Dissertation

Birational Models for Moduli of Quartic Rational Curves

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Johannes Gutenberg Universität Mainz

Autor:
Carolin Susanne Peternell
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Abstract

We study the geometry of the moduli space $M_4(\mathbb{P}^4)$ of rational normal curves of degree 4 and its compactifications in the Hilbert scheme $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$, in the moduli space of Kronecker modules of type $(4,2)$ and in the moduli space $M^{4n+2}(\mathbb{P}^4)$ of semi-stable sheaves on $\mathbb{P}^4$ with Hilbert polynomial $4n+2$. This project is motivated by the work of Ch. Lehn, M. Lehn, Ch. Sorger and D. van Straten, who constructed a family of holomorphic symplectic manifolds via a contraction of the moduli space $M_3(Y)$ of rational curves on a smooth cubic fourfold that does not contain a plane.

Contents

List of Abbreviations

E	$\mathbb{C}$-vector space of dimension 4
F	$\mathbb{C}$-vector space of dimension 2
V	$\mathbb{C}$-vector space of dimension 5 generated by linear forms $x_0, \dots, x_4$
W	set of Kronecker modules $\mathrm{Hom}(F, E \otimes V)$
W^{ss}	subset of semi-stable Kronecker modules is W
W^s	subset of stable Kronecker modules is W
K	moduli space of semi-stable Kronecker modules of type $(4, 2)$
K^s	moduli space of stable Kronecker modules of type $(4, 2)$
G	group $\mathrm{GL}_2 \times \mathrm{GL}_4/\Gamma$ acting on W
G_x	stabilizer of $x \in K$ w.r.t. the action of G
$X /\!\!/ G$	good quotient of some variety X by a group G
S_i	stable stratum consisting of Kronecker modules whose maximal minors define closed subschemes of dimension at least 2 in $\mathbb{P}^4$
P_i	stratum consisting of strictly poly-stable Kronecker modules
B_i^j resp. B_i	stable strata consisting of Kronecker modules, whose maximal minors define curves in $\mathbb{P}^4$
S	union of the stable strata S_1, S_2, S_3
P	union of the strictly poly-stable strata P_1, P_2, P_3
B	union of the stable strata B_i^j resp. B_j
$\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$	Hilbert scheme parametrising subschemes of $\mathbb{P}^4$ with Hilbert polynomial $P(n) = 4n + 1$
H_0	component of $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$ that contains the rational normal curve of degree 4 in $\mathbb{P}^4$
$P_{\mathcal{F}}(n)$	Hilbert polynomial of a coherent sheaf $\mathcal{F}$
$p_{\mathcal{F}}(n)$	reduced Hilbert polynomial of a coherent sheaf $\mathcal{F}$
$M^{4n+2}(\mathbb{P}^4)$	moduli space of semi-stable sheaves on $\mathbb{P}^4$ with Hilbert polynomial $4n + 2$
$M^{2n+1}(\mathbb{P}^4)$	moduli space of stable sheaves on $\mathbb{P}^4$ with Hilbert polynomial $2n + 1$
M_4	component in $M^{4n+2}(\mathbb{P}^4)$ containing the cokernels of the Kronecker modules whose minors define rational normal curves of degree 4 in $\mathbb{P}^4$
$S^2(M^{2n+1}(\mathbb{P}^4))$	strictly poly-stable sheaves in $M^{4n+2}(\mathbb{P}^4)$ which are direct sums of two sheaves in $M^{2n+1}(\mathbb{P}^4)$
$S^2(M^{2n+1}(\mathbb{P}^4))_0$	sheaves in $S^2(M^{2n+1}(\mathbb{P}^4))$ whose support is connected
$\mathrm{ext}^i(\mathcal{F}, \mathcal{G})$	$\dim \mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$

1

Introduction

One of the most interesting classes of complex projective manifolds are the irreducible symplectic varieties. These are connected, simply connected projective manifolds X carrying an everywhere non-degenerate holomorphic 2-form ω such that
$$H^0(X, \Omega^2_X) = \mathbb{C} \cdot \omega.$$

In dimension 2, these manifolds are exactly the (projective) K3 surfaces. In dimension at least 4 only a few examples are known so far, up to deformation: Hilbert schemes of points on K3 surfaces, generalised Kummer varieties and furthermore O'Grady constructed two sporadic examples in dimension 6 and 10 from moduli spaces of sheaves on K3 surfaces and abelian surfaces.

This thesis is part of a project aiming to understand the geometry of different compactifications of the moduli space $M_4(Y)$ of rational curves of degree 4 on a smooth cubic fourfold $Y \subset \mathbb{P}^5$. It is expected that there is a birational model of a compactification of $M_4(Y)$ which admits a fibration onto a $10-$dimensional symplectic manifold X. Moreover X should carry a Lagrangian fibration $f : X \to \mathbb{P}^5$.

The space $M_d(Y)$ of smooth rational curves of degree d on a smooth cubic fourfold $Y \subset \mathbb{P}^5$ has been intensively studied in small degrees.
In case $d = 1$, Beauville and Donagi ([2]) showed that the moduli space $M_1(Y)$ of lines is a smooth 4-dimensional holomorphic symplectic variety which is deformation equivalent to the Hilbert scheme of 0-dimensional subschemes of length 2 of a $K3$ surface.

For $d \geq 2$, $M_d(Y)$ is no longer compact. De Jong and Starr studied its compactifications $\overline{M_d(Y)}$ in the Hilbert schemes $\mathrm{Hilb}^{dn+1}(Y)$ in [4]. They show that each desingularisation of the moduli spaces $\overline{M_d(Y)}$ carries a canonical 2-form ω_d with the following properties:

For $d \geq 5$ odd, ω_d is generically non-degenerated, for $d \geq 6$, the kernel of the map $T_pM_d(Y) \to T_p^*M_d(Y)$ which is induced by the 2-form ω_d in the point p for a very general point p, has dimension 1 and for $d = 2, 3$ and 4 the kernel has dimension 3, 2 and 3 respectively.

We now consider the case $d = 3$: Let $\overline{M_3(Y)}$ be the compactification of the moduli space $M_3(Y)$ of twisted cubics on smooth cubic fourfolds in the Hilbert scheme $\mathrm{Hilb}^{3n+1}(\mathbb{P}^3)$. It was studied by Ch. Lehn, M. Lehn, Ch. Sorger and D. van Straten in [20].
One of the main theorems of [20] is:

Theorem 1.1 (Lehn, Lehn, Sorger, van Straten) *Let $Y \subset \mathbb{P}^5$ be a smooth cubic hypersurface that does not contain a plane. Then the moduli space $\overline{M_3(Y)}$ is a smooth and irreducible projective variety of dimension* 10.

Furthermore they constructed a contraction to an 8-dimensional symplectic manifold. Concretely, if we denote by ω_3 the holomorphic 2-form defined in [4] by de Jong and Starr, then their result is

Theorem 1.2 (Lehn, Lehn, Sorger, van Straten) *Let $Y \subset \mathbb{P}^5$ be a smooth cubic hypersurface that does not contain a plane. Then there is a smooth* 8-*dimensional irreducible holomorphic symplectic manifold Z and morphisms $u : \overline{M_3(Y)} \to Z$ and $j : Y \to Z$ such that the following holds*

- *(a) The symplectic structure ω in Z satisfies $u^*\omega = \omega_3$.*
- *(b) The morphism j is a closed embedding of Y as a Lagrangian submanifold in Z.*
- *(c) The morphism u factors as follows:*

$$\begin{tikzcd}
\overline{M_3(Y)} \arrow[rr, "u"] \arrow[dr, "a"'] & & Z , \\
& Z' \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

where $a : \overline{M_3(Y)} \to Z'$ is a $\mathbb{P}^2$-fibre bundle and $\sigma : Z' \to Z$ is the blow-up of Z along Y.

- *(d) The topological Euler number of Z is $e(Z) = 25650$.*

The Hilbert scheme $\mathrm{Hilb}^4(S)$ for a $K3$ surface S also has Euler number 25650 and N. Addington and M. Lehn showed in [1] that the manifold Z is deformation equivalent to the fourth Hilbert scheme of a $K3$ surface.

Lehn, Lehn, Sorger and van Straten made use in an essential way of the following result ([26]):

Theorem 1.3 (Piene, Schlessinger) *(a) The Hilbert scheme* $\mathrm{Hilb}^{3n+1}(\mathbb{P}^3)$ *consists of two irreducible components* H *and* H' *of dimension* 12 *respectively* 15*.*

(b) H *and* H' *are smooth, rational and intersect transversally in a divisor* $J \subset H$*.*

(c) The intersection J *is non-singular, rational and has dimension* 11*.*

There is an important difference between curves in $H \smallsetminus J$ and in J: Curves in $H \smallsetminus J$ are arithmetically Cohen-Macaulay and the curves in J are not. For any curve in $H \smallsetminus J$ there exists a $(3 \times 2)-$matrix, whose maximal minors generate its homogeneous ideal. All curves in $H \smallsetminus J$ are purely one-dimensional, but might be non-reduced and reducible.
The curves associated to points in J are not even Cohen-Macaulay. The homogeneous ideal of such a curve is generated by three quadrics and a cubic polynomial. These curves are plane cubic curves with a singularity and an embedded point in the singularity.

The (3×2)-matrix is a special case of a Kronecker module. In general, a Kronecker module of type $(d, 2)$ is a linear map in

$$W := \mathrm{Hom}(\mathbb{C}^d, \mathbb{C}^2 \otimes V),$$

where $V = \langle x_0, \ldots, x_d \rangle$ is the space of linear forms in $\mathbb{P}^d$. On W there is an action of the reductive group $G := (\mathrm{GL}_d \times \mathrm{GL}_2)/\Gamma$, where $\Gamma = \{tI_d, tI_2 \mid t \in \mathbb{C}^*\}$. Let W^s resp. W^{ss} denote the stable resp. semi-stable points of W with respect to the action of G.
The case $d = 3$ was studied by Ellingsrud, Piene and Strømme. They found the following connection between the moduli space of Kronecker modules of type $(3, 2)$ and the Hilbert scheme $\mathrm{Hilb}^{3n+1}(\mathbb{P}^3)$.
With the notation above, there is a geometric quotient

$$K := W^s /\!\!/ G = W^{ss} /\!\!/ G,$$

which is smooth, projective and 12-dimensional. Moreover let

$$I \subset \mathbb{P}(W) \times \mathbb{P}(W^*)$$

denote the incidence variety of all pairs (p, V) consisting of a point

$$p = \{x_0 = x_1 = x_2 = 0\}$$

on a hyperplane $V = \{x_0 = 0\}$. Ellingsrud, Piene and Strømme showed in [8] that there is an isomorphism $H \cong \mathrm{BL}_I(K)$ from the component H of the

Hilbert scheme $\mathrm{Hilb}^{3n+1}(\mathbb{P}^3)$ to the blow-up of the geometric quotient of K in I, which identifies J with the exceptional divisor.

Motivated by the importance of the theorem of Piene and Schlessinger for the work of Lehn, Lehn, Sorger and van Straten, the study the Hilbert scheme $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$ should be important for the understanding of $M_4(Y)$. However, not much is known about this Hilbert scheme so far. Martin-Deschamps and Piene described in [22] the open subset of arithmetically Cohen-Macaulay curves in the component of $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$ that contains the rational normal curves.

The first step of this project is therefore to study $M_4(\mathbb{P}^4) \subset \mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$. The reason is that any rational normal curve of degree 4 spans a hyperplane $\mathbb{P}^4$ in $\mathbb{P}^5$.

In order to find a good compactification of the set of rational normal curves of degree 4 in $\mathbb{P}^4$ and to study its properties, we consider the following three moduli spaces

- the moduli space of semi-stable Kronecker modules of type $(4, 2)$
- the Hilbert scheme of curves in $\mathbb{P}^4$ with Hilbert polynomial $4n + 1$
- the moduli space of semi-stable sheaves on $\mathbb{P}^4$ with Hilbert polynomial $4n + 2$

and study the relations between these three moduli spaces.
In Chapter 2 we describe the moduli space K of semi-stable Kronecker modules of type $(4, 2)$.
In this case, the geometric invariant theory gives a good quotient $K := W^{ss} /\!\!/ G$ of dimension 21, which is projective, normal and irreducible. But K is not a geometric quotient since there are strictly semi-stable Kronecker modules with dimension vector $(4, 2)$ that are not poly-stable forcing the existence of non-closed orbits. In the closure of any G-orbit that contains a semi-stable Kronecker module, there exists by construction exactly one orbit of a strictly poly-stable Kronecker module. In the good quotient K, a strictly semi-stable Kronecker module is mapped to the same point as the strictly poly-stable Kronecker module in the closure of its orbit. So it suffices to consider poly-stable G-orbits.

Furthermore the group $\mathrm{PGL}(V)$ acts by coordinate transformation on K. In Section 2.2 we will prove the following

Theorem 1.4 *In the good quotient $K = W^{ss} /\!\!/ G$ there are* *(a) exactly the following* 15 *orbits with respect to the action of* $\mathrm{PGL}(V)$*, that consist of stable Kronecker modules:*

$$B_0, B_1, B_2, B_3^1, B_3^2, B_4^1, B_5^1, B_5^2, B_5^3, B_6, B_7, B_{10}, S_1, S_2, S_3,$$

and furthermore one stratum B_4^2 *which is a family of stable* $\mathrm{PGL}(V)$*-orbits parametrized by* $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$*,*

(b) exactly 3 *orbits with respect to the action of* $\mathrm{PGL}(V)$ *that consist of strictly poly-stable Kronecker modules:*

$$P_1, P_2, P_3.$$

We will work out explicite representatives of any of these orbits.
Using the computer algebra system Singular ([5]) and the Luna slice theorem, we compute the dimension of the orbits.
The singular locus of K turns out to be the union P of all strictly poly-stable strata. P consists of the orbits of the block matrices of the form

$$\begin{pmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_3 & x_4 \end{pmatrix}^t, \begin{pmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_0 & x_3 \end{pmatrix}^t, \begin{pmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_0 & x_1 \end{pmatrix}^t.$$

The Kronecker modules of type $(4, 2)$ have an important geometric interpretation. The (2×2)-minors of the matrices that represent Kronecker modules define closed subschemes in $\mathbb{P}^4$. For "most" of the Kronecker modules in K, these closed subschemes will be curves in $\mathbb{P}^4$ with Hilbert polynomial $4n + 1$. In this way we define a rational map

$$\Phi_{KH} : K \dashrightarrow \mathrm{Hilb}^{4n+1}(\mathbb{P}^4).$$

More precisely, we obtain the following result:

Theorem 1.5 *Let* H_0 *be the irreducibel component of the Hilbert scheme* $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$ *of curves in* $\mathbb{P}^4$ *with Hilbert polynomial* $4n + 1$*, that contains the rational normal curves of degree* 4*. There is a rational map*

$$\Phi_{KH} : K \dashrightarrow H_0 \subset \mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$$

from the moduli space K *of Kronecker modules of type* $(4, 2)$ *into* H_0*. A Kronecker module, represented by a matrix* A*, is mapped by* Φ_{KH} *to the curve in* H_0 *which is defined by the vanishing of the* (2×2)*-minors of* A*.*
The map Φ_{KH} *is defined outside the strictly poly-stable locus* P *of* K *and the union* S *of the three strata* S_1, S_2, S_3 *of stable Kronecker modules whose maximal minors define closed subschemes of dimension* 2*.*

It is not possible to extend the rational map Φ_{KH} across S and P, but we can extend Φ_{KH} after blowing-up K along S and "partially" to the blow-up of K along P. More specifically, we study the situation locally using Luna's slice theorem. For the poly-stable stratum P_1, we construct a local model for the singularities of K in any $x \in P_1$ and also a model for a resolution of the singularity. We extend the family of curves to this blow-up. This will be worked out in Section 3.4.1.
For P_2 we give in Section 3.4.2 a general procedure for the extension of the family of curves. As P_2 lies in the intersection of the closure of the strata P_1 and all S_i, the combinatorics are much more complicated in this case.
The situation in P_3 is technically more complicated, since the stabilizer in $x \in P_3$ is SL_2 instead of $\mathbb{C}^*$. Consequently, we restrict ourselves to giving a model for the singularities of K in P_3 and omit the study of their blow-up.

Because of the difficulties to extend Φ_{KH} on the blow-up of K along $S \cup P$, we study in Chapter 4 a different compactification, namely the moduli space $M^{4n+2}(\mathbb{P}^4)$ of semi-stable sheaves on $\mathbb{P}^4$ with Hilbert polynomial $4n+2$. The motivation for choosing the Hilbert polynomial $4n+2$ is the following:
Let $Y \subset \mathbb{P}^5$ be a cubic fourfold and $\mathcal{D}(Y)$ the derived category of bounded complexes of coherent sheaves on Y. Then the sheaves $L_i := \mathcal{O}_Y(i)$ for $i = 0, 1, 2$ form an exceptional sequence, i.e. $\mathrm{RHom}(L_i, L_i) = \mathbb{C}$ and $\mathrm{RHom}(L_i, L_j) = 0$ for all $0 \leq j < i \leq 2$. Moreover, by $\mathcal{A}$ we denote the right orthogonal complement

$$\mathcal{A} := \langle L_0, L_1, L_2 \rangle^{\perp} = \{a \in \mathcal{D}(Y) \mid \mathrm{RHom}(L_i, a) = 0 \text{ for all } i = 0, 1, 2\}$$

of this exceptional sequence.
Kuznetsov showed in [19] that $\mathcal{A}$ is a 2-Calabi-Yau category, i.e. there is a Serre functor $\mathcal{S}_A$ which is given by a double shift

$$\mathrm{RHom}(a, b) = \mathrm{RHom}(b, a[2])^*$$

for all $a, b \in \mathcal{A}$. Hence the objects in the category $\mathcal{A}$ deform in the same way as coherent sheaves on a $K3$ surface. Now the moduli spaces of simple sheaves on a $K3$ surface are smooth and symplectic for the following reason (worked out by Mukai in [24]):
Since the kernel $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})_0 \cong \mathrm{Hom}(\mathcal{F}, \mathcal{F})_0^* = 0$ of the trace map

$$\mathrm{tr} : \mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) \to H^2(\mathcal{O})$$

is zero, the Kuranishi map vanishes identically and $T_{\mathcal{F}}M \cong \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$. Thus the moduli space is smooth in the point $[\mathcal{F}]$. Moreover, the symplectic form is defined via the Yoneda product:

$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \times \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to \mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) \cong \mathbb{C},\ (\alpha, \beta) \mapsto \sigma(\mathrm{tr}(\alpha \cup \beta)),$$

where σ denotes the symplectic 2-form on the $K3$ surface.

By analogy it is expected that moduli spaces of simple objects of $\mathcal{A}$, i.e. with $\mathrm{Hom}(a,a) = \mathbb{C}$ for all $a \in \mathcal{A}$, are symplectic. Such a moduli space has the dimension

$$\begin{aligned}\dim_a M &= \dim T_a M\\ &= \mathrm{ext}^1(a,a)\\ &= -(\mathrm{ext}^0(a,a) - \mathrm{ext}^1(a,a) + \mathrm{ext}^2(a,a)) + \mathrm{ext}^0(a,a) + \mathrm{ext}^2(a,a)\\ &= -\chi(a,a) + 2,\end{aligned}$$

where

$$\chi(a,b) = \sum_k (-1)^k \,\mathrm{ext}^k(a,b) = \int_Y \mathrm{ch}^\vee(a)\,\mathrm{ch}(b)\,\mathrm{td}(Y)$$

is the Euler pairing and $\mathrm{ch}^\vee(a)_k = (-1)^k \,\mathrm{ch}(a)$.
Now we define a projection functor $\Pi : \mathcal{D}(Y) \to \mathcal{A}$ as the composition

$$\Pi := \mathcal{L}_0 \circ \mathcal{L}_1 \circ \mathcal{L}_2,$$

where $\mathcal{L}_i$ is the left mutation with respect to the exceptional object L_i, i.e. for any $\mathcal{F} \in \mathcal{D}(Y)$

$$\mathcal{L}_i(\mathcal{F}) := \mathrm{cone}(L_i \otimes \mathrm{RHom}(L_i, \mathcal{F}) \to \mathcal{F})$$

is the mapping cone of the evaluation map $L_i \otimes \mathrm{RHom}(L_i, \mathcal{F}) \to \mathcal{F}$.
So if $\mathcal{F} \in \mathcal{D}(Y)$ is a locally free sheaf of degree l on a rational curve of degree d, then the dimension of the moduli space M in $a := \Pi(\mathcal{F})$ is

$$\dim_a M = 2 - \int_Y \mathrm{ch}^\vee(a)\,\mathrm{ch}(a)\,\mathrm{td}(Y) = 2 - \chi(a,a) = 6l^2 - 12l + 16.$$

So, if $l = 0$, the dimension of the moduli space M is 16 and for $l = 1$, the dimension is 10.
Now, in our situation the moduli space $M_4(Y)$ of rational curves of degree 4 on Y has dimension 13 by [4]. As we mentioned before De Jong and Starr construct in [4] a holomorphic 2-form on this moduli space. Since the kernel of this 2-form has generically dimension 3, we expect that there is a geometric contraction $M_4(Y) \dashrightarrow Z(Y)$ onto some $10-$dimensional space $Z(Y)$. This contraction corresponds to the projection $\Pi : \mathcal{D}(Y) \to \mathcal{A}$.
This analogy motivates the choice of locally free sheaves of rank 1 and degree 1 on a rational curve C of degree 4 (i.e. $l = 1$ and $d = 1$). Such sheaves have Hilbert polynomial

$$P_{\mathcal{F}}(n) = \mathrm{rank}(\mathcal{F})\deg(C) \cdot n + \deg(\mathcal{F}) + \mathrm{rank}(\mathcal{F})(1-g) = 4n + 2$$

and hence we study in Chapter 4 the moduli space $M^{4n+2}(\mathbb{P}^4)$ of semi-stable sheaves on $\mathbb{P}^4$ with Hilbert polynomial $4n+2$.

The sheaves in $M^{4n+2}(\mathbb{P}^4)$ have one-dimensional support and are pure. The space $M^{4n+2}(\mathbb{P}^4)$ contains a smooth Zariski-open subset of sheaves of the form $i_*\mathcal{O}_{\mathbb{P}^1}(1)$, where $i : \mathbb{P}^1 \to \mathbb{P}^4$ is the standard embedding of degree 4, which is given by a choice of a basis of $H^0(\mathcal{O}_{\mathbb{P}^1}(4))$. We will call the closure in $M^{4n+2}(\mathbb{P}^4)$ of this open set M_4.
The variety M_4 contains strictly poly-stable sheaves. In fact these have the form $\mathcal{O}_{Q_1} \oplus \mathcal{O}_{Q_2}$, where Q_1, Q_2 are two quadrics in $\mathbb{P}^4$ that intersect in one point. The subset of these sheaves is denoted by $S^2(M^{2n+1}(\mathbb{P}^4))_0$. Now M_4 is non-singular in any point associated to a generic strictly poly-stable sheaves in $S^2(M^{2n+1}(\mathbb{P}^4))_0$. Concretely, we will prove

Theorem 1.6 *Let* $(Q_1 \subset E_1, Q_2 \subset E_2)$ *be a pair of smooth plane quadrics intersecting in exactly one point. Then the variety* M_4 *is non-singular in the point* $[\mathcal{F}] := [\mathcal{O}_{Q_1} \oplus \mathcal{O}_{Q_2}] \in S^2(M^{2n+1}(\mathbb{P}^4))_0$.

However, it is not clear, whether the component M_4 is regular in the other points of $S^2(M^{2n+1}(\mathbb{P}^4))_0$ i.e. in points associated to sheaves $\mathcal{O}_{Q_1} \oplus \mathcal{O}_{Q_2}$ whose support consists of smooth quadrics that intersect in more than one point or of quadrics that are not smooth. The main difficulty is to decide whether the deformation of a sheaf $[\mathcal{F}] \in S^2(M^{2n+1}(\mathbb{P}^4))_0$ is still contained in the component M_4.
In the Sections 4.3 and 4.6 we will prove

Theorem 1.7 *There is a birational map* $\Phi_{KM} : K \dashrightarrow M_4 \subset M^{4n+2}(\mathbb{P}^4)$, *that is defined outside* $S \cup P$. *The inverse map* Φ_{MK} *is defined on an open subset* X *of* M_4 *that consists of sheaves* $\mathcal{F}$ *with the following properties*

(a) $\mathcal{F}$ *is globally generated by* 2 *sections,*

(b) if $\mathcal{N}$ *is the kernel of the induced map* $\mathcal{O}_{\mathbb{P}^4}^{\oplus 2}(1) \to \mathcal{F}(1)$, *then* $h^0(\mathcal{N}) = 4$,

(c) in the resolution

$$0 \longleftarrow \mathcal{F} \longleftarrow \mathcal{O}_{\mathbb{P}^4}^2 \xleftarrow{A} \mathcal{O}_{\mathbb{P}^4}(-1)^4 \longleftarrow \dots$$

of $\mathcal{F}$ *constructed from the data in (a) and (b) the Kronecker module* A *is semi-stable.*

The birational map $\Phi_{KM} : K \dashrightarrow M_4$ is obtained as follows: Consider a Kronecker module $\varphi \in K \backslash (S \cup P)$, given by a matrix A. Then φ is mapped to the cokernel of the map

$$\mathcal{O}_{\mathbb{P}^4}(-1)^4 \xrightarrow{A} \mathcal{O}_{\mathbb{P}^4}^2.$$

We will check that these cokernel sheaves are indeed contained in M_4, i.e. they have Hilbert polynomial $4n+2$ and are stable.
For the construction of an inverse map $\Phi_{MK} : M_4 \dashrightarrow K$, an important step is to prove

Proposition 1.8 *Let* $[\mathcal{F}] \in M_4$ *be a semi-stable sheaf on* $\mathbb{P}^4$ *with Hilbert polynomial* $4n+2$*. Then* $h^0\mathcal{F} = 2, h^1\mathcal{F} = 0$*. Moreover* $h^0(\mathcal{F}(1)) = 6$*.*

Furthermore we show in Lemma 4.63 that the set of cokernel sheaves of Kronecker modules in B is contained in X.

2

Moduli Space of Kronecker Modules of Type $(4, 2)$

In this Chapter we study the moduli space K of Kronecker modules of type $(4, 2)$, i.e. of (4×2)-matrices with entries in the vector space $V = \langle x_0, \ldots, x_4 \rangle$ generated by the the linear forms $x_0, \ldots, x_4$.
In Section 2.1 we recall some notions from geometric invariant theory which will be used later. In particular we discuss the notion of stability for Kronecker modules. In Section 2.2 we give a list of $\mathrm{PGL}(V)$-orbits of K and in Section 2.3 we compute a stratification of K by $\mathrm{PGL}(V)$-orbits.

1 Stability of Kronecker Modules of type $(4, 2)$

Definition 2.1 (Kronecker module) *A Kronecker module of type* (m, n) *is a linear map in* $\mathrm{Hom}(F, E \otimes V)$, *where* E *is a* $\mathbb{C}$*-vector space of dimension* m, F *is a* $\mathbb{C}$*-vector space of dimension* n *and* V *is the* 5*-dimensional* $\mathbb{C}$*-vector space generated by the linear forms* $x_0, \ldots, x_4$.

A homomorphism of Kronecker modules $\varphi_i : F_i \to E_i \otimes V$ $i = 1, 2$, *is a pair* $(f_1 : F_1 \to F_2, f_2 : E_1 \to E_2)$ *such that the following diagram commutes*

$$\begin{tikzcd}
F_1 \arrow[r, "\varphi_1"] \arrow[d, "f_1"] & E_1 \otimes V \arrow[d, "f_2 \otimes id"] \\
F_2 \arrow[r, "\varphi_2"] & E_2 \otimes V
\end{tikzcd}$$

Remark 2.2 (a) Kronecker modules are representations of the Kronecker quiver which consists of two vertices v_1 and v_2 and a finite number of arrows $a_1, \ldots, a_r : v_1 \to v_2$.

(b) for a fixed vector space V, the Kronecker modules form an abelian category.

For fixed bases of E and F a Kronecker module $\varphi \in W := \mathrm{Hom}(F, E \otimes V)$ is given by a $(m \times n)$-matrix whose entries are linear forms in $x_0, \dots, x_4$.
The group $G_1 := \mathrm{GL}_m \times \mathrm{GL}_n$ acts on $W = \mathrm{Hom}(F, E \otimes V)$ by

$$(g,h) \circ A := g \otimes id_V \cdot A \cdot h^{-1}.$$

Furthermore $\Gamma := \{(\alpha \cdot \mathrm{id}_E, \alpha \cdot \mathrm{id}_F) | \alpha \in \mathbb{C}^*\}$ acts trivially on W, so we have an action of the group $G := G_1/\Gamma$ on W. In addition we have an action of $\mathrm{GL}(V)$ on W by coordinate transformations.

For the description of a moduli space of Kronecker modules, we need some geometric invariant theory (see [25]). Hence we shortly repeat some notions.
Recall that an algebraic group G is called reductive if the radical of G (i.e. its maximal connected solvable normal subgroup) is a torus.

Definition 2.3 (stable, semi-stable point) *Let G be a reductive group that acts linearly on a vector space U. A non-zero point of U is called*

(a) unstable, if 0 is in the closure of its orbit,

(b) semi-stable, if 0 is not in the closure of its orbit,

(c) stable, if its orbit is closed and its stabilizer is finite,

(d) strictly semi-stable, if it is semi-stable, but not stable,

(e) poly-stable, if it is strictly semi-stable and the orbit is closed.

Let G be an algebraic group acting on an algebraic variety W. In the following way one can define a good categorical quotient of varieties with "good" geometric properties:

Definition 2.4 (good quotient) *A good quotient is a morphism $\pi : W \to X$ of varieties with the following properties:*

(a) π is G-invariant, affine and surjective

(b) there is an isomorphism $\mathcal{O}_X \to (\pi_* \mathcal{O}_W)^G$

(c) if $Z \subset W$ is closed and G-invariant, then $\pi(Z) \subset X$ is closed

(d) if $Z, Z' \subset W$ are closed, G-invariant and disjoint, then $\pi(Z) \cap \pi(Z') = \varnothing$.

In some situation one can define a quotient with the following "better" properties:

Definition 2.5 (geometric quotient) *A geometric quotient of W by G is a good quotient $\pi : W \to X$ such that for all $w_1, w_2 \in W$ we have $\pi(w_1) = \pi(w_2)$ if and only if $Gw_1 = Gw_2$.*

In the following, we consider Kronecker modules of type $(4, 2)$, i.e. F is a 2-dimensional $\mathbb{C}$-vector space and E is a 4-dimensional $\mathbb{C}$-vector space. Now we denote as before by W the vector space

$$W = \mathrm{Hom}(F, E \otimes V)$$

for this choice of E and F. Again for fixed bases of E and F an element in W is given by a (4×2)-matrix whose entries are linear forms in $x_0, \ldots, x_4$.
W is canonically isomorphic to $W' = \mathrm{Hom}(E^\star \otimes V^\star, F^\star)$. For a Kronecker module $A \in W$ we have $A^t \in W'$ and A is stable (resp. unstable, semi-stable or poly-stable) if and only if A^t is stable (resp. unstable, semi-stable or poly-stable).
We use the following criterion of Drézet ([6][Prop.15]) and Hulek ([14]) to describe the shape of unstable resp. strictly semi-stable resp. poly-stable matrices, which is formulated for Kronecker modules in W'.

Proposition 2.6 (Drézet, Hulek) *Let $\tau : E \otimes V \to F$ be a Kronecker module, $\tau \neq 0$. Then τ is semi-stable resp. stable if and only if for all subspaces E' and F' of E and F such that $E' \neq \{0\}$, $F' \neq F$ and $\tau(E' \otimes V) \subset F'$, the following inequality holds*

$$\frac{\dim F'}{\dim E'} \geq \frac{\dim F}{\dim E} \quad \textit{(resp. >).}$$

Since $\dim F = 2$ and $\dim E = 4$ are not coprime, there exist strictly semi-stable points, contrary to the case of Kronecker modules of type $(3, 2)$ in $\mathbb{P}^3$:

$$\frac{1}{2} = \frac{\dim F'}{\dim E'} = \frac{2}{4}.$$

Lemma 2.7 *Let A be a matrix that represents a Kronecker module in W'.*

(a) A is unstable if and only if it is equivalent under row and column operations to a matrix with two zeros in one column or more than two zeros in one row:

$$\begin{pmatrix} * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \end{pmatrix}$$

(b) A is semi-stable if and only if it is not equivalent under row and column operations to a matrix with two zeros in one column or more than two zeros in one row.

(c) *A is strictly semi-stable if and only if it is equivalent under row and column operations to a matrix with exactly two zeros in one row:*

$$\begin{pmatrix} * & * & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

(d) *A is strictly poly-stable if it is equivalent under row and column operations to a matrix of the form*

$$\begin{pmatrix} * & * & 0 & 0 \\ 0 & 0 & * & * \end{pmatrix}.$$

PROOF: (a) " $\Leftarrow$ ": If A has the form $\begin{pmatrix} * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$ with respect to an appropriate choice of bases $(e_1, \ldots, e_4)$ of E and (f_1, f_2) of F, we may choose the vector subspace $F' = \langle f_1 \rangle$ and $E' = \langle e_1, e_2, e_3 \rangle$. Then the images of $e_1, .., e_3$ by A are obviously contained in F'. Furthermore

$$\frac{1}{3} = \frac{\dim F'}{\dim E'} < \frac{\dim F}{\dim E} = \frac{1}{2}.$$

If A has the form $\begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \end{pmatrix}$ with respect to bases $(e_1, \ldots, e_4)$ of E and (f_1, f_2) of F, then we may choose $F' = \{0\}$ and $E' = \langle e_4 \rangle$. The image of e_4 by A is zero and

$$\frac{0}{1} = \frac{\dim F'}{\dim E'} < \frac{\dim F}{\dim E} = \frac{1}{2}.$$

By Proposition 2.6 the matrix A is in both cases unstable.

" $\Rightarrow$ ": Assume that A is unstable. Then again by Proposition 2.6 there exist subspaces $F' \subset F$ and $\{0\} \subsetneq E' \subset E$ such that

$$\frac{\dim F'}{\dim E'} < \frac{\dim F}{\dim E} = \frac{1}{2}$$

and $\tau(E' \otimes V) \subset F'$. So either $\dim F' = 0$ or $\dim F' = 1$.

If $\dim F' = 0$, then since $\dim E'_1 \geq 1$ we may choose any $e_1 \in E'_1$ and complete it to a basis (e_1, e_2, e_3, e_4) of E. Since at least e_1 is mapped to 0 by A, the matrix A has the form

$$\begin{pmatrix} 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$$

w.r.t. the chosen basis of E.

If $\dim F' = 1$, i.e. $F' = \langle f_1 \rangle$, then $\dim E' \geq 3$, i.e. $E' = \langle e_1, e_2, e_3 \rangle$.

Complete them again to bases (f_1, f_2) of F and (e_1, e_2, e_3, e_4) of E. Then A has the form

$$\begin{pmatrix} * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

w.r.t. these bases.

(b) Obvious.

(c) A is strictly semi-stable if and only if it is semi-stable and there exist subspaces $E' \subset E$ and $F' \subset F$ such that

$$\frac{1}{2} = \frac{\dim F'}{\dim E'} = \frac{\dim F}{\dim E} = \frac{2}{4}.$$

This follows similar to part (a).

(d) This is obvious, since in this case, A is a direct sum of Kronecker modules of type $(2, 1)$. ■

In the following we use the notation

Notation 2.8 We denote by

- W^s the set of stable,
- W^{ss} the set of semi-stable,
- W^{sss} the set of strictly semi-stable,
- W^{ps} the set of strictly poly-stable and
- W^{is} the set of unstable

Kronecker modules in $W = \mathrm{Hom}(F, E \otimes V)$.

Remark 2.9 We have the decompositions $W = W^{is} \cup W^{ss}$ and $W^{ss} = W^{sss} \cup W^s$. Furthermore $W^{ps} \subset W^{sss}$.

Proposition 2.10 *There exists a good quotient* $K := W^{ss} /\!\!/ G$. *By definition,* K *is projective, normal and irreducible. Furthermore* $\dim(K) = 21$.

Proof: For the first part, use [25] and [16].
For the second part, we use the fact that W^{ss} is a non-empty open dense subset of W. So $\dim W^{ss} = \dim W$. Furthermore,

$$\dim G = \dim \mathrm{GL}_2 + \dim \mathrm{GL}_4 - \dim \Gamma = 4 + 16 - 1 = 19.$$

Hence

$$\dim K = \dim W^{ss} /\!\!/ G = \dim W^{ss} - \dim G = 40 - 19 = 21. \blacksquare$$

Remark 2.11 This good quotient gives a moduli space for Kronecker modules of type $(4,2)$. For an appropriate linearization L, the stability in the sense of geometric invariant theory is equivalent to the stability for Kronecker modules defined in 2.3. For details see [16].

Remark 2.12 Every strictly semi-stable orbit contains in its closure a unique closed strictly poly-stable orbit. Consider the good quotient

$$p : W^{ss} \to W^{ss} /\!\!/ G.$$

Any semi-stable orbit is mapped to the same point P in the good quotient $W^{ss} /\!\!/ G$ as the poly-stable orbit in its closure. In our situation one can see this as follows: By Lemma 2.7, we may assume that the strictly semi-stable Kronecker module is represented by a matrix of the form $\begin{pmatrix} * & * & 0 & 0 \\ a & b & * & * \end{pmatrix}^t$. Now for any $t \neq 0$ we have

$$\begin{pmatrix} t & & & \\ & t & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} * & a \\ * & b \\ 0 & * \\ 0 & * \end{pmatrix} \cdot \begin{pmatrix} t^{-1} & \\ & 1 \end{pmatrix} = \begin{pmatrix} * & t \cdot a \\ * & t \cdot b \\ 0 & * \\ 0 & * \end{pmatrix}.$$

Hence the matrices $\begin{pmatrix} * & * & 0 & 0 \\ a & b & * & * \end{pmatrix}^t$ and $\begin{pmatrix} * & * & 0 & 0 \\ t \cdot a & t \cdot b & * & * \end{pmatrix}^t$ are contained in the same orbit for all $t \neq 0$. Thus the orbit of poly-stable Kronecker module given by $\begin{pmatrix} * & * & 0 & 0 \\ 0 & 0 & * & * \end{pmatrix}^t$ is in the closure of the orbit of $\begin{pmatrix} * & * & 0 & 0 \\ a & b & * & * \end{pmatrix}^t$.

2 Classification of Kronecker Modules of Type $(4,2)$

The goal of this Section is to give a classification of the types of Kronecker modules occuring in the good quotient $K = W^{ss} /\!\!/ G$.
There is a geometric interpretation for a Kronecker module $\varphi \in W$ represented by a matrix A: the ideal generated by maximal minors of A defines a closed subscheme in $\mathbb{P}^4$. In most cases this will be a curve, possibly reducible or non-reduced. We will study this geometric interpretation in Section 3.2.

Note that the stability properties of Kronecker modules in W^{ss} with respect to the action of G are invariant under the action of $\mathrm{GL}(V)$. Hence we will say, an $\mathrm{GL}(V)$-orbit is stable (resp. poly-stable, semi-stable or unstable), if one and hence all of its members are.
Clearly the action of $\mathrm{GL}(V)$ on the quotient $W^{ss} /\!\!/ G$ is the same as the action of $\mathrm{PGL}(V)$ as the $\mathbb{C}^*$-action is already taken into account by the G-action.
We will give a stratification of the action of $\mathrm{PGL}(V)$ on $W^{ss} /\!\!/ G$. Any stratum except one, which we will call B_4^2, is the orbit of a Kronecker module representing a point in $W^{ss} /\!\!/ G$ with respect to the action of $\mathrm{PGL}(V)$. In contrast, B_4^2 is the union of orbits of stable Kronecker modules parametrized by $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Any one of these orbits has codimension 5 in $W^{ss} /\!\!/ G$.
Concretely, the main result of this Section is:

Theorem 2.13 *In the good quotient $K = W^{ss} /\!\!/ G$ there are exactly the following poly-stable orbits resp. strata with respect to the action of* $\mathrm{PGL}(V)$*:*

name	*representative A*	*stability*	*codimension of the orbits resp. strata w.r.t.* $\mathrm{PGL}(V)$ *in* $W^{ss} /\!\!/ G$
B_0	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}^t$	*stable*	*0*
B_1	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & x_2 & x_3 & x_4 \end{pmatrix}^t$	*stable*	*1*
B_2	$\begin{pmatrix} x_0 & 0 & x_2 & x_3 \\ 0 & x_1 & x_3 & x_4 \end{pmatrix}^t$	*stable*	*2*

B_3^1	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & x_0 & x_3 & x_4 \end{pmatrix}^t$	*stable*	*3*
B_3^2:	$\begin{pmatrix} x_0 & 0 & x_2 & x_3 \\ 0 & x_1 & x_2 & x_4 \end{pmatrix}^t$	*stable*	*3*
B_4^1	$\begin{pmatrix} x_0 & x_1 & 0 & x_3 \\ 0 & x_0 & x_2 & x_4 \end{pmatrix}^t$	*stable*	*4*
B_4^2	*family of orbits* *represented by* $\begin{pmatrix} x_0 & 0 & x_2 & x_3 \\ 0 & x_1 & x_2 & \lambda x_3 \end{pmatrix}^t$ *for* $\lambda \neq 0, 1, \infty$	*stable*	*4*
B_5^1	$\begin{pmatrix} x_0 & x_1 & 0 & x_2 \\ 0 & x_1 & x_2 & x_3 \end{pmatrix}^t$	*stable*	*5*
B_5^2	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & x_0 & x_1 & x_4 \end{pmatrix}^t$	*stable*	*5*
B_5^3	$\begin{pmatrix} x_0 & x_1 & 0 & x_2 \\ 0 & x_0 & x_2 & x_3 \end{pmatrix}^t$	*stable*	*5*
B_6	$\begin{pmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & x_1 & x_3 \end{pmatrix}^t$	*stable*	*6*
B_7	$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & x_0 & x_1 & x_2 \end{pmatrix}^t$	*stable*	*7*

B_{10}	$\begin{pmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & x_1 & x_2 \end{pmatrix}^t$	*stable*	*10*
S_1	$\begin{pmatrix} x_0 & 0 & x_1 & x_3 \\ 0 & x_0 & x_2 & x_4 \end{pmatrix}^t$	*stable*	*8*
S_2	$\begin{pmatrix} x_0 & 0 & x_1 & x_2 \\ 0 & x_0 & x_2 & x_3 \end{pmatrix}^t$	*stable*	*9*
S_3	$\begin{pmatrix} x_0 & x_1 & 0 & x_2 \\ 0 & x_0 & x_1 & x_3 \end{pmatrix}^t$	*stable*	*9*
P_1	$\begin{pmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & x_3 \end{pmatrix}^t$	*poly-stable*	*10*
P_2	$\begin{pmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_0 & x_2 \end{pmatrix}^t$	*poly-stable*	*12*
P_3	$\begin{pmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_0 & x_1 \end{pmatrix}^t$	*poly-stable*	*18*

We begin with classifying the $\mathrm{PGL}(V)$-orbits in K:

Proposition 2.14 *In the good quotient $K = W^{ss} /\!\!/ G$ there are*

(a) *exactly the following* 15 *orbits with respect to the action of* $\mathrm{PGL}(V)$*, that consist of stable Kronecker modules:*

$$B_0, B_1, B_2, B_3^1, B_3^2, B_4^1, B_5^1, B_5^2, B_5^3, B_6, B_7, B_{10}, S_1, S_2, S_3,$$

and furthermore one stratum B_4^2 which is a family of stable $\mathrm{PGL}(V)$*-orbits parametrized by $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$,*

(b) *exactly* 3 *orbits with respect to the action of* $\mathrm{PGL}(V)$ *that consist of strictly poly-stable Kronecker modules:*

$$P_1, P_2, P_3.$$

For the proof of Proposition 2.14 we need the following preparations:

Definition 2.15 (saturation) *Let X be an irreducible reduced variety and $\mathcal{E}$ an $\mathcal{O}_X$-module. Let $\mathcal{E}'$ be a subsheaf of $\mathcal{E}$. Then we have an exact sequence*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}/\mathcal{E}' \to 0.$$

The quotient $\mathcal{E}/\mathcal{E}'$ may contain a torsion subsheaf $T(\mathcal{E}/\mathcal{E}')$. Hence we have the exact sequence

$$0 \to T(\mathcal{E}/\mathcal{E}') \to \mathcal{E}/\mathcal{E}' \to (\mathcal{E}/\mathcal{E}')/T(\mathcal{E}/\mathcal{E}') \to 0.$$

The saturation of $\mathcal{E}'$ in $\mathcal{E}$ is defined by

$$\overline{\mathcal{E}'} := \operatorname{Ker}(\mathcal{E} \to (\mathcal{E}/\mathcal{E}')/T(\mathcal{E}/\mathcal{E}')).$$

We have the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & T(\mathcal{E}/\mathcal{E}') \arrow[d] & \\
0 \arrow[r] & \mathcal{E}' \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}/\mathcal{E}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{\mathcal{E}'} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] & (\mathcal{E}/\mathcal{E}')/T(\mathcal{E}/\mathcal{E}') \arrow[r] \arrow[d] & 0 \\
 & T(\mathcal{E}/\mathcal{E}') & & 0 &
\end{tikzcd}$$

Moreover we need the following Lemma

Lemma 2.16 *There is a bijection*

$$\operatorname{Hom}_{\mathbb{C}}(F, E \otimes V) \to \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(V^* \otimes \mathcal{O}_{\mathbb{P}^1}, E \otimes \mathcal{O}_{\mathbb{P}^1}(1)),$$

where $\mathbb{P}^1 = \mathbb{P}(F^*)$.

Proof:

$$\operatorname{Hom}(V^* \otimes \mathcal{O}_{\mathbb{P}^1}, E \otimes \mathcal{O}_{\mathbb{P}^1}(1)) \cong H^0((V \otimes \mathcal{O}_{\mathbb{P}^1}) \otimes E \otimes \mathcal{O}_{\mathbb{P}^1}(1))$$

$$= V \otimes E \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \cong V \otimes E \otimes F^* \cong \operatorname{Hom}(F, E \otimes V). \blacksquare$$

Proof of 2.14: By Lemma 2.16 the problem of classifying Kronecker modules is reduced to classifying maps $\varphi : V^* \otimes \mathcal{O}_{\mathbb{P}^1} \to E \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. To do so, we decompose φ as follows

$$\begin{tikzcd}
V^* \otimes \mathcal{O}_{\mathbb{P}^1} \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & E \otimes \mathcal{O}_{\mathbb{P}^1}(1)\ , \\
\mathrm{Im}(\varphi) \arrow[r, "\beta"] & \overline{\mathrm{Im}(\varphi)} \arrow[u, "\gamma"']
\end{tikzcd}$$

where $\overline{\mathrm{Im}(\varphi)}$ is the saturation of the image of φ in $E \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. By definition, β is generically bijective, the map α is surjective and γ is injective. Moreover the dual map of γ, which is given by the transpose of the matrix for γ, is also surjective.

The sheaves $\mathrm{Im}(\varphi)$ and $\overline{\mathrm{Im}(\varphi)}$ are torsion-free since they are submodules of torsion-free $\mathcal{O}_{\mathbb{P}^1}$-modules. Since torsion-free $\mathcal{O}_X$-modules on smooth curves are locally free, all occuring sheaves in the diagram are vector bundles on $\mathbb{P}^1$. Hence we can use the classification theorem of Grothendieck for vector bundles on $\mathbb{P}^1$ [10]. Thus we have to classify commutative diagrams of the form

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^1}^5 \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & \mathcal{O}_{\mathbb{P}^1}(1)^4 \\
\mathcal{O}_{\mathbb{P}^1}^m \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{r-m} \arrow[r, "\beta"] & \mathcal{O}_{\mathbb{P}^1}^n \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{r-n} \arrow[u, "\gamma"']
\end{tikzcd}$$

for some $m, n \in \mathbb{N}$, where $n \leq m$. Here $r \leq 4$ is the rank of $\mathrm{Im}(\varphi)$ and thus also the rank of $\overline{\mathrm{Im}(\varphi)}$.

Hence to classify the maps φ, it suffices to classify the maps α, β and γ.

Classification of the maps β: First we consider maps

$$\beta : \mathcal{O}^m \oplus \mathcal{O}(1)^{r-m} \to \mathcal{O}^n \oplus \mathcal{O}(1)^{r-n},$$

where $m \geq n$. Since β is generically bijective, one can choose (local) bases of

$$\mathcal{O}^m \oplus \mathcal{O}(1)^{r-m} = \mathcal{O}^n \oplus \mathcal{O}^{m-n} \oplus \mathcal{O}(1)^{r-m} =: \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$$

and

$$\mathcal{O}^n \oplus \mathcal{O}(1)^{r-n} = \mathcal{O}^n \oplus \mathcal{O}(1)^{m-n} \oplus \mathcal{O}(1)^{r-m} =: \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3,$$

such that β is represented by a block matrix

$$\tilde{B} = \begin{array}{c} \left(\begin{array}{c|c|c} M & 0 & 0 \\ \hline 0 & \star & 0 \\ \hline 0 & 0 & N \end{array}\right) \\ \begin{array}{ccc} \underbrace{\qquad}_{n} & \underbrace{\qquad}_{m-n} & \underbrace{\qquad}_{r-m} \end{array} \end{array} \begin{array}{l} \left.\vphantom{M}\right\}n \\ \left.\vphantom{\star}\right\}m-n \\ \left.\vphantom{N}\right\}r-m \\ \vphantom{\underbrace{\qquad}_{n}} \end{array}$$

After an appropriate base change of $\mathcal{W}_3$ and $\mathcal{V}_1$, we may assume that M and N are the identity matrices, i.e.

$$\tilde{B} = \left(\begin{array}{c|c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 & 0 \\ \hline 0 & \star & 0 \\ \hline 0 & 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right).$$

So it suffices to classify maps $\tilde{\beta} : \mathcal{V}_2 = \mathcal{O}^{m-n} \to \mathcal{O}(1)^{m-n} = \mathcal{W}_2$. These maps are given by matrices of the form $H = H_1 s + H_2 t$ for some $H_1, H_2 \in M^{m-n}(\mathbb{C})$ and a basis $s, t \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$.

Lemma 2.17 *For an appropriate choice of coordinates we may assume* $H_1 = \mathrm{Id}$. *Then we can choose a basis of* $\mathcal{V}_2$ *and* $\mathcal{W}_2$ *such that* H_2 *has Jordan normal form:*

$$\begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix},$$

where $J_j = \begin{pmatrix} \lambda_j & & & \\ 1 & \lambda_j & & \\ & \ddots & \ddots & \\ & & 1 & \lambda_j \end{pmatrix}$. *Hence* $\tilde{\beta}$ *is given by a block matrix with blocks* *of the form*

$$I \cdot s + \begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & \ddots & \ddots & \end{pmatrix} t = \begin{pmatrix} s+\lambda t & & & \\ t & s+\lambda t & & \\ & \ddots & \ddots & \end{pmatrix}.$$

Proof: We may assume $\mathrm{rank}(H_1) = m - n$. In fact, otherwise, we use the fact that generically the rank of the matrix $H_1 s + H_2 t$ is $m - n$. Hence we can choose $\mu \in \mathbb{C}$ such that $H_1 + \mu H_2$ has rank $m - n$, and use the coordinate transformation $s \mapsto s + \mu t, t \mapsto t$.
Now we change bases to obtain $H_1 = \mathrm{Id}$. Afterwards a conjugation brings H_2 in Jordan normal form while leaving $H_1 = \mathrm{Id}$ invariant. ■

Classification of the map α: By definition, the map

$$\alpha : \mathcal{O}^5 \to \mathcal{O}^n \oplus \mathcal{O}^{m-n} \oplus \mathcal{O}(1)^{r-m} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$$

is surjective. For the classification of the map β, we already chose a basis of $\mathcal{O}^n \oplus \mathcal{O}^{m-n} \oplus \mathcal{O}(1)^{r-m} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$. If we change it in the following, we have to make an appropriate base change on $\mathcal{W}_1, \mathcal{W}_2$ and $\mathcal{W}_3$ such that we do not change the matrix describing β.
Since α is surjective, we may split off m copies of $\mathcal{O}$ from $\mathcal{O}^5$, such that the matrix $\tilde{A}$ associtated to α has the shape

$$\tilde{A} = \begin{array}{c} \left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ \hline & 0 & & A \\ & & & \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \ddots\\ 1\end{array}}\right\} m \\ \left.\vphantom{\begin{array}{c}0\\ A\end{array}}\right\} r-m \end{array} \\ \underbrace{\hphantom{1 \ \ddots \ 1}}_{m} \ \underbrace{\hphantom{A\ A}}_{5-m} \end{array}$$

Hence for classifying the maps α it suffices to look at surjective maps $f : \mathcal{O}^p \to \mathcal{O}(1)^q$ for $p \geq q$.

Now let $f : \mathcal{O}^p \to \mathcal{O}(1)^q$ be such a surjective map. We want to classify the possible matrices A representing f.
The kernel of f is a subsheaf of $\mathcal{O}^p$, so $\ker(f) = \bigoplus_{i=1}^{p-q} \mathcal{O}(-l_i)$ for some $l_i \geq 0$. Hence we have an exact sequence

$$0 \to \bigoplus_{i=1}^{p-q} \mathcal{O}(-l_i) \to \mathcal{O}^p \xrightarrow{A} \mathcal{O}(1)^q \to 0.$$

Now we dualize to obtain the exact sequence

$$0 \to \mathcal{O}(-1)^q \xrightarrow{A^t} \mathcal{O}^p \to \bigoplus_{i=1}^{p-q} \mathcal{O}(l_i) \to 0.$$

Taking the long exact cohomology sequence yields

$$\begin{aligned} 0 \to H^0(\mathcal{O}(-1)^q) \to H^0(\mathcal{O}^p) \to H^0(\bigoplus_{i=1}^{p-q} \mathcal{O}(l_i)) &\to H^1(\mathcal{O}(-1)^q) \\ \to H^1(\mathcal{O}^p) &\to H^1(\bigoplus_{i=1}^{p-q} \mathcal{O}(l_i)) \to \dots \end{aligned}$$

By Serre duality $H^1(\mathcal{O}^p) = 0$ and $H^1(\mathcal{O}(-1)^q) = 0$. Since furthermore

$$H^0(\mathcal{O}(-1)^q) = 0,$$

we have an isomorphism

$$\mathbb{C}^p \cong H^0(\mathcal{O}^p) \cong \bigoplus_{i=1}^{p-q} H^0(\mathcal{O}(l_i)) = \bigoplus_{i=1}^{p-q} \mathbb{C}^{l_i+1}.$$

After an appropriate change of the bases of $\mathcal{O}^{5-m}$ and $\mathcal{V}_3$, the isomorphism

$$H^0(\mathcal{O}^p) \to \bigoplus_{i=1}^{p-q} H^0(\mathcal{O}(l_i))$$

is given by a block matrix of the form

$$\begin{pmatrix} t^{l_1} & t^{l_1-1}\cdot(-s) & \dots & (-s)^{l_1} & & & & \\ & & & & \ddots & & & \\ & & & & & t^{l_{p-q}} & t^{l_{p-q}-1}\cdot(-s) & \dots & (-s)^{l_{p-q}} \end{pmatrix}$$

whose entries are sections $(-s)^i t^j$. In order to preserve the matrix representing β, when changing $\mathcal{V}_3$ we change the basis of $\mathcal{W}_3$ accordingly.
The morphism $\mathcal{O}^{l+1} \to \mathcal{O}(l)$ is given by multiplication with the sections

$$t^l, t^{l-1}\cdot(-s), \dots, (-s)^l.$$

So we can compute the kernel of the map $\mathcal{O}^p \to \bigoplus_{i=1}^{p-q} \mathcal{O}(l_i)$ by computing the syzygies of the maps (block by block) given by multiplication with the sections

$(t^l, t^{l-1}(-s), \dots, (-s)^l)$ and obtain

$$\begin{pmatrix} s & & & \\ t & s & & \\ & t & \ddots & \\ & & \ddots & s \\ & & & t \end{pmatrix}$$

in case $l > 0$. For $l = 0$ one obtains an empty block of size (1×0).
Therefore, after dualizing, we obtain $p - q$ blocks each one either of the form

$$\begin{pmatrix} s & t & & & \\ & s & t & & \\ & & \ddots & \ddots & \\ & & & s & t \end{pmatrix},$$

and of size $l_i \times (l_i + 1)$ or empty, if $l_i = 0$.

Classification of the maps γ: The map

$$\gamma : \mathcal{O}^n \oplus \mathcal{O}(1)^{m-n} \oplus \mathcal{O}(1)^{r-m} \to \mathcal{O}(1)^4$$

is injective, so its dual map

$$\gamma^* : \mathcal{O}(-1)^4 \to \mathcal{O}(-1)^{m-n} \oplus \mathcal{O}(-1)^{r-m} \oplus \mathcal{O}^n$$

is surjective. Hence we may use the same method of classification as for α. While constructing a matrix for β, we chose a basis of $\mathcal{O}^n \oplus \mathcal{O}(1)^{m-n} \oplus \mathcal{O}(1)^{r-m} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$. If we now change it, we have to modify the basis of $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{V}_3$ accordingly, such that we preserve the matrix for β. But then we also have to modify the basis of $\mathcal{O}^5$ to keep the matrix A. Thus γ^* is represented by a matrix of the form

$$\begin{array}{l} \left(\begin{array}{c|c} C & 0 \\ \hline 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right) \begin{array}{l} \left.\vphantom{C}\right\}{\scriptstyle 4-r+n} \\ \left.\vphantom{\begin{matrix}1\\ \ddots\\ 1\end{matrix}}\right\}{\scriptstyle r-n} \end{array} \\ \quad \underbrace{\qquad}_{n} \quad \underbrace{\qquad\quad}_{r-n} \end{array}$$

As in the classification of the map α the matrix, C consists of blocks

$$\begin{pmatrix} s & & & \\ t & s & & \\ & t & \ddots & \\ & & \ddots & s \\ & & & t \end{pmatrix}.$$

Altogether, with respect to the bases we chose above for

$$\tilde{A} = \left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ \hline & 0 & & A \end{array}\right),$$

$$\tilde{B} = \left(\begin{array}{ccc|c|ccc} 1 & & & & & & \\ & \ddots & & 0 & & 0 & \\ & & 1 & & & & \\ \hline & 0 & & J & & 0 & \\ \hline & & & & 1 & & \\ & 0 & & 0 & & \ddots & \\ & & & & & & 1 \end{array}\right)$$

and

$$\tilde{C} = \left(\begin{array}{c|ccc} C & & 0 & \\ \hline & 1 & & \\ 0 & & \ddots & \\ & & & 1 \end{array}\right),$$

the matrix Φ associated to φ is given by

$$\Phi = \tilde{C} \cdot \tilde{B} \cdot \tilde{A} = \left(\begin{array}{c|c|c} C & 0 & 0 \\ \hline 0 & J & 0 \\ \hline 0 & 0 & A \end{array} \right).$$

Lemma 2.18 *(a) If J has up to three pairwise distinct eigenvalues λ_1, λ_2 and λ_3, we may choose them to be $\lambda_1 = 0, \lambda_2 = \infty$ and $\lambda_3 = 1$.*

(b) If J has 4 pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_4$, we may choose the first three to be $\lambda_1 = 0, \lambda_2 = \infty$ and $\lambda_3 = 1$.

PROOF: To change the values of the λ_i we can use arbitrary automorphisms of $\mathbb{P}^1(\mathbb{C})$. Given any three pairwise distinct numbers λ_1, λ_2 and λ_3 in $\mathbb{C}$, there exists a $\rho \in \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ with $\rho(\lambda_1) = 0, \rho(\lambda_2) = \infty$ and $\rho(\lambda_3) = 1$. ∎

It remains to go through the list of all possible combinations of choices of α, β and γ and study its stability properties using Lemma 2.7. For this we refer to Appendix B. ∎

Notation 2.19 (a) In the following, we denote by P the union of the polystable strata P_1, P_2 and P_3, by S the union of S_1, S_2 and S_3 and by B the union of all other strata, i.e.

$$B_0, B_1, B_2, B_3^1, B_3^2, B_4^1, B_4^2, B_5^1, B_5^2, B_5^3, B_6, B_7, B_{10}.$$

(b) If A is a matrix that represents a Kronecker module φ in a stratum B_i resp. B_i^j resp. S_i resp. P_i, we will shortly write $A \in B_i$ resp. B_i^j resp. S_i resp. P_i.

Now we want to compute the dimension of the orbits in W^{ss} with respect to the action of G and of the strata in K with respect to the action of $\mathrm{PGL}(V)$. For this and also to check several other properties of the moduli space K, we use the Slice theorem of Luna ([18]):

Proposition 2.20 (Luna Slice Theorem) *G is a reductive algebraic group, which acts on an affine variety X algebraically. Let $q : X \to X /\!\!/ G$ be the associated quotient. Let $x \in X$ be a point with closed orbit $G.x$ and hence with reductive stabilizer G_x. Then there is a locally closed affine subvariety (slice) $S_x \subset X$ such that*

(a) $x \in S_x$,

(b) we have a cartesian diagram

$$\begin{tikzcd} X \arrow[r] & X /\!\!/ G \\ S_x \arrow[u] \arrow[r] & S_x /\!\!/ G_x \arrow[u, "\psi"'] \end{tikzcd}$$

where ψ is an étale morphism,

(c) S_x is stable under G_x.

Concretely, one can construct the Luna slice as follows ([18]): Assume that X is a vector space. Since the stabilizer G_x is a reductive group, one can compute a G_x-stable complement N_x of the tangent space $T_x(G.x)$ in x on the orbit $G.x$ in T_xX. Then the slice is an open G_x-stable neighborhood of x in the affine space $x + N_x$.

Using this one obtains the following Proposition:

Proposition 2.21 *(a) The stable points in W^{ss} with respect to the action of G have trivial stabilizer.*

(b) The stabilizer G_x of points $x \in P_1, P_2$ is $\mathbb{C}^*$, and for $x \in P_3$ the stabilizer is PGL_2.

(c) The orbits of the stable points in W^{ss} with respect to the action of G have dimension 19. The orbits of the strictly poly-stable Kronecker modules P_1 and P_2, have dimension 18 and the orbit P_3 has dimension 16.

(d) The orbits of the matrices of Theorem 2.13 in W^{ss} with respect to the joint action of $\mathrm{PGL}(V)$ and G have the following codimensions in W^{ss}:

  (i) $\mathrm{codim}(B_i^j) = i$ *for all* i, j *and* $\mathrm{codim}(B_i) = i$ *for all* i,

  (ii) $\mathrm{codim}(S_1) = 8$, $\mathrm{codim}(S_2) = \mathrm{codim}(S_3) = 9$,

  (iii) $\mathrm{codim}(P_1) = 10$, $\mathrm{codim}(P_2) = 12$, $\mathrm{codim}(P_3) = 18$.

PROOF: (a) Let $\varphi : F \to E \otimes V$ be a stable Kronecker module represented by a matrix A. Assume that $g = (G_1, G_2) \in G = (\mathrm{GL}_2 \times \mathrm{GL}_4)/\Gamma$ is contained in the stabilizer of φ. Then we obtain a non-zero morphism of Kronecker modules

$$\begin{tikzcd} F \arrow[r, "A"] \arrow[d, "G_1"] & E \otimes V \arrow[d, "G_2"] \\ F \arrow[r, "\tilde{A}"] & E \otimes V \end{tikzcd},$$

where $\tilde{A}$ also represents the Kronecker module φ. Since the Kronecker modules of type $(4,2)$ with fixed vector space V form an abelian category, we can use the Lemma of Schur for stable Kronecker modules ([28]): the homomorphism given by (G_1, G_2) is an isomorphism that is given by multiplication with a non-zero constant resp. its inverse. But this is exactly the action of Γ. Hence the stabilizer G_φ is trivial.

(b) This is a straightforward computation.

(c) This follows directly from part (a) and (b): for the orbit $G.x$ of a point x with respect to the action of G, we have $\dim G.x = \dim G - \dim G_x$.

(d) Use the following fact: Let G be a Lie group acting on a manifold M and $\mathfrak{g}$ the Lie algebra of G. Then the tangent space at $x \in M$ on the G-orbit of x is $x + \mathrm{Im}(\psi)$, where ψ is the infinitesimal action of $\mathfrak{g}$ on M. In particular, the dimension of the orbit of x equals the dimension of the tangent space at x on the G-orbit of x.

Since the joint action of $\mathrm{PGL}(V)$ and G on W^{ss} is the same as the joint action of $\mathrm{GL}(V)$ and G on W^{ss}, we will use the Lie algebra of $\mathrm{GL}(V)$ for our computations. So let $\mathfrak{g}_1, \mathfrak{g}_2$ and $\mathfrak{g}_3$ be the Lie algebras associated to $\mathrm{GL}_2(\mathbb{C})$, $\mathrm{GL}_4(\mathbb{C})$ and $\mathrm{GL}(V)$.

The infinitesimal action of

$$(g_1, g_2, g_3) \in \mathrm{Lie}(\mathrm{GL}_2 \times \mathrm{GL}_4 \times \mathrm{GL}(V)) = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

on a fixed Kronecker module represented by a matrix A is given by

$$\psi : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \to \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4 \otimes V)$$
$$(g_1, g_2, g_3) \mapsto -Ag_1 + g_2 A + \sum_{i,j,k} \left(\frac{\partial A_{i,j}}{\partial x_k} \sum_l (g_3)_{kl} x_l \cdot E_{i,j} \right),$$

where $E_{i,j}$ is the matrix with entry 1 at position (i,j) and 0 otherwise. For the concrete computations one can use the computer algebra system Singular. For the program see Appendix A.1. ∎

Remark 2.22 The lower index of B_i^j resp. the index of B_i indicates the codimension of the stratum B_i^j resp. B_i in W^{ss}.

Proposition 2.23

$$\operatorname{Sing}(W^{ss} /\!\!/ G) = P = P_1 \cup P_2 \cup P_3.$$

Proof: By Luna Slice Theorem, a point $x \in W^{ss} /\!\!/ G$ is singular if and only if x is a singular point of the quotient $S_x /\!\!/ G_x$ of the Luna slice of x on its stratum by the stabilizer of x, since $S_x /\!\!/ G_x \to W^{ss} /\!\!/ G$ is an étale morphism. The stabilizers of the stable points are trivial by Proposition 2.21(a). Thus the quotient $S_x /\!\!/ G_x$ of the slice is a vector space and $W^{ss} /\!\!/ G$ is non-singular in all stable points.

Moreover, the points in the poly-stable strata P_1, P_2 and P_3 are singular. For this, we use the Luna slice theorem again. In Section 3.4.1 we will construct a model for the quotient $S_x /\!\!/ G_x$ of the slice S_x of a point x in P_1. For all $x \in P_1$ the quotient $S_x /\!\!/ G_x$ has a singularity in x, and hence K is singular in the points of P_1. Since $P_2, P_3 \subset \overline{P_1}$, we see that K is also singular in P_2 and P_3. ∎

3 Stratification

Recall that if G is a connected algebraic group acting on a variety X, then each orbit $G.x$ is irreducible, open in its closure and its boundary $\overline{G.x} - G.x$ is a union of orbits of strictly smaller dimension. In particular, orbits of minimal dimension are closed.
In this Section we will determine for which strata A_1 and A_2 the intersection $\overline{A_1} \cap A_2 \neq \varnothing$, i.e. A_2 is contained in the closure of A_1. In this case will say that A_1 degenerates into A_2.

Lemma 2.24 *(a) B_0 degenerates into B_1.*

(b) B_1 degenerates into B_2.

(c) B_2 degenerates into B_3^1 and B_3^2.

(d) B_3^1 degenerates into B_4^1.

(e) B_3^2 degenerates into B_4^1 and B_4^2.

(f) B_4^1 degenerates into B_5^1, B_5^2 and B_5^3.

(g) B_4^2 degenerates into B_5^1.

(h) B_5^1 degenerates into B_6, S_2 and S_3.

(i) B_5^2 degenerates into B_6, S_1.

(j) B_5^3 degenerates into S_2, S_3, P_1 and B_{10}.

(k) B_6 degenerates into B_7.

(l) B_7 degenerates into B_{10} and S_3.

(m) B_{10} degenerates into P_2.

(n) S_1 degenerates into S_2.

(o) S_2 degenerates into P_2.

(p) S_3 degenerates into P_2.

(q) P_1 degenerates into P_2.

(r) P_2 degenerates into P_3.

PROOF: For the proof we refer to Appendix C. ∎

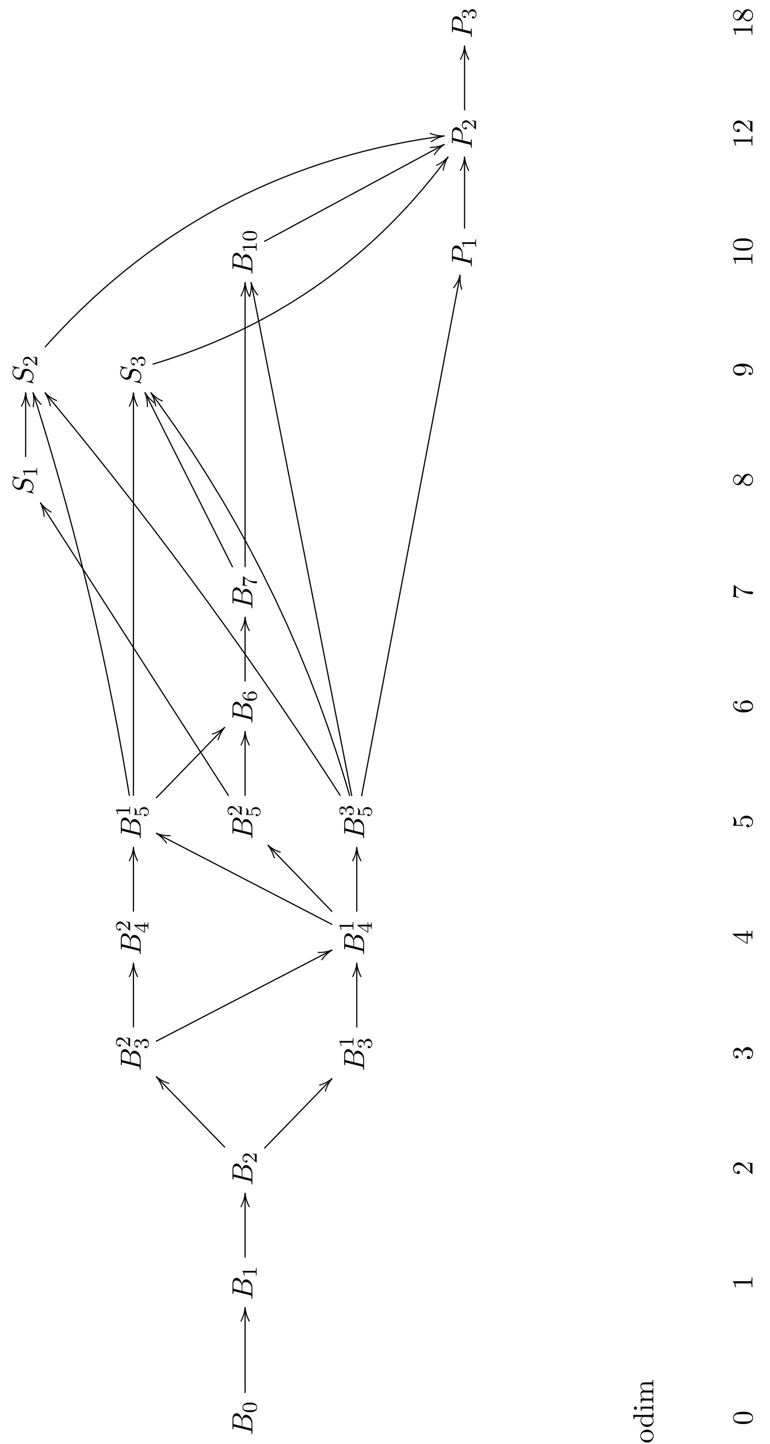


Figure 2.1: Degenerations of Kronecker modules of type $(4, 2)$

Figure 2.1 is a summary of the result of Lemma 2.24. An arrow between two strata indicates a degeneration.
There might be more degenerations, especially inside B. For the purpose of our study, the degenerations at hand suffice.
Later we also need to know, that some strata do not degenerate into some special other strata. Therefore we will prove the following lemma:

Lemma 2.25 *(a) The strata S_1, S_2 and S_3 do not degenerate into the stratum P_1.*

(b) The strata S_1, S_2, S_3 do not degenerate into B_{10}.

(c) The stratum S_1 does not degenerate into S_3.

(d) B_{10} does not degenerate into P_1.

(e) The strata B_5^1, B_6, B_7, B_5^3 do not degenerate into S_1.

Proof: The strategy to show that a stratum A_1 does not degenerate into a stratum A_2 is to construct a Luna slice S_x in an arbitrary point x of A_2 and to check that the intersection of S_x and the stratum A_1 is empty. Since all points in the stratum A_2 are projectively equivalent, it suffices to consider one particular point x in A_2. In the proof we always will work with the representative x from the list in Theorem 2.13.
To show that $S_x \cap A_1 = \varnothing$ we use rank considerations of the matrices obtained by choosing fixed parameters in the slice S_x. For that we view matrices of A_1 as (4×5)-matrices over $\mathbb{C}(s,t)$ (using the notation from Section 2.2).
We start by computing the rank of the representatives listed in Theorem 2.13 over $\mathbb{C}(s,t)$ for some strata (which is the rank of an arbitrary element of the stratum):

- P_1: $\begin{pmatrix} s+\lambda t & & & & 0\\ t & s+\lambda t & & & 0\\ & t & s+\lambda t & & 0\\ & & & s+\mu t & 0 \end{pmatrix}$ has rank 4.
- P_2: $\begin{pmatrix} s & & & & 0\\ t & & & & 0\\ & s+\lambda t & & & 0\\ & & & s+\mu t & 0 \end{pmatrix}$ has rank 3.
- P_3: $\begin{pmatrix} s & & 0 & 0 & 0\\ t & & 0 & 0 & 0\\ & s & 0 & 0 & 0\\ & t & 0 & 0 & 0 \end{pmatrix}$ has rank 2.

- B_{10}: $\begin{pmatrix} s & & & 0 & 0 \\ t & s & & 0 & 0 \\ & t & s & 0 & 0 \\ & & t & 0 & 0 \end{pmatrix}$ has rank 3.
- S_1: $\begin{pmatrix} s & & & & \\ t & & & & \\ & s & t & & \\ & & & s & t \end{pmatrix}$ has rank 3.
- S_2: $\begin{pmatrix} s & & & & 0 \\ t & & & & 0 \\ & s & t & & 0 \\ & & s & t & 0 \end{pmatrix}$ has rank 3.
- S_3: $\begin{pmatrix} s & & & & 0 \\ t & s & & & 0 \\ & t & & & 0 \\ & & s & t & 0 \end{pmatrix}$ has rank 3.

Using this we prove the assertions of the Lemma:

(a) The Luna slice at the point $x = \begin{pmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & x_3 \end{pmatrix}^t \in P_1$ is

$$S_x(t_1, \dots, t_{10}) = \begin{pmatrix} x_0 & t_1x_1 + t_2x_4 - t_4x_0 \\ x_1 & t_3x_0 + t_4x_1 + t_5x_4 \\ -t_9x_2 + t_6x_3 + t_7x_4 & x_2 \\ t_9x_3 - t_8x_2 + t_{10}x_4 & x_3 \end{pmatrix}$$

for parameters $t_1, \dots, t_{10} \in \mathbb{C}$ (computed with the Singular program in Appendix A 1).

Hence for fixed parameters $t_1, \dots, t_{10}$ the rank as a (4×5)-matrix with entries in $\mathbb{C}(s, t)$ is

$$\operatorname{rank}(S_x(t_1, \dots, t_{10})) = \operatorname{rank} \begin{pmatrix} s - t_4t & t_1t & 0 & 0 & t_2t \\ t_3t & s + t_4t & 0 & 0 & t_5t \\ 0 & 0 & t - t_9s & t_6s & t_7s \\ 0 & 0 & -t_8s & t + t_9s & t_{10}s \end{pmatrix}.$$

So we see that for all choices of parameters $t_1, \dots, t_{10}$ the matrix $S_x(t_1, \dots, t_{10})$ has rank 4. Since the representatives of S_1, S_2 and S_3 have rank 3, they are not contained in $S_x(t_1, \dots, t_{10})$, and so the strata S_1, S_2 and S_3 do not degenerate into P_1.

(b) For a fixed choice of $t_1, \dots, t_{10}$, the matrix defined by the slice $S_x(t_1, \dots, t_{10})$ at the point

$$x = \begin{pmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & x_1 & x_2 \end{pmatrix} \in B_{10}$$

is

$$S_x(t_1, \dots t_{10}) = \begin{pmatrix} x_0 - t_3x_3 - t_4x_4 & t_1x_3 + t_2x_4 \\ x_1 - t_5x_3 - t_6x_4 & x_0 + t_3x_3 + t_4x_4 \\ x_2 - t_9x_3 - t_{10}x_4 & x_1 + t_5x_3 + t_6x_4 \\ t_7x_3 + t_8x_4 & x_2 + t_9x_3 + t_{10}x_4 \end{pmatrix}.$$

Now we compute the rank of $S_x(t_1, \dots t_{10})$ in coordinates of $\mathbb{C}(s,t)$:

$$\text{rank} \begin{pmatrix} s & 0 & 0 & -t_3s + t_1t & -t_4s + t_2t \\ t & s & 0 & -t_5s + t_3t & -t_6s + t_4t \\ 0 & t & s & -t_9s + t_5t & -t_{10}s + t_6t \\ 0 & 0 & t & t_7s + t_9t & t_8s + t_{10}t \end{pmatrix} =$$

$$\text{rank} \begin{pmatrix} s & 0 & 0 & * & * \\ 0 & s & 0 & * & * \\ 0 & 0 & s & * & * \\ 0 & 0 & 0 & a_{45} & a_{55} \end{pmatrix}$$

where

$$a_{45} = t_7s + 2t_9t - 2t_5\frac{t^2}{s} + 2t_3\frac{t^3}{s^2} - t_1\frac{t^4}{s^3}$$

and

$$a_{55} = t_8s + 2t_{10}t - 2t_6\frac{t^2}{s} + 2t_4\frac{t^3}{s^2} - t_2\frac{t^4}{s^3}.$$

The latter matrix is obtained from the first one by row and column transformations in $\mathbb{C}(s,t)$.
Hence $\text{rank}(S_x(t_1, \dots, t_{10})) = 3$ if and only if $a_{45} = a_{55} = 0$. This is the case if and only if $t_1 = \dots = t_{10} = 0$, i.e. for the point $x \in B_{10}$. Since any point of the slice (except x) is a matrix of rank strictly bigger than 3 and all representatives of the strata S_1, S_2 and S_3 have rank 3, the intersection of the slice of B_{10} and S_1, S_2 resp. S_3 has to be empty. So S_1, S_2 and S_3 do not degenerate into B_{10}.

(c) We compute the slice of the point $x = \begin{pmatrix} x_0 & x_1 & 0 & x_2 \\ 0 & x_0 & x_1 & x_3 \end{pmatrix} \in S_3$ and show that it does not contain any matrix of rank 3 (again except at x).

The slice in x is

$$S_x(t_1, \dots, t_9) = \begin{pmatrix} s & 0 & -t_3s + t_1t & -t_7s + t_3t & -t_4s + t_2t \\ t & s & -t_7s + t_3t & -t_8s + t_7t & -t_9s + t_4t \\ 0 & t & -t_8s + t_7t & t_5s + t_8t & t_6s + t_9t \\ 0 & 0 & s & t & 0 \end{pmatrix}.$$

Hence for a fixed choice of the parameters $t_1, \dots, t_{10}$ we compute for the rank of $S_x(t_1, \dots, t_9)$:

$$\operatorname{rank}\begin{pmatrix} s & 0 & -t_3 s + t_1 t & -t_7 s + t_3 t & -t_4 s + t_2 t \\ t & s & -t_7 s + t_3 t & -t_8 s + t_7 t & -t_9 s + t_4 t \\ 0 & t & -t_8 s + t_7 t & t_5 s + t_8 t & t_6 s + t_9 t \\ 0 & 0 & s & t & 0 \end{pmatrix} =$$

$$\operatorname{rank}\begin{pmatrix} s & 0 & t_1 t & 2t_3 t & t_2 t \\ 0 & t & t_7 t & t_5 s + 3t_8 t & t_6 s + 2t_9 t \\ 0 & 0 & s & t & 0 \\ t & s & t_3 t & 3t_7 t & 2t_4 t \end{pmatrix}.$$

So $\operatorname{rank}(S_x(t_1, \dots, t_9)) \in \{3, 4\}$. If $\operatorname{rank}(S_x(t_1 \dots, t_9)) = 3$, the fourth row l_4 is a linear combination of the first, second and third row l_1, l_2, l_3 i.e.

$$l_4 = c_1 l_1 + c_2 l_2 + c_3 l_3,$$

where $c_i \in \mathbb{C}(s,t)$. By considering the entries in the first and second column we have $c_1 = \frac{t}{s}$ and $c_2 = \frac{s}{t}$. Considering the fifth column with this coefficients c_1 and c_2 yields $t_2 = t_4 = t_6 = t_9 = 0$. Similarly using the third and fourth column we see that $t_1 = t_3 = t_5 = t_7 = t_8 = 0$.
So

$$\operatorname{rank}(S_x(t_1, \dots t_9)) = \begin{cases} 3 & \text{if } t_1 = \dots = t_9 = 0 \\ 4 & \text{else.} \end{cases}$$

This concludes assertion (c) with the same arguments as before.

(d) This is clear, since $\dim B_{10} = \dim P_1$ and this contradicts the fact that the boundary of a stratum consists of strata of strictly smaller dimension.

(e) The matrix of a representative of S_1 contains all 5 variables $x_0, \dots, x_4$, but the matrices in the orbits of B_5^1, B_6, B_7, B_5^3 just contain 4 of these variables. ■

In particular, the following important observation follows from the previous results:

Corollary 2.26 *(a) The intersection of $\overline{S}$ and $\overline{P} = P$ is*

$$\overline{S} \cap \overline{P} = \overline{S} \cap P = \overline{P_2} = P_2 \cup P_3.$$

(b) All strata in B degenerate into B_{10}.

PROOF: (a) Since S_1 degenerates exactly into the strata S_2, P_2 and P_3 and S_3 degenerated exactly into the strata P_2 and P_3, we have $\overline{S} = S_1 \cup S_2 \cup S_3 \cup P_2 \cup P_3$. Hence the assertion is obvious.

(b) This follows directly from Lemma 2.24. ∎

3

Relation of K and the Hilbert Scheme $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$

In this Chapter we study a geometric interpretation of the Kronecker modules belonging to the moduli space K. Using this interpretation, we construct a rational map

$$\Phi_{KH} : K \dashrightarrow \mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$$

into the Hilbert scheme of curves in $\mathbb{P}^4$ with Hilbert polynomial $4n+1$. In Sections 3.3 and 3.4 we study the strata of Kronecker modules in K inside the set of indeterminacies of φ and discuss extensions of φ to appropriate blow-ups of K.

1 The Hilbert Scheme $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$

A rational normal curve of degree 4 is a smooth curve in $\mathbb{P}^4$ that is projectively equivalent to the image of $\mathbb{P}^1$ under the Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^4$ of degree 4. The Hilbert polynomial of the generic rational normal curve of degree 4 in $\mathbb{P}^4$ is $P(n) = 4n + 1$.

The vanishing of the maximal minors of the representative $\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}^t$ of a Kronecker module in B_0, i.e. the equations

$$x_0x_2 - x_1^2,\ x_0x_3 - x_1x_2,\ x_0x_4 - x_1x_3,\ x_1x_3 - x_2^2,\ x_1x_4 - x_2x_3,\ x_2x_4 - x_3^2$$

define a rational normal curve of degree 4 in $\mathbb{P}^4$.
Hence in the following we consider the Hilbert scheme $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$.
There is a component H_0 of $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$ that contains the rational normal curves of degree 4. H_0 has dimension 21:
any such curve is determined by the choice of an embedding, so

$$\dim H_0 = \dim \mathrm{GL}_5/\mathrm{GL}_2 = \dim \mathrm{GL}_5 - \dim \mathrm{GL}_2 = 5^2 - 2^2 = 21.$$

The rational normal curve C is arithmetically Cohen-Macaulay, i.e. its homogeneous coordinate ring $k[x_0, \dots, x_4]/I_C$ is Cohen-Macaulay. The subset of arithmetically Cohen-Macaulay curves in $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$ is open. In [22], Martin-Deschamps and Piene describe the structures of certain curves of this type in $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$:

Theorem 3.1 (Martin-Deschamps, Piene) *The open subset of arithmetically Cohen-Macaulay curves in* $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$ *contains the curves of the following four types:*

(a) *a rational normal curve of degree* 4 *in* $\mathbb{P}^4$,

(b) *the union of two smooth conics intersecting in one point and not contained in a hyperplane,*

(c) *a double structure on a conic, given by an ideal*

$$I_C = (x_0, x_1) \cdot (x_2, x_3) + (Q, Q'),$$

where Q *and* Q' *are quadric forms such that* $Q \in (x_2, x_3)$, $Q \notin (x_0, x_1)$ *and* $Q' \in (x_0, x_1), Q' \notin (x_2, x_3)$,

(d) *a curve given by an ideal of the form*

$$I_C = (LL_1, LL_2, LL_3, Q_1, Q_2, Q_3),$$

where L_1, L_2, L_3 *resp.* Q_1, Q_2, Q_3 *are independent linear resp. quadratic forms with* $(Q_1, Q_2, Q_3) \subset (L_1, L_2, L_3)$, *furthermore* L *is a linear form such that* (L, Q_1, Q_2, Q_3) *defines a curve* C' *of degree* 3 *and genus* 0 *in the hyperplane given by* $L = 0$ *and* $((Q_1, Q_2, Q_3) : L) \subset (L_1, L_2, L_3)$.

Additionally, Pirio and Russo found in [27] another arithmetically Cohen-Macaulay curve C in $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$ defined by the equations

$$x_0^2, x_0x_1, x_0x_2, x_1^2 - x_0x_3, 2x_1x_2 - x_0x_4, x_2^2.$$

But it is not known whether this curve is a degeneration of the rational normal curve in $\mathbb{P}^4$.

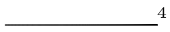
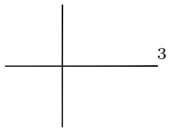
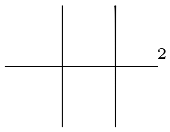
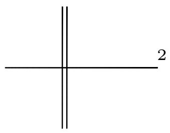
The Hilbert scheme $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$ has more than two components. For example, there is a component of dimension 21 containing elliptic curves of degree 3 in $\mathbb{P}^2$ union a disjoint line, and a component of dimension 32 containing plane smooth curves of degree 4 union 3 points and so on.

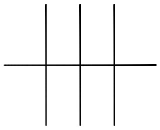
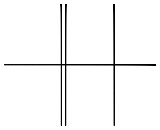
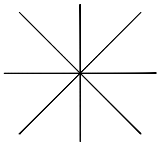
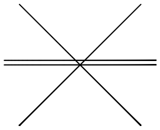
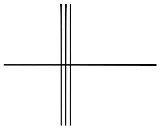
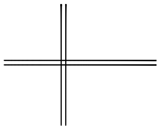
2 Construction of a Rational Map $\Phi_{KH} : K \dashrightarrow H_0$

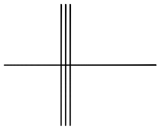
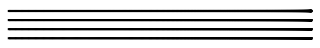
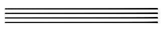
The aim of this Section is to define a rational map from the moduli space K of Kronecker modules of type $(4, 2)$ into the component H_0 of the Hilbert Scheme $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$.

For any Kronecker module represented by a matrix A, the maximal minors define a closed subscheme in $\mathbb{P}^4$. Clearly any two matrices in the same orbit with respect to the action of G define the same closed subscheme.
The action of $\mathrm{PGL}(V)$ transfers a curve C into some projectively equivalent curve C' and therefore it suffices to consider one curve in each stratum.

Concretely, ideals generated by the maximal minors of the representatives listed in Theorem 2.13 define the following ideals resp. curves. For the non-reduced curves we compute a primary decomposition with SINGULAR.

$I(B_0) = (x_0x_2 - x_1^2,\ x_0x_3 - x_1x_2,\ x_0x_4 - x_1x_3,\ x_1x_3 - x_2^2,\ x_1x_4 - x_2x_3,\ x_2x_4 - x_3^2)$ defines a rational normal curve of degree 4 in $\mathbb{P}^4$.	
$I(B_1) = (x_0x_2,\ x_0x_3,\ x_0x_4,\ x_1x_3 - x_2^2,\ x_1x_4 - x_2x_3,\ x_2x_4 - x_3^2)$ defines a twisted cubic $C_1 = \{x_1x_3 - x_2^2 = x_1x_4 - x_2x_3 = x_2x_4 - x_3^2 = 0\}$ in the $\mathbb{P}^3$ defined by $x_0 = 0$ and a line $C_2 = \{x_2 = x_3 = x_4 = 0\}$. The curves C_1 and C_2 intersect in exactly one point.	
$I(B_2) = (x_0x_1,\ x_0x_3,\ x_0x_4,\ x_1x_2,\ x_1x_3,\ x_2x_4 - x_3^2)$ defines a plane quadric $q = \{x_0 = x_1 = x_2x_4 - x_3^2 = 0\}$ and two lines $l_1 = \{x_0 = x_2 = x_3 = 0\}$ and $l_2 = \{x_1 = x_3 = x_4 = 0\}$. q intersects l_1 and l_2 each in one point, $l_1 \cap l_2 = \varnothing$.	
$I(B_3^1) = (x_0^2,\ x_0x_3, x_0x_4,\ x_1x_3 - x_0x_2,\ x_1x_4 - x_0x_3,\ x_2x_4 - x_3^2)$ defines a quadric $Q = \{x_0 = x_1 = x_3^2 - x_2x_4 = 0\}$ and a double line l, that intersect Q in one point. $l_{\text{red}} = \{x_0 = x_3 = x_4 = 0\}$ and the double structure of l is given by $x_0^2 = x_0x_2 - x_1x_3 = x_0x_3 = x_3^2 = x_4 = 0$.	

$I(B_3^2) = (x_0x_1,\ x_0x_2,\ x_0x_4, x_1x_2,\ x_1x_3,\ x_2\cdot(x_4-x_3))$, defines four lines $l_1 = \{x_0 = x_2 = x_3 = 0\}$, $l_2 = \{x_0 = x_1 = x_2 = 0\}$, $l_3 = \{x_0 = x_1 = x_3 - x_4 = 0\}$ and $l_4 = \{x_1 = x_2 = x_4 = 0\}$. The lines l_1, l_2 and l_3 are pairwise disjoint, but l_1 and l_4, resp. l_2 and l_4, resp. l_3 and l_4 intersect each in one point.	
$I(B_4^1) = (x_0^2,\ x_0x_2,\ x_0x_4,\ x_1x_2,\ x_1x_4 - x_0x_3,\ x_2x_3)$ defines two reduced lines $l_1 = \{x_0 = x_1 = x_2 = 0\}$ and $l_2 = \{x_0 = x_1 = x_3 = 0\}$ and one non-reduced line l_3. The reduced line is $(l_3)_{\mathrm{red}} = \{x_0 = x_2 = x_4 = 0\}$ and the non-reduced structure is defined by $x_0^2, x_0x_3 - x_1x_4, x_0x_4, x_2, x_4^2$. The lines l_1 and l_2 resp. l_1 and l_3 intersect in one point and $l_2 \cap l_3 = \varnothing$.	
$I(B_4^2) = (x_0x_1,\ x_0x_2,\ \lambda x_0x_3,\ x_1x_2,\ x_1x_3,\ (1-\lambda)x_2x_3)$. This ideal defines a family of curves for $\lambda \neq 0, 1, \infty$. But the curve is independent on the choice of λ. It consists of four lines intersecting in one point with the equations $l_1 = \{x_0 = x_1 = x_2 = 0\}$, $l_2 = \{x_0 = x_1 = x_3 = 0\}$, $l_3 = \{x_0 = x_2 = x_3 = 0\}$ and $l_4 = \{x_1 = x_2 = x_3 = 0\}$.	
$I(B_5^1) = (x_0x_1,\ x_0x_2,\ x_0x_3,\ x_1x_2,\ x_1x_3 - x_1x_2,\ x_2^2)$ defines two reduced lines $l_1 = \{x_0 = x_2 = x_3 = 0\}$ and $l_2 = \{x_1 = x_2 = x_3 = 0\}$ and a double line l_3, where $(l_3)_{\mathrm{red}} = \{x_0 = x_1 = x_2 = 0\}$ and the double structure is given by $x_2^2 = x_0 = x_1 = 0$. The three lines intersect in exactly one point.	
$I(B_5^2) = (x_0^2,\ x_0x_1,\ x_0x_4,\ x_1^2 - x_0x_2,\ x_1x_4 - x_0x_3,\ x_2x_4 - x_1x_3)$ defines a reduced line $l_1 = \{x_0 = x_1 = x_2 = 0\}$ and a triple line l_2. The reduced line is $(l_2)_{\mathrm{red}} = \{x_0 = x_1 = x_4 = 0\}$ and the triple structure is defined by $x_0^2 = x_0x_1 = x_1^2 - x_0x_2 = x_0x_3 - x_1x_4 = x_0x_4 = x_1^3 = x_1^2x_4 = x_1x_3 - x_2x_4 = x_1x_4^2 = x_4^3 = 0$. The lines intersect in one point.	
$I(B_5^3) = (x_0^2,\ x_0x_2,\ x_0x_3,\ x_1x_2,\ x_1x_3 - x_0x_2,\ x_2^2)$ defines two non-reduced lines l_1 and l_2 intersecting in one point. Concretely, $(l_1)_{\mathrm{red}} = \{x_0 = x_1 = x_2 = 0\}$ with a non-reduced structure given by x_0, x_1, x_2^2 and $(l_2)_{red} = \{x_0 = x_2 = x_3 = 0\}$ with a non-reduced structure given by $x_0^2,\ x_2,\ x_3$.	

$I(B_6) = (x_0^2, x_0x_1, x_0x_3, x_0x_2 - x_1^2, x_1x_3, x_2x_3)$ defines a reduced line $l_1 = \{x_0 = x_1 = x_2 = 0\}$ and a triple line l_2. The reduced line is $(l_2)_{\text{red}} = \{x_0 = x_1 = x_3 = 0\}$ with the triple structure defined by x_0^2, x_0x_1, $x_1^2 - x_0x_2$, x_1^3, x_3. The two lines intersect in one point.	
$I(B_7) = (x_0^2, x_0x_1, x_0x_2, x_0x_2 - x_1^2, x_1x_2 - x_0x_3, x_1x_3 - x_2^2)$ defines a non-reduced line l. The reduced line is $l_{\text{red}} = \{x_0 = x_1 = x_2 = 0\}$ with a non-reduced structure given by $x_0^2 = x_0x_1 = x_0x_2 = x_1x_2 - x_0x_3 = x_1^2 = x_1x_2^2 = x_2^2 - x_1x_3 = x_2^4 = 0$.	
$I(B_{10}) = (x_0^2, x_0x_1, x_0x_2, x_0x_2 - x_1^2, x_1x_2, x_2^2)$ defines a non-reduced line l. Here $l_{\text{red}} = \{x_0 = x_1 = x_2 = 0\}$ and the non-reduced structure is the first infinitesimal neighborhood $x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2$.	

Lemma 3.2 *The family of curves in $\mathbb{P}^4$ defined by the maximal minors of the representatives of the strata $B_0, B_1, B_2, B_3^1, B_3^2, B_4^1, B_4^2, B_5^1, B_5^2, B_5^3, B_6, B_7, B_{10}$ is flat over $W^s \backslash S = B$. In particular, all these curves have Hilbert polynomial $P(n) = 4n + 1$.*

Proof: The curves defined by the maximal minors of matrices representing Kronecker modules in B define a family $f : \mathcal{C} \to B$.

Because of the open nature of flatness ([12][III ex. 9.4]) there is an open subset $U \subset B$, over which this family is flat. All curves over one stratum are projectively equivalent and therefore they have the same Hilbert polynomial. By [12][III Thm 9.9] this means that if one curve is contained in the flat locus U, then the whole stratum is contained in the flat locus. The open set U is non-empty, since the family of curves over the Zariski-open dense subset $B_0 \subset B$ is flat.

If we have shown that the stratum B_{10} is contained in the flat locus of f, then the other strata $B_i \subset B$ (resp. $B_i^j \subset B$) are also contained in the flat locus:

By Corollary 2.26 all strata $B_i \subset B$ degenerate into the stratum B_{10}, hence there exists a family of Kronecker modules A_t in B_i (resp. B_i^j) such that

$$\lim_{t \to 0} A_t = A_0 \in B_{10}.$$

This means, if $B_{10} \subset U$, then the matrices A_t in some neighborhood of A_0 are also contained in U and hence $B_i \subset U$ (resp. $B_i^j \subset U$).

So it remains to show that B_{10} is contained in U. To do so, one considers the family of curves defined by the maximal minor of the family

$$A_t := \begin{pmatrix} x_1 & x_0 & x_3 & tx_2 \\ tx_2 & x_1 & tx_4 + (1-t)x_0 & x_3 \end{pmatrix}^t .$$

After some column operations, one sees that $A_t \in B_0$ for $t \neq 0$ and $A_0 \in B_{10}$. This family is contained in U, since $B_0 \subset U$ and the Hilbert polynomial of the family of curves is constant, as shown below.
With

$$I_C = (x_1^2, x_0x_1, x_1x_3, x_0^2 - x_1x_3, x_0x_3, x_3^2)$$

we denote the ideal generated by the (2×2)-minors of A_0. For A_0 we have for large n (where $(k[x_0, \ldots, x_4]/I_C)^{(n)}$ denotes the monomials of degree n in $k[x_0, \ldots, x_4]/I_C$)

$$\begin{aligned} P(n) &= \dim_k (k[x_0, \ldots, x_4]/I_C)^{(n)} \\ &= \dim_k (k[x_0, \ldots, x_4]/(x_0^2, x_0x_1, x_0x_3, x_1^2, x_1x_3, x_3^2))^{(n)} \\ &= \dim_k (k[x_2, x_4])^{(n)} + 3\dim_k (k[x_2, x_4])^{(n-1)} \\ &= (n+1) + 3n \\ &= 4n+1, \end{aligned}$$

which coincides with the Hilbert polynomial for the rational normal curves defined by A_t for $t \neq 0$. ■

Since the family of ideal sheaves of curves associated to Kronecker modules in the strata in $W^s \backslash S$ is flat, this defines a morphism

$$\tilde{\Phi}_{KH}|_{W^s \backslash S} : W^s \backslash S \to H_0 \subset \mathrm{Hilb}^{4n+1}(\mathbb{P}^4).$$

Since the morphism is G-invariant, it descends to a morphism

$$\Phi_{KH}|_{B /\!\!/ G} : B /\!\!/ G = (W^s \backslash S) /\!\!/ G \to H_0.$$

Remark 3.3 It is not possible to extend this rational map $\Phi_{KH} : K \dashrightarrow H_0$ to a morphism $K \to H_0$. The map Φ_{KH} is not defined on S and P.
The closed subscheme defined by the vanishing of maximal minors of representatives of the strata $S_1, S_2, S_3, P_1, P_2, P_3$ are closed subscheme of dimension at least 2:

(a) $I(S_1) = (x_0^2, x_0x_2, x_0x_4, x_0x_1, x_0x_3, x_1x_4 - x_2x_3)$ is a quadric $x_1x_4 - x_2x_3 = 0$ in the $\mathbb{P}^3$ given by $x_0 = 0$.

(b) $I(S_2) = (x_0^2, x_0x_2, x_0x_3, x_0x_1, x_1x_3 - x_2^2)$ is a cone over a plane quadric $x_1x_3 - x_2^2$ with an embedded point in the origin $x_0 = x_1 = x_2 = x_3 = 0$.

(c) $I(S_3) = (x_0^2, x_0x_1, x_0x_3, x_1^2, x_1x_3 - x_0x_2, x_1x_2)$ defines a plane $x_0 = x_1 = 0$ with an embedded point.

(d) $I(P_1) = (x_0x_2, x_0x_3, x_1x_2, x_1x_3)$ defines two planes $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$ in $\mathbb{P}^4$ that intersect exactly in the point $(0 : 0 : 0 : 0 : 1)$.

(e) $I(P_2) = (x_0^2, x_0x_2, x_1x_0, x_1x_2)$ defines two planes $x_0 = x_1 = 0$ and $x_0 = x_2 = 0$ in $\mathbb{P}^4$ that intersect in a line with a non-reduced structure in this line.

(f) $I(P_3) = (x_0^2, x_0x_1, x_1^2)$ defines a plane $x_0 = x_1 = 0$ in $\mathbb{P}^4$ that is non-reduced everywhere.

Now we want to see explicitly, that one can not extend Φ_{KH} to S and P. As an example we consider the stratum S_1. For the others it works analogously. We consider the situation locally. The slice in $x = \begin{pmatrix} x_0 & 0 & x_1 & x_3 \\ 0 & x_0 & x_2 & x_4 \end{pmatrix}^t$ at the stratum S_1 is

$$S_x(t_1, \ldots, t_8) =$$

$$\begin{pmatrix} x_0 - t_5x_1 - t_6x_2 - t_7x_3 - t_8x_4 & t_1x_1 + t_2x_3 + t_5x_2 + t_7x_4 \\ t_3x_2 + t_4x_4 - t_6x_1 - t_8x_3 & x_0 + t_5x_1 + t_6x_2 + t_7x_3 + t_8x_4 \\ x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

for parameters $t_1, \ldots, t_8 \in \mathbb{C}$.
We choose some fixed point $P := (p_1, \ldots, p_8)$ on the slice $S_x(t_1, \ldots, t_8)$. Above each point on the line $L \subseteq S_x(t_1, \ldots, t_8)$ connecting P with the "origin" x (which corresponds to the choice $t_i = 0$ for all i) lies a unique matrix $S_x^P(t)$ given by

$$\begin{pmatrix} x_0 + t \cdot (-p_5x_1 - p_6x_2 - p_7x_3 - p_8x_4) & t \cdot (p_1x_1 + p_2x_3 + p_5x_2 + p_7x_4) \\ t \cdot (p_3x_2 + p_4x_4 - p_6x_1 - p_8x_3) & x_0 + t \cdot (p_5x_1 + p_6x_2 + p_7x_3 + p_8x_4) \\ x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

Obviously, the family of closed subschemes defined by the vanishing of the maximal minors of these matrices is not flat, as $x \in S_1$ does not define a curve. But by construction of the slice, this family restricted to $L \backslash \{0\}$ is flat since it consists of curves with Hilbert polynomial $4n + 1$ by Lemma 3.2.
The saturation of the ideal $I = \Lambda^2(S_x^P(t))$ by the ideal (t) is defined as the union of the ideal quotients $(I : t^n)$, see [30][C1]. This saturation gives the projective closure of the family of curves over $L \backslash \{0\}$. But the choice of different lines (i.e. different fixed points P on the slice) gives different limits.

For example for $p_1 = 1$ and $p_i = 0$ otherwise, one obtains by saturation the additional equations $x_1x_3^2 = 0$, $x_1^2x_3 = 0$ and $x_1^3 = 0$ for the closed subscheme associated to the point x. The reduced structure for the curve consists of two lines intersecting in one point. But for $p_2 = 1$ and $p_i = 0$ otherwise, one obtains the additional equations $x_1x_3^2 = 0$, $x_1^2x_3 = 0$ and $x_1^3x_4 = 0$. The reduced structure of the associated curve in this case consists of three lines not all intersecting in one point. For the concrete computation we used Singular (see Appendix A 2).
Hence it is not possible to extend the rational map Φ_{KH} to the stratum S_1.

3 Extension to the Blow-up of K in S

Let $K^s := W^s /\!\!/ G$ be the moduli space of stable Kronecker modules of type $(4, 2)$ and moreover let $\mathcal{C}$ be the family of closed subschemes in $\mathbb{P}^4$ defined by the maximal minors of representatives of Kronecker modules in K^s. We have seen in Section 3.2, that restricted to $W^s \backslash S$ we obtain a flat family $\mathcal{C}$ of curves. Our aim is to prove the following:

Proposition 3.4 *One can extend $\mathcal{C}$ to a flat family $\mathcal{C}'$ of curves on* $\mathrm{BL}_S(K^s)$ *(such that the families $\mathcal{C}$ and $\mathcal{C}'$ coincide outside the exceptional divisor).*

For the proof we consider the strata S_1, S_2 and S_3 separately.

3.1 The Stratum S_1

We start with the stratum S_1.

Proof of Proposition 3.4 for S_1: We examine the situation locally.
By construction, it suffices to choose a fixed matrix in the $\mathrm{PGL}(V)$-orbit S_1. For the other matrices in this orbit the construction can be done simultaneously. The slice in the representative (of the G-action)

$$x = \begin{pmatrix} x_0 & 0 & x_1 & x_3 \\ 0 & x_0 & x_2 & x_4 \end{pmatrix}^t \in S_1$$

on the stratum S_1 is

$$S_x(t_1, \ldots, t_8) =$$
$$\begin{pmatrix} x_0 - t_5x_1 - t_6x_2 - t_7x_3 - t_8x_4 & t_1x_1 + t_2x_3 + t_5x_2 + t_7x_4 \\ t_3x_2 + t_4x_4 - t_6x_1 - t_8x_3 & x_0 + t_5x_1 + t_6x_2 + t_7x_3 + t_8x_4 \\ x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

Hence the slice is isomorphic to $\mathbb{A}^8$ with coordinates $t_1, \ldots, t_8$. By $C(t_1, \ldots, t_8)$ we denote the curve defined by the vanishing of the maximal minors of the matrices $S_x(t_1, \ldots, t_8)$ for fixed parameters $t_1, \ldots, t_8$.
Let $\mathcal{I}$ be the family of ideal sheaves on $\mathbb{P}^4 \times \mathbb{A}^8$ whose fiber over a point $(t_1, \ldots, t_8)$ in the parameter space $\mathbb{A}^8$ is given by the ideal sheaf of the curve $C(t_1, \ldots, t_8)$ in $\mathbb{P}^4$. The family is not flat, since the ideal associated to the point $(0, \ldots, 0) \in \mathbb{A}^8$ defines a 2-dimensional closed subscheme in $\mathbb{P}^4$ by Remark 3.3. Hence we want to extend the family $C(t_1, \ldots, t_8)$ of curves to a flat family of curves over the blow-up of $\mathbb{A}^8$ in the point 0. For that, recall that the blow-up $\mathrm{BL}_0(\mathbb{A}^8)$ of $\mathbb{A}^8$ with coordinates $t_1, \ldots, t_8$ in 0 is defined as the subvariety

of $\mathbb{A}^8 \times \mathbb{P}^7$ given by the equations $s_i t_j - s_j t_i = 0$ for $i, j \in \{1, \dots, 8\}$, where $s_1, \dots, s_8$ are the coordinates of $\mathbb{P}^7$.
Now we define a new family $\mathcal{J}$ of ideal sheaves on $\mathrm{BL}_0(\mathbb{A}^8) \times \mathbb{P}^4$ as follows. Let $\pi : \mathbb{P}^4 \times \mathrm{BL}_0(\mathbb{A}^8) \to \mathbb{A}^8$ be the composition of the projection on the second component with the map $\mathrm{BL}_0(\mathbb{A}^8) \to \mathbb{A}^8$. Moreover, denote by $f_1, \dots, f_6$ the maximal minors of $S_x(t_1, \dots, t_8)$ in $\mathbb{A}^8 \times \mathbb{P}^4$. Then $\mathcal{J}$ is by definition generated by $\pi^* f_1, \dots, \pi^* f_6$ and $s_i t_j - s_j t_i$ for all $i, j \in \{1, \dots, 8\}$.
Clearly, for $s_1 = \dots = s_8 = 0$, i.e. outside of the exceptional divisor, the stalks of the sheaves $\mathcal{I}$ and $\mathcal{J}$ coincide. One can easily see that the family $\mathcal{J}$ is not flat, since all fibers over the exceptional divisor define closed subschemes given by the equations $x_0 = x_1 x_4 - x_2 x_3 = 0$ in $\mathbb{P}^4$, hence 2-dimensional subschemes. However by simultaneous saturation ([30][C1]) of $\tilde{\mathcal{J}}$ by the ideal $T = (t_1, \dots, t_8)$ one obtains a flat family $\tilde{\mathcal{J}}$ of curves over $\mathrm{BL}_0(\mathbb{A}^8)$. The flatness is shown in Lemma 3.5. Concretely, one can compute $\tilde{\mathcal{J}}$ using SINGULAR (see Appendix A.3) and obtains the following equations for the curves over points on the exceptional divisor:

$$
\begin{aligned}
&0 = x_0^2 = x_0 x_1 = x_0 x_2 = x_0 x_3 = x_0 x_4, \\
&0 = x_1 x_4 - x_2 x_3, \\
&0 = p_1 = x_1 x_3^2 s_1 + x_3^3 s_2 - x_2 x_4^2 s_3 - x_4^3 s_4 + 3 x_1 x_3 x_4 s_5 \\
&\quad + 3 x_1 x_4^2 s_6 + 3 x_3^2 x_4 s_7 + 3 x_3 x_4^2 s_8 \\
&0 = p_2 = x_1^2 x_3 s_1 + x_1 x_3^2 s_2 - x_2^2 x_4 s_3 - x_2 x_4^2 s_4 + 3 x_1^2 x_4 s_5 \\
&\quad + 3 x_1 x_2 x_4 s_6 + 3 x_1 x_3 x_4 s_7 + 3 x_1 x_4^2 s_8, \\
&0 = p_3 = x_1^3 s_1 + x_1^2 x_3 s_2 - x_2^3 s_3 - x_2^2 x_4 s_4 + 3 x_1^2 x_2 s_5 + 3 x_1 x_2^2 s_6 \\
&\quad + 3 x_1^2 x_4 s_7 + 3 x_1 x_2 x_4 s_8
\end{aligned}
$$

Note, that $x_3 p_2 = x_1 p_1$ and $x_3 p_3 = x_1 p_2$. ∎

Lemma 3.5 *Using the notation from the Proof of Proposition 3.4, the saturated family* $\tilde{\mathcal{J}}$ *is a flat family of curves in* $\mathbb{P}^4$ *with Hilbert polynomial* $4n + 1$.

Proof: The main part of the proof is to show, that the family $\tilde{\mathcal{J}}$ of closed subschemes of $\mathbb{P}^4$ restricted to a family over the exceptional divisor of the blow-up $\mathrm{BL}_0(\mathbb{A}^8)$, is flat. Furthermore we show that the Hilbert polynomial of one of the curves is $4n+1$. This implies that the Hilbert polynomial is constant in the whole family parametrised by $\mathrm{BL}_0(\mathbb{A}^8)$, since we already know (by construction) that the Hilbert polynomial of any curve over points outside the exceptional divisor, is $4n + 1$.

For the proof we want to use the following Theorem ([23][Corollary of Theorem 23.1]):

Theorem 3.6 *Let X and Y be two schemes over $\mathbb{C}$ of dimension m resp. n and $f : X \to Y$ a morphism. Suppose*

- *(a) Y is regular*
- *(b) X is Cohen-Macaulay*
- *(c) f is proper*
- *(d) for every closed point $y \in Y$, the fibre $f^{-1}(y)$ is $(m-n)$-dimensional or empty.*
- *(e) X and Y are irreducible*

Then f is flat.

In our situation, X is the closed subscheme that is defined by the equations

$$x_0 = 0,\ x_1x_4 - x_2x_3 = 0,\ p_1 = p_2 = p_3 = 0$$

in $\mathbb{P}^4 \times \mathbb{P}^7$ with coordinates $x_0, \dots, x_4$ resp. $s_1, \dots, s_8$. For simplicity, we consider instead of X the closed subscheme in $\mathbb{P}^3 \times \mathbb{P}^7$ defined by the equations

$$x_1x_4 - x_2x_3 = 0, p_1 = p_2 = p_3 = 0$$

and denote it by X again. Furthermore, $Y = \mathbb{P}^7$ and $f : X \to Y$ is the projection on $\mathbb{P}^7$. Now we have to check the properties demanded in the Theorem 3.6

- (a) Obviously $Y = \mathbb{P}^7$ is regular.
- (b) We check the assertion locally on the open sets $U_{x_i} := \mathbb{A}^3 \times \mathbb{P}^7$ each defined by the equation $x_i = 1$ for $i = 1, \dots 4$.

  The ideal of the closed subscheme $X \cap U_{x_1}$ is generated by the polynomials $x_4 - x_2x_3$ and p_3. On this open set, $p_2 = x_3 \cdot p_3$ and $p_1 = x_3^2 \cdot p_3$. Clearly, $x_4 - x_2x_3$ defines a smooth hyperplane in $\mathbb{A}^3 \times \mathbb{P}^7$. The equation p_3 does not vanish on this hyperplane. Hence it defines again a hyperplane and $U_{x_1} \cap X$ is Cohen-Macaulay and $\dim(U_{x_1} \cap X) = 8$.

  On the other open sets, the situation is similar:

  - On the open set $U_{x_2} \cap X$, the ideal is defined by $x_3 - x_1x_4$ and p_3. ($p_1 = x_4^2 p_3$ and $p_2 = x_4 p_3$)

- On the open set $U_{x_3} \cap X$, the ideal is defined by $x_2 - x_1x_4$ and p_1. ($p_2 = x_1p_1$ and $p_3 = x_1^2p_1$)
- On the open set $U_{x_4} \cap X$, the ideal is defined by $x_1 - x_2x_3$ and p_1. ($p_2 = x_2p_1$ and $p_3 = x_2^2p_1$)

Altogether, X is 8-dimensional and Cohen-Macaulay (and a local complete intersection).

(c) This is clear, since f is the projection.

(d) Any fiber of f is a proper closed subset of the 2-dimensional smooth quadric defined by $x_1x_4 - x_2x_3$ in $\mathbb{P}^3$. Hence all fibers have dimension at most 1. By [12][ex II 3.22b], every irreducible component of the fiber $f^{-1}(y)$ of $y \in f(X)$ has dimension at least $\dim X - \dim Y = 1$.

(e) Assume that X is reducible, i.e. $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subset X$. Then both restricted maps $f|_{X_1} : X_1 \to Y$ and $f|_{X_2} : X_2 \to Y$ are surjective. Indeed, assume that $f|_{X_1}$ is not surjective. Then the image $f(X_1)$ of X_1 in Y has dimension at most 6. But then the generic fiber of $f|_{X_1}$ has dimension at least 2 (again by [12][ex II3.22b]), which is a contradiction to (d). Hence $f|_{X_1}$ and $f|_{X_2}$ are surjective.

Observe that there is a fiber of f that is both irreducible and reduced. For this one can choose the fiber over the point $(0:1:1:1:0:0:0:0) \in \mathbb{P}^7$. The curve associated to this point is defined by the equations

$$\begin{aligned} x_0 &= 0 \\ x_1x_4 - x_2x_3 &= 0 \\ x_3^3 - x_2x_4^2 - x_4^3 &= 0 \\ x_1x_3^2 - x_2^2x_4 - x_2x_4^2 &= 0 \\ x_1^2x_3 - x_2^3 - x_2^2x_4 &= 0. \end{aligned}$$

Whether the curve is irreducible and reduced, can be checked locally. On U_{x_1} we have $x_4 = x_2x_3$ and hence the curve is in $\mathbb{A}^2$ with coordinates x_2 and x_3 defined by the equation $p_3 = x_3 - x_2^3 - x_2^3x_3$.

Similarly, on the other open sets U_{x_2}, U_{x_3} and U_{x_4}, the curve is given by the polynomial in two variables $x_1^3x_4 - 1 - x_4$ resp. $1 - x_4^3 - x_1x_4^3$ resp. $x_3^3 - x_2 - 1$. Using [21][ex. 4.1], one can check by an easy computation that the curve is irreducible and reduced on these open sets.

Alternatively, one can easily check that this curve is smooth and hence obtained as a generic projection of a rational normal curve in $\mathbb{P}^4$ to a hyperplane. Altogether, it follows that X is irreducible.

It remains to show that the Hilbert polynomial of one of the curves over the exceptional divisor, has Hilbert polynomial $4n+1$.

For example, choose the curve defined by $x_0 = 0, x_1x_4 - x_2x_3 = 0, x_3x_4^2 = 0, x_1x_4^2 = 0, x_1x_2x_4 = 0$. An easy computation shows that the Hilbert polynomial of this curve is $4n+1$. ∎

Remark 3.7 The curves associated to points in the exceptional divisor lie in the quadric $x_1x_4 - x_2x_3$ in the $\mathbb{P}^3$ defined by $x_0 = 0$.

3.2 The Stratum S_2

The stratum S_2 is contained in the closure of S_1. Hence the slice in a point $x \in S_2$ at the stratum S_2 contains matrices from the stratum S_1. To obtain families that are flat except in the point x, we need to compute a slice in x at $\overline{S_1}$. But first we need the following Lemma as a preparation:

Lemma 3.8 *$\overline{S_1}$ is non-singular in the points of S_2.*

Proof: By Lemma 2.24 and Lemma 2.25 we have $\overline{S_1}\backslash S_1 = S_2 \cup P_2 \cup P_3$. Let $S_x(t_1, \ldots t_9)$ be the slice in

$$x = \begin{pmatrix} x_0 & 0 & x_1 & x_2 \\ 0 & x_0 & x_2 & x_3 \end{pmatrix}^t$$

transversal to the stratum S_2, i.e.

$$S_x(t_1, \ldots t_9) =$$

$$\begin{pmatrix} x_0 - t_5x_1 - t_6x_2 - t_7x_3 - t_8x_4 & t_1x_1 + t_2x_4 + t_5x_2 + t_6x_3 \\ t_3x_3 + t_4x_4 - t_6x_1 - t_7x_2 & x_0 + t_5x_1 + t_6x_2 + t_7x_3 + t_8x_4 \\ x_1 & x_2 - t_9x_4 \\ x_2 + t_9x_4 & x_3 \end{pmatrix}.$$

Now it suffices to show that x is a regular point in $S_x \cap (S_1 \cup S_2)$, since Luna's slice theorem gives an étale map $S_x /\!\!/ G_x \to W^{ss} /\!\!/ G$ and the stabilizer G_x of x with respect to the action of G is trivial for stable points.
In order to compute the intersection $S_x \cap (S_1 \cup S_2)$, we need a criterion to decide whether a point of the slice S_x is contained in S_1 resp. S_2. For that we consider as before the rank of the matrices $y \in S_x$ as (4×5)-matrices with entries in a

$H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ (where the global sections s and t form a basis): From Appendix B we know that

$$\operatorname{rank}(y) = \begin{cases} 2 & \text{if } y \in P_3, \\ 3 & \text{if } y \in S_1, S_2, S_3, P_2, B_{10}, \\ 4 & \text{otherwise.} \end{cases}$$

So

$$S_x \cap (S_1 \cup S_2) = \{y \in S_x \mid \operatorname{rank}(y) = 3\}.$$

A simple computation shows that the rank of

$$S_x(t_1, \ldots, t_9) = \begin{pmatrix} s & -t_5 s + t_1 t & -t_6 s + t_5 t & -t_5 s + t_6 t & -t_8 s + t_2 t \\ t & -t_6 s + t_5 t & -t_7 s + t_6 t & t_3 s + t_7 t & t_4 s + t_8 t \\ 0 & s & t & 0 & -t_9 t \\ 0 & 0 & s & t & t_9 s \end{pmatrix}$$

is 3 if and only if $t_1, \ldots, t_8 = 0$. Hence x is a regular point of $S_x \cap (S_1 \cup S_2)$. ∎

Lemma 3.9 *The slice of* $x = \begin{pmatrix} x_0 & 0 & x_1 & x_2 \\ 0 & x_0 & x_2 & x_4 \end{pmatrix}^t \in S_2$ *at the stratum* $\overline{S_1}$ *is*

$$\begin{pmatrix} x_0 - t_5 x_1 - t_6 x_2 - t_7 x_3 - t_8 x_4 & t_1 x_1 + t_2 x_3 + t_5 x_2 + t_6 x_4 \\ t_3 x_3 + t_4 x_4 - t_6 x_1 - t_8 x_2 & x_0 + t_5 x_1 + t_6 x_2 + t_7 x_3 + t_8 x_4 \\ x_1 & x_2 \\ x_2 & x_4 \end{pmatrix}$$

Proof: Recall from the proof of Lemma 3.8 that the slice of x transversal to the stratum S_2 is given by

$$\begin{pmatrix} x_0 - t_5 x_1 - t_6 x_2 - t_7 x_3 - t_8 x_4 & t_1 x_1 + t_2 x_3 + t_5 x_2 + t_6 x_4 \\ t_3 x_3 + t_4 x_4 - t_6 x_1 - t_8 x_2 & x_0 + t_5 x_1 + t_6 x_2 + t_7 x_3 + t_8 x_4 \\ x_1 & x_2 - t_9 x_3 \\ x_2 + t_9 x_3 & x_4 \end{pmatrix}.$$

To obtain the slice at the stratum $\overline{S_1}$, we need to remove the degenerations of x into S_1. Cleary, t_9 is a parameter of such a degeneration: In the proof of Lemma 3.8 we showed that the rank of the matrix A associated to a point on the slice has rank 3 if and only if $A \in S_1 \cup S_2$ and rank 4 otherwise. Furthermore, we saw that the rank of A is 3 if and only if $t_i = 0$ for all $i = 1, \ldots, 8$. If additionally $t_9 = 0$, then $A \in S_2$. ∎

The same construction as for S_1 in Section 3.3.1 provides the following family of curves over the exceptional divisor:

$$\begin{aligned} &x_0^2 = x_0x_2 = x_0x_4 = x_0x_1 = x_1x_4 - x_2^2 = 0 \\ &p := x_1^2s_1 + x_1x_3s_2 - x_2x_4s_3 - x_4^2s_4 + 3x_1x_2s_5 + 4x_1x_4s_6 \\ &\quad + 2x_2x_3s_7 + 3x_2x_4s_8 = 0 \end{aligned}$$

This is a family $\mathcal{C} \subset \mathbb{P}^4 \times \mathbb{P}^7$ of curves contained in the quadric $x_1x_4 - x_2^2$ in $\mathbb{P}^3$ with an embedded point in $q := (0:0:0:1:0)$.
It suffices to show that the family $\mathcal{C}' \subset \mathbb{P}^4 \times \mathbb{P}^7$ of curves defined by

$$x_0 = 0,\ x_1x_4 - x_2^2 = 0,\ p = 0$$

is flat. Then the Hilbert polynomial in the family $\mathcal{C}'$ is constant and hence also in the family $\mathcal{C}$, since we have an exact sequence

$$0 \to \mathcal{O}_q \to \mathcal{O}_{\mathcal{C}_s} \to \mathcal{O}_{\mathcal{C}'_s} \to 0$$

for all $s = (s_1 : \ldots : s_8)$.
To prove flatness in the family $\mathcal{C}'$, one can use the same method as in the Proof of Lemma 3.5. In this case, X is the closed subscheme in $\mathbb{P}^4 \times \mathbb{P}^7$ defined by the equations $x_0 = 0$, $x_1x_4 - x_2^2 = 0$, $p = 0$ and hence it is clear, that $\dim X = 8$ and X is a complete intersection in $\mathbb{P}^4 \times \mathbb{P}^7$ and hence Cohen-Macaulay. The other properties demanded in Theorem 3.6 follow in the same way as in Lemma 3.5. Again it is easy to compute the Hilbert polynomial of one of the curves in $\mathcal{C}$.

Remark 3.10 The curves associated to points on the exceptional divisor are singular curves of degree 2 lying in a cone of a plane quadric in the $\mathbb{P}^3$ defined by $x_0 = 0$ together with an embedded point in $(0:0:0:1:0)$.

3.3 The Stratum S_3

Now we consider the stratum S_3. S_2 and S_3 have the same dimension, and as we have seen in Lemma 2.25, S_3 is not contained in the closure of S_1.
In this case, a slice in the point $x = \begin{pmatrix} x_0 & x_1 & 0 & x_2 \\ 0 & x_0 & x_1 & x_3 \end{pmatrix}^t \in S_3$ is

$$\begin{pmatrix} x_0 - t_3x_2 - t_4x_4 - t_7x_3 & t_1x_2 + t_2x_4 + t_3x_3 \\ x_1 - t_7x_2 - t_8x_3 - t_9x_4 & x_0 + t_3x_2 + t_4x_4 + t_7x_3 \\ t_5x_3 + t_6x_4 - t_8x_2 & x_1 + t_7x_2 + t_8x_3 + t_9x_4 \\ x_2 & x_3 \end{pmatrix}.$$

The same construction as for S_1 in Section 3.3.1 provides the following family of curves over the exceptional divisor: a family of quartic curves

$$q = x_2^4s_1 + x_2^3x_4s_2 + 3x_2^3x_3s_3 + 2x_2^2x_3x_4s_4 - x_3^4s_5$$

$$- x_3^3x_4s_6 + 4x_2^2x_3^2s_7 + 3x_2x_3^3s_8 + 2x_2x_3^2x_4s_9$$

in the plane $x_0 = x_1 = 0$ with an embedded point $p := (0 : 0 : 0 : 0 : 1)$ of length 3.
To show that this is a flat family of curves in $\mathbb{P}^4$ parametrized by $\mathbb{P}^8$, we first show that

$$x_0 = x_1 = 0, q = 0$$

defines a flat family of curves in $\mathbb{P}^4 \times \mathbb{P}^7$. For that one uses again the methods from the Proof of Lemma 3.5. Then also $\mathcal{C}$ is a flat family of curves, since for any $s = (s_1 : \ldots : s_9)$, the kernel of $\mathcal{O}_{\mathcal{C}_s} \to \mathcal{O}_{\mathcal{C}'_s} \to 0$ is a sheaf supported on the point p which has constant Hilbert polynomial 3.
Again it is easy to compute the Hilbert polynomial of one of the curves in the family $\mathcal{C}$.

Remark 3.11 The curves associated to points on the exceptional divisor are plane singular curves of degree 4 together with an embedded point in
$(0 : 0 : 0 : 0 : 1)$.

4 Extension to the Blow-up of K in P

Recall that the singular locus of the moduli space K of semi-stable Kronecker modules of type $(4,2)$ is $P = \overline{P_1} = P_1 \cup P_2 \cup P_3$ (Prop. 2.23). These are exactly the strata of poly-stable Kronecker modules in K. There are two different types of poly-stable Kronecker modules A of type $(4,2)$: they are direct sums $A = A_1 \oplus A_2$ of Kronecker modules A_1 and A_2 of type $(2,1)$ that are either non-isomorphic (in the strata P_1 and P_2) or isomorphic (in the stratum P_3). To study possible singularities of K in the points of P, we use again the Luna slice theorem. If x is a singularity of the quotient $S_x /\!\!/ G_x$ of a certain type, then x is a singularity of $W^{ss} /\!\!/ G$ of the same type.

The maximal minors of matrices representing Kronecker modules in P define closed subschemes in $\mathbb{P}^4$ of dimension 2 (Rem. 3.3). Hence it is not possible to extend the morphism $\tilde{\Phi}_{KH} : W^s \backslash S \to \mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$ to these strata.

In the following we will study if one can extend $\tilde{\Phi}_{KH}$ to the blow-up of K in P. The main difference to the situation in Section 3.3 is that the stabilizer of points $x \in P$ with respect to the action of G is not trivial. For $x \in P_1$ or $x \in P_2$, the stabilizer is $G_x = \mathbb{C}^*$ and for $x \in P_3$, the stabilizer is $G_x = \mathrm{GL}_2$.

4.1 The Stratum P_1

Let $p : W^{ss} \to W^{ss} /\!\!/ G = K$ be the good quotient defined in Proposition 2.10. Our aim is to extend the flat family of curves over $K \backslash (S \cup P)$ defined by the vanishing of the maximal minors of matrices associated to the Kronecker modules in $K \backslash P$ to a flat family of curves over the blow-up $\mathrm{BL}_P(W^{ss} /\!\!/ G)$ of $W^{ss} /\!\!/ G$ along the poly-stable locus P (at least for points in P_1).

Let $\tilde{P}$ denote the preimage of $P \subset K$ in W^{ss} under the map p. Then by [17][Lemma 3.11], $\mathrm{BL}_{\tilde{P}}(W^{ss}) /\!\!/ G$ is the blow-up of $W^{ss} /\!\!/ G$ along $P = \tilde{P} /\!\!/ G$.

We consider the situation locally. By Luna's slice theorem, there exists for any point $x \in P_1$ a slice $S_x \subset W^{ss}$ with the following properties: The stabilizer $G_x = \mathbb{C}^*$ is a reductive group, hence there is a categorical quotient $q : S_x \to S_x /\!\!/ G_x$ and we have an étale morphism $S_x /\!\!/ G_x \to W^{ss} /\!\!/ G$.

All points $x \in P_1$ are projectively equivalent and hence the construction below can be done simultaneously in any $x \in P_1$ and we choose one fixed representative for our computations. Concretely, a Luna slice of the point

$$x = \begin{pmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & x_3 \end{pmatrix}^t \in P_1$$

is

$$S_x(t_1, \ldots, t_{10}) = \begin{pmatrix} x_0 & t_1 x_1 + t_2 x_4 - t_4 x_0 \\ x_1 & t_3 x_0 + t_4 x_1 + t_5 x_4 \\ -t_9 x_2 + t_6 x_3 + t_7 x_4 & x_2 \\ t_9 x_3 - t_8 x_2 + t_{10} x_4 & x_3 \end{pmatrix}$$

for some parameters $t_1, \ldots, t_{10} \in \mathbb{C}$, again computed with the Singular program in Appendix A 1. On this slice we have an action of the stabilizer $\mathbb{C}^*$ of x. Since this action is induced by the action of G on W^{ss}, it is given by

$$(t_1, \ldots, t_{10}) \mapsto (t \cdot t_1, \ldots, t \cdot t_5, t^{-1} \cdot t_6, \ldots, t^{-1} \cdot t_{10})$$

for $t \in \mathbb{C}^*$.

Remark 3.12 We want to give a more geometric interpretation for the slice and the blow-up of the good quotient of the slice by $\mathbb{C}^*$ along the point 0.

Therefore, by u we denote the vector $(t_1, \ldots, t_5)^t \in \mathbb{C}^5$ and by v the vector $(t_6, \ldots, t_{10})^t \in \mathbb{C}^5$. The image of the map

$$f : \mathbb{C}^5 \times \mathbb{C}^5 \to M^{5\times 5}(\mathbb{C}), (u, v) \mapsto u \cdot v^t$$

is the set $\mathrm{Im}(f) = \{A \in M^{5\times 5}(\mathbb{C}) \,|\, \mathrm{rank}(A) \leq 1\}$.
The preimage of the zero matrix in $M^{5\times 5}(\mathbb{C})$ is the set

$$\Sigma := (\mathbb{C}^5 \times \{0\}) \cup (\{0\} \times \mathbb{C}^5) \subset \mathbb{C}^5 \times \mathbb{C}^5.$$

For fixed values $(t_1, \ldots t_5, 0, \ldots, 0)$ or $(0, \ldots 0, t_6, \ldots, t_{10})$, the matrices $S_x(t_1, \ldots, t_{10})$ represent strictly semi-stable Kronecker modules.
By Lemma 2.7, if one of the t_i is non-zero, then the matrix is not poly-stable.
The map f is invariant under the action of $\mathbb{C}^*$, i.e. for any $t \in \mathbb{C}^*$, we have

$$f(t \circ (u, v)) = f(t \cdot u, t^{-1} \cdot v) = f(u, v).$$

Hence f induces a map $\tilde{f} : S_x /\!\!/ \mathbb{C}^* \to M^{5\times 5}(\mathbb{C})$. We obtain a bijective map between the slice $S_x /\!\!/ G_x$ and

$$\mathrm{Im}(f) = \{A \in M^{5\times 5}(\mathbb{C}) \,|\, \mathrm{rank}(A) \leq 1\}.$$

Indeed, assume first that u and v are not zero and $uv^t = \tilde{u}\tilde{v}^t$. W.l.o.g. $u_1 \neq 0$. Then $v_j = \frac{\tilde{u_1}\tilde{v_j}}{u_1}$ for all j. Hence $v = \frac{\tilde{u_1}}{u_1} \cdot \tilde{v}$. Similarly, $u = \frac{\tilde{v_1}}{v_1} \cdot \tilde{u}$. Hence u and $\tilde{u}$ resp. v and $\tilde{v}$ are in the same orbit with respect to the action of $\mathbb{C}^*$.
If $u = 0$ or $v = 0$, then $(u, v) \in \Sigma$, hence is is mapped to the zero matrix. In the quotient of the slice S_x by $\mathbb{C}^*$, these points are mapped to the same point as the orbit of the poly-stable matrix represented by $u = v = 0$.

The blow-up of $\mathrm{Im}(f) = \{A \in M^{5\times 5}(\mathbb{C}) \,|\, \mathrm{rank}(A) \leq 1\}$ in Σ is given by

$$\left\{ (A, H, L) \;\middle|\; \begin{array}{l} A \in \mathrm{Im}(f), \\ H \text{ a hyperplane in } \mathbb{P}^5 \text{ with } H \subset \mathrm{Ker}(A) \text{ and} \\ L \text{ a line in } \mathbb{P}^5 \text{ with } \mathrm{Im}(A) \subset L. \end{array} \right\}$$

The exceptional divisor is

$$\{(0, L, H) \,|\, L \text{ line in } \mathbb{P}^5, H \text{ hyperplane in } \mathbb{P}^5\} \cong \mathbb{P}^4 \times \mathbb{P}^4.$$

In order to see that the blow-up of K along P is smooth, we want to describe the Luna slice via Ext-groups as follows:

Lemma 3.13 *Let $A_i : F_i \to E_i \otimes V$, $i = 1, 2$ be two Kronecker modules of type $(2, 1)$, i.e. $\dim E_i = 2$ and $\dim F_i = 1$. Then*

$$\operatorname{Ext}^1(A_2, A_1) = \frac{\operatorname{Hom}(F_2, E_1 \otimes V)}{A_1 \cdot \operatorname{Hom}(E_2, E_1) - \operatorname{Hom}(F_2, F_1) \cdot A_2}.$$

Proof: The extension of a Kronecker module A_1 by A_2 is a commutative diagram

$$\begin{tikzcd}
F_1 \arrow[r, "A_1"] \arrow[d] & E_1 \otimes V \arrow[d] \\
F_1 \oplus F_2 \arrow[r, "\varphi"] \arrow[d] & (E_1 \oplus E_2) \otimes V \arrow[d] \\
F_2 \arrow[r, "A_2"] & E_2 \otimes V
\end{tikzcd}$$

where φ is given by a block matrix $\left(\begin{array}{c|c} A_1 & * \\ \hline 0 & A_2 \end{array}\right)$.

Hence giving an extension of A_1 by A_2 is the same as giving a homomorphism $F_2 \to E_1 \otimes V$.

Two extensions

$$\begin{tikzcd}
F_1 \arrow[r, "A_1"] \arrow[d] & E_1 \otimes V \arrow[d] \\
F_1 \oplus F_2 \arrow[r, "\varphi"] \arrow[d] & (E_1 \oplus E_2) \otimes V \arrow[d] \\
F_2 \arrow[r, "A_2"] & E_2 \otimes V
\end{tikzcd}$$

and

$$\begin{tikzcd}
F'_1 \arrow[r, "A_1"] \arrow[d] & E'_1 \otimes V \arrow[d] \\
F'_1 \oplus F'_2 \arrow[r, "\varphi'"] \arrow[d] & (E'_1 \oplus E'_2) \otimes V \arrow[d] \\
F'_2 \arrow[r, "A_2"] & E'_2 \otimes V
\end{tikzcd}$$

of A_1 by A_2 are isomorphic if and only if there are isomorphisms

$$F_1 \oplus F_2 \to F'_1 \oplus F'_2 \text{ and } E_1 \oplus E_2 \to E'_1 \oplus E'_2$$

given by the matrices $\left(\begin{array}{c|c} 1 & V \\ \hline 0 & 1 \end{array}\right)$ and $\left(\begin{array}{c|c} 1 & U \\ \hline 0 & 1 \end{array}\right)$ with $U \in \mathrm{Hom}(F_2, F_1)$ and $V \in \mathrm{Hom}(E_2, E_1)$ that are compatible with φ and φ'. Hence if φ' is represented by the matrix $\left(\begin{array}{c|c} A_1 & B \\ \hline 0 & A_2 \end{array}\right)$ and φ by the matrix $\left(\begin{array}{c|c} A_1 & C \\ \hline 0 & A_2 \end{array}\right)$, then

$$\left(\begin{array}{c|c} 1 & V \\ \hline 0 & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} A_1 & C \\ \hline 0 & A_2 \end{array}\right) = \left(\begin{array}{c|c} A_1 & B \\ \hline 0 & A_2 \end{array}\right) \cdot \left(\begin{array}{c|c} 1 & U \\ \hline 0 & 1 \end{array}\right)$$

and thus the two extensions are isomorphic if and only if $C = B + A_1 U - V A_2$.

■

Lemma 3.14 *For any two Kronecker modules* $A_i : F_i \to E_i \otimes V$, $i = 1, 2$ *of type* $(2, 1)$*, the dimension of* $\mathrm{Ext}^1(A_1, A_2)$ *is*

$$\dim \mathrm{Ext}^1(A_1, A_2) = \begin{cases} 5 \text{ if } A_1 \not\cong A_2 \\ 6 \text{ if } A_1 \cong A_2 \end{cases} .$$

Proof: By Lemma 3.13 there is an exact sequence

$$0 \to \mathrm{Hom}(A_1, A_2) \to \mathrm{Hom}(E_1, E_2) \oplus \mathrm{Hom}(F_1, F_2) \xrightarrow{\psi} \mathrm{Hom}(F_1, E_2 \otimes V)$$

$$\to \mathrm{Ext}^1(A_1, A_2) \to 0,$$

where the map ψ is defined by sending a pair

$$(f, g) \in \mathrm{Hom}(E_1, E_2) \oplus \mathrm{Hom}(F_1, F_2)$$

to $A_2 f - g A_1$.

Then by the Lemma of Schur for the stable Kronecker modules A_1 and A_2 it holds

$$\mathrm{Hom}(A_1, A_2) = \begin{cases} \mathbb{C} \text{ if } A_1 \cong A_2 \\ 0 \text{ otherwise.} \end{cases} .$$

Since $\dim E_i = 2$ and $\dim F_i = 1$ and hence

$$\dim(\mathrm{Hom}(E_1, E_2) \oplus \mathrm{Hom}(F_1, F_2)) = 4 + 1 = 5 \text{ and } \dim(\mathrm{Hom}(E_1, F_2 \otimes V)) = 10,$$

the assertion follows. ■

Let $[A = A_1 \oplus A_2] \in P_1$ for two Kronecker modules A_1 and A_2 of type $(2,1)$ which are non-isomorphic. The slice at A can locally be described by

$$\begin{aligned}\operatorname{Ext}^1(A,A) &= \operatorname{Ext}^1(A_1 \oplus A_2, A_1 \oplus A_2)\\ &= \operatorname{Ext}^1(A_1,A_1) + \operatorname{Ext}^1(A_1,A_2) + \operatorname{Ext}^1(A_2,A_1) + \operatorname{Ext}^1(A_2,A_2).\end{aligned}$$

Since $\operatorname{Ext}^1(A_1,A_1)$ and $\operatorname{Ext}^1(A_2,A_2)$ belong to deformations tangent to the stratum, we just consider the following part of the slice

$$S_0 = \operatorname{Ext}^1(A_1,A_2) + \operatorname{Ext}^1(A_2,A_1).$$

Since A_1 and A_2 are non-isomorphic, we have

$$\dim \operatorname{Ext}^1(A_1,A_2) = \dim \operatorname{Ext}^1(A_2,A_1) = 5.$$

So we have the following inclusion

$$(\mathbb{C}^5 \oplus \mathbb{C}^5)/\mathbb{C}^* \hookrightarrow \operatorname{End}(\mathbb{C}^5), (v,w) \mapsto v \otimes w.$$

Denote $\operatorname{Ext}(A_i,A_j)$ by A_{ij}. Let $\xi_{12} \to \mathbb{P}(A_{12}^*)$ be the tautological negative line bundle, i.e.

$$\xi_{12} = \{(v,L) \in A_{12} \times \mathbb{P}(A_{12}^*) \mid L \text{ line in } A_{12}, v \in L\}.$$

ξ_{21} is defined analogously.

Proposition 3.15
$$\xi_{12}\hat{\otimes}\xi_{21} \xrightarrow{g} (A_{12} \oplus A_{21}) /\!\!/ \mathbb{C}^*$$
defines a resolution of the singularity of the slice $(A_{12} \oplus A_{21}) /\!\!/ \mathbb{C}^*$ *in the point* 0*. In fact, this is the blow-up of* $S_x /\!\!/ \mathbb{C}^*$ *in* 0*.*

Proof: By definition

$$\xi_{12}\hat{\otimes}\xi_{21} = \left\{ (L, M, v \otimes w) \;\middle|\; \begin{array}{l} L \text{ line in } A_{12}, M \text{ line in } A_{21}, \\ v \in A_{12}, w \in A_{21}, v \otimes w \in L \otimes M \end{array} \right\}.$$

Since

$$\xi_{12}\hat{\otimes}\xi_{21} \to \mathbb{P}(A_{12}^*) \times \mathbb{P}(A_{21}^*)$$

is a line bundle and $\mathbb{P}(A_{12}^*) \times \mathbb{P}(A_{21}^*)$ is smooth, $\xi_{12}\hat{\otimes}\xi_{21}$ is also smooth. Furthermore $\dim \mathbb{P}(A_{12}^*) = \dim \mathbb{P}(A_{21}^*) = 4$ and hence we have

$$\dim \xi_{12}\hat{\otimes}\xi_{21} = 9.$$

Outside

$$\begin{aligned} g^{-1}(0) &= \big\{(L, M, v \otimes w) \in \xi_{12} \hat{\otimes} \xi_{21} \mid v = 0 \text{ or } w = 0\big\} \\ &\cong \mathbb{P}(A_{12}^*) \times \mathbb{P}(A_{21}^*) \end{aligned}$$

the map g associates to $(L, M, v \otimes w)$ the point of $S_x /\!\!/ \mathbb{C}^*$ given by the $\mathbb{C}^*$-orbit of (v, w). Obviously g is an isomorphism outside $g^{-1}(0)$, since given the $\mathbb{C}^*$-orbit of (v, w), the lines L and M are uniquely defined as $\mathbb{C}v$ and $\mathbb{C}w$.
The exceptional divisor is $g^{-1}(0) = \mathbb{P}(A_{12}^*) \times \mathbb{P}(A_{21}^*)$. Hence by the universal property of the blow-up, g is the blow-up of $(A_{12} \oplus A_{21}) /\!\!/ \mathbb{C}^*$ in the point 0.
∎

Now we consider again the family of closed subschemes in $\mathbb{P}^4$ defined by the vanishing of the maximal minors of the Kronecker modules in K. As before, by [17][Lemma 3.11], $\mathrm{BL}_\Sigma(S_x) /\!\!/ G_x$ is the blow-up of $S_x /\!\!/ G_x$ along $q(\Sigma) = 0$.

Lemma 3.16 *By U we denote the set $(\mathbb{C}^5 \times \mathbb{C}^5) \backslash \Sigma$. Let $\mathcal{C}_U \subset U \times \mathbb{P}^4$ be the family of curves defined by the maximal minors of the matrices in the slice $S_x(t_1, \dots t_{10})$ (obtained by choosing fixed values for $t_1, \dots, t_{10}$). The family $\mathcal{C}_U$ is flat and consists of curves with Hilbert polynomial $4n + 1$.*

Proof: For the proof, it is important to know that S_1, S_2 and S_3 do not degenerate into P_1. This we proved in Lemma 2.25. Using this, the assertion is clear by construction. Except the point x, any point in the slice belongs to a stratum of higher dimension in B. For these strata we know that the Hilbert polynomial of any curve defined by the maximal minors of these matrices, is $4n + 1$. ∎

Proposition 3.17 *One can extend the family $\mathcal{C}_U \subset U \times \mathbb{P}^4$ to a flat family $\mathcal{C}' \subset \mathrm{BL}_\Sigma \mathbb{C}^{10} \times \mathbb{P}^4$.*

Proof: We check the assertion in local coordinates. First one extends the flat family $\mathcal{C}_U$ of curves in $\mathbb{P}^4$ to a flat family on $(\mathrm{BL}_\Sigma \mathbb{C}^{10} \times \mathbb{P}^4) \backslash E$, where E is the exceptional divisor of the blow-up of $\mathbb{C}^{10}$ in Σ. Then simultaneous saturation gives a projective closure of this variety and hence gives an extension of $\mathcal{C}_U$ to a family $\mathcal{C}' \subset \mathrm{BL}_\Sigma \mathbb{C}^{10} \times \mathbb{P}^4$.
For the concrete computations, we use a SINGULAR program (for the listing we refer to Appendix A.4). We obtain the following family of curves associated to the points in the exceptional divisor:

1. component: $x_0 = x_1 = 0, x_3^2 s_6 + x_3 x_4 s_7 + x_2^2 s_8 - 2x_2 x_3 s_9 - x_2 x_4 s_{10},$

2. component $x_2 = x_3 = 0, x_1^2 s_1 - x_1 x_4 s_2 + x_0^2 s_3 + 2x_0 x_1 s_4 + x_0 x_4 s_5$.

These equations define a pair of plane quadrics that intersect in exactly one point for any choice of the parameters $s_1, \ldots, s_{10}$.

This family of curves is invariant under the action of $\mathbb{C}^*$, as one can easily verify using the Singular program in Appendix A 4. Hence it descends to a family on $\mathrm{BL}_\Sigma(\mathbb{C}^{10}) /\!\!/ \mathbb{C}^*$.
It remains to show that the family $\mathcal{C}'$ we constructed is flat. But this is clear, since closed subscheme associated to points on the exceptional divisor are pairs of two plane quadrics that intersect in exactly one point. All such curves are projectively equivalent and have Hilbert polynomial $4n + 1$, as we will show in Lemma 4.36. ∎

4.2 The Stratum P_2

In this section we describe the structure of P_2 and explain a possible procedure for extending the family of curves parametrized by the blow up of W^{ss} along P_1 resp. S_i for $i = 1, \ldots, 3$ to P_2.

Recall that the maximal minors of representatives of Kronecker modules in P_2 define two planes intersecting in a line with a non-reduced structure.
For the stratum P_2, the situation is more complicated as before, since it is contained in the closure of the strata S_1, S_3 and P_1 whose maximal minors do not define curves.
As a preparation we need the following

Lemma 3.18 *(a) The closure $\overline{P_1}$ of the stratum P_1 is non-singular along P_2, furthermore $\overline{S_1}$ is non-singular in points of P_2.*

(b) $\overline{S_3}$ is singular in the points of P_2.

(c) The closures of S_1 and P_1 intersect transversally. Futhermore, $\overline{P_1}$ intersects each component of $\overline{S_3}$ exactly in the point 0 and $\overline{S_1}$ intersects each component of $\overline{S_3}$ exactly in an $\mathbb{A}^2$.

Proof: As P_2 is a $\mathrm{PGL}(V)$-orbit, the computations can be done simultaneously in any point in P_2. Again we examine the situation locally on the Luna slice

$$S_x(t_1, \ldots, t_{12}) = \begin{pmatrix} x_0 - t_{11}x_2 - t_{12}x_4 & t_1x_1 + t_2x_2 + t_3x_4 \\ x_1 & t_4x_2 + t_5x_4 \\ t_6x_2 + t_7x_3 + t_8x_4 & x_0 + t_{11}x_2 + t_{12}x_4 \\ t_9x_2 + t_{10}x_4 & x_3 \end{pmatrix}$$

of the Kronecker module

$$x = \begin{pmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_0 & x_3 \end{pmatrix}^t \in P_2.$$

In coordinates of $\mathbb{C}(s,t)$ the slice is

$$S_x(t_1, \ldots, t_{12}) = \begin{pmatrix} s & t_1t & -t_{11}s + t_2t & 0 & -t_{12}s + t_3t \\ 0 & s & t_4t & 0 & t_5t \\ t & 0 & t_6s + t_{11}t & t_7s & t_8s + t_{12}t \\ 0 & 0 & t_9s & t & t_{10}s \end{pmatrix}.$$

Using rank considerations as before, one finds the following

- $S_x \cap \overline{S_1} \cong \mathbb{A}^4$ defined by the equations

  $$t_1 = t_2 = t_3 = t_6 = t_7 = t_8 = t_{11} = t_{12} = 0.$$

  If furthermore $t_9 = t_{10} = 0$ or $t_4 = t_5 = 0$ (where not all $t_i = 0$), the matrix belongs to a strictly semi-stable Kronecker modules which is identified with P_2 by the $\mathbb{C}^*$-action on the slice.

- the intersection $S_x \cap \overline{S_3}$ consists of two components that are isomorphic to $\mathbb{A}^3$ that intersect exactly in the point x: the components are given by the equations

  $$t_1 = t_2 = t_3 = t_6 = t_8 = t_9 = t_{10} = t_{11} = t_{12} = 0$$

  resp.

  $$t_2 = t_3 = t_4 = t_5 = t_6 = t_7 = t_8 = t_{11} = t_{12} = 0.$$

  As before, this components contain strictly semi-stable matrices.

- $S_x \cap \overline{P_1} \cong \mathbb{A}^2$ defined by $t_1 = \ldots = t_{10} = 0$.

Using this, we can show the assertions of the Lemma:

(a) As in the Proof of Lemma 3.8, it suffices to show that x is a regular point in $\overline{P_1} \cap S_x$ resp. $\overline{S_1} \cap S_x$. This assertions are obvious, as $\overline{P_1} \cap S_x \cong \mathbb{A}^2$ resp. $\overline{S_1} \cap S_x \cong \mathbb{A}^4$.

(b) This is obvious, as the two components of $S_3 \cap S_x$ intersect in 0.

(c) Obvious. ∎

Discussion 3.19 The aim is to explain a procedure to extend the family of curves parametrized by the blow-ups of W^{ss} along P_1, $S_1 \cup S_2$ and S_3 across P_2. Due to runtime-problems in executing this process, we postpone its explicit realization to further research.

We consider the situation locally on the Luna slice $S_x \cong \mathbb{A}^{12} =: X$ in the point

$$x = \begin{pmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_0 & x_3 \end{pmatrix}^t.$$

By [17], it suffices to compute the blow-up of the slice along the preimage of $\overline{P_1}$ under the quotient map $S_x \to S_x /\!\!/ \mathbb{C}^*$ and take the quotient after all computations.

For simplicity we introduce the following notation:

$$W := \{t_1 = \ldots = t_5 = 0\} \subset \mathbb{A}^{12} \text{ and } W' := \{t_6 = \ldots = t_{10} = 0\} \subset \mathbb{A}^{12}.$$

As explained in Section 3.4.1, the sets $W\backslash\{0\}$ and $W'\backslash\{0\}$ parametrize strictly semi-stable Kronecker modules, that are mapped to the poly-stable Kronecker module in P_1 associated to $(0,\ldots,0,t_{11},t_{12})$ under the $\mathbb{C}^*$-action on the slice (for any choice of fixed parameters t_{11} and t_{12} not both zero).

We further denote by V the subvariety

$$V=\{t_1=t_2=t_3=t_6=t_7=t_8=t_{11}=t_{12}=0\},$$

which parametrizes the points of $\overline{S_1}\cap S_x$, and by

$$U=\{t_1=t_2=t_3=t_6=t_8=t_9=t_{10}=t_{11}=t_{12}=0\}$$

and

$$U'=\{t_2=t_3=t_4=t_5=t_6=t_7=t_8=t_{11}=t_{12}=0\}$$

the subvarieties, which define the intersection of $\overline{S_3}$ with the slice S_x.

Step 1: Let $\pi_1: X_1\to X$ be the blow-up of X along W, i.e.

$$X_1=\mathrm{BL}_W(X)\subset\mathbb{A}^{12}\times\mathbb{P}^4$$

given by the equations $s_it_j-s_jt_i$, $1\le i,j\le 5$, where $s_1,\ldots,s_5$ are homogeneous coordinates of $\mathbb{P}^4$. Let W_1' be the strict transform of W' under π_1. Then W_1' is defined by the equations $t_j=0$ for $6\le j\le 10$ and $s_it_j-s_jt_i=0$ for $1\le i,j\le 5$.

Step 2: Let $\pi_2: X_2\to X_1$ be the blow-up of X_1 along the strict transform W_1'. Then $X_2\subset\mathbb{A}^{12}\times\mathbb{P}^4\times\mathbb{P}^4$ is defined by the equations $s_it_j-s_jt_i$ for $1\le i,j\le 5$ and $6\le i,j\le 10$.

Step 3: Next one has to compute the strict transform V_1 of V in X_1:

$$V_1=\mathrm{BL}_{V\cap W}(V)\subset\mathbb{A}^4\times\mathbb{P}^1,$$

where $\mathbb{A}^4\subset\mathbb{A}^{12}$ is cut out by the equations of V and $\mathbb{P}^1\subset\mathbb{P}^4$ by the equations $s_1=\ldots=s_3=0$.

Step 4: We continue by computing the strict transform V_2 of V_1 in X_2. The variety $V_2\subset V_1\times\mathbb{P}^1\subset\mathbb{A}^{12}\times\mathbb{P}^4\times\mathbb{P}^4$ is defined by $s_6=s_7=s_8=0$ and the equations of V_1. Hence V_2 is the subvariety of X_2 given by the equations $f_i:=t_i=0$ for $i=1,2,3$, $f_{j-2}:=t_j=0$ for $j=6,7,8$ and $f_7:=s_9t_{10}-s_{10}t_9=0$ and $f_8:=s_4t_5-s_5t_4=0$.

Step 5: The blow-up $X_3:=\mathrm{BL}_{V_2}(X_2)$ of X_2 along V_2 is then the subvariety of $\mathbb{A}^{12}\times\mathbb{P}^4\times\mathbb{P}^4\times\mathbb{P}^7$ defined by $r_if_j-r_jf_i=0$ for $1\le i,j\le 8$ and where r_i are homogeneous coordinates of $\mathbb{P}^7$.

Step 6: We compute the strict transform U_3 of U in X_3 in 3 substeps as before.

Step 7: Next we compute the blow-up $X_4 \subset \mathbb{A}^{12} \times \mathbb{P}^4 \times \mathbb{P}^4 \times \mathbb{P}^7 \times \mathbb{P}^8$ of X_3 along U_3.

Step 8: Then we compute the strict transform U_4' of U' in X_4 in 4 substeps as before.

Step 9: Now we compute the blow-up $X_5 \subset \mathbb{A}^{12} \times \mathbb{P}^4 \times \mathbb{P}^4 \times \mathbb{P}^7 \times \mathbb{P}^8 \times \mathbb{P}^8$ of X_4 along U_4'. In total, X_5 is given by 120 equations.

Step 10: The maximal minors of the matrices in the slice S_x define a family of ideals $\mathcal{I}$ parametrized by $\mathbb{A}^{12}$. After extending $\mathcal{I}$ to the blow-up X_5, saturation in the parameters $t_1, \ldots, t_{12}$ gives a new family $\tilde{\mathcal{I}}$.
As before, this must be done with a Computer Algebra System e.g. Singular. Then it remains to show that the associated family of closed subschemes in $\mathbb{P}^4$ is flat, i.e. any of these closed subschemes is a curve in $\mathbb{P}^4$ with Hilbert polynomial $4n + 1$. Hopefully this can be done as in the cases treated above.

4.3 The Stratum P_3

Recall that a Kronecker module in P_3 has the form $A = A_1^{\oplus 2}$ for a Kronecker module A_1 of type $(2,1)$. In order to determine the type of singularities of K in this points we again use the Luna slice theorem. As in this case the stabilizer is $\mathrm{GL}_2(\mathbb{C})$, the situation here is more complicated than in 3.4.1 and 3.3. Hence we restricted ourselves to giving a model of the singularity.
The Luna slice at A can locally be described using $\mathrm{Ext}^1(A,A)$. Note that $\dim \mathrm{Ext}^1(A_1,A_1) = 6$.
Now

$$\begin{aligned} \mathrm{Ext}^1(A,A) &= \mathrm{Ext}^1(A_1^{\oplus 2}, A_1^{\oplus 2}) \\ &= \mathrm{Ext}^1(A_1,A_1) \otimes \mathrm{End}(\mathbb{C}^2) \\ &= \mathrm{Ext}^1(A_1,A_1) \otimes \mathrm{End}(\mathfrak{gl}_2) \\ &= (\mathrm{Ext}^1(A_1,A_1) \otimes \mathbb{C} \cdot \mathrm{id}) \oplus (\mathrm{Ext}^1(A_1,A_1) \otimes \mathfrak{sl}_2). \end{aligned}$$

The deformations in $\mathrm{Ext}^1(A_1,A_1) \otimes \mathbb{C} \cdot \mathrm{id}$ are tangential to the stratum and the ones in $\mathrm{Ext}^1(A_1,A_1) \otimes \mathfrak{sl}_2$ are transversal to the stratum. Hence the Luna slice is given by $\mathrm{Ext}^1(A_1,A_1) \otimes \mathfrak{sl}_2$ and dimension $\dim \mathrm{Ext}^1(A_1,A_1) \otimes \mathfrak{sl}_2 = 6 \cdot 3 = 18$ by Lemma 3.14.
There is an action of the stabilizer of A on the slice, i.e. of

$$\mathrm{SL}_2/\{\pm 1\} = \mathrm{PGL}_2 = \mathrm{PSL}_2 \cong \mathrm{SO}_3.$$

Then $\mathbb{C}^{18} /\!\!/ \mathrm{SO}_3$ is a local model of the slice in A, i.e. there is a map

$$\mathbb{C}^{18} /\!\!/ \mathrm{SO}_3 \to W^{ss} /\!\!/ G$$

that maps a neighborhood of 0 étale onto a neighborhood of $[A]$ in $W^{ss} /\!\!/ G$.

Lemma 3.20 *There is a $(2:1)$-covering*

$$\mathbb{C}^{18} /\!\!/ \mathrm{SO}_3 \to \mathbb{C}^{18} /\!\!/ \mathrm{O}_3.$$

The following Proposition gives an explicit description of the slice at a point in P_3:

Proposition 3.21 *We have two types of invariants for the action of*

$$\mathrm{SO}_3 \cong \mathrm{PSL}_2$$

on the slice $\mathfrak{sl}_2 \times \ldots \times \mathfrak{sl}_2$: Let $X = (w_1, \ldots, w_6) \in \mathfrak{sl}_2 \times \ldots \times \mathfrak{sl}_2$, i.e. $X \in M^{3\times 6}(\mathbb{C}^6)$.

(a) *Let* $S := X^t X \in \mathrm{Sym}_6(\mathbb{C})$*. Then the entries* $s_{ij} = w_i^t w_j$ *of the matrix* S *are invariants.*

(b) *Let* $T := \Lambda^3 X \in \mathbb{C}^{\binom{3}{3}\times\binom{6}{3}} = \mathbb{C}^{20}$*. Then* $t_{ijk} := \det(w_i w_j w_k)$ *are invariants.*

There are $\binom{7}{2} = 21$ *invariants of the first type and* $\binom{6}{3} = 20$ *invariants of the second type.*
For these invariants we have the following relations:

(a) *Since* $\mathrm{rank}(S) \leq 3$*,* $\Lambda^4 S = 0$*. So the* (4×4)*-minors of* S *give* 120 *relations.*

(b) *Consider the* (9×6)*-matrix* $\left(\begin{smallmatrix} X \\ S \end{smallmatrix}\right) = \left(\begin{smallmatrix} X \\ X^t X \end{smallmatrix}\right)$*. Let* Y *be a matrix consisting of the rows* 1 *to* 3 *of* $\left(\begin{smallmatrix} X \\ S \end{smallmatrix}\right)$ *(i.e. of* X*) and another arbitrary row. By construction* $\mathrm{rang}(Y) < 4$*. That implies that* $\Lambda^4 Y = 0$*. Each entry of* $\Lambda^4 Y$ *is for some* i, j, k, l, m *of the form*

$$\begin{aligned}
&\det\begin{pmatrix} w_i^1 & w_j^1 & w_k^1 & w_l^1 \\ w_i^2 & w_j^2 & w_k^2 & w_l^2 \\ w_i^3 & w_j^3 & w_k^3 & w_l^3 \\ w_m^t w_i & w_m^t w_j & w_m^t w_k & w_m^t w_l \end{pmatrix} \\
&= -w_m^t w_i t_{jkl} + w_m^t w_j t_{ikl} - w_m^t w_k t_{ijl} + w_m^t w_l t_{ijk} \\
&= -s_{mi} t_{jkl} + s_{mj} t_{ikl} - s_{mk} t_{ijl} + s_{ml} t_{ijk}.
\end{aligned}$$

There are 90 *relations of this type.*

(c) *Let* $S_{l,m,n}^{i,j,k}$ *be the submatrix of* S *consisting of the rows* i, j, k *and the columns* l, m, n *and let* $X_{i,j,k}$ *and* $X_{l,m,n}$ *be the submatrices of* X *consisting of the columns* i, j, k *resp* l, m, n*. Then*

$$S_{l,m,n}^{i,j,k} = X_{i,j,k}^t X_{l,m,n}.$$

So

$$\begin{aligned}
\det(S_{l,m,n}^{i,j,k}) &= \det(X_{i,j,k}^t X_{l,m,n}) \\
&= t_{i,j,k} t_{k,l,m}.
\end{aligned}$$

That means we obtain the following relations:

$$t_{ijk} t_{lmn} = \det(S_{l,m,n}^{i,j,k})$$

for all i, j, k, l, m, n*. There are* 210 *Relations of this type.*

Proof: It is clear, that all the listed elements are invariants and that the listed relations hold. The main theorems of classical invariant theory for SO_3 imply that there are no more invariants and relations, see [31][Theorem 2.9A]. ∎

If we take a relation of the third kind with $i = l, j = m$ and $k = n$, we obtain

$$t_{ijk}^2 = \det(S_{ijk}^{ijk}),$$

that means up to a sign we can reconstruct the matrix T from the entries of the matrix S. (the signs of the minors of S are linked). The additional information in the $(2:1)$-covering

$$\mathbb{C}^{18} /\!\!/ \mathrm{SO}_3 \to \mathbb{C}^{18} /\!\!/ \mathrm{O}_3$$

is exactly this sign. The invariants $s_{i,j}$ are O_3-invariants, but the $t_{i,j,k}$ are just SO_3 invariants.

5 Open Problems

In order to extend the rational map $\Phi_{KH} : K \dashrightarrow H_0 \subset \mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$ to the blow-up of K in P (and especially to points over the exceptional divisor over P_3), one needs to know a model for the blow-up of K in P_3. But the combinatorics are quite complicated, since the model of the singularity in a point in P_3 has 41 invariants with 420 relations, as we have seen in Proposition 3.21. Furthermore one needs to carry out the computations from Discussion 3.19.

Furthermore it would be useful to know more about the component H_0 in the Hilbert scheme $\mathrm{Hilb}^{4n+1}(\mathbb{P}^4)$, especially about the curves that are not arithmetically Cohen-Macaulay.

4

The Moduli Space $M^{4n+2}(\mathbb{P}^4)$ of Semi-stable Sheaves

In this Chapter we first recall some facts about moduli spaces of semi-stable sheaves. In particular, we are interested in the moduli space $M^{4n+2}(\mathbb{P}^4)$ of semi-stable sheaves on $\mathbb{P}^4$ with Hilbert polynomial $4n+2$. We construct a birational map from K to some component M_4 of $M^{4n+2}(\mathbb{P}^4)$ with domain of definition as large as possible. Furthermore we study the stratum $S^2(M^{2n+1}(\mathbb{P}^4)_0)$ of strictly poly-stable sheaves in M_4 and show that M_4 is non-singular in general points of $S^2(M^{2n+1}(\mathbb{P}^4)_0)$.

1 (Semi-)stable Sheaves

In this Section we want to recall some basics about moduli spaces of semi-stable sheaves that we will use in this Chapter.
Let $\mathcal{F}$ be a coherent sheaf on a projective variety X of dimension d.

Definition 4.1 (pure) *$\mathcal{F}$ is pure of dimension d if $\dim(\mathcal{E}) = d$ for all non-trivial coherent subsheaves $\mathcal{E} \subset \mathcal{F}$.*

If X is integral and $\mathcal{F}$ a coherent sheaf on X of dimension d, then $\mathcal{F}$ is pure if and only if it is torsion free.

Definition 4.2 (reduced Hilbert polynomial) *We fix an ample line bundle $\mathcal{O}(1)$ on X. Then the Hilbert polynomial $P_{\mathcal{F}}(m) := \chi(\mathcal{F} \otimes \mathcal{O}(m))$ can be written as*

$$P_{\mathcal{F}}(m) = \sum_{i=0}^{\dim(\mathcal{F})} \alpha_i(\mathcal{F}) \frac{m^i}{i!}$$

with rational coefficients $\alpha_i(\mathcal{F})$.

The reduced Hilbert polynomial $p_{\mathcal{F}}$ *of a coherent sheaf* $\mathcal{F}$ *of dimension* d *is defined by*

$$p_{\mathcal{F}}(m) := \frac{P_{\mathcal{F}}(m)}{\alpha_d(\mathcal{F})}.$$

Definition 4.3 (semi-stable, stable, poly-stable sheaf) *(a) A coherent sheaf* $\mathcal{F}$ *of dimension* d *is semi-stable if* $\mathcal{F}$ *is pure and for any non-trivial proper subsheaf* $\mathcal{E} \subset \mathcal{F}$ *one has* $p_{\mathcal{E}} \leq p_{\mathcal{F}}$.

(b) A coherent sheaf $\mathcal{F}$ *of dimension* d *is stable if* $\mathcal{F}$ *is semi-stable and for any non-trivial proper subsheaf* $\mathcal{E} \subset \mathcal{F}$ *one has* $p_{\mathcal{E}} < p_{\mathcal{F}}$.

(c) A semi-stable sheaf $\mathcal{F}$ *is called poly-stable if* $\mathcal{F}$ *is the direct sum of stable sheaves.*

Definition 4.4 *For any integer* m*, a coherent sheaf* $\mathcal{F}$ *is called* m*-regular, if*

$$H^i(X, \mathcal{F}(m-i)) = 0 \text{ for all } i > 0.$$

If $\mathcal{F}$ is m-regular, then $\mathcal{F}$ is m'-regular for all $m' > m$. Furthermore $\mathcal{F}(m)$ is globally generated.

Definition 4.5 *A family of isomorphism classes of coherent sheaves on* X *is bounded if there is a* k*-scheme* S *of finite type and a coherent* $\mathcal{O}_{S\times X}$*-sheaf* $\mathcal{F}$ *such that the given family is contained in the set*

$$\{\mathcal{F}|_{\mathrm{Spec}(k(s))\times X} \mid s \text{ a closed point in } S\}.$$

Theorem 4.6 *The family of semi-stable sheaves with Hilbert polynomial* P *is bounded.*

PROOF: [15][Theorem 3.3.7] ∎

In the following we want to recall shortly how to construct a moduli space for semi-stable sheaves with a fixed Hilbert polynomial on some projective variety X.

Definition 4.7 *Let* $\mathcal{C}$ *be a category,* $\mathcal{C}^\circ$ *the opposite category and* $\mathcal{C}'$ *the category of functors* $\mathcal{C}^\circ \to (Sets)$*, whose morphisms are natural transformations between functors.*
A functor $\mathcal{F} \in Ob(\mathcal{C}')$ *is corepresented by* $F \in Ob(\mathcal{C})$*, if there is a* $\mathcal{C}'$*-morphism* $\alpha : \mathcal{F} \to \underline{F}$ *such that any morphism* $\alpha' : \mathcal{F} \to \underline{F'}$ *factors through a unique morphism* $\beta : \underline{F} \to \underline{F'}$*. If* α *is a* $\mathcal{C}'$*-isomorphism, we say that* $\mathcal{F}$ *is represented by* F*.*

Now we shortly recall the definition of the Quot-scheme constructed by Grothendieck. For that, we consider the category $\mathcal{C}$ of schemes over some k-scheme S of finite type. Moreover, let $f : X \to S$ be a projective morphism and $\mathcal{O}_X(1)$ an f-ample line bundle on X. Let $\mathcal{H}$ be a coherent $\mathcal{O}_X$-module and $P \in \mathbb{Q}[z]$ a fixed polynomial.
We define a functor $\mathcal{Q} : (Sch/S)^0 \to (Sets)$ as follows: for any scheme $T \to S$, $\mathcal{Q}(T)$ is the set of all T-flat coherent quotient sheaves $\mathcal{H}_T = \mathcal{O}_T \otimes \mathcal{H} \to \mathcal{F}$, such that the Hilbert polynomial of $\mathcal{H}_T$ is P. For any S-morphism $g : T' \to T$ we define the map $\mathcal{Q}(g) : \mathcal{Q}(T) \to \mathcal{Q}(T')$ by sending $\mathcal{H}_T \to F$ to $\mathcal{H}_{T'} \to g^*F$.

Theorem 4.8 *The functor $\mathcal{Q}$ is represented by a projective S-scheme*

$$\pi : Quot(\mathcal{H}, P) \to S.$$

Proof: See [15][Theorem 2.2.4]. ∎

Definition 4.9 (Jordan-Hölder Filtration) *Let $\mathcal{F}$ be a semi-stable sheaf of dimension d. A Jordan-Hölder filtration of $\mathcal{F}$ is a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_l = \mathcal{F},$$

such that for all i the factors $\mathrm{gr}_i(\mathcal{F}) := \mathcal{F}_i/\mathcal{F}_{i-1}$ are stable with reduced Hilbert polynomial $p_{\mathcal{F}}$.

Remark 4.10 (a) In the above definition all the $\mathcal{F}_i$ for $i \neq 0$ are semi-stable with reduced Hilbert polynomial $p_{\mathcal{F}}$.

(b) A Jordan-Hölder filtration need not be unique.

(c) A Jordan-Hölder filtration always exists: [15][Prop 1.5.2]

(d) The sheaf $\mathrm{gr}(\mathcal{F}) := \bigoplus \mathrm{gr}_i(\mathcal{F})$ does not depend on the choice of the Jordan-Hölder sequence: [15][Prop 1.5.2]

Definition 4.11 (S-equivalence) *Two semi-stable sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ with the same reduced Hilbert polynomial are called S-equivalent if $\mathrm{gr}(\mathcal{F}_1) \cong \mathrm{gr}(\mathcal{F}_2)$.*

Remark 4.12 Every S-equivalence class of semi-stable sheaves contains exactly one poly-stable sheaf up to isomorphism.

Let X be a projective scheme over $\mathbb{C}$, $\mathcal{O}_X(1)$ a fixed ample line bundle. Furthermore fix a polynomial $P \in \mathbb{Q}[z]$. Then a moduli functor is defined by

$$\begin{aligned}\mathcal{M} : (\mathrm{Sch}/\mathbb{C})^0 \to & \text{ Sets} \\ & S \mapsto \left\{ \begin{array}{c} \text{isomorphism classes of S-flat families of semi-} \\ \text{stable sheaves on X with Hilbert polynomial P} \end{array} \right\} \Big/ \sim\end{aligned}$$

where $\mathcal{F}_1 \sim \mathcal{F}_2$ if and only of there exists a line bundle L on S such that $\mathcal{F}_1 \cong \mathcal{F}_2 \otimes p^* L$, where $p : S \times_{\mathbb{C}} X \to S$ is the projection on the first component. Similarly, we define a moduli functor $\mathcal{M}^s$ for stable sheaves on X with Hilbert polynomial P.

Definition 4.13 *A scheme M is called a moduli space of semi-stable sheaves, if it corepresents the functor $\mathcal{M}$.*

Lemma 4.14 *(a) If $\mathcal{M}$ is corepresented by some scheme M, then S-equivalent sheaves correspond to identical points in M.*

(b) If there is a strictly semi-stable sheaf $\mathcal{F}$, then $\mathcal{M}$ cannot be represented.

We shortly repeat the idea of the construction of a moduli space for semi-stable sheaves:
The family of semi-stable sheaves on X with the fixed Hilbert polynomial P is bounded. This means that there is an $m \in \mathbb{N}$, such that $\mathcal{F}$ is m-regular and hence $\mathcal{F}(m)$ globally generated and $h^0(\mathcal{F}(m)) = P(m)$.
We define a vector space V as $V := k^{\oplus P(m)}$ and a sheaf $\mathcal{H}$ as $\mathcal{H} := V \otimes_k \mathcal{O}_X(-m)$.
The composition of the evaluation map $H^0(\mathcal{F}(m)) \otimes \mathcal{O}(-m) \to \mathcal{F}$ and the isomorphism $V \to H^0(\mathcal{F}(m))$ gives a surjective map $\rho : \mathcal{H} \to \mathcal{F}$.
This defines a closed point $[\rho : \mathcal{H} \to \mathcal{F}] \in \mathrm{Quot}(\mathcal{H}, P)$ in the open set R in the Quot scheme $\mathrm{Quot}(\mathcal{H}, P)$, where

$$R := \left\{ [\mathcal{H} \to \mathcal{E}] \in \mathrm{Quot}(\mathcal{H}, P) \,\middle|\, \begin{array}{l} \mathcal{E} \text{ semi-stable} \\ V = H^0(\mathcal{H}(m)) \cong H^0(\mathcal{E}(m)) \end{array} \right\}$$

Then R parametrizes semi-stable sheaves with Hilbert polynomial P, where we have an ambiguity given by the arbitrary choice of a basis of the vector space $H^0(\mathcal{F}(m))$.
The group $\mathrm{GL}(V)$ acts on $\mathrm{Quot}(\mathcal{H}, P)$ by composition:

$$[\rho] \circ g := [\rho \circ g]$$

for ρ an S-valued point in $\mathrm{Quot}(\mathcal{H}, P)$ and g an S-valued point is $\mathrm{GL}(V)$.

Let $p : \text{Quot}(\mathcal{H}, P) \times X \to \text{Quot}(\mathcal{H}, P)$ and $q : \text{Quot}(\mathcal{H}, P) \times X \to X$ be the projections and $\rho : q^*\mathcal{H} \to \tilde{\mathcal{F}}$ the universal quotient on $\text{Quot}(\mathcal{H}, P) \times X$. Then by [15][Prop. 2.2.5], the line bundle

$$L_l := \det(p_*(\tilde{\mathcal{F}} \otimes q^*\mathcal{O}_X(l)))$$

on $\text{Quot}(\mathcal{H}, P)$ is very ample if l is big enough and it admits an $\text{SL}(V)$-linearization.

One can show, that for $l \gg 0$, a point $[q]$ in the Quot scheme $\text{Quot}(\mathcal{H}, P)$ is (semi-)stable with respect to the line bundle L_l and the action of $\text{SL}(V)$ in the sense of Mumford's geometric invariant theory if and only if the associated sheaf $\mathcal{F}$ is (semi-)stable in the sheaf-theoretic sense.

A moduli space of semi-stable sheaves can now be constructed as the categorical quotient of the open subset $R = \overline{R}^{ss}(L_l)$ of semi-stable points in R with respect to L_l by the action of $\text{SL}(V)$. (see [15][Theorem 4.2.10])

Concretely, one obtains the following result:

Theorem 4.15 *There is a projective scheme $M_{\mathcal{O}_X(1)}(P)$ that corepresents the functor $\mathcal{M}_{\mathcal{O}_X(1)}(P)$. Closed points in $M_{\mathcal{O}_X(1)}(P)$ are in bijection with S-equivalence classes of semi-stable sheaves with Hilbert polynomial P. Moreover, there is an open subset $M^s_{\mathcal{O}_X(1)}(P)$ that corepresents the functor $\mathcal{M}^s_{\mathcal{O}_X(1)}(P)$.*

Proof: [15][Theorem 4.3.4] ∎

Proposition 4.16 *Let $\mathcal{F}$ be a stable point of the moduli space M of stable sheaves. Then the Zariski tangent space of M at $[\mathcal{F}]$ is canonically given by $T_{[\mathcal{F}]}M \cong \text{Ext}^1(\mathcal{F}, \mathcal{F})$. If $\text{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$, then M is smooth at $[\mathcal{F}]$. In general there are bounds $\text{ext}^1(\mathcal{F}, \mathcal{F}) \geq \dim_{[\mathcal{F}]} M \geq \text{ext}^1(\mathcal{F}, \mathcal{F}) - \text{ext}^2(\mathcal{F}, \mathcal{F})$.*

Proof: [15][Corollary 4.5.2] ∎

2 The Moduli Space $M^{2n+1}(\mathbb{P}^4)$

For the rest of this thesis we will consider the moduli space $M^{4n+2}(\mathbb{P}^4)$ of semi-stable sheaves on $\mathbb{P}^4$ with Hilbert polynomial $4n+2$. This moduli space contains both strictly semi-stable and stable objects. The S-equivalence class of any strictly semi-stable object contains a unique strictly poly-stable sheaf (up to isomorphism) which is the direct sum of two stable sheaves on $\mathbb{P}^4$ with Hilbert polynomial $2n+1$. Hence in order to understand the strictly semi-stable locus of the moduli space $M^{4n+2}(\mathbb{P}^4)$, we first want to study the moduli space $M^{2n+1}(\mathbb{P}^4)$ of stable sheaves on $\mathbb{P}^4$ with Hilbert polynomial $2n+1$.

The Grassmannian $\mathrm{Grass}(\mathbb{C}^5, 3)$ parametrizes planes in $\mathbb{P}^4$. The tautological sequence of $\mathrm{Grass}(\mathbb{C}^5, 3)$ is

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{B} \arrow[r] & \mathcal{O}_{\mathrm{Grass}(\mathbb{C}^5,3)} \otimes \mathbb{C}^5 \arrow[r] & \mathcal{A} \arrow[r] & 0 \end{tikzcd},$$

where $\mathcal{A}$ is a bundle of rank 3. Then $\mathbb{P}(S^2\mathcal{A}^*)$ is a $\mathbb{P}^5$-bundle over $\mathrm{Grass}(\mathbb{C}^5, 3)$, which parametrizes quadrics in a given plane $E \in \mathrm{Grass}(\mathbb{C}^5, 3)$.
Since $\dim(\mathrm{Grass}(\mathbb{C}^5, 3)) = 6$, this means that $\mathbb{P}(S^2\mathcal{A}^*)$ is smooth and of dimension 11:

$$\begin{tikzcd} M^{2n+1}(\mathbb{P}^4) & \mathbb{P}(S^2\mathcal{A}^*) \arrow[l, "\varphi"'] \arrow[d] \\ & \mathrm{Grass}(\mathbb{C}^5, 3), \end{tikzcd}$$

where φ associates to any conic C the structure sheaf $\mathcal{O}_C$ of C. The main Theorem of this Section is:

Theorem 4.17 *The map*

$$\varphi : \mathbb{P}(S^2\mathcal{A}^*) \to M^{2n+1}(\mathbb{P}^4)\,,\, C \mapsto \mathcal{O}_C,$$

which associates to a conic C the structure sheaf $\mathcal{O}_C$, is bijective.

So the sheaves in the moduli space $M^{2n+1}(\mathbb{P}^4)$ are structure sheaves of smooth plane conics, plane double lines and two lines intersecting in one point.
In the proof, the following two results will be used:

Theorem 4.18 *([11][Prop 18.9]) Let $C \subset \mathbb{P}^n$ be a reduced irreducible curve of degree d and $\langle C\rangle$ the smallest linear subspace in $\mathbb{P}^n$ that contains C. Then* $\dim(\langle C\rangle) \leq \deg(C)$ *and* $\dim(\langle C\rangle) = \deg(C)$ *if and only if C is a rational normal curve in $\langle C\rangle$.*

Proposition 4.19 *([29][III.6 Prop 6]) If $\mathcal{F}$ is a coherent algebraic sheaf on $\mathbb{P}^m$, then*

$$\deg(P_{\mathcal{F}}(n)) = \dim(\mathrm{Supp}(\mathcal{F})).$$

Proof of 4.17: We need to show that for any sheaf $\mathcal{F} \in M^{2n+1}(\mathbb{P}^4)$ there exists a curve C of one of the types mentioned above, such that $\mathcal{F}$ is the structure sheaf $\mathcal{O}_C$ of C. So consider a sheaf $\mathcal{F} \in M^{2n+1}(\mathbb{P}^4)$. By Proposition 4.19 we have $\dim(\mathrm{Supp}(\mathcal{F})) = 1$. Hence

$$(\mathrm{Supp}(\mathcal{F}))_{\mathrm{red}} = C_1 \cup \ldots \cup C_s$$

is the union of irreducible reduced curves C_i. If η_i are the generic points of the components C_i and $\alpha_i(\mathcal{F})$ the i-th coefficient of the Hilbert polynomial

$$P_{\mathcal{F}}(n) = \sum_{i=0}^{\dim(\mathcal{F})} \alpha_i(\mathcal{F}) \frac{n^i}{i!},$$

then we have

$$2 = \sum_{i=1}^{s} \mathrm{length}(\mathcal{F}_{\eta_i}) \cdot \deg(C_i) \geq \sum_{i=1}^{s} \mathrm{rank}(\mathcal{F}|_{C_i}) \deg(C_i). \tag{1}$$

If $\widetilde{C}_i$ is an irreducible component of $\mathrm{Supp}(\mathcal{F})$ and $\widetilde{C}_i$ is reduced, then the rank of the sheaf $\mathcal{F}|_{\widetilde{C}_i}$ is computed by $\mathrm{length}(\mathcal{F}_{\eta_i}) = \mathrm{rank}(\mathcal{F}|_{\widetilde{C}_i})$ and hence we have equality in equation (1), if $\mathrm{Supp}(\mathcal{F})$ is reduced.

So if $C := C_i$ is an irreducible reduced curve contained in $(\mathrm{Supp}(\mathcal{F}))_{\mathrm{red}}$, then for the rank of $\mathcal{F}|_C$ and the degree of C there are the following three possibilities:

1) $r = \mathrm{rank}(\mathcal{F}|_C) = 1$, $\deg(C) = 2$,
2) $r = 2$, $\deg(C) = 1$,
3) $r = 1$, $\deg(C) = 1$.

Before considering these cases separately, we need some preparations. For this we assume that $\mathrm{Supp}(\mathcal{F})_{\mathrm{red}}$ consists of exactly one component C.
By ν we denote the ideal sheaf $\mathcal{I}_{C/\,\mathrm{Supp}(\mathcal{F})}$ of C in $\mathrm{Supp}(\mathcal{F})$. Then the pure sheaf $\mathcal{F}$ has a filtration

$$\mathcal{F} \supset \nu\mathcal{F} \supset \nu^2\mathcal{F} \supset \cdots,$$

where the quotient sheaves $\nu^i\mathcal{F}/\nu^{i+1}\mathcal{F}$ are not necessarily pure. Hence we consider the saturations

$$\mathcal{F}_i := \overline{\nu^i\mathcal{F}}$$

of $\nu^i\mathcal{F}$ in $\mathcal{F}$. We thus obtain a filtration

$$\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots ,$$

such that the quotients $\mathcal{F}_i/\mathcal{F}_{i+1}$ are pure $\mathcal{O}_C$-modules. In fact, by definition $\overline{\nu^i\mathcal{F}}$ is the minimal subsheaf of $\mathcal{F}$ containing $\nu^i\mathcal{F}$ such that $\mathcal{F}/\overline{\nu^i\mathcal{F}}$ is pure.
The surjective maps

$$\nu^i\mathcal{F} \otimes \nu \twoheadrightarrow \nu^{i+1}\mathcal{F}$$

obviously induce maps for the saturated sheaves $\mathcal{F}_i$

$$\mathcal{F}_i \otimes \nu \longrightarrow \mathcal{F}_{i+1}$$

which still are generically surjective.
Taking quotients, these maps then give rise to morphisms

$$\mathcal{F}_i/\mathcal{F}_{i+1} \otimes \mathcal{N}^\star_{C/\operatorname{Supp}(\mathcal{F})} \longrightarrow \mathcal{F}_{i+1}/\mathcal{F}_{i+2}$$

that are also generically surjective.
Hence if $\operatorname{rank}(\mathcal{F}_i/\mathcal{F}_{i+1}) = 0$ it follows that $\operatorname{rank}(\mathcal{F}_j/\mathcal{F}_{j+1}) = 0$ for all $j \geq i$.
We obtain the following formula for the Hilbert polynomials:

$$2n + 1 = P_{\mathcal{F}}(n) = P_{\mathcal{F}_0/\mathcal{F}_1}(n) + P_{\mathcal{F}_1/\mathcal{F}_2}(n) + \cdots .$$

Hence

$$\frac{2}{\deg(C)} = \frac{2}{\alpha_1(\mathcal{O}_C)} = \operatorname{rank}(\mathcal{F}_0/\mathcal{F}_1) + \operatorname{rank}(\mathcal{F}_1/\mathcal{F}_2) \qquad (2)$$

and in particular $\mathcal{F}_i = 0$ for all $i \geq 2$.
We now consider the three cases $1, 2$ and 3 separately:

Case 1: $r = 1$ **and** $\deg(C) = 2$

If $r = 1$ and $\deg(C) = 2$, then $\dim\langle C\rangle = 2$ and hence C is a rational normal curve of degree 2 by Theorem 4.18.
As $\operatorname{length}(\mathcal{F}_\eta)\cdot\deg(C) = 2$, the support $(\operatorname{Supp}\mathcal{F})_{\text{red}}$ does not contain any other 1-dimensional component and $\operatorname{Supp}(\mathcal{F})$ is generically reduced.
Since $\alpha_1(\mathcal{O}_C) = \deg(C) = 2$, we have $\operatorname{rank}(\mathcal{F}_0/\mathcal{F}_1) = 1$ and $\operatorname{rank}(\mathcal{F}_1/\mathcal{F}_2) = 0$ in equation (2), hence $\mathcal{F}_1 = 0$. Thus $\mathcal{F} = \mathcal{F}_0$ is a locally free $\mathcal{O}_C$-module, i.e. $\mathcal{F} = \mathcal{O}_C(a)$ for some $a \in \mathbb{Z}$. Since $P_{\mathcal{F}}(n) = 2n+1$, we have $a = 0$. So $\mathcal{F} = \mathcal{O}_C$ with C a rational normal curve of degree 2 in $\mathbb{P}^2$.

Case 2: $r = 2$ **and** $\deg(C) = 1$

Second, if $r = 2$ and $\deg(C) = 1$, then $C = L$ is a line and as in a) $\mathrm{Supp}(\mathcal{F})$ does not contain another 1-dimensional component and $\mathrm{Supp}(\mathcal{F})$ is generically reduced. Hence in the equation

$$2 = \mathrm{rank}(\mathcal{F}_0/\mathcal{F}_1) + \mathrm{rank}(\mathcal{F}_1/\mathcal{F}_2),$$

we have $\mathrm{rank}(\mathcal{F}_0/\mathcal{F}_1) = 2$ and $\mathrm{rank}(\mathcal{F}_1/\mathcal{F}_2) = 0$ and so $\mathcal{F}_1 = 0$. Then $\mathcal{F} = \mathcal{F}_0$ is a locally free $\mathcal{O}_L$-module of rank 2, i.e. $\mathcal{F} = \mathcal{O}_L(a) \oplus \mathcal{O}_L(b)$ for some $a, b \in \mathbb{Z}$. As $\mathcal{F}$ is semi-stable, we have $a = b$. The Hilbert polynomial of $\mathcal{F}$ is however

$$P_{\mathcal{F}}(n) = P_{\mathcal{O}_C(a)^{\oplus 2}}(n) = 2(n + a + 1) \neq 2n + 1$$

for all $a \in \mathbb{Z}$. Hence this case does not occur.

Case 3: $r = 1$ **and** $\deg(C) = 1$

This is the most complicated case. If $r = 1$ and $\deg(C) = 1$, then C is again a line. In this case, $\mathrm{length}(\eta) \cdot \deg(C) < 2$, but since $\alpha_1(\mathcal{F}) = 2$, either $(\mathrm{Supp}(\mathcal{F}))_{\mathrm{red}}$ contains another 1-dimensional component or $\mathrm{Supp}(\mathcal{F})$ is everywhere non-reduced. We consider these two cases separately:

Case 3.1: Suppose $\mathrm{Supp}(\mathcal{F})$ contains another component C' of dimension 1. Then C' is a line, too. Furthermore $\mathrm{rank}(\mathcal{F}|_{C'}) = 1$ and then

$$\mathrm{rank}(\mathcal{F}|_C) \cdot \deg(C) + \mathrm{rank}(\mathcal{F}|_{C'}) \cdot \deg(C') = 2$$

and so $\mathrm{Supp}(\mathcal{F})$ is generically reduced.
The lines C and C' have to meet in one point q, because if they were disjoint, then

$$2n + 1 = P_{\mathcal{F}}(n) = P_{\mathcal{F}|_C}(n) + P_{\mathcal{F}|_{C'}}(n) = (n + c_1) + (n + c_2),$$

where $c_1 + c_2 = 1$. But this is a contradiction since $\mathcal{F}$ is stable.

The sheaf $\mathcal{F}$ being pure, we have the decompositions $\mathcal{F}|_C = \mathcal{O}_C(a) \oplus T$ and $\mathcal{F}|_{C'} = \mathcal{O}_{C'}(b) \oplus T'$ for sheaves T and T' supported on q.
The map

$$f : \mathcal{F} \to \mathcal{F}|_C \oplus \mathcal{F}|_{C'} \to (\mathcal{F}|_C)/T \oplus (\mathcal{F}|_{C'})/T' = \mathcal{O}_C(a) \oplus \mathcal{O}_{C'}(b)$$

is generically injective. But since $\mathcal{F}$ is pure, the kernel cannot be a subsheaf of $\mathcal{F}$ with 0-dimensional support. Hence f is injective.
Now assume that f is also surjective. Then $\mathcal{F} \cong \mathcal{O}_C(a) \oplus \mathcal{O}_{C'}(b)$. As $1 = \chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F})$, we have $h^0(\mathcal{F}) > 0$.

The stability of $\mathcal{F}$ gives $a, b < 0$, in particular $\mathcal{O}_C(a) \oplus \mathcal{O}_{C'}(b)$ does not have global sections. This is a contradiction and hence f is not surjective.

Since the cokernel of f is a sheaf supported on q with 1-dimensional stalk, there is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_C(a) \oplus \mathcal{O}_{C'}(b) \to \mathcal{O}_q \to 0$$

and therefore

$$2n + 2 + a + b = P_{\mathcal{O}_C(a) \oplus \mathcal{O}_{C'}(b)}(n) = P_{\mathcal{F}}(n) + P_{\mathcal{O}_q}(n) = (2n + 1) + 1.$$

So $a = -b$ and we can assume that $a \geq 0$ and $b \leq 0$. If we assume that $a > 0$, then $\mathcal{O}_C(a - 1)$ would be a destabilizing subsheaf of $\mathcal{F}$. Hence $a = b = 0$ and $\mathcal{F} = \mathcal{O}_{C \cup C'}$.

Case 3.2: Supp($\mathcal{F}$) is everywhere non-reduced.
By S we denote the (scheme-theoretic) support of $\mathcal{F}$. In this case C_{red} is a reduced line L and Supp($\mathcal{F}$) has no other component of dimension 1. Now rank($\mathcal{F}_0/\mathcal{F}_1$) = rank($\mathcal{F}_1/\mathcal{F}_2$) = 1. Hence there is an exact sequence

$$0 \to \mathcal{O}_L(l_1) = \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_0/\mathcal{F}_1 = \mathcal{O}_L(l_2) \to 0$$

for some $l_1, l_2 \in \mathbb{Z}$.
Then

$$\begin{aligned} 2n + 1 &= P_{\mathcal{F}}(n) \\ &= P_{\mathcal{F}_1}(n) + P_{\mathcal{F}_1/\mathcal{F}_2}(n) \\ &= n + l_1 + 1 + n + l_2 + 1 \\ &= 2n + l_1 + l_2 + 2 \end{aligned}$$

and therefore $l_1 + l_2 = -1$. Since $\mathcal{F}$ is stable, we have $l_1 < l_2$, otherwise $\mathcal{F}_1 = \mathcal{O}(l_1)$ would be a destabilizing subsheaf of $\mathcal{F}$.

Moreover, $S = \text{Supp}(\mathcal{F})$ is contained in the first infinitesimal neighborhood $4L$ of L in $\mathbb{P}^4$. Indeed, we know that $\mathcal{F}_2 = \overline{\nu^2 \mathcal{F}} = 0$. Since $\nu^2 \mathcal{F}$ is a subsheaf of the pure sheaf $\mathcal{F}$ and $\nu^2 \mathcal{F}$ coincides with $\overline{\nu^2 \mathcal{F}}$ generically, we have $\nu^2 \mathcal{F} = 0$. Obviously this implies that

$$I_{4L/\mathbb{P}^4} \subset \text{Ann}(\mathcal{F}) = \{f \in \mathcal{O}_{\mathbb{P}^4} \mid f \cdot s = 0 \text{ for all } s \in \mathcal{F}\} = I_{S/\mathbb{P}^4}$$

and therefore S is a subscheme of $4L$. In particular we have a surjective map $\mathcal{O}_{4L} \to \mathcal{O}_S$ and using that the conormal sheaf of L in $4L$ is $\mathcal{O}_L(-1)^3$, we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_L(-1)^3 \arrow[r] \arrow[d, "\widetilde{\psi}"] & \mathcal{O}_{4L} \arrow[r] \arrow[d] & \mathcal{O}_L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \nu = I_{L/S} \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[r] \arrow[d] & \mathcal{O}_L \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{3}$$

So there is a non-zero map $f : \mathcal{O}_L(l_2-1)^3 \to \mathcal{O}_L(l_1)$ defined as the composition

$$\begin{tikzcd}
\mathcal{O}_L(l_2-1)^3 = \mathcal{F}_0/\mathcal{F}_1 \otimes \mathcal{O}_L(-1)^3 \arrow[d, "\psi"] \arrow[dr, dotted, "f"] & \\
\mathcal{F}_0/\mathcal{F}_1 \otimes \mathcal{N}^\star_{L/S} \arrow[r, "\varphi"] & \mathcal{F}_1 = \mathcal{O}_L(l_1)
\end{tikzcd}$$

where the surjective map ψ is defined by $\widetilde{\psi}$ from diagram (3) and using that $\nu^2 = 0$. Hence $l_1 \geq l_2 - 1$ and then altogether, $l_1 = -1$ and $l_2 = 0$ so that $\mathcal{F}_1 = \mathcal{O}_L(-1)$ and $\mathcal{F}_0/\mathcal{F}_1 = \mathcal{O}_L$. Putting this in, the surjective map ψ reads

$$\psi : \mathcal{O}_L(-1)^3 \to \nu/\nu^2.$$

The sheaf $\nu = \mathcal{O}_L(m)$ is locally free. In fact, $\nu = \nu/\nu^2$ is an $\mathcal{O}_L$-module that is a subsheaf of $\mathcal{O}_S$. As $\mathcal{O}_S$ is pure, ν is torsion free and hence locally free on the reduced line L.
Since f factorizes via ν, we have $\nu = \mathcal{O}_L(-1)$. Hence after an appropriate coordinate change

$$\psi : \mathcal{O}_L(-1)^3 \to \nu = \mathcal{N}^*_{L/S} = \mathcal{O}_L(-1)$$

is the projection of $\mathcal{O}_L(-1)^3$ on one component and the sheaf $\mathcal{F}$ is supported on a double line in the plane.

To summarize, we have an exact sequence

$$0 \to \mathcal{O}_L(-1) \to \mathcal{F} \to \mathcal{O}_L \to 0$$

and the support of $\mathcal{F}$ lies in a plane. To conclude that $\mathcal{F} = \mathcal{O}_{2L}$ for a plane double line $2L$, we need to compute $\mathrm{ext}^1_{\mathbb{P}^2}(\mathcal{O}_L, \mathcal{O}_L(-1))$. To do this, we use the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_L \to 0.$$

The Ext-sequence yields

$$0 \to \mathrm{Hom}(\mathcal{O}_L, \mathcal{O}_L(-1) \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_L(-1)) \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_L(-1)) \to$$

$$\to \mathrm{Ext}^1(\mathcal{O}_L, \mathcal{O}_L(-1)) \to \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_L(-1)) \to \dots .$$

Since

$$\mathrm{hom}(\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_L(-1)) = h^0(\mathcal{O}_L(-1) \otimes \mathcal{O}_{\mathbb{P}^2}(1)) = 1,$$

$$\mathrm{hom}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_L(-1)) = h^0(\mathcal{O}_L(-1)) = 0$$

and

$$\mathrm{ext}^1_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_L(-1)) = h^1(\mathcal{O}_L(-1)) = 0,$$

we obtain $\mathrm{ext}^1_{\mathbb{P}^2}(\mathcal{O}_L, \mathcal{O}_L(-1)) = 1$. Since $\mathcal{F}$ is not the trivial extension, we conclude $\mathcal{F} = \mathcal{O}_{2L}$. ∎

Remark 4.20 Theorem 4.17 implies that $\dim(M^{2n+1}(\mathbb{P}^4) = 11$. Obviously $M^{2n+1}(\mathbb{P}^4)$ does not contain any strictly semi-stable sheaf.

3 Construction of a Rational Map $\Phi_{KM} : K \dashrightarrow M^{4n+2}(\mathbb{P}^4)$

In this Section we want to construct a rational map $\Phi_{KM} : K \dashrightarrow M^{4n+2}(\mathbb{P}^4)$ whose domain of definition contains all Kronecker modules in B.
Since in the moduli space $K = W^{ss} /\!\!/ G$ of Kronecker modules of type $(4, 2)$ there exist strictly semi-stable objects, there is no universal family. Hence we first define a rational map $\tilde{\Phi}_{KM} : W^{ss} \dashrightarrow M^{4n+2}(\mathbb{P}^4)$. On W^{ss} there is a tautological family

$$p_2^*\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \xrightarrow{\tilde{A}} p_2^*\mathcal{O}_{\mathbb{P}^4}^{\oplus 2},$$

where p_2 is the projection $p_2 : \mathbb{P}^4 \times W^{ss} \to \mathbb{P}^4$ on the second component. This gives a tautological sequence

$$p_2^*\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \xrightarrow{\tilde{A}} p_2^*\mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \to \mathrm{Coker}(\tilde{A}) \to 0.$$

$\mathrm{Coker}(\tilde{A})$ is a family of sheaves on $\mathbb{P}^4$ parametrized by W^{ss}. By [15][Thm 2.1.5] there exists a Zariski-open subset $U \subset W^{ss}$, on which the family is flat. Hence we can define a morphism $\tilde{\Phi}_{KM}|_U : U \to M^{4n+2}(\mathbb{P}^4)$ by $A \mapsto [\mathrm{Coker}(A^t)]$.

We want to prove that B is contained in the domain of definition of $\tilde{\Phi}_{KM}$.

Remark 4.21 Let A be a representative of a Kronecker module in one of the strata in B and denote by C the curve defined by the 2×2-minors of A. Then as a set, $\mathrm{Supp}(\mathrm{Coker}(A^t)) = C$. Indeed, we have

$$\begin{aligned} x \in \mathrm{Supp}(\mathrm{Coker}(A^t)) \quad &\Leftrightarrow \quad (\mathrm{Coker}(A^t))_x = (\mathcal{O}_{\mathbb{P}^4}^2/(\mathrm{Im}\, A^t))_x \neq 0 \\ &\Leftrightarrow \quad \mathrm{rank}((\mathrm{Im}\, A^t)_x) \neq 2. \end{aligned}$$

Now $\mathrm{rank}(\mathrm{Im}\, A^t)_x = \mathrm{rank}(A(x)) \leq 1$ if and only if $x \in C$, as the maximal minors of $A(x)$ vanish for $x \in C$.

As a preparation, we study the situation for the sheaves $\mathcal{F} = \mathrm{Coker}(A^t)$, where A is a Kronecker module in the stratum B_0, i.e. the maximal minors of A define a rational normal curve of degree 4 in $\mathbb{P}^4$.

Lemma 4.22 *$B_0 \in U$ and the map $\tilde{\Phi}_{KM}|_U$ is well-defined on B_0. In particular, for any Kronecker module $\varphi_A \in B_0$ represented by the matrix A the following holds:*

(a) The cokernel of the map

$$\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \xrightarrow{A^t} \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}$$

is

$$\operatorname{Coker}(A^t) = \mathcal{O}_C(p) \cong \mathcal{O}_{\mathbb{P}^1}(1)$$

where C is the rational normal curve defined by the maximal minors of A and p is a point on C.

(b) *The Hilbert polynomial of* $\operatorname{Coker}(A^t)$ *is* $P_{\operatorname{Coker}(A^t)}(n) = 4n + 2$.

(c) $\operatorname{Coker}(A^t)$ *is stable.*

Proof: (a) We restrict the exact sequence

$$\cdots \to \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \xrightarrow{\varphi_A^t} \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \to \mathcal{L} := \operatorname{Coker}(\varphi_A^t) \to 0$$

to C:

$$\cdots \to \mathcal{O}_C(-1)^{\oplus 4} \xrightarrow{\varphi_A^t|_C} \mathcal{O}_C^{\oplus 2} \to \mathcal{L}|_C \to 0.$$

Since C is a rational normal curve of degree 4, this sequence reads

$$\cdots \to \mathcal{O}_{\mathbb{P}^1}(-4)^{\oplus 4} \xrightarrow{\varphi_A^t|_C} \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \to \mathcal{L}|_C \to 0.$$

In coordinates of $\mathbb{P}^1$, the map $\varphi_A^t|_C$ is given by the matrix

$$A^t|_C = \begin{pmatrix} s^4 & s^3t & s^2t^2 & st^3 \\ s^3t & s^2t^2 & s^1t^3 & t^4 \end{pmatrix}.$$

By dualizing we obtain a map $\varphi : \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \xrightarrow{A|_C} \mathcal{O}_{\mathbb{P}^1}(4)^{\oplus 4}$ and compute the kernel. Now

$$\begin{aligned}
\operatorname{Ker}(\varphi) &= \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \,|\, A|_C \begin{pmatrix} f \\ g \end{pmatrix} = 0 \right\} \\
&= \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \,|\, s^4(fs+gt) = s^3t(fs+gt) = s^2t^2(fs+gt) = \right. \\
&\qquad\qquad \left. st^3(fs+gt) = t^4(fs+gt) = 0 \right\} \\
&= \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \,|\, (fs+gt) = 0 \right\} \\
&= \ker(\psi),
\end{aligned}$$

where ψ is the map

$$\psi : \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1}(1), \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} s & t \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

The map ψ is surjective, since $\mathcal{O}_{\mathbb{P}^1}(1)$ is generated by the sections s and t. So $\operatorname{Ker}\psi = \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{L}|_C = \operatorname{Coker}(\varphi_A)|_C = \mathcal{O}_{\mathbb{P}^1}(1)$.

(b) Since for large n we have

$$\chi(\mathcal{O}_C(p)(n)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1+4n)) = 4n+2,$$

the Hilbert polynomial of $\operatorname{Coker}(A^t)$ is $4n+2$.

(c) Since for any point $x \in C$, we have $\operatorname{rank}(\operatorname{Coker}(A^t)_x)) = 1$, the sheaf $\operatorname{Coker}(A^t))$ is locally free on C. Especially, restricted to C, it is torsion free. Since $\dim(\operatorname{Supp}(\operatorname{Coker}(A^t))) = 1$, this is the same as being pure. As $\operatorname{Coker}(A^t)$ is a pure sheaf of rank 1, it is stable. ∎

Proposition 4.23 *All Kronecker modules in B, i.e. whose maximal minors define a curve with Hilbert polynomial $4n+1$, are contained in U. On B the morphism $\tilde{\Phi}_{KM}|_U$ is well-defined, this means that* $\operatorname{Coker}(A^t)$ *is semi-stable and its Hilbert polynomial is $4n+2$ for any $A \in B$.*

Actually, we will show that the cokernel sheaves of Kronecker modules in B are stable.

Proof: For the proof it suffices to check the required properties for an arbitrary element in W^{ss} that is mapped to some representative of B_{10} under the quotient map $W^{ss} \to W^{ss} /\!\!/ G = K$. So let us assume that we have shown the assertions for B_{10}, which will be done in Lemma 4.24. Now we want to deduce the assertions for the other Kronecker modules in B.
If A_1 and A_2 are Kronecker modules in the same stratum B_i, there is a coordinate transformation that maps A_1 to A_2.
Hence for the Hilbert polynomials of $\operatorname{Coker}(A_1^t)$ and $\operatorname{Coker}(A_2^t)$ we have

$$P_{\operatorname{Coker}(A_1^t)}(n) = P_{\operatorname{Coker}(A_2^t)}(n)$$

and if one of the cokernel sheaves is semi-stable, then it is also true for the other.

First we show that the family of cokernel sheaves associated to Kronecker modules in B is flat.
For any family of coherent sheaves $\mathcal{F}$ on a smooth scheme $\mathbb{P}^4 \times W^{ss} \to W^{ss}$, there exists an open subscheme V of W^{ss} over which $\mathcal{F}$ is flat. This subscheme is non-empty, since the cokernel sheaves associated to B_0 form a flat family on the Zariski-open (dense) subset $B_0 \subset B$.

In the family

$$(A_s)^t := \begin{pmatrix} x_1 & x_0 & x_3 & sx_2 \\ sx_2 & x_1 & sx_4 + (1-s)x_0 & x_3 \end{pmatrix},$$

one can see after some column operations that $A_s \in B_0$ for $s \neq 0$ and $A_0 \in B_{10}$. We will show that the induced family of cokernels $\mathrm{Coker}((A_s)^t)$ is flat by showing that the Hilbert polynomial is constant in this family:
By Lemma 4.22 the cokernel of A_s, $s \neq 0$ is of the form $\mathcal{O}_C(p)$, where C is a rational normal curve of degree 4 and the Hilbert polynomial is $P_{\mathcal{O}_C(p)}(n) = 4n + 2$. And furthermore, by assumption, $P_{\mathrm{Coker}((A_0)^t)}(n) = 4n + 2$.
By Corollary 2.26, all stable Kronecker modules $A \in B$ degenerate into the stratum B_{10}, i.e. there is a family A_s, such that $A_s \in B_i$ for $s \neq 0$ and $\lim_{s \to 0} A_s = A_0 \in B_{10}$. Hence also the associated family of cokernels of the maps $\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \xrightarrow{A^t} \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}$ degenerates into a cokernel of Kronecker modules in B_{10}.
Since $\mathrm{Coker}((A_0)^t) \in V$ and V is open, there exist matrices $A_s \in B_i$ in a neighborhood of A_0, that are also contained in V and hence also the whole stratum.
But then by [15][Prop 2.1.2], the Hilbert polynomial is constant in the family of cokernel sheaves over B.

By [15][Prop 2.3.1], semi-stability is an open condition for a flat family of coherent sheaves. If we show that the cokernels of Kronecker modules in B_{10} are semi-stable, the same is true for any other cokernel of Kronecker modules in B by an argument similar to that one for the flatness.

Hence Lemma 4.24 completes the proof. ∎

Lemma 4.24 *We take the matrix*

$$A = \begin{pmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & x_1 & x_2 \end{pmatrix}^t \in W^s$$

that is mapped to a representative of the stratum B_{10} *in* K *under the quotient map* $W^{ss} \to W^{ss}/\!\!/G$*. Furthermore we denote by* $\mathcal{F}$ *the cokernel* $\mathcal{F} = \mathrm{Coker}(A^t)$*. Then* $\mathcal{F}$ *is a stable sheaf on* $\mathbb{P}^4$ *with Hilbert polynomial* $4n + 2$*.*

PROOF: I) Let L be the reduced line given by $x_0 = x_1 = x_2 = 0$ in $\mathbb{P}^4$. Recall that the maximal minors of A define the non-reduced line $4L$ which is the first infinitesimal neighborhood in $\mathbb{P}^4$.

Let $i : L \to 4L$ be the inclusion. First, we construct an exact sequence

$$0 \to i_*\mathcal{O}_L(-1)^{\oplus 2} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} i_*\mathcal{O}_L^{\oplus 2} \to 0. \tag{4}$$

To do so, let $\psi : \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \to \mathcal{O}_L^{\oplus 2}$ be the restriction to L and $\mathcal{G}_1 = \mathrm{Ker}(\psi) = \mathcal{I}_L^{\oplus 2}$. The map $\varphi : \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}$, given by the matrix A, factors through $\mathcal{G}_1$ and we have the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathcal{G}_1 \arrow[r] \arrow[d, "\mu"] & \mathcal{G}_2 \arrow[r] \arrow[d, dotted, "\alpha"] & 0 \\
\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \arrow[ur, "\lambda"] \arrow[r, "\varphi^t"] \arrow[dr, "0"'] & \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \arrow[r] \arrow[d, "\psi"] & \mathcal{F} \arrow[r] \arrow[dl, dotted, "\beta"] & 0 \\
 & i_*\mathcal{O}_L^{\oplus 2} \arrow[d] & & \\
 & 0 & &
\end{tikzcd} \tag{5}$$

where $\mathcal{G}_2 = \mathrm{Coker}(\lambda)$. The maps α and β are defined such that the diagram commutes. Then by construction, the sequence

$$0 \to \mathcal{G}_2 \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} i_*\mathcal{O}_L^{\oplus 2} \to 0$$

is exact.
Restricting to L gives $\mathcal{G}_1|_L = \mathcal{I}_L^{\oplus 2}|_L = (\mathcal{I}_L/\mathcal{I}_L^2)^{\oplus 2} = \mathcal{O}_L(-1)^{\oplus 3\cdot 2}$, and $\lambda|_L$ is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have $\mathcal{G}_2|_L = \mathcal{O}_L(-1)^{\oplus 2}$, since in the sequence

$$\mathcal{O}_L(-1)^{\oplus 4} \xrightarrow{\lambda|_L} \mathcal{O}_L(-1)^{\oplus 6} \to \mathcal{G}_2|_L \to 0$$

the map $\lambda|_L$ is injective.

We want to show, that $\mathcal{G}_2 \cong i_*\mathcal{O}_L(-1)^{\oplus 2}$. For that it remains to show that the scheme-theoretic support of $\mathcal{G}_2$ is contained in L.

To do so, we take an arbitrary local holomorphic function f that vanishes on L. If $f \cdot s = 0$ for all (local) sections s of $\mathcal{G}_2$, then the support of $\mathcal{G}_2$ is contained in L.
That means, for a preimage $\tilde{s}$ of s in $\mathcal{G}_1$ under the map $\mathcal{G}_1 \to \mathcal{G}_2$, we want to show $(f \cdot \tilde{s})_x \in \operatorname{Im}(\lambda)_x$ for all x. But since all diagrams in (5) commute, this is the same as

$$\mu((f \cdot \tilde{s})_x) \in \operatorname{Im}(\varphi^t)_{\mu(x)}.$$

The image of φ^t is

$$\operatorname{Im}(\varphi^t) = \left\{ \begin{pmatrix} u_1x_0 + u_2x_1 + u_3x_2 \\ u_2x_0 + u_3x_1 + u_4x_2 \end{pmatrix} \mid u_i \in H^0(\mathcal{O}_{\mathbb{P}^4}(-1)) \right\}.$$

A holomorphic function that vanishes on L has locally the form

$$f = f_0x_0 + f_1x_1 + f_2x_2$$

for holomorphic functions f_i. Then

$$\mu(f \cdot \tilde{s}) = f \cdot \mu(\tilde{s}) = \begin{pmatrix} \tilde{s}_1f_0x_0 + \tilde{s}_1f_1x_1 + \tilde{s}_1f_2x_2 \\ \tilde{s}_2f_0x_0 + \tilde{s}_2f_1x_1 + \tilde{s}_2f_1x_2 \end{pmatrix}.$$

But this is a section in $\operatorname{Im}(\varphi)$ by setting

$$\begin{aligned} u_1 &= -(\tilde{s}_2f_1 - \tilde{s}_1f_2)\frac{x_2}{x_0} + \tilde{s}_1f_0, \\ u_2 &= \tilde{s}_1f_1, \\ u_3 &= \tilde{s}_2f_1, \\ u_4 &= (\tilde{s}_2f_0 - \tilde{s}_1f_1)\frac{x_0}{x_2} + \tilde{s}_2f_1. \end{aligned}$$

Hence the claim follows.

II) Using the exact sequence (4) in part I) we can easily compute the Hilbert polynomial of $\mathcal{F}$:

$$\begin{aligned} P_{\mathcal{F}}(n) &= P_{i_*\mathcal{O}_L(-1)^{\oplus 2}}(n) + P_{i_*\mathcal{O}_L^{\oplus 2}}(n) \\ &= P_{\mathcal{O}_L(-1)^{\oplus 2}}(n) + P_{\mathcal{O}_L^{\oplus 2}}(n) \\ &= 2n + (2n+2) \\ &= 4n + 2. \end{aligned}$$

III) Finally we need to check that $\mathcal{F} = \operatorname{Coker}(A^t)$ is stable, where

$$A = \begin{pmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & x_1 & x_2 \end{pmatrix}.$$

In order to prove stability, consider any proper subsheaf $\mathcal{F}'$ of $\mathcal{F}$. Then there exists a map

$$f : \mathcal{F}' \to i_*\mathcal{O}_L^{\oplus 2}$$

defined as the composition of the inclusion $\mathcal{F}' \hookrightarrow \mathcal{F}$ and $\beta : \mathcal{F} \to i_*\mathcal{O}_L^{\oplus 2}$. Let

$$\mathcal{F}'' := \mathrm{Im}(f) \text{ and } \mathcal{G} := \mathrm{Ker}(f).$$

Then $\mathcal{G}$ is a subsheaf of $i_*\mathcal{O}_L(-1)^{\oplus 2}$ and $\mathcal{F}''$ a subsheaf of $i_*\mathcal{O}_L^{\oplus 2}$; both sheaves are pure (of dimension 1). So we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & i_*\mathcal{O}_L(-1)^{\oplus 2} \arrow[r] & \mathcal{F} \arrow[r] & i_*\mathcal{O}_L^{\oplus 2} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[u, phantom, "\cup\!\mid"] & \mathcal{F}' \arrow[r] \arrow[u, phantom, "\cup\!\mid"] & \mathcal{F}'' \arrow[r] \arrow[u, phantom, "\cup\!\mid"] & 0.
\end{tikzcd} \tag{6}$$

First we prove that $\mathcal{F}$ is pure, i.e. there is no subsheaf of $\mathcal{F}$ whose support consists of finitely many points. So we take $\mathcal{F}'$ to be a subsheaf of $\mathcal{F}$ with 0-dimensional support. Then clearly the support of $\mathrm{Ker}(f)$ and the support of $\mathrm{Im}(f)$ have dimension 0. But $i_*\mathcal{O}_L(-1)^{\oplus 2}$ and $i_*\mathcal{O}_L^{\oplus 2}$ are pure, thus $\mathrm{Ker}(f) = \mathrm{Im}(f) = 0$ and hence $\mathcal{F}' = 0$.

Now in order to prove stability, we consider a destabilizing subsheaf $\mathcal{F}'$ of $\mathcal{F}$, i.e. a subsheaf $\mathcal{F}'$ of $\mathcal{F}$ such that $p_{\mathcal{F}'} \geq p_{\mathcal{F}}$. Recall that the reduced Hilbert polynomial of $\mathcal{F}$ is

$$p_{\mathcal{F}}(n) = \frac{4n+2}{4} = n + \frac{1}{2}.$$

From diagram (6) we deduce that $\mathcal{F}''$ is either

- $\mathcal{F}'' = 0$ or
- $\mathcal{F}'' = i_*\mathcal{O}_L(a)$ for some $a \leq 0$ or
- $\mathcal{F}'' = i_*\mathcal{O}_L(a) \oplus i_*\mathcal{O}_L(b)$ for some $b \leq a \leq 0$.

In all cases either $\mathcal{G} = 0$ or the reduced Hilbert polynomial of $\mathcal{G}$ satisfies the inequality

$$p_{\mathcal{G}}(n) \leq p_{i_*\mathcal{O}_L(-1)^{\oplus 2}}(n) = n,$$

since $i_*\mathcal{O}_L(-1)^{\oplus 2}$ is semi-stable.

Now if $\mathcal{F}'' = 0$, then $\mathcal{F}' \cong \mathcal{G}$ and hence $p_{\mathcal{F}'}(n) \leq n < n + \frac{1}{2} = p_{\mathcal{F}}(n)$, which is a contradiction to the assumption that $\mathcal{F}'$ is a destabilizing subsheaf of $\mathcal{F}$.
If $\mathcal{F}'' \neq 0$ and $a \leq -1$, then we have $p_{\mathcal{F}''}(n) \leq p_{i_*\mathcal{O}_L(-1)}(n) = n$ and hence $p_{\mathcal{F}'}(n) \leq n < p_{\mathcal{F}}(n)$ which is again a contradiction.
So from now on we may assume that $\mathcal{F}'' = i_*\mathcal{O}_L$ or $\mathcal{F}'' = i_*\mathcal{O}_L \oplus i_*\mathcal{O}_L(b)$ for some $b \leq 0$.

Claim 4.25 $h^0(\mathcal{F}') > 0$ and $\mathcal{G}$ is one of the following

- $\mathcal{G} = 0$
- $\mathcal{G} = i_*\mathcal{O}_L(-1)$
- $\mathcal{G} = i_*\mathcal{O}_L(-1)^2$.

Proof: In case $\mathcal{G} = 0$, the assertion $h^0(\mathcal{F}') > 0$ is clear. So suppose that $\mathcal{G} \neq 0$. For any $\mathcal{G} = i_*\mathcal{O}_L(m)$ or $\mathcal{G} = i_*\mathcal{O}_L(m) \oplus i_*\mathcal{O}_L(n)$, with $m < -1$ and $n \leq -1$, then $p_{\mathcal{F}'} < p_{\mathcal{F}}$ which is a contradiction.
So we conclude that $\mathcal{G} = i_*\mathcal{O}_L(-1)^2$ or $\mathcal{G} = i_*\mathcal{O}_L(-1)$, in particular $h^0(\mathcal{G}) = h^1(\mathcal{G}) = 0$. Furthermore we know that $h^0(\mathcal{F}'') \in \{1, 2\}$ and hence it follows from the long exact cohomology sequence

$$0 \to H^0(\mathcal{G}) \to H^0(\mathcal{F}') \to H^0(\mathcal{F}'') \to H^1(\mathcal{G}) \to \dots$$

that $h^0(\mathcal{F}') > 0$. ∎

Note that $h^0(\mathcal{F}') = 2$ if and only if $\mathcal{F}'' = i_*\mathcal{O}_L^2$ and that $h^0(\mathcal{F}') = 1$ otherwise.

Claim 4.26 There are exactly the following two cases:

(a) $\mathcal{F}'' = i_*\mathcal{O}_L$ and $\mathcal{G} = i_*\mathcal{O}_L(-1)^2$

(b) $\mathcal{F}'' = i_*\mathcal{O}_L \oplus i_*\mathcal{O}_L(-1)$ and $\mathcal{G} = i_*\mathcal{O}_L(-1)$.

Proof: Since $h^0(\mathcal{F}') > 0$, the section (resp. one of the sections) of $\mathcal{F}'$ defines a map $\mathcal{O}_{\mathbb{P}^4} \xrightarrow{s} \mathcal{F}'$.
There exists a unique map $\mathcal{O}_{\mathbb{P}^4} \xrightarrow{\binom{\alpha}{\beta}} \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}$, such that the composition

$$\mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}^2 \to \mathcal{F} \to i_*\mathcal{O}_L^2$$

coincides with the given map $\mathcal{F}'' \to i_*\mathcal{O}_L^2$, i.e. the following diagram commutes:

$$\begin{tikzcd}
 & \mathcal{O}_{\mathbb{P}^4} \arrow[r, "s"] \arrow[dl, "(\alpha,\beta)"'] \arrow[d] & \mathcal{F}' \arrow[d] \\
\mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \arrow[dr] & i_*\mathcal{O}_L \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[d] \\
 & i_*\mathcal{O}_L^2 \arrow[r, equal] & i_*\mathcal{O}_L^2
\end{tikzcd}$$

Then we obtain the following commutative diagram:

$$\begin{tikzcd}
 & 0 & 0 & \\
\mathcal{O}_{\mathbb{P}^4}(-1)^4 \arrow[r, "B"] \arrow[d, equal] & \mathcal{O}_{\mathbb{P}^4} \arrow[u] \arrow[r] & \mathcal{F}/\mathcal{F}' \arrow[u] \arrow[r] & 0 \\
\mathcal{O}_{\mathbb{P}^4}(-1)^4 \arrow[r, "A"] & \mathcal{O}_{\mathbb{P}^4}^2 \arrow[u, "(\beta\ -\alpha)"] \arrow[r] & \mathcal{F} \arrow[u] \arrow[r] & 0 \\
 & \mathcal{O}_{\mathbb{P}^4} \arrow[u, "\binom{\alpha}{\beta}"] \arrow[r] & \mathcal{F}' \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where $A = \begin{pmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & x_1 & x_2 \end{pmatrix}$ and

$$B := \begin{pmatrix} \beta & -\alpha \end{pmatrix} \cdot \begin{pmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & x_1 & x_2 \end{pmatrix} = \begin{pmatrix} \beta x_0 & \beta x_1 - \alpha x_0 & \beta x_2 - \alpha x_1 & -\alpha x_2 \end{pmatrix}.$$

Hence $\mathcal{F}/\mathcal{F}' = \mathrm{Coker}(B) = i_*\mathcal{O}_L$, since the image of B is generated by x_0, x_1 and x_2, as not both α and β are zero. Therefore diagram (6) can be extended as follows:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathcal{H} \arrow[u] \arrow[r] & \mathcal{F}/\mathcal{F}' = i_*\mathcal{O}_L \arrow[u] \arrow[r] & \mathcal{E} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & i_*\mathcal{O}_L(-1)^2 \arrow[u] \arrow[r] & \mathcal{F} \arrow[u] \arrow[r] & i_*\mathcal{O}_L^2 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[u] \arrow[r] & \mathcal{F}' \arrow[u] \arrow[r] & \mathcal{F}'' \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0. \arrow[u] &
\end{tikzcd}$$

If $\mathcal{F}'' = i_*\mathcal{O}_L$, then $\mathcal{E} = i_*\mathcal{O}_L$ and so the cokernel $\mathcal{H} = 0$ and hence $\mathcal{G} \cong i_*\mathcal{O}_L(-1)^2$. This is case a).

If $\mathcal{F}'' = i_*\mathcal{O}_L \oplus i_*\mathcal{O}_L(b)$, we consider the three possible choices of $\mathcal{G}$:

- If $\mathcal{G} = 0$, then $\mathcal{H} = i_*\mathcal{O}_L(-1)^2$ which is a contradiction, since $\mathcal{H}$ is a subsheaf of $\mathcal{F}/\mathcal{F}' = i_*\mathcal{O}_L$.
- If $\mathcal{G} = i_*\mathcal{O}_L(-1)^2$, then $\mathcal{H} = 0$ and $\mathcal{E} = i_*\mathcal{O}_L$. That imples $\mathcal{F}'' = i_*\mathcal{O}_L$ which is a contradiction.
- If $\mathcal{G} = i_*\mathcal{O}_L(-1)$, then $\mathcal{H} = i_*\mathcal{O}_L(-1)$ and hence $\mathcal{E}$ is a skyscraper sheaf supported on one point. Hence $b = -1$. So we obtain case b).

∎

Finally, we rule out both possibilities obtained in Claim 4.26. First, in case a) if $\mathcal{F}'' = i_*\mathcal{O}_L$, then

$$P_{\mathcal{F}'}(n) = P_{\mathcal{G}}(n) + P_{\mathcal{F}''}(n) = P_{i_*\mathcal{O}_L(-1)^2}(n) + P_{i_*\mathcal{O}_L}(n) = 2n + n + 1$$

and hence

$$p_{\mathcal{F}'}(n) = n + \frac{1}{3} < n + \frac{1}{2} = p_{\mathcal{F}}(n),$$

which is a contradiction.
Second, in case b) if $\mathcal{F}'' = i_*\mathcal{O}_L \oplus i_*\mathcal{O}_L(-1)$, then $P_{\mathcal{F}'}(n) = 2n + n + 1$ and hence as before $p_{\mathcal{F}'} < p_{\mathcal{F}}$ which is a contradiciton.

In summary, there does not exist any proper subsheaf $\mathcal{F}'$ of $\mathcal{F}$, such that $p_{\mathcal{F}'} \geq p_{\mathcal{F}}$ and hence $\mathcal{F}$ is stable. ∎

In Proposition 4.23 we have constructed a well-defined morphism

$$\tilde{\Phi}_{KM}|_U : U \to M^{4n+2}(\mathbb{P}^4)$$

for some open subset $U \subset W^{ss}$ that contains B. As for all pairs $(g, h) \in G$, the cokernel sheaves $\mathrm{Coker}((gAh^{-1})^t)$ and $\mathrm{Coker}(A^t)$ are isomorphic, the morphism $\tilde{\Phi}_{KM}|_U$ is G-invariant. Hence $\tilde{\Phi}_{KM}|_U$ descends to a morphism

$$\Phi_{KM}|_{K'} : K' := U /\!\!/ G \to M^{4n+2}(\mathbb{P}^4).$$

Remark 4.27 The rational map $\tilde{\Phi}_{KM}$ is not defined on the strata S and P. This is clear, as the support of the cokernel $\mathcal{F}$ sheaves has dimension at least 2. Since by [29][III.6 Prop. 6] for the Hilbert polynomial $P_{\mathcal{F}}(n)$ of $\mathcal{F}$ holds $\deg(P_{\mathcal{F}}(n)) = \dim(\mathrm{Supp}(\mathcal{F}))$, it cannot be $4n + 2$. By saturation of the cokernel sheaves of the families of Kronecker modules chosen in Remark 3.3 along different directions, we see that it is not possible to extend $\tilde{\Phi}_{KM}$ to S or P.

For some strata of K one can compute the cokernel sheaves concretely.
To do so, we need the following preparation:

Lemma 4.28 *Let φ be a Kronecker module of type $(4,2)$ represented by a matrix A and C the curve defined by its maximal minors. Write $C = \bigcup_i C_i$, where C_i are the irreducible components of C.*

(a) *If after an appropriate coordinate change*

$$A^t|_{C_i} = \begin{pmatrix} x_0 & x_1 & \dots & x_{d-1} & 0 & \dots & 0 \\ x_1 & x_2 & \dots & x_d & 0 & \dots & 0 \end{pmatrix},$$

i.e. C_i is a rational normal curve of degree d in some $\mathbb{P}^d$ for $1 \leq d \leq 4$, then the cokernel of the restricted map

$$\varphi^t|_{C_i} : \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4}|_{C_i} \xrightarrow{A^t|_{C_i}} \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}|_{C_i}$$

is the sheaf $\mathcal{O}_{C_i}(p)$.

(b) *If after a coordinate transformation C_i is the line in $\mathbb{P}^4$ given by the equations $x_0 = x_1 = x_2 = 0$ and*

$$A^t|_{C_i} = \begin{pmatrix} x_3 & x_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

then $\operatorname{Coker}(A^t|_{C_i}) = \mathcal{O}_{C_i}$.

(c) *If after a coordinate transformation C_i is the line in $\mathbb{P}^4$ given by the equations $x_0 = x_1 = x_2 = 0$ and*

$$A^t|_{C_i} = \begin{pmatrix} x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and P denotes the point $x_0 = x_1 = x_2 = x_3 = 0$, then

$$\operatorname{Coker}(A^t|_{C_i}) = \mathcal{O}_{C_i} \oplus \mathbb{C}_P.$$

PROOF: (a) This can be proven analogeously to the proof of Lemma 4.22(a).

(b) Obviously $\operatorname{Im}(A^t|_{C_i}) = \mathcal{O}_{C_i} \oplus \{0\}$. The cokernel of $A^t|_{C_i}$ is

$$\operatorname{Coker}(A^t|_{C_i}) = \mathcal{O}_{C_i}^2 / \operatorname{Im}(A^t) = \mathcal{O}_{C_i}.$$

(c) We restrict the map

$$\mathcal{O}_{C_i}(-1)^{\oplus 4} \xrightarrow{\left(\begin{smallmatrix} x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)} \mathcal{O}_{C_i}^{\oplus 2}$$

to a map

$$\mathcal{O}_{C_i}(-1) \xrightarrow{(x_3\ 0)} \mathcal{O}_{C_i}^{\oplus 2}.$$

This map is injective and hence $\operatorname{Coker}(A^t|_{C_i}) = \mathcal{O}_{C_i} \oplus \mathbb{C}_P$. ∎

Example 4.29 From Lemma 4.28 one can deduce immediately:

(a) For the stratum B_0, the cokernel is $\mathcal{O}_C(p)$ for the rational normal curve C of degree 4 (as we have already seen in Lemma 4.22).

(b) B_1: Recall that the support of $\mathcal{L}$, i.e. a curve in B_1 consists of a twisted cubic C and a line L that intersect in one point. The rank of $\mathcal{L}$ is constantly 1 on the support and $\mathcal{L}|_C = \mathcal{O}_C(p)$ and $\mathcal{L}|_L = \mathcal{O}_L$.

(c) B_2: A curve in B_2 consists of a quadric Q and two lines L_1 and L_2 that each intersect Q in one point. Then $\mathcal{L}|_Q = \mathcal{O}_Q(p)$ and $\mathcal{L}|_{L_1} = \mathcal{O}_{L_1}$ resp. $\mathcal{L}|_{L_2} = \mathcal{O}_{L_2}$.

(d) B_3^2: the support of $\mathcal{L}$ consists of four lines that do not intersect in one point. $\mathcal{L}|_{L_4} = \mathcal{O}_{L_4}(p)$, and for $i = 1, 2, 3$ we have $\mathcal{L}|_{L_i} = \mathcal{O}_{L_i}$.

4 The Moduli Space $M^{4n+2}(\mathbb{P}^4)$

In this Section we want to study the moduli space $M^{4n+2}(\mathbb{P}^4)$ of semi-stable sheaves on $\mathbb{P}^4$ with Hilbert polynomial $4n + 2$.
We saw in Section 4.3 that this moduli space contains the cokernel sheaves of the Kronecker modules

$$\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}$$

of type $(4, 2)$, if the maximal minors of the matrices A representing φ define a curve.
Concretely, there is a "good" component

$$M_4 := \overline{\{i_*\mathcal{O}_{\mathbb{P}^1}(1) | i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^4 \text{ embedding } \}} \subset M^{4n+2}(\mathbb{P}^4)$$

which contains (as an open set) the cokernels of the Kronecker modules mentioned above.

Lemma 4.30 *The dimension of M_4 is $\dim M_4 = 21$.*

Proof: Since the family of rational normal curves of degree 4 in $\mathbb{P}^4$ has dimension 21, we have $\dim_{\mathcal{F}} M_4 \geq 21$ where $\mathcal{F} = \text{Coker}(A) = \mathcal{O}_C(p)$ for a Kronecker module $A \in B_0$.

Since all smooth rational normal curves C of degree 4 in $\mathbb{P}^4$ are projectively equivalent, $\text{ext}^1(\mathcal{O}_C(1), \mathcal{O}_C(1))$ is constant in the family of cokernels of smooth rational normal curves of degree 4 in $\mathbb{P}^4$. Using the computer algebra Macaulay 2 ([9]), we compute $\text{ext}^1(\mathcal{O}_C(1), \mathcal{O}_C(1)) = \text{ext}^1(\mathcal{O}_C, \mathcal{O}_C) = 21$. So we have $\dim_{\mathcal{F}} M^{4n+2}(\mathbb{P}^4) = 21$. Hence the open set

$$\{i_*\mathcal{O}_{\mathbb{P}^1}(1) \,|\, i : \mathbb{P}^1 \hookrightarrow \mathbb{P}^4 \text{ embedding}\}$$

is smooth and has dimension 21. So its closure M_4 is an irreducible component of $M^{4n+2}(\mathbb{P}^4)$ of dimension 21. ∎

In $M^{4n+2}(\mathbb{P}^4)$ there exist strictly semi-stable sheaves $\mathcal{F}$. Every S-equivalence class of semi-stable sheaves in $M^{4n+2}(\mathbb{P}^4)$ contains a unique strictly poly-stable sheaf up to isomorphism.
The strictly poly-stable sheaves are direct sums $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ of stable sheaves $\mathcal{F}_1, \mathcal{F}_2 \in M^{2n+1}(\mathbb{P}^4)$ with Hilbert polynomial $2n + 1$. Recall that we showed in Section 4.2 that $\mathcal{F}_1$ and $\mathcal{F}_2$ are structure sheaves of smooth plane conics, double lines in $\mathbb{P}^2$ or two lines intersecting in one point. Some of the strictly poly-stable sheaves are contained in the good component M_4. More concretely:

Lemma 4.31 *The intersection of $S^2(M^{2n+1}(\mathbb{P}^4))$ with the good component M_4 is the set of sheaves in $S^2(M^{2n+1}(\mathbb{P}^4))$ whose support is connected. We denote this subset by $S^2(M^{2n+1}(\mathbb{P}^4))_0$.*

Proof: First, the component M_4 contains a sheaf in $S^2(M^{2n+1}(\mathbb{P}^4))_0$: therefore consider the family of cokernel sheaves

$$p^*\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6} \xrightarrow{\varphi_t} p^*\mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \to \operatorname{Coker}(\varphi_t) \to 0,$$

where $p : \mathbb{P}^4 \times \mathbb{C} \to \mathbb{P}^4$ is the projection on the first component and the maps φ_t are represented by the matrices

$$A_t := \begin{pmatrix} x_0 & x_1 & tx_2 & tx_3 & x_2x_4 - x_3^2 & 0 \\ tx_1 & tx_2 & x_3 & x_4 & 0 & x_0x_2 - x_1^2 \end{pmatrix}.$$

For $t \neq 0$, the matrix A_t can be transformed via row and column operations to a matrix of the form

$$\begin{pmatrix} x_0 & x_1 & tx_2 & tx_3 & 0 & 0 \\ tx_1 & tx_2 & x_3 & x_4 & 0 & 0 \end{pmatrix}.$$

Hence for any $t \neq 0$, the cokernel $\operatorname{Coker}(\varphi_t)$ defines the cokernel of a Kronecker module in B_0 and for $t = 0$, the cokernel sheaf is in $S^2(M^{2n+1}(\mathbb{P}^4))_0$.

Now, the generic sheaf in $S^2(M^{2n+1}(\mathbb{P}^4))_0$ is the direct sum of two structure sheaves of two plane quadrics Q_1 and Q_2 that intersect in exactly one point. By Lemma 4.36, all such pairs of quadrics are projectively equivalent. Hence all sheaves $\mathcal{O}_{Q_1} \oplus \mathcal{O}_{Q_2} \in S^2(M^{2n+1}(\mathbb{P}^4))_0$, where Q_1 and Q_2 are two plane quadrics that intersect in exactly one point, are contained in M_4. But then the closure of this set, i.e. $S^2(M^{2n+1}(\mathbb{P}^4))_0$ is contained in M_4, as M_4 is closed. ∎

Lemma 4.32 *(a) The dimension of $S^2(M^{2n+1}(\mathbb{P}^4))_0$ is*

$$\dim S^2(M^{2n+1}(\mathbb{P}^4))_0 = 20.$$

(b) Let $\Delta(M^{2n+1}(\mathbb{P}^4)) \subset S^2(M^{2n+1}(\mathbb{P}^4))$ be the closed subscheme of $S^2(M^{2n+1}(\mathbb{P}^4))$ consisting of sheaves of the form $\mathcal{O}_Q^{\oplus 2} \in S^2(M^{2n+1}(\mathbb{P}^4))_0$ for some conic Q. Then

$$\dim \Delta(M^{2n+1}(\mathbb{P}^4)) = 11.$$

Proof: (a) First we compute the dimension of the family of smooth plane quadrics in $\mathbb{P}^4$ containing a fixed point $P \in \mathbb{P}^4$.

Let Q be a smooth quadric in $\mathbb{P}^4$ lying in some plane E. Then there is an exact sequence

$$0 \to \mathcal{N}_{Q/E} \to \mathcal{N}_{Q/\mathbb{P}^4} \to \mathcal{N}_{E/\mathbb{P}^4}|_Q \to 0. \tag{7}$$

Obviously

$$\mathcal{N}_{Q/E} = \mathcal{O}_Q(2) = \mathcal{O}_{\mathbb{P}^1}(4)$$

and furthermore

$$\mathcal{N}_{E/\mathbb{P}^4}|_Q = (\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))|_Q = \mathcal{O}_Q(1) \oplus \mathcal{O}_Q(1) = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2).$$

Since the quadric should contain a fixed point $P \in \mathbb{P}^4$, we take the tensor product of sequence (7) with the ideal sheaf $\mathcal{I}_P = \mathcal{O}_{\mathbb{P}^1}(-1)$ of P in Q. Hence

$$H^0(\mathcal{N}_{Q/\mathbb{P}^4} \otimes \mathcal{I}_P) = H^0(\mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) = 8.$$

Altogether, the dimension of the family of two smooth plane quadrics intersecting in one point, is $4 + 8 + 8 = 20$, since the choice a point in $\mathbb{P}^4$ increases the dimension by 4.

(b) The quadric Q lies in a unique plane E and

$$h^0(\mathcal{N}_{E/\mathbb{P}^4}) = h^0(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)) = 3 + 3 = 6.$$

Furthermore

$$h^0(\mathcal{N}_{Q/E}) = h^0(\mathcal{O}_Q(2)) = h^0(\mathcal{O}_{\mathbb{P}^1}(4)) = 5.$$

Hence the space of quadrics in $\mathbb{P}^4$ has dimension 11 and therefore

$$\dim \Delta(M^{2n+1}(\mathbb{P}^4)) = 11. \blacksquare$$

The following diagram summarizes the strata of the moduli space we are going to study:

$$\begin{array}{ccccc}
M^{4n+2}(\mathbb{P}^4) & \supseteq & S^2(M^{2n+1}(\mathbb{P}^4)) & \supseteq & \Delta(M^{2n+1}(\mathbb{P}^4)) \\
\cup\!| & & \cup\!| & & \| \\
M_4 & \supseteq & S^2(M^{2n+1}(\mathbb{P}^4))_0 & \supseteq & \Delta(M^{2n+1}(\mathbb{P}^4))
\end{array}$$

Again $\Delta(M^{2n+1}(\mathbb{P}^4))$ contains the sheaves of the form $\mathcal{O}_Q^{\oplus 2}$.

Lemma 4.33 *Let Q be a plane quadric in $\mathbb{P}^4$. Then*

$$\mathrm{Ext}^1_{\mathbb{P}^4}(\mathcal{O}_Q,\mathcal{O}_Q)=\mathbb{C}^{11} \text{ and } \mathrm{Ext}^2_{\mathbb{P}^4}(\mathcal{O}_Q,\mathcal{O}_Q)=\mathbb{C}^{19}.$$

Proof: Consider the Koszul resolution of $\mathcal{O}_Q$

$$0\longrightarrow \Lambda^3V\xrightarrow{\varphi_4}\Lambda^2V\xrightarrow{\varphi_3}V\xrightarrow{\varphi_2}\mathcal{O}_{\mathbb{P}^4}\xrightarrow{\varphi_1}\mathcal{O}_Q\longrightarrow 0,$$

where $V=\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^4}(-2)$ and hence $\Lambda^2V=\mathcal{O}_{\mathbb{P}^4}(-2)\oplus\mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2}$ and $\Lambda^3V=\mathcal{O}_{\mathbb{P}^4}(-4)$.
Let $K_1:=\mathrm{Ker}(\varphi_2)$ and $K_0:=\mathrm{Ker}(\varphi_1)$. The $\mathcal{O}_{\mathbb{P}^4}$-modules K_0 and K_1 need not to be locally free. We have the following exact sequences

$$0\to K_0\to\mathcal{O}_{\mathbb{P}^4}\xrightarrow{\varphi_1}\mathcal{O}_Q\to 0 \tag{8}$$

$$0\to K_1\to\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^4}(-2)\to K_0\to 0 \tag{9}$$

$$0\to\mathcal{O}_{\mathbb{P}^4}(-4)\xrightarrow{\varphi_4}\mathcal{O}_{\mathbb{P}^4}(-2)\oplus\mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2}\to K_1\to 0. \tag{10}$$

I) First consider the sequence (8). We construct the associated Ext-sequence for the $\mathcal{O}_{\mathbb{P}^4}$-module $\mathcal{O}_Q$:

$$0\to\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{O}_Q,\mathcal{O}_Q)\to\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{O}_{\mathbb{P}^4},\mathcal{O}_Q)\to\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^4}}(K_0,\mathcal{O}_Q)\to$$

$$\mathrm{Ext}^1_{\mathbb{P}^4}(\mathcal{O}_Q,\mathcal{O}_Q)\to\mathrm{Ext}^1_{\mathbb{P}^4}(\mathcal{O}_{\mathbb{P}^4},\mathcal{O}_Q)\to\cdots$$

As $\mathrm{Hom}_{\mathbb{P}^4}(\mathcal{O}_{\mathbb{P}^4},\mathcal{O}_Q)=H^0(\mathcal{O}_Q)=\mathbb{C}$, furthermore $\mathrm{Hom}(\mathcal{O}_Q,\mathcal{O}_Q)=\mathbb{C}$ and $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^4},\mathcal{O}_Q)=H^1(\mathcal{O}_Q)=0$, this exact sequence induces an isomorphism

$$\mathrm{Ext}^1(\mathcal{O}_Q,\mathcal{O}_Q)\cong\mathrm{Hom}(K_0,\mathcal{O}_Q).$$

Since $\mathrm{Ext}^i(\mathcal{O}_{\mathbb{P}^4},\mathcal{O}_Q)=H^i(\mathcal{O}_Q)=0$ for $i\geq 2$, we also have isomorphisms

$$\mathrm{Ext}^i(K_0,\mathcal{O}_Q)\cong\mathrm{Ext}^{i+1}(\mathcal{O}_Q,\mathcal{O}_Q)$$

for all $i\geq 1$.

II) Then consider the sequence (9). We take again the associated Ext-sequence for the $\mathcal{O}_{\mathbb{P}^4}$-module $\mathcal{O}_Q$:

$$0\to\mathrm{Hom}(K_0,\mathcal{O}_Q)\to\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^4}(-2),\mathcal{O}_Q)\to\mathrm{Hom}(K_1,\mathcal{O}_Q)\to$$

$$\to\mathrm{Ext}^1(K_0,\mathcal{O}_Q)\to\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^4}(-2),\mathcal{O}_Q)\to\cdots$$

Using

$$\begin{aligned}\mathrm{Hom}_{\mathbb{P}^4}(\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^4}(-2),\mathcal{O}_Q) &= \mathrm{Hom}_{\mathbb{P}^4}(\mathcal{O}_{\mathbb{P}^4},\mathcal{O}_Q\otimes(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^4}(2)))\\ &= H^0(\mathcal{O}_Q\otimes(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^4}(2))\\ &= H^0(\mathcal{O}_Q(1)^{\oplus 2}\oplus\mathcal{O}_Q(2))\\ &= H^0(\mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^1}(4))\\ &= \mathbb{C}^{11}\end{aligned}$$

and

$$\mathrm{Ext}^i(\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^4}(-2),\mathcal{O}_Q) = H^i(\mathcal{O}_Q(1)^{\oplus 2}\oplus\mathcal{O}_Q(2)) = 0 \text{ for all } i\geq 1$$

the exact sequence gives isomorphisms

$$\mathrm{Ext}^i(K_1,\mathcal{O}_Q)\cong\mathrm{Ext}^{i+1}(K_0,\mathcal{O}_Q)$$

for all $i \geq 1$.

III) Then we consider the sequence (10). We construct the associated Ext-sequence for the $\mathcal{O}_{\mathbb{P}^4}$-module $\mathcal{O}_Q$:

$$\begin{aligned}0\to\mathrm{Hom}(K_1,\mathcal{O}_Q)\to\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^4}(-2)\oplus\mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2},\mathcal{O}_Q)\to\\ \to\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^4}(-4),\mathcal{O}_Q)\to\mathrm{Ext}^1(K_1,\mathcal{O}_Q)\to\\ \to\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^4}(-2)\oplus\mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2},\mathcal{O}_Q)\to\cdots\end{aligned}$$

We compute as before

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^4}(-2)\oplus\mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2},\mathcal{O}_Q) &= H^0(\mathcal{O}_Q(2)\oplus\mathcal{O}_Q(3)^{\oplus 2})\\ &= H^0(\mathcal{O}_{\mathbb{P}^1}(4)\oplus\mathcal{O}_{\mathbb{P}^1}(6)^{\oplus 2})\\ &= \mathbb{C}^{19}\end{aligned}$$

and

$$\mathrm{Ext}^i(\mathcal{O}_{\mathbb{P}^4}(-2)\oplus\mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2},\mathcal{O}_Q) = H^i(\mathcal{O}_Q(2)\oplus\mathcal{O}_Q(3)^{\oplus 2}) = 0$$

for all $i \geq 1$ and

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^4}(-4),\mathcal{O}_Q) = \mathbb{C}^9.$$

Now we prove that

$$\mathrm{Hom}(K_1,\mathcal{O}_Q)\cong\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^4}(-2)\oplus\mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2},\mathcal{O}_Q).$$

This is the same as proving that $\mathrm{Hom}(\Lambda^2 V,\mathcal{O}_Q)\to\mathrm{Hom}(\Lambda^3 V,\mathcal{O}_Q)$ is the zero map.

The map $\mathcal{O}_{\mathbb{P}^4}(-4) \xrightarrow{\varphi_4} \mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2}$ is given by a matrix $\begin{pmatrix} q \\ l_1 \\ l_2 \end{pmatrix}$, where l_1 and l_2 define a plane and q defines a quadric q, such that $q \cap l_1 \cap l_2 = Q$.

Since

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2}, \mathcal{O}_Q) &= \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^4}, (\mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(3)^{\oplus 2})|_Q) \\ &= H^0((\mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(3)^{\oplus 2})|_Q)\end{aligned}$$

and analogously

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^4}(-4), \mathcal{O}_Q) = H^0(\mathcal{O}_{\mathbb{P}^4}(4)|_Q),$$

the map $H^0((\mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(3)^{\oplus 2})|_Q) \to H^0(\mathcal{O}_{\mathbb{P}^4}(4)|_Q)$ is given by the matrix $\begin{pmatrix} q|_Q \\ l_1|_Q \\ l_2|_Q \end{pmatrix}$. But by definition $q|_Q = l_1|_Q = l_2|_Q = 0$.

Hence

$$\mathrm{Hom}(K_1, \mathcal{O}_Q) \cong \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2}, \mathcal{O}_Q) = \mathbb{C}^{19}$$

and so

$$\mathrm{Ext}^1(K_1, \mathcal{O}_Q) \cong \mathrm{Hom}(\mathcal{O}(-4), \mathcal{O}_Q) \cong \mathbb{C}^9.$$

Now we go back to exact sequence

$$0 \to K_1 \to V \to K_0 \to 0.$$

By definition $K_0|_Q = I_Q|_Q = I_Q/I_Q^2 = N^*_{Q/\mathbb{P}^4}$. To compute the cohomology of $N^*_{Q/\mathbb{P}^4}$, we use the exact sequence

$$0 \to N_{Q/\mathbb{P}^2} \to N_{Q/\mathbb{P}^4} \to N_{\mathbb{P}^2/\mathbb{P}^4}|_Q \to 0 :$$

Then

$$\begin{aligned}\mathrm{Hom}_{\mathbb{P}^4}(K_0|_Q, \mathcal{O}_Q) &= \mathrm{Hom}_Q(N^*_{Q/\mathbb{P}^4}, \mathcal{O}_Q) \\ &= H^0(N_{Q/\mathbb{P}^4}) \\ &= H^0(N_{Q/\mathbb{P}^2}) + H^0(N_{\mathbb{P}^2/\mathbb{P}^4}|_Q) \\ &= H^0(\mathcal{O}_Q(2)) + H^0(\mathcal{O}_Q(1)^{\oplus 2}) \\ &= \mathbb{C}^5 \oplus \mathbb{C}^6 = \mathbb{C}^{11},\end{aligned}$$

as $H^1(\mathcal{O}_Q(2)) = 0$.

Altogether it follows from the sequence (8) and the considerations in part I) that

$$\mathrm{Ext}^1(\mathcal{O}_Q, \mathcal{O}_Q) \cong \mathrm{Hom}(K_0, \mathcal{O}_Q) = \mathbb{C}^{11}$$

and furthermore using part II)

$$\mathrm{Ext}^2(\mathcal{O}_Q, \mathcal{O}_Q) \cong \mathrm{Ext}^1(K_0, \mathcal{O}_Q) \cong \mathrm{Hom}(K_1, \mathcal{O}_Q) = \mathbb{C}^{19}. \blacksquare$$

Lemma 4.34 *Let Q_1 and Q_2 be two plane quadrics lying in the planes E_1 and E_2 all intersecting in exactly one point P. Then* $\operatorname{ext}^1(\mathcal{O}_{Q_1}, \mathcal{O}_{Q_2}) = 1$*.*

PROOF: We consider the Grothendieck spectral sequence defined by

$$E_2^{p,q} = H^p(\mathbb{P}^4, \mathcal{E}xt^q(\mathcal{O}_{Q_1}, \mathcal{O}_{Q_2})) \Rightarrow \operatorname{Ext}^{p+q}(\mathcal{O}_{Q_1}, \mathcal{O}_{Q_2}).$$

So we need to compute $E_2^{1,0}$ and $E_2^{0,1}$.

- $E_2^{1,0} = H^1(\mathbb{P}^4, \mathcal{H}om(\mathcal{O}_{Q_1}, \mathcal{O}_{Q_2})) = H^1(\mathbb{P}^4, 0) = 0$.
- So it remains to compute $E_2^{0,1} = H^0(\mathbb{P}^4, \mathcal{E}xt^1(\mathcal{O}_{Q_1}, \mathcal{O}_{Q_2}))$. We claim that $E_2^{0,1} \cong \mathbb{C}$. To see that, we consider the ideal sheaf sequence

  $$0 \to \mathcal{I}_{Q_1} \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{Q_1} \to 0.$$

  This gives (using [12][III Prop 6.3b]) the exact sequence

  $$0 = \mathcal{H}om(\mathcal{O}_{Q_1}, \mathcal{O}_{Q_2}) \to \mathcal{H}om(\mathcal{O}_{\mathbb{P}^4}, \mathcal{O}_{Q_2}) \xrightarrow{\Phi} \mathcal{H}om(\mathcal{I}_{Q_1}, \mathcal{O}_{Q_2})$$
  $$\to \mathcal{E}xt^1(\mathcal{O}_{Q_1}, \mathcal{O}_{Q_2}) \to \mathcal{E}xt^1(\mathcal{O}_{\mathbb{P}^4}, \mathcal{O}_{Q_2}) = 0.$$

  By [12][III Prop 6.3a] $\mathcal{H}om(\mathcal{O}_{\mathbb{P}^4}, \mathcal{O}_{Q_2}) \cong \mathcal{O}_{Q_2}$. Furthermore $\operatorname{Coker}\Phi$ is a sheaf that lives on $x = Q_1 \cap Q_2$. Hence $\operatorname{Coker}\Phi \cong \mathbb{C}_x$ and $E_2^{0,1} \cong \mathbb{C}$.

For $r \geq 2$,

$$E_{r+1}^{1,0} = \frac{\operatorname{Ker}(E_r^{1,0} \to E_r^{r+1,1-r})}{\operatorname{Im}(E_r^{1-r,r-1} \to E_r^{1,0})} = E_r^{1,0},$$

and it follows that $E_\infty^{1,0} = 0$. Hence $\operatorname{Ext}^1(\mathcal{O}_{Q_1}, \mathcal{O}_{Q_2}) = E_\infty^{0,1}$. But

$$E_{r+1}^{0,1} = \frac{\operatorname{Ker}(E_r^{0,1} \to E_r^{r,2-r})}{\operatorname{Im}(E_r^{-r,r} \to E_r^{0,1}} = \operatorname{Ker}(E_r^{0,1} \to E_r^{r,2-r}).$$

As $E_2^{2,0} = H^2(\mathbb{P}^4, \mathcal{E}xt^0(\mathcal{O}_{Q_1}, \mathcal{O}_{Q_2})) = H^2(\mathbb{P}^4, \mathcal{H}om(\mathcal{O}_{Q_1}, \mathcal{O}_{Q_2})) = 0$ and obviously $E_2^{r,2-r} = 0$ for all $r \geq 3$, it follows that $\operatorname{Ext}^1(\mathcal{O}_{Q_1}, \mathcal{O}_{Q_2}) = E_\infty^{0,1} = \mathbb{C}$. ∎

Proposition 4.35 *Let $\mathcal{F} = \mathcal{O}_{Q_1} \oplus \mathcal{O}_{Q_2} \in S^2(M^{4n+2}(\mathbb{P}^4))_0$. Then* $\operatorname{ext}^1(\mathcal{F}, \mathcal{F}) = 24$ *and* $\operatorname{ext}^2(\mathcal{F}, \mathcal{F}) \neq 0$*.*

PROOF: By Lemma 4.33 and 4.34,

$$\begin{aligned}\operatorname{ext}^1(\mathcal{F}, \mathcal{F}) &= \operatorname{ext}^1(\mathcal{O}_{Q_1} \oplus \mathcal{O}_{Q_2}, \mathcal{O}_{Q_1} \oplus \mathcal{O}_{Q_2}) \\ &= 2\operatorname{ext}^1(\mathcal{O}_{Q_1}, \mathcal{O}_{Q_2}) + \operatorname{ext}^1(\mathcal{O}_{Q_1}, \mathcal{O}_{Q_1}) + \operatorname{ext}^1(\mathcal{O}_{Q_2}, \mathcal{O}_{Q_2}) \\ &= 2 \cdot 11 + 1 + 1 = 24. \blacksquare\end{aligned}$$

5 Regularity of M_4 at Generic Points of $S^2(M^{2n+1}(\mathbb{P}^4))_0$

In this Section, let us denote by $(Q_1 \subset E_1, Q_2 \subset E_2)$ a pair of planes in $\mathbb{P}^4$ intersecting in exactly one point P together with two smooth 1-dimensional quadrics $Q_1 \subset E_1$ and $Q_2 \subset E_2$ such that Q_1 and Q_2 also intersect in P.

Lemma 4.36 *Any two pairs of non-singular quadrics* $(Q_1 \subset E_1, Q_2 \subset E_2)$ *and* $(Q'_1 \subset E'_1, Q'_2 \subset E'_2)$*, that intersect in exactly one point* P_1 *resp* P_2*, are projectively equivalent.*

PROOF: The conics Q_i are non-singular, hence they are neither a double line nor a pair of two intersecting lines.
I) There is an automorphism φ of $\mathbb{P}^4$ such that $\varphi(E'_1) = E_1$ and $\varphi(E'_2) = E_2$. Hence the pairs $(Q'_1 \subset E'_1, Q'_2 \subset E'_2)$ and $(\varphi(Q'_1) \subset E_1, \varphi(Q'_2) \subset E_2)$ are projectively equivalent. So we can assume that $E_1 = E'_1$ and $E_2 = E'_2$.

II) Next we show that for a plane E_i, a point $P_i \in E_i$ and two quadrics $Q_i, Q'_i \subset E_i$ with $P_i \in Q_i$ and $P_i \in Q'_i$ there is an automorphism ψ_i of $\mathbb{P}^2$ leaving P_i fixed with $\psi_i(Q_i) = Q'_i$. It suffices to consider the situation in affine coordinates and we may assume $E_i = \mathbb{A}^2$ and $P_i = (0,0)^t$. Then any such quadric Q_i is given by the equation

$$\begin{pmatrix} x & y \end{pmatrix} A_i \begin{pmatrix} x \\ y \end{pmatrix}$$

with a symmetric matrix A_i. Since there is a matrix $S_i \in \mathrm{GL}_2(\mathbb{C})$ with $S_i^t A'_i S_i = A_i$, the automorphism ψ_i is defined by multiplication with an invertible matrix S_i from the right resp. the transpose of S_i from the left.

III) Together, ψ_1 and ψ_2 define a morphism $\psi : E_1 \cup E_2 \to E_1 \cup E_2$, since $E_1 \cap E_2 = P$. It remains to show that we can extend this map to $\mathbb{P}^4$. But this follows from the standard linear algebra fact that if $V_1, V_2 \subset \mathbb{C}^5$ are two subspaces intersecting in a line L, and $\psi_i : V_i \to V_i$ are linear maps such that $\psi_1|_L = \psi_2|_L$, then one has a unique extension of the ψ_i to a linear map on $\mathbb{C}^5$.
∎

The aim of this Section is to prove the following Theorem:

Theorem 4.37 *Let* $(Q_1 \subset E_1, Q_2 \subset E_2)$ *be a pair of smooth plane quadrics interseting in exactly one point. Then the variety* M_4 *is non-singular in the point* $[\mathcal{F}] := [\mathcal{O}_{Q_1} \oplus \mathcal{O}_{Q_2}] \in S^2(M^{2n+1}(\mathbb{P}^4))_0$.

In the proof we denote by $\mathrm{Coker}(A^t)$ the cokernel sheaf of the map

$$\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}$$

for a $(2 \times 6)-$matrix A. The idea of the proof is to construct a 21-dimensional family of deformations $\operatorname{Coker}(A(a_i, b_i, c_i, d_i, t, \alpha, \beta, \gamma, \delta))$ of $\mathcal{F}$ inside M_4. Since $\dim M_4 = 21$, it thus suffices to show that the map

$$\tau : \mathbb{A}^{21} \to M_4$$

obtained by sending the deformation parameters $a_i, b_i, c_i, d_i, \alpha, \beta, \gamma, \delta$ and t to $\operatorname{Coker}(A(a_i, b_i, c_i, d_i, t, \alpha, \beta, \gamma, \delta))$, maps a neighborhood of 0 biholomorphic to a neighborhood of $[\mathcal{F}]$ in M_4. Since $\mathbb{A}^{21}$ is smooth, this implies that M_4 is smooth in $[\mathcal{F}]$.
By Lemma 4.36 we can assume that the planes E_1 and E_2 are given by

$$E_1 = \{(x_0 : \ldots : x_4) \in \mathbb{P}^4 \,|\, x_0 = x_1 = 0\}$$

and

$$E_2 = \{(x_0 : \ldots : x_4) \in \mathbb{P}^4 \,|\, x_2 = x_3 = 0\}$$

and that they intersect in the point $P := (0 : 0 : 0 : 0 : 1)$. Furthermore we can assume that Q_1 is the quadric defined by the additional equation

$$x_2^2 - x_3x_4 = 0$$

in the plane E_1 and Q_2 the quadric defined by the additional equation

$$x_1^2 - x_0x_4 = 0$$

in the plane E_2. Obviously these quadrics also intersect in P. Then $[\mathcal{F}]$ is given by $[\mathcal{O}_{Q_1} \oplus \mathcal{O}_{Q_2}]$ for this particular choice of Q_1 and Q_2.

We start by constructing a 20-dimensional family of deformations of $\mathcal{F}$ inside $S^2(M^{2n+1}(\mathbb{P}^4))_0$.
Since by Proposition 4.17 any sheaf in $M^{2n+1}(\mathbb{P}^4)$ is a structure sheaf supported on a curve of degree two, deformations of sheaves in $M^{2n+1}(\mathbb{P}^4)$ are given by deformations of its support. Since we want to deform $\mathcal{F}$ inside of $S^2(M^{2n+1}(\mathbb{P}^4))_0$, we need to require that the deformed quadrics and planes in the support still intersect in one point.
We obtain this by first deforming the intersection point $P = (0 : 0 : 0 : 0 : 1)$ into some point $P' = (\alpha : \beta : \gamma : \delta : 1)$. A deformation of E_1 through the point P' is given by the equations

$$f_1 := x_0 + a_1x_2 + a_2x_3 + (-\alpha - a_1\gamma - a_2\delta)x_4 = 0$$
$$f_2 := x_1 + a_3x_2 + a_4x_3 + (-\beta - a_3\gamma - a_4\delta)x_4 = 0$$

and a deformation of the quadric Q_1 through the point P' by

$$f_3 := (x_2^2 - x_3x_4) + b_1x_3^2 + b_2x_2x_3 + b_3x_2^2 + b_4x_2x_4$$

$$+ (-\gamma^2 + \delta - b_1\delta^2 - b_2\gamma\delta - b_3\gamma^2 - b_4\gamma)x_4^2 = 0.$$

Similarly we obtain for the second deformed plane and quadric the equations

$$\begin{aligned}
g_1 &:= x_2 + c_1x_0 + c_2x_1 + (-\gamma - c_1\alpha - c_2\beta)x_4 = 0\\
g_2 &:= x_3 + c_3x_0 + c_4x_1 + (-\delta - c_3\alpha - c_4\beta)x_4 = 0\\
g_3 &:= (x_1^2 - x_0x_4) + d_1x_0^2 + d_2x_0x_1 + d_3x_1^2 + d_4x_1x_4\\
&\quad + (-\beta^2 + \alpha - d_1\alpha^2 - d_2\alpha\beta - d_3\beta^2 - d_4\beta)x_4^2 = 0.
\end{aligned}$$

Hence we have a 20-dimensional parameter space, given by

$$\{a_i, b_j, c_k, d_l, \alpha, \beta, \gamma, \delta \mid i, j, k, l = 1, \dots, 4\}.$$

In short, we omit the indices and just write $(a, b, c, d, \alpha, \beta, \gamma, \delta)$.
For a given choice of parameters $(a, b, c, d, \alpha, \beta, \gamma, \delta)$, the equations

$$f_1 = f_2 = f_3 = g_1 = g_2 = g_3 = 0$$

define again a pair of quadrics and planes $(Q_1' \subset E_1', Q_2' \subset E_2')$ where the Q_i' and E_i' intersect in the point P'. Again by Lemma 4.36 any such pair is projectively equivalent to $(Q_1 \subset E_1, Q_2 \subset E_2)$. This means there is a coordinate transformation $p := p^{a,b,c,d,\alpha,\beta,\gamma,\delta}$, i.e. a projective automorphism of $\mathbb{P}^4$, with

$$\begin{aligned}
p(f_1) &= x_0\\
p(f_2) &= x_1\\
p(f_3) &= x_2^2 - x_3x_4\\
p(g_1) &= x_2\\
p(g_2) &= x_3\\
p(g_3) &= x_1^2 - x_0x_4.
\end{aligned}$$

The transformation $p^{a,b,c,d,\alpha,\beta,\gamma,\delta}$ depends analytically on the parameters a_i,b_j, c_k,d_l,α,β, γ and δ, since the family of pairs of quadrics $(Q_1' \subset E_1', Q_2' \subset E_2')$ defined by f_1,f_2,f_3,g_1,g_2 and g_3 is flat over the parameter space $\mathbb{A}^{20}$.

Next we will introduce an additional deformation parameter t. This parameter will give a deformation of $[\mathcal{F}]$ in M_4 transversal to $S^2(M^{2n+1}(\mathbb{P}^4))_0$. By

$$\begin{aligned}
&A(a, b, c, d, t, \alpha, \beta, \gamma, \delta)\\
&\qquad := \begin{pmatrix} f_1 & f_2 & t \cdot p^{-1}(x_4) & t \cdot p^{-1}(x_2) & f_3 & 0\\ t \cdot p^{-1}(x_1) & t \cdot p^{-1}(x_4) & g_1 & g_2 & 0 & g_3 \end{pmatrix}
\end{aligned}$$

we define a family of matrices. For $t = 0$, the maximal minors of $A(a, b, c, d, t, \alpha, \beta, \gamma, \delta)$ give the 20-dimensional family of deformations of $(Q_1 \subset E_1, Q_2 \subset E_2)$ defined above.

Lemma 4.38 *For $t \neq 0$ the maximal minors of $A(a,b,c,d,t,\alpha,\beta,\gamma,\delta)$ defines a rational normal curve for any choice of parameters $a,b,c,d,\alpha,\beta,\gamma,\delta$.*

Proof: $A(a,b,c,d,t,\alpha,\beta,\gamma,\delta)$ defines a rational normal curve if and only if it does after some coordinate transformation of the entries. We apply the coordinate transformation p on the entries of $A(a,b,c,d,t,\alpha,\beta,\gamma,\delta)$ and obtain a matrix

$$A(a,b,c,d,t,\alpha,\beta,\gamma,\delta)' = \begin{pmatrix} x_0 & x_1 & t\cdot x_4 & t\cdot x_2 & x_2^2 - x_3x_4 & 0 \\ t\cdot x_1 & t\cdot x_4 & x_2 & x_3 & 0 & x_1^2 - x_0x_4 \end{pmatrix}.$$

It is obvious, that the last two columns of $A(a,b,c,d,t,\alpha,\beta,\gamma,\delta)'$ are linearly dependent of the first four columns:

$$t \cdot \begin{pmatrix} x_2^2 - x_3x_4 \\ 0 \end{pmatrix} = x_2 \cdot \begin{pmatrix} t\cdot x_2 \\ x_3 \end{pmatrix} - x_3 \cdot \begin{pmatrix} t\cdot x_4 \\ x_2 \end{pmatrix}$$

and

$$t \cdot \begin{pmatrix} 0 \\ x_1^2 - x_0x_4 \end{pmatrix} = x_1 \cdot \begin{pmatrix} x_1 \\ t\cdot x_4 \end{pmatrix} - x_4 \cdot \begin{pmatrix} x_0 \\ t\cdot x_1 \end{pmatrix}.$$

Multiplying the second and third column by t gives

$$\begin{pmatrix} x_0 & t\cdot x_1 & t^2\cdot x_4 & t\cdot x_2 \\ t\cdot x_1 & t^2\cdot x_4 & t\cdot x_2 & x_3 \end{pmatrix}.$$

Hence we obtain a matrix whose maximal minors define a rational normal curve.
■

The family of matrices $A(a,b,c,d,t,\alpha,\beta,\gamma,\delta)$ induces a family of sheaves

$$\mathrm{Coker}(A(a,b,c,d,t,\alpha,\beta,\gamma,\delta)).$$

It follows from Lemma 4.38 and Lemma 4.22 that for $t \neq 0$, the sheaf $\mathrm{Coker}(A(a,b,c,d,t,\alpha,\beta,\gamma,\delta))$ has the form $\mathcal{O}_C(1)$ for some rational normal curve C of degree 4, and thus it has Hilbert polynomial $4n+2$. For $t=0$, the deformations of $\mathcal{O}_{Q_1} \oplus \mathcal{O}_{Q_2}$ in $S^2(M^{2n+1}(\mathbb{P}^4))_0$ also have Hilbert polynomial $4n+2$.
Therefore the sheaves $\mathrm{Coker}(A(a,b,c,d,t,\alpha,\beta,\gamma,\delta))$ form a flat family on the parameter space $\mathbb{A}^{21}$ by [15][Prop 2.1.2].
But this implies that we have a holomorphic map $\tau : \mathbb{A}^{21} \to M_4$ given by

$$(\alpha,\beta,\gamma,\delta,a,b,c,d,t) \mapsto [\mathrm{Coker}(A(a,b,c,d,t,\alpha,\beta,\gamma,\delta))].$$

All that remains to show is that the induced map

$$d\tau : T_0\mathbb{A}^{21} \to T_{[\mathcal{F}]}M^{4n+2}(\mathbb{P}^4)$$

is injective.

Lemma 4.39 *The induced map $d\tau : T_0\mathbb{A}^{21} \to T_{\tau(0)}M_4$ is injective, in particular τ is locally injective.*

Proof: Note that $d\tau$ maps an element $v := (v_1, \dots, v_{21}) \in T_0\mathbb{A}^{21} \cong \mathbb{A}^{21}$ to the infinitesimal deformation

$$\mathcal{F}_\varepsilon := \operatorname{Coker}(A(\varepsilon v))$$

of $\mathcal{F} = \operatorname{Coker}(A(0))$. We have to show that if $\mathcal{F}_\varepsilon$ is the trivial deformation, then $v_i = 0$ for all i.
To understand the infinitesimal deformations $\mathcal{F}_\varepsilon$ better, we use the natural one-to-one correspondence between equivalence classes of deformations of $\mathcal{F}$ over $D := \operatorname{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$ and self extensions of $\mathcal{F}$ (see [13][Theorem 2.7]). The trivial deformation corresponds to the trivial self extension of $\mathcal{F}$. We recall the construction of the extensions (see [13]).
We denote by

$$f_1 : \mathcal{F}_\varepsilon \to \mathcal{F}_\varepsilon \otimes_D \mathbb{C} = \mathcal{F}|_{\mathbb{P}^4 \times \operatorname{Spec}\mathbb{C}} \cong \mathcal{F}$$

the restriction of $\mathcal{F}_\varepsilon$ to $\mathbb{P}^4 \times \operatorname{Spec}\mathbb{C}$. Since $\mathcal{F}_\varepsilon$ is flat over D, there is a short exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{g_1} \mathcal{F}_\varepsilon \xrightarrow{f_1} \mathcal{F} \longrightarrow 0. \qquad (11)$$

Hence we have to show that if the sequence (11) is the trivial extension, then $v = 0$. We construct projective resolutions for $\mathcal{F}$ and $\mathcal{F}_\varepsilon$ and obtain the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r] & \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 2} \arrow[r, "\varphi"] \arrow[d, "g_3"] & \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \arrow[r, "\psi"] \arrow[d, "g_2"] & \mathcal{F} \arrow[r] \arrow[d, "g_1"] & 0 \\
\dots \arrow[r] & \mathcal{O}_{\mathbb{P}^4 \times D}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4 \times D}(-2)^{\oplus 2} \arrow[r, "\varphi_{\varepsilon v}"] \arrow[d, "f_3"] & \mathcal{O}_{\mathbb{P}^4 \times D}^{\oplus 2} \arrow[r, "\psi_{\varepsilon v}"] \arrow[d, "f_2"] & \mathcal{F}_\varepsilon \arrow[r] \arrow[d, "f_1"] & 0 \\
\dots \arrow[r] & \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 2} \arrow[r, "\varphi"] \arrow[d] & \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \arrow[r, "\psi"] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The map $\varphi_{\varepsilon v}$ is given by the matrix $A(\varepsilon v)$ and φ by the matrix $A(0)$.
For the left and middle vertical sequences, there exist splittings

$$s_3 : \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^4 \times D}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4 \times D}(-2)^{\oplus 2}$$

and

$$s_2 : \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^4 \times D}^{\oplus 2}.$$

Via the splittings, the map

$$\varphi_{\varepsilon v} \,:\, \left(\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 2}\right) \oplus \varepsilon \left(\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 2}\right) \to \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \oplus \varepsilon \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}$$

is given by a block matrix

$$\left(\begin{array}{c|c} A & 0 \\ \hline \Theta & A \end{array}\right).$$

Now we assume that $\mathcal{F}_\varepsilon$ is the trivial extention, i.e. there is a splitting s_1 of f_1. So

$$\mathcal{F}_\varepsilon \cong s_1(\mathcal{F}) \oplus \varepsilon g_1(\mathcal{F}) \cong \mathcal{F} \oplus \varepsilon \mathcal{F}.$$

Then $\psi_{\varepsilon v}$ is given by a matrix

$$\widetilde{\psi} := \left(\begin{array}{c|c} \psi & 0 \\ \hline u & \psi \end{array}\right)$$

for some map

$$u \,:\, \mathcal{O}_{\mathbb{P}^4}^2 \to \varepsilon \mathcal{F}.$$

Since the middle horizontal sequence in the diagram above is exact,

$$\left(\begin{array}{c|c} \psi & 0 \\ \hline u & \psi \end{array}\right) \cdot \left(\begin{array}{c|c} A & 0 \\ \hline \Theta & A \end{array}\right) = \left(\begin{array}{c|c} \psi A & 0 \\ \hline uA + \psi\Theta & \psi A \end{array}\right) = 0.$$

Hence in particular $uA + \psi\Theta = 0$.
We have to show that $\Theta = 0$. To do this we first make a base change of $\mathcal{O}_{\mathbb{P}^4 \times D}^{\oplus 2}$ as follows: The map $\psi : \mathcal{O}_{\mathbb{P}^4}^2 \to \mathcal{F}$ induces an isomorphism

$$H^0(\mathcal{O}_{\mathbb{P}^4}^2) \xrightarrow{H^0(\psi)} H^0(\mathcal{F})$$

and hence a unique map $\gamma : \mathcal{O}_{\mathbb{P}^4}^2 \to \mathcal{O}_{\mathbb{P}^4}^2$ such that $u = \psi \circ \gamma$.
We substitute $\widetilde{\psi}$ by

$$\widetilde{\psi} \cdot \left(\begin{array}{c|c} 1 & 0 \\ \hline -\gamma & 1 \end{array}\right) = \left(\begin{array}{c|c} \psi & 0 \\ \hline u & \psi \end{array}\right) \cdot \left(\begin{array}{c|c} 1 & 0 \\ \hline -\gamma & 1 \end{array}\right) = \left(\begin{array}{c|c} \psi & 0 \\ \hline u - \psi\gamma & \psi \end{array}\right) = \left(\begin{array}{c|c} \psi & 0 \\ \hline 0 & \psi \end{array}\right)$$

and consequently

$$\left(\begin{array}{c|c} A & 0 \\ \hline \Theta & A \end{array}\right)$$

by

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline \gamma & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} A & 0 \\ \hline \Theta & A \end{array}\right) = \left(\begin{array}{c|c} A & 0 \\ \hline \gamma A + \Theta & A \end{array}\right).$$

Hence we obtain the following diagram:

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 2} \arrow[rr, "A"] \arrow[d, "\binom{0}{1}"] & & \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \arrow[r, "\psi"] \arrow[d, "\binom{0}{1}"] & \mathcal{F} \arrow[r] \arrow[d, "\binom{0}{1}"] & 0 \\
\left(\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 2}\right) \oplus \varepsilon\left(\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 2}\right) \arrow[rr, "\left(\begin{array}{c|c} A & 0 \\ \hline \gamma A + \Theta & A \end{array}\right)"] \arrow[d, "\binom{1}{0}"] & & \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \oplus \varepsilon\mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \arrow[r, "\left(\begin{array}{c|c} \psi & 0 \\ \hline 0 & \psi \end{array}\right)"] \arrow[d, "\binom{1}{0}"] & \mathcal{F} \oplus \varepsilon\mathcal{F} \arrow[r] \arrow[d, "\binom{1}{0}"] & 0 \\
\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 2} \arrow[rr, "A"] \arrow[d] & & \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \arrow[r, "\psi"] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 & & 0 & 0 &
\end{tikzcd}$$

Since horizontal sequences this diagram are exact, we have

$$0 = \left(\begin{array}{c|c} \psi & 0 \\ \hline 0 & \psi \end{array}\right) \cdot \left(\begin{array}{c|c} A & 0 \\ \hline \gamma A + \Theta & A \end{array}\right) = \left(\begin{array}{c|c} \psi A & 0 \\ \hline \psi(\gamma A + \Theta) & \psi A \end{array}\right)$$

and so $\psi(\gamma A + \Theta) = 0$. This means, that

$$\operatorname{Im}(\gamma A + \Theta) \subseteq \operatorname{Ker}(\psi) = \operatorname{Im}(A).$$

To shorten notation, let $\beta := \Theta + \gamma A$.

In order to show $\Theta = 0$, we show the existence of a map λ that makes the following diagram commutative:

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 2} \arrow[r, two heads, "A"] & \operatorname{Im}(A) = \operatorname{Ker}(\psi) = \mathcal{I}_{Q_1} \oplus \mathcal{I}_{Q_2} \\ \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 2} \arrow[u, dashed, "\lambda"] \arrow[ur, "\beta"'] & \end{tikzcd} \tag{12}$$

Having such a map λ, it would follow that $\Theta + \gamma A = A \cdot \lambda$ and hence $\Theta = A\lambda - \gamma A$. But then $\Theta = 0$. Indeed, it suffices to consider deformations of the form $\mathcal{F}_\varepsilon = \operatorname{Coker}(A(\varepsilon v))$ where

$$A(\varepsilon v) = \begin{pmatrix} x_0 & x_1 & tp^{-1}(x_4) & tp^{-1}(x_2) & x_2^2 - x_3x_4 & 0 \\ tp^{-1}(x_1) & tp^{-1}(x_4) & x_2 & x_3 & 0 & x_1^2 - x_0x_4 \end{pmatrix},$$

i.e. we can assume that $\alpha = \beta = \gamma = \delta = a_i = b_i = c_i = d_i = 0$ for all $i = 1, \ldots, 4$. Otherwise, if $t = 0$ and if the other deformation parameters are chosen arbitrary, then $\operatorname{Coker}(A(\varepsilon v))$ is a deformation inside the stratum $S^2(M^{2n+1}(\mathbb{P}^4))_0$. In this case the assertion is clear for geometric reasons.

Obviously $\Theta = A\lambda - \gamma A$ if and only if $p(\Theta) = p(A\lambda - \gamma A)$ for the coordinate transformation p.

Under this transformation the matrix Θ becomes

$$\Theta = \begin{pmatrix} 0 & 0 & tx_4 & tx_2 & 0 & 0 \\ tx_1 & tx_4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$A = \begin{pmatrix} x_0 & x_1 & 0 & 0 & x_2^2 - x_3x_4 & 0 \\ 0 & 0 & x_2 & x_3 & 0 & x_1^2 - x_0x_4 \end{pmatrix}.$$

Now γ is a (2×2)-matrix with constant entries and λ a (6×6)-matrix of the form

$$\lambda = \left(\begin{array}{c|c} C_1 & L \\ \hline 0 & C_2 \end{array}\right),$$

where C_1 is a (4×4)-matrix with constant entries, C_2 a $(2 \times 2)-$matrix with constant entries and L a (4×2)-matrix whose entries are linear forms. But then, looking at the concrete matrices A and Θ, a simple computation shows that it is not possible to find any γ and λ, such that $\Theta = A\lambda - \gamma A$.

Hence it remains to show the existence of a map λ with the demanded properties.

Claim 4.40 (a) The map

$$H^0(\mathcal{O}_{\mathbb{P}^4}^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2}) = H^0(\mathcal{O}_{\mathbb{P}^4}^{\oplus 4}) \to H^0(\mathcal{I}_{Q_1}(1) \oplus \mathcal{I}_{Q_2}(1))$$

induced by A is an isomorphism.

(b) The map

$$g : H^0(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}) \to H^0(\mathcal{I}_{Q_1}(2) \oplus \mathcal{I}_{Q_2}(2))$$

induced by A is surjective.

PROOF: From the Koszul resolution

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-4) \to \mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(-2) \to \mathcal{I}_Q \to 0$$

we obtain the following two short exact sequences

$$0 \to K \to \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(-2) \to \mathcal{I}_Q \to 0 \tag{13}$$

and

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-4) \to \mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-3)^{\oplus 2} \to K \to 0. \tag{14}$$

a) Using the long exact cohomology sequence of (14) we see that $H^i(K(1)) = 0$ for all i and so by sequence (13) we have $H^0(\mathcal{I}_Q(1)) \cong H^0(\mathcal{O}_{\mathbb{P}^4}^2) = \mathbb{C}^2$.

b) For simplicity we just consider one direct summand $\mathcal{I}_Q$. Using the long exact cohomology sequence of sequence (14), we see that

$$H^0(K(2)) = \mathbb{C} \text{ and } H^i(K(2)) = 0 \text{ for all } i > 0.$$

This implies that the twist by $\mathcal{O}(2)$ of the long exact cohomology sequence of (13) gives the exact sequence

$$0 \to H^0(K(2)) \to H^0(\mathcal{O}(1)^{\oplus 2}) \oplus H^0(\mathcal{O}) \to H^0(\mathcal{I}_Q(2)) \to 0$$

and so g is surjective. ∎

After a twist by $\mathcal{O}(1)$, passing to global sections in sequence 13 gives

$$\begin{tikzcd}
H^0(\mathcal{O}_{\mathbb{P}^4}^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2}) = H^0(\mathcal{O}_{\mathbb{P}^4}^{\oplus 4}) \arrow[r, "\cong"] & H^0(\mathrm{Ker}(\psi)(1)) = H^0(\mathcal{I}_{Q_1}(1) \oplus \mathcal{I}_{Q_2}(1)) \\
H^0(\mathcal{O}_{\mathbb{P}^4}^{\oplus 4}) \arrow[u, dashed, "\exists ! \lambda_1"'] \arrow[ur, "H^0(\beta|_{\mathcal{O}(-1)^4}"'] &
\end{tikzcd}$$

and the existence of a unique map

$$\lambda_1 \, : \, \mathbb{C}^4 \cong H^0(\mathcal{O}_{\mathbb{P}^4}^{\oplus 4}) \to H^0(\mathcal{O}_{\mathbb{P}^4}^{\oplus 4}) \cong \mathbb{C}^4$$

is obvious with Claim 4.40.
Furthermore, a twist by $\mathcal{O}(2)$ gives the following diagram:

$$\begin{tikzcd}
H^0(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^4}^{\oplus 2}) \arrow[r, two heads, "g"] & H^0(\mathrm{Ker}(\psi)(2)) = H^0(\mathcal{I}_{Q_1}(2)) \oplus H^0(\mathcal{I}_{Q_2}(2)) \\
H^0(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 4}) \oplus H^0(\mathcal{O}_{\mathbb{P}^4}^{\oplus 2}) \arrow[u, dashed, "\exists \lambda_2"'] \arrow[ur, "H^0(\beta)"'] &
\end{tikzcd}$$

where the map g is surjective by Claim 4.40.

Then obviously λ_1 and λ_2 induce maps

$$\widetilde{\lambda_1} : \mathcal{O}(-1)^{\oplus 4} \to \mathcal{O}(-1)^{\oplus 4}$$

and furthermore

$$\widetilde{\lambda_2} : \mathcal{O}(-2)^{\oplus 2} \to \mathcal{O}(-2)^{\oplus 2} \oplus \mathcal{O}(-1)^{\oplus 4}.$$

So altogether it follows that there exists a map λ such that the diagram (12) commutes. This finishes the proof. ∎

6 Construction of a Rational Map $\Phi_{MK} : M_4 \dashrightarrow K$

In this Section we explicitly construct a rational map $\Phi_{MK} : M_4 \dashrightarrow K$ such that its domain of definition contains all cokernel sheaves of the Kronecker modules in B. For that we need some cohomological information about the sheaves $[\mathcal{F}] \in M_4$.

In this Section we adapt ideas of a manuscript of M. Lehn, Y. Nagai and D. van Straten from the context of degree 5 curves to the present situation.

First, we consider an exact sequence

$$0 \to A \to W \to B \to 0$$

of vector spaces, where $\dim(W) = 5$, $\dim(A) = 2$ and $\dim(B) = 3$. The surjective map $W \to B$ induces a closed immersion $\mathbb{P}(B) \subset \mathbb{P}(W)$ of codimension 2. The central projection

$$\pi : U := \mathbb{P}(W) \backslash \mathbb{P}(B) \to \mathbb{P}(A)$$

with center $\mathbb{P}(B)$ is defined as follows: For any $x \in \mathbb{P}(W) \backslash \mathbb{P}(B)$ there is a unique subspace of $\mathbb{P}(W)$ of dimension 3 that contains x and $\mathbb{P}(B)$. Then x is mapped by π to the intersection point of this 3-dimensional space with the line $L := \mathbb{P}(A)$.
This gives the open complement $U = \mathbb{P}(W) \backslash \mathbb{P}(B)$ the structure of an affine bundle $\pi : U \to L$ over the line L. Any fiber is isomorphic to $\mathbb{A}^3$.
This implies that π is an affine morphism and hence by [12][Ex II 5.17]
$U = \underline{Spec}(\mathcal{B})$ for the $\mathcal{O}_L$-algebra $\mathcal{B} := \pi_* \mathcal{O}_U$.

The choice of a splitting $s : B \to W$ gives π the structure of a vector bundle. The natural algebra structure of the vector bundle induces an isomorphism

$$\varphi : \mathcal{B} \cong S^*(B \otimes \mathcal{O}_L(-1)).$$

Any other splitting has the form $s' := s + t$ for some $t : B \to A = H^0(\mathcal{O}_L(1))$. The isomorphism φ' induced by s' differs from φ by an automorphism $\varphi' \circ \varphi^{-1}$ of the sheaf of algebras $S^*(B \otimes \mathcal{O}_L(-1))$ defined by

$$\begin{pmatrix} t \\ \mathrm{id} \end{pmatrix} : B \otimes \mathcal{O}_L(-1) \to \mathcal{O}_L \oplus (B \otimes \mathcal{O}_L(-1)) \subset S^*(B \otimes \mathcal{O}_L(-1)).$$

Let $\mathcal{N}^* := B \otimes \mathcal{O}_L(-1)$ be the conormal bundle of $\mathbb{P}(A)$ in $\mathbb{P}(W)$.

Lemma 4.41 • *Let W be a 5-dimensional vector space and $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}(W))$ a coherent sheaf on $\mathbb{P}(W)$ such that $\dim(\mathrm{Supp}(\mathcal{F})) = 1$.*

We choose for a given sheaf $\mathcal{F}$ an exact sequence $0 \to A \to W \to B \to 0$ such that $\dim B = 3, \dim A = 2$ and $\mathrm{Supp}(\mathcal{F}) \cap \mathbb{P}(B) = \varnothing$ and choose a splitting $s : B \to W$. Then one can associate to $\mathcal{F}$ a sheaf $E \in \mathrm{Coh}(\mathbb{P}(A))$ with the structure of a $\mathcal{B}$-module given by some map $\beta : E \otimes \mathcal{N}^* \to E$ with the commutativity constraint that the induced map

$$E \otimes \Lambda^2 \mathcal{N}^* \to E \otimes \mathcal{N}^* \otimes \mathcal{N}^* \to E$$

vanishes.

- *Conversely: Let $0 \to A \to W \to B \to 0$ be an exact sequence of vector spaces of dimensions $\dim W = 5, \dim A = 2$ and $\dim B = 3$ and $s : B \to W$ a splitting.*

  If E is a coherent $\mathcal{O}_L$-module with a $\mathcal{B}$-module structure as above, then E defines a coherent $\mathcal{O}_{\mathbb{P}(W)}$-module $\mathcal{F}$ with 1-dimensional support in $U = \mathbb{P}(W) \backslash \mathbb{P}(B)$.

- *The correspondence has the following properties:*

  $$H^j(E(n)) = H^j(\mathcal{F}(n)) \ \forall j \geq 0, n \in \mathbb{Z}$$

  and so

  $$P_E(n) = P_{\mathcal{F}}(n).$$

Proof: Let $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}(W))$ be an arbitrary sheaf such that

$$\dim(\mathrm{Supp}(\mathcal{F})) = 1.$$

Obviously a subspace $\mathbb{P}(B)$ can be chosen such that

$$\mathrm{Supp}(\mathcal{F}) \cap \mathbb{P}(B) = \varnothing,$$

i.e. $\mathrm{Supp}(\mathcal{F}) \subset U$. We define a map p by

$$p := \pi|_{\mathrm{Supp}(\mathcal{F})} : \mathrm{Supp}(\mathcal{F}) \to L.$$

For all $x \in L$, the set $p^{-1}(x) \subset \pi^{-1}(x) \cong \mathbb{A}^3$ is compact and so it consists of finitely many points. Hence p is a finite map. It follows that $R^i \pi_* \mathcal{F} = 0$ for all $i > 0$. Since p is finite, by [12][II 5.8.1] the direct image $\pi_* \mathcal{F} =: E$ is a coherent sheaf on L. By [12][III ex 8.1] and the fact that $R^i \pi_* \mathcal{F} = 0$ for all $i > 0$,

$$H^j(E(n)) = H^j(\mathcal{F}(n)) \ \forall j \geq 0, n \in \mathbb{Z}$$

and so

$$P_E(n) = P_{\mathcal{F}}(n).$$

Since $\mathcal{F}$ is a $\mathcal{O}_{\mathbb{P}(W)}$-module with support in U, the sheaf E has the structure of a $\pi_*\mathcal{O}_U = \mathcal{B}$-module.
The $\mathcal{B}$-module structure of E is via the isomorphism

$$\varphi : \mathcal{B} \to S^*\mathcal{N}^* \cong S^*(B \otimes \mathcal{O}_L(-1))$$

completely determined by a homomorphism

$$\beta : E \otimes \mathcal{N}^* \to E,$$

that satisfies the commutativity constraint that the induced map

$$E \otimes \Lambda^2\mathcal{N}^* \to E \otimes \mathcal{N}^* \otimes \mathcal{N}^* \to E$$

vanishes.
Replacing φ by φ' changes β to $\beta' := \beta + \mathrm{id}_E \otimes \tilde{t}$, where $\tilde{t} : B \otimes \mathcal{O}_L(-1) \to \mathcal{O}_L$ is the adjoint to $t : B \to A$.

Conversely: by [12][II ex 5.17e] there exists a sheaf $\mathcal{F}$ such that $\pi_*\mathcal{F} = E$. It remains to show that $\dim \mathrm{Supp}(\mathcal{F}) = 1$. For that, assume that $\mathrm{Supp}(\mathcal{F}) \subset \mathbb{P}(W)$ has dimension $\dim \mathrm{Supp}(\mathcal{F}) \geq 2$. Then its intersection with $\mathbb{P}(B)$ which has dimension 2 cannot be empty. ∎

A subsheaf $E' \subset E$ is a $\mathcal{B}$-submodule, if and only if the image of the map β restricted to E' is contained in E', i.e.

$$\overline{\beta} : E' \otimes \mathcal{N}^* \to E \to E/E'$$

vanishes. Then for E' the commutativity contraint mentioned above is clearly fulfilled.

Lemma 4.42 *(a) There is a bijection between subsheaves of $\mathcal{F}$ and $\mathcal{B}$-submodules of E.*

(b) $\mathcal{F}$ is (semi-)stable if and only if E is a (semi-)stable $\mathcal{B}$-module.

Proof: (a) The bijection follows from the equivalence of category of quasi-coherent $\mathcal{O}_U$-modules and quasi-coherent $\mathcal{B}$-modules. (see [12][ex II 5.17e])

(b) Assume that $\mathcal{F}$ is pure. Let T be the torsion subsheaf of E. Since $T \otimes \mathcal{N}^*$ is a torsion sheaf and E/T is torsion free, the map $\overline{\beta}$ vanishes on $T \otimes \mathcal{N}^*$. Hence T is a $\mathcal{B}$-submodule. By part (a), there is a unique subsheaf $\mathcal{F}'$ of $\mathcal{F}$ such that $\pi_* \mathcal{F}' = T$. Since π is finite and $\dim \operatorname{Supp}(T) = 0$, also $\dim \operatorname{Supp}(\mathcal{F}') = 0$. But as $\mathcal{F}$ is pure, $\mathcal{F}' = 0$ and hence $T = 0$. This implies that E is a vector bundle on L.

Conversely, if E is locally free and T a subsheaf of $\mathcal{F}$ of dimension 0, then $\pi_* T$ is a subsheaf of E with 0-dimensional support and hence $T = 0$.

Furthermore, since subsheaves $\mathcal{F}'$ of $\mathcal{F}$ and their direct images $\pi_* \mathcal{F}'$ have the same Hilbert polynomial, $\mathcal{F}$ is semi-stable if and only if E is a semi-stable sheaf with respect to the $\mathcal{B}$-submodules. ∎

Now we want to apply this to stable sheaves in the moduli space $M^{4n+2}(\mathbb{P}^4)$ and compute all possible sheaves $E = \pi_* \mathcal{F}$. So let $\mathcal{F}$ be a stable sheaf in $M^{4n+2}(\mathbb{P}^4)$. Then $E = \bigoplus_{i=1}^4 \mathcal{O}(l_i)$ for a sequence of integers $l_1 \geq \ldots \geq l_4$. We say that the integer sequence $l := (l_1, \ldots, l_4)$ has a gap if there is an i with $1 \leq i \leq 4$, such that $l_i \geq l_{i+1} + 2$.

Lemma 4.43 *If $\mathcal{F}$ is stable, then there are no gaps in the integer sequence of E.*

Proof: Assume there is an integer i such that $l_i \geq l_{i+1} + 2$. Then define E' as the subsheaf $E' := \mathcal{O}_L(l_1) \oplus \ldots \oplus \mathcal{O}_L(l_i)$. Hence $E/E' = \mathcal{O}_L(l_{i+1}) \oplus \ldots \oplus \mathcal{O}_L(l_4)$. The degrees of the summands of

$$\begin{aligned} E' \otimes \mathcal{N}^* &\cong (\mathcal{O}_L(l_1) \oplus \ldots \oplus \mathcal{O}_L(l_i)) \otimes (B \otimes \mathcal{O}_L(-1)) \\ &\cong (\mathcal{O}_L(l_1 - 1) \oplus \ldots \oplus \mathcal{O}_L(l_i - 1)) \otimes B \end{aligned}$$

are by definition strictly larger than the degrees of the summands of $E/E' = \mathcal{O}_L(l_{i+1}) \oplus \ldots \oplus \mathcal{O}_L(l_4)$. Hence the canonical map

$$E' \otimes \mathcal{N}^* = E' \otimes (B \otimes \mathcal{O}_L(-1)) \to E/E'$$

must vanish. So E' is a $\mathcal{B}$-submodule of E which is destabilizing. This is a contradiction since E is stable by Lemma 4.42(b). So the integer sequence does not have any gaps. ∎

Proposition 4.44 *For a stable sheaf $\mathcal{F} \in M^{4n+2}(\mathbb{P}^4)$, the possible integer sequences l are:*

$$l = (1, 0, -1, -2) \text{ and } l = (0, 0, -1, -1).$$

Proof: We assume that $l_1 \geq l_2 \geq l_3 \geq l_4$. Let $\sigma := \sum_{i=1}^4 l_i$, then $\sigma = -2$, as the Hilbert polynomial of $\mathcal{O}(l_1) + \ldots + \mathcal{O}(l_4)$ should be $4n + 2$.

Obviously $0 \leq l_1 < 2$. In fact, if $l_1 \leq -1$. Then $l_2, l_3, l_4 \leq -1$, and so $\sigma \leq -4$. Furthermore, if $l_1 \geq 2$, then $l_2 \geq 1, l_3 \geq 0, l_4 \geq -1$, as the integer sequence does not have a gap and so $\sigma \geq 2$.

If $l_1 = 0$, then $l_2 \in \{0, -1\}$ and if $l_2 = 0$, then $l_3 \in \{0, -1\}$ and so $l_4 = -2 - l_1 - l_2 - l_3 = -2 - l_3$. Hence we get the integer sequence $l = (0, 0, -1, -1)$. If $l_2 = -1$, then $l_3 \in \{-1, -2\}$ and $l_4 = -2 - l_1 - l_2 - l_3 = -1 - l_3$, so $l_4 \in \{0, 1\}$, which is a contradiction.

If $l_1 = 1$, then $l_2 \in \{1, 0\}$. For $l_2 = 1$ we have $l_3 \geq 0$ and $l_4 \geq -1$ and so $\sigma \geq 1$ which is a contradiction. So $l_2 = 0$ and hence $l_3 \in \{0, -1\}$. If $l_3 = 0$, then $l_4 \geq -1$ and so $\sigma \geq 0$ which is again a contradiction. So $l_3 = -1$ and we get another integer sequence $l = (1, 0, -1, -2)$. ■

To obtain cohomological information about the sheaves $\mathcal{F} \in M_4$, we want to prove that sheaves $\mathcal{F} \in M^{4n+2}(\mathbb{P}^4)$ with integer sequence $(1, 0, -1, -2)$ are not contained in the component M_4 of $M^{4n+2}(\mathbb{P}^4)$. For the proof we need the following preparation:

Lemma 4.45 *Let $\mathcal{F}$ be a stable sheaf on $\mathbb{P}^4$ with Hilbert polynomial $4n+2$ and integer sequence $l = (1, 0, -1, -2)$. Then $\mathcal{F}$ is scheme-theoretically supported on a plane.*

Proof: By assumption, E is a semi-stable $\mathcal{B}$-module on L of the form $E = \bigoplus_{i=1}^4 \mathcal{O}_L(l_i)$, where $l = (1, 0, -1, -2)$. Now consider the structure map $\beta : E \otimes \mathcal{N}^* \to E$. Since $\mathcal{N}^* = \mathcal{O}_L(-1) \otimes B$, the map β looks like

$$\begin{aligned} \beta : (\mathcal{O}_L(l_1 - 1) \oplus \mathcal{O}_L(l_2 - 1) \oplus \mathcal{O}_L(l_3 - 1) \oplus \mathcal{O}_L(l_4 - 1)) \otimes B \\ \to \mathcal{O}_L(l_1) \oplus \mathcal{O}_L(l_2) \oplus \mathcal{O}_L(l_3) \oplus \mathcal{O}_L(l_4). \end{aligned}$$

If we choose a basis $b_1, \ldots, b_3$ for the vector space B, the map β is given by three (4×4)-matrices $\beta_i = (\beta_{mn}^{(i)})_{m,n}$, where the entries are homogeneous forms in S^*A of degree $l_m - l_n + 1$. So if $l_m < l_n - 1$, then $\beta_{mn}^{(i)} = 0$. Hence for $i = 1, 2, 3$, the matrices β_i are of the form

$$\beta_i = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

Since β is defined using the isomorphism $\mathcal{B} \cong S^*\mathcal{N}^*$, the matrices β_i commute. The entry $\beta^{(i)}_{2,1}$ is a constant for $i = 1, 2, 3$. If it were 0 for all i, then $\mathcal{O}_L(1)$ would be a destabilizing $\mathcal{B}$-submodule of E, which is a contradiction. So by changing the basis of B appropriately, we may assume that $\beta^{(1)}_{2,1} = 1$ and $\beta^{(2)}_{2,1} = \beta^{(3)}_{2,1} = 0$. So

$$\beta_1 = \begin{pmatrix} * & * & * & * \\ 1 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix} \text{ and } \beta_i = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix} \text{ for } i \geq 2.$$

The entries $\beta^{(i)}_{1,1}$ are linear forms in A. If we substitute the splitting s by $s + t$, where t is the map $t : B \to A, b_i \mapsto -\beta^{(i)}_{1,1}$, we can write the β_i in the form

$$\beta_1 = \begin{pmatrix} 0 & * & * & * \\ 1 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix} \text{ and } \beta_i = \begin{pmatrix} 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix} \text{ for } i \geq 2.$$

As the matrices β_i commute, the second column of β_2 and β_3 vanishes:

$$\beta_i = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

for $i = 2, 3$.

The entry $\beta^{(1)}_{3,2}$ is a linear form. If $\beta^{(i)}_{3,2}$ vanishes for all i, then $\mathcal{O}_L(1) \oplus \mathcal{O}_L$ would be a destabilizing $\mathcal{B}$-submodule of E. Since we already know that $\beta^{(2)}_{3,2} = \beta^{(3)}_{3,2} = 0$, we can assume that $\beta^{(1)}_{3,2} =: k \neq 0$. But since for $i = 2, 3$

$$0 = [\beta_1, \beta_i] = \beta_1\beta_i - \beta_i\beta_1 = \begin{pmatrix} 0 & k \cdot \beta^{(i)}_{1,3} & * & * \\ 0 & k \cdot \beta^{(i)}_{2,3} & * & * \\ 0 & k \cdot \beta^{(i)}_{3,3} & * & * \\ 0 & k \cdot \beta^{(i)}_{4,3} & * & * \end{pmatrix},$$

it follows that $\beta^{(i)}_{1,3} = \beta^{(i)}_{2,3} = \beta^{(i)}_{3,3} = \beta^{(i)}_{4,3} = 0$ for $i = 2, 3$.

Using the same arguments, we can deduce that $\beta^{(1)}_{4,3} \neq 0$ and $\beta^{(i)}_{1,4} = \beta^{(i)}_{2,4} = \beta^{(i)}_{3,4} = \beta^{(i)}_{4,4} = 0$ for $i = 2, 3$. So $\beta_2 = \beta_3 = 0$.

This means that the scheme theoretical support of $\mathcal{F}$ is contained in the projective plane cut out by the linear forms $s(b_2)$ and $s(b_3)$ from $\mathbb{P}(W)$.

The structure of $\mathcal{F}$ is completely determined by the matrix

$$\beta_1 = \begin{pmatrix} 0 & * & * & * \\ 1 & * & * & * \\ 0 & k & * & * \\ 0 & 0 & * & * \end{pmatrix}. \blacksquare$$

Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety X. By the Hilbert Syzygy Theorem one has a resolution of $\mathcal{F}$ by locally free sheaves $\mathcal{E}_i$:

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0.$$

Definition 4.46 (Chern class for a coherent sheaf) *We define the Chern class of $\mathcal{F}$ to be*

$$c(\mathcal{F}) = \prod_{i=0}^{n} (-1)^i c(\mathcal{E}_i).$$

and the Chern character of $\mathcal{F}$ by

$$\mathrm{Ch}(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i \,\mathrm{Ch}(\mathcal{E}_i).$$

Let $\mathcal{F}$ be a stable sheaf with Hilbert polynomial $4n+2$ and integer sequence $(1, 0, -1, -2)$. The plane $\mathbb{P}^2 \subset \mathbb{P}^5$ containing its support, is unique. Otherwise $\mathrm{Supp}(\mathcal{F})$ is contained in the intersection of two planes, at best a line. Then $\mathcal{F}$ is of the form $\mathcal{F} = \bigoplus \mathcal{O}_L(a_i)$ and hence it is not stable.

Lemma 4.47 *Any stable sheaf $\mathcal{G}$ on $\mathbb{P}^2$ with Hilbert polynomial $4n+2$ has Chern classes $c_1 = 4H$ and $c_2 = 12H^2$.*

Proof: It it clear that $c_1 = aH \in H^2(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}$ and $c_2 = bH^2 \in H^4(\mathbb{P}^2, \mathbb{Z})$ for some $a, b \in \mathbb{Z}$. We use the Riemann-Roch theorem for smooth projective surfaces ([7, Thm. 14.2]):

Theorem 4.48 *If $\mathcal{G}$ is a coherent sheaf on a smooth projective surface, then*

$$\chi(\mathcal{G}) = \frac{c_1(\mathcal{G})^2 - 2c_2(\mathcal{G}) + c_1(\mathcal{G})c_1(\mathcal{T}_S)}{2} + \mathrm{rank}(\mathcal{G})\frac{c_1(\mathcal{T}_S)^2 + c_2(\mathcal{T}_S)}{12}.$$

Since by [29][III.6 Prop. 6], $\deg(\chi(\mathcal{G}(n)) = \dim(\mathrm{Supp}(\mathcal{G}))$, we have $\dim(\mathrm{Supp}(\mathcal{G})) = 1$ and hence $\mathrm{rank}(\mathcal{G}) = 0$. So we have

$$2 = \chi(\mathcal{G}) = \frac{3}{2}c_1(\mathcal{G}) + \frac{1}{2}(c_1(\mathcal{G})^2 - 2c_2(\mathcal{G}))$$

and hence $2b = 3a + a^2 - 4$.

Since for any line bundle $\mathcal{L}$ we have $c_1(\mathcal{G} \otimes \mathcal{L}) = c_1(\mathcal{G}) + rc_1(\mathcal{L})$ and $c_2(\mathcal{G} \otimes \mathcal{L}) = c_2(\mathcal{G}) - c_1(\mathcal{G})c_1(\mathcal{L})$, we get another condition for a and b by choosing $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)$: $6 = \chi(\mathcal{G} \otimes \mathcal{O}_{\mathbb{P}^2}(1)) = \frac{3}{2}a + \frac{1}{2}a^2 - b + a$, and so $6 = \frac{5}{2}a + \frac{1}{2}a^2 - b$.

Altogether: $a = 4$ and $b = 12$. ∎

By the Lemma of Schur for stable sheaves, $\hom(\mathcal{G}, \mathcal{G}) = 1$ and Serre duality gives

$$\operatorname{ext}^2(\mathcal{G}, \mathcal{G}) = \hom(\mathcal{G}, \mathcal{G}(-3)) = 0.$$

Definition 4.49 *Let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves. Then the Euler characteristic of the pair $(\mathcal{E}, \mathcal{F})$ is defined by*

$$\chi(\mathcal{E}, \mathcal{F}) := \sum (-1)^i \operatorname{ext}^i(\mathcal{E}, \mathcal{F}).$$

In the following we will use a version of the Theorem of Hirzebruch-Riemann-Roch (see [15][Lemma 6.1.3]):

Lemma 4.50 *If X is smooth and projective, then for coherent sheaves $\mathcal{E}$ and $\mathcal{F}$,*

$$\chi(\mathcal{E}, \mathcal{F}) = \int_X \operatorname{ch}^\vee(\mathcal{E}).\operatorname{ch}(\mathcal{E}).\operatorname{td}(X),$$

where $\operatorname{ch}_k^\vee(\mathcal{F}) = (-1)^k \operatorname{ch}_k(\mathcal{F})$.

Lemma 4.51 *For any sheaf $\mathcal{G} \in M^{4n+2}(\mathbb{P}^2)$, we have* $\operatorname{ext}^1(\mathcal{G}, \mathcal{G}) = 17$.

Proof:

$$\operatorname{td}(\mathcal{T}_{\mathbb{P}^2}) = 1 + \frac{1}{2}c_1(\mathcal{T}_{\mathbb{P}^2}) + \frac{1}{12}(c_1(\mathcal{T}_{\mathbb{P}^2})^2 + c_2(\mathcal{T}_{\mathbb{P}^2})) = 1 - \frac{3}{2}h + h^2,$$

$$\operatorname{ch}(\mathcal{G}) = r + c_1(\mathcal{G}) + \frac{1}{2}(c_1^2(\mathcal{G}) - 2c_2(\mathcal{G})) = 4h - 16h^2$$

and

$$\operatorname{ch}((\mathcal{G})^\vee) = r - c_1(\mathcal{G}) + \frac{1}{2}(c_1^2(\mathcal{G}) - 2c_2(\mathcal{G})) = -4h - 16h^2,$$

hence

$$\operatorname{ext}^0(\mathcal{G}, \mathcal{G}) - \operatorname{ext}^1(\mathcal{G}, \mathcal{G}) + \operatorname{ext}^2(\mathcal{G}, \mathcal{G}) = 1 - \operatorname{ext}^1(\mathcal{G}, \mathcal{G}) = \int_X \operatorname{ch}^\vee(E).\operatorname{ch}(E).\operatorname{td}(X)$$

$$= (4h - 16h^2).(-4h - 16h^2).(1 - \frac{3}{2}h + h^2) = -16. \ \blacksquare$$

By $M^{4n+2}(\mathbb{P}^2)$ we denote the moduli space of stable sheaves on $\mathbb{P}^2$ with Hilbert polynomial $4n+2$. Since for all stable sheaves $\mathcal{G} \in M^{4n+2}(\mathbb{P}^2)$, we have $\mathrm{ext}^1(\mathcal{G},\mathcal{G}) = 17$ and $\mathrm{ext}^2(\mathcal{G},\mathcal{G}) = 0$, the variety $M^{4n+2}(\mathbb{P}^2)$ is a smooth projective variety of dimension 17.

Remark 4.52 There is an open subset of $M^{4n+2}(\mathbb{P}^2)$ consisting of line bundles $\mathcal{L}$ of degree 4 on smooth plane curves of degree 4 and genus 3. Indeed, such a line bundle has Hilbert polynomial $4n+2$, since

$$\chi(\mathcal{L}) = \deg(\mathcal{L}) + (1-g) = 4 + (1-3) = 2.$$

Any smooth plane curve of degree 4 has genus 3. We consider the ideal sheaf sequence of C

$$0 \to \mathcal{I}_{C/\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}|_C \to 0. \tag{15}$$

Since $\mathcal{I}_{C/\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-4)$, we compute after a twist of sequence 15 by $\mathcal{O}_{\mathbb{P}^2}(4)$

$$h^0(\mathcal{O}_{\mathbb{P}^2}(4)|_C) = h^0(\mathcal{O}_{\mathbb{P}^2}(4)) - h^0(\mathcal{O}_{\mathbb{P}^2}) = \binom{4+2}{2} - 1 = 14$$

and hence $h^0(\mathcal{N}_{C/\mathbb{P}^2}) = h^0(\mathcal{O}_{\mathbb{P}^2}(4)|_C) = 14$.
Thus the plane curves of degree 4 and genus 3 are parametrized by an open subset in a projective space of dimension 14. The choice of $\mathcal{L} \in \mathrm{Pic}(C)$ adds another 3 dimensions: using the exponential sequence

$$0 \to H^1(C,\mathbb{Z}) \to H^1(\mathcal{O}_C) \to \mathrm{Pic}(C) \to H^2(C,\mathbb{Z}) \to \dots,$$

and the fact that $H^1(C,\mathbb{Z})$ and $H^2(C,\mathbb{Z})$ are discrete, we deduce that

$$\dim \mathrm{Pic}(C) = \dim h^1(\mathcal{O}_C) = g(C) = 3.$$

Hence the dimension of this open set is $14+3 = 17$.

We define M' to be the closed subset of $M^{4n+2}(\mathbb{P}^4)$ of stable sheaves $[\mathcal{F}]$ that are supported on a plane.

Lemma 4.53 *M' is a smooth projective variety of dimension 23.*

Proof: No stable sheaf can live on a line, hence the plane that contains the support of $[\mathcal{F}]$ is uniquely determined. This gives a map

$$M' \to \mathrm{Grass}(2,\mathbb{P}^4),$$

which has smooth fibers of dimension 17. Thus M' is a smooth projective scheme of dimension $17 + \dim(\mathrm{Grass}(2,\mathbb{P}^4)) = 17 + 3\cdot(5-3) = 23$. ∎

Proposition 4.54 *Let* $[\mathcal{F}] \in M^{4n+2}(\mathbb{P}^4)$ *be a stable sheaf with integer sequence* $(1, 0, -1, -2)$*. Then* $[\mathcal{F}]$ *is not contained in* M_4*.*

Proof: The strategy of the proof is to show that $\mathrm{ext}^1_{\mathbb{P}^4}(\mathcal{F}, \mathcal{F}) = 23 = \dim M'$. Then a neighborhood of $\mathcal{F}$ is contained in M'. Since

$$T_{\mathcal{F}} M^{4n+2}(\mathbb{P}^4) \cong \mathrm{Ext}^1_{\mathbb{P}^4}(\mathcal{F}, \mathcal{F}),$$

the sheaf $\mathcal{F}$ corresponds to a non-singular point of M' and no other component of $M^{4n+2}(\mathbb{P}^4)$ intersects M' in $\mathcal{F}$. As $M' \not\subset M_4$ and $M_4 \not\subset M'$, the assertion follows.

So we compute the dimension of the Ext-group $\mathrm{Ext}^1_{\mathbb{P}^4}(\mathcal{F}, \mathcal{F})$ now.
Let $i : \mathbb{P}^2 \to \mathbb{P}^4$ denote the closed immersion of the plane supporting $\mathcal{F}$. We have for any coherent sheaf on $\mathbb{P}^4$ an isomorphism

$$\mathrm{Hom}_{\mathbb{P}^4}(\mathcal{G}, i_* \mathcal{F}) \cong \mathrm{Hom}_{\mathbb{P}^2}(i^* \mathcal{G}, \mathcal{F})$$

that is functorial in $\mathcal{G} \in \mathrm{Coh}(\mathbb{P}^4)$. Now consider the change-of-rings-spectral sequence (see [3][XV,5])

$$E_2^{p,q} := \mathrm{Ext}^p_{\mathbb{P}^2}(\mathcal{T}or_q^{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2}), \mathcal{F}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathbb{P}^4}(\mathcal{F}, \mathcal{F})$$

and use the exact sequence of terms of low degree

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to E^2,$$

where

$$E^i = \mathrm{Ext}^i_{\mathbb{P}^4}(\mathcal{F}, \mathcal{F}) \text{ and}$$

$$E_2^{i,0} = \mathrm{Ext}^i_{\mathbb{P}^2}(\mathcal{T}or_0^{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2}), \mathcal{F}) = \mathrm{Ext}^i_{\mathbb{P}^2}(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^2}, \mathcal{F}) = \mathrm{Ext}^i_{\mathbb{P}^2}(\mathcal{F}, \mathcal{F}),$$

which we have already computed. Furthermore

$$E_2^{0,1} = \mathrm{Ext}^0_{\mathbb{P}^2}(\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2}), \mathcal{F}) = \mathrm{Hom}_{\mathbb{P}^2}(\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2}), \mathcal{F}).$$

Hence we need to compute $\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2})$. The ideal sheaf sequence of $\mathcal{O}_{\mathbb{P}^2}$ in $\mathcal{O}_{\mathbb{P}^4}$ gives rise to the exact sequence

$$0 \to \mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2}) \to \mathcal{F} \otimes \mathcal{I} \to \mathcal{F} \to \mathcal{F}|_{\mathbb{P}^2} \to 0,$$

where $\mathcal{I}$ is the ideal sheaf of $\mathbb{P}^2$ in $\mathbb{P}^4$. So using that $\mathcal{F}$ is supported on a plane, we have

$$\mathcal{T}or_1^{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^2}) = \mathcal{F} \otimes \mathcal{I}|_{\mathbb{P}^2} = \mathcal{F} \otimes \mathcal{I}/\mathcal{I}^2 = \mathcal{F} \otimes \mathcal{N}^*.$$

So we obtain an exact sequence

$$0 \to \operatorname{Ext}^1_{\mathbb{P}^2}(\mathcal{F},\mathcal{F}) \to \operatorname{Ext}^1_{\mathbb{P}^4}(\mathcal{F},\mathcal{F}) \to \operatorname{Hom}_{\mathbb{P}^2}(\mathcal{F}\otimes\mathcal{N}^*,\mathcal{F}) \to \operatorname{Ext}_{\mathbb{P}^2}(\mathcal{F},\mathcal{F}) = 0.$$

We want to compute $\operatorname{ext}^1_{\mathbb{P}^4}(\mathcal{F},\mathcal{F})$ and know that $\operatorname{ext}^1_{\mathbb{P}^2}(\mathcal{F},\mathcal{F}) = 17$. So it remains to compute the dimension of

$$\hom_{\mathbb{P}^2}(\mathcal{F}\otimes\mathcal{N}^*,\mathcal{F}) = \hom_{\mathbb{P}^2}(\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^2}(-1)^3,\mathcal{F}) = 3\hom_{\mathbb{P}^2}(\mathcal{F},\mathcal{F}(1)).$$

The multiplication by any of the coordinate functions of $\mathbb{P}^2$ gives three homomorphisms in $\hom_{\mathbb{P}^2}(\mathcal{F},\mathcal{F}(1))$. We show that there are no other homomorphisms.

$\mathcal{F}$ has the integer sequence $(1,0,-1,-2)$. Since $\mathcal{F}$ and

$$\pi_*\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-2)$$

have the same number of sections, the twisted sheaf $\mathcal{F}(-1)$ has a unique section. Since $\mathcal{F}$ is pure, the image $\mathcal{F}'$ of the map $\alpha : \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{F}$ defined by the section of $\mathcal{F}(-1)$, is pure. Assume that the kernel $K := \ker(\alpha)$ is not locally free. Denote by $\overline{K}$ the saturation of K in $\mathcal{O}_{\mathbb{P}^2}(1)$. Then the cokernel of $\overline{K} \to \mathcal{O}_{\mathbb{P}^2}(1)$ is a quotient Q of $\mathcal{F}'$ (which coincides with $\mathcal{F}'$ in all but finitely many points). But that means that the kernel of the map $\mathcal{F}' \to Q$ has a support of dimension 0. That contradicts the fact that $\mathcal{F}'$ is pure. Thus $K = \mathcal{O}_{\mathbb{P}^2}(-m)$ for some $m \geq 0$.

The Hilbert polynomial of $\mathcal{F}'$ is

$$\begin{aligned} P_{\mathcal{F}'}(n) &= P_{\mathcal{O}_{\mathbb{P}^2(1)}}(n) - P_{\mathcal{O}_{\mathbb{P}^2(-m)}}(n) \\ &= \binom{n+3}{2} - \binom{n+2-m}{2} \\ &= (m+1)n + \frac{1}{2}(m+1)(4-m). \end{aligned}$$

So the reduced Hilbert polynomial of $\mathcal{F}'$ is $n + \frac{1}{2}(4-m)$.
We assume that $\mathcal{F}'$ is a proper subsheaf of $\mathcal{F}$. Since $\mathcal{F}$ is stable and $\mathcal{F}'$ is a proper subsheaf of $\mathcal{F}$, we get the condition

$$n + \frac{1}{2}(4-m) < n + \frac{1}{2}$$

for the reduced Hilbert polynomials of $\mathcal{F}'$ and $\mathcal{F}$. Then $m \geq 4$. But that means that the Hilbert polynomial of $\mathcal{F}'$ has at least the leading coefficient 5, which is a contradiction, as $\mathcal{F}'$ is a subsheaf of a sheaf with Hilbert polynomial $4n+2$. Hence $\mathcal{F}' = \mathcal{F}$.

So we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{F} \to 0.$$

Therefore $\mathrm{Ker}(\alpha) = \mathcal{I}_C \otimes \mathcal{O}(-1)$ for some plane curve C of degree 4 and $\mathcal{F} = \mathcal{O}_C(1)$. Hence

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{F}, \mathcal{F}(1)) = \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_C(1), \mathcal{F}(1)) \subset \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{F}(1))\\ = H^0(\mathcal{F}) \cong \mathbb{C}^3.\end{aligned}$$

Hence $\mathrm{ext}^1_{\mathbb{P}^4}(\mathcal{F}, \mathcal{F}) = 17 + 6 = 23.$ ∎

Proposition 4.55 *Let* $[\mathcal{F}] \in M_4$ *be a semi-stable sheaf on* $\mathbb{P}^4$ *with Hilbert polynomial* $4n + 2$. *Then* $h^0\mathcal{F} = 2, h^1\mathcal{F} = 0$. *Moreover* $h^0(\mathcal{F}(1)) = 6$. *In particular,* $\mathcal{F}$ *is* 1*-regular.*

Proof: I) First assume that $\mathcal{F}$ is a stable sheaf. Using the notation of Lemma 4.41, set $E := \pi_*\mathcal{F}$. Then $h^j(E(n)) = h^j(\mathcal{F}(n))$ for all $j \geq 0$ and $n \in \mathbb{Z}$. By Proposition 4.54 and Proposition 4.44 we obtain for $\mathcal{F} \in M_4$ the sheaf $\pi_*\mathcal{F} = E = \mathcal{O}_L^2 \oplus \mathcal{O}_L(-1)^2$ and hence

$$h^0(\mathcal{F}) = h^0(E) = 2,\ h^1(\mathcal{F}) = h^1(E) = 0$$

and furthermore

$$h^0(E(1)) = h^0(\mathcal{O}_L(1)^{\oplus 2} \oplus \mathcal{O}_L^{\oplus 2}) = 6.$$

II) Now assume that $\mathcal{F}$ is a strictly semi-stable sheaf. In this case, there exists an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

where $\mathcal{F}'$ and $\mathcal{F}''$ have Hilbert polynomials $P_{\mathcal{F}'}(n) = P_{\mathcal{F}''}(n) = 2n + 1$. By Proposition 4.17, $\mathcal{F}'$ and $\mathcal{F}''$ are structure sheaves $\mathcal{O}_{Q'}$ resp $\mathcal{O}_{Q''}$ of curves Q and Q' of degree 2. Hence $h^1(\mathcal{F}') = h^1(\mathcal{F}'') = 0$. Since $P_{\mathcal{F}}(n) = P_{\mathcal{F}'}(n) + P_{\mathcal{F}''}(n)$, this implies that

$$h^0(\mathcal{F}) = h^0(\mathcal{O}_{Q'}) + h^0(\mathcal{O}_{Q''}) = 2.$$

Moreover, since $h^1(\mathcal{F}'(1)) = h^1(\mathcal{F}''(1)) = 0$, we have $h^0(\mathcal{O}_{Q'}(1)) = h^0(\mathcal{O}_{Q''}(1)) = P_{\mathcal{O}_{Q'}}(1) = 3$ and hence $h^0(\mathcal{F}(1)) = 3 + 3 = 6.$ ∎

Lemma 4.56 *Any semi-stable sheaf* $\mathcal{F}$ *on* $\mathbb{P}^4$ *with Hilbert polynomial* $4n+2$ *is* 2*-regular.*

Proof: Since $\dim(\mathrm{Supp}(\mathcal{F})) = 1$, it suffices to check that $H^1(\mathcal{F}(1)) = 0$ for any such sheaf $\mathcal{F}$.
First, assume that $\mathcal{F}$ is stable. Then by Proposition 4.44, the sheaf $\mathcal{F}$ has either the integer sequence $(0,0,-1,-1)$ or $(1,0,-1,-2)$. In the first case,

$$h^1(\mathcal{O}_L(1)^{\oplus 2} \oplus \mathcal{O}^{\oplus 2}) \cong h^0(\mathcal{O}_L(-4)^{\oplus 2} \oplus \mathcal{O}_L(-3)^{\oplus 2}) = 0$$

and in the second case

$$\begin{aligned} h^1(&\mathcal{O}_L(2) \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L \oplus \mathcal{O}_L(-1)) \\ &\cong h^0(\mathcal{O}_L(-4) \oplus \mathcal{O}_L(-3) \oplus \mathcal{O}_L(-2) \oplus \mathcal{O}_L(-1)) \\ &= 0. \end{aligned}$$

Hence $H^1(\mathcal{F}(1)) = 0$. Observe also that $H^1(\mathcal{F}) \neq 0$ in the second case and hence a sheaf with integer sequence $(1,0,-1,-2)$ is not 1-regular.

For a strictly semi-stable sheaf $\mathcal{F}$ we have an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

where $\mathcal{F}'$ and $\mathcal{F}''$ have Hilbert polynomial $2n+1$. As seen in the proof of Proposition 4.17, $\mathcal{F}'$ and $\mathcal{F}''$ are structure sheaves of curves of degree at most 2. So it is clear that $H^1(\mathcal{F}'(1)) = H^1(\mathcal{F}''(1)) = 0$ and hence also $H^1(\mathcal{F}(1)) = 0$.
∎

Using this cohomological data of semi-stable sheaves $\mathcal{F}$ with Hilbert polynomial $4n+2$, we want to construct a rational map $\Phi_{MK} : M_4 \dashrightarrow K$.
Since all semi-stable sheaves $\mathcal{F}$ on $\mathbb{P}^4$ with Hilbert polynomial $P_{\mathcal{F}}(n) = 4n+2$ are 2-regular, the moduli space $M^{4n+2}(\mathbb{P}^4)$ is constructed as a categorical quotient of some open subset $\tilde{X}$ of the moduli space $Q = \mathrm{Quot}(\mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 10}, P)$ by an action of the group $\mathrm{GL}(V)$, where V is a $\mathbb{C}$-vector space of dimension 10. Let $\tau : \tilde{X} \to \tilde{X} /\!\!/ \mathrm{GL}(V)$ be the good quotient of $\tilde{X}$ by the action of $\mathrm{GL}(V)$.

Since we are just interested in the component M_4 of $M^{4n+2}(\mathbb{P}^4)$, we consider the closed subset $\tau^{-1}(M_4) \subset \tilde{X}$. By definition, $\tau^{-1}(M_4)$ is $\mathrm{GL}(V)$-invariant. Thus one can construct a good quotient $M_4 = \tau^{-1}(M_4) /\!\!/ \mathrm{GL}(V)$.

Now the first step of the construction of a rational map $\Phi_{MK} : M_4 \dashrightarrow K$ is to define a rational map

$$\tilde{\Phi}_{MK} : \tau^{-1}(M_4) \dashrightarrow K.$$

Afterwards we show that $\tilde{\Phi}_{MK}$ is invariant under the action of $\mathrm{GL}(V)$ to obtain a rational map

$$\Phi_{MK} : M_4 = \tau^{-1}(M_4) /\!\!/ \mathrm{GL}(V) \dashrightarrow K.$$

We start with a flat family in $\tau^{-1}(M_4)$, i.e. a flat family

$$\mathcal{O}_{\mathbb{P}^4 \times R}(-2)^{\oplus 10} \to \tilde{\mathcal{F}} \to 0$$

of quotients of $\mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 10}$, parametrized by an open set $X \subset \tau^{-1}(M_4)$. Hence for any restriction

$$\tilde{\mathcal{F}}(x) := \tilde{\mathcal{F}}|_{\{x\} \times \mathbb{P}^4},$$

$x \in X$ we obtain $h^0(\tilde{\mathcal{F}}(x)) = 2$ and $h^1(\tilde{\mathcal{F}}(x)) = 0$ by Proposition 4.55.
Let $p : X \times \mathbb{P}^4 \to X$ and $q : X \times \mathbb{P}^4 \to \mathbb{P}^4$ denote the projections on the first resp. second component.
By φ^i we denote the natural map

$$\varphi^i(x) : R^i p_*(\tilde{\mathcal{F}}) \otimes \mathbb{C}(x) \to H^i(\{x\} \times \mathbb{P}^4, \tilde{\mathcal{F}}(x)).$$

Since $h^1(\{x\} \times \mathbb{P}^4, \tilde{\mathcal{F}}(x)) = 0$ for all x by Proposition 4.55 , the map $\varphi^1(x)$ is surjective and hence an isomorphism by [12][III. Thm 12.11a]. So $R^1 p_* \tilde{\mathcal{F}} = 0$ and $R^i p_* \tilde{\mathcal{F}} = 0$ for all $i > 1$. Hence by [12][Thm 12.11b], the map φ^0 is surjective. Now we apply again [12][Thm 12.11b] to conclude that

$$R^0 p_* \tilde{\mathcal{F}} = p_* \tilde{\mathcal{F}} =: \mathcal{A}_0$$

is a locally free $\mathcal{O}_X$-module of rank $h^0(\tilde{\mathcal{F}}|_{\{x\} \times \mathbb{P}^4}) = 2$.
Consider the evaluation map

$$\mathrm{ev} : p^* \mathcal{A}_0 = p^*(p_* \tilde{\mathcal{F}}) \to \tilde{\mathcal{F}}.$$

We set

$$X_0 := \{x \in X \mid \tilde{\mathcal{F}}(x) \text{ generated by global sections } \};$$

hence

$$\mathrm{ev}\,|_{X_0} : p^*(\mathcal{A}_0|_{X_0}) \to \tilde{\mathcal{F}}|_{p^{-1}(X_0)}$$

is surjective.

Lemma 4.57 *X_0 is an open subset of X.*

Proof: We consider the exact sequence

$$p^* \mathcal{A}_0 \overset{\mathrm{ev}}{\to} \tilde{\mathcal{F}} \to \tilde{\mathcal{F}} / \operatorname{Im}(\mathrm{ev}) \to 0.$$

The support of the coherent sheaf $\tilde{\mathcal{F}} / \operatorname{Im}(\mathrm{ev})$ is closed in $X \times \mathbb{P}^4$. Hence the image of $\operatorname{Supp}(\tilde{\mathcal{F}} / \operatorname{Im}(\mathrm{ev}))$ under the projection map p is also closed in X, since the projection p is proper. The set $p(\operatorname{Supp}(\tilde{\mathcal{F}} / \operatorname{Im}(\mathrm{ev})))$ contains exactly those points $x \in X$, for which $\tilde{\mathcal{F}}(x)$ is not globally generated. ∎

Remark 4.58 We will see in Lemma 4.63 that $X_0 \neq \varnothing$.

We define $\mathcal{N}$ to be the kernel of the evaluation map twisted by $q^*\mathcal{O}_{\mathbb{P}^4}(1)$. So $\mathcal{N}$ sits in the exact sequence

$$0 \to \mathcal{N} \to p^*(\mathcal{A}_0|_{X_0}) \otimes q^*\mathcal{O}_{\mathbb{P}^4}(1) \to \tilde{\mathcal{F}}|_{p^{-1}(X_0)} \otimes q^*\mathcal{O}_{\mathbb{P}^4}(1) \to 0. \qquad (16)$$

Since the sheaves $p^*(\mathcal{A}_0|_{X_0})$ and $\tilde{\mathcal{F}}|_{p^{-1}(X_0)}$ are flat over X_0, the family $\mathcal{N}$ is also flat over X_0, see e.g. [12][III Prop.9.1A(e)].

Lemma 4.59 *The subset*

$$X_1 := \{x \in X_0 \,|\, h^0(\mathcal{N}(x)) := h^0(\mathcal{N}|_{\{x\}\times\mathbb{P}^4}) = 4\} \subset X_0$$

is open.

Remark 4.60 Again we will see in Lemma 4.63 that $X_1 \neq \varnothing$.

Proof: First, observe that $h^0(\mathcal{N}(x)) \geq 4$ for all $x \in X_0$. Indeed, restricting the sequence (16) to $\{x\} \times \mathbb{P}^4$ gives by [21][Prop 1.2.6] an exact sequence

$$\begin{aligned} 0 \to \mathcal{N}(x) &\to (p^*(\mathcal{A}_0|_{X_0}) \otimes q^*\mathcal{O}_{\mathbb{P}^4}(1))|_{\{x\}\times\mathbb{P}^4} \\ &\to (\tilde{\mathcal{F}}|_{p^{-1}(X_0)} \otimes q^*\mathcal{O}_{\mathbb{P}^4}(1))|_{\{x\}\times\mathbb{P}^4} \to 0. \end{aligned}$$

This exact sequence gives rise to a long exact sequence of cohomology:

$$\begin{aligned} 0 \to H^0(\mathcal{N}(x)) \to &H^0((p^*(\mathcal{A}_0|_{X_0}) \otimes q^*\mathcal{O}_{\mathbb{P}^4}(1))|_{\{x\}\times\mathbb{P}^4}) \\ \to &H^0((\tilde{\mathcal{F}}|_{p^{-1}(X_0)} \otimes q^*\mathcal{O}_{\mathbb{P}^4}(1))|_{\{x\}\times\mathbb{P}^4}) \to H^1(\mathcal{N}(x)) \to 0, \end{aligned}$$

using $H^1((p^*(\mathcal{A}_0|_{X_0}) \otimes q^*\mathcal{O}_{\mathbb{P}^4}(1))|_{\{x\}\times\mathbb{P}^4}) \cong H^1(\mathcal{O}_{\mathbb{P}^4}(1))^{\oplus 2} = 0$. Since by Proposition 4.55

$$\begin{aligned} h^0((\tilde{\mathcal{F}}|_{p^{-1}(X_0)} \otimes q^*\mathcal{O}_{\mathbb{P}^4}(1))|_{\{x\}\times\mathbb{P}^4}) &= h^0(\tilde{\mathcal{F}}(x) \otimes q^*\mathcal{O}_{\mathbb{P}^4}(1)|_{\{x\}\times\mathbb{P}^4}) \\ &= h^0(\tilde{\mathcal{F}}(x) \otimes \mathcal{O}_{\{x\}\times\mathbb{P}^4}(1)) \\ &= h^0(\tilde{\mathcal{F}}(x)(1)) = 6, \end{aligned}$$

and similarly $h^0((p^*(\mathcal{A}_0|_{X_1}) \otimes q^*\mathcal{O}_{\mathbb{P}^4}(1))|_{\{x\}\times\mathbb{P}^4})) = 10$, we have $h^0(\mathcal{N}(x)) \geq 4$.

Since the family $\mathcal{N}$ is flat, it follows with semi-continuity of cohomology (see [12][Rem 12.7.1]), that the set $\{x \in X_0 \,|\, h^0(\mathcal{N}(x)) < 5\}$ is open. In total, we obtain

$$\{x \in X_0 \,|\, h^0(\mathcal{N}(x)) < 5\} = \{x \in X_0 \,|\, h^0(\mathcal{N}(x)) = 4\},$$

hence X_1 is open. ∎

Remark 4.61 The proof shows that for $x \in X_1$, the space $H^1(\mathcal{N}(x)) = 0$.

Now we construct a morphism $X_1 \to W$. Using Remark 4.61 and [12][III. Thm. 12.11] we may argue as before for $\mathcal{A}_0$ and conclude that $\mathcal{A}_1 := p_*(\mathcal{N}|_{X_1})$ is a locally free $\mathcal{O}_X$-module of rank 4. By sequence (16) we have an $\mathcal{O}_X$-linear map

$$p_*(\mathcal{N}|_{X_1}) \to p_*(p^*\mathcal{A}_0 \otimes q^*\mathcal{O}(1)).$$

Via the projection formula we obtain

$$\begin{aligned} p_*(p^*\mathcal{A}_0 \otimes q^*\mathcal{O}_{\mathbb{P}^4}(1)) &\cong \mathcal{A}_0 \otimes p_*q^*\mathcal{O}_{\mathbb{P}^4}(1) \\ &\cong \mathcal{A}_0 \otimes \mathcal{O}_X^5 \end{aligned}$$

Composing these two morphisms gives a family of Kronecker modules

$$\mathcal{A}_1 \to \mathcal{A}_0 \otimes \mathbb{C}^5.$$

This defines a morphism

$$\Phi := \tilde{\Phi}_{MK}|_{X_1} : X_1 \to W = \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_0 \otimes \mathbb{C}^5)$$

in the following way: First we define the map set-theoretically. Take a point $x \in X_1$ given by the sheaf $\mathcal{F}$. Then $\mathcal{F}$ is globally generated by two section and we have an epimorphism $\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2} \to \mathcal{F}(1) \to 0$, whose kernel $\mathcal{N}$ has exactly 4 sections. The composed map $\mathcal{O}_{\mathbb{P}^4}^{\oplus 4} \to \mathcal{N} \to \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2}$ defines a Kronecker module. By the previous considerations this map is a morphism of schemes.

Since the Kronecker modules obtained in this way might be unstable, we have to shrink X_1 once more to some open subset $X_2 \subset X_1$ to obtain a morphism

$$\tilde{\Phi}_{MK}|_{X_2} : X_2 \to W^{ss} /\!\!/ G = K.$$

Lemma 4.62 *The subset $X_2 \subset X_1$ of sheaves $\mathcal{F} \in X_1$ with $\Phi(\mathcal{F})$ a semi-stable Kronecker module, is open in X_1.*

PROOF: Being semi-stable is an open property in flat families. Hence inside $\mathrm{Im}(\Phi) \subset W$ there exists an open subset $W^{ss} \cap \mathrm{Im}(\Phi)$ of Kronecker modules that are semi-stable. Now we define X_2 to be the preimage $\Phi^{-1}(W^{ss} \cap \mathrm{Im}(\Phi)) \subset X_1$. Obviously, X_2 is an open subset of X_1. ∎

Lemma 4.63 *The cokernel sheaves*

$$\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \to \mathrm{Coker}(\varphi) \to 0$$

of Kronecker modules $\varphi \in B$ are contained in X_2. In particular, the set X_2 is non-empty.

PROOF: **Step 1:** First, it is clear that the cokernel sheaves of Kronecker modules in B are globally generated. Hence they are contained in the set X_0.

Step 2: By $\mathcal{N}$ we denote the kernel of the induced map

$$\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2} \to \mathrm{Coker}(\varphi)(1).$$

To show that the sheaves $\mathrm{Coker}(\varphi)$ are contained in X_1, it suffices to show the assertion

$$h^0(\mathcal{N}) = 4$$

for *some* Kronecker module in the stratum B_{10} because of the semi-continuity theorem of cohomology. To do this, we choose the representative

$$A = \begin{pmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & x_1 & x_2 \end{pmatrix}^t \in B_{10}.$$

Recall (Lemma 4.24) that for $\mathcal{F} := \mathrm{Coker}(A^t)$ we have the exact sequence

$$0 \to i_* \mathcal{O}_L(-1)^{\oplus 2} \to \mathcal{F} \to i_* \mathcal{O}_L^{\oplus 2} \to 0,$$

where L is the lines in $\mathbb{P}^4$ defined by $x_0 = x_1 = x_2 = 0$. In this situation we constructed the following commutative diagram (see diagram (5)):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathcal{G}_1 \arrow[r, "\beta"] \arrow[d, "\mu"] & \mathcal{G}_2 \arrow[r] \arrow[d, dotted, "\alpha"] & 0 \\
\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 4} \arrow[ur, "\tilde{\varphi}"] \arrow[r, "\varphi"] \arrow[dr, "0"'] & \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \arrow[r] \arrow[d, "\psi"] & \mathcal{F} \arrow[r] \arrow[dl, dotted] & 0 \\
 & i_* \mathcal{O}_L^{\oplus 2} \arrow[d] & & \\
 & 0 & &
\end{tikzcd}$$

Then by definition $\mathcal{N}(-1) = \operatorname{Im}(\varphi) = \operatorname{Im}(\tilde{\varphi})$. Hence there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{N} \arrow[r] & (\mathcal{I}_L \otimes \mathcal{O}_{\mathbb{P}^4}(1))^{\oplus 2} = \mathcal{G}_1(1) \arrow[r, "\beta"] \arrow[d] & i_* \mathcal{O}_L^{\oplus 2} = \mathcal{G}_2(1) \arrow[r] & 0 \\
& & (\mathcal{I}_L \otimes \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2})|_L \arrow[ur, "\gamma"] & &
\end{tikzcd}$$

The map β factorizes over $\mathcal{I}_L \otimes \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2}|_L$, since $\operatorname{Supp}(i_* \mathcal{O}_L^{\oplus 2}) = L$.
Now

$$\begin{aligned}
(\mathcal{I}_L \otimes \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2})|_L &= \mathcal{I}_L / \mathcal{I}_L^2 \otimes \mathcal{O}_L(1)^{\oplus 2} \\
&= \mathcal{N}^*_{L/\mathbb{P}^4} \otimes \mathcal{O}_L(1)^{\oplus 2} = \mathcal{O}_L(-1)^{\oplus 3} \otimes \mathcal{O}_L(1)^{\oplus 2} = \mathcal{O}_L^{\oplus 6}.
\end{aligned}$$

The map $\gamma : \mathcal{O}_L^{\oplus 6} \to \mathcal{O}_L^{\oplus 2}$ is given by a (2×6)-matrix with constant entries and is surjective. Hence $H^0(\beta)$ is surjective, too.
In order to show, that $h^0(\mathcal{N}) = 4$, we need to show that $H^0(\mathcal{I}_L \otimes \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2}) = 6$. Thus it suffices to show that

$$h^0(\mathcal{I}_L \otimes \mathcal{O}_{\mathbb{P}^4}(1)) \cong h^0((\mathcal{I}_L \otimes \mathcal{O}_{\mathbb{P}^4}(1))|_L) = 3.$$

There is an exact sequence

$$0 \to \mathcal{I}_L^2(1) \to \mathcal{I}_L \otimes \mathcal{O}_{\mathbb{P}^4}(1) \to \mathcal{I}_L \otimes \mathcal{O}_{\mathbb{P}^4}(1)|_L \to 0. \tag{17}$$

We want to use the long exact cohomology sequence of this sequence to show the assertion. To do this, we need to compute $H^0(\mathcal{I}_L^2(1))$ and $H^1(\mathcal{I}_L^2(1))$. So we consider the exact sequence

$$0 \to \mathcal{I}_L^2(1) \to \mathcal{I}_L(1) \to \mathcal{I}_L / \mathcal{I}_L^2(1) \to 0. \tag{18}$$

Since there are exactly 3 hyperplanes in $\mathbb{P}^4$ containing L, we have $h^0(\mathcal{I}_L(1)) = 3$. Then the ideal sheaf sequence of L in $\mathbb{P}^4$ gives rise to a long exact sequence

$$0 \to H^0(\mathcal{I}_L(1)) \to H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \to H^0(\mathcal{O}_L(1)) \to H^1(\mathcal{I}_L(1)) \to H^1(\mathcal{O}_{\mathbb{P}^4}(1)) = 0.$$

This implies that $H^1(\mathcal{I}_L(1)) = 0$.

Furthermore, $H^0(\mathcal{I}_L^2(1)) = 0$, since the scheme $2L$ defined by $\mathcal{I}_L^2$, does not lie in any hyperplane of $\mathbb{P}^4$. Hence the long exact cohomology sequence of the short exact sequence (18)

$$0 \to H^0(\mathcal{I}_L^2(1)) \to H^0(\mathcal{I}_L(1)) \to H^0(\mathcal{O}_L^{\oplus 3}) \to H^1(\mathcal{I}_L^2(1)) \to H^1(\mathcal{I}_L(1)) = 0,$$

implies that $H^1(\mathcal{I}_L^2(1)) = 0$.

So the assertion that $h^0(\mathcal{N} = 4)$ follows directly from the long exact cohomology sequence of the short exact sequence (17).

Step 3: It remains to show, that given a Kronecker module φ in B, then the cokernel $\mathrm{Coker}(\varphi)$ belong to X_2, not only to X_1. By definition, this means that $\tilde{\Phi}_{MK}(\mathrm{Coker}(\varphi))$ is a semi-stable Kronecker module.
By construction, $\tilde{\Phi}_{MK}(\mathrm{Coker}(\varphi))$ gives back the Kronecker module φ, which is stable by definition of B. ∎

In summary, we have established a morphism $\tilde{\Phi}_{MK}|_{X_2} : X_2 \to K$ and hence a rational map $\tilde{\Phi}_{MK} : \tau^{-1}(M_4) \dashrightarrow K$.

In order to obtain a rational map $\Phi_{MK} : M_4 \dashrightarrow K$, we need to observe that $\tilde{\Phi}_{MK}$ descends to a rational map on the good quotient $\tau^{-1}(M_4) /\!\!/ \mathrm{GL}(V)$. But this is clear since the construction is invariant under the action of $\mathrm{GL}(V)$ on $\tau^{-1}(M_4)$.

Summing up, we obtain the following result:

Theorem 4.64 *There is a birational map $\Phi_{KM} : K \dashrightarrow M_4 \subset M^{4n+2}(\mathbb{P}^4)$ defined outside $S \cup P$. The inverse map Φ_{MK} is defined on an open subset of M_4 consisting of sheaves $\mathcal{F}$ subject to the following properties*

(a) $\mathcal{F}$ is globally generated by 2 sections,

(b) if $\mathcal{N}$ is the kernel of the induced map $\mathcal{O}_{\mathbb{P}^4}^{\oplus 2}(1) \to \mathcal{F}(1)$, then $h^0(\mathcal{N}) = 4$,

(c) in the resolution

$$0 \longleftarrow \mathcal{F} \longleftarrow \mathcal{O}_{\mathbb{P}^4}^2 \xleftarrow{A} \mathcal{O}_{\mathbb{P}^4}(-1)^4 \longleftarrow \dots$$

of $\mathcal{F}$ constructed from the data in (a) and (b), the Kronecker module A is semi-stable.

Remark 4.65 Observe that $S^2(M^{2n+1}(\mathbb{P}^4)_0)$ is contained in the domain of definition of the rational map Φ_{MK}. In fact, sheaves of the form $\mathcal{O}_{Q_1} \oplus \mathcal{O}_{Q_2}$ are obviously globally generated by 2 sections and furthermore

$$h^0(\mathcal{N}) = 2 \cdot h^0(\mathcal{I}_{Q_1/\mathbb{P}^4}(1)) = 4$$

as there are two hyperplanes in $\mathbb{P}^4$ containing the quadric Q_1. It is easy to see that the Kronecker module $\Phi_{MK}(\mathcal{O}_{Q_1} \oplus \mathcal{O}_{Q_2})$ is poly-stable and its maximal minors define the unique planes spanned by Q_1 and Q_2.

7 Open Problems

I. **Question**: Is M_4 non-singular in every point, in particular in any point $x \in S^2(M^{2n+1})(\mathbb{P}^4)_0$?

(a) The regularity in generic points of $S^2(M^{2n+1})(\mathbb{P}^4)_0$, i.e. if the support of $\mathcal{O}_{Q_1} \oplus \mathcal{O}_{Q_2} \in S^2(M^{2n+1})(\mathbb{P}^4)_0$ consists of two smooth plane quadrics Q_1 and Q_2 that intersect in exactly one point, is proven in Theorem 4.37. Now one has to consider for example the special cases:

  (i) The quadrics Q_1 and Q_2 can degenerate into singular quadrics i.e. pairs of lines intersecting in one point or double lines.

  (ii) The planes E_1 and E_2 can degenerate into planes that intersect in one line or into the same plane.

  (iii) The pair of quadrics (or degenerated quadrics) can intersect in 2 or 4 points or have a common line (in case that the planes E_1 and E_2 intersect in a line or are the same).

(b) If x corresponds to a sheaf of the type $\mathcal{F} = \mathcal{O}_Q^{\oplus 2}$ for some plane quadric Q, the choice of a family of deformations of $\mathcal{F}$ inside M_4 is much more difficult than in the proof of Theorem 4.37. A crucial point in the construction of Theorem 4.37 was to describe deformations of $\mathcal{G} = \mathcal{O}_{Q_1} \oplus \mathcal{O}_{Q_2}$ inside $S^2(M^{2n+1})(\mathbb{P}^4)_0$, i.e. those deformations inside $S^2(M^{2n+1})(\mathbb{P}^4)$ that leave $\mathrm{Supp}(\mathcal{G})$ connected. Our method was to first deform the intersection point of Q_1 and Q_2 into some point p and afterwards the quadrics Q_1 and Q_2 through p in order to find a $20-$dimensional family of deformations of $\mathcal{G}$. For $\mathcal{F} = \mathcal{O}_Q^{\oplus 2}$ it is not clear how to describe the deformations leaving the support connected. Furthermore the group action of the stabilizer $\mathrm{Aut}(\mathcal{F}) = \mathrm{GL}_2(\mathbb{C})$ is more complicated.

II. **Question**: In Section 4.6 we constructed a rational map $\Phi_{MK} : M_4 \dashrightarrow K$. Its domain of definition is an open subset of M_4 consisting of sheaves $\mathcal{F}$ with the following properties:

(a) $\mathcal{F}$ is globally generated by 2 sections,

(b) if $\mathcal{N}$ is the kernel of the induced map $\mathcal{O}_{\mathbb{P}^4}^{\oplus 2}(1) \to \mathcal{F}(1)$, then $h^0(\mathcal{N}) = 4$,

(c) in the resolution

$$0 \longleftarrow \mathcal{F} \longleftarrow \mathcal{O}_{\mathbb{P}^4}^2 \xleftarrow{A} \mathcal{O}_{\mathbb{P}^4}(-1)^4 \longleftarrow \ldots$$

of $\mathcal{F}$ constructed from the data in (a) and (b) the Kronecker module A is semi-stable.

It is not clear so far, whether one can extend this rational map to a morphism on M_4, i.e. whether the set $\tau(X_2)$ is a proper open subset of M_4. So the question is whether there are sheaves $\mathcal{F}$ in M_4 that are not globally generated or sheaves $\mathcal{F}$ that are globally generated and $h^0(\mathcal{N}) > 4$. The difficulty is again to decide whether sheaves with these properties are contained in the component M_4 of the moduli space $M^{4n+2}(\mathbb{P}^4)$.

Appendix A

Source Code of the Singular Programs

The following programs used in various parts of this Thesis were developed using the computer algebra system Singular ([5]).

1 Dimensions of the Orbits and the Luna Slice

This Singular program computes the Luna Slice of a given matrix on its stratum with respect to the group action of G resp. the joint action of G and $\mathrm{PGL}(V)$. The dimension of the strata with respect to the action of G resp. G and $\mathrm{PGL}(V)$ can be read of from the length of the list WQ resp. VQ.

Listing A.1: Slice

```
LIB "matrix.lib";
int p=0;
ring P4= 0, (x(0..4)), dp;
matrix MX[5][1]=x(0..4);
int i,j;


//Input:  4x2 matrix A
//Output: 20x40 matrix, whose rows are given by the
//        action of a basis of Lie(G) on A and a
//        45x40 matrix, whose rows are given by the
//        action of a basis of Lie(G) and Lie(GL(V))
//        on A.
//        Every column of the output corresponds to
//        a Kronecker module, viewed as a 40x1
//        matrix with entries in C.
```

```
proc stabilisator(A)
{
    matrix AA[8][1];
    for (i=1;i<=2;i=i+1){ for (j=1;j<=2;j=j+1) {
        matrix EE[2][2]; EE[i,j]=1;
        matrix B[8][1]=ideal(-A*EE);
        AA=concat(AA,B); kill EE,B;
    };};
    for (i=1;i<=4;i=i+1){ for (j=1;j<=4;j=j+1) {
        matrix EE[4][4]; EE[i,j]=1;
        matrix B[8][1]=ideal(EE*A);
        AA=concat(AA,B); kill EE,B;
    };};
    matrix BB=submat(AA,1..8,2..21);

    matrix C[8][5]=transpose(lift(maxideal(1),ideal(A)));
    for (i=1;i<=5;i=i+1){ for (j=1;j<=5;j=j+1) {
        matrix EE[5][5]; EE[i,j]=1;
        matrix B[8][1]=ideal(C*EE*MX);
        AA=concat(AA,B); kill EE,B;
    };};
    AA=submat(AA,1..8,2..46);

    matrix NB[20][40]=ideal(transpose(lift(maxideal(1),ideal(transpose(BB)))));
    matrix NA[45][40]=ideal(transpose(lift(maxideal(1),ideal(transpose(AA)))));
    return(transpose(NB),transpose(NA));
};

//Input: 40xm matrix W
//Output: List of the columns of W as 4x2 matrices with entries in V = <x0,...,x4>
proc umschreiben(W)
{
    int m=ncols(W);
    matrix W2[m*8][5]=ideal(transpose(W));
    ideal J=W2*MX; // hat 8*m Eintraege
    list L;
    int j;
    for (j=1;j<=m;j=j+1){
```

```
        matrix Z[4][2]=J[(j-1)*8+1..j*8];
        L[j]=Z; kill Z;
    };
    return(L);
};



matrix UQ,TQ=stabilisator(Q);    // TQ is a 40x45 matrix
matrix WQ=syz(transpose(TQ));    // WQ is a 40x10 matrix
matrix VQ=syz(transpose(UQ));


// computes the slice with regard to the group action of G at A as list of Kronecker modules
list L=umschreiben(WQ);
for (i=1;i<=size(L);i=i+1){ print(L[i]);print(” ”);};

// computes the slice with regard to the group action of G and PGL(V) at A
list L=umschreiben(VQ);
for (i=1;i<=size(L);i=i+1){ print(L[i]);print(” ”);};
```

2 Saturation

The following program computes the saturation of the family of ideals associated to points of a line in the Luna slice of S_1.

Listing A.2: Saturation

```
LIB ”elim.lib”;
ring R=(0),(x(0..4), t),dp;

//S1

poly t(1) = 0;
poly t(2) = 1;
poly t(3) = 0;
poly t(4) = 0;
poly t(5) = 0;
poly t(6) = 0;
poly t(7) = 0;
poly t(8) = 0;


matrix M[4][2]=
x(0)+ t*(-t(5)*x(1)-t(6)*x(2)-t(7)*x(3)-t(8)*x(4)),
t*(t(1)*x(1)+t(2)*x(3)+t(5)*x(2)+t(7)*x(4)),
t*(t(3)*x(2)+t(4)*x(4)-t(6)*x(1)-t(8)*x(3)),
x(0)+t*(t(5)*x(1)+t(6)*x(2)+t(7)*x(3)+t(8)*x(4)),
x(1), x(2),
x(3), x(4);

ideal I = wedge(M,2);

ideal T = t;

ideal J = sat(I, T)[1];
ideal JT = J,T;
print(std(JT));
```

3 Extension to the Blow-up of K along S

The following program computes the saturation of the family of curves associated to points of the Luna slice of $x \in S_1$ blown up in the point x.

Listing A.3: S

```
//Extension of the curve family over K^s\S to a blow-up of S

LIB "elim.lib";
ring R=(0),(x(0..4),t(1..8),t,s(1..8)),dp;

//slice of some representative of S1 (the slice is isomorphic to A^8)
matrix M[4][2]=
x(0)-t(5)*x(1)-t(6)*x(2)-t(7)*x(3)-t(8)*x(4),
t(1)*x(1)+t(2)*x(3)+t(5)*x(2)+t(7)*x(4),
t(3)*x(2)+t(4)*x(4)-t(6)*x(1)-t(8)*x(3),
x(0)+t(5)*x(1)+t(6)*x(2)+t(7)*x(3)+t(8)*x(4),
x(1),                                          x(2),
x(3),                                          x(4);

//homogeneous ideal generated by the maximal minors of the slice M
ideal I=wedge(M,2);

//s1,...,s8 parameter of the blow-up of the A^8 along 0
matrix ST[2][8] = t(1..8), s(1..8);

//blow-up of A^8 in 0
ideal st = wedge(ST,2);


//family of curves over the blow-up
ideal IST = I, st;

ideal T = t(1..8);

//extension of the curve family by saturation in T
ideal J = sat(IST, T)[1];

//prints the extended curve family
print(std(J));
```

```


//prints the family of curves parametrised by the exceptional divisor
ideal JT = J, T;
JT = std(JT);

print(JT);
```

4 Extension to the Blow-up of K along P

The following program computes the saturation of the family of curves associated to points of the Luna slice of $x \in P_1$ blown up in the point x (respecting the $\mathbb{C}^*$-action on the slice).

Listing A.4: P

```
//Extension of the curve family over K\P to a blow-up of P

LIB "elim.lib";

ring R=(0),(x(0..4),t(1..10),s(1..10)),dp;



//slice of a representative in P1
matrix M[4][2]=x(0), t(1)*x(1)+t(2)*x(4)-t(4)*x(0),x(1), t(3)*x(0)+t(4)*x(1)+t(5)*x(4),
-t(9)*x(2)+t(6)*x(3)+t(7)*x(4), x(2),

t(9)*x(3)-t(8)*x(2)+t(10)*x(4), x(3);

//

homogeneous ideal generated by the maximal minors of the slice M

ideal I=wedge(M,2);


//blow-up of the A^{10} along some A^5 and then again along an A^5 (intersecting in the point 0)
matrix ST1[2][5] = t(1..5), s(1..5);


matrix ST2[2][5] = t(6..10),s(6..10);

ideal st1 = wedge(ST1,2);
ideal st2 = wedge(ST2,2);



```

```

//extension of the curve family via saturation

ideal IST = I, st1,st2;

ideal T1 = t(1..5);


ideal J = sat(IST, T1)[1];



ideal T2 = t(6..10);
ideal K = sat(J,T2)[1];

//prints the extended family
print(K);

ideal JT = K, T1,T2;

JT = std(JT);

//prints the extended family restricted to fibers parametrised by the exceptional divisor
print(JT);
```

Appendix B

List of Kronecker Modules

In this Appendix we systematically go through all possible matrices with entries in $K(s,t)$ occurring as Kronecker modules of type $(4,2)$.
If the matrix with entries in $K(s,t)$ has a zero row, then the associated matrix with entries in

$$V = \langle x_0, \dots, x_4 \rangle$$

also has a zero row and so it is unstable.
In the following, we will go through the remaining possible combinations. We will list the matrices M_i with entries in $K(s,t)$, together with the resulting representatives N_i of Kronecker modules as matrices with entries in V. If there are eigenvalues occurring in N_i we will apply Lemma 2.18.
By the size of the block C, we mean the number of columns of C, by the size of the block A the number of rows of A and by the size of B the number of rows of B (which is the same as the number of columns of B).

I. The size of block C is zero:

(a) The size of block B is zero:

$$M_1 = \begin{pmatrix} s & t & & & \\ & s & t & & \\ & & s & t & \\ & & & s & t \end{pmatrix}, \ N_1 = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \\ x_3 & x_4 \end{pmatrix}.$$

This is a representative of the stratum B_0.

(b) The size of block B is one:

$$M_2 = \left(\begin{array}{c|cccc} s+\lambda t & & & & \\ \hline & s & t & & \\ & & s & t & \\ & & & s & t \end{array}\right), \ N_2 = \begin{pmatrix} x_0 & \lambda x_0 \\ x_1 & x_2 \\ x_2 & x_3 \\ x_3 & x_4 \end{pmatrix}$$

and for $\lambda = 0$, this is $\begin{pmatrix} x_0 & 0 \\ x_1 & x_2 \\ x_2 & x_3 \\ x_3 & x_4 \end{pmatrix}$. This is a representative of the stratum B_1.

(c) The size of block B is two:

$$M_3 = \left(\begin{array}{cc|ccc} s+\lambda_1 t & & & & \\ & s+\lambda_2 t & & & \\ \hline & & s & t & \\ & & & s & t \end{array}\right)$$

and

$$M_4 = \left(\begin{array}{cc|ccc} s+\lambda t & & & & \\ t & s+\lambda t & & & \\ \hline & & s & t & \\ & & & s & t \end{array}\right)$$

Then $N_3 = \begin{pmatrix} x_0 & \lambda_1 x_0 \\ x_1 & \lambda_2 x_1 \\ x_2 & x_3 \\ x_3 & x_4 \end{pmatrix}$ i.e. for $\lambda_1 = \lambda_2$, $N_3^1 = \begin{pmatrix} x_0 & 0 \\ x_1 & 0 \\ x_2 & x_3 \\ x_3 & x_4 \end{pmatrix}$ is strictly semi-stable and for $\lambda_1 \neq \lambda_2$ $N_3^2 = \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \\ x_2 & x_3 \\ x_3 & x_4 \end{pmatrix}$. This is a representative of B_2.

Furthermore $N_4 = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & x_3 \\ x_3 & x_4 \end{pmatrix}$. This is a representative of B_3^1.

(d) The size of block B is three:

$$M_5 = \left(\begin{array}{ccc|cc} s+\lambda_1 t & & & & \\ & s+\lambda_2 t & & & \\ & & s+\lambda_3 t & & \\ \hline & & & s & t \end{array}\right),$$

$$M_6 = \left(\begin{array}{ccc|cc} s+\lambda_1 t & & & & \\ t & s+\lambda_1 t & & & \\ & & s+\lambda_2 t & & \\ \hline & & & s & t \end{array}\right)$$

and

$$M_7 = \left(\begin{array}{ccc|cc} s+\lambda t & & & & \\ t & s+\lambda t & & & \\ & t & s+\lambda t & & \\ \hline & & & s & t \end{array}\right).$$

Then $N_5 = \begin{pmatrix} x_0 & \lambda_1 x_0 \\ x_1 & \lambda_2 x_1 \\ x_2 & \lambda_3 x_2 \\ x_3 & x_4 \end{pmatrix}$ i.e.

(i) if the λ_i are pairwise distict: we set $\lambda_1 = 0, \lambda_2 = \infty$ and $\lambda_3 = 1$ and obtain the matrix $N_5^1 = \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \\ x_2 & x_2 \\ x_3 & x_4 \end{pmatrix}$, this is a representative of the stratum B_3^2

(ii) if $\lambda_1 = \lambda_2 \neq \lambda_3$: $N_5^2 = \begin{pmatrix} x_0 & 0 \\ x_1 & 0 \\ 0 & x_2 \\ x_3 & x_4 \end{pmatrix}$, which defines a strictly semi-stable Kronecker module. Since the situation is symmetric in λ_1, λ_2 and λ_3 we do not have to go trough other combinations, where two of the eigenvalues coincide and the third one is different.

(iii) if $\lambda_1 = \lambda_2 = \lambda_3$: $N_5^3 = \begin{pmatrix} x_0 & 0 \\ x_1 & 0 \\ x_2 & 0 \\ x_3 & x_4 \end{pmatrix}$, which defines an unstable Kronecker module

Futhermore $N_6 = \begin{pmatrix} x_0 & \lambda_1 x_0 \\ x_1 & x_0 + \lambda_1 x_1 \\ x_2 & \lambda_2 x_2 \\ x_3 & x_4 \end{pmatrix}$ i.e.

(i) if $\lambda_1 \neq \lambda_2$: $N_6^1 = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ 0 & x_2 \\ x_3 & x_4 \end{pmatrix}$ is a representative of B_4^1.

(ii) if $\lambda_1 = \lambda_2$: $N_6^2 = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & 0 \\ x_3 & x_4 \end{pmatrix}$ is strictly semi-stable and $N_7 = \begin{pmatrix} x_0 & \lambda x_0 \\ x_1 & x_0 + \lambda x_1 \\ x_2 & x_1 + \lambda x_2 \\ x_3 & x_4 \end{pmatrix}$ i.e. $\begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & x_1 \\ x_3 & x_4 \end{pmatrix}$ is a representative of B_5^2

(e) The size of block B is four: then the last column is zero.

$$M_8 = \begin{pmatrix} s+\lambda_1 t & & & & 0 \\ & s+\lambda_2 t & & & 0 \\ & & s+\lambda_3 t & & 0 \\ & & & s+\lambda_4 t & 0 \end{pmatrix},$$

$$M_9 = \left(\begin{array}{cc|ccc} s+\lambda_1 t & & & & 0 \\ & s+\lambda_2 t & & & 0 \\ \hline & & s+\lambda_3 t & & 0 \\ & & t & s+\lambda_3 t & 0 \end{array}\right),$$

$$M_{10} = \left(\begin{array}{cc|ccc} s+\lambda_1 t & & & & 0 \\ t & s+\lambda_1 t & & & 0 \\ \hline & & s+\lambda_2 t & & 0 \\ & & t & s+\lambda_2 t & 0 \end{array}\right),$$

$$M_{11} = \left(\begin{array}{c|cccc} s+\lambda_1 t & & & & 0 \\ \hline & s+\lambda_2 t & & & 0 \\ & t & s+\lambda_2 t & & 0 \\ & & t & s+\lambda_2 t & 0 \end{array}\right),$$

$$M_{12} = \begin{pmatrix} s+\lambda t & & & & 0 \\ t & s+\lambda t & & & 0 \\ & t & s+\lambda t & & 0 \\ & & t & s+\lambda t & 0 \end{pmatrix}$$

Then $N_8 = \begin{pmatrix} x_0 & \lambda_1 x_0 \\ x_1 & \lambda_2 x_1 \\ x_2 & \lambda_3 x_2 \\ x_3 & \lambda_4 x_3 \end{pmatrix}$. Again, the situation is symmetric in the eigenvalues and we just have to go through the following choices for the eigenvalues:

(i) if the λ_i are pairwise distinct: $N_8^1 = \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \\ x_2 & x_2 \\ x_3 & \lambda_4 x_3 \end{pmatrix}$ defines a family of stable matrices in the stratum B_4^2

(ii) if $\lambda_1 = \lambda_2$ and $\lambda_3, \lambda_4, \lambda_1$ pairwise distinct: $N_8^2 = \begin{pmatrix} x_0 & 0 \\ x_1 & 0 \\ 0 & x_2 \\ x_3 & x_3 \end{pmatrix}$ is strictly semi-stable

(iii) if $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4 \neq \lambda_1$: $N_8^3 = \begin{pmatrix} x_0 & 0 \\ x_1 & 0 \\ 0 & x_2 \\ 0 & x_3 \end{pmatrix}$ is the poly-stable stratum P_1

(iv) if $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$: $N_8^4 = \begin{pmatrix} x_0 & 0 \\ x_1 & 0 \\ x_2 & 0 \\ 0 & x_3 \end{pmatrix}$ is unstable

(v) if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$: $N_8^5 = \begin{pmatrix} x_0 & 0 \\ x_1 & 0 \\ x_2 & 0 \\ x_3 & 0 \end{pmatrix}$ is unstable

Then $N_9 = \begin{pmatrix} x_0 & \lambda_1 x_0 \\ x_1 & \lambda_2 x_1 \\ x_2 & \lambda_3 x_2 \\ x_3 & x_2 + \lambda_3 x_3 \end{pmatrix}$ i.e.

(i) if the λ_i are pairwise distinct: $N_9^1 = \begin{pmatrix} x_0 & 0 \\ x_1 & x_1 \\ 0 & x_2 \\ x_2 & x_3 \end{pmatrix}$ is a representative of B_5^1.

(ii) if $\lambda_1 = \lambda_2 \neq \lambda_3$: $N_9^2 = \begin{pmatrix} x_0 & 0 \\ x_1 & 0 \\ 0 & x_2 \\ x_2 & x_3 \end{pmatrix}$ is strictly semi-stable

(iii) if $\lambda_1 = \lambda_2 = \lambda_3$: $N_9^3 = \begin{pmatrix} x_0 & 0 \\ x_1 & 0 \\ x_2 & 0 \\ x_3 & x_2 \end{pmatrix}$ is unstable

(iv) if $\lambda_1 = \lambda_3 \neq \lambda_2$: $N_9^4 = \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \\ x_2 & 0 \\ x_3 & x_2 \end{pmatrix}$ is strictly semi-stable

Furthermore $N_{10} = \begin{pmatrix} x_0 & \lambda_1 x_0 \\ x_1 & x_0 + \lambda_1 x_1 \\ x_2 & \lambda_2 x_2 \\ x_3 & x_2 + \lambda_2 x_3 \end{pmatrix}$ i.e.

(i) if $\lambda_1 = \lambda_2$: $N_{10}^1 = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & 0 \\ x_3 & x_2 \end{pmatrix}$ is strictly semi-stable

(ii) if $\lambda_1 \neq \lambda_2$: $N_{10}^2 = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & x_2 \\ x_3 & x_2 + x_3 \end{pmatrix}$, this is the stratum B_5^3

Furthermore $N_{11} = \begin{pmatrix} x_0 & \lambda_1 x_0 \\ x_1 & \lambda_2 x_1 \\ x_2 & x_1 + \lambda_2 x_2 \\ x_3 & x_2 + \lambda_2 x_3 \end{pmatrix}$ i.e.

(i) if $\lambda_1 \neq \lambda_2$: $N_{11}^1 = \begin{pmatrix} x_0 & 0 \\ 0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$ is a representative of B_6

(ii) if $\lambda_1 = \lambda_2$: $N_{11}^2 = \begin{pmatrix} x_0 & 0 \\ x_1 & 0 \\ x_2 & x_1 \\ x_3 & x_2 \end{pmatrix}$ is strictly semi-stable

And $N_{12} = \begin{pmatrix} x_0 & \lambda x_0 \\ x_1 & x_0 + \lambda x_1 \\ x_2 & x_1 + \lambda x_2 \\ x_3 & x_2 + \lambda x_3 \end{pmatrix}$ i.e. $\begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & x_1 \\ x_3 & x_2 \end{pmatrix}$ is a representative of B_7

II. The size of block C is one:

(a) The size of block B is zero:

$$M_{13} = \left(\begin{array}{c|cccc} s & & & & 0 \\ t & & & & 0 \\ \hline & s & t & & 0 \\ & & s & t & 0 \end{array}\right), M_{14} = \left(\begin{array}{c|cccc} s & & & & \\ t & & & & \\ \hline & s & t & & \\ & & & s & t \end{array}\right)$$

This means $N_{13} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$ and $N_{14} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \\ x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$. N_{13} is a representative of S_2 and N_{14} a representative of S_1.

(b) The size of block B is one:

$$M_{15} = \left(\begin{array}{c|c|ccc} s & & & & 0 \\ t & & & & 0 \\ \hline & s+\lambda t & & & 0 \\ \hline & & s & t & 0 \end{array} \right),$$

and so $N_{15} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \\ x_1 & \lambda x_1 \\ x_2 & x_3 \end{pmatrix}$ resp. $\begin{pmatrix} x_0 & 0 \\ 0 & x_0 \\ x_1 & 0 \\ x_2 & x_3 \end{pmatrix}$ is strictly semi-stable

(c) The size of block B is two:

$$M_{16} = \left(\begin{array}{c|cc|cc} s & & & 0 & 0 \\ t & & & 0 & 0 \\ \hline & s+\lambda t & & 0 & 0 \\ & t & s+\lambda & 0 & 0 \end{array} \right)$$

and

$$M_{17} = \left(\begin{array}{c|cc|cc} s & & & 0 & 0 \\ t & & & 0 & 0 \\ \hline & s+\lambda_1 t & & 0 & 0 \\ & & s+\lambda_2 t & 0 & 0 \end{array} \right),$$

and so $N_{16} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \\ x_1 & \lambda x_1 \\ x_2 & x_1 + \lambda x_2 \end{pmatrix}$ resp. $\begin{pmatrix} x_0 & 0 \\ 0 & x_0 \\ x_1 & 0 \\ x_2 & x_1 \end{pmatrix}$ is strictly semi-stable.

$N_{17} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \\ x_1 & \lambda_1 x_1 \\ x_2 & \lambda_2 x_2 \end{pmatrix}$, hence

(i) if $\lambda_1 \neq \lambda_2$: $N_{17}^1 = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \\ x_1 & 0 \\ 0 & x_2 \end{pmatrix}$ is a representative of P_2

(ii) if $\lambda_1 = \lambda_2$: $N_{17}^2 = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \\ x_1 & 0 \\ x_2 & 0 \end{pmatrix}$ is unstable.

III. The size of block C is two:

(a) The size of block B is zero:

$$M_{18} = \left(\begin{array}{c|c|ccc} s & & 0 & 0 & 0 \\ t & & 0 & 0 & 0 \\ \hline & s & 0 & 0 & 0 \\ & t & 0 & 0 & 0 \end{array}\right), M_{19} = \left(\begin{array}{cc|ccc} s & & & & 0 \\ t & s & & & 0 \\ & t & & & 0 \\ \hline & & s & t & 0 \end{array}\right)$$

and so $N_{18} = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \\ x_1 & 0 \\ 0 & x_1 \end{pmatrix}$ is a representative of the poly-stable stratum P_3

and $N_{19} = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ 0 & x_1 \\ x_2 & x_3 \end{pmatrix}$ is a representative of the stratum S_3.

(b) The size of block B is one:

$$M_{20} = \left(\begin{array}{cc|ccc} s & & & 0 & 0 \\ t & s & & 0 & 0 \\ & t & & 0 & 0 \\ \hline & & s + \lambda t & 0 & 0 \end{array}\right)$$

hence $N_{20} = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ 0 & x_1 \\ x_2 & \lambda x_2 \end{pmatrix}$, resp $\begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ 0 & x_1 \\ x_2 & 0 \end{pmatrix}$ is strictly semi-stable.

IV. The size of block C is three:

$$M_{21} = \begin{pmatrix} s & & & 0 & 0 \\ t & s & & 0 & 0 \\ & t & s & 0 & 0 \\ & & t & 0 & 0 \end{pmatrix}, N_{21} = \begin{pmatrix} x_0 & 0 \\ x_1 & x_0 \\ x_2 & x_1 \\ 0 & x_2 \end{pmatrix}$$

represents the stable stratum B_{10}.

Appendix C

Degenerations of Strata

In this Appendix we figure out which strata Y degenerate into a stratum Z. This means that Z is in the closure of Y. The method will always be to write down a family of Kronecker modules A_t parametrized by $t \in \mathbb{C}$ such that $A_t \in Y$ for $t \neq 0$ and $A_0 \in Z$.
We use the enumeration of Lemma 2.24.

(a) $B_0 \to B_1$: The family $\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ tx_1 & x_2 & x_3 & x_4 \end{pmatrix}^t$ for $t \neq 0$ is contained in B_0 (multiply the first column of the representative listed in Theorem 2.13 with t and make a coordinate transformation $x_0 \mapsto \frac{1}{t}x_0$.) For $t = 0$ one obtains a representative of B_1.

(b) $B_1 \to B_2$: The stratum B_1 contains the family $\begin{pmatrix} x_0 & tx_3 & x_1 & x_2 \\ 0 & x_4 & x_2 & x_3 \end{pmatrix}^t$ for $t \neq 0$ (after a column permutation, multiplication of the second column by t and a coordinate transformation of x_4). For $t = 0$ this is a representative of B_2.

(c) $B_2 \to B_3^1$: the family $\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & tx_0 + (1-t)x_1 & x_3 & x_4 \end{pmatrix}^t$ is contained in B_2 for $t \neq 0$ and for $t = 0$ in B_3^1.
$B_2 \to B_3^2$: the family $\begin{pmatrix} x_0 & 0 & x_2 & x_3 \\ 0 & x_1 & tx_3 + (1-t)x_2 & x_4 \end{pmatrix}^t$ is contained in B_2 for $t \neq 0$ and for $t = 0$ in B_3^2.

(d) $B_3^1 \to B_4^1$: the family $\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & x_0 & tx_3 + (1-t)x_2 & x_4 \end{pmatrix}^t$ is contained in B_3^1 for $t \neq 0$ and for $t = 0$ in B_4^1.

(e) $B_3^2 \to B_4^1$: the family $\begin{pmatrix} x_0 & x_1 & 0 & x_3 \\ 0 & tx_1 + (1-t)x_0 & x_2 & x_4 \end{pmatrix}^t$ is contained in B_3^2

for $t \neq 0$ and for $t = 0$ in B_4^1.
$B_3^2 \to B_4^2$: $x_4 \to \lambda x_3$.

(f) $B_4^1 \to B_5^1$: $x_4 \to x_3$.
$B_4^1 \to B_5^2$: the family $\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & x_0 & tx_2 + (1-t)x_1 & x_4 \end{pmatrix}^t$ is contained in B_4^1 for $t \neq 0$ and for $t = 0$ in B_5^2.
$B_4^1 \to B_5^3$: $x_3 \to x_2$.

(g) $B_4^2 \to B_5^1$: in coordinates in $\mathbb{C}(s,t)$, we consider the family

$$\begin{pmatrix} s+\lambda_1 t & & & \\ & s+\lambda_2 t & & \\ & & s+\lambda_3 t & \\ & & & s+((1-c)\lambda_3 + c\lambda_4)t \end{pmatrix}$$

which is contained in B_4^2 for $c \neq 0$. For $c = 0$, it is contained in B_5^1.

(h) $B_5^1 \to B_6$: the family $\begin{pmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & tx_2 + (1-t)x_1 & x_3 \end{pmatrix}^t$ is contained in B_5^1 for $t \neq 0$ and for $t = 0$ in B_6.
$B_5^1 \to S_2$: the family $\begin{pmatrix} x_0 & t(x_1+x_2) & x_1 & x_2 \\ 0 & x_0 & x_2 & x_3 \end{pmatrix}^t$ is contained in B_5^1 for $t \neq 0$ and for $t = 0$ in S_2.
$B_5^1 \to S_3$: the family $\begin{pmatrix} x_0 & x_1 & 0 & x_3 \\ 0 & x_0 & tx_2 + (1-t)x_1 & x_2 + x_3 \end{pmatrix}^t$ is contained in B_5^1 for $t \neq 0$ and for $t = 0$ in S_3.

(i) $B_5^2 \to B_6$: $x_3 \to 0$.
$B_5^2 \to S_1$: the family $\begin{pmatrix} x_0 & tx_1 & x_2 & x_3 \\ 0 & x_0 & x_1 & x_4 \end{pmatrix}^t$ is contained in B_5^2 for $t \neq 0$ and for $t = 0$ in S_1.
$B_5^2 \to S_3$: $x_2 \to 0$.

(j) $B_5^3 \to S_2$: the family $\begin{pmatrix} x_0 & tx_1 & x_1 & x_2 \\ 0 & x_0 & x_2 & x_3 \end{pmatrix}^t$ is contained in B_5^3 for $t \neq 0$ and for $t = 0$ in S_2.
$B_5^3 \to S_3$: the family $\begin{pmatrix} x_0 & x_1 & 0 & x_2 \\ 0 & x_0 & tx_2 + (1-t)x_1 & x_3 \end{pmatrix}^t$ is contained in B_5^3 for $t \neq 0$ and for $t = 0$ in S_3.
$B_5^3 \to B_{10}$: $x_3 \to x_1$.
$B_5^3 \to P_1$: the family $\begin{pmatrix} x_0 & x_1 & 0 & tx_2 \\ 0 & tx_0 & x_2 & x_3 \end{pmatrix}^t$ is contained in B_5^3 for $t \neq 0$ and for $t = 0$ in P_1.

(k) $B_6 \to B_7$: the family $\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ 0 & x_0 & x_1 & tx_3 + (1-t)x_2 \end{pmatrix}^t$ is contained in B_6 for $t \neq 0$ and for $t = 0$ in B_7.

(l) $B_7 \to B_{10}$: $x_3 \to 0$.

$B_7 \to S_3$: the family $\begin{pmatrix} x_0 & x_1 & tx_2 & x_3 \\ 0 & x_0 & x_1 & x_2 \end{pmatrix}^t$ is contained in B_7 for $t \neq 0$ and for $t = 0$ in S_3.

(m) $B_{10} \to P_2$: the family $\begin{pmatrix} x_0 & x_1 & tx_2 & 0 \\ 0 & tx_0 & x_1 & x_2 \end{pmatrix}^t$ is contained in B_{10} for $t \neq 0$ and for $t = 0$ in P_2.

(n) $S_1 \to S_2$: $x_3 \to x_2$.

(o) $S_2 \to P_2$: $x_2 \to 0$.

(p) $S_3 \to P_2$: the family $\begin{pmatrix} x_0 & x_1 & 0 & tx_2 \\ 0 & tx_0 & x_1 & x_3 \end{pmatrix}^t$ is contained in S_3 for $t \neq 0$ and for $t = 0$ in P_2.

(q) $P_1 \to P_2$: $x_2 \to x_0$.

(r) $P_2 \to P_3$: $x_2 \to x_1$.

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