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Frobenius Polynomials for Calabi-Yau Equations

Dissertation zur Erlangung des Grades
"Doktor der Naturwissenschaften”

am Fachbereich Mathematik und Informatik
der Johannes Gutenberg-Universität Mainz,

vorgelegt von

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Mainz, den 28. Januar 2010

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Salvador Dali, Corpus Hypercubus, 1954

In algebraic geometry, it is always nice to print some pictures of the varieties one is working with. As you will see, the main geometric objects of our studies are Calabi-Yau varieties of dimension three, and it is not possible to print pictures of these. The three-dimensional cube in four-dimensional space is not a Calabi-Yau threefold, but at least, it is *some* geometric object of dimension three. Therefore, please see Dali's wonderful picture above as a modest approach to illustrate this thesis.

Introduction

Let X be a projective variety defined over a finite field $\mathbb{F}_q$, where q is a power of a prime p. For each finite algebraic extension $\mathbb{F}_{q^n}$ of $\mathbb{F}_q$, the number of points on X with coordinates in $\mathbb{F}_{q^n}$, noted by $\#X(\mathbb{F}_{q^n})$, is obviously finite. The numbers $\#X(\mathbb{F}_{q^n})$ are of great importance in studying arithmetical properties of X. They are encoded in the *zeta function* of X over $\mathbb{F}_q$, which is defined as the formal power series

$$Z(X/\mathbb{F}_q, T) := \exp\left(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n})\frac{T^n}{n}\right).$$

In 1949, Andre Weil [**57**] stated a series of conjectures concerning the zeta function of a variety over a finite field, the *Weil conjectures*. As a power series, it has coefficients in $\mathbb{Z}$, and one of Weil's conjectures is that it is an element of $\mathbb{Q}(T)$. This was proven by Dwork [**26**] in 1960, using techniques of $p-$adic analysis.

We are interested in the actual computation of the zeta function. Suppose now that we are not dealing with a single variety, but with a one-parameter family $\pi : X \to \mathbb{P}^1$ of varieties over a finite field $\mathbb{F}_q$. For each parameter value $t_0 \in \mathbb{P}^1(\mathbb{F}_q)$, the fibre X_{t_0} has a zeta function $Z(X_{t_0}/\mathbb{F}_q, T)$. The question that arises now is: *How does the zeta function vary as the parameter t_0 varies in $\mathbb{P}^1(\mathbb{F}_q)$*?

There exist several approaches to answer this question. One approach is the deformation algorithm of A. Lauder [**46**]. This algorithm was inspired by Dwork's proof of the functional equation of the zeta function of a smooth hypersurface in [**28**] and [**29**]. Lauder's algorithm computes the zeta functions of the fibres of a one-parameter family of smooth projective hypersurfaces that are deformations of a so-called *diagonal* hypersurface. To compute the zeta function, one computes the characteristic polynomial of the Frobenius endomorphism on the *Dwork cohomology* spaces. There is an explicit formula due to Dwork for the Frobenius matrix in a monomial basis of a diagonal hypersurface on a Dwork cohomology space. Dwork showed how the Picard-Fuchs equation of the family of hypersurfaces can be applied to compute the Frobenius matrix of a fibre of the family as a deformation of the Frobenius matrix of the diagonal hypersurface. This is the crucial idea behind the deformation algorithm. To perform these steps, one has to compute a monomial basis of the Dwork cohomology spaces and needs the explicit equation defining the family for the necessary reduction steps modulo the Jacobian ideal.

The author of this thesis showed that Lauder's algorithm can be extended to families of hypersurfaces in weighted projective space that are deformations of diagonal hypersurfaces

in her diploma thesis, and implemented this algorithm in MAGMA, see [**50**].
In 2000, P. Candelas, X. de la Ossa and F. Rodriguez Villegas [**14**] derived an amazing formula for the number of points on the fibres of the one-parameter family of quintic threefolds in $\mathbb{P}^4$ defined by

$$F(X,\psi)=\sum_{i=1}^{5} X_i^5 - 5\psi X_1X_2X_3X_4X_5$$

in terms of a Frobenius basis of solutions to the Picard-Fuchs equation. Let

$$\nu(\psi)=\#\{X\in\mathbb{F}_p^5, F(X,\psi)=0\},$$

and let $\lambda = 1/(5\psi)^5$. Then, $\nu(\psi)$ can be expressed in terms of the truncated (up to degree $p-1$) power series parts $f_i^{(p-1)}$ in the Frobenius basis of solutions as

$$\nu(\psi)\equiv f_0^{(p-1)}(\lambda^{p^4})+\left(\frac{p}{1-p}\right) f_1'^{(p-1)}(\lambda^{p^4})+\ldots \mod p^5.$$

Thus, the number of points on a fibre modulo p^5 can be computed explicitly by the data given by the Picard-Fuchs operator.
It was Dwork who in 1958 introduced a method to compute the complete zeta functions of the fibers of the Legendre family of elliptic curves out of the data of a differential operator (up to a sign ε), without any reference to the defining equation. Namely, he derived a formula to compute the roots of the numerator of the zeta function of a smooth, ordinary fibre from a period of the family, the holomorphic solution $\Phi_0(z)$ around $z=0$ to the Picard-Fuchs equation

$$z(1-z)\frac{d}{dz}^2\Phi+(1-2z)\frac{d}{dz}\Phi-\frac{1}{4}\Phi=0.$$

The solution $\Phi_0(z)$ is given by the hypergeometric function ${}_2F_1(1/2,1/2,1;z)$. Dwork proved that the zeta function of a smooth ordinary fibre X_{t_0}, $t_0\in\mathbb{F}_p$ of the Legendre family is then given by

$$Z(X_{t_0}/\mathbb{F}_p,T)=\frac{(1-r_{t_0}T)(1-p/r_{t_0}T)}{(1-T)(1-pT)},$$

where r_{t_0} is the $p-$adic unit given by

$$r_{t_0}=(-1)^{(p-1)/2}\frac{\Phi_0(z)}{\Phi_0(z^p)}|_{z=t}$$

for a Teichmüller lifting $t\in\mathbb{Z}_p$ of t_0. The only ingredient for the computation of the zeta function which is *not* determined by the differential operator is the constant $\varepsilon = (-1)^{(p-1)/2}$, which has to be derived by geometric considerations. Thus, Dwork found a method to compute the zeta function of a smooth ordinary fibre of the Legendre family up to a twist by a character. The numerator of the zeta function is the characteristic polynomial of the Frobenius endomorphism on the first cohomology group of any *Weil cohomology* of

the fibre. Note that it is a crucial ingredient to Dwork's method that the Picard-Fuchs operator of the Legendre family has a point of maximally unipotent monodromy at $z = 0$.
An elliptic curve is a special case of a *Calabi-Yau manifold*, namely a Calabi-Yau manifold of dimension one. Now the question arises if one can perform similar calculations in higher dimensions. What, for example, if we consider a one-parameter family $\pi : X \to S$ of smooth Calabi-Yau threefolds defined over $\mathbb{Q}$? This family has an integral model over $\mathbb{Z}$, and we assume that the reduction of the family to $\mathbb{F}_q$ is again a family of smooth Calabi-Yau threefolds $\pi_0 : X_0 \to S_0$. Let $t_0 \in S_0$ and let X_{t_0} denote the fibre over t_0. Is it possible to compute the characteristic polynomial of the relative Frobenius endomorphism, which we call the *Frobenius polynomial*, on the third crystalline cohomology $H^3_{cris}(X_{t_0})$ out of the data given by a Picard-Fuchs operator?
First of all, let us assume that the rank of $H^3_{DR}(X/S)$ is four. Then, the Picard-Fuchs operator of the family is a linear differential operator of degree four. Assume that at $z = 0$, the monodromy is maximally unipotent. Then the differential operator has special properties which are summarized in the definition of a *CY(4)-operator*. In the literature, there exists no definitive definition of CY-differential operators at the moment, but we will specify explicitly what we mean by a CY-differential operator.
If the rank of $H^3_{DR}(X/S)$ is not four, assume that there exists a rank-four submodule M of $H^3_{DR}(X(S)$ which is stable under the Gauss-Manin connection. If the monodromy at $z = 0$ is maximally unipotent, the Picard-Fuchs equation, satisfied by a holomorphic three-form generating M as a cyclic vector, is then also a CY(4)-operator.

This leads us to the main problem of this thesis, which is the following: *For the fibres X_{t_0} of a family of Calabi-Yau threefolds defined over $\mathbb{F}_q$, is it possible to give an algorithm to compute the characteristic polynomial of the relative Frobenius endomorphism, the Frobenius polynomial (maybe up to a sign ε), on (a rank four submodule M_{t_0} of) $H^3_{cris}(X_{t_0})$ out of the data given by a CY(4)-differential operator?*

Consider the situation from a $p-$adic point of view, and let $\pi : X \to S$ be a family defined over the ring of integers of a finite extension K of $\mathbb{Q}_p$. Assume that the morphism π is proper and smooth, with geometrically connected fibres. If the relative de Rham cohomology groups

$$H^i_{DR}(X/S) := \mathbb{R}^i \pi_* \Omega^\bullet_{X/S}$$

are locally free $\mathcal{O}_S-$modules, then, by a result of Berthelot [**9**], for $i \geq 0$, $H^i_{DR}(X/S)$ with its Gauss-Manin connection is an $F-$*crystal* on S.
If the family $\pi : X \to S$ is the lifting of a family $\pi_0 : X_0 \to S_0$ defined over a finite field extension k of $\mathbb{F}_p$ with $q := p^r$ elements, then for $t_0 \in S_0$, the zeta function of the fibre X_{t_0} can be expressed in terms of the characteristic polynomials of the absolute Frobenius F as

$$Z(X_{t_0}/k, T) = \prod_{i=0}^{2\dim X_{t_0}} \det(1 - TF^r | H^i_{DR}(X_t)),$$

where for $t_0 \in S_0$, t denotes the Teichmüller lifting $t \in S$. For generic t_0, the $F-$crystal $H^3_{DR}(X_t)$ is an *ordinary CY3-crystal*. This implies that if the rank of $H^3_{DR}(X/S)$ is four,

the Frobenius polynomial on $H^3_{DR}(X_t)$ has one reciprocal $p-$adic root of valuation $0, r, 2r$ and $3r$, and is determined uniquely by the reciprocal roots r_{t_0}, s_{t_0} of valuation 0 and r. Since it is a $p-$adic unit, r_{t_0} is called the *unit root* of the Frobenius polynomial. It was our goal to derive formulas to compute r_{t_0} and s_{t_0} out of the data given by the Picard-Fuchs operator on $H^3_{DR}(X/S)$.
One problem that arises if one wants to compute the Frobenius polynomial of a fibre X_{t_0} explicitly is the problem of *$p-$adic analytic continuation* to the boundary of the $p-$adic unit disc. Namely, to compute the unit root, one has to evaluate a quotient of the form

$$\frac{f_0(z)}{f_0(z^p)}$$

at a Teichmüller point, where $f_0(z)$ is the holomorphic solution to the CY(4)-differential equation. This quotient is analytic on the open $p-$adic unit disc. Dwork [**27**] constructed an explicit analytic continuation for quotients of this type, provided that the coefficients of f_0 satisfy what we call the *Dwork congruences*.

Now, a second important question arises, namely: *Can we prove the Dwork congruences for the coefficients of the power series solutions of CY(4)-differential operators?*

For the coefficients of the power series solutions of the 14 hypergeometric CY(4)-operators, Dwork proved these congruences in [**27**]. But for the majority of the CY(4)-operators, no proof of these congruences is known.
It turns out that many CY(4)-operators are Picard-Fuchs operators of families of Calabi-Yau threefolds defined by *Laurent-polynomials*. The holomorphic solution Φ_0 around $z = 0$ to the CY(4)- differential equation can be expressed in terms of a Laurent-polynomial f, namely by

$$\Phi_0(z) = \sum_{n=0}^{\infty} [f^n]_0 z^n,$$

where $[f^n]_0$ denotes the constant term in f^n. We used this fact to give a proof of a modified version of the Dwork congruences for many examples.

This thesis is structured in the following way:

In **Chapter 1** we give a short overview over the Weil conjectures and introduce some cohomology theories which were developed to provide a proof of these conjectures, like *$\ell-$adic cohomology* and *crystalline cohomology*. We review the formulas to compute the zeta function of a variety X defined over a finite field in terms of the absolute Frobenius endomorphism in crystalline and *rigid cohomology*.

In **Chapter 2** we review the theory of $F-$crystals. This theory provides the background for our computations. We are especially interested in ordinary CY3-crystals and general autodual crystals, since these objects appear as the relative crystalline cohomology groups of families of Calabi-Yau varieties.

In **Chapter 3** we give the definition of a Calabi-Yau differential operator. We review the construction of the differential module defined by a Calabi-Yau differential operator, and quote some of the properties of this differential module. This chapter contains our first result; we derive a formula for the Frobenius polynomial on an ordinary CY3-crystal with connection defined by a CY(4)-differential operator.

In **Chapter 4** we review the fact that the non-ordinary locus of an $F-$crystal, the set of zeros of the *Hasse-invariant*, can be expressed in terms of the holomorphic solution to the Picard-Fuchs equation if the coefficients of this solution satisfy the *Dwork-congruences*.

In **Chapter 5** we review Dwork's construction of an analytic continuation of a function of the type $\Phi_0(z)/\Phi_0(z^p)$ to the boundary of the $p-$adic unit disc, provided that the coefficients of the power series Φ_0 satisfy the Dwork congruences. Applying Dwork's construction, we derive explicit formulas to compute two reciprocal roots of the Frobenius polynomial on an ordinary CY3-crystal of rank 4, and hence to compute the whole Frobenius polynomial. These formulas involve the holomorphic solution to the CY(4)-differential equation defining the connection of the CY3-crystal, and the holomorphic solution to a CY(5)-differential equation which is the second exterior product of the CY(4)-equation. We give estimates of the required $p-$adic precision to recover the Frobenius polynomial correctly out of the reciprocal roots. Furthermore, we present an algorithm to compute the Frobenius polynomial out of only one of the two reciprocal roots considered above.

In **Chapter 6** we introduce a special class of CY(4)-diffential operators which are so-called *Hadamard -products*. We compute Frobenius polynomials for many of these operators; the results of our computations are documented in the appendix.

In **Chapter 7** we review the basics of the theory of *modular forms* and describe why the Frobenius polynomial is expected to factorize in a special way at the conifold points of the CY(4)-operator. We confirm this expectation by computing the Frobenius polynomial in rational conifold points of several CY(4)-operators. Some of these are Hadamard-products, as described in the previous chapter, and some are not. In each of the cases, we could identify modular forms of weight four. The results are listed in tables, part of the tables can be found in the appendix.

In **Chapter 8** we derive a weaker congruence property D3 from the Dwork conguence D2. In case that the holomorphic solution Φ_0 to a Calabi-Yau differential operator is defined by the constant terms of the powers of a Laurent-polynomial whose Newton polygon contains the origin as unique interior lattice point, we prove that the coefficients of Φ_0 satisfy the congruence D3.

In **Chapter 9** we describe an experimental approach to compute the Frobenius polynomial directly as the characteristic polynomial of some matrix, which may differ from the Frobenius matrix by some parameters. This approach worked well in the case of hypergeometric CY(4)-differential operators. We observed that the Frobenius polynomial is independent of the parameters mentioned above, and also observed that some interesting congruences involving the non-holomorphic solutions to the CY(4)-differential equation hold.

In **Chapter 10** we prove that the non-holomorphic solutions to the CY(4)-differential equation, which contain logarithmic terms, can be used to compute the unit root of the Frobenius polynomial, too.

In **Chapter 11** we describe an alternative appoach to construct an analytic continuation of a function of the type $\Phi(z)/\Phi(z^p)$ to the boundary of the $p-$adic unit disc due to Christol. We prove that this approach can be applied if the coefficients of the power series $\Phi(z)$ satisfy the Dwork congruences, and compare it to Dwork's construction.

Now, we describe the main results of this thesis in some more detail.

Our first result, see chapter 3, is the developement of an algorithm to compute the Frobenius polynomial for ordinary rank four CY3-crystals M. Let $\pi : X \to S$ be a family of smooth Calabi-Yau threefolds defined over $\mathbb{Q}$ with a flat model over $\mathbb{Z}$ such that the reduction $\pi_0 : X_0 \to S_0$ to $\mathbb{F}_p$ is again a family of smooth Calabi-Yau threefolds. Assume that there exists a rank 4 submodule M of $H^3_{DR}(X/S)$ with Picard-Fuchs operator a CY(4)-differential operator P. Let $\alpha_0 \in S_0$ and let $\alpha \in \mathbb{Z}_p$ be a Teichmüller lifting of α_0. If the CY3-crystal $M_{\alpha_0} \subset H^3_{cris}(X_{\alpha_0})$ is ordinary, the Frobenius polynomial on M_{α_0} is given by

$$\mathcal{P} := p^6T^4 + a_{\alpha_0}p^3T^3 + b_{\alpha_0}pT^2 + a_{\alpha_0}T + 1$$

and is uniquely determined by a reciprocal root r_{α_0} which is a $p-$adic unit and another reciprocal root ps_{α_0} of $p-$adic valuation 1, since the four reciprocal $p-$adic roots of $\mathcal{P}$ are given by $r_{\alpha_0}, ps_{\alpha_0}, p^2/s_{\alpha_0}$ and p^3/r_{α_0}. The roots r_{α_0} and ps_{α_0} are both eigenvalues of the Frobenius endomorphism. The $p-$adic unit r_{α_0} is the unit root of the $F-$crystal M_{α_0}, and we derive the formula

$$r_{\alpha_0} = \varepsilon \frac{f_0(z)}{f_0(z^p)}|_{z=\alpha},$$

where $\varepsilon = \pm 1$ and f_0 is the holomorphic solution around $z = 0$ to the differential equation $Pf = 0$. To derive a formula for the $p-$adic unit s_{α_0}, we use the fact that the eigenvalues of the Frobenius endomorphism on the second exterior product of M_{α_0} are products of the eigenvalues of the Frobenius endomorphism on M_{α_0}. Let Q denote the second exterior product of the differential operator P, and let g_0 denote the holomorphic solution around $z = 0$ to the differential equation $Qg = 0$. Then, we prove that s_{α_0} is given by $r_{\alpha_0}/r'_{\alpha_0}$, where r'_{α_0} can be computed as

$$r'_{\alpha_0} = \frac{g_0(z)}{g_0(z^p)}|_{z=\alpha}.$$

During the considerations in chapter 3, we see that the Frobenius matrix $A_\phi(z)$ depends on three parameters α, β, γ. But our formulas for the unit root and the root of $p-$adic valuation one prove indirectly that the Frobenius polynomial itself is *independent* of these parameters. We published this in [**51**].

If the fibre X_{α_0} is not smooth but has an ordinary double point, the Frobenius polynomial on the "limit module" M_{α_0} is expected to factorize in two factors of degree one and one factor of degree 2, which is given by $(p^3T^2 - a_pT + 1)$. The factor $(p^3T - a_pT + 1)$ is expected to be the Frobenius polynomial on $H^3_{cris}(\hat{X}_{\alpha_0})$, where $\hat{X}_{\alpha_0}$ is a rigid Calabi-Yau threefold. If p varies, by the *modularity conjecture* the coefficients a_p are the coefficients of a weight four modular form. We could compute these coefficients for many CY(4)-operators and identified the corresponding modular forms (see chapter 7).

Our next result is of a completely different character; for the proof, we only applied very elementary methods. Let f be a Laurent polynomial such that *Newton polyhedron* of f has 0 as unique interior lattice point. This is always the case if the Newton polyhedron is reflexive. As described before, the *fundamental period* of f is given by $\Phi_0(t) = \sum_{n=0}^{\infty}[f^n]_0 t^n$. It can be realized as the period of a holomorphic differential form on the toric hypersurface $\{1 - tf(X) = 0\}$, and satisfies a Picard-Fuchs equation. There is a list of CY(4)-operators that arise as Picard-Fuchs operators in this way, which we print in the appendix. We prove that the coefficients of $\Phi_0(t)$ satisfy a congruence property which can be derived from the Dwork congruences, but is slightly weaker (see chapter 8). Note that we published this result in [**52**].

Part of our next result, which was inspired by communications with P. Candelas and X. de la Ossa, is only a conjecture, but our numerous computations confirm us in believing that this conjecture holds true. Furthermore, the observations we made are quite astonishing. We use an experimental approach to compute the Frobenius polynomial directly as the characteristic polynomial of some matrix which might differ from the Frobenius matrix by some parameters. Let therefore P be a hypergeometric CY(4)-operator. Inspired by the theory of Dwork [**27**], we know that a Frobenius matrix $A_\phi(z)$ on the CY3-crystal with connection defined by P can be described in the open $p-$adic unit disc by

$$p^3 A_\phi(z)^{-1} = B(z^p)^{-1} \begin{pmatrix} p^3 & p^2\alpha & p\beta & \gamma \\ 0 & p^2 & p\alpha & \beta \\ 0 & 0 & p & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix} B(z),$$

where $B(z)$ is a *fundamental solution matrix* and the parameters α, β, γ lie in $\mathbb{Z}_p$, $\beta = \alpha^2/2$. The results of chapter 3 show indirectly that the characteristic polynomial of the matrix $A_\phi(z)$ is independent of the choice of the parameters α, β, γ. We try to give a direct explanation of this fact by considering the coefficients of the three parameters. In the case of CY(2)-operators, where the situation is similar, we prove directly that the Frobenius polynomial is indeed independent of the occuring parameter. We describe an explicit method to compute the Frobenius polynomial as the characteristic polynomial of the matrix $A_\phi(z)$ at a Teichmüller point (see chapter 9). Based on our computations, we made the following observation. Namely, that at a Teichmüller point α, the unit root can be computed by the non-holomorphic solutions to $Pf = 0$ which contain logarithms, and not only by the holomorphic solution $f_0(z)$. If the power series $f_1(z), f_2(z), f_3(z)$ are the *non-logarithmic* parts of these solutions, it turns out that we have

$$\frac{f_0^{s+1}(\alpha)}{f_0^s(\alpha)} \equiv p\frac{f_1^{s+1}(\alpha)}{f_1^s(\alpha)} \equiv p^2\frac{f_2^{s+1}(\alpha)}{f_2^s(\alpha)} \equiv p^3\frac{f_3^{s+1}(\alpha)}{f_3^s(\alpha)} \mod p^s.$$

This last observation inspired us to work out that the unit root at a Teichmüller point can be expressed in terms of the non-holomorphic solutions. Our method is straightforward; we construct fixed points of the Frobenius map involving the power series f_1, f_2 ans f_3, and use these to derive our new formulas to compute the unit root in a Teichmüller point. The crucial observation here is that some of the newly constructed functions are fixed points of the same contraction mapping as functions already appearing in the proof of theorem 2.3.1, which leads us to the canonical formula for the unit root which only involves the holomorphic solution to $Pf = 0$.

Mainz, January 2010 Kira Verena Samol

Contents

Chapter 1

The Weil conjectures and $p-$adic cohomology

In this chapter, we give a short review of the Weil conjectures and of the cohomology theories that were developed to give a proof of these conjectures. We give a very brief sketch of the properties of $\ell-$adic, crystalline and rigid cohomology.

1.1 The zeta function and the Weil conjectures

Let p be a prime and let $k = \mathbb{F}_q$ where $q = p^a$. Let X/k be a smooth, projective, geometrically connected scheme. For each finite algebraic extension k_n of k of degree n, one wants to know the number $\#X(k_n)$ of k_n-rational points of X. The values of the numbers $\#X(k_n)$ are summarized in the *Zeta function* of X/k, which is given by

$$Z(X/k, T) := \exp\left(\sum_{n=1}^{\infty} \#X(k_n)\frac{T^n}{n}\right).$$

For X/k a smooth projective curve of genus g, Weil [**57**] proved that the Zeta function $Z(X/k, T)$ satisfies the following properties:

1. $Z(X/k, T)$ is a rational function in T with integral coefficients.

2. The functional equation $Z(X/k, 1/qT) = q^{1-g}T^{2-2g}Z(X/k, T)$ holds.

3. The only poles of $Z(X/k, T)$ are 1 and $1/q$; both with multiplicity 1.

4. The complex roots of $Z(X/k, T)$ satisfy $|T| = \sqrt{q}$.

This generalizes to the following statements for smooth projective varieties X/k of dimension d, called the *Weil conjectures:*

1. *Rationality:* $Z(X/k,T)$ is a rational function.

2. *Functional equation:* $Z(X/k,1/(q^dT)) = \pm q^{dE/2}T^E Z(X/k,T)$, where E is the self-intersection number of the diagonal Δ of $X\times X$.

3. *Riemann hypothesis:* One can write

$$Z(X/k,T) = \frac{P_1(T)...P_{2d-1}(T)}{P_0(T)...P_{2d}(T)},$$

where $P_0(T) = 1-T, P_{2d}(T) = (1-q^dT)$ and for each $1 \leq i \leq 2d-1$, $P_i(T)$ is a polynomial with integer coefficients which can be written as

$$P_i(T) = \prod_{j=1}^{b_i}(1-a_{ij}T), \text{ with } |a_{ij}| = q^{i/2}.$$

4. *Betti numbers:* If X is a reduction modulo p of a non-singular variety Y defined over a number field embedded in $\mathbb{C}$, then the degree of $P_i(T)$ is the ith Betti number of the space of complex points of Y.

For an arbitrary smooth projective variety X, Dwork [**26**] proved that the Zeta function $Z(X/k,T)$ is a rational function in $\mathbb{Q}(T)$ by applying "elementary” methods from $p-$adic analysis.

For a smooth projective variety X/k, Weil already remarked in [**57**] that $Z(X/k,T)$ is a rational function if one assumes that there exists a cohomology theory for varieties over a finite field, taking values in finite dimensional vector spaces over a field of characteristic zero, in which one has a Lefschetz fixed point formula.

The necessary conditions for such a cohomology theory were formalised under the name *Weil cohomology*, see [**44**]. The construction of such a cohomology theory was one incentive for Grothendieck to develope the theory of schemes, and in particular to study their étale topology. For each prime number $\ell \neq p$, the $\ell-$adic cohomology developed by Grothendieck and his students is a Weil cohomology.

1.2 Weil conjectures and ℓ- adic cohomology

Let $\bar{k}$ be an algebraic closure of k. For a smooth projective geometrically connected scheme X/k of dimension d, let F_q denote the geometric Frobenius. If $\bar{X} = X\times_k\bar{k}$, for $0 \leq i \leq 2d$, one can define the polynomials

$$P_i := \det(1-F_qT|H^i_{et}(\bar{X},\mathbb{Q}_\ell)) \in \mathbb{Q}_\ell[T]. \quad (1.1)$$

For example, $P_0(T) = 1 - T$ and $P_{2d}(T) = 1 - q^d T$. Grothendieck proved the Lefschetz trace formula for étale cohomology in [**35**], which implies that

$$Z(X/k, T) = \frac{P_1(T)...P_{2d-1}(T)}{P_0(T)...P_{2d}(T)},$$

and thus the rationality of the Zeta function. The *functional equation* is a consequence of the Poincaré duality. Deligne [**18**] proved that the polynomials $P_i(T)$ are polynomials with coefficients in $\mathbb{Z}$, and thus independent of $\ell \neq p$, the *Riemann hypothesis* and furthermore that the a_{ij} are $\ell-$adic units for $\ell \neq p$.
In the $\ell-$adic setting, the following natural question remains open: *What are the $p-$adic valuations of the a_{ij}?*

1.3 Crystalline cohomology

The following idea is due to Grothendieck. Assume that k is a finite field of characteristic $p > 0$, and write $W = W(k)$ for the ring of Witt vectors of k. Suppose that X/k lifts to a smooth proper scheme Z/W, which means that there exists a smooth proper scheme Z/W such that

$$X = Z \times_{Spec(W)} Spec(k).$$

Now one can define the de Rham complex of Z/W and take its hypercohomology:

$$H^i_{DR}(Z/W) := \mathbb{H}^i(Z, \Omega^\bullet_{Z/W}).$$

Grothendieck [**34**] conjectured that these cohomology groups are independent of the choice of the lifting Z/W of X/k.

Let $W_n := W/p^n W$. The ith crystalline cohomology of a scheme X/k is defined to be the inverse limit

$$H^i_{cris}(X/W) := \lim_{\leftarrow} H^i(X/W_n),$$

where

$$H^i(X/W_n) := H^i((X/W_n)_{cris}, \mathcal{O}_{X/W_n})$$

is the cohomology of the crystalline site (see [**9**], [**36**]) of X/W_n with values in the sheaf of rings $\mathcal{O}_{X/W_n}$.

If X/k is the reduction of a smooth scheme Z/W_n, then we have a canonical isomorphism

$$H^i_{cris}(X/W_n) \cong H^i_{DR}(Z/W_n).$$

This is a corollary of Berthelot's theorem [[**9**], chapter V, theorem 2.3.2.]. By passing to the limit, one obtains for a scheme X/k, which is the reduction of a smooth proper scheme Z/W, a canonical isomorphism

$$H^i_{cris}(X/W) \cong H^i_{DR}(Z/W).$$

1.4 Crystalline cohomology and Frobenius

Assume that $k = \mathbb{F}_q$ where $q = p^a$ for a prime p, let $W = W(k)$ and K be the field of fractions of W. Let X/k be smooth and proper, and let σ denote the Frobenius endomorphism on k, lifting to an endomorphism σ on W.

The *absolute Frobenius* endomorphism $F : X \to X$ (which is the identity on the topological space, and the pth power map on the structure sheaf $\mathcal{O}_X$) induces a $\sigma-$linear automorphism of $H^*_{cris}(X/W) \otimes_W K$, also denoted by F. The map $F_q := F^a$ is then an automorphism of $H^*_{cris}(X/W) \otimes_W K$.
If X/k is furthermore of pure dimension d, then the $H^i_{cris}(X/W) \otimes_W K$ are finite dimensional $K-$vector spaces and zero for $i \notin [0, 2d]$.

Furthermore, $X \mapsto H^i_{cris}(X/W) \otimes_W K$ is a Weil cohomology with a Poincaré duality between H^i and H^{2d-i}, a Künneth formula and a Lefschetz trace formula.

By means of crystalline cohomology, Berthelot proved that the Zeta function $Z(X/k, T)$ is a rational function given by the formula

$$Z(X/k, T) = \prod_{i=0}^{2d} \det(1 - F_q T | H^i_{cris}(X/W) \otimes_W K)^{(-1)^{i+1}}.$$

A priori, one has $P_n(T) = \det(1 - F_q T | H^n_{cris}(X/W) \otimes_W K) \in K[T]$, but for X/k smooth and projective, the results of Deligne [**18**] allowed Katz and Messing [**42**] to prove that $P_n(T)$ is identical with the polynomial obtained by $\ell-$adic cohomology ($\ell \neq p$).

In particular, the crystalline and $\ell-$adic Betti numbers coincide:

$$b_i = \dim_{\mathbb{Q}_l} H^i_{et}(X \otimes \bar{k}, \mathbb{Q}_l) = \dim_K H^i_{cris}(X/W) \otimes_W K.$$

1.5 Rigid cohomology

We refer the reader to [**32**] for a very short introduction to rigid cohomology. For a proper introduction, see [**10**], [**11**].
The *rigid cohomology* due to Berthelot provides a Weil cohomology theory with satisfactory functorial properties for arbitrary varieties. Unlike in the case of crystalline cohomology, the varieties may also be noncomplete or singular. Just as crystalline cohomology, in many cases like smooth projective varieties, the rigid cohomology of a variety over $k = \mathbb{F}_q$ coincides with the de Rham cohomology of a lift to characteristic zero.

Let $q = p^a$ and let $K/\mathbb{Q}_p$ be a finite algebraic extension with residue field $k = \mathbb{F}_q$. For a variety X/k of dimension d, the rigid cohomology

$$H^i_{rig}(X) \text{ for } 0 \leq i \leq 2d$$

is a $K-$vector space. If X is smooth, then $H^i_{rig}(X)$ is finite dimensional for all i.

Like for singular or étale cohomology, there is also a cohomology with *compact support*, denoted by $H^i_{rig,c}(X)$, which coincides with $H^i_{rig}(X)$ if X is proper.

The $q-$power Frobenius endomorphism on X induces a $K-$linear endomorphism F on rigid cohomology. The Zeta function of X is then given by

$$Z(X/k, T) = \prod_{i=0}^{2d} \det(1 - TF|H^i_{rig,c}(X))^{(-1)^{i+1}}.$$

If X/k is smooth, let $\mathcal{X}$ denote a smooth $\mathcal{O}_K-$scheme with special fibre X. Then, the generic fibre $\mathcal{X}_K$ is a smooth $K-$variety. If there is an open immersion $\mathcal{X} \hookrightarrow \mathcal{Y}$ of $\mathcal{O}_K-$schemes such that $\mathcal{Y}$ is proper over $\mathcal{O}_K$ and the complement $\mathcal{Y}\backslash\mathcal{X}$ is a smooth relative divisor with normal crossings, then

$$H^i_{rig}(X) \cong H^i_{DR}(\mathcal{X}_K) \text{ for } 1 \leq i \leq 2d.$$

Thus, as in the case of crystalline cohomology, in the smooth case rigid cohomology can be computed in terms of the de Rham cohomology of a lift of X.

Chapter 2

F-crystals

Let $\pi : X \to S$ be a proper and smooth morphism with geometrically connected fibres defined over the ring of integers W of a finite extension K of $\mathbb{Q}_p$. If the relative de Rham cohomology groups $H^i_{DR}(X/S)$ are locally free $\mathcal{O}_S$-modules, they have the structure of an $F-$crystal. In this chapter, we provide the definitions and properties of $F-$crystals that we will need for our considerations. Most of the time, we follow the presentations of Katz [**41**] and Stienstra [**54**]. The only statement which is not mentioned there is proposition 2.4.1, we consider the second exterior product of an $F-$crystal. We will meet second exterior products of $F-$crystals when we derive formulas for the roots of the Frobenius polynomial in chapter 3.

2.1 Introduction to $F-$ Crystals

Let k be a perfect field of characteristic $p > 0$, let $W := W(k)$ be the ring of Witt vectors of k and let K be the field of fractions of W. Then W is a local ring with maximal ideal pW and residue field k, and K has characteristic 0. W is complete and separated for the $p-$adic topology, $W = \lim\limits_{\leftarrow} W/p^m W$. Let σ denote the absolute Frobenius automorphism on k, which lifts canonically to an automorphism of W, also denoted by σ.
Following [**59**], we define

Definition 2.1.1 *1. An $F-$crystal (H, F) over W is a free $W-$module H of finite rank with a $\sigma-$linear endomorphism*

$$F : H \to H$$

such that $F \otimes \mathbb{Q}_p : H \otimes \mathbb{Q}_p \to H \otimes \mathbb{Q}_p$ is an isomorphism. If F itself is an isomorphism, we call H a unit-root *$F-$crystal.*

2. *A Hodge $F-$crystal over W is an $F-$crystal H equipped with a filtration by free $W-$ submodules*

$$H = \mathrm{Fil}^0 H \supset \mathrm{Fil}^1 H \supset ... \supset \mathrm{Fil}^{N-1} H \supset \mathrm{Fil}^N H = 0$$

(called the Hodge filtration on H) which satisfies $F(\mathrm{Fil}^i H) \subset p^i H$ for all i.

Now, we come to the variation of $F-$crystals. Unlike [**41**] and [**54**], we restrict our considerations to the case of one parameter z, which will be sufficient for our purpose. Let A be the ring $W[z][g(z)^{-1}]$, where g is a polynomial in $W[z]$ not divisible by p, and let A_n be the ring $A/p^{n+1}A$. By $A_\infty := \varprojlim A/p^{n+1}A$, we denote the $p-$adic completion of A. On the characteristic p ring A_0, the Frobenius endomorphism is given by $\sigma(x) = x^p$ for all $x \in A_0$.

Notation 2.1.1 *If H is an $A_\infty-$ module, B is a $W-$module and $f : A_\infty \to B$ is a $W-$morphism, we write*

$$f^*H = B_f \otimes_{A_\infty} H,$$

where B_f is B viewed as an $B - A_\infty-$bimodule with left structure via the identity map $id : B \to B$ and right structure via $f : A_\infty \to B$.

This means that for $a \in A_\infty$ and $h \in H$, we have $1 \otimes ah = f(a) \otimes h$. The map $f^* : H \to f^*H$, given by $f^*(h) = 1 \otimes h$ is $f-$linear,

$$\begin{aligned} f^*(a_1h_1 + a_2h_2) &= 1 \otimes (a_1h_1 + a_2h_2) = 1 \otimes a_1h_1 + 1 \otimes a_2h_2 \\ &= f(a_1) \otimes h_1 + f(a_2) \otimes h_2 = f(a_1)f^*(h_1) + f(a_2)f^*(h_2). \end{aligned}$$

Definition 2.1.2 *Let H be a finitely generated free $A_\infty-$module. A connection*

$$\nabla : H \to \Omega_{A_\infty/W} \otimes_{A_\infty} H$$

is called $p-$adically topologically nilpotent if

$$\lim_{m\to\infty} \nabla(\frac{d}{dz})^m = 0$$

in the $p-$adic topology on $End_W(H)$.

Now let H be a finitely generated $A_\infty-$module with a $p-$adically topologically nilpotent connection, and let f and g be two $W-$homomorphisms $f, g : A_\infty \to B$ which are congruent modulo a divided power ideal of A_∞ (for example the ideal (p)). Then the connection provides an isomorphism (see [**19**] for a proof)

$$\chi(f, g) : f^*H \to g^*H,$$

given by

$$\chi(f, g)f^*(h) = \sum_{m\geq 0} \frac{(f(z) - g(z))^m}{m!} g^*(\nabla(\frac{d}{dz})^m h).$$

There are many different ring endomorphisms ϕ of A_∞ that restrict to σ on W and reduce to σ on A_0 modulo p. Let $\phi : A_\infty \to A_\infty$ be such a *lift of Frobenius*.

Definition 2.1.3 *An $F-$crystal $(H, \nabla, F(\phi))$ over A_∞ is a finitely generated free $A_\infty-$module H with an integrable and $p-$adically nilpotent connection*

$$\nabla : H \to \Omega_{A_\infty/W(k)} \otimes_A H$$

such that for every lift $\phi : A_\infty \to A_\infty$ of Frobenius, there exists a homomorphism of $A_\infty-$ modules

$$F(\phi) : \phi^* H \to H$$

such that the square

$$\begin{array}{ccc} H & \xrightarrow{\nabla} & \Omega^1_{A_\infty/W(k)} \otimes H \\ \downarrow{\scriptstyle F(\phi)\phi^*} & & \downarrow{\scriptstyle \phi\otimes F(\phi)\phi^*} \\ H & \xrightarrow{\nabla} & \Omega^1_{A_\infty/W(k)} \otimes H \end{array}$$

is commutative, and such that for every pair $\phi, \psi : A_\infty \to A_\infty$ of lifts of Frobenius, we have

$$F(\psi) \circ \chi(\phi, \psi) = F(\phi).$$

Moreover, we claim that $F(\phi) \otimes \mathbb{Q}_p : \phi^* H \otimes \mathbb{Q}_p \to H \otimes \mathbb{Q}_p$ is an isomorphism. If $F(\phi)$ itself is an isomorphism, we call H a unit-root crystal.

Often, especially in the next chapter, to simplify the notation, we set $F := F(\phi)\phi^*$. Let $S := Spec(A)$ and $S_0 = Spec(A_0)$. By S_∞, we denote $Spec(A_\infty)$, the $p-$adic completion of S.

Let k' be a perfect field extension of k and let $e_0 : A_0 \to k'$ be the $k-$morphism given by $e_0(z) = \alpha_0$, where $\alpha_0 \in k'$. Then e_0 defines a $k'-$valued point of S_0.

Let ϕ be a lift of Frobenius, and let $\alpha_\phi \in W(k')$ be the Teichmüller lifting of α_0 corresponding to ϕ. Then the $W-$homomorphism $e_\phi : A_\infty \to W(k')$ given by $e_\phi(z) = \alpha_\phi$ defines a $W(k')-$valued point of S_∞.

Definition 2.1.4 *The $F-$crystal on $W(k')$* induced by e_ϕ *is the $F-$crystal*

$$(e_\phi^* H, e_\phi^*(F(\phi)\phi^*)),$$

where $e_\phi^*(F(\phi)\phi^*)$ is the $\sigma-$linear map induced by the commutativity of the diagram below.

$$\begin{array}{ccc} H & \xrightarrow{F(\phi)\phi^*} & H \\ \downarrow{\scriptstyle e_\phi^*} & & \downarrow{\scriptstyle e_\phi^*} \\ e_\phi^* H & \xrightarrow{e_\phi^*(F(\phi)\phi^*)} & e_\phi^* H \end{array}$$

Let ϕ and ψ be two lifts of Frobenius and let $e_0 : A_0 \to k'$ be a $k-$morphism. Then the $F-$crystals induced by e_ϕ and e_ψ are explicitly isomorphic:

$$\begin{array}{ccc} e_\phi^* H & \xrightarrow{e_\phi^*(F(\phi)\phi^*)} & e_\phi^* H \\ \cong \downarrow{\scriptstyle \chi(e_\phi, e_\psi)} & & \cong \downarrow{\scriptstyle \chi(e_\phi, e_\psi)} \\ e_\psi^* H & \xrightarrow{e_\psi^*(F(\psi)\psi^*)} & e_\psi^* H \end{array} .$$

Thus, for every $k'-$valued point of S_0 given by e_0, we obtain an $F-$crystal (e^*H, e^*F) on $W(k')$, which is independent of the chosen lift of Frobenius. We call this $F-$crystal the $F-$crystal *induced by* e_0.

2.2 F-crystals on $W[[z-\alpha]]$

This section consists of two propositions concerning $F-$crystals over $W[[z-\alpha]]$. The first proposition 2.2.1 can be found in [**41**], but we added some details to the proof. We did not find the second proposition 2.2.2 explicitly in the literature, but we needed it to understand the proof of theorem 2.3.1.

Notation 2.2.1 *By $W << z-\alpha >>$, we denote the ring of convergent divided power series over W, i.e. the ring of the formal expressions*

$$\sum_{m\geq 0} a_m \frac{z^m}{m!},$$

where $|a_m|_p \to 0$ for $m \to \infty$.

The next proposition, which is due to Katz, relates $F-$crystals over $W[[z-\alpha]]$ to $F-$crystals over $W << z-\alpha >>$.

Proposition 2.2.1 ([41], Proposition 3.1.) *Let (H, ∇, F) be an $F-$crystal over $W[[z-\alpha]]$.*

1. *The module $W << z-\alpha >> \otimes H$ admits a basis of horizontal sections.*
2. *Every horizontal section of $W << z-\alpha >> \otimes H$ fixed by F extends to a horizontal section of H (a section defined over $W[[z-\alpha]]$).*

Proof: 1. The two $W-$homomorphisms $f, g : W[[z-\alpha]] \to W << z-\alpha >>$, where f is the natural inclusion and g is the evaluation e at α, $e(z-\alpha) = 0$, followed by the inclusion of W in $W << z-\alpha >>$, are congruent modulo the ideal $(z-\alpha)$. Thus, since $W << z-\alpha >>$ is $p-$adically complete, $\chi(f,g)$ is an isomorphism between $W << z-\alpha >> \otimes H$ with the induced connection ∇ and the module $W << z-\alpha >> \otimes_W e^*H$ with connection $d/d(z-\alpha) \otimes 1$. Any basis $\{h_i\}$ of e^*H gives a horizontal (w.r.t. $d/d(z-\alpha)\otimes 1$) basis $\{1 \otimes h_i\}$ of $W << z-\alpha >> \otimes_W e^*H$. The map $\chi(f,g)$ maps these horizontal sections to horizontal sections with regard to ∇: For any $h \in H$, we have

$$\begin{aligned} \chi(g,f)(g^*(h)) &= \sum_{m\geq 0} \frac{(-1)^m (z-\alpha)^m}{m!} f^*(\nabla(d/d(z-\alpha))^m h) \\ &= \sum_{m\geq 0} \frac{(-1)^m (z-\alpha)^m}{m!} (\nabla(d/d(z-\alpha))^m h), \end{aligned}$$

where the last equality holds since f was just the natural inclusion of $W[[z-\alpha]]$ in $W << z-\alpha >>$. We want to prove that $\nabla(d/d(z-\alpha))(\chi(g,f)(g^*(h))) = 0$ for any $h \in H$. Note that for $m \geq 1$,

$$\begin{aligned} & \nabla(d/d(z-\alpha))\left(\frac{(z-\alpha)^m}{m!}\nabla(d/d(z-\alpha))^m h\right) \\ = \quad & \frac{(z-\alpha)^{m-1}}{(m-1)!}\nabla(d/d(z-\alpha))^m h + \frac{(z-\alpha)^m}{m!}\nabla(d/d(z-\alpha))^{m+1} h. \end{aligned}$$

This implies that

$$\begin{aligned} & \nabla(d/d(z-\alpha))(\chi(g,f)(g^*(h))) \\ = \quad & \nabla(d/d(z-\alpha))h \\ + \quad & \sum_{m\geq 1}(-1)^m\left(\frac{(z-\alpha)^{m-1}}{(m-1)!}\nabla(d/d(z-\alpha))^m h + \frac{(z-\alpha)^m}{m!}\nabla(d/d(z-\alpha))^{m+1} h\right) \\ = \quad & 0, \end{aligned}$$

since the sum in the middle is a telescoping sum. Thus, for any basis $\{h_i\}$ of H, the image of the basis $\{g^*h_i\} = \{1 \otimes h_i\}$ under $\chi(g,f)$ is a horizontal basis of $f^*H = W << z-\alpha >> \otimes H$.

2. Choose a basis of the free $W[[z-\alpha]]-$module H, and let A_ϕ denote the matrix of $F(\phi) : \phi^*H \to H$. Let y be a column vector with entries in $W << z-\alpha >>$ satisfying

$$A_\phi \phi(y) = y.$$

Then, for any integer $m \geq 1$, we have

$$A_\phi \phi(A_\phi)\phi^2(A_\phi)...\phi^{m-1}(A_\phi)\phi^m(y) = y.$$

$\phi^m(y)$ is congruent to $\sigma^m(y(\alpha))$ modulo $(z-\alpha)^{pm}$, and thus we get a $(z-\alpha)-$ adic limit formula for y, namely

$$y = \lim_{m\to\infty} A_\phi \phi(A_\phi)\phi^2(A_\phi)...\phi^{m-1}(A_\phi)\sigma^m(y(\alpha)),$$

which shows that y has entries in $W[[z-\alpha]]$. $\square$

Proposition 2.2.2 *Let $e : W[[z-\alpha]] \to W$ be the $W-$homomorphism given by $e(z-\alpha) = 0$. Let (H, ∇, F) be an $F-$crystal over $W[[z-\alpha]]$ and let $e^*h \in e^*H$ satisfy $e^*F(e^*h) = e^*h$ (i.e. e^*h is a fixed point of e^*F). Let $g : W \to W << z-\alpha >>$ and $f : W[[z-\alpha]] \to W << z-\alpha >>$ be the natural inclusions of W and $W[[z-\alpha]]$ in $W << z-\alpha >>$. Then the section $\chi(g \circ e, f)(g^*e^*h)$ of the $F-$crystal $(W << z-\alpha >> \otimes H, \nabla, F)$ is a fixed point of F:*

$$F(\chi(g\circ e,f)(g^*e^*h)) = \chi(g\circ e,f)(g^*e^*h).$$

Proof: By the same argument as in the proof above, $\chi(g \circ e, f)$ is an isomorphism. Since $e^*F(e^*h) = e^*(F(h))$, it follows that

$$\chi(g \circ e, f)(g^*e^*(F(h)) = \chi(g \circ e, f)(g^*e^*h)$$

Hence, it remains to prove that

$$\chi(g \circ e, f)(g^*e^*(F(h)) = F(\chi(g \circ e, f)(g^*e^*h)). \tag{2.1}$$

Iterating the equality

$$\nabla(d/d(z-\alpha)) \circ F = p(z-\alpha)^{p-1} F \circ \nabla d/d(z-\alpha)$$

leads to

$$\nabla(d/d(z-\alpha))^m \circ F = \sum_{k=1}^{m} \frac{m!}{k!}(z-\alpha)^{kp-m} \sum_{A_{m,k}} \binom{p}{a_1} \ldots \binom{p}{a_k} F \circ \nabla(d/d(z-\alpha))^m, \tag{2.2}$$

where

$$A_{m,k} = \{(a_1, \ldots, a_k), a_1 + \ldots + a_k = m, p \geq a_i \geq 1\}.$$

But since

$$\sum_{k=m}^{pm} (-1)^m \sum_{A_{m,k}} \binom{p}{a_1} \ldots \binom{p}{a_k} = \left(\sum_{l=1}^{p} \binom{p}{l} (-1)^l \right)^m = (-1)^m$$

for the whole sum it follows that

$$\begin{aligned}
\chi(g \circ e, f)(g^*e^*(F(h))) &= \sum_{m \geq 0} (-1)^m \frac{(z-\alpha)^m}{m!} \nabla(d/d(z-\alpha))^m(F(h)) \\
&= \sum_{m \geq 0} (-1)^m \frac{(z-\alpha)^{pm}}{m!} F(\nabla(d/d(z-\alpha))^m(h)) \\
&= F\left(\sum_{m \geq 0} (-1)^m \frac{(z-\alpha)^m}{m!} \nabla(d/d(z-\alpha))^m(h) \right) \\
&= F(\chi(g \circ e, f)(g^*e^*h)),
\end{aligned}$$

and the proposition follows. □

2.3 Theorem 4.1. of Katz

In this section, we repeat the proof of Katz's theorem 4.1 in [**41**]. This theorem will be the key ingredient for our following computations. Note that we added some details to the proof, especially, we applied proposition 2.2.2.

Theorem 2.3.1 *([**41**], Theorem 4.1)*
Let $\bar{k}$ be the algebraic closure of k, and let H be an $F-$crystal over A_∞.
Assume that there exists a locally free submodule Fil^1 of H such that H/Fil^1 is free of rank one and such that $F(\mathrm{Fil}^1) \subset pH$. If for every $k-$morphism $e_0 : A_0 \to \bar{k}$ with $e_0(z) = \alpha_0$ and Teichmüller lifting $e(z) = \alpha$,$\alpha \in W(\bar{k})$, e^*H contains a direct factor of rank one, transversal to $e^*\mathrm{Fil}^1$, which is fixed by the map e^*F, then:

1. *There exists a unique unit-root F-subcrystal U of rank one of H such that $H = U \oplus \mathrm{Fil}^1$ as A_∞ modules.*

Suppose that over A_∞, U is generated by u. Write $F(u) = r(z)u$ for $r(z) \in A_\infty^*$. Then we have

2. *Let $e_0 : A_0 \to k'$ be a $k-$morphism to a perfect field extension k' of k with $e_0(z) = \alpha_0$. Let α be the Teichmüller lifting of α_0 and let $e : A_\infty \to W(k')$ be given by $e(z) = \alpha$. Then there exists an element $g_\alpha \in W(k')[[z-\alpha]]$ such that $v := g_\alpha \cdot u \in W(k')[[z-\alpha]] \otimes_{A_\infty} H$ is horizontal with regard to ∇. Furthermore, there exists a constant $c_\alpha \in W(\bar{k})$ such that $c_\alpha v$ is fixed by F and the quotient $c_\alpha g_\alpha/\phi(c_\alpha g_\alpha)$ is the power series expansion of the element $r(z)$ around α.*

Proof: *1.* Assume that H is free of rank ν. We choose a basis of H which is adapted to the filtration Fil^1. Let ϕ be the lifting of Frobenius such that α is the Teichmüller lifting of α_0 with regard to ϕ. Then the matrix of $F(\phi)$, A_ϕ, with regard to this basis is of the shape

$$A_\phi = \begin{pmatrix} pA & C \\ pB & D \end{pmatrix}$$

with $A \in M_{\nu-1,\nu-1}(A_\infty)$, $B \in M_{1,\nu-1}(A_\infty)$, $C \in M_{\nu-1,1}(A_\infty)$ and $D \in A_\infty$. Since for each Teichmüller point e, e^*H contains a unit-root subcrystal of rank 1, D must be invertible in A_∞. We have to find an element u in H such that $< u >$ is transversal to Fil^1 and such that $< u >$ is stable under $F(\phi)\phi^*$. Thus, we have to find a vector $\eta \in M_{\nu-1,1}(A_\infty)$ such that

$$\begin{pmatrix} pA & C \\ pB & D \end{pmatrix} \begin{pmatrix} \phi^*(\eta) \\ 1 \end{pmatrix} = a \begin{pmatrix} \eta \\ 1 \end{pmatrix}$$

for some $a \in A_\infty$. But

$$\begin{pmatrix} pA & C \\ pB & D \end{pmatrix} \begin{pmatrix} \phi^*(\eta) \\ 1 \end{pmatrix} = \begin{pmatrix} pA\phi^*(\eta) + C \\ pB\phi^*(\eta) + D \end{pmatrix},$$

and it follows (by a comparison of the last entry) that $a = pB\phi^*(\eta) + D$ and that $pA\phi^*(\eta) + C = (pB\phi^*(\eta) + D)\eta$. Hence it follows that

$$\eta = (pA\phi^*(\eta) + C)(1 + D^{-1}pB\phi^*(\eta))^{-1}D^{-1}.$$

The map

$$\eta \mapsto (pA\phi^*(\eta) + C)(1 + D^{-1}pB\phi^*(\eta))^{-1}D^{-1} \tag{2.3}$$

is a contraction in the $p-$adic topology of A_∞, and hence it has a unique fixed point. This determines a unique generator of U satisfying the required properties, namely the vector

$$u = \begin{pmatrix} \eta \\ 1 \end{pmatrix},$$

where η is the unique fixed point of the contraction.

2. By assumption, the module e^*H contains a fixed point of e^*F. By proposition 2.2.2, this fixed point defines a unique horizontal fixed point of F in $W(k') << z-\alpha >> \otimes H$, which extends to a horizontal fixed point in $W(k')[[z-\alpha]] \otimes H$ by proposition 2.2.1. This fixed point spans a direct factor of $W(k')[[z-\alpha]] \otimes H$, which is transversal to $Fil^1(W(k')[[z-\alpha]] \otimes H)$. Assume that a horizontal section is given by the column vector

$$\begin{pmatrix} S_\alpha \\ g_\alpha \end{pmatrix}$$

with $S_\alpha \in M_{\nu-1,1}(W(k')[[z-\alpha]])$ and $g_\alpha \in W(k')[[z-\alpha]]$. Then, the horizontal fixed point is a multiple of this column vector by a constant c_α,

$$c_\alpha \begin{pmatrix} S_\alpha \\ g_\alpha \end{pmatrix}.$$

By transversality, g_α is invertible in $W(k')[[z-\alpha]]$. Then,

$$\begin{pmatrix} pA_\alpha & C_\alpha \\ pB_\alpha & D_\alpha \end{pmatrix} \begin{pmatrix} \phi^*(c_\alpha S_\alpha) \\ \phi(c_\alpha g_\alpha) \end{pmatrix} = \begin{pmatrix} c_\alpha S_\alpha \\ c_\alpha g_\alpha \end{pmatrix},$$

which implies

$$\begin{pmatrix} pA_\alpha & C_\alpha \\ pB_\alpha & D_\alpha \end{pmatrix} \begin{pmatrix} \phi^*(S_\alpha f_\alpha^{-1}) \\ 1 \end{pmatrix} = \frac{c_\alpha g_\alpha}{\phi(c_\alpha g_\alpha)} \begin{pmatrix} S_\alpha g_\alpha^{-1} \\ 1 \end{pmatrix}. \tag{2.4}$$

Write $\eta_\alpha = S_\alpha g_\alpha^{-1}$. Then,

$$pA_\alpha \phi^*(\eta_\alpha) + C_\alpha = \eta_\alpha \frac{c_\alpha g_\alpha}{\phi(c_\alpha g_\alpha)} \text{ and } pB_\alpha \phi^*(\eta_\alpha) + D_\alpha = \frac{c_\alpha g_\alpha}{\phi(c_\alpha g_\alpha)},$$

which implies that η_α satisfies

$$\eta_\alpha = (pA_\alpha \phi^*(\eta_\alpha) + C_\alpha)(1 + D_\alpha^{-1} pB_\alpha \phi^*(\eta_\alpha))^{-1} D_\alpha^{-1}.$$

Since the endomorphism of $M_{\nu-1,1}(W(k')[[z-\alpha]])$ given by 2.3 is still a contraction in the $p-$adic topology, it follows that η_α is its unique fixed point and is hence a power series expansion of the global fixed point η around $z = \alpha$. Since $pB_\alpha \phi^*(\eta_\alpha) + D_\alpha \in W(k')[[z-\alpha]]$ is the power series expansion of $pB\phi^*(\eta) + D$ around $z = \alpha$, it follows that $c_\alpha g_\alpha / \phi(c_\alpha g_\alpha)$ is the power series expansion around $z = \alpha$ of an element in A_∞. By equation (2.4), now it follows that $c_\alpha g_\alpha / \phi(c_\alpha g_\alpha)$ is the power series expansion of the element $r(z)$. $\square$

2.4 Divisible Hodge-F-crystals

In this section, we define the notion of a divisible Hodge $F-$crystal and show explicitly that the second exterior product of a Hodge $F-$crystal is again a Hodge $F-$crystal.

Definition 2.4.1 *A* divisible Hodge $F-$crystal H of level N *is an $F-$crystal H equipped with a filtration by free $A_\infty-$submodules*

$$H = \mathrm{Fil}^0 H \supset \mathrm{Fil}^1 H \supset ... \supset \mathrm{Fil}^{N-1} H \supset \mathrm{Fil}^N H \supset \mathrm{Fil}^{N+1} H = 0$$

(called the Hodge filtration on H) which satisfies

1. $\nabla \mathrm{Fil}^i H \subset \Omega^1_{A_\infty/W(k)} \otimes_{A_\infty} \mathrm{Fil}^{i-1} H$ *(Griffiths transversality),*
2. $F(\mathrm{Fil}^i H) \subset p^i H$ *(Divisibility).*

The following proposition will be of use for us later, when we have to consider exterior products of $F-$crystals to derive formulas for the roots of the Frobenius polynomial.

Proposition 2.4.1 *Let H be a divisible Hodge $F-$crystal where $H/\mathrm{Fil}^1 H$ is free of rank one. Then $\wedge^2 H$ is a divisible Hodge $F-$crystal, with homomorphism of $A_\infty-$modules*

$$\frac{1}{p} \wedge^2 F : \wedge^2 H \to \wedge^2 H$$

and with Hodge filtration given by

$$\mathrm{Fil}^{i-1}(\wedge^2 H) = \sum_{k=0}^{i} \mathrm{Fil}^k H \wedge \mathrm{Fil}^{i-k} H$$

for $i \geq 1$.

Proof: Since $H/\mathrm{Fil}^1 H$ is of rank one, $\mathrm{Fil}^0 \wedge \mathrm{Fil}^0 = \mathrm{Fil}^0 \wedge \mathrm{Fil}^1$.
Let $a \in \mathrm{Fil}^k H$ and $b \in \mathrm{Fil}^{i-k} H$.Then, $a \wedge b \in \mathrm{Fil}^{i-1}(\wedge^2 H)$ and

$$\frac{1}{p} \wedge^2 F(a \wedge b) = \frac{1}{p} Fa \wedge Fb \in \frac{1}{p} p^k H \wedge p^{i-k} H = p^{i-1} \wedge^2 H.$$

For $i \geq 2$,

$$\nabla(a \wedge b) = \nabla(a) \wedge b + a \wedge \nabla(b) \in \Omega_{A_\infty/W(k)} \otimes_{A_\infty} \mathrm{Fil}^{i-2}(\wedge^2 H).$$

□

If (H, ∇, F) is a Hodge-$F-$crystal, then so is $(e^* H, e^* F)$.

2.5 Ordinary CY3-crystals and general autodual crystals

Following Stienstra [**54**], we introduce the notion of ordinary CY3-crystals. Crystals of that type appear as the third relative crystalline cohomology of families of ordinary Calabi-Yau threefolds. The generalization of an ordinary CY3-crystal is an ordinary autodual crystal. For the rest of this section, assume that $k = \bar{k}$ is algebraically closed.

Definition 2.5.1 *Let H be a Hodge-$F-$crystal and let*

$$0 = \mathrm{Fil}_{-1}H \subset \mathrm{Fil}_0 H \subset ... \subset \mathrm{Fil}_N H = H$$

be the finite (it can be shown that N is the same as in definition 2.4.1) increasing filtration on H defined by

$$Fil_i H = \{x \in H, p^i x \in Im(F(\phi))\}.$$

This filtration is called the conjugate filtration*.*

The conjugate filtration satisfies the Griffiths transversality condition

$$\nabla \mathrm{Fil}_i H \subset \Omega_{A_\infty/W} \otimes \mathrm{Fil}_i H.$$

Definition 2.5.2 *A divisible Hodge $F-$crystal H of level N is called* ordinary *if the graded module with the conjugate filtration, $gr_\bullet H$, is a free $A_\infty-$module and the conjugate and the Hodge-filtration are opposite,*

$$H = \mathrm{Fil}_i H \oplus \mathrm{Fil}^{i+1} H.$$

Proposition 2.5.1 *([**19**], Prop. 1.3.2) Let H be an $F-$crystal such that $gr_\bullet H$ is a free $A_\infty-$module. Then H is ordinary iff there exists a filtration by $F-$ subcrystals*

$$0 = U_{-1} \subset U_0 \subset ... \subset U_i \subset U_{i+1} \subset ...$$

such that

$$U_i/U_{i-1} \cong V_i(-i),$$

where V_i is a unit-root $F-$crystal and $(-i)$ is the Tate twist, meaning that $V_i(-i)$ is the same module with connection as V_i, but for every lift ϕ of the Frobenius, the map $F(\phi)\phi^*$ on $V_i(-i)$ is $p^i F(\phi)\phi^*$ on V_i.

Definition 2.5.3 *An* (ordinary) CY3-crystal *over A_∞ is a divisible (ordinary) Hodge-$F-$ crystal H of level 3 with a non-degenerate alternating bilinear form*

$$<,> H \times H \to A_\infty$$

such that for all $x, y \in H$ and lifts of Frobenius ϕ, we have

$$< \nabla(d/dz)x, y > + < x, \nabla(d/dz)y >= d/dz < x, y >,$$

$$< F(\phi)\phi^*(x), F(\phi)\phi^*(y) >= p^3\phi(< x, y >)$$

and the Riemann bilinear relations

$$(\mathrm{Fil}^3 H)^\perp = \mathrm{Fil}^1 H, \ (\mathrm{Fil}^2 H)^\perp = \mathrm{Fil}^2 H,$$

where $\perp$ is with regard to the bilinear form $<,>$.

If k_0 is a finite field with p^r elements, then for a CY3-crystal H which is deduced by extension of scalars from an $F-$crystal over $W(k_0)$, k_0 a perfect subfield of k, the eigenvalues of F^r on the crystal over $W(k_0)$ are distributed symmetrically. This means that there are as many eigenvalues with $p-$adic valuation 0 as eigenvalues with $p-$adic valuation 3, and as many eigenvalues with valuation 1 as eigenvalues with valuation 2 (see [**41**], section 5).
For an ordinary CY3-crystal of rank 4, on the crystal over $W(k_0)$ it follows that there is exactly one eigenvalue of F^r with $p-$adic valuation $0, 1, 2$ and 3.

The generalization of an (ordinary) CY3-crystal is an (ordinary) autodual crystal of weight N, see [**41**], section 5:

Definition 2.5.4 *An* (ordinary) autodual crystal of weight N *over* A_∞ *is a divisible (ordinary) Hodge-$F-$ crystal H of level N with a non-degenerate alternating bilinear form*

$$<,> H \times H \to A_\infty$$

such that for all $x, y \in H$ and lifts of Frobenius ϕ, we have

$$< \nabla(d/dz)x, y > + < x, \nabla(d/dz)y >= d/dz < x, y >,$$

$$< F(\phi)\phi^*(x), F(\phi)\phi^*(y) >= p^N\phi(< x, y >)$$

and the Riemann bilinear relations

$$(\mathrm{Fil}^{N+1-i} H)^\perp = \mathrm{Fil}^i H,$$

where $\perp$ is with regard to the bilinear form $<,>$.

As in the case of CY3-crystals above, the eigenvalues of F^r over $W(k_0)$ are distributed symmetrically in case of an autudoal crystal which is deduced from an $F-$crystal over $W(k_0)$ by extension of scalars.

Chapter 3

CY3-crystals and unit roots

In this chapter, we define the notion of a Calabi-Yau differential operator and relate these operators to CY3-crystals and general autodual crystals. We give a formula to determine the Frobenius-polynomial on a CY3-crystal of rank four with connection defined by a CY(4)-operator P, given that the function $f_0(z)/f_0(z^p)$ has an analytic continuation to the boundary of the $p-$adic unit disc, where f_0 is a power series solution of $Pf = 0$ around $z = 0$.

3.1 Picard-Fuchs operators of CY-type

Consider integrals of algebraically defined differential forms over certain chains in algebraic varieties. If the differential forms and chains depend on parameters, then the integrals can be considered as functions in these parameters, satisfying linear differential equations with algebraic coefficients. These differential equations are called *Picard-Fuchs equations*. It is not known in general how to determine whether a linear differential equation is a Picard-Fuchs equation or not, although there exist several conjectures about it (see [**61**]).

In this section, we review some facts about linear differential operators with maximal unipotent monodromy at $z = 0$ and define the notion of Calabi-Yau differential operators. Such operators arise as Picard-Fuchs operators of families of Calabi-Yau varieties. Our definition of CY-operators will be purely algebraic; for a general Calabi-Yau operator, we do not know that it is of geometric origin.
Let

$$P = \frac{d}{dz}^n + a_{n-1}(z)\frac{d}{dz}^{n-1} + \ldots + a_0(z) \in \mathbb{Q}(z)[\frac{d}{dz}] \tag{3.1}$$

be a linear differential operator of order n, and let $\theta := z\frac{d}{dz}$ be the logarithmic derivative. After left multiplication by z^n and then by the least common multiple of the denominators of $z^n a_{n-1}(z), ..., z^n a_0(z)$, we can rewrite P in terms of θ with polynomial coefficients and obtain

$$P = A_n(z)\theta^n + A_{n-1}(z)\theta^{n-1} + ... + A_1(z)\theta + A_0(z) \in \mathbb{Q}[z][\theta]. \tag{3.2}$$

From now on, we assume that P is regular singular at $z = 0$ and that $A_n(0) \neq 0$. We say that P has *maximal unipotent monodromy at* $z = 0$ if $A_i(0) = 0$ for $0 \leq i \leq n-1$.
We say that a differential operator P as in (3.1) has maximal unipotent monodromy at $z = 0$ (P is MUM) if the operator transformed in shape (3.2) has maximal unipotent monodromy.

Theorem 3.1.1 ([7], Theorem 4.2.2) *If P is MUM, then the subspace in $\mathbb{Q}[[z]]$ of solutions of the linear differential equation*

$$P\Phi(z) = 0$$

has dimension 1*. Moreover, every solution is defined uniquely by the value* $\Phi(0)$*.*

Definition 3.1.1 *Let P be a linear differential operator as in (3.1). Then the* formal adjoint *of P, P^*, is given by*

$$P^*\Phi(z) = \sum_{k=0}^{n} (-1)^k \frac{d}{dz}^k (a_k(z)\Phi(z)).$$

Definition 3.1.2 *We call a linear differential operator P of order n as in (3.1) a* CY(n)-operator *if*

1. *P has maximal unipotent monodromy at* 0 *(MUM).*
2. *P is self-dual in the sense that*

$$P = (-1)^n \exp(-\frac{2}{n}\int a_{n-1}(z)dz) \circ P^* \circ \exp(\frac{2}{n}\int a_{n-1}(z)dz),$$

   where "$\circ$" means the composition of differential operators.

3. *The power series solution $\Phi_0(z)$ to the differential equation*

$$P\Phi(z) = 0$$

   with $\Phi_0(0) = 1$ satisfies $\Phi_0(z) \in \mathbb{Z}[[z]]$.

The first condition in definition 3.1.2 (MUM) implies that the operator P is irreducible and can (after writing P as in (3.2)) be written in the form

$$\theta^n + zP_1(\theta) + z^2 P_2(\theta) + \ldots + z^d P_d(\theta),$$

for some positive integer d, where $P_i(\theta) \in \mathbb{Q}[\theta]$ is a polynomial in θ of degree $\leq n$.
The second condition in definition 3.1.2 is equivalent to the condition that the transformed operator

$$\tilde{P} = \exp(\frac{1}{n}\int a_{n-1}(z)dz) \circ P \circ \exp(-\frac{1}{n}\int a_{n-1}(z)dz)$$

satisfies

$$\tilde{P} = (-1)^n \tilde{P}^*$$

which translates into $\lfloor (n-1)/2 \rfloor$ differential-polynomial conditions on the coefficients a_i. For $n = 4$ one finds the condition of [**4**]:

$$a_1 = \frac{1}{2}a_2a_3 - \frac{1}{8}a_3^3 + a_2' - \frac{3}{4}a_3a_3 - \frac{1}{2}a_3''. \tag{3.3}$$

In [**2**] one finds a list with more than 350 examples of such CY(4)-operators. Note that these operators satisfy additional integrality properties, namely that the genus zero instanton numbers are integral. Because of the MUM-condition, by Theorem 3.1.1 the solution $\Phi_0(x)$ from the third condition in definition 3.1.2 is unique and conversely determines the operator P.

3.2 CY- differential equations and ordinary autodual crystals

In this section, we repeat some results by M. Bogner [**12**] and J.-D. Yu [**60**] concerning differential modules defined by CY-differential operators.
Let p be a prime, let $q = p^a$ and let k be the field with q elements. Let $W := W(k)$ denote the ring of Witt vectors of k, and let K be the fraction field of W.

Let P be a CY(n)-differential operator, and let M_P be the differential module defined by

$$M_P := K(z)[\theta]/K(z)[\theta]P$$

with generator ω defined by

$$\begin{array}{rcl} K(z) & \rightarrow & M_P \\ 1 & \mapsto & \omega, \end{array}$$

where the map is the natural projection. M_P is then a free $K(z)-$module of rank n with cyclic vector ω and a basis given by $\{\omega, \theta\omega, \theta^2\omega, ..., \theta^{n-1}\omega\}$. We define a filtration on M_p by setting

$$\text{Fil}^i M_P =< \{\theta^j\omega\}_{j=0}^{n-1-i} >_{K(z)},$$

the $K(z)-$ module spanned by $\{\omega, ..., \theta^{n-1-i}\omega\}$. Let

$$Y_n = \exp\left(\frac{2}{n}\int a_{n-1}dz/z\right). \tag{3.4}$$

Then Bogner [**12**] (or Yu [**60**], Theorem 1.2.) prove the following

Theorem 3.2.1 *If $Y_n \in K(z)$, there exists a non-degenerate alternating bilinear form*

$$<,>: M_P \times M_P \rightarrow K(z),$$

uniquely determined by setting $< \omega, \theta^{n-1}\omega >= cY_n$ for some constant $c \in K$, satisfying

$$< \theta(x), y > + < x, \theta(y) >= \theta(< x, y >) \text{ for } x, y \in M_P \tag{3.5}$$

and

$$< \mathrm{Fil}^{n-i} M_P, \mathrm{Fil}^i M_P >= 0. \tag{3.6}$$

Following [**60**], we define

$$\Gamma_P := K[[z]][\theta]/K[[z]][\theta]P,$$

which is a $K[[z]]-$lattice in $M_P \otimes_{K[T]} K[[T]]$. By $\mathrm{Fil}^i\Gamma_P$, we denote the filtration induced by the filtration $\mathrm{Fil}^i M_p$ on M_P. Then, we have the following theorem:

Theorem 3.2.2 *([**60**], Theorem 1.4.) There exists a unique increasing filtration*

$$0 = U_{-1} \subset U_0 \subset ... \subset U_{n-1} = \Gamma_P$$

of $K[[z]][\theta]-$ submodules of Γ_P such that, for $-1 \leq i \leq n-1$,

$$\Gamma_P = U_i + \mathrm{Fil}^{i+1}\Gamma_P$$

and the U_i/U_{i-1} are trivial $K[[z]][\theta]-$modules.

Now suppose that there exists a constant $c \in K^\times$ such that $cY_n \in W[[z]]$. We define a connection ∇ on Γ_P by setting

$$\nabla(\theta)(\omega) = \theta(\omega).$$

To give Γ_P the structure of an ordinary autodual F-crystal (Γ_P, ∇, F) of weight $n-1$ over $W[[z]]$, we assume that for every lifting of Frobenius ϕ, we have a horizontal map $F(\phi)\phi^*$ giving a conjugate Filtration Fil_i as in definition 2.5.1 which, by the theorems 3.2.2 and 2.5.1, is opposite to the Hodge filtration,

$$\Gamma_P = \mathrm{Fil}_i\Gamma_P \oplus \mathrm{Fil}^{i+1}\Gamma_P \tag{3.7}$$

satisfying

$$F(\phi)\phi^*\mathrm{Fil}^i\Gamma_P \subset p^i\Gamma_P \tag{3.8}$$

and

$$< F(\phi)\phi^*(x), F(\phi)\phi^*(y) >= p^{n-1}\phi(< x, y >) \tag{3.9}$$

for all $x, y \in \Gamma_P$. Thus, by theorem 3.2.1 it follows that (Γ_P, ∇, F) is an ordinary autodual $F-$crystal over $W[[z]]$.

3.3 Horizontal sections for CY differential operators

In this section, we give a formula for horizontal sections in the differential module Γ_P, where P is a CY(4)- or CY(5)- differential operator.
Let P be a CY(4)-operator. The differential equation $Pf = 0$ can be written in the form

$$f^{(4)} + a_3 f^{(3)} + a_2 f^{(2)} + a_1 f^{(1)} + a_0 f = 0,$$

where the coefficients a_i satisfy the following relation:

$$a_1 = \frac{1}{2}a_2a_3 - \frac{1}{8}a_3^3 + a_2' - \frac{3}{4}a_3a_3' - \frac{1}{2}a_3''. \tag{3.10}$$

Proposition 3.3.1 *(see* [**59**]*) Let P be a CY(4) differential operator and let (H, ∇) be a $K(z)/K$ differential module. Let $\omega \in H$ such that*

$$\nabla^4\omega + a_3\nabla^3\omega + a_2\nabla^2\omega + a_1\nabla\omega + a_0\omega = 0$$

(where $\nabla(\omega) := \nabla(d/dz)(\omega)$) and let $f_0 \in K[[z]]$ be a formal solution to the differential equation $Pf = 0$ around $z = 0$. If $Y_4 = \exp\left(1/2\int a_3\right) \in K[[z]]$, then the following element $u_4 \in H \otimes_{K[z]} K[[z]]$ is horizontal with regard to ∇ :

$$\begin{aligned} u_4 &= Y_4[f_0\nabla^3(\omega) - f_0'\nabla^2(\omega) + f_0''\nabla(\omega) - f_0'''\omega] + (Y_4a_3 - Y_4')[f_0\nabla^2(\omega) - f_0''\omega] \\ &+ (Y_4a_2 - (Y_4a_3)' + Y'')[f_0\nabla(\omega) - f_0'\omega]. \end{aligned} \tag{3.11}$$

Proof: See [**59**]. The proof is by direct computation, using (3.10).

Now let Q be a CY(5)-operator. The differential equation $Qf = 0$ can be written in the form

$$f^{(5)} + b_4f^{(4)} + b_3f^{(3)} + b_2f^{(2)} + b_1f^{(1)} + b_0f = 0.$$

Proposition 3.3.2 *The operator Q satisfies the second condition for CY(5) of the introduction, if and only if the coefficients $b_i(z)$ satisfy the relations*

$$b_2 = \frac{3}{5}b_3b_4 - \frac{4}{25}b_4^3 + \frac{3}{2}b_3' - \frac{6}{5}b_4b_4' - b_4'' \tag{3.12}$$

and

$$\begin{aligned} b_0 &= \frac{1}{2}b_1' - \frac{2}{125}b_3b_4^3 + \frac{1}{5}b_1b_4 - \frac{1}{10}b_3b_4'' + \frac{2}{5}b_4'''b_4 + \frac{4}{5}b_4''b_4' + \frac{16}{125}b_4'b_4^3 \\ &+ \frac{12}{25}(b_4')^2b_4 - \frac{3}{10}b_3''b_4 + \frac{8}{25}b_4^2b_4'' - \frac{3}{10}b_3'b_4' - \frac{3}{25}b_4^2b_3' - \frac{1}{4}b_3''' + \frac{16}{3125}b_4^5 \\ &+ \frac{1}{5}b_4'''' - \frac{3}{25}b_3b_4'b_4. \end{aligned} \tag{3.13}$$

Proof: By direct calculation, for details we refer to [**12**].

Proposition 3.3.3 *Let Q be a CY(5) differential operator and let (H, ∇) be a $K(z)/K$ differential module. Let $\eta \in H$ such that*

$$\nabla^5\eta + b_4\nabla^4\eta + b_3\nabla^3\eta + b_2\nabla^2\eta + b_1\nabla\eta + b_0\eta = 0$$

(where $\nabla(\eta) := \nabla(d/dz)(\eta)$) and let $f_\alpha \in K[[z]]$ be a formal solution to the differential equation $Qf = 0$ around $z = 0$. If $Y_5 = \exp\left(2/5\int b_4\right) \in K[[z]]$, then the following

element $u_5 \in H \otimes_{K[z]} K[[z]]$ *is horizontal with regard to* ∇*:*

$$\begin{aligned} u_5 &= Y_5[f_0\nabla^4(\eta) - f_0'\nabla^3(\eta) + f_0''\nabla^2(\eta) - f_0'''\nabla(\eta) + f_0''''\eta] \\ &+ (Y_5b_4 - Y_5')[f_0\nabla^3(\eta) - \frac{1}{3}f_0'\nabla^2(\eta) - \frac{1}{3}f_0''\nabla(\eta) + f_0'''\eta] \\ &+ (Y_5b_3 - (Y_5b_4)' + Y_5'')[f_0\nabla^2(\eta) + f_0''\eta] + (\frac{4}{3}((Y_5b_4)' - Y_5'') - \alpha b_3)f_0'\nabla(\eta) \\ &+ (\frac{1}{2}((Y_5b_3)' - \frac{4}{3}((Y_5b_4)'' - Y_5''')))[f_0'\eta + f_0\nabla(\eta)] \\ &+ (Y_5b_1 - \frac{1}{2}((Y_5b_3)' - \frac{4}{3}((Y_5b_4)' - Y_5'''')))f_0\eta, \end{aligned} \tag{3.14}$$

Proof: Applying the identities (3.12) and (3.14), one directly verifies that u_5 satisfies $\nabla(u_5) = 0$.

3.4 Calabi-Yau varieties

In this section, we give a very brief introduction to Calabi-Yau varieties. Especially, we are interested in Calabi-Yau threefolds, since families of these correspond to CY(4)-operators, the main subjects of our studies.

Definition 3.4.1 *A* Calabi-Yau variety *is a smooth complex projective variety of dimension* m *satisfying*

1. $H^i(X, \mathcal{O}_X) = 0$ *for every* $0 < i < m$ *and*
2. *The canonical bundle* $K_X := \Omega_X^n$ *of* X *is trivial,* $K_X \cong \mathcal{O}_X$.

Let $H^q(\Omega_X^p)$ be the (p, q)th *Hodge cohomology group* of X with *Hodge number* $h^{p,q}(X) := \dim_{\mathbb{C}} H^q(\Omega_X^p)$. By complex conjugation, we have $H^q(\Omega_X^p) = H^p(\Omega_X^q)$, and by Serre duality, it follows that $H^q(\Omega_X^p) = H^{m-q}(\Omega_X^{m-p})$.
This implies directly that there is a symmetry in the Hodge-numbers:

$$h^{p,q}(X) = h^{q,p}(X) \text{ and } h^{p,q}(X) = h^{m-p,m-q}(X).$$

The number $h^k(X) := \dim_{\mathbb{C}} H^k(X, \mathbb{C})$ is called the k*th Betti number* of X. By the Hodge decomposition of $H^k(X, \mathbb{C})$,

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^q(\Omega_X^p),$$

it follows that

$$h^k(X) = \sum_{p+q=k} h^{p,q}(X) = \sum_{i=0}^{k} h^{i,k-i}(X).$$

The Calabi-Yau conditions assert that $h^{i,0}(X) = 0$ for $0 < i < m$ and $h^{0,0}(X) = h^{m,0}(X) = 1$.

The Hodge numbers can be displayed in a *Hodge diamond*. For a Calabi-Yau variety of dimension $m = 3$, the Hodge diamond looks as follows,

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1}(X) & & 0 & \\ 1 & & h^{1,2}(X) & & h^{1,2}(X) & & 1 \\ & 0 & & h^{1,1}(X) & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

since $h^{1,1}(X) = h^{2,2}(X)$ and $h^{2,1}(X) = h^{1,2}(X)$. Now let X be a Calabi-Yau variety defined over $\mathbb{Q}$. Then, X has an integral model over $\mathbb{Z}$. Let $k := \mathbb{F}_p$ where p is a prime. The reduction X_0 of X over k is a Calabi-Yau variety over k if it is a smooth variety.

3.5 CY-differential equations and families of Calabi-Yau varieties

In this section, following Yu [**60**], we explain where CY(n)-operators arise in geometry as Picard-Fuchs operators of families of Calabi-Yau varieties. For example, CY(2)-operators arise from families of elliptic curves, CY(3)-operators arise from families of $K3$ surfaces with Picard-number 19 with a point of maximal degeneration (type III in the terminology of [**31**]) and CY(4)-operators arise from families of Calabi-Yau threefolds with $h^{12} = 1$ that are studied in mirror symmetry, [**17**].

In this section, we will talk about so-called *logarithmic structures*. The basic definitions and properties of logarithmic structures can be found in [**38**]. Let $\pi : X \to \mathbb{P}^1$ be a flat projective pencil whose generic fibre is smooth, and assume that each singular fibre of π is a union of reduced normal crossing divisors. We equip X and $\mathbb{P}^1$ with the natural smooth logarithmic structures associated to the union of the singular fibres in X and the critical values in $\mathbb{P}^1$. By $\omega^i := \omega^i_{X/\mathbb{P}^1}$, we denote the sheaf of relative differential $i-$forms with logarithmic poles with respect to the logarithmic structures on X.

If the generic fibre of π is an irreducible Calabi-Yau variety of dimension $m \geq 1$, we call a pencil π as above a *nice pencil of Calabi-Yau varieties of dimension* m. Then the sheaf $\pi_*\omega^m$ is an invertible sheaf on $\mathbb{P}^1$.

Now suppose that there exists a locally direct factor M of rank $m+1$ of $R^m\pi_*\omega^\bullet$ which is stable under the Gauss-Manin connection and contains $\pi_*\omega^m$.
Let $a \in \mathbb{P}^1_{\mathbb{C}}$ be a $\mathbb{C}-$valued point and let N denote the logarithm of the local monodromy around a. Then N acts on the stalk M_a and is nilpotent. If the monodromy is maximally unipotent, $N^m \neq 0$ on M_a , then M is the unique irreducible locally direct factor of

$R^m\pi_*\omega^\bullet$ containing $\pi_*\omega^m$ which is stable under the Gauss-Manin connection. The following theorem by Yu relates nice families of Calabi-Yau varieties to CY-differential operators.

Theorem 3.5.1 *(see* [**60**], *Corollary 2.2) Let $\pi : X \to \mathbb{P}^1$ be a nice pencil of Calabi-Yau varieties of dimension m such that there exists a locally direct factor M with maximally unipotent monodromy at 0. Let η be a basis of local sections of $\pi_*\omega^m$ at 0 and let P be the Picard-Fuchs operator of η. Then P is a CY(m+1)-operator.*

Now, we consider the whole situation from a $p-$adic point of view. Let therefore p be a prime, let $q = p^a$, and assume that $p > m$. Let $k = \mathbb{F}_q$, let W the ring of Witt vectors of k and let K be the fraction field of W. Let $\pi : X \to \mathbb{P}^1$ be a nice pencil of Calabi-Yau varieties of dimension m over K with totally degenerate fibre at 0. We say that π has *nice reduction* if π has a flat model over W such that the reduction $\bar{\pi} : \bar{X} \to \mathbb{P}^1$ over k is also a nice pencil of Calabi-Yau varieties of dimension m, and the logarithmic structures of π and $\bar{\pi}$ are induced by a smooth logarithmic structure on the flat model over W.

Lemma 3.5.1 *(see* [**60**], *Lemma 3.1) Suppose that the pencil π has a nice reduction and that the factor M with maximally unipotent monodromy at 0 satisfies that M_0 is stable under the absolute Frobenius. Then the Frobenius action on M_0 is ordinary and there exists a dense open subset S in $\mathbb{P}^1$ such that for the restriction $\pi : X \to S$, the crystal M is an ordinary CY3-crystal.*

3.6 Dwork's deformation from $z = 0$

Now that we know a formula for the horizontal sections in $F-$crystals defined by CY(4) and CY(5)- differential operators, we want to derive a formula for the element $r(z)$ appearing in theorem 2.3.1 for both cases.

For the rest of this chapter, we choose the lifting of Frobenius given by $\phi(z) = z^p$. The section u_n defined in equation (3.11) if $n = 4$ and equation (3.14) if $n = 5$ and all constant multiples of this section are horizontal with regard to ∇. With regard to the basis $\{\omega, \nabla(\omega), \nabla^2(\omega), ..., \nabla^{n-1}(\omega)\}$ of Γ_P, where $\nabla(\omega) := \nabla(d/d(z-\alpha))(\omega)$, horizontal sections, written as column vectors, are constant multiples of

$$Y_n \begin{pmatrix} N_0 \\ f_0 \end{pmatrix},$$

where N_0 is a $(n-1)\times 1-$matrix with entries in $W[[z]]$. Hence it follows by theorem 2.3.1 that there exists a constant c_0 such that

$$c_0 Y_n \begin{pmatrix} N_0 \\ f_0 \end{pmatrix} = c_0 Y_n f_0 \begin{pmatrix} f_0^{-1} N_0 \\ 1 \end{pmatrix}$$

is a fixed point of F, and the element

$$\frac{c_0 f_0(z)}{c_0^\sigma f_0(z^p)}$$

is the power series expansion of $(Y_n(z^p)/Y_n(z))r(z)$ around $z = 0$.

Let $\mathcal{M}(z)$ be the connection matrix representing $\nabla(d/dz)$ with regard to our choice of basis, which is given by

$$\mathcal{M}(z) := \begin{pmatrix} 0 & \dots & 0 & -a_0 \\ 1 & & \vdots & -a_1 \\ \vdots & \ddots & 0 & \vdots \\ 0 & \dots & 1 & -a_{n-1} \end{pmatrix},$$

Consider the matrix differential equation

$$\frac{d}{dz}\mathcal{C}(z) + \mathcal{M}(z)\mathcal{C}(z) = 0$$

and let $\mathcal{C}(z)$ be the solution to this differential equation given by

$$\mathcal{C}(z) = Y_n \begin{pmatrix} \mathcal{N}_0 & -\mathcal{N}_1 & \dots & (-1)^{n-1}\mathcal{N}_{n-1} \\ y_0 & -y_1 & \dots & (-1)^{n-1}y_{n-1} \end{pmatrix},$$

where $y_0(z), ..., y_{n-1}(z)$ is a Frobenius basis of solutions to the differential equation

$$P_n y = 0$$

around $z = 0$ such that $y_0(z) = f_0(z)$. Note that the highest power of a logarithm occuring in $y_k(z)$ is $\log(z)^k$. For example, a Frobenius basis of solutions to a CY(4)-differential equation is given by

$$\begin{aligned} y_0(z) &= f_0(z), \\ y_1(z) &= \log(z)f_0(z) + f_1(z), \\ y_2(z) &= \frac{1}{2}\log^2(z)f_0(z) + \log(z)f_1(z) + f_2(z), \\ y_3(z) &= \frac{1}{6}\log^3(z)f_0(z) + \frac{1}{2}\log^2(z)f_1(z) + \log(z)f_2(z) + f_3(z), \end{aligned}$$

where $f_0(0) = 1$ and $f_i(0) = 0$ for $1 \le i \le 3$. Let $C(z)$ denote the non-logarithmic part of $\mathcal{C}(z)$, i.e. in each entry of $\mathcal{C}(z)$, we formally set "$\log(z) = 0$". Thus, for example in the case of a CY(4)-operator, the matrix $C(z)$ is given by

$$C(z) = Y_4 \begin{pmatrix} N_0 & -N_1 & N_2 & -N_3 \\ f_0 & -f_1 & f_2 & -f_3 \end{pmatrix}$$

for some $n - 1 \times 1$ matrices N_i containing no logarithmic terms.

Let N be the logarithm of the monodromy. Because of the MUM-condition, N is given by

$$N := \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ & & & 1 \\ 0 & \dots & & 0 \end{pmatrix}$$

It is our goal to determine the constant $c := c_0/c_0^\sigma$. Let G denote the Gram matrix of the pairing $<,>$ on Γ_P, which is, by theorem 3.2.1, given by

$$G = Y_n T$$

for a scalar matrix T satisfying

$$TN + N^t T = 0. \tag{3.15}$$

By equations (3.7) and (3.8), there exists a constant matrix $\mathfrak{A}_n$ satisfying

$$p\mathfrak{A}_n N = N\mathfrak{A}_n, \tag{3.16}$$

and furthermore, by equation 3.9,

$$p^{n-1}T = \mathfrak{A}_n T \mathfrak{A}_n, \tag{3.17}$$

such that as in [**27**], Lemma 6.2., the absolute Frobenius matrix A_ϕ of $F(\phi)$ with regard to our choice of basis is given by

$$A_\phi(z) = C(z)\mathfrak{A}_n C(z^p)^{-1} \tag{3.18}$$

on the open disc pW.
By conditions 3.16, 3.15 and 3.17 it follows that

$$\mathfrak{A}_n = \begin{pmatrix} \varepsilon & \alpha_1 & \dots & \alpha_{n-1} \\ 0 & \ddots & & \\ 0 & 0 & p^{n-2}\varepsilon & p^{n-2}\alpha_1 \\ 0 & 0 & 0 & p^{n-1}\varepsilon \end{pmatrix},$$

and $\varepsilon = \pm 1$.
By equation (3.18), it follows that

$$A_\phi(z)C(z^p) = C(z)\mathfrak{A}_n,$$

which implies

$$A_\phi(z) \begin{pmatrix} f_0(z)^{-1}N_0(z^q) \\ 1 \end{pmatrix} = \varepsilon \frac{Y_n(z) f_0(z)}{Y_n(z^p) f_0(z^p)} \begin{pmatrix} f_0(z^p)^{-1}N_0(z^p) \\ 1 \end{pmatrix}.$$

This determines the constant c as $c = \varepsilon$ and leads to the following proposition:

Proposition 3.6.1 *The formal power series*

$$\varepsilon \frac{f_0(z)}{f_0(z^p)}$$

is the power series expansion around $z = 0$ *of the element* $(Y_n(z^p)/Y_n(z))r(z)$*, where* $\varepsilon \in \{\pm 1\}$.

By the conditions 3.16, 3.15 and 3.17, for a CY(4)-operator, $\mathfrak{A}_4$ is given by

$$\begin{pmatrix} \varepsilon & \alpha & \beta & \gamma \\ 0 & p\varepsilon & p\alpha & p\beta \\ 0 & 0 & p^2\varepsilon & p^2\alpha \\ 0 & 0 & 0 & p^3\varepsilon \end{pmatrix},$$

where $\beta = \alpha^2/2$, and is thus determined up to two parameters $\alpha, \gamma \in W$. This indicates that we can derive a formula for the absolute Frobenius matrix A_ϕ at a Teichmüller point up to two parameters that remain undetermined. But although the Frobenius matrix itself depends on these parameters, we will see in the end of this chapter that the Frobenius polynomial $\det(1 - TA_\phi^a)$ at a Teichmüller point is *independent* of these parameters.

Theorem 3.6.1 *Let x be a Teichmüller point satisfying $x^{p^a} = x$ and P a CY(4)-operator. Then, if it can be evaluated in $z = x$, the Frobenius polynomial corresponding to P at this point, $\det(1 - TA_\phi(z)^a)|_{z=x}$, is independent of the parameters α, β and γ.*

Note that by the expression $\det(1 - TA_\phi(z)^a)|_{z=x}$, we mean that there exists an *analytic continuation* (see chapter 5) of the power series $\det(1 - TA_\phi(z)^a) \in W[T][[z]]$ to a neighborhood of x, and that we evaluate this analytic continuation there.

3.7 Example: The Legendre family of elliptic curves

In the case of a CY(2)-operator and a family of elliptic curves, the formula given in proposition 3.6.1 is almost enough to compute the Frobenius polynomial of a smooth ordinary fibre of the family explicitly. We demonstrate this in the example of the Legendre family of elliptic curves.

For a prime $p > 2$, let H be the polynomial

$$h(z) := \sum_{j=0}^{p-1} \left(\frac{\left(\frac{1}{2}\right)_j}{j!} \right)^2 z^j,$$

and let S be the $\mathbb{Z}_p-$scheme $S := Spec(\mathbb{Z}_p[z][(z(1-z)h(z))^{-1}])$. By X/S_∞, we denote the Legendre family of elliptic curves, whose affine equation is given by

$$X_z : y^2 = x(x-1)(x-z),$$

where $z \neq 0, 1$ and h is its Hasse-invariant.

The relative de Rham cohomology $H := H^1_{DR}(X/S_\infty)$ of the family is free of rank 2, and the Hodge filtration $\mathrm{Fil}^1 H$ is generated by the differential

$$\omega := \frac{dx}{y}.$$

Let ∇ be the Gauss-Manin connection on H. Let $\omega' := \nabla(d/dz)(\omega)$. Then ω satisfies the differential equation

$$z(1-z)\omega'' + (1-2z)\omega' - \frac{1}{4}\omega = 0,$$

and the cup-product is given by

$$<\omega, \omega> = <\omega', \omega'> = 0 \text{ and } <\omega, \omega'> = -<\omega', \omega> = -2/(z(1-z)).$$

For any α_0 in S_0, the curve X_{α_0} is ordinary. Let $e_0 : A_0 \mapsto k$, $e_0(z) = \alpha_0$, and let e be a Teichmüller lifting of e_0. Then the $F-$crystal $e^*H \cong H^1_{DR}(X_\alpha/W)$ is an ordinary Hodge-$F-$crystal, and thus H satisfies the conditions of theorem 2.3.1.

The Zeta function $Z(X_{\alpha_0}/\mathbb{F}_p, T)$ is of the form

$$Z(X_{\alpha_0}/\mathbb{F}_p, T) = \frac{(1-r(\alpha)T)(1-p/r(\alpha)T)}{(1-T)(1-pT)},$$

and we have to find a formula to compute $r(\alpha)$.

Let f_0 be the unique solution in $W[[z]]$ to the above differential equation with constant term 1. The horizontal sections around $z = 0$ are constant multiples of

$$Yf_0'\omega - Yf_0\nabla(\omega),$$

where $Y = z(1-z)$.
Then, by theorem 2.3.1, there exists a constant c_0 such that $\frac{c_0 Y(z) f_0(z^p)}{c_0^\sigma Y(z^p) f_0(z^p)}$ is the power series expansion of the element $r(z)$.
As in section 3.6, the constant $c = \frac{c_0}{c_0^\sigma}$ is equal to $\varepsilon = \pm 1$, and a power series expansion of $(Y(z^p)/Y(z))r(z)$ around the origin is given by

$$\varepsilon \frac{f_0(z)}{f_0(z^p)}, \text{ where } \varepsilon \in \{1, -1\}.$$

It was proven by Dwork [**27**] that there exists a function $F(z)$, analytic on

$$\mathcal{D} := \{x \in W | |H(x)|_p = 1\},$$

which coincides with $f_0(z)/f_0(z^p)$ on the open unit disc pW. Thus, a formula for $r(\alpha)$ is given by

$$r(\alpha) = \varepsilon \frac{Y(\alpha)}{Y(\alpha^p)} F(\alpha) = \varepsilon F(\alpha),$$

where the last equality holds since $Y(\alpha) = Y(\alpha^p)$. Hence, one has a formula for the unit root of the Frobenius polynomial of a fibre of the Legendre family modulo the constant ε. By a geometric argument, it turns out that $\varepsilon = (-1)^{(p-1)/2}$. The geometrical origin of ε lies in the geometry of the singular fibre X_0 which has a node with tangent cone $x^2 + y^2 = 0$, that splits over $\mathbb{F}_p$ precisely when $\varepsilon = 1$.

3.8 Frobenius polynomials for CY3-crystals of rank four

At the moment, by proposition 3.6.1, we have a formula for the function $r(z)$ from theorem 2.3.1 for ordinary CY3-crystals of rank four with Picard- Fuchs operator of CY(4)-type, and for ordinary autodual crystals of rank five with Picard-Fuchs operator of CY(5)-type. For $\alpha_0 \in k$, where k is a field with $q = p^a$ elements, the function $r(z)$, evaluated at the Teichmüller lifting α of α_0, computes the unit root

$$r_{\alpha_0} = r(\alpha)r(\alpha^p)...r(\alpha^{p^{a-1}})$$

of the corresponding crystal. But it was our goal to compute the complete Frobenius polynomial for a CY3-crystal of rank four, and not only its unit root. In this section, we will derive a formula for the reciprocal root of $p-$adic valuation 1 in terms of the unit root of the ordinary CY3-crystal with connection specified by a CY(4)-operator P, and the unit root of the ordinary autodual $F-$crystal with connection specified by a CY(5)-operator $Q = \wedge^2 P$.

Let k be the finite field with $q = p^a$ elements, let $W := W(k)$ be the ring of Witt vectors and let K be the field of fractions of W. Let P be a CY(4)-operator, and let $s(z) \in \mathbb{Z}[z]$ be the polynomial sucht that the singular points of P are the roots of $s(z)$. Let (H, ∇, F) be a corresponding $CY3-$crystal over A_∞, where $A = W[z][(s(z)h(z))^{-1}]$. Let $S_0 = Spec(A_0)$, $S_\infty := Spec(A_\infty)$ and let $\alpha_0 \in S_0$.
Let ϕ be a lifting of Frobenius and let $e : A_\infty \to W(k)$, $e(z) = \alpha$ be the Teichmüller lifting of $e_0 : A_0 \to k$, $e_0(z) = \alpha_0$. We assume that for each $\alpha_0 \in S_0$, the CY3-crystal e^*H is ordinary (this is a condition on the polynomial $h(z)$).
For the Frobenius polynomial $\det(1 - Te^*F^a|e^*H)$, this means that there exists $p-$adic units $r_{\alpha_0}, s_{\alpha_0}$ such that

$$\det(1 - Te^*F^a) = (1 - r_{\alpha_0}T)(1 - qs_{\alpha_0}T)(1 - q^2/s_{\alpha_0}T)(1 - q^3/r_{\alpha_0}T), \qquad (3.19)$$

under the further assumption that $\mathcal{P}(T) := \det(1 - Te^*F^a)$ suits the Weil conjecture *functional equation*:

$$\mathcal{P}(T) = T^4/q^6\mathcal{P}(1/(q^3T)).$$

Then there exist (p-adic) integers a_{α_0} and $b_{\alpha_0} \in \mathbb{Z}$ such that $\mathcal{P}$ is symmetric in the following way:

$$\mathcal{P}(T) = 1 + a_{\alpha_0}T + b_{\alpha_0}qT + a_{\alpha_0}q^3T^3 + q^6T^4.$$

It is our goal to derive formulas to compute the $p-$adic units r_{α_0} and s_{α_0}.

By assumption, the crystal H satisfies the conditions of theorem 2.3.1. The $p-$adic unit r_{α_0} is just the element $r(\alpha)r(\alpha^p)...r(\alpha^{p^{a-1}})$, so we have to derive a formula to evaluate the element $r(z) \in A_\infty$ at a Teichmüller point α.

Now consider the $p-$adic unit s_α. In general, if $f : V \to V$ is a homomorphism of vector spaces, then the eigenvalues of $\wedge^2 f : \wedge^2 V \to \wedge^2 V$ are given by products ab, where a and b are eigenvalues of f corresponding to linearly independent eigenvectors.

By proposition 2.4.1, the Frobenius endomorphism on each fiber $\wedge^2 e^*H$ of the crystal $\wedge^2 H$ is given by $\frac{1}{p}\wedge^2(e^*F)$.

The rank $6 = \binom{4}{2}$ $A_\infty-$ module $\wedge^2 H$ is a direct sum of an $A_\infty-$ module G of rank 5 and a rank 1 module. The rank 1 module is generated by a section that corresponds to the pairing $< -, - >$ and is horizontal with respect to ∇.
We construct a 5th order differential operator Q on the submodule G by choosing Q to be the differential operator of minimal order such that for any two linearly independent solutions $y_1(z), y_2(z)$ of the differential equation $Py = 0$,

$$w := z \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

is a solution of $Qw = 0$.

Proposition 3.8.1 *The operator Q satisfies the first and the second condition of CY(5).*

Proof: The statement that Q satisfies the first condition of CY(5) is the content of [**4**], Proposition 4. A direct computation shows that since P is a CY(4)-operator, the coefficients of Q satisfy the equations (3.14) and (3.12), so the second condition of CY(5) holds.□

In all examples it was found that the operator Q also has an integral power series solution, and thus satisfies the third condition of CY(5). For the moment, however, we are unable to prove this in general so we

Conjecture 3.8.1 *The differential operator Q, constructed from a CY(4)-operator P as above, satisfies the third condition of CY(5).*

So if conjecture 3.8.1 holds true, the differential operator Q is a CY(5)-operator.

The operator Q can be expressed in terms of $\wedge^2 P(\theta, z)$ as

$$Q(\theta, z) = \wedge^2 P(\theta - 1, z).$$

For the differential operators P and Q, we use the same notation with coefficients a_i and b_i as in section 3.3.

Proposition 3.8.2 *Let Q be the CY(5)-operator constructed above, let $\nabla := \nabla(d/dz)$ and let $\omega \in H$ such that*

$$\nabla^4\omega + a_3\nabla^3\omega + a_2\nabla^2\omega + a_1\nabla\omega + a_0\omega = 0.$$

Then, the element $\eta := z\omega \wedge \nabla\omega \in G$ satisfies

$$\nabla^5\eta + b_4\nabla^4\eta + b_3\nabla^3\eta + b_2\nabla^2\eta + b_1\nabla\eta + b_0\eta = 0.$$

Proof: The proposition follows by a straightforward calculation, applying the relations between the coefficients a_i of the CY(4)-operator P and the coefficients b_i of the CY(5)-operator Q listed in [**1**].□

Just as for the $F-$crystal (H, ∇, F), we assume for the $F-$crystal $(G, \nabla, 1/p \wedge^2 F)$ that for each $\alpha_0 \in S_0$, the $F-$crystal e^*G is an autodual ordinary $F-$crystal of weight 4 (this is a condition on the polynomial $h(z)$, too).

The eigenvalues of the relative Frobenius on $\wedge^2 e^*H$ are of the form $u_{\alpha_0}v_{\alpha_0}/q$, where u_{α_0} and v_{α_0} are eigenvalues of e^*F^a on e^*H.
Thus, if r_{α_0} is the unit root of the $F-$crystal e^*H, and r'_{α_0} is the unit root of e^*G, then the reciprocal roots of the Frobenius polynomial $\mathcal{P}(T)$ on e^*H are given by

$$r_{\alpha_0}, qr'_{\alpha_0}/r_{\alpha_0}, q^2 r_{\alpha_0}/r'_{\alpha_0}, q^3/r_{\alpha_0}, \tag{3.20}$$

and it follows that

$$s_{\alpha_0} = r'_{\alpha_0}/r_{\alpha_0}.$$

Since the $F-$crystal G satisfies the conditions of theorem 2.3.1, the element r'_{α_0} is just the element $r(\alpha)r(\alpha^p)...r(\alpha^{p^{a-1}})$ (this time w.r.t. G) evaluated at $z = \alpha$.

3.9 Formulas for r_{α_0} and r'_{α_0}

In this section, we put the results of the preceding sections together to give formulas for the $p-$adic units r_{α_0} and r'_{α_0}.

Let $f_0(z)$ be the unique power series solution to the differential equation $Pf = 0$ around $z = 0$ satisfying $f_0(0) = 1$, and let g_0 be the unique solution to $Qg = 0$ around $z = 0$ satisfying $g_0(0) = 1$. Assume that $Y_4(z)$ and $Y_5(z)$ are rational functions.

Theorem 3.9.1 *Let $\alpha_0 \in S_0$, and let α be a Teichmüller lifting of α_0. Let $pW \subset \mathfrak{D} \subset W$ be a domain containing α, and assume that there exist analytic elements $F(z)$ and $G(z)$ of support $\mathfrak{D}$ coinciding with $f_0(z)/f_0(z^p)$ and $g_0(z)/g_0(z^p)$ on pW. Then*

$$r_{\alpha_0} = \varepsilon^a F(\alpha)...F(\alpha^{p^{a-1}}) \text{ and } r'_{\alpha_0} = G(\alpha)...G(\alpha^{p^{a-1}}),$$

where $\varepsilon \in \{\pm 1\}$.

Proof: The first statement follows directly by proposition 3.6.1, since $\alpha = \alpha^{p^a}$ and thus

$$\frac{Y_4(\alpha)}{Y_4(\alpha^p)}\frac{Y_4(\alpha)^p}{Y_4(\alpha^{p^2})}\cdots\frac{Y_4(\alpha^{p^{a-1}})}{Y_4(\alpha^{p^a})} = \frac{Y_4(\alpha)}{Y_4(\alpha^{p^a})} = 1$$

and, concerning the second statement,

$$\frac{Y_5(\alpha)}{Y_5(\alpha^p)}\frac{Y_5(\alpha)^p}{Y_5(\alpha^{p^2})}\cdots\frac{Y_5(\alpha^{p^{a-1}})}{Y_5(\alpha^{p^a})} = \frac{Y_5(\alpha)}{Y_5(\alpha^{p^a})} = 1.$$

It only remains to be proven that $r'_{\alpha_0} = G(\alpha)$ is independent of ε. On the rank four $F-$crystal H, the matrix $\mathfrak{A}_4$ has diagonal entries $\varepsilon, ..., p^3\varepsilon$. Thus, the matrix $\mathfrak{A}_5$ on the rank 5 $F-$crystal $G \subset \wedge^2 H$ has diagonal entries which are products of two distinct diagonal entries of $\mathfrak{A}_4$ divided by p (by construction of the $F-$crystal $\wedge^2 H$), and hence the diagonal entries are $1, p, p^2, p^3, p^4$. The $(5,5)-$th entry of $\mathfrak{A}_5$ is 1, and the statement follows. □

Proof of theorem 3.6.1:
The formulas for the two roots of the Frobenius polynomial given in theorem 3.9.1 determine the Frobenius polynomial up to the constant $\varepsilon = \pm 1$, given that there exists analytic elements $F(z)$ and $G(z)$ that can be evaluated in the Teichmüller point. Both formulas are completely independent of the parameters α, β and γ. Thus, it follows that if the Frobenius polynomial can be computed by the formulas in theorem 3.9.1, then it is independent of the parameters.

Now the question of the construction of analytic elements $F(z)$ and $G(z)$ still remains unanswered. We will consider this problem in the following chapters to derive explicit formulas for the computation of r_{α_0} and r'_{α_0}.

Chapter 4

The Hasse-invariant

One of the problems that remains to be solved is to determine the element $h(z)$ defininig $A := W[z][(s(z)h(z))^{-1}]$. The zero set of $h(z)$ should be the locus over which the $F-$crystal H becomes non-ordinary. In the example of the Legendre family, $h(z)$ is the Hasse-invariant of the family. The roots of the Hasse-invariant are the points over which the fibers become supersingular.
Following [**39**], we describe some properties of the Hasse-invariant and give an explicit formula for the Hasse-invariant for families of hypergeometric Calabi-Yau threefolds.

4.1 The Hasse-invariant and the Picard-Fuchs equation

Let $A_0 = \mathbb{F}_p[z][s(z)^{-1}]$, $S_0 = Spec(A_0)$ and let $f : X_0 \to S_0$ be a family of Calabi-Yau threefolds. Assume that there exists a smooth lifting $f : X \to S_\infty$ of the family to characteristic 0. Assume furthermore that the CY3-crystal $H^3_{cris}(X/S_\infty)$ contains a CY3-subcrystal H of rank four where the Gauss-Manin connection ∇ is specified by a CY(4)-differential operator P. We are interested in the locus over which the $F-$crystal H becomes non-ordinary.

Let F denote the absolute Frobenius on S_0; F is the identity map on the underlying topological space and the pth power map on the structure sheaf $\mathcal{O}_{S_0}$.

By HW, we denote the *Hasse-Witt operation*

$$HW : F^* R^3 f_*(\mathcal{O}_{X_0}) \to R^3 f_*(\mathcal{O}_{X_0}).$$

For a definition of HW, see [**39**], section (2.3.). The following propositions relate the Hasse-Witt operation to the non-ordinary locus of the $F-$crystal H.

Proposition 4.1.1 *([**39**], proposition (2.3.4.1.5) and its corollary)*
In order to have a direct sum decomposition

$$H := \mathrm{Fil}^1 H \oplus \mathrm{Fil}_0 H,$$

the Hasse-Witt operation has to be an isomorphism.

Since $R^3 f_*(\mathcal{O}_{X_0}) \subset H$ is a free $\mathcal{O}_{S_0}$-module of rank 1, the Hasse-Witt operation can be represented by a $1 \times 1-$matrix over $\Gamma(S_0, \mathcal{O}_{S_0})$, i.e. by an element $h(z) \in \Gamma(S_0, \mathcal{O}_{S_0})$. By proposition 4.1.1, the zero set of $h(z)$ is the non-ordinary locus of the CY3-crystal H.

Proposition 4.1.2 *(Igusa, Manin,* [**39**]*, proposition (2.3.6.3)) The element $h(z)$ satisfies*

$$Ph(z) = 0 \mod p.$$

Thus $h(z)$ is a solution to the Picard-Fuchs equation modulo p. This fact provides us with the means to determine $h(z)$ for a wide class of examples.
For many CY(4)-operators P, the coefficients of the integral solution $f_0(z)$ to the differential equation $Pf = 0$ satisfy certain congruence properties, the so-called *Dwork-congruences* (see definition 5.1.2). For these operators, the following proposition provides us with the means to determine the Hasse-invariant.

Proposition 4.1.3 *Let $f_0(z)$ be the unique power series solution to the differential equation $Pf = 0$ around $z = 0$ satisfying $f_0(0) = 1$. Assume that the coefficients of f_0 satisfy the Dwork congruences. Then $h(z)$ is a polynomial of degree at most $p-1$,*

$$h(z) = f_0^{<p-1}(z) \mod p,$$

where $f_0^{<p-1}$ denotes the truncation of f_0 up to degree $p-1$.

Proof: See [**60**], corollary 3.7. and the preceding propositions. □

4.2 The Hasse-invariant for hypergeometric CY-threefolds

For the case of hypersurfaces, Katz [**39**] also gives a formula to compute the Hasse-invariant which does not involve the differential equation and its holomorphic solution, but the defining equations of the family. This formula can be extended to all of the 14 families of hypergeometric CY-threefolds, which are either families of weighted projective hypersurfaces or complete intersections in weighted projective spaces.

Let $T = Spec(\mathbb{F}_p[\psi][s(\psi)^{-1}])$, and let

$$f : X \to T$$

be a family of Calabi-Yau threefolds with fibres in the weighted projective space

$$\mathbb{P}^n(w_1, ..., w_{n+1}),$$

defined by a regular sequence of weighted homogeneous polynomials $F_1, ..., F_r$, $\deg(F_i) = d_i$ in $\Gamma(\mathcal{O}_T, T)[X_1, ..., X_{n+1}]$, where weight$(X_i) = w_i$, such that $d := d_1 + ... + d_r$ satisfies

$$d = w_1 + ... + w_{n+1}.$$

Note that for all $\lambda \in \mathbb{F}_p$ with $s(\lambda) = 0$, the family would become singular.

Proposition 4.2.1 *Let $f : X \to T$ be defined as above. Then the Hasse-invariant of X/T is given by the coefficient of*

$$(X_1 ... X_{n+1})^{p-1}$$

in $(F_1 \cdot ... \cdot F_r)^{p-1}$.

Proof: Similar to the arguments in [**39**], section 2.3.7., where the case of hypersurfaces is worked out in detail. □

All 14 families of hypergeometric families satisfy the conditions above and are listed in table 4.1. In each of these cases, the modulo p Hasse-invariant can be expressed in terms of the *Hasse polynomial*

$$H(z) = \sum_{n=0}^{p-1} a(n) z^n,$$

where the coefficients $a(n)$ of these polynomials are listed in table 4.1.
Namely, in each case there exists an integer k such that

$$h(\psi) \equiv \psi^{k(p-1)} H(z) \mod p.$$

We will demonstrate this in two examples.

Example 1.

We have (in the notation of table 4.1)

$$V_6(z) : F := X_1^6 + X_2^6 + X_3^6 + X_4^6 + X_5^3 - 6\psi X_1 X_2 X_3 X_4 X_5 = 0.$$

Let $z = (1/6\psi)^6$ and let $h(\psi)$ be the Hasse-invariant. Then

$$h(\psi) \equiv \psi^{p-1} H(z) \mod p.$$

Since $h(\psi)$ is the coefficient of $(X_1 X_2 X_3 X_4 X_5)^{p-1}$ in F^{p-1}, it follows that

$$\begin{aligned} h(\psi) &= \sum_{i \geq 0} (-6\psi)^{p-1-6i} \binom{p-1}{i} \binom{p-1-i}{i} \binom{p-1-2i}{i} \binom{p-1-3i}{i} \binom{p-1-4i}{2i} \\ &= (-6\psi)^{p-1} \sum_{i \geq 0} (-6\psi) \frac{(p-1)!}{i!^4 (2i)! (p-1-6i)!} \\ &\equiv \psi^{p-1} \sum_{i=0}^{p-1} (6\psi)^{-6i} \frac{(6i)!}{i!^4 (2i)!} \mod p \\ &= \psi^{p-1} H(z). \end{aligned}$$

Example 2

We have (in the notation of table 4.1)

$V_{3,3}(z) : F_1 := X_1^3+X_2^3+X_3^3-3\psi X_4X_5X_6 = 0, F_2 := X_4^3+X_5^3+X_6^3-3\psi X_1X_2X_3 = 0,$

and let $z = (1/3\psi)^6$. Let $h(\psi)$ be the Hasse-invariant. Then

$$h(\psi) \equiv \psi^{2(p-1)}H(z) \mod p.$$

Since $h(\psi)$ is the coefficient of $(X_1x_2X_3X_4X_5X_6)^{p-1}$ in $(F_1F_2)^{p-1}$, it follows that

$$\begin{aligned} h(\psi) &= \sum_{i\geq 0}(-3\psi)^{2(p-1)-6i}\left(\binom{p-1}{i}\binom{p-1-i}{i}\binom{p-1-2i}{i}\right) \\ &= (-3\psi)^{2(p-1)}\sum_{i\geq 0}(-3\psi)^{-6i}\left(\frac{(p-1)!}{i!^3(p-1-3i)!}\right)^2 \\ &\equiv \psi^2\sum_{i=0}^{p-1}(3\psi)^{-6i}\left(\frac{(3i)!}{i!^3}\right)^2 \mod p \\ &= \psi^{2(p-1)}H(z). \end{aligned}$$

threefold	ambient space	$a(n)$
V_5	$\mathbb{P}^4$	$\frac{(5n)!}{n!^5}$
V_6	$\mathbb{P}^4(1,1,1,1,2)$	$\frac{(6n)!}{n!^4(2n)!}$
V_8	$\mathbb{P}^4(1,1,1,1,4)$	$\frac{(8n)!}{n!^4(4n)!}$
V_{10}	$\mathbb{P}^4(1,1,1,2,5)$	$\frac{(10n)!}{n!^3(2n)!(5n)!}$
$V_{3,3}$	$\mathbb{P}^5$	$\left(\frac{(3n)!}{n!^3}\right)^2$
$V_{2,4}$	$\mathbb{P}^5$	$\binom{2n}{n}\frac{(4n)!}{n!^4}$
$V_{2,2,3}$	$\mathbb{P}^6$	$\binom{2n}{n}^2\frac{(3n)!}{n!^3}$
$V_{2,2,2,2}$	$\mathbb{P}^7$	$\binom{2n}{n}^4$
$V_{3,4}$	$\mathbb{P}^5(1,1,1,1,1,2)$	$\binom{3n}{n}\frac{(4n)!}{n!^4}$
$V_{4,4}$	$\mathbb{P}^5(1,1,1,1,2,2)$	$\left(\frac{(4n)!}{n!^4}\right)^2$
$V_{2,6}$	$\mathbb{P}^5(1,1,1,1,1,3)$	$\binom{2n}{n}\frac{(6n)!}{n!^2(2n)!}$
$V_{4,6}$	$\mathbb{P}^5(1,1,1,2,2,3)$	$\binom{4n}{n}\frac{(6n)!}{n!^2(2n)!^2}$
$V_{6,6}$	$\mathbb{P}^5(1,1,2,2,3,3)$	$\left(\frac{(6n)!}{n!(2n)!(3n)!}\right)^2$
$V_{2,12}$	$\mathbb{P}^5(1,1,1,1,4,6)$	$\binom{2n}{n}\frac{(12n)!}{n!^2(4n)!(6n)!}$

(4.1)

By $V_{d_1,...,d_r}$, we denote the family whose fibres are complete intersections in weighted projective space defined by polynomial equations $F_1, ..., F_r$ of weighted degree $d_1, ..., d_r$.

Chapter 5

Analytic continuation and computations

In the previous chapters, we were faced with the problem of finding an analytic continuation of a quotient of the form

$$\frac{f_0(z)}{f_0(z^p)}$$

to the boundary of the $p-$adic unit disc, since such an analytic continuation would give us the means to evaluate the function $r(z)$, which is necessary to compute unit roots.
In this chapter, we consider this question for an analytic continuation. We describe Dwork's [**27**] analytic continuation method for the case that the coefficients $a(n)$ of the power series

$$f_0(z) = \sum_{n=0}^{\infty} a(n) z^n$$

satisfy certain congruence properties and review a class of examples already considered by Dwork.
Then, we give an explicit algorithm to compute the Frobenius polynomial corresponding to a CY(4)-operator by computing the $p-$adic units r_α and r'_α up to a given $p-$adic precision. We investigate in the $p-$adic precision required to recover the coefficients a_α and b_α of the Frobenius polynomial correctly.

5.1 Dwork congruences and analytic continuation

In this section, we repeat a special case of Dwork's theorem 2. from [**27**] and its proof and the application of this theorem to the contruction of an analytic continuation of some function defined on the open $p-$ adic unit disc. First of all, we recall some definitions from Krasner's theory of uniform analytic functions (see [**45**]).

Definition 5.1.1 *Let Ω be a complete field of characteristic zero with non-archimedian valuation having a countable value group and a countable residue class field.*

1. *A set* $\mathfrak{D} \subset \Omega \cup \{\infty\}$ *is called* ultra open *about* $\alpha \in \Omega$ *if for all* $\xi \in \mathfrak{D}$*, the set*

$$\{|x-\alpha|; x \in \Omega \cup \{\infty\} \setminus \mathfrak{D}, |x-\alpha| < |\xi - \alpha|\}$$

is finite. $\mathfrak{D}$ *is called* quasi-connected *if* $\mathfrak{D}$ *is ultra open about any* $\alpha \in \mathfrak{D} \cap \Omega$.

2. *A family* $\mathfrak{F}$ *of subsets of* $\Omega \cup \{\infty\}$ *is called* chained *if for all* $U, V \in \mathfrak{F}$ *there exists elements* $F_1, ..., F_n \in \mathfrak{F}$ *such that* $U = F_1$, $V = F_n$ *and* $F_i \cap F_{i+1} \neq \emptyset$ *for* $1 \leq i \leq n-1$.

3. *Let* $\mathfrak{D}$ *be quasi-connected. An* analytic element f of support $\mathfrak{D}$ *is a mapping* $f : \mathfrak{D} \to \Omega$ *which lies in the closure under the topology of uniform convergence on* $\mathfrak{D}$ *of the set of rational functions with no poles in* $\mathfrak{D}$.

4. *Two analytic elements* f_1, f_2 *are called* equivalent *if there exists a sequence* $g_1, ..., g_m$ *of analytic elements such that* $f_1 = g_1$, $f_2 = g_m$ *, the intersection of the supports of* g_i *and* g_{i+1} *is nonempty for* $1 \leq i \leq m-1$ *and such that* g_i *and* g_{i+1} *conicide on the intersection of their supports.*

5. *Let* F *be an equivalence class of analytic elements, and let* $\mathfrak{D}(F)$ *be the union of the supports of the elements of* F. *For all* $x \in \mathfrak{D}(F)$, $f(x)$ *is independent of* f *if* f *ranges over all elements in* F *whose support contains* x. *Hence,* F *is a function on* $\mathfrak{D}(F)$. *We call* F *a* uniform analytic function *of support* $\mathfrak{D}(F)$.

Theorem 5.1.1 (Uniqueness Theorem [45]) *If* f_1 *and* f_2 *are analytic elements with supports* $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$ *such that* $\mathfrak{D}_1 \cap \mathfrak{D}_2 \neq \emptyset$, *then* f_1 *and* f_2 *coincide on* $\mathfrak{D}_1 \cap \mathfrak{D}_2$ *if they coincide on a subset having a limit point in* $\mathfrak{D}_1 \cap \mathfrak{D}_2$.

Definition 5.1.2 *Let* $(a(n))_{n \in \mathbb{N}_0}$ *be a sequence with values in* $\mathbb{Z}_p$ *with* $a(0) = 1$. *We say that* $(a(n))_n$ *satisfies the* Dwork congruences *if for all* $s, m \in \mathbb{N}_0$ *and all* $n < p^{s+1}$, *we have*

1. $$\frac{a(n+mp^{s+1})}{a([n/p]+mp^s)} \equiv \frac{a(n)}{a([n/p])} \mod p^{s+1}$$

   ,

2. $a(n)/a([n/p]) \in \mathbb{Z}_p$.

Remark 5.1.1 *If* $(a(n))_n$ *satisfies 1. of the Dwork congruences, then*

$$\frac{a(n+mp^{s+1})}{a([n/p]+mp^s)} \equiv \frac{a(n)}{a([n/p])} \mod p^{s+1}$$

for arbitrary $n \in \mathbb{N}_0$.

Proof: Let $n = n_0 + n_1 p^{s+1}$. Then

$$\frac{a(n+mp^{s+1})}{a([n/p]+mp^s)} = \frac{a(n_0+(n_1+m)p^{s+1})}{a([n_0/p]+(n_1+m)p^s)} \equiv \frac{a(n_0)}{a([n_0/p])} \mod p^{s+1}$$

and

$$\frac{a(n)}{a([n/p])} = \frac{a(n_0+n_1p^{s+1})}{a([n_0/p]+n_1p^s)} \equiv \frac{a(n_0)}{a([n_0/p])} \mod p^{s+1}$$

and the statement follows directly by combining the two equations. □

The following theorem, due to B. Dwork, provides us with the main ingredient to construct an explicit analytic continuation to the $p-$adic unit disc for a class of functions analytic (in the sense of Krasner) in the open $p-$adic unit disc. Since this is a key theorem, we also include a proof. This proof is essentially the same as in [**27**], we only changed some notation.

Theorem 5.1.2 ([27],Theorem 2.) *Let $(a(n))_n$ be a $\mathbb{Z}_p$-valued sequence satisfying the Dwork congruences. Let*

$$\Phi(z) = \sum_{n=0}^{\infty} a(n)z^n.$$

Then for all $m \geq 0$, $s \geq 0$,

$$\Phi(z)\sum_{j=mp^s}^{(m+1)p^s-1} a(j)z^{pj} \equiv \Phi(z^p)\sum_{j=mp^{s+1}}^{(m+1)p^{s+1}-1} a(j)z^j \mod a(m)p^{s+1}[[z]]. \tag{5.1}$$

Proof: Let $n = pN + r$, where $0 \leq r \leq p-1$. The coefficient of z^n on the lefthand side of (5.1) is

$$\sum_{j=mp^s}^{(m+1)p^s-1} a(r+p(N-j))a(j),$$

while the coefficient on the righthand side of (5.1) is

$$\sum_{j=mp^s}^{(m+1)p^s-1} a(N-j)a(r+pj).$$

We prove the Theorem by proving that for all $m, s, N \geq 0$,

$$\sum_{j=mp^s}^{(m+1)p^s-1} a(r+p(N-j))a(j) - a(N-j)a(r+pj) \equiv 0 \mod p^{s+1}a(m) \tag{5.2}$$

To prove (5.2) for $s = 0$, remark that by the $\mod p$ Dwork congruences

$$\frac{a(r+p(N-m))}{a(N-m)} \equiv a(r) \equiv \frac{a(r+pm)}{a(m)} \mod p,$$

and thus

$$a(r+p(N-m))a(m)-a(N-m)a(r+pm)\equiv 0 \mod a(m)a(N-m)p.$$

Now, we proceed by induction on s. Write the induction hypothesis

$$(\alpha)_s: \text{ (5.2) holds for all } m\geq 0, N\in\mathbb{N}_0 \text{ and for } 0,...,s-1.$$

To prove $(\alpha)_{s+1}$, we will first prove that for $0\leq t\leq s$,

$$(\beta)_{s,t}: \sum_{j=mp^s}^{(m+1)p^s-1} a(r+p(N+mp^s-j))a(j)-a(N+mp^s-j)a(r+pj)$$
$$\equiv$$
$$\sum_{j=0}^{p^{s-t}-1}\frac{a(j+mp^{s-t})}{a(j)}\sum_{k=jp^t}^{(j+1)p^t-1} a(r+p(N-k))a(k)-a(N-k)a(r+pk)$$
$$\mod p^{s+1}a(m).$$

To prove $(\beta)_{s,0}$ remark that for the lefthand side, we have

$$\sum_{j=mp^s}^{(m+1)p^s-1} a(r+p(N+mp^s-j))a(j)-a(N+mp^s-j)a(r+pj)$$
$$=\sum_{j=0}^{p^s-1} a(r+p(N-j))a(j+mp^s)-a(N-j)a(r+pj+mp^{s+1}),$$

while the righthand side of $(\beta)_{s,0}$ is

$$\sum_{j=0}^{p^s-1} a(r+p(N-j))a(j+mp^s)-\frac{a(N-j)a(r+pj)a(j+mp^s)}{a(j)}.$$

Apply part 1. of the Dwork congruences

$$\frac{a(r+pj)a(j+mp^s)}{a(j)}\equiv a(r+pj+mp^{s+1}) \mod p^{s+1}a(j+mp^s)$$

to each of the summands on the righthand side of $(\beta)_{s,0}$ to obtain $(\beta)_{s,0}$ "modulo $p^{s+1}a(j+mp^s)$".
Since $j<p^s$, by part 2. of the Dwork congruence $a(j+mp^s)/a(m)\in\mathbb{Z}_p$, and thus any congruence modulo $p^{s+1}a(j+mp^s)$ implies a congruence modulo $p^{s+1}a(m)$ and $(\beta)_{s,0}$ follows.

To prove $(\beta)_{s,t}$ for arbitrary $0\leq t\leq s-1$, the next step is to prove that $(\beta)_{s,t}$ and $(\alpha)_s$ imply $(\beta)_{s,t+1}$.
Therefore, write $j:=\mu+ip$ and write the righthand side of $(\beta)_{s,t}$ as the double sum

$$\sum_{\mu=0}^{p-1}\sum_{i=0}^{p^{s-t-1}-1}\frac{a(\mu+ip+mp^{s-t})}{a(\mu+ip)}\sum_{k=(\mu+ip)p^t}^{(\mu+ip+1)p^t-1} a(r+p(N-k))a(k)-a(N-k)a(r+pk).$$

By part 1. of the Dwork congruences, there exists an $X_{i,\mu} \in \mathbb{Z}_p$such that

$$a(\mu + ip + mp^{s-t}) = \frac{a(\mu + ip)a(i + mp^{s-t-1})}{a(i)} + X_{i,\mu}p^{s-t}a(i + mp^{s-t-1}),$$

If we define

$$Y_{i,\mu} := X_{i,\mu}p^{s-t}\frac{a(i + mp^{s-t})}{a(\mu + ip)} \sum_{k=(\mu+ip)p^t}^{(\mu+ip+1)p^t-1} a(r + p(N - k))a(k) - a(N - k)a(r + pk),$$

then each summand in the double sum is of the form

$$\frac{a(i + mp^{s-t-1})}{a(i)} \sum_{k=(\mu+ip)p^t}^{(\mu+ip+1)p^t-1} a(r + p(N - k))a(k) - a(N - k)a(r + pk) + Y_{i,\mu}.$$

Since $t < s$, we can apply $(\alpha)_s$ to see that the sum appearing in $Y_{i,\mu}$ is congruent to zero $\mod p^{t+1}a(\mu + ip)$. But this implies that $Y_{i,\mu} \equiv 0 \mod p^{s+1}a(i + mp^{s-t-1})$, and furthermore, since $i < p^{s-t-1}$, that $Y_{i,\mu} \equiv 0 \mod p^{s+1}a(m)$ by part 2. of the Dwork congruences.
Hence, we have $(\beta)_{s,t+1}$:

$$\begin{aligned}
&\sum_{j=mp^s}^{(m+1)p^s-1} a(r + p(N + mp^s - j))a(j) - a(N + mp^s - j)a(r + pj)\\
\equiv\ &\sum_{\mu=0}^{p-1}\sum_{i=0}^{p^{s-t-1}-1} \frac{a(i + mp^{s-t-1})}{a(i)} \sum_{k=(\mu+ip)p^t}^{(\mu+ip+1)p^t-1} a(r + p(N - k))a(k) - a(N - k)a(r + pk)\\
=\ &\sum_{i=0}^{p^{s-t-1}-1} \frac{a(i + mp^{s-t-1})}{a(i)} \sum_{k=ip^{t+1}}^{(i+1)p^{t+1}-1} a(r + p(N - k))a(k) - a(N - k)a(r + pk),
\end{aligned}$$

where the congruence is modulo $p^{s+1}a(m)$ and the equality is obtained by writing the double sum as a single sum.

Now let N be minimal such that

$$\sum_{j=0}^{p^s-1} a(r + p(N - j))a(j) - a(N - j)a(r + jp) \not\equiv 0 \mod p^{s+1}.$$

Then $(\beta)_{s,s}$ implies for arbitrary $m \geq 1$ that modulo $p^{s+1}a(m)$,

$$\begin{aligned}
&\sum_{j=mp^s}^{(m+1)p^s-1} a(r + p(N - j))a(j) - a(N - j)a(r + jp) \equiv\\
&a(m)\sum_{j=0}^{p^s-1} a(r + p(N - mp^s - j))a(j) - a(N - mp^s - j)a(r + pj),
\end{aligned}$$

where the righthand side is congruent to 0 modulo $p^{s+1}a(m)$ by the minimality of N. Thus, for $m \geq 0$, we have

$$\sum_{j=mp^s}^{(m+1)p^s-1} a(r+p(N-j))a(j) - a(N-j)a(r+jp) \equiv 0 \mod p^{s+1}a(m). \quad (5.3)$$

Choose T such that $Tp^s > N$, and observe that

$$\sum_{j=0}^{Tp^s-1} a(r+p(N-j))a(j) - a(N-j)a(r+pj)$$

is the coefficient of z^n in

$$\Phi(z) \sum_{j=0}^{Tp^s-1} a(j)z^{pj} - \Phi(z^p) \sum_{j=0}^{p^{s+1}-1} a(j)z^j$$

for $n = r + pN$ which is zero since $n < p^{s+1}T$.
Since for $1 \leq m \leq T-1$, equation (5.3) holds, it follows that

$$\sum_{j=0}^{p^s-1} a(r+p(N-j))a(j) - a(N-j)a(r+jp) \equiv 0 \mod p^{s+1},$$

a contradiction to the choice of N. Thus, we have proven (5.3) for all $m \geq 0$, and hence $(\alpha)_{s+1}$ follows.□

With regard to the question of analytic continuation, it turns out that we only need the equality (5.1) for $m = 0$. But for the prove of (5.1) for $m = 0$ and arbitrary s, it seems to be necessary to prove the statement for arbitrary m.

In the next theorem, an explicit analytic continuation of the function $\Phi(z)/\Phi(z^p)$ (analytic on the open $p-$adic unit disc) to a domain in the closed $p-$adic unit disc is constructed. It is this explicit analytic continuation that we will apply in our computations of the unit root r_{α_0}.

Let k be the finite field with $q = p^a$ elements, and let $W := W(k)$ be the ring of Witt vectors of k.

Theorem 5.1.3 ([27], Theorem 3.) *Let $(a(n))_n$ be a $W-$valued sequence satisfying the Dwork congruences. Let*

$$\Phi(z) = \sum_{n=0}^{\infty} a(n)z^n$$

and

$$\Phi^s(z) = \sum_{n=0}^{p^s-1} a(n)z^n.$$

Let $\mathfrak{D}$ be the region in W

$$\mathfrak{D} := \{x \in W, |\Phi^1(x)| = 1\}.$$

Then $\Phi(z)/\Phi(z^p)$, which is a uniform analytic function on pW, is the restriction to pW of an analytic element f of support $\mathfrak{D}$:

$$f(x) = \lim_{s\to\infty} \Phi^{s+1}(x)/\Phi^s(x^p).$$

Proof: The function $\Phi(z)/\Phi(z^p)$ converges on pW since $\Phi(z)$ and $\Phi(z^p)$ converge there and since $\Phi(z^p)$ assumes only nonzero values there. Take an infinte sequence of open discs

$$\mathfrak{D}_n := \{x \in W, |x| \leq 1 - 1/n\}$$

which form a chained family and let Φ_n be the restriction of Φ to $\mathfrak{D}_n$. Then $\{\Phi_n\}_{n\geq 2}$ lies in an equivalence class of analytic elements, and thus Φ is a uniform analytic function of support pW. It follows that $\Phi(z)/\Phi(z^p)$ is a uniform analytic function of support pW.
By theorem 5.1.1, it follows that for $s \geq 0$,

$$\Phi(z)\Phi^s(z^p) \equiv \Phi(z^p)\Phi^{s+1}(z) \mod p^{s+1}W[[z]].$$

Since $a(0) = 1$, $\Phi(z)$ and $\Phi^s(z)$ are units in $W[[z]]$, which implies

$$\frac{\Phi(z)}{\Phi(z^p)} \equiv \frac{\Phi^{s+1}(z)}{\Phi^s(z^p)} \mod p^{s+1}W[[z]].$$

Now, we prove that $|\Phi^s(x)| = 1$ for $x \in \mathfrak{D}$. Since equation (5.1) holds for $s = 0$ and $m = 0$, we obtain

$$\frac{\Phi^{s+1}(z)}{\Phi^s(z^p)} \equiv \Phi^1(z) \mod pW[[z]],$$

and thus

$$\Phi^{s+1}(z) \equiv \Phi^s(z^p)F^1(z) \mod pW[[z]].$$

Since $|\Phi^1(x)| = |\Phi^1(x^p)| = 1$ for all $x \in \mathfrak{D}$ by definition of $\mathfrak{D}$, it now follows by induction on s that $|\Phi^s(x)| = 1$ for all $x \in \mathfrak{D}$.
By equation (5.1), it also follows that

$$\frac{\Phi^{s+1}(z)}{\Phi^s(z^p)} \equiv \frac{\Phi^s(z)}{\Phi^{s-1}(z^p)} \mod p^sW[[z]],$$

and hence that

$$\Phi^{s+1}(z)\Phi^{s-1}(z^p) \equiv \Phi^s(z)\Phi^s(z^p) \mod p^sW[z].$$

This equality can be specialized to any $x \in W$, and if $x \in \mathfrak{D}$, then every factor is a unit and we obtain

$$f_s(x) \equiv f_{s-1}(x) \mod p^s$$

for

$$f_s(x) := \frac{\Phi^{s+1}(x)}{\Phi^s(x^p)}.$$

But this shows that the sequence $\{f_s\}_{s\geq 0}$ converges uniformly on $\mathfrak{D}$, and

$$f(z) := \lim_{s\to\infty} \frac{F^{s+1}(z)}{F^s(z^p)}$$

is an analytic function on $\mathfrak{D}$. A specialization of equation (5.1) to $x \in pW$ shows that

$$f_s(x) \equiv \frac{\Phi(x)}{\Phi(x^p)} \mod p^{s+1},$$

and the theorem follows. □

5.2 Dwork congruences for hypergeometric CY(4)-operators

In [**27**], Dwork proves that these congruences hold for sequences of so-called *binomial type numbers*. It turns out that the coefficients of the power series solutions of the 14 hypergeometric CY(4)-operators are of binomial type, and hence that the Dwork congruences hold in the 14 hypergeometric examples. This is the first class of examples for which we computed the unit root r_{α_0}.

In [**27**], we find a stronger version of the following

Theorem 5.2.1 *([**27**], Corollary 2) Let $\theta_1, ..., \theta_r$ be positive integers. For $n \in \mathbb{Z}_+$, let*

$$A(n) = \prod_{i=1}^{r} \frac{(\theta_i)_n}{n!}, B(n) = \prod_{i=1}^{r} \frac{(\lceil\theta_i/p\rceil)_n}{n!}.$$

Then

1. $$A(n)/B([n/p]) \in \mathbb{Z}_p,$$

2. $$\frac{A(n+mp^{s+1})}{B([n/p]+mp^s)} \equiv \frac{A(n)}{B([n/p])} \mod p^{s+1}$$

   for all primes p and $m, s \in \mathbb{Z}_+$.

A direct consequence of theorem 5.2.1 is the following

Corollary 5.2.1 *Let $\theta_1, ..., \theta_r$ and $k_1, ..., k_r$ be positive integers. For $n \in \mathbb{Z}_+$, let*

$$A(n) = \prod_{i=1}^{r} \frac{(\theta_i)_{k_i n}}{(k_i n)!}, B(n) = \prod_{i=1}^{r} \frac{(\lceil\theta_i/p\rceil)_{k_i n}}{(k_i n)!}.$$

Then 1. and 2. of the above theorem hold for all primes p and $m, s \in \mathbb{Z}_+$.

For the coefficients $a(n)$ (listed in table 4.1) of the power series solutions of the 14 hypergeometric CY(4)-operators, it follows directly by theorem 5.2.1 that the Dwork congruences hold. Namely, we can express $a(n)$ and $a([n/p])$ as $A(n)$ and $B(n)$ for certain integers $\theta_1, ..., \theta_r$ in all 14 cases.
For example, we have

$$a(n) := \frac{(5n)!}{n!^5} = \frac{(1)_n}{n!}\frac{(n+1)_n}{n!}\frac{(2n+1)_n}{n!}\frac{(3n+1)_n}{n!}\frac{(4n+1)_n}{n!}$$

and thus

$$a(n) = A(n)$$

for $\theta_1 = 1, \theta_2 = n+1, ..., \theta_5 = 4n+1$ and

$$a([n/p]) := \frac{(5[n/p])!}{([n/p])!^5} =$$
$$\frac{(1)_{[n/p]}}{[n/p]!}\frac{(\lceil(n+1)/p\rceil)_{[n/p]}}{[n/p]!}\frac{(\lceil(2n+1)/p\rceil)_{[n/p]}}{[n/p]!}\frac{(\lceil(3n+1)/p\rceil)_{[n/p]}}{[n/p]!}\frac{(\lceil(4n+1)/p\rceil)_{[n/p]}}{[n/p]!},$$

and thus

$$a([n/p]) = b([n/p]).$$

In the table below, we list the numbers $\theta_1, ..., \theta_r$ and $k_1, ..., k_r$ for all of the 14 cases.

$a(n)$	$\theta_1, ..., \theta_r$	$k_1, ..., k_r$
$\frac{(5n)!}{n!^5}$	$1, n+1, 2n+1, 3n+1, 4n+1$	$1, 1, 1, 1, 1$
$\frac{(6n)!}{(2n)!n!^4}$	$1, 2n+1, 3n+1, 4n+1, 5n+1$	$2, 1, 1, 1, 1$
$\frac{(8n)!}{(4n)!n!^4}$	$1, 4n+1, 5n+1, 6n+1, 7n+1$	$4, 1, 1, 1, 1$
$\frac{(10n)!}{(5n)!(2n)!n!^3}$	$1, 5n+1, 7n+1, 8n+1, 9n+1$	$5, 2, 1, 1, 1$
$\left(\frac{(3n)!}{n!^3}\right)^2$	$1, n+1, 2n+1, 1, n+1, 2n+1$	$1, 1, 1, 1, 1, 1$
$\binom{2n}{n}\frac{(4n)!}{n!^4}$	$1, n+1, 1, n+1, 2n+1, 3n+1$	$1, 1, 1, 1, 1, 1$
$\binom{2n}{n}^2\frac{(3n)!}{n!^3}$	$1, n+1, 1, n+1, 1, n+1, 2n+1$	$1, 1, 1, 1, 1, 1, 1$
$\binom{2n}{n}^4$	$1, n+1, 1, n+1, 1, n+1, 1, n+1$	$1, 1, 1, 1, 1, 1, 1, 1$
$\binom{3n}{n}\frac{(4n)!}{n!^4}$	$1, 2n+1, 1, 2n+1, 3n+1$	$2, 1, 1, 1, 1, 1$
$\left(\frac{(4n)!}{n!^4}\right)^2$	$1, n+1, 2n+1, 3n+1, 1, n+1, 2n+1, 3n+1$	$1, 1, 1, 1, 1, 1, 1, 1$
$\binom{2n}{n}\frac{(6n)!}{(2n)!^2n!^2}$	$1, n+1, 1, 2n+1, 4n+1, 5n+1, 6n+1$	$1, 1, 2, 2, 1, 1, 1, 1$
$\binom{4n}{n}\frac{(6n)!}{(2n)!^2n!^2}$	$1, 3n+1, 1, 2n+1, 4n+1, 5n+1, 6n+1$	$3, 1, 2, 2, 1, 1, 1, 1$
$\left(\frac{(6n)!}{(3n)!(2n)!n!}\right)^2$	$1, 3n+1, 5n+1, 1, 3n+1, 5n+1$	$3, 2, 1, 3, 2, 1$
$\binom{2n}{n}\frac{(12n)!}{(6n)!(4n)!n!^2}$	$1, n+1, 1, 6n+1, 10n+1, 11n+1$	$1, 1, 6, 4, 1, 1$

Hence it follows that the Dwork congruences hold for the coefficients of the power series solutions to the 14 hypergeomteric CY(4)-operators.

5.3 Explicit formulas

In this section, we give explicit $p-$adic formulas to compute the unit root r_{α_0} and the $p-$adic unit defining the reciprocal root of $p-$adic valuation 1, r'_{α_0}, of the Frobenius polynomial out of the data given by a CY(4)-operator. These formulas can be applied directly to compute r_{α_0} and r'_{α_0} in practice.

Let p be a prime and let k be the field with $q = p^a$ elements. By $W := W(k)$, we denote the ring of Witt vectors of k.

Let P be a CY(4)-differential operator which is a Picard-Fuchs operator, let $f_0(z)$ be the unique power series solution to the equation

$$Pf = 0$$

around $z = 0$ satisfying $f_0(0) = 1$. Assume that the coefficients of f_0 satisfy the Dwork congruences.

According to proposition 4.1.3, the Hasse-invariant is given by $h_4(z) := f_0^1(z)$, and is thus a polynomial of degree $\leq p-1$.
Let Q be the corresponding CY(5)-operator and let $g_0(z)$ be the unique power series solution to

$$Qg = 0$$

around $z = 0$ satisfying $g_0(0) = 1$. As for f_0, we assume that the coefficients of g_0 satisfy the Dwork congruences. According to proposition 4.1.3, the Hasse invariant is given by $h_5(z) = g_0^1(z)$.

To ensure that our CY3-crystal is ordinary, i.e.

$$\text{rank}(\text{Fil}^0/\text{Fil}^1) = \text{rank}(\text{Fil}^1/\text{Fil}^2) = \text{rank}(\text{Fil}^2/\text{Fil}^3) = \text{rank}(\text{Fil}^3) = 1$$

in case that P is a Picard-Fuchs operator which becomes singular in the locus defined by $s(z) = 0$, we set $h(z) := h_4(z)h_5(z)$, $A := W[z][(s(z)h(z))^{-1}]$ and define (H, ∇, F) to be the rank 4 CY3-crystal over A_∞ with connection specified by the operator P.
Let $\alpha_0 \in S_0 = Spec(A_0)$, and let α be a Teichmüller lifting of α_0. Let $e : A_\infty \to W$, $e(z) = \alpha$. By theorem 5.1.3 combined with theorem 3.9.1, it follows that up to a constant $\varepsilon \in \{\pm 1\}$, the two $p-$adic units r_{α_0} and r'_{α_0} determining the Frobenius polynomial on e^*H can be computed with $p-$adic precision p^s by the formulas

$$r_{\alpha_0} \equiv \left(\varepsilon \frac{f_0^s(\alpha)}{f_0^{s-1}(\alpha^p)} \right)^{1+\sigma+\ldots+\sigma^{a-1}} \mod p^s \tag{5.4}$$

and

$$r'_{\alpha_0} \equiv \left(\frac{g_0^s(\alpha)}{g_0^{s-1}(\alpha^p)} \right)^{1+\sigma+\ldots+\sigma^{a-1}} \mod p^s. \tag{5.5}$$

These explicit formulas allow us to perform computations in practice.

5.4 Required $p-$adic precision

In the previous section, we gave explicit formulas to compute the $p-$adic units r_{α_0} and r'_{α_0}. Now, we give estimates for the $p-$adic accuracy with which r_{α_0} and r'_{α_0} have to be computed to recover the integral coefficients a_{α_0} and $b_{\alpha_0} \in \mathbb{Z}$ of the Frobenius polynomial

$$\mathcal{P} = p^6T^4 + p^3a_{\alpha_0}T^3 + pb_{\alpha_0}T^2 + a_{\alpha_0}T + 1$$

correctly.

Let therefore be $k = \mathbb{F}_p$, the field with p elements. By the Weil conjectures (Riemann hypothesis), the complex absolute value of the reciprocal complex roots of $\mathcal{P}$ is $p^{3/2}$. For the integers a_{α_0} and b_{α_0}, this implies

$$|a_{\alpha_0}| \leq 4 \cdot p^{3/2} \text{ and } |b_{\alpha_0}| \leq 6 \cdot p^2,$$

where $|\cdot|$ denotes the complex absolute value. Since

$$4p^{3/2} < \frac{p^3}{2}$$

for all $p \geq 5$ and

$$6p^2 < \frac{p^3}{2}$$

for all $p \geq 13$, it follows that for all $p \geq 13$, r_{α_0} and r'_{α_0} have to be computed modulo p^3 to recover a_α and b_α. For $p \in \{5, 7, 11\}$, we have to compute modulo p^4, and for $p = 3$, we have to compute modulo p^5. Thus, the $p-$adic precision up to which we have to compute r_{α_0} and r'_{α_0} is rather low, and in the general case ($p \geq 13$), we only have to compute the first $p^3 - 1$ coefficients of the power series f_0 and g_0.

5.5 A bound for the number of possible Frobenius polynomials

Since the Frobenius polynomial at a smooth point α_0 satisfies the Weil conjectures, it is a symmetric polynomial of the shape

$$\mathcal{P} := p^6T^4 + p^3aT^3 + pbT^2 + aT + 1$$

for some integers a, b depending on α_0. Note that apart from this section, we always write a_{α_0} and b_{α_0} instead of a and b. Let $X := p^{3/2}T$. Then, as a polynomial in X, we have

$$\mathcal{P} = X^4 + \alpha X^3 + \beta X^2 + \alpha X + 1,$$

where $\alpha := p^{-3/2}a$ and $\beta := p^{-2}b$. Let u be a complex root of $\mathcal{P}(X)$. Then, in general, all four complex roots of the polynomial $\mathcal{P}(X)$ can be described in terms of u as

$$u, \bar{u}, u^{-1}, \bar{u}^{-1},$$

where by $\bar{u}$, we denote the complex conjugate of u. Now, we want to find out which inequalities have to be satisfied such that these four roots lie on the unit circle. In general, the four roots are pairwise distinct.
Assume that the roots do lie on the unit circle. There are three "limit cases" that can occur before two (or four) of the roots move away from the unit circle.

1. In the case that all four roots lie in $\mathbb{C} \setminus \mathbb{R}$ and move out of the unit circle, we obtain two pairs of complex roots u, u^{-1} and $\bar{u}, \bar{u}^{-1}$, such that u and u^{-1} lie on the same line through the origin and $\bar{u}$ and $\bar{u}^{-1}$ lie on the same line through the origin. The limit case is then the case $u = u^{-1}$ and $\bar{u} = \bar{u}^{-1}$.

2. In the case that two of the roots become positive real numbers and move out of the unit circle, the limit case is that two of the roots are equal to 1.

3. In the case that two of the roots become negative real numbers and move out of the unit circle, the limit case is that two of the roots are equal to -1.

This has the following consequences for the polynomial $\mathcal{P}$. Let $v := u + \bar{u}$. In the limit cases, $\mathcal{P}$ can be written as

1.
$$\begin{aligned}(X-u)^2(X-\bar{u})^2 &= (X^2 - vX + 1)^2 \\ &= X^4 - 2vX^3 + (2+v^2)X^2 - 2vX + 1.\end{aligned}$$

2.
$$\begin{aligned}(X-u)(X-\bar{u})(X-1)^2 &= (X^2 - vX + 1)(X-1)^2 \\ &= X^4 - (2+v)X^3 + (2-2v)X^2 - (2+v)X + 1.\end{aligned}$$

3.
$$\begin{aligned}(X-u)(X-\bar{u})(X+1)^2 &= (X^2 - vX + 1)(X+1)^2 \\ &= X^4 - (v-2)X^3 + (2-2v)X^2 - (v-2)X + 1.\end{aligned}$$

Thus, it follows that

1. $\alpha = -2v, \beta = 2 + v^2$,
2. $\alpha = -(2+v), \beta = 2 - 2v$,
3. $\alpha = -(v-2), \beta = 2 - 2v$.

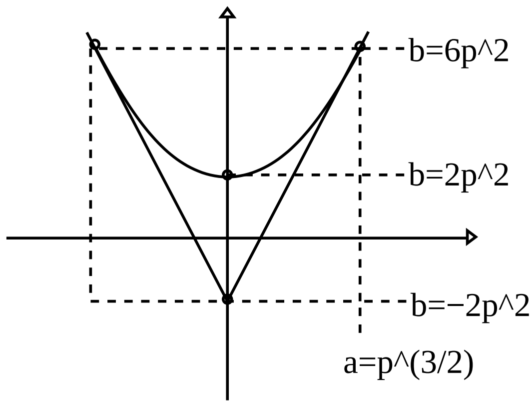
This leads us to the following inequalities for α and β:

$$\begin{aligned}\beta &\leq 2 + \frac{1}{4}\alpha^2, \\ \beta &\geq -2 + 2\alpha, \\ \beta &\geq -2 - 2\alpha.\end{aligned}$$

Hence, for a and b it follows that

$$\begin{aligned} b &\leq 2p^2 + \frac{a^2}{4p}, \\ b &\geq -2p^2 + 2ap^{1/2}, \\ b &\geq -2p^2 - 2ap^{1/2}. \end{aligned}$$

These three inequalities enclose a domain in $\mathbb{R}^2$, as can be seen in the picture.



For $p >> 0$, a reasonable approximation for the number of possible Frobenius polynomials is given by the enlosed area, which can be determined by integration. Thus, we obtain the asymptote $\frac{32}{3}p^{7/2}$ for the number of possible Frobenius polynomials. The exact numbers of possible Frobenius polynomials and the estimated numbers are listed for some primes in the table beyond. We also give the number of possible irreducible Frobenius polynomials.

p	2	3	5	7
exact number	129	511	3001	9703
estimate number	120	498	2981	9679
irreducible	83	384	2631	8932

5.6 Example

Now, we describe the computational steps we performed in MAGMA for one specific example. We consider the operator

$$P := \theta^4 - 4\,x\,(2\,\theta+1)^2\left(7\,\theta^2+7\,\theta+2\right) - 128\,x^2\,(2\,\theta+1)^2\,(2\,\theta+3)^2\,,$$

which is nr. 45 from the list [**2**]. Note that in the notation of chapter 6, this operator is the operator $A * a$.

We compute the Frobenius polynomial for $p = 7$ and $\alpha_0 = 2 \in \mathbb{F}_7$ with 4 digits of $7-$adic precision, i.e. modulo 7^4. Since $2 \neq -\frac{1}{16}$ and $2 \neq \frac{1}{128}$ in $\mathbb{F}_7$, α_0 is not a singular point of the differential equation.

First of all, we computed the truncated power series solution $f_0^{s+1}(z)$ to the differential equation

$$Pf = 0,$$

and obtained

$$f_0^4(z) = 1 + 8z + 360z^2 + 22400z^3 + 1695400z^4 + 143011008z^5 + ...$$

Thus, $f_0^1(\alpha_0) = 1 \in \mathbb{F}_7$ is nonzero. Let $\alpha^{(4)}$ be the Teichmüller lifting of α_0 with $7-$adic accuracy of 4 digits. Evaluating f_0 in this point, we obtain

$$f_0^4(\alpha^{(4)}) \equiv 1709 \mod 7^4$$

and

$$f_0^3((\alpha^{(4)})^7) \equiv 1814 \mod 7^4.$$

Thus, the unit root of the Frobenius polynomial is

$$r_{\alpha_0} \equiv \frac{f_0^4(\alpha^{(4)})}{f_0^3((\alpha^{(4)})^7)} = 582 \mod 7^4.$$

To compute the second root of the Frobenius polynomial, we compute the truncated power series solution $g_0^4(z)$ of the fifth order differential equation

$$Qg = 0,$$

where Q is the second exterior power of the differential operator P, given by

$$\begin{aligned} Q &= \theta^5 - z(44 + 260\theta + 628\theta^2 + 792\theta^3 + 560\theta^4 + 224\theta^5) \\ &+ z^2(-6512 + 400\theta + 44160\theta^2 + 71040\theta^3 + 42240\theta^4 + 8448\theta^5) \\ &+ z^3(4177920 + 131809280\theta + 165888000\theta^2 + 105676800\theta^3 + 34406400\theta^4 \\ &+ 4587520\theta^5) \\ &+ z^4(100663296 + 2852126720\theta + 3103784960\theta^2 + 1635778560\theta^3 + 419430400\theta^4 \\ &+ 41943040\theta^5). \end{aligned}$$

The solution is given by

$$g_0^4 = 1 + 44z + 3652z^2 + 337712z^3 + 33909700z^4 + 3567877424z^5 + ...,$$

$g_0^1(\alpha_0) = 2 \in \mathbb{F}_7$ is nonzero and we compute

$$g_0^4(\alpha^{(4)}) \equiv 51 \mod 7^4$$

and

$$g_0^3((\alpha^{(4)})^7) \equiv 1387 \mod 7^4.$$

Thus,

$$r'_{\alpha_0} \equiv \frac{g_0^{(7^4-1)}(\alpha^{(4)})}{g_0^{(7^3-1)}((\alpha^{(4)})^7)} = 1101 \mod 7^4.$$

Since the Frobenius polynomial (with $7-$adic accuracy 4) is given by

$$\mathcal{P}(T) = (1 - r^4T)(1 - 7\hat{r}^4/r^4T)(1 - 7^2r^4/\hat{r}^4T)(1 - 7^3/r^4T),$$

we finally obtain

$$\mathcal{P}(T) = 7^6T^4 - 7^3 \cdot 8T^3 + 7 \cdot 2T^2 - 8T + 1.$$

As expected, the complex roots of $\mathcal{P}$ do have complex absolute value $7^{-3/2}$.

5.7 An algorithm to compute the Frobenius polynomial from one root

With regard to the list of CY(4) operators [**2**], it turns out that, as mentioned above, in most cases, the coefficients of the power series solution f_0 satisfy the Dwork congruences. But unfortunately, the same is not true for the coefficients of the power series solutions to the differential equations of order 5 given by the second exterior products. Thus, in many cases we can compute the unit root r_{α_0}, but not the $p-$adic unit r'_{α_0}.

To avoid a computation of r'_{α_0}, we give an algorithm which computes the Frobenius polynomial out of the root r_{α_0} alone in case that the Frobenius polynomial is irreducible. For this algorithm, the root r_{α_0} has to be computed with a higher $p-$adic accuracy.

Since the two integers a_{α_0} and b_{α_0} defining

$$\mathcal{P} := p^6T^4 + p^3a_\alpha T^3 + pb_\alpha T^2 + a_\alpha T + 1$$

satisfy

$$|a_{\alpha_0}| \leq 4p^{3/2} \text{ and } |b_{\alpha_0}| \leq 6p^2,$$

it follows that we may write a_{α_0} and b_{α_0} in digits

$$a_{\alpha_0} = a_0 + a_1p + a_2p^2, b_{\alpha_0} = b_0 + b_1p + b_2p^2,$$

where either

$$0 \leq a_0, a_1 \leq p - 1, 0 \leq a_2 \leq 1 \text{ or } 1 - p \leq a_0, a_1, 0, -1 \leq a_2 \leq 0$$

and either

$$0 \leq b_0, b_1 \leq p - 1, 0 \leq b_2 \leq 7 \text{ or } 1 - p \leq b_0, b_1 \leq 0, -6 \leq b_2 \leq 0,$$

depending on whether a_{α_0} and b_{α_0} are positive or negative. Thus, there are 6 indeterminates modulo p and two signs (the sign of a_{α_0} and b_{α_0}) to determine. This means that to compute a_{α_0} and b_{α_0} from the given $p-$adic unit r_{α_0}, we need 8 linear equations in

$a_0, a_1, a_2, b_0, b_1, b_2$, which means that we have to determine r_{α_0} modulo p^8.
Let $u_{\alpha_0} = 1/r_{\alpha_0}$. Then, u_{α_0} satisfies

$$p^6 u_{\alpha_0}^4 + p^3 a_{\alpha_0} u_{\alpha_0}^3 + p b_{\alpha_0} u_{\alpha_0}^2 + a_{\alpha_0} u_{\alpha_0} + 1 = 0. \tag{5.6}$$

Let $u_{\alpha_0} \mod p^8$ be given by the $p-$adic digits

$$u_{\alpha_0}^8 = u_0 + u_1 p + u_2 p^2 + u_3 p^3 + u_4 p^4 + u_5 p^5 + u_6 p^6 + u_7 p^7. \tag{5.7}$$

Combining equations (5.6) and (5.7),we obtain the following algorithm to determine a_{α_0} and b_{α_0}:

Algorithm

Step 1: Solve $a_0 \equiv 1/u_0 \mod p$. Then, $a_0 \in \{1/u_0 \mod p, p - (1/u_0 \mod p)\}$.

Step 2: Find all admissible (a_0, a_1, b_0) satisfying

$$h_1(a_0, a_1, b_0) := p(a_0u_1 + a_1u_0 + b_0u_0^2) + a_0u_0 + 1 \equiv 0 \mod p^2.$$

This requires $(2p-1)^2$ comparisons.

Step 3: Find all admissible $(a_0, a_1, a_2, b_0, b_1)$ satisfying

$$\begin{aligned} h_2(a_0, a_1, a_2, b_0, b_1) &:= p^2(a_0u_2 + a_1u_1 + a_2u_0 + 2b_0u_0u_1 + b_1u_0^2) \\ &+ h_1(a_0, a_1, b_0) \equiv 0 \mod p^3. \end{aligned}$$

This requires $(2p-1) \cdot 9$ comparisons.

Step 4: Find all admissible $(a_0, a_1, a_2, b_0, b_1, b_2)$ satisfying

$$\begin{aligned} &p^3(a_0u_0^3 + a_0u_3 + a_1u_2 + a_2u_1 + 2b_0u_0u_2 + b_0u_1^2 + 2b_1u_0u_1 + b_2u_0^2) \\ + \quad &h_2(a_0, a_1, a_2, b_0, b_1) \equiv 0 \mod p^4. \end{aligned}$$

This requires 12 comparisons.

Step 5: For all $(a_0, a_1, a_2, b_0, b_1, b_2)$ determined by *Step 4*, check if for the corresponding $(a_{\alpha_0}, b_{\alpha_0})$,

$$p^6(u_{\alpha_0}^8)^4 + p^3a_{\alpha_0}(u_{\alpha_0}^8)^3 + pb_{\alpha_0}(u_{\alpha_0}^8)^2 + a_{\alpha_0}u_{\alpha_0}^8 + 1 \equiv 0 \mod p^8. \tag{5.8}$$

There exists exactly one tuple $(a_{\alpha_0}, b_{\alpha_0})$ satisfying equation (5.8).

To compute the unit root r_{α_0} modulo p^8 is extremely time-consuming, and becomes impossible (at the moment) for all primes ≥ 5. Thus, to accelerate the algorithm, it is sensible to find means to compute the Frobenius polynomial from the unit root r_{α_0} modulo a power $p^n < p^8$. Therefore, we check if the tuples determined in *Step 4* satisfy the following two conditions:

Condition 1: The polynomial $\mathcal{P}_\alpha := p^6T^4 + a_{\alpha_0}p^3T^3 + b_{\alpha_0}pT^2 + a_{\alpha_0}T + 1$ is irreducible in $\mathbb{Q}[T]$.

Condition 2: The absolute values of the reciprocal complex roots of $\mathcal{P}_\alpha$ are equal to $p^{3/2}$.

Experimentally, it turns out that under these conditions, as expected, we can determine the correct tuple $(a_{\alpha_0}, b_{\alpha_0})$ from $r_{\alpha_0} \mod p^6$.

5.8 Example

In the tables below, we list the results we computed for the operator P which is no.101 from the list [**2**] for several primes with the Algorithm described above. P is given by

$$\begin{aligned} P &:= \theta^4 - z(124\theta^4 + 242\theta^3 + 187\theta^2 + 66\theta + 9) \\ &+ z^2(123\theta^4 - 246\theta^3 - 787\theta^2 - 554\theta - 124) \\ &+ z^3(123\theta^4 + 738\theta^3 + 689\theta^2 + 210\theta + 12) \\ &- z^4(124\theta^4 + 254\theta^3 + 205\theta^2 + 78\theta + 12) + z^5(\theta+1)^4. \end{aligned}$$

In the notation of chapter 6, P is the operator $b * b$. For this example, the coefficients of the solution g_0 to the 5th order differential equation do not satisfy the Dwork congruences. Hence, we could not compute the $p-$adic unit r'_{α_0} as described in section 5.3.
In some cases, we were not able to determine $(a_{\alpha_0}, b_{\alpha_0})$ uniquely out of lack of precision. In these cases, we give the set of possible tuples $(a_{\alpha_0}, b_{\alpha_0})$. Note that $\alpha_0 = 1$ and $\alpha_0 = p - 1$ are singular points of the differential equation, and the Frobenius polynomial is not irreducible there.
The number c in the entry $(a_{\alpha_0}, b_{\alpha_0}), c$ indicates the number of possible tuples $(a_{\alpha_0}, b_{\alpha_0})$ without the two conditions applied. For $p = 5$, we computed the unit root r_{α_0} modulo p^6.

$\alpha_0 = 2$	$\alpha_0 = 3$
$(4, 6), 3$	$(4, 6), 3$

The entry "-" means that for this parameter value, we could not compute the unit root r_{α_0}, because over this point, the $F-$crystal is not ordinary. For $p = 7$, we computed r_{α_0} modulo p^5.

$\alpha_0 = 2$	$\alpha_0 = 3$	$\alpha_0 = 4$	$\alpha_0 = 5$
$(25, 40), 16$	$-$	$(25, 40), 16$	$-$

For $p = 11$, we computed r_{α_0} modulo p^4.

$\alpha_0 = 2$ $(102, 472), 103$	$\alpha_0 = 3$ $\{(61, 226), (-38, 18), (-27, 189)\}, 105$
$\alpha_0 = 4$ see $\alpha_0 = 3$	$\alpha_0 = 5$ $(-34, 89), 106$
$\alpha_0 = 6$ $(102, 472), 103$	$\alpha_0 = 7$ $\{(8, 61), (19, 91), (63, 211), (-14, 1)\}, 104$
$\alpha_0 = 8$ see $\alpha_0 = 7$	$\alpha_0 = 9$ $-$

The entry "n.i." means that in this point, the Frobenius polynomial is not irreducible. For $p = 13$, we computed r_{α_0} modulo p^4.

$\alpha_0 = 2$ $(20, -115), 104$	$\alpha_0 = 3$ $\{(-55, 180), (-29, 109)\}, 103$
$\alpha_0 = 4$ n.i.	$\alpha_0 = 5$ n.i.
$\alpha_0 = 6$ $(70, 400), 108$	$\alpha_0 = 7$ see $\alpha_0 = 2$
$\alpha_0 = 8$ n.i.	$\alpha_0 = 9$ see $\alpha_0 = 3$
$\alpha_0 = 10$ n.i.	$\alpha_0 = 11$ see $\alpha_0 = 6$

Chapter 6

Some special Picard-Fuchs equations: Hadamard products

In this chapter, we apply the method explained in the previous chapter to compute Frobenius polynomials for some special fourth order operators. These operators belong to the list [**2**]. The first class of operators we consider are the hypergeometric operators (No. $1-14$ in the list). A typical example of the second class of operators is operator 45 from that list:

$$\theta^4 - 4\,x\,(2\,\theta+1)^2\left(7\,\theta^2+7\,\theta+2\right) - 128\,x^2\,(2\,\theta+1)^2\,(2\,\theta+3)^2\,.$$

This operator is a so-called *Hadamard product* of two second order operators.
We mention how to solve the problem of fixing the constant $\varepsilon = \pm 1$ occuring in equation (5.4) for a Hadamard product.

6.1 Hadamard products

The *Hadamard product* of two power series $f(x) := \sum_n a_n x^n$ and $g(x) = \sum_n b_n x^n$ is the power-series defined by the coefficient-wise product:

$$f * g(x) := \sum_n a_n b_n x^n.$$

It is a classical theorem, due to Hurwitz, that if f and g satisfy linear differential equations P and Q resp., then $f * g$ satisfies a linear differential equation $P * Q$. Only in very special cases, the Hadamard product of two CY-operators will again be CY, but it is a general fact that if f and g satisfy differential equations of *geometrical origin*, then so does $f * g$. For a proof, we refer to [**5**], chapter 2. Here we sketch the idea. The multiplication map

$$m : \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}^*, (s,t) \mapsto s.t$$

can be compactified to a map

$$\mu : \widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathbb{P}^1$$

by blowing-up the two points $(0,\infty)$ and $(\infty,0)$ of $\mathbb{P}^1\times\mathbb{P}^1$. Given two families $X\longrightarrow\mathbb{P}^1$ and $Y\longrightarrow\mathbb{P}^1$ over $\mathbb{P}^1$, we define a new family $X*Y\longrightarrow\mathbb{P}^1$, as follows. The cartesian product $X\times Y$ maps to $\mathbb{P}^1\times\mathbb{P}^1$ and can be pulled back to $X*Y$ over $\widetilde{\mathbb{P}^1\times\mathbb{P}^1}$. Via the map μ we obtain a family over $\mathbb{P}^1$. If n resp. m is the fibre dimension of $X\longrightarrow\mathbb{P}^1$ resp. $Y\longrightarrow\mathbb{P}^1$, then $X*Y\longrightarrow\mathbb{P}^1$ has fibre dimension $n+m+1$. The critical points of $X*Y\longrightarrow\mathbb{P}^1$ are, apart from 0 and ∞, the products of the critical values of the factors. In down-to-earth terms, if $X\longrightarrow\mathbb{P}^1$ and $Y\longrightarrow\mathbb{P}^1$ are defined by say Laurent polynomials $F(x)$ and $G(y)$ resp., then the fibre of $X*Y\longrightarrow\mathbb{P}^1$ over u is defined by the equations

$$F(x)=s, G(y)=t, s.t=u.$$

If the period functions for $X\longrightarrow\mathbb{P}^1$ and $Y\longrightarrow\mathbb{P}^1$ are represented as

$$f(s)=\int_\gamma Res(\frac{\omega}{F(x)-s})=\sum_n a_n s^n,$$

$$g(t)=\int_\delta Res(\frac{\eta}{G(y)-t})=\sum_m b_m t^m,$$

then

$$\begin{aligned}\int_{T_\gamma\times T_\delta\times S^1\times S^1}\frac{\omega\wedge\eta\wedge ds\wedge dt}{(F(x)-s)(G(y)-t)(st-u)} &= \int_{S^1\times S^1}\sum a_n s^n b_m t^m\frac{ds\wedge dt}{st-u}\\ &= \sum a_n b_n u^n = f(u)*g(u),\end{aligned}$$

where T_γ and T_δ are the Leray coboundaries of γ and δ, is a period of $X*Y\longrightarrow\mathbb{P}^1$.
For example, if we apply this construction to the rational elliptic surfaces $X=Y$ with singular fibres of Kodaira type I_9 over 0 and I_1 over ∞ and two further fibres of type I_1, we obtain a family $X*Y\longrightarrow\mathbb{P}^1$, with generic fibre a Calabi-Yau 3-fold with $h^{12}=1$ and $\chi=-36$.

6.2 Some special $CY(2)$-operators

We will use Hadamard-products of some very special CY(2)-operators appearing in [**4**] from which we also take the names. These operators are all associated to *extremal rational elliptic surfaces* $X\longrightarrow\mathbb{P}^1$ with non-constant j-function. Such a surface has three or four singular fibres, [**48**]. The six cases with three singular fibres fall into four isogeny-classes and each of these gives rise to a Picard-Fuchs operator of hypergeometric type (named A,B,C,D) and one obtained by performing a Möbius transformation interchanging ∞ with the singular point $\neq 0$ (named e,h,i,j).

Name	Operator	a_n
A	$\theta^2-4x(2\theta+1)^2$	$\frac{(2n)!^2}{n!^4}$
B	$\theta^2-3x(3\theta+1)(3\theta+2)$	$\frac{(3n)!}{n!^3}$
C	$\theta^2-4x(4\theta+1)(4\theta+3)$	$\frac{(4n)!}{(2n)!n!^2}$
D	$\theta^2-12x(6\theta+1)(6\theta+5)$	$\frac{(6n)!}{(3n)!(2n)!n!}$

Name	Operator	a_n
e	$\theta^2 - x(32\theta^2 + 32\theta + 12) + 256x^2(\theta+1)^2$	$16^n \sum_k (-1)^k \binom{-1/2}{k}\binom{-1/2}{n-k}^2$
h	$\theta^2 - x(54\theta^2 + 54\theta + 21) + 729x^2(\theta+1)^2$	$27^n \sum_k (-1)^k \binom{-2/3}{k}\binom{-1/3}{n-k}^2$
i	$\theta^2 - x(128\theta^2 + 128\theta + 52) + 4096x^2(\theta+1)^2$	$64^n \sum_k (-1)^k \binom{-3/4}{k}\binom{-1/4}{n-k}^2$
j	$\theta^2 - x(864\theta^2 + 864\theta + 372) + 18664x^2(\theta+1)^2$	$432^n \sum_k (-1)^k \binom{-5/6}{k}\binom{-1/6}{n-k}^2$

The six cases with four singular fibres are the Beauville surfaces ([**8**]) and also form four isogeny classes and lead to the six Zagier-operators, called (a, b, c, d, f, g).
These are also of the form

$$\theta^2 - x(a\theta^2 + a\theta + b) - cx^2(\theta+1)^2,$$

but now the discriminant $1 - ax - cx^2$ is not a square, so the operator has four singular points.

Name	Operator	a_n
a	$\theta^2 - x(7\theta^2 + 7\theta + 2) - 8x^2(\theta+1)^2$	$\sum_k \binom{n}{k}^3$
c	$\theta^2 - x(10\theta^2 + 10\theta + 3) + 9x^2(\theta+1)^2$	$\sum_k \binom{n}{k}^2\binom{2k}{k}$
g	$\theta^2 - x(17\theta^2 + 17\theta + 6) + 72x^2(\theta+1)^2$	$\sum_{i,j} 8^{n-i}(-1)^i\binom{n}{i}\binom{i}{j}^3$
d	$\theta^2 - x(12\theta^2 + 12\theta + 4) + 32x^2(\theta+1)^2$	$\sum_k \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}$
f	$\theta^2 - x(9\theta^2 + 9\theta + 3) + 27x^2(\theta+1)^2$	$\sum_k (-1)^k 3^{n-3k}\binom{n}{3k}\frac{(3k)!}{k!^3}$
b	$\theta^2 - x(11\theta^2 + 11\theta + 3) - x^2(\theta+1)^2$	$\sum_k \binom{n}{k}^2\binom{n+k}{n}$

The ten products $A * A$, etc. form 10 of the 14 hypergeometric families from [**2**]. The 16 products $A * e$ etc. are not hypergeometric, but also have three singular fibres. The 24 operators $A * a$ etc. have, apart from 0 and ∞ two further singular fibres. The operators $a * a$ etc. have four singular fibres apart from 0 and ∞.

Concerning the Dwork congruences for the solutions of Calabi-Yau differential equations which are Hadamard-products as described above, we observed the following:
1) The Dwork congruences hold for the operators $a, b, \ldots, j$. For the Apery-sequence (case b) this was also conjectured in [**59**] (it follows from [**27**] that A, B, C, D satisfy the Dwork congruences). It follows that the Dwork congruences hold for all fourth order Hadamard products within this group.
2) For the hypergeometric cases $A * A$ etc, and the cases $A * a$, etc. the Dwork congruences also hold for the associated fifth order operator, although even for the simplest examples like the quintic threefold, this is not at all obvious. In the case of the quintic, the holomorphic solution around $z = 0$ to the fifth order differential equation is given by the formula $F_0(z) = \sum_{n=0}^{\infty} A_n z^n$, where

$$A_n := \sum_{k=0}^{n} \frac{(5k)!}{k!^5}\frac{5(n-k)!}{(n-k)!^5}(1 + k(-5H_k + 5H_{n-k} + 5H_{5k} - 5H_{5(n-k)}))$$

and H_k is the harmonic number $H_k = \sum_{j=1}^{k} \frac{1}{j}$. Thus, by the formula it is not even obvious that the coefficients A_n are integers.
3) In fact, the Dwork congruences hold for *almost all* fourth order operators from the list [**2**]. It is an interesting problem to try to prove these experimental facts. We do this for the operators from the list [**2**] that are related to Laurent polynomials whose Newton polygons have 0 as unique interior lattice point.
On the other hand, it is clear that they cannot hold in general for differential operators of geometrical origin: if we multiply f_0 with a rational function of x we obtain a (much more complicated) CY-operator for which the congruences in general will not hold.

6.3 A geometric example: $b * b$

In this section, we describe the geometry of the family of Calabi-Yai threefolds corresponding to the Hadamard product $b * b$.
The CY(2)-differetial operator b corresponds to a semi-stable family of elliptic curves

$$\pi : X \to \mathbb{P}^1,$$

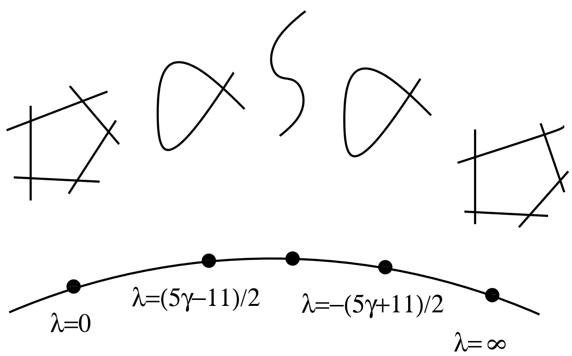
i.e. X is a smooth surface and the singular fibres have type I_5, I_5, I_1, I_1, i.e. they are given by a union of 5 rational curves configured as a $5-$gon (I_5) or an irreducible nodal rational curve (I_1). X is obtained from the singular surface $\bar{X} \subset \mathbb{P}^2 \times \mathbb{P}^1$ given by the homogeneous equation

$$X_1(X_1 - X_3)(X_2 - X_3) = \lambda X_2 X_3 (X_1 - X_2),$$

where the fibration $\bar{\pi} : \bar{X} \to \mathbb{P}^1$ is given by the projection to $\mathbb{P}^1$. By resolving $\bar{X}$, we obtain X. The singular fibres of X and the types of the singular fibres are given in the following table:

λ	∞	0	$\frac{5\sqrt{5}-11}{2}$	$\frac{-5\sqrt{5}-11}{2}$
	I_5	I_5	I_1	I_1

In the picture, let $\gamma := \sqrt{5}$.



The product $\bar{X} * \bar{X} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ is given by the equations

$$\begin{aligned} \bar{X}_\lambda : X_1(X_1 - X_3)(X_2 - X_3) &= \lambda X_2 X_3 (X_1 - X_2) \\ \bar{X}_{u/\lambda} : Y_1(Y_1 - Y_3)(Y_2 - Y_3) &= \frac{u}{\lambda} Y_2 Y_3 (Y_1 - Y_2) \end{aligned}$$

which can be written as

$$X_1(X_1 - X_3)(X_2 - X_3) Y_1(Y_1 - Y_3)(Y_2 - Y_3) = u X_2 X_3 (X_1 - X_2) Y_2 Y_3 (Y_1 - Y_2).$$

The fibration $\bar{\mu} : \bar{X} * \bar{X} \to \mathbb{P}^1$ is given by projection to $\mathbb{P}^1$. Now let u be fixed, and let $(X * X)_u$ denote the fibre $\mu^{-1}(u)$. Let

$$A_1 := \{\lambda : X_\lambda \text{ is singular}\} = \{0, \infty, (5\sqrt{5} - 11)/2, (-5\sqrt{5} - 11)/2\}$$

and

$$A_2 := \{\lambda : X_{u/\lambda} \text{ is singular}\} = \{0, \infty, 2u/(5\sqrt{5} - 11), 2u/(-5\sqrt{5} - 11)\}.$$

Let b_λ^1 and b_λ^2 denote the number of components of the fibres X_λ and $X_{u/\lambda}$ respectively. If

$$u \notin \{0, \infty, -1, 1, 123 - 55/2\sqrt{5}, 123 + 55/2\sqrt{5}\},$$

by [**37**], proposition 1.1 and proposition 1.2., the threefold $(X * X)_u$ admits a projective small resolution $\widehat{(X * X)}_u$. We compute the Hodge numbers $h^{1,1}(\widehat{(X * X)}_u)$ and $h^{2,1}(\widehat{(X * X)}_u)$. By [**37**], section 1, the Hodge numbers are given by the formulas

$$h^{1,1}(\widehat{(X * X)}_u) = \sum_{\lambda \in A_1 \cup A_2} (b_\lambda^1 b_\lambda^2 - 1) + 19 - \sum_{\lambda \in A_1} (b_\lambda^1 - 1) - \sum_{\lambda \in A_2} (b_\lambda^2 - 1)$$

and

$$h^{1,2}(\widehat{(X * X)}_u) = 19 - \sum_{\lambda \in A_1 \cap A_2} (b_\lambda^1 + b_\lambda^2 - 1).$$

Thus, in our example, it follows that

$$h^{1,1}(\widehat{(X * X)}_u) = 51 \text{ and } h^{1,2}(\widehat{(X * X)}_u) = 1,$$

and hence the Euler characteristic is given by $\chi = -100$.

6.4 The constant ε in equation (5.4)

In this section, we repeat a result of Yu [**59**] to determine the constant ε in equation (5.4) for differential operators that are Hadamard products of CY(2)-operators.

Let $X \to \mathbb{P}^1$ and $Y \to \mathbb{P}^1$ be two pencils of elliptic curves over $\mathbb{F}_p$ with totally degenerate fibres X_0 and Y_0 at the origin. In the 14 hypergeometric examples like $A * A$ and in the 24

examples corresponding to operators which are Hadamard products like $A * a$, X_0 and Y_0 are simple normal crossing divisors.
If X_0 is split multiplicative (the slopes of the tangent lines at the singularity lie in $\mathbb{F}_p$), then $\varepsilon_X = 1$, while if X_0 is non-split, $\varepsilon_X = -1$. The same holds for ε_Y. Let P and Q be the CY(2)-operators corresponding to the families X and Y, and let ε be the constant in the formula of the unit root corresponding to the CY(4)-operator $P * Q$. Then the following Lemma holds.

Lemma 6.4.1 *(*[**59**]*, Lemma 4.1) The constant ε in given by $\varepsilon = \varepsilon_X \cdot \varepsilon_Y$.*

This puts us in a position to compute the Frobenius polynomials for CY(4)-operators like $A * a$ explicitly.

6.5 The constant ε in the hypergeometric cases

In the 14 hypergeometric cases of CY(4)-operators, it is possible to prove that there exist families of Calabi-Yau threefolds for which the constant ε satisfies $\varepsilon = 1$. Namely, for each of the 14 hypergeometric CY(4)-operators P, there exists a family of complete intersections in weighted projective space such that P is the Picard-Fuchs operator on a rank-4-submodule of the relative H^3_{DR}. With the help of the defining equations of these families of Calabi-Yau threefolds, we prove that $\varepsilon = 1$.

As in section 4.2, let $T = Spec(\mathbb{F}_p[\psi][s(\psi)^{-1}])$, and let $f : X \to T$ be a family of hypergeometric Calabi-Yau threefolds with defining polynomials $F_1(X, \psi), ..., F_r(X, \psi)$. Let $z := (k_1\psi)^{-k_2}$, where k_1 and k_2 are positive integers depending on the family. By $z(\alpha)$, we denote $z(\alpha) := (k_1\alpha)^{-k_2}$.

Proposition 6.5.1 *Let $H(z)$ be the polynomial of X, and let $\alpha \in \mathbb{F}_p$. If r_{α_0} is the unit root of X_α, then*

$$r_\alpha \equiv H(z(\alpha)) \mod p.$$

Proof: Let N_α be the number of points on X_α with coordinates in $\mathbb{F}_p$, and let $N'_\alpha = \{x \in \mathbb{F}_p^{d+1}, F_i(x, \alpha) = 0, 1 \leq i \leq r\}$, where d is the dimension of the ambient space of X_α and k is the number of the defining equations.
Then $N_\alpha = \frac{N'_\alpha - 1}{p-1}$ and $N_\alpha \equiv 1 - N'_\alpha \mod p$.
Let

$$P_i(T) = \det(1 - FT | H^i_{et}(X_\alpha \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, \mathbb{Q}_l))$$

Then, by the Weil conjectures, the Zeta function $Z(X_\alpha/\mathbb{F}_p, T)$ is given by

$$\begin{aligned}
Z(X_\alpha/\mathbb{F}_p, T) &= \frac{P_1(T)P_3(T)P_5(T)}{P_0(T)P_2(T)P_4(T)P_6(T)} \\
&= \frac{P_3(T)}{(1-T)(1-pT)(1-p^2T)(1-p^3T)} \\
&\equiv \frac{1-aT+O(T^2)}{(1-T)(1-pT)(1-p^2T)(1-p^3T)} \mod T^2 \\
&\equiv \frac{1-aT}{1-T} \mod (T^2, p) \\
&\equiv 1+(1-a)T \mod (T^2, p).
\end{aligned}$$

Remark that $r_{\alpha_0} \equiv a \mod p$. On the other hand, by definition $Z(X_\alpha/\mathbb{F}_p, T)$ is given by

$$\begin{aligned}
Z(X/\mathbb{F}_p, T) &= \exp(N_\alpha T + O(T^2)) \\
&\equiv 1 + N_\alpha T \mod T^2.
\end{aligned}$$

It follows that $a \equiv N'_\alpha \mod p$. Obviously, the following equality holds:

$$N'_\alpha = \sum_{x \in \mathbb{F}_p^{d+1}} (1 - F_1(x,\alpha)^{p-1})...(1 - F_r(x,\alpha)^{p-1}).$$

Since

$$\sum_{x \in \mathbb{F}_p} x^k = \begin{cases} -1 & \text{if } (p-1)|k \\ 0 & \text{otherwise} \end{cases},$$

it follows that

$$\sum_{x \in \mathbb{F}_p^{d+1}} (1 - F_1(x,\alpha)^{p-1})...(1 - F_r(x,\alpha)^{p-1}) \equiv C_{p-1} \mod p,$$

where C_{p-1} is the coefficient of $(X_1...X_{d+1})^{p-1}$ in $(F_1(x,\alpha)...F_r(x,\alpha))^{p-1}$.
Thus, $a \equiv N'_\alpha \equiv C_{p-1} \mod p$, and it follows by section 4.2 that

$$C_{p-1} \equiv \alpha^{k(p-1)} H(z(\alpha)) \equiv H(z(\alpha)) \mod p$$

for some $k \in \mathbb{N}$, where the last equality holds since $\alpha^{q-1} = 1$.
Thus,

$$r_\alpha \equiv a \equiv C_{p-1} \equiv H(z(\alpha)) \mod p.$$

□

Now, we can apply proposition 6.5.1 to prove that the constant ε in equation (5.4) is equal to 1 in the hypergeometric cases.

Proposition 6.5.2 *In the* 14 *hypergeometric cases, we have* $\varepsilon = 1$.

Proof: By equation (5.4), we have $r_\alpha \equiv \varepsilon f_0^1(\alpha) \mod p$. Since $H(z) = f_0^1(z)$, the proposition follows with proposition 6.5.1. □

6.6 Computations

In the hypergeometric cases we reproduced the results obtained in [**56**]. In the appendix, the results of our calculations on the 24 operators which are Hadamard products like $A * a$ etc. are collected. We computed coefficients $(a_{\alpha_0}, b_{\alpha_0})$ of the Frobenius polynomial

$$P(T) = 1 + a_{\alpha_0}T + b_{\alpha_0}pT^2 + a_{\alpha_0}p^3T^3 + p^6T^4$$

for all primes p between 3 and 17 and for all possible values of $\alpha_0 \in \mathbb{F}_p^*$. To generate the tables of coefficients in the appendix, we used the programming language MAGMA. We computed with an overall $p-$adic accuracy of 500 digits. This was necessary, since in the computation of the power series solutions to the differential equations $Pf = 0$ and $Qg = 0$, denominators divisible by large powers of p occured during the calculations (although the solutions themselves have integral coefficients). The occurance of large denominators reduces the $p-$adic accuracy in MAGMA, and thus we had to compute with such a high overall accuracy to obtain correct results in the end. For the unit roots themselves, we computed the ratio

$$\frac{f_0^3(z)}{f_0^2(z^p)}|_{z=\alpha} \mod p^3$$

with $p-$adic accuracy modulo p^3. We checked our results for the tuples $(a_{\alpha_0}, b_{\alpha_0})$ determined the absolute values of the complex roots of the Frobenius polynomial, which by the Weil conjectures should have absolute value $p^{-3/2}$. Needless to say, this was always fulfilled.

Chapter 7

Modular forms

If the fibre X_s of a family $\pi : X \longrightarrow \mathbb{P}^1$ of Calabi-Yau threefolds over $s \in \mathbb{P}^1(\mathbb{Q})$ acquires an ordinary double point, then we expect the Frobenius polynomial to factor as

$$\mathcal{P}(T) = (1 - \chi(p)T)(1 - p\chi(p)T)(1 - a_pT + p^3T^2)$$

for some character χ. The factor $(1 - a_pT + p^3T^2)$ is the Frobenius polynomial on the two dimensional pure part of H^3. This part can be identified with the H^3 of a small resolution $\hat{X}_s$, which then is a rigid Calabi-Yau 3-fold.
By the modularity conjecture, the numbers a_p are Fourier coefficients of a weight four modular form for some congruence subgroup $\Gamma_0(N)$.
In this chapter, we describe the above phenomenon and identify the modular forms for several examples of CY(4)-operators.

7.1 Basic definitions

In this section, we give the basic definitions of modular forms and formulate the modularity conjecture for rigid Calabi-Yau threefolds. For a more detailed presentation of the subject, see [**47**].

The group $\Gamma := \mathrm{SL}(2, \mathbb{Z})$ is called the *full modular group*. For $N \in \mathbb{N}$, the subgroups

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, c \equiv 0 \mod N \right\}$$

of finite index in Γ are called *Hecke subgroups* of Γ.

An *unrestricted modular form of weight* $k \in \mathbb{Z}$ and *level* $N \in \mathbb{N}$ is an analytic function on the upper half plane $\mathbb{H}$ satisfying

$$f\left(\frac{a\tau + c}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N), \tau \in \mathbb{H}.$$

Let $q = e^{2\pi i\tau}$. The function f has a $q-$expansion

$$f(q) = \sum_{n=-\infty}^{\infty} c_n q^n.$$

It is called a *modular form* if $c_n = 0$ for $n < 0$, and a *cusp form* if $c_n = 0$ for $n \leq 0$. Let $M_k(\Gamma_0(N))$ denote the set of modular forms of weight k and level N. It is a finite dimensional vector space, and the set of cusp forms, denoted by $S_k(\Gamma_0(N))$, is a subspace of $M_k(\Gamma_0(N))$.
A cusp form is called an eigenform if it is the eigenvector of a so-called *Hecke operator*. If $r_1r_2|N$ and f is an eigenform for $\Gamma_0(N/r_1r_2)$, then $f(r_1\tau)$ is an eigenform for $\Gamma_0(N)$ and is called an *oldform*. The oldforms span a subspace $S_k^{old}(\Gamma_0(N))$, whose orthogonal complement is denoted by $S_k^{new}(\Gamma_0(N))$. An eigenform in $S_k^{new}(\Gamma_0(N))$ is called a *newform*.

Let X be a Calabi-Yau threefold defined over $\mathbb{Q}$, and let p be a prime of *good reduction* for X, i.e. the reduction of X modulo p, $\bar{X}$, is again a Calabi-Yau threefold. Let F_p denote the geometric Frobenius on $\bar{X}$. We define

$$a_p(X) := tr(F_p|H^3_{et}(X \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_l).$$

A Calabi-Yau threefold X is called *rigid* if $h^{2,1}(X) = 0$, and thus $h^3(X) = 2$. The modularity conjecture [**49**] for rigid Calabi-Yau threefolds states the following:

Conjecture 7.1.1 *Let X be a rigid Calabi-Yau threefold defined over $\mathbb{Q}$. Then X is* modular*, i.e. there exists a newform*

$$f(q) = \sum_{k=1}^{\infty} b_m q^m$$

of weight 4 for $\Gamma_0(N)$ such that $a_p(X) = b_p$ for all primes of good reduction for X. The level N is only divisible by primes of bad reduction for X.

The modularity conjecture has been proven by Yui und Gouvea in full generality in [**33**], based on the results of Dieulefait in [**23**] and [**24**]. Recently, Dieulefait [**21**] provided another proof of the conjecture. By a result of Serre [**53**] and Dieulefait [**22**], the exponent e_p of a prime p dividing N is bounded by $e_p \leq 2$ for $p > 3$, $e_3 \leq 5$ and $e_2 \leq 8$.
In the following, to identify weight four newforms for $\Gamma_0(N)$, we will use the notation of [**47**]. By N/m, we denote the mth newform for $\Gamma_0(N)$. A *twist* of a newform is a newform which only differs from the original form by a character.

7.2 The Frobenius polynomial at an ordinary double point

Let $X \to \mathbb{P}^1$ be a family with generic fibre a CY-threefold, and let $\Sigma \subset \mathbb{P}^1$ be the set of points where the fibres become singular. We set $S := \mathbb{P}^1 \setminus \Sigma$.
Assume that over the point $s \in \Sigma$, the fibre X_s aquires an ordinary double point. Assume furthermore that for a smooth fibre X_t, we have $h^{1,2}(X_t) = 1$. $H^3(X_t)$ is a pure Hodge

structure of weight 3.
Let T be the monodromy operator around s and let $N := \log T/2\pi i$. Since s is an ordinary double point, N is a nilpotent map of rank 1 satisfying $N^2 = 0$.
Let H be the limiting mixed Hodge structure. Since the map

$$N^l : Gr^W_{3+l}H \to Gr^W_{3-l}H$$

is an isomorphism for all $l \geq 0$, it follows that the rank of $Gr^W_2 H$ and $Gr^W_4 H$ is equal to one. Thus, it follows that $Gr^W_3 H$ is a pure Hodge structure weight 3 of rank $2 = 1 + 1$, while $Gr^W_2 H$ and $Gr^W_4 H$ are pure Hodge structures of weight 2 and 4.

$$\begin{array}{ccccccc} & & & & 1 & & \\ 1\ 1\ 1\ 1 & \to & 1 & 0 & & 0 & 1 \\ & & & & 1 & & \end{array}$$

pure Hodge structure *mixed Hodge structure on the limit*

Now consider the consequences of this $p-$adically. Let p be a prime, $k = \mathbb{F}_p$ and let $W := W(k)$ be the ring of Witt vectors. Let $X \to \mathbb{P}^1$ be a pencil of CY-varieties having a flat model over W, such that the reduction over k, $\pi_0 : X_0 \to \mathbb{P}^1$, is a pencil of Calabi-Yau varieties. Assume that over $s \in \mathbb{P}^1(k)$, the fibre X_s aquires an ordinary double point. Then, according to the limiting mixed Hodge structure, the Frobenius polynomial of X_s on the "limit module" is expected to factor in the following way:

$$\mathcal{P}(T) = (1 - \chi(p)T)(1 - p\chi(p)T)(1 - a_pT + p^3T^2)$$

where χ is a character, the factors $(1 - \chi(p)T)$ and $(1 - p\chi(p)T)$ correspond to the rank-1-modules Gr^W_2 and Gr^W_4 of weight 2 and 4 respectively, and the factor $(1-a_pT+p^3T^2)$ corresponds to Gr^W_3. The pure Hodge structure Gr^W_3 has Hodge numbers $h^{0,3} = 1$, $h^{1,2} = 0$, which are the Hodge numbers of the rigid Calabi-Yau threefold $\hat{X}_s$ which is a small resolution of the fibre X_s. We talk about the Frobenius polynomial on the "limit module" since for the singular fibre X_s, neither crystalline cohomology nor rigid cohomology provide the right framework, and our formula for the unit root of the Frobenius polynomial is only valid in smooth points. But still, it turns out that by evaluating the quotients

$$\varepsilon \frac{f_0(z)}{f_0(z^p)} \text{ and } \frac{g_0(z)}{g_0(z^p)}$$

at a Teichmüller lifting of a singular point, we could compute polynomials that factored in the way just described above.

7.3 Modular forms of weight four

In some cases, for primes ≤ 23, the $p-$adic accuracy modulo p^3 for the computations described in section 6.6 was too low, and we had to compute mod p^4. This happened in the case where the parameter $\alpha_0 \in \mathbb{F}_p$ was a critical point of the differential equation. But it is somewhat of a miracle that our calculation made sense at the critical points at all. At an ordinary double point, the Frobenius polynomial is expected to factor as

$$\mathcal{P}(T) = (1 - \chi(p)T)(1 - p\chi(p)T)(1 - a_pT + p^3T^2)$$

for some character χ. Since the factor $(1 - a_pT + p^3T^2)$ is the Frobenius polynomial on the two dimensional H^3 of a rigid Calabi-Yau 3-fold, according to the Weil conjectures, a_p satisfies $|a_p| \leq 2p^{3/2}$, where $|\cdot|$ denotes the complex absolute value. For the integral coefficients a_{α_0} and b_{α_0} of the Frobenius polynomial

$$\mathcal{P} = p^6T^4 + p^3a_{\alpha_0}T^3 + pb_{\alpha_0}T^2 + a_{\alpha_0}T + 1,$$

we derive the bounds

$$|a_{\alpha_0}| \leq p + p^2 + 2p^{3/2} \text{ and } |b_{\alpha_0}| \leq 2(p^{3/2} + p^2 + p^{5/2}).$$

Since

$$p + p^2 + 2p^{3/2} < \frac{p^4}{2}$$

for all $p \geq 3$ and

$$2(p^{3/2} + p^2 + p^{5/2}) < \frac{p^4}{2}$$

for all $p \geq 5$, it follows that by computing the $p-$adic units r_{α_0} and r'_{α_0} modulo p^4, we recover the Frobenius polynomial correctly. Note that by the estimates above, it also follows that for primes ≥ 29, it is enough to compute modulo p^3 in the singular case, too.

According to the modularity conjecture for such Calabi-Yau 3-folds, the coefficients a_p are Fourier coefficients of a weight four modular form for some congruence subgroup $\Gamma_0(N)$.

This is exactly the phenomenon that occurs at the singular points of our differential equations. For the hypergeometric cases we refind the results of [**56**]. For 16 of the 24 operators $A*a$ etc, we have two rational critical values. In 31 of the cases we are able to conjecturally identify the modular form. We say conjecturally, since we only computed the coefficients a_p for $p \leq 23$.

We remark that the critical points of the operators are reciprocal integers and the level of the corresponding modular form divides that integer. For the cases involving the operator c one usually has equality and so the modular form for $D * c$ presumably has level 3888, which was outside the range of our table. Remark that all levels appearing only involve primes 2 and 3.

Case	Point	Form	Twist of	Point	Form	Twist of
$A*a$	$-1/16$	$8/1$	–	$1/128$	$64/5$	$8/1$
$B*a$	$-1/27$	$27/2$	$27/1$	$1/126$	$54/2$	–
$C*a$	$-1/64$	$32/3$	$32/2$	$1/512$	$256/3$	–
$D*a$	$-1/432$	$216/4$	$216/2$	$1/3456$	$1728/16$	$216/1$
$A*c$	$1/144$	$48/1$	$24/1$	$1/16$	$16/1$	$8/1$
$B*c$	$1/243$	$243/1$	–	$1/27$	$27/1$	–
$C*c$	$1/576$	$576/3$	$94/4$	$1/64$	$64/3$	$32/2$
$D*c$	$1/3888$		$1944/5$	$1/432$	$432/9$	$216/2$
$A*d$	$1/128$	$64/4$	$32/1$	$1/64$	$32/2$	–
$B*d$	$1/216$	$9/1$	–	$1/108$	$108/4$	$108/2$
$C*d$	$1/512$	$256/1$	–	$1/256$	$128/4$	$128/1$
$D*d$	$1/3456$	$576/8$	$288/1$	$1/1728$	$864/3$	$864/1$
$A*g$	$1/144$	$24/1$	–	$1/128$	$64/1$	$8/1$
$B*g$	$1/243$	$243/2$	$243/1$	$1/216$	$54/4$	$54/2$
$C*g$	$1/576$	$288/10$	$96/4$	$1/512$	$256/4$	$256/3$
$D*g$	$1/3888$	$1944/6$	$1944/5$	$1/3456$	$1728/15$	–

7.4 An Algorithm to compute coefficients of modular forms

In this section, we describe how to compute the factor $(p^3T^2 - a_pT + 1)$ of the Frobenius polynomial in an ordinary double point if the unit root r_{α_0} is known modulo p^3. Note that it is not necessary to know the second root, and thus r'_{α_0}, of the degree four Frobenius polynomial for these computations.

Since the coefficient a_p determining the factor $(p^3T^2 - a_pT + 1)$ of the Frobenius polynomial in an ordinary double point satisfies

$$|a_p| \leq 2p^{3/2}$$

according to the Weil conjetcures, for all primes $p \geq 5$, it follows that

$$|a_p| < p^2.$$

Thus, we can write

$$a_p = a_0 + pa_1,\ 0 \leq a_0, a_1 \leq p - 1 \text{ or } 1 - p \leq a_0, a_1 \leq 0.$$

Let r_{α_0} be the unit root of the Frobenius polynomial, and let $u_{\alpha_0} = 1/r_{\alpha_0}$. Then, u_{α_0} is a root of the polynomial $p^3T^2 - a_pT + 1$. Assume that we have computed u_{α_0} modulo p^3,

$$u_{\alpha_0} = u_0 + pu_1 + p^2u_2, 0 \leq u_0, u_1, u_2 \leq p - 1 \text{ or } 1 - p \leq u_0, u_1, u_2 \leq 0.$$

Then, since

$$p^3u_{\alpha_0}^2 - a_p u_{\alpha_0} + 1 \equiv 0 \mod p^3,$$

we can recover a_0 and a_1 by the following formulas:

$$\begin{aligned} a_0 &\equiv 1/u_0 \mod p \\ a_1 &\equiv \frac{1-u_0a_0}{pu_0} - a_0u_1/u_0 \mod p \end{aligned}$$

and

$$p(u_0 + pu_1 + p^2u_2) * (a_0 + pa_1) \equiv 1 \mod p^3.$$

We applied these formulas to compute the coeffcients a_p out of the unit root for several examples found in the following sections.

Modular forms for the operators $a * a$ etc.

In this section, we give the results of our computations for CY(4)-operators that are Hadamard products of some CY(2)-operators with themselves.

We also computed the coefficients of the conjectured modular forms (conjectured since we computed a_p only for $p \leq 23$) in singular points that are not rational. Note that in the case $b * b$, we were not able to identify the modular forms in the singular points $123 - 55/2\sqrt{5}$ and $123 + 55/2\sqrt{5}$ since we could not compute the necessary amount of coefficients a_p to determine the modular forms in these points.

Case	Point	Form	Twist of	Point	Form	Twist of
$a * a$	1	$21/2$	–	$1/8$	$14/2$	–
	$-1/8$	$6/1$	–	$1/64$	$21/2$	–
$b * b$	1	$22/2$	–	-1	$5/1$	–
$c * c$	1	$10/1$	–	$1/9$	$18/1$	$6/1$
	$-1/9$	$180/5$	$60/1$	$1/81$	$10/1$	–
$d * d$	$1/16$	$12/1$	–	$1/32$	$16/1$	$8/1$
	$-1/32$	$48/3$	$6/1$	$1/64$	$12/1$	–
$f * f$	$1/27$	$9/1$	–	$-1/27$	$54/4$	$54/2$
	$1/54 - 1/54\sqrt{-3}$	$27/1$	–	$1/54 - 1/54\sqrt{-3}$	$27/1$	–
$g * g$	$1/64$	$17/1$	–	$1/72$	$18/1$	$6/1$
	$1/81$	$17/1$	–	$-1/72$	$306/8$	$102/3$

Modular forms for operators that are no Hadamard products

In this section, we give our results of the computations of conjectured modular forms for conifold points of CY(4)-operators that are, unlike the operators considered in the previous sections, no Hadamard-products of operators of lower degree.

An explicit descriprion of the differential operators we considered, together with the first coefficients a_p (up to $p = 19$) of the conjectured modular forms can be found in appendix A.2.

In this table, we list some operators with one conifold point. Note that the only primes occuring in the factorizations of the levels of the modular forms are $2, 3, 5, 7$.

Case	Point	Form	Twist of
20	$1/54$	$108/2$	–
23	$1/32$	$32/3$	$32/2$
73	$1/432$	$432/13$	$54/2$
116	$1/256$	$72/2$	$24/1$
119	$1/54$	$108/2$	–
255	$1/81$	$225/4$	$5/1$
266	$1/192$	$882/14$	$126/2$
291	$1/512$	$192/4$	$6/1$
292	$1/432$	?	$392/2$

In this table, we list some operators with two rational conifold points. The only primes occuring in the factorizations of the levels of the modular forms are $2, 3, 5, 7, 11$.

Case	Point	Form	Twist of	Point	Form	Twist of
28	$1/64$	$14/2$	–	1	$6/1$	–
33	$1/1024$	$28/1$	–	$1/16$	$28/1$	–
55	$-1/64$	$5/1$	–	$1/256$	$40/2$	–
182	$1/27$	$33/2$	–	$1/16$	$22/3$	–
183	$1/64$	$16/1$	$8/1$	$1/48$	$72/1$	–
205	$1/32$	$32/3$	$32/2$	$1/27$	$15/2$	–
293	$1/1296$	$720/5$	$5/1$	$1/16$	$80/4$	$5/1$
296	$-1/27$	$99/1$	$33/1$	$1/512$	$44/1$	–
297	$1/512$	$80/4$	$5/1$	$1/432$	$180/5$	$60/1$
299	$-1/16$	$72/2$	$24/1$	$1/32$	$96/3$	$96/2$
301	$1/864$	$288/11$	$96/4$	$1/64$	$16/1$	$8/1$
303	$-1/432$	$108/4$	$108/2$	$1/3456$	$432/8$	$108/1$
305	$-1/64$	$56/2$	–	$1/1728$	$504/1$	$168/1$

In this table, we list some operators with three rational conifold points. the primes occuring in the factorizations of the levels of the modular forms are $2, 3, 5, 7, 11, 17, 19, 23$.

Case	Point	Form	Twist of	Point	Form	Twist of	Point	Form	Twist of
21	$-1/4$	$8/1$	–	$1/32$	$112/4$	$28/2$	$1/4$	$56/2$	–
34	$1/25$	$30/1$	–	$1/9$	$6/1$	–	1	$6/1$	–
59	$1/54$	$684/5$	$684/4$	$1/16$	$228/2$	–	$1/4$	$12/1$	–
268	$-1/27$	$15/2$	–	$-1/36$	$324/2$	$324/1$	$1/108$	$60/1$	–
269	$-1/16$	$176/1$	$88/2$	$-1/27$	?	$33/1$	$1/48$	$432/11$	$216/1$
283	$1/108$	?	$552/2$	$1/16$	$23/1$	–	$1/12$	$216/4$	$216/2$
298	$-1/4$	$68/1$	–	$-1/36$	$12/1$	–	$1/64$	$34/2$	–

Chapter 8

Dwork congruences for reflexive polyhedra

In this chapter, we prove congruence properties of the coefficients of power series related to Laurent polynomials. Let $F(X,t) = 1 - tf(X)$, where $f(X) = f(X_1, X_2, X_3, X_4) \in \mathbb{Z}[X_1^{\pm 1}, ..., X_4^{\pm 1}]$ is a Laurent polynomial. Assume that the zeros of $F(X,t)$ define an affine Calabi-Yau threefold in $\mathbb{T} \cong (\mathbb{C}^*)^4$. It turns out that many CY(4)-operators arise as Picard-Fuchs operators of families of smooth Calabi-Yau compactifications of affine toric Calabi-Yau threefolds as described above.
For example, take the Laurent polynomial

$$f(X_1, X_2, X_3, X_4) = X_1 + X_2 + X_3 + X_4 + 1/X_1X_2X_3X_4.$$

In this case, the zeros of $F(X,t)$ define the affine Calabi-Yau threefold in $\mathbb{T}$ whose smooth Calabi-Yau compactification is mirror symmetric to the quintic threefold in $\mathbb{P}^5$ defined by the equation

$$X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 - 5tX_1X_2X_3X_4X_5 = 0.$$

Let $z = t^5$. In our example, the Picard-Fuchs operator is then given by

$$\theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4),$$

and the holomorphic solution is

$$\Phi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{n!^5} z^n.$$

The holomorphic solutions of differential operators related to Laurent polynomials as described above can be explicitly expressed in terms of the Laurent polynomial $f(X)$. We prove that the coefficients of these power series solutions satisfy a modified version of the Dwork congruences. The methods we apply for the proof are completely elementary.

8.1 Laurent polynomials and the congruence D3

We prove a lemma about integral polyhedra with 0 as unique interior lattice point and derive a "weaker" congruence property D3 from the Dwork congruence D2. The lemma will be the key ingredient for the proof of the congruence D3 for coefficients of power series that are related to integral polyhedra with only one interior point.

Let $(a(n))_n$ be a sequence satisfying the Dwork congruences D1 and D2. By a cross-multiplication, the congruence D2 becomes

$$D3 : a(n + mp^{s+1})a([n/p]) \equiv a([n/p] + mp^s)a(n) \mod p^{s+1}.$$

So if we write

$$n = n_0 + n_1 p + n_2 p^2 + ... + n_s p^s$$

and set $n_{s+1} := m$, where $n_0, ..., n_s$ satisfy $0 \le n_i \le p - 1$ and n_{s+1} is an arbitrary non-negative integer, then D3 is equivalent to

$$\begin{aligned} & a(n_0 + ... + n_{s+1}p^{s+1})a(n_1 + ... + n_s p^{s-1}) \\ \equiv \; & a(n_0 + ... + n_s p^s)a(n_1 + ... + n_{s+1}p^s) \mod p^{s+1}. \end{aligned}$$

In the following sections, we will prove this congruence for sequences $a(n)$ which are given implicitly by Laurent polynomials whose Newton polyhedra contain 0 as unique interior lattice point.

We will use the familiar multi-index notation for monomials and exponents

$$X^{\mathbf{a}} = X_1^{a_1} X_2^{a_2} \dots X_n^{a_n}, \quad \mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$$

to write a general Laurent-polynomial as

$$f = \sum_{\mathbf{a}} c_{\mathbf{a}} X^{\mathbf{a}} \in \mathbb{Z}[X_1, X_1^{-1}, X_2, X_2^{-1}, \dots, X_n, X_n^{-1}].$$

The *support* of f is the set of exponents $\mathbf{a}$ occuring in f, i.e.

$$\text{supp}(f) := \{\mathbf{a} \in \mathbb{Z}^n \mid c_{\mathbf{a}} \neq 0\}$$

The *Newton polyhedron* $\Delta(f) \subset \mathbb{R}^n$ of f is defined as the convex hull of its support

$$\Delta(f) := \text{convex}(\text{supp}(f)).$$

When the support of f consists of m monomials, we can put the information of the polyhedron $\Delta := \Delta(f)$ in an $n \times m$ matrix $\mathcal{A} \in Mat(m \times n, \mathbb{Z})$, whose columns $\mathbf{a}_j$, $j = 1, 2, \dots, m$ are the exponents of f;

$$\mathcal{A} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m) = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ \vdots & & & \vdots \\ a_{n,1} & a_{1,2} & \dots & a_{n,m} \end{pmatrix}$$

so that we can write

$$f = \sum_{j=1}^{m} c_j X^{\mathbf{a}_j} = \sum_{j=1}^{m} c_j \prod_{i=1}^{n} X^{a_{i,j}}$$

The polyhedron Δ is the image of the standard simplex Δ_m under the map

$$\mathbb{R}^m \xrightarrow{\mathcal{A}} \mathbb{R}^n.$$

Lemma 8.1.1 *The interior points (i.e. the points that do not lie on the boundary) of a polytope Δ with 0 as interior lattice point are combinations*

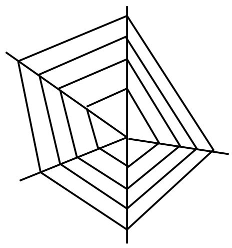
$$\alpha_1\mathbf{a}_1 + ... + \alpha_m\mathbf{a}_m$$

of the columns of $\mathcal{A}$ with $\sum_{j=1}^{m} \alpha_j < 1$.

Proof:
Assume that there exist $(\alpha_1, ..., \alpha_m)$ with $\sum_{j=1}^{m} \alpha_j = \varepsilon < 1$ such that

$$P := \alpha_1\mathbf{a}_1 + ... + \alpha_m\mathbf{a}_m$$

lies on the boundary of Δ. Then $Q := \frac{1}{\varepsilon}(\alpha_1\mathbf{a}_1 + ... + \alpha_m\mathbf{a}_m)$ also lies on the boundary of the polytope (since $\frac{1}{\varepsilon}\sum_{j=1}^{m} \alpha_j = 1$), and lies on the same line through the origin 0, $\overline{0P}$ as the point P.



But since P and Q both lie on $\overline{0Q}$ and on the boundary, the boundary contains a line through the origin. This is a contradiction since 0 is an interior point of Δ. $\square$

The following lemma will play a key role in the sequel.

Lemma 8.1.2 *Let Δ be an integral polyhedron with 0 as unique interior point. Then for all non-negative integral vectors $(\ell_1, \ell_2, \ldots, \ell_m) \in \mathbb{Z}^m$ such that*

$$\sum_{i=1}^{m} a_{i,j}\ell_j \neq 0$$

for some $1 \leq i \leq n$, one has

$$\gcd_{i=1,\ldots,n} (\sum_{j=1}^{m} a_{i,j}\ell_j) \leq \sum_{j=1}^{m} \ell_j.$$

Proof: Assume that there exists a non-negative integral vector $\ell = (\ell_1, ..., \ell_m) \in \mathbb{Z}^m$ such that $\sum_{i=1}^{m} a_{i,j}\ell_j \neq 0$ for some $1 \leq i \leq n$ and

$$g := \gcd_{i=1,\ldots,n} (\sum_{j=1}^{m} a_{i,j}\ell_j) > \sum_{j=1}^{m} \ell_j.$$

We have

$$\mathbf{a}_1\ell_1 + ... + \mathbf{a}_m\ell_m = \mathcal{A} \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_m \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{m} a_{1,j}\ell_j \\ \vdots \\ \sum_{j=1}^{m} a_{n,j}\ell_j \end{pmatrix}.$$

The components of the vector at the right hand side are all divisible by g so that after division by g we obtain a non-zero lattice point

$$v := \frac{\ell_1}{g}\mathbf{a_1} + ... + \frac{\ell_m}{g}\mathbf{a}_m \in \mathbb{Z}^n$$

of Δ with

$$\sum_j \frac{\ell_j}{g} < 1.$$

By lemma 8.1.1, the interior points of Δ (i.e. the points that do not lie on the boundary) consist of the combinations

$$\alpha_1\mathbf{a}_1 + ... + \alpha_m\mathbf{a}_m$$

of the columns of $\mathcal{A}$ with $\sum_{j=1}^{m} \alpha_j < 1$. As 0 was assumed to be the only interior lattice point of Δ we arrive at a contradiction. □

We remark that the above statement applies in particular to *reflexive polyhedra*, which have 0 as unique interior lattice point.

8.2 The fundamental period

Every Laurent polynomial defines implicitly the power series whose coefficients are the constant terms in the powers of the Laurent polynomial. It turns out that this power series can be seen as a *period* on the toric hypersurface defined by the Laurent polynomial. By a period, we mean an integral of an algebraically defined differential form over a chain in some algebraic variety.
In this section, we state the theorem that the coefficients of this period satisfy the modified Dwork congruence D3. This theorem is the main result of this chapter, and will be proven step by step in the following sections.

Notation 8.2.1 *For a Laurent-polynomial we denote by* $[f]_0$ *the* constant term, *that is, the coefficient of the monomial* X^0.

Definition 8.2.1 *The* fundamental period *of* f *is the series*

$$\Phi(t) := \sum_{k=0}^{\infty} a(k)t^k, \quad a(k) := [f^k]_0.$$

Note that the function $\Phi(t)$ can be interpreted as the period of a holomorphic differential form on the hypersurface $X_t := \{t.f = 1\} \subset (\mathbb{C}^*)^n$, as one has

$$\begin{aligned} \Phi(t) &= \sum_{k=0}^{\infty}[f^k]_0 t^k \\ &= \sum_{k=0}^{\infty} \frac{1}{(2\pi i)^n} \int_T f^k t^k \Omega \\ &= \frac{1}{(2\pi i)^n} \int_T \sum_{k=0}^{\infty} f^k t^k \Omega \\ &= \frac{1}{(2\pi i)^n} \int_T \frac{1}{1-tf}\Omega \\ &= \int_{\gamma_t} \omega_t. \end{aligned}$$

Here $\Omega := \frac{dX_1}{X_1}\frac{dX_2}{X_2}\ldots\frac{dX_n}{X_n}$, T is the cycle given by $|X_i| = \epsilon_i$ and homologous to the Leray coboundary of $\gamma_t \in H_{n-1}(X_t)$ and

$$\omega_t = Res_{X_t}(\frac{1}{1-tf}\Omega).$$

In particular, $\Phi(t)$ is a solution of a Picard-Fuchs equation; the coefficients $a(k)$ satisfy a linear recursion relation. This is the point where the CY(4)-operators appear again; for many Laurent-polynomials $f(X)$ in four variables, the period $\Phi(t)$ is the solution to a CY(4)-differential equation. Examples for this can be found in the list in the appendix.

Theorem 8.2.1 *Let* $f \in \mathbb{Z}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$. *Assume that the Newton polyhedron* $\Delta(f)$ *has* 0 *as its unique interior lattice point.*
Then the coefficients $a(n) = [f^n]_0$ *of the fundamental period satisfy for each prime number* p *and* $s \in \mathbb{N}$ *the congruence*

$$a(n_0 + ... + n_s p^s)a(n_1 + ... + n_{s-1}p^{s-2}) \equiv$$

$$a(n_0 + ... + n_{s-1}p^{s-1})a(n_1 + ... + n_s p^{s-1}) \mod p^s, \tag{8.1}$$

where $0 \leq n_i \leq p-1$ *for* $0 \leq i \leq s-1$.

We remark that already for the simplest cases where the Newton polyhedron contains more than one interior lattice point, like $f = X^2 + X^{-1}$, the coefficients $a(n)$ do not satisfy such simple congruences.

8.3 Proof for the congruence $\mod p$

The most simple part of theorem 8.2.1 to prove is the congruence modulo p. But part of the crucial ideas behind the proof of the congruence modulo arbitrary p^s, $s \geq 1$, already apply in this simple case, without complicated technical details.

To prove D3 for $s = 1$ we have to show that for all $n_0 \leq p-1$,

$$a(n_0 + n_1 p) \equiv a(n_0)a(n_1) \mod p.$$

The proof we will give is completely elementary; the key ingredient is lemma 8.1.2, which states that for all non-negative integral $\ell = (\ell_1, ..., \ell_m)$ one has

$$\gcd_{i=1,\dots,n} (\sum_{j=1}^{m} a_{i,j}\ell_j) \leq \sum_{j=1}^{m} \ell_j.$$

Proposition 8.3.1 *Let f be a Laurent polynomial as above and $n_0 < p$. Then*

$$[f^{n_0} f^{n_1 p}]_0 \equiv [f^{n_0}]_0 \, [f^{n_1}]_0 \mod p.$$

Proof: As f has integral coefficients, we have $f^{n_1 p}(X) \equiv f^{n_1}(X^p) \mod p$. So the congruence is implied by the equality

$$[f^{n_0}(X) f^{n_1}(X^p)]_0 = [f^{n_0}(X)]_0 \, [f^{n_1}(X)]_0 \,,$$

which means: the product of a monomial from $f^{n_0}(X)$ and a monomial from $f^{n_1}(X^p)$ can never be constant, unless the two monomials are constant themselves. It is this statement that we will prove now.

For the product of a non-constant monomial from $f^{n_0}(X)$ and a non-constant monomial from $f^{n_1}(X^p)$ to be constant, the monomial coming from $f^{n_0}(X)$ has to be a monomial in $X_1^p, ..., X_n^p$, since all monomials in $f^{n_1}(X^p)$ are monomials in $X_1^p, ..., X_n^p$.

A monomial

$$M := X^{\ell_1 \mathbf{a}_1 + \ell_2 \mathbf{a}_2 + ... + \ell_m \mathbf{a}_m} = \prod_{j=1}^{m} X_1^{a_{1,j}\ell_j} ... X_n^{a_{n,j}\ell_j}$$

appearing in $f^{n_0}(X)$ corresponds to a partition

$$n_0 = \ell_1 + ... + \ell_m$$

of n_0 in non-negative integers ℓ_i. If M were a monomial in $X_1^p, ..., X_n^p$, then we would have the divisibility

$$p \mid \sum_{j=1}^{m} a_{i,j}\ell_j \text{ for } 1 \leq i \leq n,$$

and hence

$$p \mid \gcd_{i=1,\dots,n} (\sum_{j=1}^{m} a_{i,j}\ell_j).$$

On the other hand, by lemma 8.1.2 we have

$$\gcd_{i=1,\dots,n} (\sum_{j=1}^{m} a_{i,j}\ell_j) \leq \sum_{j=1}^{m} \ell_j = n_0 < p.$$

So we conclude that $\sum_{i=1}^{m} a_{i,j}\ell_j = 0$ for $1 \leq j \leq n$ and that the monomial M is the constant monomial X^0. Hence it follows that

$$[f^{n_0}(X)f^{n_1}(X^p)]_0 = [f^{n_0}(X)]_0 [f^{n_1}(X^p)]_0 ,$$

and since

$$[f^{n_1}(X^p)]_0 = [f^{n_1}(X)]_0 ,$$

the proposition follows. □

8.4 Strategy for higher s

The idea for the higher congruences is basically the *same as for* $s = 1$, but is combinatorially more involved. Surprisingly, one does not need any statements stronger than 8.1.2. To prove the congruence (8.1), we have to show that

$$\left[\prod_{k=0}^{s} f^{n_k p^k}\right]_0 \left[\prod_{k=1}^{s-1} f^{n_k p^{k-1}}\right]_0 \equiv \left[\prod_{k=0}^{s-1} f^{n_k p^k}\right]_0 \left[\prod_{k=1}^{s} f^{n_k p^{k-1}}\right]_0 \mod p^s. \tag{8.2}$$

To do this, we will use the following expansion of $f^{np^s}(X)$:

Proposition 8.4.1 *We can write*

$$f^{np^s}(X) = \sum_{k=0}^{s} p^k g_{n,k}(X^{p^{s-k}}),$$

where $g_{n,k}$ is a polynomial of degree np^k in the monomials of f, independent of s, defined inductively by $g_{n,0}(X) = f^n(X)$ and

$$p^k g_{n,k}(X) := f(X)^{np^k} - \sum_{j=0}^{k-1} p^j g_{n,j}(X^{p^{k-1-j}}). \tag{8.3}$$

Proof: We have to prove that the right-hand side of equation (8.3) is divisible by p^k. This is proved by induction on k and an application of the congruence

$$f(X)^{p^m} \equiv f(X^p)^{p^{m-1}} \mod p^m. \tag{8.4}$$

For $k = 1$, the divisibility follows directly by (8.4). Assume that the statement is true for $m \leq k-1$. Write $f(X)^{np^{k-1}} = \sum_{j=0}^{k-1} p^j g_{n,j}(X^{p^{k-1-j}})$. Then, $\sum_{j=0}^{k-1} p^j g_{n,j}(X^{p^{k-j}}) = f(X^p)^{np^{k-1}} \equiv f(X)^{np^k} \mod p^n$, and thus $f(X)^{np^k} - \sum_{j=0}^{k-1} p^j g_{n,j}(X^{p^{k-j}}) \equiv 0 \mod p^n$.□

The congruences involve constant term expressions of the form

$$\left[\prod_{k=a}^{b} f^{n_k p^k}\right]_0 = \left[\prod_{k=a}^{b}\sum_{j=0}^{k} p^j g_{n_k,j}(X^{p^{k-j}})\right]_0$$
$$= \sum_{i_a\le a}\cdots\sum_{i_b\le b} p^{\sum_{k=a}^{b} i_k}\left[\prod_{k=a}^{b} g_{n_k,i_k}(X^{p^{k-i_k}})\right]_0 . \qquad (8.5)$$

Thus, equation (8.2) translates into

$$\sum_{i_0\le 0}\cdots\sum_{i_s\le s}\sum_{j_1\le 0}\cdots\sum_{j_{s-1}\le s-2} p^{\sum_{k=0}^{s} i_k+\sum_{k=1}^{s-1} j_k}\left[\prod_{k=0}^{s} g_{n_k,i_k}(X^{p^{k-i_k}})\right]_0\left[\prod_{k=1}^{s-1} g_{n_k,j_k}(X^{p^{k-1-j_k}})\right]_0$$
$$\equiv$$
$$\sum_{i_0\le 0}\cdots\sum_{i_{s-1}\le s-1}\sum_{j_1\le 0}\cdots\sum_{j_s\le s-1} p^{\sum_{k=0}^{s-1} i_k+\sum_{k=1}^{s} j_k}\left[\prod_{k=0}^{s-1} g_{n_k,i_k}(X^{p^{k-i_k}})\right]_0\left[\prod_{k=1}^{s} g_{n_k,j_k}(X^{p^{k-1-j_k}})\right]_0$$
$$\bmod p^s \qquad (8.6)$$

Since this congruence is supposed to hold modulo p^s, on the left-hand side, only the summands with $\sum_{k=0}^{s} i_k + \sum_{k=1}^{s-1} l_k \le s-1$ contribute, and on the right-hand side, only those with $\sum_{k=0}^{s-1} i_k + \sum_{k=1}^{s} l_k \le s-1$ play a role.
Now, we proceed by comparing these summands on both sides of equation (8.2). We will prove that each summand on the right-hand side is equal to exactly one summand on the left-hand side and vice versa.

8.5 Splitting positions

So we are led to study for $a \le b$ expressions of the type

$$G(a,b;I) := \left[\prod_{k=a}^{b} g_{n_k,i_k}(X^{p^{k-i_k}})\right]_0$$

where the $0 \le n_k \le p-1$ are fixed for $a \le k \le b$ and $I := (i_a, ..., i_b)$ is a sequence with $0 \le i_k \le k$.

Definition 8.5.1 *We say that* $G(a,b;I)$ splits at ℓ *if*

$$G(a,b;I) = G(a,\ell-1;I)G(\ell,b;I).$$

The number of entries of I is determined implicitly by a and b so that by $G(a,\ell-1;I)$ we mean the expression corresponding to the sequence $(i_a, ..., i_{\ell-1})$, while by $G(\ell,b;I)$ we mean the expression corresponding to $(i_\ell, ..., i_b)$. Note that $\ell = a$ represents a trivial splitting, but splitting at $\ell = b$ is a non-trivial property.

Proposition 8.5.1 *If $k - i_k \geq \ell$ for all $k \geq \ell$, then $G(a,b;I)$ splits at ℓ.*

Proof: A monomial $\prod_{j=1}^{m}(X^{p^{k-i_k}})^{\mathbf{a}_j\beta_{j,k}}$ occuring in $g_{n_k,i_k}(X^{p^{k-i_k}})$ corresponds to a partition

$$\beta_{1,k} + ... + \beta_{m,k} = p^{i_k}n_k \leq p^{i_k+1} - p^{i_k}$$

of the number $p^{i_k}n_k$ in non-negative integers $\beta_{1,k},...,\beta_{m,k}$. So we have

$$p^{k-i_k}(\beta_{1,k} + ... + \beta_{m,k} \leq p^{k+1} - p^k.$$

It follows from the assumptions that the product $G(\ell,b;I) = \prod_{k=\ell}^{b} g_{n_k,i_k}(X^{p^{k-i_k}})$ is a Laurent-polynomial in X^p. As a consequence, the product of a monomial in $G(a,\ell-1;I) = \prod_{k=a}^{\ell-1} g_{n_k,i_k}(X^{p^{k-i_k}})$ and a monomial of $G(\ell,b;I)$ can be constant only if the sum

$$m_i := \sum_{j=1}^{m} p^{a-i_a}a_{i,j}\beta_{j,a} + ... + \sum_{j=1}^{m} p^{\ell-1-i_{\ell-1}}a_{i,j}\beta_{j,\ell-1}$$

is divisible by p^ℓ for $1 \leq i \leq n$.
Set

$$\gamma_j := p^{a-i_a}\beta_{j,a} + ... + p^{\ell-1-i_{\ell-1}}\beta_{j,\ell-1}$$

so that

$$\sum_{j=1}^{m} a_{i,j}\gamma_j = m_i.$$

It follows that

$$\sum_{j=1}^{m}\gamma_j = \sum_{j=1}^{m} p^{a-i_a}\beta_{j,a} + ... + \sum_{j=1}^{m} p^{\ell-1-i_{\ell-1}}\beta_{j,\ell-1} \leq p^{a+1} - p^a + ... + p^\ell - p^{\ell-1} = p^\ell - p^a < p^\ell.$$

Hence, it follows that

$$p^\ell \mid \gcd_{i=1,...,n}(\sum_{j=1}^{m} a_{i,j}\gamma_j) \leq \sum_{j=1}^{m}\gamma_j < p^\ell,$$

where the first inequality follows from Theorem 8.1.2. This implies $\sum_{j=1}^{m} a_{i,j}\gamma_j = 0$ for $1 \leq i \leq n$. But this means that the monomial in $\prod_{k=t}^{s-1} g_{n_k,i_k}(X^{p^{k-i_k}})$ is itself constant. □

Now that we know that we can split up expressions $G(a,b;I)$ satisfying the condition given in proposition 8.5.1, we proceed by proving that all the summands on both sides of equation 8.6 that do not have a coefficient divisible by p^s satisfy this splitting condition.

8.6 Three combinatorial lemmas

In this section, we prove three simple combinatorial lemmas which will be applied to split up expressions $G(0,s;I)G(1,s-1;J+1)$ that occur in the congruence (8.2).

Definition 8.6.1 *Let $a \leq b$ and $I = (i_a, i_{a+1}, \ldots, i_b)$ a sequence with $0 \leq i_k \leq k$ for all k with $a \leq k \leq b$. We say that ℓ is a* splitting index for I *if $\ell > a$ and for $k \geq \ell$ one has*

$$i_k \leq k - \ell.$$

Remark that for a splitting index ℓ one can apply proposition 8.5.1 and that $i_\ell = 0$.

Lemma 8.6.1 *Let I as above and assume that*

$$\sum_{k=a}^{b} i_k \leq b - a - 1.$$

Then there exists at least one splitting index for I.

Proof: Let $\mathcal{N} := \{k \ \ | i_k = 0\}$ be the set of all indices k such that the corresponding i_k is zero. Since the sum has $b - a + 1$ summands i_k, the set $\mathcal{N}$ has at least two elements. So there exists at least one index $k \neq a$ such that $i_k = 0$.
We will show by contradiction that one of these zero-indices is a splitting index.
We say that $\nu > k$ is a *violating index* with respect to $k \in \mathcal{N}$ if $i_\nu > \nu - k$. Assume now that all $k \in \mathcal{N}$ possess a violating index.
It follows directly that for each violating index ν, $i_\nu \geq 2$. Furthermore, if ν is a violating index for m different zero-indices $k_1 < ... < k_m$, it follows that $i_\nu \geq m + 1$.
Now assume that we have μ different violating indices $\nu_1, ..., \nu_\mu$ and that ν_j is a violating index for all $j \in \mathcal{N}_j$, where we partition $\mathcal{N}$ into disjoint subsets

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup ... \cup \mathcal{N}_\mu.$$

Then $\sum_{j=1}^{\mu} i_{\nu_j} \geq \sum_{j=1}^{\mu} (\#\mathcal{N}_j + 1) = \#\mathcal{N} + \mu$, and

$$\sum_{k=a+1}^{b} i_k \geq \#\mathcal{N} \cdot 0 + \sum_{j=1}^{\mu} i_{\nu_j} + (b - a - (\#\mathcal{N} + \mu)) \cdot 1 = b - a > b - a - 1,$$

a contradiction.□

We can sharpen lemma 8.6.1 to:

Lemma 8.6.2 *Let I be as above and assume that*

$$\sum_{k=a}^{b} i_k = b - a - m.$$

Then there exist at least m different splitting indices for I.

Proof: We proceed by induction on m. The case $m = 1$ is just lemma 8.6.1. Assume that for all $n \leq m$, we have proven the statement.
Now assume $\sum_{k=a}^{b} i_k = b - a - (m + 1)$. Since $m + 1 > 1$, there exists a splitting index ν. We can split up the set of indices $\{i_a, ..., i_b\} = \{i_a, ..., i_{\nu-1}\} \cup \{i_\nu, ..., i_b\}$ in position ν such that $\sum_{k=a}^{\nu-1} i_k = N_\nu$ and $\sum_{k=\nu}^{b} i_k = b - a - m - 1 - N_\nu$. Depending on N_ν, we have to distinguish between the following cases:

1. $N_\nu > (\nu-1)-a-1$. It follows that $b-a-m-1-N_\nu < b-a-m-((\nu-1)-a-1) = b-m-(\nu-1)$, and thus $\sum_{k=\nu}^{b} i_k \leq b-\nu-m$. By induction, there exist at least m splitting indices in $(i_\nu, ..., i_b)$, and thus for the whole $(i_a, ..., i_b)$, there exist at least $m+1$ such indices.

2. The case $N_\nu \leq (\nu-1)-a-1$ splits up into two cases:

   (a) $N_\nu \leq (\nu-1)-a-m$. By induction, $(i_a, ..., i_{\nu-1})$ has at least m splitting indices, and the whole $(i_a, ..., i_b)$ has at least $m+1$ such indices.

   (b) $N_\nu = (\nu-1)-a-n$, where $1 \leq n \leq m$. Since $\sum_{k=a}^{\nu-1} i_k = (\nu-1)-a-n$, by induction for $(i_a, ..., i_{\nu-1})$ exist at least n splitting indices. Since $\sum_{k=\nu}^{b} i_k = b-\nu-(m-n)$, for $(i_\nu, ..., i_b)$, there exist at least $m-n$ splitting indices. Thus, for the whole $(i_a, ..., i_b)$ exist at least $n+(m-n)+1 = m+1$ splitting indices.

□

Lemma 8.6.3 *1. Let $I = (i_0, ..., i_s)$ and $J = (j_1, ..., j_{s-1})$ with*

$$\sum_{k=0}^{s} i_k + \sum_{k=1}^{s-1} j_k \leq s-1.$$

Let S_I be the set of splitting indices of I and S_J be the set of splitting indices of J. Then

$$S_I \cap (S_J \cup \{1, s\}) \neq \emptyset.$$

2. Let $I = \{i_0, ..., i_{s-1}\}$ and $J = (j_1, ..., j_s)$ with

$$\sum_{k=0}^{s-1} i_k + \sum_{k=1}^{s} j_k \leq s-1.$$

Let S_I be the set of splitting indices of I and S_J be the set of splitting indices of J. Then

$$(S_I \cup \{s\}) \cap (S_J \cup \{1\}) \neq \emptyset.$$

Proof:

1. Note that since $S_I \cup S_J \cup \{1, s\} \subset \{1, 2, ..., s\}$, it follows that $\#(S_I \cup S_J \cup \{1, s\}) \leq s$. Note that $\sum_{k=0}^{s} i_k \geq s - \#S_I$ by lemma 8.6.2. This implies that $\sum_{k=1}^{s-1} j_k \leq s-2-(s-(\#S_I+1))$, and hence that $\#S_J \geq s-(\#S_I+1)$ by lemma 8.6.2. But $\#S_I + \#S_J + 2 = \#S_I + s - (\#S_I+1) + 2 = s+1 > s$, which implies $\#(S_I \cap (S_J \cup \{1, s\})) \geq 1$, and thus the statement follows.

2. Note that since $(S_I \cup \{s\}) \cup (S_J \cup \{1\}) \subset \{1, ..., s\}$, it follows that $\#(S_I \cup \{s\}) \cup (S_J \cup \{1\}) \leq s$. Now $\sum_{k=0}^{s-1} i_k \geq s-1-\#S_I$, which implies $\sum_{k=1}^{s} j_k \leq s-1-(s-\#S_I-1)$, and $\#S_J \geq s-\#S_I-1$. But $\#S_I+1+\#S_J+1 \geq \#S_I+1+s-\#S_I = s+1 > s$, which implies that $\#((S_I \cup \{s\}) \cap (S_J \cup \{1\})) \geq 1$, and the statement follows.

□

8.7 Proof for higher s

We will use the combinatorical lemmas on splitting indices from the last section to prove the congruence (8.2) modulo p^s.
For a sequence $I = (i_a, ..., i_b)$, we write

$$p^I := p^{\sum_{k=a}^{b} i_k}.$$

For a sequence $J = (j_a, ..., j_b)$, we define $J + 1 := (j_a + 1, ..., j_b + 1)$.
Note that if $k - j_k > 0$ for $a \leq k \leq b$, then we have

$$G(a, b; J + 1) = G(a, b; J), \tag{8.7}$$

since the constant term of a Laurent-polynomial $f(X)$ is the same as the constant term of the Laurent-polynomial $f(X^p)$.
Let

$$p^{I+J} G(0, s; I) G(1, s - 1; J + 1)$$

be a summand on the left-hand side of (8.6) defined by the tuple (I, J) with $\sum_{k=0}^{s} i_k + \sum_{k=1}^{s-1} j_k \leq s - 1$, and let $1 \leq \nu \leq s$ be such that $G(0, s; I)$ splits in position ν and either $G(1, s - 1; J + 1)$ splits in position ν or $\nu \in \{1, s\}$. Such a ν exists by lemma (8.6.3). Define $I' = (i'_0, ..., i'_{s-1})$ and $J' = (j'_1, ..., j'_s)$ by

$$\begin{aligned} i'_k &= i_k \text{ for } k \leq \nu - 1, \\ i'_k &= j_k \text{ for } k \geq \nu, \\ j'_k &= j_k \text{ for } k \leq \nu - 1, \\ j'_k &= i_k \text{ for } k \geq \nu. \end{aligned}$$

To show that $p^{I'+J'} G(0, s - 1; I') G(1, s; J' + 1)$ is in fact a summand on the right-hand side of (8.6), we have to explain why $i'_k \leq k$ and $j'_k \leq k - 1$. Note that $j_k \leq k - 1$ for $1 \leq k \leq s - 1$ and $i_k \leq k$ for $0 \leq k \leq s$. Furthermore, we have $i_k \leq k - 1$ for $k \geq \nu$ since $i_\nu = 0$ and $G(0, s; I)$ splits in position ν, which means that $k - i_k \geq \nu \geq 1$ for $k \geq \nu$.
By definition of j'_k and i'_k, it now follows that $j'_k \leq k - 1$ for $1 \leq k \leq s$, and $i'_k \leq k$ for $0 \leq k \leq s - 1$.
Now that we know that $p^{I'+J'} G(0, s - 1; I') G(1, s; J' + 1)$ is in fact a summand on the right-hand side of congruence (8.6), we prove the following proposition. Remark that we obviously have $p^{I+J} = p^{I'+J'}$.

Proposition 8.7.1 *Let I, J, I' and J' be defined as above. Then,*

$$G(0, s, I) G(1, s - 1; J + 1) = G(0, s - 1; I') G(1, s; J' + 1).$$

Thus, we can identify each summand on the left-hand side of (8.6) with a summand on the right-hand side.

Proof: By a direct computation:

$$\begin{aligned}
& G(0,s;I)G(1,s-1;J+1) \\
= \;& G(0,\nu-1;I)G(\nu,s;I)G(1,\nu-1;J+1)G(\nu,s-1;J+1) \text{ by lemma 8.6.3} \\
= \;& G(0,\nu-1;I)G(\nu,s;I+1)G(1,\nu-1;J+1)G(\nu,s-1;J) \text{ by } (8.7) \\
= \;& G(0,\nu-1;I)G(\nu,s-1;J)G(1,\nu-1;J+1)G(\nu,s;I+1) \text{ (commutation)} \\
= \;& G(0,\nu-1;I')G(\nu,s-1;I')G(1,\nu-1;J'+1)G(\nu,s;J'+1) \text{ by definition of } I',\, J' \\
= \;& G(0,s-1;I')G(1,s;J'+1) \text{ by lemma 8.6.3,}
\end{aligned}$$

the statement follows. Note that the last equality follows since by definition of I' and J', $i'_\nu = j'_\nu = 0$, $k - i'_k \geq \nu$ and $k - j'_k \geq \nu$ for $k > \nu$. Thus, $G(0, s-1; I')$ and $G(1, s; J'+1)$ both split at ν.□

Since by P proposition 8.7.1, we can identify every summand on the left-hand side of equation (8.6) satisfying $I + J \leq s - 1$ with a summand on the right-hand side, both sides are equal modulo p^s and the proof of theorem 8.2.1 is complete.

Remark: The above arguments to prove the congruence $D3$ can be slightly simplified, as was shown to us by A. Mellit.

8.8 Examples

In this section, we give some examples for which we could apply theorem 8.2.1.

No. 24 from the list of Batyrev and Kreuzer

Let f be the Laurent-polynomial No. 24 in the list of Batyrev and Kreuzer [**6**], which is given by

$$\begin{aligned}
f: \;=\; & 1/X_4 + X_2 + 1/X_1X_4 + 1/X_1X_3X_4 + 1/X_1X_2X_3X_4 + 1/X_3 + X_1/X_3 \\
+ \; & X_2/X_3X_4 + X_1/X_3X_4 + X_1X_2/X_3X_4 + X_2/X_4 + 1/X_2X_4 + 1/X_1X_2X_4 \\
+ \; & 1/X_1X_2 + 1/X_1 + 1/X_2X_3X_4 + X_4 + 1/X_2 + X_1 + X_1/X_4 + 1/X_3X_4 \\
+ \; & X_3 + 1/X_2X_3.
\end{aligned}$$

The first 9 coefficients $a(n) := [f^n]_0$ in this example are:
$a(0) = 1, a(1) = 0, a(2) = 18, a(3) = 168, a(4) = 2430, a(5) = 37200, a(6) = 605340, a(7) = 10342080, a(8) = 182788830.$

The Newton polyhedron $\Delta(f)$ is reflexive (see [**6**]), and hence by theorem 8.2.1, the coefficients $a(n)$ satisfy the congruence (8.1) modulo p^s for arbitrary s.
Note that the power series $\Phi(t) = \sum_{n=0}^{\infty} a(n)t^n$ is a solution to a fourth order linear differential equation $PF = 0$, where the differential operator P is a CY(4)-operator and is given

by

$$\begin{aligned} P &:= 885010054\theta^4 + t(9123382\theta(-291 - 1300\theta - 2018\theta^2 + 1727\theta^3) + ... \\ &+ 3461674786667136t^{11}(\theta+1)(\theta+2)(\theta+3)(\theta+4), \end{aligned}$$

where $\theta := t\partial/\partial t$.

No. 41 from the list of Batyrev and Kreuzer

Let f be the Laurent-polynomial No. 41 in the list of Batyrev and Kreuzer [**6**], which is given by

$$\begin{aligned} f &:= X_4 + X_1 + X_1X_4 + 1/X_1X_3X_4 + 1/X_2X_3 + 1/X_1X_2X_4 + 1/X_1X_3 \\ &+ 1/X_1 + X_3X_4 + 1/X_1X_2 + 1/X_1X_2X_3X_4 + 1/X_1X_2X_3 \\ &+ X_1X_2X_3X_4 + X_2X_3X_4 + 1/X_1X_4 + X_2X_3 + 1/X_2 + 1/X_3 + 1/X_4 \\ &+ X_1X_3X_4 + X_3 + 1/X_3X_4 + X_2 + 1/X_2X_4 + 1/X_2X_3X_4 \end{aligned}$$

The first 9 coefficients $a(n) := [f^n]_0$ in this example are: $a(0) = 1, a(1) = 0, a(2) = 20, a(3) = 186, a(4) = 2940, a(5) = 46680, a(6) = 803990, a(7) = 14453460, a(8) = 269264380$.
As in the example before, the Newton polyhedron $\Delta(f)$ is reflexive, and hence the coefficients $a(n)$ satisfy congruence (8.1) for arbitrary s. Note that also in this case, $\Phi(t)$ is a solution to a fourth order Calabi-Yau differential equation, where the CY(4)-operator P is given by

$$\begin{aligned} P &:= 82281\theta^4 + 91t\theta(-273 - 1210\theta - 1874\theta^2 + 782\theta^3) + ... \\ &- 21292817700t^{11}(\theta+1)(\theta+2)(\theta+3)(\theta+4). \end{aligned}$$

No. 39 from the list of Batyrev and Kreuzer

Let f be the Laurent-polynomial No. 39 in the list of Batyrev and Kreuzer [**6**], which is given by

$$\begin{aligned} f &:= X_1/X_2X_3 + 1/X_1X_2X_3X_4 + X_3X_4 + X_4 + 1/X_1X_2 + 1/X_1X_3 \\ &+ 1/X_1X_2X_4 + X_2 + X_1X_2/X_3X_4 + X_1/X_3 + X_2/X_4 + X_3 \\ &+ 1/X_4 + X_1 + X_1/X_2 + 1/X_2X_3X_4 + X_1/X_3X_4 \\ &+ X_2/X_3X_4 + 1/X_1 + 1/X_2 + 1/X_1X_3X_4 + 1\mathbb{X}_2X_4 + 1/X_3. \end{aligned}$$

The first 9 coefficients $a(n) := [f^n]_0$ in this example are: $a(0) = 1, a(1) = 0, a(2) = 20, a(3) = 168, a(4) = 2652, a(5) = 40080, a(6) = 666920, a(7) = 11536560, a(8) = 207013660$.
As in the examples before, the Newton polyhedron $\Delta(f)$ is reflexive, and hence the coefficients $a(n)$ satisfy congruence (8.1) for arbitrary s. Note that also in this case, $\Phi(t)$ is

solution to a fourth order Calabi-Yau differential equation, where the CY(4)-operator P is given by

$$\begin{aligned} P &:= 16\theta^4 - 4t\theta(12 + 53\theta + 82\theta^2 + 2\theta^3)... \\ &- 621920t^{11}(\theta + 1)(\theta + 2)(\theta + 3)(\theta + 4). \end{aligned}$$

No. 62 from the list of Batyrev and Kreuzer

Let f be the Laurent-polynomial No. 62 in the list of Batyrev and Kreuzer [**6**], which is given by

$$f := X_1 + X_2 + X_3 + X_4 + \frac{1}{X_1X_2} + \frac{1}{X_1X_3} + \frac{1}{X_1X_4} + \frac{1}{X_1^2X_2X_3X_4}.$$

Then the coefficients $a(n)$ are given by $a(n) = 0$ if $n \neq 0 \mod 3$ and

$$a(3n) = \frac{(3n)!}{n!^3} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}.$$

The Newton polyhedron $\Delta(f)$ is reflexive (see [**6**]), and hence by theorem 8.2.1, the coefficients $a(n)$ satisfy the congruence (8.1) modulo p^s for arbitrary s.
The power series $\Phi(t) = \sum_{n=0}^{\infty} a(3n)t^n$ is solution to a fourth order linear differential equation $PF = 0$, where the differential operator P is of Calabi-Yau type and is given by

$$\begin{aligned} P &:= \theta^4 - 3t(3\theta + 2)(3\theta + 1)(11\theta^2 + 11\theta + 3) \\ &- 9t^2(3\theta + 5)(3\theta + 2)(3\theta + 4)(3\theta + 1). \end{aligned}$$

Since in this example (as in many others), only the coefficients $a(n)$ with $n = 3k$ are nonzero, it would be good to prove the following congruence for this example:

$$\begin{aligned} & a(3(n_0 + n_1p + ... + n_sp^s))a(3(n_1 + ... + n_{s-1}p^{s-2})) \\ \equiv\ & a(3(n_0 + ... + n_{s-1}p^{s-1}))a(3(n_1 + ... + n_sp^{s-1})) \mod p^s. \end{aligned}$$

8.9 Behaviour under covering

The last example raises the question after a congruence among the $k-$fold coefficients if $a(n) \neq 0$ implies $k|n$. Let f be a Laurent-polynomial corresponding to a reflexive polyhedron, let $\mathcal{A}$ be the exponent matrix corresponding to f, and consider the vectors with integral entries in the Kernel of $\mathcal{A}$. If there exists a positive integer k such that

$$\ell := \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_m \end{pmatrix} \in \ker(\mathcal{A}) \Rightarrow k|(\ell_1 + ... + \ell_m),$$

then it follows that

$$a(n) := [f^n]_0 \neq 0 \Rightarrow k|n,$$

since for $l \in \mathbb{N}$,

$$[f^l]_0 = \sum_{(\ell_1,\ldots,\ell_m)\in A_{f,l}} c_{\mathbf{a}_1}^{\ell_1}\ldots c_{\mathbf{a}_m}^{\ell_m}\binom{l}{\ell_1}\binom{l-\ell_1}{\ell_2}\ldots\binom{l-\ell_1-\ldots.-\ell_{m-2}}{\ell_{m-1}},$$

where

$$\begin{aligned} A_{f,l} &:= \{(\ell_1,\ldots,\ell_m)\in\mathbb{N}_0^m, \ell_1+\ldots+\ell_m = l, \sum_{j=1}^m a_{i,j}\ell_j = 0 \text{ for } 1\leq i\leq n\} \\ &= \{(\ell_1,\ldots,\ell_m)\in\mathbb{N}_0^m, \ell_1+\ldots+\ell_m = l, \mathcal{A}\cdot(\ell_1,\ldots,\ell_m)^T = 0\} \\ &= \ker(\mathcal{A})\cap\{(\ell_1,\ldots,\ell_m)\in\mathbb{N}_0^m, \ell_1+\ldots+\ell_m = l.\}. \end{aligned}$$

We are interested in the congruences

$$\begin{aligned} &a(k(n_0+\ldots+n_sp^s))a(k(n_1+\ldots+n_{s-1}p^{s-2}))\equiv \\ &a(k(n_0+\ldots+n_{s-1}p^{s-1}))a(k(n_1+\ldots+n_sp^{s-1})) \mod p^s, \end{aligned} \tag{8.8}$$

which we will prove in general for $s = 1$, and which we will prove for the example No. 62 from the list of Batyrev and Kreuzer by proving that the following condition is satisfied in this example:

Condition 8.9.1 *For a tuple* $(\ell_1,\ldots,\ell_m)$ *with*

$$\ell_1+\ldots+\ell_m = k\mu \leq k(p-1),$$

it follows that

$$p|\gcd(\sum_{j=1}^m a_{i,1}\ell_1,\ldots,\sum_{j=1}^m a_{j,n}\ell_j) \Rightarrow \sum_{j=1}^m a_{i,1}\ell_j = \ldots = \sum_{j=1}^m a_{j,n}\ell_j = 0.$$

First of all, we give a general proof of (8.8) for $s = 1$.

Proposition 8.9.1 *Let* $a(n), n\in\mathbb{N}$ *be an integral sequence satisfying*

$$a(n_0+n_1p+\ldots+n_sp^s)\equiv a(n_0)a(n_1)\ldots a(n_s) \mod p$$

for $0\leq n_i\leq p-1$ *and* $a(n)\neq 0$ *iff* $k|n$. *Then*

$$a(k(n_0+n_1p+\ldots+n_sp^s))\equiv a(kn_0)a(kn_1)\ldots a(kn_s) \mod p.$$

Proof: If $kn_i < p$ for $1\leq i\leq s$, then the proposition follows directly. Hence assume that there exists an n_i such that $kn_i = n_i' + n_i''p > p-1$. We may assume that $kn_j < p$ for all $j < i$. Then

$$\begin{aligned} a(k(n_0+n_1p+\ldots+n_sp^s)) &= a(kn_0+\ldots+kn_{i-1}p^{i-1}+n_i'p^i+\ldots) \\ &\equiv a(kn_0)\ldots a(kn_{i-1})a(n_i')\ldots \mod p. \end{aligned}$$

Since $n_i < p$ and $kn_i = n_i' + n_i''p \geq p$, it follows that $k \not| n_i'$ and $a(n_i') = 0$ by assumption. Hence it follows that

$$a(k(n_0 + n_1 p + ... + n_a p^s)) \equiv 0 \mod p.$$

On the other hand, $a(kn_1) = a(n_1' + n_1''p) \equiv a(n_1')a(n_1'') \mod p$ where $a(n_i') = 0$, and thus $a(kn_i) \equiv 0 \mod p$ and

$$a(kn_0)...a(kn_s) \equiv 0 \mod p$$

and the proposition follows. □

Corollary 8.9.1 *Let $f(X)$ be a Laurent polynomial such that the corresponding polytope P_f is reflexive and such that $[f^n]_0 \neq 0 \Rightarrow k|n$. Then*

$$[f^{k(n_0+n_1p+...+n_sp^s)}]_0 \equiv [f^{kn_0}]_0...[f^{kn_s}]_0 \mod p.$$

No. 62 from the list of Batyrev and Kreuzer again

In this example, the exponent matrix is

$$\mathcal{A} := \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

A basis of $\ker(\mathcal{A})$ is given by

$$\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\},$$

and thus it follows that $[f^n]_0 \neq 0 \Rightarrow 3|n$ and $k = 3$. We prove that condition 8.9.1 is satisfied in this example. Assume that $p \neq 3$ and that

$$p|\gcd(\sum_{j=1}^{8} a_{1,j}\ell_j, ..., \sum_{j=1}^{8} a_{4,j}\ell_j) \text{ for } \ell_1 + ... + \ell_8 = 3\mu \leq 3(p-1).$$

This means that there exist $x_1, x_2, x_3, x_4 \in \mathbb{Z}$ such that

$$\begin{aligned} \ell_1 &= \ell_5 + \ell_6 + \ell_7 + 2\ell_8 + x_1 p, \\ \ell_2 &= \ell_5 + \ell_8 + x_2 p, \\ \ell_3 &= \ell_6 + \ell_8 + x_3 p, \\ \ell_4 &= \ell_7 + \ell_8 + x_4 p, \end{aligned}$$

which implies

$$3(\ell_5+\ell_6+\ell_7+2\ell_8)+(x_1+x_2+x_3+x_4)p=3\mu\leq 3(p-1).$$

Thus, it follows that $(x_1+...+x_4)=3z$ for some $z\in\mathbb{Z}$ and that

$$\ell_5+\ell_6+\ell_7+2\ell_8+zp=\mu\leq p-1.$$

Since $\ell_5,...,\ell_8$ are non-negative integers, it follows directly that $z\leq 0$. Now, consider the following cases:

1. $z=0$: Then
$$\ell_5+\ell_6+\ell_7+2\ell_8\leq p-1 \tag{8.9}$$
Assume that $x_i<0$, i.e. $x_i\leq -1$ for some $1\leq i\leq 4$. Since $\ell_1,...,\ell_4$ are non-negative integers, it follows that either $\ell_5+\ell_6+\ell_7+2\ell_8\geq p$ or $\ell_j+\ell_8\geq p$ for some $5\leq j\leq 7$, a contradiction to (8.9). Thus, since $x_1+x_2+x_3+x_4=0$, it follows that $x_1=x_2=x_3=x_4=0$ and that
$$\sum_{j=1}^{8}a_{1,j}\ell_j=...=\sum_{j=1}^{8}a_{4,j}\ell_j=0$$
in this example.

2. $z<0$: Assume that $\ell_5+\ell_6+\ell_7+2\ell_8<(-z+1)p$. Since $\ell_1\geq 0$, it follows that $x_1>z-1$, and since x_1 is integral, we have $x_1\geq z$. Since $x_1+x_2+x_3+x_4=3z$, it follows that $x_2+x_3+x_4\leq 2z$. Now assume that $x_i\geq z$ for $2\leq i\leq 4$. Then $x_2+x_3+x_4\geq 3z$, a contradiction. Hence there exists an index i such that $x_i<z$, and hence $x_i\leq z-1$. Since $\ell_i\geq 0$, it follows that $\ell_{i+2}+\ell_8\geq(-z+1)p$, a contradiction since $\ell_{i+2}+\ell_8\leq\ell_5+\ell_6+\ell_7+2\ell_8<(-z+1)p$ by assumption. Thus, we have $\ell_5+\ell_6+\ell_7+2\ell_8\geq(-z+1)p$, which implies $p\leq\ell_5+\ell_6+\ell_7+2\ell_8+zp\leq p-1$, a contradiction.

Thus, it follows that the only possible case is $z=0$, and $x_1=x_2=x_3=x_4=0$, which proves that condition 8.9.1 is satisfied in this example.

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In this second example, the Laurent polynomial f is given by

$$f:=X_1+X_2+X_3+X_4+\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{X_2X_3}{X_1}+\frac{1}{X_2X_3X_4},$$

and the exponent matrix is

$$\mathcal{A}:=\begin{pmatrix}1&0&0&0&-1&0&0&-1&0\\0&1&0&0&0&-1&0&1&-1\\0&0&1&0&0&0&-1&1&-1\\0&0&0&1&0&0&0&0&-1\end{pmatrix}.$$

A basis of $\ker(\mathcal{A})$ is given by

$$\{\begin{pmatrix}1\\0\\0\\0\\1\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\\0\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\1\\0\\0\\0\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\\1\\0\\0\\0\\0\\1\end{pmatrix}\},$$

and thus it follows that $[f^n]_0 \neq 0 \Rightarrow 2|n$ and $k = 2$. For this example, no formula for $a(2n)$ is known. We prove that condition 8.9.1 is satisfied in this example.
Assume that $p \neq 2$ and that

$$p|\gcd(\sum_{i=1}^{9} k_i^1\alpha_1, ..., \sum_{i=1}^{9} k_i^4\alpha_i) \text{ for } \alpha_1 + ... + \alpha_9 = 2\mu \leq 2(p-1).$$

This means that there exist $x_1, x_2, x_3, x_4 \in \mathbb{Z}$ such that

$$\begin{aligned}\alpha_1 &= \alpha_5 + \alpha_8 + x_1 p,\\ \alpha_2 &= \alpha_6 - \alpha_8 + \alpha_9 + x_2 p,\\ \alpha_3 &= \alpha_7 - \alpha_8 + \alpha_9 + x_3 p,\\ \alpha_4 &= \alpha_9 + x_4 p,\end{aligned}$$

which implies

$$2(\alpha_5 + \alpha_6 + \alpha_7 + 2\alpha_9) + (x_1 + x_2 + x_3 + x_4)p = 2\mu \leq 2(p-1).$$

Thus, it follows that $(x_1 + x_2 + x_3 + x_4) = 2z$ for some $z \in \mathbb{Z}$ and that

$$\alpha_5 + \alpha_6 + \alpha_7 + 2\alpha_9 + zp = \mu \leq p - 1.$$

Since $\alpha_5, ..., \alpha_9$ are non-negative integers, it follows directly that $z \leq 0$. Now, consider the following two possible cases:

1. $z = 0$: Then
$$\alpha_5 + \alpha_6 + \alpha_7 + 2\alpha_9 \leq p - 1. \tag{8.10}$$
Assume that $x_i < 0$, i.e. $x_i \leq -1$ for some $2 \leq i \leq 4$. Since $\alpha_1, ..., \alpha_4$ and α_8 are non-negative integers, it follows that either $\alpha_6 + \alpha_9 \geq p$, $\alpha_7 + \alpha_9 \geq p$ or $\alpha_9 \geq p$, a contradiction to (8.10).
Assume that $x_1 < 0$, i.e. $x_1 \leq -1$. Then $\alpha_5 + \alpha_8 \geq -x_1 p$ and thus $\alpha_8 \geq -x_1 p - \alpha_5$. But since α_2 and α_3 are non-negative, this implies $\alpha_5 + \alpha_6 + \alpha_9 \geq (x_2 + x_1)p$ and $\alpha_5 + \alpha_7 + \alpha_9 \geq (x_3 + x_1)p$. If $x_2 + x_1 < 0$, i.e. $x_2 + x_1 \leq -1$, then

$\alpha_5 + \alpha_6 + \alpha_9 \geq p$, a contradiction to (8.10). The same holds for $x_3 + x_1$ and $\alpha_5 + \alpha_7 + \alpha_9$. Thus, $x_2 \geq -x_1$ and $x_3 \geq -x_1$, which implies $x_4 \leq x_1 \leq -1$ since $x_1 + x_2 + x_3 + x_4 = 0$ and thus $\alpha_9 \geq p$, a contradiction to (8.10). Hence, the only possible case is $x_1 = x_2 = x_3 = x_4 = 0$, which implies

$$\sum_{i=1}^{8} k_i^1 \alpha_i = ... = \sum_{i=1}^{8} k_i^4 \alpha_i = 0$$

in this example.

2. $z < 0$: Then

$$\alpha_5 + \alpha_6 + \alpha_7 + 2\alpha_9 + zp \leq p - 1. \tag{8.11}$$

Assume that $\alpha_5+\alpha_6+\alpha_7-\alpha_8+2\alpha_9 < (-z+1)p$. Then, since α_1, α_2 and α_3 are non-negative integers, it follows that $x_1+x_2+x_3 \geq z$. Since $x_1+x_2+x_3+x_4 = 2z$, this induces $x_4 \leq z$ and thus $\alpha_9 \geq -zp$, since α_4 is non-negative. Thus, $2\alpha_9 \geq -2zp$, which implies $-zp \leq 2\alpha_9 + zp \leq \alpha_5 + \alpha_6 + \alpha_7 + 2\alpha_9 + zp$, a contradiction to (8.11). Thus, $p \leq \alpha_5 + \alpha_6 + \alpha_7 - \alpha_8 + 2\alpha_9 + zp \leq \alpha_5 + \alpha_6 + \alpha_7 + 2\alpha_9 + zp$, which is also a contradiction to (8.11). Thus, it follows that the case $z < 0$ can not occur.

In the whole, it follows that $z = 0$ and $x_1 = x_2 = x_3 = x_4 = 0$, which proves that condition 8.9.1 is satisfied in this example.

8.10 The statement D1

For the proof of congruence (8.1), the coefficients $c_{\mathbf{a}}$ of

$$f(X) = \sum_{\mathbf{a}} c_{\mathbf{a}} X^{\mathbf{a}}$$

did not play a role. This is different if one is interested in the proof of part D1 of the Dwork congruences. Let $n \in \mathbb{N}$, and write $n = n_0 + pn_1$, where $n_0 \leq p - 1$. Then to prove D1 for the sequence $a(n) := [f^n]_0$ means that one has to prove that

$$\frac{[f^{n_0+n_1p}]_0}{[f^{n_1}]_0} \in \mathbb{Z}_p. \tag{8.12}$$

Sticking to the notation of the previous sections, we write

$$f^{n_0+n_1p}(X) = f^{n_0}(X) f^{n_1}(X^p) + p f^{n_0}(X) g_{n-1,1}(X). \tag{8.13}$$

Assume that $p^k | [f^{n_1}]_0$. To prove (8.12), one has to prove that $p^k | [f^{n_0+n_1p}]_0$. By (8.13), this is equivalent to proving that $p^{k-1} | [f^{n_0} g_{n_1,1}(X)]_0$. Thus, the proof of part D1 of the Dwork congruences requires an investigation in the $p-$adic orders of the constant terms of f^{n_1} and $g_{n_1,1}$ for arbitrary n_1, and methods that are completely different from the methods we applied to prove the congruence (8.1).

Chapter 9

An experimental approach: matrix computations

In this chapter, we describe an approach we tried to compute the Frobenius polynomial for hypergeometric Calabi-Yau differential operators. The idea was to compute the characteristic polynomial of the Frobenius matrix with a given precision by computing the characteristic polynomial of a matrix which might be different from the Frobenius matrix, but has the same characteristic polynomial.
This method worked out well in practice for hypergeometric CY(4)-differential operators, and we obtained the same results as by computing the unit root and the root of $p-$adic valuation 1 of the Frobenius polynomial. If the differential operators were not of hypergeometric type, the analytic continuation method which we applied experimentally did not work, and thus in these cases, we could not perform any computations.

9.1 Matrix computations for CY(4)-operators

First, we describe the crucial idea behind the method. Assume that P is a hypergeometric CY(4)-differential operator. Let P be given by

$$P = A_4(z)\theta^4 + A_3(z)\theta^3 + ... + A_0(z)$$

with $A_4(0) \neq 0$. Let

$$\begin{aligned}
y_0(z) &= f_0(z), \\
y_1(z) &= \log(z) f_0(z) + f_1(z), \\
y_2(z) &= \frac{1}{2}\log^2(z) f_0(z) + \log(z) f_1(z) + f_2(z), \\
y_3(z) &= \frac{1}{6}\log^3(z) f_0(z) + \frac{1}{2}\log^2(z) f_1(z) + \log(z) f_2(z) + f_3(z),
\end{aligned}$$

where $f_0(0) = 1$ be a Frobenius basis of solutions to a CY(4)-differential equation

$$Py = 0. \tag{9.1}$$

Let $\mathcal{B}(z)$ be the matrix

$$\mathcal{B}(z) = \begin{pmatrix} y_3 & \theta(y_3) & \theta^2(y_3) & \theta^3(y_3) \\ y_2 & \theta(y_2) & \theta^2(y_2) & \theta^3(y_2) \\ y_1 & \theta(y_1) & \theta^2(y_1) & \theta^3(y_1) \\ y_0 & \theta(y_0) & \theta^2(y_0) & \theta^3(y_0) \end{pmatrix}.$$

Let $\mathcal{M}(z)$ be the connection matrix with regard to the basis $\{\omega, \nabla(\omega), \nabla^2(\omega), \nabla^3(\omega)\}$ for $\nabla := \nabla(\theta)$, where $\theta := zd/dz$ is the logarithmic derivative. Then, $\mathcal{B}(z)$ is a solution to the differential equation

$$z\frac{d}{dz}\mathcal{B}(z) - \mathcal{B}(z)\mathcal{M}(z) = 0,$$

which is the "dual" to the differential equation

$$z\frac{d}{dz}\mathcal{C}(z) + \mathcal{M}(z)\mathcal{C}(z) = 0$$

occuring in section 3.6.
As in section 3.6, let $\mathfrak{A}_4$ denote the matrix

$$\mathfrak{A}_4 = \begin{pmatrix} \varepsilon & \alpha & \beta & \gamma \\ 0 & \varepsilon p & p\alpha & p\beta \\ 0 & 0 & \varepsilon p^2 & p^2\alpha \\ 0 & 0 & 0 & \varepsilon p^3 \end{pmatrix},$$

where $\beta = \alpha^2/2$.
We introduce the following notation:

$$\theta^l(Y_k) := \theta^l(y_k)|_{\log(z)=0} \text{ for } 0 \le l \le 3, 0 \le k \le 3,$$

i.e. $\theta^l(Y_k)(z)$ is the non-logarithmic part of $\theta^l(y_k)(z)$, such that, for example, $Y_1(z) = f_1(z)$.
Using this notation we define the matrix $B(z)$ as

$$B(z) := \begin{pmatrix} Y_3 & \theta(Y_3) & \theta^2(Y_3) & \theta^3(Y_3) \\ Y_2 & \theta(Y_2) & \theta^2(Y_2) & \theta^3(Y_2) \\ Y_1 & \theta(Y_1) & \theta^2(Y_1) & \theta^3(Y_1) \\ Y_0 & \theta(Y_0) & \theta^2(Y_0) & \theta^3(Y_0) \end{pmatrix}.$$

In terms of the power series f_0, f_1, f_2, f_3, B can be written as

$$B(z) = \begin{pmatrix} f_3 & \theta(f_3) + f_2 & \theta^2(f_3) + 2\theta(f_3) + f_1 & \theta^3(f_3) + 3\theta^2(f_2) + 3\theta(f_1) + f_0 \\ f_2 & \theta(f_2) + f_1 & \theta^2(f_2) + 2\theta(f_1) + f_0 & \theta^3(f_2) + 3\theta^2(f_1) + 3\theta(f_0) \\ f_1 & \theta(f_1) + f_0 & \theta^2(f_1) + 2\theta(f_0) & \theta^3(f_1) + 3\theta^2(f_0) \\ f_0 & \theta(f_0) & \theta^2(f_0) & \theta^3(f_0) \end{pmatrix}.$$

Since $\mathcal{B}(z)$ is a solution to the “dual” differential equation, it follows for the Frobenius matrix $A_\phi(z)$ that

$$p^3 A_\phi(z)^{-1} = B(z^p)^{-1} p^3 \mathfrak{A}_4^{-1} B(z)$$

on $p\mathbb{Z}_p$. Since, by the Weil conjectures, the characteristic polynomial of $p^3 A_\phi(z)^{-1}$ coincides with the characteristic polynomial of $A_\phi(z)$ at a Teichmüller point, we want to compute the characteristic polynomial of $p^3 A_\phi(z)^{-1}$ explicitly. For simplicity, define

$$A(z) := p^3 A_\phi(z)^{-1}.$$

We assume that the constant $\varepsilon = \pm 1$ occuring in $\mathfrak{A}_4$ is 1 in this case, since P is a hypergeometric differential operator and for these operators, there exist families of CY-threefolds such that $\varepsilon = 1$.

Thus, we have the means to compute the Frobenius matrix on $p\mathbb{Z}_p$ up to two parameters α, γ. By previous results, see section 5.7, we know that the Frobenius polynomial is indeed independent of α and γ. In this section, we try to provide another explanation of this fact. To compute the characteristic polynomial of the Frobenius matrix in a Teichmüller point, we have a problems to solve:

Find an analytic continuation of the matrix $A(z)$ to the Teichmüller points on the boundary of the $p-$adic unit disc.

During the computations we will describe in the following, we "guessed” a method of analytic continuation that worked out well in the hypergeometric examples.

We could compute the Frobenius polynomials in Teichmüller points for hypergeometric CY(4)-operators, and the results we obtained coincided with the results we computed by the unit-root-method.

To compute the Frobenius polynomial in a Teichmüller point x, we want to evaluate the matrix

$$A(z) = B(z^p)^{-1} p^3 \mathfrak{A}_4^{-1} B(z)$$

in the point $z = x$.

Our approach to do this was to translate Dworks’ analytic continuation method to compute the unit root to the case of matrices in the most obvious way:

Let $B^s(z)$ be the matrix $B(z)$ truncated up to degree $p^s - 1$, i.e. the entries of $B^s(z)$ are the power series entries of $B(z)$ truncated after degree $p^s - 1$, which are polynomials of degree $p^s - 1$.

Let

$$d^s(z) := \det(B^s(z)).$$

We define

$$\mathcal{D} := \{x \in \mathbb{Z}_p, |f_0^1(x)| = |d^1(x)| = 1\}.$$

The translation of the analytic continuation method of Dwork to the case of matrices implies the following

Conjecture 9.1.1 *Let $x \in \mathcal{D}$ be a Teichmüller point. Then*

$$\det(I - TA(z))|_{z=x} \equiv \det(I - TA^s(x)) \mod p^s$$

where

$$A^s(x) = B^s(x)^{-1} p^3 \mathfrak{A}_4^{-1} B^{s+1}(x).$$

Our computations with hypergeometric CY(4)-operators confirm this conjecture. First of all, we computed with the parameter values $\alpha = \beta = \gamma = 0$ and the result coincided with the result of the unit root method. Then we tried different parameter values, but, as expected, the result stayed the same. This obervation suggested the following

Conjecture 9.1.2 *Let $x \in \mathcal{D}$ be a Teichmüller point. If the characteristic polynomial $\det(I - TA^s(x))$ of the matrix*

$$A^s(x) := B^s(x)^{-1} p^3 \mathfrak{A}_4^{-1} B^{s+1}(x)$$

converges $p-$adically to a polynomial in $\mathbb{Z}_p[T]$, it is independent of the choice of the parameters α, β and γ.

Instead of direct a proof, we can only give some arguments, different from the arguments in section 5.7 that might explain why this conjecture holds true. Therefore, we state some congruences that might help to explain conjecture 9.1.2.

During our computations, we observed the following congruences. First of all, since the element $d^{s+1}(x)/d^s(x)$ is the coefficient of p^6T^4 in $\det(I - TA^s(x))$, we conjecture that

Conjecture 9.1.3 *For a Teichmüller point $x \in \mathcal{D}$,*

$$\frac{d^{s+1}(x)}{d^s(x)} \equiv 1 \mod p^{s+1}.$$

Conjecture 9.1.4 *For $0 \leq k \leq 1$, $0 \leq n, m \leq 3$ and a Teichmüller point $x \in \mathcal{D}$, we have the congruences*

$$\frac{\theta^n(Y_k)^{s+1}(x)}{\theta^m(Y_k)^{s+1}(x)} \equiv \frac{\theta^n(Y_k)^s(x)}{\theta^m(Y_k)^s(x)} \mod p^s.$$

It was the following conjecture that led us to the investigations we made in chapter 10. It seems as though these congruences are true in all exapmles where the coefficients of f_0 satisfy the Dwork congruences.

Conjecture 9.1.5 *For a Teichmüller point $x \in \mathcal{D}$, the following congruences hold:*

$$\frac{f_0^{s+1}(x)}{f_0^s(x^p)} \equiv p\frac{f_1^{s+1}(x)}{f_1^s(x^p)} \equiv p^2\frac{f_2^{s+1}(x)}{f_2^s(x^p)} \equiv p^3\frac{f_3^{s+1}(x)}{f_3^s(x^p)} \mod p^s.$$

Explanation of conjecture 9.1.2: The coefficients of α, β, γ in $\det(I - TA^s(x))$ are of the form

$$\begin{aligned}\frac{p^{2+i}}{d^s(x)} \quad &\cdot \quad (\theta^{\sigma(0)}(Y_i)^{s+1}(x)\theta^{\sigma(1)}(Y_i)^s(x) - \theta^{\sigma(0)}(Y_i)^s(x)\theta^{\sigma(1)}(Y_i)^{s+1}(x))\\ &\cdot \quad (\theta^{\sigma(2)}(Y_j)^{s+1}(x)\theta^{\sigma(3)}(Y_k)^s(x) - \theta^{\sigma(3)}(Y_j)^{s+1}(x)\theta^{\sigma(2)}(Y_k)^s(x)),\end{aligned}$$

where σ is a permutation on $\{0,1,2,3\}$, $0 \le i \le 2$ and $0 \le j,k \le 3$, $i \ne j \ne k \ne i$. Since this looks rather complicated, take for example

$$\begin{aligned}\frac{p^2}{d^s(x)} \quad &\cdot \quad (\theta^2(Y_0)^{s+1}(x)\theta(Y_0)^s(x) - \theta^2(Y_0)^s(x)\theta(Y_0)^{s+1}(x))\\ &\cdot \quad (Y_1^{s+1}(x)\theta^3(Y_2)^s(x) - \theta^3(Y_1)^{s+1}(x)Y_2^s(x)),\end{aligned}$$

which can be written as

$$\begin{aligned}\frac{p^2}{d^s(x)} \quad &\cdot \quad \theta^2(Y_0)^s(x)\theta^2(Y_0)^{s+1}(x)\left(\frac{\theta(Y_0)^s(x)}{\theta^2(Y_0)^s(x)} - \frac{\theta(Y_0)^{s+1}(x)}{\theta^2(Y_0)^{s+1}(x)}\right)\\ &\cdot \quad (Y_1^{s+1}(x)\theta^3(Y_2)^s(x) - \theta^3(Y_1)^{s+1}(x)Y_2^s(x)).\end{aligned}$$

Now if conjecture 9.1.4 holds true, it follows that *if*

$$p^2\theta^2(Y_0)^s(x)\theta^2(Y_0)^{s+1}(x)(Y_1^{s+1}(x)\theta^3(Y_2)^s(x) - \theta^3(Y_1)^{s+1}(x)Y_2^s(x))/d^s(x)$$

is a $p-$adic unit, then the whole term is congruent to 0 modulo p^s. A similar argument applies for all coefficients of α, β and γ. At the moment, we have not yet proven that the elements in question are in fact $p-$adic units, and thus we are not in a position to prove conjecture 9.1.2 now with this approach.

9.2 Matrix computations for CY(2)-operators

In this section, we describe the matrix approach for CY(2)-operators. In this case, which is much simpler that the case of CY(4)- operators, we were able to prove that the characteristic polynomial of the Frobenius matrix at a Teichmüller point is indeed independent of the choice of the parameter α occuring in the matrix $\mathfrak{A}_2$ directly, by application of certain congruences. Note that the fact that the Frobenius polynomial is independent of α follows directly from the general theory, since it is completely determined by the unit root, which does not depend on α in any way.
Let P be a CY(2)-operator. A Frobenius basis of solutions to the differential equation $Py = 0$ is then given by

$$\begin{aligned} y_0 &= f_0\\ y_1 &= f_1 + f_0\log(z),\end{aligned}$$

where $f_i \in \mathbb{Q}[[z]]$.

In the case of a CY(2)-operator, the matrix $B(z)$ is given by

$$B(z) := \begin{pmatrix} f_1 & \theta(f_1)+f_0 \\ f_0 & \theta(f_0) \end{pmatrix}.$$

Let

$$d^s(z) := \det(B^s(z)).$$

The matrix $\mathfrak{A}_2$ is of the shape

$$\mathfrak{A}_2 = \begin{pmatrix} p & \alpha \\ 0 & 1 \end{pmatrix},$$

and thus only depending on one parameter α.
We define

$$\mathcal{D} := \{x \in \mathbb{Z}_p, |f_0^1(x)| = |d^1(x)| = 1\}.$$

As in the case of CY(4)- operators, our computations implied that the translation of Dworks analytic continuation method works out well to compute the Frobenius polynomial at a Teichmüller point x. Let $A(z) := pA_\phi(z)^{-1}$.

Conjecture 9.2.1 *Let $x \in \mathcal{D}$ be a Teichmüller point. Then*

$$\det(I - TA(z))|_{z=x} \equiv \det(I - TA^s(x)) \mod p^s$$

where

$$A^s(x) = B^s(x)^{-1} p \mathfrak{A}_2^{-1} B^{s+1}(x).$$

Unlike in the CY(4)-case, in the CY(2)-case we can actually prove the independence of the Frobenius polynomial at a Teichmüller point x of the parameter α.

Proposition 9.2.1 *Let $x \in \mathcal{D}$ be a Teichmüller point. If the the characteristic polynomial* $\det(I - TA^s(x))$ *of the matrix*

$$A^s(x) := B^s(x^p)^{-1} p \mathfrak{A}_2^{-1} B^{s+1}(x)$$

converges $p-$adically to a polynomial in $\mathbb{Z}_p[T]$,

$$\det(I - TA^s(x)) \equiv \det(I - TA^{s+1}(x)) \mod p^s,$$

it is independent of the choice of the parameter α.

Proof: The coefficient of T^2 in $\det(I - TA^s(x))$ is given by

$$pd^{s+1}(x)/d^s(x^p),$$

and is thus independent of the choice of α. Furthermore, since

$$\frac{d^s(x)}{d^{s-1}(x)} \equiv \frac{d^{s+1}(x)}{d^s(x)} \mod p^s,$$

it follows that $|d^s(x)| = 1$ for all s.
The coefficient of T is given by

$(-pf_1^{s+1}(x)\theta(f_0)^s(x) - f_1^s(x)\theta(f_0)^{s+1}(x) + p(\theta(f_1)^{s+1}(x) + f_0^{s+1}(x))f_0^s(x) + (\theta(f_1)^s(x) + f_0^s(x))f_0^{s+1}(x) - p\alpha f_0^{s+1}(x)\theta(f_0)^s(x) + p\alpha f_0^s(x)\theta(f_0)^{s+1}(x))/d^s(x)$.

Thus, it is independent of the choice of the parameter α iff

$$\frac{pf_0^{s+1}(x)f_0^s(x)\left(\frac{\theta(f_0)^{s+1}(x)}{f_0^{s+1}(x)} - \frac{\theta(f_0)^s(x)}{f_0^s(x)}\right)}{d^s(x)} \equiv 0 \mod p^{s+1}.$$

Since $|d^s(x)| = |f_0^s(x)| = 1$ by assumption, this is equivalent to

$$\frac{\theta(f_0)^{s+1}(x)}{f_0^{s+1}(x)} \equiv \frac{\theta(f_0)^s(x)}{f_0^s(x)} \mod p^s.$$

But this is true by [**27**], Lemma 3.4.(ii), and the proposition follows.□

Note that our computations suggested the following

Conjecture 9.2.2 *Let $x \in \mathcal{D}$ be a Teichmüller point. Then*

$$d^{s+1}(x)/d^s(x^p) \equiv 1 \mod p^{s+1}.$$

Furthermore, during our numerous computations we observed that

Conjecture 9.2.3 *Let $x \in \mathcal{D}$ be a Teichmüller point. Then*

$$p\frac{f_1^{s+1}(x)}{f_1^s(x^p)} \equiv \frac{f_0^{s+1}(x)}{f_0^s(x^p)} \mod p^s.$$

This conjecture seems to be true for all CY(2)-operators and led us to the investigations we performed in chapter 10.

Chapter 10

Some new formulas to compute the unit root

During the explicit calculations described in the last chapter, we discovered experimentally that for a Frobenius basis of solutions to a CY(4)-differential equation, the following congruences hold in a Teichüller point x:

$$\frac{f_0^{s+1}(x)}{f_0(x)} \equiv \frac{pf_1^{s+1}(x)}{f_1^s(x)} \equiv \frac{p^s f_2^{s+1}(x)}{f_2^s(x)} \equiv \frac{p^3 f_3^{s+1}(x)}{f^s(x)} \mod p^s.$$

This led us to the question: *Is it possible to compute the unit root in terms of the other solutions of the CY-differential equation?* In this chapter, we prove by very down-to-earth methods that in general the answer is *yes*. In a Teichmüller point, the analytic element $r(z)$ from theorem 2.3.1 coincides with an analytic function that can be expressed in terms of the power series $f_1(z), f_2(z)$ and $f_3(z)$ occuring in the logarithmic solutions.

10.1 Fixed points involving the other solutions

As in the previous chapter, let P be a CY(4)-differential operator. Remember that around $z = 0$, the differential equation $\mathcal{P}y = 0$ has a Frobenius basis of solutions with at most logarithmic singularities. In this section, we derive formulas for fixed points of the Frobenius matrix A_ϕ in terms of the power series $f_1(z), f_2(z), f_3(z)$ that occur in the logarithmic solutions y_1, y_2 and y_3.

As in section 3.2, let

$$Y_4 = \exp\left(\frac{1}{2}\int a_3 \frac{dz}{z}\right). \tag{10.1}$$

We assume that $Y_4(z)$ has a power series expansion in $\mathbb{Z}_p[[z]]$. As in the previous chapter, we choose the basis $\{\omega, \nabla(\omega), \nabla^2(\omega), \nabla^3(\omega)\}$ where $\nabla := \nabla(\theta)$ and let $\mathcal{M}(z)$ denote the connection matrix with regard to this basis. As in section 3.6, let $C(z)$ denote the non-logarithmic part of the matrix $\mathcal{C}(z)$ satisfying

$$z\frac{d}{dz}\mathcal{C}(z) + \mathcal{M}(z)\mathcal{C}(z) = 0$$

and

$$\mathcal{C}(z) = Y_4(z)\begin{pmatrix} \mathcal{N}_0 & -\mathcal{N}_1 & \mathcal{N}_2 & -\mathcal{N}_3 \\ y_0 & -y_1 & y_2 & -y_3 \end{pmatrix}$$

for some $3 \times 1-$ matrices $\mathcal{N}_i$. This special shape of $\mathcal{C}(z)$ implies that $C(z)$ satisfies

$$C(z) = Y_4(z)\begin{pmatrix} N_0 & -N_1 & N_2 & -N_3 \\ f_0 & -f_1 & f_2 & -f_3 \end{pmatrix}$$

for some 3×1 matrices N_i with power series entries, which can be seen as a starting condition for the differential equation above. Let $A_\phi(z)$ denote the Frobenius matrix with regard to the chosen basis, and write

$$A_\phi(z) = \begin{pmatrix} pA_0 & C_0 \\ pB_0 & D_0 \end{pmatrix}.$$

As in section 3.6, on the open $p-$adic unit disc, we have the equality

$$A_\phi(z) = C(z)\begin{pmatrix} \varepsilon & \alpha & \beta & \gamma \\ 0 & \varepsilon p & \alpha p & \beta p \\ 0 & 0 & \varepsilon p^2 & \alpha p^2 \\ 0 & 0 & 0 & \varepsilon p^3 \end{pmatrix} C(z^p)^{-1}. \tag{10.2}$$

A direct application of equation 10.2 leads us to the following formulas for sections that are mapped to constant multiples of themselves by the Frobenius mapping:

Proposition 10.1.1 *Let*

$$x_1 := \frac{\varepsilon\alpha}{p-1},\ x_2 := \frac{\varepsilon\beta + \varepsilon\alpha x_1}{p^2 - 1}$$

and

$$x_3 := \frac{\varepsilon\gamma + \varepsilon\beta x_1 + \varepsilon\alpha x_2}{p^3 - 1}.$$

Then, for $1 \le i \le 3$, *we have*

$$\begin{aligned} & A_\phi(z)Y_4(z^p)\left(\begin{pmatrix} N_i(z^p) \\ f_i(z^p) \end{pmatrix} + x_1\begin{pmatrix} N_{i-1}(z^p) \\ f_{i-1}(z^p) \end{pmatrix} + \ldots + x_i\begin{pmatrix} N_0(z^p) \\ f_0(z^p) \end{pmatrix}\right) \\ = \;& \varepsilon p^i Y_4(z)\left(\begin{pmatrix} N_i(z) \\ f_i(z) \end{pmatrix} + x_1\begin{pmatrix} N_{i-1}(z) \\ f_{i-1}(z) \end{pmatrix} + \ldots + x_i\begin{pmatrix} N_0(z) \\ f_0(z) \end{pmatrix}\right). \end{aligned}$$

Proof: The proof is by a direct calculation, applying equation (10.2) and then solving the equations for x_1, x_2 and x_3. $\square$

10.2 New formulas for the unit root in Teichmüller points

In chapter 3 we derived that the analytic element $r(z)$, which, evaluated at a Teichmüller point, is the unit root in this point, coincides with the analytic function

$$\frac{\varepsilon f_0(z)}{f_0(z^p)}$$

on the open $p-$adic unit disc. In this section, we use the formulas for the sections that are mapped to constant multiples of themselves derived in proposition 10.1.1 to construct analytic elements $F_0^1(z), F_0^2(z), F_0^3(z)$ on the open $p-$adic unit disc whose analytic continuations to the boundary of the $p-$adic unit disc coincide with $r(z)$ at the Teichmüller points and involve the power series f_0, f_1, f_2, f_3.

By proposition 10.1.1, it follows that for $0 \leq i \leq 3$,

$$\begin{aligned} &\begin{pmatrix} pA_0 & C_0 \\ pB_0 & D_0 \end{pmatrix} Y_4(z^p) \begin{pmatrix} N_i(z^p) + x_1 N_{i-1}(z^p) + ... + x_i N_0(z^p) \\ f_i(z^p) + x_1 f_{i-1}(z^p) + ... + x_i f_0(z^p) \end{pmatrix} \\ = \; & p^i \varepsilon Y_4(z) \begin{pmatrix} N_i(z) + x_1 N_{i-1}(z) + ... + x_i N_0(z) \\ f_i(z) + x_1 f_{i-1}(z) + ... + x_i f_0(z) \end{pmatrix}. \end{aligned} \tag{10.3}$$

If we define

$$\eta_0^i(z) := \frac{N_i(z) + x_1 N_{i-1}(z) + ... + x_i N_0(z)}{f_i(z) + x_1 f_{i-1}(z) + ... + x_i f_0(z)}$$

and

$$F_0^i(z) := \frac{p^i \varepsilon (f_i(z) + x_1 f_{i-1}(z) + ... + x_i f_0(z))}{f_i(z^p) + x_1 f_{i-1}(z^p) + ... + x_i f_0(z^p)},$$

then equation (10.3) leads to

$$\begin{pmatrix} pA_0 & C_0 \\ pB_0 & D_0 \end{pmatrix} \begin{pmatrix} \eta_0^i(z^p) \\ 1 \end{pmatrix} = Y_4(z)/Y_4(z^p) \begin{pmatrix} \eta_0^i(z) F_0^i(z) \\ F_0^i(z) \end{pmatrix}. \tag{10.4}$$

But this implies that

$$pA_0 \eta_0^i(z^p) + C_0 = \eta_0^i(z) Y_4(z)/Y_4(z^p) F_0^i(z) \tag{10.5}$$
$$pB_0 \eta_0^i(z^p) + D_0 = Y_4(z)/Y_4(z^p) F_0^i(z), \tag{10.6}$$

and since D_0 is invertible, it follows that $\eta_0^i(z)$ satisfies

$$\eta_0^i(z) = D_0^{-1} \frac{pA_0 \eta_0^i(z^p) + C_0}{1 + pD_0^{-1} B_0 \eta_0^i(z^p)}. \tag{10.7}$$

Proposition 10.2.1 *Let $0 \leq i \leq 3$. If we write $f_i = \sum_{k=1}^{\infty} a_k^i z^k$, then $|a_k^i|_p \leq p^{i(s-1)}$ if $k < p^s$.*

Proof: Since $f_0(z) \in \mathbb{Z}_p[[z]]$, the statement is true for f_0. For $i > 0$, we have

$$\begin{aligned} & Y_4(z^p)(pB_0(N_i(z^p) + x_1 N_{i-1}(z^p) + ... + x_i N_0(z^p)) \\ + \quad & D_0(f_i(z^p) + x_1 f_{i-1}(z^p) + ... + x_i f_0(z^p))) \\ = \quad & p^i \varepsilon Y_4(z)(f_i(z) + x_1 f_{i-1}(z) + ... + x_i f_0)., \end{aligned} \tag{10.8}$$

where $Y_4(z)$ has a power series expansion in $\mathbb{Z}_p[[z]]$. Now assume that the statement holds for all $j < i$. By induction on s, we will prove that the statement holds for the coefficients of f_i. Since $a_0^i = 0$, we have $|a_0^i| \le p^{-1}$, and thus $|a_k^i| \le p^{-1}$ for all $k < p^0 = 1$.
Assume that the statement holds for $k < p^{s-1}$, and let $p^{s-1} \le k < p^s$. Since the entries of the matrix N_i are sums of derivatives of f_i and the f_j for $j < i$, equation (10.8) translates into an equality of power series that leads to the following equality for a_k^i:

$$p^i a_k^i = \sum_{l \le [k/p]} c_l a_l + u_k^i,$$

where $c_l \in \mathbb{Z}_p$ and $|u_k^i| < p^{(i-1)s-i}$. Since by assumption $|a_l^i| < p^{i(s-1)}$ for $l \le [k/p] < p^{s-1}$, it follows that $|a_k^i| < p^{is}$.□

Proposition 10.2.2 *If $x_i \neq 0$, the formal power series entries of the matrix $\eta_0^i(z)$ have radius of convergence $r = 1$. If $x_i = 0$, the Laurent series entries of $\eta_0^i(z)$ converge in the open annulus with inner radius of convergence $r_1 = 0$ and outer radius of convergence $r_2 = 1$.*

Proof: Assume first that $x_i \neq 0$. The coefficients of the power series entries $N_i^j(z) = \sum_{k=0}^{\infty} c_k^{(i,j)} z^k, 1 \le j \le 4$ of $N_i(z)$ satisfy $|c_k^{i,j}|_p \le p^{i(s-1)}$ for $k < p^s$ by proposition 10.2.1, and thus the radius of convergence of $N_i^j(z)$ is $r = 1$. The same holds for the power series $f_i(z)$. Since $f_i(0) + x_1 f_{i-1}(0) + ... + x_i f_0(0) = x_i \neq 0$, the quotient $\eta_0^i(z)$ converges in an open disc of radius $1 \ge r > 0$ around 0.
By equation (10.7), it follows that if $\eta_0^i(z^p)$ converges in the open disc of radius r, then $\eta_0^i(z)$ converges in the open disc of radius r.
Conversely, if $\eta_0^i(z)$ converges in the open disc of radius r, then $\eta_0^i(z^p)$ converges in the open disc of radius $r^{1/p}$. Iterating these two arguments, we obtain that for every $n \in \mathbb{N}$, $\eta_0^i(z)$ converges in the open disc of radius r^{1/p^n}. Since $\lim_{n \to \infty} r^{1/p^n} = 1$, the proposition follows.
If $x_i = 0$, then the quotient $\eta_0^i(z)$ is a matrix whose entries are Laurent series that converge in an annulus with inner radius $r_1 = 0$ and outer radius $r_2 > 0$. Similar to the case $x_i \neq 0$, one proves by iteration that $r_2 = 1$. □

Now, as in the previous sections, let $h(z) := f_0^1(z)$, let $s(z)$ be the polynomial whose zero set are the singular points of the differential operator P_n and define $A := \mathbb{Z}_p[z][(s(z)h(z))^{-1}]$. As before, let $S_0 = Spec(A_0)$ and $S_\infty = Spec(A_\infty)$.

Proposition 10.2.3 *Let $x_0 \in S_\infty$. Then the mapping*

$$x \mapsto D^{-1}(x_0)\frac{pA(x_0)x + C(x_0)}{1 + pD^{-1}(x_0)B(x_0)x}$$

is a contraction mapping in $\mathbb{Q}_p$.

Proof: Let $x, y \in \mathbb{Q}_p$. Then

$$\begin{aligned}
& \left| D^{-1}(x_0)\frac{pA(x_0)x + C(x_0)}{1 + pD^{-1}(x_0)B(x_0)x} - D^{-1}(x_0)\frac{pA(x_0)y + C(x_0)}{1 + pD^{-1}(x_0)B(x_0)y} \right|_p \\
= & \left| D^{-1}(x_0)\frac{(pC(x_0)B(x_0) - pA(x_0)D(x_0))(x - y)}{p^2B(x_0)^2xy + pB(x_0)D(x_0)(x + y) + D(x_0)^2} \right|_p \\
= & \left| D^{-1}(x_0)\frac{(C(x_0)B(x_0) - A(x_0)D(x_0))}{p^2B(x_0)^2xy + pB(x_0)D(x_0)(x + y) + D(x_0)^2} \right|_p |p(x - y)|_p \\
\leq & \ |p(x - y)|_p = \frac{|x - y|_p}{p},
\end{aligned}$$

since $D(x_0)$ is a unit in $\mathbb{Z}_p$. $\Box$

Proposition 10.2.4 *Let η^i be an analytic element of support S_∞ coinciding with η_0^i on $p\mathbb{Z}_p$ (or $p\mathbb{Z}_p \setminus \{0\}$). Then on S_∞, we have*

$$\eta^i(z) = D(z)^{-1}\frac{pA(z)\eta^i(z^p) + C(z)}{1 + pD(z)^{-1}B(z)\eta^i(z^p)}.$$

Proof: The function $D(z)^{-1}\frac{pA(z)\eta^i(z^p)+C(z)}{1+pD(z)^{-1}B(z)\eta^i(z^p)}$ is an analytic function on S_∞ (or $S_\infty \setminus \{0\}$) coinciding with $\eta_0^i(z)$ on the open subset $p\mathbb{Z}_p$ (or $p\mathbb{Z}_p \setminus \{0\}$). Since $\eta^i(z)$ is an analytic function on S_∞ (or $S_\infty \setminus \{0\}$) coinciding with $\eta_0^i(z)$ on $p\mathbb{Z}_p$ (or $p\mathbb{Z}_p \setminus \{0\}$), the statement follows by the uniqueness theorem 5.1.1.$\Box$

Theorem 10.2.1 *Let $1 \leq i \leq n - 1$, and let $x_0 \in S_\infty$ be a Teichmüller point satisfying $x_0^p = x_0$, and let η^i be an analytic element of support S_∞ (or $S_\infty \setminus \{0\}$) conciding with η_0^i on $p\mathbb{Z}_p$ (or $p\mathbb{Z}_p \setminus \{0\}$). Let η^0 be an analytic element of support S_∞ coinciding with η_0^0 on $p\mathbb{Z}_p$. Then $\eta^i(x_0) = \eta^0(x_0)$.*

Proof: Since $x_0 = x_0^p$, it follows that $\eta^i(x_0) = \eta^i(x_0^p)$. By proposition 10.2.4, it follows that $\eta^i(x_0)$ satisfies

$$\eta^i(x_0) = D^{-1}(x_0)\frac{pA(x_0)\eta^i(x_0) + C(x_0)}{1 + pD^{-1}(x_0)B(x_0)\eta^i(x_0)}.$$

Thus, $\eta^i(x_0)$ is a fixed point of the contraction described in proposition 10.2.3. Since by the proof of theorem 2.3.1, $\eta^0(x_0)$ is a fixed point of this contraction, too, it follows that $\eta^i(x_0) = \eta^0(x_0)$. $\Box$

Since the element $F^0(z)$ appearing beyond is nothing but $r(z)$ by chapter 3, the following corollary now proves the identity of $r(z)$ with several analytic functions involving the power series f_1, f_2, f_3 in Teichmüller points.

Corollary 10.2.1 *Let $F^0(z)$ be an analytic function on S_∞, coinciding with $F_0^0(z)$ on $p\mathbb{Z}_p$. Then for each $x_0 \in S_\infty$ (or $S_\infty \setminus \{0\}$) satisfying $x_0^p = x_0$, we have $F^i(x_0) = F^0(x_0)$, where $F^i(z)$ is an analytic function on S_∞ (or $S_\infty \setminus \{0\}$) coinciding with $F_0^i(z)$ on $p\mathbb{Z}_p$ (or $p\mathbb{Z}_p \setminus \{0\}$).*

Proof: We have

$$F^0(x_0) = pB(x_0)\eta^0(x_0) + D(x_0)$$

and

$$F^i(x_0) = pB(x_0)\eta^i(x_0) + D(x_0).$$

Since $\eta^i(x_0) = \eta^0(x_0)$ by theorem 10.2, the statement follows. □

10.3 An explicit construction of the analytic continuation of $\mu_i(z)$ and $F_i(z)$

In this section, we consider the question for an explicit $p-$adic analytic continuation for the elements $F_0^i(z)$ to the boundary of the $p-$adic unit disc. These functions are analytic on open $p-$adic unit disc. For the rest of this section, for $1 \leq i \leq 3$ we assume that

$$f_i(z) = \sum_{k=1}^{\infty} a_k^i z^k \in z\mathbb{Q}[[z]]$$

satisfies $|a_1^i|_p = 1$. We derive a condition on the coefficients of the power series f_i to construct an explicit $p-$adic analytic continuation F^i of the power series F_0^i to the boundary of the $p-$ adic unit disc. In the end, we prove that *if* there exists an analytic continuation F^i of F_0^i, then at a Teichmüller point x_0, we have the equality

$$F^i(x_0) \equiv \varepsilon p^i \frac{f_i^{s+1}(x_0)}{f_i^s(x_0)} \mod p^s,$$

which explains the observations we made during our computations.

Definition 10.3.1 *For $1 \leq i \leq 3$, let $\mathfrak{D}_i$ be the region*

$$\mathfrak{D}_i := p\mathbb{Z}_p \cup \{x \in \mathbb{Z}_p, |f_i^1(x)|_p = 1, |f_i^2(x)|_p = p^i\},$$

and let $\mathfrak{D}_i^o$ be the region $\mathfrak{D}_i \setminus \{0\}$.

Proposition 10.3.1 *For all $s \geq 2$, $l \geq 1$ and $1 \leq i \leq 3$, we have*

$$p^i f_i(zp^l)\frac{f_i^{s-1}((zp^l)^p)}{p^{lp}} \equiv \frac{f_i((zp^l)^p)}{p^{lp}}p^i f_i^s(zp^l) \mod p^{s-1}\mathbb{Z}_p[[z]]. \tag{10.9}$$

Proof: We will prove the proposition by comparing the coefficients of z^n on both sides of equation (10.9).
For $n \leq p^s - 1$, the coefficients of z^n on both sides are equal in $\mathbb{Z}_p$. Let $n \geq p^s$, $p^{t-1} \leq n \leq p^t$ for some $t \geq s+1$. Write $n = n_0 + pn_1$. The coefficient of z^n on the lefthand side of equation (10.9) is

$$p^{l(n-p)+i}\sum_{j=0}^{p^s-1} a^i_{n-pj}a^i_j,$$

while the coefficient on the righthand side is

$$p^{l(n-p)+i}\sum_{j=0}^{p^s-1} a^i_{n_1-j}a^i_{n_0+pj}.$$

We will prove that both coefficients are congruent to 0 modulo p^{s-1} and start with the coefficient on the lefthand side.
Since $|a^i_{n-pj}|_p \leq p^{i(t-1)}$ and $|a^i_j|_p \leq p^{i(s-1)}$, we have to prove that

$$l(n-p) + i - i(t-1) - i(s-1) \geq s-1 \text{ for all } s \geq 2.$$

But

$$l(n-p) + i - i(t-1) \geq l(p^{t-1} - p) + i - i(t-1) \geq (i+1)(t-2) \geq (i+1)(s-1)$$

for all $t \geq 2$, and thus the coefficient on the lefthand side is congruent to 0 modulo p^{s-1}.
Considering the coefficient on the righthand side, we have $|a^i_{n_1-j}|_p \leq p^{i(t-2)}$ and $|a^i_{n_0+pj}|_p \leq p^{is}$. This leads to the same inequality as on the righthand side. □

Corollary 10.3.1 *For all $x_0 \in p\mathbb{Z}_p \setminus \{0\}$, we have*

$$x_0^p\frac{p^i f_i(x_0)}{f_i(x_0^p)} \equiv x_0^p\frac{p^i f_i^s(x_0)}{f_i^{s-1}(x_0^p)} \mod p^{s-1}\mathbb{Z}_p.$$

Proof: Write $x_0 = p^l u$, where u is a unit in $\mathbb{Z}_p$. Since the radius of convergence of the formal power series $f_i(z)$ is $r = 1$, we may evaluate f_i in $x_0 = up^l$ and the proposition implies

$$p^i f_i(up^l)\frac{f_i^{s-1}((up^l)^p)}{p^{lp}} \equiv \frac{f_i((up^l)^p)}{p^{lp}}p^i f_i^s(up^l) \mod p^{s-1}\mathbb{Z}_p.$$

Since $\frac{f_i^{s-1}((up^l)^p)}{p^{lp}}$ and $\frac{f_i((up^l)^p)}{p^{lp}}$ are units in $\mathbb{Z}_p$, we may divide by them, multiply both sides by the unit u^p, and the statement follows. □

Now, we will consider the question about the analytic continuation of the functions $F_0^i(z)$.

Proposition 10.3.2 *If $\nu_i :=$ord$_p(x_i) < pl-1$, then for $s \geq 2$ and $l \geq 1$,*

$$p^i(f_i(zp^l)+x_1pf_{i-1}(zp^l)+...+x_if_0(zp^l))\frac{f_i^{s-1}((zp^l)^p)+x_1pf_{i-1}^{s-1}((zp^l)^p)+...+x_if_0^{s-1}((zp^l)^p)}{p^{\nu_i}}$$

$$\equiv$$

$$\frac{f_i((zp^l)^p)+x_1f_{i-1}((zp^l)^p)+...+x_if_0((zp^l)^p)}{p^{\nu_i}}p^i(f_i^s(zp^l)+x_1f_{i-1}^s(zp^l)+...+x_if_0^s(zp^l))$$

$$\mod p^{s-1}\mathbb{Z}_p[[z]].$$

Otherwise,

$$p^i(f_i(zp^l)+x_1f_{i-1}(zp^l)+...+x_if_0(zp^l))\frac{f_i^{s-1}((zp^l)^p)+x_1f_{i-1}^{s-1}((zp^l)^p)+...+x_if_0^{s-1}((zp^l)^p)}{p^{pl}}$$

$$\equiv$$

$$\frac{f_i((zp^l)^p)+x_1f_{i-1}((zp^l)^p)+...+x_if_0((zp^l)^p)}{p^{pl}}p^i(f_i^s(zp^l)+x_1f_{i-1}^s(zp^l)+...+x_if_0^s(zp^l))$$

$$\mod p^{s-1}\mathbb{Z}_p[[z]].$$

Proof: Literally the same as for proposition 10.3.1. In the case $\nu_i < pl-1$, we may only divide by p^{ν_i} (and not by p^{pl}) since $f_i(zp^l)+x_1f_{i-1}(zp^l)+...+x_if_0(zp^l)$ has the constant term x_i of $p-$adic valuation ν_i. □

Corollary 10.3.2 *For $x_0 \in p\mathbb{Z}_p$, we have*

$$x_0^p\frac{p^i(f_i(x_0)+x_1f_{i-1}(x_0)+...+x_ipf_0(x_0))}{f_i(x_0^p)+x_1f_{i-1}(x_0)+...+x_if_0(x_0^p)} \equiv$$

$$x_0^p\frac{p^i((f_i^s(x_0)+x_1f_{i-1}^s(x_0)+...+x_if_0^s(x_0))}{f_i^{s-1}(x_0^p)+x_1f_{i-1}^{s-1}(x_0^p)+...+x_if_0^{s-1}(x_0^p)} \mod p^{s-1}.$$

By $F^{i,s}(z)$, we denote the quotient

$$F_i^s(z) = \frac{p^i(f_i^s(z)+x_1f_{i-1}^s(z)+...+x_if_0^s(z))}{f_i^{s-1}(z^p)+x_1f_{i-1}^s(z^p)+...+x_if_0^s(z^p)}.$$

In the following, we will always assume that the next condition is satisfied.

Condition 10.3.1 *For $1 \leq i \leq 3$, we have*

$$p^{2i(s-1)+i}f_i^s(z)f_i^s(z^p) \equiv p^{2i(s-1)+i}f_i^{s+1}(z)f_i^{s-1}(z^p) \mod p^s\mathbb{Z}_p[z].$$

Proposition 10.3.3 *Let $1 \leq i \leq 3$. If condition 10.3.1 is satisfied for all $s \geq 2$ then*

$$p^{2i(s-1)+i}(f_i^s(z) + x_1 f_{i-1}^s(z) + ... + x_i f_0^s(z))(f_i^s(z^p) + x_1 f_{i-1}^s(z^p) + ... + x_i f_0^s(z^p))$$

$$\equiv$$

$$p^{2i(s-1)+i}(f_i^{s+1}(z)+x_1 f_{i-1}^{s+1}(z)+...+x_i f_0^{s+1}(z))(f_i^{s-1}(z^p)+x_1 f_{i-1}^{s-1}(z^p)+...+x_i f_0^{s-1}(z^p))$$

$$\mod p^{s-1}\mathbb{Z}_p[z].$$

Proof: Since for $1 \leq i \leq 3$ and $j < i$ we have $p^{i(s-1)} f_j^s(z) \in p^{(i-j)(s-1)}\mathbb{Z}_p[z]$, the statement follows directly since condition 10.3.1 holds.□

Proposition 10.3.4 *Let $x_0 \in \mathbb{Z}_p$ such that*

$$|f_i^1(x_0)|_p = 1, \ |f_i^2(x_0)|_p = p^i.$$

If condition 10.3.1 is satisfied for all $s \geq 2$, then

$$|f_i^s(x_0)|_p = p^{i(s-1)}.$$

Proof: First of all, we prove that $|f_i^s(x_0)|_p = p^{i(s-1)}$ implies that $|f_i^s(x_0^p)|_p = p^{i(s-1)}$. Let $|f_i^s(x_0)|_p = p^{i(s-1)}$. Then $p^{i(s-1)} f_i^s(x_0) \neq 0 \mod p$. Since $p^{i(s-1)} f_i^s(z^p) \in \mathbb{Z}_p[z]$, $p^{i(s-1)} f_i^s(x_0) \equiv p^{i(s-1)} f_i^s(x_0^p) \mod p$, and the statement follows.
Now, we proceed by induction on s. Assume that $|f_i^{s-1}(x_0)|_p = p^{i(s-2)}$ and $|f_i^s(x_0)|_p = p^{i(s-1)}$. Since

$$p^{2i(s-1)+i} f_i^s(x_0) f_i^s(x_0^p) \equiv p^{2i(s-1)+i} f_i^{s+1}(x_0) f_i^{s-1}(x_0^p) \mod p^s$$

and since the lefthand side of the equation has $p-$adic order i, it follows that $f_i^{s+1}(x_0)$ must have $p-$ adic order $-is$ such that both sides of the equation have the same p- adic order. □

Proposition 10.3.5 *Assume that for all $s \geq 2$, condition 10.3.1 is satisfied. Let $x_0 \in \mathfrak{D}_i$, $|x_0|_p = 1$. Then*

$$F_i^{s+1}(x_0) \equiv F_i^s(x_0) \mod p^{s-1}.$$

Proof: Applying proposition 10.3.3 and specializing to $z = x_0$ leads to the equation

$$p^{2i(s-1)+i}(f_i^s(x_0) + x_1 f_{i-1}^s(x_0) + ... + x_i f_0^s(x_0))(f_i^s(x_0^p) + x_1 f_{i-1}^s(x_0^p) + ... + x_i f_0^s(x_0^p))$$

$$\equiv$$

$$p^{2i(s-1)+i}(f_i^{s+1}(x_0)+x_1 f_{i-1}^{s+1}(x_0)+...+x_i f_0^{s+1}(x_0))(f_i^{s-1}(x_0^p)+x_1 f_{i-1}^{s-1}(x_0^p)+...+x_i f_0^{s-1}(x_0^p))$$

$$\mod p^s.$$

By proposition 10.3.4, $p^{i(s-1)} f_i^s(x_0^p)$ and $p^{i(s-2)} f_i^{s-1}(x_0^p)$ are $p-$adic units, and it follows directly that the same holds for $p^{i(s-1)}(f_i^s(x_0^p) + x_1 f_{i-1}^s(x_0^p) + ... + x_i f_0^s(x_0^p))$ and

$p^{i(s-2)}(f_i^{s-1}(x_0^p)+x_1 f_{i-1}^{s-1}(x_0^p)+...+x_i f_0^{s-1}(x_0^p))$. Thus, division by these two elements leads to

$$p^i F_i^{s+1}(x_0) \equiv p^i F_i^s(x_0) \mod p^s,$$

and the statement follows. □

Proposition 10.3.6 *Let the assumptions be as in proposition 10.3.5. Then*

$$F_i^s(x_0) \equiv \frac{p^i f_i^s(x_0)}{f_i^{s-1}(x_0)} \mod p^{s-1}.$$

Proof: This follows directly by a cross-multiplication, since

$$p^{i(s-1)} f_j^s(x_0) \equiv 0 \mod p^{s-1}$$

for all $j < i$ by proposition 10.3.4. □

The preceding propositions and corollarys lead to the following theorem:

Theorem 10.3.1 *Assume that for all $s \geq 2$, condition 10.3.1 is satisfied. Then $F_0^i(z)$, which is an analytic function on $p\mathbb{Z}_p$ (or $p\mathbb{Z}_p \setminus \{0\}$ if $x_i = 0$), is the restriction to $p\mathbb{Z}_p$ (or $p\mathbb{Z}_p \setminus \{0\}$) of an analytic element of support $\mathfrak{D}_i$ (or $\mathfrak{D}_i^o$):*

$$F^i(z) = \lim_{s\to\infty} \frac{p^i(f_i^s(z)+x_1 f_{i-1}^s(z)+...+x_i f_0^s(z))}{f_i^{s-1}(z^p)+x_1 f_{i-1}^{s-1}(z^p)+...+x_i f_0^{s-1}(z^p)}.$$

Proof: By corollary 10.3.2 and proposition 10.3.5, the sequence $\{F^{i,s}(z)\}$ converges uniformly on $\mathfrak{D}_i$ and coincides with $F_0^i(z)$ on $p\mathbb{Z}_p$ (or $p\mathbb{Z}_p \setminus \{0\}$). □

Now, our observations concerning the computation of the unit root by the power series f_1, f_2 and f_3 can be explained by the following corollary, which is a direct consequence of theorem 10.3.1 and proposition 10.3.6.

Corollary 10.3.3 *Let the assumptions be as in the theorem above, and let $x_0 \in \mathfrak{D}_i$ (or $\mathfrak{D}_i^o$) be a Teichmüller point. Then the unit root can be computed with the power series f_i for $1 \leq i \leq 3$ by*

$$F^i(x_0) \equiv \frac{p^i f_i^s(x_0)}{f_i^{s-1}(x_0)} \mod p^{s-1}.$$

Chapter 11

An alternative Method for analytic continuation

In this chapter, we describe a method due to G. Christol (see [**15**]) to evaluate a fraction of the form

$$\frac{f(z)}{f(z^p)}$$

at a Teichmüller point x, where

$$f(z) = \sum_{n=0}^{\infty} a(n)z^n$$

is a power series which is a so-called algebraic $A-$element, i.e. an element in the $A-$algebra $D(A)$. The key ingredient of the method is to construct, for a given Teichmüller point x, an analytic continuation of the functions $f(z)$ and $f(z^p)$ *themselves* to the open neighborhood $B(x, 1)$, where $B(x, 1)$ denotes the open neighborhood of radius 1 of x.

If $f \in D(A)$ is continuable, in [**15**] and [**16**], Christol only gives a proof of the existence of an explicit analytic continuation, but no estimates for the $p-$adic precision. We analyse the special case that the coefficients of f are integral and satisfy the Dwork congruences, and compare the continuation method of Christol to the continuation method of Dwork.

Unfortunately, if the Dwork congruences are not satisfied, we were not able to apply Christols method in practice, since we could not derive any estimates for the $p-$adic precision in this case.

11.1 The $A-$algebra $D(A)$ and the Dwork congruences

Let A be the ring of integers of a finite algebraic exentsion of $\mathbb{Q}_p$ and let $\mathbb{C}_p$ denote the completion of the algebraic closure of $\mathbb{Q}_p$.

An $A-$*function f* is an analytic function on the open disc $B(0, 1)$ whose power series expansion

$$f(z) = \sum_{n=0}^{\infty} a(n)z^n$$

around $z = 0$ has coefficients $a(n) \in A$.
An $A-$function f is called *algebraic* if there exists a nonzero polynomial G in $\mathbb{C}_p[z, y]$ such that

$$G(z, f(z)) = 0 \text{ for all } z \in B(0, 1).$$

Definition 11.1.1 *By $D(A)$, we denote the closure of the set of algebraic $A-$functions under the topology of uniform convergence on the open disc $B(0, 1)$.*

In [**16**], the following criterion for a function to be an element of $D(A)$ is proven:

Theorem 11.1.1 ([15], Theorem 2) *An $A-$function f is an element of $D(A)$ if and only if for all $k \geq 1$, the number of functions*

$$f_{n,h}(z) = \sum_{m=0}^{\infty} a(n + mp^h)z^m, 0 \leq n < p^h$$

modulo p^k is finite.

We prove that any f whose coefficients $a(n)$ satisfy the Dwork congruences lies in $D(A)$ by proving the following theorem and then applying theorem 11.1.1:

Theorem 11.1.2 *Let*

$$f(z) = \sum_{n=0}^{\infty} a(n)z^n$$

be a power series with coefficients $a(n) \in \mathbb{Z}$ satisfying the Dwork congruences. Then, for each $k \geq 1$, the number of functions

$$f_{n,h}(z) = \sum_{m=0}^{\infty} a(n + mp^h)z^m, 0 \leq n < p^h$$

modulo p^k is finite.

Proof: Since the sequence $(a(n))_n$ satisfies the Dwork congruences, for all $h \geq k$, we have

$$\frac{a(n + mp^h)}{a([n/p] + mp^{h-1})} = \frac{a(n + mp^{h-k}p^k)}{a(n + mp^{h-k}p^{k-1})} \equiv \frac{a(n)}{a([n/p])} \mod p^k,$$

and thus

$$a(n + mp^h) \equiv a([n/p] + mp^{h-1})\frac{a(n)}{a([n/p])} \mod p^k.$$

Applying the Dwork congruences $h - k$ times in this way, we obtain that

$$a(n + mp^h) \equiv a([n/p^{h-k}] + mp^k)\frac{a(n)}{a([n/p^{h-k}])} \mod p^k. \tag{11.1}$$

But this means that

$$f_{n,h}(z) = \sum_{m=0}^{\infty} a(n + mp^h)z^m \equiv \frac{a(n)}{a([n/p^{h-k}])} \sum_{m=0}^{\infty} a([n/p^{h-k} + mp^k])z^m \mod p^k,$$

where $[n/p^{h-k}] < p^k$.
For each n and $k \leq h$, the fraction $a(n)/a([n/p^{h-k}])$ lies in $\mathbb{Z}_p$, and thus

$$\#\{\frac{a(n)}{a([n/p^{h-k}])} \mod p^k; n < p^h, k \leq h\} \leq p^k,$$

since $\mathbb{Z}_p/p^k\mathbb{Z}_p$ contains p^k elements. Since $[n/p^{h-k}] < p^k$, it follows that

$$\#\{\sum_{m=0}^{\infty} a([n/p^{h-k}] + mp^k)z^m\} \leq p^k.$$

Thus modulo p^k, $f_{n,h}(z)$ takes at most p^{2k} different values if $k \leq h$.
For $h < k$, we have $n < p^h < p^k$, and thus

$$\#\{f_{n,h}(z), 0 \leq h < k, 0 \leq n < p^h\} < kp^k.$$

This completes the proof of the theorem. □

Now, we consider the power series $f(z^p)$ and prove that if the coefficients of $f(z)$ satisfy the Dwork congruences, $f(z^p)$ belongs to $D(A)$, too. As a power series in z, $f(z^p)$ can be written as

$$g(z) := f(z^p) = \sum_{n=0}^{\infty} a(n)z^{pn} = \sum_{n=0}^{\infty} b(n)z^n,$$

where

$$b(n) := \begin{cases} a(n/p) \text{ if } p|n \\ 0 \text{ otherwise.} \end{cases}$$

Corollary 11.1.1 *Let $f(z)$ be a power series with coefficients in $\mathbb{Z}$ satisfying the Dwork congruences. Then, for each $k \geq 1$, the number of functions $g_{n,h}(z)$, $n < p^h$ modulo p^k is finite.*

Proof: If p does not divide n, $g_{n,h}(z) = 0$ for arbitrary h. Otherwise,

$$g_{n,h}(z) = \sum_{m=0}^{\infty} b(n + mp^h)z^m = \sum_{m=0}^{\infty} a(n/p + mp^{h-1})z^m = f_{n/p,h-1},$$

and we can apply Theorem 11.1.2. □

Theorem 11.1.2 and corollary 11.1.1 combined with theorem 11.1.1 now prove that $f(z)$ and $f(z^p)$ are elements of the $A-$algebra $D(A)$.

11.2 Explicit construction of an analytic continuation

For the so-called continuable elements in $D(A)$, in [**15**], an explicit formula for an analytic continuation in a neighborhood of a fixed point x on the boundary of the $p-$adic unit disc is given.
We define a new notion of continuability and prove that all $f \in D(A)$ whose coefficients satisfy the Dwork congruences are continuable in the new way. Furthermore, we prove that Christol's analytic continuation method also applies to functions that are continauble in the new sense.
In [**15**], section 5, Christol defines the notion of continuability of an element $f \in D(A)$ as follows:

Definition 11.2.1 *f is continuable in a point x with $|x|_p = 1$ if*

$$\lim_{h\to\infty} \sum_{k=1}^{p^h-1} a(n + kp^h)x^{n+kp^h} = 0$$

for all $n \in \mathbb{Z}$.

In the following, we will say that a function which is continuable as defined in definition (11.2.1) is *continuable in the sense of Christol*. We found out that for the construction of an analytic continuation, it is *not* always necessary that a function is continuable in the sense of Christol. For our purposes, we modify the definition in the following way:

Definition 11.2.2 *f is continuable if for all $h \geq 1$ and $k \geq 1$,*

$$a(kp^h - 1) \equiv 0 \mod p^h.$$

We will say that a function which is continuable as defined in definition 11.2.2 is *continuable in the weak sense of Christol*.

Proposition 11.2.1 *Let $(a(n))_n$ satisfy the Dwork congruences and $a(p-1) \equiv 0 \mod p$. Then, f is continuable in the weak sense of Christol.*

Proof: First of all, we prove that $a(p^h - 1) \equiv 0 \mod p^h$ by induction on h.
By the Dwork congruences, we have

$$\frac{a(p^2-1)}{a(p-1)} \equiv a(p-1) \mod p,$$

and thus $\frac{a(p^2-1)}{a(p-1)} \equiv 0 \mod p$ and $a(p^2-1) \equiv 0 \mod p^2$. Now let $h > 2$. Then

$$\frac{a(p^h-1)}{a(p^{h-1}-1)} \equiv \frac{a(p^{h-1}-1)}{a(p^{h-2}-1)} \mod p^h$$

by the Dwork congruences. By induction hypothesis, $\frac{a(p^{h-1}-1)}{a(p^{h-2}-1)} \equiv 0 \mod p$, and thus it follows that

$$\frac{a(p^h - 1)}{a(p^{h-1} - 1)} \equiv 0 \mod p. \tag{11.2}$$

Since $a(p^{h-1} - 1) \equiv 0 \mod p^{h-1}$, it follows that $a(p^h - 1) \equiv 0 \mod p^h$.
Now, consider $a(kp^h - 1)$ for $k \geq 2$. Since $a(kp - 1) = a((k - 1)p + p - 1)$, we have

$$\frac{a(kp - 1)}{a(k - 1)} \equiv a(p - 1) \mod p$$

by the Dwork congruences, and thus $a(kp - 1) \equiv 0 \mod p$ for any $k \geq 2$. Since $a(kp^h - 1) = a((k - 1)p^h + p^h - 1)$, it follows by the Dwork congruences that

$$\frac{a(kp^h - 1)}{a(kp^{h-1} - 1)} \equiv \frac{a(p^h - 1)}{a(p^{h-1} - 1)} \mod p^h.$$

By equation (11.2), it follows that $\frac{a(kp^h-1)}{a(kp^{h-1}-1)} \equiv 0 \mod p$. Since $a(kp^{h-1} - 1) \equiv 0 \mod p^{h-1}$ by induction, we obtain $a(kp^h - 1) \equiv 0 \mod p^h$ and the proposition follows. □

Definition 11.2.3 *For $f \in D(A)$, we define*

$$P(f, h) := \sum_{k=1}^{p^h-1} \sum_{n=0}^{kp^h-1} a(n).$$

For a Teichmüller point x, remark that

$$P(f(xz), h) = \sum_{k=1}^{p^h-1} \sum_{n=0}^{kp^h-1} a(n)x^n.$$

Definition 11.2.4 *A sequence $h(n)$ tends multiplicatively to infinity if $\lim_{n\to\infty} h(n) = \infty$ and $h(n)|h(n + 1)$ for all $n \geq 1$.*

In [**15**] and [**16**], Christol proves the following key lemma for the construction of the analytic continuation:

Lemma 11.2.1 *For $f \in D(A)$ and x a Teichmüller point, the sequence $P(f(xz), h)$ converges if h tends multiplicatively to infinity.*

We set $P(f(xz)) := \lim_{h\to\infty} P(f(xz), h)$. Note that in the statement of the lemma, nothing is said about the speed of convergence of the sequence $P(f(xz), h)$. Unfortunately, from the proof of the lemma, we were not able to deduce any formula of the form

$$P(f(xz), h) \equiv P(f(xz)) \mod p^{F(h)} \tag{11.3}$$

for some function F. Thus, we can make no statement about the $p-$adic precision of the truncation $P(f(xz), h)$.
In [**15**], section 5, Christol proves the following proposition:

Proposition 11.2.2 *For $f \in D(A)$ and a Teichmüller point x, $|x| = 1$, the function in y, defined by $f(x+y) := P(f(x+y)z)$ for $|y| < 1$ is an analytic function for y in the open disc $B(0,1)$.*

We call $f(x+y)$ the continuation of f to the open disc $B(x,1)$ of radius 1 centered at x.
If f is continuable in the weak sense of Christol, the continuation of f satisfies the following properties:

Theorem 11.2.1 (similar to [15], Theorem 7) *If f is continuable in the weak sense of Christol in a Teichmüller point x, then for all y in the open disc $B(0,1)$, we have*

$$(zf)(x+y) = (x+y)f(x+y)$$

and

$$(\frac{\partial f}{\partial z})(x+y) = \frac{\partial f(x+y)}{\partial y}.$$

Proof: Since

$$P((zf)((x+y)z), h) = (x+y)P(f((x+y)z), h) - \sum_{k=1}^{p^h-1} (x+y)^{kp^h} a(kp^h - 1),$$

and f is continuable in the weak sense of Christol, we have

$$P((zf)((x+y)z), h) \equiv (x+y)P(f((x+y)z), h) \mod p^h,$$

and thus for the limit we obtain

$$(zf)(x+y) = (x+y)f(x+y).$$

Concerning the second statement, since

$$P((\frac{\partial f}{\partial z})((x+y)z), h) = \frac{\partial}{\partial y}P(f(x+y)z, h) + \sum_{k=1}^{p^h-1} kp^h(x+y)^{kp^h-1}a(kp^h),$$

it follows directly that

$$P((\frac{\partial f}{\partial z})((x+y)z), h) \equiv \frac{\partial}{\partial y}P(f(x+y)z, h) \mod p^h$$

and thus for the limit we obtain

$$(\frac{\partial f}{\partial z})(x+y) = \frac{\partial f(x+y)}{\partial y}$$

and the theorem follows. □

Corollary 11.2.1 *Let $f \in D(A)$ be continuable in the weak sense of Christol. If f is the solution of a linear differential equation*

$$\mathcal{P}(z, \partial/\partial z) f(z) = 0,$$

then the continuation $f(x+y)$ around a point x with $|x| = 1$ is a solution of the differential equation

$$\mathcal{P}(x+y, \partial/\partial y) f(x+y) = 0.$$

By the (modified) results of Christol, we know by the corollary that if $f \in D(A)$ is continuable in the sense of Christol or in the weak sense of Christol, the limit $P(f(xz))$ computes an analytic continuation of f in the point x with $|x| = 1$. But since we are lacking a formula of the type of formula 11.3, we can not apply this analytic continuation in practice to actually compute $f(z)|_{z=x}$ for a Teichmüller point x.
Now, for the rest of this section, we analyse the whole quotient

$$\frac{P(f(xz), h)}{P(f(x^p z^p), h)}$$

and its convergence properties depending on h in case that the coefficients of the power series f satisfy the Dwork congruences. Note that for $g(z) := f(z^p)$, we have

$$P(f(x^p z^p), h) = \sum_{k=1}^{p^h-1} \sum_{n=0}^{kp^{h-1}-1} a(n) x^{pn} = \sum_{n=0}^{p^{2h-1}-p^{h-1}-1} (p^h - 1 - [n/p^{h-1}]) a(n) x^{pn}.$$

As in chapter 5, let $f_k^s(z)$ denote the truncation

$$f_k^s(z) := \sum_{n=kp^s}^{(k+1)p^s-1} a(n) z^n.$$

Furthermore, let $\hat{f}_k^s(z)$ denote the truncation

$$\hat{f}_k^s(z) = \sum_{n=0}^{kp^s-1} a(n) z^n = \sum_{n=0}^{k-1} f_n^s(z). \tag{11.4}$$

It follows by a direct computation that

$$P(f(xz), h) = \sum_{k=1}^{p^h-1} \hat{f}_k^h(x) = \sum_{k=1}^{p^h-1} \sum_{n=0}^{k-1} f_n^h(x) \tag{11.5}$$

and

$$P(f(x^p z^p), h) = \sum_{k=1}^{p^h-1} \hat{f}_k^{h-1}(x^p) = \sum_{k=1}^{p^h-1} \sum_{n=0}^{k-1} f_n^{h-1}(x^p). \tag{11.6}$$

Let X denote a variable. By Theorem 5.1,we have

$$f(X)f_n^{h-1}(X^p) \equiv f(X^p)f_n^h(X) \mod p^h\mathbb{Z}_p[[X]],$$

and applying the formulas (11.5) and (11.6), we obtain by a direct computation that

$$f(X)P(f(X^pz^p),h) \equiv f(X^p)P(f(Xz),h) \mod p^h\mathbb{Z}_p[[X]]. \tag{11.7}$$

Thus in the same way as theorem 5.1.3 follows from theorem 5.1, the following theorem follows from equation (11.7):

Theorem 11.2.2 *Let $f \in D(A)$ such that the sequence $(a(n))_n$ satisfies the Dwork congruences. Let*

$$\mathfrak{D} := \{x \in \mathbb{Z}_p, |P(f(xz),1)| = |P(f(x^pz^p),1)| = 1\}.$$

Then $f(z)/f(z^p)$, which is a uniform analytic function on $p\mathbb{Z}_p$, is the restriction to $p\mathbb{Z}_p$ of an analytic element F of support $\mathfrak{D}$:

$$F(x) = \lim_{h\to\infty} \frac{P(f(xz),h)}{P(f(x^pz^p),h)}.$$

Proof: Just like the proof of theorem 5.1.3.□

This leads us to the following formula for explicit computations:

$$\frac{f(z)}{f(z^p)}|_{z=x} \equiv \frac{P(f(xz),h)}{P(f(x^pz^p),h)} \mod p^h. \tag{11.8}$$

Thus, to compute the quotient $\frac{f(z)}{f(z^p)}|_{z=x}$ modulo p^h, one has to compute $p^{2h} - p^h - 1$ of the coefficients $a(n)$ of f, whereas for the formula given by Dwork, one only has to compute $p^h - 1$ of the coefficients, which makes the formula we derived from Christols considerations rather useless in practice.

As a final remark, note that by way of computation we found out that, for general f satisfying the Dwork congruences and x such that $|P(f(xz),1)| = 1$, it is not true that

$$P(f(x,z),h) \equiv P(f(xz),h+1) \mod p^h.$$

Hence, even in case that the Dwork congruences are satisfied, we are not in a position to evaluate $f(z)$ at a point x up to a given precision by the method proposed by Christol.

Chapter A

Appendix

A.1 Hadamard products

In this appendix we collect the results of our calculations on the 24 operators which are Hadamard products $A*a$, etc. We computed coefficients (a,b) of the Frobenius polynomial

$$P(T) = 1 + aT + bpT^2 + ap^3T^3 + p^6T^4$$

for all primes p between 3 and 17 and for all possible values of $z \in \mathbb{F}_p^*$. If there occurs a "-" in the table instead of the tuple (a,b), then the corresponding $z \in \mathbb{F}_p$ is either a zero of $f_0^1(z)$ or of g_0^1 or of both, where f_0 is the power series solution of the fourth order differential equation and g_0 is the solution of the fifth order equation. The appearance of $(a,b)'$ means that the polynomial is *reducible*. The appearance of $(a,b)^*$ means that the corresponding z is a singular point of the differential equation.

The Case $A*a$

This is operator nr. 45 from the list [**2**]:

$$\theta^4 - 4\,x\,(2\,\theta+1)^2\left(7\,\theta^2+7\,\theta+2\right) - 128\,x^2\,(2\,\theta+1)^2\,(2\,\theta+3)^2$$

$p = 3$

z	1	2
	$-$	$-$

$p = 5$:

z	1	2	3	4
	$(6,-6)'$	$(28,38)*$	$-$	$(32,62)*$

$p = 7$:

z	1	2	3	4	5	6
	(2,-46)	(-8,2)	(32,-94)*	(80,290)*	(10,50)'	-

$p = 11$:

z	1	2	3	4	5	6	7	8
	(56,290)'	-	(-16,2)'	(6,26)	(16,98)	(12,114)'	(26,106)	-

z	9	10
	(-8,2)	(-36,210)'

$p = 13$:

z	1	2	3	4	5	6	7	8
	(-8,270)'	(20,-106)	(-4,86)	(-204,646)*	(22,-30)	(-160,30)*	(-34,50)	(-16,302)

z	9	10	11	12
	(58,146)	(18,34)	(84,406)	(56,206)'

$p = 17$:

z	1	2	3	4	5	6	7	8
	(256,-322)*	(256,-322)*	(-24,542)	(44,166)	(210,1218)	(24,-178)'	(-100,278)	(22,50)

z	9	10	11	12	13	14	15	16
	(-4,70)	(52,470)	(-84,342)'	-	(22,-334)'	(18,258)	(184,974)	(-56,302)'

The Case $B * a$

This is operator nr. 15 from the list [**2**]:

$$\theta^4 - 3\,x\,(3\,\theta+1)\,(3\,\theta+2)\left(7\,\theta^2+7\,\theta+2\right) - 72\,x^2\,(3\,\theta+1)\,(3\,\theta+2)\,(3\,\theta+4)\,(3\,\theta+5)$$

$p = 3$:

z	1	2
	$(2,4)$	$(8,13)$

$p = 5$:

z	1	2	3	4
	$(-18,-22)$	$-$	$(3,-22)$	$(6,41)$

$p = 7$:

z	1	2	3	4	5	6
	(-31,-102)	(-13,60)	-	(20,12)	-	-

$p = 11$:

z	1	2	3	4	5	6	7	8
	(36,170)	(-147,422)	(-15,152)	(21,170)	(45,224)	(-24,71)	(-3,-28)	(-72,-478)

z	9	10
	(51,170)	(-12,8)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(23,60)	(20,192)	(-13,72)	(23,330)	(-103,-768)	-	(50,285)	(14,-138)

z	9	10	11	12
	(17,144)	(56,228)	-	(-202,618)

$p = 17$:

z	1	2	3	4	5	6	7	8
	(-12,128)	(105,488)	(93,254)	(21,-250)	(-234,-718)	(-60,-25)	(-39,38)	-

z	9	10	11	12	13	14	15	16
	(-132,668)	(-414,2522)	(108,362)	(117,524)	(-39,-142)	(-21,488)	-	(15,-196)

The Case $C * a$

This is operator nr. 68 from the list [**2**]:

$$\theta^4 - 4\,x\,(4\,\theta+1)\,(4\,\theta+3)\left(7\,\theta^2+7\,\theta+2\right) - 128\,x^2\,(4\,\theta+1)\,(4\,\theta+3)\,(4\,\theta+5)\,(4\,\theta+7)$$

$p = 3$:

z	1	2
	$(2,-2)$	$-$

$p = 5$:

z	1	2	3	4
	$-$	$(6,6)$	$(-18,-22)*$	$(8,38)$

$p = 7$:

z	1	2	3	4	5	6
	(24,-158)*	-	(4,2)’	(22,50)	(-2,66)’	(72,226)*

$p = 11$:

z	1	2	3	4	5	6	7	8
	(10,158)	(124,146)*	-	(-10,-38)	-	(92,-238)*	(28,34)	(-32,122)

z	9	10
	-	(14,50)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(232,1038)*	(-4,150)	(-32,62)’	(-46,146)’	(46,126)	(-2,210)	(58,290)	(162,58)*

z	9	10	11	12
	(12,-50)	(64,206)	(-6,86)	(24,262)

$p = 17$:

z	1	2	3	4	5	6	7	8
	(-60,246)	(-30,162)	(52,226)	-	(8,134)	-	(178,962)	-

z	9	10	11	12	13	14	15	16
	(404,2342)*	-	-	(-32,-190)	(336,1118)*	(24,-142)	(-24,254)	(66,506)

The Case $D * a$

This is operator nr. 62 from the list [**2**]:

$$\theta^4 - 12\,x\,(6\,\theta+1)\,(6\,\theta+5)\left(7\,\theta^2+7\,\theta+2\right) - 1152\,x^2\,(6\,\theta+1)\,(6\,\theta+5)\,(6\,\theta+7)\,(6\,\theta+11)$$

$p = 3$:

z	1	2
	$(2,4)$	$(8,13)$

$p = 5$:

z	1	2	3	4
	$(34,74)*$	$(29,44)*$	$-$	$-$

$p = 7$:

z	1	2	3	4	5	6
	(5,-4)	(4,-40)	(59,122)*	(65,170)*	(22,92)	-

$p = 11$:

z	1	2	3	4	5	6	7	8
	-	(12,96)	(-9,-46)	(25,14)	(59,296)	(-160,578)*	(-115,38)*	(29,184)

z	9	10
	(8,-142)	(-24,15)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(67,276)	(56,374)	(5,-100)’	(-138,-278)*	(4,-12)	(-193,492)*	(38,350)	(47,188)

z	9	10	11	12
	(-23,8)’	(-3,322)’	(-36,199)	(8,-160)

$p = 17$:

z	1	2	3	4	5	6	7	8
	(67,284)	(18,-79)	(-131,728)	(45,218)	(19,-388)’	(80,490)	(262,-214)*	(72,164)

z	9	10	11	12	13	14	15	16
	(-150,822)	(55,-70)’	(160,863)	(250,-430)*	(-15,150)	(11,-278)	(141,768)	(-16,56)

The Case $A * b$

This is operator nr. 25 from the list [**2**]:

$$\theta^4 - 4\,x\,(2\,\theta+1)^2\left(11\,\theta^2+11\,\theta+3\right) - 16\,x^2\,(2\,\theta+1)^2\,(2\,\theta+3)^2$$

$p = 3$:

z	1	2
	$-$	$(5,14)$

$p = 5$:

z	1	2	3	4
	$(13,16)$	$-$	$(2,26)'$	$(-3,16)$

$p = 7$:

z	1	2	3	4	5	6
	(10,50)’	(25,74)	-	(-10,82)	(10,-30)	(-15,26)

$p = 11$:

z	1	2	3	4	5	6	7	8
	(39,262)	(112,2)*	(2,58)’	(-26,42)	(15,166)	(10,-134)	(39,142)	(47,78)

z	9	10
	(152,482)*	(-26,122)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(-60,166)’	-	-	-	-	(-30,90)	(20,214)	(35,120)

z	9	10	11	12
	(50,98)	(60,246)	(35,-40)	(15,12)

$p = 17$:

z	1	2	3	4	5	6	7	8
	-	(115,744)	(20,86)’	(-10,50)	(-15,368)	(-25,60)	(140,790)’	(35,-56)

z	9	10	11	12	13	14	15	16
	-	(-15,208)	(20,-394)	(-20,-330)	-	(55,540)	(75,632)	(60,134)

The Case $B * b$

This is operator nr. 24 from the list [**2**]:

$$\theta^4 - 3\,x\,(3\,\theta+1)\,(3\,\theta+2)\left(11\,\theta^2+11\,\theta+3\right) - 9\,x^2\,(3\,\theta+1)\,(3\,\theta+2)\,(3\,\theta+4)\,(3\,\theta+5)$$

$p = 3$:

z	1	2
	$(5,7)$	$(5,19)$

$p = 5$:

z	1	2	3	4
	$-$	$(8,-4)$	$-$	$(7,-4)$

$p = 7$:

z	1	2	3	4	5	6
	-	(25,113)	(25,86)	-	(-15,-11)	(15,25)

$p = 11$:

z	1	2	3	4	5	6	7	8
	-	(105,-82)	(-29,152)	(10,-127)	(-3,62)	(37,197)	(15,188)	(36,107)

z	9	10
	(150,458)	-

$p = 13$:

z	1	2	3	4	5	6	7	8
	(90,319)	(15,112)	-	(-35,-4)	-	(45,142)	-	(-5,-151)

z	9	10	11	12
	(20,210)	(35,252)	(-85,447)	(-45,49)

$p = 17$:

z	1	2	3	4	5	6	7	8
	(50,-115)	(-30,524)	(10,362)	(-10,83)	(65,470)	(165,947)	(30,362)	(10,362)

z	9	10	11	12	13	14	15	16
	(80,407)	(45,83)	-	(110,461)	(-120,569)	(-25,-196)	(-40,38)	(-50,56)

The Case $C * b$

This is operator nr. 51 from the list [**2**]:

$$\theta^4 - 4\,x\,(4\,\theta+1)\,(4\,\theta+3)\left(11\,\theta^2+11\,\theta+3\right) - 16\,x^2\,(4\,\theta+1)\,(4\,\theta+3)\,(4\,\theta+5)\,(4\,\theta+7)$$

$p = 3$: $p = 5$:

z	1	2
	$(5,14)$	$(5,2)$

z	1	2	3	4
	$(3,-4)'$	$(12,46)$	$-$	$-$

$p = 7$:

z	1	2	3	4	5	6
	(40,122)	(-10,18)	-	(-5,90)	(15,26)	-

$p = 11$:

z	1	2	3	4	5	6	7	8
	(24,130)	(39,162)'	-	(-5,-74)	(-64,-574)*	(-144,386)*	(30,206)	(-4,162)

z	9	10
	(-26,122)	(19,130)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(80,430)	(75,282)	(-15,96)	(45,228)	(-30,-38)	(-5,-190)	(30,166)	(-80,282)

z	9	10	11	12
	(30,202)	(10,202)	(30,122)	-

$p = 17$:

z	1	2	3	4	5	6	7	8
	(10,522)	(35,292)	(70,626)	(90,554)	(-70,382)	(50,-110)'	(90,326)	(-25,188)

z	9	10	11	12	13	14	15	16
	-	(65,514)	(15,-150)'	(65,450)	(-50,162)	(115,410)'	(15,124)	(-100,326)

The Case $D * b$

This is operator nr. 63 from the list [**2**]:

$$\theta^4 - 12\,x\,(6\,\theta+1)\,(6\,\theta+5)\left(11\,\theta^2+11\,\theta+3\right) - 144\,x^2\,(6\,\theta+1)\,(6\,\theta+5)\,(6\,\theta+7)\,(6\,\theta+11)$$

$p = 3$: $p = 5$:

z	1	2
	$(5,7)$	$(5,19)$

z	1	2	3	4
	$-$	$(24,76)$	$(4,1)$	$(-1,-4)$

$p = 7$:

z	1	2	3	4	5	6
	(15,47)	-	(5,31)	-	(-5,62)	(25,95)

$p = 11$:

z	1	2	3	4	5	6	7	8
	(39,142)	(13,137)	(-2,87)	(104,-94)*	(23,4)	(8,129)	(169,686)*	-

z	9	10
	-	(-41,157)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(15,139)	(85,410)'	(40,86)	(75,275)	(-5,-268)	-	(-55,355)	(-25,189)

z	9	10	11	12
	(20,293)	(15,-120)	(-40,305)	(-15,-180)

$p = 17$:

z	1	2	3	4	5	6	7	8
	(30,88)	(-10,206)	(15,236)	(20,111)	(90,239)	(140,749)	-	(10,-231)

z	9	10	11	12	13	14	15	16
	-	(-30,410)	(5,-41)	(105,698)	-	(-10,542)	(-140,684)	(50,-137)

The Case $A * c$

This is operator nr.58 from the list [**2**]:

$$\theta^4 - 4\,x\,(2\,\theta+1)^2\left(10\,\theta^2+10\,\theta+3\right) + 144\,x^2\,(2\,\theta+1)^2\,(2\,\theta+3)^2$$

$p = 3$:

z	1	2
	$(8,2)*$	$-$

$p = 5$:

z	1	2	3	4
	$(-28,38)*$	$-$	$(-2,-14)$	$(16,-34)*$

$p = 7$:

z	1	2	3	4	5	6
	(12,2)	(32,-94)*	(26,50)	(80,290)*	-	(12,82)

$p = 11$:

z	1	2	3	4	5	6	7	8
	(-160,578)*	(-60,290)	(-12,-78)	(4,-14)	(20,178)	(14,122)'	(-46,170)	(-4,-14)

z	9	10
	-	(36,98)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(-108,-698)*	(-14,-70)	(16,126)	(32,158)	(8,62)	(42,202)'	(16,62)	(42,10)

z	9	10	11	12
	(-204,646)*	(16,126)	(2,314)	(-16,254)'

$p = 17$:

z	1	2	3	4	5	6	7	8
	(-76,278)	(-8,-178)	(-134,562)'	(8,302)	(-24,-178)'	(-142,706)	(-32,110)	(168,942)

z	9	10	11	12	13	14	15	16
	-	(76,278)	(66,2)	(-38,178)	(-12,-234)	-	(224,-898)*	(-356,1478)*

The Case $B * c$

This is operator nr.70 from the list [**2**]:

$$\theta^4 - 3\,x\,(3\,\theta+1)\,(3\,\theta+2)\left(10\,\theta^2+10\,\theta+3\right) + 81\,x^2\,(3\,\theta+1)\,(3\,\theta+2)\,(3\,\theta+4)\,(3\,\theta+5)$$

$p = 3$:

z	1	2
	$(5,10)'$	$(-4,-2)'$

$p = 5$:

z	1	2	3	4
	$(-9,-4)$	$(-27,32)*$	$(11,-16)'$	$(-3,32)$

$p = 7$:

z	1	2	3	4	5	6
	(17,54)'	(2,30)	(-46,18)*	-	(11,-24)	(-31,-102)*

$p = 11$:

z	1	2	3	4	5	6	7	8
	(-144,386)*	(18,89)'	(3,-100)	-	-	(-6,26)	-	(-72,350)

z	9	10
	(147,422)*	(27,62)'

$p = 13$:

z	1	2	3	4	5	6	7	8
	(-202,618)*	(62,198)	(-190,450)*	(-34,147)	(5,150)'	-	(20,30)	-

z	9	10	11	12
	(20,30)	(-31,203)	-	(41,240)

$p = 17$:

z	1	2	3	4	5	6	7	8
	-	(174,947)'	(-33,20)	(39,326)	(-16,57)	(6,362)	(-180,-1690)*	(-18,2)

z	9	10	11	12	13	14	15	16
	(-18,-358)	(-63,200)	-	(234,-718)*	(-39,-214)	(-81,92)	(-135,776)'	(-144,866)

The Case $C * c$

This is operator nr. 69 from the list [**2**]:

$$\theta^4 - 4\,x\,(4\,\theta+1)\,(4\,\theta+3)\left(10\,\theta^2+10\,\theta+3\right) + 144\,x^2\,(4\,\theta+1)\,(4\,\theta+3)\,(4\,\theta+5)\,(4\,\theta+7)$$

$p = 3$:

z	1	2
	$(-4,-14)*$	$(-4,10)$

$p = 5$:

z	1	2	3	4
	$(-32,62)*$	$(-8,2)'$	$(-4,26)$	$-$

$p = 7$:

z	1	2	3	4	5	6
	(40,-30)*	-	(-4,-54)	(44,2)*	(36,118)	-

$p = 11$:

z	1	2	3	4	5	6	7	8
	(32,130)	(16,2)'	(72,-478)*	(-20,-46)	(-172,722)*	(12,54)	(-40,218)	(-20,182)

z	9	10
	(28,82)	(-28,50)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(-20,-138)	(4,218)	-	(40,206)'	(40,2)	(72,290)	-	-

z	9	10	11	12
	(-12,70)	(140,-250)*	(60,334)	(132,-362)*

$p = 17$:

z	1	2	3	4	5	6	7	8
	(-24,-82)	(-72,110)	(-12,26)	(-276,38)*	(-76,122)	(148,734)	(88,218)	(316,758)*

z	9	10	11	12	13	14	15	16
	(-28,-58)	(-176,962)	(-112,386)’	(-28,470)	(-120,462)	(24,210)’	(-4,-266)	(64,382)’

The Case $D * c$

This is operator nr. 64 from the list [**2**]:

$$\theta^4 - 12\,x\,(6\,\theta+1)\,(6\,\theta+5)\left(10\,\theta^2+10\,\theta+3\right) + 1296\,x^2\,(6\,\theta+1)\,(6\,\theta+5)\,(6\,\theta+7)\,(6\,\theta+11)$$

$p = 3$: $p = 5$:

z	1	2
	$(5,10)'$	$(-4,-2)'$

z	1	2	3	4
	$-$	$(19,-16)*$	$(-31,56)*$	$-$

$p = 7$:

z	1	2	3	4	5	6
	(-6,-50)	(31,128)’	(47,26)*	-	(86,338)*	-

$p = 11$:

z	1	2	3	4	5	6	7	8
	(-49,238)	(-75,350)	(31,76)	(115,38)*	(-21,60)	(8,-98)	(-18,-7)	-

z	9	10
	(-136,290)*	(14,122)’

$p = 13$:

z	1	2	3	4	5	6	7	8
	(-198,562)*	-	(-44,222)	(-31,8)	-	(75,310)	(25,140)	(45,160)

z	9	10	11	12
	(-138,-278)*	(22,75)	(44,254)	(-7,-4)

$p = 17$:

z	1	2	3	4	5	6	7	8
	(16,-94)	(121,520)	(-111,444)	-	(-362,1586)*	-	(79,488)	(-2,250)

z	9	10	11	12	13	14	15	16
	-	(236,-682)*	(-6,-342)	(95,392)	(63,254)	-	(-162,851)	(-83,368)

The Case $A * d$

This is operator nr. 36 from the list [**2**]:

$$\theta^4 - 16\,x\,(2\,\theta+1)^2\left(3\,\theta^2+3\,\theta+1\right) + 512\,x^2\,(2\,\theta+1)^2\,(2\,\theta+3)^2$$

$p = 3$: $p = 5$:

z	1	2
	$(4,-14)$	$-$

z	1	2	3	4
	$(8,46)$	$(-8,-82)$	$-$	$-$

$p = 7$:

z	1	2	3	4	5	6
	(40,-30)	(-8,-30)	(-12,34)	-	-	-

$p = 11$:

z	1	2	3	4	5	6	7	8
	(-80,322)	(-8,162)	(-28,146)	(-20,82)	(172,722)	-	(16,-30)	-

z	9	10
	(24,-62)	(-4,-142)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(-36,86)	(28,118)	(56,270)	(36,230)	(-48,254)	(-200,590)	(72,398)	-

z	9	10	11	12
	-	(-18,8)	(60,214)	(-132,-362)

$p = 17$:

z	1	2	3	4	5	6	7	8
	(44,-90)	(-212,-1114)	(-76,598)	(-276,38)	-	(28,326)	-	(-84,422)

z	9	10	11	12	13	14	15	16
	(-4,6)	(112,606)	(-16,-162)	(-8,-50)	(-44,598)	(44,-42)	(20,470)	(124,774)

The Case $B * d$

This is operator nr. 48 from the list [**2**]:

$$\theta^4 - 12\,x\,(3\,\theta+1)\,(3\,\theta+2)\left(3\,\theta^2+3\,\theta+1\right) + 288\,x^2\,(3\,\theta+1)\,(3\,\theta+2)\,(3\,\theta+4)\,(3\,\theta+5)$$

$p = 3$:

z	1	2
	$(-1,-8)$	$(-7,16)$

$p = 5$:

z	1	2	3	4
	$-$	$(21,-4)*$	$-$	$-$

$p = 7$:

z	1	2	3	4	5	6
	(-5,32)	(-11,32)	(-5,38)	(-8,62)	(-55,90)*	(36,-62)*

$p = 11$:

z	1	2	3	4	5	6	7	8
	-	(-29,152)	(37,80)	(-89,386)	(69,-514)*	(8,-145)	(50,98)’	-

z	9	10
	(-40,170)	(-1,98)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(36,49)	(21,-44)	(18,322)’	-	(-112,-642)*	(58,98)	-	(-21,334)

z	9	10	11	12
	(27,-56)	(-154,-54)*	(33,166)	(-24,106)

$p = 17$:

z	1	2	3	4	5	6	7	8
	(88,614)	(-32,326)	(234,-718)*	(-11,128)	(-14,-286)	(109,362)’	(-35,146)	(105,308)

z	9	10	11	12	13	14	15	16
	(15,20)	-	-	(18,155)	(88,569)	(-5,506)	(-71,452)	(-20,-250)

The Case $C * d$

This is operator nr. 38 from the list [**2**]:

$$\theta^4 - 16\,x\,(4\,\theta+1)\,(4\,\theta+3)\left(3\,\theta^2+3\,\theta+1\right) + 512\,x^2\,(4\,\theta+1)\,(4\,\theta+3)\,(4\,\theta+5)\,(4\,\theta+7)$$

$p = 3$:

z	1	2
	$(-10,10)$	$(2,-22)$

$p = 5$:

z	1	2	3	4
	$(36,86)$	$-$	$-$	$-$

$p = 7$:

z	1	2	3	4	5	6
	-	(36,-62)	(-12,-2)	(-4,66)	(-10,10)	(-2,26)

$p = 11$:

z	1	2	3	4	5	6	7	8
	(10,122)	(150,458)	(12,-78)	(-118,74)	(-64,306)	(20,146)	(-42,122)	-

z	9	10
	(-98,434)	(-4,-30)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(-16,126)	(32,158)	(236,1094)	-	-	(-16,158)	-	-

z	9	10	11	12
	(-14,-86)	(12,54)	(2,42)	(62,346)

$p = 17$:

z	1	2	3	4	5	6	7	8
	(-240,-610)	(96,382)	(62,314)	(-24,14)	(8,78)	(-24,402)	(94,354)	(20,294)

z	9	10	11	12	13	14	15	16
	(-396,2198)	(44,438)	(-58,162)	(-4,354)	(76,230)	(6,-158)	(40,590)	(12,22)

The Case $D * d$

This is operator nr. 65 from the list [**2**]:

$$\theta^4 - 48\,x\,(6\,\theta+1)\,(6\,\theta+5)\left(3\,\theta^2+3\,\theta+1\right) + 4608\,x^2\,(6\,\theta+1)\,(6\,\theta+5)\,(6\,\theta+7)\,(6\,\theta+11)$$

$p = 3$:

z	1	2
	$(-1,-8)$	$(-7,16)$

$p = 5$:

z	1	2	3	4
	$(-26,26)*$	$(-11,-64)*$	$(-1,-2)$	$(14,23)$

$p = 7$:

z	1	2	3	4	5	6
	(-3,-4)	(-12,54)	-	(-9,40)	(-11,66)	(43,-6)*

$p = 11$:

z	1	2	3	4	5	6	7	8
	(67,-538)*	(-19,-58)	(-17,-62)	(-48,106)	(-13,104)'	-	(79,324)	(-62,282)

z	9	10
	(-5,-128)	(30,173)

$p = 13$:

z	1	2	3	4	5	6	7	8
	-	(22,178)	-	(24,-70)	(87,428)	(-164,86)*	-	(-35,142)

z	9	10	11	12
	(-58,179)	(47,276)	(33,-86)	(-126,-446)*

$p = 17$:

z	1	2	3	4	5	6	7	8
	(16,30)	(-31,364)	(-23,94)	(40,281)	(22,99)	(59,-24)	(-410,2450)*	(43,592)

z	9	10	11	12	13	14	15	16
	(109,472)	(-25,158)	(15,230)	(110,690)	(5,552)	(-198,-1366)*	(-40,342)	(20,-50)

The Case $A * f$

This is operator nr. 133 from the list [**2**]:

$$\theta^4 - 12\,x\,(2\,\theta+1)^2\left(3\,\theta^2+3\,\theta+1\right) + 432\,x^2\,(2\,\theta+1)^2\,(2\,\theta+3)^2$$

$p = 3$:

z	1	2
	$(2,10)'$	$(-1,-2)$

$p = 5$:

z	1	2	3	4
	$(3,44)$	$(-6,-6)'$	$(-3,28)$	$(-18,42)'$

$p = 7$:

z	1	2	3	4	5	6
	(48,34)*	(9,26)	(1,26)	(17,26)	(64,162)*	(9,26)

$p = 11$:

z	1	2	3	4	5	6	7	8
	(-48,210)	-	-	(3,158)	(-36,18)	(-36,82)	(27,70)	(54,266)

z	9	10
	(21,-58)	(-54,122)

$p = 13$:

z	1	2	3	4	5	6	7	8
	-	(38,146)	(-47,48)	(-18,-38)	(-192,478)*	(133,660)	(-11,84)	(-34,146)

z	9	10	11	12
	(-18,-166)	(58,242)	(-192,478)*	(50,98)

$p = 17$:

z	1	2	3	4	5	6	7	8
	(48,350)	-	-	(-48,286)	(-9,-260)	-	(72,494)	(-111,524)

z	9	10	11	12	13	14	15	16
	(72,622)	(-81,268)	(6,42)	(-48,334)	(42,-54)	(-18,-54)	(-126,570)	-

The Case $B * f$

This is operator nr. 134 from the list [**2**]:

$$\theta^4 - 9\,x\,(3\,\theta+1)\,(3\,\theta+2)\left(3\,\theta^2+3\,\theta+1\right) + 243\,x^2\,(3\,\theta+1)\,(3\,\theta+2)\,(3\,\theta+4)\,(3\,\theta+5)$$

$p = 3$:

z	1	2
	$(-4,13)$	$(5,4)$

$p = 5$:

z	1	2	3	4
	$(-24,71)$	$(3,17)$	$-$	$(-3,-31)$

$p = 7$:

z	1	2	3	4	5	6
	(11,75)	-	(5,-12)	(-34,-78)*	(-34,-78)*	(5,60)'

$p = 11$:

z	1	2	3	4	5	6	7	8
	(15,218)	(-78,296)	(-12,2)	(-36,194)	(-3,-79)	(69,263)	(-36,113)	-

z	9	10
	(-24,107)	(-9,131)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(-1,-171)	(-133,-348)*	(23,114)	(41,159)	(-25,165)	(-109,450)	(-133,-348)*	(32,-48)

z	9	10	11	12
	(98,495)	(-55,99)	(50,33)	(44,306)’

$p = 17$:

z	1	2	3	4	5	6	7	8
	(-12,-322)	(-135,695)	(-105,506)	(-63,227)	(30,434)	(-24,-286)	(45,254)	(-156,857)

z	9	10	11	12	13	14	15	16
	(42,92)	(15,-25)	(30,-142)	(12,362)	(-6,236)	(108,641)	(15,461)	(-84,587)

The Case $C * f$

This is operator nr. 135 from the list [**2**]:

$$\theta^4 - 12\,x\,(4\,\theta+1)\,(4\,\theta+3)\,\left(3\,\theta^2+3\,\theta+1\right) + 432\,x^2\,(4\,\theta+1)\,(4\,\theta+3)\,(4\,\theta+5)\,(4\,\theta+7)$$

$p = 3$:

z	1	2
	$(-4,-2)'$	$(5,10)'$

$p = 5$:

z	1	2	3	4
	$(-12,22)$	$(-3,34)$	$(6,26)$	$-$

$p = 7$:

z	1	2	3	4	5	6
	(5,-46)	(60,130)*	(52,66)*	(9,58)	(1,62)	-

$p = 11$:

z	1	2	3	4	5	6	7	8
	-	(-24,146)	(-18,38)	(-48,146)	(51,202)	(15,50)	(-24,26)	-

z	9	10
	(-78,322)	(-27,-30)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(14,2)	(-54,142)	(44,230)	(11,-124)	(5,-190)	(210,730)*	(-22,-118)	-

z	9	10	11	12
	(44,198)’	(99,436)	(154,-54)	-

$p = 17$:

z	1	2	3	4	5	6	7	8
	-	(-111,688)	(90,494)	(-39,44)	(6,-358)	(42,322)	(-138,810)	-

z	9	10	11	12	13	14	15	16
	(-105,412)	(135,698)	-	(6,534)	(-72,622)	(-39,74)	(12,262)	(-36,582)’

The Case $D * f$

This is operator nr. 136 from the list [**2**]:

$$\theta^4 - 36\,x\,(6\,\theta+1)\,(6\,\theta+5)\,\left(3\,\theta^2+3\,\theta+1\right) + 3888\,x^2\,(6\,\theta+1)\,(6\,\theta+5)\,(6\,\theta+7)\,(6\,\theta+11)$$

$p = 3$:

z	1	2
	$(-4,13)$	$(5,4)$

$p = 5$:

z	1	2	3	4
	$-$	$-$	$(-6,-7)$	$(-21,67)$

$p = 7$:

z	1	2	3	4	5	6
	(15,-1)	(52,66)*	(5,80)’	-	(9,44)	(60,130)*

$p = 11$:

z	1	2	3	4	5	6	7	8
	(-27,123)	(-9,137)	(-24,62)	(-6,-163)	(-54,246)	(-36,254)	(48,208)	-

z	9	10
	(51,126)	(-63,289)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(35,162)	(35,86)	(64,243)	(-60,310)	(-207,688)*	-	-	(20,5)

z	9	10	11	12
	(93,383)	-	(-207,688)*	(5,-259)

$p = 17$:

z	1	2	3	4	5	6	7	8
	(-90,265)	(-99,653)	(6,375)	(-132,580)	(48,230)	(36,394)	(87,335)	(-156,760)

z	9	10	11	12	13	14	15	16
	(72,415)	(-45,342)	(-48,74)	(12,-94)	(33,-201)	(15,478)	(9,-225)	(-36,74)

The Case $A * g$

This is operator nr. 137 from the list [**2**]:

$$\theta^4 - 4\,x\left(17\,\theta^2 + 17\,\theta + 6\right)(2\,\theta + 1)^2 + 1152\,x^2\,(2\,\theta + 1)^2\,(2\,\theta + 3)^2$$

$p = 3$:

z	1	2
	$-$	$(8, 2)*$

$p = 5$:

z	1	2	3	4
	$-$	$(-32, 62)*$	$(-6, 42)'$	$(16, -34)*$

$p = 7$:

z	1	2	3	4	5	6
	(6,50)	(80,290)*	(8,2)	(32,-94)*	(16,2)	(6,34)

$p = 11$:

z	1	2	3	4	5	6	7	8
	(-104,-94)*	(-8,98)'	(2,170)	(-64,194)	(-32,2)	(8,2)	-	-

z	9	10
	(12,114)'	-

$p = 13$:

z	1	2	3	4	5	6	7	8
	(-108,-698)*	(14,146)	-	(-56,174)'	-	(-160,30)*	(36,278)	(36,118)

z	9	10	11	12
	(66,322)	(-36,22)	(16,-114)	(24,206)

$p = 17$:

z	1	2	3	4	5	6	7	8
	(-88,494)'	(-356,1478)*	(-40,14)	(92,326)'	(4,-154)	(88,350)	(10,-430)'	(6,-174)

z	9	10	11	12	13	14	15	16
	(6,210)	(-148,854)'	-	(56,206)	(-92,566)	(-182,1010)	(224,-898)*	(64,62)

The Case $B * g$

This is operator nr. 138 from the list [**2**]:

$$\theta^4 - 3\,x\,(3\,\theta + 1)\,(3\,\theta + 2)\left(17\,\theta^2 + 17\,\theta + 6\right) + 648\,x^2\,(3\,\theta + 1)\,(3\,\theta + 2)\,(3\,\theta + 4)\,(3\,\theta + 5)$$

$p = 3$:

z	1	2
	$(-4,-2)'$	$(5,10)'$

$p = 5$:

z	1	2	3	4
	$(18,-22)*$	$(-33,68)*$	$(-9,14)$	$-$

$p = 7$:

z	1	2	3	4	5	6
	(5,-66)	(32,96)	(-46,18)*	(23,96)	(-13,-12)	-

$p = 11$:

z	1	2	3	4	5	6	7	8
	(-120,98)*	(6,-37)	(-24,89)	(60,206)'	-	-	-	(72,-478)*

z	9	10
	(-9,-10)	(-39,134)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(-31,6)'	-	(-190,450)*	(86,321)	(-103,-768)*	(-4,222)	(-1,288)'	(-16,66)'

z	9	10	11	12
	(-16,210)	-	(14,33)	(41,294)'

$p = 17$:

z	1	2	3	4	5	6	7	8
	(-18,506)	(45,344)	(63,146)	(36,83)	(-171,902)	-	(-432,2846)*	(-150,812)

z	9	10	11	12	13	14	15	16
	(-66,164)	(414,2522)*	(-15,-160)	(-57,146)	(-36,-241)	(3,524)	-	(-6,182)

The Case $C * g$

This is operator nr. 139 from the list [**2**]:

$$\theta^4 - 4\,x\,(4\,\theta+1)\,(4\,\theta+3)\left(17\,\theta^2+17\,\theta+6\right) + 1152\,x^2\,(4\,\theta+1)\,(4\,\theta+3)\,(4\,\theta+5)\,(4\,\theta+7)$$

$p = 3$:

z	1	2
	$(-4,10)$	$(-4,-14)*$

$p = 5$:

z	1	2	3	4
	$(-28,38)*$	$-$	$(18,-22)*$	$(-4,14)$

$p = 7$:

z	1	2	3	4	5	6
	(88,354)*	(2,-46)	-	(68,194)*	(-18,22)	(-6,58)'

$p = 11$:

z	1	2	3	4	5	6	7	8
	(-14,194)	(-140,338)*	(72,-478)*	(50,290)	(-8,130)	(-58,202)	(10,-70)	(-24,-6)

z	9	10
	(16,106)	(-24,2)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(-20,294)	(2,-126)	-	-	-	(32,134)	(38,174)	(202,618)*

z	9	10	11	12
	(30,-62)	(224,926)*	(-22,38)	(20,270)

$p = 17$:

z	1	2	3	4	5	6	7	8
	-	(-22,338)	(-128,746)	(-44,86)	(-50,74)	(-44,74)	(-52,14)	(316,758)*

z	9	10	11	12	13	14	15	16
	(-208,1186)*	(40,110)	(-22,-94)	(164,818)	(8,-370)	(-52,218)	(-182,1010)	(-64,302)

The Case $D * g$

This is operator nr. 140 from the list [**2**]:

$$\theta^4 - 12\,x\,(6\,\theta+1)\,(6\,\theta+5)\left(17\,\theta^2+17\,\theta+6\right) + 10368\,x^2\,(6\,\theta+1)\,(6\,\theta+5)\,(6\,\theta+7)\,(6\,\theta+11)$$

$p = 3$:

z	1	2
	$(-4,-2)'$	$(5,10)'$

$p = 5$:

z	1	2	3	4
	$(-26,26)*$	$(19,-16)*$	$-$	$-$

$p = 7$:

z	1	2	3	4	5	6
	(24,64)	-	(53,74)*	(29,86)	(26,-142)*	(9,-4)

$p = 11$:

z	1	2	3	4	5	6	7	8
	(13,50)	(-17,2)	(8,-58)	-	(14,83)	(160,578)*	(-68,257)	(-32,172)

z	9	10
	(-128,194)*	(-37,38)

$p = 13$:

z	1	2	3	4	5	6	7	8
	(-198,562)*	(-4,-202)	(-4,218)	(49,294)	-	(-193,492)*	(84,386)	(24,163)

z	9	10	11	12
	(-9,214)	(24,211)	(19,-36)	(54,170)

$p = 17$:

z	1	2	3	4	5	6	7	8
	(-10,-394)	(71,368)'	(157,908)'	(112,431)	(-47,122)	(3,-144)	(-350,1370)*	(-154,716)

z	9	10	11	12	13	14	15	16
	(-38,-76)	(236,-682)*	(-63,-24)	-	-	(5,-394)	(49,320)	(-38,338)

A.2 Modular forms

In this section, we list the CY(4)-differential operators for which we computed the coefficients of modular forms in conifold points. For each conifold point, we list the coefficients a_p of the conjectured modular form up to $p = 19$. Note that we can only conjecture the identity of the modular form since we only computed the first few coefficients.
The numbers of the Calabi-Yau operators are the numbers used in the paper [**2**]. AESZ 20 denotes operator number 20 from the list in [**2**]. If these numbers differ from the numbers in the CY-database [**3**], we also mention the number in the database in brackets ().

A.2.1 Operators with one rational conifold point

AESZ 20

The differential operator is given by

$$\begin{aligned}\theta^4 &- -3z(48\theta^4+60\theta^3+53\theta^2+23\theta+4)+3^2z^2(873\theta^4+1980\theta^3+2319\theta^2+1344\theta+304)\\ &- 2\cdot 3^4z^3(1269\theta^4+3888\theta^3+5250\theta^2+3348\theta+800)\\ &+ 2^2 3^6 z^4(891\theta^4+3240\theta^3+4653\theta^2+2952\theta+688)\\ &- 2^3 3^{11} z^5(\theta+1)^2(3\theta+2)(3\theta+4).\end{aligned}$$

The first coefficients of the modular form in the singular point $1/54$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/54$	$108/2$	0	0	-9	-1	-63	-28	-72	98

AESZ 23

The differential operator is given by

$$\begin{aligned}3^2\theta^4 &- 2^2 3z(64\theta^4+80\theta^3+73\theta^2+33\theta+6)\\ &+ 2^7z^2(194\theta^4+440\theta^3+527\theta^2+315\theta+75)\\ &- 2^{12}z^3(94\theta^4+288\theta^3+397\theta^2+261\theta+66)\\ &+ 2^{17}z^4(22\theta^4+80\theta^3+117\theta^2+77\theta+19)-2^{23}z^5(\theta+1)^4.\end{aligned}$$

The first coefficients of the modular form in the singular point $1/32$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/32$	$32/3$	0	-8	-10	-16	40	-50	-30	-40

AESZ 73

The differential operator is given by

$$\begin{aligned}\theta^4 &- 2\cdot 3^2z(42\theta^4+60\theta^3+45\theta^2+15\theta+2)+2^2 3^5 z^2(180\theta^4+432\theta^3+453\theta^2+222\theta+40)\\ &- 2^4 3^9 z^3(2\theta+1)^2(13\theta^2+29\theta+20)+2^6 3^{12} z^4(2\theta+1)^2(2\theta+3)^2.\end{aligned}$$

The first coefficients of the modular form in the singular point $1/432$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/432$	$432/13$	0	0	12	7	-60	-79	-108	-11

AESZ 116

The differential operator is given by

$$\begin{aligned}\theta^4 &- 2^5 z(10\theta^4+26\theta^3+20\theta^2+7\theta+1)+2^8z^2(52\theta^4+472\theta^3+832\theta^2+492\theta+103)\\ &+ 2^{16}z^3(14\theta^4+12\theta^3-96\theta^2-105\theta-29)-2^{18}z^4(2\theta+1)(56\theta^3+468\theta^2+646\theta+249)\\ &- 2^{24}z^5(2\theta+1)(4\theta+3)(4\theta+5)(2\theta+3).\end{aligned}$$

The first coefficients of the modular form in the singular point $1/256$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/256$	$72/2$	0	0	-14	-24	28	-74	-82	92

AESZ 119

The differential operator is given by

$$\begin{aligned}\theta^4 &- 3z(48\theta^4+60\theta^3+53\theta^2+23\theta+4)+3^2z^2(873\theta^4+1980\theta^3+2319\theta^2+1344\theta+304)\\ &- 2\cdot 3^4z^3(1269\theta^4+3888\theta^3+5259\theta^2+3348\theta+800)\\ &+ 2^2 3^6 z^4(891\theta^4+3240\theta^3+4653\theta^2+2952\theta+688)-2^3 3^{11} z^5(\theta+1)^2(3\theta+2)(3\theta+4).\end{aligned}$$

The first coefficients of the modular form in the singular point $1/54$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/54$	$108/2$	0	0	-9	-1	-63	-28	-72	98

AESZ 194 (DB 255)

The differential operator is given by

$$\begin{aligned}17^2\theta^4 &- 17z(1465\theta^4+2768\theta^3+2200\theta^2+816\theta+119)\\ &+ 2z^2(62015\theta^4+131582\theta^3+125017\theta^2+65926\theta+15300)\\ &- 2\cdot 3^3z^3(43250\theta^4+10914\theta^3+12803\theta^2+7446\theta+1700)\\ &+ 3^6z^4(265\theta^4+836\theta^3+1118\theta^2+700\theta+168)-3^{10}z^5(\theta+1)^4.\end{aligned}$$

The first coefficients of the modular form in the singular point $1/81$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/81$	$225/4$	-4	0	0	-6	-32	38	26	100

AESZ 214 (DB 266)

The differential operator is given by

$$\begin{aligned}\theta^4 &- 2z(90\theta^4+188\theta^3+141\theta^2+47\theta+6)-2^2z^2(564\theta^4+1520\theta^3+1705\theta^2+934\theta+192)\\ &- 2^4z^3(2\theta+1)(286\theta^3+813\theta^2+851\theta+294)-2^6 3z^4(2\theta+1)(4\theta+3)(4\theta+5)(2\theta+3).\end{aligned}$$

The first coefficients of the modular form in the singular point $1/192$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/192$	$882/14$	-2	0	6	0	-30	-2	66	52

AESZ 220 (DB 291)

The differential operator is given by

$$\begin{aligned}\theta^4 &- 2^4z(20\theta^4+56\theta^3+38\theta^2+10\theta+1)-2^{10}z^2(84\theta^4+240\theta^3+261\theta^2+134\theta+25)\\ &- 2^{16}z^3(2\theta+1)^2(23\theta^2+55\theta+39)-2^{23}z^4(2\theta+1)^2(2\theta+3)^2.\end{aligned}$$

The first coefficients of the modular form in the singular point $1/512$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/512$	$192/4$	0	3	-6	-16	-12	-38	-126	-20

AESZ 221 (DB 292)

The differential operator is given by

$$\begin{aligned}5^2\theta^4 &- 2^25z(404\theta^4+1096\theta^3+773\theta^2+225\theta+25)\\ &- 2^4z^2(668960\theta^4+1374080\theta^3+1010960\theta^2+528000\theta+11625)\\ &- 2^815z^3(2\theta+1)(56720\theta^3+95000\theta^2+84220\theta+2689)\\ &- 2^{15}3^2z^4(2\theta+1)(1208\theta^3+2892\theta^2+2842\theta+969)\\ &- 2^{20}3^3z^5(2\theta+1)(6\theta+5)(6\theta+7)(2\theta+3).\end{aligned}$$

The first coefficients of the modular form in the singular point $1/432$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/432$	?	0	0	-12	$-$	12	-76	-8	-100

A.2.2 Operators with two rational conifold points

AESZ 28

The differential operator is given by

$$\theta^4-z(65\theta^4+130\theta^3+105\theta^2+40\theta+6)+2^2z^2(4\theta+3)(\theta+1)^2(4\theta+5).$$

The first coefficients of the modular forms in the singular points $1/64$, 1 are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/64$	$14/2$	2	-2	-12	7	48	56	-114	2
1	$6/1$	-2	-3	6	-16	12	38	-126	20

AESZ 33

The differential operator is given by

$$\begin{aligned}\theta^4 &- 2^2z(324\theta^4+456\theta^3+321\theta^2+93\theta+10)+2^9z^2(584\theta^4+584\theta^3+4\theta^2-71\theta-13)\\ &- 2^{16}z^3(324\theta^4+192\theta^3+123\theta^2+48\theta+7)+2^{24}z^4(2\theta+1)^4.\end{aligned}$$

The first coefficients of the modular forms in the singular points $1/1024$, $1/16$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/1024$	$28/1$	0	-10	-8	-7	-40	-12	-58	26
$1/16$	$28/1$	0	-10	-8	-7	-40	-12	-58	26

AESZ 55

The differential operator is given by

$$\begin{aligned} 3^2\theta^4 &- 2^2 3z(208\theta^4+224\theta^3+163\theta^2+51\theta+6) \\ &+ 2^9 z^2(32\theta^4-928\theta^3-1606\theta^2-837\theta-141) \\ &+ 2^{16}z^3(144\theta^4+576\theta^3+467\theta^2+144\theta+15)-2^{24}z^4(2\theta+1)^4. \end{aligned}$$

The first coefficients of the modular forms in the singular points $-1/64$, $1/256$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$-1/64$	$5/1$	-4	2	-5	6	32	-38	26	100
$1/256$	$40/2$	0	-6	-5	-34	16	58	-70	4

AESZ 182

The differential operator is given by

$$\theta^4-z(43\theta^4+86\theta^3+77\theta^2+34\theta+6)+2^2 3z^2(\theta+1)^2(6\theta+5)(6\theta+7).$$

The first coefficients of the modular forms in the singular points $1/27$, $1/16$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/27$	$33/2$	-5	3	-14	-32	-11	-38	-2	72
$1/16$	$22/3$	2	1	-3	-10	11	-16	42	116

AESZ 183

The differential operator is given by

$$\theta^4-2^2z(2\theta+1)^2(7\theta^2+7\theta+3)+2^4 3z^2(2\theta+1)(4\theta+3)(4\theta+5)(2\theta+3).$$

The first coefficients of the modular forms in the singular points $1/64$, $1/48$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/64$	$16/1$	0	4	-2	-24	44	22	50	-44
$1/48$	$72/1$	0	0	16	-12	64	58	32	-136

AESZ 205

The differential operator is given by

$$\theta^4 - z(59\theta^4 + 118\theta^3 + 105\theta^2 + 46\theta + 8) + 2^5 3 z^2(\theta+1)^2(3\theta+2)(3\theta+4).$$

The first coefficients of the modular forms in the singular points $1/32$, $1/27$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/32$	$32/3$	0	-8	-10	-16	40	-50	-30	-40
$1/27$	$15/2$	1	3	5	-24	52	22	-14	-20

AESZ 229 (DB 293)

The differential operator is given by

$$\begin{aligned}\theta^4 &- 2^2 z(256\theta^4 + 728\theta^3 + 506\theta^2 + 142\theta + 15)\\ &- 2^4 3^2 z^2(2336\theta^4 + 2336\theta^3 - 1768\theta^2 - 1176\theta - 189)\\ &- 2^9 3^4 z^3(512\theta^4 - 432\theta^3 - 404\theta^2 - 108\theta - 9) + 2^{12}3^8 z^4(2\theta+1)^4.\end{aligned}$$

The first coefficients of the modular forms in the singular points $1/1296$, $1/16$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/1296$	$720/5$	0	0	5	-6	32	-38	-26	-100
$1/16$	$80/4$	0	-2	-5	-6	-32	-38	26	-100

AESZ 232 (DB 296)

The differential operator is given by

$$\begin{aligned}5^2\theta^4 &- 5z(26617\theta^4 + 4658\theta^3 + 3379\theta^2 + 1050\theta + 120)\\ &+ 2^6 3 z^2(673\theta^4 - 4871\theta^3 - 10282\theta^2 - 5410\theta - 860)\\ &+ 2^{10}3^2 z^3(955\theta^4 + 4320\theta^3 + 3477\theta^2 + 1020\theta + 100)\\ &- 2^{17}3^3 z^4(3\theta+1)(2\theta+1)^2(3\theta+2).\end{aligned}$$

The first coefficients of the modular forms in the singular points $-1/27$, $1/512$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$-1/27$	$99/1$	1	0	4	-26	-11	-32	-74	-60
$1/512$	$44/1$	0	-5	-7	-26	-11	52	46	-96

AESZ 233 (DB 297)

The differential operator is given by

$$\begin{aligned}\theta^4 &- 2^4 z(83\theta^4 + 94\theta^3 + 71\theta^2 + 24\theta + 3) + 2^{11}3 z^2(101\theta^4 + 191\theta^3 + 174\theta^2 + 71\theta + 10)\\ &- 2^{16}3^2 z^2(203\theta^4 + 432\theta^3 + 333\theta^2 + 102\theta + 11) + 2^{23}3^3 z^4(3\theta+1)(2\theta+1)^2(3\theta+2).\end{aligned}$$

The first coefficients of the modular forms in the singular points $1/512$, $1/432$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/512$	$80/4$	0	-2	-5	-6	-32	-38	26	-100
$1/432$	$180/5$	0	0	5	-28	24	-70	-102	20

AESZ 235 (DB 299)

The differential operator is given by

$$\begin{aligned} 7^2\theta^4 &- 14z\theta(46\theta^3+52\theta^2+33\theta+7)\\ &- 2^2z^2(73332\theta^4+288488\theta^3+426333\theta^2+266670\theta+6272)\\ &- 2^4z^3(28600\theta^4+447760\theta^3+1204833\theta^2+1112279\theta+35098)\\ &+ 2^9z^4(22300\theta^4+59200\theta^3-741\theta^2-65509\theta-3049)\\ &+ 2^{14}z^5(1740\theta^4+13200\theta^3+1971\theta^2+1095\theta+190)\\ &- 2^{19}z^6(22\theta^4+24\theta^3-9\theta^2-21\theta-7)-2^{25}z^7(\theta+1)^4. \end{aligned}$$

The first coefficients of the modular forms in the singular points $-1/16$, $1/32$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$-1/16$	$72/2$	0	0	-14	-24	28	-74	-82	92
$1/32$	$96/3$	0	3	-14	-36	-36	54	-22	36

AESZ 237 (DB 301)

The differential operator is given by

$$\begin{aligned} \theta^4 &- 2^4z(46\theta^4+128\theta^3+91\theta^2+27\theta+3)-2^93z^2(74\theta^4-16\theta^3-231\theta^2-127\theta-20)\\ &+ 2^{14}3^2z^3(14\theta^4+216\theta^3+175\theta^2+51\theta+5)+2^{19}3^3z^4(3\theta+1)(2\theta+1)^2(3\theta+2). \end{aligned}$$

The first coefficients of the modular forms in the singular points $1/864$, $1/64$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/864$	$288/11$	0	0	-2	12	-60	-42	-10	132
$1/64$	$16/1$	0	4	-2	-24	44	22	50	-44

AESZ 239 (DB 303)

The differential operator is given by

$$\begin{aligned} \theta^4 &+ 2^43z(9\theta^4-198\theta^3-131\theta^2-32\theta-3)-2^{11}3^2z^2(486\theta^4+1215\theta^3+81\theta^2-27\theta-5)\\ &- 2^{16}3^5z^3(891\theta^4+972\theta^3+675\theta^2+216\theta+25)-2^{23}3^8z^4(4\theta+1)^2(3\theta+2)^2. \end{aligned}$$

The first coefficients of the modular forms in the singular points $-1/432$, $1/3456$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$-1/432$	$108/4$	0	0	9	-1	63	-28	72	98
$1/3456$	$432/8$	0	0	0	37	0	-19	0	163

AESZ 241 (DB 305)

The differential operator is given by

$$\begin{aligned}\theta^4 &- 2^4z(152\theta^4+160\theta^3+110\theta^2+30\theta+3)+2^{10}3z^2(428\theta^4+176\theta^3-299\theta^2-170\theta-25)\\ &- 2^{17}3^2z^3(136\theta^4-216\theta^3-180\theta^2-51\theta-5)-2^{24}3^3z^4(3\theta+1)(2\theta+1)^2(3\theta+2).\end{aligned}$$

The first coefficients of the modular forms in the singular points $-1/64$, $1/1728$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$-1/64$	$56/2$	0	-2	-16	-7	24	-68	54	-46
$1/1728$	$504/1$	0	0	2	7	-12	-66	70	-92

A.2.3 Operators with three rational conifold points

AESZ 21

The differential operator is given by

$$\begin{aligned}5^2\theta^4 &- 2^25z(36\theta^4+84\theta^3+72\theta^2+30\theta+5)-2^4z^2(181\theta^4+268\theta^3+71\theta^2-70\theta-35)\\ &+ 2^8z^3(\theta+1)(37\theta^3+248\theta^2+375\theta+165)\\ &+ 2^{10}z^4(39\theta^4+198\theta^3+331\theta^2+232\theta+59)+2^{15}z^5(\theta+1)^4.\end{aligned}$$

The first coefficients of the modular forms in the singular points $-1/4$, $1/32$, $1/4$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$-1/4$	$8/1$	0	-4	-2	24	-44	22	50	44
$1/32$	$112/4$	0	-4	6	-7	12	-82	-30	-68
$1/4$	$56/2$	0	-2	-16	-7	24	-68	54	-46

AESZ 34

The differential operator is given by

$$\theta^4-z(35\theta^4+70\theta^3+63\theta^2+28\theta+5)+z^2(\theta+1)^2(259\theta^2+518\theta+285)-15^2z^3(\theta+1)^2(\theta+2)^2.$$

The first coefficients of the modular forms in the singular points $1/25$, $1/9$, 1 are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/25$	$30/1$	-2	3	5	32	-60	-34	42	-76
$1/9$	$6/1$	-2	-3	6	-16	12	38	-126	20
1	$6/1$	-2	-3	6	-16	12	38	-126	20

AESZ 59

The differential operator is given by

$$\begin{aligned} 7^2\theta^4 \quad &- \quad 14z(257\theta^4 + 520\theta^3 + 435\theta^2 + 175\theta + 28) \\ &+ \quad 2^2z^2(134970\theta^4 + 555360\theta^3 + 81222\theta^2 + 503370\theta + 11396) \\ &- \quad 2^3z^3(17201\theta^4 + 114996\theta^3 + 248466\theta^2 + 202629\theta + 55412) \\ &- \quad 2^4z^4(5762\theta^4 + 29668\theta^3 + 48150\theta^2 + 31741\theta + 7412) \\ &- \quad 2^5 \cdot 3z^5(4\theta + 5)(3\theta + 2)(3\theta + 4)(4\theta + 3). \end{aligned}$$

The first coefficients of the modular forms in the singular points $1/54$, $1/16$, $1/4$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/54$	$684/5$	0	0	-18	-32	-46	-72	8	19
$1/16$	$228/2$	0	-3	-7	21	-37	26	-33	-19
$1/4$	$12/1$	0	3	-18	8	36	-10	18	-100

AESZ 216 (DB 268)

The differential operator is given by

$$\begin{aligned} \theta^4 \quad &- \quad 3z\theta(27\theta^3 + 18\theta^2 + 11\theta + 2) - 2 \cdot 3^3z^2(72\theta^4 + 414\theta^3 + 603\theta^2 + 330\theta + 64) \\ &+ \quad 2^2 3^5 z^3(93\theta^4 - 720\theta^2 - 708\theta - 184) \\ &+ \quad 2^3 3^7 z^4(2\theta + 1)(54\theta^3 + 405\theta^2 + 544\theta + 200) \\ &- \quad 2^4 3^{10} z^5(2\theta + 1)(3\theta + 2)(3\theta + 4)(2\theta + 3). \end{aligned}$$

The first coefficients of the modular forms in the singular points $-1/27$, $-1/36$, $1/108$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$-1/27$	$15/2$	1	3	5	-24	52	22	-14	-20
$-1/36$	$324/2$	0	0	-3	-4	24	-25	21	-52
$1/108$	$60/1$	0	-3	-5	-28	-24	-70	102	20

AESZ 217 (DB 269)

The differential operator is given by

$$\begin{aligned} 7^2\theta^4 \quad &+ \quad 7z\theta(13\theta^3 - 118\theta^2 - 73\theta - 14) \\ &- \quad 2^3 3z^2(3378\theta^4 + 13446\theta^3 + 18869\theta^2 + 11158\theta + 2352) \\ &- \quad 2^4 3^3 z^3(3628\theta^4 + 17920\theta^3 + 31668\theta^2 + 22596\theta + 5383) \\ &- \quad 2^8 3^3 z^4(2\theta + 1)(572\theta^3 + 2370\theta^2 + 2896\theta + 1095) \\ &- \quad 2^{10} 3^4 z^5(2\theta + 1)(6\theta + 5)(6\theta + 7)(2\theta + 3). \end{aligned}$$

The first coefficients of the modular forms in the singular points $-1/16$, $-1/27$, $1/48$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$-1/16$	$176/1$	0	1	-7	6	11	-40	-78	-36
$-1/27$	?	0	0	4	-26	$-$	-32	-74	-60
$1/48$	$432/11$	0	0	4	-3	-28	-11	44	-29

AESZ 226 (DB 283)

The differential operator is given by

$$\begin{aligned} 5^2\theta^4 &- 10z(328\theta^4 + 692\theta^3 + 551\theta^2 + 205\theta + 30) \\ &+ 2^2 3z^2(53520\theta^4 + 254160\theta^3 + 383870\theta^2 + 230200\theta + 4860) \\ &- 2^4 3^3 z^3(3520\theta^4 + 45200\theta^3 + 121080\theta^2 + 102050\theta + 2630) \\ &- 2^6 3^3 z^4(2\theta+1)(5860\theta^3 + 30390\theta^2 + 39470\theta + 1527) \\ &- 2^8 3^4 z^5(2\theta+1)(6\theta+5)(6\theta+7)(2\theta+3). \end{aligned}$$

The first coefficients of the modular forms in the singular points $1/108$, $1/16$, $1/12$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$1/108$	?	0	0	14	2	-58	-50	76	60
$1/16$	$23/1$	-2	-5	-6	-8	34	-57	-80	-70
$1/12$	$216/4$	0	0	0	-9	-17	-44	56	-94

AESZ 234 (DB 298)

The differential operator is given by

$$\begin{aligned} 7^2\theta^4 &- 14z\theta(192\theta^3 + 60\theta^2 + 37\theta + 7) \\ &- 2^2 z^2(176080\theta^4 + 1151440\theta^3 + 1667150\theta^2 + 945560\theta + 18816) \\ &+ 2^4 3^2 z^3(202880\theta^4 + 572880\theta^3 + 275240\theta^2 - 74550\theta - 5026) \\ &- 2^6 3^5 z^4(2\theta+1)(4580\theta^3 - 6570\theta^2 - 17990\theta - 846) \\ &- 2^{12} 3^8 z^5(2\theta+1)(\theta+1)^2(2\theta+3). \end{aligned}$$

The first coefficients of the modular forms in the singular points $-1/4$, $-1/36$, $1/64$ are

Point	Form	a_2	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
$-1/4$	$68/1$	0	-2	-8	-12	-10	-38	-17	4
$-1/36$	$12/1$	0	3	-18	8	36	-10	18	-100
$1/64$	$34/2$	-2	-2	-18	-10	-6	74	17	-88

A.3 Laurent polynomials

In the tables below, we list Laurent polynomials whose fundamental periods satisfy Calabi-Yau differential equations of order 4. These polynomials were found by D. van Straten and G. Almkvist in the list of Batyrev and Kreuzer, [**6**]. The Newton polyhedra of all of these Laurent polynomials contain the origin as unique interior lattice point, and thus they provide examples for the Laurent polynomials considered in chapter 8. The numbers of the CY(4)-operators are the numbers used in the list in [**2**]. If these numbers differ from the number in the database [**3**], or if the operators do not appear in the list [**2**], we also give the number of the operator in the database.

AESZ 3

The differential operator is given by
$\theta^4 - 256\,t^2\,(\theta+1)^4\,,$
and a Laurent polynomial is
$X^{-1} + \frac{T}{X} + \frac{TY}{X} + \frac{Y}{X} + \frac{ZY}{X} + \frac{YZT}{X} + \frac{ZT}{X} + \frac{Z}{X} + \frac{X}{TY} + \frac{X}{Y} + X + \frac{X}{T} + \frac{X}{ZT} + \frac{X}{Z} + \frac{X}{ZY} + \frac{X}{YZT}.$
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{2n}]_0$.

AESZ 4

The differential operator is given by
$\theta^4 - 729\,t^3\,(\theta+1)^2\,(\theta+2)^2$
and a Laurent polynomial is
$X^{-1} + \frac{T}{X} + \frac{X^2}{YT} + \frac{X^2}{ZT} + \frac{YT}{X} + \frac{ZT}{X} + \frac{Y}{X} + \frac{Z}{X} + \frac{X^2}{YZT}.$
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{3n}]_0$.

AESZ 5

The differential operator is given by
$\theta^4 - 108\,t^3\,(\theta+2)\,(\theta+1)\,(2\,\theta+3)^2$
and Laurent polynomials are
$X^{-1} + \frac{T}{X} + \frac{TY}{X} + \frac{Y}{X} + \frac{ZY}{X} + \frac{ZYT}{X} + \frac{ZT}{X} + \frac{Z}{X} + \frac{X^2}{ZYT}$
or
$X^{-1} + \frac{T}{X} + \frac{Y}{X} + \frac{ZY}{X} + \frac{ZT}{X} + \frac{X^2}{ZY} + \frac{Z}{X} + \frac{X^2}{ZYT}.$
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{3n}]_0$.

AESZ 6

The differential operator is given by
$\theta^4 - 1024\,t^4\,(\theta+1)\,(\theta+2)^2\,(\theta+3)$
and Laurent polynomials are

$\frac{1}{XYT} + Z + Y + X + T + \frac{1}{XYZ}$
or
$X^{-1} + \frac{T}{X} + \frac{X}{Y} + \frac{ZT}{X} + \frac{Z}{X} + \frac{Y^2}{XT} + \frac{Y^2}{X} + \frac{Y^2}{XZ} + \frac{Y^2}{XZT}$.
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{4n}]_0$.

AESZ 8

The differential operator is given by
$\theta^4 - 6912\, t^6\, (\theta+5)\,(\theta+1)\,(\theta+3)^2$
and a Laurent polynomial is
$\frac{T^2Z}{XY} + \frac{TZ}{XY} + \frac{X}{Y} + \frac{TZ^2}{XY} + \frac{Y^2}{XT} + \frac{Y^2}{XZT} + \frac{Y^2}{XZ}$.
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{6n}]_0$.

AESZ 10

The differential operator is given by
$\theta^4 - 4096\, t^4\, (\theta+3)^2\,(\theta+1)^2$
and a Laurent polynomial is
$\frac{XT}{Y} + \frac{X}{Y} + \frac{Z}{X} + \frac{Y^2}{XZT} + X^{-1} + \frac{T}{X} + \frac{XZ}{Y} + \frac{XY}{ZT}$.
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{4n}]_0$.

AESZ 11

The differential operator is given by
$\theta^4 - 192\, t^4\, (3\,\theta+8)\,(3\,\theta+4)\,(\theta+3)\,(\theta+1)$
and a Laurent polynomial is
$\frac{T}{X} + X^{-1} + \frac{XZ}{Y} + \frac{X}{Y} + \frac{XT}{Y} + \frac{Z}{X} + \frac{Y^2}{XZT}$.
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{4n}]_0$.

AESZ 12

The differential operator is given by
$\theta^4 - 6912\, t^6\, (\theta+5)\,(2\,\theta+3)\,(2\,\theta+9)\,(\theta+1)$
and the Laurent polynomial is
$\frac{1}{X^2YZT} + \frac{1}{XY^2ZT} + T + Y + Z + X$.
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{6n}]_0$.

AESZ 14

The differential operator is given by
$\theta^4 - 6912\, t^6\, (\theta + 5)\, (\theta + 1)\, (\theta + 3)^2$
and a Laurent polynomial is
$\frac{1}{X^3 YT} + Z + Y + X + T + \frac{1}{X^3 YZ}$.
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{6n}]_0$.

AESZ 24

The differential operator is given by
$\theta^4 - 27\, t^3\, (\theta + 2)\, (\theta + 1)\left(11\,\theta^2 + 33\,\theta + 27\right) - 729\, t^6\, (\theta + 5)\, (\theta + 4)\, (\theta + 2)\, (\theta + 1)$
and Laurent polynomials are
$\frac{1}{YT} + Z + T + \frac{1}{XY} + \frac{1}{XZ} + Y + X$
or
$\frac{1}{XT} + Z + Y + X + T + \frac{1}{ZX} + \frac{1}{YX} + \frac{1}{X^2 YZT}$.
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{3n}]_0$.

AESZ 25

The differential operator is given by
$\theta^4 - 16\, t^2 \left(11\,\theta^2 + 22\,\theta + 12\right)(\theta + 1)^2 - 256\, t^4\, (\theta + 3)^2\, (\theta + 1)^2$
and a Laurent polynomial is
$X^{-1} + \frac{T}{X} + \frac{YT}{X} + \frac{Y}{X} + \frac{YZ}{X} + \frac{YZT}{X} + \frac{ZT}{X} + \frac{Z}{X} + \frac{X}{YT} + \frac{X}{Y} + \frac{X}{ZT} + \frac{X}{Z} + \frac{X}{YZ} + \frac{X}{YZT}$.
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{2n}]_0$.

AESZ 26

The differential operator is given by
$\theta^4 - 2\, t\, (2\,\theta + 1)^2 \left(13\,\theta^2 + 13\,\theta + 4\right) - 12\, t^2\, (2\,\theta + 1)\, (3\,\theta + 2)\, (3\,\theta + 4)\, (2\,\theta + 3)$
and Laurent polynomials are
$\frac{YT}{X} + \frac{ZY}{X} + \frac{Y}{X} + \frac{X}{YT} + \frac{X}{ZY} + X^{-1} + \frac{ZT}{X} + \frac{Z}{X} + \frac{T}{X} + \frac{X}{Z} + \frac{X}{T}$
or
$T^{-1} + X + Z + \frac{Z}{XT} + Y^{-1} + \frac{T}{ZY} + X^{-1} + T + Y + \frac{Y}{XT} + Z^{-1}$
or
$\frac{YZT}{X} + \frac{T}{X} + \frac{TZ}{X} + \frac{X}{Z} + \frac{X}{T} + XY + X^{-1} + \frac{Z}{XY} + \frac{1}{XY} + \frac{T}{XY} + \frac{X}{TZ} + \frac{X}{YZT}$.
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{2n}]_0$.

AESZ 29

The differential operator is given by
$\theta^4 - 2\,t\,(2\,\theta+1)^2\left(17\,\theta^2+17\,\theta+5\right)+4\,t^2\,(2\,\theta+1)\,(\theta+1)^2\,(2\,\theta+3)$
and Laurent polynomials are
$X^{-1}+\frac{X}{ZY}+\frac{T}{X}+\frac{Y}{X}+\frac{YT}{X}+\frac{X}{TZY}+\frac{X}{YT}+\frac{TZY}{X}+\frac{ZY}{X}+\frac{TZ}{X}+\frac{X}{Y}+\frac{Z}{X}+\frac{X}{TZ}$
or
$\frac{YT}{X}+\frac{YZ}{X}+\frac{Y}{X}+\frac{X}{YT}+\frac{X}{YZ}+\frac{X}{Y}+\frac{Z}{X}+\frac{TZ}{X}+\frac{T}{X}+X^{-1}+\frac{X}{Z}+\frac{X}{T}.$
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{2n}]_0$.

AESZ 51

The differential operator is given by
$\theta^4 - 64\,t^4\,(\theta+3)\,(\theta+1)\left(11\,\theta^2+44\,\theta+48\right)-4096\,t^8\,(\theta+5)\,(\theta+3)\,(\theta+7)\,(\theta+1)$
and a Laurent polynomial is
$\frac{1}{X^2T}+Z+Y+X+T+\frac{1}{X^2Z}+\frac{1}{X^2Y}+\frac{1}{X^4YZT}.$
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{4n}]_0$.

AESZ 185

The differential operator is given by
$\theta^4 - 24\,t^2\left(3\,\theta^2+6\,\theta+4\right)(\theta+1)^2-432\,t^4\,(\theta+1)\,(\theta+2)^2\,(\theta+3)$
and Laurent polynomials are
$\frac{1}{YZT}+Y^{-1}+T+Z^{-1}+X^{-1}+X+Y+Z+\frac{ZY}{X}$
or
$\frac{1}{TXZ}+T+X^{-1}+Z^{-1}+Y+Z+Y^{-1}+\frac{Z}{XY}+\frac{1}{TYX}+X$
or
$T+X^{-1}+Z^{-1}+Y^{-1}+\frac{1}{TXZ}+\frac{1}{TYX}+\frac{1}{TZY}+Y+Z+X.$
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{2n}]_0$.

AESZ 214 (DB 266)

The differential operator is given by
$\theta^4 + t\left(-12-94\,\theta-282\,\theta^2-376\,\theta^3-180\,\theta^4\right)+$
$t^2\left(-768-3736\,\theta-6820\,\theta^2-6080\,\theta^3-2256\,\theta^4\right)$
$-16\,t^3\,(2\,\theta+1)\left(286\,\theta^3+813\,\theta^2+851\,\theta+294\right)$
$-192\,t^4\,(2\,\theta+1)\,(4\,\theta+3)\,(4\,\theta+5)\,(2\,\theta+3)$
and a Laurent polynomial is

$$X^{-1}+\frac{T}{X}+\frac{TY}{X}+\frac{Y}{X}+\frac{YZ}{X}+\frac{TYZ}{X}+\frac{TZ}{X}+\frac{Z}{X}+\frac{X}{TY}+\frac{X}{Y}+\frac{X}{T}+\frac{X}{TZ}+\frac{X}{Z}+\frac{X}{YZ}.$$

The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{2n}]_0$.

AESZ 218 (DB 270)

The differential operator is given by
$49\,\theta^4 + t\left(-588 - 4410\,\theta - 12726\,\theta^2 - 16632\,\theta^3 - 8064\,\theta^4\right) +$
$t^2\left(29232 + 145824\,\theta + 245172\,\theta^2 + 140832\,\theta^3 + 14256\,\theta^4\right) +$
$t^3\left(-111888 - 413532\,\theta - 373140\,\theta^2 + 54432\,\theta^3 + 57456\,\theta^4\right)$
$-\,1296\,t^4\,(2\,\theta+1)\left(36\,\theta^3 + 306\,\theta^2 + 421\,\theta + 156\right)$
$-\,5184\,t^5\,(2\,\theta+1)\,(3\,\theta+2)\,(3\,\theta+4)\,(2\,\theta+3)$
and a Laurent polynomial is

$$\frac{1}{XY} + \frac{YTZ}{X} + \frac{X}{Z} + \frac{TZ}{X} + \frac{XY}{Z} + \frac{T}{XY} + \frac{T}{X} + \frac{Z}{XY} + \frac{X}{YTZ} + \frac{X}{T} + \frac{XY}{T} + \frac{X}{TZ} + \frac{Z}{X}.$$

The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{2n}]_0$.

AESZ 209 (DB 290)

The differential operator is given by
$289\,\theta^4 + t\left(-4046 - 31790\,\theta - 94826\,\theta^2 - 126072\,\theta^3 - 64668\,\theta^4\right) +$
$t^2\left(-22644 - 96424\,\theta - 40116\,\theta^2 + 274304\,\theta^3 + 249632\,\theta^4\right) +$
$t^3\left(-19176 - 71196\,\theta - 83140\,\theta^2 - 132192\,\theta^3 - 264720\,\theta^4\right) +$
$128\,t^4\,(2\,\theta+1)\left(196\,\theta^3 + 498\,\theta^2 + 487\,\theta + 169\right) - 4096\,t^5\,(2\,\theta+1)\,(\theta+1)^2\,(2\,\theta+3)$
and a Laurent polynomial is
$X^{-1} + \frac{T}{X} + \frac{YT}{X} + \frac{Y}{X} + \frac{YZ}{X} + \frac{YZT}{X} + \frac{ZT}{X} + \frac{Z}{X} + \frac{X}{YT} + \frac{X}{Y} + \frac{X}{T} + \frac{X}{ZT} + \frac{X}{Z} + \frac{X}{YZ} + \frac{X}{YZT}.$
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{2n}]_0$.

DB 287A

The differential operator is given by
$441\,\theta^4 + t\left(-4410 - 33516\,\theta - 97545\,\theta^2 - 128058\,\theta^3 - 69069\,\theta^4\right) +$
$t^2\left(-272580 - 1348200\,\theta - 2137700\,\theta^2 - 923360\,\theta^3 + 154240\,\theta^4\right) +$
$t^3\left(97440 + 1861776\,\theta + 6723376\,\theta^2 + 7894656\,\theta^3 + 1706176\,\theta^4\right)$
$-1280\,t^4\,(2\,\theta+1)\left(1916\,\theta^3 + 2622\,\theta^2 + 1077\,\theta + 91\right) - 102400\,t^5\,(2\,\theta+1)\,(\theta+1)^2\,(2\,\theta+3)$
and Laurent polynomials are
$\frac{Z}{X} + \frac{TZY}{X} + \frac{T}{X} + \frac{TZ}{X} + \frac{X}{Z} + \frac{X}{T} + XY + X^{-1} + \frac{Z}{XY} + \frac{1}{XY} + \frac{T}{XY} + \frac{X}{TZ} + \frac{X}{TZY}$
or
$X^{-1} + \frac{X}{YZ} + \frac{T}{X} + \frac{Z}{X} + \frac{TZ}{X} + \frac{X}{YTZ} + \frac{X}{T} + \frac{YTZ}{X} + \frac{YZ}{X} + \frac{YT}{X} + \frac{X}{Z} + \frac{Y}{X} + \frac{X}{YT}.$
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n) = [f^{2n}]_0$.

DB 309 A

The differential operator is given by
$81\,\theta^4 + t\left(-972 - 7452\,\theta - 21933\,\theta^2 - 28962\,\theta^3 - 17937\,\theta^4\right) +$
$t^2\left(9504 + 89280\,\theta + 391648\,\theta^2 + 805888\,\theta^3 + 559552\,\theta^4\right) +$

$t^3\left(-539136-3186432\,\theta-7399680\,\theta^2-8902656\,\theta^3-6046720\,\theta^4\right)+$
$32768\,t^4\,(2\,\theta+1)\left(340\,\theta^3+618\,\theta^2+455\,\theta+129\right)$
$-\,4194304\,t^5\,(2\,\theta+1)\,(\theta+1)^2\,(2\,\theta+3)$
and Laurent polynomials are
$\frac{YT}{X}+X+\frac{YT}{XZ}+\frac{XZ}{YT}+X^{-1}+\frac{X}{Y}+\frac{X}{YT}+\frac{T}{XZ}+\frac{X}{YZ}+\frac{T}{X}+\frac{1}{XZ}+\frac{YZ}{X}+\frac{Y}{X}+\frac{XZ}{T}$
or
$X^{-1}+\frac{T}{X}+\frac{YT}{X}+\frac{Y}{X}+\frac{YZ}{X}+\frac{TZY}{X}+\frac{TZ}{X}+\frac{Z}{X}+\frac{X}{YT}+\frac{X}{Y}+\frac{X}{T}+\frac{X}{Z}+\frac{X}{YZ}+\frac{X}{TZY}.$
The coefficient $a(n)$ of t^n of the solution to the differential equation is then given by $a(n)=[f^{2n}]_0$.

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Summary

Let $\pi : X \to S$ be a one-parameter family of smooth Calabi-Yau threefolds defined over $\mathbb{Z}$, and assume that there exists a submodule $M \subset H^3_{DR}(X/S)$ of rank four which is stable under the Gauss-Manin connection, such that the Picard-Fuchs operator P on M is what we call a *Calabi-Yau* operator of order 4.

Let k be a finite field of characterstic p. For the ordinary fibres X_{t_0}, $t_0 \in S_0$ of the reduction $\pi_0 : X_0 \to S_0$ over k, we derive an explicit formula to compute the characteristic polynomial of the Frobenius endomorphism, the *Frobenius polynomial*, on the corresponding submodule M_{cris} of the third crystalline cohomology $H^3_{cris}(X_{t_0})$ by computing two of its roots.

Let $f_0(z)$ be the holomorphic solution to the differential equation $Pf = 0$ around $z = 0$. Since the unit root of the Frobenius polynomial at a Teichmüller point t is given by $f_0(z)/f_0(z^p)|_{z=t}$, a crucial step of the computation of the Frobenius polynomial is the construction of a $p-$adic analytic continuation of the quotient $f_0(z)/f_0(z^p)$ to the boundary of the $p-$adic unit disc. In case that $f_0(z)$ can be expressed in terms of the constant terms in the powers of a Laurent polynomial whose Newton polyhedron contains the origin as unique interior lattice point, we prove that the coefficients of $f_0(z)$ satisfy certain congruence properties that are crucial to construct the analytic continuation.

If the fibre X_{t_0} aquires an ordinary double point, we expect that the limit Frobenius polynomial factors in a specific way, and that there exists one factor of degree two which is determined by one coefficient a_p. As p varies, we expect that there exists a modular form of weight four with coefficients a_p by the modularity theorem. We could confirm this expectation by our numerous computations.

Furthermore, we derive formulas to compute the Frobenius polynomial in terms of the non-holomorphic solutions to the differential equation $Pf = 0$ around $z = 0$.

Zusammenfassung

Sei $\pi : X \to S$ eine über $\mathbb{Z}$ definierte Familie von Calabi-Yau Varietäten der Dimension drei. Es existiere ein unter dem Gauss-Manin Zusammenhang invarianter Untermodul $M \subset H^3_{DR}(X/S)$ von Rang vier, sodass der Picard-Fuchs Operator P auf M ein sogenannter ***Calabi-Yau*** Operator von Ordnung vier ist.

Sei k ein endlicher Körper der Charaktetristik p, und sei $\pi_0 : X_0 \to S_0$ die Reduktion von π über k. Für die gewöhnlichen (ordinary) Fasern X_{t_0} der Familie leiten wir eine explizite Formel zur Berechnung des charakteristischen Polynoms des Frobeniusendomorphismus, des *Frobeniuspolynoms*, auf dem korrespondierenden Untermodul $M_{cris} \subset H^3_{cris}(X_{t_0})$ her.

Sei nun $f_0(z)$ die Potenzreihenlösung der Differentialgleichung $Pf = 0$ in einer Umgebung der Null. Da eine reziproke Nullstelle des Frobeniuspolynoms in einem Teichmüller-Punkt t durch $f_0(z)/f_0(z^p)|_{z=t}$ gegeben ist, ist ein entscheidender Schritt in der Berechnung des Frobeniuspolynoms die Konstruktion einer $p-$adischen analytischen Fortsetzung des Quotienten $f_0(z)/f_0(z^p)$ auf den Rand des $p-$adischen Einheitskreises. Kann man die Koeffizienten von f_0 mithilfe der konstanten Terme in den Potenzen eines Laurent-Polynoms, dessen Newton-Polyeder den Ursprung als einzigen inneren Gitterpunkt enthält, ausdrücken, so beweisen wir gewisse Kongruenz-Eigenschaften unter den Koeffizienten von f_0. Diese sind entscheidend bei der Konstruktion der analytischen Fortsetzung.

Enthält die Faser X_{t_0} einen gewöhnlichen Doppelpunkt, so erwarten wir im Grenzübergang, dass das Frobeniuspolynom in zwei Faktoren von Grad eins und einen Faktor von Grad zwei zerfällt. Der Faktor von Grad zwei ist dabei durch einen Koeffizienten a_p eindeutig bestimmt. Durchläuft nun p die Menge aller Primzahlen, so erwarten wir aufgrund des Modularitätssatzes, dass es eine Modulform von Gewicht vier gibt, deren Koeffizienten durch die Koeffizienten a_p gegeben sind. Diese Erwartung hat sich durch unsere umfangreichen Berechnungen bestätigt.

Darüberhinaus leiten wir weitere Formeln zur Bestimmung des Frobeniuspolynoms her, in welchen auch die nicht-holomorphen Lösungen der Differentialgleichung $Pf = 0$ in einer Umgebung der Null eine Rolle spielen.

Lebenslauf

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Ich wurde am 20.04.1982 in Bochum geboren. Von 1988 bis 1992 besuchte ich die Grundschule in Koblenz-Lay. Darauf folgend besuchte ich von 1992 bis 2001 das Bischöfliche Cusanus-Gymnasium in Koblenz.

Nach dem Abitur 2001 (Note *sehr gut*, 1.0) begann ich im Oktober 2001 mein Mathematikstudium in Mainz. Im Nebenfach studierte ich bis zum Vordiplom Informatik und Betriebswirtschaftlehre, nach dem Vordiplom dann Betriebswirtschaftslehre. Von 2003 bis 2006 war ich am Fachbereich Mathematik als wissenschaftliche Hilfskraft beschäftigt. Während des gleichen Zeitraums war ich Stipendiatin der Studienstiftung des Deutschen Volkes. 2006 schloss ich mein Studium im Oktober mit der Note *mit Auszeichnung* ab. Der Titel meiner Diplomarbeit lautet *Die Deformationsmethode von Lauder und Dwork für gewichtet projektive Hyperflächen*, die Arbeit wurde von Prof. Dr. D. van Straten betreut.

Von November 2006 bis Dezember 2009 war ich wissenschaftliche Mitarbeiterin an der Johannes Gutenberg-Universität Mainz. Seit Juli 2007 wurde ich dabei im Rahmen des Sonderforschungsbereiches SFB Transregio 45 der DFG beschäftigt und hatte das Amt der Doktorandensprecherin des SFB inne.

Im September 2007 war ich für einen Monat zu Gast im Max Planck-Institut für Mathematik in Bonn, als 1. Preis für die Diplomarbeit, erhalten auf der Studierendenkonferenz der DMV in Berlin im März 2007. Im Dezember 2008 und im Januar 2009 besuchte ich jeweils für eine Woche die Universität Oxford.