Mathematical Aspects of Feynman Integrals

Dissertation

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"What can be said at all can be said clearly; and whereof one cannot speak thereof one must be silent."
(Ludwig Wittgenstein)

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CHAPTER 1

Introduction

1.1. General Introduction

The idea that matter consists of smallest elements belongs to human culture since the ancient Greeks. The physics of the last few decades has given deep insight into the properties of these supposedly elementary particles. This illumination of the microcosmos is a story of success for two scientific concepts: the *scattering experiment* and *quantum field theory*.

Scattering experiments, from Rutherford's scattering of α-particles off gold atoms in 1909 to the collision of highly accelerated protons at the Large Hadron Collider, expected to begin in 2009, have provided crucial information on the building blocks of matter. The success of a physical theory of elementary particles has to be judged by its capability to predict the values of quantities which can be measured in scattering experiments. The typical quantity to be measured in this kind of experiments is the so-called cross-section, from which in turn characteristic properties of the particles involved and their interaction can be deduced.

The evolution of physical theories, probed on these experimental data, led to a model for the fundamental strong, electromagnetic and weak interactions of the known elementary particles. This so-called Standard Model is in good agreement with a wealth of observed phenomena. Nevertheless, the model is expected to be extended, depending on eagerly expected results from the Large Hadron Collider in the near future[1].

The Standard Model is a highly successful quantum field theory whose application to the precise evaluation of an observable is usually far from trivial. In order to apply the model to the quantitative prediction of a cross-section one requires a *perturbative* formulation. The basic idea of perturbation theory is the assumption, that the interaction energies between the particles are relatively small compared to the energy of their free motion. The strength of an interaction is scaled by a so-called coupling parameter, which is assumed to be a relatively small quantity. An observable is then evaluated as a power series in the coupling parameter. This infinite series is truncated at a certain order and the precision of the result depends on the number of orders to be taken into account.

The evaluation of the coefficients of this power series is in general highly elaborate. With increasing order of the perturbative expansion it becomes more and more difficult and it is therefore for many cases possible only to consider the first one or two orders. At higher orders the computational techniques of present days, involving the use of advanced computer programs, always have reached the limits of computer capabilities very soon. However, because of the increasing precision of the experiments and the nature of the physical effects to be observed, the consideration of higher orders becomes relevant.

The topics of this dissertation are related to calculational problems which hinder higher-order calculations. The complications to be considered here do not depend on the underlying physical model. They hamper the evaluations in the Standard Model but

[1]By the time of writing, the first collisions at the Large Hadron Collider are expected for the fall of 2009. A possible discovery of the Higgs boson is expected to take place not before 2010.

they are also present in Quantum Electrodynamics and even in simple toy models like the so-called ϕ^4-theory.

The difficulties to be treated lie in the nature of Feynman integrals and therefore in the nature of the common perturbative formulation of quantum field theories in general. A good understanding of this particular kind of integrals is inevitable for the efficient calculation of observables. An additional motivation to study the properties of these integrals arose over the last ten years from pure mathematics, as it turned out that Feynman integrals tend to evaluate to certain numbers which are significant in algebraic geometry. The achievements of this dissertation are related to both aspects and motivated from both points of view.

For this reason we want this dissertation to be readable for the physicist and the mathematician. Feynman integrals will be considered as objects of mathematical interest in their own right, without referring to a particular physical model or even a physical process. Thanks to Richard Feynman one does not need to know quantum field theory in order to obtain these integrals. They can be constructed from certain intuitive graphs, the Feynman graphs, using a set of so-called Feynman rules which incorporate the properties of the chosen physical model. One does not need to be a physicist to study Feynman integrals. A physicist just has the strongest reason to do so.

1.2. Outline

In the following we apply (algebraic) graph theory, algebraic geometry and number theory to Feynman integrals. Not all aspects to be discussed in the following are closely related to each other. However, Ariadne's thread through the dissertation may be seen in the course of the calculation of a Feynman integral:

The above general introduction and the present outline form chapter 1.

Chapter 2: We begin with a brief introduction of Feynman integrals and Feynman rules. The framework of dimensional regularization is explained and some further standard techniques are described, such that the problem of calculating an arbitrary Feynman integral is reduced to the evaluation of regularized, scalar Feynman integrals in the convenient Feynman parametric representation.

Chapter 3: Not all Feynman integrals really need to be calculated separately. Most of them can be expressed in terms of simpler integrals, so-called master integrals, by the so-called IBP-identities (*integration-by-parts* identities). We briefly review this technique and give a handy formulation of some of these identities in the Feynman parametric representation.

Chapter 4: For many cases the Feynman parametric representation is a convenient starting point for the actual evaluation of a master integral. In this representation the integrand is expressed in terms of two certain polynomials, the Symanzik polynomials. We dedicate chapter 4 to a graph theoretic study of these graph polynomials and derive a novel relation for them. This is achieved by the use of a generalized theorem of the so-called matrix-tree-type. We furthermore explain the correspondence between the Symanzik polynomials and the multivariate Tutte polynomial.

Chapter 5: A method for the numerical evaluation of an arbitrary Feynman integral, possibly a master integral, in a certain momentum region was given by the sector decomposition algorithm of Binoth and Heinrich, which is briefly reviewed. We expose the problem that this algorithm does not terminate in the general case and solve this problem by mapping the combinatorics of the algorithm to the abstract polyhedra game of Hironaka. We extend the algorithm by use of so-called winning strategies of this game and obtain a version which always terminates. We briefly report on an implementation of this

improved algorithm. Furthermore we add an explanation in terms of algebraic geometry on the relation between sector decomposition and the so-called resolution of singularities by a sequence of blow-ups, formalized by Hironaka's game. The work presented in this chapter was previously published in joint work with Stefan Weinzierl [**BW08**].

Chapter 6: Having finally obtained a result for a Feynman integral, one may be interested in the mathematical nature of the result. Chapter 6 is dedicated to the classification of numbers and functions one obtains in the Laurent coefficients of a dimensionally regularized Feynman integral. We briefly discuss zeta values, multiple zeta values, polylogarithms and elliptic integrals and their presence in loop calculations. Then we prove a statement on the general nature of the Laurent coefficients: Under certain (rather weak) assumptions, they evaluate to special numbers, called periods. Our proof of this statement uses the improved sector decomposition algorithm. The mentioned theorem and its proof are joint work with Stefan Weinzierl, previously published in the article [**BW09**].

Chapter 7 contains the conclusions of the dissertation. In appendix A we provide a collection of auxiliary mathematical definitions. Appendix B consists of a detailed discussion of a special example of a Feynman graph, the so-called two-loop equal mass sunrise graph. In appendix C we discuss the arithmetic nature of certain prefactors of Feynman integrals.

The ordering of the chapters could have been chosen in a different way. We believe that a reader who is familiar to the preliminaries discussed in chapter 1 can continue with any of the remaining chapters without losing necessary information for his understanding[2].

Let us emphasize that chapters 4, 5 and 6 contain the main achievements of the dissertation, which may be very briefly summarized as:

- A detailed study of graph theoretic properties of the Symanzik polynomials;
- A solution to the termination problem of sector decomposition;
- The proof of a theorem on the arithmetic nature of Feynman integrals.

A more general goal of our dissertation is to give a contribution to the dialogue between mathematicians and physicists on the subject of Feynman integrals, as this turns out to be a subject of interest for both communities. The aspects to be considered are related to physically relevant, explicit calculations on the one hand and to topics of active branches of research in pure mathematics on the other hand. We hope that our presentation is accessible to physicists and mathematicians alike.

1.2.1. A Remark on Wittgenstein and Singularities. The heraldic motto of our thesis, *"What can be said at all can be said clearly; and whereof one cannot speak thereof one must be silent"* stems from the preamble of Ludwig Wittgenstein's "Tractatus logico philosophicus", which is an important work of the Austrian philosopher. With this sentence, which refers to the limits of language and thinking itself, Wittgenstein intended to abstract the essence of his "Tractatus". Of course, we slightly abuse the deep statement by placing it in the more worldly context of this dissertation, and the same is true for further quotations of Wittgenstein at the beginnings of our chapters. Let us at least give a brief remark on the patron of our text.

Wittgenstein's viewpoint on language has surely influenced the modern sciences[3]. Wittgenstein himself was highly influenced by the English mathematician and philosopher Bertrand Russel and in an important way by physics. As a young man he read

[2]An exception might be the proof of theorem 68 in chapter 6 which requires notions and concepts of sector decomposition, described in chapter 5.

[3]For example, Alan Turing, a founding father of computer sciences, regularly attended Wittgenstein's lectures in Cambridge.

a book on classical mechanics, "Die Prinzipien der Mechanik in neuem Zusammenhange dargestellt" by Heinrich Hertz. Hertz felt uncomfortable with the concept of force in Newton's mechanics and he therefore suggested to reformulate mechanics without the Newtonian force. In such a formulation, according to Hertz, questions around the nature of forces would not be answered, they would instead not even arise[4].

Wittgenstein tried to adopt this idea to philosophy. He believed that philosophical puzzles which seemingly can not be answered are simply misconceptions, arising from the use of an inappropriate language. As he believed, in an appropriate language without logical defects, these questions could not even be asked in a meaningful way. To say it in his words: *"The riddle does not exist. If a question can be put at all, then it can also be answered."*

The physicists of today are less puzzled by questions of classical mechanics than by the problems of quantized theories. In the context presented in this dissertation we may be disturbed by the presence of ill-defined integrals at an early stage of the calculation, causing the necessity of regularization and renormalization procedures. The divergent Feynman integral may be seen as one of the riddles of perturbative quantum field theory.

Wittgenstein (who died three years after Feynman developed the path integral formalism) might have suggested to drop such a theory in favour of a formulation, where no divergent integrals arise. This dissertation obviously does not follow such an advice. We already mentioned the reason. The mainstream framework in which our work is settled is known to be successful. It yields significant finite numbers which agree with experimental data.

Nevertheless, we want to keep Wittgenstein's words in our text, as the contradiction to his paradigm might not be too strong, because studying the divergent integrals should make sense to both: To someone who speaks the well-developed language despite the presence of the riddle, and to someone who tries to develop a language, where the riddle is absent. Both of them need to know the riddle.

[4]This formulation did not become accepted in general; the Newtonian force remained a standard concept in physics.

CHAPTER 2

Preliminaries on Feynman Integrals

"The aspects of things that are most important to us are hidden because of their simplicity and familiarity."
(Ludwig Wittgenstein)

We want to study Feynman integrals in their own right, but with this chapter we want to give at least an indication on how they arise from quantum field theory. We begin with a glance at their general role in perturbative calculations and their construction from Feynman rules. As we will see, the integrals as obtained from these rules are often divergent. We briefly introduce dimensional regularization, which is a standard approach to handle these divergences. Furthermore we have to introduce a few standard techniques by which we can reduce the problem of calculating a general Feynman integral to the calculation of scalar integrals in a certain parameterization. These integrals then will be the objects of interest for the remaining chapters.

The reader who is familiar to Feynman integrals and the reader who does not care *why* certain integrals are of any interest is invited to jump to the next chapter. The reader who is instead very interested in the topics of the present chapter finds much more detailed introductions in physics textbooks [**BDJ01, IZ06, PS95**], lecture notes [**Wei06**] and in a more mathematical language in the recent volumes [**CM08, Zei06a, Zei09**].

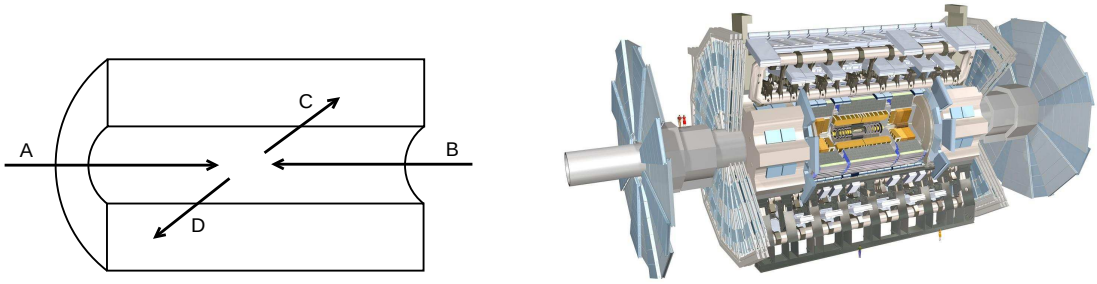
2.1. Theories, Graphs and Rules

2.1.1. What do Feynman integrals stand for? Let us for a moment think of the typical situation in a scattering experiment as depicted in figure 2.1.1 (a). Some particles A and B are produced, accelerated and then brought to collision. In the region of the collision, the kinetic energy and the energy which is equivalent to the masses of A and B is transferred to the production of new particles, say for example two particles C and D. These new particles then leave the region of the collision and enter the detectors where some of their properties are measured, usually by reactions with the detector material in which the particles are destroyed or captured[1]. In the modern scattering experiments (cf. figure 2.1.1 (b)) a huge number of such collisions is conducted and the resulting particles can be very different from collision to collision.

We may consider three stages of the experiment:

- Before the collision, we have the particles A and B which are far away from each other, such that their interaction can be neglected. We know which type of particles they are, so we know characteristic properties like their mass and their

[1]Let us remark, that many particles do not reach the detector and therefore their properties have to be deduced from the measurement of the other particles involved. For example the top quark lives too short to reach the detectors. It decays after less than 10^{-24} seconds, a time which is even too short to constitute a meson or baryon. The lighter quarks and the gluons form jets of hadrons after a very short time, which are then measured by the detector. The neutrino instead can pass through the detector material without being noticed.



(a) (b)

FIGURE 2.1.1. (a) A simplistic sketch of a scattering experiment. (b) An image of one of the detectors at the Large Hadron Collider (see [**E**$^+$**08**]): The ATLAS Experiment at CERN (http://atlas.ch). The detector is 46 meters long and it weighs 7000 tons. For details we refer to reference [**A**$^+$**08**].

charge. We furthermore have information on their kinematical properties, like their momentum. In a quantum theory we formalize this known information on the physical state by vectors in a Hilbert space. Let us denote the vector of this *initial state* before the collision by $|\alpha\rangle$.

- After the collision we may assume again to have no interaction between the new particles C and D and we obtain their properties, at least partly, from their detection. We encode this known information by $\langle\beta|$, a vector in the dual Hilbert space which describes the so-called *final state*.
- When the particles A and B collide and their interactions can not be neglected, one *does not know* what happens exactly. In particular we may imagine, that the collision of A and B at first leads to some intermediate particles which then in turn produce C and D. These intermediate particles which are only assumed to be present in the region of the collision for a short time and which are neither produced nor detected by the experimentalist are called *virtual* particles. There are infinitely many possibilities of how the particles C and D are obtained from A and B via intermediate creation and destruction of virtual particles. A quantum field theory includes a set of assumptions on these intermediate possibilities, as we will see below.

EXAMPLE 1. One of the most important experimental sites in particle physics was the Large Electron-Positron Collider (LEP) at CERN in Geneva, operating in the years from 1989 to 2000. The initial particles A and B in this experiment were electrons e^- and positrons e^+. A typical result of this collision would be the production of a muon μ^- and an anti-muon μ^+. A common way to denote this case is $e^- + e^+ \to \mu^- + \mu^+$. It can be successfully described by the theory of Quantum Electrodynamics (QED). This model does not include the possibility of a direct production of μ^- and μ^+ from e^- and e^+. Instead one assumes at least one virtual particle, which in this case must be a photon. This possibility can be depicted by the intuitive diagram of figure 2.1.2 (a). We can read this diagram from the left to the right[2]: after the collision of e^- and e^+ we assume the

[2]The arrows on the straight lines should not be confused with the direction of the momenta of the particles. In a Feynman graph, by the usual convention, arrows such as in figure 2.1.2 show the direction

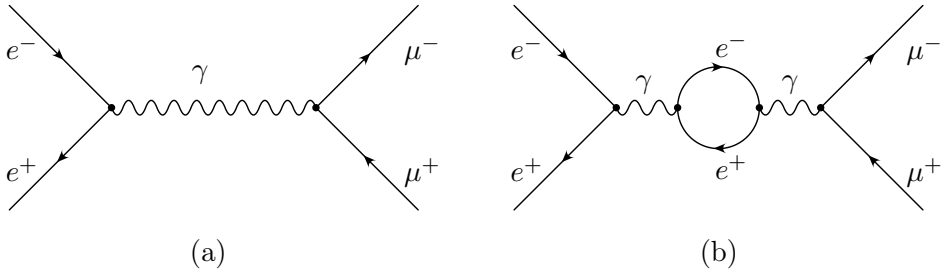


FIGURE 2.1.2. Two examples of Feynman graphs for the physical process $e^- + e^+ \to \mu^- + \mu^+$.

intermediate existence of a virtual photon γ and finally we have the creation of μ^- and μ^+. QED allows for further possibilities, for example an intermediate existence of another pair of e^- and e^+ as shown in figure 2.1.2 (b). The diagrams shown in these figures are *Feynman graphs*. They depict possibilities in the collision region, allowed by the theory.

The transition from a given initial state $|\alpha\rangle$ to a final state $\langle\beta|$ is called a *physical process*. The above $e^- + e^+ \to \mu^- + \mu^+$ is an example for such a process. By conduction of a large number of collisions one can measure the probability for a process to occur. The quantity by which this probability is conveniently expressed is the so-called cross section. The non-trivial part of its calculation involves a scalar product

$$\langle\beta|\,\Omega\,|\alpha\rangle\,,$$

where Ω is a transition operator according to the applied quantum field theory. This operator encodes all the possibilities in the collision region, which the theory allows for. The theory is successful, if the calculated cross section agrees with the measured one.

In order to predict a cross section *precisely*, we would have to take *all possibilities* into account which the given quantum field theory allows to occur in the region close to the collision, where the virtual particles have to be assumed[3]. Assuming that the interaction energies are small in comparison to the kinetic energies we expand the operator Ω as a power series in the coupling constants which scale the interaction energies. As a result we obtain the above scalar product and the cross section as a power series in these constants, which is then truncated at a certain order. With this truncation we take into account only a finite number of possibilities, corresponding to the lowest powers of the coupling constants.

As we will see below in more detail, a vertex in a Feynman graph contributes a certain power of a coupling constant to the operator Ω. For example in QED each vertex is assigned the coupling constant g. Therefore the graph of figure 2.1.2 (a) contributes a term with g^2 and the graph of figure 2.1.2 (b) contributes a term with g^4 to Ω. The number of vertices and the power of the coupling constant is not equal, in general, for example in Quantum Chromodynamics (QCD) we have vertices which are assigned the

of the so-called fermion number flow. The momentum of a particle will instead always be depicted by an additional arrow parallel to the corresponding edge (see e. g. figure 2.1.3).

[3]This important feature of a quantized theory can already be found in the quantum mechanical description of the famous double-slit experiment: It is not helpful to think that a particle has in fact passed through either the left or the right slit. The interference pattern can only be explained by taking *both* possibilities into account.

square of the coupling constant. Nevertheless, by truncating Ω at a finite power of the coupling constants, we consider a finite number of Feynman graphs which depend on the theory.

Now let us give a brief outline on how a quantum field theory determines the transition operator Ω.

2.1.2. Quantum Field Theories in Minkowski Space. A quantum field theory describes particles by *field operators*. These are operator-valued functions of *space-time*. The common quantum field theories do not incorporate gravity. They are defined on Minkowski space, which is the space-time of special relativity. In special relativity we usually denote space-time points x and momenta p by four-component vectors

$$x = \left(x^0, x^1, x^2, x^3\right), \qquad p = \left(p^0, p^1, p^2, p^3\right).$$

The metric of Minkowski space is given by the metric tensor g of rank two, which we may denote by the matrix

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.1.1}$$

The common notation distinguishes between a vector and its dual with respect to this metric by use of upper and lower indices, in the sense that

$$x^\mu = \sum_{\nu=0}^{3} g^{\mu\nu} x_\nu. \tag{2.1.2}$$

Using Einstein's sum convention we avoid to explicitely write the summation over an index which appears twice. In this sense we write a scalar product as

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} p_\mu g^{\mu\nu} q_\nu \equiv p^\mu q_\mu. \tag{2.1.3}$$

We furthermore make use of the notation

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}. \tag{2.1.4}$$

A quantum field theory is determined by the so-called *Lagrange density* $\mathcal{L}$, which is a functional of field operators. $\mathcal{L}$ determines the properties of the particles via the so-called *Euler-Lagrange equations* which are obtained from a variational principle. The particles are formalized by the solutions of these differential equations.

Let us give two examples for a quantum field theory:

Example 2. Quantum Electrodynamics is determined by the Lagrange density

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - eA^\mu\overline{\psi}\gamma_\mu\psi \tag{2.1.5}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.1.6}$$

where the field operators are A^μ for the photon and $\overline{\psi}$ and ψ for the description of the fermions, like for example the above mentioned particles e^-, e^+, μ^- or μ^+. Here it is understood, that the field operators depend on the space-time, $A^\mu \equiv A^\mu(x)$, $\psi \equiv \psi(x)$, $\overline{\psi} \equiv \overline{\psi}(x)$. m is the mass and e the charge of the fermion under consideration. γ^μ denote the so-called Dirac matrices. As Euler-Lagrange equations one obtains the famous

Maxwell equations and the Dirac equation, which describe the properties of photons and fermions and their interaction.

QED provides a description of certain physical processes and it is contained in the more general Standard Model. The fermions and photons described by this theory are particles with a certain degree of freedom, called *spin*. For the description of this property the fields A as well as $\overline{\psi}$ and ψ each need to have more than one component.

If we instead assume to have only one kind of particle which has no spin, we can use simpler quantum field theories, which may be considered as toy models. An example for such a theory with a single one-component field ϕ is the so-called ϕ^4-theory.

EXAMPLE 3. The ϕ^4-theory is determined by the Lagrange density

$$\mathcal{L}_{\phi^4} = \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{m^2}{2}\phi^2 - \frac{g}{4!}\phi^4 \tag{2.1.7}$$

where g is a coupling constant and m the mass of the only kind of particles, which is described by the one-component field ϕ.

The cross section of a physical process is calculated by use of the Green's function of the Euler-Lagrange differential equations of the theory. For a process where the number of incoming and outgoing particles is n one speaks of the *n-point Green's function*. We will refer to the incoming and outgoing particles, i. e. the particles which are produced and detected by the experimentalist, as *external* particles. The product $\langle\beta|\,\Omega\,|\alpha\rangle$ provides the *truncated n-point Green's function*. Let $G^{(n)}$ be the n-point Green's function, defined by the differential operator of the Euler-Lagrange equations. The *truncated* n-point Green's function $G^{(n)}_{\text{trunc}}$ is defined as the product of $G^{(n)}$ with the inverse 2-point functions of each of the external edges:

$$G^{(n)}_{\text{trunc}}\left(p_1,\,...,\,p_n\right) = \prod_{i=1}^{n}\left(G^{(2)}\left(p_i,\,-p_i\right)\right)^{-1} G^{(n)}\left(p_1,\,...,\,p_n\right). \tag{2.1.8}$$

Here p_1, ..., p_n are the momenta of the n external particles. In the case of the above mentioned process $e^- + e^+ \to \mu^- + \mu^+$ we have four external particles and we therefore would calculate the corresponding truncated 4-point Green's function.

The transition operator Ω is determined by the so-called time evolution operator

$$e^{i\int d^4x \mathcal{L}_{\text{int}}[\phi(x)]}$$

where $\mathcal{L}_{\text{int}}$ is the part of the Lagrange density which formalizes the interaction of the particles. For example for QED $\mathcal{L}_{\text{int}}$ is equal to $-eA^\mu\overline{\psi}\gamma_\mu\psi$ and for ϕ^4-theory it is equal to $-\frac{g}{4!}\phi^4$. The above exponential function is series expanded and by use of a theorem of Wick, one obtains each coefficient of the perturbative expansion of $\langle\beta|\,\Omega\,|\alpha\rangle$ as a sum of Feynman integrals. We do not want to show this calculation here and refer to section 6-1-1 of **[IZ06]** instead, where Wick's theorem is discussed in detail. One can see from the calculation presented there, that all the terms in the sum can be obtained by drawing all possible graphs of a certain type and then constructing a Feynman integral for each of these graphs, according to certain rules. These graphs are the *Feynman graphs* and the rules are *Feynman rules*.

2.1.3. Feynman Graphs and Rules. Let us discuss the set of Feynman graphs and Feynman rules for the simple example of ϕ^4-theory. Feynman graphs are very intuitive pictures of the possibilities in the region of the collision. The free movement of particles is depicted by edges, and the interactions, corresponding to destructions and creations of particles, are depicted by vertices. As we have only one kind of particle and one kind

of interaction in ϕ^4-theory, we need only one type of edge and one type of vertex, which sits at the end of four edges. Note, that the vertex has four incident edges because four copies of the field ϕ appear in the interaction term $-\frac{g}{4!}\phi^4$. (In the case of ϕ^3-theory there would be a vertex with three edges.)

From these edges and vertices we build up the Feynman graphs and construct the terms of the Green's function according to the Feynman rules as follows (cf. **[IZ06]**). One example for a Feynman graph of ϕ^4-theory is shown in figure 2.1.3. Up to a certain power of the coupling constant, assigned to the vertices, (or alternatively up to a certain loop-number, defined in chapter 4) we draw all graphs with n external edges, to each of them assigned one of the momentum variables p_1, ..., p_n. We ignore the graphs with a line which is connected to the same vertex at both ends (i. e. a self-loop, see 4). The internal edges, corresponding to the virtual particles, are assigned momentum variables k_1, ..., k_I. All the momenta are directed, which means that each of the momentum variables is *incoming* with respect to one of the vertices connected to the edge, and *outgoing* with respect to the other vertex. (The language of graphs will be more thoroughly introduced in chapter 4). The Feynman integral corresponding to a graph like this is obtained from the following Feynman rules in momentum space:

(1) To the jth external edge, we assign the factor
$$\frac{i}{p_j^2 - m^2 + i\rho}.$$
(2) To the lth internal edge, we formally assign
$$\frac{dk_l^0 dk_l^1 dk_l^2 dk_l^3}{(2\pi)^4} \frac{i}{k_l^2 - m^2 + i\rho}.$$
(3) To each vertex, we assign the distribution
$$-ig(2\pi)^4 \delta^4 \left(\sum p_i + \sum k_j \right)$$
where the sum in the argument runs through all incoming momenta of the vertex.
(4) We integrate over each of the real valued components of the variables k_1, ..., k_I from $-\infty$ to ∞ the product of all these contributions. We will use the notation
$$\int d^4 k_l \quad \equiv \quad \int_{-\infty}^{\infty} dk_l^0 ... \int_{-\infty}^{\infty} dk_l^3$$
where each of the integrations on the right-hand side is over the real axis.
(5) Depending on the graph we have to divide the integral by a certain integer, the so-called symmetry factor (as defined e. g. in **[IZ06]**).

The Green's function up to the desired order is obtained as a sum of all these terms. The functions $i/\left(p_j^2 - m^2 + i\rho\right)$ and $i/\left(k_l^2 - m^2 + i\rho\right)$ are the *propagators* of the corresponding particles. We have tacitly introduced a small, real valued parameter ρ which assures that the propagators are well-defined at $p_j^2 = m^2$ and $k_l^2 = m^2$ respectively. In the end one takes the limit $\rho \to 0$ and we will suppress the $i\rho$-term for brevity.

Example 4. As an example let us consider the graph in figure 2.1.3. In this figure, the assigned momenta fulfill the momentum conservation condition, which is imposed by the δ-distributions of the above rule (3). According to the above Feynman rules we obtain the Feynman integral
$$I\left(p^2, m^2, g\right) =$$

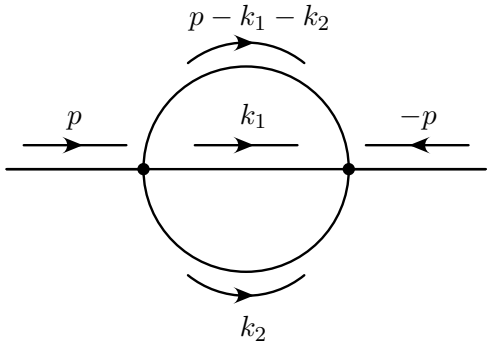


FIGURE 2.1.3. A Feynman graph of ϕ^4-theory: the two-loop sunrise graph.

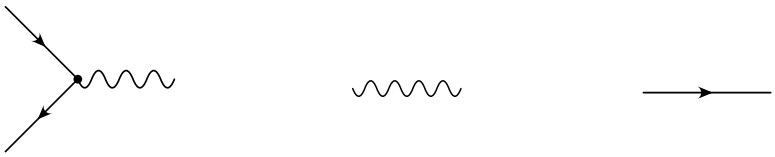
$$\frac{(-ig)^2}{3!}\left(\frac{i}{p^2-m^2}\right)^2 \int d^4k_1 d^4k_2 \frac{i^3}{\left(k_1^2-m^2\right)\left(k_2^2-m^2\right)\left((p-k_1-k_2)^2-m^2\right)} \tag{2.1.9}$$

where we suppressed the term $i\rho$. $1/3!$ is the symmetry factor. $I\left(p^2,\, m^2,\, g\right)$ is a term of the sum in the 2-point Green's function. We note that in the truncated Green's function the term $\left(\frac{i}{p^2-m^2}\right)^2$ cancels due to equation (2.1.8). In dimensional regularization (see below) the first complete evaluation of the above Feynman integral was achieved by Laporta and Remiddi [**LR05**]. A detailed discussion of the graph of figure 2.1.3 is given in appendix B.

Let us intuitively observe that in the above way to obtain a Green's function from graphs and rules we take all possibilities of the interaction region into account, which are allowed by the given theory. By the choice of the Lagrangian we choose the appearing kinds of particles, each corresponding to a kind of edge and propagator. Furthermore the interaction term $\mathcal{L}_{\text{int}}$ determines the allowed vertices which correspond to the way the particles can interact. In this way the Lagrangian $\mathcal{L}$ determines the possible topologies of the graphs. At each vertex we have momentum conservation which we impose by the appropriate δ-distributions in rule (3). We have to take into account, that the momentum k of a virtual particle, unlike an external momentum, may take values such that $k^2 \neq m^2$ where m is the mass of the particle. In the case of a graph with loops, we have momenta of virtual particles, which are not determined by external momenta via momentum conservation. In order to take all possible configurations of momenta into account we therefore have to integrate over the momenta of the virtual particles as demanded by the above rule (4).

We remark that according to Heisenberg's uncertainty principle we have to decide whether we consider the space-time point of a particle or the momentum as we can not determine both precisely. Throughout this dissertation we work in momentum-space as this is the most common choice for practical calculations.

We have just discussed the Feynman rules of ϕ^4-theory, which is a relatively simple toy model. The Feynman rules of other quantum field theories, like QED or QCD, follow the same pattern: propagators and powers of coupling constants are assigned to the edges and vertices respectively and each Feynman integral is obtained as the integral over the momenta of the virtual particles. For each kind of particle to be described one needs to associate a different propagator. Accordingly, the coupling constants depend on the different kinds of interaction to be described by different vertices. The Feynman rules for the Standard Model can for example be found in the appendix of [**BDJ01**].



(a) (b) (c)

FIGURE 2.1.4. The building blocks for the Feynman graphs of QED: (a) the vertex of QED, (b) the photon line and (c) the fermion line.

Let us instead consider QED again. In this theory we have fermions and photons, usually depicted by straight and wiggly edges respectively, and one kind of vertex. The edges and the vertex of the Feynman graphs of QED are shown in figure 2.1.4. We note that the vertex of QED is incident to three edges which are two fermion lines and one photon line. This form of the vertex corresponds to the interaction term in the Lagrange density of QED, $eA^\mu\overline{\psi}\gamma_\mu\psi$ (cf. equation (2.1.5)), which contains two factors for the fermion field and one for the photon field. In this simple manner one can read off the building blocks of the Feynman graphs directly from the Lagrange density of the theory.

Just like in the above rules of ϕ^4-theory we have to draw all possible graphs of QED with the elements of figure 2.1.4 to a certain order and assign certain components of which the integrands will consist of. The main difference to the integrands of ϕ^4-theory and other scalar toy models is an additional tensor structure. This comes from the structure of the propagators of particles with spin. These propagators are of the form

$$i\frac{k_\mu\gamma^\mu + m}{k^2 - m^2}$$

for fermions and

$$-i\frac{g_{\mu\nu}}{k^2}$$

for photons (after a convenient choice of a gauge parameter, according to the so-called Feynman gauge). Obviously this structure is an additional complication, but we will see in section 2.4 how to write a tensor-like Feynman integral in terms of scalar Feynman integrals.

2.1.4. Wick Rotation and Spherical Coordinates. There is a more simple complication which we would like to cure before: The scalar products of momenta like k^2 or p^2 in the above propagators are defined with respect to the Minkowski metric of equation (2.1.1). Let f be the integrand of a Feynman integral, containing propagators of the form

$$\frac{i}{p^2 - m^2 + i\rho}.$$

The small term $i\rho$, which we usually suppress in our notation, shifts the poles of the propagator away from the real axis. One can find a closed contour in the complex k^0-plane, which does not contain the poles. From the vanishing of an integral over such a contour, one can deduce that

$$\int_{-\infty}^{\infty} dk^0 f\left(k^0\right) \quad = \quad -\int_{-i\infty}^{i\infty} dk^0 f\left(k^0\right).$$

Therefore in a Feynman integral, the integration over the real k^0-axis can be replaced by the integration over the imaginary axis. This replacement is referred to as *Wick rotation*.

A simple change of parameters

$$\begin{aligned} k^0 &= iK^0, \\ k^j &= K^j \text{ for } j = 1, 2, 3 \end{aligned}$$

leads to

$$\begin{aligned} k^2 &= -K^2, \\ d^4k &= id^4K \end{aligned}$$

where K^2 is an ordinary Euclidean scalar product and the K^0-integration is over the real axis. We obtain

$$\int d^4k f\left(-k^2\right) = i\int d^4K f\left(K^2\right).$$

The right-hand side is an integral over Euclidean space. Following these steps, we can write any Feynman integral as an integral over Euclidean space where all the scalar products in the integrand are Euclidean.

We can apply Wick rotation as well for higher dimensional spaces with a Minkowski-like metric. Therefore let us consider the more general case of n-dimensional vectors K in Euclidean space, where n is some natural number. For an integral of a function $f\left(K^2\right)$ it may be useful to change to spherical coordinates. The generalized spherical coordinates are given by

$$\begin{aligned} K^0 &= K\cos\left(\theta_1\right), \\ K^1 &= K\sin\left(\theta_1\right)\cos\left(\theta_2\right), \\ &... \\ K^{n-2} &= K\sin\left(\theta_1\right)...\sin\left(\theta_{n-2}\right)\cos\left(\theta_{n-1}\right), \\ K^{n-1} &= K\sin\left(\theta_1\right)...\sin\left(\theta_{n-2}\right)\sin\left(\theta_{n-1}\right), \end{aligned}$$

where $K = \sqrt{K^2}$ and θ_{n-1} is the azimuthal angle while all remaining θ's are polar angles. The integration over the angles gives

$$\int_0^\pi d\theta_1 \sin^{n-2}\left(\theta_1\right)...\int_0^\pi d\theta_{n-2}\sin\left(\theta_{n-2}\right)\int_0^{2\pi} d\theta_{n-1} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \tag{2.1.10}$$

where

$$\Gamma(x) = \int_0^\infty dt e^{-t}t^{x-1}$$

is Euler's Gamma function, which is defined for $\mathrm{Re}(x) > 0$. It is important to notice, that the left hand side of equation (2.1.10) is obviously defined for $n \in \mathbb{N}$ while the right-hand side is a function which could be evaluated for complex values of n.

For the integral of $f\left(K^2\right)$ we obtain

$$\int_{-\infty}^\infty dK^0...\int_{-\infty}^\infty dK^{n-1} f\left(K^2\right) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\int_0^\infty dK K^{n-1} f\left(K^2\right). \tag{2.1.11}$$

Here again, the function on the right-hand side is defined for complex n. In the following we will see how this fact allows for the method of dimensional regularization.

2.2. Dimensional Regularization

2.2.1. The Need for Regularization. It is very common, that a Feynman integral is divergent. In the previous section, we have seen that Feynman integrals are constructed from Feynman rules which are not sensitive to the question whether the resulting integral converges or not. Of course the problem already lies in the definition of the Green's function to be considered. Despite this problem, it is for many theories[4] possible to obtain a finite cross section from these functions. In the standard approach, which we want to consider, this is achieved after the application of two procedures, which can be treated separately: *regularization* and *renormalization*.

By regularization we mean that instead of the ill-defined integral I, as we obtain it from Feynman rules, we consider an integral $I(\epsilon)$ which depends on an additional parameter ϵ, the regularization parameter, such that in some range of this parameter the integral is well-defined and in the limit $\epsilon \to 0$ the integral $I(\epsilon)$ coincides with the original ill-defined integral I:

$$\lim_{\epsilon \to 0} I(\epsilon) = I.$$

Then $I(\epsilon)$ is called the regularized Feynman integral. We possibly evaluate the integral, assuming ϵ to be in the region where it is well-defined, and then we can study the divergent behaviour of the result, which we obtain as a function in ϵ. Then we can separate the result into a divergent part and a finite part.

Renormalization usually is a re-definition of parameters, like the coupling constants and particle masses, such that the divergences which we obtained from the previous step are absorbed and the Green's function of the new parameters is finite. These new definitions are equivalent to the consideration of a different Lagrange density, which differs from the original one by so-called *counter terms*. Such re-definitions have to be made with each order of the perturbation series which is included in the result. At this point we find it necessary to emphasize, that renormalization is *not* a subject of this dissertation. We discuss properties of regularized Feynman integrals and methods for their calculation, knowing that their evaluation by subsequent renormalization procedures yield the renormalized Green's function and, in the end, the finite cross section. However, we do not discuss the step of renormalization any further.

Let us give a simple example for a divergent Feynman integral:

Example 5. The integral

$$I\left(p^2,\, m^2\right) \;=\; \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2)\left((k-p)^2-m^2\right)}$$

is the Feynman integral of the graph of figure 3.2.1, arising from ϕ^3-theory. Here and in the following we usually omit trivial prefactors like multiples of i, coupling constants and symmetry factors, as they are irrelevant for issues of integration.

In the region where k^2 is much larger than m^2 and p^2 the integrand behaves like $\frac{1}{k^4}$. Applying Wick rotation and spherical coordinates we may formally write

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \;=\; i \int \frac{d^4K}{(2\pi)^4} \frac{1}{K^4} = i2\pi^2 \int_0^\infty dK K^{-1}, \tag{2.2.1}$$

[4] These are so-called renormalizable theories, for example ϕ^4-theory, QED and the Standard Model.

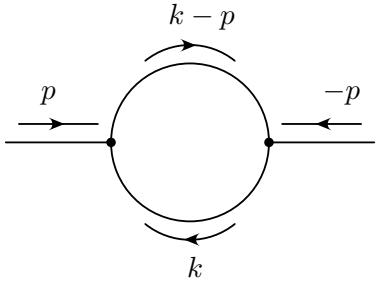


FIGURE 2.2.1. One-loop selfenergy graph.

but now we see that the integral is ill-defined. From the region of large K we obtain a logarithmic divergence because of

$$\lim_{B\to\infty}\int_A^B dK K^{-1} \quad = \quad \lim_{B\to\infty}\left(\ln\left(B\right)+const.\right),\ A>0.$$

Therefore we see, that $I\left(p^2,\, m^2\right)$ has a logarithmic divergence from the large momentum region.

Now let us in the sense of equation (2.1.11) rewrite equation (2.2.1) for a n-dimensional integral:

$$\int \frac{d^n k}{(2\pi)^n}\frac{1}{k^4} \quad = \quad i\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\int_0^\infty dK K^{n-5}.$$

For $n<4$ we have

$$\lim_{B\to\infty}\int_A^B dK K^{n-5} \quad = \quad \lim_{B\to\infty}\left(\frac{B^{n-4}}{n-4}+const.\right)=const.$$

and therefore we have no divergence from the large momentum region.

The example shows, that the divergent behaviour of a Feynman integral depends on the fact that it is defined with respect to four dimensional Minkowski space. If we instead consider the corresponding integral in an n-dimensional space, we may find an n such that the integral converges. But such a consideration only makes sense, if we obtain the physically relevant integral of the four dimensional space in a limit. To this end, the integer n needs to be replaced by a continuous variable. This leads to the idea of dimensional regularization, which was introduced to quantum field theory by t'Hooft and Veltman [**tV72**][5], Bollini and Giambiagi [**BG72**], Cicuta and Montaldi [**CM72**], Ashmore [**Ash72**] and in a more abstract setting by Speer and Westwater [**SW71**]. We use the most common version of dimensional regularization which was formulated by Collins [**Col84**].

2.2.2. Definition of Integrals over D Dimensions. We introduce the dimension parameter $D\in\mathbb{C}$, an integer $\lambda\in\mathbb{Z}$ and a parameter $\epsilon\in\mathbb{C}$, satisfying

$$D \quad = \quad 2\lambda-2\epsilon.$$

The parameter λ will be important in section 2.4, but for the moment let us fix $\lambda=2$, such that the limit $D\to 4$ is equivalent to $\epsilon\to 0$.

[5]In the year 1999, t'Hooft and Veltman received the Nobel Prize in Physics for their work on the mathematical foundations of electroweak interactions, including reference [**tV72**].

Now for an arbitrary Feynman integral we want to obtain the corresponding regularized integral by the formal replacement

$$\int \frac{d^4k}{(2\pi)^4} \quad \rightarrow \quad \int \frac{d^Dk}{(2\pi)^D} \tag{2.2.2}$$

for all integration momenta k. Then, assuming the value of D (or equivalently of ϵ) to be in a region such that the integral converges, we want to possibly evaluate the regularized integral $I(\epsilon)$. Then we expand $I(\epsilon)$ as a Laurent series in ϵ. We may separate the principal part of the series from the part which is finite in $\epsilon \rightarrow 0$. This separation can subsequently be used for renormalization procedures, for example so-called *minimal subtraction* (MS) **[t'H73, Wei73]** or *modified minimal subtraction* ($\overline{\text{MS}}$) **[BBDM78]**.

An integral over a D-dimensional space where D can be any complex number is just a formal notation at first. In order to make any sense of this notation and the replacement given by equation (2.2.2), one needs to find a definition in terms of ordinary, integer-dimensional integration, fulfilling the desired properties of an integral in D dimensions. Let us refer to the definition of Collins **[Col84]**:

We consider a Euclidean space $\mathbb{E}$ and let $f(k, p_1, ..., p_r)$ be a scalar function, only depending on scalar products of the finitely many vectors k, p_1, ..., p_r in $\mathbb{E}$. We want to give a meaning to the object

$$\int d^Dk f(k, p_1, ..., p_r).$$

In order to do so, we assume $\mathbb{E}$ to be an infinite dimensional vector space and each of the vectors k, p_1, ..., p_r to have an infinite sequence of components. As the number of these vectors r is finite, this space has a n-dimensional subspace $\mathbb{E}_\parallel$, $n \in \mathbb{N}$ with $n \leq r$, containing the vectors p_1, ..., p_r. The orthogonal component of this subspace shall be denoted $\mathbb{E}_\perp$, such that $\mathbb{E} = \mathbb{E}_\parallel \times \mathbb{E}_\perp$. In this sense we decompose the vector k as

$$k \quad = \quad k_\parallel + k_\perp$$

with $k_\parallel \in \mathbb{E}_\parallel$ and $k_\perp \in \mathbb{E}_\perp$. $k_\parallel$ has n components, n being finite, and we therefore may write

$$\int d^Dk f(k, p_1, ..., p_r) \quad = \quad \int d^nk_\parallel \int d^{D-n}k_\perp f(k, p_1, ..., p_r)$$

where the integration

$$\int d^nk_\parallel \equiv \int_{-\infty}^{\infty} dk^1 ... \int_{-\infty}^{\infty} dk^n$$

is well-defined. As the function $f(k, p_1, ..., p_r)$ depends only on scalar products of the vectors, it does not depend on the direction of $k_\perp$. Therefore it makes sense to define

$$\int d^{D-n}k_\perp f(k, p_1, ..., p_r) \quad := \quad \frac{2\pi^{(D-n)/2}}{\Gamma\left(\frac{D-n}{2}\right)} \int_0^\infty dk_\perp k_\perp^{D-n-1} f(k, p_1, ..., p_r). \tag{2.2.3}$$

On the right-hand side we have an ordinary integral, which we may think of as obtained from formally introducing spherical coordinates in $D-n$ dimensions, as we see by comparison with equation (2.1.11). We see from equation (2.1.11) that the given definition of the symbol $\int d^Dk$ contains the ordinary integration over integer-dimensional spaces as a special case. Therefore it is a meaningful extension of ordinary integration.

In order to obtain a meaningful regularized integral for an arbitrary Feynman integral, the definition given by equation (2.2.3) still needs to be extended to tensor-like functions. Furthermore, for certain cases it is necessary to extend the definition to regions of D where the real part of D is negative. Moreover one has to ask for uniqueness and certain special

properties of the D-dimensional integrals in order to provide a useful regularization. For a detailed discussion of all of these subjects we refer to Collins' book **[Col84]**. In the following subsection we summarize some of the most important properties which are proven in the same book.

The above definition, using infinite dimensional spaces, is important to obtain all of the mentioned properties in a consistent way, but we will not make any explicit use of the definition in the following. In particular, we will not consider momenta in an infinite dimensional space. Instead we follow the usual approach to just use the definition as a justification for speaking of $\int d^D k f(k, p_1, ..., p_r)$ as an integral over a *D-dimensional space* and thinking of the arguments k, p_1, ..., p_r as vectors in such a space.

2.2.3. Properties of Dimensionally Regularized Integrals. Let $g(k)$ be any given function of the D-dimensional vector k. Dimensionally regularized integrals, as introduced above, satisfy the following three axioms of Wilson **[Wil73]**. For any a, b, $s \in \mathbb{C}$ and any D-dimensional vector q we have:

- Additivity and linearity:

$$\int d^D k \left(a g(k) + b g(k)\right) = a \int d^D k g(k) + b \int d^D k g(k), \tag{2.2.4}$$

- Scaling:

$$\int d^D k g(sk) = s^{-D} \int d^D k g(k), \tag{2.2.5}$$

- Translation invariance:

$$\int d^D k g(k+q) = \int d^D k g(k). \tag{2.2.6}$$

Moreover one can verify the following useful properties:

- We can interchange the order of integrations as

$$\int d^D k \int d^D q g(k, q) = \int d^D q \int d^D k g(k, q), \tag{2.2.7}$$

  where g is any function of two vectors k and q.
- We can interchange the order of integration and differentiation with respect to different vectors as

$$\frac{\partial}{\partial q} \int d^D k g(k, q) = \int d^D k \frac{\partial}{\partial q} g(k, q). \tag{2.2.8}$$

- For $D = 2\lambda - 2\epsilon$ and any a, $b \in \mathbb{Z}$ and k-independent terms u and f we have

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{\left(-k^2\right)^a}{\left(-uk^2 + f\right)^b} = \frac{\Gamma(\lambda + a - \epsilon)\,\Gamma(b - \lambda - a + \epsilon)}{\Gamma(\lambda - \epsilon)\,\Gamma(b)} \frac{u^{-\lambda - a + \epsilon}}{f^{b - \lambda - a + \epsilon}}. \tag{2.2.9}$$

  Let us observe, that on the right-hand side of this equation each a appears together with $\lambda - \epsilon = D/2$. Hence raising a by 1 gives the same result as raising λ by 1, i. e. D by 2, apart from a prefactor $\lambda - \epsilon$.
- For differentiation with respect to the components of a vector we have

$$\int d^D k \frac{\partial}{\partial k^\mu} q^\mu g(k) = 0. \tag{2.2.10}$$

- If h is a function of k^2 we have

$$\int d^D k k^\mu h\left(k^2\right) = 0. \tag{2.2.11}$$

- If we extend the metric tensor of Minkowski space to the D-dimensional case, we obtain for its contraction

$$g^{\mu\nu}g_{\mu\nu} \quad = \quad D. \tag{2.2.12}$$

There are further properties of dimensionally regularized Feynman integrals which give good reasons to prefer dimensional regularization to other regularization methods. For example so-called cut-off regularization, where the upper boundary of the integration is limited, introduces a parameter which causes a non-trivial behaviour under Lorentz transformation. This is not the case for dimensional regularization. Furthermore t'Hooft and Veltman have shown in reference [**tV72**] that for certain theories[6] the so-called Ward identities, obtained from the invariance of a theory under so-called gauge transformations, are fulfilled by dimensionally regularized Green's functions. The method of dimensional regularization is compatible with the behaviour of crucial objects of the theory under Lorentz and gauge transformations.

For us there is in addition a far more obvious reason to work in dimensional regularization: All the other techniques discussed in the present chapter like Wick rotation and Feynman parameterization can be combined with dimensional regularization. Moreover the tensor reduction to be discussed in section 2.4 is based on dimensionally regularized integrals.

2.3. Feynman Parameters

The Feynman parameter technique is extensively discussed in the literature (see e. g. [**IZ06, Wei06, Nak71**]) and we just want to highlight the basic steps of the calculation and then concentrate on how to obtain the resulting form of the integral in a convenient way, directly from the graph.

Let us consider a generic, dimensionally regularized Feynman integral, obtained from the Feynman rules of a scalar theory, like for instance ϕ^4-theory. As we have seen, the propagators of such a scalar theory are of the form

$$\frac{1}{k_j^2 - m_j^2} \quad \text{or} \quad \frac{1}{p_j^2 - m_j^2},$$

where the momenta k_j and p_j are the momenta assigned to internal and external edges respectively and the parameters m_j are the masses. For convenience, let Λ denote the set of all external momenta p and masses m. For each vertex we have a term $\delta^D\left(\sum p_i + \sum k_j\right)$ (written in D dimensions). Let N_v denote the number of vertices. By integration over N_v of the momenta, eliminating the δ-distributions, we obtain the Feynman integral

$$I\left(D, \Lambda\right) \quad = \quad \int d^D k_1 ... d^D k_L \frac{1}{\prod_{j=1}^{N} \left(-P_j\right)^{\nu_j}} \tag{2.3.1}$$

where

$$P_j = q_j^2 - m_j^2 \tag{2.3.2}$$

where the momenta q_j are linear combinations of k_1, ..., k_L and the external momenta p_1, ..., p_{N_e}. The exponents ν_j are positive integers, N is the number of internal edges and N_e the number of external edges of the Feynman graph. L is the loop-number, given by

$$L \quad = \quad N - N_v + 1. \tag{2.3.3}$$

[6]The Ward-identities do not hold in the D-dimensional case if the theory requires the consideration of the so-called chirality operator γ_5. Nevertheless, in such a case the identities can be obtained after a finite renormalization.

For the evaluation of further momentum integrations we would like to use equation (2.2.9). To this end we need to write the Feynman integral in the form

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{\left(-k^2\right)^a}{\left(-uk^2+f\right)^b} \tag{2.3.4}$$

with k being one of the integration momenta. The terms u and f must be independent of k. This form can be obtained by repeated application of the so-called Feynman trick and by use of the translation invariance of D-dimensional integrals as in equation (2.2.6).

The *Feynman trick* in its simplest form is the formula

$$\frac{1}{P_1 P_2} = \int_0^1 dx \frac{1}{\left(xP_1+(1-x)P_2\right)^2}.$$

We simply rewrite a product in the denominator as a power of a sum at the cost of introducing an integration. In its general form, the Feynman trick reads

$$\prod_{i=1}^n \frac{1}{\left(-P_i\right)^{\nu_i}} = \frac{\Gamma(\nu)}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^1 \left(\prod_{i=1}^n dx_i x_i^{\nu_i-1}\right) \frac{\delta\left(1-\sum_{i=1}^n x_i\right)}{\left(-\sum_{i=1}^n x_i P_i\right)} \tag{2.3.5}$$

with $\nu = \sum_{i=1}^n \nu_i$. The newly introduced integration variables x_i are called *Feynman parameters*.

Starting from equation (2.3.1) we apply Wick rotation and the Feynman parameterization of equation (2.3.5) to $I(D, \Lambda)$. Then, by subsequent use of translation invariance and completion of squares, we can write the denominator in the form as in equation (2.3.4) with respect to one of the integration momenta k. Then we use equation (2.2.9) to evaluate the integration over k. We continue to apply these steps until all momentum integrals are evaluated and we remain with an integral only over the Feynman parameters:

$$I(D, \Lambda) = i^L \pi^{LD/2} \frac{\Gamma(\nu - LD/2)}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^1 d^N x \delta\left(1-\sum_{i=1}^N x_i\right) \left(\prod_{i=1}^N x_i^{\nu_i-1}\right) \frac{\mathcal{U}^{\nu-(L+1)D/2}}{\mathcal{F}(\Lambda)^{\nu-LD/2}}, \tag{2.3.6}$$

where the functions $\mathcal{U}$ and $\mathcal{F}(\Lambda)$ are polynomials in the Feynman parameters and $\mathcal{F}$ furthermore depends on the external momenta and the masses. We will refer to the right-hand side of equation (2.3.1) as the *momentum space representation* and to the right-hand side of equation (2.3.6) as the *Feynman parametric representation* of $I(D, \Lambda)$.

The explicit derivation of the form of equation (2.3.6) from equation (2.3.1) can be very laborious. However, there is a very useful shortcut. The functions $\mathcal{U}$ and $\mathcal{F}(\Lambda)$ can be directly constructed from the topology of the graph as follows (cf. **[Wei06]**):

Let G be the Feynman graph corresponding to the integral $I(D, \Lambda)$. We say that a connected graph is a *tree*, if it does not contain any loops. We believe that the terms *tree* and *loop* are intuitively clear, but we nevertheless give a more precise definition of these terms in chapter 4. To each internal edge e_j we associate a directed momentum q_j, a mass m_j and Feynman parameter x_j. We remember that L is the loop-number as defined above by equation (2.3.3). There are different possibilities of removing L edges of G such that we obtain a tree. The set of all possible trees obtained from removing L edges of G is denoted by $\mathcal{T}_1$. For each tree $T \in \mathcal{T}_1$, the *chord* $\mathcal{C}(T, G)$ is defined as the set of edges e_j, which belong to G but not to T. The function $\mathcal{U}$ is defined to be the sum of

all monomials, which are products of all the Feynman parameters of edges *not* belonging to a tree in $\mathcal{T}_1$:

$$\mathcal{U} := \sum_{T \in \mathcal{T}_1} \prod_{e_j \in \mathcal{C}(T,G)} x_j. \tag{2.3.7}$$

The sum runs through all trees in $\mathcal{T}_1$.

Now we take a tree $T \in \mathcal{T}_1$ and remove one more line. The result is a set of two trees (T_1, T_2), called a *2-forest.* The set of all 2-forests obtained in this way is denoted by $\mathcal{T}_2$. For each such 2-forest (T_1, T_2) the chord $\mathcal{C}((T_1, T_2), G)$ is the set of edges belonging to G but not to (T_1, T_2). We take the sum of all the momenta associated to the lines in this set and then take the square of this sum:

$$s_{(T_1,T_2)} = \left(\sum_{e_j \in \mathcal{C}((T_1,T_2),G)} q_j \right)^2,$$

where the orientation of the momenta q_j is chosen such that they are incoming at the vertices of T_1(cf. equation (4.1.10) below). Momentum conservation implies, that the sum of all incoming momenta at each vertex is zero. By use of this rule, we can express each $s_{(T_1,T_2)}$ as a square of a sum of only external momenta. These parameters $s_{(T_1,T_2)}$ are in the following referred to as the kinematical invariants[7]. Let us define an auxiliary function $\mathcal{F}_0$ to be the sum of monomials corresponding to the 2-forests (T_1, T_2), multiplied with the kinematical invariants:

$$\mathcal{F}_0 := \sum_{(T_1,T_2) \in \mathcal{T}_2} \left(\prod_{e_j \in \mathcal{C}((T_1,T_2),G)} x_j \right) \left(-s_{(T_1,T_2)}\right). \tag{2.3.8}$$

The polynomial $\mathcal{F}$ is the sum of $\mathcal{F}_0$ and a product of $\mathcal{U}$ with a sum of a product of the squared masses with the corresponding Feynman parameters:

$$\mathcal{F} := \mathcal{F}_0 + \mathcal{U} \sum_{j=1}^{N} x_j m_j^2. \tag{2.3.9}$$

We note that both functions $\mathcal{U}$ and $\mathcal{F}$ are homogeneous polynomials in the Feynman parameters. $\mathcal{U}$ is linear in each of the Feynman parameters while $\mathcal{F}$ can be quadratic in a single parameter if the corresponding mass is non-zero. $\mathcal{F}$ is linear in the m_j^2 and the kinematical invariants.

Let us derive the functions $\mathcal{U}$ and $\mathcal{F}_0$ for two examples:

Example 6. (a) We consider the one-loop triangle graph of figure 2.3.1 (a) with the incoming external momenta p_1, p_2, p_3 and the Feynman parameters as assigned in the figure. From equations (2.3.7) and (2.3.8) we obtain

$$\begin{aligned} \mathcal{U} &= x_1 + x_2 + x_3, \\ \mathcal{F}_0 &= x_1 x_2 \left(-p_3^2\right) + x_2 x_3 \left(-p_1^2\right) + x_1 x_3 \left(-p_2^2\right). \end{aligned}$$

(b) For the two-loop graph of figure 2.3.1 (a) we have only one external momentum p and five Feynman parameters as assigned in the figure. Equations (2.3.7) and (2.3.8) yield

$$\begin{aligned} \mathcal{U} &= (x_1 + x_2 + x_3 + x_4)\, x_5 + (x_1 + x_4)\,(x_2 + x_3), \\ \mathcal{F}_0 &= \left((x_1 + x_4)\, x_2 x_3 + (x_2 + x_3)\, x_1 x_4 + (x_1 + x_2)\,(x_3 + x_4)\, x_5\right)\left(-p^2\right). \end{aligned}$$

[7] They are invariant under Lorentz transformations.

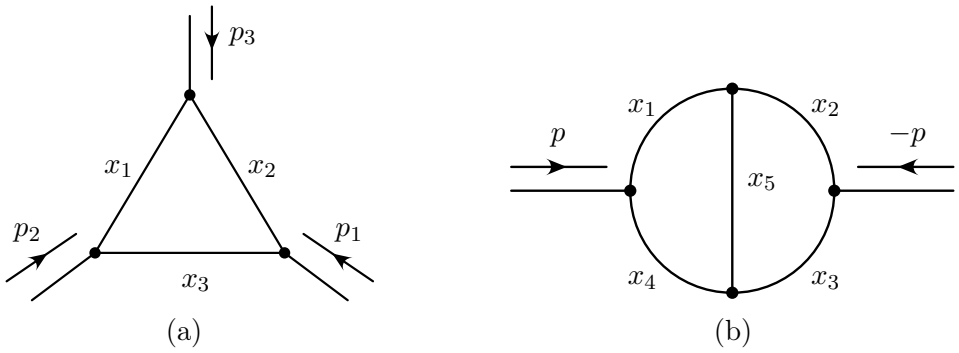


FIGURE 2.3.1. (a) The one-loop triangle. (b) The two-loop bubble.

The consistency of equation (2.3.6) with the definitions given by equations (2.3.7) and (2.3.8) is shown in detail in **[IZ06, KRTW08, Nak71]** and for the sake of brevity, we do not want to repeat these derivations here. Let us just highlight a key point of the calculation: With respect to the internal momenta q_j of the graph let $B^\star$ be the matrix defined by

$$B^\star{}_{ij} = \begin{cases} 1 \text{ if the momentum } q_j \text{ is outgoing from the } i\text{th vertex,} \\ -1 \text{ if the momentum } q_j \text{ is incoming to the } i\text{th vertex,} \\ 0 \text{ if the } j\text{th line is not connected to the } i\text{th vertex.} \end{cases} \tag{2.3.10}$$

$B^\star$ is called the *incidence matrix of the oriented graph*. For each vertex v_i we can write the condition of momentum conservation by use of $B^\star$. Let P_{v_i} denote the sum of external momenta which are incoming at v_i. Then due to momentum conservation we obtain

$$P_{v_i} - \sum_{j=1}^{N} B^\star{}_{ij} q_j = 0$$

for each vertex v_i where the sum runs through the N internal edges of the graph. Therefore the momentum space Feynman integral can be expressed by use of δ-distributions

$$\delta\left(P_{v_i} - \sum_{j=1}^{N} B^\star{}_{ij} q_j \right).$$

Then, after the introduction of Feynman parameters (or equivalently Schwinger parameters as in **[IZ06]**) the Feynman integral can still be expressed by use of $B^\star$. After the evaluation of the momentum integrations, one can use relations between $B^\star$ and the functions $\mathcal{U}$ and $\mathcal{F}$ in order to write the Feynman integral in terms of these polynomials. In chapter 4 we will define the incidence matrix B of a graph which is not necessarily oriented. (In contrast to $B^\star$ this matrix B will have only non-negative entries.) We will discuss relations between a matrix closely related to B and the functions $\mathcal{U}$ and $\mathcal{F}$.

In the above discussions we have used certain terms of graph theory. Terms like for example *loop*, *tree*, *forest* and the *orientation* of edges can be understood intuitively. The more precise definitions of these terms would be beyond the purpose of the present chapter, which we intend to be a brief introduction. These precise definitions require a rather detailed introduction of basic terms of graph theory and we want to provide this introduction in chapter 4 in order to avoid possible misunderstandings. After having introduced the mentioned terms properly, we will also formulate the construction of $\mathcal{U}$ and

$\mathcal{F}$ once again. Furthermore, chapter 4 includes a detailed discussion of further properties of $\mathcal{U}$ and $\mathcal{F}$.

2.4. From Tensors to Scalars

We have seen in section 2.1 that Feynman integrals of a theory describing particles with spin can have a tensor structure. We would like to express such tensor-like integrals in terms of scalar integrals. Let us start with a trivial example:

EXAMPLE 7. Let k be a momentum vector with components k^μ in D-dimensional space. Let $f\left(k^2\right)$ be a scalar valued function of k^2. Using $g_{\mu\nu}g^{\mu\nu} = D$ one easily verifies the identity

$$\int \frac{d^Dk}{i\pi^{D/2}} k^\mu k^\nu f\left(k^2\right) \quad = \quad -\frac{1}{D} g^{\mu\nu} \int \frac{d^Dk}{i\pi^{D/2}} \left(-k^2\right) f\left(k^2\right). \tag{2.4.1}$$

Let us note, that the integrand on the right-hand side in comparison to $f\left(k^2\right)$ has an additional power of $\left(-k^2\right)$. We have already seen in equation (2.2.9), how raising the power of $\left(-k^2\right)$ by 1 corresponds to raising λ by 1, which is the same as raising D by 2. Therefore we obtain

$$\int \frac{d^Dk}{i\pi^{D/2}} k^\mu k^\nu f\left(k^2\right) \quad = \quad -\frac{1}{2} g^{\mu\nu} \int \frac{d^{D+2}k}{i\pi^{(D+2)/2}} f\left(k^2\right) \equiv -\frac{1}{2} g^{\mu\nu} \mathbf{D}^+ \int \frac{d^Dk}{i\pi^{D/2}} f\left(k^2\right),$$

defining $\mathbf{D}^+$ as the operator which substitutes D by $D+2$.

On the left-hand side of equation (2.4.1) we have a tensor-like integral where the integrand is a tensor of rank 2 while on the right-hand side we have a k-independent tensor times a scalar integral. In dimensional regularization it is always possible to write an arbitrary tensor-like integral $I^{\mu_1\mu_2\cdots\mu_n}$ of rank $n \in \mathbb{N}$ as a linear combination

$$I^{\mu_1\mu_2\ldots\mu_n} \quad = \quad \sum_{i=1}^{r} t_i^{\mu_1\mu_2\ldots\mu_n} I_i \tag{2.4.2}$$

where the $t_i^{\mu_1\mu_2\ldots\mu_n}$ are simple tensors consisting of external momenta and the metric tensor. The integrals I_i on the right-hand side are scalar integrals.

For one-loop Feynman integrals, such a decomposition can be obtained from the reduction method of Passarino and Veltman in reference [**PV79**]. A method to obtain such a reduction for dimensionally regularized Feynman integrals with an arbitrary loop-number was developed by Tarasov in [**Tar96, ?**]:

We consider a general tensor-like integral

$$I^{\mu_1\ldots\sigma_{n_N}}\left(D,\,\Lambda\right) \quad = \quad \int d^Dk_1 \ldots d^Dk_L \frac{1}{\prod_{j=1}^N P_j^{\nu_j}} \prod_{r=1}^{n_1} q_r^{\mu_r} \ldots \prod_{s=1}^{n_N} q_s^{\sigma_s}$$

which differs from the scalar integral

$$I^{\text{scalar}}\left(D,\,\Lambda\right) \quad = \quad \int d^Dk_1 \ldots d^Dk_L \frac{1}{\prod_{j=1}^N P_j^{\nu_j}}$$

(cf. equation (2.3.1)) by the tensor $\prod_{r=1}^{n_1} q_r^{\mu_r} \ldots \prod_{s=1}^{n_N} q_s^{\sigma_s}$. Using the same notations as above q_j denotes the momentum along the jth edge of the graph, being a linear combination of internal and external momenta, and we have $P_j = q_j^2 - m_j^2$. For the reduction

of an integral $I^{\mu_1 \cdots \sigma_{n_N}}(D, \Lambda)$ to scalar integrals, Tarasov derives a relation which we can symbolically write as

$$I^{\mu_1 \cdots \sigma_{n_N}}(D, \Lambda) = T\left(\{q_i\}, \left\{\frac{\partial}{\partial m_i^2}\right\}, \mathbf{D}^+\right) I^{\text{scalar}}(D, \Lambda) \tag{2.4.3}$$

where $T\left(\{q_i\}, \left\{\frac{\partial}{\partial m_i^2}\right\}, \mathbf{D}^+\right)$ is an operator, acting on $I^{\text{scalar}}(D, \Lambda)$. As we will not make explicit use of a reduction as in equation (2.4.3), we refer to **[Tar96]** for the precise definition of $T\left(\{q_i\}, \left\{\frac{\partial}{\partial m_i^2}\right\}, \mathbf{D}^+\right)$. Detailed examples are given in **[?]**.

However, we will need to know which kind of scalar integrals will be obtained on the right-hand side of equation (2.4.3). $T\left(\{q_i\}, \left\{\frac{\partial}{\partial m_i^2}\right\}, \mathbf{D}^+\right)$ is a polynomial in the operator $\mathbf{D}^+$ which we introduced in example 7 and in the differential operators $\frac{\partial}{\partial m_i^2}$ where m_i are the masses associated to the internal edges. Let us observe that the derivative $\frac{\partial}{\partial m_i^2}$ acts on the integrand of $I^{\text{scalar}}(D, \Lambda)$ as

$$\frac{\partial}{\partial m_i^2} \frac{1}{\prod_{j=1}^N P_j^{\nu_j}} = \frac{1}{\prod_{j=1}^N P_j^{\nu_j + \delta_{ij}}},$$

with the Kronecker δ

$$\delta_{ij} = \begin{cases} 1 \text{ for } i = j, \\ 0 \text{ for } i \neq j. \end{cases}$$

The derivatives increase the powers ν_j of the propagators. Furthermore we see that the operator $\mathbf{D}^+$, as defined in the above example 7 raises the value of D by 2 or, equivalently, of λ by 1.

From these considerations we see that all scalar integrals on the right-hand side of Tarasov's equation (2.4.3) can be expressed in the form of equation (2.3.6) with the ν's and the λ in

$$D = 2\lambda - 2\epsilon$$

being arbitrary integers. Integrals of this type will therefore be the main objects of interest in the remaining chapters of the dissertation. As we mentioned above, we will refer to equation (2.3.1) as the momentum space representation and to equation (2.3.6) as the Feynman parametric representation of $I(D, \Lambda)$ in the following chapters.

CHAPTER 3

Integration-by-Parts for Feynman Integrals

"Calculation is not an experiment."
(Ludwig Wittgenstein)

The number of Feynman integrals to be taken into account in the calculation of a cross section increases rapidly with the desired order in the perturbation series. For example a calculation at one-loop order requires just a few integrals while the two-loop or three-loop precision already may require the consideration of several hundred Feynman integrals. Usually the evaluation of a single two- or three-loop integral is already very difficult, so the separate evaluation of hundreds or thousands of these integrals would practically be impossible without automation.

Therefore it is necessary to use relations among the Feynman integrals in order to reduce the number of integrals which actually need to be evaluated. The standard approach for such a reduction is the use of so-called IBP-identities[1] (integration-by-parts) **[CT81, Tka81]**, which are obtained from the general property of dimensionally regularized integrals, given by equation (2.2.10). These identities are used to express Feynman integrals as a linear combination of simpler ones. By this procedure, a large number of integrals can be expressed in terms of a considerably smaller number of usually simpler integrals. If these can not be reduced any further they are called *master integrals*.

In section 3.1 we give a brief review on IBP-identities in the momentum space representation, where they are usually derived. We discuss the Laporta algorithm, which is the most common way to combine these equations to an efficient reduction algorithm. In section 3.2 we demonstrate the translation of IBP-identities to the Feynman parametric representation. We derive a handy formula for the one-loop case in terms of the polynomials $\mathcal{U}$ and $\mathcal{F}$.

Our motivation to discuss IBP-identities in this dissertation is two-fold. Firstly, due to the importance of these widely used identities for practical calculations, their systematic combination to an efficient reduction algorithm is an active field of research. It seems that the optimal way to obtain such a reduction algorithm is yet unknown. If the reduction is not efficient, one is often confronted with a too large number of identities, hindering the evaluation. A detailed knowledge of the properties of IBP-identities may help for improvements of the efficiency of the reductions. Secondly, to our knowledge and surprise, the important identities have not yet been of any explicit use in the approach to Feynman integrals in terms of algebraic geometry, where the geometric properties of the zero-sets of $\mathcal{U}$ and $\mathcal{F}$ are considered (see related remarks in the chapters 5 and 6). We will emphasize in the subsequent chapters, that certain mathematical properties of a Feynman integral can be deduced from the properties of the so-called master integrals, obtained from a

[1]The IBP-identities provide the most common approach for the reduction of Feynman integrals. We want to mention, that there is a more general set of identities, considered by Tarasov in reference **[Tar96]**, which contains the IBP-identities as a subset. Furthermore, very recently, Tarasov presented a new kind of functional relations in the article **[Tar08]**.

reduction by IBP-identities. Seen from this point of view, IBP-identities may be useful in more mathematically oriented approaches.

3.1. The IBP-Approach

3.1.1. Trivial Algebraic Reduction. Before we start to discuss the essential part of the approach, let us introduce a convenient notation and some preliminary manipulations of the original Feynman integral.

We consider dimensionally regularized, scalar Feynman integrals in their momentum space representation, which are slightly more general than the one in equation (2.3.1), as we want to allow for scalar products of momenta in the numerator for the moment. Such scalar products will occur from the reduction mechanism to be discussed. Therefore the integrals to be considered have the form

$$I_{B,\,N}\,(D,\,\Lambda) \;=\; \int d^D k_1 ... d^D k_L \frac{\prod_{i=1}^{B} S_i^{\beta_i}}{\prod_{j=1}^{N} P_j^{\alpha_j}} \tag{3.1.1}$$

where for convenience the sign of the propagators is different from the one in equation (2.3.1). We furthermore apply the following notations:

- L is the loop-number, defined by equation (2.3.3), and N is the number of internal edges of the associated Feynman graph[2].
- k_1, ..., k_L are the loop momenta. The sum of all incoming external momenta is zero by momentum conservation, therefore we can write the integral as a function of $\mathcal{E}$ external momenta p_1, ..., $p_{\mathcal{E}}$, where the number $\mathcal{E}$ is the number of external edges minus one.
- Each S_i with $i = 1, 2, ..., B$ is a scalar product of two of the above momentum vectors, at least one of them being a loop momentum. B is the number of all such products, which is given by
$$B \;=\; L\mathcal{E} + \frac{1}{2}L(L+1).$$
- $P_j = q_j^2 - m_j^2$ is the inverse of the propagator associated to the jth internal edge, where m_j is the mass and q_j is the momentum associated to this edge. Each q_j is a linear combination of the above momentum vectors. (The masses m_j may be equal and they also may be zero.)
- The α_j and β_i are non-negative integers.
- We omitted the general prefactor $(2\pi)^{L(2-D)}$.

Example 8. As a very simple example let us consider the integral

$$\int d^D k \frac{kp}{(k^2 - m^2)\left((k-p)^2 - m^2\right)},$$

associated to the topology of the one-loop self-energy graph. Note that this integral differs from the one in example 5 by the scalar product kp in the numerator. We have

[2]As we described in chapter 2, a given quantum field theory determines the kind of vertices which constitute the corresponding Feynman graphs. Therefore, for a Feynman graph constructed by the Feynman rules of a given theory, the numbers L and N are not arbitrary. They fulfill certain relations. Nevertheless, in the process of the reduction by IBP-identities, one furthermore needs to consider graphs, which result from the original graph by the contraction of edges and which do not need to belong to the set of graphs, which can be constructed from the Feynman rules of the given theory. For this reason we want to avoid the use of any additional relations, which depend on the choice of a theory in the way as just described.

two momenta k, p and all the possible scalar products of these two vectors with at least one loop-momentum are $S_1 = k^2$ and $S_2 = kp$. The inverse propagators can trivially be expressed by use of these products as $P_1 = k^2 - m^2$ and $P_2 = k^2 - 2kp + p^2 - m^2$. We notice that in the general case the inverse propagators are linearly independent over the S_i and that they are sums of a linear combination of the S_i and terms which do not depend on the loop momenta (like m^2 and p^2 in the present example).

In our example the scalar product kp appears in the numerator and in the denominator. By completing the square we obtain

$$\int d^Dk \frac{kp}{P_1P_2} = -\frac{1}{2}\int d^Dk \frac{1}{P_1} + \frac{1}{2}\int d^Dk \frac{1}{P_2} + \frac{p^2}{2}\int d^Dk \frac{1}{P_1P_2}.$$

In the integrands on the right-hand side, kp appears just in the denominator, namely in P_2. We have used simple algebraic relations between the S_i and P_j in order to reduce an integrand as a linear combination of integrands which do *not* contain loop-momentum dependent scalar products in *both*, numerator and denominator.

Let us translate this simple step to the general case (cf. **[Lap00]**). We denote an arbitrary inverse propagator by $P_j = \sum_{h=1}^{B} C_h S_h + f(\{p\}, \{m\})$ with C_h being integer coefficients and $f(\{p\}, \{m\})$ a function of external momenta and particle masses. If P_j contains C_iS_i we have the identity

$$\frac{S_i}{P_j} = \frac{1}{C_i}\left(1 - \frac{P_j - C_iS_i}{P_j}\right) \tag{3.1.2}$$

where on the right-hand side, after the subtraction, the numerator does not contain S_i. By subsequent application of this relation for the fractions in the integrand we can express any Feynman integral of equation (3.1.1) as a linear combination of Feynman integrals depending on $B - N$ different loop-momentum-dependent scalar products, which we can not rewrite in terms of inverse propagators, and of course still depending on the N inverse propagators. In the case of $B \leq N$, as in our example above, all the scalar products are rewritten.

Let us subsume this calculation by the term *algebraic reduction* and let us call the remaining $B - N$ scalar products $\tilde{S}_1$, ..., $\tilde{S}_{B-N}$ *algebraically irreducible.* The inverse propagators together with the irreducible scalar products form a set of B functions $\mathcal{D}_1 = \tilde{S}_1$, ..., $\mathcal{D}_{B-N} = \tilde{S}_{B-N}$, $\mathcal{D}_{B-N+1} = P_1$, ..., $\mathcal{D}_B = P_N$ which are linearly independent over the S_i. Each S_i which is contained in a propagator can be reduced, therefore, for a given set of scalar products, the set of irreducible scalar products is uniquely determined by the propagators. Furthermore any S_i of the given momenta can be expressed in terms of the $\mathcal{D}_j$. Therefore the $\mathcal{D}_j$ form a complete basis.

After algebraic reduction we can, instead of equation (3.1.1), consider the *algebraically irreducible* Feynman integrals

$$\int d^Dk_1 ... d^Dk_L \frac{1}{\prod_{j=1}^{N} P_j^{\alpha_j}} \qquad \text{for } B \leq N, \tag{3.1.3}$$

$$\int d^Dk_1 ... d^Dk_L \frac{\prod_{i=1}^{B-N} \tilde{S}_i^{\beta_i}}{\prod_{j=1}^{N} P_j^{\alpha_j}} \qquad \text{for } B > N, \tag{3.1.4}$$

or equivalently

$$\int d^Dk_1 ... d^Dk_L \frac{1}{\prod_{i=1}^{B} \mathcal{D}_i^{\nu_i}}, \tag{3.1.5}$$

the ν_i for $i = 1, ..., B$ now being both positive or negative integers.

We want to remark, that the integrals with irreducible scalar products as in equation (3.1.4) can be furthermore expressed by integrals where the numerator is one as in equation (3.1.3). This is achieved by the use of Tarasov's method as described in [**Tar96**], which we already used with equation (2.4.3) for the reduction of general tensor integrals. The method yields integrals with a changed space time dimension. Hence, by use of Tarasov's method and allowing for shifts in the dimension, we can later on justify to restrict our attention to integrals of the form of equation (3.1.3) with $\alpha_j \geq 0$.

The identities being discussed in the following will relate different integrals of the kind of equation (3.1.5) to each other. Apart from prefactors not depending on integration parameters they will only differ by the tuple of integer exponents $(\nu_1, ..., \nu_B)$. Following Lee's point of view in [**Lee08**] we can consider this tuple to be a set of coordinates representing the integral as a point in $\mathbb{Z}^B$. Obviously some regions in this space correspond to families of integrals with different complexity. A point in a region of $\mathbb{Z}^B$ where some of the ν_i are zero represents an integral with less than B factors in the denominator. Usually, such an integral is less difficult to calculate than the corresponding one with B factors. In a reduction procedure, one always tries to express the integral with B factors in the denominator by integrals of the regions with less than B factors. Let us introduce the identities which provide such a reduction.

3.1.2. Integration-by-Parts Identities. The entire approach discussed in this chapter is based on the well-known property of dimensionally regularized Feynman integrals, which we have already seen in equation (2.2.10): Let v be any vector in D-dimensional momentum space, k a loop momentum vector and $f(k)$ a scalar valued function of k and possibly depending on other momenta. Extending Gauss' theorem to D-dimensional integrals one obtains the identity

$$\int d^D k \frac{\partial}{\partial k^\mu} v^\mu f(k) \quad = \quad 0. \tag{3.1.6}$$

For v one can take any of the internal or external momenta of a given Feynman integral $\int d^D k f(k)$. Then evaluating the differentiation by the product rule leads to relations among different Feynman integrals, the so-called *IBP-identities* (integration-by-parts identities). They were stated and applied for the first time in [**CT81, Tka81**].

More explicitly, let us consider the algebraically irreducible Feynman integral

$$I(\nu_1, ..., \nu_B) = \int d^D k_1 ... d^D k_L \frac{1}{\prod_{j=1}^B \mathcal{D}_j^{\nu_j}}, \tag{3.1.7}$$

which we consider as a function of the exponents $\nu_1, ..., \nu_B$. Of course it is still understood, that they also depend on the external momenta and the masses, but for the moment let us omit these dependencies in our notation. For each integral $I(\nu_1, ..., \nu_B)$ we can apply equation (3.1.6) to obtain the set of identities

$$\int d^D k_1 ... d^D k_L \frac{\partial}{\partial k_i^\mu} q_h^\mu \frac{1}{\prod_{j=1}^B \mathcal{D}_j^{\nu_j}} \quad = \quad 0 \tag{3.1.8}$$

where the q_h is one of the momentum vectors $k_1, ..., k_L, p_1, ..., p_\mathcal{E}$. We will refer to these identities as the *basic* IBP-identities of $I(\nu_1, ..., \nu_B)$. The set consists of $L(L+\mathcal{E})$ identities. Of course these basic identities can be combined to infinitely many linear combinations, as v in equation (3.1.6) may be any linear combination of the vectors $k_1, ..., k_L, p_1, ..., p_\mathcal{E}$.

Evaluating the differentiation in equation (3.1.8) and performing trivial algebraic reduction we obtain a linear combination of integrals on the left-hand side:

$$\sum_{i=1}^{n} c_i I\left(\nu_{i1},\, ...,\, \nu_{iB}\right) \;=\; 0$$

where the c_i are rational functions of D, squared external momenta and particle masses and depending on the integers ν_j. The integrals $I\left(\nu_{i1},\, ...,\, \nu_{iB}\right)$ are all of the form of equation (3.1.7) where possibly exponents are decreased and increased by one. More precisely, they fall into three types of integrals:

(1) All exponents remain the same: $\nu_{ij} = \nu_j$ for all $j \in \{1,\, ...,\, B\}$.
(2) Just one exponent is increased: $\nu_{il} = \nu_l + 1,\ \nu_{ij} = \nu_j$ for all $j \in \{1,\, ...,\, B\}\backslash l$.
(3) One exponent is decreased and another one is increased: $\nu_{il} = \nu_l + 1,\, \nu_{ik} = \nu_k - 1,\ \nu_{ij} = \nu_j$ for all $j \in \{1,\, ...,\, B\}\backslash\{k,\, l\}$.

Let us briefly explain, why we have these three cases. We consider a differentiation like $\frac{\partial}{\partial k_j}\frac{1}{\mathcal{D}_i^{\nu_i}}$. If $\mathcal{D}_i$ depends on k_j the exponent ν_j is increased at first. In addition, the differentiation yields a certain factor in the numerator, possibly depending on k_j and other momenta. By algebraic reduction as described above, this factor can lead to cancellations, either with $\mathcal{D}_i$, such that the ν_j is lowered again, which yields an integral of case 1 in the above enumeration, or with any other factor $\mathcal{D}$ in the denominator, leading to case 3. Moreover algebraic reduction may lead to a term with a constant in the numerator, such that no factor $\mathcal{D}$ cancels. Such a term corresponds to case 2 above. Examples will be shown in the next subsections.

Let us similarly to **[Lee08]** and **[Smi06]** define the operators A_α and B_α for $\alpha = 1,\, ...,\, B$ as follows

$$\begin{aligned} \left(A_\alpha I\right)\left(\nu_1,\, ...,\, \nu_B\right) &= \nu_\alpha I\left(\nu_1,\, ...,\, \nu_\alpha + 1,\, ...,\, \nu_B\right), \\ \left(B_\alpha I\right)\left(\nu_1,\, ...,\, \nu_B\right) &= I\left(\nu_1,\, ...,\, \nu_\alpha - 1,\, ...,\, \nu_B\right). \end{aligned} \tag{3.1.9}$$

In the literature these operators are often denoted by the bold number of the exponent to be decreased or increased. For example $\mathbf{3}^- = B_3$, $\mathbf{5}^+ = \frac{1}{\nu_5}A_5$. With the notation as in equations (3.1.9), any IBP-identity is written (see **[Lee08]**) as

$$\left(\sum_{\alpha=1}^{B}\left(\sum_{\beta=1}^{B} a_{\alpha\beta}A_\alpha B_\beta + b_\alpha A_\alpha\right) + c\right) I\left(\nu_1,\, ...,\, \nu_B\right) \;=\; 0 \tag{3.1.10}$$

where the $a_{\alpha\beta}$, b_α, c are again rational functions of D and physical parameters. The three kinds of terms in this sum, c, $b_\alpha A_\alpha$ and $a_{\alpha\beta}A_\alpha B_\beta$, correspond to the three types of integrals we just mentioned.

Obviously the IBP-identities can be used to express one $I\left(\nu_{i1},\, ...,\, \nu_{iB}\right)$ in terms of other ones, which are possibly less hard to evaluate or which can again be expressed in terms of other ones and so on. In the following subsections we discuss at first an intuitive and then a more systematic approach for such a reduction.

3.1.3. A Two-Loop Example and the Triangle Rule. Before we focus on a more systematic treatment let us discuss the reduction by IBP-identities for an instructive case. (For this and many other examples see **[Smi06]**.)

Example 9. We consider the generic scalar integral of the massless two-loop propagator topology of figure (3.1.1), which is

$$I\left(\nu_1,\, \nu_2,\, \nu_3,\, \nu_4,\, \nu_5\right) \;=\; \int\int d^Dk_1 d^Dk_2\; j\left(\nu_1,\, \nu_2,\, \nu_3,\, \nu_4,\, \nu_5\right)$$

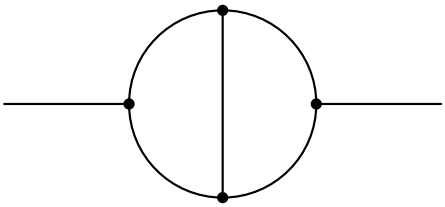


FIGURE 3.1.1. A two-loop propagator topology.

with

$$\begin{aligned} j\left(\nu_1,\,\nu_2,\,\nu_3,\,\nu_4,\,\nu_5\right) &= \frac{1}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}\mathcal{D}_4^{\nu_4}\mathcal{D}_5^{\nu_5}} \\ &= \frac{1}{\left(k_1^2\right)^{\nu_1}\left(\left(p-k_1\right)^2\right)^{\nu_2}\left(k_2^2\right)^{\nu_3}\left(\left(p-k_2\right)^2\right)^{\nu_4}\left(\left(k_1-k_2\right)^2\right)^{\nu_5}}, \end{aligned}$$

I depends on D and p^2 and the ν_i are positive integers. For this integral we obtain six basic IBP-identities by setting $q_\mu = k_{1\mu},\, k_{2\mu},\, p_\mu$ in equation (3.1.8). Let us show the simple evaluation explicitly, at least for one of these identities. We consider the identity

$$\int\int d^D k_1 d^D k_2 \frac{\partial}{\partial k_{1\mu}} k_{1\mu} j\left(\nu_1,\,\nu_2,\,\nu_3,\,\nu_4,\,\nu_5\right) \;=\; 0.$$

We apply the product rule and algebraic reduction to the integrand:

$$\frac{\partial}{\partial k_{1\mu}} k_{1\mu} j\left(\nu_1,\,\nu_2,\,\nu_3,\,\nu_4,\,\nu_5\right) =$$

$$j\left(\nu_1,\,\nu_2,\,\nu_3,\,\nu_4,\,\nu_5\right)\left(\frac{\partial}{\partial k_{1\mu}} k_{1\mu} + \left(k_1^2\right)^{\nu_1} k_{1\mu}\frac{\partial}{\partial k_{1\mu}}\left(k_1^2\right)^{-\nu_1}\right.$$

$$\left. + \left(\left(p-k_1\right)^2\right)^{\nu_2} k_{1\mu}\frac{\partial}{\partial k_{1\mu}}\left(\left(p-k_1\right)^2\right)^{-\nu_2} + \left(\left(k_1-k_2\right)^2\right)^{\nu_5} k_{1\mu}\frac{\partial}{\partial k_{1\mu}}\left(\left(k_1-k_2\right)^2\right)^{-\nu_5}\right) =$$

$$j\left(\nu_1,\,\nu_2,\,\nu_3,\,\nu_4,\,\nu_5\right)\left(D - 2\nu_1 + \nu_2\frac{2k_1p - 2k_1^2}{\left(p-k_1\right)^2} - \nu_5\frac{2k_1^2 - 2k_1k_2}{\left(k_1-k_2\right)^2}\right) =$$

$$j\left(\nu_1,\,\nu_2,\,\nu_3,\,\nu_4,\,\nu_5\right)\left(D - 2\nu_1 - \nu_2\left(1 + \frac{k_1^2}{\left(p-k_1\right)^2} - \frac{p^2}{\left(p-k_1\right)^2}\right)\right.$$

$$\left. -\nu_5\left(1 + \frac{k_1^2}{\left(k_1-k_2\right)^2} - \frac{k_2^2}{\left(k_1-k_2\right)^2}\right)\right) =$$

$$\frac{D - 2\nu_1 - \nu_2 - \nu_5}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}\mathcal{D}_4^{\nu_4}\mathcal{D}_5^{\nu_5}} - \frac{\nu_2}{\mathcal{D}_1^{\nu_1-1}\mathcal{D}_2^{\nu_2+1}\mathcal{D}_3^{\nu_3}\mathcal{D}_4^{\nu_4}\mathcal{D}_5^{\nu_5}} + \frac{\nu_2 p^2}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2+1}\mathcal{D}_3^{\nu_3}\mathcal{D}_4^{\nu_4}\mathcal{D}_5^{\nu_5}}$$

$$- \frac{\nu_5}{\mathcal{D}_1^{\nu_1-1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}\mathcal{D}_4^{\nu_4}\mathcal{D}_5^{\nu_5+1}} + \frac{\nu_5}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3-1}\mathcal{D}_4^{\nu_4}\mathcal{D}_5^{\nu_5+1}}.$$

Therefore we have

$$\frac{\partial}{\partial k_{1\mu}} k_{1\mu} I \;=\; \left(D - 2\nu_1 - \nu_2 - \nu_5 - A_2\left(B_1 - p^2\right) - A_5\left(B_1 - B_3\right)\right) I = 0.$$

All three different cases, explained in the previous section, are present in this example.

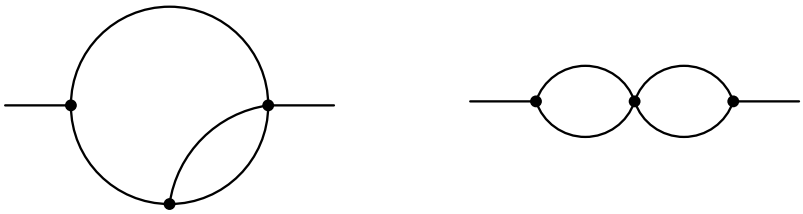


FIGURE 3.1.2. Two-loop graphs, resulting from the contraction of an edge in figure 3.1.1.

By a similar calculation we obtain a second basic IBP-identity

$$k_{2\mu}\frac{\partial}{\partial k_{1\mu}}I = \left(\nu_5 - \nu_1 - A_1\left(B_3 - B_5\right) - A_2\left(B_1 + B_4 - B_5 - p^2\right) - A_5\left(B_1 - B_3\right)\right) I$$
$$= 0$$

and we can derive the other four basic identities in the same way.

Let us see what we can achieve with these first two identities only. We consider the difference of the two identities (cf. the first example in **[CT81]**)

$$\left(k_{2\mu}\frac{\partial}{\partial k_{1\mu}} - \frac{\partial}{\partial k_{1\mu}}k_{1\mu}\right) I =$$

$$\left(\nu_1 + \nu_2 + 2\nu_5 - D - A_1\left(B_3 - B_5\right) - A_2\left(B_4 - B_5\right)\right) I = 0, \tag{3.1.11}$$

which we can write as

$$I = \frac{A_1B_3I - A_1B_5I + A_2B_4I - A_2B_5I}{\nu_1 + \nu_2 + 2\nu_5 - D}.$$

We see that in each integral on the right-hand side one of the exponents ν_3, ν_4, ν_5 is decreased by one and none of these exponents is increased. Hence repeated use of this equation will enable us to express I in terms of integrals where one of the exponents ν_3, ν_4, ν_5 is zero. These are integrals corresponding to the graphs shown in figure 3.1.2. They can be considered less complex than I, so for this example equation (3.1.11) is sufficient for a reduction of I to simpler integrals. Only two of the initial six basic IBP-identities have been used.

Of course, for most Feynman integrals, a useful reduction requires a lot of more effort and the consideration of more than two IBP-identities. With the above example we have considered a particularly easy case, where in a few steps we obtained the useful equation (3.1.11) due to the fact that the underlying Feynman graph has a triangle as in figure 3.1.3 as a sub-graph and the propagators are massless. For general graphs with these properties one can construct identities equivalent to equation (3.1.11). These so-called *triangle rules* **[vRS00]** are derived as follows.

Consider a Feynman graph containing the triangle graph in figure 3.1.3 as a sub-graph, such that the edges with the momenta q_1 and q_2 are internal edges. The Feynman integral for the entire graph is

$$I\left(\nu_1,\, \nu_2,\, \nu_3,\, ...\right) = \int d^Dk \prod d^D\tilde{k} j\left(\nu_1,\, \nu_2,\, \nu_3,\, ...\right)$$
$$= \int d^Dk \prod d^D\tilde{k} \frac{1}{\left(\left(k+q_1\right)^2 - m_1^2\right)^{\nu_1}\left(\left(k+q_2\right)^2 - m_2^2\right)^{\nu_2}\left(k^2\right)^{\nu_3}\prod \tilde{P}},$$

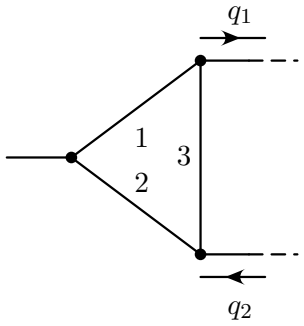


FIGURE 3.1.3. A triangle as a sub-graph.

where we have written the terms depending on the loop-momentum k of the triangle explicitly and indicated the presence of further propagators and integrations by terms with a tilde. While in example 9 all propagators were massless we consider a more general case here. We have assigned arbitrary masses m_1, m_2 to the edges 1 and 2 respectively. The propagator of edge 3 is massless. One of the basic IBP-identities for I is

$$\int d^D k \frac{\partial}{\partial k^\mu} k^\mu j\left(\nu_1, \nu_2, \nu_3, ...\right) \quad = \quad 0$$

which evaluates to

$$\begin{aligned} I \quad = \quad & \frac{1}{D-\nu_1-\nu_2-2\nu_3}\left(A_1 B_3 I - A_1 \int d^D k \prod d^D \tilde{k} \left(q_1^2 - m_1^2\right) j\left(\nu_1, \nu_2, \nu_3, ...\right) \right. \\ & \left. + A_2 B_3 I - A_2 \int d^D k \prod d^D \tilde{k} \left(q_2^2 - m_2^2\right) j\left(\nu_1, \nu_2, \nu_3, ...\right) \right). \end{aligned}$$

If the edges labelled by q_1 and q_2 are assigned masses m_1 and m_2 respectively, then their inverse propagators appear in $\prod \tilde{P}$ as $\left(q_1^2 - m_1^2\right)^{\nu_4} \left(q_2^2 - m_2^2\right)^{\nu_5}$ and we have a cancellation with the corresponding terms in the above identity. Hence these terms can be expressed by the lowering operators B_4 and B_5 respectively, so in this case we obtain

$$I \quad = \quad \frac{A_1 B_3 I - A_1 B_4 I + A_2 B_3 I - A_2 B_5 I}{D-\nu_1-\nu_2-2\nu_3}. \tag{3.1.12}$$

Equation (3.1.12) is a *triangle rule*. We see that on the right-hand side the powers of the propagators of the edges 1 and 2 are raised and the powers of the edges 3, 4 and 5 are lowered, due to the presence of the operators B_3, B_4 and B_5. The relation can be used repeatedly until one of the exponents ν_3, ν_4, ν_5 turns zero in a term on the right hand side. Therefore repeated use of equation (3.1.12) always leads to a reduction of the original integral I to integrals where at least one of the edges 3, 4 and 5 is contracted. The latter integral can be considered easier as it has less propagators.

Let us repeat the properties of an initial Feynman graph G, which are necessary to obtain equation (3.1.12). The graph G contains a triangle as in figure 3.1.3 as a sub-graph, such that two edges are external to the triangle but internal edges of G. In figure 3.1.3 these are the two edges with the momenta q_1 and q_2. Furthermore, the mass m_1 is assigned to the edge 1 in the figure and to the edge of q_1. Correspondingly, an arbitrary mass m_2 is assigned to edge 2 and to the edge of q_2. This choice of masses is essential to obtain the desired equation as it leads to the cancellation of the terms $\left(q_1^2 - m_1^2\right)$ and $\left(q_1^2 - m_2^2\right)$. By these cancellations the powers of the corresponding propagators are lowered, so we obtain the operators B_4 and B_5 in equation (3.1.12). We furthermore assume that edge 3 is massless and obtain the operator B_3.

We see, that we need to make some assumptions in order to obtain a recurrence relation as equation (3.1.12) in such an easy way. In the following we concentrate on reduction methods for the general case, where such tricks possibly do not apply.

3.1.4. The Laporta Algorithm. The algorithm of Laporta [**Lap00**] generates a system of finitely many basic IBP-identities and uses conventional Gauss-elimination to successively express all integrals in the system in terms of a finite and small number of integrals, which can not be reduced any further. The latter integrals are the so-called *master integrals*. The Feynman integrals which belong to the set of master integrals are determined by the system of identities.

Of course, in each step of this procedure, an integral shall possibly be expressed in terms of simpler ones. A human being can decide from case to case, which of two given Feynman integrals is, for whatever reasons, easier to handle than the other. However, for a fully automated calculation one needs a universal ordering on all the Feynman integrals under consideration. It is a key point of Laporta's algorithm, that it incorporates such an ordering, which we want to define now.

As in equation (3.1.4) we consider integrands of the form

$$j\,(n,\,u,\,\alpha,\,\beta) \;=\; \frac{\prod_{i=1}^{B-n}\tilde{S}_i^{\beta_i}}{\prod_{j=1}^{n}P_{u_j}^{\alpha_j}} \tag{3.1.13}$$

where n is the number of propagators, $B-n$ is the number of algebraically irreducible scalar products. By $\alpha = (\alpha_1,\,...,\,\alpha_n)$ and $\beta = (\beta_1,\,...,\,\beta_{B-n})$ we denote the ordered sets of the exponents and by $u = (u_1,\,...,\,u_n)$ the ordered set of indices of the inverse propagators. u is just a set of indices encoding which propagators are present in the integrand. For a given integrand of this form let us define the numbers

$$\begin{aligned} M_p &= \sum_{i=1}^{B-n}\beta_i, \\ M_d &= \sum_{i=1}^{n}(\alpha_i - 1), \end{aligned}$$

i. e. the sum of the exponents of the irreducible scalar products and the sum of the exponents of the inverse propagators minus their number respectively. We consider the set of integrands of the form of equation (3.1.13) characterized by given values of n, M_p, M_d but differing by individual values of the α_j and β_i. We denote such a set by $\left[n;\;\begin{matrix}M_p\\M_d\end{matrix}\right]$.

We can sort the integrands by the use of *lexicographical ordering,* which is defined as follows*:* A sequence of numbers $(a_1,\,...,\,a_n)$ is *larger* than another sequence $(b_1,\,...,\,b_l)$, which we denote

$$(a_1,\,...,\,a_n) \;\succ\; (b_1,\,...,\,b_l)\,,$$

if either $n > l$ or if $n = l$ and if furthermore there exists an m between 0 and n such that $a_m > b_m$ and $a_i = b_i$ for all $0 < i < m$. For example $(4,\,4,\,4,\,1) \succ (4,\,4,\,2,\,8)$ but $(4,\,4,\,2,\,8) \succ (4,\,4,\,4)$. With these notations we can now phrase Laporta's ordering of the integrands of equation (3.1.13): For two integrands $j\,(N,\,u_1,...,\,u_n,\,\alpha_1,...,\alpha_n\,\beta_1,...,\,\beta_{B-n})$ and $j'\left(N',\,u'_1,...,\,u'_l,\,\alpha'_1,...,\alpha'_l\,\beta'_1,...,\,\beta'_{B-l}\right)$ of the above type we say that j is *larger* than j', denoted

$$j \;\succ_L\; j',$$

if

$$(N, u_1, ..., u_n, \alpha_1, ..., \alpha_n, \beta_1, ..., \beta_{B-n}) \succ \left(N', {u'}_1, ..., {u'}_l, {\alpha'}_1, ..., {\alpha'}_l, {\beta'}_1, ..., {\beta'}_{B-l}\right). \tag{3.1.14}$$

Now let us describe Laporta's algorithm. We consider a given set of N inverse propagators P_1, ..., P_N and B scalar products. The number of loop-momenta is L. A set $u = (u_1, ..., u_n)$ with $u_1 < u_2 < ... < u_n$ is chosen to be an ordered subset of (1, ..., N) (see step 2 of the algorithm below). Such a subset fixes one of $\begin{pmatrix} N \\ n \end{pmatrix}$ possibilities to pick n inverse propagators $\{P_{u_1}, ..., P_{u_n}\}$ of the given ones. After this choice an integrand j as in equation (3.1.13) is selected by the additional choice of exponents α and β (see line 6 below). Laporta's reduction algorithm **[Lap00]** can be formulated as follows:

ALGORITHM 10. *(Laporta)*

1. *Let S be a set of identities and J a set of integrands. S and J are empty sets at the beginning. For each integer n from L to N do*
2. *For each of the $\begin{pmatrix} N \\ n \end{pmatrix}$ combinations $u = (u_1, ..., u_n)$ with $u_1 < ... < u_n$ do*
3. *Choose two non-negative integers a_u, b_u.*
4. *For each M_d from 0 to a_u do*
5. *For each M_p from 0 to b_u do*
6. *Choose one j $(n, u, \alpha, \beta) \in \begin{bmatrix} n; & M_p \\ & M_d \end{bmatrix}$ and create the corresponding basic IBP-identities (as in equation (3.1.8)).*
7. *For each of these new identities, which can not be obtained as a linear combination of identities of S, do*
8. *Substitute the integrands of the new identity which belong to J by use of the identities of S.*
9. *Extend the set J by the largest integrand of the new identity.*
10. *Extend the set S by the new identity.*

The resulting set S is the desired system of IBP-identities for the reduction of the given class of integrals.

In step 9 of the algorithm, *large* is meant in the sense of the above ordering. We see that the algorithm uses the ordering in two ways. Firstly, as Laporta emphasizes in reference **[Lap00]**, the computation time depends strongly on the order, in which the IBP-identities are considered. Therefore the above algorithm starts (in step 6) with the basic IBP-identities of the smallest integrands in the sense of the above ordering and proceeds to the identities of the larger integrands. Secondly, an IBP-identity is always used to express the largest of its integrands by the smaller ones, as we see in the steps 8 and 9. If we use an ordering of the integrands which is different from the one defined by equation (3.1.14) then the system of equations generated by the algorithm may be different and one in general obtains a different set of master integrals.

REMARK 11. Laporta's algorithm is a widely used and very successful method. Nevertheless its heuristic nature gives room to possible improvements. The main problem is, that the number of identities considered by the algorithm increases rapidly with N, which makes the reduction very elaborate at high loop orders. An impressive example for this problem can be found in Toedtli's dissertation **[Toe09]**, where in section 4.4 it is estimated that Laporta's algorithm would need to produce more than 180 000 identities for the calculation of a certain Green's function at three-loop order.

The algorithm usually generates far more identities than necessary. Therefore one may in addition use procedures to select the useful identities, as for example in Toedtli's dissertation. A very recent approach to such a selection was given by Lee [**Lee08**]. He reveals an underlying group structure of the operators which determine the IBP-identities. From this structure he deduces criteria for a selection of identities to be considered in the reduction[3].

Another problem may be seen in the mentioned fact that the information, which of the Feynman integrals under consideration are master integrals, is usually not clear from the beginning. In a different reduction approach initiated by Baikov [**Bai96**], this problem is supposed to be absent. We also want to mention another approach by a decomposition in terms of Gröbner bases, as introduced to Feynman integrals by Tarasov [**Tar98**], for example using a generalized Buchberger algorithm [**SS06**]. The great advantage of the Laporta algorithm in comparison to these alternatives is its applicability to the general case, given by its simplicity.

3.2. Translation to Feynman Parameters

We have reviewed the approach of IBP-identities in the momentum space representation. In the present section we will discuss the same identities in the Feynman parametric representation[4]. We will see that IBP-identities can be written in a convenient way by the use of the Feynman graph polynomials $\mathcal{U}$ and $\mathcal{F}$, defined in chapter 2 (and extensively discussed in chapter 4).

3.2.1. IBP-Operators for Feynman Parametric Integrals. We have seen that all IBP-identities in momentum space can be conveniently formulated by the use of the operators A_i and B_i as defined in equation (3.1.9). In the following let us consider the case where all scalar products in the problem are algebraically reducible (i. e. they can be expressed in terms of the propagators), such that the operators A_i and B_j raise and lower the exponents of inverse propagators (and not of scalar products). We mentioned in subsection 3.1.1 that this is not a severe restriction, as integrals with algebraically irreducible scalar products in the numerator of the integrand can be rewritten in terms of integrals where the numerator is one and where the space-time dimension D is changed (see the method of Tarasov in reference [**Tar96**]).

Each such scalar Feynman integral can be written in terms of Feynman parameters as

$$I_G = \int d^D k_1 ... d^D k_L \frac{1}{i^L \pi^{LD/2}} \frac{1}{\prod_{j=1}^N (-P_j)^{\nu_j}} = \int_0^1 d^N x j_G(\nu_1, ..., \nu_N) \tag{3.2.1}$$

with

$$j_G(\nu_1, ..., \nu_N) = \frac{\Gamma(\nu - LD/2)}{\prod_{j=1}^N \Gamma(\nu_j)} \left(\prod_{j=1}^N x_j^{\nu_j - 1} \right) \delta \left(1 - \sum_{i=1}^N x_i \right) \frac{\mathcal{U}_G^{\nu-(L+1)D/2}}{\mathcal{F}_G^{\nu-LD/2}}.$$

and $\nu = \sum_{j=1}^N \nu_j$, $d^N x = \prod_{k=1}^N dx_k$. Here we have assigned the index G to the integral, the integrand and the two polynomials $\mathcal{U}$ and $\mathcal{F}$. G denotes the underlying Feynman

[3] The implementation FIRE [**Smi08**] makes practical use of Lee's ideas.

[4] Integration-by-parts was applied to Feynman parametric integrals e. g. in [**BDK94**]. Unlike the method given there, we do not obtain identities from differentiation with respect to Feynman parameters but instead just re-write the above IBP-identities, obtained from momentum space representation.

graph, from which these functions are determined. We denote the jth edge of G by e_j. Each inverse propagator P_j and each Feynman parameter x_j is assigned to an edge e_j.

Let us observe how the operators A_i and B_i act on the Feynman parametric integrals. A_i simply increases ν_i in the above equation by one if $\nu_i \geq 1$. (If $\nu_i = 0$, the IBP-identities obtained from this integral will not contain A_i.) The exponents of the graph polynomials $\mathcal{U}_G$ and $\mathcal{F}_G$ are increased by A_i, but the polynomials themselves remain unchanged. B_i decreases ν_i in the equation, if $\nu_i > 1$. In the case of $\nu_i = 1$ the operator B_i cancels the inverse propagator P_i. This is equivalent to the contraction of the corresponding edge in the graph. G/e_i shall denote the graph which we obtain from the graph G by contraction of the edge[5] e_i. The inverse propagator P_i and the Feynman parameter x_i are assigned to e_i. In the Feynman parametric representation I_{G/e_i} is obviously not obtained from I_G just by lowering an exponent. I_G and I_{G/e_i} differ furthermore by the number of integration parameters and by their polynomials. Therefore we see that for a formulation in the Feynman parametric representation we have to treat the case of $\nu_i = 1$ separately.

We have

$$\begin{aligned} (A_i I_G)(\nu_1, ..., \nu_N) &= \nu_i I_G(\nu_1, ..., \nu_i + 1, ..., \nu_N) \text{ for } \nu_i > 0, \\ (B_i I_G)(\nu_1, ..., \nu_N) &= \begin{cases} I_G(\nu_1, ..., \nu_i - 1, ..., \nu_N) \text{ for } \nu_i > 1, \\ I_{G/e_i}(\nu_1, ..., \nu_{i-1}, \nu_{i+1}, ..., \nu_N) \text{ for } \nu_i = 1. \end{cases} \end{aligned} \tag{3.2.2}$$

For the moment let us restrict to the case, where just exponents are raised or lowered, i. e. where no B_j acts on an integral with $\nu_j = 1$. For this case it makes sense to introduce operators $\tilde{A}_i$ and $\tilde{B}_j$ acting on the integrand of I_G as follows:

$$\begin{aligned} (A_i I_G)(\nu_1, ..., \nu_N) &= \int_0^1 d^N x \tilde{A}_i j_G(\nu_1, ..., \nu_N) \text{ for } \nu_i > 0, \\ (B_i I_G)(\nu_1, ..., \nu_N) &= \int_0^1 d^N x \tilde{B}_i j_G(\nu_1, ..., \nu_N) \text{ for } \nu_i > 1. \end{aligned}$$

We have

$$\begin{aligned} &\tilde{A}_i j_G(\nu_1, ..., \nu_N) = j_G(\nu_1, ..., \nu_i + 1, ..., \nu_N) \\ &= \frac{\nu_i \Gamma(\nu + 1 - LD/2)}{(\nu_i)! \prod_{\substack{j=1 \\ j \neq i}}^N \Gamma(\nu_j)} x_i \left(\prod_{j=1}^N x_j^{\nu_j - 1} \right) \delta\left(1 - \sum_{i=1}^N x_i \right) \frac{\mathcal{U}_G^{\nu+1-(L+1)D/2}}{\mathcal{F}_G^{\nu+1-LD/2}} \\ &= x_i (\nu - LD/2) \frac{\mathcal{U}_G}{\mathcal{F}_G} j_G(\nu_1, ..., \nu_N) \end{aligned} \tag{3.2.3}$$

and similarly

$$\begin{aligned} \tilde{B}_i j_G(\nu_1, ..., \nu_N) &= j_G(\nu_1, ..., \nu_i - 1, ..., \nu_N) \\ &= \frac{\nu_i}{x_i} \frac{1}{\nu - LD/2} \frac{\mathcal{F}_G}{\mathcal{U}_G} j_G(\nu_1, ..., \nu_N). \end{aligned} \tag{3.2.4}$$

According to equation (3.1.10) the lowering operators B_i in IBP-identities always appear in combination with raising operators, as in $A_i B_j$. In this combination the obvious cancellations lead to

$$\begin{aligned} \tilde{A}_i \tilde{B}_j j_G(\nu_1, ..., \nu_N) &= j_G(\nu_1, ..., \nu_i + 1, ..., \nu_j - 1, ..., \nu_N) \\ &= \frac{x_i \nu_j}{x_j} j_G(\nu_1, ..., \nu_N). \end{aligned} \tag{3.2.5}$$

[5] A contraction of e_i is achieved by joining the vertices at the ends of e_i and subsequently removing e_i. In chapter 4 we will discuss the contraction of edges in more detail.

In the following we apply these equations to provide a form of the IBP-identities in terms of Feynman parameters and $\mathcal{U}$ and $\mathcal{F}$.

3.2.2. Generic Equations for One-Loop Sub-graphs. Let us return to an important aspect in the derivation of the triangle rule (equation (3.1.12)). We have seen, that the rule is a recurrence relation for a graph which contains a triangle as in figure 3.1.3 as a sub-graph. For example let us consider the two-loop graph of figure (3.1.1). It contains the triangle of figure (3.1.3) as a sub-graph. There are two edges (labelled with q_1 and q_2) which are external edges of the triangle and internal edges of the two-loop graph. The key point in the derivation of the triangle rule was a cancellation of the inverse propagators of these two edges. The example of the triangle rule shows that it can be useful to consider the basic IBP-identities corresponding to a one-loop sub-graph (the triangle) in order to find a useful recurrence relation for the entire graph (the two-loop graph). Therefore let us in the following give a formulation for the IBP-identities corresponding to one-loop sub-graphs in terms of Feynman parameters[6].

Let G be a Feynman graph with at least one loop. Let G_i be the sub-graph of G which consists of the edges whose momenta depend on the loop-momentum k_i, of all vertices at the ends of these edges and of all further edges, which these vertices are incident to. The latter edges are external with respect to G_i, but each of them may be either external or internal with respect to G. We already mentioned an example: If G is the two-loop graph of figure (3.1.1) a one-loop sub-graph G_i is the triangle of figure (3.1.3). An external edge of G_i is either external or internal with respect to G.

Each basic IBP-identity (obtained from equation (3.1.8)) of the Feynman integral of G is derived by differentiating with respect to just one loop momentum. Hence for the derivation of a basic IBP-identity obtained from $\frac{\partial}{\partial k_i^\mu} q^\mu$ it is useful to focus on the sub-graph G_i. Let the external edges of G_i have assigned the momenta p_1, p_2, ... The squares of these momenta appear in the IBP-identities derived from $\frac{\partial}{\partial k_i^\mu} q^\mu$.

If one p_i is external with respect to G we just keep it as a parameter. If it is internal with respect to G, it can be expressed in terms of external momenta and of inverse propagators of G. The latter propagators then yield raising and lowering operators on edges which belong to another loop of G. For example in the triangle rule of equation (3.1.12) the operators B_4 and B_5 lower powers of inverse propagators whose edges are external with respect to the triangle but internal with respect to a different loop.

In the following we will separately consider the structure of IBP-identities of massless one-loop graphs with external momenta p_1, p_2, ... Because of the arguments just mentioned, these identities then can be used to study the IBP-identities for a general L-loop graph G which contains the considered one-loop graph as a sub-graph.

At first we discuss the examples of the massless one-loop graphs with two and three edges and then we extend our observations to n edges.

Example 12. We consider the Feynman graph G on the left hand side of figure (3.2.1) where the numbers inside the loop stand for the indices i of the edges e_i, corresponding to inverse propagators P_i and Feynman parameters x_i. The exponents ν_i of the inverse propagators are assumed to be larger than one. The Feynman integral for this graph reads

$$I_G = \int \frac{d^D k_1}{i\pi^{D/2}} \frac{1}{(k^2)^{\nu_1} \left((k-p)^2\right)^{\nu_2}} = \int_0^1 dx_1 dx_2 j_G(\nu_1, \nu_2)$$

[6]We want to mention, that the IBP-identities of one-loop integrals are of course studied in detail in the literature. A systematic method for the reduction of possibly tensor-like one-loop integrals with an arbitrary number of external edges was given in **[GG04, GGZ04]**.

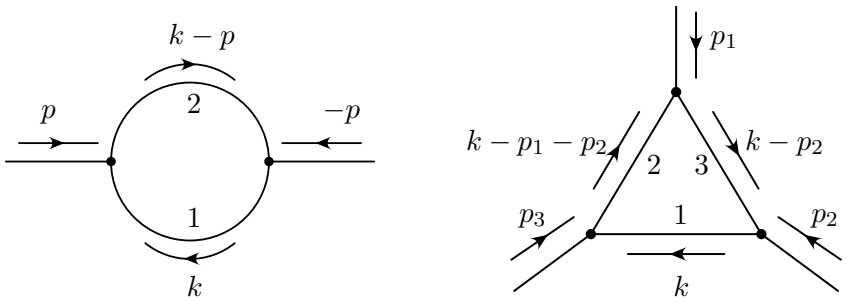


FIGURE 3.2.1. One-loop selfenergy graph and triangle graph

with

$$\begin{aligned} j_G(\nu_1, \nu_2) &= \frac{\Gamma(\nu_1+\nu_2-D/2)}{\nu_1!\nu_2!} x_1^{\nu_1-1} x_2^{\nu_2-1} \delta(1-x_1-x_2) \frac{\mathcal{U}_G^{\nu_1+\nu_2-D}}{\mathcal{F}_G^{\nu_1+\nu_2-D/2}}, \\ \mathcal{U}_G &= x_1+x_2, \\ \mathcal{F}_G &= x_1 x_2 \left(-p^2\right). \end{aligned}$$

The operators for the derivation of the basic IBP-identities as in equation (3.1.8) are $\frac{\partial}{\partial k^\mu} k^\mu$ and $p^\mu \frac{\partial}{\partial k^\mu}$. Instead of the latter operator let us consider $\frac{\partial}{\partial k^\mu}(k^\mu - p^\mu)$. From $\frac{\partial}{\partial k^\mu} k^\mu$ we obtain

$$(D-2\nu_1-\nu_2) I_G = \left(A_2 B_1 + \left(-p^2\right) A_2\right) I_G$$

and from $\frac{\partial}{\partial k^\mu}(k^\mu - p^\mu)$ we obtain

$$(D-\nu_1-2\nu_2) I_G = \left(A_1 B_2 + \left(-p^2\right) A_1\right) I_G.$$

Due to the given assumption that the ν_i are larger than one, we can write these equations in terms of operators $\tilde{A}_i$, $\tilde{B}_j$ which act on the Feynman parametric integrand $\tilde{j}_G(\nu_1, \nu_2)$ as in the equations (3.2.3) and (3.2.5). We obtain

$$\begin{aligned} (D-2\nu_1-\nu_2)\int_0^1 dx j_G &= \int_0^1 d^N x \left(x_2 \frac{\nu_1}{x_1} + \left(-p^2\right) x_2 (\nu - D/2) \frac{\mathcal{U}_G}{\mathcal{F}_G}\right) j_G, \\ (D-\nu_1-2\nu_2)\int_0^1 dx j_G &= \int_0^1 d^N x \left(x_1 \frac{\nu_2}{x_2} + \left(-p^2\right) x_1 (\nu - D/2) \frac{\mathcal{U}_G}{\mathcal{F}_G}\right) j_G. \end{aligned}$$

We can write the terms in parentheses on the right-hand side as

$$\begin{aligned} x_2 \frac{\nu_1}{x_1} + \left(-p^2\right) x_2 (\nu - D/2) \frac{\mathcal{U}_G}{\mathcal{F}_G} &= \mathcal{U}_G|_{x_1=0} \frac{\nu_1}{x_1} + \frac{\partial \mathcal{F}_G}{\partial x_1} (\nu - D/2) \frac{\mathcal{U}_G}{\mathcal{F}_G}, \\ x_1 \frac{\nu_2}{x_2} + \left(-p^2\right) x_1 (\nu - D/2) \frac{\mathcal{U}_G}{\mathcal{F}_G} &= \mathcal{U}_G|_{x_2=0} \frac{\nu_2}{x_2} + \frac{\partial \mathcal{F}_G}{\partial x_2} (\nu - D/2) \frac{\mathcal{U}_G}{\mathcal{F}_G}. \end{aligned}$$

We will observe that the IBP-identities for general massless one-loop graphs can be written in a similar form. But first let us see yet one more example.

EXAMPLE 13. We consider G to be the massless triangle graph on the right-hand side of figure (3.2.1), again assuming ν_1, ν_2, ν_3 to be greater than one. We apply the

operators $\frac{\partial}{\partial k^\mu} v_i^\mu$ with $v_1^\mu = k^\mu$, $v_2^\mu = k^\mu - p_1^\mu - p_2^\mu$, $v_3^\mu = k^\mu - p_2^\mu$ and obtain in this order the identities

$$\begin{aligned}
(D - 2\nu_1 - \nu_2 - \nu_3)\, I_G &= \left(A_3 B_1 + A_2 B_1 + \left(-p_2^2\right) A_3 + \left(-p_3^2\right) A_2\right) I_G, \\
(D - \nu_1 - 2\nu_2 - \nu_3)\, I_G &= \left(A_1 B_2 + A_3 B_2 + \left(-p_3^2\right) A_1 + \left(-p_1^2\right) A_3\right) I_G, \\
(D - \nu_1 - \nu_2 - 2\nu_3)\, I_G &= \left(A_1 B_3 + A_2 B_3 + \left(-p_2^2\right) A_1 + \left(-p_1^2\right) A_2\right) I_G.
\end{aligned}$$

The translation to Feynman parameters yields

$$\begin{aligned}
(D - 2\nu_1 - \nu_2 - \nu_3)\, I_G &= \int_0^1 d^N x \left(\mathcal{U}_G|_{x_1=0} \frac{\nu_1}{x_1} + \frac{\partial \mathcal{F}_G}{\partial x_1} (\nu - D/2) \frac{\mathcal{U}_G}{\mathcal{F}_G} \right) j_G, \\
(D - \nu_1 - 2\nu_2 - \nu_3)\, I_G &= \int_0^1 d^N x \left(\mathcal{U}_G|_{x_2=0} \frac{\nu_2}{x_2} + \frac{\partial \mathcal{F}_G}{\partial x_2} (\nu - D/2) \frac{\mathcal{U}_G}{\mathcal{F}_G} \right) j_G, \\
(D - \nu_1 - \nu_2 - 2\nu_3)\, I_G &= \int_0^1 d^N x \left(\mathcal{U}_G|_{x_3=0} \frac{\nu_3}{x_3} + \frac{\partial \mathcal{F}_G}{\partial x_3} (\nu - D/2) \frac{\mathcal{U}_G}{\mathcal{F}_G} \right) j_G.
\end{aligned}$$

3.2.3. The Polynomial Structure of IBP-Identities. We see that in the case of the two examples above a basic set of linearly independent IBP-identities can be obtained by a very simple formula, which in Feynman parameters can be written in a compact way by the use of $\mathcal{U}$ and $\mathcal{F}$:

If I_G is the Feynman integral of either example 12 or example 13, the basic IBP-identities (as defined by equation (3.1.8)) can be written as

$$(D - \nu - \nu_i)\, I_G = \int_0^1 d^N x \left(\mathcal{U}_G|_{x_i=0} \frac{\nu_i}{x_i} + \frac{\partial \mathcal{F}_G}{\partial x_i} (\nu - D/2) \frac{\mathcal{U}_G}{\mathcal{F}_G} \right) j_G, \tag{3.2.6}$$

where i runs through the indices of the internal edges. This notation always assumes all $\nu_i > 1$. In the case of some ν_i equal to 1 we would have to use a slightly more complicated notation according to the last case in equation (3.2.2).

Now we want to show, that equation (3.2.6) remains true if I_G is an arbitrary massless one-loop graph, where the exponents of the inverse propagators are larger than 1:

PROPOSITION 14. *Let I_G be a massless one-loop Feynman integral with*

$$\begin{aligned}
I_G &= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{\left((k - p_1)^2\right)^{\nu_1} \left((k - p_1 - p_2)^2\right)^{\nu_2} \ldots (k^2)^{\nu_N}} \\
&= \int d^N x j_G
\end{aligned}$$

with arbitrary N, j_G being the integrand in Feynman parametric representation and with all the ν_i larger than one. Then I_G satisfies the identities

$$(D - \nu - \nu_i)\, I_G = \int_0^1 d^N x \left(\mathcal{U}_G|_{x_i=0} \frac{\nu_i}{x_i} + \frac{\partial \mathcal{F}_G}{\partial x_i} (\nu - D/2) \frac{\mathcal{U}_G}{\mathcal{F}_G} \right) j_G.$$

PROOF. G is a one-loop graph with N internal edges e_1, e_2, ..., e_N. At each of the N vertices we have an incoming external momentum[7] p_i^μ. The loop-momentum is k^μ and to each internal edge e_i is assigned an internal momentum

$$V_i^\mu = k^\mu - \sum_{j=1}^{i} p_j^\mu$$

[7]The external momentum at one vertex may be the sum of the momenta which are assigned to more than one external edge at this vertex.

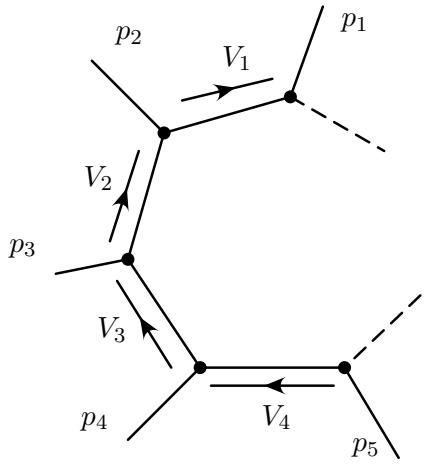


FIGURE 3.2.2. One-loop graph with N internal edges

as shown in figure (3.2.2). (Note that by momentum conservation we have $V_N^\mu = k^\mu - \sum_{j=1}^N p_j^\mu = k^\mu$.) The inverse propagator of the edge e_i is $V_i^\mu V_{i\,\mu} = V_i^2$. We have the Feynman integral

$$\begin{aligned} I_G &= \int \frac{d^D k}{i\pi^{D/2}} \tilde{j}_G \\ &= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{\left((k-p_1)^2\right)^{\nu_1} \left((k-p_1-p_2)^2\right)^{\nu_2} ... (k^2)^{\nu_N}} \\ &= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{\left(V_1^2\right)^{\nu_1} \left(V_2^2\right)^{\nu_2} ... \left(V_N^2\right)^{\nu_N}}. \end{aligned}$$

and we consider the N linearly independent IBP-identities from the operators $\frac{\partial}{\partial k^\mu} V_i^\mu$ for $i = 1, ..., N$, acting on the integrand $\tilde{j}_G$. Note that $\tilde{j}_G$ denotes the integrand in the momentum space representation. We derive

$$\begin{aligned} \frac{\partial}{\partial k^\mu} V_i^\mu j_G &= \tilde{j}_G \left(\frac{\partial}{\partial k^\mu} V_i^\mu + \sum_{j=1}^N \left(V_j^2\right)^{\nu_j} V_i^\mu \frac{\partial}{\partial k^\mu} \left(V_j^2\right)^{-\nu_j} \right) \\ &= \tilde{j}_G \left(D + \sum_{j=1}^N \nu_j \frac{-2 V_i^\mu V_{j\,\mu}}{V_j^2} \right) \\ &= \tilde{j}_G \left(D + \sum_{j=1}^N \nu_j \frac{(V_i - V_j)^2 - V_i^2 - V_j^2}{V_j^2} \right) \\ &= \tilde{j}_G \left(D - \underbrace{\sum_{j=1}^N \nu_j}_{=:\nu} - \nu_i + \sum_{\substack{j=1 \\ j \neq i}}^N \nu_j \frac{(V_i - V_j)^2 - V_i^2}{V_j^2} \right). \end{aligned}$$

The inverse propagators V_i^2 and V_j^2 can be considered as resulting from the action of raising and lowering operators respectively, which leads to

$$(D-\nu-\nu_i)\,I_G \;=\; \sum_{\substack{j=1\\ j\neq i}}^{N} \Big(-(V_i-V_j)^2\,A_j + A_j B_i\Big)\,I_G. \tag{3.2.7}$$

These are IBP-identities of I_G in the momentum space representation.
Now let j_G be the integrand of the same integral in the Feynman parametric representation:

$$I_G \;=\; \int d^N x\, j_G.$$

Then, due to the assumption $\nu_i > 1$ for $i = 1, 2, ..., N$, equation (3.2.7) translates to

$$(D-\nu-\nu_i)\,I_G =$$

$$\int d^N x \sum_{\substack{j=1\\ j\neq i}}^{N} \left(-(V_i-V_j)^2\,x_j\,(\nu - D/2)\,\frac{\mathcal{U}_G}{\mathcal{F}_G} + \nu_i \frac{x_j}{x_i}\right) j_G. \tag{3.2.8}$$

$V_i - V_j$ is the incoming external momentum of a sub-graph of G, which is obtained as one of the connected components after deleting e_i and e_j. Hence the term $-(V_i - V_j)^2$ is the coefficient of $x_i x_j$ in the polynomial $\mathcal{F}_G$. For this reason and the fact that $\mathcal{U}_G = \sum_{j=1}^{N} x_j$, we see that the equations (3.2.8) and (3.2.6) are the same. □

For the sake of brevity we have restricted the discussions of the present section mainly to the case where all exponents ν_i are larger than one. The treatment of the case where some of the exponents are equal to one would be more complicated. In equation (3.2.2) we have seen, that the lowering of an exponent which is equal to one corresponds to the contraction of an edge. Therefore the corresponding IBP-identities would contain the polynomials $\mathcal{U}$ and $\mathcal{F}$ of the graph G and furthermore of graphs which are obtained from G by the contraction of an edge. Such an IBP-identity would obviously take a less compact form than equation (3.2.6) above and we dispense with a more detailed consideration of this case.

3.2.4. A Concluding Remark on IBP-Identities. In this chapter we gave a brief review on IBP-identities in the momentum space representation. We furthermore formulated a simple translation to the Feynman parametric representation and derived the convenient equation (3.2.6) for one-loop Feynman integrals I_G with the powers ν_j of the inverse propagators being larger than one. We explained, that the consideration of such a one-loop graph can be helpful to study the IBP-identities of larger graphs, which contain the one-loop graph as a sub-graph. An example was given by the well-known triangle rule.

IBP-identities provide a powerful and widely used approach to write Feynman integrals in terms of simpler ones. For an arbitrary Feynman integral the Laporta algorithm combines IBP-identities to a reduction in terms of master integrals. Publicly available implementations of the Laporta algorithm are the computer programs AIR [**AL04**] and FIRE [**Smi08**]. We also mentioned alternative reduction mechanisms which are not applicable to the general case. Focusing on the Laporta algorithm we mentioned, that the

algorithm generates far more identities than necessary for the reduction and that one usually does not know beforehand, which integrals will be the master integrals of a reduction. A solution of these two problems, particularly of the first one, would be an important step towards a more efficient evaluation of Feynman integrals. Recent developments towards a diminishment of the number of identities have been made in the dissertation of Toedtli [**Toe09**] and in a more abstract way in the work of Lee [**Lee08**]. Very recently, a new, efficient implementation of the Laporta algorithm was used by Bonciani, Ferroglia, Gehrmann and Studerus in [**BFGS09**]. The article refers to a future publication of this new computer program by Studerus, which is currently in preparation.

Moreover, let us remark that IBP-identities are used in an approach by Remiddi and collaborators [**CCLR98, GR00, Rem97**] (generalizing an idea of Kotikov [**Kot91**]) for the evaluation of master integrals. In this approach Feynman integrals are considered as solutions of certain differential equations, which are usually constructed by the use of IBP-identities. To our understanding, as it lies in the nature of a master integral to have relatively few propagators, the problem of handling a very large set of equations typically appears in the earlier step of the reduction to master integrals, but not in the evaluation of the master integral itself.

In the remainder of our dissertation we will not make explicit use of IBP-identities. Nevertheless, for some of the following considerations it will be useful to keep in mind, that by use of IBP-identities, sets of infinitely many Feynman graphs can be expressed by finitely many master integrals. For example let M be a set of infinitely many Feynman integrals where each integral can be reduced to a finite set of master integrals $\{I_1, ..., I_n\}$. Then if we evaluate the master integrals $I_1, ..., I_n$, we can in principle use the IBP-identities to evaluate any other integral of M. More generally, if the master integrals fulfill a certain property, one might find, by the use of IBP-identities, that each integral of M fulfills the same property. For example if all the master integrals are linear combinations of a certain class of functions, say polylogarithms (see chapter 6), then the same must be true for each integral in M.

The use of the polynomials $\mathcal{U}$ and $\mathcal{F}$ in equation (3.2.6) may furthermore give a reason for the detailed study of these polynomials which we aim at in the following chapter.

CHAPTER 4

Feynman Graph Polynomials

"The limits of my language mean the limits of my world."
(Ludwig Wittgenstein)

The main purpose of the present chapter is to demonstrate, how the language of algebraic graph theory can be applied to describe the combinatorial structure of Feynman integrals. Feynman integrals are most accessible to the terminology of graph theory in their Feynman parametric representation (see equation (2.3.6)). The two so-called Symanzik polynomials arising from this representation are our main objects of interest.

It appears to us that relevant new theorems and a more convenient notation have been established in graph theory since Nakanishi's comprehensive book on the subject from 1971 [**Nak71**]. At the same time, in mathematical physics the interest in Symanzik polynomials has grown, particularly within the past few years due to studies of the zero-sets of these polynomials from the point of view of algebraic geometry[1]. Likewise, in the evaluation of multi-loop integrals, Symanzik polynomials play a key role, as successful calculational methods like sector decomposition (see chapter 5 and references given there) start from parametric representations of Feynman integrals. However, there are powerful techniques which do not rely on using the most characteristic properties of Symanzik polynomials discussed in this chapter. On the other hand, very recently, a new kind of algorithm using these properties to a large extent was presented by Brown [**Bro09**] and serves for the evaluation of a special class of integrals where the results are expressed by multiple zeta values. These recent developments on both, the graph theoretical and the physical side, motivate the recapitulation of Symanzik polynomials which we attempt to give with this chapter.

A second aim is motivated by our observation that in the literature very often only one of the two Symanzik polynomials is discussed, due to its property of being a so-called Kirchhoff polynomial (see section 4.1.3 below). This polynomial will be called the *first* Symanzik polynomial and the other one will be called the *second* one (precise definitions given below). In terms of Feynman graphs, ignoring the second Symanzik polynomial corresponds to the restriction to vacuum graphs. A relation between the first and the second Symanzik polynomial is well known, and therefore in some restricted approaches of this kind the extension to the general case might be simple. Nevertheless we want to bring the second Symanzik polynomial (and Feynman graphs with external legs) closer to the combinatorial treatments. To this end we state a new relation involving both of the Symanzik polynomials. Our proof of this relation uses the all-minors-matrix-tree-theorem [**Cha82, Che82, Moo94**].

In the end of the chapter we point out the relation between Symanzik polynomials and the multivariate Tutte polynomial. To our understanding the latter object receives a lot of attention in today's research on topics of abstract combinatorics and is related to

[1] See [**Blo07, Mar09**] for introductions and e. g. [**And08, AM08a, AM08b, AM09b, AM09a, BB03a, BEK06, Mar08, MR08**] for further details.

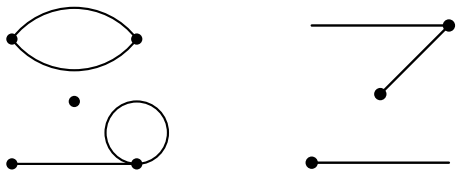


FIGURE 4.1.1. (a) A graph. (b) Lines and dots, but not a graph.

a variety of subjects in mathematics and physics alike (see e. g. [**KRTW08, Sok05**]). A deeper study of its possible use for the calculation of Feynman integrals might be fruitful in the future.

4.1. Graphs, Feynman Graphs and Associated Matrices

4.1.1. Basic Notions of Graph Theory. Attempting to give a self-consistent introduction, we start with the definition of some rudimentary objects of standard graph theory. (For more detailed introductions, see e. g. [**Bol98, Die05, Nak71**]).

For a set H we denote the number of its elements by $|H|$. A set is called *minimal* with respect to a certain property, if it fulfills the property, but any proper subset does not.

A *graph* G consists of a set of edges $E_G = \{e_1, e_2, ..., e_n\}$, a set of vertices $V_G = \{v_1, v_2, ..., v_m\}$ and a map ϕ_G from E_G to pairs of vertices in V_G. ϕ_G is called *incidence relation* and the vertices $v_i, v_j \in V_G$ are said to be incident to the edge $e_l \in E_G$, if

$$\phi_G(e_l) = \{v_i, v_j\}. \tag{4.1.1}$$

The pair of vertices on the right-hand side is not ordered and the case $i = j$ is allowed. v_i, v_j are then said to be *end-points* of e_l. In the typical pictorial representation of a graph $G = (E_G, V_G, \phi_G)$, vertices (dots) are drawn at the ends of the edges (lines) to which they are incident.

According to the definition, a graph may contain vertices, which are not incident to an edge. On the other hand, an edge where one or two ends have no incident vertex is not allowed. Two examples are depicted in figure (4.1.1), a graph in (a) and a collection of lines and dots which does not represent a graph in (b). As a direct consequence of the definition, if a graph has no vertices, it can not have any edges. We may call the unique graph given by $V_G = \emptyset$ the *empty graph* (but we will not make use of this notion in the following).

For some purposes we may introduce an orientation on a graph by replacing ϕ_G by a map from E_G to *ordered* pairs of vertices,

$$\tilde{\phi}_G(e_l) = (v_i, v_j), \tag{4.1.2}$$

with (v_i, v_j) now being an ordered set. Then $G = \left(E_G, V_G, \tilde{\phi}_G\right)$ is said to be an *oriented graph*. In equation (4.1.2) we call v_i the *head* and v_j the *tail* of e_l. In an oriented graph a vertex is a *root*, if it is not a tail of any edge. An oriented graph can have more than one root.

In addition to the notion of edges being incident to vertices it is also useful to have a word expressing that two vertices are end-points of the same edge. Any two vertices of V_G which are incident to the same edge of E_G are said to be *adjacent*. The information which is encoded by the map ϕ_G can alternatively be expressed in terms of incidence

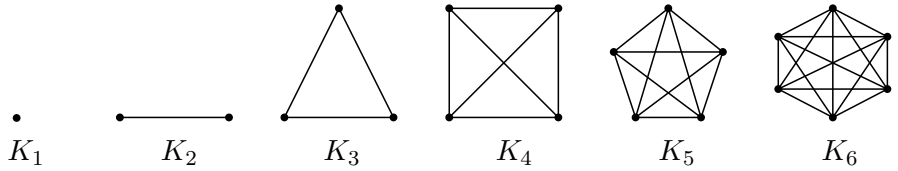


FIGURE 4.1.2. The complete graphs K_n for $n = 1, ..., 6$.

or adjacency. We will see below how to use both concepts to encode the information in terms of matrices.

A *sub-graph* G' of a graph G is a graph with $E_{G'} \subseteq E_G$, $V_{G'} \subseteq V_G$ and with $\phi_{G'}$ being the restriction of ϕ_G to $E_{G'}$. We can always obtain a sub-graph by deleting edges and vertices in the original graph, with the condition that the remaining edges still must have end-points at both of their ends. A sub-graph G' of G is a *spanning* sub-graph of G, if G and G' have the same vertices.

For convenience, let us give names to some special kinds of graphs. If a vertex v_i is incident to itself, meaning that there is an edge e_l which satisfies $\phi_G(e_l) = \{v_i, v_i\}$, we call the graph consisting of v_i and e_l a *self-loop*. If for two vertices v_i, v_j the relation $\phi_G(e_l) = \{v_i, v_j\}$ is fulfilled by more than one edge, we call the graph consisting of v_i, v_j and all those edges a *multiple edge*. If two edges fulfill the relation, we may speak of a *double edge*. If a vertex is not an end-point of any edge we call it *isolated*. For example, the graph in figure (4.1.1) (a) has a self-loop, a double edge and an isolated vertex as sub-graphs.

Now let us consider a graph which has n vertices, no self-loops, no multiple edges and where every vertex is adjacent to each of the other vertices. For each n there is one unique graph with these properties, called the *complete* graph with n vertices, denoted by K_n. As an example, the complete graphs with less than seven vertices are shown in figure 4.1.2. Any graph G' without self-loops and multiple edges and with $|V_{G'}| \leq n$ can be obtained as a sub-graph of K_n.

The self-loop, the double edge and the complete graph with 3 vertices K_3 are special cases of so-called *loops*. Before we give a general definition of a loop it is useful to introduce the degree of a vertex and the concept of connectedness. For a vertex v_i let $S[v_i]$ denote the set of edges incident to v_i and $L[v_i] \subset S[v_i]$ the set of edges which are incident to v_i at both of their ends. Then the *degree* of v_i is defined by the sum of the numbers of elements in these sets:

$$\deg(v_i) = |S[v_i]| + |L[v_i]|. \tag{4.1.3}$$

In other words, v_i sits at $\deg(v_i)$ ends of edges. More generally, for a subset of edges $H \subset E_G$ we define the *relative degree* of a vertex v_i to be

$$\deg(v_i, H) = |S[v_i] \cap H| + |L[v_i] \cap H|. \tag{4.1.4}$$

EXAMPLE 15. As an example let us consider the graph G of figure 4.1.3, consisting of the vertices $V_G = \{v_1, v_2, v_3, v_4\}$ and the edges $E_G = \{e_1, e_2, e_3, e_4\}$. None of the edges is incident to the same vertex at bot of its ends (i. e. we have no self-loops). Therefore the sets $L[v_i]$ are empty and we have $\deg(v_i) = |S[v_i]|$ for all $i = 1, ..., 4$. v_1 and v_2 are end-points of two edges, v_3 is the end-point of three edges and v_4 of one edge. Therefore we obtain $\deg(v_1) = 2$, $\deg(v_2) = 2$, $\deg(v_3) = 3$ and $\deg(v_4) = 1$. Now let us consider the sub-graph G' of G, consisting of the vertices $V_{G'} = \{v_1, v_2, v_3\}$ and the edges $E_{G'} = \{e_1, e_2, e_3\}$. (This sub-graph is the complete graph K_3, cf. figure 4.1.2). We may be interested in the relative degree of the vertices with respect to the set $E_{G'}$. According

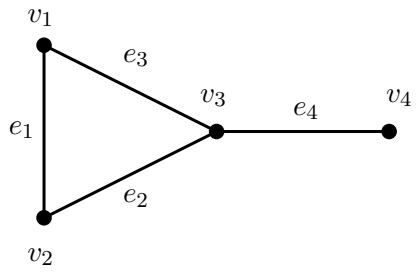


FIGURE 4.1.3. A graph G, considered in example 15.

to equation (4.1.4) we obtain $\deg(v_1, E_{G'}) = 2$, $\deg(v_2, E_{G'}) = 2$, $\deg(v_3, E_{G'}) = 2$ and $\deg(v_4, E_{G'}) = 0$.

Two vertices v_i and v_j are called *connected*, if there is a sequence of vertices $v_i = u_0, u_1, ..., u_k = v_j$ such that u_h and u_{h+1} are adjacent for $h = 1, 2, ..., k-1$. The relation "*is connected to*" is an equivalence relation, decomposing the set V_G into equivalence classes. By definition, no edge is incident to two vertices which are not connected. Hence also the edges are decomposed into corresponding equivalence classes by this relation. A sub-graph of G consisting of an equivalence class of V_G with respect to being connected and the corresponding edges is called a *connected component* of G. A graph is called connected, if all its vertices are connected. A connected graph has exactly one connected component. Let us denote the number of connected components of a graph G by κ_G. For example the graph in figure 4.1.1 (a) consists of three connected components.

This terminology allows for a convenient definition of loops:

DEFINITION 16. A sub-graph G' of a graph G is a *loop* if G' is connected and $\deg(v_i, E_{G'}) = 2$ for each $v_i \in V_{G'}$.

For instance the sub-graph G' in the above example 15 is a loop. Every loop can be depicted by drawing a circle and distributing n vertices on it. For example with $n = 1, 2, 3$ we obtain the self-loop, the double edge and K_3 respectively. The above definition yields exactly the graphs which are known as loops in the physics literature[2].

On the other hand it is important to notice, that the number of loops in a graph is in general not what physics literature means by the *loop-number*. For a connected graph G we define the *loop-number* $\imath_G$, in agreement with physics literature, to be

$$\imath_G = |E_G| - |V_G| + \kappa_G, \tag{4.1.5}$$

where κ_G is the number of connected components of G. We remark that in some references the loop-number $\imath_G$ is also called the *cyclomatic number*.

To give an example, the graph in figure (4.1.4) has three sub-graphs which are loops, but its loop-number is two. It is important to notice this difference. A corresponding Feynman graph of this topology would be said to be a two-loop graph. In the context of Feynman integrals the loop-number is the number of integration momenta $L = \imath_G$, as for example in equation (2.3.1). We remark that equation (4.1.5), defining $\imath_G$, and equation (2.3.3), defining L, are the same equations, if the considered graph is connected[3]. This is

[2]Graph theorists instead often use the word 'loop' only for what we defined to be a self-loop, which is in turn known as 'tadpole' to physicists.

[3]Note that equation (2.3.3) was formulated in the description of Feynman graphs, where external edges are half-edges and external vertices, as defined in the present chapter, do not exist.

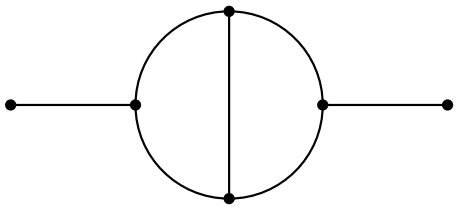


FIGURE 4.1.4. The two-loop bubble has three loops as sub-graphs and loop-number $\ell_G = 2$.

the case for Feynman graphs throughout this dissertation. A general graph instead may have more than one connected component.

In the following we want to use the terms *loop* and *loop-number* as we just defined them, in agreement with the terminology of the physics literature. We want to avoid a possible confusion between the loop-number ℓ_G and the number of sub-graphs, which are loops. Therefore we will explicitely use the symbol ℓ_G (or in the case of Feynman integrals the letter L) whenever we refer to the loop-number.

As we mentioned, all edges of a graph are incident to vertices at both ends, as in figure (4.1.4). We do not allow for loose ends. Nevertheless, we distinguish between *external* and *internal edges* as follows: We define a vertex v_i with $\deg(v_i) \leq 1$ to be an *external vertex*[4] and we call an edge incident to an external vertex an *external edge*. All other vertices and edges are said to be *internal*. Let us write V_G^{ext}, V_G^{int}, E_G^{ext}, E_G^{int} for the set of external vertices, internal vertices, external edges and internal edges respectively. A sub-graph $\tilde{G}$ of a graph G is called the *core* of G, if it is obtained from G by deleting all external edges and vertices. In the above example 15, G has only one external vertex v_4 and one external edge e_4 and the sub-graph G' is the core of G.

Note that in many descriptions of Feynman graphs, an additional type of edge is introduced for the external edges. These are edges with a vertex at only one of the ends, called half-edge or flag. We have used such half-edges for the external edges in the figures of the previous chapters, but we do not want to use this concept here. It is sufficient and convenient for the following treatment to use the standard edges of graphs as defined above, with a vertex at both ends. Of course, the half-edges of the other notation can be translated one-to-one to our external edges of the present chapter, because in both descriptions of Feynman graphs there is nothing like an internal edge which is incident to a vertex of degree one.

A graph which has no loop as a sub-graph is called a *forest*. If a forest has k connected components we speak of a k-forest. A 1-forest is called a *tree*. A *path* is a tree with at least one edge and whose vertices are all of degree not greater than 2. We will sometimes consider forests and trees which are spanning sub-graphs of a considered graph and which we will therefore call spanning forests and spanning trees respectively[5].

For Feynman integrals it is sufficient to restrict to the consideration of connected graphs, which do not fall into pieces after deletion of just one edge. We call an edge $e_j \in E_G$ a *cut-edge* of G if G' has more connected components than G, where G' is

[4]We note that according to this definition an isolated vertex is external. However, isolated vertices are not important in our consideration of Feynman graphs below and we could as well have called them internal.

[5]Note that in the literature very often 'forest of G' (or 'tree of G') already means a spanning sub-graph of G. For us instead a forest of G (or a tree of G) is a sub-graph of G which is not necessarily spanning G. We will explicitely write 'spanning' where we mean it.

obtained from G by deleting only e_j. We call a graph *one-edge-irreducible* (or one-particle-irreducible according to the physics literature) if it is connected and has no cut-edge.

The step from graphs to Feynman graphs will be done in the next subsection by adding more properties to graphs and we will do so by attaching variables to the edges. In general we say that a set H *labels* a set H' if $|H| = |H'|$ and if there is a one-to-one correspondence between the elements of H and H'. (For example, in chapter 2 we labelled the internal edges with a set of Feynman parameters.) A graph with labelled edges or vertices is called a *labelled graph*. Feynman graphs in the following sections will be considered to be labelled graphs with some additional properties. The sets of labels will comprise properties like particle momenta, their masses etc.

4.1.2. Scalar Feynman Graphs and Polynomials. In the physics literature, Feynman graphs are usually introduced in the framework of quantum field theory, instead of being defined in their own right. They have a unique purpose in physics, namely the construction of the associated Feynman integral. Depending on the underlying physical theory, the Feynman graphs need to contain a certain amount of information which is incorporated into the Feynman integrals in a defined way, given by the Feynman rules.

In order to build a bridge from pure graph theory to Feynman integrals, let us start from the following definition:

DEFINITION 17. An *abstract Feynman graph* G_F is a connected, oriented graph, none of whose vertices is of degree two and with a finite number of sets H_1, H_2, ..., H_n, each of them labelling the edges.

The physical information, in particular momentum conservation, still needs to be imposed in terms of properties of the sets of labels H_1, H_2, ..., H_n. In order to define such sets explicitely let us restrict to a special class of Feynman integrals to be constructed and let us accordingly impose properties on the sets of labels. As we have seen in chapter 2, every dimensionally regularized Feynman integral can be decomposed to a series of tensors with scalar-valued integrals as coefficients [**Tar96, ?**]. These scalar integrals can be interpreted as Feynman integrals as well, constructable from Feynman rules of scalar toy-models. We call the corresponding kind of Feynman graph *scalar*.

The scalar integrals are of the form

$$I(D, \Lambda) = \int \prod_{i=1}^{L} \frac{d^D k_i}{i\pi^{D/2}} \prod_{j=1}^{N} \frac{1}{\left(-q_j^2 + m_j^2\right)^{\nu_j}}, \tag{4.1.6}$$

with L, N being natural numbers, Λ the set of masses and external momenta and D a complex number (cf. equations (2.3.1) and (3.2.1)). The terms $\frac{1}{\left(-q_j^2+m_j^2\right)^{\nu_j}}$ are scalar propagators, raised to integer powers ν_j. m_1, ..., m_N are real valued parameters representing the particle masses. k_1, ..., k_L, q_1, ..., q_N are Lorentz-vectors, of which the q are linear combinations of the k_i and possibly a number of external vectors p_1, ..., p_J, $J \in \mathbb{N}$. These linear combinations are due to momentum conservation, which is conveniently defined on graphs.

Let us consider the class of Feynman graphs which is sufficient for the construction of the integrals of equation (4.1.6). Let G_F be an (abstract) Feynman graph, with N internal and S external edges and loop-number $\mathfrak{l}_{G_F} = L$. Let us for simplicity number the edges such that e_1, ..., e_N are internal and e_{N+1}, ..., e_{N+S} are external. Furthermore, in order to simplify the notation, we assign to each label the number of the corresponding edge as an index, such that for example a mass m_i is associated to the edge e_i for each

$i = 1, ..., N + S$. Let the edges of G_F be labelled by the following three sets:

$$\begin{aligned} H_1 &= \left\{ m_1, m_2, ..., m_N, \underbrace{0, ..., 0}_{\text{labeling external edges}} \right\}, \\ H_2 &= \left\{ \nu_1, \nu_2, ..., \nu_N, \underbrace{0, ..., 0}_{\text{labeling external edges}} \right\}, \\ H_3 &= \{q_1 = k_1, ..., q_L = k_L, q_{L+1}, ..., q_N, \\ &\quad q_{N+1} = p_{N+1 N+1}, ..., q_{N+S} = p_{N+S}\}, \end{aligned} \tag{4.1.7}$$

where the labelling is chosen such that there are $\imath_{G_F} = L$ loops in the graph, each of which has at least one edge labelled by one of the variables k_i. (We remind ourselves, that $\imath_{G_F}$ is the loop-number, defined by equation (4.1.5), and that the number of sub-graphs of G_F which are loops might be larger than $\imath_{G_F}$.)

To complete the construction, we have to impose momentum conservation. For a sub-graph G' of G and edges $e_i \in E_G$ let us define the auxiliary function α by

$$\alpha(G', e_i) = \begin{cases} 0 \text{ if the endpoints of } e_i \text{ are both } \in G' \text{ or both } \notin G', \\ 1 \text{ if a } v \in V_{G'} \text{ is the tail and a } u \notin V_{G'} \text{ is the head of } e_i, \\ -1 \text{ if a } v \in V_{G'} \text{ is the head and a } u \notin V_{G'} \text{ is the tail of } e_i. \end{cases}$$

Momentum conservation requires that for each *internal* vertex $v_j \in V_{G_F}$ the equation

$$\sum_{e_i \text{ incident to } v_j} \alpha(v_j, e_i) q_i = 0 \tag{4.1.8}$$

must be fulfilled by the momenta q_i. The orientation of the graph represents the directions of the momenta q_i and by equation (4.1.8) we impose the condition, that the sum of incoming momenta at each internal vertex must be zero. This condition determines $q_{L+1}, ..., q_N$ as linear combinations of the momenta k_i and p_i. Imposing the property of momentum conservation completes the construction of the scalar Feynman graph G_F. The labelled graph G_F together with the imposed properties of the labels provides the sufficient information for the construction of scalar Feynman integrals as in equation (4.1.6).

A complication of the representation of Feynman integrals by equation (4.1.6) is the presence of Lorentz-vectors. This can be avoided by the change to Feynman parameters (cf. chapter 2). In the Feynman parametric representation, the scalar Feynman integrals of equation (4.1.6) read

$$I(D, \Lambda) = f_{G_F} \int_0^1 \left(\prod_{e_i \in E_{\tilde{G}_F}} dx_i x_i^{\nu_i} \right) \delta \left(1 - \sum_{e_i \in E_{\tilde{G}_F}} x_i \right) \frac{\left(\mathcal{U}_{G_F}\right)^{\sum_{e_i \in E_{\tilde{G}_F}} \nu_i - (L+1)D/2}}{\left(\mathcal{F}_{G_F}\right)^{\sum_{e_i \in E_{\tilde{G}_F}} \nu_i - LD/2}}, \tag{4.1.9}$$

with

$$f_{G_F} = \frac{\Gamma\left(\sum_{e_i \in E_{\tilde{G}_F}} \nu_i - LD/2\right)}{\prod_{e_i \in E_{\tilde{G}_F}} \Gamma(\nu_i)}.$$

We have assigned the numbers of the edges to the corresponding variables as indices. $\tilde{G}_F$ is the core of G_F and $L = \mathfrak{l}_{G_F}$ is the loop-number.

The polynomial functions $\mathcal{U}_{G_F}$, $\mathcal{F}_{G_F}$ are defined as follows (cf. chapter 2 and **[Wei06]**). Let $\mathcal{T}_1\left(\tilde{G}_F\right)$ be the set of spanning trees and $\mathcal{T}_2\left(\tilde{G}_F\right)$ the set of spanning 2-forests of $\tilde{G}_F$. We write $(T_i, T_j) \in \mathcal{T}_2\left(\tilde{G}_F\right)$, where T_i, T_j are the two connected components of the 2-forest (T_i, T_j). For a sub-graph G' of a graph G let $\mathcal{C}(G', G)$ be the set of edges of G not belonging to G'. For each two-forest $(T_k, T_l) \in \mathcal{T}_2(\tilde{G}_F)$ there is a *kinematical invariant*

$$s_{(T_k,T_l)} = \left(\sum_{e_j \in \mathcal{C}((T_k,T_l),\tilde{G}_F)} \alpha(T_k, e_j) q_j\right)^2. \tag{4.1.10}$$

By the use of momentum conservation, each $s_{(T_k,T_l)}$ can be expressed in terms of scalar products of only the *external* momenta p_i. Let us denote this set of scalar products by P.

The polynomials in equation (4.1.9) are defined as

$$\mathcal{U}_{G_F}(x_1, ..., x_N) = \sum_{T \in \mathcal{T}_1(\tilde{G}_F)} \prod_{e_i \in \mathcal{C}(T,\tilde{G}_F)} x_i, \tag{4.1.11}$$

$$\mathcal{F}_{0,G_F}(x_1, ..., x_N, P) = \sum_{(T_k,T_l)\in\mathcal{T}_2(\tilde{G}_F)} \left(\prod_{e_i \in \mathcal{C}((T_k,T_l),\tilde{G}_F)} x_i\right)(-1)s_{(T_k,T_l)}. \tag{4.1.12}$$

$$\mathcal{F}_{G_F} = \mathcal{F}_{0,G_F} + \mathcal{U}_{G_F} \sum_{e_i \in \tilde{G}_F} x_i m_i^2. \tag{4.1.13}$$

We have already given these definitions by equations (2.3.7), (2.3.8) and (2.3.9) in chapter 2. All integration variables and functions in equation (4.1.9) are scalar valued. Therefore it should be possible to obtain the Feynman integral in equation (4.1.9) without explicit reference to Lorentz vectors as variables. In fact, it is possible to define equation (4.1.9) based on a set of formal scalar variables. This is done as follows. We introduce the following simple changes on the labels of G_F:

- The set of vectors H_3 is replaced by a set of formal real valued scalar variables q'_i :

$$H'_3 = \left\{q'_1, ..., q'_N, q'_{N+1} = p'_{N+1}, ..., q'_{N+S} = p'_{N+S}\right\}. \tag{4.1.14}$$

- The restriction due to momentum conservation can be imposed by the condition

$$\sum_{e_i \text{ incident to } v_j} \alpha(v_j, e_i) q'_i = 0 \tag{4.1.15}$$

  for each internal vertex v_j.
- We introduce an additional set of real valued labels

$$H_4 = \left\{x_1, x_2, ..., x_N, \underbrace{0, ..., 0}_{\text{labeling external edges}}\right\},$$

  the Feynman parameters.

Now let us define

$$s'_{(T_k,\,T_l)} = \left(\sum_{e_j \in \mathcal{C}((T_k,T_l),\,\tilde{G}_F)} \alpha(T_k,\,e_j) q'_j \right)^2 ,$$

where $\tilde{G}_F$ is the core of G_F and let us use the equations (4.1.15) to express these functions in terms of products of the scalar variables $p' \in H'_3$, which are the labels of external edges. We obtain

$$s'_{(T_k,\,T_l)} = \left(\sum_{p'_i \in E^{\text{incident}}_{G_F}(T_k)} p'_i \right) \left(\sum_{p'_j \in E^{\text{incident}}_{G_F}(T_l)} p'_j \right) \tag{4.1.16}$$

where $E^{\text{incident}}_{G_F}(T_k)$ denotes the set of external edges of G_F which are incident to T_k.

Let us denote the set of products of the variables p'_i by P'. There is an obvious one-to-one correspondence between P' and P and we obtain the above functions $s_{(T_k,\,T_l)}$ by the formal replacements

$$s_{(T_k,\,T_l)} \;=\; s'_{(T_k,\,T_l)}\Big|_{p'_i p'_j = p_i p_j} \quad \text{for all } p'_i\, p'_j \in P'. \tag{4.1.17}$$

Now we have all ingredients to obtain the above polynomials and the integral (4.1.9) from the Feynman graph G_F in the obvious way. In the following discussions we will use real valued variables as in H'_3 for the momenta and functions $s'_{(T_k,\,T_l)}$ for the kinematical invariants, omitting the primes of these objects in our notation, as we expect no confusion to arise.

Let us briefly summarize what we have done in the present subsection. First we have repeated the arguments allowing for the restriction to scalar integrals. We defined Feynman graphs to be connected, oriented, labelled graphs without vertices of degree two. A class of Feynman graphs G_F was equipped with labels such that a scalar momentum-space integral of equation (4.1.6) can be obtained from a graph of this class in the described way. Then, by minor changes, we obtained the class of Feynman graphs which is suitable for a direct construction of the Feynman parametric representation (4.1.9). The main advantage of a restriction to scalar valued variables will be obvious in a later subsection, where we consider these labels to be elements of certain matrices.

4.1.3. Symanzik and Kirchhoff Polynomials. Let G be an oriented, connected graph labelled by the formal scalar variables introduced above. The number of internal edges is N. In addition to the Feynman parameters x_1, x_2, ..., x_N we label the internal edges also by auxiliary parameters y_1, y_2, ..., y_N, for a reason which will be obvious immediately (i. e. in equations (4.1.18) and (4.1.19)). We remind ourselves that the core of G is denoted by $\tilde{G}$.

Definition 18. The Symanzik polynomials of G are

$$\begin{aligned} U_G\,(y_1,\,...,\,y_N) \;&=\; \sum_{T \in \mathcal{T}_1(\tilde{G})} \prod_{e_i \in T} y_i, \\ F_{0,\,G}\,(y_1,\,...,\,y_N,\,p_{N+1},\,...,\,p_{N+S}) \;&=\; \sum_{(T_k,\,T_l) \in \mathcal{T}_2(\tilde{G})} \left(\prod_{e_i \in (T_k,\,T_l)} y_i \right) (-1) s_{(T_k,\,T_l)}. \end{aligned}$$

We refer to U_G as the *first* and to $F_{0,\,G}$ as the *second* Symanzik polynomial of G.

The second Symanzik polynomial $F_{0,G}$ and its coefficients $s_{(T_k,T_l)}$ as functions of the variables p_i are defined via the variables q_1, ..., q_{N+S} fulfilling the momentum conservation condition as in equation (4.1.15). Note that the orientation of the graph and momentum conservation are not needed to define the first Symanzik polynomial. We can define $U_G(y_1, ..., y_N)$ in the above way for any connected, not necessarily oriented graph.

We have defined the Symanzik polynomials such that the terms of the sums are monomials in the variables y_i, whose edges belong to the corresponding forest. The polynomials $\mathcal{U}_G$ and $\mathcal{F}_{0,G}$ instead were defined by the edges which do *not* belong in the corresponding forest. We obtain the simple relations

$$\mathcal{U}_G(x_1, ..., x_N) = U_G(y_1, ..., y_N)\big|_{y_i=\frac{1}{x_i}\forall e_i\in E_{\tilde{G}}} \cdot \prod_{e_i\in E_{\tilde{G}}} x_i, \tag{4.1.18}$$

$$\mathcal{F}_{0,G}(x_1, ..., x_N, P) = F_{0,G}(y_1, ..., y_N, P)\big|_{y_i=\frac{1}{x_i}\forall e_i\in E_{\tilde{G}}} \cdot \prod_{e_i\in E_{\tilde{G}}} x_i. \tag{4.1.19}$$

We note, that in the literature sometimes $\mathcal{U}_G$ and $\mathcal{F}_{0,G}$ are referred to as Symanzik polynomials. In the remainder of this chapter it will be convenient to consider U_G and $F_{0,G}$ instead, keeping in mind, that these polynomials are closely related to $\mathcal{U}_G$ and $\mathcal{F}_{0,G}$ by the simple equations (4.1.18) and (4.1.19). Relations like (4.1.18) and (4.1.19) play a central role in the context of dual graphs as discussed in subsection 4.1.6 below.

Definition 19. The *Kirchhoff polynomial* of G is

$$\mathcal{K}_G(y_1, ..., y_{|E_G|}) = \sum_{T\in\mathcal{T}_1(G)} \prod_{e_i\in T} y_i.$$

Note that in contrast to U_G, the terms in the Kirchhoff polynomial contain also the labels of external edges. The first Symanzik polynomial U_G of a graph G is in fact the Kirchhoff polynomial of the core of G:

$$U_G = \mathcal{K}_{\tilde{G}}.$$

Remark 20. Apart from the well known appearance in the Feynman parametric representation of Feynman integrals, the Kirchhoff polynomial is regarded a lot in the literature. It appears in the context of electric networks [**Che82, Kir47**], of statistical mechanics (Potts-model) [**Sok05**] and in a series of articles on a conjecture of Kontsevich about its zeroes [**BB03a, Kon97, Sta98, Ste98**]. The Kirchhoff polynomial and its zero-set have recently become crucial links from perturbative quantum field theory to certain topics of algebraic geometry, like the theory of motives (see e. g. [**AM08b, AM08a, AM09b, And08, BB03a, BEK06, Mar08**]). A very important property of the Kirchhoff polynomial, and hence also of U_G, is its relation to determinants of certain matrices, given by so-called matrix-tree-theorems which we review below. It is one purpose of the present chapter to prove a relation between the polynomial F_{0,G_F} and such matrices.

Note that each of the external edges belongs to all the spanning trees of a graph. If G has external edges, meaning that the number of edges $|E_G|$ is larger than the number of internal edges N, then $\mathcal{K}_G$ is different from U_G, but we can easily use $\mathcal{K}_G$ to obtain $\mathcal{U}_G$ by

$$\mathcal{U}_G(x_1, ..., x_N) = \mathcal{K}_G(y_1, ..., y_{|E_G|})\big|_{y_i=\frac{1}{x_i}\forall e_i\in E_G} \cdot \prod_{e_i\in E_G} x_i.$$

In this equation all the labels y_i from external edges are removed by multiplication with the corresponding label x_i. (Note the difference to equation (4.1.18) where the product of the variables x_i ran through $e_i \in E_{\tilde{G}}$.)

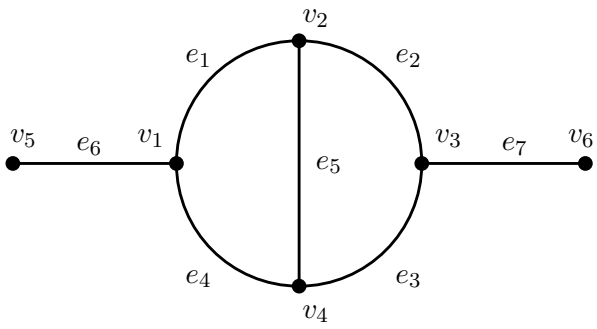


FIGURE 4.1.5. A two-loop graph.

4.1.4. Associated Matrices. From here on we want to assume that all graphs under consideration have no self-loops, unless we explicitely allow for them. We just want to remark that in principle, many of the objects and results to be discussed in the following could be extended to the case of graphs with self-loops (or would even remain unchanged in this case). Nevertheless, for us it is more desirable to present the main ideas in a rather simple, consistent notation, which would be disturbed by repeated distinction between the cases with and without self-loops.

Let us consider a graph G with n vertices and m edges. We define the *incidence matrix* $B_G = (B_{ij})_{n\times m}$ of G as the $n \times m$-matrix with the components

$$B_{ij} := \begin{cases} 1 \text{ if } v_i \in e_j, \\ \\ 0 \text{ otherwise.} \end{cases} \tag{4.1.20}$$

Note that unlike the incidence matrix of an oriented graph, $B^\star$, defined in equation (2.3.10), the matrix B does not contain information about a possible orientation of the edges.

We define the *adjacency matrix* $A_G = (A_{ij})_{n\times n}$ of G as the $n \times n$-matrix with the components

$$A_{ij} := \sum_{\{e_k \in E_G | e_k \text{ incident to } v_i, v_j\}} 1. \tag{4.1.21}$$

The matrix element A_{ij} is equal to the number of edges in E_G incident to both distinct vertices v_i and v_j.

Let us furthermore define the matrix D_G as the diagonal matrix $(D_{ij})_{n\times n}$ with D_{ii} being the degree of $v_i \in V_G$ and $D_{ij} = 0$ for $i \neq j$:

$$D_{ij} := \begin{cases} \deg(v_i) \text{ if } i = j, \\ \\ 0 \text{ otherwise.} \end{cases} \tag{4.1.22}$$

Moreover, we have (cf. **[Die05]**, Proposition 1.9.8)

$$B_G B_G^T = D_G + A_G.$$

The matrix $D_G - A_G$ is independent of the orientation of the graph G, it is symmetric, positive semi-definite and it is called the *Laplacian* of G:

DEFINITION 21. Let G be a graph. The *Laplacian* of G is defined to be

$$L_G = D_G - A_G. \tag{4.1.23}$$

EXAMPLE 22. For the graph G of figure (4.1.5), the matrices A_G, D_G and L_G are 6×6-matrices. The rows and columns from top to bottom and from left to right respectively correspond to the vertices from v_1 to v_6. With this ordering we obtain

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \; L_G = \begin{pmatrix} 3 & -1 & 0 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & -1 & 3 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Let M be any symmetric matrix with rows and columns labelled by the set of vertices V_G and let $S_1 \subseteq V_G$ and $S_2 \subseteq V_G$. Then we denote by $M\left[(S_1,\, S_2)\right]$ the sub-matrix of M obtained by deleting the rows with indices in S_1 and columns with indices in S_2. If $S_1 = S_2$ we use the abbreviation $M\left[S_1\right] \equiv M\left[(S_1,\, S_1)\right]$. Choosing one arbitrary vertex v_i of G and deleting the corresponding row and column in L_G we obtain an $(n-1)\times(n-1)$-matrix $L_G[v_i]$, called *reduced Laplacian* of G. For a Laplacian L_G, we can obviously construct $|V_G|$ reduced Laplacians. All of these $|V_G|$ reduced Laplacians have the same determinant, as we will see below.

For certain problems in graph theory it is useful to have a generalized version of the Laplacian. In **[GR01d]** Godsil and Royle define a generalized Laplacian of a graph G with n vertices to be a symmetric $n\times n$-matrix $M = (M_{ij})$ where $M_{ij} < 0$ if v_i and v_j are adjacent vertices of G and $M_{ij} = 0$ if v_i and v_j are distinct (i. e. $i \neq j$) and not adjacent. There are no constraints on the diagonal elements of the matrix. To give an example, the matrix $-A_G$ is a generalized Laplacian of G.

Let us consider a labelled graph G with a set $Y = \left\{y_1,\, ...,\, y_{|E_G|}\right\}$ of formal scalar variables labelling the edges. Using these variables we define a matrix, which for positive values of the y_i is a generalized Laplacian in the above sense of Godsil and Royle:

DEFINITION 23. (**[Sta98, Tut84]**) The *generic Laplacian (or Kirchhoff matrix)* $\mathcal{L}_G(Y) = (\mathcal{L}_{ij}(Y))$ of G is the $n \times n$-matrix defined by

$$\mathcal{L}_{ij}(Y) = \begin{cases} \sum_{\{e_k \in E_G | e_k \text{ incident to } v_i,\, v_j\}} (-y_k) & \text{if } i \neq j, \\ \\ \sum_{\{e_k \in E_G | e_k \text{ incident to } v_i\}} y_k & \text{if } i = j, \end{cases}$$

for all v_i, $v_j \in V_G$. A sum over the empty set shall be zero.

EXAMPLE 24. Let G be the graph of figure (4.1.5). Then the generic Laplacian reads

$$\mathcal{L}_G(Y) = \begin{pmatrix} y_1+y_4+y_6 & -y_1 & 0 & -y_4 & -y_6 & 0 \\ -y_1 & y_1+y_2+y_5 & -y_2 & -y_5 & 0 & 0 \\ 0 & -y_2 & y_2+y_3+y_7 & -y_3 & 0 & -y_7 \\ -y_4 & -y_5 & -y_3 & y_3+y_4+y_5 & 0 & 0 \\ -y_6 & 0 & 0 & 0 & y_6 & 0 \\ 0 & 0 & -y_7 & 0 & 0 & y_7 \end{pmatrix}.$$

We can formulate a relation for $\mathcal{L}_G$ which is the analogue of equation (4.1.23) for L_G. To this end we define the *generic adjacency matrix* $\mathcal{A}_G = (\mathcal{A}_{ij})_{n\times n}$ of G by

$$\mathcal{A}_{ij} := \sum_{e_k = v_i v_j \in E} y_k,$$

and furthermore the matrix $(\mathcal{D}_{ij})_{n\times n}$ with $\mathcal{D}_{ii} = \sum_{e_k \in v_i} y_k$ and $\mathcal{D}_{ij} = 0$ for $i \neq j$. With these definitions we obtain

$$\mathcal{L}_G = \mathcal{D}_G - \mathcal{A}_G,$$

where we suppressed the dependence on the set Y in our notation. Note that the three matrices $\mathcal{L}_G$, $\mathcal{D}_G$, $\mathcal{A}_G$ yield the classical matrices L_G, D_G, A_G by setting all parameters $y_k = 1$.

The generic Laplacian where the row and column corresponding to some vertex v_i are removed is denoted $\mathcal{L}_G(Y)[v_i]$ and is sometimes called the *reduced* generic Laplacian. Some authors refer to the generic Laplacian as the *Kirchhoff matrix* (cf. [**Tut84**]). The matrix and the related matrix-tree-theorem which will be discussed below stems from Kirchhoff's work on electric networks (see [**Kir47**]). In this context, the labels y_i would be interpreted as conductivities. Before we come to Kirchhoff's important theorem and it's extensions, let us first introduce two standard operations on a graph.

4.1.5. Deletion and Contraction. We define deletion and contraction of edges of a graph G and describe their effect on the generic Laplacian. For a subset of edges $E'_G \subseteq E_G$ we define $G\backslash E'_G$ to be the graph consisting of the same vertices as G (i. e. $V_{G\backslash E'_G} = V_G$) and of the set of edges of G without the edges in E'_G (i. e. $E_{G\backslash E'_G} = E_G\backslash E'_G$). The incidence relation $\phi_{G\backslash E'_G}$ is the restriction of ϕ_G to $E_G\backslash E'_G$. We say $G\backslash E'_G$ is obtained from G by *deletion* of E'_G. $G\backslash E'_G$ is always a sub-graph of G.

For a subset of vertices $V'_G \subset V_G$ and one vertex $v_i \in V'_G$ we define G/V'_G to be the graph where the vertices V'_G are removed from V_G, except for v_i (i. e. $V_{G/V'_G} = V_G\backslash (V'_G\backslash v_i)$) and with the same edges as in G ($E_{G/V'_G} = E_G$). The incidence relation is $\phi_{G/V'_G}(e_k) = \phi_G(e_k)|_{v_j = v_i \,\forall v_j \in V'_G}$. We say G/V'_G is obtained from G by *identification of the vertices* in V'_G.

If we want to consider the deletion of only one edge $e_k \in E_G$ we will write $G\backslash e_k \equiv G\backslash\{e_k\}$. Furthermore we define G/e_k to be the graph obtained from G by identifying the end-points of e_k and subsequently deleting e_k. We say G/e_k is obtained from G by *contraction* of e_k. The deletion and the contraction of an edge are two important operations in graph theory. In the following, we want to study the changes in the Laplacian under these operations.

Let us assume a graph G with edges labelled by Y and an edge $e_k \in E_G$ with $\phi_G(e_k) = (v_i, v_j)$ such that none of the graphs G, $G\backslash e_k$, G/e_k has a self-adjacent vertex. Then we consider the corresponding generic Laplacians and obtain the following relations. For the deletion we obtain, rather obviously:

$$\mathcal{L}_{G\backslash e_k} \quad = \quad \mathcal{L}_G|_{y_k = 0}\,. \tag{4.1.24}$$

If we reduce $\mathcal{L}_{G\backslash e_k}$ by the column and line corresponding to the remaining vertex v_i of e_k and if we define $\mathcal{E}^{(i,k)}$ as the $n \times n$-matrix with $\mathcal{E}^{(i,k)}_{ii} = y_k$ and all other entries zero, then we can write for the reduced Laplacian

$$\mathcal{L}_{G\backslash e_k}[v_i] = \mathcal{L}_G - \mathcal{E}^{(i,k)}. \tag{4.1.25}$$

The construction of $\mathcal{L}_{G/e_k}$ is just a little less simple. For convenience we label the vertices such that $v_j = v_n$, i.e. the last line and column of $\mathcal{L}_G$ corresponds to v_n. Let

us, in addition, assume that no other edge than e_k is incident to both v_i and v_n, which means that the contraction of this edge does not lead to a self-loop. In the graph G/e_k, the edges which were incident to v_n in G will be additional incident edges to v_i, except for e_k itself. Hence starting from $\mathcal{L}_G$ we have to add the column j to the column i and delete column n and then do the same with the rows. Note that the y_k-term will cancel out in the diagonal term $\left(\mathcal{L}_{G/e_k}\right)_{ii}$ because of the different signs in the diagonal and off-diagonal terms of $\mathcal{L}_G$. Therefore the $(n-1)\times(n-1)$-matrix $\mathcal{L}_{G/e_k}$ is obtained as

$$\left(\mathcal{L}_{G/e_k}\right)_{ab} = \begin{cases} (\mathcal{L}_G)_{ab} \text{ if } a,\, b \notin \{i,\, n\}, \\ (\mathcal{L}_G)_{nb} + (\mathcal{L}_G)_{ib} \text{ if } a = i \text{ and } b \neq i, \\ (\mathcal{L}_G)_{an} + (\mathcal{L}_G)_{ai} \text{ if } b = i \text{ and } a \neq i, \\ (\mathcal{L}_G)_{ii} + (\mathcal{L}_G)_{ni} + (\mathcal{L}_G)_{in} + (\mathcal{L}_G)_{nn} \text{ if } a = b = i. \end{cases} \tag{4.1.26}$$

The only difference to $\mathcal{L}_G$ lies in the columns and lines numbered by i and n, the latter ones not existing anymore in $\mathcal{L}_{G/e_k}$. Hence, if we reduce $\mathcal{L}_{G/e_k}$ by the columns and lines i, we obtain a sub-matrix of $\mathcal{L}_G$ (cf. **[GR01d]**):

$$\mathcal{L}_{G/e_k}[v_i] = \mathcal{L}_G[\{v_i,\, v_n\}]. \tag{4.1.27}$$

For the classical Laplacians L_G, L_{G/e_k}, $L_{G\setminus e_k}$ we obtain relations similar to the equations (4.1.24), (4.1.25), (4.1.26), (4.1.27) by just setting all parameters $y_k = 1$.

Now, as we have discussed deletion and contraction, let us mention that the Kirchhoff polynomial fulfills a remarkable property with respect to these operations. Let e_i be an edge of G which is neither a cut-edge nor a self-loop. Then we have (see e.g. **[KRTW08]**)

$$\mathcal{K}_G = y_i \mathcal{K}_{G/e_i} + \mathcal{K}_{G\setminus e_i}. \tag{4.1.28}$$

This property is sometimes referred to as *deletion/contraction relation* (or algorithm), as one can use it to express the Kirchhoff polynomial of any graph as linear combinations of Kirchhoff polynomials of graphs G' which consist only of cut-edges and self-loops. The Kirchhoff polynomial of such a graph is in fact a monomial, namely

$$\mathcal{K}_{G'}(Y) = \prod_{e_i \text{ cut-edge of } G'} y_i.$$

In section 4.3 of this chapter, we will mention a deletion/contraction relation similar to equation (4.1.28) for a different, more general polynomial, the so-called Tutte polynomial.

4.1.6. Duality. In this subsection let us consider connected graphs which are allowed to have self-loops. We remember, that we defined the first Symanzik polynomial U_G such that it is the Kirchhoff polynomial of the core of G. The polynomial $\mathcal{U}_G$, which appears in the Feynman integral, is related to U_G, as we mentioned, by

$$\mathcal{U}_{G_F}(x_1,\, ...,\, x_N) = U_{G_F}(y_1,\, ...,\, y_N)|_{y_i = \frac{1}{x_i} \forall e_i \in E_{\tilde{G}_F}} \cdot \prod_{e_i \in E_{\tilde{G}_F}} x_i. \tag{4.1.29}$$

Now we may ask, whether $\mathcal{U}_G(x_1,\, ...,\, x_N)$ is as well a Kirchhoff polynomial of a graph. In the following we want to give an answer to this question for the case where G is a so-called *planar* graph. In this case, as we will see, we can associate a dual graph to G and then the polynomial $\mathcal{U}_G$ is the Kirchhoff polynomial of this dual graph[6].

[6]Instead, for graphs G which are *not* planar, we are not aware of a method to decide whether $\mathcal{U}_G$ is a Kirchhoff polynomial or not.

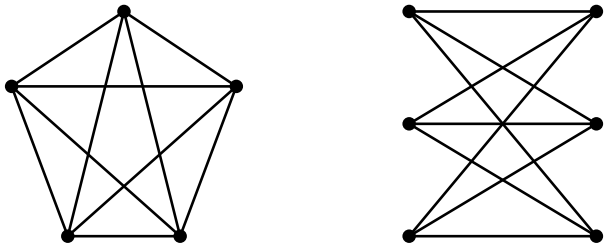


FIGURE 4.1.6. The 'smallest' non-planar graphs.

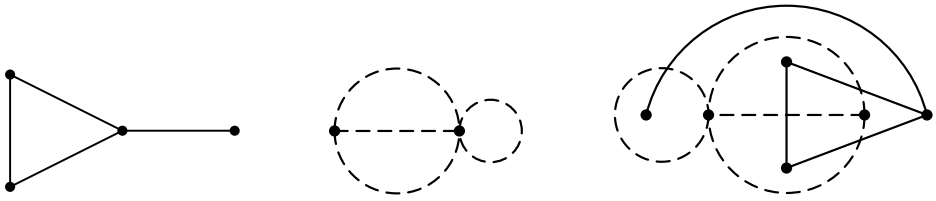


FIGURE 4.1.7. (a) A graph G. (b) The dual graph $G^\star$ of G. (c) The construction of $G^\star$ from G (or vice versa).

A graph is called *planar* if it can be embedded in a plane without crossings of edges. This is the case for all graphs we have discussed up to now in this dissertation, but for example for the graphs in figure (4.1.6) it is not the case. The graphs in figure (4.1.6), called K_5 (the complete 5-vertex graph) and $K_{3,3}$, can not be drawn on a piece of paper without crossings of edges. For complicated graphs it might be hard to find out, if such an embedding in the plane exists. For these cases the following theorem of Wagner[7] is useful:

THEOREM 25. *(Wagner* **[Wag37]***, cf.* **[Die05]***) A graph G is planar if and only if none of the graphs obtained from G by a (possibly empty) sequence of contractions of edges contains K_5 or $K_{3,3}$ as a sub-graph.*

In other words, for any non-planar graph G we either find K_5 or $K_{3,3}$ (or both) as a sub-graph, or we can obtain a graph G' from G by contractions of edges, such that G' contains K_5 or $K_{3,3}$ (or both) as a sub-graph. As a direct consequence, all proper sub-graphs of K_5 and $K_{3,3}$ and all graphs obtained from them by contractions must be planar. In this sense, K_5 and $K_{3,3}$ are the 'smallest' of all non-planar graphs.

Each planar graph G has a *dual graph* $G^\star$ which can be obtained as follows (for an example see figure (4.1.7)):

- Draw the graph G in a plane, such that no edges intersect. In this way, the graph divides the plane into open subsets, called *faces*.
- Draw a vertex inside each face. These are the vertices of $G^\star$.
- For each edge e_i of E_G draw a new edge between the two vertices of the faces, which are separated by e_i. The new edges are the edges of $G^\star$.

[7]Wagner's result is an analogue of an earlier theorem by Kuratowski **[Kur30]**.

The Kirchhoff polynomial fulfills the identity

$$\mathcal{K}_{G^\star}(x_1, ..., x_N) = \mathcal{K}_G(y_1, ..., y_N)|_{y_i=\frac{1}{x_i}\forall e_i\in E_G} \cdot \prod_{e_i\in E_G} x_i, \tag{4.1.30}$$

where N is the number of edges in G, which is as well the number of edges in $G^\star$.

We note from the construction of the dual graph, that for each external edge in G there is a self-loop in $G^\star$ and of course for each self-loop in G there is an external edge in $G^\star$. An external edge is contained in every spanning tree and a self-loop in no spanning tree. Therefore, if e_i is an external edge in G, we see that its label y_i is contained in each term of $\mathcal{K}_G$. The corresponding x_i labels a self-loop in $G^\star$ and is contained in no term of $\mathcal{K}_{G^\star}$. If instead e_i is a self-loop in G, then no term in $\mathcal{K}_G$ contains y_i but and each term in $\mathcal{K}_{G^\star}$ contains x_i. For these reasons, a comparison of equations (4.1.29) and (4.1.30) yields

$$\mathcal{U}_G(x_1, ..., x_N) = \mathcal{K}_{G^\star}(x_1, ..., x_N)$$

for planar graphs G. $\mathcal{U}_G$ is the Kirchhoff polynomial of the dual graph of the planar graph G.

Remark 26. It is important to note that the above construction of the dual graph $G^\star$ depends on the way, how G is drawn in the plane and therefore, the same graph G can have different dual graphs (for an example we refer to [**AM08b**]). These dual graphs have the same Kirchhoff polynomial. The fact that different graphs can have the same Kirchhoff polynomial is related to a theorem of Whitney on matroids of graphs [**Whi33**] (also see [**Oxl86**]).[8]

The results of Brown's algorithm [**Bro09**] for a certain class of Feynman graphs suggest, that planarity, or more precisely the crossing number, plays an important role for the arithmetic properties of the integrals. We refer to the theorems in [**Bro09**] for the detailed statements. Furthermore we refer to recent work of Aluffi and Marcolli [**AM08b**], where properties of certain associated motives are deduced by the detailed consideration of the dual graphs of certain Feynman graphs.

4.2. Symanzik Polynomials and Theorems of the Matrix-Tree-Type

In the present section, if not stated otherwise, we consider connected graphs, such that the corresponding Kirchhoff polynomials are well-defined.

4.2.1. Theorems of the Matrix-Tree-Type. The Laplacian plays a role in an important result on counting the spanning trees of a graph, which goes back to Kirchhoff [**Kir47**]:

Theorem 27. *(classical matrix-tree-theorem) Let G be a graph with Laplacian matrix L_G and let v_i be an arbitrary vertex of G. Then the determinant of the reduced classical Laplacian,* $\det(L_G[v_i])$*, is equal to the number of spanning trees of G.*

A first generalization of this theorem concerns the generic Laplacian:

[8] A matroid is a very general mathematical structure which can be used to encode the linear dependence of elements, for example of vectors, in a given set. The information encoded in a graph can be formulated in terms of matroids. We refer to [**Oxl03, Oxl06**] for detailed introductions to matroids. Whitney's theorem classifies graphs, such that the associated matroids of each class are isomorphic. The multivariate Tutte polynomial, to be discussed below, encodes the entire information of a corresponding matroid and on the other hand, it yields the Kirchhoff polynomial in a certain limit, as we will see.

THEOREM 28. *(generic matrix-tree-theorem)* (Tutte [**Tut84**], theorem VI.29) *Let G be a graph with N edges, all of them labelled by the set $Y = \{y_1, ..., y_N\}$ and let v_i be an arbitrary vertex of G. Let $\mathcal{L}_G(Y)$ be the generic Laplacian and $\mathcal{K}_G(Y)$ the Kirchhoff polynomial. Then we have*

$$\mathcal{K}_G(Y) = \det\left(\mathcal{L}_G(Y)[v_i]\right).$$

The Kirchhoff polynomial evaluated at $y_i = 1$ for all i, $\mathcal{K}_G(Y)|_{y_i=1 \forall i=1,...,N}$, is equal to the number of spanning trees of G. In this sense, we can consider the Kirchhoff polynomial as a generalization of the number of spanning trees.

Using the above result we can express the first Symanzik polynomial $U_G(Y)$ by the determinant of the reduced generic Laplacian, namely

$$U_G(Y) \quad = \quad \det\left(\mathcal{L}_{\tilde{G}}(Y)[v_i]\right) \tag{4.2.1}$$

where $\tilde{G}$ is the core of G and where it is understood, that U_G depends just on the labels of the internal edges.

By use of a further generalization, which we want to discuss in the following, we can relate both Symanzik polynomials to a generic Laplacian in one formula. The generalization is called the *all-minors-matrix-tree-theorem* and was stated in the compact form which we will refer to by Moon [**Moo94**]. It relies on previous developments by Chen [**Che82**] and Chaiken [**Cha82**].

Before we can state the theorem, we need to introduce some auxiliary objects. Let us consider a graph G with n vertices $V = \{v_1, v_2, ..., v_n\}$. It will be convenient to use an ordered set of integers $S_n = (1, 2, ..., n)$ to represent the vertices. Let $A = (a_1, ..., a_m)$ and $B = (b_1, ..., b_m)$ be non-empty ordered subsets of S_n. A and B shall have the same number of elements $m = |A| = |B|$, $m \leq n$, and the ordering of the elements in A and B shall be the same as in S_n. For example we might have $S_n = (1, 2, 3, 4)$, $A = (2, 3)$, $B = (2, 4)$. Let V_A be the subset of V with $v_i \in V_A$ if $i \in A$. V_B is defined the same way. For our example this means $V_A = \{v_2, v_3\}$, $V_B = \{v_2, v_4\}$.

Now for given n, A and B, let $\mathcal{F}_{G,A,B}$ be the set of all possible forests F with the following properties:

- F is a spanning sub-graph of G.
- F consists of m trees $T_1, T_2, ..., T_m$.
- Each tree of F has exactly one external vertex in V_A and exactly one vertex (internal or external) in V_B. (The vertex in V_A and the vertex in V_B may be the same.)

Each forest F of this type shall correspond to a term in a sum, as defined below. The sign of the term will depend on which of the vertices of V_A and V_B belong to the same tree in F. This dependence is encoded in the following way. For each forest $F \in \mathcal{F}_{G,A,B}$ we define a permutation map

$$\sigma_F \quad : \quad S_n \mapsto S^F$$

where the set of integers $S^F = (s_1, ..., s_n)$ is a permutation of the set S_n, such that

$$s_j = i$$

if $a_i \in V_A$ and $b_j \in V_B$ belong to the same tree in F. The number of transpositions in the permutation map σ_F is denoted by N_F. In this way, the forest F determines the number N_F.

In addition, let us define the auxiliary function

$$\epsilon_{A,B} \quad = \quad (-1)^{\sum_{i\in A} i + \sum_{j\in B} j}.$$

For an $n \times n$ matrix $\mathcal{L}$ we denote by $\mathcal{L}[A, B]$ the sub-matrix, which is obtained from $\mathcal{L}$ by deleting the ith rows and the jth columns for all $i \in A$, $j \in B$.

With these notations we state the all-minors-matrix-tree-theorem as follows:

THEOREM 29. *(all-minors-matrix-tree-theorem) (cf.* **[Moo94]***, Theorem 3.1) Let G be a graph and let $\mathcal{L}_G(Y)$ be its generic Laplacian. Then*

$$\det\left(\mathcal{L}_G(Y)[A, B]\right) = \epsilon(A, B) \sum_{F \in \mathcal{F}_{G, A, B}} (-1)^{N_F} \prod_{e_i \in E_F} y_i. \tag{4.2.2}$$

This formulation is obtained from Moon ([**Moo94**], Theorem 3.1) by some minor changes, which are mainly notational. In his version, Moon states the theorem for a matrix which does not necessarily need to be defined by a graph, but it is identical to the generic Laplacian $\mathcal{L}_{K_n}(Y)$ of the complete graph with n vertices. We obtain Moons formulation, if we set $G = K_n$ in the above theorem. Let us convince ourselves, that we are allowed to state the theorem for a general graph G with n vertices (still without self-loops), as we just did. Assume that equation (4.2.2) is true for $G = K_n$, as it was proven by Moon. Let G' be a graph obtained from K_n by deletion of an edge $e_i \in E_{K_n}$. Then the forests $\mathcal{F}_{G', A, B}$ are obtained from $\mathcal{F}_{K_n, A, B}$ by removing all forests containing e_i. On the right-hand side of equation (4.2.2) this has the same effect as setting $y_i = 0$. On the left hand side setting $y_i = 0$ in $\mathcal{L}_{K_n}$ gives $\mathcal{L}_{G'}$. Hence the equation remains true for G'.

Now let G'' be the graph obtained from K_n by adding an edge $\tilde{e}$ to the graph such that it is incident to the same vertices as $e_i \in E_{K_n}$. G'' then contains a double-edge. For each forest in $\mathcal{F}_{K_n, A, B}$ which contains e_i we have to add the same forest with e_i replaced by $\tilde{e}$. (Obviously there is no forest containing both of these edges.) In this way we obtain $\mathcal{F}_{G'', A, B}$. This replacement on the right-hand side of equation (4.2.2) corresponds to the replacement of y_i by $y_i + \tilde{y}$ on the right-hand side of equation (4.2.2), where $\tilde{y}$ is the label of $\tilde{e}$. On the left hand side the same replacement in $\mathcal{L}_{K_n}$ gives $\mathcal{L}_{G''}$ and we see, that the equation remains true.

All other graphs with n vertices can be obtained by repeated use of these two operations. Therefore the proof of the above theorem is given by the proof in [**Moo94**], which we do not repeat here[9].

In the following subsection we will only need a particular case of the theorem, namely the case where $A = B$. For this case the above defined permutation σ_F is always the identical map and therefore we have $N_F = 0$ and $(-1)^{N_F} = 1$ for every F. Furthermore we have $\epsilon(A, A) = 1$ and therefore equation (4.2.2) simplifies to

$$\det\left(\mathcal{L}_G(Y)[A, A]\right) = \sum_{F \in \mathcal{F}_{G, A, A}} \prod_{e_i \in E_F} y_i. \tag{4.2.3}$$

4.2.2. Relations between the Symanzik Polynomials. In the beginning of the chapter we have introduced Feynman graphs as graphs with certain labels and properties. Then we have derived the Symanzik polynomials from these labels and we have furthermore used the labels for the definition of the generic Laplacian. The generic-matrix-tree-theorem relates the generic Laplacian to the first Symanzik polynomial, as we have seen in equation (4.2.1). In the following we want to use the all-minors-matrix-tree-theorem to establish a relation between a generic Laplacian and both Symanzik polynomials.

[9]This would require to introduce more terminology, which will not be of help for the remainder of the chapter.

In the following let us consider a Feynman graph G_F with non-empty sets of internal and external edges. $N \neq 0$ is the number of internal and $S \neq 0$ the number of external edges. It will be convenient to use the sets of labels

$$\begin{aligned} Z &= \{z_1, ..., z_{N+S}\}, \\ &= Z^{\text{int}} \cup Z^{\text{ext}} \\ Z^{\text{int}} &= \{y_1, ..., y_N\}, \\ Z^{\text{ext}} &= \{p_{N+1}, ..., p_{N+S}\}, \end{aligned}$$

where Z^{int} labels the internal and Z^{ext} the external edges of G_F.

From the generic matrix-tree-theorem we know (cf. equation (4.2.1))

$$U_{G_F}\left(Z^{\text{int}}\right) = \det\left(\mathcal{L}_{\tilde{G}_F}\left(Z^{\text{int}}\right)[v_i]\right)$$

where $\mathcal{L}_{\tilde{G}_F}\left(Z^{\text{int}}\right)[v_i]$ is the matrix obtained from the generic Laplacian of the core $\tilde{G}$ by removing the row and the column corresponding to the vertex v_i, which can be arbitrarily chosen.

Now let us consider the matrix $\mathcal{L}_{G_F}(Z)\left[V_{G_F}^{\text{ext}}\right]$, which is obtained from the generic Laplacian of G_F by removing the rows and columns corresponding to all the external vertices of G_F. This matrix has as many rows and columns as G_F has internal vertices. Let n be the number of internal vertices. The elements of Z^{ext} appear only in the diagonal entries of the $n \times n$-matrix. More precisely, the matrix can be written as

$$\mathcal{L}_{G_F}(Z)\left[V_{G_F}^{\text{ext}}\right] = \mathcal{L}_{\tilde{G}_F}\left(Z^{\text{int}}\right) + \mathcal{Q}_{G_F}\left(Z^{\text{ext}}\right)$$

where $\mathcal{Q}_{G_F}$ is defined to be the diagonal $n \times n$-matrix with the entries

$$\mathcal{Q}_{ij}\left(Q^{\text{ext}}\right) = \begin{cases} \sum_{\{e_k \in E_{G_F}^{\text{ext}} | e_k \text{ incident to } v_i\}} p_k \text{ for } i = j, \\ \\ 0 \text{ for } i \neq j, \end{cases}$$

with sums over empty sets being zero. To say it again in words, the matrix $\mathcal{L}_{G_F}(Z)\left[V_{G_F}^{\text{ext}}\right]$ is obtained from the generic Laplacian of the core of G_F by adding, for each external edge, the label to the diagonal entry corresponding to the vertex, which the edge is incident to.

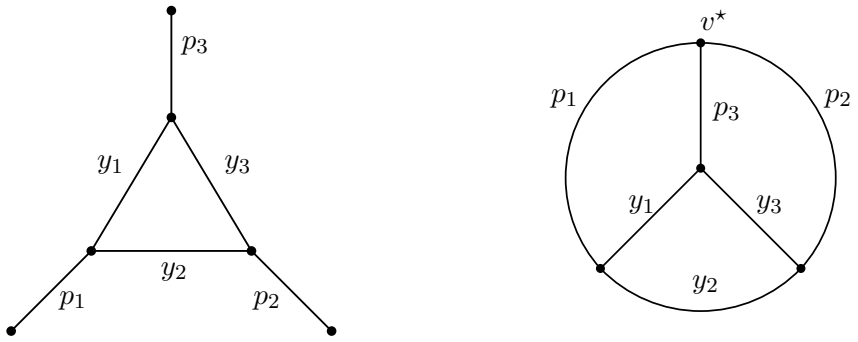
Now we will see, that this matrix is the reduced Laplacian of a related graph. We consider the graph $G_F/V_{G_F}^{\text{ext}}$, which is defined to be the graph obtained from G_F by identification of all the external vertices to one vertex, which we denote $v^\star$ here. In the generic Laplacian $\mathcal{L}_{G_F/V_{G_F}^{\text{ext}}}$ of this graph, the elements of Z^{ext} appear in the diagonal entries in the same way as in the corresponding entries of $\mathcal{L}_{G_F}(Z)\left[V_{G_F}^{\text{ext}}\right]$ and furthermore in the one row and the column, which is associated to the vertex $v^\star$. If this row and this column is removed, we obviously obtain $\mathcal{L}_{G_F}(Z)\left[V_{G_F}^{\text{ext}}\right]$. Hence we have the relation

$$\mathcal{L}_{G_F}(Z)\left[V_{G_F}^{\text{ext}}\right] = \mathcal{L}_{G_F/V_{G_F}^{\text{ext}}}[v^\star]. \tag{4.2.4}$$

Therefore, as the graph $G_F/V_{G_F}^{\text{ext}}$ is identical with its own core, we have the following direct consequence of equation (4.2.4):

LEMMA 30. *We have*

$$\begin{aligned} \det\left(\mathcal{L}_{G_F}(Z)\left[V_{G_F}^{ext}\right]\right) &= \mathcal{K}_{G_F/V_{G_F}^{ext}}(Z) \\ &= U_{G_F/V_{G_F}^{ext}}(Z). \end{aligned}$$



(a) (b)

FIGURE 4.2.1. The triangle and the wheel with three spokes, considered in example 31.

EXAMPLE 31. As an example let us consider the triangle graph G_F in figure 4.2.1 (a). Then the graph where all external vertices of G_F are identified, i. e. $G_F/V_{G_F}^{\text{ext}}$, is the so-called wheel with three spokes, shown in figure 4.2.1 (b). The generic Laplacian of the triangle G_F is the matrix

$$\mathcal{L}_{G_F}(Z) = \begin{pmatrix} y_1+y_2+p_2 & -y_2 & -y_1 & -p_1 & 0 & 0 \\ -y_2 & y_2+y_3+p_2 & -y_3 & 0 & -p_2 & 0 \\ -y_1 & -y_3 & y_1+y_3+p_3 & 0 & 0 & -p_3 \\ -p_1 & 0 & 0 & p_1 & 0 & 0 \\ 0 & -p_2 & 0 & 0 & p_2 & 0 \\ 0 & 0 & -p_3 & 0 & 0 & p_3 \end{pmatrix}.$$

The generic Laplacian of the wheel with three spokes $G_F/V_{G_F}^{\text{ext}}$ is

$$\mathcal{L}_{G_F/V_{G_F}^{\text{ext}}}(Z) = \begin{pmatrix} y_1+y_2+p_2 & -y_2 & -y_1 & -p_1 \\ -y_2 & y_2+y_3+p_2 & -y_3 & -p_2 \\ -y_1 & -y_3 & y_1+y_3+p_3 & -p_3 \\ -p_1 & -p_2 & -p_3 & p_1+p_2+p_3 \end{pmatrix}.$$

The matrix $\mathcal{L}_{G_F}(Z)\left[V_{G_F}^{\text{ext}}\right]$ is obtained from $\mathcal{L}_{G_F}(Z)$ by cancellation of the last three rows and columns. The matrix $\mathcal{L}_{G_F/V_{G_F}^{\text{ext}}}[v^\star]$ is obtained from $\mathcal{L}_{G_F/V_{G_F}^{\text{ext}}}(Z)$ by cancellation of the last row and column. Obviously the matrices $\mathcal{L}_{G_F}(Z)\left[V_{G_F}^{\text{ext}}\right]$ and $\mathcal{L}_{G_F/V_{G_F}^{\text{ext}}}[v^\star]$ are the same, as stated in equation (4.2.4).

Now let us state the main theorem of the present chapter.

THEOREM 32. *We have*

$$U_{G_F/V_{G_F}^{ext}}(Z) = U_{G_F}\left(Z^{int}\right)\sum_{p_i\in Z^{ext}} p_i + F_{0,G_F}\left(Z^{int}, Z^{ext}\right) + \mathcal{O}\left(p^3\right)$$

where $\mathcal{O}\left(p^3\right)$ is the sum of all terms in $U_{G_F/V_{G_F}^{\text{ext}}}(Z)$ with more than two factors in Z^{ext}.

PROOF. According to lemma 30 we have

$$U_{G_F/V_{G_F}^{\text{ext}}}(Z) = \det\left(\mathcal{L}_{G_F}(Z)\left[V_{G_F}^{\text{ext}}\right]\right). \tag{4.2.5}$$

Furthermore according to equation (4.2.3), which we obtained as a special case of the all-minors-matrix-tree-theorem, we can write the right-hand side of equation (4.2.5) as

$$\det\left(\mathcal{L}_{G_F}(Z)\left[V_{G_F}^{\text{ext}}\right]\right) \quad = \quad \sum_{F\in\mathcal{F}_{G_F,\,V_{G_F}^{\text{ext}}}} \prod_{e_i\in E_F} z_i,$$

where $\mathcal{F}_{G_F,\,V_{G_F}^{\text{ext}}}$ is the set of all forests F such that F is a spanning sub-graph of G_F, F consists of $\left|V_{G_F}^{\text{ext}}\right|$ trees T_1, T_2, ..., $T_{\left|V_{G_F}^{\text{ext}}\right|}$ and each tree of F has exactly one external vertex in $V_{G_F}^{\text{ext}}$. (In our formulation of the all-minors-matrix-tree-theorem above (theorem 29), the corresponding class of forests was denoted $\mathcal{F}_{G,\,A,\,B}$.)

Now let us sort the terms of this polynomial as

$$\sum_{F\in\mathcal{F}_{G_F,\,V_{G_F}^{\text{ext}}}} \prod_{e_i\in E_F} z_i \quad = \quad u_0+u_1+u_2+u_{>2},$$

where u_0 consists of all the terms which contain no element of Z^{ext} as a factor, u_1 the terms with exactly one element of Z^{ext}, and accordingly u_2 and $u_{>2}$ the terms with exactly two and more than two elements of Z^{ext} respectively. In order to prove the theorem we have to show that $u_0=0$, $u_1=U_{G_F}\left(Z^{\text{int}}\right)\sum_{p_{e_i}\in Z^{\text{ext}}} p_i$ and $u_2=F_{0,G_F}\left(Z^{\text{int}},\,Z^{\text{ext}}\right)$.

Each tree in each forest $F\in\mathcal{F}_{G_F,\,V_{G_F}^{\text{ext}}}$ contains exactly one vertex in $V_{G_F}^{\text{ext}}$. Therefore the trees fall into two types of trees: Firstly the type we want to call t_0, which is the type of trees with no edges at all (i.e. isolated vertices of $V_{G_F}^{\text{ext}}$). Secondly the type we call t_1, being the type of trees with exactly one of the external edges of G_F. Note that the above sum can be written as

$$\sum_{F\in\mathcal{F}_{G_F,\,V_{G_F}^{\text{ext}}}} \prod_{T\in F}\prod_{e_i\in T} z_i,$$

where T is a tree of the type t_1. Trees of the type t_0 do not contain edges and therefore they do not contribute to the above sum.

For T being a tree of type t_1 in a forest $F\in\mathcal{F}_{G_F,\,V_{G_F}^{\text{ext}}}$ the product $\prod_{e_i\in T} z_i$ contains exactly one element of Z^{ext}. The set $\mathcal{F}_{G_F,\,V_{G_F}^{\text{ext}}}$ contains only one forest, where all the trees of the forest are of type t_0. The corresponding term in the sum is zero. Therefore $u_0=0$.

For the consideration of u_1 and u_2 let us remember, that the forests in $\mathcal{F}_{G_F,\,V_{G_F}^{\text{ext}}}$ are spanning forests of G_F. Each of these forests consists, as we just said, of trees of type t_1 and of isolated vertices (trees of type t_0) which are external in G_F. Hence if in such a forest F we remove from each t_1-tree the vertex and the edge, which are external in G_F and furthermore remove the t_0-trees, then we obtain a forest, which is a spanning forest of the core $\tilde{G}_F$. A forest obtained from F in this way shall be denoted by $\tilde{F}$. For a t_1-tree T in F, the tree which is obtained by deleting the vertex and the edge external in G_F shall be denoted $\tilde{T}$.

The terms in u_1 correspond to the forests, which contain one tree of type t_1 and $S\equiv\left|V_{G_F}^{\text{ext}}\right|$ trees of type t_0. In such a forest the t_1-tree T must contain all vertices which are internal in G_F. Therefore $\tilde{T}$ is a spanning tree of G_F. Obviously there are S forests contributing to u_1, which contain the same $\tilde{T}$. Hence we have S products $\prod_{T\in F}\prod_{e_i\in T} z_i$

in u_1, which differ just by an element of Z^{ext}. Therefore we have

$$u_1 = \sum_{\tilde{T} \text{ spanning tree of } \tilde{G}_F} \left(\prod_{e_j \in \tilde{T}} z_j \left(\sum_{p_i \in Z^{\text{ext}}} p_i \right) \right) = U_{G_F}\left(Z^{\text{int}}\right) \sum_{p_i \in Z^{\text{ext}}} p_i.$$

Similar arguments apply for u_2. We can write u_2 as a sum of spanning 2-forests $\left(\tilde{T}_k, \tilde{T}_l\right)$ of $\tilde{G}_F$. Then each term of this sum contains firstly a sum over the elements of Z^{ext} whose edges are incident to $\tilde{T}_k$, and secondly a similar term for $\tilde{T}_l$. Let us denote the set of external edges of G_F which are incident to $\tilde{T}_k$ by $E_{G_F}^{\text{incident}}\left(\tilde{T}_k\right)$. Then we have

$$u_2 = \sum_{\left(\tilde{T}_k, \tilde{T}_l\right) \in \mathcal{T}_2\left(\tilde{G}_F\right)} \left(\prod_{e_j \in \left(\tilde{T}_k, \tilde{T}_l\right)} z_j \left(\sum_{p_i \in E_{G_F}^{\text{incident}}\left(\tilde{T}_k\right)} p_i \right) \left(\sum_{p_j \in E_{G_F}^{\text{incident}}\left(\tilde{T}_l\right)} p_j \right) \right).$$

Now we see that according to equation (4.1.16) the product

$$\left(\sum_{p_i \in E_{G_F}^{\text{incident}}\left(\tilde{T}_k\right)} p_i \right) \left(\sum_{p_j \in E_{G_F}^{\text{incident}}\left(\tilde{T}_l\right)} p_j \right)$$

is the kinematical invariant associated to the spanning 2-forest $\left(\tilde{T}_k, \tilde{T}_l\right)$.

Let us state this argument in more detail by following the lines of subsection 4.1.2. We introduce an auxiliary set

$$H = \{q_1, ..., q_N, q_{N+1} = p_{N+1}, ..., q_{N+S} = p_{N+S}\}$$

of scalar labels on the edges of G_F, where the last S elements on the right are exactly the elements of Z^{ext}. Then we assume that H satisfies

$$\sum_{e_i \text{ incident to } v_j} \alpha(v_j, e_i) q_i = 0$$

at each internal vertex and we obtain

$$s_{(T_k, T_l)} = \left(\sum_{p_i \in E_{G_F}^{\text{incident}}(T_k)} p_i \right) \left(\sum_{p_j \in E_{G_F}^{\text{incident}}(T_l)} p_j \right).$$

Therefore we have

$$u_2 = F_{0,G_F}\left(Z^{\text{int}}, Z^{\text{ext}}\right).$$

This proves the theorem. □

The theorem provides a relation between the Symanzik polynomials of G_F and of $G_F/V_{G_F}^{\text{ext}}$. It can furthermore be used as a simple construction rule for F_{0,G_F}. Let us give an example:

EXAMPLE 33. As in example 31 we again want to consider the triangle G_F in figure 4.2.1 (a) and the corresponding graph $G_F/V_{G_F}^{\text{ext}}$ which is the wheel with three spokes

in figure 4.2.1 (b). With the labels as assigned in figure 4.2.1 (b) we obtain the first Symanzik polynomial of the wheel with three spokes:

$$\begin{aligned} U_{G_F/V_{G_F}^{\text{ext}}}(Z) &= \det\left(\mathcal{L}_{G_F/V_{G_F}^{\text{ext}}}[v^\star]\right) \\ &= \det\left(\mathcal{L}_{G_F}(Z)\left[V_{G_F}^{\text{ext}}\right]\right) \\ &= (y_1y_2+y_1y_3+y_2y_3)(p_1+p_2+p_3) \\ &\quad +y_1p_2(p_1+p_3)+y_2p_3(p_1+p_2)+y_3p_1(p_2+p_3). \end{aligned}$$

We convince ourselves, that the first term of the sum contains the first Symanzik polynomial of G_F:

$$U_{G_F}\left(Z^{\text{int}}\right) = y_1y_2+y_1y_3+y_2y_3.$$

If we impose momentum conservation in the remaining terms of the sum, then we obtain the second Symanzik polynomial of G_F:

$$\begin{aligned} F_{0,G_F}\left(Z^{\text{int}}, Z^{\text{ext}}\right) &= y_1p_2(p_1+p_3)+y_2p_3(p_1+p_2)+y_3p_1(p_2+p_3) \\ &= y_1\left(-p_2^2\right)+y_2\left(-p_3^2\right)+y_3\left(-p_1^2\right). \end{aligned}$$

We see, that theorem 32 allows us to obtain the Symanzik polynomials U_{G_F} and F_{0,G_F} just from the determinant of a matrix and without the explicit construction of spanning trees or 2-forest. This can be useful for the automatic evaluation of Symanzik polynomials of more complicated graphs. Matrices are obviously standard objects in today's computer-algebra-systems (CAS). On the contrary, graphs and procedures like the construction of all sub-graphs with certain properties are usually not implemented in a standard CAS. Instead of constructing spanning trees and 2-forests by hand, or, in more complicated cases, by the use of specific computer programs, we may simply construct the Laplacian $\mathcal{L}_{G_F}(Z)$, cancel the rows and columns of the external vertices and use a (standard) CAS or even pocket-calculator to evaluate the determinant. Then we can directly read the Symanzik polynomials off the determinant, as shown above.[10]

Let us, for completeness, add a relation between the Symanzik polynomials, which again goes back to Kirchhoff's work [**Kir47**]. This relation identifies terms in the second Symanzik polynomial to the first Symanzik polynomials of another graph. Let G_F be a Feynman graph and let v_1 and v_2 be distinct internal vertices in G_F. Then $G_F(v_1, v_2)$ shall denote the graph, which is obtained from G_F by adding a new edge $\tilde{e}$ incident to v_1 and v_2 and then contracting this edge. $G_F(v_1, v_2)$ therefore has as many edges as G_F, but one vertex less. Let p_{v_i} be the element of Z^{ext} labelling the external edge, which is incident to v_i. If no external edge is incident to v_i then p_{v_i} is zero. This notation is convenient to state the relation:

PROPOSITION 34. *For a Feynman graph G_F we have*

$$F_{0,G_F}\left(Z^{int}, Z^{ext}\right) = -\frac{1}{2}\sum_{\substack{v_1 \neq v_2 \\ v_1,\, v_2 \text{ internal}}} p_{v_1}p_{v_2}U_{G_F(v_1,v_2)}\left(Z^{int}\right).$$

The factor $\frac{1}{2}$ has to appear, because the sum contains each pair of internal vertices twice. The relation is stated and proven in [**KRTW08**].

Whenever v_1 and v_2 are adjacent in G_F the graph $G_F(v_1, v_2)$ contains a self-loop. We did not allow for self-loops in the previous subsections. For the relation studied here

[10] A CAS might fail to provide the determinant of a very large matrix in reasonable computation time, but for example the evaluation of the determinant of a 10×10-matrix, corresponding to a graph with 10 internal vertices, still takes only a few seconds by the use of Maple 9.5 on a standard PC.

it is sufficient to be aware that a self-loop belongs to any spanning tree of a graph and therefore does not contribute to the first Symanzik polynomial $U_{G_F(v_1,\,v_2)}$. Therefore, the relation is not affected by possible self-loops in $G_F\,(v_1,\,v_2)$.

4.2.3. A Remark on Characteristic Polynomials. Because of the special form of the matrix $\mathcal{L}_{G_F}\left(\tilde{Y}\right)\left[V_{G_F}^{\text{ext}}\right]$ we did not need to exploit the full power of the all-minors-matrix-tree-theorem (29) for our proof. Hence it might be interesting to relate our statement to predecessors of this theorem. We want to point out a link to an identity obtained by Kelmans and Chelnokov which can be seen as an intermediate step towards the above used all-minors-matrix-tree-theorem [11].

Kelmans and Chelnokov [**KC74, Kel67**] consider the *characteristic polynomial of the Laplacian* L_G of a graph G. For G being a graph with n vertices this is the polynomial

$$\begin{aligned}\Psi(\lambda,\,G) &= \frac{1}{\lambda}\det(\lambda\mathbb{1}_{n\times n}-L_G)\\ &= \prod_{i=1}^{n-1}(\lambda-\lambda_i(G))\\ &= \lambda^{n-1}-b_1(G)\lambda^{n-2}+\cdots+(-1)^i b_i(G)\lambda^{n-1-i}+\cdots+(-1)^{n-1}b_{n-1}(G),\end{aligned}$$

where λ_i are the eigenvalues of L_G and the coefficients $b_i(G)$ are non-negative integers. Let G' be a sub-graph of G and let G/G' be the graph obtained from G by identifying all vertices of G' and subsequently removing the loops created by this identification. In other words, the sub-graph G' is contracted to one vertex $v^\star \in V_G$. Let $\tau(G)$ denote the number of spanning trees of G. Kelmans and Chelnokov derive the identity

$$b_i(G) = \sum_{G'\subseteq G,\,|V_{G'}|=n-i}\tau(G/G'),\quad i=0,\,1,\,...,\,n-1, \tag{4.2.6}$$

stating that the b_i count all the spanning trees of all possible sub-graphs of G resulting from identification of $k=n-i$ vertices.

Let us assume, that the graph G is the core $\tilde{G}_F = G$ of a Feynman graph G_F. Then let us compare the matrix $\lambda\mathbb{1}_{n\times n}-L_{\tilde{G}_F}$ with

$$\mathcal{L}_{G_F}\,(Z)\left[V_{G_F}^{\text{ext}}\right] = \mathcal{L}_{\tilde{G}_F}\left(Z^{\text{int}}\right)+\mathcal{Q}_{G_F}\left(Z^{\text{ext}}\right)$$

from above. The generic Laplacian $\mathcal{L}_{\tilde{G}_F}\left(Z^{\text{int}}\right)$ is a generalization of $L_{\tilde{G}_F}$. In the same sense we can consider the matrix $\mathcal{Q}_{G_F}\left(Z^{\text{ext}}\right)$ to be a generalization of $\lambda\mathbb{1}_{n\times n}$. Here the set $\left\{0,\,Z^{\text{ext}}\right\}$ replaces the set $\{\lambda\}$. Hence we can interpret the matrix $\mathcal{L}_{G_F}\,(Z)\left[V_{G_F}^{\text{ext}}\right]$ as a possible generalization of $\lambda\mathbb{1}_{n\times n}-L_G$. Furthermore the polynomial $U_{G_F/V_{G_F}^{\text{ext}}}(Z) = \det\left(\mathcal{L}_{G_F}\,(Z)\left[V_{G_F}^{\text{ext}}\right]\right)$ is a generalization of the polynomial $\Psi(\lambda,\,G)\cdot\lambda$. In this sense equation (4.2.6) is related to the all-minors-matrix-tree-theorem and to our theorem (30). From this point of view a closer study of the characteristic polynomials and eigenvalues of Laplacians might be useful. For an introduction to the latter topic we refer to chapter 13 of [**GR01d**] and the survey [**Moh91**].

[11] Further predecessors of the theorem are for instance the matrix-tree-theorem itself (28) and a theorem on rooted forests by Fiedler and Sedláček in [**FS58**].

4.2.4. Dodgson's Rule on Determinants. The above mentioned theorems of the matrix-tree-type allow us to use properties of determinants in calculations concerning graph polynomials. Apart from the obvious properties, determinants of arbitrary $n \times n$-matrices fulfill an identity discovered by Reverend Charles Lutwidge Dodgson[12] under the name of "condensation of determinants" in [**Dod66**].

Let $n \in \mathbb{N}$ and I and J be subsets of $\{1, 2, ..., n\}$. For sets with one element we write the element itself. Let A be an $n \times n$ matrix and let $A\left[(I, J)\right]$ again denote the matrix obtained from A by removing the ith rows and the jth columns for all $i \in I$ and $j \in J$. Again we write $A[(I, I)] = A[I]$ for convenience. Dodgson's rule is

$$\det\left(A\right)\det\left(A\left[\{1, n\}\right]\right) \quad = \quad \det\left(A\left[n\right]\right)\det\left(A\left[1\right]\right) - \det\left(A\left[(n, 1)\right]\right)\det\left(A\left[(1, n)\right]\right).$$

A proof was given by Zeilberger in [**Zei97**].

The equation can be used to find relations among Kirchhoff polynomials of different graphs. As an example we quote the following result by Stembridge, which was proven using Dodgson's identity.

THEOREM 35. *(Stembridge* [**Ste98**]*) Let G be a connected graph with edges labelled by Y and let e, $e' \in E_G$ be distinct edges. Let $\Delta_{G,\,e,\,e'}$ be defined as*

$$\Delta_{G,\,e,\,e'} \quad = \quad \mathcal{K}_{(G/e)\backslash e'}\mathcal{K}_{(G/e')\backslash e} - \mathcal{K}_{(G/e)/e'}\mathcal{K}_{(G\backslash e)\backslash e'},$$

the dependence on Y being understood.
(a) If u and v are the endpoints of e and u' and v the endpoints of e', then

$$\Delta_{G,\,e,\,e'} \quad = \quad \left(\det\left(\mathcal{L}_G\left[\left(\{u, v\}, \{u', v\}\right)\right]\right)\right)^2.$$

(b) If u and v are the endpoints of e, u' and v' the endpoints of e' and all four vertices are distinct, then

$$\Delta_{G,\,e,\,e'} \quad = \quad \left(\det\left(\mathcal{L}_G\left[\left(\{u, v\}, \{u', v\}\right)\right]\right) - (-1)^r \det\left(\mathcal{L}_G\left[\left(\{u, v\}, \{v', v\}\right)\right]\right)\right)^2,$$

where r denotes the distance between the rows u' and v' in $\mathcal{L}_G\left[(v, v)\right]$.

In the context of Feynman integrals, Dodgson's rule was applied to the Kirchhoff polynomial first in [**BEK06**]. In [**Bro09**] Brown uses the identity (as in part (a) of Stembridge's theorem) at a crucial point of his method for the evaluation of a special class of Feynman integrals to multiple zeta values.

4.3. The Multivariate Tutte Polynomial

Now that we have discussed Symanzik polynomials and their relations, we give a brief outlook on the so-called *multivariate Tutte polynomial*. The multivariate Tutte polynomial generalizes the standard *Tutte polynomial* (defined below), which was introduced by W. T. Tutte [**Tut47, Tut54, Tut67**] and is known to be a useful generalization of many graph invariants. It is closely related to the so-called chromatic polynomial, the flow polynomial, the reliability polynomial and many others. For a detailed review of the standard Tutte polynomial and its interrelations we refer to the reviews by Ellis-Monaghan and Merino [**EMM08a, EMM08b**].

In a sense the multivariate generalization of the Tutte polynomial which we want to discuss in the following contains the Symanzik polynomials and is related to a variety of problems in graph theory and physics alike. Most of the material given in this section is based on Sokal's comprehensive introduction [**Sok05**].

[12] Dodgson's even more famous literary work contains for example the novel "Alice in wonderland" which he wrote using the pseudonym Lewis Carroll. This novel has inspired Dominik Zeillinger's presentation of his winning strategy for Hironaka's polyhedra game in [**Zei05**]. The latter strategy is a topic of chapter 5 of this dissertation.

4.3.1. Basic Definition. Let us consider a graph G with the edges labelled by $Y = \{y_1, y_1, ..., y_{|E_G|}\}$. We will also need one further formal variable q. For convenience let us introduce the symbol $\sqsubseteq$, such that $G' \sqsubseteq G$ expresses, that G' is a spanning sub-graph of G. Note that the spanning sub-graphs we considered before were usually trees and 2-forests, but in general, spanning sub-graphs of course may contain loops and they may consist of more than one (or two) connected components. We remind ourselves, that we denote the number of connected components of G by κ_G, the loop-number by $\imath_G$ and that we have the relation

$$\imath_G = |E_G| - |V_G| + \kappa_G. \tag{4.3.1}$$

Definition 36. (Sokal, **[Sok05]**) The *multivariate Tutte polynomial* of G is defined as

$$Z_G\left(q, y_1, ..., y_{|E_G|}\right) = \sum_{G' \sqsubseteq G} q^{\kappa_{G'}} \prod_{e_i \in E_{G'}} y_i. \tag{4.3.2}$$

We see, that the sum runs through all spanning sub-graphs of G, possibly having loops and possibly being disconnected. Each term of the sum corresponds to one spanning sub-graph G' and is a product of $q^{\kappa_{G'}}$ and all the edge-labels of G'.

The fact that the graphs G' are spanning graphs of G and therefore $|V_{G'}| = |V_G|$ and equation (4.3.1) can be used to write the multivariate Tutte polynomial as

$$Z_G\left(q, y_1, ..., y_{|E_G|}\right) = q^{|V_G|} \sum_{G' \sqsubseteq G} q^{\imath_{G'}} \prod_{e_i \in E_{G'}} \frac{y_i}{q}. \tag{4.3.3}$$

The *standard Tutte polynomial* **[Tut47, Tut54, Tut67]** is defined as

$$T_G(u, v) = \sum_{G' \sqsubseteq G} (u-1)^{\kappa_{G'} - \kappa_G} (v-1)^{\imath_{G'}}. \tag{4.3.4}$$

The multivariate Tutte polynomial as a function $Z_G(q, y)$ of q and a one-element set $Y = \{y\}$ is related to $T_G(u, v)$ by the identity

$$T_G(u, v) = (u-1)^{-\kappa(G)} (v-1)^{-|V_G|} Z_G\left((u-1)(v-1), v-1\right). \tag{4.3.5}$$

4.3.2. $q \longrightarrow 0$ Limits. In contrast to the above definition of the multivariate Tutte polynomial we have defined the Symanzik polynomials without the use of the variable q. Furthermore, in the case of U_G the sum runs through the spanning trees of the core and in the case of $F_{0,G}$ the sum runs through a certain subset of the spanning 2-forests. The multivariate Tutte polynomial instead considers all spanning sub-graphs and therefore it contains in general considerably more terms than the Symanzik polynomials.

If we write down the multivariate Tutte polynomial explicitly for a given graph, we can of course select the terms corresponding to certain sub-graphs by hand (and possibly recombine them to Symanzik polynomials). More systematically, we can select terms corresponding to certain sub-graphs by taking the limit $q \longrightarrow 0$ in an appropriate way. The general strategy is to take a limit $\lim_{q \longrightarrow 0} q^a Z_G$, such that the terms in Z_G which shall be selected have the form $q^{-a} \prod_{e_i \in E_{G'}} y_i$. By cancellation of the q-prefactors, the corresponding $\prod_{e_i \in E_{G'}} y_i$ do not vanish in the limit. On the other hand all terms with a factor $q^{-b} \prod_{e_i \in E_{G'}} y_i$ with $b < a$ turn to zero.

We consider three possibilities:

(1) We select the sub-graphs with a minimal number of connected components. This is done by taking the limit $q \longrightarrow 0$ of $q^{-\kappa_G} Z_G(q, Y)$ while keeping all y_i fixed:

$$\lim_{q \longrightarrow 0} q^{-\kappa_G} Z_G\left(q, y_1, ..., y_{|E_G|}\right) = C_G\left(y_1, ..., y_{|E_G|}\right).$$

The result is the generating polynomial of maximally spanning sub-graphs, defined as

$$C_G\left(y_1,\, ...,\, y_{|E_G|}\right) \;=\; \sum_{\substack{G' \sqsubseteq G \\ \kappa_{G'} = \kappa_G}} \prod_{e_i \in E_{G'}} y_i .$$

Note that the sum runs through sub-graphs which are allowed to have loops but not more connected components than G itself. To convince ourselves that this is the result, we can see in equation (4.3.2) that the q-dependent factors in $q^{-\kappa_G} Z_G$ are $q^{\kappa_{G'}-\kappa_G}$, which tend to one for $q \longrightarrow 0$ only for G' with $\kappa_{G'} = \kappa_G$.

(2) We select the sub-graphs which are forests by taking the limit $q \longrightarrow 0$ of

$$q^{-|V_G|} Z_G\left(q,\, qy'_1,\, ...,\, qy'_{|E_G|}\right)$$

while keeping all y'_i fixed:

$$\lim_{q \longrightarrow 0} q^{-|V_G|} Z_G\left(q,\, qy'_1,\, ...,\, qy'_{|E_G|}\right) \;=\; Fo_G\left(y'_1,\, ...,\, y'_{|E_G|}\right)$$

where the polynomial

$$Fo_G\left(y'_1,\, ...,\, y'_{|E_G|}\right) \;=\; \sum_{\substack{G' \sqsubseteq G \\ \mathfrak{l}_{G'} = 0}} \prod_{e_i \in E_{G'}} y'_i$$

is called the generating polynomial of spanning forests. It can be seen from equation (4.3.3) that keeping all $\frac{y_{e_i}}{q}$ fixed, the q-dependent factor $q^{\mathfrak{l}_{G'}}$ goes to one for G' with $\mathfrak{l}_{G'} = 0$ and to zero for the other sub-graphs.

(3) Now let us define the generating polynomial of maximal spanning forests to be

$$\mathcal{K}_G\left(y_1,\, ...,\, y_{|E_G|}\right) \;=\; \sum_{\substack{G' \sqsubseteq G \\ \kappa_{G'} = \kappa_G \\ \mathfrak{l}_{G'} = 0}} \prod_{e_i \in G'} y_i .$$

Note that the sum runs through a set of graphs which is the intersection of the sets where the sums in C_G and in Fo_G run through. It is the set of spanning forests with the minimal number of connected components. If G is a connected graph, these forests are spanning trees and we obtain exactly the Kirchhoff polynomial of G.

To obtain $\mathcal{K}_G$ from C_G we have to select the sub-graphs without loops, which in the sum in C_G are the sub-graphs with $|V_G| - \kappa_{G'}$ edges. We impose this selection by replacing all variables y_i by λy_i and taking the following limit:

$$\lim_{\lambda \longrightarrow 0} \lambda^{\kappa_G - |V_G|} C_G\left(\lambda y_1,\, ...,\, \lambda y_{|E_G|}\right) \;=\; \mathcal{K}_G\left(y_1,\, ...,\, y_{|E_G|}\right).$$

Alternatively we can obtain $\mathcal{K}_G$ from Fo_G by replacing all variables y'_i by $\lambda y'_i$ and taking the limit $\lambda \longrightarrow \infty$:

$$\lim_{\lambda \longrightarrow \infty} \lambda^{\kappa_G - |V_G|} Fo_G\left(\lambda y'_1,\, ...,\, \lambda y'_{|E_G|}\right) \;=\; \mathcal{K}_G\left(y'_1,\, ...,\, y'_{|E_G|}\right).$$

As a third possibility we can directly obtain $\mathcal{K}_G$ from the multivariate Tutte polynomial in one step. By the substitutions $t_i = \frac{y_i}{q^\alpha}$ with $0 < \alpha < 1$ in equation

(4.3.2) we obtain

$$Z_G\left(q,\, q^\alpha t_1,\, ...,\, q^\alpha t_{|E_G|}\right) \;=\; q^{\alpha|V_G|}\sum_{G'\sqsubseteq G} q^{\alpha\mathfrak{l}_{G'}+(1-\alpha)\kappa_{G'}}\prod_{e_i\in E_{G'}} t_i.$$

The graphs for which $\alpha\mathfrak{l}_{G'}+(1-\alpha)\kappa_{G'}$ takes the smallest possible value $(1-\alpha)\kappa_G$ are the desired spanning forests with $\kappa_{G'}=\kappa_G$. Hence for these graphs, the limit

$$\lim_{q\longrightarrow 0} q^{(1-\alpha)\kappa_G} q^{\alpha\mathfrak{l}_{G'}+(1-\alpha)\kappa_{G'}}$$

is one and for all other graphs zero. So we obtain $\mathcal{K}_G$ taking the limit

$$\lim_{q\longrightarrow 0} q^{-\alpha|V_G|+(1-\alpha)\kappa_G} Z_G\left(q,\, q^\alpha t_1,\, ...,\, q^\alpha t_{|E_G|}\right) \;=\; \mathcal{K}_G\left(t_1,\, ...,\, t_{|E_G|}\right).$$

It is clear that the latter limit relates the multivariate Tutte polynomial to the Symanzik polynomials via the relations between the Kirchhoff and the Symanzik polynomials, which we discussed in the previous section.

The fact that multivariate Tutte polynomials and Symanzik polynomials are closely related has been emphasized by Krajewski, Rivasseau, Tanasa and Wang in **[KRTW08]**[13].

4.3.3. Further Properties. (Multivariate) Tutte polynomials are objects with some very particular and well-studied properties, of which some might eventually be useful in the context of Feynman integrals. We want to end this chapter by quoting just some of the most striking properties, all of them (and many more) being discussed in more detail in Sokal's review **[Sok05]**.

- If $e\in E_G$ and $G\backslash e$ is the graph obtained from G by deleting e and G/e is the graph obtained from G by contracting e, we have

$$Z_G\left(q,\, Y\right) \;=\; Z_{G\backslash e}\left(q,\, Y\backslash e\right) + y_e Z_{G/e}\left(q,\, Y\backslash e\right). \tag{4.3.6}$$

  This kind of identity is referred to as *deletion/contraction relation*. We had such a property for the Kirchhoff polynomial with equation (4.1.28).
- If G consists of two connected components G_1 and G_2, we have

$$Z_G\left(q,\, Y\right) \;=\; Z_{G_1}\left(q,\, Y\right) Z_{G_2}\left(q,\, Y\right).$$

- If G consists of sub-graphs G_1 and G_2 which have exactly one vertex and no edge in common (i.e. $V_{G_1}\cup V_{G_2}=V_G$, $E_{G_1}\cup E_{G_2}=E_G$, $|V_{G_1}\cap V_{G_2}|=1$, $|E_{G_1}\cap E_{G_2}|=0$) then we have

$$Z_G\left(q,\, Y\right) = \frac{Z_{G_1}\left(q,\, Y\right) Z_{G_2}\left(q,\, Y\right)}{q}.$$

- If G is a planar graph and $G^\star$ is the corresponding dual graph we have

$$Z_{G^\star}\left(q,\, y_1, ...,\, y_{|E_G|}\right) \;= q^{1-|V_G|}\quad\left(\prod_{e_i\in E_G} y_i\right) Z_G\left(q,\, q/y_1, ...,\, q/y_{|E_G|}\right).$$

- If we restrict q to be a positive integer, one can write the multivariate Tutte polynomial as

$$Z_G\left(q\in\mathbb{N},\, Y\right) \;=\; \sum_{\sigma:\, V_G\to\{1,\,2,\,...,\,q\}}\;\prod_{e_i\in E_G}\left(1+y_i\delta\left(\sigma_{v_1(e_i)},\, \sigma_{v_2(e_i)}\right)\right)$$

[13] In this work the authors furthermore introduce a new kind of Symanzik polynomials, suitable for quantum field theories on non-commutative spacetime, and they show that these are related to the so-called Bollobas-Riordan polynomial, which is a generalization of the multivariate Tutte polynomial.

where δ is the Kronecker delta, $v_1(e_i)$ and $v_2(e_i)$ are the vertices at the ends of the edge e and the sum runs over all possible maps σ from the vertices of G to the integers $\{1, 2, ..., q\}$. By replacing the variables y_i in an appropriate manner by Boltzmann weights, this polynomial gives the partition function of the q-state Potts model (see e.g. **[Wu83]**) of statistical physics.

- The multivariate Tutte polynomial can be defined on a matroid[14] (instead of a graph) and it then contains *all* the information about the matroid.

We mentioned the recent work of Krajewski et al. **[KRTW08]** where to our knowledge the relation between the multivariate Tutte polynomial and Symanzik polynomials was mentioned for the first time in the context of Feynman integrals. We believe that because of this relation and the special properties we just mentioned, Tutte polynomials might be a helpful tool for the investigation of the combinatorial properties of Feynman integrals in the future. To our knowledge, a first explicit application of a property of the (classical) Tutte polynomial to so-called motives which are associated to Feynman graphs was given in a very recent work by Aluffi and Marcolli **[AM09a]**. In this article the authors prove a deletion/contraction relation for a certain polynomial which is closely related to the associated motives.

Many methods for the evaluation of Feynman integrals usually do not yet make use of most of the combinatorial properties of Symanzik polynomials which we discussed in the present chapter. In the following chapter, we will give a detailed discussion of one particular technique where Symanzik polynomials are present in the calculation. We will see in detail, how the zero-sets of the Symanzik polynomials determine the singularities of the Feynman integral. The consideration of these zero-sets as geometrical objects provides the link to a technique of algebraic geometry, which will be used in the following.

[14] See **[Oxl03, Oxl86, Oxl06]** for an introduction to matroids and their relation to graphs.

CHAPTER 5

Sector Decomposition and Resolution of Singularities

"Logic takes care of itself; all we have to do is to look and see how it does it."
(Ludwig Wittgenstein)

In the present chapter we introduce an improvement of the widely used sector decomposition algorithm of Binoth and Heinrich [**BH00, BH04, Hei08**]. Sections 5.1 and 5.2 follow the lines of the joint work with Stefan Weinzierl [**BW08**], where the improvement was presented for the first time. The latter publication includes the first publicly available implementation of the sector decomposition algorithm[1]. In section 5.3 we give a more abstract geometrical point of view.

Sector decomposition is a method for the systematic disentanglement of the singularities of Feynman integrals by dispartment and transformation of integration domains. The mentioned algorithm of Binoth and Heinrich iterates these steps systematically. It serves for the numerical evaluation of multi-loop integrals in the Euclidean momentum region for Feynman graphs with an *arbitrary* loop-number and *arbitrary* topology. In section 5.1 we give a brief review on the algorithm and expose a problem of the original version of the algorithm described in [**BH00, BH04, Hei08**]. There are possible cases for which the algorithm does not terminate. We give an example leading to infinite recursion. By mapping the iterative step of the algorithm to a purely combinatorial problem, known as Hironaka's polyhedra game [**Hir67**], we find an extension of the algorithm, such that termination is guaranteed. Hironaka's game and its solutions [**EH02, Spi83, Zei05, Zei06b**] are discussed in section 5.2. Our extended version of the algorithm includes these solutions and therefore terminates. Hironaka's polyhedra game was originally given as the formulation of a combinatorial problem of algebraic geometry, which is the choice of so-called blow-ups for a resolution of singularities [**Hir64**]. In section 5.3 we explain how this important problem of mathematics is connected to the sector decomposition of Feynman integrals.

For the reader of the previous chapters the application of a technique from algebraic geometry to the calculation of Feynman integrals might not come as a great surprise. In chapter 4 we have extensively discussed the Symanzik polynomials, arising from the parametric representation of a Feynman integral. The zero-sets of these polynomials give rise to the possible singularities of the integral. As soon as we consider the zero-sets as geometrical objects, we are automatically in touch with algebraic geometry. As we already mentioned, we observe that a lot of present day's efforts in mathematical physics is invested into a better understanding of the correspondence between Feynman integrals and concepts of algebraic geometry. The common goal of these efforts appears to be an improvement of the highly conjectural theory of so-called motives. In this field of research a purely mathematical problem is addressed with the use of a tool developed in physics. It seems that algebraic geometry in a sense profits from quantum field theory, which provides useful examples for objects of mathematical interest.

[1] In the meantime also the program FIESTA by Smirnov and Tentyukov [**ST09**] (using Mathematica and C) has become available. Ueda and Fujimoto report on a further program [**UF09**] (using FORM).

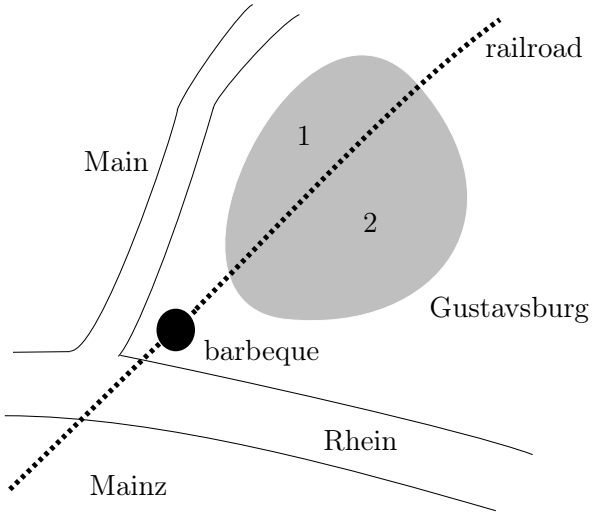


FIGURE 5.1.1. A barbeque at the river mouth.

Therefore a natural question should arise: Can the physicist profit from the correspondence as well and can he or she address his or her own problems by use of the knowledge of algebraic geometers? How can well developed tools of algebraic geometry help in the calculation of Feynman integrals? We believe, that the present chapter contributes one possible answer to this question. A very specific problem, originating from the calculation of multi-loop integrals, is solved with tools of algebraic geometry.

Moreover we believe, that the following treatment should convince the reader, that we did not intentionally try to attach algebraic geometry to sector decomposition just in order to present a particularly fancy approach. Instead of that, Hironaka's polyhedra game and its solutions appear to be the natural way to solve the termination problem of sector decomposition. This example may give rise to the hope, that algebraic geometry will contribute further improvements for the evaluation of Feynman integrals in the future.

5.1. Sector Decomposition

5.1.1. The Basic Idea of Sector Decomposition. Figure 5.1.1 shows a rough sketch of the town Gustavsburg (grey area) from bird's eye view. Let us assume that we have friends in Gustavsburg and we want to invite them to a barbeque at the place which we marked with a black spot. We may tell our friends: "Walk to the area, where the rivers Rhein and Main meet". With respect to the rivers, this information has *two* pieces: They have to go as close as they can to the Rhein and as close as they can to the Main. Our friends need to keep both pieces of information in mind, if they want to find us.

But if we know Gustavsburg, we know that there is a railway track right across the town, dividing it into two regions (the dotted line in figure 5.1.1). Let us assume, that our friends would never get the idea of walking across this railway track on their way. Then we can just tell the friends living in region 1 to come as close as possible to the Rhein. Without crossing the railway track, this can only mean to walk to the place where the barbeque is. Accordingly we tell our friends in region 2 to come as close as possible to the Main. Both groups of friends will find the barbeque. We have separated the two pieces of information, such that each of our friends needs to remember only *one* river. For our forgetful friends, this simplifies the problem of finding us.

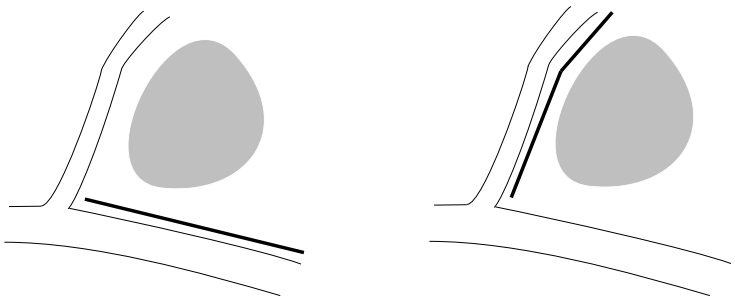


FIGURE 5.1.2. Barbeques at the riverbanks.

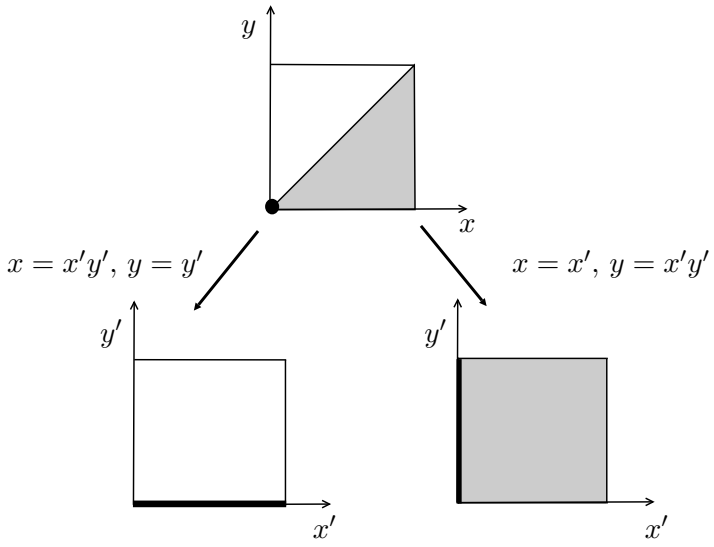


FIGURE 5.1.3. A simple sector decomposition.

Our friends in region 1 know, that they will find the barbeque, if only they approach the river Rhein. So for them it is *as easy* to find it, *as if* there was no railway track and the barbeque was everywhere along the river Rhein (see figure 5.1.2 (a)). Correspondingly, for our friends in region 2 it is the same situation as living in a town without railways and the barbeque being all along the river Main (see figure 5.1.2 (b)). This simplification is what sector decomposition and resolution of singularities is about[2].

Let us speak slightly more scientific again and instead of barbeques and outskirts of Mainz care about singularities and domains of integrals.

EXAMPLE 37. Let us consider the integral

$$I \quad = \quad \int_0^1 dx \int_0^1 dy \frac{1}{P(x,\, y)} x^{a+b\epsilon} y^{c+d\epsilon} \tag{5.1.1}$$

where a, b, c, d are some fixed integers, $\epsilon \in \mathbb{C}$ is a regularization parameter and P is the polynomial

$$P(x,\, y) \quad = \quad x + xy + y.$$

[2]We may say, that what we have just seen is a sector decomposition of Gustavsburg, or equivalently, a resolution of a barbeque.

Singularities of the integral can arise from terms in the integrand, which are not bounded in the integration domain. For example there are values of ϵ where for $a < 0$ the factor $x^{a+b\epsilon}$ goes to infinity as x approaches zero. Accordingly for $c < 0$ the factor $y^{c+d\epsilon}$ goes to infinity if y approaches zero. These possible singularities are independent in the sense that the first case affects only the x-integration and the second one only the y-integration. If these were the only possible singularities, we could care at first about the x-integration without worrying about the factor $y^{c+d\epsilon}$ and then, in a second step, care about the y-integration.

The term $\frac{1}{P(x,y)}$ instead goes to infinity if *both* variables go to zero. In this case we speak of an *overlapping* singularity. Such singularities obviously make the treatment of the integral more difficult. Therefore we want to disentangle the overlapping singularity by the following decomposition of the integration domain. We write

$$I = I_1 + I_2 \tag{5.1.2}$$

with

$$I_1 = \int_0^1 dx \int_0^1 dy \frac{1}{P(x,\,y)} x^{a+b\epsilon} y^{c+d\epsilon} \theta(x \geq y), \tag{5.1.3}$$

$$I_2 = \int_0^1 dx \int_0^1 dy \frac{1}{P(x,\,y)} x^{a+b\epsilon} y^{c+d\epsilon} \theta(y \geq x) \tag{5.1.4}$$

where θ denotes the Heaviside-function. The supports of the integrands of I_1 and I_2 are shown in figure (5.1.3). Both regions have the shape of a triangle. We deform them into quadratic regions by the following transformations. For I_1 we substitute

$$\begin{aligned} x &= x', \\ y &= x'y' \end{aligned}$$

and write I_1 in terms of the new parameters x', y'. We obtain

$$I_1 = \int_0^1 dx' \int_0^1 dy' \frac{1}{P_1'(x',\,y')} x'^{a+c+1+(b+d)\epsilon} y'^{c+d\epsilon}.$$

We have used that $\theta(x' \geq x'y') = 1$ everywhere in the integration domain. The polynomial is

$$\begin{aligned} P_1'(x',\,y') &= x' + x'^2 y' + x'y' \\ &= x' \left(1 + x'y' + y'\right). \end{aligned}$$

Since x' is factored out, we obtain

$$I_1 = \int_0^1 dx' \int_0^1 dy' \frac{1}{1 + x'y' + y'} x'^{a+c+(b+d)\epsilon} y'^{c+d\epsilon}.$$

We see, that the fraction $\frac{1}{1+x'y'+y'}$ does not go to infinity anywhere in the integration domain and therefore it does not yield a singularity of the integral. The only possible singularities of I_1 are disentangled ones, arising from the monomial $x'^{a+c+(b+d)\epsilon} y'^{c+d\epsilon}$.

Geometrically speaking, the zero-set of the polynomial $1 + x'y' + y'$ does not intersect with the integration domain. This is the desired case. The much simpler zero-set of the monomial still intersects the integration domain and this is what the singularity structure of I_1 still depends on. These singularities are not overlapping.

A similar treatment is applied to I_2. Here we substitute

$$\begin{aligned} x &= x'y', \\ y &= y' \end{aligned}$$

and obtain

$$I_2 \quad = \quad \int_0^1 dx' \int_0^1 dy' \frac{1}{x' + x'y' + 1} x'^{a+b\epsilon} y'^{a+c+(b+d)\epsilon}.$$

Here we have factored out y'. The zero-set of the polynomial $x'+x'y'+1$ does not intersect the integration domain and therefore we have no overlapping singularities.

Let us consider the decomposition $I = I_1 + I_2$ and observe, what has happened to the critical point $x = 0$, $y = 0$ in the new coordinates. In the domain of I_1, by the given substitutions, the point is transformed into the line where $x' = 0$. Accordingly in the domain of I_2 the critical point has become the line $y' = 0$. In figure (5.1.3) we have marked these regions by thick black lines. Comparing the figures (5.1.2) and (5.1.3) we see the similarity to our barbeque example from above. In both examples a problematic region, here a point, is associated to regions of higher dimension. Then in order to approach the latter regions, one needs to consider one coordinate less than in the original problem. The dependence on the coordinates has been disentangled, or as we may say, resolved. We will come back to the present example in section 5.3 in order to reconsider it from a geometrical point of view.

EXAMPLE 38. As a slightly more difficult example let us consider the integral I of equation (5.1.1), where this time the polynomial is

$$P(x, y) \quad = \quad x + xy + y^2.$$

Similar to the previous example, we decompose $I = I_1 + I_2$. In the first sector, after the substitutions $x = x'$ and $y = x'y'$, we obtain

$$P_1'(x', y') \quad = \quad x' \left(1 + x'y' + x'y'^2\right).$$

The polynomial in the bracket has the desired property of being non-zero in the domain of integration, so the possible singularities of I_1 are disentangled.

However, in the second sector we have a different situation this time. After the substitutions $x = x'y'$ and $y = y'$ we obtain

$$P_2'(x', y') \quad = \quad y' \left(x' + x'y' + y'\right).$$

Therefore we have

$$I_2 \quad = \quad \int_0^1 dx' \int_0^1 dy' \frac{1}{x' + x'y' + y'} x'^{a+b\epsilon} y'^{a+c+(b+d)\epsilon}.$$

The zero-set of the polynomial $x'+x'y'+y'$ intersects with the domain of integration. We see, that the latter polynomial already appeared in the starting point of example 37. So we can furthermore decompose $I_2 = I_{2,1} + I_{2,2}$, according to the steps of example 37. As a result, the singularities in $I_{2,1}$ and $I_{2,2}$ are disentangled. So we obtain the decomposition $I = I_1 + I_{2,1} + I_{2,2}$ where none of the integrals on the right-hand side has overlapping singularities.

For the above concept to be useful for the calculation of more general integrals we need to allow for more integration variables and for an arbitrary number of decompositions. This is provided by the widely used sector decomposition algorithm of Binoth and Heinrich [**BH00, BH04, Hei08**], which we review below.

A sector decomposition was already used by Hepp [**Hep66**] to disentangle overlapping ultraviolet singularities in his famous proof of the Bogoliubov-Parasiuk-theorem [**BP57**]

on renormalization[3]. Later, in a practical calculation of Feynman integrals, the concept was used for the extraction of singularities of certain one-loop-integrals by Denner and Roth [**DR96**].

The algorithm by Binoth and Heinrich was the first algorithm which *systematically iterates* the decomposition of integration domains in order to provide (numerical) results for Feynman integrals with an *arbitrary* loop-number. In practical applications, there are of course restrictions given by the realizations on the computer, but the algorithm itself can be applied to integrals of arbitrarily complicated Feynman graphs. An overview of the wide range of applications of the algorithm can be found in [**Hei08**].

We want to indicate what the main problem of the following treatment will be. With equation (5.1.2) we decomposed the original integration domain of the above example into two regions, such that in one region we have $x \geq y$ and in the other one we have $y \geq x$. In the following we want to decompose higher dimensional domains such that in each resulting piece one of the variables is greater or equal to *some* of the other variables, possibly not all of them. This leaves a certain freedom of choice. We want to formalize this choice as follows:

Consider an integral of the form

$$I = \int_0^1 dx_1 \int_0^1 dx_2 ... \int_0^1 dx_n f(x_1, ..., x_n)$$

with some function f. The domain of integration is an n-dimensional cube. We choose a subset of the numbers from 1 to n,

$$S = \{\alpha_1, ..., \alpha_k\} \subseteq \{1, ..., n\},$$

containing k elements where $2 \leq k \leq n$. Then the domain is decomposed into k regions, such that in the lth region x_{α_l} is greater or equal the other x_{α_i} with $\alpha_i \in S$.

Example 39. Let the original domain be the unitary cube in $\mathbb{R}^3$. Then we have four possible choices for S, namely $\{1, 2\}$, $\{2, 3\}$, $\{1, 3\}$ and $\{1, 2, 3\}$ (see figure 5.1.4). In the case of $S = \{1, 2\}$ we have two regions, namely one where $x_1 \geq x_2$ and one where $x_2 \geq x_1$. Choosing $\{2, 3\}$ or $\{1, 3\}$ we obtain two regions as well. The choice $S = \{1, 2, 3\}$ splits the domain into three regions: one where $x_1 \geq x_2$ and $x_1 \geq x_3$, one where $x_2 \geq x_1$ and $x_2 \geq x_3$ and a third one where $x_3 \geq x_1$ and $x_3 \geq x_2$.

The difficulty will be, that not all possible choices of S are equally helpful for the disentanglement of the singularities. In fact, some ways of choosing S can even lead to a never ending iteration, as we will see.

5.1.2. The Algorithm of Binoth and Heinrich. We consider polynomials P in real valued variables $x_1, x_2, ..., x_n$. Let X denote the set $\{x_1, x_2, ..., x_n\}$. Writing $P(X)$ we mean, that P depends on some of the elements of X, not necessarily on all of them. $\mathbb{R}^n_+$ shall denote the subset of $\mathbb{R}^n$, where $x_j \geq 0$ for all the x_j. $\mathbb{R}^n_{+\setminus 0}$ denotes the subset where $x_j > 0$ for all the x_j.

We can write each polynomial as

$$P(X) = \sum_{j=1}^{p} k_j \prod_{i=1}^{n} x_i^{m_{ij}}$$

[3]The theorem and Hepp's proof are, together with Zimmermann's work [**Zim69**], the fundaments of the well-known BPHZ-renormalization. A first proof was already provided by Bogoliubov and Parasiuk, but according to Hepp it was "hard to find two theoreticians whose understanding of the essential steps of the proof is isomorphic".

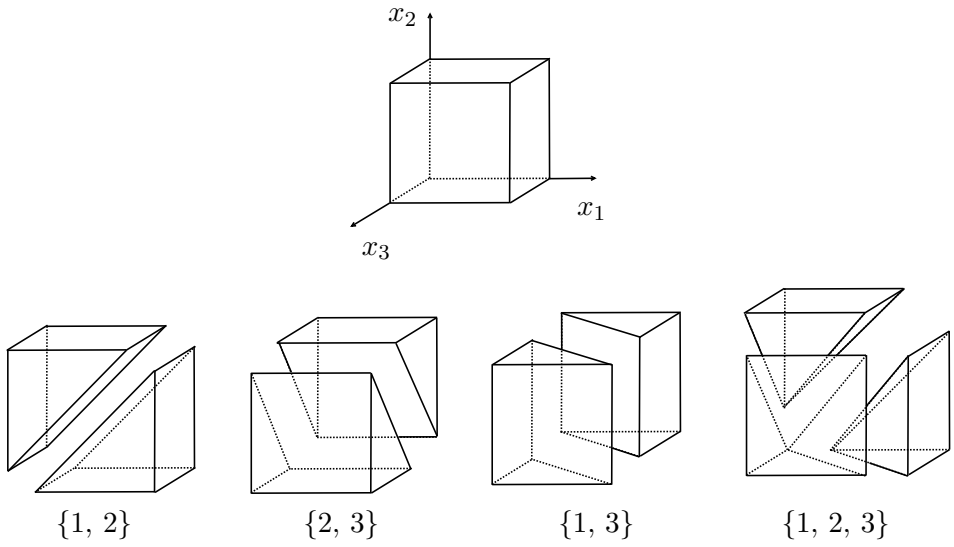


FIGURE 5.1.4. Four ways of cutting a cube into pieces.

where the m_{ij} are integers and the k_j are nonzero coefficients with real values. Then obviously $P(X)$ is nonzero in $\mathbb{R}^n_{+\backslash 0}$ if we have $k_j > 0$ for all $j = 1, 2, ..., p$. In the following algorithm, we will allow only for such polynomials.

A polynomial is nonzero in $\mathbb{R}^n_+$, if all the coefficients are positive and furthermore in one of the terms all the exponents m_{ij} are zero, such that this term consists only of the nonzero coefficient. An example is $P_1(x_1, x_2) = k_1 x_1 x_2 + k_2 x_2^2 + k_3$ with k_1, k_2, k_3 greater than zero.

Now let us call a polynomial $P(X)$ *monomialized*, if we can write it as

$$P(X) = \tilde{P}(X) \prod_{i=1}^{n} x_i^{m_i} \tag{5.1.5}$$

such that in $\tilde{P}(X)$ one term of the sum consists of a nonzero coefficient. An example is $P_2(x_1, x_2) = k_1 x_1^3 x_2^2 + k_2 x_1^2 x_2^3 + k_3 x_1^2 x_2 = \tilde{P}(x_1, x_2) x_1^2 x_2$ with the above P_1 as $\tilde{P}$. In different words, a polynomial is called monomialized, if one of its monomials can be factored out. We see that if P is monomialized and all its coefficients are positive then $\tilde{P}(X)$ is nonzero in $\mathbb{R}^n_+$.

Starting point: The algorithm is applied to an integral

$$I = \int_{x_j \geq 0} d^n x \delta \left(1 - \sum_{i=1}^{n} x_i\right) \left(\prod_{i=1}^{n} x_i^{a_i + \epsilon b_i}\right) \prod_{j=1}^{r} \left(P_j(X)\right)^{c_j + \epsilon d_j} \tag{5.1.6}$$

where the integration domain is understood to be $\mathbb{R}^n_+$. All the parameters a_i, b_i, c_i and d_i are integers and $\epsilon \in \mathbb{C}$ is the regularization parameter. For the algorithm to be applicable, the polynomials in the integrand need to have the important property of being *nonzero in* $\mathbb{R}^n_{+\backslash 0}$, *but not necessarily nonzero in* $\mathbb{R}^n_+$. In other words, they are not allowed to vanish *inside* the integration domain, but they may vanish on the boundary.

Note that a general scalar Feynman integral (cf. equation (2.3.6))

$$I_G = \int_{x_j \geq 0} d^N x \delta \left(1 - \sum_{i=1}^{N} x_i\right) \left(\prod_{i=1}^{N} x_i^{\nu_i - 1}\right) \frac{{\mathcal{U}_G}^{\nu - (L+1)D/2}}{{\mathcal{F}_G}^{\nu - LD/2}}, \tag{5.1.7}$$

(where we omitted a trivial prefactor) is obtained as a special case of the above integral I by setting $n = N$, $r = 2$, $a_i = \nu_i - 1$, $b_i = 0$, $P_1 = \mathcal{U}_G$, $P_2 = \mathcal{F}_G$, $c_1 = \nu - 2(L+1)$, $d_1 = L+1$, $c_2 = \nu - 2L$, $d_2 = L$. N is the number of internal edges and L is the loop-number of the Feynman graph. For every Feynman graph G the polynomial $\mathcal{U}_G$ is nonzero in $\mathbb{R}^n_{+\backslash 0}$ as all coefficients are equal to 1. Whether the condition is fulfilled for $\mathcal{F}_G$ depends on the external momenta and the masses. In the *Euclidean momentum region* where all masses are positive or zero and all kinematical invariants are negative or zero, $\mathcal{F}_G$ is nonzero in $\mathbb{R}^n_{+\backslash 0}$. For this region of masses and external momenta, the algorithm can be applied to the Feynman integral I_G.

It is understood that the integrals I and I_G in the above equations (5.1.6) and (5.1.7) are functions of the regularization parameter and possibly of variable coefficients of the terms in the polynomials, considered in the region of positive values, such that each polynomial P_j is nonzero in $\mathbb{R}^n_{+\backslash 0}$. In particular the Feynman integral I_G depends on squared masses and kinematical invariants (cf. the definition of $\mathcal{F}_G$ in equation (2.3.9)). Here, for the sake of brevity, we avoid to always denote these dependences explicitely.

Step 0: Homogenization. Due to the presence of $\delta\left(1 - \sum_{i=1}^n x_i\right)$ we can multiply terms in the integrand with $\sum_{i=1}^n x_i$ without changing the integral. In this way we can obtain homogeneous polynomials. For each polynomial P_j in the integrand we determine its highest degree h_j in all the variables x_i and multiply all its terms of a lower degree with an appropriate power of $\sum_{i=1}^n x_i$, such that the resulting polynomial is homogeneous.

We give an example for an integral over two parameters x_1 and x_2 with $\delta(1 - x_1 - x_2)$. For $P = x_1^3 + x_2$ in the integrand the highest degree is 3 and the present step of the algorithm yields a substitution of P by the homogeneous polynomial $P' = x_1^3 + (x_1 + x_2)^2 x_2 = x_1^3 + x_1^2 x_2 + 2x_1 x_2^2 + x_2^3$.

Note that for Feynman integrals nothing needs to be done here, as the polynomials $\mathcal{U}_G$ and $\mathcal{F}_G$ are homogeneous already.

From here on let h_j denote the degree of the polynomial P_j.

Step 1: Primary decomposition. Now let $\mathcal{I}$ denote the integrand of I. We decompose I into n terms, the so-called *primary* sectors, as

$$I = \int_{x_j \geq 0} d^n x \mathcal{I} \quad = \quad \sum_{l=1}^n \int_{x_j \geq 0} d^n x \prod_{i=1,\, i\neq l} \theta(x_l \geq x_i) \mathcal{I}.$$

Then we make a substitution for each term of this sum: In the l-th term we substitute

$$\begin{aligned} x_j &= x'_l x'_j \text{ for } j \neq l, \\ x_l &= x'_l. \end{aligned}$$

The set of the new parameters x'_1, ..., x'_n is denoted by X' and the substitutions yield

$$\begin{aligned} I \quad = \quad & \sum_{l=1}^n \int_{x'_j \geq 0} \int_{x_l \geq 0} d^{n-1} x' dx'_l \delta \left(1 - x'_l - x'_l \sum_{i=1,\, i\neq l}^n x'_i \right) \prod_{i=1,\, i\neq l} \theta(x'_l \geq x'_l x'_i) \\ & \cdot {x'_l}^{-c} \left(\prod_{i=1,\, i\neq l}^n {x'_i}^{a_i + \epsilon b_i} \right) \prod_{j=1}^r \left(P'_j \left(X' \right) \right)^{c_j + \epsilon d_j} \end{aligned}$$

with

$$-c = n - 1 + \sum_{i=1}^n (a_i + \epsilon b_i) - \sum_{j=1}^r h_j (c_j + \epsilon d_j)$$

and where we have factored out powers of x_l as

$$P_j(X) = {x'}_l^{\sum_{j=1}^r h_j(c_j+\epsilon d_j)} {P'}_j\left(X'\right).$$

Note that the new polynomials ${P'}_j(X')$ are independent of ${x'}_l$, because of the homogeneity of the P_j. Therefore it is now easy to evaluate the ${x'}_l$-integration as

$$\int_{{x'}_l\geq 0} d{x'}_l \delta\left(1-{x'}_l-{x'}_l\sum_{i=1,\,i\neq l}^{n}{x'}_i\right)\prod_{i=1,\,i\neq l}^{n}\theta({x'}_l\geq {x'}_l{x'}_i){x'}_l^{-c} =$$

$$\prod_{i=1,\,i\neq l}^{n}\theta(1\geq {x'}_i)\left(1+\sum_{i=1,\,i\neq l}^{n}{x'}_i\right)^c.$$

In the remaining integrals over x'_i the product of θ-functions restricts the support of the integrand to the $n-1$-dimensional unit-cube. We therefore obtain

$$I \;=\; \sum_{l=1}^{n}\int_0^1 d^{n-1}x'\left(\prod_{i=1,\,i\neq l}^{n}{x'}_i^{a_i+\epsilon b_i}\right)\left(1+\sum_{i=1,\,i\neq l}^{n}{x'}_i\right)^c\prod_{j=1}^{r}\left({P'}_j\left(X'\right)\right)^{c_j+\epsilon d_j}.$$

Omitting the primes and considering $\left(1+\sum_{i=1,\,i\neq l}^{n}{x'}_i\right)^c$ as just one more polynomial in the integrand, we see that the terms we obtain in the sum are integrals of the form[4]

$$\int_0^1 d^nx\left(\prod_{i=1}^{n}x_i^{a_i+\epsilon b_i}\right)\prod_{j=1}^{r}\left(P_j(X)\right)^{c_j+\epsilon d_j}. \tag{5.1.8}$$

At this stage, all polynomials $P_j(X)$ are non-negative in $\mathbb{R}_+^n$ and not necessarily homogeneous.

Step 2: Iterative decomposition. The following step is crucial for the disentanglement of the singularities. At first let us consider one primary sector and one polynomial $P(X)$ of its integrand, assuming that $P(X)$ is not monomialized. (If all the polynomials are monomialized, there is nothing to do and one proceeds with step 3.) The algorithm determines a set $S=\{\alpha_1,\,...,\,\alpha_k\}\subseteq\{1,\,...,\,n\}$, containing at least two elements. The choice depends on the polynomial $P(X)$. A set of rules which defines how to obtain S from $P(X)$ shall be called a *strategy*. Let us for the moment assume, that we have a given strategy at hand. We will give a detailed discussion on various possible strategies below.

According to the chosen set S we decompose the integral of the sector under consideration as:

$$\int_0^1 d^nx \;=\; \sum_{l=1}^{k}\int_0^1 d^nx\prod_{i=1,\,i\neq l}^{k}\theta\left(x_{\alpha_l}\geq x_{\alpha_i}\right). \tag{5.1.9}$$

Here the sum and the product run through the indices 1, ..., k of the elements in S. The terms of the sum are called *sub-sectors*. (Such decompositions were considered in the examples 37, 38 and 39.)

[4]In order to avoid a possible source of confusion we remark, that, as one integration was carried out, the integrals are of the form $\int_0^1 d^{n-1}x...$, but as the integrand in the l-th primary sector does not depend on x_l we may also write $\int_0^1 d^nx...$ because $\int_0^1 dx_l=1$. Anyway it will not be important for the following, whether we have $n-1$ or n integrations.

In each sub-sector we make different substitutions: In the l-th sub-sector we substitute

$$x_i = \begin{cases} {x'}_{\alpha_l} \text{ for } i = \alpha_l, \\ {x'}_i {x'}_{\alpha_l} \text{ for } i \neq \alpha_l \text{ and } i \in S, \\ x'_i \text{ for } i \neq S. \end{cases} \tag{5.1.10}$$

Let us denote the resulting polynomial in the l-th sub-sector by $P'_l(X')$. An appropriate choice of S will lead to a factorization in $P'_l(X')$, such that we can factor out ${x'}_l$ to a certain power:

$$P'_l(X') = {x'}^m_{\alpha_l} \tilde{P}'_l(X').$$

The substitution furthermore yields a Jacobian factor $x_{\alpha_l}^{k-1}$ in the l-th sector.

Now we iterate this step in the following sense: For each newly obtained $\tilde{P}_l$ which is not monomialized we choose a new set S_l according to the strategy and decompose the l-th sub-sector in the above sense into further sub-sectors and perform the corresponding substitutions. The iteration stops if all the polynomials which are generated in this way are monomialized.

We continue with this iterative decomposition for each non-monomialized polynomial in all the sectors. As a result we obtain a decomposition for I, where each term in the sum is an integral of the type of equation (5.1.8) such that all the polynomials in all the integrands are monomialized.

We want to illustrate this crucial step of the algorithm by one more example:

EXAMPLE 40. Let us assume $P(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_2^2x_3$ is a polynomial in the integrand of a primary sector. Obviously it is not monomialized. Let us furthermore assume the strategy tells us to choose $S = \{1, 2\}$ for this polynomial. Then there are two sub-sectors.

For the first sub-sector the substitutions are $x_1 = {x'}_1$, $x_2 = {x'}_1{x'}_2$ and $x_3 = {x'}_3$. We obtain $P'_1({x'}_1, {x'}_2, {x'}_3) = {x'}_1^2\left(1 + {x'}_2 + {x'}_2^2{x'}_3\right)$. This polynomial is monomialized. If in the same integral there are no more polynomials which are not monomialized, this sub-sector is not decomposed any further.

In the second sub-sector the substitutions are $x_1 = {x'}_1{x'}_2$, $x_2 = {x'}_2$ and $x_3 = {x'}_3$. We obtain $P'_2({x'}_1, {x'}_2, {x'}_3) = {x'}_2^2\left({x'}_1^2 + {x'}_1 + {x'}_3\right) =: {x'}_2^2\tilde{P}'_2$. This polynomial is not monomialized. The zero-set of the term $\tilde{P}'_2$ intersects with the integration domain. Therefore the iteration must continue. The next set S_2 is chosen with respect to the polynomial $\tilde{P}'_2$ and the sub-sector is further decomposed.

Step 3: Pole separation. At the end of step 2, the integral I is decomposed into a sum, where each term corresponds to a sub-sector and where all the appearing integrals are still of the form

$$\int_0^1 d^n x \left(\prod_{i=1}^n x_i^{a_i+\epsilon b_i}\right) \prod_{j=1}^r \left(P_j(X)\right)^{c_j+\epsilon d_j},$$

but now all the polynomials are non-zero in the integration domain. Therefore it depends exclusively on the monomial $\prod_{i=1}^n x_i^{a_i+\epsilon b_i}$ whether the integral diverges or not. A singularity may arise from the integration over any x_j which appears with a negative power $a_j < 0$ in the monomial.

Let us for the moment consider the integral in only one such variable x_j. The integrand is denoted $\mathcal{I}(x_j,\,\epsilon)$. We Taylor-expand $\mathcal{I}(x_j,\,\epsilon)$ around $x_j = 0$:

$$\int_0^1 dx_j x_j^{a_j+b_j\epsilon}\mathcal{I}(x_j,\,\epsilon) \;=\; \int_0^1 dx_j x_j^{a_j+b_j\epsilon}\left(\sum_{p=0}^{|a_j|-1}\frac{x_j^p}{p!}\mathcal{I}^{(p)}(\epsilon)+\mathcal{I}^{(R)}(x_j,\,\epsilon)\right) \tag{5.1.11}$$

where $\mathcal{I}^{(p)}(\epsilon)$ denotes the p-th derivative in x_j and $\mathcal{I}^{(R)}(x_j,\,\epsilon)$ the remainder of the expansion:

$$\begin{aligned}\mathcal{I}^{(p)}(\epsilon) &= \frac{\partial}{\partial x_j^p}\,\mathcal{I}(x_j,\,\epsilon)|_{x_j=0}\,,\\ \mathcal{I}^{(R)}(x_j,\,\epsilon) &= \mathcal{I}(x_j,\,\epsilon)-\sum_{p=0}^{|a_j|-1}\frac{x_j^p}{p!}\mathcal{I}^{(p)}(\epsilon).\end{aligned}$$

The x_j-integral over $\mathcal{I}^{(R)}(x_j,\,\epsilon)$ does not lead to any ϵ-poles. Therefore, equation (5.1.11) separates the pole-part, which is the integral over the sum of derivatives. The x_j-integration in this part can now be evaluated analytically:

$$\int_0^1 dx_j x_j^{a_j+b_j\epsilon}\frac{x_j^p}{p!}\mathcal{I}^{(p)}(\epsilon) \;=\; \frac{1}{a_j+b_j\epsilon+p+1}\frac{\mathcal{I}^{(p)}(\epsilon)}{p!}. \tag{5.1.12}$$

For all the variables x_j with $a_j < 0$ the separations and analytical integrations are carried out in this way.

We want to emphasize, that these simple analytical evaluations of the integrations which yield the pole terms are possible due to the disentanglement of the singularities, achieved by the monomialization of the polynomials in the previous step 2. If there still were overlapping singularities, arising from the intersections of zero-sets of polynomials (appearing to negative powers) with the integration domain, the present step 3 would not be possible.

Step 4: Series expansion. After the previous steps, all remaining integrals in the decomposition of I are finite. The coefficients of these integrals are ϵ-dependent rational functions, as we see from equation (5.1.12). Now we can expand I as a Laurent series in ϵ

$$I \;=\; \sum_{i=A}^{B} C_i\epsilon^i + \mathcal{O}\left(\epsilon^B\right)$$

with $A,\, B \in \mathbb{Z}$ and truncate the series to a desired order.

Step 5: Numerical integration. All the previous steps are done analytically and the functions C_i in terms of the remaining, finite integrals are obtained exactly. However, the analytical evaluation of these integrals is in general unknown. For this reason, the parameters for the masses and the kinematical invariants have to be fixed at some point and the remaining integrals are computed numerically, for example by use of Monte Carlo methods.

Example 41. Let us consider the so-called *massless planar on-shell double box*, which is the graph of figure 5.1.5 where we set $p_i^2 = 0$ for $i = 1,\, 2,\, 3,\, 4$ and all masses equal to zero. The corresponding Feynman integral is

$$I_G(\epsilon,\,s,\,t) \;=\; g(\epsilon)\int d^7x\delta\left(1-\sum_{i=1}^{7}x_i\right)\mathcal{U}^{1+3\epsilon}\mathcal{F}^{-3-2\epsilon} \tag{5.1.13}$$

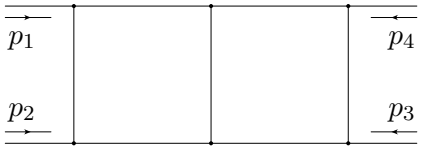


FIGURE 5.1.5. The planar double box.

with the prefactor

$$g(\epsilon) \quad = \quad (-1)^7 \Gamma(3+2\epsilon)$$

and the polynomials

$$\mathcal{U} = (x_1+x_2+x_3)(x_5+x_6+x_7) + x_4(x_1+x_2+x_3+x_5+x_6+x_7), \tag{5.1.14}$$

$$\mathcal{F} = (x_2x_3(x_4+x_5+x_6+x_7) + x_5x_6(x_1+x_2+x_3+x_4)$$

$$+x_2x_4x_6 + x_3x_4x_5)(-s) + x_1x_4x_7(-t). \tag{5.1.15}$$

The polynomial $\mathcal{F}$ depends on the kinematical invariants $s := (p_1+p_2)^2$ and $t := (p_2+p_3)^2$.

The above algorithm can be straightforwardly applied for negative values of s and t. By use of the original strategy which is defined in subsection 5.1.3, the algorithm produces 293 sectors. For $s=-1$ and $t=-1$ Binoth and Heinrich [**BH00**] obtain

$$I_G(\epsilon,-1,-1) \quad \approx \quad g(\epsilon)\left(-\frac{2.0000}{\epsilon^4}+\frac{6.0000}{\epsilon^3}+\frac{4.9188}{\epsilon^2}-\frac{11.495}{\epsilon}-13.811+\mathcal{O}(\epsilon)\right).$$

These and other results from reference [**BH00**] were important checks for the analytical evaluations of double box integrals by Smirnov [**Smi99**], Tausk [**Tau99**] and Gehrmann and Remiddi [**GR01c, GR01b**]. We also used these numbers for cross-checks with our implementation [**BW08**]. The precision of the numerical result depends exclusively on the method which is chosen for the numerical integrations in step 5. One can in principle always improve the precision at the cost of calculation time.

5.1.3. The termination problem. In the above description of the algorithm we consciously left a gap in the crucial step 2. The choice of a set

$$S = \{\alpha_1, ..., \alpha_k\} \subseteq \{1, ..., n\}$$

according to a given non-monomialized polynomial $P(X)$ determines, how the corresponding integration domain is decomposed. As we already agreed, we want to call a set of rules to determine S a *strategy*. Now let us introduce the original strategy by Binoth and Heinrich:

Strategy X. (Binoth and Heinrich [**BH00**]) Choose $S = \{\alpha_1, ..., \alpha_k\}$ of step 2 to be a minimal set, such that everywhere in the region given by $x_\alpha = 0$ for all $\alpha \in S$ the polynomial is zero.

Here *minimal* means that there is no proper subset $S' \subset S$ such that the polynomial is zero in the entire region given by $x_\alpha = 0$ for all $\alpha \in S'$.

As a first observation we notice, that Strategy X still leaves a freedom of choice in many cases. For example the polynomial $P(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_1$ according to Strategy X allows for the choices $\{1, 2\}$ and $\{1, 3\}$. ($\{1, 2, 3\}$ is not allowed, as it is not

minimal with respect to the mentioned property.) Of course a freedom of choice is not a problem for an algorithm. On the contrary, it may allow for a certain flexibility.

More important is the following observation:

EXAMPLE 42. We consider the polynomial $P(x_1, x_2, x_3) = x_1x_3^2 + x_2^2 + x_2x_3$. An allowed choice according to Strategy X is $S = \{1, 2\}$, because $P(0, 0, x_3) = 0$ and S is minimal in the above sense. In the first sector created by this choice, the substitutions are $x_1 = x'_1$, $x_2 = x'_1x'_2$, $x_3 = x'_3$ such that we obtain

$$\begin{aligned} P(x_1, x_2, x_3) &= x'_1x'^2_3 + x'^2_1x'^2_2 + x'_1x'_2x'_3 = x'_1\left(x'^2_3 + x'_1x'^2_2 + x'_2x'_3\right) \\ &= x'_1\tilde{P}_1\left(x'_1, x'_2, x'_3\right). \end{aligned}$$

As the polynomial $\tilde{P}_1(x'_1, x'_2, x'_3) = x'^2_3 + x'_1x'^2_2 + x'_2x'_3$ is not monomialized, we need to continue with the iterative decomposition and choose a new set S_1 corresponding to this polynomial. But we see that the new polynomial is in fact the old one:

$$\tilde{P}_1\left(x'_1, x'_2, x'_3\right) = P\left(x'_1, x'_3, x'_2\right).$$

The decomposition has in this sense not changed the situation and Strategy X again allows for a similar choice of S_l, such that again the same polynomial is obtained after the next decomposition, and so on ad infinitum. We have an infinite recursion.

This example shows, that the above algorithm with the use of Strategy X does not terminate in every case. Let us emphasize, that the example is not a pathological one. The chosen polynomial is very simple and there is no reason why such a polynomial should not appear at some stage of the decomposition of a multi-loop integral.

We formulate the following task:

PROBLEM 43. Determine a strategy for the choice of S, such that the algorithm is guaranteed to terminate for all possible cases.

In the remainder of the present chapter we discuss solutions to this problem. We will explain three different strategies, suitable to replace Strategy X in order to guarantee for the termination of the algorithm. These strategies have been found in the context of the combinatorial problem known as Hironaka's polyhedra game. We will demonstrate, that the combinatorics of step 2 in the algorithm of Binoth and Heinrich can be mapped to this problem.

REMARK 44. Despite the observed defect, Strategy X turned out to be very useful in practical calculations. For the Feynman integrals mentioned in subsection (5.2.7) below and many other examples, Strategy X did not lead to an infinite recursion. Moreover in these cases it turns out to be more efficient than other strategies discussed below, in the sense that it produces a smaller number of sub-sectors (see subsection 5.2.7). Our implementation [**BW08**] works with Strategy X as default strategy and allows the user to replace Strategy X by one of three further strategies discussed below.

Given the efficiency of Strategy X, the following two improvements, avoiding its replacement, were proposed by Heinrich in [**Hei08**]:

(1) The infinite recursion in example 42 is due to the presence of quadratic terms in the polynomial. At the very beginning of the sector decomposition of a Feynman integral, nonlinear terms can only occur in the polynomial $\mathcal{F}$ due to the presence of massive propagators, as we see from the equation $\mathcal{F} = \mathcal{F}_0 + \mathcal{U}\sum_{j=1}^{N} x_jm_j^2$. The polynomial $\mathcal{U}$ is linear in the x's, therefore the term $\mathcal{U}\sum_{j=1}^{N} x_jm_j^2$ is quadratic

in individual parameters x_j if $m_j \neq 0$. Therefore an infinite recursion at this early stage of the decomposition can be excluded by imposing an additional rule: The set S of the first decomposition of step 2 is chosen to be the indices of the Feynman parameters belonging to the edges of the massive propagarors. In other words, one chooses S such that $\alpha \in S$ if and only if $m_\alpha \neq 0$. Then, as a consequence of factorization, at least one of the quadratic powers does not appear in the new polynomial. Therefore the new polynomial does not resemble its predecessor, so an infinite recursion can not appear at this stage. Nevertheless, quadratic and higher order terms can of course occur again at later stages of the decomposition.

(2) In order to avoid infinite recursions in certain cases at later stages, S can be chosen to be the maximal set of variables occurring with the same power in the polynomial.

Both extensions are heuristic in nature and have shown to be useful in many applications [**Hei08**]. They are not known to guarantee for the termination in the general case.

5.1.4. The combinatorics of sector decomposition. In order to solve problem 43 let us formalize the combinatorics of step 2 of the algorithm.

At first we observe, that a polynomial which is monomialized can not lose this property in further decompositions. Therefore the algorithm can monomialize the polynomials in sequence. If one polynomial is monomialized in every sub-sector, the algorithm can proceed with the next polynomial, and so on. Therefore, if the algorithm is known to terminate for an integral I which has only one polynomial in the integrand, we can also be sure that it terminates for integrals with a product of many polynomials. For this reason, it is sufficient to study the combinatorics of the algorithm for the case of only one polynomial.

Let us consider a polynomial depending on n parameters x_1, ..., x_n and consisting of p terms, denoted

$$P(X) = \sum_{i=1}^{p} k_i x_1^{m_1^{(i)}} x_2^{m_2^{(i)}} ... x_n^{m_n^{(i)}}.$$

The exponents $m_j^{(i)}$ are non-negative integers and the coefficients k_i are positive. We remember that one step in the iterative decomposition consists of:

- The choice of the set $S = \{\alpha_1, ..., \alpha_k\} \subseteq \{1, ..., n\}$ according to $P(X)$ by a strategy.
- The substitution

$$x_j = \begin{cases} x'_{\alpha_l} \text{ for } j = \alpha_l, \\ x'_j x'_{\alpha_l} \text{ for } j \neq \alpha_l \text{ and } j \in S, \\ x'_j \text{ for } j \neq S. \end{cases} \tag{5.1.16}$$

in the l-th term of the sum of equation (5.1.9).

Let us consider the l-th term of this sum, i. e. the l-th sector, and let $u_j^{(i)}$ denote the exponents of the new parameters x'_j, defined by

$$\begin{aligned} P(X) &= \sum_{i=1}^{p} k_i x_1^{m_1^{(i)}} x_2^{m_2^{(i)}} ... x_n^{m_n^{(i)}}. \\ &= \sum_{i=1}^{p} k_i x'^{u_1^{(i)}}_1 x'^{u_2^{(i)}}_2 ... x'^{u_n^{(i)}}_n = P'_l\left(X'\right). \end{aligned} \tag{5.1.17}$$

Of course in the integral the substitution additionally creates a Jacobian prefactor, but this does not affect the termination.

It is easy to observe, that the substitution of equation (5.1.16) can equivalently be described by the equations

$$u_j^{(i)} = \begin{cases} \sum_{\alpha \in S} m_\alpha^{(i)} \text{ if } j = \alpha_l, \\ m_j^{(i)} \text{ if } j \neq \alpha_l \end{cases} \tag{5.1.18}$$

for all terms of the polynomial, $i = 1, 2, ..., p$.

In the desirable case, $P'_l(X')$ factorizes in ${x'}_{\alpha_l}$ to some power c:

$$P'_l(X') = {x'}_{\alpha_l}^c \tilde{P}_l(X'). \tag{5.1.19}$$

If we denote the exponents in $\tilde{P}_l$ by $v_j^{(i)}$ such that

$$\tilde{P}_l(X') = \sum_{i=1}^{p} k_i {x'}_1^{v_1^{(i)}} {x'}_2^{v_2^{(i)}} ... {x'}_n^{v_n^{(i)}},$$

we can equivalently denote the substitution of equation (5.1.16) as

$$v_j^{(i)} = \begin{cases} \sum_{\alpha \in S} m_\alpha^{(i)} - c \text{ if } j = \alpha_l, \\ m_j^{(i)} \text{ if } j \neq \alpha_l \end{cases} \tag{5.1.20}$$

with the same number $c \in \mathbb{N}$ for all $i = 1, 2, ..., p$. We emphasize, that $\tilde{P}_l(X')$ is obviously monomialized if and only if $P'_l(X')$ is monomialized. So the condition for the algorithm to terminate can be formulated with both polynomials equivalently.

Which condition must be fulfilled by the $u_j^{(i)}$ such that $P'_l(X')$ is monomialized? $P'_l(X')$ is monomialized if its sum contains a term whose monomial ${x'}_1^{u_1^{(q)}} {x'}_2^{u_2^{(q)}} ... {x'}_n^{u_n^{(q)}}$ can be factored out from $P'_l(X')$, such that

$$P'_l(X') = {x'}_1^{u_1^{(q)}} {x'}_2^{u_2^{(q)}} ... {x'}_n^{u_n^{(q)}} \left(k_q + \sum_{i=1,\, i \neq q}^{p} k_i {x'}_1^{u_1^{(i)} - u_1^{(q)}} {x'}_2^{u_2^{(i)} - u_2^{(q)}} ... {x'}_n^{u_n^{(i)} - u_n^{(q)}} \right)$$

and all powers $u_j^{(i)} - u_j^{(q)}$ are non-negative. k_q can take any positive value. In other words, there must be an integer q in the set $\{1, 2, ..., p\}$ such that

$$u_j^{(q)} \leq u_j^{(i)} \tag{5.1.21}$$

for all $i = 1, 2, ..., p$ and all $j = 1, 2, ..., n$.

Now let us translate the substitution and the condition of being monomialized into a geometric formulation. Again we consider the region $\mathbb{R}_+^n$ of the n-dimensional real space, where all coordinates are non-negative, but now let the points in this space not be given by the x_j-parameters, but by *their exponents*. Let the exponent vector $m^{(i)} = \left(m_1^{(i)}, m_2^{(i)}, ..., m_n^{(i)}\right)$ of each term in $P(X)$ define a point in $\mathbb{R}_+^n$. Then the polynomial $P(X)$ with p terms determines a set $M = \left\{m^{(1)}, m^{(2)}, ..., m^{(p)}\right\}$ of p points. $m_j^{(i)}$ denotes the j-th coordinate of the i-th point. Remember that all the $m_j^{(i)}$ are non-negative integers.

Now we may compare the points in M given by $P(X)$ to the points in a set $U_{\alpha_l} = \left\{u^{(1)}, u^{(2)}, ..., u^{(p)}\right\}$, which is the set of exponent vectors of $P'_l(X')$, such that $u_j^{(i)}$ as in equation (5.1.17) is the j-th coordinate of the i-th point in U_{α_l}. We see from equation (5.1.18), that the points in U_{α_l} differ from the points in M only in the α_l-th coordinate.

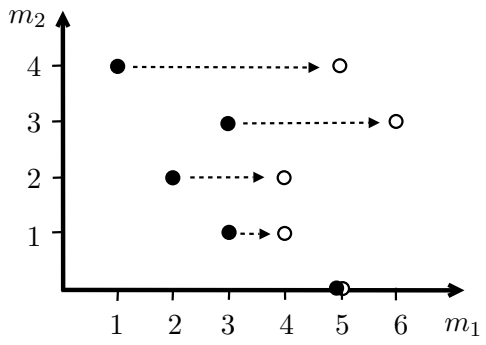


FIGURE 5.1.6. A set of points, shifted according to a decomposition step.

A point $u^{(i)}$ is the result of shifting $m^{(i)}$ along the α_l-th coordinate-axis. Moreover if we additionally consider the set of points $V_{\alpha_l} = \left\{v^{(1)}, v^{(2)}, ..., v^{(p)}\right\}$, obtained in the same manner from $\tilde{P}_l\left(X'\right)$, we see that V_{α_l} is obtained from U_{α_l}, by shifting each point by c along the α_l-th axis.

EXAMPLE 45. We consider the polynomial

$$P(x_1, x_2) = x_1x_2^4 + x_1^2x_2^2 + x_1^3x_2 + x_1^3x_2^3 + x_1^5.$$

The corresponding set M consists of the five points (1, 4), (2, 2), (3, 1), (3, 3) and (5, 0). We assume $S = \{1, 2\}$ and consider the first sector, where we substitute $x_1 = x'_1$ and $x_2 = x'_1x'_2$. We obtain

$$P'_1(x'_1, x'_2) = x'^5_1x'^4_2 + x'^4_1x'^2_2 + x'^4_1x_2 + x'^6_1x'^3_2 + x'^5_1.$$

The corresponding set of points U contains (5, 4), (4, 2), (4, 1), (6, 3) and (5, 0). Figure (5.1.6) shows the sets M (full points) and U (empty points).

It will be useful to consider an auxiliary object, called the *positive convex hull* of a set of points.

DEFINITION 46. A region in $\mathbb{R}^n$ is said to be *convex*, if every pair of two points in this region can be joined by a straight line segment, all of whose points belong to the region. The *convex hull* of a set of points M is the minimal convex set containing M. The *positive convex hull* Δ_M of the set M is defined to be the convex hull of the set $\cup_{m\in M}\left(m + \mathbb{R}^n_+\right)$.

It is much simpler to introduce Δ_M by pictorial examples, as given by the grey areas in figure 5.1.7 (a) and (b). Δ_M is sometimes called the *Newton polyhedron* of M. A point $m \in M$ is called a *generator* of the positive convex hull Δ_M, if Δ_M is different from $\Delta_{M\backslash m}$ where $M\backslash m$ denotes the set of points in M without the point m. In other words, the generators are the *corner points* of the Newton polyhedron, and if we leave one of them away, the polyhedron looks different. For example in figure 5.1.7 (a) the point $m^{(1)}$ is a generator of Δ_M, while the point $m^{(2)}$ is not.

If only one point m is a generator of Δ_M then we have $\Delta_M = \left(m + \mathbb{R}^n_+\right)$. An example is shown in figure 5.1.7 (b). This situation resembles the case of a monomialized polynomial:

LEMMA 47. *The polynomial $P\left(X\right)$ is monomialized, if and only if the positive convex hull Δ_M of the corresponding set of points M is generated by only one point.*

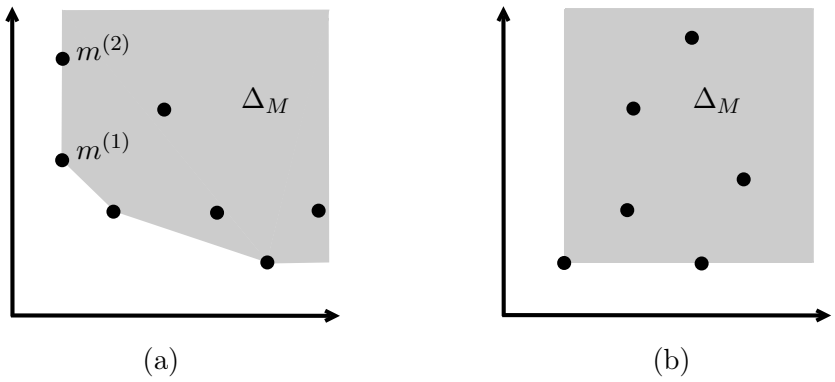


FIGURE 5.1.7. The positive convex hull Δ_M, generated by three points in (a) and generated by one point in (b).

PROOF. If $P(X)$ is monomialized there must be a q in $\{1, 2, ..., p\}$ such that $m_j^{(q)} \leq m_j^{(i)}$ for all $i = 1, 2, ..., p$ and all $j = 1, 2, ..., n$. Then Δ_M is the positive convex hull of $\left(m^{(q)} + \mathbb{R}_+^n\right)$ and therefore it is generated only by $m^{(q)}$. This proves one part of the statement. If Δ_M is generated by only one point, meaning that there is one point m in $\mathbb{R}_+^n$ such that Δ_M is the convex hull of $\left(m + \mathbb{R}_+^n\right)$, then m must by the definition of Δ_M be a point in M. As m is the only point in M generating Δ_M, we have $m_j \leq m_j^{(i)}$ for all $i = 1, 2, ..., p$ and all $j = 1, 2, ..., n$, which is the condition for $P(X)$ to be monomialized. This proves the second part of the statement. □

Now we have described a decomposition step and the condition for the termination of the decomposition in terms of sets of points. Let us summarize the entire step 2 of the algorithm of Binoth and Heinrich in this formulation:

- We start with a given polynomial $P(X)$, which corresponds to a given set of points M in $\mathbb{R}_+^n$. The coordinates are natural numbers.
- The choice of the set S according to a given strategy corresponds to a choice of coordinate directions in $\mathbb{R}_+^n$.
- For each element α_l in S a new set of points U_{α_l} is created. It is obtained from M by shifts depending on the element α_l under consideration (i. e. the sector) and on the entire set S. These shifts are given by equation (5.1.18).
- We iterate the procedure in the sense that $P'_l(X')$ (or $\tilde{P}_l(X')$) is the newly given polynomial and correspondingly U_{α_l} (or V_{α_l}) the given set of points.
- The iteration stops, if the polynomial is monomialized, that is, if the positive convex hull of the corresponding set of points is generated by only one point.
- The algorithm terminates, if the latter condition is fulfilled in all the created sectors after a finite number of iterations. Problem 43 is the task to find a strategy, such that this is always the case.

What we just described here is in fact the problem of Hironaka's polyhedra game. This was for the first time noticed by Müller-Stach and Weinzierl and then elaborated in our work with Weinzierl [**BW08**]. In the following we will discuss this game and three winning strategies, which will be suitable strategies to guarantee for the termination of the algorithm.

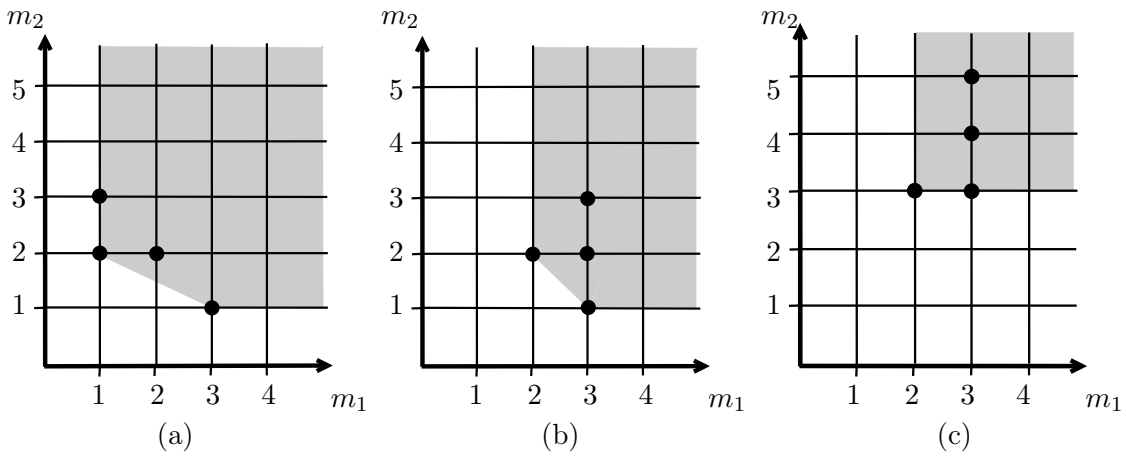


FIGURE 5.2.1. A run of the polyhedra game: Player A always chooses $\{1, 2\}$. Player B chooses 1 in the first move and 2 in the second move. Player A wins after two moves.

5.2. Hironaka's Polyhedra Game

5.2.1. The Game. Hironaka's polyhedra game [**Hir67**] is a simple combinatorial game for two players. Let us call[5] them player A and player B. To play the game they just need the notion of a set of points $M = \left\{m^{(1)}, m^{(2)}, ..., m^{(p)}\right\}$ in $\mathbb{N}^n$ and the corresponding positive convex hull $\Delta_M \subset \mathbb{R}^n_+$. We remember, that Δ_M is defined to be the convex hull of the set $\cup_{m\in M}\left(m + \mathbb{R}^n_+\right)$. Δ_M is said to be generated by only one point, if it is the convex hull of the set $\left(m + \mathbb{R}^n_+\right)$ where m is just one point in $\mathbb{R}^n_+$.

The game is the following:

Player A and player B are given an initial set of points $M = \left\{m^{(1)}, m^{(2)}, ..., m^{(p)}\right\}$ in $\mathbb{N}^n$. One move of the game is:

- Player A chooses a non-empty subset $S \subseteq \{1, ..., n\}$.
- Player B chooses one element α_l out of this subset.

Then, according to the choices of the players, all the points $m^{(i)} = \left(m^{(i)}_1, ..., m^{(i)}_n\right) \in M$ with $i = 1, ..., p$, are replaced by new points $m'^{(i)} = \left(m'^{(i)}_1, ..., m'^{(i)}_n\right)$ given by

$$m'^{(i)}_j \quad = \quad \begin{cases} \sum_{\alpha\in S} m^{(i)}_\alpha - c \text{ if } j = \alpha_l, \\ m^{(i)}_j \text{ if } j \neq \alpha_l \end{cases}$$

where $c \in \mathbb{N} \cup \{0\}$ is fixed and small enough, such that the new coordinates are non-negative. This defines the new set of points $M' = \left\{m'^{(1)}, m'^{(2)}, ..., m'^{(p)}\right\}$. If the positive convex hull $\Delta_{M'}$ of M' is generated by only one point, then the game ends and player A wins. Otherwise one sets $M = M'$ to be the initial set of points and goes on with the next move. If this game goes on forever, player B wins.

Let us give a simple example for one run of the game:

[5]A reader to whom 'player A' and 'player B' are too anonymous concepts finds two more prominent players in Zeillingers dissertation [**Zei05**] on pages 19 to 21. Having Lewis Carroll's novel in mind, Zeillinger pictures the game in Alice's wonderland to be the adjusting of the hand of the rabbit's (player A's) watch, done by the capricious hatter (player B).

EXAMPLE 48. Let us fix $c = 1$ and let the initial points be (1, 2), (1, 3), (2, 2), (3, 1), shown in figure 5.2.1 (a). For such a two dimensional setup, player A has not much of a choice. Choices of S with only one element never improve A's situation. So we let him choose $S = \{1, 2\}$. Player B decides to choose $\alpha_l = 1$. Then the points are moved according to $m'^{(i)}_1 = m^{(i)}_1 + m^{(i)}_2 - 1$ and $m'^{(i)}_2 = m^{(i)}_2$. The resulting points are (2, 2), (3, 3), (3, 2), (3, 1), shown in figure 5.2.1 (b). The positive convex hull of this set is generated by more than one point, so the game goes on. In the second move player A chooses again $\{1, 2\}$ and player B this time chooses 2. Now the points are moved accordingly along the second axis. The resulting set of points, given in figure 5.2.1 (c), has a positive convex hull which is generated by only one point and player A therefore wins the game.

The game is rather simple in the two-dimensional case. If player A always chooses $\{1, 2\}$ he wins the game after finitely many moves, no matter how B decides. But how does player A have to choose his set S in the higher-dimensional case?

PROBLEM 49. Determine a strategy for the choice of S, such that player A wins the game after a finite number of moves, for any initial set M and no matter which choices player B takes.

Such a strategy is called a *winning strategy* of the game. Despite the simplicity of the game, finding a winning strategy turned out to be a non-trivial but solvable problem.

Comparing the above game with our discussion in subsection 5.1.4 it is now obvious, that Hironaka's polyhedra game encodes the combinatorics of iterative sector decomposition. Hence the problems 43 and 49 are the same. A winning strategy of Hironaka's polyhedra game is suitable to replace Strategy X in the above algorithm and to guarantee for the termination.

We want to remark that in the literature sometimes a version of the game is considered where the number c is fixed at a certain value. Zeillinger, in his work to be discussed below, fixes $c = 0$ while Spivakovsky for his strategy sets $c = 1$. We have seen in subsection 5.1.4 that this difference is irrelevant for our termination problem, because of the simple fact that either both of the polynomials in equation (5.1.19) are monomialized, or none of them. We can observe the same fact by looking at the positive convex hull Δ_M. A change of c just leads to a translation of Δ_M, but the shape of Δ_M does not depend on c. Therefore we can map our problem to versions of the game with any $c \in \mathbb{N} \cup \{0\}$.

Let us emphasize, that the winning strategy must work for all possible choices player B can take. The choice of player B, selecting an element $\alpha_l \in S$, corresponds to the consideration of one of the created sectors in sector decomposition. The algorithm will only terminate, if in *all the created sectors* the polynomial is monomialized after finitely many steps. In this sense, sector decomposition considers all the possible choices of player B.

Finding a winning strategy for Hironaka's polyhedra game is an important problem of mathematics, as the game encodes the combinatorics for a certain type of resolution of singularities. We want to give a brief insight into resolution of singularities in the context of sector decomposition in section 5.3.

The problem remained unsolved for more than 15 years until a first winning strategy was found by Spivakovsky [**Spi83**]. We will discuss this one and two other winning strategies in the following. Implementations of these three strategies are contained in our computer program [**BW08**]. In addition we will review another strategy given by Smirnov and Tentyukov which is implemented in their program FIESTA [**ST09**].

5.2.2. Winning Strategies. How do we know, whether a strategy is a winning strategy? We may at first distinguish between two cases in which player A would *lose* the game. In the first case a set of points M appears again after a number of moves, such that the same moves can be repeated infinitely many times. This resembles the situation of example 42, by which we have shown that Strategy X is not a winning strategy. To be sure that this first case can not occur, it would be sufficient to prove, that the strategy under consideration in some sense 'improves' the set of points with every move. As a second case we may imagine, that player A's strategy fulfills this condition, and the set of points is changed with each move in a way, such that a set can not appear repeatedly, but nevertheless the winning condition is never reached. To exclude this case we have to be sure, that it is not possible to have infinitely many 'improvements'.

In order to formalize the concept of an 'improvement' we use the notion of lexicographical ordering of ordered sets of numbers. We already used lexicographical ordering in chapter 3. Let us remind ourselves that with respect to *lexicographical ordering* a sequence of numbers $(a_1, ..., a_n)$ is *larger* than another sequence $(b_1, ..., b_l)$, denoted as

$$(a_1, ..., a_n) \quad \succ \quad (b_1, ..., b_l)\,,$$

if either $n > l$ or if $n = l$ and there exists an m between 0 and n such that $a_m > b_m$ and $a_i = b_i$ for all $0 < i < m$. Correspondingly, with respect to lexicographical ordering, we use the symbols $\succeq$, $\preceq$ for greater or equal and smaller or equal, respectively.

For a given set of points M let us denote the set after n moves of the game by M^n. Assume that we can find a map τ from a set of points M to an ordered set of numbers

$$\tau(M) \quad = \quad a^M = \left(a_1^M, ..., a_k^M\right).$$

Then if two such sets of numbers a^M and $a^{M'}$ are different, the sets of points M and M' must be different. Hence, if we find a τ, such that at each stage of the game we have

$$\tau(M^n) \quad \succ \quad \tau(M^{n+1}), \tag{5.2.1}$$

then we know that the same set of points can not appear repeatedly. If this equation is true for all n and all possible choices player B can take, then we say that the sequence of numbers *decreases*. In this case, the same sequence of numbers can not appear more than once in the game and neither can the corresponding set of points. Now it just remains to make sure, that the sequence of numbers can only decrease in finitely many steps and the lexicographically smallest possible sequence resembles the winning condition for player A. For example we may have a map τ such that all the numbers a_j^M are non-negative integers and if Δ_M is generated by only one point, all numbers in $\tau(M)$ are zero. Then, if equation (5.2.1) is always true, there are always finitely many moves.

As a consequence, we have a proof that a strategy is a winning strategy, if we find a map τ, such that the sequence decreases lexicographically with every move and it can not decrease infinitely often. This concept is used to prove, that the following three strategies are winning strategies. For each of these strategies there is a τ, fulfilling the above conditions. We start with the discussion of Zeillinger's winning strategy, which is simpler than the other ones, because each chosen S contains only two numbers. Its proof is simpler as well, because the sequence given by his τ consists just of three non-negative integers.

5.2.3. Zeillinger's Strategy. In the following let D_{S,α_l} denote the map of one move of the game, determined by the choice of S and α_l:

$$\begin{aligned} D_{S,\alpha_l} : \mathbb{N}^n \quad &\rightarrow \quad \mathbb{N}^n, \\ m^{(i)} \quad &\mapsto \quad m'^{(i)}, \end{aligned}$$

given by

$$m'^{(i)}_j = \begin{cases} \sum_{\alpha\in S} m^{(i)}_\alpha - c \text{ if } j = \alpha_l, \\ m^{(i)}_j \text{ if } j \neq \alpha_l \end{cases}$$

where $c \in \mathbb{N} \cup \{0\}$ is fixed. D_{S,α_l} is simply the shift of the points in one move. We denote the set of points before the move by M and after the move by M'.

Now let M_C denote the set of generators of Δ_M. We remember, that these are the corner points of Δ_M. Let us consider a point $m^{(q)} \in M$ which is not a corner point of Δ_M, i. e. $m^{(q)}$ lies somewhere in the interior of Δ_M. What can we say about its shift $m'^{(q)} = D_{S,\alpha_l}\left(m^{(q)}\right)$? Can the new point $m'^{(q)} \in M'$, obtained from $m^{(q)}$, be a corner point of the new Newton polyhedron $\Delta_{M'}$? Intuitively one may already see (cf. figure 5.1.6) that this can not be the case. Indeed, the generators M'_C of the new $\Delta_{M'}$ are shifts of generators of Δ_M, but not of points from the interior of Δ_M. Therefore, the number of corner points can either decrease or remain the same after one move:

LEMMA 50. *Let M and M' be non-empty sets of points with $M' = D_{S,\alpha_l}(M)$ and let M_C and M'_C be the sets of generators of Δ_M and $\Delta_{M'}$ respectively. Then for any non-empty $S \subseteq \{1, ..., n\}$, $\alpha_l \in S$, $c \in \mathbb{N}$ we have*

$$\begin{aligned} M'_C &\subseteq D_{S,\alpha_l}(M_C), \\ \left|M'_C\right| &\leq \left|M_C\right|. \end{aligned}$$

$|M'_C|$ and $|M'_C|$ denote the number of elements in M'_C and M'_C respectively. A proof of lemma 50 is given in **[Zei05]**.

We remember, that the set of points after p moves is denoted by M^p. Player A wins the game if there is a finite p such that $\left|M^p_C\right| = 1$. As a direct consequence of the above lemma we see that all points in M^p_C for any p are images of points in M_C. Hence, the termination of the game depends just on M_C and its images. This fact is used in Zeillinger's proof of his winning strategy, as we will see. Now let us prepare the formulation of Zeillinger's strategy.

For a given $M_C \subseteq M$ we define W_{M_C} to be the set of vectors connecting the corner points:

$$W_{M_C} = \left\{ w = m^{(i)} - m^{(j)} \mid m^{(i)}, m^{(j)} \in M_C \right\},$$

where $m^{(i)}$ and $m^{(j)}$ may be the same point. All components w_i of these vectors are integers, possibly positive, negative or zero. If all the components of a vector w are non-negative we write $w \in \mathbb{N}^n_0$, and if all the components of w are non-positive we write $w \in -\mathbb{N}^n_0$. The vector all of whose components are zero is in $\mathbb{N}^n_0$ and as well in $-\mathbb{N}^n_0$. We observe, that if there is a vector $w \in W_{M_C}$ which has as well positive and negative components, the Newton polyhedron Δ_M is generated by more than one point and player A has not won yet. If after p moves the images of all the vectors of W_{M_C} are either in $\mathbb{N}^n_0$ or in $-\mathbb{N}^n_0$, then the Newton polyhedron of the new set of points is generated by only one point and player A wins.

For each vector let us define two functions $L(w)$ and $N(w)$. We define $L(w)$ of the vector $w \in W$ by

$$L(w) = \max_{1\leq i\leq n} w_i - \min_{1\leq i\leq n} w_i.$$

This is the difference between the largest and the smallest component of w. For example for the vector $w = (4, 7, -2)$ the function evaluates to $L(w) = 9$.

We define the function $N(w)$ of $w \in W$ by

$$N(w) \quad = \quad \left| \left\{ j \left| 1 \leq j \leq n,\, w_j = \min_{1 \leq i \leq n} w_i \text{ or } w_j = \max_{1 \leq i \leq n} w_i \right. \right\} \right|,$$

which is the number of components of w which are either equal to the minimal or to the maximal number in w. For example for $w = (1,\, 5,\, 3,\, 5,\, 5,\, -8,\, 2,\, -8)$ we have $N(w) = |\{j \,|1 \leq j \leq n,\, w_j = -8 \text{ or } w_j = 5\}| = 2 + 3 = 5$. Note that the possible values of the functions $L(w)$ and $N(w)$ are non-negative integers.

Now we can formulate Zeillinger's winning strategy:

THEOREM 51. *(Zeillinger* **[Zei05, Zei06b]***) The following is a winning strategy for Hironaka's polyhedra game:*

For the given set of points M in $\mathbb{N}_0^n$ let $\tilde{w}$ be a vector in W_{M_C}, for which the sequence $(L(\tilde{w}),\, N(\tilde{w}))$ is minimal with respect to lexicographical ordering, meaning that

$$(L(\tilde{w}),\, N(\tilde{w})) \quad \preceq \quad (L(w),\, N(w)),\, \forall w \in W_{M_C}.$$

Player A chooses $S = \{k,\, l\}$, where k and l are defined by

$$\tilde{w}_k = \min_{1 \leq i \leq n} \tilde{w}_i \quad \textit{and} \quad \tilde{w}_l = \max_{1 \leq i \leq n} \tilde{w}_i.$$

PROOF. In his dissertation **[Zei05]**, Zeillinger presents the following proof. As we have explained in subsection 5.2.2, a strategy is proven to be a winning strategy for Hironaka's polyhedra game, if there is a map τ from the sets of points M, M', ..., M^p to sequences of numbers, which decrease lexicographically with the ongoing game, but which can not decrease infinitely often.

For Zeillinger's strategy there is a map τ_{Z}, defined by

$$\tau_{\mathrm{Z}}(M) \quad = \quad \begin{cases} (0,\, 0,\, 0) \text{ if } \Delta_M \text{ is generated by only one point,} \\ (|M_C|,\, L(\tilde{w}),\, N(\tilde{w})) \text{ otherwise.} \end{cases}$$

One just needs to prove that for any M we have

$$\tau_{\mathrm{Z}}\left(M'\right) \quad \prec \quad \tau_{\mathrm{Z}}(M) \tag{5.2.2}$$

where $M' = D_{S,\alpha_l}(M)$ with S choosen according to the strategy and α_l any element of S. As $|M_C|,\, L(\tilde{w}),\, N(\tilde{w})$ are non-negative integers, it is immediately clear, that the sequence can not decrease infinitely often.

According to lemma 50 we always have $|{M'}_C| \leq |M_C|$. If $|{M'}_C| < |M_C|$, equation (5.2.2) is obviously fulfilled.

Let us assume that $|{M'}_C| = |M_C|$. Let $\tilde{w}$ be the vector in the set of vectors before the move, i. e. in W_{M_C}, for which the sequence $(L(\tilde{w}),\, N(\tilde{w}))$ is minimal in W_{M_C}, i. e. $(L(\tilde{w}),\, N(\tilde{w})) \preceq (L(w),\, N(w)),\, \forall w \in W_{M_C}$. Furthermore let $\tilde{v}$ be the vector in the set of vectors after the move, i. e. in $W_{{M'}_C}$, for which the sequence $(L(\tilde{v}),\, N(\tilde{v}))$ is minimal in $W_{{M'}_C}$, i. e. $(L(\tilde{v}),\, N(\tilde{v})) \preceq (L(w'),\, N(w')),\, \forall w' \in W_{{M'}_C}$. We have to compare just these two vectors $\tilde{w}$ and $\tilde{v}$. It remains to be proven that

$$(L(\tilde{v}),\, N(\tilde{v})) \quad \prec \quad (L(\tilde{w}),\, N(\tilde{w})). \tag{5.2.3}$$

From $|{M'}_C| = |M_C|$ it follows, that for each vector $w = m^{(i)} - m^{(j)} \in W_{M_C}$ there is a vector $w' = D_{S,\alpha_l}\left(m^{(i)}\right) - D_{S,\alpha_l}\left(m^{(j)}\right) = {m'}^{(i)} - {m'}^{(j)} \in W_{{M'}_C}$. Let $\tilde{w}'$ be the vector obtained from $\tilde{w}$. If we can show that $(L(\tilde{w}'),\, N(\tilde{w}')) \prec (L(\tilde{w}),\, N(\tilde{w}))$ then we have automatically $(L(\tilde{v}),\, N(\tilde{v})) \prec (L(\tilde{w}),\, N(\tilde{w}))$ by the definition of $\tilde{v}$.

Without loss of generality we can say that the first component of $\tilde{w}$ is a minimal one and the second is a maximal one: $\tilde{w}_1 = \min_{1\leq i\leq n} \tilde{w}_i$ and $\tilde{w}_2 = \max_{1\leq i\leq n} \tilde{w}_i$. By definition, $\tilde{w}_1$ and $\tilde{w}_2$ can not both be non-negative or non-positive and therefore

$$\tilde{w}_1 \quad < \tilde{w}_1 + \tilde{w}_2 < \quad \tilde{w}_2.$$

The choice of player A is $S = \{1, 2\}$ according to the strategy. Let player B choose $\alpha_l = 1$. Then we obtain $\tilde{w}' = (\tilde{w}_1 + \tilde{w}_2, \tilde{w}_2, ...)$. If $\tilde{w}_1$ is the only component of $\tilde{w}$ with the value $\min_{1\leq i\leq n} \tilde{w}_i$, then we have $\min_{1\leq i\leq n} \tilde{w}_i < \min_{1\leq j\leq n} \tilde{w}'_j$ and therefore $L(\tilde{w}') < L(\tilde{w})$. Then equation (5.2.3) is fulfilled. Otherwise, if there are other components in $\tilde{w}$ with the minimal value, then $L(\tilde{w}') = L(\tilde{w})$, but the number of components with the minimal value is smaller in $\tilde{w}'$ and therefore we have $N(\tilde{w}') < N(\tilde{w})$, which leads to equation (5.2.3). Similar arguments apply to the case where player B chooses $\alpha_l = 2$.

Therefore, we have either $|M'_C| < |M_C|$ or we have $|M'_C| = |M_C|$ and for the latter case we have shown, that equation (5.2.3) is always true. Hence equation (5.2.2) is always true and this proves the theorem. □

With the above winning strategy Zeillinger has succeeded in finding a solution which is very easy to apply and which can be proven in a few steps. But for our purpose the strategy also has a drawback. When we use it for sector decomposition, we are likely to generate a large number of sectors, much larger than in the case of Strategy X, as we observe for example in the case of the integrals mentioned in subsection 5.2.7 below. An implementation in a computer program will therefore be slow. Therefore we consider two further winning strategies in the following, which are more complicated than Zeillinger's strategy, but more efficient.

5.2.4. Spivakovsky's Strategy. Before we can formulate Spivakovsky's winning strategy, we need to introduce a number of auxiliary objects. All of them are determined from a given set $\Delta \subset \mathbb{R}^n_+$, which is defined to be the positive convex hull of a set of points whose coordinates are non-negative rational numbers. (By allowing for rational numbers we obtain Δ as a generalization of the above Δ_M.) Points in Δ shall be denoted by $\nu = (\nu_1, ..., \nu_n)$.

- We define $\omega(\Delta)$ to be the vector whose n components ω_i are the minima of the individual coordinates of elements in Δ:

  $$\omega_i \quad = \quad \min\{\nu_i \mid \nu \in \Delta\},\ i = 1, ..., n.$$

  Obviously all ω_i are coordinates of the generators of Δ.
- Let $\tilde{\nu} = (\tilde{\nu}_1, ..., \tilde{\nu}_n)$ be the point obtained from ν by subtracting $\omega(\Delta)$, i. e.

  $$\tilde{\nu}_i \quad = \quad \nu_i - \omega_i,\ i = 1, ..., n.$$

  Then let $\tilde{\Delta}$ denote the set of all the $\tilde{\nu}$ obtained from all $\nu \in \Delta$. We write

  $$\tilde{\Delta} \quad = \quad \Delta - \omega(\Delta).$$

- For a subset $\Gamma \subseteq \{1, ..., n\}$ we define

  $$d_\Gamma(\Delta) = \min\left\{\sum_{j\in\Gamma} \nu_j \mid \nu \in \Delta\right\} \quad \text{and} \quad d(\Delta) = d_{\{1,...,n\}}(\Delta).$$

  In other words, for some points in Δ the value of the sum $\sum_{j\in\Gamma} \nu_j$ is minimal and then it defines $d_\Gamma(\Delta)$. Note that the points for which $\sum_{j\in\Gamma} \nu_j$ is minimal belong to the boundary of Δ and there is at least one generator of Δ among them.

- We define the sequence of sets

$$(I_0, \Delta_0, I_1, \Delta_1, ..., I_r, \Delta_r), \qquad r \in \mathbb{N} \tag{5.2.4}$$

with

$$I_0 = \{1, ..., n\}, \qquad \Delta_0 = \Delta$$

and where the remaining sets in the sequence are defined recursively from I_0 and Δ_0 as follows.

- For each Δ_k we define a set H_k by

$$H_k \;=\; \left\{ j \in I_k \mid \exists \nu \in \Delta_k \text{ such that } \sum_{i \in I_k} \nu_i = d(\Delta_k) \text{ and } \tilde{\nu}_j \neq 0 \right\}.$$

Then we define

$$I_{k+1} \;=\; I_k \backslash H_k.$$

According to this definition H_k and I_{k+1} are two complementary subsets of I_k.

- We define $\mathbb{R}_+^{I_k}$ to be the $|I_k|$-dimensional subspace of $\mathbb{R}_+^n$, spanned by all the j-th coordinate axes with $j \in I_k$. For each I_k and its complementary subsets H_k and I_{k+1} we define

$$M_{H_k} \;=\; \left\{ \nu \in \mathbb{R}_+^{I_k} \mid \sum_{j \in H_k} \nu_j < 1 \right\}.$$

Furthermore for a point $\nu \in \mathbb{R}_+^{I_k}$ let α denote the vector of its i-th coordinates with $i \in I_{k+1}$ and β the vector of its j-th coordinates with $j \in H_k$. Using these notations we define the projection

$$\begin{aligned} P_{H_k} \;:\; & M_{H_K} \to \mathbb{R}_+^{I_{k+1}}, \\ & P_{H_k}(\alpha, \beta) = \frac{\alpha}{1 - |\beta|},\ \alpha \in \mathbb{R}_+^{I_{k+1}},\ \beta \in \mathbb{R}_+^{H_k},\ |\beta| = \sum_{j \in H_k} \beta_j. \end{aligned}$$

Then Δ_{k+1} is defined by

$$\Delta_{k+1} \;=\; P_{H_k}\left(M_{H_k} \cap \left(\frac{\tilde{\Delta}}{d\left(\tilde{\Delta}_k\right)} \cup \Delta_k \right) \right).$$

Here $\frac{\tilde{\Delta}}{d\left(\tilde{\Delta}_k\right)}$ is defined to be the set of points which is obtained from $\tilde{\Delta}$ by multiplying each coordinate of each point with the number $\frac{1}{d\left(\tilde{\Delta}_k\right)}$. All sets Δ_{k+1} are positively convex.

- The recursion stops when either $\Delta_r = \emptyset$ or $d\left(\tilde{\Delta}_r\right) = 0$. By convention we set $\tilde{\emptyset} = \emptyset$ and $d(\emptyset) = \infty$.

At this point we are well advised to give an example.

Example 52. Let us consider the case of figure (5.2.2) where $\Delta \subset \mathbb{R}_+^2$ is generated by the two points $\nu^{(1)} = (1, 2)$ and $\nu^{(2)} = (4, 0)$. We have $\omega(\Delta) = (1, 0)$, $d(\Delta) = 3$, $\tilde{\nu}^{(1)} = (0, 2)$, $\tilde{\nu}^{(2)} = (3, 0)$. The sequence $(I_0, \Delta_0, I_1, \Delta_1, ..., I_r, \Delta_r)$ begins with $I_0 = \{1, 2\}$ and $\Delta_0 = \Delta$. We obtain $H_0 = \{2\}$, because there is the point $\nu^{(1)}$ with $\sum_{i \in I_0} \nu_i^{(1)} = 1 + 2 = 3 = d(\Delta)$ and $\tilde{\nu}_2^{(1)} \neq 0$. Therefore we directly obtain $I_1 = I_0 \backslash H_0 = \{1\}$. The set M_{H_0} is the set of all points in $\mathbb{R}_+^2$ where the second component is smaller than one. The

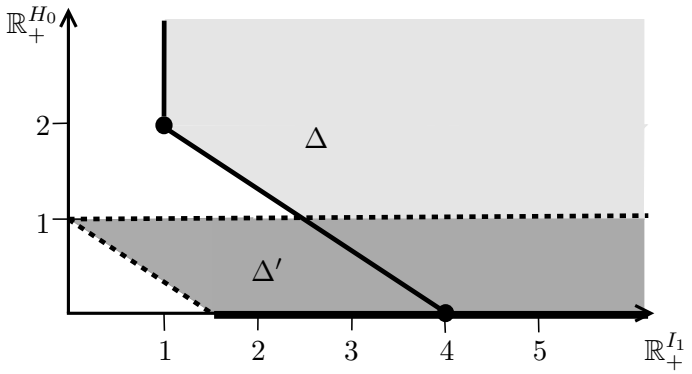


FIGURE 5.2.2. A two-dimensional example for Spivakovsky's auxiliary objects.

set $\Delta' := M_{H_0} \cap \left(\frac{\tilde{\Delta}}{d(\tilde{\Delta}_0)} \cup \Delta_0 \right)$ with $d\left(\tilde{\Delta}_0\right) = 2$ is the dark grey region in figure (5.2.2) and Δ_1 is its projection from (1, 1) to $\mathbb{R}_+^{I_1}$, drawn as a thick line on the axis.

Now we formulate Spivakovsky's winning strategy:

THEOREM 53. *(Spivakovsky* **[Spi83]***) The following is a winning strategy for Hironaka's polyhedra game:*

For the given set of points M in $\mathbb{N}_0^n$ we consider the sequence

$$(I_0,\, \Delta_0,\, I_1,\, \Delta_1,\, ...,\, I_r,\, \Delta_r)$$

obtained from $\Delta_0 = \Delta_M$.
If $\Delta_r = \emptyset$, player A chooses $S = \{1,\, ...,\, n\} \setminus I_r$.
If $\Delta_r \neq \emptyset$, player A first chooses a minimal subset $\Gamma_r \subseteq I_r$, such that $\sum_{j \in \Gamma_r} \nu_j \geq 1$ for all $\nu \in \Delta_r$ and then he sets $S = (\{1,\, ...,\, n\} \setminus I_r) \cup \Gamma_r$.

The proof is given in reference **[Spi83]**. The lexicographically decreasing sequence of numbers found by Spivakovsky is

$$\tau_{\mathrm{S}}(M) \;=\; \left\{ d\left(\tilde{\Delta}_0\right),\, |I_1|,\, d\left(\tilde{\Delta}_1\right),\, ...,\, |I_r|,\, d\left(\tilde{\Delta}_r\right),\, d(\Delta_r) \right\}$$

with $\Delta_0 = \Delta_M$. Spivakovsky proves that

$$\tau_{\mathrm{S}}\left(M'\right) \;\prec\; \tau_{\mathrm{S}}(M)$$

is true at any stage of the game, and that the sequence can not decrease infinitely often.

5.2.5. A Strategy due to Encinas and Hauser. Let us give one more winning strategy for the game. It is based on recent work by Encinas and Hauser **[EH02]** and is a variant of Spivakovsky's strategy. We can use the notation of the previous subsection and define just a few additional objects. Instead of the sequence of equation (5.2.4) we need to construct the sequence

$$(c_{-1},\, I_0,\, \Delta_0,\, c_0,\, I_1,\, \Delta_1,\, ...,\, c_{r-1},\, I_r,\, \Delta_r),\, r \in \mathbb{N}$$

starting from

$$c_{-1} = 0,\, I_0 = \{1,\, ...,\, n\},\, \Delta_0 = \Delta.$$

Again the set H_k is defined as

$$H_k \;=\; \left\{ j \in I_k \mid \exists \nu \in \Delta_k \text{ such that } \sum_{i \in I_k} \nu_i = d(\Delta_k) \text{ and } \tilde{\nu}_j \neq 0 \right\}.$$

However, Δ_k and I_k are obtained in a different way this time. At first we define

$$c_k \;=\; d\left(\tilde{\Delta}_k\right).$$

Then for each $\tilde{\Delta}_k$ we obtain a companion set $\tilde{\Delta}_k^C$ by

$$\tilde{\Delta}_k^C \;=\; \begin{cases} \tilde{\Delta}_k \cup \left(\frac{c_k}{c_{k-1}-c_k}\omega\left(\Delta_k\right) + \mathbb{R}_+^{I_k}\right) & \text{if } 0 < c_k < c_{k-1}, \\ \tilde{\Delta}_k \text{ otherwise.} \end{cases}$$

Now let $T_k \subseteq H_k$ be a subset of H_k which is chosen according to a rule R: $T_k = R(H_k)$. The rule R can be freely chosen by player A with the only obstruction, that it is a deterministic rule, meaning that if $T_k = R(H_k)$ and $T' = R(H'_k)$ and $H_k = H'_k$, then $T_k = T'_k$. This additional freedom is a difference to Spivakovsky's strategy. Using the chosen set T_k we now define

$$\Delta_{T_k} \;=\; c_k P_{T_k}\left(M_{T_k} \cap \frac{\tilde{\Delta}_k^C}{c_k}\right),$$

P being the projection as defined above. Then we set

$$\begin{aligned} I_{k+1} &= I_k \backslash T_k, \\ \Delta_{k+1} &= \Delta_{T_k}. \end{aligned}$$

Again we stop the iterative construction of the sequence as soon as $d\left(\tilde{\Delta}_k\right) = 0$ or $\Delta_r = \emptyset$. With these notations we formulate the winning strategy:

THEOREM 54. *The following is a winning strategy for Hironaka's polyhedra game: For the given set of points M in $\mathbb{N}_0^n$ we consider the sequence*

$$(c_{-1},\, I_0,\, \Delta_0,\, c_0,\, I_1,\, \Delta_1,\, ...,\, c_{r-1},\, I_r,\, \Delta_r)$$

obtained from $\Delta_0 = \Delta_M$.
If $\Delta_r = \emptyset$, player A chooses $S = \{1, ..., n\} \backslash I_r$.
If $\Delta_r \neq \emptyset$, player A first chooses a minimal subset $\Gamma_r \subseteq I_r$, such that $\sum_{j\in\Gamma_r} \nu_j \geq c_{r-1}$ for all $\nu \in \Delta_r$ and then he sets $S = (\{1, ..., n\} \backslash I_r) \cup \Gamma_r$.

A compact formulation of a proof is provided in [**BW08**]. It arises from the proof by Encinas and Hauser [**EH02**] for Hironaka's theorem on the resolution of singularities (see theorem 58 below). The decreasing sequence for the strategy reads

$$\tau_{\mathrm{EH}}(M) \;=\; \left\{d\left(\tilde{\Delta}_0\right),\, |I_0| - |H_0|,\, d\left(\tilde{\Delta}_1\right),\, ...,\, |I_{r-1}| - |H_{r-1}|,\, d\left(\tilde{\Delta}_r\right),\, d\left(\Delta_r\right)\right\}.$$

5.2.6. Strategy S in FIESTA: An Extension by Smirnov and Tentyukov. In [**ST09**] Smirnov and Tentyukov present the program FIESTA, which is their implementation of the sector decomposition. It differs from our implementation by a new strategy, Strategy S, for the choice of the set S, coming with slight changes in step 2 of the sector decomposition algorithm. Strategy S tells player A at first to choose a vector $v = (v_1, ..., v_n)$ such that it is a normal vector to a *facet* of Δ_M, meaning that it is orthogonal to some of the vectors connecting the corner points of Δ_M (as considered in Zeillinger's strategy). Then S is chosen to be the set of indices α_i where the corresponding component v_{α_i} is non-zero:

$$S \;=\; \{\alpha_i \mid v_{\alpha_i} \neq 0\}.$$

Then step 2 is changed in the following way: Instead of decomposing the integration domain exclusively along planar subspaces as in equation (5.1.9), FIESTA decomposes the domain as

$$\int_0^1 d^n x \;=\; \sum_{l=1}^{k} \int_0^1 d^n x \prod_{i=1,\, i\neq l}^{k} \theta\left(x_{\alpha_l}^{a_{\alpha_l}} \geq x_{\alpha_i}^{a_{\alpha_i}}\right)$$

where the exponents a_{α_i} are defined by

$$\begin{pmatrix} a_{\alpha_1} \\ a_{\alpha_2} \\ a_{\alpha_3} \\ \vdots \\ a_{\alpha_k} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & ... & 1 \\ 1 & 0 & 1 & ... & 1 \\ 1 & 1 & 0 & ... & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & ... & 0 \end{pmatrix}^{-1} \begin{pmatrix} v_{\alpha_1} \\ v_{\alpha_2} \\ v_{\alpha_3} \\ \vdots \\ v_{\alpha_k} \end{pmatrix}$$

and where $\alpha_i \in S$ for $i = 1, ..., k$. Then, instead of the replacements given by equations (5.3.1), FIESTA substitutes

$$x_i \;=\; \begin{cases} {x'}_{\alpha_l}^{v_{\alpha_l}} \text{ for } i = \alpha_l, \\ x'_i {x'}_{\alpha_l}^{v_{\alpha_l}} \text{ for } i \neq \alpha_l \text{ and } i \in S, \\ x'_i \text{ for } i \neq S. \end{cases}$$

According to Smirnov and Tentyukov this yields cubic integration domains again.

In order to guarantee for the termination of FIESTA, Strategy S is combined with Zeillinger's strategy. Whenever Zeillinger's sequence $\tau_{\mathrm{Z}}(M)$ is not decreased by the choice of Strategy S, this choice is replaced by the choice due to Zeillinger's strategy. Smirnov and Tentyukov observe empirically, that this is the case for less than five percent of the decompositions. In [**SS08**] A. V. Smirnov and V. A. Smirnov furthermore observe, that for many cases Strategy S produces the same number of sectors as a decomposition to so-called *Speer sectors*[6].

5.2.7. Implementation and Efficiency. Our implementation of the sector decomposition algorithm is joint work with Stefan Weinzierl, published in reference [**BW08**]. The computer program can be freely obtained from the webpage
`http://wwwthep.physik.uni-mainz.de/~stefanw/software.html`.
The implementation is a library, written in C++ and GiNaC [**BFK02**], and uses the package VEGAS [**Lep78, Lep80**] for the numerical Monte Carlo integration. The analytical and the numerical part of the evaluation are combined in one single program. It was the first publicly available implementation of sector decomposition.

Details on the installation and the use of the library are provided in reference [**BW08**] and we do not want to repeat these technical instructions here. Let us just very briefly summarize the main properties of the implementation. A possible user will write a C++ program, incorporating the function `do_sector_decomposition` which is the main routine of the library. The arguments of the function are arranged in the three structures `integration_data`, `integrand`, `monte_carlo_parameters` and an optional parameter, which determines the amount of information to be printed by the program during the run. The argument `integration_data` contains the variables which are used as Feynman parameters, the regularization parameter ϵ and it determines which term of the Laurent

[6] Like the mentioned sectors used by Hepp [**Hep66**] (so-called Hepp sectors), Speer sectors [**Spe68, Spe77**] were successfully used in proofs of theorems on renormalization. Speer sectors are obtained from a particular choice, which directly depends on the topology of the Feynman graph.

series is computed. The structure `integrand` encodes all the information of the integrand (cf. equation (5.1.6))

$$\left(\prod_{i=1}^{n} x_i^{a_i+\epsilon b_i}\right) \prod_{j=1}^{r} \left(P_j\left(X\right)\right)^{c_j+\epsilon d_j}$$

as a list of the exponents $a_i + \epsilon b_i$, a list of the polynomials $P_j\left(X\right)$ and a list of the exponents of the polynomials $c_j + \epsilon d_j$. The structure `monte_carlo_parameters` contains the input parameters for the numerical Monte Carlo integration with VEGAS. The latter parameters determine the desired precision of the result.

The user is free to choose between four strategies for the iterative step 2 of the sector decomposition algorithm. A strategy is chosen by simply setting a global variable `CHOICE_STRATEGY` equal to either `STRATEGY_A` for Zeillinger's winning strategy, `STRATEGY_B` for Spivakovsky's winning strategy, `STRATEGY_C` for the winning strategy due to Encinas and Hauser or `STRATEGY_X` for Strategy X of Binoth and Heinrich.

The computation time first of all depends strongly on the considered Feynman integral and the chosen precision of the numerical integration, determined by the number of function evaluations. Furthermore, as we already mentioned, the running time depends on the chosen strategy. An indication of this dependence is the number of sectors being produced.

Let us as an example once again consider the massless planar double box, shown in figure 5.1.5 and considered in example 41 above. The input data in the structure `integrand` is given by the corresponding Feynman integral

$$\int d^7x \delta\left(1 - \sum_{i=1}^{7} x_i\right) \mathcal{U}^{1+3\epsilon} \mathcal{F}^{-3-2\epsilon} \tag{5.2.5}$$

where we suppressed a trivial prefactor (cf. equation (5.1.13)) and where $\mathcal{U}$ and $\mathcal{F}$ are given by equations (5.1.14) and (5.1.15). Using Strategy X we let our program compute the integral at the values $s = -1$ and $t = -1$ for the kinematical invariants in $\mathcal{F}$. After an appropriate choice of Monte Carlo parameters (for details see [**BW08**]), the calculation takes about 40 minutes on a standard PC and produces the output:

```
Order eps^(-4):  2.00001 +/- 9.25208e-05
Order eps^(-3):  -5.99992 +/- 0.000359897
Order eps^(-2):  -4.91623 +/- 0.00157598
Order eps^(-1):  11.4958 +/- 0.00681643
Order 1:  13.8236 +/- 0.0207286
```

In the column after the +/- sign, the estimated error of the numerical integration is displayed. By a change of the Monte Carlo parameters one could easily achieve a more precise result at the cost of a longer calculation time. We see that the above output agrees with the results of Binoth and Heinrich in reference [**BH00**], quoted in example 41 above, and with the analytical result by Smirnov [**Smi99**].

The use of Strategy X led to 293 sub-sectors for the above integral. In table 1 we show the numbers of generated sub-sectors for further integrals and all four different strategies. In this table *Bubble* stands for the Feynman integral of the massless selfenergy graph of figure 3.2.1, *Triangle* is the massless triangle graph as e. g. in figure `2.3.1` (a), *Tbubble* is the massless two-loop graph of figure `2.3.1` (b) as evaluated in [**BW03**], *Planar double-box* is the integral we just discussed and *Non-planar double* box refers to the two-loop integral evaluated analytically by Tausk in [**Tau99**].

Integral	Zeillinger	Spivakovsky	Encinas/Hauser	Strategy X
Bubble	2	2	2	2
Triangle	3	3	3	3
Tbubble	58	48	48	48
Planar double-box	755	586	586	293
Non-planar double-box	1138	698	698	395

TABLE 1. The number of generated sub-sectors for different Feynman integrals by use of different strategies.

For the numbers given here and in all examples known to us, Zeillinger's strategy leads to the highest number of sectors. It is followed by the strategies of Spivakovsky and Encinas/Hauser, which both lead to the same second largest number of sectors (despite the mentioned differences). The most efficient choice in our examples is Strategy X. Further examples, including Strategy S of FIESTA can be obtained from the references **[ST09, SS08]**. We furthermore want to mention a new, efficient method of decomposition, presented very recently by Kaneko and Ueda in reference **[KU09]**.

One may be interested in the development of an optimally efficient strategy, which produces as few sectors as possible. At the moment one does not have a method to determine the minimal number of sectors that have to be produced for a given Feynman integral. The development of such a method or an optimally efficient strategy would presumably be helpful for the understanding of sector decomposition and the underlying combinatorics. Nevertheless, in the practical calculations it would probably not diminish the calculation time significantly, as with Strategy X and Strategy S one already has relatively efficient strategies at hand, as the examples given here and in **[ST09, SS08]** show.

5.3. Resolution of Singularities

In the previous sections we have seen, that the singular behaviour of a Feynman integral is essentially determined by the polynomials in the integrand. In the following we want to take the point of view of algebraic geometry by considering the zero-sets of these polynomials as geometrical objects.

We have already seen that sector decomposition, by monomialization and factorization, yields polynomials whose zero-sets do not intersect the integration domain of the Feynman integral. By the monomialization of a polynomial, the possible singularities of the integral were disentangled. Now let us for a moment forget about integrals and consider just the zero-set X of a polynomial P. For example let us imagine that X is a two-dimensional surface in a three-dimensional space. Let us assume, that X is not everywhere smooth. It may have edges and cusps. These are the *singular points* of X. (A more precise definition of singular points is given below.)

One may whish to find a hypersurface Y, possibly of higher dimension than X, which is everywhere smooth, and a map σ (with certain properties), such that X is the image

of Y:

$$\sigma : Y \to X.$$

Such a map will be called a *resolution* (to be defined below). Usually it is difficult to find such a map and one needs to construct Y and σ stepwise. One tries to find a sequence X, X', X'', ..., Y, such that X' possibly has a less complicated singular behaviour than X, X'' has simpler singularities than X' and so on. The single steps in this procedure are so-called *blow-ups*.

It turns out, that sector decomposition and the resolution of singularities are closely related. We have seen, that the combinatorics of the algorithm by Binoth and Heinrich is encoded in Hironaka's polyhedra game. This game was originally defined in order to study the combinatorics of the resolution of singularities. In the present section we want to give an indication on the following correspondences:

- A single step in iterative sector decomposition corresponds to a *blow-up*.
- The sector decomposition of a Feynman integral corresponds to the *resolution* of singularities by a sequence of blow-ups.

A reader speaking the language of algebraic geometry may have already recognized *blow-ups* and *resolutions* in the above treatment. We now attempt to give an explanation of these terms to the reader to whom this branch of mathematics is less familiar. We want to introduce the precise definition of *blow-ups* and *resolutions of singularities* and to this end we need to introduce some basic terminology of algebraic geometry. In order not to speak much more abstract than necessary, we try to restrict ourselves as much as possible to the particular part of the mathematical theory which is sufficient to show the mentioned correspondences explicitely.

Before we start with the technicalities it might be instructive to have some illustrations in mind:

- We may think of an entangled ball of wool. If we want to disentangle the ball in order to obtain a single, smooth string, we need to loosen the knot by carefully pulling here and there. We need to pull the string apart at the points where it interlaces. This careful, stepwise process has a lot in common with the resolution of singularities (cf. [**Hau03**]).
- The use of the term 'blow-up' indicates descriptively what is done in a single step of this process. A balloon which is not inflated has a crumpled surface and we can make it smooth by pulling apart the surface or by blowing up the balloon. However, it may be more precise to think of a blow-up as a smooth object, whose shadow, i. e. whose projection, looks crumpled or entangled. For example we may imagine a smooth piece of wire, held in the light in such a way that its shadow is a curve which has kinks or which intersects with itself, i. e. a curve with singular points.

  We have just introduced a *resolution* as a map σ from the smooth object Y to the singular object X, and not vice versa from X to Y as one might think. As a reason we can imagine that the map from the smooth to the singular object is a well-defined projection, or a sequence of projections, while a map in the other direction is more complicated to be defined properly, as we will see.
- In the sense of projections we can again think of the example of the barbeque from the beginning of the present chapter. The point-like barbeque at the river-mouth has in a sense been pulled apart and we have obtained lines along the rivers. The black spot in figure 5.1.1 can be seen as the projection of the thick black lines in figure 5.1.2.

5.3.1. Varieties and Rational Maps. The following introduction of basic terminology of algebraic geometry is to a large extent geared to the book of Harris [**Har95**]. The basic objects of algebraic geometry are usually defined with respect to a field K which is not further specified[7]. We may think of the field of complex numbers $\mathbb{C}$. We will consider the n-dimensional Euclidean vector space K^n with respect to the field K. Furthermore we will consider the *affine space* $\mathbb{A}^n_K$ which is understood to be the space K^n but without a distinguished point, which serves as the origin[8]. For convenience we write $\mathbb{A}^n_K \equiv \mathbb{A}^n$. For example if our field is $\mathbb{C}$ we can think of $\mathbb{A}^n$ as the Euclidean space $\mathbb{C}^n$, where we do not care, which point is the origin. $K[x_1, ..., x_n]$ denotes the set of polynomials[9] (forming a ring) of the variables x_i with values in K.

The basic geometrical object we want to consider is the common zero-set of polynomials. Such a set of points is called a *variety* in algebraic geometry[10]. More precisely, an *affine variety* $V(\underline{P}) \subset \mathbb{A}^n$ is the common zero-set of a set of polynomials $\underline{P} = \{P_1, ..., P_r\}$ with $P_i \in K[x_1, ..., x_n]$ for $i = 1, ..., r$. For the moment, r may be any natural number. Later on we only need the simpler case, where the variety is the zero-set of only one polynomial. For simplicity let us denote such a variety by $V(P)$, where $P \in K[x_1, ..., x_n]$. A subset of a variety is called *sub-variety*, if it is as well the common zero-set of polynomials. (Each affine variety $V(\underline{P}) \subset \mathbb{A}^n$ is a sub-variety of $\mathbb{A}^n$.)

The *projective space* over K denoted by $\mathbb{P}^n$ is defined to be the set of one-dimensional sub-spaces of K^{n+1}. A point in the space $\mathbb{P}^n$ is a straight line in K^{n+1} through the origin. We denote a point in the projective space $\mathbb{P}^n$ by $[x_0, x_1, ..., x_n]$, where the entries x_i are the components of the vector $(x_0, x_1, ..., x_n)$, which spans the corresponding line through the origin in K^{n+1}. A *projective variety* $\tilde{V}(\underline{P}) \subset \mathbb{P}^n$ is the common zero-set of a set of *homogeneous* polynomials $\underline{P} = \{P_1, ..., P_r\}$ with $P_i \in K[x_0, x_1, ..., x_n]$ for $i = 1, ..., r$. We see that the use of the term 'projective' makes sense for the zero-set of homogeneous polynomials, as each line through the origin of K^{n+1} obviously contains the origin, and the zero-set of an arbitrary homogeneous polynomial contains the origin as well. If P is the only element of $\underline{P}$ we write $\tilde{V}(P)$.

A variety can be made a *topological space* by defining, which subsets of the variety are said to be *open* and which are said to be *closed*. By defining the sub-varieties of a variety to be its closed sets and their complements to be the open sets we introduce the so-called *Zariski topology* on the variety. In the following the adjective *open* (or *closed*) shall refer to this topology. A topological space is called *reducible*, if it is the union of two proper closed subsets. Otherwise it is called *irreducible*. As the closed subsets of a variety are the sub-varieties according to Zariski topology, a variety is irreducible, if it is not the union of two proper sub-varieties. For many purposes it makes sense to restrict ones attention to the irreducible varieties.

As an example we consider an affine variety $V(P)$ given by one polynomial P. Assume that P is a product $P = P_1 P_2$ of two polynomials P_1 and P_2. Obviously P is zero at each point where P_1 is zero and in addition at each point where P_2 is zero. Hence the variety $V(P)$ is the union of the corresponding sub-varieties: $V(P) = V(P_1) \cup V(P_2)$. Therefore it is reducible.

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[7] For the definition of a *field* see appendix A.

[8] A more detailed definition of the affine space is given in appendix A for completeness.

[9] For the purpose of this section we could fix the field $K = \mathbb{C}$ and we might even think about polynomials with only real-valued variables. But it appears more instructive to us to keep K not specified in order to obtain a formulation which is close to the standard textbooks on algebraic geometry.

[10] We prefer the rather straightforward point of view on varieties given here (cf. [**Har95**]) to more abstract definitions as for example given in [**Dan94**].

For the definition of blow-ups we still require the notion of certain functions and maps. Let $U \subset V$ be an open subset of an affine variety and $p \in U$ a point. Then a function f on U is *regular* at p if in some neighborhood of p it can be written as a fraction $\frac{P_1}{P_2}$ with polynomials P_1, $P_2 \in K[x_1, ..., x_n]$ and with $P_2(p) \neq 0$. f is regular on U, if it is regular at every $p \in U$. More generally, a map

$$\begin{aligned} \phi : V &\rightarrow \mathbb{A}^n, \\ p &\mapsto (f_1(p), ..., f_n(p)) \text{ for } p \in V \end{aligned}$$

from an affine variety V to $\mathbb{A}^n$ is a *regular map*, if the components f_i for $i = 1, ..., n$ are regular functions on V. A regular map from an affine variety V_1 to another affine variety V_2 is simply a regular map where the image is contained in V_2. The *graph* Γ_ϕ *of the regular map* ϕ is the closed subset of the Cartesian product $V \times W$, given by

$$\Gamma_\phi = \{(p, \phi(p)) \mid p \in V\}.$$

Γ_ϕ is closed with respect to Zariski topology. The regular functions on V form a ring $A(V)$, which is called the *coordinate ring*[11].

A little less fundamental and more subtle is the definition of *rational maps*. We will think of a rational map ψ as given by a tuple $(f_1(p), ..., f_n(p))$ where the components f_i for $i = 1, ..., n$ are fractions $\frac{g_i^{(1)}}{g_i^{(2)}}$ of regular functions $g_i^{(1)}$, $g_i^{(2)}$. Since some of the $g_i^{(2)}$ can be zero in some p, the f_i are not necessarily well-defined. These points p are not excluded here, as they were in the definition of regular maps. Therefore, the tuple $(f_1(p), ..., f_n(p))$ is not a proper definition of a map at all the points p which we want to consider. In fact, a proper definition as in **[Har95]**, given in appendix A, refers to a *rational map* not as a map but as an equivalence class of maps. For the purpose of the following it will be sufficient to speak less abstract and instead think of a rational map as $(f_1(p), ..., f_n(p))$, keeping in mind, that it is not regular everywhere. In order to distinguish rational maps from regular maps we adopt the convention of using a dashed arrow. For a rational map ψ from a variety V to a variety W we write

$$\psi : V \dashrightarrow W.$$

The graph Γ_ψ of the rational map $\psi : V \dashrightarrow W$, is defined by use of a subset $U \subset V$, where $\psi|_U$ is a regular map. At first, the graph $\Gamma_{\psi|_U} \subset U \times W$ is the graph of a regular map, given by the definition of the graph of a regular map above. Then the *graph* Γ_ψ *of the rational map* ψ is defined to be the *closure* of $\Gamma_{\psi|_U}$ in $V \times W$ with respect to Zariski topology. Γ_ψ is independent of the choice of U. Blow-ups are such graphs of rational maps together with a projection. We will give a basic example in the following subsection.

5.3.2. Blow-ups. We want to compare a simple example of a blow-up to a simple example of sector decomposition. Let us at first recall our initial example 37 from the beginning of the chapter: We have an integral

$$I = \int_0^1 dx \int_0^1 dy j(x, y)$$

where the integrand $j(x, y)$ gives rise to an overlapping singularity. The polynomial in the integrand is monomialized by decomposing the two-dimensional integration domain D, consisting of the points $(x, y) \in \mathbb{R}^2$ with $x \leq 1$, $0 \leq y \leq 1$, into two triangle-shaped regions, such that in one region $D_{x \geq y}$ we have $x \geq y$ and in the other we have $y \geq x$.

[11]Given $I(V) := \{f \in K[x_1, ..., x_n] \mid f(p) = 0 \forall p \in V\}$ to be the ideal of polynomials vanishing on V, the coordinate ring is defined to be the quotient $A(X) = K[x_1, ..., x_n]/I(V)$.

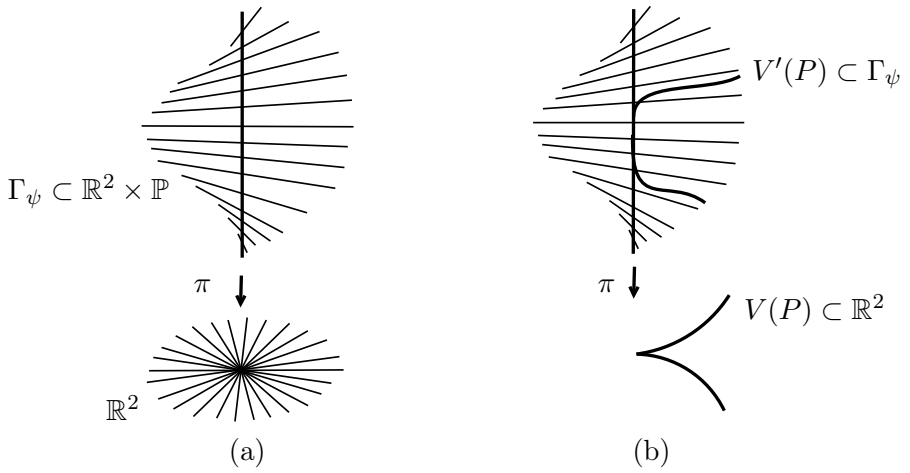

FIGURE 5.3.1. Blow-up of $\mathbb{R}^2$ at (0, 0) as in example 55.

Then the integral over the first region $I_1 = \int_{D_{x\geq y}} dxdyj(x, y)$ is expressed in terms of new parameters x' and y' according to

$$x = x' \quad \text{and} \quad y = x'y'. \tag{5.3.1}$$

With respect to the new parameters we obtain $I_1 = \int_{D_1} dxdyj'(x', y')$ with the square-shaped domain consisting of $(x', y') \in \mathbb{R}^2$ with $0 \leq x' \leq 1$, $0 \leq y' \leq 1$. Here the line $\{(x', y') \in \mathbb{R}^2 \mid x' = 0,\ 0 \leq y \leq 1\} \subset D_1$ corresponds to the point $(0, 0) \in D$, where the overlapping divergence arose. Figure (5.1.3) shows this essential correspondence. Similar steps apply to the second sector.

Now, in comparison, let us give an example from algebraic geometry (cf. [**Har95**]) for a rational map which is not regular everywhere:

EXAMPLE 55. We consider the rational map ψ given by

$$\begin{aligned} \psi : \mathbb{A}^2 &\dashrightarrow \mathbb{P}^1, \\ (x, y) &\mapsto [x, y] \text{ for } (x, y) \neq (0, 0). \end{aligned}$$

All the points $(x, y) \in \mathbb{A}^2$ which lie on a common line through (0, 0) are mapped to the same point in $\mathbb{P}^1$, denoted $[x, y]$. Therefore it is clear, that ψ is not a regular map on (0, 0), as this point alone can not define a line.

How does the graph Γ_ψ look like? Figure 5.3.1 (a) shows a sketch, where the vertical direction resembles $\mathbb{P}^1$, perpendicular to the x-y-plane $\mathbb{A}^2$. The points of Γ_ψ with the same coordinate in $\mathbb{P}^1$ are the ones whose orthographic projection down to $\mathbb{A}^2$ lie on the same line through (0, 0). Therefore, the graph Γ_ψ looks like a twisted plane, or if we draw just some of the lines, like a twisted staircase as in our drawing.

This is a simple example of a blow-up. The pair of the graph Γ_ψ together with the projection map

$$\pi : \Gamma_\psi \to \mathbb{A}^2$$

is called the *blow-up of* $\mathbb{A}^2$ *at the point* (0, 0). As we mentioned, π projects the horizontal lines in Γ_ψ to the lines through (0, 0) in $\mathbb{A}^2$. The only vertical line in Γ_ψ is projected to the point (0, 0). It is important to note, that this is the only region, where π is *not* an isomorphism. In the language of algebraic geometry, (0, 0) is called the *center* of the

blow-up and the vertical line in Γ_ψ is called the *exceptional locus*. It is the only region in Γ_ψ, where we can not parameterize Γ_ψ by x and y.

What is the correspondence to the sector decomposition example? Let us consider a set of points $U_{x\neq 0}$ which is defined as the space $\mathbb{R}^2$ without the x-axis, i. e. $U_{x\neq 0} = \left\{(x,\, y) \in \mathbb{R}^2 \mid x \neq 0\right\}$. The restriction of ψ to this set gives a graph which we want to denote by

$$\Gamma_{\psi,x\neq 0} = \left\{(x,\, y,\, \psi\,(x,\, y)) \mid (x,\, y) \in U_{x\neq 0}\right\}.$$

The blow-up Γ_ψ is the closure of this graph.

A parameterization of $\Gamma_{\psi,x\neq 0}$ in $\mathbb{R}^3$ can be given by the points $(x,\, y,\, z) \in \mathbb{R}^3$ with $z = \frac{y}{x}$, which is well-defined because we have $x \neq 0$ everywhere in $U_{x\neq 0}$ by definition. In this way the two-dimensional set $\Gamma_{\psi,x\neq 0}$ is parameterized by x and y. But we alternatively may consider the set to be parametrized by two new parameters $x' \equiv x$ and $y' \equiv z$. This yields

$$x = x' \text{ and } y = x'y'$$

on the graph. These are exactly the equations (5.3.1), defining the substitutions in the sector of I_1 in the above example for sector decomposition. The y'-axis is not contained in $\Gamma_{\psi,x\neq 0}$. The blow-up Γ_ψ is obtained from $\Gamma_{\psi,x\neq 0}$ by simply including the y'-axis. As we said, Γ_ψ is the closure of $\Gamma_{\psi,x\neq 0}$. The projection of the y'-axis is the point $(0,\, 0)$. This is the analogue of the correspondence between the line $\left\{(x',\, y') \in \mathbb{R}^2 \mid x' = 0,\, 0 \leq y \leq 1\right\} \subset D_1$ and $(0,\, 0)$ in the above example.

The geometrical description for the second sector I_2 is analogous and is obtained from the above by exchanging x and y.

Our comparison of the examples shows, that in the language of algebraic geometry the single decompositions in the algorithm of Binoth and Heinrich are blow-ups of integration domains along certain centers. As centers we choose common zero-sets of monomials. For example the center of the above example, the point $(0,\, 0)$, is the zero-set of the monomial x and the monomial y ($x = x_1$, $y = x_2$). In other words, the centers which we choose in sector decomposition are intersections of the coordinate hypersurfaces (which in the case of the above example are the x- and y-axis). Such centers are determined by the choice of the set S. In the above example the center $(0,\, 0)$ corresponds to the choice $S = \{1,\, 2\}$. Accordingly in n variables the choice $S = \{\alpha_1,\, ...,\, \alpha_k\} \subset \{1,\, ...,\, n\}$ determines the center of the blow-up to be the subset where $x_{\alpha_1} = 0,\, ...,\, x_{\alpha_k} = 0$.

For completeness let us give a more general definition of blow-ups according to Harris [**Har95**]:

Definition 56. Let $V \in \mathbb{A}^n$ be an affine variety with its ring of regular functions $A(V)$. Furthermore let $f_0,\, ...,\, f_k \in A(X)$ generate the ideal defining a sub-variety $W \subset V$. We consider the rational map

$$\psi : V \quad \dashrightarrow \quad \mathbb{P}^n$$

given by $\psi(p) = [f_0,\, ...,\, f_n]$. Then the graph Γ_ψ together with the projection $\Gamma_\psi \to V$ is called the *blow-up of V along W*.

Without going into further detail on the determination of varieties by ideals, we note that the choice of the generators $f_0,\, ...,\, f_k$ which determines the center W corresponds to the choice of the set S in sector decomposition, which was discussed in detail in the previous sections. As we have seen, the right way to choose S in order to obtain a terminating sector decomposition is given by winning strategies of Hironaka's polyhedra game. In the geometrical language, the winning strategies give the choice of centers W for sequences of blow-ups which *resolve* certain singularities. Now let us see what that means.

5.3.3. Resolution of Singularities. For simplicity let us assume to have only one polynomial P in the parameters x_1, ..., x_n with its affine variety $V(P) \subset \mathbb{A}^n$, being its zero-set. A point on $V(P)$ where all the partial derivatives $\frac{\partial P}{\partial x_i}$, $i = 1, ..., n$, are zero is a *singular point* of $V(P)$. The singular points form the kinks in one-dimensional curves, edges in two-dimensional subspaces and so on. A variety without singular points is called *smooth*.

Let us consider a blow-up of the space $\mathbb{A}^n$ along some variety W and let the graph of the corresponding rational map ψ again be denoted by Γ_ψ. Then $\mathbb{A}^n$ is a projection of Γ_ψ and each point in $V(P) \subset \mathbb{A}^n$ is the projection of some subset of Γ_ψ. A singular point in $V(P)$ may be the projection of a smooth subset[12] of Γ_ψ. If this is the case, we can say, that the singular behaviour of $V(P)$ has been improved under the blow-up.

EXAMPLE 57. Let us consider the polynomial $P = x^2 - y^3$. The variety $V(P)$ is the so-called cusp in $\mathbb{A}^2$, shown in lower part of figure 5.3.1 (b). It contains the point (0, 0) as a singular point. If we choose this point to be the center of a blow-up we obtain the above example 55, which was the blow-up of $\mathbb{A}^2$ along the point (0, 0), shown in figure 5.3.1 (a). The improvement under this blow-up for the curve can be observed in figure 5.3.1 (b). The vertical line in the graph Γ_ψ is the exceptional locus, whose projection is the singular point. There is a sub-variety $V'(P)$ in Γ_ψ whose projection is $V(P)$. In contrast to the original $V(P)$ its blow-up $V'(P)$ is smooth everywhere. It is tangential to the vertical line.

In simple cases it is easy to find a blow-up, such that the according subset of Γ_ψ is smooth. For the general case the task is harder and one usually needs a sequence of blow-ups to obtain all singular points as projections of smooth regions. If this is achieved, then the sequence of blow-ups gives a *resolution of the singularities*. Hironaka's famous theorem states the existence of such resolutions for a very general case:

THEOREM 58. *(Hironaka* **[Hir64]***, cf.* **[Har95]***) Let X be any variety over a field of characteristic 0. Then there exists a smooth variety Y and a regular birational map $\sigma : Y \to X$.*

The latter map is called a *resolution of singularities* of X.

The *characteristic* of a ring is defined as follows: If 1_R is the identity element ('the one') and 0_R is the additive identity element ('the zero') of a ring R and if we can write 0_R as a sum of terms 1_R then the characteristic of R is the smallest natural number n such that

$$\underbrace{1_R + ... + 1_R}_{n \text{ terms}} = 0_R.$$

If such a relation does not hold for any natural number n, the characteristic of the ring is zero. The fields $\mathbb{R}$ and $\mathbb{C}$ are examples of fields of characteristic zero, because $\sum_{i=1}^{n} 1 \neq 0$ for any $n \in \mathbb{N}$. For a field of non-zero characteristic, the problem of resolution of singularities is more difficult and the above discussed version of Hironaka's polyhedra game is not sufficient to describe the combinatorial situation anymore.

For his work on the resolution of singularities including the original proof **[Hir64]** of the above theorem (formulated on more than 200 pages) Hironaka was awarded the Fields Medal in 1970. In addition to the existence of a resolution for general varieties he showed for the zero-set of a single polynomial that a winning strategy to his polyhedra game

[12]We already mentioned a simple (Gedanken)experiment: A piece of wire held underneath our desk lamp can be bended in a way such that it remains smooth but its shadow on our desk is a curve with a kink.

determines the right choice of centers of blow-ups, such that a resolution is obtained. As we mentioned, the first winning strategy [**Spi83**] was then found by Spivakovsky in 1982. An instructive approach to the problem of the resolution of singularities and Hironaka's theorem is given by Hauser in [**Hau03**]. This introduction reviews a recent proof by Encinas and Hauser [**EH02**], which is much shorter than the original one by Hironaka. This work of Encinas and Hauser contains a constructive method for the resolution of the singularities of a variety, possibly given by the zero-set of more than one polynomial. For the case of only one polynomial, the method provides the strategy described in subsection 5.2.5 above.

We want to remark, that Hironaka's theorem states the existence of a resolution for a case which is more general than the one considered in our treatment of sector decomposition. It is more general in the following three ways:

- The algorithm works only for polynomials whose zero-set does not intersect with the interior of $\mathbb{R}^n_+$. We remind ourselves, that in the case of Feynman integrals this condition was fulfilled by each polynomial $\mathcal{U}$ and by the polynomial $\mathcal{F}$ in the Euclidean momentum space. Hironaka's theorem does not assume the mentioned property of the polynomials.
- In section 5.1 about sector decomposition we argued, that we only need to consider one polynomial at a time. The variety in Hironaka's theorem may be the common zero-set of many polynomials.
- The theorem allows for a field of characteristic zero, which is not necessarily $\mathbb{R}$.

These differences might indicate possible extensions of the sector decomposition algorithm, possibly without the need of changing the combinatorics of step 2, given by the polyhedra game. Maybe the extension to the non-Euclidean momentum space would be the most desirable one[13]. In order to construct such an extension it could be helpful to study in more detail which of the technical constraints of the algorithm are not necessary in the geometrical description and therefore might be discarded. For example the fact that the integration domains are n-dimensional cubes at each stage of the algorithm is very convenient for the iteration, but it is not necessary for the disentanglement of singularities.

Let us think of other alternatives for the procedure of the disentanglement of singularities of Feynman integrals. We have seen in the section about Hironaka's polyhedra game, that there is more than one winning strategy for the game and we have observed in examples, that their application usually leads to different numbers of sectors. In general, the way to obtain a resolution of singularities is not unique. We can obtain a resolution by different sequences of blow-ups. Moreover, for certain cases, these sequences do not necessarily need to be constructed by the use of a winning strategy of Hironaka's game, as we already know from the successful applications of Strategy X. The winning strategies are useful to find a resolution for the general case, but for certain cases of integrals we may find different strategies, which yield a resolution. In this context we want to mention a sequence of blow-ups which was used to disentangle the singularities of certain Feynman integrals in [**BEK06**], created by the use of a rule for the selection of the centers, which is not deduced from Hironaka's game.

Moreover, one may not only consider different ways to construct a sequence of blow-ups, but also different kinds of blow-ups. Let us just very briefly mention the alternative

[13]For special cases one can address the non-Euclidean region by a simple substitution before the decomposition starts. The trick is described in appendix C of reference [**ST09**].

approach of the so-called *Nash blow-up*[14] (see e. g. chapter 17 of [**Har95**]). A Nash blow-up is the graph of a map which replaces the variables of the given polynomial by partial derivatives of the polynomial. In a sense the Nash blow-up associates to each non-singular point of the variety its tangent space at this point. The singularities of arbitrary curves can always be resolved by a sequence of Nash blow-ups, but for arbitrary varieties, a proof of the corresponding statement is not known. Nevertheless it may be interesting to find an application of Nash blow-ups for Feynman integrals. An advantage to the approach discussed above might be that there is no center which would have to be chosen and therefore one might possibly obtain each resolution by a unique sequence of blow-ups.

In the beginning of the chapter we mentioned, that the idea of disentangling singularities of Feynman integrals by a decomposition of the integration domain at first arose in Hepp's proof [**Hep66**] of the Bogoliubov-Parasiuk-theorem, which led to BPHZ renormalization. Let us conclude the chapter with the remark, that the theory of renormalization might as well profit from the application of tools from the theory of the resolution of singularities. Very recent progress in this field is made by the application of blow-up sequences in the context of Epstein-Glaser renormalization in position space [**BBK09**]. For a different approach to renormalization with many sophisticated applications of algebraic geometry we refer to recent work of Marcolli [**Mar08**].

[14]The Nash blow-up, sometimes referred to as Nash modification or Nash transform, was proposed to Hironaka by John Forbes Nash for the resolution of singularities. Nash was awarded the Nobel Prize in Economics for his work in game theory and he is known to a large non-scientific audience as the subject of the Hollywood movie "A beautiful mind", which is based on his biography.

CHAPTER 6

Periods

"One often makes a remark and only later sees how true it is."
(Ludwig Wittgenstein)

In the above chapters we have discussed topics of relevance for different stages of the evaluation of a Feynman integral. Before we continue with a discussion of the arithmetic nature of the results, the final stage, let us give a brief summary on what we have seen up to this point. We started in chapter 2 with the dimensional regularization (introducing the parameter $D = 2\lambda - 2\epsilon$) of an arbitrary Feynman integral and its expansion in terms of scalar Feynman integrals. In chapter 3 we have seen how IBP relations are used to express a Feynman integral as a linear combination of possibly simpler ones, the master integrals. Having reduced the problem so far and considering some scalar (master) integral I_G, one needs to decide for an appropriate method for the evaluation of I_G as a Laurent series in ϵ. I_G is possibly a function of particle masses and external momenta. Let us denote the set of these *external parameters* by Λ_G.

The focus of our dissertation lies on the method of sector decomposition, which starts from the Feynman parametric representation. Chapter 4 contains a detailed treatment of the polynomials of this representation of the integral. The sector decomposition method, discussed in chapter 5, yields the Laurent series

$$I_G\left(\Lambda_G, \epsilon\right) \quad = \quad \sum_{j=-2L}^{\infty} C_j\left(\Lambda_G\right) \epsilon^j$$

(L being the loop-number) where the coefficients $C_j\left(\Lambda_G\right)$ are linear combinations of finite integrals. As the analytical evaluation of these integrals is unknown in the general case one often has to fix the values of the external variables and integrate numerically. This is a drawback of the method compared to purely analytical approaches.

On the other hand, sector decomposition has a great advantage. In contrast to many other approaches, the method can be applied to arbitrary Feynman integrals. Therefore it can be used to study general properties of Feynman integrals. The present chapter exhibits such a general property, stated below by the main theorem of the dissertation. Our theorem says, that the Laurent coefficients $C_j\left(\Lambda_G\right)$ of an arbitrary scalar Feynman integral evaluate to so-called *periods* at algebraic values of the external parameters in the Euclidean momentum region. This is the region where all masses are non-negative and all kinematical invariants are non-positive. We will give a constructive proof by use of the improved sector decomposition algorithm.

The set of periods as defined by Kontsevich and Zagier (see below) is a special set of numbers. It is a subset of the complex numbers and contains the whole set of algebraic numbers but also transcendental numbers. Periods play a central role in modern algebraic geometry and related fields of mathematics. Kontsevich and Zagier give a survey of correspondences which are far beyond the scope of this dissertation. We just want to give

a glance at some of these topics and refer to further instructive literature in an outlook at the end of the chapter.

The more straightforward point of view we want to take in this chapter is the following: Interesting functions and numbers are known to be present in Laurent coefficients of many Feynman integrals. In particular, polylogarithms and multiple zeta values are omnipresent in today's multi-loop calculations. Giving a brief introduction to these functions and numbers in section 6.1 we will easily see that multiple zeta values and values of polylogarithms at algebraic arguments belong to the set of periods. On the other hand the presence of these functions and numbers in the results is not known to follow a general pattern. Furthermore very recent results include elliptic integrals, which are not known to have a representation in terms of polylogarithms.

This motivates our focus on the more general objects: the periods. We state and prove our theorem in section 6.2, closely following the joint work with Stefan Weinzierl [**BW09**], where this work was presented for the first time. The theorem is a very general statement on the result one has to expect in a multi-loop calculation. It tells us, that no matter which Feynman integral we consider, what we obtain in the end will not be beyond a set of functions, whose evaluations in the mentioned region will belong to this particular set of numbers.

6.1. Periods, Nested Sums and Iterated Integrals

6.1.1. Periods: Definition and Basic Properties. Kontsevich and Zagier give the following definition of periods:

Definition 59. (Kontsevich and Zagier [**KZ**]) A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

We denote the set of periods by $\mathcal{P}$. Kontsevich and Zagier state that one obtains the same set of numbers if one replaces the word "rational" by "algebraic" in the definition[1].

Let us remind ourselves of the meaning of the terms in the definition. The set of rational numbers is denoted by $\mathbb{Q}$. The algebraic numbers over $\mathbb{Q}$ are defined to be solutions x of polynomial equations with rational coefficients:

$$x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \quad = \quad 0 \text{ with all } a_j \in \mathbb{Q}.$$

The set of algebraic numbers over $\mathbb{Q}$ is denoted $\overline{\mathbb{Q}}$. The set $\overline{\mathbb{Q}}$ obviously contains $\mathbb{Q}$. Numbers which are not algebraic are called *transcendental*. As we mentioned, the set of periods is a proper subset of the complex numbers $\mathbb{C}$ and contains the algebraic numbers $\overline{\mathbb{Q}}$ as a proper subset:

$$\begin{array}{ccccccc} \mathbb{N} & \subset & \mathbb{Z} & \subset & \mathbb{Q} & \subset & \overline{\mathbb{Q}} \\ & & & & & & \cap \\ & & & & \cap & & \mathcal{P} \\ & & & & & & \cap \\ & & & & \mathbb{R} & \subset & \mathbb{C} \end{array}$$

We remind ourselves, that $f(x)$ is a *rational function* if it can be written as a fraction of polynomials in x. *Algebraic functions* are solutions of equations of the form

$$p_0(x) + p_1(x) f(x) + ... + p_r(x) \left(f(x)\right)^r \quad = \quad 0$$

[1] Further definitions are discussed in [**Fri05**].

with p_0, ..., p_r being polynomials in x. A domain in $\mathbb{R}^n$ given by polynomial inequalities with algebraic coefficients shall in the following be called a *semi-algebraic set*.

We furthermore remember, that for a complex valued function $f(x)$ and a domain of integration A the integral $\int_A dx f(x)$ is said to be *absolutely convergent* if the integral of the absolute value of the integrand, i. e. $\int_A dx\,|f(x)|$, is convergent.

EXAMPLE 60. The numbers

$$\sqrt{2} = \int_{2x^2\leq 1} dx,\ \pi = \int\int_{x^2+y^2\leq 1} dxdy \quad \text{and} \quad \ln(2) = \int_1^2 \frac{dx}{x}$$

are examples for periods. As we see, they can be written as absolutely convergent integrals over semi-algebraic sets and the integrands are algebraic functions. Note that the domain of integration of

$$\ln(q) = \int_1^q \frac{dx}{x}$$

is a semi-algebraic set as long as q is algebraic.

$\sqrt{2}$ is an algebraic number while $\ln(2)$ and π are examples for periods which are transcendental. It is much more difficult to find a complex number which is *not* a period. We can see from the definition that the set $\mathcal{P}$ is countable, while we know that $\mathbb{C}$ is not countable, so there must be an uncountable set of complex numbers which are not periods. It is conjectured in **[KZ]**, that the number $1/\pi$ and two prominent constants of Euler, $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and $\gamma_E = \lim_{n\to\infty}\left(1+\frac{1}{2}+...+\frac{1}{n}-\ln(n)\right)$, do not belong to $\mathcal{P}$. Very recently Yoshinaga constructed the first real number which is known to be not a period **[Yos08]**.

The set of periods forms an algebra over $\overline{\mathbb{Q}}$ (see **[Fri05, KZ]**). In particular, sums and products of periods are again periods. This can be observed as follows: Let a and b be periods, given by

$$a = \int_{G_1} d^n x f(x), \qquad b = \int_{G_2} d^m y g(y)$$

where f and g are rational functions and $G_1 \subset \mathbb{R}^n$ and $G_2 \subset \mathbb{R}^m$ are semi-algebraic sets. Then their product is the period

$$a \cdot b \ = \ \int_{G_1\times G_2} d^n x d^m y f(x) g(y). \tag{6.1.1}$$

Their sum can be written

$$a + b \ = \ \int_G d^n x d^m y \left((1-t)\, f(x) + t g(y)\right) \tag{6.1.2}$$

with

$$G \ = \ G_1 \times \{0\} \times [0,\,1]^m \cup [0,\,1]^n \times \{1\} \times G_2 \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m, \tag{6.1.3}$$

where t is the variable of the middle factor in this Cartesian product, taking only the discrete values 0 and 1. Note that t is used to define G, but nevertheless, G is a set of dimension $n+m$. The set G, the disjoint union of semi-algebraic sets, is a semi-algebraic set itself and therefore $a+b$ is a period. (The subsets $[0,\,1]^m$ and $[0,\,1]^n$ can be replaced by any other semi-algebraic subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, without changing the integral.)

In the proof of our theorem below, we will consider integrals with sums of products of logarithms in the integrand, and we will have to decide, whether they are periods. At

first let us observe, that if $G \in \mathbb{R}^n$ is a semi-algebraic set and $f(x)$ and $g(x)$ are rational functions in $x \in G$ with rational coefficients such that

$$I = \int_G d^n x f(x) \ln(g(x)) \tag{6.1.4}$$

is absolutely convergent, then I is a period. We see this writing

$$\int_G d^n x f(x) \ln(g(x)) = \int_G d^n x f(x) \int_0^1 dt \frac{g(x)-1}{(g(x)-1)t+1} = \int_{G'} d^n x dt \frac{f(x)(g(x)-1)}{(g(x)-1)t+1}$$

with G' being the set in $\mathbb{R}^{n+1}$ consisting of the points $(x_1, ..., x_n, t)$ with $(x_1, ..., x_n) \in G$ and $t \in [0, 1]$. G' is a semi-algebraic set and the integrand is a rational function.

In the same way we can rewrite integrands with sums and products of logarithms. For a sum we have

$$\int_G d^n x \left(f_1(x) \ln(g_1(x)) + f_2(x) \ln(g_2(x))\right) =$$

$$\int_{G''} d^n x dt_1 dt_2 \left(\frac{f_1(x)(g_1(x)-1)}{(g_1(x)-1)t_1+1} + \frac{f_2(x)(g_2(x)-1)}{(g_2(x)-1)t_2+1}\right)$$

and for a product we have

$$\int_G d^n x f(x) \ln(g_1(x)) \ln(g_2(x)) = \int_{G''} d^n x dt_1 dt_2 f(x) \frac{g_1(x)-1}{(g_1(x)-1)t+1} \frac{g_2(x)-1}{(g_2(x)-1)t+1}$$

where f_1, f_2, g_1, g_2 are rational functions and G'' is the set of points $(x_1, ..., x_n, t_1, t_2) \in \mathbb{R}^{n+2}$ with $(x_1, ..., x_n) \in G$ and $t_1, t_2 \in [0, 1]$.

We can apply these steps in combination to construct the more general integral

$$J = \int_G d^n x \sum_{i=1}^{r} f_i(x) \prod_{j=1}^{s} (\ln(g_j(x)))^{m_j} \tag{6.1.5}$$

where $G \in \mathbb{R}^n$ is a semi-algebraic set, all the f's and g's are rational functions, r and s are natural numbers and all the exponents m_j are integers. The above equations prove the following:

LEMMA 61. *An absolutely convergent integral of the form J (of equation (6.1.5)) is a period.*

The following motivational introduction to our theorem follows a general paradigm:

Principle: (Kontsevich and Zagier **[KZ]***). Whenever you meet a new number, and have decided (or convinced yourself) that it is transcendental, try to figure out whether it is a period.*

Starting from the rather general notions of nested sums and iterated integrals we briefly introduce zeta values, multiple zeta values, classical polylogarithms and harmonic polylogarithms and we give indications on their presence in loop calculations. They will be easily recognized to be periods, or functions whose values at algebraic arguments are periods.

6.1.2. Nested Sums. The numbers and functions to be introduced in the following will be defined in terms of sums. At first let us introduce the most general type of a sum we will need. For integers k, n, m_1, ..., m_k and complex numbers z_1, ..., z_k we define

Z-sums [**MUW02**] by

$$\begin{aligned} Z(n) &= \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases} , \\ Z(n; m_1, ..., m_k; z_1, ..., z_k) &= \sum_{i=1}^{n} \frac{z_1^i}{i^{m_1}} Z(i-1; m_2, ..., m_k; z_2, ..., z_k). \end{aligned}$$

Equivalently we can write

$$Z(n; m_1, ..., m_k; z_1, ..., z_k) = \sum_{n \geq i_1 > i_2 > ... > i_k > 0} \frac{z_1^{i_1}}{i_1^{m_1}} \cdots \frac{z_k^{i_k}}{i_k^{m_k}}. \tag{6.1.6}$$

We will allow for sums with infinitely many terms (i. e. $n \to \infty$), denoted by $Z(\infty; ...)$.

As an alternative to Z-sums it might be convenient to consider *S-sums*, defined by

$$\begin{aligned} S(n) &= \begin{cases} 1, & n > 0, \\ 0, & n \leq 0, \end{cases} \\ S(n; m_1, ..., m_k; z_1, ..., z_k) &= \sum_{i=1}^{n} \frac{z_1^i}{i^{m_1}} S(i; m_2, ..., m_k; z_2, ..., z_k), \end{aligned}$$

equivalently written as

$$S(n; m_1, ..., m_k; z_1, ..., z_k) = \sum_{n \geq i_1 \geq i_2 \geq ... \geq i_k \geq 1} \frac{z_1^{i_1}}{i_1^{m_1}} \cdots \frac{z_k^{i_k}}{i_k^{m_k}}. \tag{6.1.7}$$

As both types of sums are closely related, they can be converted into each other using relations given in the article [**MUW02**]. The authors of reference [**MUW02**] furthermore point out, that Z-sums and S-sums are direct generalizations of two prominent sets of sums over finitely many terms: *Euler-Zagier sums* [**Zag94**] and *harmonic sums*. One can define Euler-Zagier sums as Z-sums with all variables z_i equal to one:

$$Z^{EZ}(n; m_1, ..., m_k) = Z(n; m_1, ..., m_k; 1, ..., 1), \tag{6.1.8}$$

and harmonic sums as *S-sums* where the variables z_i take fixed values in $\{-1, 1\}$:

$$S^H(n; m_1, ..., m_k; \pm 1, ..., \pm 1) = S(n; m_1, ..., m_k; \pm 1, ..., \pm 1). \tag{6.1.9}$$

6.1.3. Iterated Integrals. As we will discuss below, multiple zeta values and polylogarithms can be written as sums and as well as integrals. These integrals are called *iterated integrals* or *Chen iterated integrals* [**Che77**] (also see [**Kas95**]) of complex valued differential 1-forms. They are defined as follows:

Let $f_i(t)$ for $i = 1, ..., n$ and $t \in \mathbb{R}$ be complex valued functions and $\omega_i = f_i(t)dt$ define complex valued differential 1-forms on a real interval $[a, b]$ with $a, b \in \mathbb{R}$. Then the iterated integral denoted by $\int_a^b \omega_1 ... \omega_n$ is inductively defined by

$$\int_a^b \omega_1 = \int_a^b f_1(t)dt, \tag{6.1.10}$$

$$\int_a^b \omega_1 ... \omega_n = \int_a^b f_1(t) \left(\int_a^t \omega_2 ... \omega_n \right) dt. \tag{6.1.11}$$

To give an example, this definition allows us to write

$$\int_a^b \frac{1}{t} \left(\int_a^t \frac{1}{1-t'} dt' \right) dt \equiv \int_a^b \frac{dt}{t} \frac{dt}{1-t}. \tag{6.1.12}$$

We can alternatively write the general iterated integral as

$$\int_a^b \omega_1 ... \omega_n \quad = \quad \int_{b \geq t_1 \geq ... \geq t_{n-1} \geq t_n \geq a} f_1(t_1) f_2(t_2) ... f_n(t_n) \, dt_n ... dt_2 dt_1. \tag{6.1.13}$$

If such an integral is absolutely convergent, a and b are rational numbers and the product $f_1(t_1) f_2(t_2) ... f_n(t_n)$ is a rational function with rational coefficients, then we see that the value of the integral is a period. Therefore we will see that multiple zeta values and values of polylogarithms at algebraic arguments are periods.

Let us introduce the most general type of iterated integrals we will need to represent multiple zeta values and polylogarithms:

Iterated integrals with all differential 1-forms of the form

$$\omega_i \quad = \quad \frac{dt}{t - z_i}$$

are called *hyperlogarithms*. They were investigated by Lappo-Danilevsky [**LD11**] and reconsidered by Goncharov in the article [**Gon01**] where they are defined[2] as

$$\mathrm{MP}(z_0, z_1, ..., z_r) \quad = \quad \int_0^{z_0} \frac{dt}{z_1 - t} \frac{dt}{z_2 - t} ... \frac{dt}{z_r - t}. \tag{6.1.14}$$

Kummer, Poincaré and Lappo-Danilevsky treated them as analytic functions of just one variable z_0. Goncharov considers these functions where some of the arguments may become zero and writes them as functions of all non-zero variables z_i as:

$$\mathrm{MP}_{s_1, ..., s_r}(z_0, z_1, ..., z_r) =$$

$$\int_0^{z_0} \underbrace{\frac{dt}{t} ... \frac{dt}{t} \frac{dt}{z_1 - t}}_{s_1 \text{ times}} \underbrace{\frac{dt}{t} ... \frac{dt}{t} \frac{dt}{z_2 - t}}_{s_2 \text{ times}} ... \underbrace{\frac{dt}{t} ... \frac{dt}{t} \frac{dt}{z_r - t}}_{s_r \text{ times}}. \tag{6.1.15}$$

He calls these functions *multiple polylogarithms*. We will give their representation as sums below.

6.1.4. (Multiple) Zeta Values. The numbers to be discussed in the following are (generalizations of) values of the famous *Riemann zeta function*. For a variable $s \in \mathbb{C}$ with a real part larger than one, the Riemann zeta function $\zeta(s)$ is defined by the absolutely convergent sum[3]

$$\zeta(s) \quad = \quad \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{6.1.16}$$

In order to give an indication on the importance of this function[4] we want to mention two famous statements around the mysterious distribution of prime numbers in the natural numbers:

[2]Note that Goncharov defines the iterated integrals in a reversed ordering, such that his function I is $I(z_r, , ..., z_1, z_0) = (-1)^r \mathrm{MP}(z_0, z_1, ..., z_r)$.

[3]We learned a mnemonic for Riemann's zeta function from Herbert Gangl: It is a sum, which counts. But it does not count $1+2+3+...$ or $1 + \frac{1}{2} + \frac{1}{3} + ...$ because these sums diverge. In order not to diverge, it uses a number s for counting $1 + \frac{1}{2^s} + \frac{1}{3^s} + ...$. One may speculate if it is a coincidence, that this function is denoted 'zeta' while the German word for counting (to count: zählen) begins with the letter 'z'.

[4]For an introduction to this function and its relations to physics we refer to chapter 6 of Zeidler's book [**Zei06a**].

- *Euler's prime number theorem:* For each real number $s > 1$ one has

$$\zeta(s) = \prod_{\text{all primes } p} \left(1 - \frac{1}{p^s}\right)^{-1} \tag{6.1.17}$$

where the product runs through all the existing prime numbers. The proof was given by Euler in 1737.

- Riemann showed in 1859, that $\zeta(s)$ can be extended to the whole complex plane as the solution of the functional equation

$$\frac{2^{s-1}\pi^s}{\Gamma(s)}\zeta(1-s) = \zeta(s)\cos\left(\frac{\pi s}{2}\right). \tag{6.1.18}$$

The extension is meromorphic in $\mathbb{C}$ with only one pole at $s = 1$. Furthermore Riemann has proven a formula relating the solutions of $\zeta(s) = 0$ with $s \in \mathbb{C}$, i. e. the *zeros* of ζ, to the number of prime numbers between 1 and a chosen number **[Rie60]**. ζ is known to be zero at $s = -2, -4, -6, ...$, the so-called *trivial zeros*. Riemann conjectured, that the real part of all the other zeros in $\mathbb{C}$ is equal to $\frac{1}{2}$. This conjecture is known as *Riemann hypothesis*, assumed to be one of the most important open problems in mathematics[5]. A detailed introduction to the problem is given by Bombieri in **[Bom]**.

We are interested in values of $\zeta(s)$ at integer values of s, where s is larger than one. We will refer to these values as *zeta values*.

Some of these zeta values are well studied, for example Euler evaluated

$$\zeta(2) = \frac{\pi^2}{6} \quad \text{and} \quad \zeta(4) = \frac{\pi^4}{90}$$

and one furthermore knows, that

$$\zeta(2m) = \frac{\pi^{2m}(-1)^{m-1}2^{2m-1}B_{2m}}{(2m)!}$$

for $m = 1, 2, 3, ...$ and B_i being the Bernoulli numbers. The values $\zeta(2m+1)$ are more difficult. For example one does not know whether these numbers are transcendental or not, even though zeta values are often referred to as 'transcendentals' in the literature. In fact they are not even known to be irrational, except for $\zeta(3)$ whose irrationality was proven by Apéry **[Ape79]**.

An important generalization of zeta values are *multiple zeta values* (MZVs) defined by Zagier in the article **[Zag94]** as[6]

$$\zeta(s_1, ..., s_r) = \sum_{n_1>...>n_r>0} \frac{1}{n_1^{s_1}...n_r^{s_r}} \text{ for integers } s_1 \geq 2,\ s_i \geq 1,\ i = 2, ..., r. \tag{6.1.19}$$

For the special case of only one argument we obviously obtain the above zeta values. Using the notation of Z-sums we can write

$$\zeta(s_1, ..., s_r) = Z(\infty; s_1, ..., s_r; 1, ..., 1). \tag{6.1.20}$$

By comparison with equation (6.1.8) MZVs are generalizations of Euler-Zagier sums to sums with infinitely many terms.

[5] It is one of the seven millennium problems of the Clay Mathematics Institute. Its solution is supposed to be awarded one million dollars.

[6] Note that in our notation the ordering of the arguments is reversed in comparison to Zagier's definition. We adopt our notation from the more recent physics literature.

The same article by Zagier presents the integral representation of MZVs as found by Kontsevich:

$$\zeta(s_1, ..., s_r) = \int_0^1 \underbrace{\frac{dt}{t} ... \frac{dt}{t} \frac{dt}{1-t}}_{s_1 \text{ times}} \underbrace{\frac{dt}{t} ... \frac{dt}{t} \frac{dt}{1-t}}_{s_2 \text{ times}} ... \underbrace{\frac{dt}{t} ... \frac{dt}{t} \frac{dt}{1-t}}_{s_r \text{ times}}. \tag{6.1.21}$$

By comparison with equation (6.1.15) we see that they are special values of multiple polylogarithms, where all the arguments z_i are fixed to be either 0 or 1 and where the upper integration limit is 1. Such integrals are also known as *Drinfel'd integrals*. From the above integral representation we see that MZVs are periods.

Concerning their presence in Feynman integral calculations we want to refer to two famous examples, which we already met in previous chapters.

EXAMPLE 62. Our first example (cf. [**Bie05**]) is the dimensionally regularized scalar Feynman integral of the one-loop selfenergy graph of figure 3.2.1 in chapter 2. We name the external momentum q and we set the masses equal to zero. The exponents of the inverse propagators are positive integers ν_1 and ν_2. Up to a trivial prefactor we obtain the integral

$$I\left(D, \nu_1, \nu_2, q^2\right) = \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{\left(k_1\right)^{\nu_1} \left(\left(q-k_2\right)^2\right)^{\nu_2}}.$$

Its evaluation (see e. g. appendix C of [**Bie05**]) yields

$$I\left(D, \nu_1, \nu_2, q^2\right) =$$

$$\left(q^2\right)^{m-\epsilon-\nu_1-\nu_2} \frac{\Gamma\left(\nu_1+\nu_2-\lambda+\epsilon\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)} \frac{\Gamma\left(\lambda-\epsilon-\nu_1\right)\Gamma\left(\lambda-\epsilon-\nu_2\right)}{\Gamma\left(2\lambda-2\epsilon-\nu_1-\nu_2\right)} \tag{6.1.22}$$

with $D = 2\lambda - 2\epsilon$. Euler's gamma function can be expanded in ϵ as

$$\Gamma(1-\epsilon) = \exp\left(\gamma_E \epsilon\right) \exp\left(\sum_{n=2}^{\infty} \frac{\zeta(n)}{n} \epsilon^n\right) \text{ for } |\epsilon| < 1. \tag{6.1.23}$$

Furthermore we can expand

$$\left(q^2\right)^x = \exp\left(\ln\left(\left(q^2\right)^x\right)\right) = 1 + x \ln\left(q^2\right) + \mathcal{O}\left(x^2\right).$$

From these equations we see, that the Laurent coefficients of I in ϵ apart from γ_E, rational numbers and powers of $\ln\left(q^2\right)$ only contain zeta values $\zeta(n)$.

A similar result with respect to MZVs is known for the following much more involved example.

EXAMPLE 63. We consider the integral of the two-loop graph of figure 2.3.1 (b) of chapter 2 (which we referred to as *Tbubble* in chapter 5). The external momentum is denoted p and all masses equal to zero. We obtain

$$I\left(D, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, p^2\right) =$$

$$\left(-p^2\right)^{\sum_{i=1}^5 \nu_i - 2m + 2\epsilon} \int \frac{d^D k_1}{i\pi^{D/2}} \int \frac{d^D k_2}{i\pi^{D/2}} \frac{1}{\left(-k_1^2\right)^{\nu_1} \left(-k_2^2\right)^{\nu_2} \left(-k_3^2\right)^{\nu_3} \left(-k_4^2\right)^{\nu_4} \left(-k_5^2\right)^{\nu_5}}.$$

In [**BW03**] Bierenbaum and Weinzierl present a way to expand the function to all orders in ϵ. This method proves their following theorem:

THEOREM 64. *(Bierenbaum and Weinzierl* [**BW03**]*) Multiple zeta values are sufficient for the integral* $I\left(\lambda-\epsilon,\,\nu_1,\,\nu_2,\,\nu_3,\,\nu_4,\,\nu_5,\,p^2\right)$*, if all powers of the propagators are of the form* $\nu_j = n_j + a_j\epsilon$*, where* n_j *are positive integers and* a_j *are non-negative real numbers.*

Motivated by these results, one may ask for the following:

PROBLEM 65. Determine the maximal set F_{MZV} of Feynman integrals, whose Laurent coefficients in dimensional regularization can be expressed as linear combinations of multiple zeta values with rational coefficients (up to powers of γ_E).

To our knowledge, this problem is far from being solved[7]. Nevertheless, from the IBP-identities as discussed in chapter 3 we see that one knows a whole class of integrals to belong to F_{MZV} if they can be reduced to master integrals which lie in F_{MZV}. Therefore the above results have implications for a whole family of integrals.

Very recent progress towards an answer to problem 65 was made in Brown's article [**Bro09**]. We also refer to early work of Broadhurst and Kreimer [**BK95, BK97**], who used knot-theory to predict the right multiple zeta values in results of certain vacuum-graph integrals and then checked their predictions by precise numerical computations. Both approaches were capable to evaluate certain families of integrals to MZVs and both seem to fail for integrals with a large loop-number[8].

REMARK 66. We just want to add a very brief remark on algebraic relations between MZVs, which is an important field of mathematical research. Due to the representations as nested sums and iterated integrals, a product of two MZVs always yields two linear combinations of MZVs. From the representation by sums, one obtains a linear combination

$$\zeta\left(s\right)\zeta\left(k\right) \;=\; \sum_{l\in\Delta(s,\,k)} \zeta(l)$$

where $s = (s_1,\,...,\,s_n)$ and $k = (k_1,\,...,\,k_m)$ are the tuples of arguments on the left hand side and the sum on the right-hand side runs through a set of tuples $\Delta(s,\,k)$, which is obtained from the so-called *quasi shuffle product* of s and k. On the other hand, from the representation of the MZVs by iterated integrals one obtains a similar relation, where the sum on the right-hand side instead runs through a set of tuples, which is obtained from the so-called *shuffle product*. For the sake of brevity we only want to refer to [**Hof00**] for the definition of these products.

Nevertheless, let us give an example, taken from [**BBBL01**]: The product $\zeta\left(2,\,1\right)\zeta\left(2\right)$ can be evaluated in two ways. Using the representation by sums one obtains

$$\zeta\left(2,\,1\right)\zeta\left(2\right) = 2\zeta\left(2,\,2,\,1\right) + \zeta\left(4,\,1\right) + \zeta\left(2,\,3\right) + \zeta\left(2,\,1,\,2\right)$$

and from the representation by integrals one obtains

$$\zeta\left(2,\,1\right)\zeta\left(2\right) = 6\zeta\left(3,\,1,\,1\right) + 3\zeta\left(2,\,2,\,1\right) + \zeta\left(2,\,1,\,2\right).$$

[7] A closely related open problem was formulated by Marcolli in conjecture 1.1 in the article [**Mar09**]. In this conjecture, formulated as a question, Marcolli asks whether all Feynman integrals of scalar field theories provide periods of so-called mixed Tate motives in their finite part, no matter which procedure is used for regularization and renormalization. According to a conjecture of Tate, periods of mixed Tate motives are always multiple zeta values (see e. g. the introduction [**And08**]). We add a brief remark on motives at the end of this chapter.

[8] A vacuum integral where these approaches fail might simply not belong to F_{MZV}. We believe to understand from [**And08**] and [**Bro09**] that such an integral would be interesting for the theory of motives as it might be associated to an object beyond the particular class of mixed Tate motives.

As a result, we obtain that the right-hand side of the first equation is equal to the right hand side of the second one.

As we see in this example, combining shuffle and quasi shuffle relations yields new relations between MZVs, which can otherwise not be obtained. A detailed introduction to such relations can be found in [**Hof00, Sou08**]. We want to mention, that in the article [**MUW02**] quasi shuffle relations are used in algorithms for the systematic evaluation of Feynman integrals in terms of Z-sums. Very recently, a large compendium of relations between MZVs was provided in reference [**BBV09**].

The so-called main Diophantine conjecture implies, that shuffle and quasi shuffle relations should in a certain sense be sufficient to generate all algebraic relations between MZVs. It is furthermore believed, that there are no algebraic relations between the zeta values of odd arguments, $\zeta(2m+1)$, $m = 1, 2, ...$ A precise formulation of both of these conjectures is given in reference [**Wal00**].[9]

6.1.5. Classical and Harmonic Polylogarithms. Similarly, as MZVs are the generalizations of zeta values, there are generalizations of the logarithm, called polylogarithms. By a Taylor expansion of $\ln(z)$ around 1 we can write the logarithm as the sum

$$-\ln(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \text{ for } |z| < 1. \tag{6.1.24}$$

Substituting n by n^2 in the denominator, we obtain the *dilogarithm*, which was first considered by Euler [**Eul68**] (also see Zagier's lectures [**Zag88**]):

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \text{ for } |z| < 1. \tag{6.1.25}$$

By generalizing the denominator to n^s we obtain the *classical polylogarithms* (see e. g. [**Lew81**])

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \text{ for } |z| < 1,\ s = 1, 2, ... \tag{6.1.26}$$

In a further generalization we allow for more than one complex argument and obtain the already mentioned *multiple polylogarithms* of Goncharov [**Gon01**]:

$$\mathrm{Li}_{s_1, s_2, .., s_r}(z_1, z_2, ..., z_r) = \sum_{n_1 > n_2 > ... > n_r} \frac{z_1^{n_1} z_2^{n_2} ... z_r^{n_r}}{n_1^{s_1} n_2^{s_2} ... n_r^{s_r}} \text{ for } |z_i| < 1,\ s_i = 1, 2, ...$$

We easily can identify these sums as Z-sums for $|z_i| < 1$:

$$-\ln(1-z) = Z(\infty; 1; z), \tag{6.1.27}$$

$$\mathrm{Li}_2(z) = Z(\infty; 2; z), \tag{6.1.28}$$

$$\mathrm{Li}_s(z) = Z(\infty; s; z), \tag{6.1.29}$$

$$\mathrm{Li}_{s_1, s_2, .., s_r}(z_1, z_2, ..., z_r) = Z(\infty; s_1, s_2, ..., s_r; z_1, z_2, ..., z_r). \tag{6.1.30}$$

[9] For a wealth of further references on MZVs we may refer the reader to a regularly updated literature list on Michael Hoffman's webpage [**Hof**].

All of these functions have a representation in terms of iterated integrals:

$$-\ln(1-z) = \int_0^z \frac{dt}{1-t}, \tag{6.1.31}$$

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t}dt = \int_0^z \frac{dt}{t}\frac{dt}{1-t}, \tag{6.1.32}$$

$$\mathrm{Li}_s(z) = -\int_0^z \frac{\mathrm{Li}_{s-1}(t)}{t}dt = \int_0^z \underbrace{\frac{dt}{t}...\frac{dt}{t}\frac{dt}{1-t}}_{s \text{ times}} \quad \text{for } s>2, \tag{6.1.33}$$

$$\mathrm{Li}_{s_1,s_2,..,s_r}(z_1,\,z_2,\,...,\,z_r) = \int_0^1 \underbrace{\frac{dt}{t}...\frac{dt}{t}\frac{dt}{(z_1)^{-1}-t}}_{s_1 \text{ times}} ... \underbrace{\frac{dt}{t}...\frac{dt}{t}\frac{dt}{(z_1z_2...z_r)^{-1}-t}}_{s_r \text{ times}}$$

$$\equiv \mathrm{MP}_{s_1,...,s_r}\left(1,\,(z_1)^{-1},\,(z_1z_2)^{-1},\,...,\,(z_1z_2...z_r)^{-1}\right). \tag{6.1.34}$$

Equation (6.1.34) was obtained by Goncharov in [**Gon01**]. From this result we see, that all mentioned polylogarithms evaluate to periods at algebraic values of the z_i. The integral representations given in the equations (6.1.31) to (6.1.34) are well-defined beyond arguments $|z_i|<1$, i. e. in a larger domain than the Z-sums of the equations (6.1.27) to (6.1.30). For example, $\mathrm{Li}_2(z)$ is defined everywhere in the complex plane except for a cut starting at $z=1$ (see e. g. [**Zag88**]).

Classical polylogarithms occur in Laurent coefficients of Feynman integrals. The arguments of these functions are usually algebraic functions of the external parameters. The logarithm and the dilogarithm appeared already in the general analytic result of the scalar one-loop integrals as evaluated by t'Hooft and Veltman in [**tV79**][10]. Among the examples for the presence of higher polylogarithms we may refer to the analytic result for the massless double box by Smirnov [**Smi99**]. The compact formula given there contains classical polylogarithms ln, Li_2, Li_3, Li_4, evaluated at fractions of the kinematical invariants. Of course, this is just one of many possible examples.

A successful method for the evaluation of Feynman integrals to polylogarithms is outlined in Weinzierl's talk [**Wei07**]: By using the Mellin transform of the Feynman integral, one obtains hypergeometric functions. Subsequently, these are expanded in terms of Z-sums and these can be expressed in terms of polylogarithms. (We have already seen a very simple variant of such a calculation in example 62.) Powerful algorithms for the expansion of hypergeometric functions in terms of Z-sums have been developed by Moch, Uwer and Weinzierl in [**MUW02**].

Another family of polylogarithms has turned out to be particularly useful for loop-calculations. In reference [**RV00**] Remiddi and Vermaseren introduced *harmonic polylogarithms* (HPLs) $H(n_w,\,...,\,n_1;\,z)$ in terms of iterated integrals. They are functions of a complex argument z and arguments $n_1,\,...,\,n_w$ which take values in $\{-1,\,0,\,1\}$. The number w is called the *weight* of the function. HPLs are defined recursively using the rational functions $f(n_i;\,x)$:

$$f(0;\,t)=\frac{1}{t},\;f(1;\,t)=\frac{1}{1-t},\;f(-1;\,t)=\frac{1}{1+t}.$$

[10] A simplified result for the box-graph was given by Denner, Nierste and Scharf in [**DNS91**] (also see section 4 of the review [**Den93**]).

For $w = 1$ Remiddi and Vermaseren define

$$H(0;\, z) = \int_1^z \frac{dt}{t} = \ln(z), \tag{6.1.35}$$

$$H(1;\, z) = \int_0^z \frac{dt}{1-t} = -\ln(1-z), \tag{6.1.36}$$

$$H(-1;\, z) = \int_0^z \frac{dt}{1+t} = \ln(1+z). \tag{6.1.37}$$

For $w > 1$ they recursively define

$$\begin{aligned} H(0,\, ...,0\,;\, z) &= \frac{1}{w!}\ln^w(z), \\ H(n_w,\, ...,\, n_1;\, z) &= \int_0^z dz' f(n_w;\, z') H(n_{w-1},\, ...,\, n_1;\, z'), \end{aligned} \tag{6.1.38}$$

where in the last equation at least one n_i is not equal to zero.

In some cases harmonic polylogarithms can be expressed by classical polylogarithms and so-called *Nielsen polylogarithms* [**Nie09**] which are generalizations of the dilogarithm. Remiddi and Vermaseren point out that such expressions can not be found for all HPLs.

A difference between HPLs and all the iterated integrals we mentioned before arises from equation (6.1.35) where the lower integration limit is 1. By recursion this is the case for all the HPLs with $n_w = 0$. Such HPLs are said to have a *trailing zero*. Let us denote HPLs with $n_w \neq 0$ by $H^\star$. For this subset of HPLs Remiddi and Vermaseren introduce a notation[11] by lower indices s_i. This notation is similar to the above Li-notation, except that the indices s_i may be negative. In Remiddi and Vermaseren's notation the HPLs without trailing zeroes are defined by:

$$H^\star_{m_1,\, ...,\, m_k}(z) =$$

$$\int_0^z f(\mathrm{sign}(m_1);\, t)dt \underbrace{\frac{dt}{t} ... \frac{dt}{t} f(\mathrm{sign}(m_2);\, t)dt}_{|m_2| \text{ times}} ... \underbrace{\frac{dt}{t} ... \frac{dt}{t} f(\mathrm{sign}(m_k);\, t)dt}_{|m_k| \text{ times}} \tag{6.1.39}$$

where the parameters m_i can be positive or negative integers and where $\mathrm{sign}(m_i) = 1$ for positive and $\mathrm{sign}(m_i) = -1$ for negative m_i. Using shuffle identities and integration by parts, Remiddi and Vermaseren show how to express an arbitrary harmonic polylogarithm in terms of harmonic polylogarithms without trailing zeroes (up to the exceptions $H(0;\, z) = \ln(z)$, $H(0,\, 0;\, z)$, $H(0,\, 0,\, 0;\, z)$ and so on).

A first generalization[12] of HPLs was given in [**GR02**] by Gehrmann and Remiddi, who extended the set of allowed rational functions in the integrand to

$$g(0;\, t) = \frac{1}{t},\ g(1;\, t) = \frac{1}{t-1},\ g(1-u;\, t) = \frac{1}{t+u-1},\ g(-u;\, t) = \frac{1}{t+u},$$

where u is an additional independent variable. By a recursive definition similar to equation (6.1.38) one obtains the *two-dimensional harmonic polylogarithms* (2dHPLs). This set of functions obviously contains the HPLs (up to possible signs arising from the different conventions). Gehrmann and Remiddi note, that the 2dHPLs without trailing zeroes (defined as in the case of HPLs) are contained in the set of multiple polylogarithms. Therefore they evaluate to periods at algebraic arguments.

[11] For consistency with our above notation we have a reversed order of indices compared to [**RV00**].

[12] We want to mention that further extensions were given by Aglietti and Bonciani in the article [**AB04**] and by Birthwright, Glover and Marquard in reference [**BGM04**].

Let us add the remark that an automated evaluation of the ubiquitous multiple zeta values, classical polylogarithms, harmonic polylogarithms and multiple polylogarithms has become important for practical multi-loop calculations. The numerical evaluation of HPLs and 2dHPLs was provided by Gehrmann and Remiddi in [**GR01a, GR02**] and an implementation for all the functions just mentioned, including multiple polylogarithms, was developed by Vollinga and Weinzierl in reference [**VW05**]. The latter implementation is integrated in the computer algebra package GiNaC [**BFK02**] from version 1.3 onwards.

In a thorough introduction to polylogarithms, less focused on periods and Feynman integrals, there would be much more to say about these functions. For a much more detailed introduction we may refer the reader to reference [**Wal00**], the survey [**GZ00**] and to the wealth of references given in the talk [**Wei07**].

6.1.6. Elliptic Integrals. We already mentioned that it is not known in general which Feynman integrals have only Laurent coefficients which are linear combinations of MZVs with rational coefficients (see problem 65), and the same is true for classical and harmonic polylogarithms. Many known analytical results, but remarkably not all of them, can be expressed in terms of MZVs and polylogarithms at present.

As an example let us consider the analytical result for the so called equal mass two-loop sunrise graph by Laporta and Remiddi [**LR05**]. This graph was our first example for a graph of ϕ^4-theory in chapter 2 (see figure 2.1.3) and we will discuss it in more detail in appendix B. In the result by Remiddi and Laporta, the Laurent coefficients of the dimensionally regularized Feynman integral are expressed in terms of the *complete elliptic integral of the first kind* which is defined as

$$K(\lambda) \quad = \quad \int_0^1 \frac{dy}{\sqrt{(1-y^2)\,(1-\lambda y^2)}}.$$

This function was already applied in the consideration of the same Feynman graph by configuration space techniques in [**GP00**] (also see [**Alm67**]). An expression of this function in terms of polylogarithms is not known. Nevertheless, we can easily observe, that the function evaluates to periods. The domain of integration is a semi-algebraic set and the integrand is an algebraic function and therefore, we obtain a period for algebraic values of λ.

The result of Laporta and Remiddi seems to indicate an arithmetic property of Feynman integrals. The requirement of elliptic integrals suggests that the set of MZVs and the set of polylogarithms might not provide all the functions which are necessary to represent the Laurent coefficients of dimensionally regularized Feynman integrals. On the other hand, the example shows that the set of periods is not excluded by the presence of such elliptic integrals. Therefore it makes sense to consider the set of functions which evaluate to periods at algebraic arguments. In the following section we prove that this set of functions, at least in the Euclidean momentum region, is sufficient to express the Laurent coefficients of any Feynman integral.

We want to mention that the result for the two-loop sunrise graph with equal masses is not the only example where elliptic integrals are required. Further elliptic integrals, more complicated than the elliptic integral of the first kind, are used in analytic results by Laporta in [**Lap08**] for the three-loop and four-loop sunrise topology. The latter results were partially obtained by numerical fits. Laporta emphasizes that for the integrals used there as well expressions in terms of polylogarithms are not known.

6.2. Periods and Feynman Integrals

The theorems to be discussed in the following state that the Laurent coefficients of Feynman integrals under weak assumptions to be specified below evaluate to periods. The statements are trivial for convergent and non-trivial for divergent Feynman integrals. Let us consider a general Feynman integral in Feynman parametric representation (cf. equation (2.3.6))

$$I_G(D) = \int_{x_j \geq 0} d^n x \delta \left(1 - \sum_{i=1}^{n} x_i \right) \left(\prod_{i=1}^{n} x_i^{\nu_i - 1} \right) \frac{\mathcal{U}_G{}^{\nu-(L+1)D/2}}{\mathcal{F}_G{}^{\nu-LD/2}}, \tag{6.2.1}$$

where a prefactor is suppressed and the dependence on masses and kinematical invariants is not explicitly denoted. If the integral converges for $D \to 4$, then it follows directly from the definition of periods that I_G is a period in this limit if the masses and kinematical invariants take algebraic values. The following non-trivial results apply to Feynman integrals which are possibly divergent.

6.2.1. A Theorem of Belkale and Brosnan. In **[BB03b]** Belkale and Brosnan prove a theorem on the Laurent coefficients of so-called *Igusa local zeta functions.* Let $\Delta_n \subset \mathbb{R}^n$ be the so-called *standard n-simplex,* defined as the set

$$\Delta_n = \left\{ (x_1, ..., x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i \leq 1 \right\} \tag{6.2.2}$$

and let f be a polynomial function $f \in \mathbb{R}[x_1, ..., x_n]$, which is non-negative on Δ_n. Then we consider the function

$$I(\alpha) = \int_{\Delta_n} f^\alpha d^n x \tag{6.2.3}$$

where $\alpha \in \mathbb{C}$ is understood to be an exponent. The function $I(\alpha)$ is called Igusa local zeta function. It is meromorphic in $\mathbb{C}$ (see **[Ati70, BG69]**). Belkale and Brosnan prove the following:

THEOREM 67. *(Belkale and Brosnan* **[BB03b]***) Suppose that $f \in \mathbb{Q}[x_1, ..., x_n]$ is a polynomial with rational coefficients and let α_0 be an integer. Let*

$$I(\alpha) = \sum_{i=A}^{\infty} C_i (\alpha - \alpha_0)^i \tag{6.2.4}$$

be the Laurent series expansion of $I(\alpha)$ at α_0 with $A \in \mathbb{Z}$. Then all C_i are periods.

The result is related to Feynman integrals as follows. Note that the above function $I(\alpha)$ is an integral with one polynomial to an arbitrary complex power in the integrand. A Feynman integral of this type is

$$I_G(D) = \int_{\mathbb{R}_+^{n+1}} d^{n+1} x \delta \left(1 - \sum_{i=1}^{n+1} x_i \right) \mathcal{U}^{-D/2}, \tag{6.2.5}$$

which we obtain for the case of a graph without external lines or all kinematical invariants zero, all masses equal to the mass-unit μ and all propagators occurring to power 1. Evaluating one integration by elimination of the δ-distribution yields

$$I_G(D) = \int_{\Delta^n} d^n x \, \mathcal{U}'^{-D/2}, \tag{6.2.6}$$

with $\mathcal{U}'(x_1, ..., x_n) = \mathcal{U}(x_1, ..., x_n, x_{n+1} = 1 - \sum_{i=1}^{n} x_i)$. We know from the properties of the polynomial $\mathcal{U}$ that $\mathcal{U}'$ is positive everywhere in the domain Δ^n and all of its coefficients are rational, namely 1 or -1. Therefore theorem 67 applies to the Feynman integral I_G of equation (6.2.5) and implies that its Laurent coefficients are periods.

Note that the integral $I_G(D)$ in equation (6.2.5) is a very special type of a Feynman integral. In reference **[BB03b]** Belkale and Brosnan expressed their belief, that it should be possible to extend their theorem to general Feynman integrals. Our main theorem to be stated in the following is such an extension to a class of integrals which is even more general and which contains general Feynman integrals as a special case. We want to emphasize, that the polynomial f in their theorem is assumed to be non-negative in the integration domain and to have rational coefficients. We will not use the theorem of Belkale and Brosnan in our proof, but we will have to adopt the same assumption with respect to all polynomials appearing. The assumption of non-negative polynomials will allow for the use of the sector decomposition in our proof. The decomposition will yield functions whose arguments consist of the coefficients of the polynomials. If we evaluate these functions at algebraic arguments, we will obtain periods.

6.2.2. The Main Theorem. Adopting our notation from chapter 5 we consider polynomials P in real valued variables $x_1, x_2, ..., x_n$. X denotes the set $\{x_1, x_2, ..., x_n\}$. The notation $P(X)$ means, that P depends on some of the elements of X, not necessarily on all of them. Let us consider the integral

$$J(\epsilon) = \int_{x_j \geq 0} d^n x \delta \left(1 - \sum_{i=1}^{n} x_i\right) \left(\prod_{i=1}^{n} x_i^{a_i + \epsilon b_i}\right) \prod_{j=1}^{r} \left(P_j(X)\right)^{c_j + \epsilon d_j} \tag{6.2.7}$$

where the integration domain is understood to be $\mathbb{R}_+^n$ and all the variables a_i, b_i, c_i and d_i are integers. We assume that the coefficients of the polynomials are algebraic numbers and that the polynomials are nonzero in the region $\mathbb{R}_{+\setminus 0}^n$, but not necessarily nonzero in the domain of integration $\mathbb{R}_+^n$. For simplicity let us assume that all polynomials are positive inside the integration region, which is of course not a restriction, as a general sign in front of the integral is irrelevant for our treatment.

The integral $J(\epsilon)$ has a Laurent expansion

$$J(\epsilon) = \sum_{j=A}^{\infty} C_j \epsilon^j, \; A \in \mathbb{Z}. \tag{6.2.8}$$

Our main theorem, as worked out with Stefan Weinzierl in reference **[BW09]**, is the following:

THEOREM 68. *The coefficients C_j of the Laurent expansion of the integral $J(\epsilon)$ are periods.*

PROOF. We give a constructive proof based on the sector decomposition algorithm of Binoth and Heinrich as described in chapter 5. As a strategy for the choice of the set S we take one of the winning strategies of Hironaka's polyhedra game. Therefore the algorithm terminates and gives the coefficients C_j of the Laurent expansion of $J(\epsilon)$. We follow the steps of the algorithm, referring to our above description, and we show, that the integrals C_j obtained after step 4 of the algorithm are periods. Step 5, the numerical evaluation of these integrals, is irrelevant for the proof. Let us proceed through the relevant steps of the algorithm:

Step 1 is the decomposition to primary sectors which expresses J as a sum of finitely many integrals of the form

$$\int_0^1 d^n x \left(\prod_{i=1}^n x_i^{a_i+\epsilon b_i} \right) \prod_{j=1}^r \left(P_j\left(X\right) \right)^{c_j+\epsilon d_j} . \tag{6.2.9}$$

The polynomials $P_j\left(X\right)$ are not necessarily monomialized yet. We note that from now on each integral at the end of a step is over a n-dimensional cubic domain of integration, which is clearly a semi-algebraic set.

Step 2 is the iterative sector decomposition. At the end of this step, J is expressed as a sum of finitely many integrals which are still of the form of equation (6.2.9), but now, after monomialization and factorization, all of the polynomials are non-zero in the entire domain of integration.

Step 3 uses the fact that the singular behaviour of the integrals at this stage depends just on the terms $\prod_{i=1}^n x_i^{a_i+\epsilon b_i}$. Considering each of the integrals, we remind ourselves that for a parameter x_j which appears to a negative power $a_j < 0$ in the monomial, we have to Taylor-expand the integrand around $x_j = 0$:

$$\int_0^1 dx_j x_j^{a_j+b_j\epsilon} \mathcal{I}\left(x_j,\, \epsilon\right) = \int_0^1 dx_j x_j^{a_j+b_j\epsilon} \left(\sum_{p=0}^{|a_j|-1} \frac{x_j^p}{p!} \mathcal{I}^{(p)}(\epsilon) + \mathcal{I}^{(R)}\left(x_j,\, \epsilon\right) \right) \tag{6.2.10}$$

with

$$\mathcal{I}^{(p)}(\epsilon) = \frac{\partial}{\partial x_j^p} \mathcal{I}\left(x_j,\, \epsilon\right)|_{x_j=0} , \tag{6.2.11}$$

$$\mathcal{I}^{(R)}\left(x_j,\, \epsilon\right) = \mathcal{I}\left(x_j,\, \epsilon\right) - \sum_{p=0}^{|a_j|-1} \frac{x_j^p}{p!} \mathcal{I}^{(p)}(\epsilon). \tag{6.2.12}$$

Here it is understood, that in general the functions $\mathcal{I}\left(x_j,\, \epsilon\right), \mathcal{I}^{(p)}(\epsilon), \mathcal{I}^{(R)}\left(x_j,\, \epsilon\right)$ are integrals over further variables. The integral

$$\int_0^1 dx_j x_j^{a_j+b_j\epsilon} \mathcal{I}^{(R)}\left(x_j,\, \epsilon\right) \tag{6.2.13}$$

does not contribute to ϵ poles by the x_j-integration, but possibly by integrations over other variables x_i. For the pole-part of the x_j-integration, we evaluate

$$\int_0^1 dx_j x_j^{a_j+b_j\epsilon} \frac{x_j^p}{p!} \mathcal{I}^{(p)}(\epsilon) = \frac{1}{a_j + b_j\epsilon + p + 1} \frac{\mathcal{I}^{(p)}(\epsilon)}{p!}. \tag{6.2.14}$$

One continues for all other variables x_i with $a_i < 0$, as we already described in chapter 5.

At the end of this procedure we obtain J as a sum of finitely many integrals of the form

$$K(\epsilon) = \frac{1}{g(\epsilon)} \int_0^1 d^n x F(X,\, \epsilon) \tag{6.2.15}$$

with

$$F(X,\,\epsilon) \;=\; \sum_{j=1}^{N} f_j(X,\,\epsilon), \tag{6.2.16}$$

$$f_j(X,\,\epsilon) \;=\; g_j(\epsilon)\prod_{i=1}^{n} x_i^{a_i^j+\epsilon b_i}\prod_{k=1}^{r}\left(P_k^j\,(X)\right)^{c_k^j+\epsilon d_k} \tag{6.2.17}$$

(with j everywhere understood to be an index, not an exponent). The numbers a_i^j, b_i, c_k^j, d_k are integers for all $i = 1, ..., n$ and $j = 1, ..., N$. The functions $g(\epsilon)$ and $g_j(\epsilon)$ are polynomials in ϵ with integer coefficients. The functions $1/g(\epsilon)$ are the products of the prefactors $1/(a_j+b_j\epsilon+p+1)$ from the right-hand side of equation (6.2.14). The functions $P_k^j\,(X)$ are polynomials with rational coefficients and they are non-zero in the domain of integration. In each single variable the integrals $\int_0^1 d^n x F(X,\,\epsilon)$ are of the form

$$\int_0^1 dx x^{\epsilon b} R(x,\,\epsilon). \tag{6.2.18}$$

Such integrals are absolutely convergent for all ϵ with $|\epsilon| < |1/b|$ if the function $R(x,\,\epsilon)$ is non-singular and therefore bounded in the domain of integration. Therefore we see that there is a neighborhood around $\epsilon = 0$ where the integrands $f_j(X,\,\epsilon)$ of equation (6.2.16) are bounded. Hence the integrals $\int_0^1 d^n x F(X,\,\epsilon)$ of equation (6.2.15) are absolutely convergent in a neighborhood around $\epsilon = 0$.

In Step 4 we expand $J(\epsilon)$ as a Laurent series in $\epsilon = 0$. We expand the functions $1/g(\epsilon)$ and $F(X,\,\epsilon)$ of equation (6.2.15) as follows:

$$\frac{1}{g(\epsilon)} = \sum_{r=A}^{\infty} g_r \epsilon^r, \quad F(X,\,\epsilon) = \sum_{r=0}^{\infty} F_r\,(X)\,\epsilon^r. \tag{6.2.19}$$

As a result we obtain the expansion of the integral of equation (6.2.15) as

$$K(\epsilon) \;=\; \sum_{r=A}^{\infty} K_r \epsilon^r \tag{6.2.20}$$

with

$$K_r \;=\; \sum_{s=A}^{r} g_s \int_0^1 d^n x F_{r-s}\,(X)\,. \tag{6.2.21}$$

The prefactors g_s are rational numbers. The integrands $F_{r-s}\,(X)$, as we observe from equation (6.2.17), consist of the coefficients of the expansions

$$x^{a+\epsilon b} \;=\; x^a \sum_{k=0}^{\infty} \frac{b^k}{k!}\,(\ln\,(x))^k\,\epsilon^k, \tag{6.2.22}$$

$$(P\,(X))^{c+\epsilon d} \;=\; (P\,(X))^c \sum_{k=0}^{\infty} \frac{d^k}{k!}\,(\ln\,(P\,(X)))^k\,\epsilon^k. \tag{6.2.23}$$

Now we are ready to observe that the Laurent coefficients C_j of $J(\epsilon)$ are periods, due to the following arguments: The coefficients C_j are sums of finitely many terms of the form

K_r. As the sum of finitely many periods is again a period and as the prefactors g_s are just rational numbers, it remains to show, that the integrals

$$\int_0^1 d^n x F_{r-s}(X) \tag{6.2.24}$$

are periods. The domain of integration is clearly a semi-algebraic set. Furthermore we see that the integrals are absolutely convergent: In each variable we have integrals of the form

$$\int_0^1 dx \left(\ln(x)\right)^k R(x), \qquad k \in \mathbb{N} \cup \{0\}, \tag{6.2.25}$$

where the function $R(x)$ contains the terms $\left(\ln\left(P(X)\right)\right)^k$ of equation (6.2.23). In the domain of integration, the polynomials $P(X)$ are non-zero and therefore the logarithms $\ln(P(X))$ do not diverge. Hence the functions $R(x)$ are bounded in the domain of integration and the integrals of the equations (6.2.25) and (6.2.24) are absolutely convergent[13].

Now that we obtained absolutely convergent integrals over semi-algebraic domains, it remains to check whether the integrands $F_{r-s}(X)$ fulfill the remaining conditions for the K_r to be periods. We see from equation (6.2.22) and (6.2.23) that these integrands are sums of products of rational functions with logarithms of rational functions, all of their coefficients being algebraic numbers. We note, that the latter numbers are algebraic, because we assumed the coefficients of the polynomials $P_j(X)$ to be algebraic numbers.

Therefore the integrals of equation (6.2.24) are of the type of equation (6.1.5), satisfying the conditions of lemma 61 from the beginning of the present chapter. Hence, as a consequence of lemma 61, these integrals are periods. This completes the proof. □

Now let us show how theorem 68 applies to Feynman integrals. We consider the general scalar multi-loop integral as in equation (6.2.1)

$$I_G = \left(\mu^2\right)^{\nu - LD/2} \int_{x_j \geq 0} d^n x \delta\left(1 - \sum_{i=1}^N x_i\right) \left(\prod_{i=1}^N x_i^{\nu_i - 1}\right) \frac{{\mathcal{U}_G}^{\nu-(L+1)D/2}}{{\mathcal{F}_G}^{\nu - LD/2}} \tag{6.2.26}$$

where we remember, that μ is the mass scale. Note that I_G and $\mathcal{F}_G$ may be functions in kinematical invariants s_i and masses m_i. Let λ be an integer and set $D = 2\lambda - 2\epsilon$. Then this integral has a Laurent series expansion in ϵ

$$I_G = \sum_{j=-2L}^{\infty} C_j \epsilon^j.$$

With respect to this expansion we obtain:

Corollary 69. *If all kinematical invariants s_i are non-positive algebraic numbers and all masses m_i and μ are non-negative algebraic numbers, the coefficients C_j of the Laurent expansion of the integral I_G are periods.*

The statement is a direct consequence of theorem 68 as the integral I_G is a special case of the integral J. As we have already seen in chapter 5 we obtain I_G by setting $n = N$, $r = 2$, $a_i = \nu_i - 1$, $b_i = 0$, $P_1 = \mathcal{U}_G$, $P_2 = \mathcal{F}_G$, $c_1 = \nu - 2(L+1)$, $d_1 = L + 1$, $c_2 = \nu - 2L$, $d_2 = L$ in J.

[13]The power of $\ln(x)$ does not cause singularities at the lower boundary as we have $\lim_{a\to 0} \int_a^1 dx \ln(x) = \lim_{a\to 0} (-1 - a\ln(a) + a) = -1$.

In the literature, very often Feynman integrals are considered, which differ from the integral I_G as used here by a general prefactor. In appendix C we observe, that the Laurent coefficients of these prefactors are periods as well, up to the presence of powers of $1/\pi$.

Theorem 68 and corollary 69 provide a general statement on the arithmetic nature of Feynman integrals. In the present chapter we have seen that multiple zeta values and polylogarithms are present in multi-loop calculations, but it is not known whether all Feynman integrals can be expressed in terms of these objects. The set of periods instead turns out to be a set of numbers which includes all Laurent coefficients of all Feynman integrals under the mentioned assumptions of masses and kinematical invariants taking algebraic values in the Euclidean momentum region. We mentioned that the restriction to the Euclidean region is required by the sector decomposition algorithm which is used in the proof of our theorem.

Periods, as we mentioned in the beginning of the chapter, play a central role in algebraic geometry and particularly in the theory of so-called motives. In the following outlook we want to add some partly speculative remarks on the correspondence between periods and Feynman integrals in this geometric context.

6.3. Outlook on Periods and Feynman Integrals

6.3.1. Motives and the Classification of Feynman Integrals. At several places in this dissertation we have mentioned a connection from Feynman integrals to the theory of so-called motives, without explaining what motives are. In fact motives are highly abstract objects considered in algebraic geometry and a precise introduction to their theory would be beyond the scope of this dissertation. Nevertheless, there are certain statements related to motives which can partly be understood from the background of the previous treatments and which might have applications for the practical calculation of Feynman integrals. In this outlook we want to refer to these statements on motives and speculate on possible applications.

In chapter 5 we have considered the zero-sets of polynomials in the integrand of a Feynman integral and we have introduced the notion of a variety for such zero-sets, which is a notion from algebraic geometry. In algebraic geometry one can study properties of varieties by considering the cohomology of the variety. There are different kinds of cohomologies which may be considered, i. e. so-called de Rham cohomology, Betti cohomology etc. and these different cohomologies of a given variety have common properties. Motives are objects which should 'motivate' these common properties. More precisely, the theory of motives is an attempt to unify the different cohomologies to one 'universal' cohomology theory. Therefore a motive which is associated to a variety shall incorporate certain properties of the corresponding cohomologies.

There are different approaches of associating motives to Feynman integrals[14] and without referring to details we just want to focus on the following important point:

According to a conjecture of Grothendieck, a motive is determined by certain associated periods (see e. g. [**And08**]). Furthermore, certain periods which are given by Feynman integrals can be interpreted as such periods of motives, as it was shown for the first time in detail in the work of Bloch, Esnault and Kreimer [**BEK06**] (also see [**Blo07**]). For an introduction to periods of motives and the connection to Feynman integrals we refer to [**And08**] and [**Mar09**].

Moreover there is a special class of motives which are better understood than motives in the general case: the class of so-called mixed Tate motives. According to a conjecture

[14] We refer to [**AM08b, AM08a, AM09b, AM09a, And08, BB03a, BEK06, CM08, Mar08, Mar09**].

of Tate, these are exactly the motives whose periods are multiple zeta values (MZVs) [**And08**] (also see [**Gon01, Ter02**]). To our knowledge, all motives which have been associated to Feynman integrals up to now belong to the class of mixed Tate motives. In problem 65 of the present chapter, we have asked for the set F_{MZV} of Feynman integrals, whose Laurent coefficients are linear combinations of MZVs. Possibly these Laurent coefficients could be related to mixed Tate motives. We may summarize the correspondences just mentioned in the following simplistic diagram:

$$\begin{array}{ccccc} \text{motives} & \leftrightarrow & \text{periods} & \leftrightarrow & \text{Feynman integrals} \\ \cup & & \cup & & \cup \\ \text{mixed Tate motives} & \leftrightarrow & \text{MZVs} & \leftrightarrow & F_{\mathrm{MZV}} \end{array}$$

However we have to mention that for a general divergent Feynman integral, the situation seems to be more difficult than we just suggested, as to our knowledge up to now only finite integrals have been interpreted as periods of mixed Tate motives, but not the Laurent coefficients of divergent integrals. Nevertheless, the possibility of such an association to Laurent coefficients of a divergent integral is indicated in a remark in [**AM09b**].

From the point of view given here we may hope, that the theory of motives can provide a classification of Feynman integrals, which would be useful for practical calculations. In particular we may hope that, by use of the conjecture of Tate, we may find a way to determine the set F_{MZV} of Feynman integrals which evaluate to MZVs. Let us try to give this hope a more explicit formulation in the following.

Let $A \subset \mathcal{P}$ be a subset of the set of periods and let F_A be the set of Feynman integrals whose Laurent coefficients are linear combinations of elements of A. For example A may be the set of MZVs (and then $F_A = F_{\mathrm{MZV}}$) or the set of certain values of polylogarithms. Let I be a given Feynman integral. Then we may hope that we can construct an object which is associated to the Feynman integral I and from which we can directly read off whether I belongs to F_A or not. In the case of MZVs, the conjecture of Tate may be helpful for such a construction. We may call such an object an *indicator* for I to belong to F_A. For example a function

$$\mu_A(I) = \begin{cases} 1 \text{ if } I \in F_A, \\ \\ 0 \text{ if } I \notin F_A \end{cases}$$

would be such a desired indicator. To our knowledge, such an indicator does not exist for general Feynman integrals and any significant set A, such as MZVs or values of polylogarithms.

In this context let us mention the approach of Kreimer to predict the presence and the weight of MZVs in Feynman integrals of special classes via a correspondence to knot theory (see [**Kre00**] and references given there). The articles [**BK97, BK95**] of Broadhurst and Kreimer show that this way of indication is successful up to a certain loop-number. Such a method of indication would be highly desirable for a more general class of integrals, and it is our hope that the theory of motives may be the key to such a method.

Asking for even more, we may hope that for a subset of periods A one day an algorithm, say ALG_A, can be developed, which can be used to evaluate any Feynman integral in F_A analytically. If we would have such an algorithm and an indicator μ_A at hand, we could systematically use the indicator μ_A to select the integrals which can be evaluated by the algorithm ALG_A. Such a procedure would be particularly useful if the evaluation of the indicator takes less computation time than a usual run of the algorithm, or, for example, if the algorithm in the case of $I \notin F_A$ would lead to an infinite recursion. We remember that for each finite set of integrals we could evaluate with ALG_A, we would

obtain an infinite set of integrals which can be calculated in terms of these results by use of the IBP-identities as discussed in chapter 3.

Our wish-list, consisting of a useful indicator μ_A and a powerful algorithm ALG_A, might seem to be utopian for the near future. However, there is recent research in the field between Feynman integrals and motives which in our opinion might be useful for the development of such indicators and algorithms. In the case of the development of the indicator we would be interested in objects which contain valuable information about a Feynman integral and whose explicit calculation is easier than the calculation of the integral itself. For example in reference [**AM08b**] Aluffi and Marcolli explicitly compute certain class invariants for a family of Feynman graphs (i. e. massless n-loop sunrise graphs) which encode information on their singularity structure and on the corresponding motives. Furthermore we can interpret Brown's work in reference [**Bro09**] as a valuable step towards the development of a desired algorithm $\mathrm{ALG}_{\mathrm{MZV}}$, i. e. an algorithm for the evaluation of any $I \in F_{\mathrm{MZV}}$. We may speculate, that a successful development of such an algorithm $\mathrm{ALG}_{\mathrm{MZV}}$ could then in turn give rise to the right ideas for an extension to the set of Feynman integrals which evaluate to polylogarithms. Such a technique would be a great improvement for the efficient evaluation of physical cross-sections to high precision.

6.3.2. Differential Equations of Geometric Origin. In the present chapter we have discussed many different functions which have a common property: they evaluate to periods if their arguments take algebraic values. In corollary 69 we have seen, that also the Laurent coefficients of a general Feynman integral I_G of equation (6.2.26) are functions with the same property, at least in the Euclidean momentum region.

In reference [**KZ**] Kontsevich and Zagier point out, that functions of this type, integrals which depend on parameters and which evaluate to periods, satisfy linear differential equations of a very special type. The property was first observed by Fuchs and the differential equations are known as *Picard-Fuchs differential equations* or members of *Gauss-Manin systems.* Other authors refer to these equations simply as differential equations of geometric origin. (For introductions see [**And89, Kat68**].)

Like the theory of motives, the theory of Picard-Fuchs equations lies beyond what we can present in this dissertation. Let us just imagine a variety, given as the zero set of some polynomials whose coefficients are fixed numbers. We furthermore may imagine an integral over a closed path on this variety. If instead of polynomials with fixed coefficients we consider polynomials which depend on additional variables (just like the polynomial $\mathcal{F}$ depends on masses and kinematical invariants) then the variety depends on these parameters as well, or in other words, we consider in fact a continuum of varieties. With the variation of the additional parameters, the variety changes and accordingly the integral defined by the closed path on the variety changes as well. In such a sense we can imagine that the integral fulfills a differential equation with respect to such a parameter which reflects a variation of a geometrical object.

Given this point of view let us come back to Feynman integrals. From the physics literature, Feynman integrals are well known to satisfy certain differential equations. The so-called *differential equations approach* originating in the works of Remiddi and collaborators [**CCLR98, GR00, Rem97**], generalizing an idea of Kotikov [**Kot91**], considers Feynman integrals as solutions of linear differential equations in the particle masses or kinematical invariants. Therefore also the Laurent coefficients are known to be solutions of certain differential equations. The approach provides a method for the construction of these differential equations, partially using IBP-identities. We may ask for the geometric origin of these equations in order to provide a further bridge to algebraic geometry and possibly improve the differential equations approach by use of such a connection.

Let us again formulate our hope more explicitly. From the given point of view it may be desirable to try to decide whether the differential equations given by the method of Remiddi et al. are Picard-Fuchs equations and accordingly can be interpreted as originating from a geometrical object. There is a complication which stands in the way of such an approach. As Kontsevich and Zagier mention in reference [**KZ**], it is not known in general how to determine, whether a given differential equation is a Picard-Fuchs equation. Nevertheless, there are certain conjectural criteria and Kontsevich and Zagier summarize three conjectures stating conditions under which a differential equation is of the Picard-Fuchs type. To my opinion it would be interesting and possible to check whether differential equations for Feynman integrals due to Remiddi et al. fulfill the criteria of at least one of these conjectures.

CHAPTER 7

Conclusions

In this dissertation we studied combinatorial and arithmetic properties of general scalar Feynman integrals in dimensional regularization. The starting point of most of our considerations was the Feynman parametric representation of these integrals. Let us summarize the main achievements of the dissertation, which we aimed for in the chapters 4, 5 and 6.

In chapter 4 we provided a detailed graph theoretic discussion of the two Symanzik polynomials which arise in the Feynman parametric representation. We focused on the construction of these polynomials by the generic Laplacian matrix of a graph. While the first of the Symanzik polynomials is known to be obtained from the determinant of a minor of this matrix due to the matrix-tree-theorem, we used the more general all-minors-matrix-tree-theorem to prove a novel relation which also includes the second Symanzik polynomial. We ended the chapter explaining the relationship of the Symanzik polynomials with the multivariate Tutte polynomial, which we expect to be useful in future considerations of combinatorial properties of Feynman integrals.

In chapter 5, following joint work with Stefan Weinzierl, we considered the sector decomposition algorithm of Binoth and Heinrich for the numerical evaluation of arbitrary scalar Feynman integrals in the Euclidean momentum region. By giving a counterexample, we exhibited the fact that this powerful algorithm in its original version does not terminate for every possible situation. We showed that the iterative step of the algorithm can be mapped to the abstract polyhedra game of Hironaka. By use of the existing winning strategies of this combinatorial game we extended the sector decomposition algorithm such that the resulting version is known to terminate for the general case. We implemented this new version, combining the analytical and the numerical part of the calculation into one program, which is the first publicly available implementation of sector decomposition. Hironaka's game was originally applied in the context of the resolution of singularities of varieties in algebraic geometry and we ended chapter 5 by a geometric explanation of the analogies between sector decomposition and such resolutions.

The correspondence between Feynman integrals and topics in algebraic geometry furthermore gave a motivation for the consideration of the set of periods in chapter 6. We provided a brief review on periods and further sets of numbers and functions which are periods or which evaluate to periods at algebraic arguments respectively. We emphasized the importance of these objects in today's multi-loop calculations and we pointed out that up to now no subset of periods is known to be sufficient to express the Laurent coefficients of general dimensionally regularized Feynman integrals. This discussion served as a motivational introduction to the main theorem of this dissertation, obtained in joint work with Stefan Weinzierl. Our theorem implies, that all the Laurent coefficients of an arbitrary dimensionally regularized Feynman integral are periods if the masses and kinematical invariants take algebraic values in the Euclidean momentum region. The theorem is in fact formulated for a more general class of integrals, allowing for an arbitrary number of polynomials in the integrand. Our constructive proof relies on the improved sector decomposition algorithm.

The work we just summarized provides the main body of this dissertation while chapters 2 and 3 are of a more supplementary character. In chapter 2 we introduced the standard techniques and representations which were required for the understanding of all subsequent discussions. We particularly tried to give an introduction which is comprehensible also for a reader who is not familiar with the use of Feynman integrals in perturbative quantum field theory. In chapter 3 we reviewed the technique of IBP-identities for expressing Feynman integrals in terms of simpler ones. We focused on the systematic reduction given by the Laporta algorithm and we furthermore considered IBP-identities in the Feynman parametric representation and derived a convenient formula for the one-loop case. We pointed out the importance of combining IBP-identities in order to obtain an efficient reduction for practical calculations, and the fact that the optimal way to do so is still to be found.

A central point of this dissertation is the use of Hironaka's polyhedra game in the algorithm of Binoth and Heinrich, by which the close relationship between the methods of sector decomposition and resolution of singularities in algebraic geometry becomes visible. This connection may be seen as a very explicit example for the usefulness of the field of research which started to evolve in between Feynman integrals and algebraic geometry over the last ten years. In outlooks and remarks at the end of some of the previous chapters we expressed our hope and belief, that this field of research will not only be a valuable source of examples for the theory of motives but moreover yield further improvements for the understanding and the efficient calculation of Feynman integrals.

To this end it seems to be inevitable for physicists and mathematicians to explain techniques for the evaluation of Feynman integrals and sophisticated tools of algebraic geometry to each other. To no surprise, the language of the phenomenologist and the language of the algebraic geometer are so different that they might not even realise when they speak about the same subject. The future success of the mentioned field of research may to a large extent rely on the ability of both communities to find a common language.

It was Ludwig Wittgenstein who knew a useful rule of thumb, which we seem to forget sometimes:

> "*Telling someone something he does not understand is pointless, even if you add that he will not be able to understand it.*"

Let us try to communicate in a language which we both understand. Let us say clearly what we can say, and whereof we cannot speak, thereof let us be silent.

Appendix A: Auxiliary Definitions

In chapter 5 we have introduced several notions of algebraic geometry. Let us supplement the explanations given there, by the following additional definitions.

DEFINITION 70. (*Ring*, cf. [**Lan05**]) A *ring* A is a set, together with two laws of composition called multiplication and addition respectively, satisfying the following conditions:

(a) With respect to addition, A is a commutative group.

(b) The multiplication is associative, and has a unit element.

(c) For all x, y, $z \in A$ we have distributivity: $(x+y)z = xz+yz$ and $z(x+y) = zx+zy$.

A is called *commutative ring*, if it is commutative with respect to the multiplication.

DEFINITION 71. (*Field*, cf. [**Lan05**]) A *field* K is a commutative ring such that the unit element of the multiplication, the one, is not equal to the neutral element of the addition, the zero, i. e. $1 \neq 0$, and such that each non-zero $x \in K$ has a multiplicative inverse in K.

DEFINITION 72. *(Affine space,* cf. [**DS01**]*)* Let K be a field. Let furthermore A be a set and let there be a map from $A \times A$ to a vector space L over K. The image of an element $(a, b) \in A \times A$ is denoted $\overrightarrow{ab}$ and is called the vector with beginning in a and end in b. Let the map have the following properties:

(a) For any fixed element $a \in A$ the map $x \to \overrightarrow{ax}$, $x \in A$ is a bijection of A on L.

(b) For any elements a, b, $c \in A$ we have $\overrightarrow{ab} + \overrightarrow{bc} + \overrightarrow{ca} = \overrightarrow{0}$, where $\overrightarrow{0}$ is the zero vector in L.

Then A is called affine space over K, its elements are called points and its dimension is defined to be the dimension of L.

In chapter 5 we considered rational maps as given by tuples of fractions of regular functions. A more abstract but precise definition of a rational map is the following:

DEFINITION 73. *(Rational map,* [**Har95**]*)* For an irreducible variety V and any variety W a *rational map*

$$\psi : V \quad \dashrightarrow \quad W$$

is defined to be an equivalence class of pairs (U, ϕ) with $U \subset V$ a dense open subset and $\phi : U \to W$ a regular map, where two such pairs (U_1, ϕ_1) and (U_2, ϕ_2) are equivalent, if the regular maps coincide on the intersection of the subsets: $\phi_1|_{U_1 \cap U_2} = \phi_2|_{U_1 \cap U_2}$. This defines the use of the dashed arrow "$\dashrightarrow$".

Appendix B: The Two-Loop Equal Mass Sunrise Graph

In chapter 6 we mentioned the example of the two-loop sunrise graph, shown in figure 7.0.1, where all masses are set equal and non-zero. We already considered the graph in chapter 2 as an example for a graph of ϕ^4-theory. Despite the simple form of the graph it turns out that the corresponding Feynman integral is difficult to evaluate. The first complete analytical result was derived by Laporta and Remiddi in the article [**LR05**]. As we mentioned above, this result remarkably contains a kind of integrals, elliptic integrals of the first kind, which are not known to be expressible in terms of polylogarithms. For this reason we want to give a brief review on the result of Laporta and Remiddi in this appendix.

We consider the two-loop Feynman graph G of figure 7.0.1, where the external momentum is denoted p, the momenta assigned to the internal edges are k_1, k_2 and $p - k_1 - k_2$, according to momentum conservation, and where furthermore the same mass variable m is assigned to all three internal edges. If we allow for arbitrary integer powers ν_1, ν_2, ν_3 of the inverse propagators, we obtain an infinite set of Feynman integrals for this graph. By the use of IBP-identities derived by Tarasov in reference [**Tar97**], all integrals of this family can be expressed in terms of only two of these integrals, namely

$$I_i\left(D,\, p^2,\, m^2\right) \;=\; \Omega(D) \int \frac{d^D k_1 d^D k_2}{\left(k_1^2 + m^2\right)^i \left(k_2^2 + m^2\right) \left(\left(p - k_1 - k_2\right)^2 + m^2\right)}$$

with the power $i = 1,\, 2$ and

$$\Omega(D) = \left(\Gamma\left(3 - D/2\right)\right)^{-2} 16^{-1} \pi^{-D},$$

where for the prefactor $\Omega(D)$ we adopt the convention of reference [**LR05**]. Furthermore Laporta and Remiddi find that I_2 can be expressed as a linear combination of $I_1\left(D,\, p^2,\, m^2\right)$ and $\frac{d}{d(p^2)} I_1\left(D,\, p^2,\, m^2\right)$. Therefore the consideration of I_1 is sufficient in order to provide a result for the entire family of integrals given by the graph G with equal, non-zero masses.

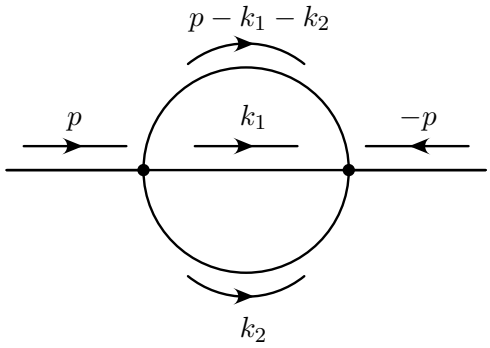

FIGURE 7.0.1. The two-loop sunrise graph.

In the following the mass m is set equal to one, which means that the result will be obtained in units of the arbitrary mass m, so this is not a restriction of the generality. For simplicity let us furthermore denote $p^2 = z$ and in order to adopt the notation of reference [**LR05**] we will write

$$S(D, z) = I_1(D, z, 1).$$

In order to avoid possible confusion we furthermore adopt the convention of Laporta and Remiddi to consider the Laurent expansion in a parameter $\sigma = D - 4 = 2\epsilon$:

$$S(D, z) = \sum_{n=-2}^{\infty} S^{(n)}(4, z)\,\sigma^n = \sum_{n=-2}^{\infty} 2^n S^{(n)}(4, z)\,\epsilon^n.$$

The authors furthermore define $\eta = D - 2$ and consider the series

$$S(D, z) = \sum_{n=0}^{\infty} S^{(n)}(2, z)\,\eta^n.$$

For the coefficients at $n = -2, 1, 0$ they obtain the relations:

$$S^{(-2)}(4, z) = -\frac{3}{8}, \tag{7.0.1}$$

$$S^{(-1)}(4, z) = \frac{1}{32}(z + 18), \tag{7.0.2}$$

$$\begin{aligned} S^{(0)}(4, z) &= \frac{1}{12}(z+1)(z+9)\left(1 + (z-3)\frac{d}{dz}\right) S^{(0)}(2, z) \\ &\quad -\frac{1}{128}(72 + 13z) \end{aligned} \tag{7.0.3}$$

and a fourth relation which expresses $S^{(1)}(4, z)$ in terms of $S^{(0)}(2, z)$, $S^{(1)}(2, z)$ and $\frac{d}{dz}S^{(0)}(2, z)$. The latter three functions are analytically evaluated in the main body of the article [**LR05**].

For the sake of brevity let us restrict our review to $S^{(0)}(2, z)$ which determines $S^{(0)}(4, z)$ by equation (7.0.3). From our main theorem 68 it follows that $S^{(0)}(4, z)$ and $S^{(0)}(2, z)$ are periods at negative algebraic values of z. Let us try to re-derive this fact from the explicit result of Laporta and Remiddi, under which conditions $S^{(0)}(2, z)$ is a period.

Laporta and Remiddi find that $S^{(0)}(2, z)$ fulfills the following differential equation:

$$\left(\frac{d^2}{dz^2} + \left(\frac{1}{z} + \frac{1}{z+1} + \frac{1}{z+9}\right)\frac{d}{dz} + \left(\frac{1}{3z} - \frac{1}{4(z+1)} - \frac{1}{12(z+9)}\right)\right) S^{(0)}(2, z)$$

$$= \frac{3}{8z(z+1)(z+9)}.$$

By the method of variation of constants they give the solution

$$\begin{aligned} S^{(0)}(2, z) &= \Psi_1(z)\left(\Psi_1^{(0)} - \frac{1}{24}\int_0^z dw \Psi_1(w)\right) \\ &\quad +\Psi_2(z)\left(\Psi_2^{(0)} + \frac{1}{24}\int_0^z dw \Psi_2(w)\right) \end{aligned} \tag{7.0.4}$$

where $\Psi_1^{(0)}$ and $\Psi_2^{(0)}$ are integration constants and the functions $\Psi_1(z)$ and $\Psi_2(z)$ are two independent solutions of the homogeneous differential equation

$$\left(\frac{d^2}{dz^2}+\left(\frac{1}{z}+\frac{1}{z+1}+\frac{1}{z+9}\right)\frac{d}{dz}+\left(\frac{1}{3z}-\frac{1}{4(z+1)}-\frac{1}{12(z+9)}\right)\right)\Psi(z)=0. \tag{7.0.5}$$

We see that the coefficients of equation (7.0.5) become singular at $z = 0$, -1, -9 and ∞. Laporta and Remiddi give solutions $\Psi_1(z)$ and $\Psi_2(z)$ in the neighborhoods of these singular points. The radius of convergence is 1 for the solutions around $z = 0$ and $z = -1$ and it is 8 for the solution around $z = 9$. Then, in order to provide an analytical solution in the whole range $-\infty < z < \infty$, they give interpolating solutions, which are valid in the open intervals $(0 > z > -1)$, $(-1 > z > -9)$ and $(-9 > z > -\infty)$.

The solutions $\Psi_1(z)$ and $\Psi_2(z)$ in each of these regions are obtained as linear combinations of elliptic integrals of the form

$$J^+_{a,\,b}(u)=\int_{a(u)}^{b(u)}\frac{dx}{\sqrt{R_4(u,\,x)}}\text{ or }J^-_{a,\,b}(u)=\int_{a(u)}^{b(u)}\frac{dx}{\sqrt{-R_4(u,\,x)}} \tag{7.0.6}$$

with $u = -z$, with the polynomial

$$R_4(u,\,x)=x(x-4)\left(x-(\sqrt{u}-1)^2\right)\left(x-(\sqrt{u}+1)^2\right)$$

and where the boundaries $a(u)$ and $b(u)$ are contained in the set $\{0,\,4,\,(\sqrt{u}-1)^2,\,(\sqrt{u}+1)^2\}$. It is easy to see, that the values of any integral of the form of $J^+_{a,\,b}(u)$ or $J^-_{a,\,b}(u)$ are periods if the integrals are absolutely convergent and if the argument u is algebraic. Therefore we know from the result of Laporta and Remiddi, that the factors $\Psi_1(z)$ and $\Psi_2(z)$ in equation (7.0.4) evaluate to periods at algebraic z in the regions $(0 > z > -1)$, $(-1 > z > -9)$ and $(-9 > z > -\infty)$.

The integrals of equation (7.0.6) are related to the *complete elliptic integral of the first kind*

$$K(\lambda)\;=\;\int_0^1\frac{dy}{\sqrt{(1-y^2)\,(1-\lambda y^2)}}.$$

The relation to this integral is given as follows. Consider

$$I^+\,(a,\,b)=\int_a^b\frac{dx}{\sqrt{R_4(x)}}\text{ or }I^-\,(a,\,b)=\int_a^b\frac{dx}{\sqrt{-R_4(x)}}$$

with

$$R_4(x)=(x-x_1)\,(x-x_2)\,(x-x_3)\,(x-x_4)\,,$$

$x_1 < x_2 < x_3 < x_4$ and $a\,b \in \{x_1,\,x_2,\,x_3,\,x_4\}$. By appropriate changes of variables one obtains

$$I^-\,(x_1,\,x_2)=I^-\,(x_3,\,x_4)=\frac{2}{\sqrt{(x_4-x_2)\,(x_3-x_1)}}K(\lambda),$$

$$I^+\,(x_2,\,x_3)=\frac{2}{\sqrt{(x_4-x_2)\,(x_3-x_1)}}K(1-\lambda),$$

with

$$\lambda=\frac{(x_2-x_1)\,(x_4-x_3)}{(x_4-x_2)\,(x_3-x_1)}.$$

Let us discuss the remaining terms in equation (7.0.4). Laporta and Remiddi obtain

$$\Psi_1^{(0)}=\frac{\sqrt{3}}{12}\int_0^1 du\int_0^{(\sqrt{u}-1)^2}\frac{dx}{\sqrt{-R_4(u,\,x)}}\;\text{ and }\;\Psi_2^{(0)}=0.$$

They furthermore obtain the identity

$$\int_0^1 du \int_0^{(\sqrt{u}-1)^2} \frac{dx}{\sqrt{-R_4(u,\, x)}} = -\frac{1}{2} \int_0^1 \frac{dx}{\sqrt{x(4-x)}} \ln x. \tag{7.0.7}$$

The integral of equation (7.0.7) is clearly a period as on the left-hand side the integrand is an algebraic function and the domain of integration is a semi-algebraic set. Hence $\Psi_1^{(0)}$ is a period.

The above solution for $S^{(0)}\,(2,\, z)$ furthermore contains integrals $\int_0^z dw \Psi_i(w)$, $i = 1,\, 2$, which take the form

$$\int_0^z du \int_{a(u)}^{b(u)} \frac{dx}{\sqrt{R_4(u,\, x)}} \text{ or } \int_0^z du \int_{a(u)}^{b(u)} \frac{dx}{\sqrt{-R_4(u,\, x)}}.$$

We can of course write

$$\int_0^z du \int_{a(u)}^{b(u)} \frac{dx}{\sqrt{\pm R_4(u,\, x)}} = \int_G du\, dx \frac{1}{\sqrt{\pm R_4(u,\, x)}}$$

with $G = \{(u,\, x) \mid a(u) \leq x \leq b(u);\, 0 \leq u \leq z\}$. As Laporta and Remiddi consider each of the solutions in its region of absolute convergence and as $a(u)$ and $b(u)$ are either algebraic constants of algebraic functions of u, the integrals are periods if they are evaluated at algebraic z. So we can conclude, that $S^{(0)}\,(2,\, z)$ is a period for algebraic values of z in the whole range $-\infty < z < \infty$.

The elliptic integral of the first kind was as well applied to two-loop sunrise graphs by Groote and Pivovarov in reference [**GP00**]. Furthermore, as we mentioned above, for the three-loop and four-loop case Laporta presents analytical results in terms of further elliptic integrals in reference [**Lap08**]. Aluffi and Marcolli consider sunrise graphs in reference [**AM08b**] in the context of motives.

Appendix C: Periods and Prefactors of Feynman Integrals

With corollary 69 of chapter 6, we have applied our main theorem 68 to the general Feynman integral I_G of equation (6.2.26). This integral differs from what the literature very often considers as a Feynman integral by a general prefactor. Let us see whether this factor affects the statement of corollary 69.

In the momentum representation the integral I_G of equation (6.2.26) reads

$$I_G = \frac{\prod_{j=1}^{N}\Gamma(\nu_j)}{\Gamma(\nu - LD/2)}\left(\mu^2\right)^{\nu-LD/2}\int\prod_{r=1}^{L}\frac{d^Dk}{i\pi^{D/2}}\prod_{j=1}^{N}\frac{1}{\left(-q_j^2+m_j^2\right)^{\nu_j}}$$

where the dependence on D, masses and kinematical invariants is understood and where the k_j are loop-momenta and the q_j are linear combinations of loop-momenta and external momenta. We may take this form as the definition of a generic Feynman integral, but the more common form in practical applications is

$$\tilde{I}_G = g^{2L}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{L\epsilon}\left(\mu^2\right)^{\nu-LD/2}\int\prod_{r=1}^{L}\frac{d^Dk}{(2\pi)^D}\prod_{j=1}^{N}\frac{1}{\left(q_j^2+m_j^2\right)^{\nu_j}}.$$

We included a physical coupling factor g^2 and a factor $e^{\gamma_E}/4\pi$ which is conveniently used in calculations in the so-called $\overline{MS}$-renormalization-scheme[1] **[BBDM78]**. Both factors appear to the power L which is the loop-number. By the comparison of $I_G$and $\tilde{I}_G$,

$$\tilde{I}_G = K_G I_G,$$

we obtain the relative prefactor

$$K_G = g^{2L}(-1)^{\nu}i^{n+L}(4\pi)^{-LD/2}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{L\epsilon}\frac{\Gamma(\nu-LD/2)}{\prod_{j=1}^{N}\Gamma(\nu_j)}.$$

Corollary 69 applies to $\tilde{I}_G$, if the Laurent coefficients of K_G are periods. Let us briefly observe whether this is the case, keeping in mind that according to conjectures mentioned above, the numbers $1/\pi$ and γ_E do not belong to the set of periods.

With $D = 2m - 2\epsilon$ with some $m \in \mathbb{Z}$ let us write

$$K_G = K_{G1}K_{G2}K_{G3}$$

with

$$\begin{aligned} K_{G1} &= \frac{(-1)^{\nu}i^{n+L}}{\prod_{j=1}^{N}\Gamma(\nu_j)}, \\ K_{G2} &= e^{L\gamma_E\epsilon}\Gamma(\nu - Lm + L\epsilon), \\ K_{G3} &= \left(\frac{g^2}{(4\pi)^m}\right)^L \end{aligned}$$

[1] $\overline{MS}$ stands for modified minimal subtraction.

and let us discuss the three factors separately.

Because of $\nu_j \in \mathbb{N}$ for all $j = 1, ..., N$ we have $\Gamma(\nu_j) = (\nu_j - 1)!$ and therefore K_{G1} is a complex number whose real and imaginary part are rational numbers. Hence K_{G1} is clearly a period.

K_{G2} can be trivially rewritten as

$$K_{G2} = e^{L\gamma_E \epsilon} \Gamma(1 + L\epsilon) \frac{\Gamma(\nu - Lm + L\epsilon)}{\Gamma(1 + L\epsilon)}$$

and we observe by use of the functional equation $\Gamma(x+1) = x\Gamma(x)$ that the term

$$\frac{\Gamma(\nu - Lm + L\epsilon)}{\Gamma(1 + L\epsilon)}$$

is a rational function in ϵ with rational coefficients and therefore its Laurent-coefficients are periods. We furthermore use the expansion

$$\Gamma(1 + \epsilon) = \exp\left(-\gamma_E \epsilon + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) \epsilon^n \right)$$

to obtain

$$e^{L\gamma_E \epsilon} \Gamma(1 + L\epsilon) = \exp\left(\sum_{n=2}^{\infty} \frac{(-L)^n}{n} \zeta(n) \epsilon^n \right).$$

Note that the γ_E-terms exactly cancel each other. By Taylor-expansion of the exponential function on the right-hand side we obtain powers of zeta-values with rational prefactors as coefficients of ϵ^n. These terms are periods, as we know from our above treatment, so the Laurent coefficients of K_{G2} are periods.

The remaining term $K_{G3} = \left(g^2/(4\pi)^m\right)^L$ is just a simple constant. We want to briefly comment on the presence $1/\pi$, which is assumed not to be a period:

- In the physics literature it is common practice to consider results of perturbative calculations as series expanded in $\frac{g^2}{(4\pi)^m}$. For $D = 4 - 2\epsilon$ and $\alpha = \frac{g^2}{(4\pi)}$a quantity σ to be calculated is often expressed as

  $$\sigma = \sigma_0 \left(1 + \frac{\alpha}{4\pi} c_1 + \left(\frac{\alpha}{4\pi} \right)^2 c_2 + ... \right)$$

  such that the powers of $\frac{g^2}{(4\pi)^2}$ are explicitely factored out. Then powers of $1/\pi$ are not contained in the coefficients c_1, c_2, ... Therefore, under the assumptions of corollary 69, the contributions of Laurent coefficients of the integral $\tilde{I}_G$ to these c_1, c_2, ... are periods.
- Kontsevich and Zagier point out in that it makes sense for many purposes to consider the *extended period ring* $\hat{\mathcal{P}} = \mathcal{P}[\pi]$ which is the set of polynomials in $1/\pi$ with coefficients in $\mathcal{P}$. The extension is convenient in geometrical contexts (see chapter 4 of **[KZ]**). We see that under the assumptions of corollary 69 (and for g^2 assumed to be an algebraic number) the Laurent coefficients of the integral $\tilde{I}_G$ are members of the extended period ring $\hat{\mathcal{P}}$.

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