z + \delta)^{-1}$ for $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in G_D$ and $Z \in \mathcal{H}_g$. One can show that G_D is a discrete subgroup of $Sp_{2g}(\mathbb{R})$ and that any such subgroup G acts properly and discontinuously on $\mathcal{H}_g$. This implies that the quotient

$$\mathcal{A}'_D := \mathcal{H}_g / G_D$$

is a normal complex analytic space of dimension $\frac{g(g+1)}{2}$.
Hence the last proposition translates to:

Theorem 7.28. *The normal complex analytic space $\mathcal{A}'_D := \mathcal{H}_g/G_D$ is a moduli space for polarised abelian varieties of type D.*

There is another formulation: For any commutative ring R with 1 of characteristic 0 define the group

$$Sp^D_{2g}(R) = \{M \in R^{2g\times 2g} \mid M \left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right) M^\top = \left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)\}.$$

The map

$$\sigma_D\colon Sp^D_{2g}(\mathbb{R}) \longrightarrow Sp_{2g}(\mathbb{R}), \quad \sigma_D(M) := \left(\begin{smallmatrix} 1_g & 0 \\ 0 & D \end{smallmatrix}\right)^{-1} M \left(\begin{smallmatrix} 1_g & 0 \\ 0 & D \end{smallmatrix}\right)$$

is an isomorphism of groups that maps $\Gamma_D := Sp^D_{2g}(\mathbb{Z})$ to G_D. Both groups G_D and Γ_D are often called the *paramodular group.*
The associated action of $Sp^D_{2g}(\mathbb{R})$ in $\mathcal{H}_g$ is given by

$$M\langle Z\rangle = (aZ + bD)(D^{-1}cZ + D^{-1}dD)^{-1} \text{ for all } M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Sp^D_{2g}(\mathbb{R}).$$

Corollary 7.29. *The normal complex analytic space $\mathcal{A}_D := \mathcal{H}_g/\Gamma_D$ is a moduli space for polarised abelian varieties of type D.*

7.4.3 Level D-structure

In the last sections we saw that $\mathcal{H}_g/\Gamma_D$ is a moduli space for polarised abelian varieties of type D, while $\mathcal{H}_g$ itself is a moduli space for polarised abelian varieties of type D with symplectic basis. A symplectic basis cannot be given in algebraic terms, but one can consider several additional structures to (X, H), either because they reduce the number of automorphisms and allow the construction of a "finer" moduli-space or because they carry interesting geometric information. The more additional structure we want to encode, the smaller is the subgroup of Γ_D acting on $\mathcal{H}_G$. A *level D-structure* is kind of the closest replacement for the notion of a symplectic basis.

Let $(X = V/\Lambda, H)$ be a polarised abelian variety of type $D = diag(d_1, \dots, d_g)$. Recall the (multiplicative) alternating form $e^H\colon K(H) \times K(H) \longrightarrow \mathbb{C}^*$, $e^H(\bar{v}, \bar{w}) = \exp(-2\pi i \operatorname{Im} H(v, w))$. In Section 2.7.3 we introduced the group $K(D) = (\mathbb{Z}^g/D\mathbb{Z}^g)^2$ and the (multiplicative) alternating form $e^D\colon K(D) \times K(D) \longrightarrow \mathbb{C}^*$. A *level D-structure* on (X, H) is by definition a symplectic isomorphism $\bar{b}\colon K(H) \longrightarrow K(D)$. The symplectic isomorphism $\bar{b}\colon K(H) \longrightarrow K(D)$ can be identified with the ordered set $\{\bar{b}^{-1}(f_1), \dots, \bar{b}^{-1}(f_{2g})\}$ where $f_1, \dots, f_{2g}$ denotes the standard generators of $K(D)$. This is a basis of $K(L)$.
So we can define an *isomorphism of polarised abelian varieties with level D-structure* similarly to the definition we made in Section 7.4.1 as an isomorphism of polarised abelian varieties that maps the j-th element of the given basis of $K(L)$ to the corresponding element of $K(L')$.
Given a symplectic isomorphism $\bar{b}$ there is a symplectic basis $\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g$ of Λ for H such that $\bar{b}(\overline{\tfrac{1}{d_i}\lambda_i}) = f_i$ and $\bar{b}(\overline{\tfrac{1}{d_i}\mu_i}) = f_{g+i}$ for $1 \le i \le g$.

Every $Z \in \mathcal{H}_g$ determines a polarised abelian variety of type D with level D-structure:

$$Z \longmapsto \Big(X_Z, H_Z, \{\overline{\tfrac{1}{d_1}\lambda_1}, \dots, \overline{\tfrac{1}{d_g}\lambda_g}, \overline{\tfrac{1}{d_1}\mu_1}, \dots, \overline{\tfrac{1}{d_g}\mu_g}\}\Big)$$

where $(X_Z, H_Z, \{\lambda_1, \dots, \mu_g\})$ is the polarised abelian variety of type D with symplectic basis of Proposition 7.26.

By what we said above it is clear that every polarised abelian variety with level D-structure is isomorphic to one of these. One can show that

$$\varphi\colon \Big(X_Z, H_Z, \{\overline{\tfrac{1}{d_1}\lambda_1}, \ldots, \overline{\tfrac{1}{d_g}\mu_g}\}\Big) \longrightarrow \Big(X_{Z'}, H_{Z'}, \{\overline{\tfrac{1}{d_1}\lambda'_1}, \ldots, \overline{\tfrac{1}{d_g}\mu'_g}\}\Big)$$

is an isomorphism of polarised abelian varieties with level D-structure if and only if the matrix R of the rational representation of φ is an element of the group

$$\Gamma_D(D) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_D \mid a - 1_g \equiv b \equiv c \equiv d - 1_g \equiv 0 \mod D\}$$

where we write $a \equiv 0 \mod D$ for $a \in D \cdot \mathbb{Z}^{g \times g}$.
It might not be clear from the definition, but $\Gamma_D(D)$ is a normal subgroup of finite index in Γ_D. As a subgroup of Γ_D it also acts properly and discontinuously on $\mathcal{H}_g$ and we obtain

Theorem 7.30. *The normal complex analytic space $\mathcal{A}_D(D) := \mathcal{H}_g/\Gamma_D(D)$ is a moduli space for polarised abelian varieties of type $D = diag(d_1, \ldots, d_g)$ with level D-structure. The embedding $\Gamma_D(D) \hookrightarrow \Gamma_D$ induces a holomorphic map $\mathcal{A}_D(D) \to \mathcal{A}_D$ of finite degree.*

7.4.4 Generalised level n-structure

A *level n-structure* on a principally polarised abelian variety (X, H) is by definition a level $(n1_g)$-structure on the polarised abelian variety (X, nH) in the sense of Section 7.4.3, i.e. a symplectic basis of the n-division points X_n. We want to generalise the notion of a level n-structure to a polarised abelian variety (X, H) of arbitrary type D. But in this case H does in general not induce a nondegenerate multiplicative alternating form on X_n.
Let $(X = V/\Lambda, H)$ be a polarised abelian variety of type D. A symplectic basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ of Λ for H determines a basis of the group X_n, namely $\overline{\frac{1}{n}\lambda_1}, \ldots, \overline{\frac{1}{n}\lambda_g}, \overline{\frac{1}{n}\mu_1}, \ldots, \overline{\frac{1}{n}\mu_g}$. A *generalised level n-structure* for (X, H) is defined to be a basis of X_n coming from a symplectic basis in this way. We call two such triplets $(X, H, \{x_1, \ldots, x_{2g}\})$ and $(X', H', \{x'_1, \ldots, x'_{2g}\})$ isomorphic, if there is an isomorphism $\varphi\colon (X, H) \longrightarrow (X', H')$ as polarised abelian varieties, and $\varphi(x_i) = x'_i$ for all $1 \leq i \leq 2g$.
For every $Z \in \mathcal{H}_g$

$$\Big(X_Z, H_Z, \{\overline{\tfrac{1}{n}\lambda_1}, \ldots, \overline{\tfrac{1}{n}\lambda_g}, \overline{\tfrac{1}{n}\mu_1}, \ldots, \overline{\tfrac{1}{n}\mu_g}\}\Big)$$

is a polarised abelian variety with level n-structure, where $(X_Z, H_Z, \{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\})$ is the polarised abelian variety of type D with symplectic basis as in Proposition 7.26. Conversely, it is clear that every polarised abelian variety with level n-structure is isomorphic to one of these and again we have to analyse when two of them are isomorphic.

With an argument similar to the one in the last section, one can see that Z and $Z' \in \mathcal{H}_g$ determine isomorphic polarised abelian varieties with generalised level n-structure if and only if $Z' = R\langle Z\rangle$ and R is an element of the group

$$\Gamma_D(n) = \{R \in \Gamma_D \mid R \equiv 1_{2g} \mod n\}.$$

Theorem 7.31. *The normal complex analytic space $\mathcal{A}_D(n) := \mathcal{H}_g/\Gamma_D(n)$ is a moduli space for polarised abelian varieties of type $D = diag(d_1, \ldots, d_g)$ with generalised level n-structure. The embedding $\Gamma_D(n) \hookrightarrow \Gamma_D$ induces a holomorphic map $\mathcal{A}_D(n) \to \mathcal{A}_D$ of finite degree.*

7.4.5 Decomposition of the lattice

In Section 2.4 we defined a *decomposition of* Λ *(for H)* to be a decomposition

$$\Lambda = \Lambda_1 \oplus \Lambda_2$$

with isotropic subgroups $\Lambda_1, \Lambda_2 \subseteq \Lambda$ for $\mathrm{Im}(H)$. Any symplectic basis $\lambda_1, \ldots, \lambda_g$, $\mu_1, \ldots, \mu_g$ of Λ for H determines such a decomposition via

$$\Lambda_1 = \langle \lambda_1, \ldots, \lambda_g\rangle, \quad \Lambda_2 = \langle \mu_1, \ldots, \mu_g\rangle.$$

Thus every $Z \in \mathcal{H}_g$ determines a polarised abelian variety of type D with a decomposition, namely

$$Z \longmapsto (X_Z, H_Z, \Lambda_Z = \Lambda_1 \oplus \Lambda_2),$$

where $\Lambda_1 := Z\mathbb{Z}^g$ and $\Lambda_2 := D\mathbb{Z}^g$. An isomorphism of polarised abelian varieties with a decomposition $\varphi\colon (X, H, \Lambda = \Lambda_1 \oplus \Lambda_2) \longrightarrow (X', H', \Lambda = \Lambda_1' \oplus \Lambda_2')$ is defined to be an isomorphism φ of polarised abelian varieties, such that $\rho_r(\varphi)(\Lambda_1) = \Lambda_1'$ and $\rho_r(\varphi)(\Lambda_2) = \Lambda_2'$. It follows from the proof of the elementary divisor theorem that every polarised abelian variety with a decomposition is of this form.
It turns out that $(X_Z, H_Z, \Lambda_Z = \Lambda_1 \oplus \Lambda_2)$ and $(X_{Z'}, H_{Z'}, \Lambda' Z = \Lambda_1' \oplus \Lambda_2')$ are isomorphic if and only if $Z' = R\langle Z\rangle$ for $R \in \Delta_D$ with

$$\Delta_D := \left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_D \mid b = c = 0\right\}.$$

Proposition 7.32. *The normal complex analytic space $\mathcal{A}_D^\Delta := \mathcal{H}_g/\Delta_D$ is a moduli space for polarised abelian varieties with a decomposition.*

The embedding $\Delta_D \hookrightarrow \Gamma_D$ yields holomorphic maps $\mathcal{H}_g \xrightarrow{\pi_1} \mathcal{A}_D^\Delta \xrightarrow{\pi_2} \mathcal{A}_D$ which both have infinite fibres.

Bibliography

[Arn89] V. I. Arnold. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, [1989?]. Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition.

[Aud96] Michèle Audin. *Spinning tops*, volume 51 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1996. A course on integrable systems.

[Aud08] Michèle Audin. *Hamiltonian systems and their integrability*, volume 15 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2008. Translated from the 2001 French original by Anna Pierrehumbert, Translation edited by Donald Babbitt.

[AvM82] M. Adler and P. van Moerbeke. The algebraic integrability of geodesic flow on SO(4). *Invent. Math.*, 67(2):297–331, 1982.

[AvMV04] Mark Adler, Pierre van Moerbeke, and Pol Vanhaecke. *Algebraic integrability, Painlevé geometry and Lie algebras*, volume 47 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004.

[Bar87] Wolf Barth. Abelian surfaces with $(1,2)$-polarization. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 41–84. North-Holland, Amsterdam, 1987.

[BHM87] Wolf Barth, Klaus Hulek, and Ross Moore. Degenerations of Horrocks-Mumford surfaces. *Math. Ann.*, 277(4):735–755, 1987.

[BL95] Ch. Birkenhake and H. Lange. Moduli spaces of abelian surfaces with isogeny. In *Geometry and analysis (Bombay, 1992)*, pages 225–243. Tata Inst. Fund. Res., Bombay, 1995.

[BL04] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*, volume 302 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2004.

[BLvS89] C. Birkenhake, H. Lange, and D. van Straten. Abelian surfaces of type $(1, 4)$. *Math. Ann.*, 285(4):625–646, 1989.

[DGR82] B. Dorizzi, B. Grammaticos, and A. Ramani. Painlevé conjecture revisited. *Phys. Rev. Lett.*, 49(21):1539–1541, 1982.

[GP01] Mark Gross and Sorin Popescu. Calabi-Yau threefolds and moduli of abelian surfaces. I. *Compositio Math.*, 127(2):169–228, 2001.

[Gri94] Valeri Gritsenko. Irrationality of the moduli spaces of polarized abelian surfaces. *Internat. Math. Res. Notices*, (6):235 ff., approx. 9 pp., 1994.

[GvS10] Mauricio D. Garay and Duco van Straten. Classical and quantum integrability. *Mosc. Math. J.*, 10(3):519–545, 661, 2010.

[Hai83] L. Haine. Geodesic flow on SO(4) and abelian surfaces. *Math. Ann.*, 263(4):435–472, 1983.

[HH64] Michel Hénon and Carl Heiles. The applicability of the third integral of motion: Some numerical experiments. *Astronom. J.*, 69:73–79, 1964.

[HKW93] Klaus Hulek, Constantin Kahn, and Steven H. Weintraub. *Moduli spaces of abelian surfaces: compactification, degenerations, and theta functions*, volume 12 of *De Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1993.

[HM73] G. Horrocks and D. Mumford. A rank 2 vector bundle on $\mathbf{P}^4$ with $15,000$ symmetries. *Topology*, 12:63–81, 1973.

[HR00] Klaus Hulek and Kristian Ranestad. Abelian surfaces with two plane cubic curve fibrations and Calabi-Yau threefolds. In *Complex analysis and algebraic geometry*, pages 275–316. de Gruyter, Berlin, 2000.

[Hud90] R. W. H. T. Hudson. *Kummer's quartic surface*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1990. With a foreword by W. Barth, Revised reprint of the 1905 original.

[Igu72] Jun-ichi Igusa. *Theta functions*. Springer-Verlag, New York-Heidelberg, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 194.

[Knö86] Horst Knörrer. Integrable Hamiltonsche Systeme und algebraische Geometrie. *Jahresber. Deutsch. Math.-Verein.*, 88(2):82–103, 1986.

[Kov89] Sophie Kovalevskaya. Sur le probleme de la rotation d'un corps solide autour d'un point fixe. *Acta Math.*, 12(1):177–232, 1889.

[Lio55] J. Liouville. Note sur l'intégration des équations différentielles de la dynamique. *Ann. Math. Pures Appl.*, 20:137–138, 1855.

[Man88] Nicolae Manolache. Syzygies of abelian surfaces embedded in $\mathbf{P}^4(\mathbf{C})$. *J. Reine Angew. Math.*, 384:180–191, 1988.

[MP04] Andrzej J. Maciejewski and Maria Przybylska. All meromorphically integrable 2D Hamiltonian systems with homogeneous potential of degree 3. *Phys. Lett. A*, 327(5-6):461–473, 2004.

[MP05] Andrzej J. Maciejewski and Maria Przybylska. Darboux points and integrability of Hamiltonian systems with homogeneous polynomial potential. *J. Math. Phys.*, 46(6):062901, 33, 2005.

[MR99] Juan J. Morales Ruiz. *Differential Galois theory and non-integrability of Hamiltonian systems*. Modern Birkhäuser Classics. Birkhäuser/Springer, Basel, 1999. [2013] reprint of the 1999 edition [MR1713573].

[MRR01] Juan J. Morales-Ruiz and Jean Pierre Ramis. A note on the non-integrability of some Hamiltonian systems with a homogeneous potential. *Methods Appl. Anal.*, 8(1):113–120, 2001.

[Mum83] David Mumford. *Tata lectures on theta. I*, volume 28 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1983. With the assistance of C. Musili, M. Nori, E. Previato and M. Stillman.

[Mum84] David Mumford. *Tata lectures on theta. II*, volume 43 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1984. Jacobian theta functions and differential equations, With the collaboration of C. Musili, M. Nori, E. Previato, M. Stillman and H. Umemura.

[Mum91] David Mumford. *Tata lectures on theta. III*, volume 97 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1991. With the collaboration of Madhav Nori and Peter Norman.

[Prz07] Maria Przybylska. Finiteness of integrable n-dimensional homogeneous polynomial potentials. *Phys. Lett. A*, 369(3):180–187, 2007.

[Prz09a] M. Przybylska. Darboux points and integrability of homogeneous Hamiltonian systems with three and more degrees of freedom. *Regul. Chaotic Dyn.*, 14(2):263–311, 2009.

[Prz09b] M. Przybylska. Darboux points and integrability of homogeneous Hamiltonian systems with three and more degrees of freedom. Nongeneric cases. *Regul. Chaotic Dyn.*, 14(3):349–388, 2009.

[Ram85] S. Ramanan. Ample divisors on abelian surfaces. *Proc. London Math. Soc. (3)*, 51(2):231–245, 1985.

[Rie57] B. Riemann. Theorie der Abel'schen Functionen. *J. Reine Angew. Math.*, 54:115–155, 1857.

[Sem12] Michael Semmel. *The geometry of Lagrangian fibres.* PhD thesis, Johannes Gutenberg-Universität Mainz, 2012.

[SvS13] Michael Semmel and Duco van Straten. An abelian surface with $(1,6)$-polarization. *Nonlinearity*, 26(11):2973–2992, 2013.

[Van96] Pol Vanhaecke. *Integrable systems in the realm of algebraic geometry*, volume 1638 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1996.

[Van08] Pol Vanhaecke. Algebraic integrability: a survey. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 366(1867):1203–1224, 2008.

[Zig82] S. L. Ziglin. Bifurcation of solutions and the nonexistence of first integrals in Hamiltonian mechanics. I. *Funktsional. Anal. i Prilozhen.*, 16(3):30–41, 96, 1982.

[Zig83] S. L. Ziglin. Bifurcation of solutions and the nonexistence of first integrals in Hamiltonian mechanics. II. *Funktsional. Anal. i Prilozhen.*, 17(1):8–23, 1983.