Moduli spaces of (G, h)–constellations

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Abstract

Given a reductive group G acting on an affine scheme X over $\mathbb{C}$ and a Hilbert function $h\colon \operatorname{Irr} G \to \mathbb{N}_0$, we construct the moduli space $M_\theta(X)$ of θ–stable (G,h)–constellations on X, which is a common generalisation of the invariant Hilbert scheme after Alexeev and Brion [AB05] and the moduli space of θ–stable G–constellations for finite groups G introduced by Craw and Ishii [CI04]. Our construction of a morphism $M_\theta(X) \to X/\!\!/G$ makes this moduli space a candidate for a resolution of singularities of the quotient $X/\!\!/G$.

Furthermore, we determine the invariant Hilbert scheme of the zero fibre of the moment map of an action of Sl_2 on $(\mathbb{C}^2)^{\oplus 6}$ as one of the first examples of invariant Hilbert schemes with multiplicities. While doing this, we present a general procedure for the realisation of such calculations. We also consider questions of smoothness and connectedness and thereby show that our Hilbert scheme gives a resolution of singularities of the symplectic reduction of the action.

Zusammenfassung

Für eine reduktive Gruppe G, die auf einem affinen $\mathbb{C}$–Schema X wirkt, und eine Hilbertfunktion $h\colon \operatorname{Irr} G \to \mathbb{N}_0$ konstruieren wir den Modulraum $M_\theta(X)$ der θ–stabilen (G,h)–Konstellationen auf X, der eine gemeinsame Verallgemeinerung des invarianten Hilbertschemas nach Alexeev und Brion [AB05] und des von Craw und Ishii [CI04] eingeführten Modulraumes von θ–stabilen G–Konstellationen für endliche Gruppen G ist. Unsere Konstruktion eines Morphismus $M_\theta(X) \to X/\!\!/G$ macht diesen Modulraum zu einem Kandidaten einer Auflösung der Singularitäten des Quotienten $X/\!\!/G$.

Außerdem bestimmen wir das invariante Hilbertschema der Nullfaser der Impulsabbildung einer Wirkung von Sl_2 auf $(\mathbb{C}^2)^{\oplus 6}$ als eines der ersten Beispiele von invarianten Hilbertschemata mit Multiplizitäten. Dabei beschreiben wir eine allgemeine Vorgehensweise für derartige Berechnungen. Ferner zeigen wir, dass unser Hilbertschema glatt und zusammenhängend ist und daher eine Auflösung der Singularitäten der symplektischen Reduktion der Wirkung darstellt.

Résumé

Nous construisons l'espace de modules $M_\theta(X)$ des (G, h)–constellations θ–stables sur X pour un groupe réductif G qui agit sur un schéma affine X sur $\mathbb{C}$ et pour une fonction de Hilbert $h\colon \operatorname{Irr} G \to \mathbb{N}_0$. Cet espace de modules est une généralisation commune du schéma de Hilbert invariant d'après Alexeev et Brion [AB05] et de l'espace de modules des G–constellations θ–stables pour un groupe fini G introduit par Craw et Ishii [CI04]. Notre construction d'un morphisme $M_\theta(X) \to X/\!\!/G$ fait de cet espace de modules un candidat pour une résolution des singularités du quotient $X/\!\!/G$.
De plus, nous déterminons le schéma de Hilbert invariant de la fibre en zéro de l'application moment d'une action de Sl_2 sur $(\mathbb{C}^2)^{\oplus 6}$. C'est un des premiers exemples d'un schéma de Hilbert invariant avec multiplicités. Ceci nous amène à décrire une façon générale de procéder pour effectuer de tels calculs. En outre, nous démontrons que notre schéma de Hilbert invariant est lisse et connexe : Cet exemple est donc une résolution des singularités de la réduction symplectique de l'action.

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Introduction

Hilbert schemes play an important role in the search for resolutions of singularities, in particular for symplectic or, more generally, crepant ones: If X is a smooth surface, then by Fogarty [Fog68, Theorem 2.4] the Hilbert scheme of points $\mathrm{Hilb}^n(X)$ is a resolution of the singularities of the symmetric product S^nX for every $n \in \mathbb{N}$. In the case where X carries a symplectic structure, this is even a symplectic resolution by Beauville [Bea83, Proposition 5]. Further, if one considers the action of a finite group G on a variety X, there is Ito and Nakamura's G–Hilbert scheme $G\text{-Hilb}(X)$ [IN96, IN99, Nak01]. In the case where X is a non–singular quasiprojective variety and $G \subset \mathrm{Aut}(X)$ a finite group such that the canonical bundle ω_X is a locally trivial G–sheaf, Bridgeland, King and Reid [BKR01] give a sufficient condition assuring that the irreducible component of the G–Hilbert scheme containing the free G–orbits is a crepant resolution of the quotient X/G. Moreover, they prove that up to dimension 3 this orbit component is the whole of $G\text{-Hilb}(X)$. Hence, if X is a variety of dimension at most 3, the G–Hilbert scheme itself is a crepant resolution of X/G.

There exist two generalisations of the G–Hilbert scheme: To find a complete list of resolutions for finite group quotients, Craw and Ishii introduce the moduli space of θ–stable G–constellations [CI04]. They show that for finite abelian groups $G \subset Sl_3(\mathbb{C})$, every projective crepant resolution of $\mathbb{C}^3/G$ can be obtained as such a moduli space. On the other hand, to deal with quotients for reductive instead of finite groups Alexeev and Brion provide the invariant Hilbert scheme [AB04, AB05]. The main goal of this thesis is to construct a common generalisation of these, the moduli space $M_\theta(X)$ of θ–stable (G, h)–constellations for a reductive group G and a map $h\colon \mathrm{Irr}\, G \to \mathbb{N}_0$, which replaces the regular representation occurring in [CI04]. The following paragraphs summarize more precisely the approaches of Craw and Ishii and of Alexeev and Brion and our contribution to the subject.

Given a finite group $G \subset Gl_n(\mathbb{C})$ acting on $\mathbb{C}^n$, the notion of G–constellation introduced in [CI04] generalises the concept of G–clusters from G–invariant quotients of $\mathcal{O}_{\mathbb{C}^n}$ with isotypic decomposition isomorphic to the regular representation R of G to G–equivariant coherent $\mathcal{O}_{\mathbb{C}^n}$–modules with this given isotypic decomposition. Such a G–constellation $\mathcal{F}$ is θ–stable for some $\theta \in \operatorname{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q})$ if $\theta(\mathcal{F}) = 0$ and if for every non–zero proper G–equivariant coherent subsheaf $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$ one has $\theta(\mathcal{F}') > 0$. In this situation, Craw and Ishii construct the moduli space M_θ of θ–stable G–constellations as the GIT–quotient of the space of quiver representations associated to G by the group of G–equivariant automorphisms of R as described by King in [Kin94]. For a special choice of θ they recover $M_\theta = G\text{-Hilb}(\mathbb{C}^n)$.

As a second generalisation of the G–Hilbert scheme, Alexeev and Brion fix a complex reductive group G and a map $h\colon \operatorname{Irr} G \to \mathbb{N}_0$ on the set $\operatorname{Irr} G = \{\rho\colon G \to Gl(V_\rho)\}$ of isomorphy classes of irreducible representations of G. Then for any affine G–scheme X, in [AB04, AB05] the authors define the invariant Hilbert scheme $\operatorname{Hilb}_h^G(X)$, whose closed points parameterise all G–invariant subschemes of X whose coordinate rings have isotypic decomposition isomorphic to $\bigoplus_{\rho\in\operatorname{Irr} G} \mathbb{C}^{h(\rho)} \otimes_{\mathbb{C}} V_\rho$, or equivalently all quotients $\mathcal{O}_X/\mathcal{I}$, where $\mathcal{I}$ is an ideal sheaf in $\mathcal{O}_X$, with this prescribed isotypic decomposition.

Our contribution to these constructions of moduli spaces is to unify the ideas of [CI04] and [AB04, AB05]: For a complex reductive group G, an affine G–scheme X and a map $h\colon \operatorname{Irr} G \to \mathbb{N}_0$ we define the notion of (G,h)–constellation, which is a G–equivariant coherent $\mathcal{O}_X$–module with isotypic decomposition given by h as above. Then we introduce θ–stability analogously to the case of G–constellations. This stability condition is more delicate than the one of Craw and Ishii since it involves infinitely many parameters. We locate finitely many of them which control the others. Then we construct the moduli space of θ–stable (G,h)–constellations by means of geometric invariant theory and invariant Quot schemes in a parallel way to the construction of the moduli space of stable vector bundles of Simpson [Sim94] as presented in [HL10] by Huybrechts and Lehn. As a generalisation of the Hilbert–Chow morphism we moreover construct a morphism $M_\theta(X) \to X/\!\!/G$. Further studies of $M_\theta(X)$ have to be made in order to decide whether this morphism gives a resolution of singularities.

The structure of this thesis is as follows:

Since very little is known about invariant Hilbert schemes and there is a lack of examples in the symplectic setting up to now, in Chapter 1 we determine an example of an invariant Hilbert scheme, namely of the zero fibre of the moment map of an action of Sl_2 on $(\mathbb{C}^2)^{\oplus 6}$. It is one of the first examples of invariant Hilbert schemes with multiplicities. In addition to the examination of the example, we present a general procedure for the realisation of such calculations in Section 1.3. We determine our Hilbert scheme to be

$$Sl_2\text{-Hilb}(\mu^{-1}(0)) = \{(A,W) \in \overline{\mathcal{O}}_{[2^2,1^2]} \times \mathrm{Grass}_{iso}(2,\mathbb{C}^6) \mid \mathrm{im}\, A^t \subset W\}.$$

Additionally, we show that it is smooth and connected. Hence it is a resolution of singularities of the symplectic reduction of the action. The contents of this chapter have been published as an article on its own in *Transformation Groups* [Bec11].

In Chapter 2 we introduce the notions of (G,h)–constellation, θ–semistability and θ–stability analogously to the case of G–constellations and we define the corresponding moduli functors $\overline{\mathcal{M}}_\theta(X)$ and $\mathcal{M}_\theta(X)$. Then we show that every θ–stable (G,h)–constellation is generated as an $\mathcal{O}_X$–module by its components indexed by a certain finite subset $D_- \subset \mathrm{Irr}\, G$, so that each θ–stable (G,h)–constellation is a quotient of a fixed coherent sheaf $\mathcal{H}$ and hence an element of the invariant Quot scheme $\mathrm{Quot}^G(\mathcal{H},h)$. With a slightly more restrictive choice of θ, the same holds for θ-semistability. At the end of this chapter we show that if h is chosen such that the value on the trivial representation ρ_0 is 1 and θ_{ρ_0} is the only negative value of θ, then the moduli functor $\mathcal{M}_\theta(X)$ equals the Hilbert functor $\mathcal{H}ilb_h^G(X)$.

In Chapter 3 we deal with the geometric invariant theory of the invariant Quot scheme $\mathrm{Quot}^G(\mathcal{H},h)$ in order to construct a moduli space of (G,h)–constellations as its GIT–quotient: The invariant Quot scheme is equipped with a certain ample line bundle $\mathscr{L}$ coming from the embedding into a product of Grassmannians as established in Section 3.1. Considering the gauge group Γ, we examine GIT–stability and GIT–semistability on $\mathrm{Quot}^G(\mathcal{H},h)$ with respect to the induced linearisation on $\mathscr{L}$ twisted by a certain character χ. Thus, on the set of GIT–semistable quotients $\mathrm{Quot}^G(\mathcal{H},h)^{ss}$ we obtain the categorical quotient $\mathrm{Quot}^G(\mathcal{H},h)^{ss}/\!/_{\mathscr{L}_\chi}\Gamma$, which turns out to be a moduli space of GIT–semistable (G,h)–constellations in Chapter 5.

In Chapter 4 we establish a correspondence of (G,h)–constellations and G–equivariant quotients $[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \mathrm{Quot}^G(\mathcal{H},h)$ and a correspondence of their respective subobjects. This allows us to introduce another (semi)stability condition $\widetilde{\theta}$ which is equivalent to GIT–(semi)stability but resembles very much θ–(semi)stability. We show that if $\mathcal{F}$ is θ–stable, then it is also $\widetilde{\theta}$–stable and hence any corresponding point $[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}]$ in $\mathrm{Quot}^G(\mathcal{H},h)$ is GIT–stable. This allows us to realise the functor $\mathcal{M}_\theta(X)$ of flat families of θ–stable (G,h)–constellations as a subfunctor of the functor $\mathcal{M}_{\chi,\kappa}(X)$ of flat families of GIT–stable (G,h)–constellations.

In Chapter 5 we consider properties of these functors. First, we show that $\overline{\mathcal{M}}_{\chi,\kappa}(X)$ and $\mathcal{M}_{\chi,\kappa}(X)$ are corepresented by $\mathrm{Quot}^G(\mathcal{H},h)^{ss}/\!\!/_{\mathscr{L}_\chi}\,\Gamma$ and $\mathrm{Quot}^G(\mathcal{H},h)^s/\Gamma$, respectively. In the same way, $\mathcal{M}_\theta(X)$ is corepresented by its subset $\mathrm{Quot}^G(\mathcal{H},h)^s_\theta/\Gamma$, where $\mathrm{Quot}^G(\mathcal{H},h)^s_\theta$ is the set of θ–stable elements in $\mathrm{Quot}^G(\mathcal{H},h)$. We call

$$M_\theta(X) := \mathrm{Quot}^G(\mathcal{H},h)^s_\theta/\Gamma$$

the moduli space of θ–stable (G,h)–constellations. Furthermore, we prove that θ–stability is open in flat families. From this fact we deduce that $M_\theta(X)$ is an open subscheme of $\mathrm{Quot}^G(\mathcal{H},h)^s/\Gamma$ and hence a quasiprojective scheme. We define the scheme $\overline{M}_\theta(X)$ as its closure in $\mathrm{Quot}^G(\mathcal{H},h)^{ss}/\!\!/_{\mathscr{L}_\chi}\,\Gamma$. Finally, we construct a morphism from $\overline{M}_\theta(X)$ to the quotient $X/\!\!/G$ corresponding to the Hilbert–Chow morphism.
As an outlook, at the end of this thesis we discuss some further aspects of the moduli spaces $M_\theta(X)$ and $\overline{M}_\theta(X)$, which are worth being pursued in the future.
There are two appendices: In Appendix A we work out a G–equivariant version of frame bundles, which we need in Section 5.1 to interpret the functors of (G,h)–constellations with various stability conditions as quotients of the functors of (semi)stable quotients modulo the choice of a particular quotient map. In Appendix B we construct the relative invariant Quot scheme, which is a generalisation of the invariant Quot scheme constructed by Jansou in [Jan06]. We need this relative version in order to show that θ–stability is an open property in flat families of (G,h)–constellations in Section 5.2, so that eventually the moduli space of θ–stable (G,h)–constellations can be obtained as an open subscheme of the geometric quotient $\mathrm{Quot}^G(\mathcal{H},h)^s/\Gamma$.

Notation and conventions

In this thesis, G will always be a complex connected reductive algebraic group and X an affine G–scheme over $\mathbb{C}$, that is an affine scheme $X = \operatorname{Spec} R$ over $\mathbb{C}$ such that G acts on X and on its coordinate ring $\mathbb{C}[X] = R$.

We work over the category $(\mathrm{Sch}/\mathbb{C})$ of noetherian schemes over $\mathbb{C}$. In the definition of contravariant functors we denote by $(\mathrm{Sch}/\mathbb{C})^{\mathrm{op}}$ its opposite category. For a scheme $Y \in (\mathrm{Sch}/\mathbb{C})$, there is the functor of points $\underline{Y}\colon (\mathrm{Sch}/\mathbb{C})^{\mathrm{op}} \to (\mathrm{Set})$ to the category (Set) of sets, given by $\underline{Y}(S) = \mathrm{Hom}(S,Y)$. For $S \in (\mathrm{Sch}/\mathbb{C})$ let further (Sch/S) be the category of noetherian schemes over S.

1. An example of an Sl_2–Hilbert scheme with multiplicities

Let G be a complex connected reductive algebraic group and X an affine G–scheme over $\mathbb{C}$. Denote by $\mathrm{Irr}(G)$ the set of isomorphism classes of irreducible representations of G and let $h\colon \mathrm{Irr}(G) \to \mathbb{N}_0$ be a map, called *Hilbert function* in the following. In this setting, Alexeev and Brion define in [AB05] the invariant Hilbert scheme $\mathrm{Hilb}_h^G(X)$ parameterising G–invariant subschemes of X whose modules of global sections all have the same isotypic decomposition $\bigoplus_{\rho\in\mathrm{Irr}\,G} \mathbb{C}^{h(\rho)} \otimes_{\mathbb{C}} V_\rho$ as G–modules. Their definition relies on the work of Haiman and Sturmfels on multigraded Hilbert schemes [HS04] and generalises the G–Hilbert scheme of Ito and Nakamura [IN96, IN99, Nak01].
In the case where the Hilbert function h is multiplicity–free, i.e. $\mathrm{im}\, h \subset \{0,1\}$, several examples of invariant Hilbert schemes have been determined by Jansou [Jan07], Bravi and Cupit-Foutou [BCF08] and Papadakis and van Steirteghem [PvS10], which all turn out to be affine spaces. Jansou and Ressayre [JR09] give some examples of invariant Hilbert schemes with multiplicities, which are also affine spaces. There are some more involved examples of invariant Hilbert schemes by Brion (unpublished) and Budmiger [Bud10]. Here we present a more substantial example, where X is a 9–dimensional singular variety, whose quotient is additionally equipped with a symplectic structure. The group we consider is Sl_2 and the Hilbert function is the one of its regular representation

$$h\colon \mathbb{N}_0 \to \mathbb{N},\ d \mapsto d+1. \tag{1.1}$$

The knowledge of such examples where the Hilbert scheme is not an affine space is important for understanding general properties of invariant Hilbert schemes: Which conditions have to be fulfilled so that the invariant Hilbert scheme is connected or smooth? Is the invariant Hilbert scheme a resolution of singularities of the quotient $X/\!\!/G$? This is for example the case for the G–Hilbert scheme where G is finite, X is quasiprojective and non–singular and has dimension at most 3 [BKR01].

Our example of an invariant Hilbert scheme for Sl_2 will be smooth and connected and it will even be a resolution of singularities, but it does not inherit the additional structure of symplectic variety of the quotient.

Now we present the setting of our example. Consider the action of Sl_2 on the vector space $(\mathbb{C}^2)^{\oplus 6} = \mathrm{Mat}_{2\times 6}(\mathbb{C})$ arising as symplectic double from the action of Sl_2 on $(\mathbb{C}^2)^{\oplus 3}$ via multiplication on the left.

The moment map $\mu\colon (\mathbb{C}^2)^{\oplus 6} \to \mathfrak{sl}_2$, $M \mapsto MQM^tJ$ defines the symplectic reduction $(\mathbb{C}^2)^{\oplus 6}/\!/\!/Sl_2 := \mu^{-1}(0)/\!/Sl_2$, where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. In [Bec10] we obtained its description as a nilpotent orbit closure $\mu^{-1}(0)/\!/Sl_2 \cong \overline{\mathcal{O}}_{[2^2,1^2]}$ in the orthogonal Lie algebra $\mathfrak{so}_6$ for the quadratic form given by the matrix $Q = \left(\begin{smallmatrix} 0 & I_3 \\ I_3 & 0 \end{smallmatrix}\right)$. Writing $(\mathbb{C}^2)^{\oplus 6} = \mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^6$ we see that we have a symmetric situation with an action of $SO_6 = SO(Q)$ by multiplication from the right. Moreover, μ is invariant for this action, so that SO_6 acts on the zero fibre $\mu^{-1}(0)$. As both actions commute, SO_6 also acts on the quotient by Sl_2. The quotient map $\nu\colon \mu^{-1}(0) \to \mu^{-1}(0)/\!/Sl_2$ is given by mapping M to M^tJMQ. In fact, the quotient map of the Sl_2–action is the moment map of the SO_6–action and vice versa. The SO_6–action will play an important role while analysing $\mu^{-1}(0)/\!/Sl_2$ and the corresponding Hilbert scheme.

The symplectic variety $\overline{\mathcal{O}}_{[2^2,1^2]}$ has two well–known symplectic resolutions of singularities, namely the cotangent bundle $T^*\mathbb{P}^3 \cong \{(A,L) \in Y \times \mathbb{P}^3 \mid \operatorname{im} A^t \subset L\}$ and its dual $(T^*\mathbb{P}^3)^* \cong \{(A,H) \in Y \times (\mathbb{P}^3)^* \mid H \subset \ker A^t\}$. Here we identify $\mathfrak{sl}_4 \cong \mathfrak{so}_6$, so that we have $Y := \{A \in \mathfrak{sl}_4 \mid \operatorname{rk} A \leq 1\} \cong \overline{\mathcal{O}}_{[2^2,1^2]}$. We want to know if there is a natural (symplectic) resolution. Since Hilbert schemes of points and G–Hilbert schemes are often candidates for (symplectic) resolutions [Fog68, Bea83, BKR01], we hope that this is also true for invariant Hilbert schemes. Indeed, with the choice of the Hilbert function h in (1.1), in our example we find

Theorem 1.1 *The invariant Hilbert scheme* $Sl_2\text{-Hilb}(\mu^{-1}(0)) := \mathrm{Hilb}_h^{Sl_2}(\mu^{-1}(0))$ *of the zero fibre of the moment map of the action of* Sl_2 *on* $(\mathbb{C}^2)^{\oplus 6}$ *is the scheme*

$$\{(A,W) \in \overline{\mathcal{O}}_{[2^2,1^2]} \times \mathrm{Grass}_{iso}(2,\mathbb{C}^6) \mid \operatorname{im} A^t \subset W\}, \tag{1.2}$$

where $\mathrm{Grass}_{iso}(2,\mathbb{C}^6)$ *is the Grassmannian of 2–dimensional isotropic subspaces of* $\mathbb{C}^6$ *with respect to the quadratic form given by* Q*. Moreover,* $Sl_2\text{-Hilb}(\mu^{-1}(0))$ *is smooth and connected, and thus a resolution of singularities of the symplectic reduction* $\mu^{-1}(0)/\!/Sl_2$*.*

Remark. Sl_2-Hilb$(\mu^{-1}(0))$ is not a *symplectic* resolution of $\mu^{-1}(0)/\!\!/Sl_2$ since it is not a semismall resolution. However, making use of the isomorphism

$$Sl_2\text{-Hilb}(\mu^{-1}(0)) \to \{(A, L, H) \in Y \times \mathbb{P}^3 \times (\mathbb{P}^3)^* \mid \operatorname{im} A^t \subset L \subset H \subset \ker A^t\}$$

given by the assignments $(A, L \wedge H) \leftarrow\!\shortmid (A, L, H)$ and $(A, W) \mapsto (A, L_W, H_W)$ with $L_W := \{v \in \mathbb{C}^4 \mid \dim(v \wedge W) = 0\}$, $H_W := \{v \in \mathbb{C}^4 \mid \dim(v \wedge W) \leq 1\}$, it dominates the two symplectic resolutions:

$$\begin{array}{ccccc} & & Sl_2\text{-Hilb}(\mu^{-1}(0)) & & \\ & \swarrow & \downarrow & \searrow & \\ T^*\mathbb{P}^3 & & & & (T^*\mathbb{P}^3)^* \\ & \searrow & & \swarrow & \\ & & \mu^{-1}(0)/\!\!/Sl_2 & & \end{array} \qquad \begin{array}{ccccc} & & (A, W) & & \\ & \swarrow & \downarrow & \searrow & \\ (A, L_W) & & & & (A, H_W) \\ & \searrow & & \swarrow & \\ & & A & & \end{array}$$

The Sl_2–Hilbert scheme Sl_2-Hilb$(\mu^{-1}(0))$ consists of points of two different types:

Theorem 1.2 *The subscheme $Z_{A,W} \subset \mu^{-1}(0)$ corresponding to the point (A, W) in Sl_2-Hilb$(\mu^{-1}(0))$ is*

$$Z_{A,W} \cong \begin{cases} Sl_2, & \text{if } A \in \mathcal{O}_{[2^2,1^2]}, \\ \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \middle| ad - bc = 0 \right\}, & \text{if } A = 0. \end{cases}$$

This chapter is organised as follows: In Section 1.1 we introduce the invariant Hilbert scheme as defined by Alexeev and Brion in [AB05]. First, we give their definition of the invariant Hilbert functor, which is represented by the invariant Hilbert scheme. Then we introduce the Hilbert–Chow morphism and analyse which conditions on the Hilbert function have to be satisfied so that this morphism, or at least its restriction to a certain component, is proper and birational. These are, besides smoothness of the scheme, the important properties for being a resolution. With regard to this, we define the orbit component $\mathrm{Hilb}_h^G(X)^{orb}$, which is the unique component mapping birationally to the set of closed G–orbits. If the invariant Hilbert scheme is not irreducible, this component is still a candidate for a resolution.
Afterwards, we turn to our example in Section 1.2, where we compute the general fibre of the quotient in order to determine the right Hilbert function which guarantees birationality.

Section 1.3 is the heart of this chapter. First, we develop a general method to find generators of the sheaves of covariants occurring in the definition of the invariant Hilbert functor. Then we construct an embedding of the invariant Hilbert scheme into a product of Grassmannians following ideas by Brion and based on the embedding constructed in [HS04]. Thus this chapter does not only give an involved example of an invariant Hilbert scheme with multiplicities of a variety which is not an affine space, but it can also be consulted as a guidance for the determination of further examples. While describing the general process, we always switch to its application to the example at the end of each step. As a result, we obtain the orbit component in our example as (1.2).
To conclude the proof of Theorem 1.1, i.e. to find out if the orbit component coincides with the whole Hilbert scheme, in Section 1.4 we show that the latter is smooth by considering the tangent space to the invariant Hilbert scheme and we prove that it is connected.

1.1. The invariant Hilbert scheme after Alexeev and Brion

Before passing to the specific example of an invariant Hilbert scheme, we present the general construction of the invariant Hilbert scheme introduced by Alexeev and Brion in [AB04, AB05]. For further details on invariant Hilbert schemes consult Brion's survey [Bri11].

Fix a complex reductive algebraic group G and an affine G–scheme X over $\mathbb{C}$. We denote by $\operatorname{Irr} G$ the set of isomorphism classes of irreducible representations $\rho\colon G \to Gl(V_\rho)$ of G and by $\rho_0 \in \operatorname{Irr} G$ the trivial representation.
As G is reductive, every G–module W decomposes as the sum of its isotypic components $W \cong \bigoplus_{\rho\in\operatorname{Irr} G} W_{(\rho)} = \bigoplus_{\rho\in\operatorname{Irr} G} W_\rho \otimes_{\mathbb{C}} V_\rho$, where $W_\rho = \operatorname{Hom}_G(V_\rho, W)$.
We call the dimension of $\operatorname{Hom}_G(V_\rho, W)$ the *multiplicity* of ρ in W. If each irreducible representation occurs with finite multiplicity, i.e. $h_W(\rho) := \dim \operatorname{Hom}_G(V_\rho, W) < \infty$ for all $\rho \in \operatorname{Irr} G$, then $h_W\colon \operatorname{Irr} G \to \mathbb{N}_0$ is called the *Hilbert function* of W. It is said to be *multiplicity–free* if $h_W(\rho) \in \{0, 1\}$ for all $\rho \in \operatorname{Irr} G$. In this thesis we will call any map $h\colon \operatorname{Irr} G \to \mathbb{N}_0$ a Hilbert function. Unless stated otherwise, in this chapter we will always assume that $h(\rho_0) = 1$.
If $\mathcal{F}$ is a G–equivariant coherent $\mathcal{O}_{X\times S}$–module over some noetherian basis S where G acts trivially and $p\colon X \times S \to S$ is the projection then there is also an isotypic decomposition $p_*\mathcal{F} \cong \bigoplus_{\rho\in\operatorname{Irr} G} \mathcal{F}_\rho \otimes_{\mathbb{C}} V_\rho$, where the sheaves of covariants $\mathcal{F}_\rho = \mathcal{H}om_G(V_\rho, \mathcal{F})$

are coherent $\mathcal{O}_S$–modules. They are locally free if and only if $\mathcal{F}$ is flat over S. In this case denote by $h_{\mathcal{F}}(\rho) := \operatorname{rk} \mathcal{F}_\rho$ their rank.

Definition 1.1.1 [AB05, Definition 1.5] For any function $h\colon \operatorname{Irr} G \to \mathbb{N}_0$, the associated functor

$$\begin{aligned} \mathcal{H}ilb_h^G(X)\colon (\mathrm{Sch}/\mathbb{C})^{\mathrm{op}} &\to (\mathrm{Set}) \\ S &\mapsto \left\{ \begin{array}{c|l} Z \subset X \times S & Z \text{ a } G\text{–invariant closed subscheme,} \\ \quad \searrow^{p} \;\; \downarrow & p \text{ flat,} \\ \qquad\quad S & h_{\mathcal{O}_Z} = h \end{array} \right\}, \\ (f\colon T \to S) &\mapsto (Z \mapsto (id_X \times f)^* Z) \end{aligned}$$

is called the *invariant Hilbert functor.*

Notation. We denote the sheaves of covariants in the isotypic decomposition of $p_*\mathcal{O}_Z$ by $\mathcal{F}_\rho = \mathcal{H}om_G(V_\rho, p_*\mathcal{O}_Z)$. By the condition $h_{\mathcal{O}_Z} = h$ in the definition, they are locally free $\mathcal{O}_S$–modules of rank $h(\rho)$.

Remark. In analogy to the case of finite G the coordinate ring of every fibre $Z(s)$ of the projection $p\colon Z \to S$ of a closed point $s \in S$ satisfies

$$\mathbb{C}[Z(s)] = \Gamma(Z(s), \mathcal{O}_{Z(s)}) = (p_*\mathcal{O}_Z)(s) \cong \bigoplus_{\rho \in \operatorname{Irr} G} \mathbb{C}^{h(\rho)} \otimes_{\mathbb{C}} V_\rho$$

since the fibre $\mathcal{F}_\rho(s)$ is a $\mathbb{C}$–vector space of dimension $h(\rho)$. This can be considered as $h(\rho)$ copies of V_ρ for every $\rho \in \operatorname{Irr} G$, so we write $\bigoplus_{\rho \in \operatorname{Irr} G} V_\rho^{\oplus h(\rho)}$ instead. In particular, the only invariants of $\mathbb{C}[Z(s)]$ are the elements of the isotypical component of the trivial representation ρ_0, i.e. $h(\rho_0)$ copies of the constants.

Proposition 1.1.2 *[HS04, AB04, AB05] There exists a quasiprojective scheme representing $\mathcal{H}ilb_h^G(X)$, the* invariant Hilbert scheme $\operatorname{Hilb}_h^G(X)$.

We are interested in the relation between the invariant Hilbert scheme and the quotient $X/\!/G = \operatorname{Spec} \mathbb{C}[X]^G$ parameterising the closed orbits of the action of G on X. There is an analogue of the Hilbert–Chow morphism, the quotient–scheme map

$$\eta\colon \operatorname{Hilb}_h^G(X) \to \operatorname{Hilb}^{h(\rho_0)}(X/\!/G),\ Z \mapsto Z/\!/G,$$

described in [Bri11, Section 3.4]. It is proper and even projective [Bri11, Proposition 3.12]. As we assume $h(\rho_0) = 1$, we have $\eta\colon \operatorname{Hilb}_h^G(X) \to \operatorname{Hilb}^1(X/\!/G) = X/\!/G$.
For this morphism or at least its restriction to some component of $\operatorname{Hilb}_h^G(X)$ to be birational, one has to choose the Hilbert function h_F of the general fibre F of the quotient map $\nu\colon X \to X/\!/G$:

$$\Gamma(F, \mathcal{O}_X) = \bigoplus_{\rho \in \operatorname{Irr} G} V_\rho^{\oplus h_F(\rho)}.$$

Lemma 1.1.3 *Suppose X is irreducible. Then $\operatorname{Hilb}_{h_F}^G(X)$ has an irreducible component $\operatorname{Hilb}_{h_F}^G(X)^{orb}$ such that the restriction of the Hilbert–Chow morphism to this component $\eta\colon \operatorname{Hilb}_{h_F}^G(X)^{orb} \to X/\!/G$ is birational.*

Proof. By an independent result of Brion [Bri11, Proposition 3.15] and Budmiger [Bud10, Theorem I.1.1], if $\nu\colon X \to X/\!/G$ is flat, then $X/\!/G$ represents the Hilbert functor $\mathcal{H}ilb_{h_F}^G(X)$, thus $X/\!/G \cong \operatorname{Hilb}_{h_F}^G(X)$. In the non–flat case let $U \subset X/\!/G$ be a non–empty open affine subset such that $\nu^{-1}(U) \to U$ is flat. Since all fibres of $\nu^{-1}(U) \to U$ have the same Hilbert function h_F as the general fibre of ν, the invariant Hilbert scheme $\operatorname{Hilb}_{h_F}^G(\nu^{-1}(U)) = \eta^{-1}(U)$ is an open subscheme of $\operatorname{Hilb}_{h_F}^G(X)$ and U is isomorphic to $\eta^{-1}(U)$. Thus the restriction of η to its closure $\operatorname{Hilb}_{h_F}^G(X)^{orb} := \overline{\eta^{-1}(U)}$ is birational.
If X and hence $X/\!/G$ is irreducible, so are U and $\eta^{-1}(U) \cong U$. Hence there is an irreducible component $C \subset \operatorname{Hilb}_{h_F}^G(X)$ containing $\eta^{-1}(U)$. The morphism $\eta|_C\colon C \to X/\!/G$ is dominant and the fibres of an open subset of $X/\!/G$ are finite (indeed the preimage of each element in U is a point). This means that $\dim C = \dim X/\!/G$, hence $\overline{\eta^{-1}(U)} = C$ is an irreducible component. □

Definition 1.1.4 The variety $\operatorname{Hilb}_{h_F}^G(X)^{orb}$ constructed in the Lemma is called the *orbit component* or *main component* of $\operatorname{Hilb}_{h_F}^G(X)$.

Remark. 1. The orbit component corresponds to the coherent component for toric Hilbert schemes. It is the principal component in the sense that it is birational to $X/\!/G$.
2. The map $\eta|_{\operatorname{Hilb}_{h_F}^G(X)^{orb}}$ is dominant and proper and $\operatorname{Hilb}_{h_F}^G(X)^{orb} \subset \operatorname{Hilb}_{h_F}^G(X)$ is closed, so $\eta|_{\operatorname{Hilb}_{h_F}^G(X)^{orb}}$ is even surjective. Thus it is a natural candidate for a resolution of singularities of $X/\!/G$.

Remark 1.1.5 If the general fibre of $\nu\colon X \to X/\!/G$ happens to be the group G itself then the Hilbert function is $h_G(\rho) = \dim(V_\rho)$ since we have $\Gamma(G, \mathcal{O}_G) = \mathbb{C}[G] \cong$

$\bigoplus_{\rho\in\operatorname{Irr} G} V_\rho^* \otimes_{\mathbb{C}} V_\rho$ and $\dim(V_\rho^*) = \dim(V_\rho)$. In analogy to the case of finite groups, in this situation we write

$$G\text{-Hilb}(X) := \operatorname{Hilb}^G_{h_G}(X) \qquad \text{and} \qquad G\text{-Hilb}(X)^{orb} := \operatorname{Hilb}^G_{h_G}(X)^{orb}.$$

1.2. Determination of the Hilbert function

1.2.1. The quotient related to the Hilbert scheme

We consider the action of Sl_2 on $(\mathbb{C}^2)^{\oplus 6}$ via multiplication from the left. There is a symplectic structure on $(\mathbb{C}^2)^{\oplus 6}$ given by the matrix $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, namely the bilinear form $(\mathbb{C}^2)^{\oplus 6} \times (\mathbb{C}^2)^{\oplus 6} \to \mathbb{C}$, $(M,N) \mapsto \operatorname{tr}(M^tJN)$. To obtain a quotient to which this symplectic structure descends, one considers the moment map (cf. [MFK94, Chapter 8]) and the quotient of its zero fibre, called *symplectic reduction* or *Marsden–Weinstein reduction*. As shown in [Bec10], in our case the moment map $\mu\colon (\mathbb{C}^2)^{\oplus 6} \to \mathfrak{sl}_2$ is given by $M \mapsto MQM^tJ$, where $Q = \left(\begin{smallmatrix} 0 & I_3 \\ I_3 & 0 \end{smallmatrix}\right)$. The symplectic reduction $(\mathbb{C}^2)^{\oplus 6}/\!/\!/Sl_2 = \mu^{-1}(0)/\!/Sl_2$ can be described as a nilpotent orbit closure

$$\mu^{-1}(0)/\!/Sl_2 \cong \overline{\mathcal{O}}_{[2^2,1^2]} = \{A \in \mathfrak{so}_6 \mid A^2 = 0,\ \operatorname{rk} A \leq 2,\ \operatorname{Pf}_4(QA) = 0\},$$

where $\operatorname{Pf}_4(QA)$ denotes the Pfaffians of the 15 skew–symmetric 4×4–minors of QA. Under the adjoint action of SO_6 this variety consists of two orbits of matrices of rank 2 and 0, respectively: $\overline{\mathcal{O}}_{[2^2,1^2]} = \mathcal{O}_{[2^2,1^2]} \cup \{0\}$. The quotient map is $\nu\colon \mu^{-1}(0) \to \overline{\mathcal{O}}_{[2^2,1^2]}$, $M \mapsto M^tJMQ$.

In coordinates $M = \left(\begin{smallmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \end{smallmatrix}\right)$ we have

$$\begin{aligned} M^tJMQ &= \begin{pmatrix} (-x_{2,i}x_{1,3+j} + x_{1,i}x_{2,3+j})_{ij} & (-x_{2,i}x_{1,j} + x_{1,i}x_{2,j})_{ij} \\ (-x_{2,3+i}x_{1,3+j} + x_{1,3+i}x_{2,3+j})_{ij} & (-x_{2,3+i}x_{1,j} + x_{1,3+i}x_{2,j})_{ij} \end{pmatrix} \\ &= \begin{pmatrix} (\Lambda^{i,3+j})_{ij} & (\Lambda^{i,j})_{ij} \\ (\Lambda^{3+i,3+j})_{ij} & (\Lambda^{j,3+i})_{ij} \end{pmatrix}, \end{aligned}$$

where i and j always range from 1 to 3 and $\Lambda^{s,t} = \det(x^{(s)}, x^{(t)})$ is the 2×2–minor of the s–th and t–th column in M. Thus the fibres of ν consist of those M with fixed 2×2–minors. A further condition is $M \in \mu^{-1}(0)$, i.e.

$$0 = MQM^t = \begin{pmatrix} 2\cdot \sum_{i=1}^{3} x_{1,i}x_{1,3+i} & \sum_{i=1}^{3} (x_{1,i}x_{2,3+i} + x_{1,3+i}x_{2,i}) \\ \sum_{i=1}^{3} (x_{1,i}x_{2,3+i} + x_{1,3+i}x_{2,i}) & 2\cdot \sum_{i=1}^{3} x_{2,i}x_{2,3+i} \end{pmatrix}.$$

1.2.2. The general fibre of the quotient

In order to determine the Hilbert function h_F of the general fibre F of the quotient map $\nu\colon \mu^{-1}(0) \to \mu^{-1}(0)/\!/Sl_2$, so that $\mathrm{Hilb}^{Sl_2}_{h_F}(\mu^{-1}(0))$ is birational to $\mu^{-1}(0)/\!/Sl_2$, we have to compute F first. Therefore we need to know the locus where ν is flat.

Proposition 1.2.1 *The quotient map ν restricted to the preimage of the open orbit of the SO_6–action $\nu^{-1}(\mathcal{O}_{[2^2,1^2]}) \to \mathcal{O}_{[2^2,1^2]}$ is flat. Therefore, the fibres over all points in the orbit $\mathcal{O}_{[2^2,1^2]}$ are isomorphic.*

Proof. $\mu^{-1}(0)$ is equipped with an action of SO_6 via multiplication on the right, which induces the adjoint action on $\mu^{-1}(0)/\!/Sl_2 = \overline{\mathcal{O}}_{[2^2,1^2]}$. Since the map $\nu\colon \mu^{-1}(0) \to \overline{\mathcal{O}}_{[2^2,1^2]}$ is SO_6–equivariant, ν is flat over the whole SO_6–orbit $\mathcal{O}_{[2^2,1^2]}$ or over no point of this orbit. By Grothendieck's Lemma on generic flatness and since $\overline{\mathcal{O}}_{[2^2,1^2]} \setminus \mathcal{O}_{[2^2,1^2]} = \{0\}$, the second case cannot occur. By equivariance, all fibres over this orbit are isomorphic.
□

As a consequence, for computing the general fibre it is enough to determine the fibre over one point A_0 in the flat locus $\mathcal{O}_{[2^2,1^2]}$. We choose $A_0 = (a_{ij})$ with $a_{15} = -a_{24} = 1$ and $a_{ij} = 0$ otherwise. For $M \in \nu^{-1}(A_0)$ this corresponds to $\Lambda^{1,2} = 1 = -\Lambda^{2,1}$ and $\Lambda^{i,j} = 0$ otherwise. Thus

$$1 = \Lambda^{1,2} = x_{11}x_{22} - x_{12}x_{21}, \quad \text{hence } x_{11} \neq 0 \neq x_{22} \text{ or } x_{12} \neq 0 \neq x_{21}.$$

Without loss of generality assume $x_{11} \neq 0$. Then $x_{22} = \dfrac{1 + x_{12}x_{21}}{x_{11}}$.
For $j = 3, \ldots, 6$ we have

$$\begin{aligned}
0 = \Lambda^{1,j} = x_{11}x_{2j} - x_{1j}x_{21} \quad &\Rightarrow \quad x_{2j} = \frac{x_{1j}x_{21}}{x_{11}}, \\
0 = \Lambda^{2,j} = x_{12}x_{2j} - x_{1j}x_{22} \quad &\Rightarrow \quad x_{12}\frac{x_{1j}x_{21}}{x_{11}} = x_{1j}\frac{1 + x_{12}x_{21}}{x_{11}} \\
&\qquad\qquad\qquad = \frac{x_{1j}}{x_{11}} + \frac{x_{1j}x_{12}x_{21}}{x_{11}} \\
&\Rightarrow \quad x_{1j} = 0 \quad \text{for } j = 3, \ldots, 6, \\
&\Rightarrow \quad x_{2j} = \frac{x_{1j}x_{21}}{x_{11}} = 0 \quad \text{for } j = 3, \ldots, 6.
\end{aligned}$$

This implies
$$\begin{aligned}
&x_{11}x_{14} + x_{12}x_{15} + x_{13}x_{16} = 0, \\
&x_{11}x_{24} + x_{12}x_{25} + x_{13}x_{26} + x_{14}x_{21} + x_{15}x_{22} + x_{16}x_{23} = 0, \\
&x_{21}x_{24} + x_{22}x_{25} + x_{23}x_{26} = 0,
\end{aligned}$$

so $M \in \mu^{-1}(0)$ is automatic. This shows that the general fibre is

$$F := \nu^{-1}(A_0) = \left\{ \left(\begin{smallmatrix} x_{11} & x_{12} & 0 & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 & 0 & 0 \end{smallmatrix}\right) \in (\mathbb{C}^2)^{\oplus 6} \mid x_{11}x_{22} - x_{12}x_{21} = 1 \right\} \cong Sl_2. \tag{1.3}$$

This justifies the notation Sl_2-Hilb$(\mu^{-1}(0)) = \mathrm{Hilb}_{h_F}^{Sl_2}(\mu^{-1}(0))$ as introduced in Remark 1.1.5.

Remark. Analogous calculations over 0 show that the fibre $\nu^{-1}(0)$ has dimension 5, so ν is not flat over 0 and $\mathcal{O}_{[2^2,1^2]}$ is the maximal flat locus.

1.2.3. The Hilbert function of the general fibre

The Hilbert function is determined by the isotypic decomposition of the general fibre. The irreducible representations of Sl_2 are parameterised by the natural numbers including zero: $\mathrm{Irr}(Sl_2) \cong \mathbb{N}_0$, $V_d \leftrightarrow d$, where $V_d = \mathbb{C}[x,y]_d$ consists of homogeneous polynomials of degree d so that $\dim V_d = d+1$. By Remark 1.1.5 the coordinate ring of Sl_2 decomposes as

$$\mathbb{C}[Sl_2] \cong \bigoplus_{d\in\mathbb{N}_0} V_d^{\oplus \dim V_d} = \bigoplus_{d\in\mathbb{N}_0} V_d^{\oplus(d+1)},$$

so in this case the Hilbert function is given by the dimension $h_{Sl_2}(d) = \dim V_d = d+1$. For the Hilbert scheme this means that the sheaves of covariants $\mathcal{F}_d$ have to be locally free of rank $d+1$.

1.3. Determination of the orbit component

Our idea to identify Sl_2-Hilb$(\mu^{-1}(0))$ is to determine generators for the sheaves of covariants $\mathcal{F}_d$ and to use them to embed the Sl_2–Hilbert scheme into the product of $\mu^{-1}(0)/\!\!/Sl_2$ and some Grassmannians. First, in Section 1.3.1 we describe the sheaves $\mathcal{F}_\rho$ in general by giving a space F_ρ of generators as an $\mathcal{O}_{\mathrm{Hilb}_h^G(X)}$–module for each $\rho \in \mathrm{Irr}\, G$. We calculate $\mathcal{F}_1$ in our example. In Section 1.3.2 we describe how to obtain a map η_ρ to the Grassmannian of quotients of F_ρ of dimension $h(\rho)$. We show that one can embed $\mathrm{Hilb}_h^G(X)$ into a product of finitely many of these Grassmannians. Afterwards, for Sl_2-Hilb$(\mu^{-1}(0))$ we calculate the map η_1 corresponding to the standard representation. We show that this single representation is enough to give an embedding of the orbit component into $\mu^{-1}(0)/\!\!/Sl_2 \times \mathrm{Grass}(F_1, h(1))$. Then we determine a strict subset of this which contains the image. Finally, by writing the Grassmannian as a homogeneous space we prove in

Section 1.3.3 that the embedding is even an isomorphism. This allows us to determine explicitly the elements of Sl_2-Hilb$(\mu^{-1}(0))$ as subschemes of $\mu^{-1}(0)$ in Section 1.3.4 and thus prove Theorem 1.2.

1.3.1. The sheaves of covariants $\mathcal{F}_\rho$

To describe the invariant Hilbert scheme or at least its orbit component, we have to determine the locally free sheaves $\mathcal{F}_\rho$ of rank $h(\rho)$ on $\mathrm{Hilb}_h^G(X)$. For the trivial representation we have the following result by Brion [Bri11, Proof of Proposition 3.15], for which we give a more detailed proof.

Lemma 1.3.1 *If $h(\rho_0) = 1$ then for any scheme S and every $Z \in \mathcal{H}ilb_h^G(X)(S)$ we have $\mathcal{F}_{\rho_0} \cong \mathcal{O}_S$. In particular, for the universal subscheme this yields $\mathcal{F}_{\rho_0} \cong \mathcal{O}_{\mathrm{Hilb}_h^G(X)}$.*

Proof. Taking invariants, the defining equation of the sheaves of covariants $\mathcal{F}_\rho$ yields the isomorphism $p_*\mathcal{O}_Z^G \cong \bigoplus_{\rho\in\mathrm{Irr}\,G} \mathcal{F}_\rho \otimes_{\mathbb{C}} V_\rho^G$. But the trivial representation is the only irreducible representation admitting invariants, and all of its elements are invariants. Thus $\bigoplus_{\rho\in\mathrm{Irr}\,G} \mathcal{F}_\rho \otimes_{\mathbb{C}} V_\rho^G = \mathcal{F}_{\rho_0}$. There is a morphism $p^\#\colon \mathcal{O}_S = \mathcal{O}_S^G \to p_*\mathcal{O}_Z^G \cong \mathcal{F}_{\rho_0}$ induced by p, which is injective since p is surjective. The $\mathcal{O}_S$–modules $\mathcal{O}_S$ and $\mathcal{F}_{\rho_0}$ are both locally free of rank one. Over each closed point $s \in S$ the fibres are $\mathcal{O}_S(s) = \mathbb{C}$ and $\mathcal{F}_{\rho_0}(s) = (p_*\mathcal{O}_Z)^G(s) = (p_*\mathcal{O}_Z)^G \otimes_{\mathbb{C}} k(s) = (p_*\mathcal{O}_Z \otimes_{\mathbb{C}} k(s))^G = \mathbb{C}[Z(s)]^G$, and $\mathbb{C}[Z(s)]^G = V_{\rho_0} \cong \mathbb{C}$. So by Nakayama's Lemma, $p^\#$ is an isomorphism, hence $\mathcal{O}_S \cong \mathcal{F}_{\rho_0}$. □

For general ρ we additionally observe what happens if there is an action on X by another complex connected reductive group H commuting with the G–action. By [Bri11, Proposition 3.10], such an action also induces an action on $X/\!\!/G$ and on $\mathrm{Hilb}_h^G(X)$, such that the quotient map and the Hilbert–Chow morphism are H–equivariant.
Consider the isotypic decomposition $\mathbb{C}[X] \cong \bigoplus_{\rho\in\mathrm{Irr}\,G} \mathbb{C}[X]_\rho \otimes_{\mathbb{C}} V_\rho$, where H acts by the induced action on $\mathbb{C}[X]_\rho = \mathrm{Hom}_G(V_\rho, \mathbb{C}[X])$ and trivially on V_ρ.

Proposition 1.3.2 *For every $\rho \in \mathrm{Irr}\,G$, the $\mathbb{C}[X]^G$–module $\mathbb{C}[X]_\rho$ is finitely generated. Hence there is a finite dimensional H–module F_ρ and an H–equivariant surjection $\mathbb{C}[X]^G \otimes_{\mathbb{C}} F_\rho \twoheadrightarrow \mathbb{C}[X]_\rho$. For any element $Z \in \mathcal{H}ilb_h^G(X)(S)$ and any scheme S, the space F_ρ generates the sheaf of covariants $\mathcal{F}_\rho = \mathrm{Hom}(V_\rho, p_*\mathcal{O}_Z)$ as an $\mathcal{O}_S$–module, so that the morphism of $\mathcal{O}_S$–H–modules $\mathcal{O}_S \otimes_{\mathbb{C}} F_\rho \to \mathcal{F}_\rho$ is surjective.*

Proof. The space $\mathbb{C}[X]_\rho = \mathrm{Hom}_G(V_\rho, \mathbb{C}[X])$ is finitely generated as an $\mathbb{C}[X]^G$–module, see [Dol03, Corollary 5.1]. Thus we can choose finitely many generators and define F_ρ to be the H–module generated by them. This gives an H–equivariant surjection $\mathbb{C}[X]^G \otimes_{\mathbb{C}} F_\rho \twoheadrightarrow \mathbb{C}[X]_\rho$.
To determine generators for $\mathcal{F}_\rho$ we use the universal subscheme $\mathrm{Univ}_h^G(X)$. Then we obtain the result for an arbitrary scheme S and every element $Z \in \mathcal{H}ilb_h^G(X)(S)$ by pulling it back. We have

$$\begin{array}{ccccc} \mathrm{Univ}_h^G(X) & \subset & X \times \mathrm{Hilb}_h^G(X) & \longrightarrow & X \\ & \searrow^{p} & \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle \nu} \\ & & \mathrm{Hilb}_h^G(X) & \xrightarrow{\ \eta\ } & X/\!\!/G \end{array}$$

The action of H on X, $X/\!\!/G$ and $\mathrm{Hilb}_h^G(X)$ induces an action of H on the fibred product $X \times_{X/\!\!/G} \mathrm{Hilb}_h^G(X)$ and on $\mathrm{Univ}_h^G(X)$ such that all morphisms in the diagram are H–equivariant. By [Bri11, Proposition 3.15], in this situation we have an embedding $\mathrm{Univ}_h^G(X) \hookrightarrow X \times_{X/\!\!/G} \mathrm{Hilb}_h^G(X)$. This yields a surjective H–equivariant morphism

$$\mathcal{O}_{\mathrm{Hilb}_h^G(X)} \otimes_{\mathbb{C}[X]^G} \mathbb{C}[X] \twoheadrightarrow p_*\mathcal{O}_{\mathrm{Univ}_h^G(X)}.$$

By definition, we have $p_*\mathcal{O}_{\mathrm{Univ}_h^G(X)} \cong \bigoplus_{\rho \in \mathrm{Irr}\, G} \mathcal{F}_\rho \otimes_{\mathbb{C}} V_\rho$ with an induced action of H on each $\mathcal{F}_\rho$ and the trivial action on V_ρ. Furthermore, we can consider the isotypic decomposition $\mathcal{O}_{\mathrm{Hilb}_h^G(X)} \otimes_{\mathbb{C}[X]^G} \mathbb{C}[X] \cong \bigoplus_{\rho \in \mathrm{Irr}\, G} \mathcal{O}_{\mathrm{Hilb}_h^G(X)} \otimes_{\mathbb{C}[X]^G} \mathbb{C}[X]_\rho \otimes_{\mathbb{C}} V_\rho$ as G–modules. Together, we obtain H–equivariant surjections

$$\mathcal{O}_{\mathrm{Hilb}_h^G(X)} \otimes_{\mathbb{C}[X]^G} \mathbb{C}[X]_\rho \twoheadrightarrow \mathcal{F}_\rho$$

for every $\rho \in \mathrm{Irr}\, G$. This shows that the $\mathcal{O}_{\mathrm{Hilb}_h^G(X)}$–$H$–module $\mathcal{F}_\rho$ is generated by $\mathbb{C}[X]_\rho$, which is in turn generated by F_ρ over $\mathbb{C}[X]^G$. This yields

$$\mathcal{O}_{\mathrm{Hilb}_h^G(X)} \otimes_{\mathbb{C}} F_\rho \twoheadrightarrow \mathcal{O}_{\mathrm{Hilb}_h^G(X)} \otimes_{\mathbb{C}[X]^G} \mathbb{C}[X]_\rho \twoheadrightarrow \mathcal{F}_\rho. \tag{1.4}$$

□

Remark 1.3.3 In place of the invariant Hilbert scheme one may more generally consider the invariant Quot scheme $\mathrm{Quot}^G(\mathcal{H}, h)$ for a fixed coherent sheaf $\mathcal{H}$ on X, constructed by Jansou in [Jan06]. It parameterises quotients $\mathcal{H} \twoheadrightarrow \mathcal{F}$ with isotypic decomposition of $H^0(\mathcal{F})$ isomorphic to $\bigoplus_{\rho \in \mathrm{Irr}\, G} V_\rho^{\oplus h(\rho)}$. The invariant Quot scheme generalises the

invariant Hilbert scheme: $\mathrm{Hilb}^G_h(X) = \mathrm{Quot}^G(\mathcal{O}_X, h)$, where the quotients $\mathcal{F}$ are just structure sheaves $\mathcal{O}_Z$ of the subschemes Z of X.
A generalisation of Proposition 1.3.2 also holds for the invariant Quot scheme if one considers the decomposition $p_*(\pi^*\mathcal{H}) = \bigoplus_{\rho \in \mathrm{Irr}\, G} \mathcal{H}_\rho \otimes_{\mathbb{C}} V_\rho$ over any scheme S, where $\pi\colon X \times S \to X$ and $p\colon X \times S \to S$, and one replaces $\mathbb{C}[X]_\rho$ by $\mathcal{H}_\rho$ and F_ρ by suitably chosen spaces H_ρ which generate $\mathcal{H}_\rho$ as an $\mathbb{C}[X]^G$–module and $\mathcal{F}_\rho$ as an $\mathcal{O}_S$–module. We present a different construction of this in Proposition 3.1.2.

Application to $\mathcal{F}_1$

We already know that $\mathcal{F}_0 = \mathcal{O}_{Sl_2\text{-Hilb}(\mu^{-1}(0))}$ is free of rank 1 by Lemma 1.3.1. We continue with the standard representation $V_1 = \mathbb{C}^2$ and determine $\mathcal{F}_1$. It will turn out in Proposition 1.3.5 that at least the orbit component $Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb}$ is already completely determined by this sheaf.
There is an action of SO_6 on $\mu^{-1}(0)$ via multiplication from the right and the induced action on $\overline{\mathcal{O}}_{[2^2,1^2]}$ by conjugation. The induced action on $Sl_2\text{-Hilb}(\mu^{-1}(0))$ is also by multiplication from the right. Following Proposition 1.3.2 we obtain

Proposition 1.3.4 *The six projections $p_i|_{\mu^{-1}(0)}\colon \mu^{-1}(0) \to \mathbb{C}^2$, $i = 1, \dots, 6$ generate $\mathcal{F}_1$. Hence we may take $F_1 \cong \mathbb{C}^6$ to be the standard representation of SO_6.*

Proof. By the proof of Proposition 1.3.2, $\mathcal{F}_1$ is generated by $\mathrm{Hom}_{Sl_2}(\mathbb{C}^2, \mathbb{C}[\mu^{-1}(0)])$, which is isomorphic to $\mathrm{Mor}_{Sl_2}(\mu^{-1}(0), \mathbb{C}^2)$ because of the self–duality of the standard representation of Sl_2. The inclusion $\mu^{-1}(0) \subset (\mathbb{C}^2)^{\oplus 6}$ induces a surjective morphism $\mathrm{Mor}_{Sl_2}((\mathbb{C}^2)^{\oplus 6}, \mathbb{C}^2) \twoheadrightarrow \mathrm{Mor}_{Sl_2}(\mu^{-1}(0), \mathbb{C}^2)$ by shrinking morphisms to $\mu^{-1}(0)$. According to [How95], the space of Sl_2–equivariant morphisms $\mathrm{Mor}_{Sl_2}((\mathbb{C}^2)^{\oplus 6}, \mathbb{C}^2)$ is a free module of rank 6 over the ring of invariants $\mathbb{C}[(\mathbb{C}^2)^{\oplus 6}]^{Sl_2}$, generated by the projections $p_i\colon (\mathbb{C}^2)^{\oplus 6} \to \mathbb{C}^2$ to the i–th component.
The restrictions $p_i|_{\mu^{-1}(0)}\colon \mu^{-1}(0) \to \mathbb{C}^2$ still span a 6–dimensional space: Consider for example the matrices M_i where each column except the i–th one is 0. Then $M_i Q M_i^t = 0$ for $i = 1, \dots, 6$, so $M_i \in \mu^{-1}(0)$. In turn, the identity $p_j(M_i) = \delta_{ij}\binom{x_{1j}}{x_{2j}}$ shows that the $p_i|_{\mu^{-1}(0)}$ are linearly independent. Thus $\mathrm{Mor}_{Sl_2}(\mu^{-1}(0), \mathbb{C}^2) \cong \mathrm{Hom}_{Sl_2}((\mathbb{C}^2)^{\oplus 6}, \mathbb{C}^2)$ and $F_1 = \langle p_i \mid i = 1, \dots, 6\rangle \cong \mathbb{C}^6$.
The SO_6–equivariant identification $\mathbb{C}^6 \cong \mathrm{Hom}_{Sl_2}((\mathbb{C}^2)^{\oplus 6}, \mathbb{C}^2)$, $e_i \mapsto p_i$ induces the inner product $\langle p_i, p_j\rangle = \delta_{i+3,j} + \delta_{j+3,i}$ on $\langle p_1, \dots, p_6\rangle$. For this reason we can also write $\langle p, q\rangle = p^t Q q$ for all maps $p, q \in F_1$ and we see that F_1 is the standard representation. $\square$

1.3.2. Embedding the Hilbert scheme into a product of Grassmannians

As remarked in the proof of Proposition 1.3.2, every map $S \to \mathrm{Hilb}_h^G(X)$ gives us a map $\mathcal{O}_S \otimes_{\mathbb{C}} F_\rho \to \mathcal{F}_\rho$ by pulling back the morphism (1.4). Since $\mathcal{F}_\rho$ is a locally free quotient of $\mathcal{O}_S \otimes_{\mathbb{C}} F_\rho$ of rank $h(\rho)$, this in turn corresponds to a map $S \to \mathrm{Grass}(F_\rho, h(\rho))$ into the Grassmannian of quotients of F_ρ of dimension $h(\rho)$. In particular, taking $S = \mathrm{Hilb}_h^G(X)$, we obtain a map of schemes

$$\eta_\rho \colon \mathrm{Hilb}_h^G(X) \to \mathrm{Grass}(F_\rho, h(\rho)).$$

In the situation of Proposition 1.3.2 this map is again H–equivariant. Evaluating at a closed point $s \in S$ yields

$$\begin{array}{lllll} (S \to \mathrm{Hilb}_h^G(X)) & \longmapsto & (\mathcal{O}_S \otimes_{\mathbb{C}} F_\rho \to \mathcal{F}_\rho) & \longmapsto & (S \to \mathrm{Grass}(F_\rho, h(\rho))), \\ (s \mapsto Z_s) & \longmapsto & (f_{\rho,s} \colon F_\rho \to \mathcal{F}_\rho(s)) & \longmapsto & (s \mapsto \mathcal{F}_\rho(s)), \end{array} \tag{1.5}$$

where the fibres $\mathcal{F}_\rho(s)$ are vector spaces of dimension $h(\rho)$. Hence we have

$$\eta_\rho \colon \mathrm{Hilb}_h^G(X) \to \mathrm{Grass}(F_\rho, h(\rho)),\ Z \mapsto \mathcal{F}_\rho(Z).$$

As $\mathbb{C}[X]_\rho = \mathrm{Hom}_G(V_\rho, \mathbb{C}[X]) \cong \mathrm{Mor}_G(X, V_\rho^*)$, the elements of the generating space F_ρ are G–equivariant morphisms from X to V_ρ^* and evaluating at an element $Z \in \mathrm{Hilb}_h^G(X)$ means restricting $\mathrm{Mor}_G(X, V_\rho^*) \to \mathrm{Mor}_G(Z, V_\rho^*)$, so in (1.5) we have

$$f_{\rho,Z} \colon F_\rho \twoheadrightarrow \mathcal{F}_\rho(Z),\ p \mapsto p|_Z. \tag{1.6}$$

The map η_{ρ_0} does not yield any information since $\mathrm{Grass}(F_{\rho_0}, h(\rho_0)) = \mathrm{Grass}(\mathbb{C}, 1)$ is only a point. The product of the Hilbert–Chow morphism and the η_ρ defines a map

$$\mathrm{Hilb}_h^G(X) \to X /\!\!/ G \times \prod_{\substack{\rho \in \mathrm{Irr}\, G \\ \rho \neq 0}} \mathrm{Grass}(F_\rho, h(\rho)). \tag{1.7}$$

This map is a closed immersion, even if we replace the right hand side by a product over a suitably chosen finite subset of $\mathrm{Irr}\, G$ only: Indeed, let $B = TU$ be a Borel subgroup of G, where T is a maximal torus and U the unipotent radical. Assigning to V_ρ its highest weight gives a one–to–one correspondence between $\mathrm{Irr}\, G$ and the set of dominant weights Λ^+ in the weight lattice Λ of T. Extend h to Λ by 0. Let V be a finite–dimensional T–module containing $X /\!\!/ U$. By [AB05, Theorem 1.7, Lemma 1.6], we have closed embeddings $\mathrm{Hilb}_h^G(X) \hookrightarrow \mathrm{Hilb}_h^T(X /\!\!/ U) \hookrightarrow \mathrm{Hilb}_h^T(V)$ and each module $\mathbb{C}[V]_\rho$ is generated by some $\mathbb{C}$–vector space E_ρ over $\mathbb{C}[V]^T$. The E_ρ can be chosen as lifts of F_ρ, so that we have $E_\rho \twoheadrightarrow F_\rho$ under $\mathbb{C}[V] \twoheadrightarrow \mathbb{C}[X]$. As shown by [HS04, Theorem 2.2, 2.3], the map

$$\mathrm{Hilb}_h^T(V) \hookrightarrow \prod_{\rho\in D} \mathrm{Grass}_{V/\!/T}(E_\rho, h(\rho))$$

is a closed immersion for a suitably chosen finite subset $D \subset \Lambda$. Since h vanishes outside Λ^+ we even obtain $D \subset \operatorname{Irr} G$ in our case. Every quotient of F_ρ of dimension $h(\rho)$ is also a quotient of E_ρ of dimension $h(\rho)$, so for any $\rho \in D$ we have an embedding $\mathrm{Grass}_{X/\!/G}(F_\rho, h(\rho)) \hookrightarrow \mathrm{Grass}_{V/\!/T}(E_\rho, h(\rho))$. Further, every element in $\mathrm{Hilb}_h^T(V)$ coming from $\mathrm{Hilb}_h^T(X/\!/U)$ is already generated by F_ρ. This means that the composite morphism $\mathrm{Hilb}_h^T(X/\!/U) \hookrightarrow \prod_{\rho\in D} \mathrm{Grass}_{V/\!/T}(E_\rho, h(\rho))$ factors through $\prod_{\rho\in D} \mathrm{Grass}_{X/\!/G}(F_\rho, h(\rho))$, so that we obtain

$$\begin{array}{ccccc}
 & & \prod_{\rho\in D} \mathrm{Grass}_{X/\!/G}(F_\rho, h(\rho)) & \hookrightarrow & \prod_{\rho\in D} \mathrm{Grass}_{V/\!/T}(E_\rho, h(\rho)) \\
 & \nearrow & \uparrow & & \uparrow \\
\mathrm{Hilb}_h^G(X) & \hookrightarrow & \mathrm{Hilb}_h^T(X/\!/U) & \hookrightarrow & \mathrm{Hilb}_h^T(V) \\
\downarrow & & \downarrow & & \downarrow \\
X/\!/G & = & (X/\!/U)/\!/T & \hookrightarrow & V/\!/T
\end{array}$$

In fact, $\mathrm{Hilb}_h^G(X)$ embeds into $X/\!/G \times \prod_{\rho\in D} \mathrm{Grass}(F_\rho, h(\rho))$ because the relative Grassmannian $\mathrm{Grass}_{X/\!/G}(F_\rho, h(\rho))$ is isomorphic to the product $X/\!/G \times \mathrm{Grass}(F_\rho, h(\rho))$ and for $Z \in \mathrm{Hilb}_h^G(X)$, $\mathbb{C}[Z] = \bigoplus_{\rho\in \operatorname{Irr} G} \mathcal{F}_\rho \otimes_{\mathbb{C}} V_\rho$, the elements $[F_\rho \twoheadrightarrow \mathcal{F}_\rho] \in \mathrm{Grass}_{X/\!/G}(F_\rho, h(\rho))$, $\rho \in \operatorname{Irr} G$, all map to the same point $Z/\!/G$ in $X/\!/G$.

This suggests the following procedure to determine the invariant Hilbert scheme: One can start with any representation. Call it ρ_1. If $\eta \times \eta_{\rho_1}$ is not a closed immersion, add another representation ρ_2. If $\eta \times \eta_{\rho_1} \times \eta_{\rho_2}$ is not a closed immersion, continue. There will we a number $s \in \mathbb{N}$ such that $\eta \times \eta_{\rho_1} \times \ldots \times \eta_{\rho_s}$ is a closed immersion. Then identify the image of this immersion.

Remark. Replacing F_ρ by H_ρ as in Remark 1.3.3, all steps except the last one also apply to the invariant Quot scheme, so that for some finite subset $D \subset \operatorname{Irr} G$ we obtain a (not necessarily closed) embedding

$$\mathrm{Quot}^G(\mathcal{H}, h) \to \prod_{\rho\in D} \mathrm{Grass}(H_\rho, h(\rho)).$$

We construct this morphism in Section 3.1.

Determination of η_1

The knowledge of F_1 gives us an SO_6–equivariant map

$$\eta_1 \colon Sl_2\text{-Hilb}(\mu^{-1}(0)) \to \mathrm{Grass}(F_1, \dim V_1) = \mathrm{Grass}(\mathbb{C}^6, 2),\ Z \mapsto \mathcal{F}_1(Z).$$

The fibre $\mathcal{F}_1(Z)$ of the sheaf $\mathcal{F}_1$ is generated by the restrictions of the projections $p_i \colon \mu^{-1}(0) \to \mathbb{C}^2$ to the subscheme $Z \subset \mu^{-1}(0)$.

Proposition 1.3.5 *1. The map $\eta \times \eta_1$ is given by*

$$Sl_2\text{-Hilb}(\mu^{-1}(0)) \to \mu^{-1}(0)/\!\!/Sl_2 \times \mathrm{Grass}(2, \mathbb{C}^6),\ Z \mapsto (Z/\!\!/Sl_2, \ker(f_{1,Z})^{\perp}).$$

2. The image of $\eta \times \eta_1$ restricted to the orbit component $Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb}$ is contained in $Y := \{(A, U) \in \overline{\mathcal{O}}_{[2^2,1^2]} \times \mathrm{Grass}_{iso}(2, \mathbb{C}^6) \mid \mathrm{im}\, A^t \subset U\}$.

Proof. 1. To describe the morphism $\eta_1 \colon Sl_2\text{-Hilb}(\mu^{-1}(0)) \to \mathrm{Grass}(\mathbb{C}^6, 2)$ explicitly, we analyse the map $f_{1,Z} \colon F_1 \to \mathcal{F}_1(Z)$ defined in (1.6). As it is surjective, we have $\mathcal{F}_1(Z) \cong F_1/\ker(f_{1,Z})$. Now we can identify the Grassmannian of quotients with the Grassmannian of subspaces via the canonical isomorphism $\mathrm{Grass}(\mathbb{C}^6, 2) \to \mathrm{Grass}(2, \mathbb{C}^6)$, $F_1/\ker(f_{1,Z}) \mapsto \ker(f_{1,Z})^{\perp}$. Thus η_1 is the morphism $Sl_2\text{-Hilb}(\mu^{-1}(0)) \to \mathrm{Grass}(2, \mathbb{C}^6)$, $Z \mapsto \ker(f_{1,Z})^{\perp}$.

2. Over $\mathcal{O}_{[2^2,1^2]}$, the morphism $\eta \times \eta_1 \colon \eta^{-1}(\mathcal{O}_{[2^2,1^2]}) \to \mathcal{O}_{[2^2,1^2]} \times \mathrm{Grass}(2, \mathbb{C}^6)$ is given by $Z_A \mapsto (A, \ker(f_{1,Z_A})^{\perp})$. For analysing the image, we choose the special point $A_0 \in \mathcal{O}_{[2^2,1^2]}$ again. Description (1.6) combined with (1.3) shows that $\ker(f_{1,Z_{A_0}}) = \langle p_3, p_4, p_5, p_6 \rangle$ with orthogonal complement $\ker(f_{1,Z_{A_0}})^{\perp} = \langle p_4, p_5 \rangle$ by definition of the inner product above. Since $p_4^t Q p_4 = p_4^t Q p_5 = p_5^t Q p_5 = 0$, this space is isotropic. Thus for every point A in the open orbit, $\ker(f_{1,Z_A})^{\perp}$ is isotropic. As being isotropic is a closed condition, $\eta \times \eta_1$ maps the closure of the preimage of $\mathcal{O}_{[2^2,1^2]}$ under η, the orbit component, to the isotropic Grassmannian:

$$\eta \times \eta_1 \colon \overline{\eta^{-1}(\mathcal{O}_{[2^2,1^2]})} = Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb} \to \overline{\mathcal{O}}_{[2^2,1^2]} \times \mathrm{Grass}_{iso}(2, \mathbb{C}^6).$$

For the additional condition we only need to examine $A_0 = \begin{pmatrix} 0 & \begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \\ 0 & 0 \end{pmatrix}$ again. We can consider A_0 and its transpose A_0^t as maps

$$A_0 \colon F_1 \to F_1,\ p_4 \mapsto -p_2,\ p_5 \mapsto p_1,\ p_i \mapsto 0 \text{ for } i = 1, 2, 3, 6,$$
$$A_0^t \colon F_1 \to F_1,\ p_1 \mapsto p_5,\ p_2 \mapsto -p_4,\ p_i \mapsto 0 \text{ for } i = 3, 4, 5, 6.$$

Thus we have $\mathrm{im}(A_0^t) = \langle p_4, p_5 \rangle = \ker(f_{1,Z_{A_0}})^{\perp}$. Since $\eta \times \eta_1$ is SO_6–equivariant, the equality $\mathrm{im}(A^t) = \ker(f_{1,Z_A})^{\perp}$ holds for every A in the orbit $\mathcal{O}_{[2^2,1^2]}$ and we obtain

$$\eta \times \eta_1(\eta^{-1}(\mathcal{O}_{[2^2,1^2]})) \subset Y' := \{(A, U) \in \mathcal{O}_{[2^2,1^2]} \times \mathrm{Grass}_{iso}(2, \mathbb{C}^6) \mid \mathrm{im}\, A^t = U\}.$$

If $A \in \overline{\mathcal{O}}_{[2^2,1^2]} \setminus \mathcal{O}_{[2^2,1^2]}$, its rank is smaller than 2 (indeed $A = 0$), and so is $\dim(\mathrm{im}\, A^t)$. Hence the closure of Y' in $\overline{\mathcal{O}}_{[2^2,1^2]} \times \mathrm{Grass}_{iso}(2, \mathbb{C}^6)$ is Y. □

We will see in the further examination that $\eta \times \eta_1$ actually is an isomorphism (Proposition 1.3.7), even on the whole invariant Hilbert scheme (Proposition 1.4.4).

1.3.3. The Grassmannian as a homogeneous space

For a further analysis of the image, we consider the isotropic Grassmannian as a homogeneous space $\mathrm{Grass}_{iso}(2, \mathbb{C}^6) = SO_6/P$, where $P = (SO_6)_{W_0}$ is the isotropy group of an arbitrary point $W_0 \in \mathrm{Grass}_{iso}(2, \mathbb{C}^6)$. We choose $W_0 = \langle p_1, p_2 \rangle$. If $g_W \in SO_6$ is chosen such that $W = g_W W_0$, the isomorphism is

$$\mathrm{Grass}_{iso}(2, \mathbb{C}^6) \to SO_6/P,\ W \mapsto g_W P = [g_W],\ gW_0 \leftarrow\!\shortmid [g].$$

The projection $f\colon Y \xrightarrow{pr_2} \mathrm{Grass}_{iso}(2, \mathbb{C}^6) \cong SO_6/P,\ (A, U) \mapsto U \mapsto [g_U]$ makes Y a fibre bundle with typical fibre $E := f^{-1}([I_6]) = pr_2^{-1}(W_0)$. It can be witten as an associated SO_6–bundle, i.e. $Y \cong SO_6 \times^P E := SO_6 \times E/\!\sim$ with relation $(g, A) \sim (gp^{-1}, pAp^{-1})$.

Lemma 1.3.6 *The fibre $E = \{A \in \overline{\mathcal{O}}_{[2^2,1^2]} \mid \mathrm{im}\, A^t \subset W_0\}$ is one–dimensional.*

Proof. Let $A^t = (a_{ij})$, i.e. $A^t p_i = \sum a_{ji} p_j$. We have

- $\mathrm{im}\, A^t \subset W_0 = \langle p_1, p_2 \rangle$, thus $a_{ij} = 0$ if $i = 3, 4, 5, 6$,
- by duality, $W_0^{\perp} = \langle p_1, p_2, p_3, p_6 \rangle \subset \ker A^t$, which implies $a_{ij} = 0$ if $j = 1, 2, 3, 6$.

There only remain a_{14}, a_{24}, a_{15} and a_{25}. But

- $A^t \in \mathfrak{so}_6$ implies $a_{14} = a_{25} = 0$ and $a_{24} = -a_{15}$.

Thus E is isomorphic to $\mathbb{A}^1_{\mathbb{C}}$. □

Connecting this to the Hilbert scheme, we have

$$\begin{tikzcd} & \mu^{-1}(0)/\!\!/Sl_2 & \\ Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb} \arrow[ur, "\eta"] \arrow[rr, "\eta\times\eta_1"] \arrow[dr, "f'=f\circ(\eta\times\eta_1)"'] & & Y \cong SO_6 \times^P E \arrow[ul, "pr_1"'] \arrow[dl, "f"] \\ & SO_6/P & \end{tikzcd}$$

The existence of f' means that $Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb}$ can be written as an associated SO_6–bundle with fibre $F := f'^{-1}([I_6])$ and combining the two SO_6–bundles we obtain

$$\begin{tikzcd} SO_6 \times^P F \arrow[rr, "\cong"] \arrow[d, "(\eta\times\eta_1)'"'] \arrow[ddr] & & Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb} \arrow[d, "\eta\times\eta_1"] \arrow[ddl, "f'"'] \\ SO_6 \times^P E \arrow[rr, "\cong"] \arrow[dr] & & Y \arrow[dl, "f"] \\ & SO_6/P & \end{tikzcd}$$

As $\eta \times \eta_1$ is birational and proper, restricting $(\eta \times \eta_1)'$ to the fibre over any point of SO_6 yields a birational and proper morphism $\psi\colon F \to E$. Since E is isomorphic to the affine line, ψ must be an isomorphism. As a consequence, we get an explicit description of $Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb}$:

Proposition 1.3.7 *The orbit component of the Sl_2–Hilbert scheme is isomorphic to Y:*

$$Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb} \cong \{(A,U) \in \overline{\mathcal{O}}_{[2^2,1^2]} \times \mathrm{Grass}_{iso}(2,\mathbb{C}^6) \mid \mathrm{im}\, A^t \subset W\}.$$

1.3.4. The points of $\mathrm{Hilb}^G_h(X)^{orb}$ as subschemes of X

To identify the points of $\mathrm{Hilb}^G_h(X)^{orb}$ as subschemes of X, we assume there is an embedding

$$\mathrm{Hilb}^G_h(X)^{orb} \hookrightarrow X/\!\!/G \times \prod_{\rho\in M} \mathrm{Grass}(F_\rho, h(\rho)),\ Z \to (Z/\!\!/G, (\mathcal{F}_\rho(Z))_{\rho\in M})$$

where $M \subset \mathrm{Irr}\, G$ is a suitable finite subset and $\mathcal{F}_\rho(Z) = F_\rho / \ker(f_{\rho,Z})$ with the restriction map $f_{\rho,Z}\colon F_\rho \to \mathcal{F}_\rho(Z)$. This embedding gives us the invariant part and the ρ–parts of the ideal I_Z of Z as

$$(I_Z)^G = I_{Z/\!\!/G}$$
$$(I_Z)_\rho = \big(\ker(f_{\rho,Z})\big).$$

Thus $I_Z \supset I_M := \langle I_{Z/\!/G}, \ker(f_{\rho,Z}) \mid \rho \in M\rangle$. If I_M already has Hilbert function h, then I_Z has no further generators and we obtain $I_Z = I_M$.

The points of Sl_2-Hilb$(\mu^{-1}(0))^{orb}$ as subschemes of $\mu^{-1}(0)$

We are now ready to prove Theorem 1.2 for the points of the orbit component. With regard to Proposition 1.4.4, the following proposition is in fact Theorem 1.2:

Proposition 1.3.8 *The subscheme $Z_{A,W} \subset \mu^{-1}(0)$ corresponding to $(A,W) \in Y$ is*

$$Z_{A,W} \cong \begin{cases} Sl_2, & \text{if } A \in \mathcal{O}_{[2^2,1^2]}, \\ \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \middle| ad - bc = 0\right\}, & \text{if } A = 0. \end{cases}$$

Proof. Considering $\eta \times \eta_1$, which embeds the orbit component Sl_2-Hilb$(\mu^{-1}(0))^{orb}$ into $\mu^{-1}(0)/\!/Sl_2 \times \mathrm{Grass}(2,\mathbb{C}^6)$ via $Z \mapsto (Z/\!/Sl_2, \ker(f_{1,Z})^\perp)$, we have to compute $Z_{A,W} = (\eta \times \eta_1)^{-1}(A,W)$ or its ideal $I_{A,W}$. The action of SO_6 on the Hilbert scheme and on Y reduces this to the calculation of one $Z_{A,W}$ for every orbit of Y: Since $\eta \times \eta_1$ is SO_6–equivariant, all points in the preimage of one orbit are isomorphic. Y decomposes into two SO_6–orbits $\{(A, \mathrm{im}\, A^t) \mid A \in \mathcal{O}_{[2^2,1^2]}\} \cong \mathcal{O}_{[2^2,1^2]}$ and $\{0\} \times \mathrm{Grass}_{iso}(2,\mathbb{C}^6)$, because the action on $\mathrm{Grass}_{iso}(2,\mathbb{C}^6)$ is transitive.
First we consider $A \in \mathcal{O}_{[2^2,1^2]}$. Since η is an isomorphism of schemes over the flat locus $\mathcal{O}_{[2^2,1^2]}$, we already know that $Z_{A,W} = \eta^{-1}(A) = \nu^{-1}(A) \cong Sl_2$ by Section 1.2.2.
Now let $A \in \overline{\mathcal{O}_{[2^2,1^2]}} \setminus \mathcal{O}_{[2^2,1^2]} = \{0\}$. Then $Z_{0,W}/\!/Sl_2 = 0$, so all 2×2–minors of elements in $Z_{0,W}$ vanish, i.e. $(I_{0,W})^{Sl_2} = (\Lambda^{i,j} \mid i,j = 1,\dots,6)$.
We calculate the subscheme $Z_{0,W}$ explicitly for $W = W_0 = \langle p_1, p_2\rangle$. Consider the map $f_{1,Z_{0,W_0}} \colon F_1 \to \mathcal{F}_1(Z_{0,W_0})$, $q \mapsto q|_{Z_{0,W_0}}$. We know that $W_0 = \ker(f_{1,Z_{0,W_0}})^\perp$. If $q = \sum_{i=1}^6 a_i p_i \in \ker(f_{1,Z_{0,W_0}})$, we have $0 = q(M) = \sum_{i=1}^6 a_i \binom{x_{1i}}{x_{2i}}$ for every $M \in Z_{0,W_0}$. Thus, the component of I_{0,W_0} corresponding to the standard representation is $(I_{0,W_0})_1 = (\sum_{i=1}^6 a_i x_{1i},\ \sum_{i=1}^6 a_i x_{2i} \mid q \in W_0^\perp)$ and for the induced subscheme $Z'_{0,W_0} := \mathrm{Spec}(\mathbb{C}[\mu^{-1}(0)]/((I_{0,W_0})^{Sl_2} + (I_{0,W_0})_1)) \supset Z_{0,W_0}$ we have

$$Z'_{0,W_0} = \left\{ M \in (\mathbb{C}^2)^{\oplus 6} \,\middle|\, \begin{array}{l} MQM^t = 0, \Lambda^{i,j} = 0 \;\forall\, i,j, \\ \sum_{i=1}^6 a_i x_{1i} = 0 = \sum_{i=1}^6 a_i x_{2i} \;\forall\, q \in W_0^\perp \end{array} \right\}.$$

In our case, $W_0^\perp = \langle p_1, p_2, p_3, p_6\rangle$, thus letting q be each of these generators yields the equations $x_{1i} = 0 = x_{2i}$ if $i = 1,2,3,6$. This means that in Z'_{0,W_0} we have $M =$

$\left(\begin{smallmatrix} 0 & 0 & 0 & x_{14} & x_{15} & 0 \\ 0 & 0 & 0 & x_{24} & x_{25} & 0 \end{smallmatrix}\right)$ and $0 = \Lambda^{45} = x_{14}x_{25} - x_{15}x_{24}$. Then the equation $MQM^t = 0$ is automatically fulfilled. So we obtain

$$Z'_{0,W_0} = \left\{ \begin{pmatrix} 0 & 0 & 0 & x_{14} & x_{15} & 0 \\ 0 & 0 & 0 & x_{24} & x_{25} & 0 \end{pmatrix} \in (\mathbb{C}^2)^{\oplus 6} \middle| x_{14}x_{25} - x_{15}x_{24} = 0 \right\}.$$

Since this is a flat deformation of Sl_2, the corresponding ideal has the correct Hilbert function, which means that we obtain $I_{0,W_0} = ((I_{0,W_0})^{Sl_2} + (I_{0,W_0})_1)$ and $Z_{0,W_0} = Z'_{0,W_0}$. □

1.4. Properties of the invariant Hilbert scheme

Up to now we have characterised the orbit component of the Sl_2–Hilbert scheme only. To complete the description of the Sl_2–Hilbert scheme, we analyse some of its properties. We prove that $Sl_2\text{-Hilb}(\mu^{-1}(0))$ is smooth at every point of $Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb}$ in Section 1.4.1, so that the orbit component is a smooth connected component of $Sl_2\text{-Hilb}(\mu^{-1}(0))$. By showing that $Sl_2\text{-Hilb}(\mu^{-1}(0))$ is connected and hence coincides with the orbit component, Section 1.4.2 concludes the proof of Theorem 1.1, namely $Sl_2\text{-Hilb}(\mu^{-1}(0)) = \{(A, W) \in \overline{\mathcal{O}}_{[2^2,1^2]} \times \text{Grass}_{iso}(2, \mathbb{C}^6) \mid \text{im}\, A^t \subset W\}$.

1.4.1. Smoothness

One way to examine smoothness of the Hilbert scheme is to calculate its tangent space. If the dimension of the tangent space equals the dimension of the Hilbert scheme at every point of the orbit component, the latter is smooth and one concludes that there is no additional component of the invariant Hilbert scheme intersecting it, so $\text{Hilb}_h^G(X)^{orb}$ is a connected component of the invariant Hilbert scheme.

Let $Z \in \text{Hilb}_h^G(X)$, $R := \Gamma(X, \mathcal{O}_X)$ and $\mathcal{I}_Z$ be the ideal of Z in $\mathcal{O}_X$ with space of global sections I_Z. Here is a formula to compute the tangent space of the invariant Hilbert scheme at the point Z:

Proposition 1.4.1 *[AB05, Proposition 1.13] The tangent space of the invariant Hilbert scheme is given by*

$$\begin{aligned} T_Z \text{Hilb}_h^G(X) &= \text{Hom}_R(I_Z, R/I_Z)^G = \text{Hom}_{R/I_Z}(I_Z/I_Z^2, R/I_Z)^G \\ &= H^0(\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z))^G. \end{aligned}$$

Remark 1.4.2 Consider the regular part Z_{reg} of Z. If Z is reduced, restricting morphisms to Z_{reg} yields injections

$$\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_{Z_{reg}}}(\mathcal{I}_{Z_{reg}}/\mathcal{I}_{Z_{reg}}^2, \mathcal{O}_{Z_{reg}}) \qquad \text{and}$$
$$\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z)^G \hookrightarrow \mathrm{Hom}_{\mathcal{O}_{Z_{reg}}}(\mathcal{I}_{Z_{reg}}/\mathcal{I}_{Z_{reg}}^2, \mathcal{O}_{Z_{reg}})^G.$$

Taking global sections we obtain

$$\mathrm{Hom}_{R/I_Z}(I_Z/I_Z^2, R/I_Z)^G \hookrightarrow H^0(Z_{reg}, \mathcal{H}om_{\mathcal{O}_{Z_{reg}}}(\mathcal{I}_{Z_{reg}}/\mathcal{I}_{Z_{reg}}^2, \mathcal{O}_{Z_{reg}}))^G.$$

All these maps are isomorphisms if Z is normal. In this case one can determine the global sections of the normal sheaf $(\mathcal{I}_{Z_{reg}}/\mathcal{I}_{Z_{reg}}^2)^\vee = \mathcal{H}om_{\mathcal{O}_{Z_{reg}}}(\mathcal{I}_{Z_{reg}}/\mathcal{I}_{Z_{reg}}^2, \mathcal{O}_{Z_{reg}})$ in order to obtain a description of the tangent space.

The tangent space of Sl_2-Hilb$(\mu^{-1}(0))$

In the case of Sl_2, we use the previous description of the tangent space of the invariant Hilbert scheme in order to show that the orbit component of Sl_2-Hilb$(\mu^{-1}(0))$ is smooth and connected:

Proposition 1.4.3 *For every point $Z \in Sl_2$-Hilb$(\mu^{-1}(0))^{orb}$ the dimension of the tangent space is*

$$\dim T_Z Sl_2\text{-Hilb}(\mu^{-1}(0)) = 6 = \dim Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb}.$$

Therefore, the orbit component is a smooth connected component of the invariant Hilbert scheme.

Proof. As before, we only have to consider one point of each SO_6–orbit because the dimension of the tangent space is stable in every orbit of the SO_6–action. Over the open orbit there is nothing to show, because we know that $\eta^{-1}(\mathcal{O}_{[2^2,1^2]}) \cong \mathcal{O}_{[2^2,1^2]}$ is smooth. Over the origin we consider

$$\begin{aligned} Z := Z_{0,W_0} &= \left\{ \begin{pmatrix} 0 & 0 & 0 & x_{14} & x_{15} & 0 \\ 0 & 0 & 0 & x_{24} & x_{25} & 0 \end{pmatrix} \middle| x_{14}x_{25} - x_{15}x_{24} = 0 \right\} \\ &\cong \left\{ \begin{pmatrix} \lambda x & \lambda y \\ \mu x & \mu y \end{pmatrix} \middle| x, y \in \mathbb{C}, [\lambda : \mu] \in \mathbb{P}^1 \right\}. \end{aligned}$$

Our strategy for computing the dimension of the tangent space at this point is the following: First, we give an explicit description of the ideal $\mathcal{I}$ of Z and of the vector space $\mathbb{C}[\mu^{-1}(0)]/\mathcal{I}$ and of the dual of the normal sheaf $\mathcal{I}/\mathcal{I}^2$. The scheme Z is normal since it is a complete intersection and the codimension of $Z \setminus Z_{reg} = \{0\}$ in Z is greater than 2, namely 3. Hence using Remark 1.4.2 we reduce the computation of the tangent space to the examintation of Z_{reg}. We give an explicit description of the structure sheaf and the normal sheaf of this non–affine scheme on an open covering. In order to simplify this, we further reduce from the consideration of Sl_2–linearised sheaves on Z_{reg} to B–linearised sheaves on $\mathbb{C}^2 \setminus \{0\}$ for a Borel subgroup B of Sl_2. After describing the B–linearised sheaves corresponding to the structure sheaf and the normal sheaf of Z_{reg} on an open covering of $\mathbb{C}^2 \setminus \{0\}$, we compute their global sections. Finally, the number of B–invariants of these global sections is the dimension of the tangent space.

Explicit description of the structure sheaf and the dual of the normal sheaf of Z

We have $Z \subset \mu^{-1}(0)_{sing}$: If $M \in Z$ then all of its 2×2–minors vanish. This shows that $M \in V(X^tJX) = \mu^{-1}(0)_{sing}$, where $X = \left(\begin{smallmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \end{smallmatrix}\right)$ describes the coordinates in $\mathbb{C}[x_{11}, \dots, x_{26}]$.

From now on we also write $a := x_{14}$, $b := x_{15}$, $c := x_{24}$ and $d := x_{25}$. Let $\mathcal{I}$ be the ideal of Z in $R := \mathbb{C}[\mu^{-1}(0)] = \mathbb{C}[x_{11}, \dots, x_{26}]/(XQX^t)$. Setting $z := x_{14}x_{25} - x_{15}x_{24}$ we have

$$\mathcal{I} = (x_{11}, x_{12}, x_{13}, x_{16}, x_{21}, x_{22}, x_{23}, x_{26}, z),$$
$$R/\mathcal{I} = \mathbb{C}[a, b, c, d]/(ad - bc).$$

Then $\mathcal{I}/\mathcal{I}^2 = R\langle x_{11}, x_{12}, x_{13}, x_{16}, x_{21}, x_{22}, x_{23}, x_{26}, z\rangle$ with relations $XQX^t = 0$:

$$\begin{aligned} 0 &= x_{11}x_{14} + x_{12}x_{15} + x_{13}x_{16} \equiv x_{11}a + x_{12}b && \mod \mathcal{I}^2 \\ 0 &= x_{11}x_{24} + x_{12}x_{25} + x_{13}x_{26} + x_{14}x_{21} + x_{15}x_{22} + x_{16}x_{23} \\ &\equiv x_{11}c + x_{12}d + x_{21}a + x_{22}b && \mod \mathcal{I}^2 \\ 0 &= x_{21}x_{24} + x_{22}x_{25} + x_{23}x_{26} \equiv x_{21}c + x_{22}d && \mod \mathcal{I}^2. \end{aligned}$$

Reduction to Z_{reg}

We analyse the tangent space $T_Z Sl_2\text{-Hilb}(\mu^{-1}(0))$ of the invariant Hilbert scheme by reducing to

$$\dot{Z} := Z_{reg} = Z \setminus \{0\} = \{(\lambda v, \mu v) \mid v \in \mathbb{C}^2 \setminus \{0\}, [\lambda : \mu] \in \mathbb{P}^1\}.$$

Let $\dot{\mathcal{I}}$ be the ideal sheaf of $\dot{Z}$. As Z is normal, by Remark 1.4.2 we have

$$\begin{aligned} T_Z Sl_2\text{-Hilb}(\mu^{-1}(0)) &= H^0(Z, \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z))^{Sl_2} \\ &\cong H^0(\dot{Z}, \mathcal{H}om_{\mathcal{O}_{\dot{Z}}}(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2, \mathcal{O}_{\dot{Z}}))^{Sl_2}. \end{aligned}$$

Consider the covering of $\dot{Z}$ by the open affine sets $\dot{Z}_a = \operatorname{Spec} R_a$, $\dot{Z}_b = \operatorname{Spec} R_b$, $\dot{Z}_c = \operatorname{Spec} R_c$ and $\dot{Z}_d = \operatorname{Spec} R_d$, where

$$\begin{aligned} R_a &= (\mathbb{C}[a,b,c,d]/(ad-bc))_a = \mathbb{C}[a,a^{-1},b,c,d]/(ad-bc) = \mathbb{C}[a,a^{-1},b,c], \\ R_b &= (\mathbb{C}[a,b,c,d]/(ad-bc))_b = \mathbb{C}[a,b,b^{-1},d], \\ R_c &= (\mathbb{C}[a,b,c,d]/(ad-bc))_c = \mathbb{C}[a,c,c^{-1},d], \\ R_d &= (\mathbb{C}[a,b,c,d]/(ad-bc))_d = \mathbb{C}[b,c,d,d^{-1}]. \end{aligned}$$

In order to describe the sheaf $\dot{\mathcal{I}}/\dot{\mathcal{I}}^2$, we compute it on each set of this covering. As $\dot{\mathcal{I}} = \mathcal{I}|_{\dot{Z}}$ and $\mathcal{I}$ coincide on an open subset, $\dot{\mathcal{I}}/\dot{\mathcal{I}}^2$ is generated by x_{11}, x_{12}, x_{13}, x_{16}, x_{21}, x_{22}, x_{23}, x_{26}, z with relations

$$\begin{aligned} 0 &= x_{11}a + x_{12}b \\ 0 &= x_{11}c + x_{12}d + x_{21}a + x_{22}b \\ 0 &= x_{21}c + x_{22}d. \end{aligned}$$

Since a is invertible in R_a, the first relation yields $x_{11} = -\frac{b}{a}x_{12}$. The second relation becomes $0 = -\frac{b}{a}x_{12}c + x_{12}\frac{bc}{a} + x_{21}a + x_{22}b = x_{21}a + x_{22}b$, thus $x_{21} = -\frac{b}{a}x_{22}$. Then the third equation $0 = -\frac{b}{a}x_{22}c + x_{22}\frac{bc}{a}$ is automatically fulfilled and gives no more information. Denoting $\dot{\mathcal{I}}_a := \dot{\mathcal{I}}|_{\dot{Z}_a}$, this shows that

$$\dot{\mathcal{I}}_a/\dot{\mathcal{I}}_a^2 = R_a\langle x_{12}, x_{13}, x_{16}, x_{22}, x_{23}, x_{26}, z\rangle$$

is free of rank 7. This means that $\dot{\mathcal{I}}/\dot{\mathcal{I}}^2$ is locally free of rank 7, since we obtain analogously

$$\begin{aligned} \dot{\mathcal{I}}_b/\dot{\mathcal{I}}_b^2 &= R_b\langle x_{11}, x_{13}, x_{16}, x_{21}, x_{23}, x_{26}, z\rangle, \\ \dot{\mathcal{I}}_c/\dot{\mathcal{I}}_c^2 &= R_c\langle x_{12}, x_{13}, x_{16}, x_{22}, x_{23}, x_{26}, z\rangle, \\ \dot{\mathcal{I}}_d/\dot{\mathcal{I}}_d^2 &= R_d\langle x_{11}, x_{13}, x_{16}, x_{21}, x_{23}, x_{26}, z\rangle. \end{aligned}$$

Let $\dot{Z}_{ab} = \operatorname{Spec} R_{ab}$. We obtain

$$\begin{aligned} R_{ab} &= \mathbb{C}[a,a^{-1},b,b^{-1},c,d]/(ad-bc) = \mathbb{C}[a,a^{-1},b,b^{-1},c] = \mathbb{C}[a,a^{-1},b,b^{-1},d], \\ \dot{\mathcal{I}}_{ab}/\dot{\mathcal{I}}_{ab}^2 &= \mathbb{C}[a,a^{-1},b,b^{-1},c]\langle x_{12}, x_{13}, x_{16}, x_{22}, x_{23}, x_{26}, z\rangle \\ &= \mathbb{C}[a,a^{-1},b,b^{-1},d]\langle x_{11}, x_{13}, x_{16}, x_{21}, x_{23}, x_{26}, z\rangle \end{aligned}$$

with $d = \frac{b}{a}c$ and base change $x_{11} = -\frac{b}{a}x_{12}$ and $x_{21} = -\frac{b}{a}x_{22}$.

Reduction of Sl_2–linearised sheaves to sheaves linearised with respect to a Borel subgroup

To compute $H^0(\dot{Z}, \mathcal{H}om(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2, \mathcal{O}_{\dot{Z}}))^{Sl_2}$, we reduce the Sl_2–linearised sheaf $\dot{\mathcal{I}}/\dot{\mathcal{I}}^2$ on $\dot{Z}$ to a B–linearised sheaf on $\mathbb{C}^2 \setminus \{0\}$, where $B = \left\{ \left(\begin{smallmatrix} t & u \\ 0 & t^{-1} \end{smallmatrix} \right) \middle| \, t \in \mathbb{C}^*, u \in \mathbb{C} \right\}$ is the Borel subgroup of upper triangular matrices of Sl_2.

Claim *$\dot{Z}$ is an associated Sl_2–bundle:*

$$\dot{Z} \cong Sl_2 \times^B E, \quad \textit{where} \;\; E = \{(\lambda e_1, \mu e_1) \mid [\lambda : \mu] \in \mathbb{P}^1\} \cong \mathbb{C}^2 \setminus \{0\} \;\textit{and}\; e_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right).$$

Proof. There is a natural map

$$\varphi \colon \dot{Z} \to \mathbb{P}^1 \times \mathbb{P}^1, \; (\lambda v, \mu v) \mapsto ([v], [\lambda : \mu]).$$

Since $g \cdot (\lambda v, \mu v) = (\lambda g v, \mu g v)$ for every $g \in Sl_2$, the map φ is equivariant for the action $g \cdot ([v], [\lambda : \mu]) = ([gv], [\lambda : \mu])$ on $\mathbb{P}^1 \times \mathbb{P}^1$. This yields an equivariant projection

$$\pi \colon \dot{Z} \to \mathbb{P}^1, \; (\lambda v, \mu v) \mapsto [v].$$

Further, there is an isomorphism $Sl_2/B \xrightarrow{\cong} \mathbb{P}^1$, $\left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right) \cdot B \mapsto [g_{11} : g_{21}]$.
We have $E = \pi^{-1}([e_1])$. The action of B on E induced by the action of Sl_2 on $\dot{Z}$ is $b \cdot (\lambda e_1, \mu e_1) = (t\lambda e_1, t\mu e_1)$. Thus on $\mathbb{C}^2 \setminus \{0\}$ we have $b \cdot (\lambda, \mu) = (t\lambda, t\mu)$, i.e. the action of B on $\mathbb{C}^2 \setminus \{0\}$ coincides with the action of $\mathbb{C}^*$. This proves the claim.

Now an Sl_2–linearised sheaf $\mathcal{F}$ on $\dot{Z}$ corresponds to a B–linearised sheaf $\mathcal{G}$ on $\mathbb{C}^2 \setminus \{0\}$ as well as their duals correspond to each other. If $j \colon \mathbb{C}^2 \setminus \{0\} \hookrightarrow \dot{Z}$ denotes the inclusion and $e = I_2 \cdot B \in Sl_2/B \cong \mathbb{P}^1$ we obtain $\mathcal{G}$ as the fibre $\mathcal{F}(e) = j^*\mathcal{F}$. In the other direction we have $\mathcal{F} = Sl_2 \times^B \mathcal{G}$.
The invariant global sections of corresponding sheaves coincide:

$$H^0(\dot{Z}, \mathcal{H}om_{\mathcal{O}_{\dot{Z}}}(\mathcal{F}, \mathcal{O}_{\dot{Z}}))^{Sl_2} = H^0(\mathbb{C}^2 \setminus \{0\}, \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^2 \setminus \{0\}}}(\mathcal{G}, \mathcal{O}_{\mathbb{C}^2 \setminus \{0\}}))^B.$$

We take $\mathcal{F} = \mathcal{I}/\mathcal{I}^2$ and are interested in determining the dual of $j^*\mathcal{F}$.
As $\mathcal{O}_{\mathbb{C}^2} = \mathbb{C}[\lambda, \mu]$ and $\mathbb{C}^2 \setminus \{0\} = \mathbb{C}^2 \setminus \{0\}_\lambda \cup \mathbb{C}^2 \setminus \{0\}_\mu$, the structure sheaf is given by

$$\mathcal{O}_{\mathbb{C}^2 \setminus \{0\}}(\mathbb{C}^2 \setminus \{0\}_\lambda) = \mathbb{C}[\lambda, \lambda^{-1}, \mu],$$
$$\mathcal{O}_{\mathbb{C}^2 \setminus \{0\}}(\mathbb{C}^2 \setminus \{0\}_\mu) = \mathbb{C}[\lambda, \mu, \mu^{-1}].$$

In our case the inclusion is $j\colon \mathbb{C}^2 \setminus \{0\} \to \dot{Z},\ (\lambda,\mu) \mapsto \left(\begin{smallmatrix} \lambda & \mu \\ 0 & 0 \end{smallmatrix}\right)$, so on the level of rings we have $a \mapsto \lambda$, $b \mapsto \mu$, $c \mapsto 0$ and $d \mapsto 0$. This means that $j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)$ is given by

$$\begin{aligned} j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)(\mathbb{C}^2 \setminus \{0\}_\lambda) &= \mathbb{C}[\lambda,\lambda^{-1},\mu]\langle x_{12}, x_{13}, x_{16}, x_{22}, x_{23}, x_{26}, z\rangle, \\ j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)(\mathbb{C}^2 \setminus \{0\}_\mu) &= \mathbb{C}[\lambda,\mu,\mu^{-1}]\langle x_{11}, x_{13}, x_{16}, x_{21}, x_{23}, x_{26}, z\rangle, \\ j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)(\mathbb{C}^2 \setminus \{0\}_{\lambda\mu}) &= \mathbb{C}[\lambda,\lambda^{-1},\mu,\mu^{-1}]\langle x_{12}, x_{13}, x_{16}, x_{22}, x_{23}, x_{26}, z\rangle \\ &= \mathbb{C}[\lambda,\lambda^{-1},\mu,\mu^{-1}]\langle x_{11}, x_{13}, x_{16}, x_{21}, x_{23}, x_{26}, z\rangle \end{aligned}$$

with base change $x_{11} = -\frac{\mu}{\lambda}x_{12}$ and $x_{21} = -\frac{\mu}{\lambda}x_{22}$.

To compute the dual $j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee = \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^2\setminus\{0\}}}(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2, \mathcal{O}_{\mathbb{C}^2\setminus\{0\}})$, denote by (y_{ij}, w) the basis dual to (x_{ij}, z), i.e. $y_{ij}(x_{kl}) = \delta_{(ij)(kl)}$, $y_{ij}(z) = 0$, $w(x_{ij}) = 0$, $w(z) = 1$. Then we have

$$\begin{aligned} j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee(\mathbb{C}^2 \setminus \{0\}_\lambda) &= \mathbb{C}[\lambda,\lambda^{-1},\mu]\langle y_{12}, y_{13}, y_{16}, y_{22}, y_{23}, y_{26}, w\rangle, \\ j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee(\mathbb{C}^2 \setminus \{0\}_\mu) &= \mathbb{C}[\lambda,\mu,\mu^{-1}]\langle y_{11}, y_{13}, y_{16}, y_{21}, y_{23}, y_{26}, w\rangle, \\ j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee(\mathbb{C}^2 \setminus \{0\}_{\lambda\mu}) &= \mathbb{C}[\lambda,\lambda^{-1},\mu,\mu^{-1}]\langle y_{12}, y_{13}, y_{16}, y_{22}, y_{23}, y_{26}, w\rangle \\ &= \mathbb{C}[\lambda,\lambda^{-1},\mu,\mu^{-1}]\langle y_{11}, y_{13}, y_{16}, y_{21}, y_{23}, y_{26}, w\rangle \end{aligned}$$

with base change $y_{11} = -\frac{\lambda}{\mu}y_{12}$ and $y_{21} = -\frac{\lambda}{\mu}y_{22}$.

Computation of the global sections

The global sections $H^0(\mathbb{C}^2 \setminus \{0\}, j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee)$ are the kernel of the map

$$\begin{aligned} H^0(\mathbb{C}^2\setminus\{0\}_\lambda, j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee) \oplus H^0(\mathbb{C}^2\setminus\{0\}_\mu, j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee) &\xrightarrow{\varphi} H^0(\mathbb{C}^2\setminus\{0\}_{\lambda\mu}, j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee), \\ (p,q) &\mapsto p|_{\mathbb{C}^2\setminus\{0\}_{\lambda\mu}} - q|_{\mathbb{C}^2\setminus\{0\}_{\lambda\mu}}. \end{aligned}$$

Let

$$\begin{aligned} p &= p_1y_{12} + p_2y_{13} + p_3y_{16} + p_4y_{22} + p_5y_{23} + p_6y_{26} + p_7w, \quad p_i \in \mathbb{C}[\lambda,\lambda^{-1},\mu], \\ q &= q_1y_{11} + q_2y_{13} + q_3y_{16} + q_4y_{21} + q_5y_{23} + q_6y_{26} + q_7w, \quad q_i \in \mathbb{C}[\lambda,\mu,\mu^{-1}]. \end{aligned}$$

Denote $p_i = \frac{p_i^N}{p_i^D}$ and $q_i = \frac{q_i^N}{q_i^D}$ with $p_i^N, q_i^N \in \mathbb{C}[\lambda,\mu]$, $p_i^D \in \mathbb{C}[\lambda]$ and $q_i^D \in \mathbb{C}[\mu]$, p_i^N, p_i^D relatively prime, as well as q_i^N, q_i^D. In $\mathbb{C}[\lambda,\lambda^{-1},\mu,\mu^{-1}]$ we have

$$q = -\frac{\lambda}{\mu}q_1y_{12} + q_2y_{13} + q_3y_{16} - \frac{\lambda}{\mu}q_4y_{22} + q_5y_{23} + q_6y_{26} + q_7w.$$

Thus if $i \in \{2,3,5,6,7\}$, for p and q to be equal in $\mathbb{C}[\lambda,\lambda^{-1},\mu,\mu^{-1}]$ we must have $p_i = q_i$, i.e. $p_i^N \cdot q_i^D = p_i^D \cdot q_i^N$. As p_i^N and p_i^D have no common factor, p_i^D must divide q_i^D. But

p_i^D is a polynomial in λ while q_i^D is a polynomial in μ. This forces p_i^D to be constant, without loss of generality $p_i^D = 1$. This immediately implies $q_i^D = 1$ since q_i^N and q_i^D are coprime. We obtain $p_i^N = p_i = q_i = q_i^N \in \mathbb{C}[\lambda, \mu]$.
If $i = 1$ or 4, we see $p_i = -\frac{\lambda}{\mu} q_i$, or $\mu p_i = -\lambda q_i$. Thus $p_i^N = \lambda \tilde{p}_i^N$, $q_i^N = -\mu \tilde{p}_i^N$ with some $\tilde{p}_i^N \in \mathbb{C}[\lambda, \mu]$ and $p_i^D = 1 = q_i^D$ as before. This yields

$$\begin{aligned}
&H^0(\mathbb{C}^2 \setminus \{0\}, \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^2\setminus\{0\}}}(j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2), \mathcal{O}_{\mathbb{C}^2\setminus\{0\}})) = \ker\varphi \\
&= \{(\lambda p_1 y_{12} + p_2 y_{13} + p_3 y_{16} + \lambda p_4 y_{22} + p_5 y_{23} + p_6 y_{26} + p_7 w, \\
&\quad - \mu p_1 y_{11} + p_2 y_{13} + p_3 y_{16} - \mu p_4 y_{21} + p_5 y_{23} + p_6 y_{26} + p_7 w) \mid p_i \in \mathbb{C}[\lambda, \mu]\} \\
&= \mathbb{C}[\lambda, \mu]\langle \lambda y_{12}, y_{13}, y_{16}, \lambda y_{22}, y_{23}, y_{26}, w \rangle,
\end{aligned}$$

which is a free module of rank 7.

Computation of invariants

Let us now consider the actions of Sl_2 and B on these modules. Let $g = \left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right)$. Then

$$\begin{aligned}
g \cdot x_{1i} &= g_{11} x_{1i} + g_{12} x_{2i}, & g \cdot x_{2i} &= g_{21} x_{1i} + g_{22} x_{2i}, \\
g \cdot a &= g_{11} a + g_{12} c, & g \cdot c &= g_{21} a + g_{22} c, \\
g \cdot b &= g_{11} b + g_{12} d, & g \cdot d &= g_{21} b + g_{22} d,
\end{aligned}$$

$$\begin{aligned}
g \cdot z &= g(x_{14} x_{25} - x_{15} x_{24}) \\
&= (g_{11} x_{14} + g_{12} x_{24})(g_{21} x_{15} + g_{22} x_{25}) - (g_{11} x_{15} + g_{12} x_{25})(g_{21} x_{14} + g_{22} x_{24}) \\
&= (g_{11} g_{22} - g_{12} g_{21})(x_{14} x_{25} - x_{15} x_{24}) = z.
\end{aligned}$$

The action on the dual is determined by

$$\begin{aligned}
&g \cdot y_{1i}(x_{1i}) = y_{1i}(g^{-1} x_{1i}) = y_{1i}(g_{22} x_{1i} - g_{12} x_{2i}) = g_{22}, \\
&g \cdot y_{1i}(x_{2i}) = y_{1i}(g^{-1} x_{2i}) = y_{1i}(-g_{21} x_{1i} + g_{11} x_{2i}) = -g_{21} \\
&\qquad \Rightarrow \quad g \cdot y_{1i} = g_{22} y_{1i} - g_{21} y_{2i}, \\
&g \cdot y_{2i}(x_{1i}) = y_{2i}(g^{-1} x_{1i}) = y_{2i}(g_{22} x_{1i} - g_{12} x_{2i}) = -g_{12}, \\
&g \cdot y_{2i}(x_{2i}) = y_{2i}(g^{-1} x_{2i}) = y_{2i}(-g_{21} x_{1i} + g_{11} x_{2i}) = g_{11} \\
&\qquad \Rightarrow \quad g \cdot y_{2i} = -g_{12} y_{1i} + g_{11} y_{2i}, \\
&g \cdot w(z) = w(g^{-1} z) = w(z) \qquad \Rightarrow \quad g \cdot w = w.
\end{aligned}$$

Correspondingly, over $\mathbb{C}^2 \setminus \{0\}$, the action of $g = \left(\begin{smallmatrix} t & u \\ 0 & t^{-1} \end{smallmatrix}\right)$ is

$$\begin{array}{lll} g \cdot \lambda = t\lambda, & g \cdot x_{1i} = tx_{1i} + ux_{2i}, & g \cdot y_{1i} = t^{-1}y_{1i}, \\ g \cdot \mu = t\mu, & g \cdot x_{2i} = t^{-1}x_{2i}, & g \cdot y_{2i} = -uy_{1i} + ty_{2i}, \\ & g \cdot z = z, & g \cdot w = w. \end{array}$$

Considering the decomposition $B = TU$ with torus $T = \left\{\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)\right\}$ and unipotent radical $U = \{\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\}$, we can compute the B–invariants stepwise:

$$\mathbb{C}[\lambda, \mu]\langle \lambda y_{12}, y_{13}, y_{16}, \lambda y_{22}, y_{23}, y_{26}, w\rangle^B = (\mathbb{C}[\lambda, \mu]\langle \lambda y_{12}, y_{13}, y_{16}, \lambda y_{22}, y_{23}, y_{26}, w\rangle^U)^T.$$

Let $\underline{u} = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$. We have

$$\left.\begin{array}{ll} \underline{u} \cdot \lambda = \lambda, & \underline{u} \cdot \lambda y_{12} = \lambda y_{12}, \\ \underline{u} \cdot \mu = \mu, & \underline{u} \cdot y_{13} = y_{13}, \\ \underline{u} \cdot w = w, & \underline{u} \cdot y_{16} = y_{16}, \end{array}\right\} \text{invariants}$$

$$\left.\begin{array}{l} \underline{u} \cdot \lambda y_{22} = \lambda(-uy_{12} + y_{22}) = -u\lambda y_{12} + \lambda y_{22}, \\ \underline{u} \cdot y_{23} = -uy_{13} + y_{23}, \\ \underline{u} \cdot y_{26} = -uy_{16} + y_{26} \end{array}\right\} \begin{array}{l} \text{cannot be combined} \\ \text{to form invariants.} \end{array}$$

So we gain

$$\mathbb{C}[\lambda, \mu]\langle \lambda y_{12}, y_{13}, y_{16}, \lambda y_{22}, y_{23}, y_{26}, w\rangle^U = \mathbb{C}[\lambda, \mu]\langle \lambda y_{12}, y_{13}, y_{16}, w\rangle.$$

To compute the T–invariants, let $\underline{t} = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$. We obtain

$$\begin{array}{lll} \text{degree 1:} & \text{invariants:} & \text{degree } -1: \\ \underline{t} \cdot \lambda = t\lambda, & \underline{t} \cdot w = w, & \underline{t} \cdot y_{13} = t^{-1}y_{13}, \\ \underline{t} \cdot \mu = t\mu, & \underline{t} \cdot \lambda y_{12} = t\lambda t^{-1} y_{12} = \lambda y_{12}, & \underline{t} \cdot y_{16} = t^{-1}y_{16}. \end{array}$$

This yields the invariants w, λy_{12}, λy_{13}, μy_{13}, λy_{16} and μy_{16}. So we have computed

$$\begin{aligned} H^0(\dot{Z}, \mathrm{Hom}(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2, \mathcal{O}_{\dot{Z}}))^{Sl_2} &= H^0(\mathbb{C}^2 \setminus \{0\}, \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathbb{C}^2\setminus\{0\}}))^B \\ &= \mathbb{C}\langle \lambda y_{12}, \lambda y_{13}, \mu y_{13}, \lambda y_{16}, \mu y_{16}, w\rangle. \end{aligned}$$

This means that $T_Z Sl_2$-Hilb$(\mu^{-1}(0))$ is 6–dimensional and therefore the orbit component of the invariant Hilbert scheme is a smooth connected component. □

1.4.2. Connectedness

To examine connectedness we look at $\mathbb{C}^*$–actions:
If there is a $\mathbb{C}^*$–action on X which commutes with the G–action, it descends to a $\mathbb{C}^*$–action on $X/\!\!/G$ so that the quotient map $X \to X/\!\!/G$ is $\mathbb{C}^*$–equivariant. In this case, one way to investigate whether the invariant Hilbert scheme is connected is to compute the induced $\mathbb{C}^*$–action on $\mathrm{Hilb}_h^G(X)$ and to determine all fixed points of $\mathbb{C}^*$ in $X/\!\!/G$. The Hilbert–Chow morphism is proper and $\mathbb{C}^*$–equivariant, therefore for every fixed point x in the image there is at least one fixed point in every connected component of the fibre $\eta^{-1}(x)$.

Remark. Let $(X/\!\!/G)_*$ denote the flat locus of the quotient map. Since $\eta|_{\eta^{-1}((X/\!\!/G)_*)}$ is an isomorphism, every irreducible component of the invariant Hilbert scheme different from $\mathrm{Hilb}_h^G(X)^{orb} = \overline{\eta^{-1}((X/\!\!/G)_*)}$ only contains points of the fibres over $X/\!\!/G \setminus (X/\!\!/G)_*$. If one can show that all connected components of these fibres meet the orbit component, and additionally one knows the orbit component to be smooth, then there cannot be any further component. In this case $\mathrm{Hilb}_h^G(X) = \mathrm{Hilb}_h^G(X)^{orb}$ is connected.

Connectedness of $Sl_2\text{-Hilb}(\mu^{-1}(0))$

The next proposition shows that $Sl_2\text{-Hilb}(\mu^{-1}(0))$ is connected. This is the remaining step to conclude the proof of Theorem 1.1 because we have already shown that the connected component $Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb}$ is smooth.

Proposition 1.4.4 *The invariant Hilbert scheme $Sl_2\text{-Hilb}(\mu^{-1}(0))$ is connected, hence it coincides with its orbit component and we have*

$$\begin{aligned} Sl_2\text{-Hilb}(\mu^{-1}(0)) &= Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb} \\ &= \{(A, W) \in \overline{\mathcal{O}}_{[2^2,1^2]} \times \mathrm{Grass}_{iso}(2, \mathbb{C}^6) \mid \mathrm{im}\, A^t \subset W\}. \end{aligned}$$

Proof. We consider the action of $\mathbb{C}^*$ on $\mu^{-1}(0)$ by scalar multiplication and the induced action on $\mu^{-1}(0)/\!\!/Sl_2 = \overline{\mathcal{O}}_{[2^2,1^2]}$. For $t \in \mathbb{C}$ and $M \in \mu^{-1}(0)$ we have $(tM)^t J(tM)Q = t^2(M^t JMQ)$, thus the action on the quotient is multiplication with t^2. Then the only $\mathbb{C}^*$–invariant element $A \in \overline{\mathcal{O}}_{[2^2,1^2]}$ is $A = 0$, so all fixed points of $Sl_2\text{-Hilb}(\mu^{-1}(0))$ map to 0.
The induced action on $Sl_2\text{-Hilb}(\mu^{-1}(0))$ maps Z to tZ. If Z is an Sl_2–invariant subscheme of $\mu^{-1}(0)$, then tZ is also Sl_2–invariant because the action of Sl_2 commutes

with scalar multiplication. Secondly, the global sections of Z and tZ and their isotypic decompositions coincide, so indeed $tZ \in Sl_2\text{-Hilb}(\mu^{-1}(0))$.
The following Lemma shows that the set of $\mathbb{C}^*$–fixed points in $Sl_2\text{-Hilb}(\mu^{-1}(0))$ is $\mathrm{Grass}_{iso}(2, \mathbb{C}^6)$, the fibre of $Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb}$ over zero. Consequently, $\eta^{-1}(0)$ has no further components, and the same is true for $Sl_2\text{-Hilb}(\mu^{-1}(0))$. □

Lemma 1.4.5 *The set of fixed points in* $Sl_2\text{-Hilb}((\mathbb{C}^2)^{\oplus 6})$ *under the* $\mathbb{C}^*$*–action is isomorphic to the Grassmannian* $\mathrm{Grass}(2, \mathbb{C}^6)$ *of 2–dimensional subspaces of* $\mathbb{C}^6$. *Its subset of* $\mathbb{C}^*$*–fixed points in* $Sl_2\text{-Hilb}(\mu^{-1}(0))$ *is given by* $\mathrm{Grass}_{iso}(2, \mathbb{C}^6)$.

Proof. Let $Z \subset (\mathbb{C}^2)^{\oplus 6}$ be a $\mathbb{C}^*$–fixed point in $Sl_2\text{-Hilb}((\mathbb{C}^2)^{\oplus 6})$ for the first assertion and in $Sl_2\text{-Hilb}(\mu^{-1}(0))$ for the second one. Equivalently, its corresponding ideal $\mathcal{I}$ is homogeneous. Then the Hilbert–Chow morphism maps Z to 0, so all 2×2–minors of each element in Z vanish. Hence $\mathcal{I}$ contains all the 15 minors $\Lambda^{i,j}$.
Now let us analyse the homogeneous invariant ideals $\mathcal{I}$ in $R = \mathbb{C}[x_{11}, \ldots, x_{26}]$, containing all $\Lambda^{i,j}$, with isotypic decomposition $R/\mathcal{I} \cong \bigoplus_{d \in \mathbb{N}_0} V_d^{\oplus(d+1)}$, where $V_d = \mathbb{C}[x, y]_d$. Afterwards we will restrict to ideals containing XQX^t, which are the fixed points of $Sl_2\text{-Hilb}(\mu^{-1}(0))$.
The representation $(\mathbb{C}^2)^{\oplus 6} = \mathrm{Hom}(\mathbb{C}^6, \mathbb{C}^2)$ consists of 6 copies of V_1, so that its coordinate ring R is isomorphic to $\bigoplus_{n \in \mathbb{N}_0} S^n(V_1^{\oplus 6})$. Since $R \cong \bigoplus_{n \in \mathbb{N}_0} S^n(\mathrm{Hom}(\mathbb{C}^6, \mathbb{C}^2)^*)$ is graded and $\mathcal{I}$ is homogeneous, $R/\mathcal{I}$ is still a graded object. The invariance of $\mathcal{I}$ guarantees that $\mathcal{I}_1$ is a subrepresentation of $\mathrm{Hom}(\mathbb{C}^6, \mathbb{C}^2)^*$, i.e. there is a subspace $V \subset \mathbb{C}^6$ such that $\mathcal{I}_1 = \mathrm{Hom}(V, \mathbb{C}^2)^*$. The isotypic decompostion of $R/\mathcal{I}$ requires exactly two copies of V_1, and they must already come from $R_1/\mathcal{I}_1$, since no such copy can be contributed or killed by generators of higher degree. If the dimension of V were 5 or 6 then $R_1/\mathcal{I}_1$ would consist of one or zero copies of V_1, respectively, hence it would be too small. If $\dim V \leq 3$ then $R_1/\mathcal{I}_1$ would be too big because it would contain at least three copies of V_1. Thus we know that $\dim V = 4$, so that after a transformation of coordinates we can write $\mathcal{I} \supset \mathcal{J} = (x_3, y_3, x_4, y_4, x_5, y_5, x_6, y_6, x_1y_2 - y_1x_2)$, since the other 2×2–minors $x_iy_j - y_jx_i$ do not contribute to the generation of the ideal. Then $R/\mathcal{J} \cong \mathbb{C}[x_1, y_1, x_2, y_2]/(x_1y_2 - y_1x_2)$ is the coordinate ring of a flat deformation of Sl_2 and has isotypic decomposition $\bigoplus_{d \in \mathbb{N}_0} V_d^{\oplus(d+1)}$ as desired. Hence we need no further generators and $\mathcal{I} = \mathcal{J}$.
So the fixed points in $Sl_2\text{-Hilb}((\mathbb{C}^2)^{\oplus 6})$ under the $\mathbb{C}^*$–action correspond to the choice of a 4–dimensional subspace of $\mathbb{C}^6$. Hence it is parameterised by the Grassmannian

$\mathrm{Grass}(4, \mathbb{C}^6)$, which is isomorphic to $\mathrm{Grass}(\mathbb{C}^6, 2)$ and $\mathrm{Grass}(2, \mathbb{C}^6)$.
For Z to be contained in $\mu^{-1}(0)$ we have to pick only those ideals which contain XQX^t, so that we have $MQM^t = 0$ for every $M \in Z$. We interpret $M \in (\mathbb{C}^2)^{\oplus 6}$ as a map $\mathbb{C}^6 \to \mathbb{C}^2$. The fact $M \in Z = \mathrm{Spec}(R/\mathcal{I})$ means that M vanishes on V, so we can interpret it as a map $\mathbb{C}^6/V \to \mathbb{C}^2$. As the inner product on $(\mathbb{C}^2)^{\oplus 6}$ is induced by the inner product on $\mathbb{C}^6$, the condition $MQM^t = 0$ for every $M \in Z$ is equivalent to the vanishing of v^tQv for all $v \in \mathbb{C}^6/V$. This shows that $\mathcal{I} \supset (XQX^t)$ if and only if $\mathbb{C}^6/V$ is an isotropic subspace of $\mathbb{C}^6$. □

Remark. Since $\mu^{-1}(0) \subset (\mathbb{C}^2)^{\oplus 6}$, the invariant Hilbert scheme $Sl_2\text{-Hilb}(\mu^{-1}(0))$ is a subscheme of $Sl_2\text{-Hilb}((\mathbb{C}^2)^{\oplus 6})$. The calculation of the fixed points suggests that the fibre over 0 of $Sl_2\text{-Hilb}((\mathbb{C}^2)^{\oplus 6})$ contains the whole Grassmannian. Indeed one has $Sl_2\text{-Hilb}((\mathbb{C}^2)^{\oplus 6}) = \{(\mathbb{C}^2)^{\oplus 6}/\!\!/Sl_2 \times \mathrm{Grass}(2, \mathbb{C}^6) \mid \mathrm{im}\, A^t \subset W\}$ as a forthcoming work by Terpereau will show.

2. (G, h)–constellations

In this chapter, we generalise the notion of G–constellation, originally introduced by Craw and Ishii in [CI04] for finite groups, to the case of reductive groups. In our definition, we replace the isotypic decomposition of the regular representation by an isotypic decomposition given by a prescribed Hilbert function h. Further, we adapt Craw and Ishii's notion of θ–stability and θ–semistability and we introduce the moduli functors $\mathcal{M}_\theta(X)$ and $\overline{\mathcal{M}}_\theta(X)$ of θ–stable and θ–semistable (G, h)–constellations, respectively. Then in Section 2.2 we show that θ–semistable (G, h)–constellations satisfy a certain finiteness condition. Afterwards, we examine flat families of (G, h)–constellations and reduce the verification of the θ–(semi)stability condition to finitely many subsheaves only. The aim is to constuct a moduli space of θ–stable (G, h)–constellations representing $\mathcal{M}_\theta(X)$, which, for a special choice of θ, recovers the invariant Hilbert scheme. Indeed, in Section 2.3 we show that if $h(\rho_0) = 1$ and θ is chosen appropriately, then $\mathcal{M}_\theta(X)$ coincides with the invariant Hilbert functor.

2.1. Definitions

As in the previous chapter, let G be a reductive group, X an affine G–scheme and $h\colon \operatorname{Irr} G \to \mathbb{N}_0$ a Hilbert function, where $\operatorname{Irr} G$ denotes the set of isomorphy classes of irreducible representations $\rho\colon G \to Gl(V_\rho)$.

Definition 2.1.1

1. Let $R_h := \bigoplus_{\rho \in \operatorname{Irr} G} \mathbb{C}^{h(\rho)} \otimes_{\mathbb{C}} V_\rho$ be the G–module with multiplicities given by h. A (G, h)*–constellation* on X is a G–equivariant coherent $\mathcal{O}_X$–module $\mathcal{F}$ such that $H^0(\mathcal{F})$ is isomorphic to R_h as a representation of G.
2. Given a scheme S, a *family of* (G, h)*–constellations* over S is a coherent sheaf $\mathscr{F}$ on a family of affine G–schemes $\mathcal{X}$ over S in the sense of [AB05, Definition 1.1], i.e. on a scheme $\mathcal{X}$ equipped with an action of G and an affine G–invariant

morphism $\mathcal{X} \to S$ of finite type, such that the restrictions $\mathscr{F}(s) = \mathscr{F}|_{\mathcal{X}(s)}$ are (G, h)–constellations on the fibres $\mathcal{X}(s) := \mathcal{X} \times_S \operatorname{Spec}(k(s))$.

We would like to represent the functor that assigns to a scheme S the set of families of (G, h)–constellations on a scheme X. In general, the set of (G, h)–constellations on X is too large to be parameterised by a scheme. Hence, to construct a moduli space of these objects, we restrict ourselves to (G, h)–constellations satisfying a certain stability condition $\theta \in \operatorname{Hom}(\operatorname{Irr} G, \mathbb{Q}) \cong \mathbb{Q}^{\operatorname{Irr} G}$. To define such a stability condition, we first need to associate to θ a function on the representation ring $R(G) = \bigoplus_{\rho \in \operatorname{Irr} G} \mathbb{Z} \cdot \rho$ and on the category $\operatorname{Coh}^G(X)$ of G–equivariant coherent $\mathcal{O}_X$–modules:

Definition 2.1.2 If $\theta \in \mathbb{Q}^{\operatorname{Irr} G}$, we define a function $\theta \colon R(G) \to \mathbb{Q} \cup \{\infty\}$ by

$$\theta(W) := \langle \theta, h_W \rangle := \sum_{\rho \in \operatorname{Irr} G} \theta_\rho \cdot \dim W_\rho$$

where $W = \bigoplus_{\rho \in \operatorname{Irr} G} W_\rho \otimes_{\mathbb{C}} V_\rho$ is the isotypic decomposition of W.
In order to consider θ as a function $\theta \colon \operatorname{Coh}^G(X) \to \mathbb{Q} \cup \{\infty\}$ we set

$$\theta(\mathcal{F}) := \theta(H^0(\mathcal{F})) = \sum_{\rho \in \operatorname{Irr} G} \theta_\rho \cdot \dim \mathcal{F}_\rho$$

with $H^0(\mathcal{F}) = \bigoplus_{\rho \in \operatorname{Irr} G} \mathcal{F}_\rho \otimes_{\mathbb{C}} V_\rho$. In particular, if $\mathcal{F}$ is a (G, h)–constellation, then $\theta(\mathcal{F}) = \sum_{\rho \in \operatorname{Irr} G} \theta_\rho h(\rho)$.

We are now in the position to define the stability condition we need on (G, h)–constellations:

Definition 2.1.3 A (G, h)–constellation $\mathcal{F}$ is called θ*–semistable* if $\theta(\mathcal{F}) = 0$ and if for all G–equivariant coherent subsheaves $\mathcal{F}' \subset \mathcal{F}$ we have $\theta(\mathcal{F}') \geq 0$. Moreover, $\mathcal{F}$ is called θ*–stable* if $\theta(\mathcal{F}) = 0$ and if for all non–zero proper G–equivariant coherent subsheaves $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$ we have $\theta(\mathcal{F}') > 0$.
For convenience, we replace the similar conditions for stability and semistability by setting everything concerning semistability in parentheses and we introduce the symbol "$\underset{(-)}{>}$": A (G, h)–constellation $\mathcal{F}$ is called θ*–(semi)stable* if $\theta(\mathcal{F}) = 0$ and if for all non–zero proper G–equivariant coherent subsheaves $\mathcal{F}' \subset \mathcal{F}$ we have $\theta(\mathcal{F}') \underset{(-)}{>} 0$. In the same way, "$\underset{(-)}{<}$" stands for "$\leq$" in the case of semistability and "$<$" in the case of stability.

Remark 2.1.4 Every G–equivariant subsheaf $\mathcal{F}'$ of $\mathcal{F}$ induces a G–equivariant quotient $\mathcal{F}'' := \mathcal{F}/\mathcal{F}'$ of $\mathcal{F}$. Conversely, every G–equivariant quotient $\alpha\colon \mathcal{F} \twoheadrightarrow \mathcal{F}''$ induces a G–equivariant subsheaf $\mathcal{F}' := \ker\alpha$ of $\mathcal{F}$. In both cases the corresponding Hilbert functions satisfy $h_{\mathcal{F}'} + h_{\mathcal{F}''} = h$, so that $\theta(\mathcal{F}) = \theta(\mathcal{F}') + \theta(\mathcal{F}'')$. Thus a (G,h)–constellation $\mathcal{F}$ is θ–semistable if and only if $\theta(\mathcal{F}) = 0$ and if for all non–zero proper G–equivariant quotients $\mathcal{F} \twoheadrightarrow \mathcal{F}''$ we have $\theta(\mathcal{F}'') < 0$, and $\mathcal{F}$ is θ–stable if and only if $\theta(\mathcal{F}) = 0$ and if for all G–equivariant quotients $\mathcal{F} \twoheadrightarrow \mathcal{F}''$ we have $\theta(\mathcal{F}'') \leq 0$.

Now we define the moduli functors that we will consider in the following:

Definition 2.1.5 The moduli functor of θ–semistable (G,h)–constellations on X is

$$\begin{aligned}
&\overline{\mathcal{M}}_\theta(X)\colon (\mathrm{Sch}/\mathbb{C})^{\mathrm{op}} \to (\mathrm{Set})\\
&S \mapsto \{\mathscr{F} \text{ an } S\text{–flat family of } \theta\text{–semistable } (G,h)\text{–constellations on } X \times S\}/_{\cong},\\
&(f\colon S' \to S) \mapsto \big(\overline{\mathcal{M}}_\theta(X)(S) \to \overline{\mathcal{M}}_\theta(X)(S'), \mathscr{F} \mapsto (\mathrm{id}_X \times f)^*\mathscr{F}\big).
\end{aligned}$$

The moduli functor of θ–stable (G,h)–constellations on X is

$$\begin{aligned}
&\mathcal{M}_\theta(X)\colon (\mathrm{Sch}/\mathbb{C})^{\mathrm{op}} \to (\mathrm{Set})\\
&S \mapsto \{\mathscr{F} \text{ an } S\text{–flat family of } \theta\text{–stable } (G,h)\text{–constellations on } X \times S\}/_{\cong},\\
&(f\colon S' \to S) \mapsto \big(\mathcal{M}_\theta(X)(S) \to \mathcal{M}_\theta(X)(S'), \mathscr{F} \mapsto (\mathrm{id}_X \times f)^*\mathscr{F}\big).
\end{aligned}$$

2.2. Finiteness

Our strategy to construct the moduli space $M_\theta(X)$ of θ–stable (G,h)–constellations is to show that all θ–(semi)stable (G,h)–constellations are quotients of a certain coherent $\mathcal{O}_X$–module $\mathcal{H}$ and to obtain our moduli space by considering the invariant Quot scheme $\mathrm{Quot}^G(\mathcal{H},h)$ and its GIT–quotient.
In order to do that fix $\theta \in \mathbb{Q}^{\mathrm{Irr}\,G}$ such that $\theta_\rho < 0$ for only finitely many $\rho \in \mathrm{Irr}\,G$. This induces a decomposition

$$\mathrm{Irr}\,G = D_+ \cup D_0 \cup D_- \qquad \text{such that} \qquad \theta_\rho \begin{cases} > 0, & \rho \in D_+,\\ = 0, & \rho \in D_0,\\ < 0, & \rho \in D_-.\end{cases}$$

By the assumption on θ, the set D_- is finite. Since $\theta(\mathcal{F})$ is supposed to be 0 for any θ–semistable (G,h)–constellation $\mathcal{F}$, the values of θ have to be chosen such that $\langle\theta,h\rangle = 0$. In particular, the series $\sum_{\rho\in\mathrm{Irr}\,G}\theta_\rho h(\rho)$ is convergent.

Remark 2.2.1 If $\theta = 0$ or at least $\theta_\rho = 0$ whenever $h(\rho) \neq 0$, then every (G,h)–constellation is θ–semistable, but there are no θ–stable (G,h)–constellations. This case is not of any interest. To avoid this, in the following we will always assume that there is an irreducible representation ρ such that $\theta_\rho \neq 0$ and $h(\rho) \neq 0$. In particular, $D_- \cap \operatorname{supp} h$ and $D_+ \cap \operatorname{supp} h$ are assumed to be non–empty.

Let $\mathcal{F}$ be a θ–(semi)stable (G,h)–constellation and $\mathcal{F}'$ a G–equivariant coherent subsheaf of $\mathcal{F}$. Let $H^0(\mathcal{F}') \cong \bigoplus_{\rho\in\operatorname{Irr} G} \mathcal{F}'_\rho \otimes_{\mathbb{C}} V_\rho$ be the isotypic decomposition of its global sections. Then we have $h'(\rho) := \dim \mathcal{F}'_\rho \leq h(\rho)$ for every $\rho \in \operatorname{Irr} G$. Since D_- is finite, $\theta(\mathcal{F}')$ is also a convergent series and we have

$$\theta(\mathcal{F}') = \sum_{\rho\in\operatorname{Irr} G} \theta_\rho h'(\rho) = \underbrace{\sum_{\rho\in D_-} \underbrace{\theta_\rho}_{<0} \underbrace{h'(\rho)}_{\geq 0}}_{\leq 0} + \underbrace{\sum_{\rho\in D_+} \underbrace{\theta_\rho}_{>0} \underbrace{h'(\rho)}_{\geq 0}}_{\geq 0} \overset{!}{\underset{(-)}{>}} 0.$$

As a philosophy, if $\mathcal{F}$ is to be θ–(semi)stable, the values $h'(\rho)$ should be as large as possible in D_+ and as small as possible in D_-. This means that all subsheaves of $\mathcal{F}$ should be similar to $\mathcal{F}$ in positive parts and they should nearly vanish in negative parts. In other words, the most destabilising subsheaf of $\mathcal{F}$ is the subsheaf of $\mathcal{F}$ generated by its summands in D_-.
We have the following finiteness result:

Theorem 2.2.2 *If $\mathcal{F}$ is a θ–stable (G,h)–constellation on X, then it is generated by $\bigoplus_{\rho\in D_-} \mathcal{F}_\rho \otimes_{\mathbb{C}} V_\rho$ as an $\mathcal{O}_X$–module.*

Proof. Consider the $\mathcal{O}_X$–submodule $\mathcal{F}'$ of $\mathcal{F}$ generated by $\bigoplus_{\rho\in D_-} \mathcal{F}_\rho \otimes_{\mathbb{C}} V_\rho$. Then we have:

$$\begin{aligned} h'(\rho) &= h(\rho) \quad \text{for } \rho \in D_-, \\ h'(\rho) &\leq h(\rho) \quad \text{for } \rho \in D_+ \cup D_0. \end{aligned}$$

This implies

$$\theta(\mathcal{F}') = \sum_{\rho\in D_-} \theta_\rho h'(\rho) + \sum_{\rho\in D_+} \theta_\rho h'(\rho) \leq \sum_{\rho\in D_-} \theta_\rho h(\rho) + \sum_{\rho\in D_+} \theta_\rho h(\rho) = \theta(\mathcal{F}) = 0.$$

Since $\mathcal{F}$ is θ–stable this means that $\mathcal{F}' = \mathcal{F}$, because otherwise $\mathcal{F}'$ would destabilise $\mathcal{F}$. This shows that every θ–stable sheaf $\mathcal{F}$ is generated by $\bigoplus_{\rho\in D_-} \mathcal{F}_\rho \otimes_{\mathbb{C}} V_\rho$. □

Definition 2.2.3 If a (G,h)–constellation $\mathcal{F}$ on X is generated by $\bigoplus_{\rho\in D_-} \mathcal{F}_\rho \otimes_{\mathbb{C}} V_\rho$ as an $\mathcal{O}_X$–module, we say *$\mathcal{F}$ is generated in D_-*.

Remark 2.2.4 If we even have $\theta \in (\mathbb{Q} \setminus \{0\})^{\operatorname{Irr} G}$ then the theorem also holds for θ–semistable (G,h)–constellations $\mathcal{F}$. For then in the proof θ–semistability yields $\theta(\mathcal{F}') = 0$ and hence $h'(\rho) = h(\rho)$ for every $\rho \in D_+$. Since $D_0 = \emptyset$ in this case, this already gives $\mathcal{F}' = \mathcal{F}$.

This finiteness result causes us to define the following free $\mathcal{O}_X$–module of finite rank:

$$\mathcal{H} := \bigoplus_{\rho \in D_-} \mathbb{C}^{h(\rho)} \otimes_{\mathbb{C}} V_\rho \otimes_{\mathbb{C}} \mathcal{O}_X \cong \mathcal{O}_X^{\sum_{\rho \in D_-} h(\rho) \dim V_\rho}. \tag{2.1}$$

Then by Theorem 2.2.2 it follows that every θ–(semi)stable (G,h)–constellation can be obtained as a quotient of $\mathcal{H}$ (if $\theta \in (\mathbb{Q} \setminus 0)^{\operatorname{Irr} G}$). We will establish this in more detail in Section 4.1. Consequently, we may consider $\operatorname{Quot}^G(\mathcal{H}, h)$ to construct the moduli space of θ–stable (G,h)–constellations.

Another consequence of the consideration of D_- is that θ–(semi)stability can be proven by checking finitely many subsheaves only, as the following sequence of propositions and lemmas shows.

Proposition 2.2.5 *A (G,h)–constellation $\mathcal{F}$ is θ–(semi)stable if $\theta(\mathcal{F}) = 0$ and for all non–zero proper G–equivariant subsheaves $\widetilde{\mathcal{F}} \subset \mathcal{F}$ generated in D_- we have $\theta(\widetilde{\mathcal{F}}) \underset{(-)}{>} 0$.*

Proof. Assume that $\theta(\widetilde{\mathcal{F}}) \underset{(-)}{>} 0$ for every proper G–equivariant subsheaf $\widetilde{\mathcal{F}} \subset \mathcal{F}$ generated in D_- and let $\mathcal{F}'$ be a G–equivariant subsheaf of $\mathcal{F}$. Consider the subsheaf $\mathcal{F}^*$ of $\mathcal{F}'$ generated by the $\mathcal{F}'_\rho$, $\rho \in D_-$, so that we have $h^*(\rho) := \dim \mathcal{F}^*_\rho = h'(\rho)$ for $\rho \in D_-$ and $h^*(\rho) \leq h'(\rho)$ for $\rho \in \operatorname{Irr} G \setminus D_-$. Since $\mathcal{F}^*$ is generated in D_-, we have

$$\begin{aligned} \theta(\mathcal{F}') &= \sum_{\rho \in D_-} \theta_\rho h'(\rho) + \sum_{\rho \in \operatorname{Irr} G \setminus D_-} \underbrace{\theta_\rho}_{\geq 0} h'(\rho) \\ &\geq \sum_{\rho \in D_-} \theta_\rho h^*(\rho) + \sum_{\rho \in \operatorname{Irr} G \setminus D_-} \theta_\rho h^*(\rho) = \theta(\mathcal{F}^*) \underset{(-)}{>} 0. \end{aligned}$$

□

Lemma 2.2.6 *The family of pairs*

$$\left\{ (\mathcal{F}, \mathcal{F}') \,\middle|\, \begin{array}{l} \mathcal{F} \text{ a } (G,h)\text{–constellation generated in } D_-, \\ \mathcal{F}' \subset \mathcal{F} \text{ a } G\text{–equivariant coherent subsheaf generated in } D_- \end{array} \right\} \tag{2.2}$$

is bounded, i.e. there is a noetherian scheme Z, a coherent sheaf of $\mathcal{O}_{X \times Z}$–modules $\mathscr{F}$ and a G–equivariant coherent subsheaf $\mathscr{F}'$ of $\mathscr{F}$ such that the family (2.2) *is contained in the set $\{(\mathscr{F}|_{X \times \operatorname{Spec}(k(z))}, \mathscr{F}'|_{X \times \operatorname{Spec}(k(z))}) \mid z \text{ a closed point in } Z\}$.*

Proof. The set of (G, h)–constellations $\mathcal{F}$ generated in D_- is parameterised by a subset of the noetherian scheme $\mathrm{Quot}^G(\mathcal{H}, h)$. For a fixed $\mathcal{F}$ the subsheaves $\mathcal{F}' \subset \mathcal{F}$ generated in D_- are determined by the choice of subspaces $\mathcal{F}'_\rho \subset \mathcal{F}_\rho$ for $\rho \in D_-$. Hence they are parameterised by a subset of $\prod_{\rho \in D_-} \coprod_{k=0}^{h(\rho)} \mathrm{Grass}(k, \mathbb{C}^{h(\rho)})$. Thus the set (2.2) is parameterised by a subset of $\mathrm{Quot}^G(\mathcal{H}, h) \times \prod_{\rho \in D_-} \coprod_{k=0}^{h(\rho)} \mathrm{Grass}(k, \mathbb{C}^{h(\rho)})$. This is a noetherian scheme, so the family (2.2) is bounded by the universal family of its functor of points. □

Remark. Our notion of boundedness differs from [HL10, Definition 1.7.5] in the requirement on Z not to be of finite type but noetherian only. This is enough for later use.

Proposition 2.2.7 *There is a finite set of Hilbert functions $\{h_1, \dots, h_n\}$ such that for any θ–stable (G, h)–constellation $\mathcal{F}$ and any G–equivariant coherent subsheaf $\mathcal{F}'$ of $\mathcal{F}$ generated in D_-, the Hilbert function h' of $\mathcal{F}'$ is one of the $h_1, \dots, h_n$.*

Proof. Since any θ–stable (G, h)–constellation is generated in D_- by Theorem 2.2.2, Lemma 2.2.6 says that the family of pairs $(\mathcal{F}, \mathcal{F}')$ with $\mathcal{F}$ a θ–stable (G, h)–constellation and $\mathcal{F}'$ a G–equivariant coherent subsheaf of $\mathcal{F}$ generated in D_- is parameterised by a noetherian basis Z and bounded by a pair of coherent sheaves $(\mathscr{F}, \mathscr{F}')$ on $X \times Z$. The family $(\mathscr{F}, \mathscr{F}')$ is not necessarily flat on Z, but we can use [Gro61, Lemme 3.4] to obtain a flattening stratification of Z, that is a finite decomposition $Z = \coprod_{i=1}^n Z_i$ of Z into a disjoint union of locally closed subschemes $Z_i \subset Z$ such that $(\mathscr{F}|_{Z_i}, \mathscr{F}'|_{Z_i})$ is a flat family on Z_i. Then for all $z \in Z_i$ the fibres $\mathscr{F}(z)$ have the same Hilbert function h_i.
Lemme 3.4 in [Gro61] is only formulated in the case where $\mathcal{O}_{X \times Z}$ and $(\mathscr{F}, \mathscr{F}')$ are graded over $\mathbb{N}_0$, $\mathcal{O}_{X \times Z}$ is generated by $(\mathcal{O}_{X \times Z})_1$ and h is a polynomial. We reduce our situation to this setting as follows: Define a map $a\colon \operatorname{Irr} G \cong \mathbb{N}_0^{\operatorname{rk} G} \to \mathbb{N}_0$ via $\rho = \sum_{\rho_i \in \operatorname{Irr} G} n_i \rho_i \mapsto \sum_{\rho \in \operatorname{Irr} G} n_i$, where all but finitely many n_i vanish. Then $\mathcal{O}_{X \times Z}$ is graded over $\mathbb{N}_0$ with $(\mathcal{O}_{X \times Z})_n = \bigoplus_{a(\rho)=n} (\mathcal{O}_{X \times Z})_\rho$. The same holds for $\mathscr{F}$ and $\mathscr{F}'$. The function $p\colon \mathbb{N}_0 \to \mathbb{N}_0$, $p(n) = \sum_{a(\rho)=n} h(\rho)$ describes the rank of the $\mathscr{F}_n$ and analogously we have p' for $\mathscr{F}'$. Further, there is a degree d such that the ring $\mathcal{O}_{X \times Z}^{(d)} := \bigoplus_{n \in \mathbb{N}_0} (\mathcal{O}_{X \times Z})_{nd}$ is generated by $(\mathcal{O}_{X \times Z}^{(d)})_1 = (\mathcal{O}_{X \times Z})_d$. For $i = 1, \dots, d-1$ set $\mathscr{F}^i := \bigoplus_{n \in \mathbb{N}_0} \mathscr{F}_{i+nd}$, so that $\mathscr{F} = \mathscr{F}^0 \oplus \ldots \oplus \mathscr{F}^{d-1}$. Then all the $\mathscr{F}^i$ are $\mathcal{O}_{X \times Z}^{(d)}$–modules and each corresponding function p^i with $p^i(n) = \operatorname{rk} \mathscr{F}_n^i$ is a polynomial. By [Gro61, Lemme 3.4] we find a flattening stratification for each $\mathscr{F}^i$. In the same way we obtain a flattening stratification for the $(\mathscr{F}')^i$. Their common refinement yields a flattening stratification for $(\mathscr{F}, \mathscr{F}')$. □

Corollary 2.2.8 *With the notation of Proposition 2.2.7, a (G,h)–constellation $\mathcal{F}$ is θ–(semi)stable if $\theta(\mathcal{F}) = 0$ and for all $i = 1,..,n$ with h_i actually occurring as a Hilbert function of some non–zero proper G–equivariant subsheaf of $\mathcal{F}$ generated in D_-, we have* $\langle \theta, h_i \rangle \underset{(=)}{>} 0$.

2.3. The invariant Hilbert scheme as a moduli space of (G,h)–constellations

For recovering the invariant Hilbert functor (cf. Definition 1.1.1) and the invariant Hilbert scheme, one has to choose θ such that D_- consists of the trivial representation only:

Proposition 2.3.1 *If $h(\rho_0) = 1$ and θ is chosen such that $D_- = \{\rho_0\}$, then the moduli functor of θ–stable (G,h)–constellations coincides with the invariant Hilbert functor:*

$$\mathcal{M}_\theta(X) = \mathcal{H}ilb_h^G(X).$$

Proof. Let S be a noetherian scheme over $\mathbb{C}$, $s \in S$ a point and $\mathcal{F} = \mathscr{F}(s)$ a fibre of a flat family $\mathscr{F}$ of (G,h)–constellations on $X \times S$ generated in D_-. The condition $D_- = \{\rho_0\}$ means $\theta_{\rho_0} h(\rho_0) = -\sum_{\rho \in D_+} \underbrace{\theta_\rho}_{>0} \underbrace{h(\rho)}_{>0} < 0$. For any G–equivariant subsheaf $\mathcal{F}' \subset \mathcal{F}$ we have $\theta(\mathcal{F}') = \sum_{\rho \in \operatorname{Irr} G} \theta_\rho \underbrace{h'(\rho)}_{\leq h(\rho)}$. Taking into account that $h(\rho_0) = 1$ there are two cases for $h'(\rho_0)$:

- $h'(\rho_0) = 1 = h(\rho_0)$: In this case
$$\theta(\mathcal{F}') = \theta_{\rho_0} \cdot 1 + \sum_{\substack{\rho \in \operatorname{Irr} G \\ \rho \neq \rho_0}} \theta_\rho h'(\rho) = \sum_{\substack{\rho \in \operatorname{Irr} G \\ \rho \neq \rho_0}} \underbrace{\theta_\rho}_{\geq 0} \underbrace{(h'(\rho) - h(\rho))}_{\leq 0} \leq 0,$$
so for stable $\mathcal{F}$ this case cannot occur.
- Hence for stable $\mathcal{F}$ we have $h'(\rho_0) = 0$, so that no proper subsheaf of $\mathcal{F}$ contains V_{ρ_0}. Thus the $\mathcal{O}_X$–module generated by V_{ρ_0} is $\mathcal{F}$, i.e. $\mathcal{F}$ is cyclic. Hence it is isomorphic to a quotient of $\mathcal{O}_X$ and we have $\mathcal{F} \cong \mathcal{O}_{Z_s}$ for some $Z_s \in \operatorname{Hilb}_h^G(X)$. This means that $\mathscr{F} \cong \mathcal{O}_{\mathcal{Z}}$ for $\mathcal{Z} = \{(Z_s, s) \mid s \in S\} \in \mathcal{H}ilb_h^G(X)(S)$.

Conversely, consider an element $\mathcal{Z} \in \mathcal{H}ilb_h^G(X)(S)$. Every fibre $\mathcal{O}_{\mathcal{Z}}(s)$ of its structure sheaf is generated by the image of $1 \in \mathcal{O}_X$, which is an invariant. Therefore, every proper G–equivariant subsheaf $\mathcal{F}'$ of $\mathcal{O}_Z$ satisfies $h'(\rho_0) = 0$ and hence $\theta(\mathcal{F}') > 0$. So $\mathcal{O}_{\mathcal{Z}}(s)$ is θ–stable for every $s \in S$, which means $\mathcal{O}_{\mathcal{Z}} \in \mathcal{M}_\theta(X)(S)$. □

Corollary 2.3.2 *If $h(\rho_0) = 1$ and θ is chosen such that $D_- = \{\rho_0\}$, then $\mathcal{M}_\theta(X)$ is representable and the moduli space of θ–stable (G, h)–constellations is $M_\theta(X) = \mathrm{Hilb}_h^G(X)$.*

3. Geometric Invariant Theory of the invariant Quot scheme

In the last chapter we have shown that every θ–(semi)stable (G, h)–constellation is a quotient of $\mathcal{H} := \bigoplus_{\rho \in D_-} \mathbb{C}^{h(\rho)} \otimes_{\mathbb{C}} V_\rho \otimes_{\mathbb{C}} \mathcal{O}_X$. Now we consider the invariant Quot scheme $\mathrm{Quot}^G(\mathcal{H}, h)$ parameterising all G–equivariant quotient maps $[q \colon \mathcal{H} \twoheadrightarrow \mathcal{F}]$, where $\mathcal{F}$ is a G–equivariant coherent $\mathcal{O}_X$–module whose module of global sections is isomorphic to $R_h := \bigoplus_{\rho \in \mathrm{Irr}\, G} V_\rho^{\oplus h(\rho)}$. In Section 3.1 we construct an embedding of the invariant Quot scheme into a product of Grassmannians generalising the embedding (1.7). This equips $\mathrm{Quot}^G(\mathcal{H}, h)$ with an ample line bundle $\mathscr{L}$. Thereafter we discuss the geometric invariant theory (GIT) of $\mathrm{Quot}^G(\mathcal{H}, h)$ in order to obtain a categorical quotient $\mathrm{Quot}^G(\mathcal{H}, h)^{ss} /\!\!/_{\mathscr{L}_\chi} \Gamma$ of GIT–semistable quotients and its subset of stable objects, the geometric quotient $\mathrm{Quot}^G(\mathcal{H}, h)^{s} /\!\!/_{\mathscr{L}_\chi} \Gamma = \mathrm{Quot}^G(\mathcal{H}, h)^s / \Gamma$. Its subset which contains the θ–stable (G, h)–constellations will be our candidate for the moduli space of θ–stable (G, h)–constellations. Here, Γ denotes the gauge group of $\mathcal{H}$ and $\mathscr{L}_\chi$ is the ample line bundle $\mathscr{L}$ with linearisation depending on the choice of a character χ of Γ. We describe these parameters in Section 3.2. Afterwards, in Section 3.3 we examine 1–parameter subgroups of Γ and establish their description via filtrations of the vector space $\bigoplus_{\rho \in D_-} \mathbb{C}^{h(\rho)}$ in order to obtain Mumford's numerical criterion for GIT–(semi)stability in Section 3.4. Out of this we eventually establish a condition for GIT–(semi)stability by considering subspaces of $\bigoplus_{\rho \in D_-} \mathbb{C}^{h(\rho)}$ instead of filtrations. This condition will be used to compare GIT–(semi)stability to θ–(semi)stability in Chapter 4.

3.1. Embeddings of the invariant Quot scheme

Let $\mathcal{H}$ be any coherent G–equivariant $\mathcal{O}_X$–module with isotypic decomposition $H^0(\mathcal{H}) = \bigoplus_{\rho \in \mathrm{Irr}\, G} \mathcal{H}_\rho \otimes_{\mathbb{C}} V_\rho$ and $h \colon \mathrm{Irr}\, G \to \mathbb{N}_0$ a Hilbert function satisfying $h(\rho_0) = 1$. Then we consider the invariant Quot scheme $\mathrm{Quot}^G(\mathcal{H}, h)$ as constructed in [Jan06]. Before we address ourselves to the geometric invariant theory of the invariant Quot scheme, we first construct an embedding of $\mathrm{Quot}^G(\mathcal{H}, h)$ into a finite product $\prod_{\sigma\in D} \mathrm{Grass}(H_\sigma, h(\sigma))$ of Grassmannians generalising the closed immersion (1.7).

First, we construct an embedding of the invariant Quot scheme into a finite product of ordinary Quot schemes:

Proposition 3.1.1 *There is a finite subset $D \subset \mathrm{Irr}\, G$ such that*

$$\mathrm{Quot}^G(\mathcal{H}, h) \longrightarrow \prod_{\rho\in D} \mathrm{Quot}(\mathcal{H}_\rho, h(\rho)),\ [q\colon \mathcal{H} \to \mathcal{F}] \longmapsto (q|_{\mathcal{H}_\rho}\colon \mathcal{H}_\rho \to \mathcal{F}_\rho) \qquad (3.1)$$

is injective.

Proof. Let $[u\colon \mathcal{H} \boxtimes \mathcal{O}_{\mathrm{Quot}^G(\mathcal{H},h)} \twoheadrightarrow \mathcal{U}] \in \mathcal{Q}uot^G(\mathcal{H}, h)(\mathrm{Quot}^G(\mathcal{H}, h))$ be the universal quotient. Denote by $\mathcal{K} := \ker u$ its kernel. If $p\colon X \times \mathrm{Quot}^G(\mathcal{H}, h) \to \mathrm{Quot}^G(\mathcal{H}, h)$ is the projection onto the second factor, then we consider the isotypic decomposition $p_*\mathcal{K} = \bigoplus_{\rho\in\mathrm{Irr}\, G} \mathcal{K}_\rho \otimes_{\mathbb{C}} V_\rho$. Let D be a finite set such that $\mathcal{K}$ is generated by the $\mathcal{K}_\rho \otimes_{\mathbb{C}} V_\rho$, $\rho \in D$ as an $\mathcal{O}_{X\times\mathrm{Quot}^G(\mathcal{H},h)}$–module.
First we show that the universal quotient can be reconstructed from the $\mathcal{O}_{\mathrm{Quot}^G(\mathcal{H},h)}$–module homomorphisms $\eta_\rho\colon \mathcal{H}_\rho \boxtimes \mathcal{O}_{\mathrm{Quot}^G(\mathcal{H},h)} \twoheadrightarrow \mathcal{U}_\rho$ for $\rho \in D$.
Since G is reductive, we have $\mathcal{K}_\rho = \ker(\mathcal{H}_\rho \boxtimes \mathcal{O}_{\mathrm{Quot}^G(\mathcal{H},h)} \twoheadrightarrow \mathcal{U}_\rho)$ for every $\rho \in \mathrm{Irr}\, G$. Thus, if we are given η_ρ for $\rho \in D$ we also have $\mathcal{K}_\rho$ for $\rho \in D$. Hence we obtain $\mathcal{K}$, since it is generated by the $\mathcal{K}_\rho \otimes_{\mathbb{C}} V_\rho$, $\rho \in D$. Therefore we can reconstruct $\mathcal{U} := \mathrm{coker}(\mathcal{K} \to \mathcal{H})$.
Now if S is an arbitrary noetherian scheme and $[q\colon \mathcal{H}\boxtimes\mathcal{O}_S \twoheadrightarrow \mathcal{F}] \in \mathcal{Q}uot^G(\mathcal{H}, h)(S)$ then there exists a unique morphism $\alpha\colon S \to \mathrm{Quot}^G(\mathcal{H}, h)$ such that $[q]$ is the pull–back of the universal quotient: $\alpha^*u = q\colon \mathcal{H} \boxtimes \mathcal{O}_S \to \alpha^*\mathcal{U} = \mathcal{F}$. Since $\mathcal{U}$ is flat over $\mathrm{Quot}^G(\mathcal{H}, h)$, the functor α^* is exact. Hence we have an exact sequence of $\mathcal{O}_{X\times S}$–modules

$$0 \longrightarrow \alpha^*\mathcal{K} \longrightarrow \mathcal{H} \boxtimes \mathcal{O}_S \xrightarrow{q} \alpha^*\mathcal{U} \longrightarrow 0.$$

Therefore, $\ker q = \alpha^*\mathcal{K}$ is generated in the degrees in D, so that it can be reconstructed if $\ker q_\rho$ for $\rho \in D$ is given.
This shows that the map of functors $\mathcal{Q}uot^G(\mathcal{H}, h) \to \prod_{\rho\in D} \mathcal{Q}uot(\mathcal{H}_\rho, h(\rho))$ is a monomorphism. Then this also holds for the morphism of schemes (3.1). □

The next step is to embed each Quot scheme $\mathrm{Quot}(\mathcal{H}_\rho, h(\rho))$ into a certain Grassmannian:

Proposition 3.1.2 *For each $\rho \in \operatorname{Irr} G$ there is a finite dimensional vector space H_ρ and a surjection $\mathbb{C}[X/\!/G] \otimes_{\mathbb{C}} H_\rho \twoheadrightarrow \mathcal{H}_\rho$ which induces an embedding*

$$\operatorname{Quot}(\mathcal{H}_\rho, h(\rho)) \hookrightarrow \operatorname{Grass}(H_\rho, h(\rho)). \tag{3.2}$$

Proof. Denote $Q_\rho := \operatorname{Quot}(\mathcal{H}_\rho, h(\rho))$. Let $[u_\rho \colon \mathcal{H} \boxtimes \mathcal{O}_{Q_\rho} \twoheadrightarrow \mathcal{U}_\rho] \in \mathcal{Q}uot(\mathcal{H}_\rho, h(\rho))(Q_\rho)$ be the universal quotient. By the definition of the Quot scheme, the $\mathcal{O}_{Q_\rho}$–module $\mathcal{U}_\rho$ is locally free of rank $h(\rho)$. Hence there is a finite dimensional $\mathbb{C}$–vector space $U_\rho \subset \mathcal{H}_\rho$ such that the restriction $u_\rho|_{\mathcal{O}_{U_\rho \otimes_{\mathbb{C}} Q_\rho}} \colon U_\rho \otimes_{\mathbb{C}} \mathcal{O}_{Q_\rho} \to \mathcal{U}_\rho$ is surjective. Taking the fibres at every point of $\operatorname{Quot}(\mathcal{H}_\rho, h(\rho))$, this yields a morphism

$$\operatorname{Quot}(\mathcal{H}_\rho, h(\rho)) \longrightarrow \operatorname{Grass}(U_\rho, h(\rho)).$$

This morphism need not be injective. In order to obtain an embedding, we possibly have to enlarge U_ρ. Therefore we use the following finiteness results:
1. It is a well–known fact that the module of covariants $\mathcal{H}_\rho = \operatorname{Hom}(V_\rho, \mathcal{H})$ is finitely generated as a $\mathbb{C}[X/\!/G]$–module, see [Dol03, Corollary 5.1]. Let W_ρ be a $\mathbb{C}$–vector space generated by such generators. Then there is a surjective map $\mathbb{C}[X/\!/G] \otimes_{\mathbb{C}} W_\rho \twoheadrightarrow \mathcal{H}_\rho$.
2. The kernel $\mathcal{K}_\rho := \ker u_\rho$ of the universal quotient is a coherent $\mathbb{C}[X/\!/G] \otimes_{\mathbb{C}} \mathcal{O}_{Q_\rho}$–module. Hence there is a finite–dimensional $\mathbb{C}$–vector space $K_\rho \subset \mathcal{K}_\rho$ which generates $\mathcal{K}_\rho$ as a $\mathbb{C}[X/\!/G] \otimes_{\mathbb{C}} \mathcal{O}_{Q_\rho}$–module. For every $k \in K_\rho$ we write $k = \sum f_i \otimes m_{ik}$ with finitely many elements $f_i \in \mathcal{O}_{Q_\rho}$ and $m_{ik} \in \mathcal{H}_\rho$. Let M_ρ be the $\mathbb{C}$–vector space spanned by all the m_{ik}.
Define $H_\rho := (W_\rho + U_\rho) \oplus M_\rho$. We claim that the morphism

$$\begin{aligned} \eta_\rho \colon \operatorname{Quot}(\mathcal{H}_\rho, h(\rho)) &\longrightarrow \operatorname{Grass}(H_\rho, h(\rho)), \\ [q \colon \mathcal{H}_\rho \twoheadrightarrow \mathcal{F}_\rho] &\longmapsto [(q|_{W_\rho + U_\rho}, q|_{M_\rho}) \colon H_\rho \twoheadrightarrow \mathcal{F}_\rho] \end{aligned}$$

constructed this way is injective. In order to prove this we have to reconstruct u_ρ if we are given a morphism $f_\rho \colon H_\rho \otimes_{\mathbb{C}} \mathcal{O}_{Q_\rho} \to \mathcal{U}_\rho$ in the image of η_ρ. Let $A_\rho := \ker f_\rho$. Since $f_\rho \in \operatorname{im} \eta_\rho$, we have $A_\rho \supset (1 \otimes_{\mathbb{C}} K_\rho) \otimes_{\mathbb{C}} \mathcal{O}_{Q_\rho}$. This means that A_ρ generates $\mathcal{K}_\rho$ as a $\mathbb{C}[X/\!/G]$–module and we obtain u as the cokernel of $\mathcal{K}_\rho \to \mathcal{H}_\rho \boxtimes \mathcal{O}_{Q_\rho}$.
As in the proof of Proposition 3.1.1, the injectivity for an arbitrary scheme S and an element $[\mathcal{H}_\rho \boxtimes \mathcal{O}_S \to \mathcal{F}_\rho] \in \mathcal{Q}uot(\mathcal{H}_\rho, h(\rho))(S)$ can be shown by pulling back the universal quotient. Then the result also holds pointwise. □

Together, these embeddings yield an embedding of the invariant Quot scheme into a product of finitely many Grassmannians:

Corollary 3.1.3 *The composition of the embedding* (3.1) *with the embeddings* (3.2) *for* $\rho \in D$ *yields an embedding*

$$\eta \colon \operatorname{Quot}^G(\mathcal{H}, h) \hookrightarrow \prod_{\rho \in D} \operatorname{Grass}(H_\rho, h(\rho)). \tag{3.3}$$

3.2. The parameters needed for GIT

Now let $\mathcal{H}$ be as defined in (2.1). In this section we introduce a group action on the invariant Quot scheme of $\mathcal{H}$, for which we want to obtain the GIT–quotient. In order to determine this quotient, we need to find an ample line bundle on $\operatorname{Quot}^G(\mathcal{H}, h)$, which can be linearised with respect to the group action. The linearisation depends on a character of the group.

In the definition of $\mathcal{H}$, we write $A_\rho := \mathbb{C}^{h(\rho)}$, i.e. $\mathcal{H} := \bigoplus_{\rho \in D_-} A_\rho \otimes_{\mathbb{C}} V_\rho \otimes_{\mathbb{C}} \mathcal{O}_X$. For every $[q \colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \operatorname{Quot}^G(\mathcal{H}, h)$, the sheaf $\mathcal{F} = q(\mathcal{H})$ is generated by the finitely many components $q(A_\rho \otimes_{\mathbb{C}} V_\rho \otimes_{\mathbb{C}} \mathcal{O}_X)$, $\rho \in D_-$, as an $\mathcal{O}_X$–module.

Certainly, these components are in general not identical with the isotypic components $\mathcal{F}_{(\rho)} := \mathcal{F}_\rho \otimes_{\mathbb{C}} V_\rho = q(\mathcal{H}_{(\rho)})$, since following Steinberg's formula [Hum72, Section 24.4] the isotypic component $\mathcal{H}_{(\rho)}$ may contain components of the form $A_{\rho'} \otimes_{\mathbb{C}} \mathbb{C}[X]_{\rho''} \otimes_{\mathbb{C}} V_\rho$ in addition to $A_\rho \otimes_{\mathbb{C}} \mathbb{C}[X]^G \otimes_{\mathbb{C}} V_\rho$, namely if V_ρ occurs as a summand in the decomposition $V_{\rho'} \otimes_{\mathbb{C}} V_{\rho''} = \bigoplus_{\sigma \in \operatorname{Irr} G} V_\sigma^{\oplus m^\sigma_{\rho'\rho''}}$.

3.2.1. The line bundle $\mathscr{L}$ and the weights κ

In the last section we showed that there is a finite subset $D \subset \operatorname{Irr} G$ and an embedding η of $\operatorname{Quot}^G(\mathcal{H}, h)$ into a product of Grassmannians $\prod_{\sigma \in D} \operatorname{Grass}(H_\sigma, h(\sigma))$, where H_σ is a $\mathbb{C}$–vector space with generators as in the proof of Proposition 3.1.2. Composing η with the Plücker embedding π_σ for every occurring Grassmannian we have

$$\operatorname{Quot}^G(\mathcal{H}, h) \overset{\eta}{\hookrightarrow} \prod_{\sigma \in D} \operatorname{Grass}(H_\sigma, h(\sigma)) \overset{(\pi_\sigma)_\sigma}{\hookrightarrow} \prod_{\sigma \in D} \mathbb{P}(\Lambda^{h(\sigma)} H_\sigma). \tag{3.4}$$

For any set containing D we also obtain an embedding. For example, adding further representations if necessary, we may assume $D_- \subset D$. Since $\operatorname{Grass}(H_\sigma, h(\sigma))$ is a point if $h(\sigma) = 0$ and hence it does not contribute to the embedding, we will always suppose $D \subset D_- \cup D_+$.

In the following discussion of the geometric invariant theory, different choices of D lead to different notions of GIT–(semi)stability. We will take advantage of the variation of D and the corresponding stability condition in Chapter 4.3.

For every choice of $\kappa \in \mathbb{N}_0^D$, the ample line bundles $\mathcal{O}_\sigma(1)$ on $\mathbb{P}(\Lambda^{h(\sigma)} H_\sigma)$ give a line bundle $\bigotimes_{\sigma \in D}(\pi_\sigma^* \mathcal{O}_\sigma(1))^{\kappa_\sigma} = \bigotimes_{\sigma \in D}(\det \mathcal{W}_\sigma)^{\kappa_\sigma}$ on the product of the Grassmannians, where $\mathcal{W}_\sigma$ denotes the universal family of $\mathrm{Grass}(H_\sigma, h(\sigma))$. It is ample if $\kappa_\sigma \geq 1$ for every $\sigma \in D$. This in turn induces an ample line bundle

$$\mathscr{L} = \eta^* \bigotimes_{\sigma \in D} (\pi_\sigma^* \mathcal{O}_\sigma(1))^{\kappa_\sigma} = \bigotimes_{\sigma \in D} (\det \mathcal{U}_\sigma)^{\kappa_\sigma} \tag{3.5}$$

on $\mathrm{Quot}^G(\mathcal{H}, h)$, where $p_* \mathcal{U} = \bigoplus_{\sigma \in \mathrm{Irr}\, G} \mathcal{U}_\sigma \otimes_{\mathbb{C}} V_\sigma$ is the isotypic decomposition of the universal quotient $[\pi^* \mathcal{H} \twoheadrightarrow \mathcal{U}]$ on $X \times \mathrm{Quot}^G(\mathcal{H}, h)$. Here, $\pi\colon X \times \mathrm{Quot}^G(\mathcal{H}, h) \to X$ and $p\colon X \times \mathrm{Quot}^G(\mathcal{H}, h) \to \mathrm{Quot}^G(\mathcal{H}, h)$ denote the projections.

Remark. In Chapter 4.3 we will also consider $\mathscr{L}$ with weights $\kappa_\sigma \in \mathbb{Q}_{>0}$. To give this a meaning, let k be the common denominator of all the κ_σ, $\sigma \in D$. Then we have $k\kappa_\sigma \in \mathbb{N}$ for all $\sigma \in D$ and $\mathscr{L}^k$ is an ample line bundle on $\mathrm{Quot}^G(\mathcal{H}, h)$, which defines an embedding as above.

3.2.2. The gauge group Γ and the character χ

For giving concrete surjections $\mathcal{H} \twoheadrightarrow \mathcal{F}$ rather than only coherent $\mathcal{O}_X$–modules $\mathcal{F}$ which are quotients of $\mathcal{H}$, we have to choose a map $A_\rho \to \mathcal{F}_\rho$ for every $\rho \in D_-$. In order to obtain a moduli space parameterising sheaves $\mathcal{F}$ independent of this choice, we need to divide it out and therefore consider the natural action of the gauge group $\Gamma' := \prod_{\rho \in D_-} Gl(A_\rho)$ on $\mathcal{H}$ by multiplication from the left on the constituent components. Since the scalar matrices act trivially, we actually consider the action of $\Gamma := \big(\prod_{\rho \in D_-} Gl(A_\rho)\big)/\mathbb{C}^*$. This action induces a natural action on $\mathrm{Quot}^G(\mathcal{H}, h)$ from the right: Let $\gamma = (\gamma_\rho)_{\rho \in D_-}$ and $[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \mathrm{Quot}^G(\mathcal{H}, h)$. Then $[q] \cdot \gamma$ is the map

$$[q] \cdot \gamma\colon \mathcal{H} \twoheadrightarrow \mathcal{F},\ a_\rho \otimes v_\rho \otimes f \mapsto q(\gamma_\rho a_\rho \otimes v_\rho \otimes f).$$

Further, this action induces a natural linearisation on some power $\mathscr{L}^k$ of $\mathscr{L}$ (compare to the remark after Lemma 4.3.2 in [HL10]). Replacing κ_σ by $k\kappa_\sigma$ for every $\sigma \in D$, we can assume that $\mathscr{L}$ itself carries a Γ–linearisation. Additionally, we can twist this linearisation with respect to a charater χ, where $\chi(\gamma) = \prod_{\rho \in D_-} \det(\gamma_\rho)^{\chi_\rho}$ and $\chi \in \mathbb{Z}^{D_-}$

such that $\sum_{\rho\in D_-}\chi_\rho h(\rho)=0$. We write $\mathscr{L}_\chi$ for the line bundle $\mathscr{L}$ equipped with the linearisation twisted by the character χ.

3.3. One–parameter subgroups and filtrations

To construct the GIT–quotient, we examine 1–parameter subgroups of Γ in order to apply Mumford's numerical criterion and hence deduce a condition for GIT–(semi)stability. Let $[q\colon \mathcal{H}\twoheadrightarrow\mathcal{F}]\in \mathrm{Quot}^G(\mathcal{H},h)$ and $\lambda\colon \mathbb{C}^*\to\Gamma$ be a 1–parameter subgroup. Then λ induces a grading and a descending filtration on $A:=\bigoplus_{\rho\in D_-}A_\rho$, so that for every $\rho\in D_-$ we have

$$A_\rho=\bigoplus_{n\in\mathbb{Z}}A_\rho^n,\qquad A_\rho^{\geq n}=\bigoplus_{m\geq n}A_\rho^m,$$

where $A_\rho^n=\{a\in A_\rho\mid \lambda(t)\cdot a=t^na\}$ is the subspace of A_ρ on which λ acts with weight n. This induces a grading

$$\mathcal{H}=\bigoplus_{n\in\mathbb{Z}}\mathcal{H}^n,\quad \text{where}\quad \mathcal{H}^n=\bigoplus_{\rho\in D_-}A_\rho^n\otimes_\mathbb{C}V_\rho\otimes_\mathbb{C}\mathcal{O}_X,$$

and the corresponding filtration is

$$\mathcal{H}^{\geq n}=\bigoplus_{m\geq n}\mathcal{H}^m=\bigoplus_{\rho\in D_-}A_\rho^{\geq n}\otimes_\mathbb{C}V_\rho\otimes_\mathbb{C}\mathcal{O}_X.$$

This in turn induces a filtration of $\mathcal{F}$ by

$$\mathcal{F}^{\geq n}:=q(\mathcal{H}^{\geq n}),$$

and we define graded pieces

$$\mathcal{F}^{[n]}:=\mathcal{F}^{\geq n}/\mathcal{F}^{\geq n+1}.$$

Remark 3.3.1 Clearly, only finitely many A_ρ^n are non–zero for every $\rho\in D_-$, so the same holds for $\mathcal{H}^n$ and $\mathcal{F}^{[n]}$. Further, only finitely many $\mathcal{H}^{\geq n}$ and $\mathcal{F}^{\geq n}$ are different from 0 or $\mathcal{H}$ and $\mathcal{F}$, respectively.

The graded object corresponding to the filtration of $\mathcal{F}$ is

$$\overline{\mathcal{F}}:=\bigoplus_{n\in\mathbb{Z}}\mathcal{F}^{[n]}=\bigoplus_{n\in\mathbb{Z}}\mathcal{F}^{\geq n}/\mathcal{F}^{\geq n+1}.$$

For the sheaves of covariants of $\overline{\mathcal{F}}$ we have $\overline{\mathcal{F}}_\sigma = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}_\sigma^{[n]}$ for every $\sigma \in \operatorname{Irr} G$. Since G is reductive, the sequences

$$0 \to \mathcal{F}_\sigma^{\geq n+1} \to \mathcal{F}_\sigma^{\geq n} \to \mathcal{F}_\sigma^{[n]} \to 0$$

are exact for every $\sigma \in \operatorname{Irr} G$ and every $n \in \mathbb{Z}$, so that $\dim \mathcal{F}_\sigma^{[n]} = \dim \mathcal{F}_\sigma^{\geq n} - \dim \mathcal{F}_\sigma^{\geq n+1}$. Let $M, N \in \mathbb{Z}$ such that $\dim \mathcal{F}_\sigma^{[n]} = 0$ for every $n > M$, $n < -N$. Then $\mathcal{F}_\sigma^{\geq -N} = \mathcal{F}_\sigma$ and $\mathcal{F}_\sigma^{\geq M+1} = 0$ and we have

$$\begin{aligned}\dim \overline{\mathcal{F}}_\sigma = \sum_{n\in\mathbb{Z}} \dim \mathcal{F}_\sigma^{[n]} &= \sum_{n=-N}^{M} \big(\dim \mathcal{F}_\sigma^{\geq n} - \dim \mathcal{F}_\sigma^{\geq n+1}\big)\\ &= \dim \mathcal{F}_\sigma^{\geq -N} - \dim \mathcal{F}_\sigma^{\geq M+1} = \dim \mathcal{F}_\sigma.\end{aligned}$$

Therefore $\overline{\mathcal{F}}$ has the same Hilbert function as $\mathcal{F}$, so that the sum of the graded pieces $[q_n \colon \mathcal{H}^n \twoheadrightarrow \mathcal{F}^{[n]}]$ yields a point $[\overline{q} = \oplus_n q_n \colon \mathcal{H} \twoheadrightarrow \overline{\mathcal{F}}] \in \operatorname{Quot}^G(\mathcal{H}, h)$. It has the property that it is the limit of the action of $\lambda(t)$ on $[q]$ when t tends to infinity:

Lemma 3.3.2 $[\overline{q}] = \lim_{t\to 0}[q] \cdot \lambda(t)^{-1} = \lim_{t\to\infty}[q] \cdot \lambda(t)$.

Proof. We proceed analogously to [HL10, Lemma 4.4.3]. We will construct a quotient $[Q \colon \mathcal{H} \otimes_\mathbb{C} \mathbb{C}[T] \twoheadrightarrow \mathscr{F}] \in \mathcal{Q}uot^G(\mathcal{H}, h)(\mathbb{A}^1)$ over $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[T]$ with fibres $[Q(0)] = [\overline{q}]$ and $[Q(t)] = [q] \cdot \lambda(t)^{-1}$ for every $t \neq 0$. As $Q(0)$ is the limit of the $Q(t)$ this gives the assertion. Define

$$\mathscr{F} := \bigoplus_{n\in\mathbb{Z}} \mathcal{F}^{\geq n} \otimes_\mathbb{C} T^{-n} \subset \mathcal{F} \otimes_\mathbb{C} \mathbb{C}[T, T^{-1}].$$

As $\mathcal{F}^{\geq n} = 0$ for $n \gg 0$, only finitely many summands with negative exponent of T are non–zero. So let M be a positive integer such that $\mathcal{F}^{\geq n} = 0$ and $\mathcal{H}^{\geq n} = 0$ for all $n > M$. Thus $\mathscr{F} = \bigoplus_{n\leq M} \mathcal{F}^{\geq n} \otimes_\mathbb{C} T^{-n} \subset \mathcal{F} \otimes_\mathbb{C} T^{-M}\mathbb{C}[T]$. Analogously, we define

$$\mathscr{H} := \bigoplus_{n\in\mathbb{Z}} \mathcal{H}^{\geq n} \otimes_\mathbb{C} T^{-n} \subset \mathcal{H} \otimes_\mathbb{C} T^{-M}\mathbb{C}[T]$$

and q induces a surjection $[q' \colon \mathscr{H} \twoheadrightarrow \mathscr{F}]$ of $\mathbb{A}^1$–flat coherent sheaves on $\mathbb{A}^1 \times X$.
Let $A_V = \bigoplus_{\rho\in D_-} A_\rho \otimes_\mathbb{C} V_\rho$. There is a map $\nu \colon A_V \otimes_\mathbb{C} \mathbb{C}[T] \to \bigoplus_{n\in\mathbb{Z}} A_V^{\geq n} \otimes_\mathbb{C} T^{-n}$ defined by $\nu|_{A_V^m \otimes_\mathbb{C} 1} = \operatorname{id}_{A_V^m} \otimes_\mathbb{C} T^{-m}$, i.e. for $v \in A_V^m$ we have $\nu(v \otimes T^k) = v \otimes T^{k-m}$. Then we have indeed $v \in A_V^{\geq -(k-m)} = A_V^{\geq (m-k)}$. The map ν is an isomorphism because every element $v \otimes T^{-n}$ with $v \in A_V^m$ and $m \geq n$ has a unique preimage $v \otimes T^{m-n}$.

The surjection $Q = q' \circ (\nu \otimes 1)$ makes the following diagram commutative:

$$\begin{array}{ccccc} A_V \otimes_{\mathbb{C}} \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[T] & \xrightarrow[\nu\otimes 1]{\cong} & \mathscr{H} & \hookrightarrow & \mathcal{H} \otimes_{\mathbb{C}} T^{-M}\mathbb{C}[T] \\ \downarrow Q & & \downarrow q' & & \downarrow q\otimes 1 \\ \bigoplus_{n\in\mathbb{Z}} F^{\geq n} \otimes_{\mathbb{C}} T^{-n} & = & \mathscr{F} & \hookrightarrow & \mathcal{F} \otimes_{\mathbb{C}} T^{-M}\mathbb{C}[T] \end{array}$$

On the special fibre $\{0\} \times X$ we have

$$\begin{aligned} \mathscr{F}(0) = \mathscr{F}/(T\cdot\mathscr{F}) &= \Big(\bigoplus_{n\in\mathbb{Z}} \mathcal{F}^{\geq n} \otimes T^{-n}\Big)\Big/\Big(\bigoplus_{n\in\mathbb{Z}} \mathcal{F}^{\geq n} \otimes T^{-n+1}\Big) \\ &= \Big(\bigoplus_{n\in\mathbb{Z}} \mathcal{F}^{\geq n} \otimes T^{-n}\Big)\Big/\Big(\bigoplus_{n\in\mathbb{Z}} \mathcal{F}^{\geq n+1} \otimes T^{-n}\Big) \\ &= \bigoplus_{n\in\mathbb{Z}} \mathcal{F}^{\geq n}/\mathcal{F}^{\geq n+1} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}^{[n]} = \overline{\mathcal{F}} \end{aligned}$$

and in the same way $\mathscr{H}(0) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}^n = \mathcal{H}$, so $Q(0) = \oplus_n q_n = \overline{q}$. Restricting to the open complement $\mathbb{A}^1 \setminus \{0\}$ corresponds to inverting the variable T, so that all horizontal arrows in the diagram above become isomorphisms:

$$\begin{array}{ccc} \mathcal{H} \otimes_{\mathbb{C}} \mathbb{C}[T,T^{-1}] & \xrightarrow{\nu\otimes 1} & \mathcal{H} \otimes_{\mathbb{C}} \mathbb{C}[T,T^{-1}] \\ \downarrow Q & & \downarrow q\otimes 1 \\ \mathscr{F} \otimes_{\mathbb{C}[T]} \mathbb{C}[T,T^{-1}] & \xrightarrow{\cong} & \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}[T,T^{-1}] \end{array}$$

For fixed $t \in \mathbb{C}$, $\nu(t)|_{A_V^m}$ is just multiplication with $\lambda(t)^{-1}|_{A_V^m} = t^{-m}$ on every weight space A_V^m. Hence $Q(t)$ is just $[q] \cdot \lambda(t)^{-1}$. □

The description of $[\overline{q}]$ as a limit of $[q] \cdot \lambda(t)$ yields that it is a fixed point of the action of λ. Hence there is an action of λ on the fibre

$$\mathscr{L}_\chi([\overline{q}]) = \bigotimes_{\sigma\in D} \det(\overline{\mathcal{F}}_\sigma)^{\kappa_\sigma} = \bigotimes_{\sigma\in D} \det\Big(\bigoplus_{n\in\mathbb{Z}} \mathcal{F}_\sigma^{[n]}\Big)^{\kappa_\sigma} = \bigotimes_{\sigma\in D}\bigotimes_{n\in\mathbb{Z}} \det(\mathcal{F}_\sigma^{[n]})^{\kappa_\sigma}.$$

We examine this action in the following in order to gain some criteria for the GIT–(semi)stability of $[q]$.

3.4. GIT–(semi)stability

For understanding the (semi)stability condition in the GIT–sense as defined in [MFK94, Definition 1.7], we consider the weight of the action of 1–parameter subgroups on $\mathscr{L}_\chi$.

Since this weight plays an important role in the following, we adopt Mumford's definition [MFK94, Definition 2.2] to our situation:

Definition 3.4.1 For $[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \mathrm{Quot}^G(\mathcal{H}, h)$ and every 1–parameter subgroup λ we define $\mu_{\mathscr{L}_\chi}(q,\lambda)$ as the weight of λ on $\mathscr{L}_\chi([\overline{q}])$.

Thus, in our situation, Mumford's numerical criterion [MFK94, Theorem 2.1] can be formulated as follows:

Proposition 3.4.2 (Mumford's numerical criterion)
The point $[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \mathrm{Quot}^G(\mathcal{H}, h)$ is GIT–(semi)stable with respect to the twisted line bundle $\mathscr{L}_\chi$ if and only if for every non–trivial 1–parameter subgroup $\lambda\colon \mathbb{C}^* \to \Gamma$ we have $\mu_{\mathscr{L}_\chi}(q,\lambda) \underset{(-)}{>} 0$.

Remark. In the case of vector bundles, GIT–(semi)stability is equivalent to the condition $\mu(q,\lambda) \underset{(-)}{<} 0$ [HL10, Theorem 4.2.11] when μ is defined via the weight of λ on the fibre of $\mathscr{L}$ at the limit at zero. As we consider the limit at infinity, or equivalently the limit of the inverse 1–parameter subgroup at zero, we have GIT–(semi)stability exactly when the negative weight is $\underset{(-)}{<} 0$, i.e. $\mu_{\mathscr{L}_\chi}(q,\lambda) \underset{(-)}{>} 0$.

Now we establish some expressions for $\mu_{\mathscr{L}_\chi}(q,\lambda)$ in terms of κ and χ:

Lemma 3.4.3 *The weight of the action of $\mathbb{C}^*$ via λ on $\mathscr{L}_\chi[\overline{q}]$ is*

$$\begin{aligned}\mu_{\mathscr{L}_\chi}(\overline{q},\lambda) &= \sum_{n\in\mathbb{Z}} n\Big(\sum_{\sigma\in D} \kappa_\sigma \cdot \dim_{\mathbb{C}}(\mathcal{F}_\sigma^{[n]}) + \sum_{\rho\in D_-} \chi_\rho \cdot \dim_{\mathbb{C}}(A_\rho^n)\Big)\\ &= \sum_{n\in\mathbb{Z}} n\big(\kappa(\mathcal{F}^{[n]}) + \chi(A^n)\big).\end{aligned}$$

Proof. The weight $\mu_{\mathscr{L}_\chi}(\overline{q},\lambda)$ is the exponent in the identity

$$\lambda(t)|_{\mathscr{L}_\chi([\overline{q}])} = \lambda(t)|_{\bigotimes_{\sigma\in D}\bigotimes_{n\in\mathbb{Z}} \det(\mathcal{F}_\sigma^{[n]})^{\kappa_\sigma}} = t^{\mu_{\mathscr{L}_\chi}(\overline{q},\lambda)} \cdot \mathrm{id}_{\mathscr{L}_\chi([\overline{q}])}\,.$$

This number splits into a sum $\mu_{\mathscr{L}_\chi}(\overline{q},\lambda) = m + m_\chi$, where m is the weight on the fibre of the original line bundle $\mathscr{L}([\overline{q}])$ and m_χ comes from the twist with the character χ. Since the weight of λ on $\mathcal{F}_\sigma^{[n]}$ is n, for its weight on the determinant $\det(\mathcal{F}_\sigma^{[n]})^{\kappa_\sigma}$ we obtain $n \cdot \dim(\mathcal{F}_\sigma^{[n]}) \cdot \kappa_\sigma$. The weights on the factors of the tensor products over D and $\mathbb{Z}$ translate to a sum of the weights, so we obtain $m = \sum_{\sigma\in D}\sum_{n\in\mathbb{Z}} n \cdot \kappa_\sigma \cdot \dim \mathcal{F}_\sigma^{[n]}$.

The $\lambda(t)_\rho$ are diagonal matrices of size $(\dim A_\rho) \times (\dim A_\rho)$ with entries t^n according to the decomposition $A_\rho = \bigoplus_{n\in\mathbb{Z}} A_\rho^n$. The twist by the character χ is given by taking the product of the determinants of the $\lambda(t)_\rho$ to the χ_ρ's power. Thus we have

$$t^{m_\chi} = \prod_{\rho\in D_-} \det(\lambda(t)_\rho)^{\chi_\rho} = \prod_{\rho\in D_-} \prod_{n\in\mathbb{Z}} t^{n\cdot\dim(A_\rho^n)\cdot\chi_\rho},$$

and $m_\chi = \sum_{\rho\in D_-} \sum_{n\in\mathbb{Z}} n \cdot \chi_\rho \cdot \dim(A_\rho^n)$.
Together, this yields

$$\mu_{\mathscr{L}_\chi}(\overline{q}, \lambda) = \sum_{n\in\mathbb{Z}} n \left(\sum_{\sigma\in D} \kappa_\sigma \cdot \dim \mathcal{F}_\sigma^{[n]} + \sum_{\rho\in D_-} \chi_\rho \cdot \dim A_\rho^n \right).$$

□

Generalising the calculation before Proposition 3.1 in [Kin94], we obtain another formula for $\mu_{\mathscr{L}_\chi}(\overline{q}, \lambda)$:

Proposition 3.4.4 *In terms of the filtration corresponding to a 1–parameter subgroup λ, we have*

$$\mu_{\mathscr{L}_\chi}(\overline{q}, \lambda) = \sum_{n=-N+1}^{M} \big(\kappa(\mathcal{F}^{\geq n}) + \chi(A^{\geq n})\big) - N \cdot \kappa(\mathcal{F}),$$

where $-N$ is the minimal and M the maximal occurring weight.

Proof. By the assumption on N and M we have $\mathcal{F}^{\geq n} = 0$, $A^{\geq n} = 0$ for $n > M$ and $\mathcal{F}^{\geq n} = \mathcal{F}$, $A^{\geq n} = A$ for $n \leq -N$, so we can use Lemma 3.4.5 twice, setting $B = \mathcal{F}$, $\varphi = \kappa$ and $B = A$, $\varphi = \chi$, respectively. This yields

$$\begin{aligned}\sum_{n\in\mathbb{Z}} n\big(\kappa(\mathcal{F}^{[n]}) + \chi(A^n)\big) &\overset{(3.6)}{=} \sum_{n=-N+1}^{M} \big(\kappa(\mathcal{F}^{\geq n}) + \chi(A^{\geq n})\big) - N \cdot \big(\kappa(\mathcal{F}) + \underbrace{\chi(A)}_{=0}\big) \\ &= \sum_{n=-N+1}^{M} \big(\kappa(\mathcal{F}^{\geq n}) + \chi(A^{\geq n})\big) - N \cdot \kappa(\mathcal{F}).\end{aligned}$$

□

In the proof we used the next lemma, which gives an explicit connection between the values of a function applied on a filtered object and the values of the same function applied on the graded pieces of this object:

Lemma 3.4.5 *Let $B = \bigoplus_{n\in\mathbb{Z}} \bigoplus_\tau B_\tau^n$ be a graded object such that for some integers N, M we have $B^n := \bigoplus_\tau B_\tau^n = 0$ for every $n < -N$, $n > M$. Denote the corresponding* *filtered objects by* $B^{\geq n} = \bigoplus_{n\geq m}\bigoplus_\tau B_\tau^m$, *so that* $B^{\geq M+1} = 0$ *and* $B^{\geq -N} = B$. *To every collection of rational numbers* $\varphi = (\varphi_\tau)$ *we can assign the rational number* $\varphi(B) = \sum_\tau \varphi_\tau \dim(B_\tau)$. *In this situation, we have*

$$\sum_{n\in\mathbb{Z}} n\cdot\varphi(B^n) = \sum_{n=-N+1}^{M}\varphi(B^{\geq n}) - N\cdot\varphi(B). \tag{3.6}$$

Proof. Since $B_\tau^n = B_\tau^{\geq n}/B_\tau^{\geq n+1}$ for every τ and since the dimension is additive on quotients, we obtain

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} n\cdot\varphi(B^n) &= \sum_{n=-N+1}^{M} n\cdot\sum_\tau \varphi_\tau \dim(B_\tau^{\geq n}/B_\tau^{\geq n+1})\\
&= \sum_{n=-N}^{M} n\cdot\sum_\tau \varphi_\tau\big(\dim(B_\tau^{\geq n}) - \dim(B_\tau^{\geq n+1})\big)\\
&= \sum_{n=-N}^{M} n\cdot\varphi(B^{\geq n}) - \sum_{n=-N+1}^{M+1}(n-1)\cdot\varphi(B^{\geq n})\\
&= \sum_{n=-N+1}^{M}\varphi(B^{\geq n}) + (-N)\cdot\varphi(\underbrace{B^{\geq N}}_{=B}) - (M+1-1)\cdot\varphi(\underbrace{B^{\geq M+1}}_{=0})\\
&= \sum_{n=-N+1}^{M}\varphi(B^{\geq n}) - N\cdot\varphi(B).
\end{aligned}$$

□

For later use we prove that GIT–(semi)stability is invariant under the action of Γ:

Proposition 3.4.6 *If* $[q]\in \mathrm{Quot}^G(\mathcal{H},h)$ *is GIT–(semi)stable, then so is* $[q]\cdot\gamma$ *for every* $\gamma\in\Gamma$.

Proof. Let $[q]\in\mathrm{Quot}^G(\mathcal{H},h)$, $\gamma\in\Gamma$. If λ is a 1–parameter subgroup, then so is $\widetilde{\lambda} := \gamma^{-1}\lambda\gamma$. For $\lim_{t\to\infty}[q]\cdot\lambda(t) = [\overline{q}]$ we have $\lim_{t\to\infty}([q]\cdot\gamma)\cdot\gamma^{-1}\lambda(t)\gamma = [\overline{q}]\cdot\gamma$. The grading on the A_ρ induced by $\widetilde{\lambda}$ is $\widetilde{A}_\rho^n = \gamma_\rho^{-1}A_\rho^n$, so that

$$\begin{aligned}
(q\cdot\gamma)\big(\bigoplus_{\rho\in D_-}\widetilde{A}_\rho^{\geq n}\otimes_\mathbb{C} V_\rho\otimes_\mathbb{C}\mathcal{O}_X\big) &= (q\cdot\gamma)\big(\bigoplus_{\rho\in D_-}\gamma_\rho^{-1}A_\rho^{\geq n}\otimes_\mathbb{C} V_\rho\otimes_\mathbb{C}\mathcal{O}_X\big)\\
&= q\big(\bigoplus_{\rho\in D_-}\gamma_\rho\gamma_\rho^{-1}A_\rho^{\geq n}\otimes_\mathbb{C} V_\rho\otimes_\mathbb{C}\mathcal{O}_X\big)\\
&= q\big(\bigoplus_{\rho\in D_-}A_\rho^{\geq n}\otimes_\mathbb{C} V_\rho\otimes_\mathbb{C}\mathcal{O}_X\big) = \mathcal{F}^{\geq n}.
\end{aligned}$$

This shows that $\mu_{\mathscr{L}_\chi}(q\cdot\gamma,\gamma^{-1}\lambda\gamma)=\mu_{\mathscr{L}_\chi}(q,\lambda)$. Hence we have $\mu_{\mathscr{L}_\chi}(q,\lambda)\geq 0$ for every 1–parameter subgroup λ if and only if $\mu_{\mathscr{L}_\chi}(q\cdot\gamma,\lambda)\geq 0$ for every 1–parameter subgroup λ. □

3.4.1. 1–step filtrations

Next we analyse the stability condition for 1–step filtrations in order to simplify the condition for GIT–(semi)stability:

Let $A\supsetneq A'\supsetneq 0$ be a 1–step filtration and A'' a complement of A' in A. Then for any 1–parameter subgroup of Γ acting with some weight n' on A' and n'' on A'', the weights have to fulfill $n'\cdot\dim A'+n''\cdot\dim A''=0$. Therefore, up to a multiple in $\frac{1}{\gcd(\dim A',\dim A)}\mathbb{Z}$ we have $n'=\dim A''=\dim A-\dim A'$ and $n''=-\dim A'$. We denote the 1–parameter subgroup associated to A' in this way by λ'. We have

$$\begin{aligned}
A^{\dim A-\dim A'}&=A',\\
A^{-\dim A'}&=A''\cong A/A',\\
\mathcal{F}^{\dim A-\dim A'}&=q\Big(\bigoplus_{\rho\in D_-}A'_\rho\otimes_{\mathbb{C}}V_\rho\otimes_{\mathbb{C}}\mathcal{O}_X\Big)=:\mathcal{F}',\\
\mathcal{F}^{-\dim A'}&=q\Big(\bigoplus_{\rho\in D_-}(A'_\rho\oplus A''_\rho)\otimes_{\mathbb{C}}V_\rho\otimes_{\mathbb{C}}\mathcal{O}_X\Big)/\mathcal{F}^{\dim A-\dim A'}=\mathcal{F}/\mathcal{F}'.
\end{aligned}$$

This yields

$$\begin{aligned}
\mu_{\mathscr{L}_\chi}(q,\lambda')&=(\dim A-\dim A')\cdot\big(\kappa(\mathcal{F}')+\chi(A')\big)-\dim A'\cdot\big(\kappa(\mathcal{F}/\mathcal{F}')+\chi(A/A')\big)\\
&=(\dim A-\dim A')\cdot\big(\kappa(\mathcal{F}')+\chi(A')\big)\\
&\qquad\qquad-\dim A'\cdot\big(\kappa(\mathcal{F})-\kappa(\mathcal{F}')+\underbrace{\chi(A)}_{=0}-\chi(A')\big)\\
&=\dim A\cdot\big(\kappa(\mathcal{F}')+\chi(A')\big)-\dim A'\cdot\kappa(\mathcal{F}).
\end{aligned}$$

Thus we obtain the following criterion for $\mu_{\mathscr{L}_\chi}(q,\lambda')$ to be positive:

$$\begin{aligned}
\mu_{\mathscr{L}_\chi}(q,\lambda')\underset{(-)}{>}0\quad&\iff\quad\mu(A'):=\dim A\cdot\big(\kappa(\mathcal{F}')+\chi(A')\big)-\dim A'\cdot\kappa(\mathcal{F})\underset{(-)}{>}0\\
&\iff\quad\dim A\cdot\big(\kappa(\mathcal{F}')+\chi(A')\big)\underset{(-)}{>}\dim A'\cdot\kappa(\mathcal{F})\\
&\iff\quad\frac{\big(\kappa(\mathcal{F}')+\chi(A')\big)}{\dim A'}\underset{(-)}{>}\frac{\kappa(\mathcal{F})}{\dim A}.
\end{aligned}\tag{3.7}$$

Here we have $\dim A\neq 0$ since $D_-\neq\emptyset$ by Remark 2.2.1 and $\dim A'\neq 0$ by the assumption $A'\neq 0$.

The next lemma shows that it is enough to consider 1–step filtrations to examine GIT–(semi)stability:

Lemma 3.4.7 *A point* $[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \mathrm{Quot}^G(\mathcal{H}, h)$ *is GIT–(semi)stable* $\iff$ *for every* 1*–step filtration* $A \supsetneq A' \supsetneq 0$ *we have* $\mu(A') \underset{(-)}{>} 0$.

Proof. "$\Rightarrow$": Considering the 1–parameter subgroup corresponding to the filtration, this follows from Mumford's numerical criterion.
"$\Leftarrow$": Let λ be any non–trivial 1–parameter subgroup. By Mumford's numerical criterion we have to show that $\mu_{\mathscr{L}_X}(q, \lambda) \underset{(-)}{>} 0$. Let $-N$ denote the minimal and M the maximal occurring weight. Then for every $n \in \{-N+1, \dots, M\}$ the sequence $A \supsetneq A^{\geq n} \supsetneq 0$ is a 1–step filtration. Thus we have $\kappa(\mathcal{F}^{\geq n}) + \chi(A^{\geq n}) \underset{(-)}{>} \frac{\dim A^{\geq n}}{\dim A} \cdot \kappa(\mathcal{F})$ by (3.7). This yields

$$
\begin{aligned}
\mu_{\mathscr{L}_X}(q, \lambda) &= \sum_{n=-N+1}^{M} \big(\kappa(\mathcal{F}^{\geq n}) + \chi(A^{\geq n})\big) - N \cdot \kappa(\mathcal{F}) \\
&\underset{(-)}{>} \sum_{n=-N+1}^{M} \dim A^{\geq n} \cdot \frac{\kappa(\mathcal{F})}{\dim A} - N \cdot \kappa(\mathcal{F}) \\
&= N \cdot \dim A \cdot \frac{\kappa(\mathcal{F})}{\dim A} - N \cdot \kappa(\mathcal{F}) = 0,
\end{aligned}
$$

since by Lemma 3.4.5 with $B = A$, $\varphi \equiv 1$ we have

$$
\sum_{n=-N+1}^{M} \dim A^{\geq n} = \underbrace{\sum_{n \in \mathbb{Z}} n \cdot \dim A^n}_{=0} + N \cdot \dim A = N \cdot \dim A.
$$

This shows that $[q]$ is GIT–(semi)stable. □

Thus we have established the following criterion for GIT–(semi)stability:

Corollary 3.4.8 *An element* $[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \mathrm{Quot}^G(\mathcal{H}, h)$ *is GIT–(semi)stable if and only if for every graded subspace* $0 \neq A' \subsetneq A$ *and* $\mathcal{F}' := q\big(\bigoplus_{\rho \in D_-} A'_\rho \otimes_{\mathbb{C}} V_\rho \otimes_{\mathbb{C}} \mathcal{O}_X\big)$ *the inequality* $\mu(A') := \dim A \cdot \big(\kappa(\mathcal{F}') + \chi(A')\big) - \dim A' \cdot \kappa(\mathcal{F}) \underset{(-)}{>} 0$ *holds.*

4. The connection between the stability conditions

As we want to construct the moduli space of θ–stable (G,h)–constellations on an affine G–scheme X as an open subset of the GIT–quotient $\mathrm{Quot}^G(\mathcal{H},h)^{ss}/\!\!/_{\mathscr{L}_\chi}\Gamma$, first of all we determine the elements in $\mathrm{Quot}^G(\mathcal{H},h)$ originating from (G,h)–constellations in Section 4.1. It turns out that every GIT–semistable quotient can indeed be obtained from a (G,h)–constellation in a particular way, so that we can define a functor $\mathcal{M}_{\chi,\kappa}(X)$ of flat families of GIT–stable (G,h)–constellations. We compare $\mathcal{M}_{\chi,\kappa}(X)$ with the functor $\mathcal{M}_\theta(X)$ of flat families of θ–stable (G,h)–constellations. Therefore, in Section 4.2 we establish a correspondence of the G–equivariant coherent subsheaves generated in D_- of a (G,h)–constellation $\mathcal{F}$ and the graded subspaces of $A=\bigoplus_{\rho\in D_-}A_\rho$ defining subsheaves of $\mathcal{H}$. This leads us to the definition of a new stability condition $\widetilde{\theta}$ on (G,h)–constellations which coincides with GIT–stability for (G,h)–constellations generated in D_-. This reduces our examination of the stability conditions to a comparison of θ and $\widetilde{\theta}$, which look very similar for a certain choice of the GIT–parameters χ and κ. Indeed, in Section 4.3 we show that θ is a limit of the $\widetilde{\theta}$, when the finite subset $D\subset\operatorname{Irr}G$ in the definition of $\widetilde{\theta}$ varies. Furthermore, we find out that θ–stability implies $\widetilde{\theta}$–stability and hence GIT–stability, so that the functor of θ–stable (G,h)–constellations is a subfunctor of the functor of GIT–stable (G,h)–constellations.

4.1. Quotients originating from (G,h)–constellations

To determine the points in the invariant Quot scheme which originate from θ–semistable (G,h)–constellations, we analyse the quotient map for these elements first.
From Section 2.2 we deduce that all θ–semistable (G,h)–constellations $\mathcal{F}$ are quotients of

$$\mathcal{H}:=\bigoplus_{\rho\in D_-}A_\rho\otimes_{\mathbb{C}}V_\rho\otimes_{\mathbb{C}}\mathcal{O}_X,$$

where $A_\rho = \mathbb{C}^{h(\rho)}$ and D_- is the finite subset of $\operatorname{Irr} G$ where θ takes negative values: Since $\mathcal{F}_\rho = \mathcal{H}om_G(V_\rho, \mathcal{F})$ we have natural evaluation maps

$$ev_\rho \colon \mathcal{F}_\rho \otimes_\mathbb{C} V_\rho \otimes_\mathbb{C} \mathcal{O}_X \to \mathcal{F},\ \alpha \otimes v \otimes f \mapsto \alpha(v) \cdot f$$

and $\mathcal{F}$ is generated as an $\mathcal{O}_X$–module by the images of ev_ρ, $\rho \in D_-$ by Theorem 2.2.2. Choosing a basis of each $\mathcal{F}_\rho$, i.e. fixing an isomorphism $\psi_\rho \colon A_\rho \to \mathcal{F}_\rho$, and composing it with the evaluation map, we obtain

$$q_\rho \colon A_\rho \otimes_\mathbb{C} V_\rho \otimes_\mathbb{C} \mathcal{O}_X \to \mathcal{F},\ a \otimes v \otimes f \mapsto \psi_\rho(a)(v) \cdot f \tag{4.1}$$

and the q_ρ add up to the whole of $\mathcal{F}$:

$$q := \underset{\rho \in D_-}{\oplus} q_\rho \colon \mathcal{H} = \bigoplus_{\rho \in D_-} A_\rho \otimes_\mathbb{C} V_\rho \otimes_\mathbb{C} \mathcal{O}_X \to \mathcal{F}.$$

This gives us a point $[q \colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \operatorname{Quot}^G(\mathcal{H}, h)$ with the property that the map

$$\varphi_\rho \colon A_\rho \to \mathcal{F}_\rho = \mathcal{H}om_G(V_\rho, \mathcal{F}),\ a \mapsto (v \mapsto q(a \otimes v \otimes 1)), \tag{4.2}$$

is just the isomorphism ψ_ρ since for $a \in A_\rho$ and $v \in V_\rho$ we have

$$\varphi_\rho(a)(v) = q(a \otimes v \otimes 1) = \psi_\rho(a)(v) \cdot 1 = \psi_\rho(a)(v).$$

The point $[q \colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \operatorname{Quot}^G(\mathcal{H}, h)$ constructed this way depends on the choice of the isomorphisms ψ_ρ. Any other choice differs from ψ_ρ by an element in $Gl(A_\rho)$, so that a (G,h)–constellation can be seen as an element in the quotient of $\operatorname{Quot}^G(\mathcal{H}, h)$ by $\Gamma := \left(\prod_{\rho \in D_-} Gl(A_\rho)\right)/\mathbb{C}^*$. We will make this more precise in Chapter 5.

Conversely, for any element $[q \colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \operatorname{Quot}^G(\mathcal{H}, h)$, the quotient $\mathcal{F}$ is a G–equivariant coherent $\mathcal{O}_X$–module with isotypic decomposition isomorphic to R_h, so it is a (G,h)–constellation. However, the induced maps φ_ρ need not be isomorphisms so that $[q]$ need not originate from a (G,h)–constellation as above even if $\mathcal{F}$ is θ–stable. Since we want to determine a moduli space $M_\theta(X)$ of θ–stable (G,h)–constellations as a subscheme of $\operatorname{Quot}^G(\mathcal{H}, h)^{ss} /\!\!/_{\mathscr{L}_\chi} \Gamma$, we are interested in exploring which quotient maps q do indeed arise from a (G,h)–constellation.
The next lemma shows that for a general point $[q] \in \operatorname{Quot}^G(\mathcal{H}, h)$ the maps φ_ρ are isomorphisms if $[q]$ is GIT–semistable.

Lemma 4.1.1 *Let $[q \colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \operatorname{Quot}^G(\mathcal{H}, h)$ be GIT–semistable. If $\chi_\rho < \frac{\kappa(\mathcal{F})}{\dim A}$ for some $\rho \in D_-$, then $\varphi_\rho \colon A_\rho \to \mathcal{F}_\rho$ is an isomorphism.*

Proof. Fix $\rho \in D_-$ and let $K_\rho := \ker\varphi_\rho$. If φ_ρ is not injective, then $A \supset K_\rho \supsetneq 0$ is a 1–step filtration. For the induced sheaf we obtain $\mathcal{F}' = q(K_\rho \otimes_{\mathbb{C}} V_\rho \otimes_{\mathbb{C}} \mathcal{O}_X) = \varphi_\rho(K_\rho)\cdot\mathcal{O}_X = 0$, so that

$$\begin{aligned}\mu(K_\rho) &= \dim A \cdot \big(\kappa(0) + \chi(K_\rho)\big) - \dim K_\rho \cdot \kappa(\mathcal{F})\\ &= \dim A \cdot \chi_\rho \dim K_\rho - \dim K_\rho \cdot \kappa(\mathcal{F})\\ &= \dim K_\rho \cdot (\dim A \cdot \chi_\rho - \kappa(\mathcal{F})) < 0\end{aligned}$$

by the assumption on χ_ρ.
This is a contradiction to semistability, so $\ker\varphi_\rho$ has to be 0. As A_ρ and $\mathcal{F}_\rho$ have the same dimension $h(\rho)$, this implies that φ_ρ is an isomorphism. □

This means that for every GIT–semistable quotient $[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \mathrm{Quot}^G(\mathcal{H},h)$ the q_ρ are of the form (4.1) for $\rho \in D_-$. In this sense, $[q]$ arises from a (G,h)–constellation.

If for a (G,h)–constellation $\mathcal{F}$ and a choice of isomorphisms $(\psi_\rho)_{\rho\in D_-}$ the corresponding point is GIT–(semi)stable, then the same is true for any other choice of isomorphisms by Proposition 3.4.6. Thus it makes sense to deal with GIT–(semi)stable (G,h)–constellations:

Definition 4.1.2 A (G,h)–constellation $\mathcal{F}$ is *GIT–(semi)stable*, if for some and hence any choice of isomorphisms $(\psi_\rho)_{\rho\in D_-}$ the corresponding point as defined in (4.1) is GIT–(semi)stable. Let

$$\begin{aligned}&\overline{\mathcal{M}}_{\chi,\kappa}(X)\colon (\mathrm{Sch}/\mathbb{C})^{\mathrm{op}} \to (\mathrm{Set})\\ &S \mapsto \{\mathscr{F} \text{ an } S\text{–flat family of GIT–semistable } (G,h)\text{–constellations on } X\times S\}/_{\cong}\\ &(f\colon S'\to S) \mapsto \big(\overline{\mathcal{M}}_{\chi,\kappa}(X)(S) \to \overline{\mathcal{M}}_{\chi,\kappa}(X)(S'), \mathscr{F} \mapsto (\mathrm{id}_X \times f)^*\mathscr{F}\big),\end{aligned}$$

and

$$\begin{aligned}&\mathcal{M}_{\chi,\kappa}(X)\colon (\mathrm{Sch}/\mathbb{C})^{\mathrm{op}} \to (\mathrm{Set})\\ &S \mapsto \{\mathscr{F} \text{ an } S\text{–flat family of GIT–stable } (G,h)\text{–constellations on } X\times S\}/_{\cong}\\ &(f\colon S'\to S) \mapsto \big(\mathcal{M}_{\chi,\kappa}(X)(S) \to \mathcal{M}_{\chi,\kappa}(X)(S'), \mathscr{F} \mapsto (\mathrm{id}_X \times f)^*\mathscr{F}\big)\end{aligned}$$

be the moduli functors of GIT–semistable and GIT–stable (G,h)–constellations on X generated in D_-, respectively.

From the discussion above we expect that $\mathrm{Quot}^G(\mathcal{H},h)^{ss}/\!/_{\mathscr{L}_\chi}\,\Gamma$ and $\mathrm{Quot}^G(\mathcal{H},h)^{s}/\Gamma$ corepresent these functors. We will see this in Section 5.1.

4.2. Correspondence between graded subspaces of A and G–equivariant subsheaves of $\mathcal{F}$

If the map $A_\rho \to \mathcal{F}_\rho$ is injective and hence an isomorphism, we may establish a correspondence between subsheaves of the (G,h)–constellation $\mathcal{F}$ and graded subspaces of A. By Lemma 4.1.1 this correspondence applies to GIT–semistable elements. First we begin with some graded subspace $A' \subset A$, i.e. we have subspaces $A'_\rho \subset A_\rho$ for every $\rho \in D_-$. Let

$$\mathcal{F}' := q\big(\bigoplus_{\rho\in D_-} A'_\rho \otimes_{\mathbb{C}} V_\rho \otimes_{\mathbb{C}} \mathcal{O}_X\big) = \big(\bigoplus_{\rho\in D_-} \varphi_\rho(A'_\rho)\big)\cdot \mathcal{O}_X \tag{4.3}$$

be the sub–$\mathcal{O}_X$–module of $\mathcal{F}$ generated by the $\varphi_\rho(A'_\rho)$, $\rho \in D_-$. Since $\varphi_\rho|_{A'_\rho}$ is injective we have $\dim A'_\rho \le \dim \mathcal{F}'_\rho$ for every $\rho \in D_-$.
Further, we define

$$\widetilde{A}'_\rho := \varphi_\rho^{-1}(\mathcal{F}'_\rho), \qquad \widetilde{A}' := \bigoplus_{\rho\in D_-} \widetilde{A}'_\rho.$$

Then we have

- $\dim \widetilde{A}'_\rho = \dim \mathcal{F}'_\rho =: h'(\rho)$ since φ_ρ is an isomorphism,
- $\widetilde{A}'_\rho = \varphi_\rho^{-1}\big(\big[\big(\bigoplus_{\sigma\in D_-}\varphi_\sigma(A'_\sigma)\big)\cdot\mathcal{O}_X\big]_\rho\big) \supset \varphi_\rho^{-1}\big(\varphi_\rho(A'_\rho)\big) = A'_\rho$,
- $q\big(\bigoplus_{\rho\in D_-}\widetilde{A}'_\rho \otimes_{\mathbb{C}} V_\rho \otimes_{\mathbb{C}} \mathcal{O}_X\big) = \big(\bigoplus_{\rho\in D_-}\varphi_\rho(\widetilde{A}'_\rho)\big)\cdot\mathcal{O}_X = \mathcal{F}'$, since $\varphi_\rho(\widetilde{A}'_\rho) = \mathcal{F}'_\rho$ if $\rho\in D_-$ and $\mathcal{F}'$ is generated in D_-.

For this reason, $\widetilde{A}'$ is called the saturation of A'.

Conversely, if we start with some subsheaf $\mathcal{F}' \subset \mathcal{F}$, we can proceed in the same way to obtain the saturation $\widetilde{\mathcal{F}}'$ of $\mathcal{F}'$: Let

$$\widetilde{A}'_\rho := \varphi_\rho^{-1}(\mathcal{F}'_\rho), \qquad \widetilde{A}' := \bigoplus_{\rho\in D_-}\widetilde{A}'_\rho,$$
$$\widetilde{\mathcal{F}}' := q\big(\bigoplus_{\rho\in D_-}\widetilde{A}'_\rho\otimes_{\mathbb{C}} V_\rho\otimes_{\mathbb{C}}\mathcal{O}_X\big) = \big(\bigoplus_{\rho\in D_-}\varphi_\rho(\widetilde{A}'_\rho)\big)\cdot\mathcal{O}_X = \big(\bigoplus_{\rho\in D_-}\widetilde{\mathcal{F}}'_\rho\big)\cdot\mathcal{O}_X,$$

As before we have $\dim\widetilde{A}'_\rho \le \dim\widetilde{\mathcal{F}}'_\rho$ and $\varphi_\rho^{-1}(\widetilde{\mathcal{F}}'_\rho) \supset \widetilde{A}'_\rho$ for every $\rho\in D_-$ as well as $q\big(\bigoplus_{\rho\in D_-}\varphi_\rho^{-1}(\widetilde{\mathcal{F}}'_\rho)\otimes_{\mathbb{C}} V_\rho\otimes_{\mathbb{C}}\mathcal{O}_X\big) = \big(\bigoplus_{\rho\in D_-}\varphi_\rho(\varphi_\rho^{-1}(\widetilde{\mathcal{F}}'_\rho))\big)\cdot\mathcal{O}_X = \widetilde{\mathcal{F}}'$. Moreover, $\widetilde{\mathcal{F}}'$ is the $\mathcal{O}_X$–module generated by the $\mathcal{F}'_\rho$, $\rho\in D_-$, so we have

$$\mathcal{F}'_\rho = \widetilde{\mathcal{F}}'_\rho \quad \text{for every } \rho\in D_-,$$
$$\mathcal{F}'_\rho \supset \widetilde{\mathcal{F}}'_\rho \quad \text{for every } \rho\in \operatorname{Irr} G\setminus D_-.$$

Thus if $\mathcal{F}'$ is generated in D_- then $\mathcal{F}' = \widetilde{\mathcal{F}}'$ and $\dim \widetilde{A}'_\rho = \dim \mathcal{F}'_\rho = h'(\rho)$.

Inspired by this correspondence we define a new function, which describes GIT–(semi)stability in terms of the $\mathcal{F}'$ instead of the A':

Definition 4.2.1 Let $\mathcal{F}$ be any (G,h)–constellation, $\mathcal{F}' \subset \mathcal{F}$ a G–equivariant coherent subsheaf, $h'(\rho) := \dim \mathcal{F}'_\rho$. Let $\widetilde{\theta}\colon \operatorname{Coh}^G(X) \to \mathbb{Q}$ be the function

$$\widetilde{\theta}(\mathcal{F}') := \sum_{\rho \in D_-} \Big(\kappa_\rho + \chi_\rho - \frac{\kappa(\mathcal{F})}{\dim A}\Big) h'(\rho) \quad + \sum_{\sigma \in D \setminus D_-} \kappa_\sigma h'(\sigma).$$

In the above setting if $\mathcal{F}'$ is generated in D_- we have $h'(\rho) = \dim \widetilde{A}'_\rho$. Comparing this definition to the expression (3.7) we find

$$\dim A \cdot \widetilde{\theta}(\mathcal{F}') = \mu(\widetilde{A}'). \tag{4.4}$$

Remark. Since the notion of GIT–stability on $\operatorname{Quot}^G(\mathcal{H}, h)$ depends on the embedding into a product of Grassmannians, the definition of $\widetilde{\theta}$ depends on the choice of the finite subset $D \subset \operatorname{Irr} G$. If there is any ambiguity about D we write $\widetilde{\theta}_D$ instead of $\widetilde{\theta}$.

The next theorem reduces the examination of the relation between θ–(semi)stability and GIT–(semi)stability to the comparison of θ and $\widetilde{\theta}$ for sheaves generated in D_-.

Theorem 4.2.2 *Let $\chi_\rho \le \frac{\kappa(\mathcal{F})}{\dim A}$. Then $[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \operatorname{Quot}^G(\mathcal{H}, h)$ is GIT–(semi)stable if and only if $\mathcal{F}$ is a $\widetilde{\theta}$–(semi)stable (G,h)–constellation.*

Proof. "$\Rightarrow$": Let $\mathcal{F}'$ be a G–equivariant subsheaf of $\mathcal{F}$. Consider the subsheaf $\mathcal{F}''$ of $\mathcal{F}'$ generated by the $\mathcal{F}'_\rho$, $\rho \in D_-$, so that we have $h''(\rho) := \dim \mathcal{F}''_\rho = h'(\rho)$ for $\rho \in D_-$ and $h''(\rho) \le h'(\rho)$ for $\rho \in \operatorname{Irr} G \setminus D_-$. We define $\widetilde{A}'' = \bigoplus_{\rho \in D_-} \varphi_\rho^{-1}(\mathcal{F}''_\rho)$ as above. As $\mathcal{F}''$ is generated in D_-, we have $\widetilde{\theta}(\mathcal{F}'') = \frac{\mu(\widetilde{A}'')}{\dim A} \underset{(-)}{>} 0$ by GIT–(semi)stability. For $\mathcal{F}'$ this yields

$$\begin{aligned} \widetilde{\theta}(\mathcal{F}') &= \sum_{\rho \in D_-} \Big(\kappa_\rho + \chi_\rho - \frac{\kappa(\mathcal{F})}{\dim A}\Big) \underbrace{h'(\rho)}_{=h''(\rho)} \; + \sum_{\sigma \in D \setminus D_-} \underbrace{\kappa_\sigma}_{>0} \underbrace{h'(\sigma)}_{\ge h''(\sigma)} \\ &\ge \sum_{\rho \in D_-} \Big(\kappa_\rho + \chi_\rho - \frac{\kappa(\mathcal{F})}{\dim A}\Big) h''(\rho) \; + \sum_{\sigma \in D \setminus D_-} \kappa_\sigma h''(\sigma) = \widetilde{\theta}(\mathcal{F}'') \underset{(-)}{>} 0. \end{aligned}$$

"$\Leftarrow$": Let $A' \subset A$ be a graded subspace. As in (4.3) we construct $\mathcal{F}'$ and $\widetilde{A}' \supset A'$. By θ–(semi)stability we have $\mu(\widetilde{A}') = \dim A \cdot \widetilde{\theta}(\mathcal{F}') \underset{(-)}{>} 0$. Further, we obtain

$$\begin{aligned}\chi(\widetilde{A}') - \chi(A') = \chi(\widetilde{A}'/A') &= \sum_{\rho \in D_-} \chi_\rho \cdot \dim(\widetilde{A}'/A')_\rho \\ &\leq \sum_{\rho \in D_-} \frac{\kappa(\mathcal{F})}{\dim A} \cdot \dim(\widetilde{A}'/A')_\rho = \frac{\kappa(\mathcal{F})}{\dim A} \cdot \sum_{\rho \in D_-} \dim(\widetilde{A}'/A')_\rho \\ &= \frac{\kappa(\mathcal{F}) \cdot \dim(\widetilde{A}'/A')}{\dim A} = \frac{\dim \widetilde{A}' - \dim A'}{\dim A} \cdot \kappa(\mathcal{F}).\end{aligned}$$

Separating $\widetilde{A}'$ and A' and multiplying by $\dim A$ this yields

$$\dim A \cdot \chi(\widetilde{A}') - \dim \widetilde{A}' \cdot \kappa(\mathcal{F}) \leq \dim A \cdot \chi(A') - \dim A' \cdot \kappa(\mathcal{F}),$$

so that

$$\begin{aligned}\mu(A') &= \dim A \cdot (\kappa(\mathcal{F}') + \chi(A')) - \dim A' \cdot \kappa(\mathcal{F}) \\ &\geq \dim A \cdot (\kappa(\mathcal{F}') + \chi(\widetilde{A}')) - \dim \widetilde{A}' \cdot \kappa(\mathcal{F}) = \mu(\widetilde{A}') \underset{(-)}{>} 0.\end{aligned}$$

□

If we could show that

$$\widetilde{\theta}(\mathcal{F}') \underset{(-)}{>} 0 \quad \Longleftrightarrow \quad \theta(\mathcal{F}') \underset{(-)}{>} 0 \tag{4.5}$$

for every G–equivariant subsheaf $\mathcal{F}'$ of a (G,h)–constellation $\mathcal{F}$, then in consideration of the theorem and Proposition 2.2.5, we would also have that a (G,h)–constellation is θ–(semi)stable if and only if it is GIT–(semi)stable. Therefore it would even be enough to show (4.5) for $\mathcal{F}$ and $\mathcal{F}'$ generated in D_- by Proposition 2.2.5 and the proof of the above theorem. The equivalence (4.5) might be asking too much for, but in the following section we show at least that θ–stability implies GIT–stability (Corollary 4.3.6). As the Theorem suggests, we therefore compare θ and $\widetilde{\theta}$ and we show that θ–stability implies $\widetilde{\theta}$–stability.

4.3. Comparison of θ and $\widetilde{\theta}$

We have defined two functions on $\mathrm{Coh}^G(X)$:

$$\begin{aligned}\theta(\mathcal{F}') &= \sum_{\rho \in D_-} \theta_\rho h'(\rho) \quad + \sum_{\sigma \in D \setminus D_-} \theta_\sigma h'(\sigma) \quad + \sum_{\tau \in \mathrm{Irr}\, G \setminus D} \theta_\tau h'(\tau), \\ \widetilde{\theta}(\mathcal{F}') &= \sum_{\rho \in D_-} \Big(\kappa_\rho + \chi_\rho - \frac{\kappa(\mathcal{F})}{\dim A}\Big) h'(\rho) \quad + \sum_{\sigma \in D \setminus D_-} \kappa_\sigma h'(\sigma).\end{aligned}$$

The main difference is that θ is defined as the sum over infinitely many elements while the number of summands in $\widetilde{\theta}$ is finite. We define the part outside D of θ by

$$S_D := \sum_{\tau \in \operatorname{Irr} G \setminus D} \theta_\tau h(\tau).$$

To compare θ and $\widetilde{\theta}$ we make the following approach for choosing the character χ and the weights κ in the definition of our ample line bundle $\mathscr{L}$:

$$\begin{aligned} \chi_\rho &= \theta_\rho - \kappa_\rho + \frac{\kappa(\mathcal{F})}{\dim A} && \text{for } \rho \in D_-, \\ \kappa_\rho &> 0 \quad \text{arbitrary} && \text{for } \rho \in D_-, \\ \kappa_\sigma &= \theta_\sigma + \frac{S_D}{d \cdot h(\sigma)} && \text{for } \sigma \in D \setminus D_-, \end{aligned} \tag{4.6}$$

where $d := \#(D \setminus D_-)$ is the number of summands in the second sum in the definition of $\widetilde{\theta}$. Since $D \subset D_- \cup D_+$ we have $\theta_\sigma > 0$ for all $\sigma \in D \setminus D_-$. Furthermore, the inequality $S_D \geq 0$ holds, so that we always have $\kappa_\sigma > 0$.

Remark. Since $\theta_\rho < 0$ and $\kappa_\rho > 0$ for every $\rho \in D_-$, we automatically have

$$\chi_\rho = \theta_\rho - \kappa_\rho + \frac{\kappa(\mathcal{F})}{\dim A} < \frac{\kappa(\mathcal{F})}{\dim A},$$

so the prerequisites of Lemma 4.1.1 and Theorem 4.2.2 are always satisfied with the choice (4.6) of χ and κ.

The following two lemmas substantiate why the choice (4.6) for χ and κ is natural:

Lemma 4.3.1 *Let $\mathcal{F}$ be a (G, h)–constellation. With Ansatz (4.6) of χ and κ for any G–equivariant coherent subsheaf $\mathcal{F}'$ of $\mathcal{F}$ we have*

$$\widetilde{\theta}(\mathcal{F}') = \sum_{\rho \in D_-} \theta_\rho h'(\rho) \quad + \sum_{\sigma \in D \setminus D_-} \theta_\sigma h'(\sigma) \quad + \quad \frac{S_D}{d} \sum_{\sigma \in D \setminus D_-} \frac{h'(\sigma)}{h(\sigma)},$$

in particular $\widetilde{\theta}(\mathcal{F}) = \theta(\mathcal{F})$.

Proof. We have

$$\begin{aligned} \widetilde{\theta}(\mathcal{F}') &= \sum_{\rho \in D_-} \Big(\kappa_\rho + \chi_\rho - \frac{\kappa(\mathcal{F})}{\dim A}\Big) h'(\rho) \quad + \sum_{\sigma \in D \setminus D_-} \kappa_\sigma h'(\sigma) \\ &= \sum_{\rho \in D_-} \theta_\rho h'(\rho) \quad + \sum_{\sigma \in D \setminus D_-} \Big(\theta_\sigma + \frac{S_D}{d \cdot h(\sigma)}\Big) h'(\sigma) \\ &= \sum_{\rho \in D_-} \theta_\rho h'(\rho) \quad + \sum_{\sigma \in D \setminus D_-} \theta_\sigma h'(\sigma) \quad + \quad \frac{S_D}{d} \sum_{\sigma \in D \setminus D_-} \frac{h'(\sigma)}{h(\sigma)}, \end{aligned}$$

so

$$\begin{aligned}
\widetilde{\theta}(\mathcal{F}) &= \sum_{\rho\in D_-} \theta_\rho h(\rho) \quad + \sum_{\sigma\in D\setminus D_-} \theta_\sigma h(\sigma) \quad + \quad \frac{S_D}{d} \sum_{\sigma\in D\setminus D_-} \frac{h(\sigma)}{h(\sigma)} \\
&= \sum_{\rho\in D_-} \theta_\rho h(\rho) \quad + \sum_{\sigma\in D\setminus D_-} \theta_\sigma h(\sigma) \quad + \quad S_D \\
&= \theta(\mathcal{F}).
\end{aligned}$$

□

Remark. If the support $D_- \cup D_+$ of θ is finite, then one may take $D = D_- \cup D_+$. In this case the summand S_D vanishes and the lemma yields

$$\widetilde{\theta}(\mathcal{F}') = \sum_{\rho\in\operatorname{supp}\theta} \theta_\rho h'(\rho) = \theta(\mathcal{F}').$$

In particular, if G is a finite group, θ–(semi)stability and GIT–(semi)stability coincide as in the construction of Craw and Ishii [CI04]. But for a reductive group G, the support of θ will be infinite in general for otherwise the (G,h)–constellations which are θ–semistable but not θ–stable might not be quotients of $\mathcal{H}$ by Remark 2.2.4.

Lemma 4.3.2 *If χ and κ are defined as in* (4.6)*, χ is an admissible character if and only if $\theta(\mathcal{F}) = 0$.*

Proof. A character χ of $\prod_{\rho\in D_-} Gl(A_\rho)$ is a character of $\prod_{\rho\in D_-} Gl(A_\rho)/\mathbb{C}^*$ if and only if $\sum_{\rho\in D_-} \chi_\rho h(\rho) = 0$. We have

$$\begin{aligned}
\sum_{\rho\in D_-} \chi_\rho h(\rho) &= \sum_{\rho\in D_-} \Big(\theta_\rho - \kappa_\rho + \frac{\kappa(\mathcal{F})}{\dim A}\Big) h(\rho) \\
&= \sum_{\rho\in D_-} \theta_\rho h(\rho) \; - \sum_{\rho\in D_-} \kappa_\rho h(\rho) \; + \; \frac{\kappa(\mathcal{F})}{\dim A} \cdot \underbrace{\sum_{\rho\in D_-} h(\rho)}_{=\dim A} \\
&= \sum_{\rho\in D_-} \theta_\rho h(\rho) \; - \sum_{\rho\in D_-} \kappa_\rho h(\rho) \; + \; \underbrace{\kappa(\mathcal{F})}_{=\sum_{\rho\in D}\kappa_\rho h(\rho)} \\
&= \sum_{\rho\in D_-} \theta_\rho h(\rho) \; + \sum_{\rho\in D\setminus D_-} \kappa_\rho h(\rho) \\
&= \sum_{\rho\in D_-} \theta_\rho h(\rho) \; + \sum_{\rho\in D\setminus D_-} \Big(\theta_\rho + \frac{S_D}{d\cdot h(\rho)}\Big) h(\rho) \\
&= \sum_{\rho\in D_-} \theta_\rho h(\rho) \; + \sum_{\rho\in D\setminus D_-} \theta_\rho h(\rho) \; + \; S_D \quad = \quad \theta(\mathcal{F}).
\end{aligned}$$

$\square$

For comparing θ to $\widetilde{\theta}$, we consider $\widetilde{\theta} = \widetilde{\theta}_D$ when the finite subset $D \subset \operatorname{Irr} G$ varies. We obtain the following error term:

Proposition 4.3.3 *If $D_- \cup D_+ \supset \widetilde{D} \supset D$, then for any G–equivariant subsheaf $\mathcal{F}'$ of a (G,h)–constellation $\mathcal{F}$ we have*

$$\widetilde{\theta}_{\widetilde{D}}(\mathcal{F}') - \widetilde{\theta}_D(\mathcal{F}') = \sum_{\tau \in \widetilde{D} \setminus D} \left(\theta_\tau h(\tau) + \frac{S_{\widetilde{D}}}{\widetilde{d}} \right) \left(\frac{h'(\tau)}{h(\tau)} - \frac{1}{d} \sum_{\sigma \in D \setminus D_-} \frac{h'(\sigma)}{h(\sigma)} \right),$$

where $\widetilde{d} := \#(\widetilde{D} \setminus D_-)$, and

$$\theta(\mathcal{F}') - \widetilde{\theta}_D(\mathcal{F}') = \sum_{\tau \in \operatorname{Irr} G \setminus D} \theta_\tau h(\tau) \left(\frac{h'(\tau)}{h(\tau)} - \frac{1}{d} \sum_{\sigma \in D \setminus D_-} \frac{h'(\sigma)}{h(\sigma)} \right).$$

Proof. By Lemma 4.3.1, we have

$$\widetilde{\theta}_D(\mathcal{F}') = \sum_{\rho \in D_-} \theta_\rho h'(\rho) \; + \sum_{\sigma \in D \setminus D_-} \theta_\sigma h'(\sigma) \; + \frac{S_D}{d} \sum_{\sigma \in D \setminus D_-} \frac{h'(\sigma)}{h(\sigma)} \qquad \text{and}$$

$$\widetilde{\theta}_{\widetilde{D}}(\mathcal{F}') = \sum_{\rho \in D_-} \theta_\rho h'(\rho) \; + \sum_{\sigma \in \widetilde{D} \setminus D_-} \theta_\sigma h'(\sigma) \; + \frac{S_{\widetilde{D}}}{\widetilde{d}} \sum_{\sigma \in \widetilde{D} \setminus D_-} \frac{h'(\sigma)}{h(\sigma)}.$$

So the difference is

$$\begin{aligned}
\widetilde{\theta}_{\widetilde{D}}(\mathcal{F}') - \widetilde{\theta}_D(\mathcal{F}') &= \sum_{\sigma \in \widetilde{D} \setminus D} \theta_\sigma h'(\sigma) \; + \frac{S_{\widetilde{D}}}{\widetilde{d}} \sum_{\sigma \in \widetilde{D} \setminus D_-} \frac{h'(\sigma)}{h(\sigma)} \; - \; \frac{S_D}{d} \sum_{\sigma \in D \setminus D_-} \frac{h'(\sigma)}{h(\sigma)} \\
&= \sum_{\sigma \in \widetilde{D} \setminus D} \theta_\sigma h'(\sigma) \; + \frac{S_{\widetilde{D}}}{\widetilde{d}} \sum_{\sigma \in \widetilde{D} \setminus D} \frac{h'(\sigma)}{h(\sigma)} \; + \left(\frac{S_{\widetilde{D}}}{\widetilde{d}} - \frac{S_D}{d} \right) \sum_{\sigma \in D \setminus D_-} \frac{h'(\sigma)}{h(\sigma)} \\
&\overset{(*)}{=} \sum_{\tau \in \widetilde{D} \setminus D} \left(\theta_\tau h(\tau) + \frac{S_{\widetilde{D}}}{\widetilde{d}} \right) \frac{h'(\tau)}{h(\tau)} \; - \; \frac{1}{d} \sum_{\tau \in \widetilde{D} \setminus D} \left(\frac{S_{\widetilde{D}}}{\widetilde{d}} + \theta_\tau h(\tau) \right) \sum_{\sigma \in D \setminus D_-} \frac{h'(\sigma)}{h(\sigma)} \\
&= \sum_{\tau \in \widetilde{D} \setminus D} \left(\theta_\tau h(\tau) + \frac{S_{\widetilde{D}}}{\widetilde{d}} \right) \left(\frac{h'(\tau)}{h(\tau)} - \frac{1}{d} \sum_{\sigma \in D \setminus D_-} \frac{h'(\sigma)}{h(\sigma)} \right),
\end{aligned}$$

since for $(*)$ we calculate

$$\begin{aligned}
\frac{S_{\widetilde{D}}}{\widetilde{d}} - \frac{S_D}{d} &= \frac{1}{\widetilde{d}} \sum_{\tau \in \operatorname{Irr} G \setminus \widetilde{D}} \theta_\tau h(\tau) \;-\; \frac{1}{d} \sum_{\tau \in \operatorname{Irr} G \setminus D} \theta_\tau h(\tau) \\
&= \left(\frac{1}{\widetilde{d}} - \frac{1}{d} \right) \sum_{\tau \in \operatorname{Irr} G \setminus \widetilde{D}} \theta_\tau h(\tau) \;-\; \frac{1}{d} \sum_{\tau \in \widetilde{D} \setminus D} \theta_\tau h(\tau) \\
&= -\frac{\widetilde{d} - d}{\widetilde{d} d} S_{\widetilde{D}} \;-\; \frac{1}{d} \sum_{\tau \in \widetilde{D} \setminus D} \theta_\tau h(\tau) \\
&= -\frac{1}{d} \sum_{\tau \in \widetilde{D} \setminus D} \frac{S_{\widetilde{D}}}{\widetilde{d}} \;-\; \frac{1}{d} \sum_{\tau \in \widetilde{D} \setminus D} \theta_\tau h(\tau) \\
&= -\frac{1}{d} \sum_{\tau \in \widetilde{D} \setminus D} \left(\frac{S_{\widetilde{D}}}{\widetilde{d}} + \theta_\tau h(\tau) \right).
\end{aligned}$$

The calculation of the second error term is the following:

$$\begin{aligned}
\theta(\mathcal{F}') - \widetilde{\theta}_D(\mathcal{F}') &= \sum_{\tau \in \operatorname{Irr} G \setminus D} \theta_\tau h'(\tau) \;-\; \frac{S_D}{d} \sum_{\sigma \in D \setminus D_-} \frac{h'(\sigma)}{h(\sigma)} \\
&= \sum_{\tau \in \operatorname{Irr} G \setminus D} \theta_\tau h'(\tau) \;-\; \frac{1}{d} \sum_{\tau \in \operatorname{Irr} G \setminus D} \theta_\tau h(\tau) \sum_{\sigma \in D \setminus D_-} \frac{h'(\sigma)}{h(\sigma)} \\
&= \sum_{\tau \in \operatorname{Irr} G \setminus D} \theta_\tau \left(h'(\tau) - \frac{1}{d} \sum_{\sigma \in D \setminus D_-} \frac{h'(\sigma)}{h(\sigma)} h(\tau) \right) \\
&= \sum_{\tau \in \operatorname{Irr} G \setminus D} \theta_\tau h(\tau) \left(\frac{h'(\tau)}{h(\tau)} - \frac{1}{d} \sum_{\sigma \in D \setminus D_-} \frac{h'(\sigma)}{h(\sigma)} \right).
\end{aligned}$$

□

The set $\mathcal{D} = \{D \subset \operatorname{Irr} G \mid D_- \cup D_+ \supset D \supset D_-\}$ of all subsets of $D_- \cup D_+$ containing D_- is directed with respect to inclusion. In this sense, we can take the limit over these sets. This allows us to reveal the relation between θ and $\widetilde{\theta}$:

Corollary 4.3.4 *The function θ is the pointwise limit of the functions $\widetilde{\theta}_D$ as D converges to the whole support of θ:*

$$\theta(\mathcal{F}') = \lim_{D \in \mathcal{D}} \widetilde{\theta}_D(\mathcal{F}') \quad \forall\, \mathcal{F}' \subset \mathcal{F}.$$

Proof. Since $\theta(\mathcal{F}) = \sum_{\tau \in \operatorname{Irr} G} \theta_\tau h(\tau)$ is convergent, the sum $\sum_{\tau \in \operatorname{Irr} G \setminus D} \theta_\tau h(\tau)$ converges to 0 when D becomes larger. Further, for every $\tau \in D_- \cup D_+$ we have $0 \leq \frac{h'(\tau)}{h(\tau)} \leq 1$, so $\left| \frac{h'(\tau)}{h(\tau)} - \frac{1}{d} \sum_{\sigma \in D \setminus D_-} \frac{h'(\sigma)}{h(\sigma)} \right| \leq 1$. □

In general, equality will only hold in the limit, but not for finite D. We use this corollary to show that every θ–stable (G,h)–constellation is also $\widetilde{\theta}$–stable.

Proposition 4.3.5 *There is a finite subset $D \subset D_- \cup D_+$ such that the following holds: If $\mathcal{F}$ is a θ–stable (G,h)–constellation and $\mathcal{F}'$ a G–equivariant subsheaf of $\mathcal{F}$, both generated in D_-, then for every finite set $\widetilde{D}$ containing D we have $\widetilde{\theta}_{\widetilde{D}}(\mathcal{F}') > 0$.*

Proof. By Proposition 2.2.7, the set

$$\{\theta(\mathcal{F}'') \mid \mathcal{F}'' \subset \mathcal{F} \text{ a } G\text{–equivariant subsheaf generated in } D_-\}$$

is finite. Let θ_0 be its minimum. In particular, $\theta(\mathcal{F}') \geq \theta_0$.
If we fix $\varepsilon > 0$, by Corollary 4.3.4 there is a subset $D = D(\varepsilon, \mathcal{F}') \subset D_- \cup D_+$ such that $|\theta(\mathcal{F}') - \widetilde{\theta}_{\widetilde{D}}(\mathcal{F}')| < \varepsilon$ for every $\widetilde{D} \supset D$. Since by Proposition 2.2.7 the functions $\theta(\mathcal{F}')$ and $\widetilde{\theta}_{\widetilde{D}}(\mathcal{F}')$ take only finitely many values when $\mathcal{F}'$ varies, D can be chosen simultaneously for all the $\mathcal{F}'$. Now if we choose $\varepsilon < \theta_0$, we obtain $D = D(\varepsilon)$ such that for every $\widetilde{D} \supset D$ we have

$$\widetilde{\theta}_{\widetilde{D}}(\mathcal{F}') > |\theta(\mathcal{F}') - \varepsilon| \geq \theta_0 - \varepsilon > 0.$$

□

Now we summarise:

Corollary 4.3.6 *Let $\theta \in \mathbb{Q}^{\operatorname{Irr} G}$ be a stability condition on the set of (G,h)–constellations on X with $\langle \theta, h \rangle = 0$. For $\mathcal{H} := \bigoplus_{\rho \in D_-} \mathbb{C}^{h(\rho)} \otimes_{\mathbb{C}} V_\rho \otimes_{\mathbb{C}} \mathcal{O}_X$ we consider the invariant Quot scheme $\operatorname{Quot}^G(\mathcal{H}, h)$ and the ample line bundle $\mathscr{L} = \bigotimes_{\sigma \in D} (\det \mathcal{U}_\sigma)^{\kappa_\sigma}$ on $\operatorname{Quot}^G(\mathcal{H}, h)$ with $D \subset \operatorname{Irr} G$ large enough in the sense of Proposition 4.3.5, $\kappa_\rho > 0$ arbitrary for $\rho \in D_- := \{\rho \in \operatorname{Irr} G \mid \theta_\rho < 0\}$ and $\kappa_\sigma = \theta_\sigma + \frac{S_D}{d \cdot h(\sigma)}$ for $\sigma \in D \setminus D_-$. Let the natural linearisation of $\left(\prod_{\rho \in D_-} Gl(\mathbb{C}^{h(\rho)})\right)/\mathbb{C}^*$ on $\mathscr{L}$ be twisted by the character $\chi \colon \operatorname{Irr} G \to \mathbb{Z}$, $\chi_\rho = \theta_\rho - \kappa_\rho + \frac{\kappa(\mathcal{F})}{\dim A}$. We set $\widetilde{\theta}(\mathcal{F}') = \sum_{\rho \in D_-} \left(\kappa_\rho + \chi_\rho - \frac{\kappa(\mathcal{F})}{\dim A}\right) h'(\rho) + \sum_{\sigma \in D \setminus D_-} \kappa_\sigma h'(\sigma)$. With these choices of D, κ, χ and $\widetilde{\theta}$, every θ–stable (G,h)–constellation is $\widetilde{\theta}$–stable and hence GIT–stable.*

Proof. This is a direct consequence of Proposition 4.3.5 and Theorem 4.2.2, if the set D in the embedding (3.4) and the definition of the line bundle (3.5) is chosen large enough in the sense of the proof above. □

On the level of functors, we obtain the following:

Corollary 4.3.7 *With the same notation and choices as in Corollary 4.3.6, the moduli functor $\mathcal{M}_\theta(X)$ of θ–stable (G,h)–constellations on X is a subfunctor of the moduli functor $\mathcal{M}_{\chi,\kappa}(X)$ of GIT–stable (G,h)–constellations on X.*

5. The moduli space of θ–stable (G,h)–constellations

In this chapter, we use the notation and assumptions of Corollary 4.3.6. The preceding chapters leave us with the following situation: We have

$$\begin{aligned}
\mathrm{Quot}^G(\mathcal{H},h)^{ss} &:= \{[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \mathrm{Quot}^G(\mathcal{H},h) \mid [q] \text{ is GIT–semistable}\}\\
\mathrm{Quot}^G(\mathcal{H},h)^{s} &:= \{[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \mathrm{Quot}^G(\mathcal{H},h) \mid [q] \text{ is GIT–stable}\}\\
\mathrm{Quot}^G(\mathcal{H},h)^{s}_\theta &:= \{[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \mathrm{Quot}^G(\mathcal{H},h) \mid \mathcal{F} \text{ is } \theta\text{–stable}\}
\end{aligned}$$

and inclusions

$$\mathrm{Quot}^G(\mathcal{H},h)^s_\theta \subset \mathrm{Quot}^G(\mathcal{H},h)^s \subset \mathrm{Quot}^G(\mathcal{H},h)^{ss}.$$

Forgetting the choice of the particular quotient map, this yields inclusions

$$\left\{\begin{matrix}\theta\text{–stable}\\ (G,h)\text{–constellations}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{GIT–stable}\\ (G,h)\text{–constellations}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{GIT–semistable}\\ (G,h)\text{–constellations}\end{matrix}\right\}.$$

On the level of functors this translates into a sequence

$$\mathcal{M}_\theta(X) \subset \mathcal{M}_{\chi,\kappa}(X) \subset \overline{\mathcal{M}}_{\chi,\kappa}(X).$$

In Section 5.1 we show that $\overline{\mathcal{M}}_{\chi,\kappa}(X)$, $\mathcal{M}_{\chi,\kappa}(X)$ and $\mathcal{M}_\theta(X)$ are corepresented by the categorical quotient $\mathrm{Quot}^G(\mathcal{H},h)^{ss}/\!/_{\mathscr{L}_\chi}\Gamma$, the geometric quotient $\mathrm{Quot}^G(\mathcal{H},h)^s/\Gamma$ and its subscheme $M_\theta(X) := \mathrm{Quot}^G(\mathcal{H},h)^s_\theta/\Gamma$, respectively. Thus, we obtain

$$\begin{array}{ccccc}
\mathrm{Quot}^G(\mathcal{H},h)^s_\theta & \subset & \mathrm{Quot}^G(\mathcal{H},h)^s & \subset & \mathrm{Quot}^G(\mathcal{H},h)^{ss}\\
\downarrow & & \downarrow & & \downarrow\\
M_\theta(X) & \subset & \mathrm{Quot}^G(\mathcal{H},h)^s/\Gamma & \subset & \mathrm{Quot}^G(\mathcal{H},h)^{ss}/\!/_{\mathscr{L}_\chi}\Gamma.
\end{array}$$

$M_\theta(X)$ is the moduli space of θ–stable (G,h)–constellations. It generalises the invariant Hilbert scheme as we have shown in Section 2.3.

In Section 5.2 we show that $M_\theta(X)$ is an open subscheme of $\mathrm{Quot}^G(\mathcal{H}, h)^s/\Gamma$ and is therefore quasiprojective.
To conclude the construction of $M_\theta(X)$ as a moduli space over the quotient $X/\!/G$, in Section 5.3 we construct the desired morphism $M_\theta(X) \to X/\!/G$ generalising the Hilbert–Chow morphism.

5.1. Corepresentability

Let $R := \mathrm{Quot}^G(\mathcal{H}, h)^{ss}$, $R^s := \mathrm{Quot}^G(\mathcal{H}, h)^s$ and $R^s_\theta := \mathrm{Quot}^G(\mathcal{H}, h)^s_\theta$ be the subsets of $\mathrm{Quot}^G(\mathcal{H}, h)$ of GIT–semistable, GIT–stable and θ–stable quotients of $\mathcal{H}$, respectively. For elements $[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \mathrm{Quot}^G(\mathcal{H}, h)$ the sheaf $\mathcal{F} = q(\mathcal{H}) = q(\bigoplus_{\rho \in D_-} A_\rho \otimes_{\mathbb{C}} V_\rho) \cdot \mathcal{O}_X$ is automatically generated in D_-. Moreover, for $[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in R$ the maps $\varphi_\rho\colon A_\rho \to \mathcal{F}_\rho$, $a \mapsto (v \mapsto q(a \otimes v \otimes 1))$ are isomorphisms for every $\rho \in D_-$ by Lemma 4.1.1 and since the inequality $\chi_\rho < \frac{\kappa(\mathcal{F})}{\dim A}$ holds. As presented in Subsection 3.2.2, the choice of these isomorphisms is described by the action of the group $\Gamma' := \prod_{\rho \in D_-} Gl(A_\rho)$, which acts on $\mathrm{Quot}^G(\mathcal{H}, h)$ from the right by left multiplication on the components of $\mathcal{H}$. The subsets R and R^s are invariant under this action by Proposition 3.4.6. The same holds for R^s_θ since the action of an element in Γ' does not change $\mathcal{F}$.
To deal with the ambiguity of the choice of the φ_ρ, we have the following relation between the moduli problem and quotients of this group action:

Proposition 5.1.1 *A morphism $R \to M$ is a categorical quotient of the action of $\Gamma' = \prod_{\rho \in D_-} Gl(A_\rho)$ on R if and only if M corepresents $\overline{\mathcal{M}}_{\chi,\kappa}(X)$. In the same way, a morphism $R^s \to M^s$ is a categorical quotient of the Γ'–action on R^s if and only if M^s corepresents $\mathcal{M}_{\chi,\kappa}(X)$, and a morphism $R^s_\theta \to M^s_\theta$ is a categorical quotient of the Γ'–action on R^s_θ if and only if M^s_θ corepresents $\mathcal{M}_\theta(X)$.*

Proof. We proceed analogously to [HL10, Lemma 4.3.1].
Let S be a noetherian scheme over $\mathbb{C}$ and $\mathscr{F}$ a flat family of (G, h)–constellations generated in D_- which is parameterised by S, so that for every $s \in S$ the fibre $\mathscr{F}(s)$ is a (G, h)–constellation on X. Let $p\colon X \times S \to S$ denote the projection. We look at the isotypic decomposition

$$p_*\mathscr{F} \cong \bigoplus_{\rho \in D_-} \mathscr{F}_\rho \otimes_{\mathbb{C}} V_\rho.$$

The conditions that G is reductive, p is affine and $\mathscr{F}$ is flat over S yield that the $\mathscr{F}_\rho$ are locally free $\mathcal{O}_S$–modules of rank $h(\rho)$ and that we have $(\mathscr{F}_\rho)(s) = \mathscr{F}(s)_\rho$. We define the

$\mathcal{O}_S$–submodule

$$V_{\mathscr{F}}^- := \bigoplus_{\rho \in D_-} (p_*\mathscr{F})_{(\rho)} = \bigoplus_{\rho \in D_-} \mathscr{F}_\rho \otimes_{\mathbb{C}} V_\rho \subset p_*\mathscr{F}. \tag{5.1}$$

The pullback of the inclusion $i\colon V_{\mathscr{F}}^- \hookrightarrow p_*\mathscr{F}$ composed with the natural surjection $\alpha\colon p^*p_*\mathscr{F} \twoheadrightarrow \mathscr{F}$ corresponding to the identity under the adjunction $\mathrm{Hom}(p^*p_*\mathscr{F}, \mathscr{F}) \cong \mathrm{Hom}(p_*\mathscr{F}, p_*\mathscr{F})$ yields a morphism

$$\varphi_{\mathscr{F}} := \alpha \circ p^*i\colon p^*V_{\mathscr{F}}^- \to \mathscr{F}.$$

Fibrewise, $\varphi_{\mathscr{F}}(s)\colon (p^*V_{\mathscr{F}}^-)(s) \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathscr{F}(s)$ is surjective since each $\mathscr{F}(s)$ is generated in D_- as an $\mathcal{O}_X$–module. So $\varphi_{\mathscr{F}}$ is also surjective.
With the notation $A_V := \bigoplus_{\rho \in D_-} A_\rho \otimes_{\mathbb{C}} V_\rho$ we consider the G–equivariant frame bundle $\pi\colon \mathbb{I}(\mathscr{F}) := \mathbb{I}\mathrm{som}_G(A_V \otimes_{\mathbb{C}} \mathcal{O}_S, V_{\mathscr{F}}^-) \to S$ associated to $V_{\mathscr{F}}^-$ as described in Appendix A. It parameterises G–equivariant isomorphisms $A_V \otimes_{\mathbb{C}} \mathcal{O}_S \to V_{\mathscr{F}}^-$ and gives us a canonical morphism $\alpha\colon A_V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{I}(\mathscr{F})} \to \pi^*V_{\mathscr{F}}^-$.
Now we consider $\pi_X := \mathrm{id}_X \times \pi\colon X \times \mathbb{I}(\mathscr{F}) \to X \times S$ and the universal trivialisation $\alpha \otimes_{\mathbb{C}} \mathrm{id}_X\colon A_V \otimes_{\mathbb{C}} \mathcal{O}_{X \times \mathbb{I}(\mathscr{F})} = \mathcal{H} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{I}(\mathscr{F})} \to (p \circ \pi_X)^*V_{\mathscr{F}}^-$ on $X \times \mathbb{I}(\mathscr{F})$. Thus we obtain a canonically defined quotient

$$[\pi_X^*\varphi_{\mathscr{F}} \circ (\mathrm{id}_X \otimes_{\mathbb{C}} \alpha)\colon \mathcal{H} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{I}(\mathscr{F})} \to \pi_X^* p^* V_{\mathscr{F}}^- \to \pi_X^*\mathscr{F}] \in \mathcal{Q}uot^G(\mathcal{H}, h)(\mathbb{I}(\mathscr{F})),$$

which in turn yields a classifying morphism

$$\phi_{\mathscr{F}}\colon \mathbb{I}(\mathscr{F}) \longrightarrow \mathrm{Quot}^G(\mathcal{H}, h),\ \psi \longmapsto [q_\psi\colon \mathcal{H} \twoheadrightarrow (\pi_X^*\mathscr{F})(\psi) = \mathscr{F}(\pi(\psi))].$$

As discussed in Appendix A, the gauge group Γ' acts on $\mathbb{I}(\mathscr{F})$ from the right and $\pi\colon \mathbb{I}(\mathscr{F}) \to S$ is a principal Γ'–bundle. By construction, $\phi_{\mathscr{F}}$ is Γ'–equivariant and we have $\phi_{\mathscr{F}}^{-1}(R) = \pi^{-1}(S^{ss})$, where $S^{ss} = \{s \in S \mid \mathscr{F}(s) \text{ GIT–semistable}\}$. If S parameterises GIT–semistable sheaves, we even have $\phi_{\mathscr{F}}^{-1}(R) = \pi^{-1}(S) = \mathbb{I}(\mathscr{F})$, hence $\phi_{\mathscr{F}}(\mathbb{I}(\mathscr{F})) = \phi_{\mathscr{F}}(\phi_{\mathscr{F}}^{-1}(R)) \subset R$. This means that in fact we have $\phi_{\mathscr{F}}\colon \mathbb{I}(\mathscr{F}) \to R$. This morphism induces a transformation of functors

$$\underline{\mathbb{I}(\mathscr{F})}/\underline{\Gamma'} \to \underline{R}/\underline{\Gamma'}.$$

Since $\pi\colon \mathbb{I}(\mathscr{F}) \to S$ is a principal Γ'–bundle, S is a categorical quotient of $\mathbb{I}(\mathscr{F})$, so that we obtain an element in $(\underline{R}/\underline{\Gamma'})(S)$. Thus we have constructed a transformation $\overline{\mathcal{M}}_{\chi,\kappa}(X) \to \underline{R}/\underline{\Gamma'}$.

Denoting $p_R\colon X \times R \to R$, the universal family $[q\colon p_R^*\mathcal{H} \twoheadrightarrow \mathcal{U}]$ on R yields an inverse by mapping $(\underline{R}/\underline{\Gamma'})(S)$ to $\overline{\mathcal{M}}_{\chi,\kappa}(X)(S) = (\mathrm{id}_X \times \xi)^*\mathcal{U}$, where $\xi\colon S \to R$ is the unique classifying morphism.

Altogether this means that a scheme M corepresents $\overline{\mathcal{M}}_{\chi,\kappa}(X)$ if and only if it corepresents $\underline{R}/\underline{\Gamma'}$, hence if and only if it is a categorical quotient of R by Γ'.

The same proof literally goes through replacing GIT–semistability by GIT–stability and R, M and $\overline{\mathcal{M}}_{\chi,\kappa}(X)$ by R^s, M^s and $\mathcal{M}_{\chi,\kappa}(X)$, respectively, as well as replacing GIT–semistability by θ–stability and R, M and $\overline{\mathcal{M}}_{\chi,\kappa}(X)$ by R^s_θ, M^s_θ and $\mathcal{M}_\theta(X)$. □

Corollary 5.1.2 *The categorical quotient* $\mathrm{Quot}^G(\mathcal{H},h)^{ss}/\!/_{\mathscr{L}_\chi}\Gamma$ *corepresents the functor* $\overline{\mathcal{M}}_{\chi,\kappa}(X)$*, the geometric quotient* $\mathrm{Quot}^G(\mathcal{H},h)^s/\Gamma$ *corepresents* $\mathcal{M}_{\chi,\kappa}(X)$ *and its subscheme* $\mathrm{Quot}^G(\mathcal{H},h)^s_\theta/\Gamma$ *corepresents* $\mathcal{M}_\theta(X)$.

Proof. The quotients by Γ and Γ' coincide since multiples of the identity act trivially. For Γ' and $\overline{\mathcal{M}}_{\chi,\kappa}(X)$ and $\mathcal{M}_{\chi,\kappa}(X)$ the assertion is an immediate consequence of the proposition. Since $\mathrm{Quot}^G(\mathcal{H},h)^s/\Gamma$ is even a geometric quotient and $\mathrm{Quot}^G(\mathcal{H},h)^s_\theta$ is a G–invariant subset of $\mathrm{Quot}^G(\mathcal{H},h)^s$, $\mathrm{Quot}^G(\mathcal{H},h)^s_\theta/\Gamma$ is also a geometric quotient and the assertion follows immediately from Proposition 5.1.1. □

Definition 5.1.3 The scheme $M_\theta(X) := \mathrm{Quot}^G(\mathcal{H},h)^s_\theta/\Gamma$ is called the *moduli space of θ–stable (G,h)–constellations.*

Remark. In Section 2.3 we have already seen (Corollary 2.3.2) that if $h(\rho_0) = 1$ and if θ is chosen such that $D_- = \{\rho_0\}$, we recover the invariant Hilbert scheme:

$$M_\theta(X) = \mathrm{Hilb}^G_h(X).$$

5.2. Openness of θ–stability

In order to show that the moduli space $M_\theta(X)$ is an open subscheme of $\mathrm{Quot}^G(\mathcal{H},h)^s/\Gamma$, we prove that the properties of being θ–stable and θ–semistable are open in flat families of (G,h)–constellations:

Proposition 5.2.1 *Being θ–stable and θ–semistable is an open property in flat families of (G,h)–constellations.*

Proof. We proceed analogously to [HL10, Proposition 2.3.1]. Let $f\colon \mathcal{X} \to S$ be a family of affine G–schemes and $\mathscr{F}$ a flat family of (G,h)–constellations on $\mathcal{X}$. Let

$$\begin{aligned} H \quad &:= \left\{ h'' \text{ a Hilbert function} \;\middle|\; \begin{array}{l} \exists\, s \in S \text{ and a surjection } \alpha(s)\colon \mathscr{F}(s) \to \mathcal{F}'' \\ \text{with } \ker \alpha(s) \text{ generated in } D_- \text{ and } h_{\mathcal{F}''} = h'' \end{array} \right\}, \\ H^{ss} &:= \{h'' \in H \mid \langle \theta, h'' \rangle > 0\}, \\ H^{s} \;\, &:= \{h'' \in H \mid \langle \theta, h'' \rangle \geq 0\}. \end{aligned}$$

By Proposition 2.2.7 and Remark 2.1.4, H is finite. For each Hilbert function h'' in H we consider the relative invariant Quot scheme $\pi_{h''}\colon \operatorname{Quot}^G_{\mathcal{X}/S}(\mathscr{F}, h'') \to S$ with fibres $\operatorname{Quot}^G(\mathscr{F}(s), h'')$ over $s \in S$. Since the multiplicities of the $\mathscr{F}(s)$ are finite, the map $\pi_{h''}$ is projective by Proposition B.5. Thus its image is a closed subset of S. Remark 2.1.4 says that $\mathscr{F}(s)$ is θ–semistable if and only if the Hilbert function h'' of every quotient of $\mathscr{F}(s)$ satisfies $\langle \theta, h'' \rangle < 0$ and θ–stable if $\langle \theta, h'' \rangle \leq 0$. Accordingly, $\mathscr{F}(s)$ is θ–(semi)stable if and only if s is not contained in the finite, hence closed, union $\bigcup_{h'' \in H^{(s)s}} \operatorname{im}(\pi_{h''})$. □

The openness of the property of being θ–stable transfers to the scheme $M_\theta(X)$:

Proposition 5.2.2 *The moduli space of θ–stable (G,h)–constellations $M_\theta(X)$ is an open subscheme of* $\operatorname{Quot}^G(\mathcal{H}, h)^s/\Gamma$.

Proof. To show this, we consider the inclusion $\operatorname{Quot}^G(\mathcal{H}, h)_\theta \subset \operatorname{Quot}^G(\mathcal{H}, h)^s$. The scheme $\operatorname{Quot}^G(\mathcal{H}, h)^s$ represents the functor $\mathcal{Q}uot^G(\mathcal{H}, h)^s$. Let

$$\mathscr{F} \in \mathcal{Q}uot^G(\mathcal{H}, h)^s\big(\operatorname{Quot}^G(\mathcal{H}, h)^s\big)$$

be the universal family in $\operatorname{Quot}^G(\mathcal{H}, h)^s$, so that the fibre $\mathscr{F}(\mathcal{F})$ equals $\mathcal{F}$. Since by Proposition 5.2.1 the property of being θ–stable is open in flat families, the set

$$\operatorname{Quot}^G(\mathcal{H}, h)^s_\theta = \{[q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \operatorname{Quot}^G(\mathcal{H}, h)^s/\Gamma \mid \mathscr{F}(\mathcal{F}) \text{ is } \theta\text{–stable}\}$$

is open in $\operatorname{Quot}^G(\mathcal{H}, h)^s$.
Moreover, the quotient map $\nu\colon \operatorname{Quot}^G(\mathcal{H}, h)^s \to \operatorname{Quot}^G(\mathcal{H}, h)^s/\Gamma$ is open. Thus, its image $M_\theta(X) = \nu(\operatorname{Quot}^G(\mathcal{H}, h)^s_\theta)$ is open in $\operatorname{Quot}^G(\mathcal{H}, h)^s/\Gamma$. □

Since $M_\theta(X)$ is an open subscheme of $\operatorname{Quot}^G(\mathcal{H}, h)^{ss}/\!\!/_{\mathscr{L}_\chi} \Gamma$, it is a quasiprojective scheme. We additionally consider its closure:

Definition 5.2.3 The closure of $M_\theta(X)$ in $\operatorname{Quot}^G(\mathcal{H}, h)^{ss}/\!\!/_{\mathscr{L}_\chi} \Gamma$ is denoted by $\overline{M}_\theta(X)$.

5.3. The map into the quotient $X/\!/G$

For the invariant Quot scheme, Jansou [Jan06, Page 13] constructed an analogue of the Hilbert–Chow morphism

$$\gamma\colon \operatorname{Quot}^G(\mathcal{H}, h) \longrightarrow \operatorname{Quot}(\mathcal{H}^G, h(\rho_0)),\ [q\colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \longmapsto [q|_{\mathcal{H}^G}\colon \mathcal{H}^G \twoheadrightarrow \mathcal{F}^G].$$

In the case where $h(\rho_0) = 1$, we extend the restriction $\gamma|_{\operatorname{Quot}^G(\mathcal{H},h)^{ss}}$ to a morphism to $X/\!/G$:

Theorem 5.3.1 *If $h(\rho_0) = 1$, there is a morphism $\operatorname{Quot}^G(\mathcal{H}, h)^{ss} \to X/\!/G$, which yields a morphism*

$$\eta\colon \overline{M}_\theta(X) \to X/\!/G,\ \mathcal{F} \mapsto \operatorname{supp} \mathcal{F}^G.$$

Proof. Let S be a noetherian scheme over $\mathbb{C}$ and $[q\colon \pi^*\mathcal{H} \twoheadrightarrow \mathscr{F}] \in \mathcal{Q}uot^G(\mathcal{H}, h)^{ss}(S)$, where $\pi\colon X \times S \to X$. Then we have $\gamma_S(q)\colon \mathcal{O}_S \otimes_{\mathbb{C}} \mathcal{H}^G \to \mathscr{F}^G$. Since every fibre $q(s)$ is GIT–semistable, the morphism $\varphi_{\rho_0}\colon A_{\rho_0} \to \mathscr{F}^G(s)$ defined in (4.2) is an isomorphism for every $s \in S$. Hence $\gamma_S(q)$ restricted to the subset $\mathcal{O}_S \otimes_{\mathbb{C}} \mathcal{O}_X^G \cong \mathcal{O}_S \otimes_{\mathbb{C}} A_{\rho_0} \otimes_{\mathbb{C}} \mathcal{O}_X^G$ of $\mathcal{O}_S \otimes_{\mathbb{C}} \mathcal{H}^G$ maps surjectively to $\mathscr{F}^G$. Consider the composite morphism

$$\psi\colon \mathcal{O}_S \xrightarrow{\operatorname{id}\otimes 1} \mathcal{O}_S \otimes_{\mathbb{C}} \mathcal{O}_X^G \twoheadrightarrow \mathscr{F}^G.$$

The image of $s \otimes 1 \in \mathcal{O}_S \otimes_{\mathbb{C}} \mathcal{O}_X^G$ is a function $f(s)$. If it were 0 for some $s \in S$, the map $\mathcal{O}_S \otimes_{\mathbb{C}} \mathcal{O}_X^G \to \mathscr{F}^G$ would not be surjective on the fibre $\mathscr{F}^G(s)$, so this cannot happen. Thus ψ is nowhere 0. The $\mathcal{O}_S$–modules $\mathcal{O}_S$ and $\mathscr{F}^G$ are both locally free of rank 1, so ψ is an isomorphism. This shows that $\mathscr{F}^G$ corresponds to a subscheme $Z \subset S \times X/\!/G$. With the notation

$$\begin{array}{ccccc} Z & \overset{i}{\hookrightarrow} & S \times X/\!/G & \xrightarrow{pr_2} & X/\!/G \\ & \underset{p}{\searrow}\ \cong & \downarrow & & \\ & & S & & \end{array}$$

we obtain a morphism

$$pr_2 \circ i \circ p^{-1}\colon S \to X/\!/G.$$

This construction is compatible with base change. Indeed, let $g\colon T \to S$ be a morphism of noetherian schemes over $\mathbb{C}$. Denoting by $\pi_T\colon X \times T \to X$ the projection to X, we get $[(id_X \times g)^*q\colon \pi_T^*\mathcal{H} \twoheadrightarrow (id_X \times g)^*\mathscr{F}] \in \mathcal{Q}uot^G(\mathcal{H}, h)^{ss}(T)$. The invariants satisfy

$$((id_X \times g)^*\mathscr{F})^G = g^*\mathscr{F}^G \cong g^*\mathcal{O}_S \cong \mathcal{O}_T.$$

Hence, the subscheme corresponding to $((id_X \times g)^* \mathscr{F})^G$ is $Z_T = Z \times_S T$ and we have

$$\begin{array}{ccccccc} Z_T & \overset{j}{\hookrightarrow} & T \times X/\!/G & \longrightarrow & S \times X/\!/G & \longrightarrow & X/\!/G \\ & \underset{p_T}{\overset{\cong}{\searrow}} & \downarrow & & \multicolumn{3}{c}{\underbrace{\qquad\qquad\qquad}_{pr_2 \circ (g \times \mathrm{id}_{X/\!/G})}} \\ & & T & & & & \end{array}$$

The above construction yields a morphism

$$(pr_2 \circ (g \times \mathrm{id}_{X/\!/G})) \circ j \circ p_T^{-1} \colon T \to X/\!/G.$$

We have the following commuting diagram:

$$\begin{array}{ccccc} T \times X/\!/G & \overset{j}{\hookleftarrow} & Z_T & \overset{p_T}{\longrightarrow} & T \\ \Big\downarrow{\scriptstyle g \times \mathrm{id}_{X/\!/G}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle g} \\ S \times X/\!/G & \overset{i}{\hookleftarrow} & Z & \overset{p}{\longrightarrow} & S \end{array}$$

In particular, we have $i \circ p^{-1} \circ g = (g \times \mathrm{id}_{X/\!/G}) \circ j \circ p_T^{-1}$, so that the morphism for T is the composition of the morphism for S with g:

$$\begin{array}{ccc} T & \xrightarrow{(pr_2 \circ (g \times \mathrm{id}_{X/\!/G})) \circ j \circ p_T^{-1}} & X/\!/G \\ \Big\downarrow{\scriptstyle g} & \nearrow_{pr_2 \circ i \circ p^{-1}} & \\ S & & \end{array}$$

Thus, we have constructed a morphism of functors

$$\mathcal{Q}uot^G(\mathcal{H}, h)^{ss} \to \mathrm{Mor}(\cdot, X/\!/G).$$

Plugging in $\mathrm{Quot}^G(\mathcal{H}, h)^{ss}$, this gives a morphism of schemes

$$\eta \colon \mathrm{Quot}^G(\mathcal{H}, h)^{ss} \to X/\!/G.$$

By construction, the subscheme of $X/\!/G$ corresponding to $\mathcal{F}^G = \mathcal{O}_X^G / I_{\mathcal{F}}$ for a point $[q \colon \mathcal{H} \twoheadrightarrow \mathcal{F}] \in \mathrm{Quot}^G(\mathcal{H}, h)^{ss}$ is just its support

$$\operatorname{supp} \mathcal{F}^G = \{\mathfrak{p} \in \mathcal{O}_X^G \mid \mathfrak{p} \supset I_{\mathcal{F}}\} = \left\{ \sqrt{I_{\mathcal{F}}} \right\}.$$

It only consists of one point since $\dim \mathcal{F}^G = h(\rho_0) = 1$.
Since $\mathcal{F}^G$ does not depend on the choice of a basis of $\mathcal{H}$, the morphism η is Γ–invariant. Hence it descends to $\mathrm{Quot}^G(\mathcal{H}, h)^{ss}/\!/_{\mathscr{L}_\chi} \Gamma$. Restricting it to $\overline{M}_\theta(X)$ we eventually obtain a morphism

$$\eta \colon \overline{M}_\theta(X) \to X/\!/G, \ \mathcal{F} \mapsto \operatorname{supp} \mathcal{F}^G$$

and the same for $\mathrm{Quot}^G(\mathcal{H}, h)^s/\Gamma$ and $M_\theta(X)$. □

Thus we have constucted an analogue of the Hilbert–Chow morphism for $M_\theta(X)$ and $\overline{M}_\theta(X)$, which relates these moduli spaces to the quotient $X/\!\!/G$.

Remark. In Proposition 3.1.1 we constructed morphisms

$$\gamma_\rho\colon\ \mathrm{Quot}^G(\mathcal{H},h)\longrightarrow \mathrm{Quot}(\mathcal{H}_\rho,h(\rho)),\ [q\colon \mathcal{H}\to\mathcal{F}]\longmapsto [q|_{\mathcal{H}_\rho}\colon \mathcal{H}_\rho\to\mathcal{F}_\rho],$$

where γ_{ρ_0} is the Hilbert–Chow morphism γ. Therefore one may adopt the proof of Theorem 5.3.1 to this more general situation and obtain morphisms

$$\eta_\rho\colon \overline{M}_\theta(X)\longrightarrow \mathrm{S}^{h(\rho)}(X/\!\!/G),\ \mathcal{F}\longmapsto \operatorname{supp}\mathcal{F}_\rho$$

for an arbitrary Hilbert function h and for every $\rho\in\operatorname{Irr}G$.

6. Outlook

In this thesis we constructed the moduli space $M_\theta(X)$ of θ–stable (G,h)–constellations and a morphism $\eta\colon M_\theta(X) \to X/\!\!/G$. Further, we determined an involved example of an invariant Hilbert scheme for the group Sl_2 acting on a symplectic variety X, which is a special case of a moduli space $M_\theta(X)$. The determination of further examples would be interesting in order to get an idea of the properties of these moduli spaces, e.g. concerning smoothness, connectedness and, for symplectic varieties $X/\!\!/G$, symplecticity of $M_\theta(X)$. In particular, we would like to find out how our example is related to the moduli space of the same Sl_2–action on X for different parameters of θ. In general, the variation of θ is also a noteworthy topic. Moreover, some questions concerning the closure of $M_\theta(X)$ and the properties of η still have to be investigated.
Here we discuss some ideas which are worth being pursued in the future.

6.1. The geometric meaning of points in $\overline{M}_\theta(X)$

We defined the moduli space $\overline{M}_\theta(X)$ as the closure of $M_\theta(X)$ in $\mathrm{Quot}^G(\mathcal{H},h)^{ss}/\!\!/_{\mathscr{L}_\chi}\,\Gamma$ without explicitly describing its elements geometrically. A natural question is

Question 6.1.1 *Does the scheme $\overline{M}_\theta(X)$ corepresent the moduli functor $\overline{\mathcal{M}}_\theta(X)$ of θ–semistable (G,h)–constellations?*

First of all, one has to face the question if every θ–semistable (G,h)–constellation is also GIT–semistable. Secondly, it would be interesting to determine the values of θ for which the notions of θ–stability and θ–semistability coincide. In this case we obtain $\overline{M}_\theta(X) = M_\theta(X)$. For example this is true for the invariant Hilbert scheme. Since $h(\rho_0) = 1$, any subsheaf $\mathcal{F}'$ of a (G,h)–constellation $\mathcal{F}$ has a Hilbert function h' with $h'(\rho_0) = 0$ or $h'(\rho_0) = 1$. In the first case, $\theta(\mathcal{F}')$ is strictly positive and in the second case, $\mathcal{F}' = \mathcal{F}$ by Section 2.3. Hence there are no θ–semistable (G,h)–constellations which are not θ–stable.

In the construction of Craw and Ishii [CI04] and King [Kin94] θ only consists of finitely many components. In their case, θ–semistability and GIT–semistability are even equivalent, as well as θ–stability and GIT–stability. It would be interesting to know if this also holds in our case. With regard to Theorem 4.2.2 this is equivalent to the following question:

Question 6.1.2 *Let $\mathcal{F}$ be a (G,h)–constellation generated in D_- and $\mathcal{F}'$ a G–equivariant coherent subsheaf of $\mathcal{F}$ which is also generated in D_-. Choose $\theta \in \mathbb{Q}^{\operatorname{Irr} G}$ with $\theta(\mathcal{F}) = 0$ and $\widetilde{\theta}$ as in Definition 4.2.1 with values* (4.6) *of χ and κ. In this setting, do we have*

$$\theta(\mathcal{F}') \underset{(=)}{>} 0 \quad \Longleftrightarrow \quad \widetilde{\theta}(\mathcal{F}') \underset{(=)}{>} 0 \quad ?$$

If not, are there additional assumptions on θ under which this equivalence holds?

6.2. Theory of Hilbert functions

Regarding the error estimate in Proposition 4.3.3, Question 6.1.2 is equivalent to

Question 6.2.1 *Let $\mathcal{F}$ be a (G,h)–constellation generated in D_- and let h' be one of the finitely many possible Hilbert functions of its G–equivariant coherent subsheaves generated in D_- as established in Lemma 2.2.7. Fix $\varepsilon > 0$. Does there exist a finite subset $D \subset \operatorname{Irr} G$ such that for every finite set $\widetilde{D} \supset D$ one has*

$$\sum_{\tau \in \operatorname{Irr} G \setminus \widetilde{D}} \theta_\tau h(\tau) \left(\frac{h'(\tau)}{h(\tau)} - \frac{1}{d} \sum_{\sigma \in \widetilde{D} \setminus D_-} \frac{h'(\sigma)}{h(\sigma)} \right) < \varepsilon \quad ?$$

To answer this question one has to study the properties of Hilbert functions extensively. In particular, one has to answer the following questions:

Question 6.2.2 *Let $h\colon \operatorname{Irr} G \to \mathbb{N}_0$ be the Hilbert function of some G–module such that h is determined by the values $h(\rho)$ for ρ in some finite subset $D_- \subset \operatorname{Irr} G$.*

1. *Which kinds of functions are possible for h?*

2. *Let $h'\colon \operatorname{Irr} G \to \mathbb{N}_0$ be a function determined by the values $h'(\rho)$ for ρ in D_- and $h'(\rho) \leq h(\rho)$ for every $\rho \in \operatorname{Irr} G$. If h' occurs as a Hilbert function of a G–equivariant coherent subsheaf of a (G,h)–constellation, what are the possible values of h'?*

6.3. Resolution of singularities

The original purpose of our construction of $M_\theta(X)$ was the search for resolutions of singularities, especially in the symplectic setting. Therefore, one would have to investigate the following:

Question 6.3.1 *Is $\overline{M}_\theta(X)$ or $M_\theta(X)$ smooth or does there exist a smooth connected component?*

Question 6.3.2 *Is $\eta\colon M_\theta(X) \to X/\!/G$ projective?*

Further, we want to know:

Question 6.3.3 *Is the map $\eta\colon \overline{M}_\theta(X) \to X/\!/G$ or its restriction to a smooth connected component a resolution of singulaties? If this is the case and if $X/\!/G$ is a symplectic variety, is η even a symplectic resolution?*

Conversely, inspired by the situation for finite G examined in [CI04], we can ask:

Question 6.3.4

1. *Is every crepant resolution of singularities of $X/\!/G$ a component of some moduli space of θ–stable (G, h)–constellations $M_\theta(X)$ for an appropriate choice of θ?*
2. *What is the relation between the spaces $M_\theta(X)$ for different choices of θ? For example, is there a chamber structure in the space $\mathbb{Q}^{\operatorname{Irr} G}$ such that for θ in any chamber and θ' in an adjacent wall there is a map $M_{\theta'}(X) \to M_\theta(X)$ and for every wall–crossing the involved moduli spaces are related by a flop?*
3. *Is there a distinguished choice of θ so that $M_\theta(X)$ dominates any other $M_{\theta'}(X)$? Are there minimal choices which give symplectic resolutions?*

In particular, in the situation of our example

$$\begin{array}{ccccc} & & Sl_2\text{-Hilb}(\mu^{-1}(0)) & & \\ & \swarrow & \downarrow & \searrow & \\ T^*\mathbb{P}^3 & & & & (T^*\mathbb{P}^3)^* \\ & \searrow & \downarrow & \swarrow & \\ & & \mu^{-1}(0)/\!/Sl_2 & & \end{array}$$

in Chapter 1, we know that Sl_2-Hilb$(\mu^{-1}(0)) = M_\theta(X)$ for $\theta \in \mathbb{Q}^{\operatorname{Irr} G}$ such that θ_{ρ_0} is the only negative value.

Question 6.3.5 *Are the symplectic resolutions $T^*\mathbb{P}^3$ and $(T^*\mathbb{P}^3)^*$ also of the form $M_\theta(X)$ and if so, which is the correct choice for θ?*

A. G–equivariant frame bundles

We carry over the construction of frame bundles in [HL10, Example 4.2.3] to the G–equivariant setting.

Let S be a scheme over $\mathbb{C}$ with trivial G–action and $\mathcal{E}$ a G–equivariant $\mathcal{O}_S$–module with isotypic decomposition

$$\mathcal{E} = \bigoplus_{\rho \in E} \mathcal{E}_\rho \otimes_{\mathbb{C}} V_\rho$$

for some finite subset $E \subset \operatorname{Irr} G$, where the $\mathcal{E}_\rho$ are locally free $\mathcal{O}_S$–modules of rank r_ρ. Let $r := \sum_{\rho \in E} r_\rho$. We write $A_\rho := \mathbb{C}^{r_\rho}$ and $A_V := \bigoplus_{\rho \in E} A_\rho \otimes_{\mathbb{C}} V_\rho$. For $\rho \in E$ we consider the geometric vector bundles

$$\pi_\rho \colon \mathbb{H}\mathrm{om}(A_\rho \otimes_{\mathbb{C}} \mathcal{O}_S, \mathcal{E}_\rho) := \underline{\mathrm{Spec}}(\mathrm{S}^* \mathcal{H}om(A_\rho \otimes_{\mathbb{C}} \mathcal{O}_S, \mathcal{E}_\rho)^\vee) \to S$$

as defined in [Har77, Exercise II.5.18]. They parameterise $\mathcal{O}_S$–module homomorphisms $f_\rho \colon A_\rho \otimes_{\mathbb{C}} \mathcal{O}_S \to \mathcal{E}_\rho$. The construction of these bundles yields canonical morphisms $\pi_\rho^* \mathcal{H}om(A_\rho \otimes_{\mathbb{C}} \mathcal{O}_S, \mathcal{E}_\rho)^\vee \to \mathcal{O}_{\mathbb{H}\mathrm{om}(A_\rho \otimes_{\mathbb{C}} \mathcal{O}_S, \mathcal{E}_\rho)}$. Let further

$$\begin{aligned} \mathbb{H}(\mathcal{E}) := \mathbb{H}\mathrm{om}_G(A_V \otimes_{\mathbb{C}} \mathcal{O}_S, \mathcal{E}) &:= \underline{\mathrm{Spec}}(\mathrm{S}^* \bigoplus_{\rho \in E} \mathcal{H}om(A_\rho \otimes_{\mathbb{C}} \mathcal{O}_S, \mathcal{E}_\rho)^\vee) \\ &= \prod_{\rho \in E} \mathbb{H}\mathrm{om}(A_\rho \otimes_{\mathbb{C}} \mathcal{O}_S, \mathcal{E}_\rho), \end{aligned}$$

where the product is taken over the base scheme S.

Since $\bigoplus_{\rho \in E} \mathcal{H}om(A_\rho \otimes_{\mathbb{C}} \mathcal{O}_S, \mathcal{E}_\rho) \cong \mathcal{H}om_G(\bigoplus_{\rho \in E} A_\rho \otimes_{\mathbb{C}} V_\rho \otimes_{\mathbb{C}} \mathcal{O}_S, \bigoplus_{\rho \in E} \mathcal{E}_\rho \otimes_{\mathbb{C}} V_\rho)$, the geometric vector bundle $\pi \colon \mathbb{H}(\mathcal{E}) \to S$ parameterises G–equivariant $\mathcal{O}_S$–module homomorphisms $f \colon A_V \otimes_{\mathbb{C}} \mathcal{O}_S \to \mathcal{E}$. Over any point $s \in S$ its elements are $k(s)$–linear maps $f(s) \colon A_V \otimes_{\mathbb{C}} k(s) \to \mathcal{E}(s)$. Here, the canonical morphism is

$$\alpha \colon \pi^* \mathcal{H}om_G(A_V \otimes_{\mathbb{C}} \mathcal{O}_S, \mathcal{E})^\vee \to \mathcal{O}_{\mathbb{H}(\mathcal{E})}.$$

Dualising it, we obtain a morphism $\alpha^\vee \colon \mathcal{O}_{\mathbb{H}(\mathcal{E})} \to \pi^* \mathcal{H}om_G(A_V \otimes_{\mathbb{C}} \mathcal{O}_S, \mathcal{E})$. It is determined by the image of $1 \in \mathcal{O}_{\mathbb{H}(\mathcal{E})}$, so that giving α is equivalent to giving a G–equivariant

homomorphism $\alpha'\colon \pi^*(A_V \otimes_{\mathbb{C}} \mathcal{O}_S) = A_V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{H}(\mathcal{E})} \to \pi^*\mathcal{E}$ or a collection of homomorphisms $(\alpha'_\rho\colon A_\rho \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{H}(\mathcal{E})} \to \pi^*\mathcal{E}_\rho)_{\rho\in E}$. For every homomorphism $f \in \mathbb{H}(\mathcal{E})$ we have $\alpha'(f)\colon A_V \otimes_{\mathbb{C}} k(f) \to \pi^*\mathcal{E}(f) = \mathcal{E}(s) \otimes_{k(s)} k(f)$.
The canonical morphism α has the universal property that for any pair of morphisms $(u\colon T \to S,\ a\colon u^*\mathcal{H}om_G(A_V \otimes_{\mathbb{C}} \mathcal{O}_S, \mathcal{E})^\vee \to \mathcal{O}_T)$ there exists a classifying morphism $\Psi_{u,a}\colon T \to \mathbb{H}(\mathcal{E})$ satisfying $\Psi_{u,a} \circ \pi = u$ and

$$\Psi_{u,a}^*\alpha = a\colon \Psi_{u,a}^*\pi^*\mathcal{H}om_G(A_V \otimes_{\mathbb{C}} \mathcal{O}_S, \mathcal{E})^\vee = u^*\mathcal{H}om_G(A_V \otimes_{\mathbb{C}} \mathcal{O}_S, \mathcal{E})^\vee \to \mathcal{O}_T.$$

Equivalently, for a G–equivariant $\mathcal{O}_T$–module homomorphism $a'\colon A_V \otimes_{\mathbb{C}} \mathcal{O}_T \to u^*\mathcal{E}$, the morphism $\Psi_{u,a}$ satisfies $\Psi_{u,a}^*\alpha' = a'\colon A_V \otimes_{\mathbb{C}} \mathcal{O}_T \to \Psi_{u,a}^*\pi^*\mathcal{E} = u^*\mathcal{E}$.

The open subscheme

$$\mathbb{I}\mathrm{som}_G := \mathbb{I}\mathrm{som}_G(A_V \otimes_{\mathbb{C}} \mathcal{O}_S, \mathcal{E}) := \{f \in \mathbb{H}(\mathcal{E}) \mid \det \alpha'(f) \neq 0\}$$

of G–equivariant isomorphisms $A_V \otimes_{\mathbb{C}} \mathcal{O}_S \to \mathcal{E}$ is called the *G–equivariant frame bundle associated to $\mathcal{E}$*. Here, the canonical map $\alpha'\colon A_V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{I}\mathrm{som}_G} \to \pi^*\mathcal{E}$ is a G–equivariant isomorphism. For any morphism of schemes $u\colon T \to S$ together with an isomorphism $a'\colon A_V \otimes_{\mathbb{C}} \mathcal{O}_T \to u^*\mathcal{E}$, there exists a unique morphism $\Psi_{u,a}\colon T \to \mathbb{I}\mathrm{som}_G$ such that $\Psi_{u,a} \circ \pi = u$ and $\Psi_{u,a}^*\alpha' = a'\colon A_V \otimes_{\mathbb{C}} \mathcal{O}_T \to \Psi_{u,a}^*\pi^*\mathcal{E} = u^*\mathcal{E}$.

There is an action of $\Gamma' := \prod_{\rho\in E} Gl_{r_\rho}$ on $\mathbb{H}(\mathcal{E})$ from the right by left multiplication on the components A_ρ: For closed points $s \in \underline{S}(\mathbb{C})$ and $g = (g_\rho) \in \underline{\Gamma'}(\mathbb{C})$ and $f = \oplus f_\rho$ consisting of homomorphisms $f_\rho\colon A_\rho \otimes_{\mathbb{C}} k(s) = A_\rho \to \mathcal{E}_\rho(s)$ we have $(f \cdot g)_\rho(a) = f_\rho(g_\rho a)$. If f is an isomorphism then so is $f \cdot g$, hence $\mathbb{I}\mathrm{som}_G$ is invariant under this action and $\mathbb{I}\mathrm{som}_G \to S$ is even a Zariski–locally trivial principal Γ'–bundle. In particular, the geometric quotient $\mathbb{I}\mathrm{som}_G/\Gamma'$ exists. Its elements are G–equivariant $\mathcal{O}_S$–modules isomorphic to $A_V \otimes_{\mathbb{C}} \mathcal{O}_S$ without a particular choice of an isomorphism.

B. Relative invariant Quot schemes

In the proof of Proposition 5.2.1 we need a relative version of the invariant Quot scheme. The absolute case has been studied by Jansou [Jan06] building upon the multigraded Quot scheme of Haiman and Sturmfels [HS04]. The passage from the absolute to the relative situation is standard.
Let $S \in (\mathrm{Sch}/\mathbb{C})$ and $\mathcal{X}$ a family of affine G–schemes over S. Denote $p\colon \mathcal{X} \to S$.

Definition B.1 For any G–equivariant coherent $\mathcal{O}_{\mathcal{X}}$–module $\mathcal{H}$, the *relative invariant Quot functor* is the functor

$$\begin{aligned}
&\mathcal{Q}uot^G_{\mathcal{X}/S}(\mathcal{H},h)\colon (\mathrm{Sch}/S)^{\mathrm{op}} \to (\mathrm{Set})\\
&(g\colon T\to S) \mapsto \left\{ q\colon (id_X \times g)^*\mathcal{H} \twoheadrightarrow \mathscr{F} \;\middle|\; \begin{array}{l} q \text{ a } G\text{–equivariant morphism,}\\ \mathscr{F} \text{ is } T\text{–flat,}\\ h_{\mathscr{F}} = h \end{array} \right\}\\
&\left(\begin{array}{ccc} T' & \xrightarrow{\tau} & T \\ & \searrow^{g'} & \downarrow g \\ & & S \end{array} \right) \mapsto \left(\begin{array}{rcl} \mathcal{Q}uot^G_{\mathcal{X}/S}(\mathcal{H},h)(T) & \to & \mathcal{Q}uot^G_{\mathcal{X}/S}(\mathcal{H},h)(T') \\ q & \mapsto & (id_X \times \tau)^* q \end{array} \right).
\end{aligned}$$

As in the absolute case, the invariant Quot functor is represented by a quasiprojective scheme over S:

Proposition B.2 *There is a scheme Q over S representing $\mathcal{Q}uot^G_{\mathcal{X}/S}(\mathcal{H},h)$, i.e. there exists a morphism of schemes $f\colon Q \to S$ and a universal quotient $u \in \mathcal{Q}uot^G_{\mathcal{X}/S}(\mathcal{H},h)(Q)$ such that for every morphism $g\colon T \to S$ together with a quotient $q \in \mathcal{Q}uot^G_{\mathcal{X}/S}(\mathcal{H},h)(T)$ there is a unique morphism $a\colon T \to Q$ of schemes over S satisfying $f \circ a = g$ and $(id_X \times a)^*u = q$.*

Proof. We proceed in several steps each beginning with a claim written in italic letters followed by its proof.

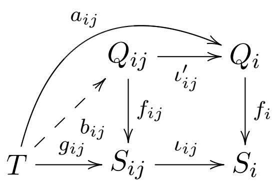
1. The construction is local in the basis: Let $S = \bigcup S_i$ with open affine schemes S_i. For every i, suppose $\mathcal{Q}uot^G_{\mathcal{X}|_{S_i}/S_i}(\mathcal{H}|_{S_i}, h)$ is represented by a scheme $f_i \colon Q_i \to S_i$ over S_i with universal quotient

$$[u_i \colon (\mathrm{id}_X \times f_i)^*(\mathcal{H}|_{S_i}) \to \mathscr{F}] \in \mathcal{Q}uot^G_{\mathcal{X}|_{S_i}/S_i}(\mathcal{H}|_{S_i}, h)(Q_i).$$

Let $S_{ij} := S_i \cap S_j$. Then for every i and j we have

$$\begin{array}{ccc} Q_{ij} := Q_i \times_{U_i} S_{ij} & \xrightarrow{\iota'_{ij}} & Q_i \\ \downarrow{\scriptstyle f_{ij}} & & \downarrow{\scriptstyle f_i} \\ S_{ij} & \xrightarrow{\iota_{ij}} & S_i \end{array}$$

Then Q_{ij} represents the functor $\mathcal{Q}uot^G_{\mathcal{X}|_{S_{ij}}/S_{ij}}(\mathcal{H}|_{S_{ij}}, h)$ with universal quotient given by $u_{ij} := (id_X \times \iota'_{ij})^* u_i \colon (id_X \times f_{ij})^*(\mathcal{H}|_{S_{ij}}) \twoheadrightarrow \mathscr{F}$. Indeed, let $g_{ij} \colon T \to S_{ij}$ be a scheme over S_{ij} and $q_{ij} \in \mathcal{Q}uot^G_{\mathcal{X}|_{S_{ij}}/S_{ij}}(\mathcal{H}|_{S_{ij}}, h)(T)$. Then T is also a scheme over S_i and we have $\mathcal{Q}uot^G_{\mathcal{X}|_{S_{ij}}/S_{ij}}(\mathcal{H}|_{S_{ij}}, h)(T) = \mathcal{Q}uot^G_{\mathcal{X}|_{S_i}/S_i}(\mathcal{H}|_{S_i}, h)(T)$. Now since Q_i represents $\mathcal{Q}uot^G_{\mathcal{X}|_{S_i}/S_i}(\mathcal{H}|_{S_i}, h)$ there is a map $a_{ij} \colon T \to Q_i$ such that $f_i \circ a_{ij} = \iota_{ij} \circ g_{ij}$ and $(id_X \times a_{ij})^* u_i = q_{ij}$. Then by the universal property of the fibred product there is a map $b_{ij} \colon T \to Q_{ij}$ satisfying $f_{ij} \circ b_{ij} = g_{ij}$ and $\iota'_{ij} \circ b_{ij} = a_{ij}$:



Thus we also have

$$\begin{aligned}(id_X \times b_{ij})^* u_{ij} &= (id_X \times b_{ij})^*(id_X \times \iota'_{ij})^* u_i = ((id_X \times \iota'_{ij}) \circ (id_X \times b_{ij}))^* u_i \\ &= (id_X \times (\iota'_{ij} \circ b_{ij}))^* u_i = (id_X \times (a_{ij}))^* u_i = q_{ij}.\end{aligned}$$

Hence Q_{ij} represents $\mathcal{Q}uot^G_{\mathcal{X}|_{S_{ij}}/S_{ij}}(\mathcal{H}|_{S_{ij}}, h)$. The same holds for Q_{ji}. Therefore, there exists a unique isomorphism $\varphi_{ij} \colon Q_{ij} \to Q_{ji}$. By its uniqueness the cocycle condition is satisfied, so that the Q_i can be glued to a scheme Q over S, which represents $\mathcal{Q}uot^G_{\mathcal{X}/S}(\mathcal{H}, h)$.

2. We can assume $\mathcal{X} = X \times S$ is a product: By step 1 we can assume that S is affine. Consider the isotypic decomposition $p_* \mathcal{O}_{\mathcal{X}} = \bigoplus_{\rho \in \mathrm{Irr}\, G} \mathcal{F}_\rho \otimes_{\mathbb{C}} V_\rho$, where $p \colon \mathcal{X} \to S$. As p is a morphism of finite type, $p_* \mathcal{O}_{\mathcal{X}}$ is finitely generated as an $\mathcal{O}_S$–algebra. Hence there are finitely many $\mathcal{F}_\rho \otimes_{\mathbb{C}} V_\rho$ such that $\mathrm{S}^*_{\mathcal{O}_S}\big(\bigoplus_{\rho \in D_{\mathcal{X}}} \mathcal{F}_\rho \otimes_{\mathbb{C}} V_\rho\big) \twoheadrightarrow p_*\mathcal{O}_{\mathcal{X}}$ is a G–equivariant surjection. Since S is affine and each $\mathcal{F}_\rho$ is coherent, there is even a free module of generators

$$\mathrm{S}^*_{\mathcal{O}_S}\Big(\bigoplus_{\rho \in D_{\mathcal{X}}} V_\rho \otimes_{\mathbb{C}} \mathcal{O}_S^{k(\rho)}\Big) = \mathcal{O}_S \otimes_{\mathbb{C}} \mathrm{S}^*\Big(\bigoplus_{\rho \in D_{\mathcal{X}}} V_\rho^{k(\rho)}\Big) \twoheadrightarrow p_*\mathcal{O}_{\mathcal{X}}.$$

Geometrically, this corresponds to an embedding $i\colon \mathcal{X} \hookrightarrow S \times X$ over S, where $X = \operatorname{Spec} \mathrm{S}^*(\bigoplus_{\rho \in D_{\mathcal{X}}} V_\rho^{k(\rho)}) \cong \mathbb{A}^{\sum_{\rho \in D_{\mathcal{X}}} k(\rho) \dim(V_\rho)}$. Hence replacing $\mathcal{H}$ by $i_*\mathcal{H}$, we can reduce to the product case.

3. We can assume that there is a G–equivariant coherent sheaf $\mathcal{H}'$ on X and a map $\nu\colon \pi^*\mathcal{H}' \twoheadrightarrow \mathcal{H}$, where $\pi\colon X \times S \to X$ is the projection: We consider the isotypic decomposition $p_*\mathcal{H} = \bigoplus_{\rho \in \operatorname{Irr} G} \mathcal{H}_\rho \otimes_{\mathbb{C}} V_\rho$. The $\mathcal{H}_\rho$ are locally free and by Step 1 we can even assume the $\mathcal{H}_\rho$ to be free. Since $\mathcal{H}$ is a coherent $\mathcal{O}_{X\times S}$–module, there is a finite subset $D_{\mathcal{H}} \subset \operatorname{Irr} G$ and for each $\rho \in D_{\mathcal{H}}$ there is an $\mathcal{O}_S$–submodule $\mathcal{U}_\rho \subset \mathcal{H}_\rho$ of finite rank such that $\mathcal{H}$ is generated by the $p^*\mathcal{U}_\rho \otimes_{\mathbb{C}} V_\rho$, $\rho \in D_{\mathcal{H}}$. Hence for every $\rho \in D_{\mathcal{H}}$ we find a surjection $\mathcal{O}_S^{m(\rho)} \twoheadrightarrow \mathcal{U}_\rho$ with $m(\rho) \in \mathbb{N}$. On $X \times S$ we obtain

$$\begin{array}{ccc}
\pi^*\Big(\bigoplus_{\rho \in D_{\mathcal{H}}} \mathcal{O}_X^{m(\rho)} \otimes_{\mathbb{C}} V_\rho\Big) = & \bigoplus_{\rho \in D_{\mathcal{H}}} \mathcal{O}_{X\times S}^{m(\rho)} \otimes_{\mathbb{C}} V_\rho & \\
& \downarrow & \searrow \\
& \bigoplus_{\rho \in D_{\mathcal{H}}} p^*\mathcal{U}_\rho \otimes_{\mathbb{C}} V_\rho & \twoheadrightarrow \mathcal{H}
\end{array}$$

Thus every quotient of $\mathcal{H}$ is also a quotient of $\pi^*\mathcal{H}'$ with $\mathcal{H}' := \bigoplus_{\rho \in D_{\mathcal{H}}} \mathcal{O}_X^{m(\rho)} \otimes_{\mathbb{C}} V_\rho$.

4. $\mathcal{Q}uot^G_{(X\times S)/S}(\mathcal{H}, h)$ is a subfunctor of $\mathcal{Q}uot^G(\mathcal{H}', h) \times S$: For a scheme T over S, we have the following commuting diagram

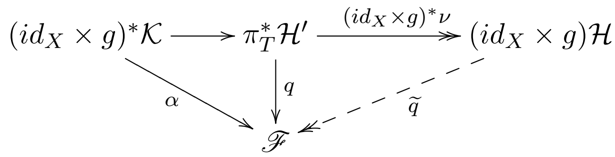
$$\begin{array}{ccccc}
X \times T & \xrightarrow{id_X \times g} & X \times S & \xrightarrow{\pi} & X \\
\downarrow{\scriptstyle p_T} & & \downarrow{\scriptstyle p} & & \\
T & \xrightarrow{g} & S & &
\end{array}
\qquad (\pi_T = \pi \circ (id_X \times g)\colon X \times T \to X)$$

If $[q\colon (id_X \times g)^*\mathcal{H} \twoheadrightarrow \mathscr{F}] \in \mathcal{Q}uot^G_{(X\times S)/S}(\pi^*\mathcal{H}', h)(T)$ then the $\mathcal{O}_{X\times T}$–module $\mathscr{F}$ is also a quotient of $(id_X \times g)^*\pi^*\mathcal{H}' = \pi_T^*\mathcal{H}'$. Therefore, we define a natural transformation $\mathcal{Q}uot^G_{(X\times S)/S}(\mathcal{H}, h) \to \mathcal{Q}uot^G(\mathcal{H}', h) \times \underline{S}$ via

$$\begin{aligned}
&\mathcal{Q}uot^G_{(X\times S)/S}(\mathcal{H}, h)(T) \to \mathcal{Q}uot^G(\mathcal{H}', h)(T) \times \underline{S}(T),\\
&[q\colon (id_X \times g)^*\mathcal{H} \twoheadrightarrow \mathscr{F}] \mapsto ([\nu \circ q\colon \pi_T^*\mathcal{H}' \to \mathscr{F}], [g\colon T \to S]).
\end{aligned}$$

5. $\mathcal{Q}uot^G_{(X\times S)/S}(\mathcal{H},h)$ is represented by a closed subscheme of $\mathrm{Quot}^G(\mathcal{H}',h)\times S$*:* By [Jan06], the scheme $\mathrm{Quot}^G(\mathcal{H}',h)$ represents the invariant Quot functor $\mathcal{Q}uot^G(\mathcal{H}',h)$. Thus, $\mathcal{Q}uot^G(\mathcal{H}',h)\times \underline{S}$ is represented by $\mathrm{Quot}^G(\mathcal{H}',h)\times S$, which is an S–scheme via the projection $\mathrm{Quot}^G(\mathcal{H}',h)\times S\to S$. So $\mathcal{Q}uot^G_{(X\times S)/S}(\mathcal{H},h)$ is represented by a closed subscheme over S if the natural transformation given in Step 3 is a closed embedding for every S–scheme T.

To show this, let $[q\colon \pi_T^*\mathcal{H}\twoheadrightarrow \mathscr{F}]\in \mathcal{Q}uot^G(\mathcal{H}',h)(T)$ for some S–scheme T. Denoting $\mathcal{K}:=\ker(\nu\colon \pi^*\mathcal{H}'\twoheadrightarrow\mathcal{H})$, we have the following diagram:



where $\widetilde{q}$ exists if and only if $\alpha := q|_{(id_X\times g)^*\mathcal{K}}=0$. Analogously to Step 2 we can find a surjection $\nu_{\mathcal{K}}\colon \pi_T^*\mathcal{K}'\twoheadrightarrow (id_X\times g)^*\mathcal{K}$ with $\mathcal{K}'=\bigoplus_{\rho\in D_{\mathcal{K}}}\mathcal{O}_X^{n(\rho)}\otimes_{\mathbb{C}} V_\rho$ and $n(\rho)\in\mathrm{Irr}\,G$ for ρ in some finite set $D_{\mathcal{K}}\subset\mathrm{Irr}\,G$. Let $\alpha':=\alpha\circ\nu_{\mathcal{K}}$. Since $\nu_{\mathcal{K}}$ is surjective, we have $\alpha=0$ if and only if $\alpha'=0$. This is the case if and only if $((p_T)_*\alpha')_\rho\colon \mathcal{O}_X^{n(\rho)}\to((p_T)_*\mathscr{F})_\rho$ vanishes for every $\rho\in D_{\mathcal{K}}$. By the following lemma, the vanishing of $((p_T)_*\alpha')_\rho$ gives us a unique closed subscheme $T_\rho\subset T$ for each ρ. Thus we obtain a closed subscheme $T_0:=\bigcap_{\rho\in D_{\mathcal{K}}}T_\rho\subset T$ describing the vanishing of $(p_T)_*\alpha'$. Applying this construction to $T=\mathrm{Quot}^G(\mathcal{H}',h)\times S$ and q the universal quotient on this scheme, we obtain a closed subscheme $Q\subset\mathrm{Quot}^G(\mathcal{H}',h)\times S$ over S such that every morphism $T'\to T$ factors through Q if and only if every quotient in $(\mathrm{Quot}^G(\mathcal{H}',h)\times S)(T')$ comes from an element in $\mathcal{Q}uot^G_{(X\times S)/S}(\mathcal{H},h)(T)$. This shows that Q represents $\mathcal{Q}uot^G_{(X\times S)/S}(\mathcal{H},h)$. □

Lemma B.3 *Let T be a scheme and $\beta\colon\mathcal{E}\to\mathcal{F}$ an $\mathcal{O}_T$–module homomorphism with $\mathcal{F}$ locally free. Then there exists a unique closed subscheme $T_0\subset T$ such that any morphism $f\colon T'\to T$ factors through T_0 if and only if $f^*\beta=0$.*

Proof. We include the proof of this well–known lemma for the convenience of the reader. Since $\mathcal{F}$ is locally free, β corresponds to a morphism $\beta'\colon\mathcal{F}^\vee\otimes_{\mathbb{C}}\mathcal{E}\to\mathcal{O}_T$. Denote the image of β', which is an ideal in $\mathcal{O}_T$, by $\mathcal{I}$. Then $f^*\beta=0\Leftrightarrow f^*\beta'=0\Leftrightarrow f^{-1}\mathcal{I}=0$ and $T_0:=V(\mathcal{I})$ has the required property. □

Definition B.4 The scheme $\mathrm{Quot}^G_{\mathcal{X}/S}(\mathcal{H},h):=Q$ over S is called the *relative invariant Quot scheme.*

Proposition B.5 *If the $\mathcal{O}_S$–G–module $p_*\mathcal{H}$ has finite multiplicities, then the relative invariant Quot scheme* $\mathrm{Quot}^G_{\mathcal{X}/S}(\mathcal{H}, h)$ *is projective over S.*

Proof. We proceed analogously to [Jan06, Proposition 1.12]. As a closed subscheme of the quasiprojective scheme $\mathrm{Quot}^G(\mathcal{H}', h) \times S$ over S, the relative invariant Quot scheme is quasiprojective over S. Thus, in order to show that the morphism $\mathrm{Quot}^G_{\mathcal{X}/S}(\mathcal{H}, h) \to S$ is projective, it suffices to show that it is proper. Therefore we use the valuative criterion of properness [Har77, Theorem II.4.7].

Let D be a discrete valuation ring over S and K its field of fractions. We denote by $p_K\colon \mathcal{X} \times_S \operatorname{Spec} K \to \operatorname{Spec} K$ and $p_D\colon \mathcal{X} \times_S \operatorname{Spec} D \to \operatorname{Spec} D$ the projections to the base schemes. We have to show that whenever there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{\ \phi\ } & \mathrm{Quot}^G_{\mathcal{X}/S}(\mathcal{H}, h) \\ \downarrow & \overset{\bar{g}}{\searrow} & \downarrow \\ \operatorname{Spec} D & \xrightarrow{\ g\ } & S \end{array}$$

then there exists a unique extension $\widetilde{\phi}\colon \operatorname{Spec} D \to \mathrm{Quot}^G_{\mathcal{X}/S}(\mathcal{H}, h)$ such that the diagramm commutes.

Such a morphism ϕ corresponds to an element in $\mathcal{Q}uot^G_{\mathcal{X}/S}(\mathcal{H}, h)(K)$, i.e. to a surjective morphism $[q\colon \mathcal{H} \otimes_{\mathcal{O}_S} K \twoheadrightarrow \mathcal{F}_K]$ of $\mathcal{O}_X \otimes_{\mathcal{O}_S} K$–modules such that in the decomposition $p_{K*}\mathcal{F}_K = \bigoplus_{\rho \in \operatorname{Irr} G} (\mathcal{F}_K)_\rho \otimes_{\mathbb{C}} V_\rho$ the sheaves of covariants $(\mathcal{F}_K)_\rho$ are K–vector spaces of dimension $h(\rho)$. Thus we have an exact sequence

$$0 \to B_K \to \mathcal{H} \otimes_{\mathcal{O}_S} K \to \mathcal{F}_K \to 0.$$

The inclusion $\mathcal{H} \otimes_{\mathcal{O}_S} K \supset \mathcal{H} \otimes_{\mathcal{O}_S} D$ allows us to define a subsheaf $B' := B_K \cap (\mathcal{H} \otimes_{\mathcal{O}_S} D)$ of $\mathcal{H} \otimes_{\mathcal{O}_S} D$, which yields a quotient $\mathcal{F}' = (\mathcal{H} \otimes_{\mathcal{O}_S} D)/B'$ of $\mathcal{H} \otimes_{\mathcal{O}_S} D$. Since G is reductive we have

$$\mathcal{F}'_\rho = (\mathcal{H}_\rho \otimes_{\mathcal{O}_S} D)/B'_\rho = (\mathcal{H}_\rho \otimes_{\mathcal{O}_S} D)/((B_K)_\rho \cap (\mathcal{H}_\rho \otimes_{\mathcal{O}_S} D)). \tag{B.1}$$

Now let $(F_D)_\rho := \mathcal{F}'_\rho/(\mathit{torsion})$. The kernel $(B_D)_\rho = (\mathcal{H}_\rho \otimes_{\mathcal{O}_S} D)/(\mathcal{F}_D)_\rho$ is the saturation of B'_ρ. We have exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B'_\rho & \longrightarrow & \mathcal{H}_\rho \otimes_{\mathcal{O}_S} D & \longrightarrow & \mathcal{F}'_\rho & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & (B_D)_\rho & \longrightarrow & \mathcal{H}_\rho \otimes_{\mathcal{O}_S} D & \longrightarrow & (\mathcal{F}_D)_\rho & \longrightarrow & 0, \end{array}$$

which become equal after tensoring with K. The $(\mathcal{F}_D)_\rho$ are torsion–free and hence they are flat D–modules. Since $\mathcal{H}_\rho$ has finite multiplicities, each $(\mathcal{F}_D)_\rho$ is finitely generated and locally free of rank $h(\rho)$.

The direct sum $B_D := p_D^*\big(\bigoplus_{\rho\in \operatorname{Irr} G}(B_D)_\rho \otimes_{\mathbb{C}} V_\rho\big)$ is a submodule of $\mathcal{H}\otimes_{\mathcal{O}_S} D$. Indeed, by the construction, B' is a submodule. Let $f \in \mathcal{O}_X$ be a function mapping B'_ρ to B'_σ. We have to show that it maps $(B_D)_\rho$ to $(B_D)_\sigma$. We have the following diagramm:

$$\xymatrix{ B'_\rho \ar[d]_{\cdot f} & \subset & (B_D)_\rho \ar@{-->}[d]^{\alpha} \ar[r] \ar@/^2pc/[rrd]^{\psi} & \mathcal{H}_\rho \otimes_{\mathcal{O}_S} D \ar[d] & \\ B'_\sigma \ar@/_2pc/[rrr]_{\varphi} & \subset & (B_D)_\sigma \ar[r] & \mathcal{H}_\sigma \otimes_{\mathcal{O}_S} D \ar[r] & (F_D)_\sigma }$$

Since $(B_D)_\sigma = \ker(\mathcal{H}_\sigma \otimes_{\mathcal{O}_S} D \to (F_D)_\sigma)$, the morphism φ is the zero map, and the same holds for the composition $\varphi \circ f$. Hence ψ factors through $(B_D)_\rho/B'_\rho$. This module is torsion since $(B_D)_\rho$ is the saturation of B'_ρ. In contrast to this, $(\mathcal{F}_D)_\sigma$ is torsion–free by its definition. Hence the image of ψ is 0. This shows that α exists and multiplication with f maps $(B_D)_\rho$ to $(B_D)_\sigma$.

Thus, the quotient $\mathcal{F}_D = (\mathcal{H}_\rho \otimes_{\mathcal{O}_S} D)/B_D$ of $\mathcal{H}\otimes_{\mathcal{O}_S} D$ is an element in $\mathcal{Q}uot^G_{\mathcal{X}/S}(\mathcal{H},h)(D)$, which corresponds to a morphism $\widetilde{\phi}\colon \operatorname{Spec} D \to \operatorname{Quot}^G_{(X\times S)/S}(\mathcal{H},h)$. Because of (B.1) we obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_D \otimes_D K & \longrightarrow & \mathcal{H}\otimes_{\mathcal{O}_S} D \otimes_D K & \longrightarrow & \mathcal{F}_D \otimes_D K & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & B_K & \longrightarrow & \mathcal{H}\otimes_{\mathcal{O}_S} K & \longrightarrow & \mathcal{F}_K & \longrightarrow & 0. \end{array}$$

Hence the restriction of $\widetilde{\phi}$ to $\operatorname{Spec} K$ is ϕ. □

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