On differential operators of Calabi-Yau type

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Introduction

The study of differential equations related to geometry is an old but still very active subject. One of its origins lies in the studies of the hypergeometric differential equation

$$x(1-x)\frac{d^2y}{dx^2} + (c-(a+b+1)x)\frac{dy}{dx} - aby = 0$$

of Euler [Eul69], Gauß [Gau12] and Kummer [Kum36], as well as in Riemann's work [Rie57] concerning the analytic continuation of local solutions of this equation.
Another important example is *Legendre's equation*

$$(1-k^2)\frac{d^2y}{dk^2} + \frac{1-3k^2}{k}\frac{dy}{dk} - y = 0,$$

which is satisfied by the elliptic integrals

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad \text{and} \quad \int_1^{1/k} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

considered as functions in k^2. The study of these integrals lead to the notion of Riemann surfaces of genus one, such that for fixed k each of the given elliptic integrals can be written as integrals of a holomorphic one-form over a one-cycle. Integrals of this type are called *periods* and determine this Riemann surface uniquely up to biholomorphy.
If we consider families of elliptic curves over a one-dimensional parameter space, it was observed by Fricke and Klein in [FK90] that an appropriate associated family of periods $y(t)$ fulfills the hypergeometric differential equation

$$\frac{d^2y(t)}{dJ(t)^2} + \frac{1}{J(t)}\frac{dy(t)}{dJ(t)} + \frac{(31/144)J(s) - 1/36}{J(t)^2(J(t)-1)^2}y(t) = 0$$

with respect to the J-function associated to the family.
The study of differential equations satisfied by periods - so-called *Picard-Fuchs equations* or *Picard-Fuchs operators* - was generalized to families of algebraic manifolds with higher dimensional fibers by Picard and Poincaré. Griffiths incorporated those studies into 20th century language of mathematics, see e.g. [Gri70] and [Gri84], which leads to *variations of Hodge structure* and *Gauß-Manin systems.*
As Riemann surfaces of genus one are Calabi-Yau manifolds, it is natural to study Picard-Fuchs equations for higher dimensional families of Calabi-Yau manifolds. Contrary to the one-dimensional case, in general there is no universal way to describe these equations for higher dimensional families as Fricke and Klein did.

The case of three dimensional fibers is of particular interest in mirror symmetry. In [COGP92], P. Candelas and his collaborators made an amazing discovery: They studied the family of quintics $X \subset \mathbb{P}^4$. It was discovered by H. Schubert in [Sch86] that such a manifold contains 2875 lines and in fact, a parameter count suggests that X contains for each degree d a finite number n_d of rational curves of degree d. Candelas et al. were able to predict these numbers by studying the Picard-Fuchs equation

$$\left(z\frac{d}{dz}\right)^4 - 5z\left(5z\frac{d}{dz}+1\right)\left(5z\frac{d}{dz}+2\right)\left(5z\frac{d}{dz}+3\right)\left(5z\frac{d}{dz}+4\right)$$

of the "mirror manifold".
These remarkable observations led to an enormous mathematical activity which both tried to explain the occurring phenomena and to establish similar results for other families of Calabi-Yau threefolds. To mention just a few of the many milestones achieved, V. Batyrev described in [Bat94] a way to construct the mirror of Calabi-Yau hypersurfaces in toric varieties via dual reflexive polytopes and a proof of the so called Mirror theorem was given by A. Givental in [Giv96] and B. Lian, K. Liu and S.-T. Yau in [LLY97]. Further Picard-Fuchs equations related to families of Calabi-Yau threefolds were constructed by V. Batyrev, D. van Straten and others in [BS95] and [BCFKS98].
Motivated by those studies, G. Almkvist, C. van Enckevort, D. van Straten and W. Zudilin started to collect differential operators which have similar algebraic properties and are potential Picard-Fuchs equations of families of Calabi-Yau threefolds. These operators are called *of CY-type*. The resulting first version of their list [AESZ05, Appendix A] contained 306 of such operators. As they fulfill very restrictive integrality properties, they are hard to find without a family at hand. In fact, only very few of them are constructed as Picard-Fuchs equations of a given family. An often applied method to find operators of this type is the following:
Take an integral hypergeometric expression C_m and compute, e.g. with Zeilbergers algorithm in `MAPLE`, a recurrence relation $r_k(m)C_{m+k} + \cdots + r_0(m)C_m$ for the coefficients C_m. This recurrence can be transformed into a differential operator L, such that $f = \sum_{m=0}^{\infty} C_m z^m$ is a solution of L. As observed by D. van Straten, some solutions of this type are *Hadamard products* of power series, i.e. they can be written as

$$\sum_{m=0}^{\infty} C_m z^m = \sum_{m=0}^{\infty} A_m B_m z^m =: \sum_{m=0}^{\infty} A_m z^m \star \sum_{m=0}^{\infty} B_m z^m,$$

where $\sum_{m=0}^{\infty} A_m z^m$ and $\sum_{m=0}^{\infty} B_m z^m$ fulfill differential equations of lower degree.
Having these observations in mind, we pose the following questions:

- Which of the operators given in [AESZ05, Appendix A] are Picard-Fuchs operators of families of Calabi-Yau threefolds?
- Is there a suitable way to classify Picard-Fuchs operators related to families of Calabi-Yau manifolds from a purely algebraic point of view?
- Which of the potential Picard-Fuchs operators can be constructed as Hadamard products of operators of lower degree?

In this thesis, we mainly concentrate on the third question but will also touch the first two by the approach we take. As pioneered by Riemann, the key idea to answer this question is to look at the solutions of the differential operators and their behavior under analytic continuation. In the case he studied first, i.e. for hypergeometric differential equations, it turns out that analytic continuation of local solutions even provides a global understanding of them. In modern language, this is rephrased by saying that the associated local system of solutions is *linearly rigid*. It is a remarkable result by N. Katz, see [Kat96], that each linearly rigid local system can be reduced to a local system of rank one - which is completely determined by a collection of points - via applications of tensor products and *middle convolutions* with Kummer sheaves. In fact, as these operations are invertible, this provides a method to construct rigid local systems. A translation to the level of fuchsian differential systems in an explicit way was established by M. Dettweiler and S. Reiter in [DR07]. We can turn a differential systems into a differential operator by choosing a cyclic vector. Moreover, we can use *middle Hadamard products* instead of middle convolutions, which correspond to the Hadamard product of power series on the level of holomorphic solutions. Hence, we detect those rigid local systems which are potentially induced by differential Calabi-Yau operators, construct related differential operators via tensor products and middle Hadamard products and investigate which of them are of CY-type. As operators of CY-type correspond to very special cyclic vectors, we rather translate tensor and middle Hadamard products to the level of differential operators directly in an appropriate way. In particular, the translation provides that resulting operators have a potential geometric background, i.e. they are of *geometric origin* in the sense of [And89]. We also turn our interest to local systems which are not linearly rigid but for which corresponding differential operators can be constructed by similar methods as used before.

To be more precise, this thesis is organized as follows:

In the **first chapter**, we briefly recall basic facts of the formal algebraic theory of linear homogeneous differential operators. This discussion particularly covers their relation to differential modules, i.e. systems of first order differential equations, as well as the formal theory of solutions. A key fact is that the category of differential modules over a field is equivalent to the category of representations of an algebraic group scheme and hence a so called Tannaka category. We also take a closer look at fuchsian differential operators which naturally appear in the geometric situation we consider.

In the **second chapter**, we briefly recall the notion of local systems, connections and Picard-Fuchs operators. We also introduce those families of Calabi-Yau manifolds which are of particular interest of our studies. Most importantly, each of these families can be realized over the rational numbers and admits a point of maximally unipotent monodromy.

In the **third chapter**, we describe algebraic properties of Picard-Fuchs operators for these special families of Calabi-Yau threefolds. Potential Picard-Fuchs operators by means of that description are called of CY-type. These operators underly quite restrictive integrality properties, which are for a given operator mostly only checked numerically.

The **fourth chapter** is devoted to the discussion of CY-type operators of degree up to five and relations amongst them which can be seen as reflection of exceptional isomorphisms between their differential Galois groups.

In the **fifth chapter**, we discuss the notion of linearly rigid local systems and extend it to arbitrary reduced algebraic subgroups of $\mathrm{GL}_n(\mathbb{C})$ via the rigidity index. Furthermore, we review the tensor product and the middle convolution of local systems with Kummer sheaves. To make those operations as explicit as possible, we relate local systems to generators of their induced monodromy representation via appropriate choices concerning the underlying topological space. This gives tuples of invertible matrices $(T_1, \dots, T_{r+1})$ such that

$$\prod_{i=1}^{r+1} T_i = \mathbb{1}$$

holds. All operations are reviewed on the level of tuples of matrices of this type, which we refer to as m-tuples.

The translation of the middle convolution and the middle Hadamard product to the level of fuchsian differential operators is done in an appropriate way in **chapter six**.

In **chapter seven**, we classify and construct those linearly rigid local systems which are potentially induced by differential operators of CY-type. This classification - which is in fact a special case of the classification of linearly rigid local systems by C. Simpson in [Sim91] - yields four families of potential CY-type operators. However, it turns out that each family seems to contain only a finite number of CY-type operators. For degree up to five, we obtain all previously known CY-type operators with linearly rigid monodromy tuple by this method.

In **chapter eight**, we investigate local systems of rank four whose rigidity index in $\mathrm{Sp}_4(\mathbb{C})$ is zero. By considering also symmetric and exterior powers, we are again able to reduce each local system of this type to a local system of rank one and construct two families of potential operators of CY-type. Similar to the linearly rigid case, it seems that each family only contains a finite number of them.

Finally, **chapter nine** is devoted to the special case of rank four local systems on $\mathbb{P}^1$ minus four points with rigidity index two in $\mathrm{Sp}_4(\mathbb{C})$. Considering additional operations, we are able to reduce each local system of this type to a local system of rank two which is induced by a Heun equation. Inverting the operations, we end up with five families of potential operators of CY-type. Again, each family seems to contain a finite number of CY-type operators, which seem to be precisely those which are induced by Heun equations of CY-type. Via this method, we find CY-type operators of degree four which were previously unknown.

The **Appendix** provides tables of Jordan matrices and additional information concerning the families of differential operators which are constructed in chapters seven, eight and nine.

Several computations in this thesis were carried out in Maple 12 which is a trandemark of Maplesoft.

Contents

Chapter 1

Differential modules and differential operators

In this chapter, we briefly recall some basic facts and constructions concerning differential operators and differential modules from an algebraic point of view. We only work over rings which contain the rational numbers. In the first section, we investigate the category of differential modules over fields and especially recall tensor operations. The second section is devoted to solution spaces of differential modules. The link to differential operators via cyclic vectors and related operations on the level of differential operators is stated in section three. In the fourth section, we consider differential modules and differential operators over the rings of formal power series and polynomials as well as over their quotient fields. Those objects can be classified via their local behaviour of solutions which is done in section five. In particular, we discuss properties of so-called regular singular and fuchsian differential modules and differential operators, as those are the ones which appear in our underlying geometric framework in the later chapters.

A more detailed account as well as further information and omitted proofs to most of the results stated here can be found in [Sin96], [PS02] and [Inc56].

Throughout the whole section, we denote by R a simple commutative differential ring, i.e. each ideal $I \subset R$ which is closed under the derivation is trivial, with quotient field k and derivation $(\cdot)'$. For simplicity, we denote the n-th differential of an element $a \in R$ by $a^{(n)}$. Furthermore, we assume that the ring of constants C is an algebraically closed field of characteristic zero. In particular, the quotient field k of R has the same field of constants then. We only investigate differential fields for which $C \neq k$ holds.

1.1 Differential modules

We first introduce the ring of differential operators over R which is one of the main objects we are dealing with.

1.1.1 Definition Let $R\{\partial\}$ be the free R-algebra generated by ∂ and $J \subset R\{\partial\}$ be the ideal generated by all elements $a\partial - \partial a - a'$ for $a \in R$. We call

$$R[\partial] := R\{\partial\}/J$$

the *ring of differential operators* over R.

One can show that the ring of differential operators $R[\partial]$ is a simple non-commutative left and right Euclidean ring. Therefore, each element of $L \in R[\partial]$ is of the form $L = \sum_{i=0}^{n} a_i \partial^i$, where the $a_i \in R$ are uniquely determined. The *degree* of L is given by

$$\deg(L) := \max\{j \in \mathbb{N} \mid a_j \neq 0\}.$$

Furthermore, each left and each right ideal of $R[\partial]$ is principal and we define the *greatest common right divisor* GCRD and the *least common left multiple* LCLM of a finite number of operators in the obvious way. We also have the natural notion of *reducibility* of operators.

Next, we introduce differential modules over R.

1.1.2 Definition A *differential R-module* is a tuple (M, ∂), where M is a finitely generated R-module and $\partial \colon M \to M$ is a map satisfying

(i) $\partial(m+n) = \partial(m) + \partial(n)$ for every $m, n \in M$.

(ii) $\partial(fm) = f'm + f\partial(m)$ for every $f \in R$ and every $m \in M$.

A *differential morphism* between differential R-modules (M, ∂) and (N, ∂) is a morphism of R-modules $\psi \colon M \to N$ satisfying

$$\partial\psi(m) = \psi(\partial m)$$

for every $m \in M$. We denote the C-vector space of differential morphisms from (M, ∂) to (N, ∂) by $\mathrm{Hom}_{R[\partial]}(M, N)$. With those settings and the obvious composition rules between morphisms, differential R-modules form a category which we denote by Diff_R.

We often denote a differential R-module (M, ∂) just by M.

Free differential R-modules have a very natural link to systems of first order differential equations over R:

Given a differential R-module (M, ∂) and an R-basis $\mathcal{E} = \{e_1, \ldots, e_n\}$, we can express the action of ∂ on M with respect to this basis via

$$\partial v = v' - A_{\mathcal{E}} v,$$

with $v' = (v_1', \ldots, v_n')^T$ and a unique $A_{\mathcal{E}} \in \mathrm{Mat}_n(R)$. We call $A_{\mathcal{E}}$ the *representation matrix* of ∂ with respect to $\mathcal{E}$. For $B \in \mathrm{GL}_n(R)$, a change of the basis $\mathcal{E}$ to $B\mathcal{E} = \{Be_1, \ldots, Be_n\}$ induces the so-called *gauge transformation*

$$A_{B\mathcal{E}} = BAB^{-1} - B^{-1}B'$$

on $A_{\mathcal{E}}$. On the other hand, a given matrix differential equation $v' = Av$ with $A \in \mathrm{Mat}_n(R)$ clearly determines a free differential R-module.

The category Diff_R is abelian. We describe some constructions in this category explicitly.

1.1.3 Definition Let $(M, \partial), (M_1, \partial), (M_2, \partial)$ be differential R-modules.

(i) A *differential submodule* of (M, ∂) is a subspace N for which $\partial N \subset N$ holds. If the submodules of M are just the trivial ones, we call M *irreducible*.

(ii) The differential R-module (S, ∂) with $S = M_1 \oplus M_2$ and

$$\partial(m_1 + m_2) = \partial(m_1) + \partial(m_2)$$

for each $m_1 \in M_1$ and $m_2 \in M_2$ is called the *direct sum* of (M_1, ∂) and (M_2, ∂).

(iii) The differential R-module (T, ∂) with $T = M_1 \otimes_R M_2$ and

$$\partial(m_1 \otimes m_2) = \partial(m_1) \otimes m_2 + m_1 \otimes \partial(m_2)$$

for each $m_1 \in M_1$ and $m_2 \in M_2$ is called the *tensor product* of (M_1, ∂) and (M_2, ∂). Symmetric and exterior products of M are defined similarly.

(iv) The differential module (H, ∂_H) with $H := \mathrm{Hom}_R(M_1, M_2)$ and

$$(\partial\varphi)(m) := \partial(\varphi(m)) - \varphi(\partial m)$$

for each $\varphi \in \mathrm{Hom}_R(M_1, M_2)$ and $m \in M_1$ is called the module of *inner morphisms* from M_1 to M_2. The *dual module* $M^\vee$ to M is the differential R-module $(\mathrm{Hom}_R(M, R), \partial)$ of inner morphisms from M to R.

Given two free differential R-modules M_1 and M_2, many of the natural isomorphisms on the level of free R-modules - such as for instance $(\mathrm{Hom}_R(M_1, M_2), \partial) \cong (M_1^\vee \otimes_R M_2, \partial)$ and $(M, \partial) \cong \left(((M^\vee)^\vee, \partial\right)$ - can be regarded as differential isomorphisms.

1.2 Solutions

A very important tool to describe the category Diff_k of differential modules over a field is the covariant solution space. Thinking of a differential module as a first order matrix differential equation $v' - Av$, a solution in a finite extension $S \supset k$ of differential fields is nothing else than a vector $w \in S^n$, for which $w' - Aw = 0$ holds. The translation to the notion of a differential module yields the following:

1.2.1 Definition Consider a differential k-module (M, ∂) and a simple differential ring S, such that $S \supset k$ is an extension of differential rings and the quotient field of S again has constants $C = C(R)$. We call the C-vector space $\ker(\partial, S \otimes_k M)$ the *S-valued covariant solution space* of (M, ∂).

Given a differential k-module (M, ∂), a classical Wronskian argument reveals that we have

$$\dim_C(\ker(\partial, S \otimes_k M) \leq \dim_k(M)$$

for each appropriate extension $S \supset R$. We look for a minimal extension of k such that the covariant solution space of M has maximal dimension. Similar to the algebraic closure of a field, one can even construct a differential ring $\mathcal{F}$ such that the covariant solution space of every differential k-module has maximal dimension. To be more precise, we state:

1.2.2 Definition A differential ring $\mathcal{F} \supset k$ is called an *universal Picard-Vessiot ring* of k, if

(i) $\mathcal{F}$ is simple and its field of constants is C.

(ii) $\dim_C \ker(\partial, \mathcal{F} \otimes_k M) = \dim_k(M)$ for any differential k-module M.

(iii) $\mathcal{F}$ is minimal in the sense that every other simple differential ring S satisfying the two properties above contains a differential subring isomorphic to $\mathcal{F}$.

Its quotient field is called a *universal Picard-Vessiot field* of k. A *Picard-Vessiot ring* of M is a with respect to inclusion minimal subring $\mathrm{PV} \subset \mathcal{F}$ such that $\dim_C \ker(\partial, \mathrm{PV} \otimes_k M) = \dim_k(M)$ holds.

It is shown in [PS02, Chapter 10] that each differential field k admits a universal Picard-Vessiot ring which is unique up to isomorphism. Therefore, we may speak of the universal Picard-Vessiot ring of k and the Picard-Vessiot ring of a differential k-module.

1.2.3 Remark. (i) For each differential k-module M with Picard-Vessiot ring PV, we call $\ker(\partial, \mathrm{PV} \otimes_k M)$ just the solution space of M. Furthermore, the functor $\ker(\partial, \mathcal{F} \otimes_k \cdot)$ - where $\mathcal{F}$ is the universal Picard-Vessiot ring of k - from Diff_R to the category of finite dimensional C-vector spaces is covariant exact.

(ii) For each differential k-module M with Picard-Vessiot ring PV, there is a canonical isomorphism

$$\mathrm{PV} \otimes_C \ker(\partial, \mathrm{PV} \otimes_k M) \cong \mathrm{PV} \otimes_k M.$$

In particular, each C-basis of $\ker(\partial, \mathrm{PV} \otimes_k M)$ can be extended to a PV-basis of $\mathrm{PV} \otimes_k M$. We call such a basis a *horizontal basis* of $\mathrm{PV} \otimes_k M$.

Similar to the case for ordinary field extensions, we may attach a Galois group to a differential module. This group encodes important algebraic properties of both the differential module and its solutions.

1.2.4 Definition For a differential k-module M with Picard-Vessiot ring PV, we call the group

$$\mathrm{Gal}(M) := \mathrm{Aut}_{k[\partial]}(\mathrm{PV})$$

its *differential Galois group*. Further, we call the group $\mathrm{Aut}_{k[\partial]}(\mathcal{F})$, where $\mathcal{F}$ denotes the universal Picard-Vessiot ring of k, the *absolute differential Galois group* of k.

The differential Galois group $G = \mathrm{Gal}(M)$ of a given differential k-module M with Picard-Vessiot ring PV acts on its covariant solution space via

$$G \circlearrowleft \ker(\partial, \mathrm{PV} \otimes_k M), \ \gamma(r \otimes m) := \gamma(r) \otimes m,$$

for each $r \in \mathrm{PV}$ and each $m \in M$. Therefore, G has a representation in $\mathrm{GL}_n(C)$ for $n = \dim_k M$. This representation is faithful and G is an algebraic group.
By [PS02, Theorem 2.33], tensor operations on objects of Diff_k are in one to one correspondence to representations of their differential Galois groups.

1.2.5 Theorem *Let $\mathcal{F} \supset k$ be the universal Picard-Vessiot ring and G the absolute Galois group of k. Denote the category of finite dimensional G-modules over C by $\mathrm{Repr}_G(C)$. The attachment*

$$\mathrm{Diff}_k \to \mathrm{Repr}_G(C), \ M \mapsto \ker(\partial, \mathcal{F} \otimes_R M)$$

gives rise to a covariant exact functor which commutes with taking submodules, quotients, duals, direct sums and all tensor operations. Moreover, this functor induces an equivalence of categories.

In more sophisticated terms, Theorem 1.2.5 states that Diff_k is a *neutral Tannaka category*. This theorem also allows us to transport notions of representation theory to differential modules in the obvious way.

1.3 Differential operators

We link differential k-modules and differential operators in $k[\partial]$. One direction of this link is immediate: For a differential operator $L \in k[\partial]$, the quotient $M_L := k[\partial]/k[\partial]L$ has a natural structure as a differential k-module. The other direction is provided by choosing special vectors in a differential module.

1.3.1 Definition Given a differential k-module M, we consider for each element $m \in M$ the evaluation map

$$ev_m \colon k[\partial] \to M, \sum_{i=0}^{n} a_i \partial^i \mapsto \sum_{i=0}^{n} a_i \partial^i m.$$

We call the monic generator of the kernel of ev_m as a left ideal the *minimal operator* of m over $k[\partial]$. Furthermore, we call m a *cyclic vector* of M if the degree of its minimal operator equals the k-dimension of M, i.e. the set $\{m, \partial m, \ldots, \partial^{\dim_k(M)-1} m\}$ is a k-basis of M. We call a pair (M, e) consisting of a differential module M and a cyclic vector $e \in M$ a *marked differential module*.

For our purposes, we can use the following result by N. Katz:

1.3.2 Proposition *Suppose that the field of constants C of k is algebraically closed. Then each differential k-module M has a cyclic vector. In particular, there is a differential operator $L \in k[\partial]$ such that M is isomorphic to $k[\partial]/k[\partial]L$.*

In fact, this proposition implies that there is a one to one correspondence between monic differential operators $L \in k[\partial]$ and marked differential modules (M, e). This allows us to transport constructions and notions for differential modules to the level of differential operators. We first investigate differential isomorphisms. Given two marked differential k-modules $(M_1, e_1), (M_2, e_2)$ together with a differential isomorphism $\varphi \colon M_1 \to M_2$, it is clear that $(M_2, \varphi(e_1))$ is also a marked differential k-module. The corresponding base change leads to the following relation of the minimal operators of e_1 and e_2:

1.3.3 Proposition *For each two differential operators $L_1, L_2 \in k[\partial]$, the differential k-modules $k[\partial]/k[\partial]L_1$ and $k[\partial]/k[\partial]L_2$ are isomorphic if and only if $\deg(L_1) = \deg(L_2)$ and there exist non-zero $P, Q \in k[\partial]$ with $\deg(P) < \deg(L_1)$, $\deg(Q) < \deg(L_1)$, $L_1 P = Q L_2$ and $\mathrm{GCRD}(P, L_2) = 1$.*

Next, we transport operations for differential modules to the level of differential operators in a natural way. We start with the tensor operations.

1.3.4 Definition Consider monic differential operators $L_1, L_2, L \in k[\partial]$ with associated marked differential modules $(M_1, e_1), (M_2, e_2)$ and (M, e). The *tensor product* $L_1 \otimes L_2 \in k[\partial]$ of L_1 and L_2 is the minimal operator of $e_1 \otimes e_2 \in M_1 \otimes M_2$. Similarly, the *$n$-th exterior power* $\bigwedge^n L \in k[\partial]$ of L is the minimal operator of $e \wedge \partial e \wedge \cdots \wedge \partial^{n-1} e \in \bigwedge^n M$. and the *$n$-th symmetric power* $\mathrm{Sym}^n L \in k[\partial]$ of L is the minimal operator of $e \cdot \cdots \cdot e \in \mathrm{Sym}^n M$.

In general, $(M_1 \otimes M_2, e_1 \otimes e_2)$ is not a marked differential module and therefore the degree of $L_1 \otimes L_2$ may be smaller than the dimension of $M_1 \otimes M_2$. The same observation holds for exterior and symmetric powers of a differential operator. The situation for the dual module is as follows:

1.3.5 Proposition *Consider a marked differential k-module (M, e) with corresponding minimal operator $L_e = \partial^n + \sum_{i=0}^{n-1} a_i \partial^i$. Furthermore, write $\mathcal{E} := \{e, \partial e, \ldots, \partial^{n-1} e\}$ and denote by $\mathcal{E}^\vee := \{e^\vee, \ldots, (\partial^{n-1} e)^\vee\} \subset M^\vee$ the associated dual k-basis of $M^\vee$ as a k-vector space. Then*

(i) *setting $(\partial^{-1} e)^\vee := 0$, we have*

$$\partial \left(\partial^i e\right)^\vee = a_i \left(\partial^{n-1} e\right)^\vee - \left(\partial^{i-1} e\right)^\vee$$

for every $i = 0, \ldots, n-1$.

(ii) $\left(M^\vee \left(\partial^{n-1} e\right)^\vee\right)$ *is a marked differential k-module with corresponding minimal operator*

$$L_{(\partial^{n-1} e)^\vee} = \partial^n + \sum_{i=0}^{n-1} (-1)^{n+i} \partial^i a_i \in k[\partial].$$

The proposition above yields the following notion of the dual of a differential operator.

1.3.6 Definition Given $L = \sum_{i=0}^{n} a_i \partial^i \in k[\partial]$ we call

$$L^\vee := \sum_{i=0}^{n} (-1)^i \partial^i a_i \in k[\partial]$$

its *dual operator*.

For a monic differential operator $L = \partial^n + \sum_{i=0}^{n-1} a_i \partial^i \in k[\partial]$, we have a very intuitive notion of a solution, namely an element f in any differential extension of k such that $L(f) = f^{(n)} + \sum_{i=0}^{n-1} a_i f^{(i)} = 0$ holds. We formalize this notion further by introducing another type of solution spaces for differential modules.

1.3.7 Definition Given a differential k-module M and a simple differential ring S such that $S \supset k$ is an extension of differential rings and the quotient field of S again has constants C, we call the C-vector space $\mathrm{Hom}_{k[\partial]}(M, S)$ the *S-valued contravariant solution space* of M. For $m \in M$ with minimal operator L, we call the image of the map

$$\chi_m \colon \mathrm{Hom}_{k[\partial]}(M, S) \to S, \ \chi_m(\psi) := \psi(m)$$

the *S-valued solution space* $\mathrm{Sol}_L(S)$ of L. If S is the Picard-Vessiot ring of M, we just write Sol_L for the S-valued solution space of L.

As for the covariant solution space, we have the inequalities $\dim_C(\mathrm{Hom}_{k[\partial]}(M, S)) \leq \dim_k(M)$ and $\dim_C(\mathrm{Sol}_L(S)) \leq \deg(L)$, where equality holds if S is the universal Picard-Vessiot ring of M.

1.3.8 Definition For $L \in k[\partial]$ monic with corresponding marked differential module (M, e), we define its *Picard-Vessiot ring* to be the Picard-Vessiot ring of M and its *differential Galois group* to be the differential Galois group of M.

Given a differential k-module M with Picard-Vessiot ring PV, its differential Galois group $G = \mathrm{Gal}(M)$ acts on its contravariant solution space via

$$G \circlearrowleft \mathrm{Hom}_{k[\partial]}(M, \mathrm{PV}), \;\; \gamma.\varphi(m) := \gamma(\varphi(m)).$$

The co- and contravariant solution space of M are dual to each other as G-modules.
It is natural to ask which subsets of the universal Picard-Vessiot ring are solution spaces of differential operators. The answer is given by the following:

1.3.9 Lemma *Consider the universal Picard-Vessiot ring $\mathcal{F}$ of k with absolute differential Galois group G and a finite dimensional C-vector space $V \subset \mathcal{F}$. Then there is a differential operator $L \in k[\partial]$ whose solution space is V if and only if the action of G on $\mathcal{F}$ leaves V invariant as a set.*

With this result at hand, we can reinterpret tensor constructions for differential modules on the level of their solution spaces:

1.3.10 Proposition *Consider monic differential operators $L_1, L_2, L \in k[\partial]$ with associated marked differential modules $(M_1, e_1), (M_2, e_2)$ and (M, e). Then*

(i) *the solution space of $L_1 \otimes L_2$ is spanned by $\{y_1 y_2 \mid y_1 \in \mathrm{Sol}_{L_1}, y_2 \in \mathrm{Sol}_{L_2}\}$.*

(ii) *the solution space of $\bigwedge^n L$ is spanned by*

$$\{\mathrm{Wr}(y_1, \dots, y_n) \mid y_i \in \mathrm{Sol}_L \text{ for all } i = 1, \dots, n\},$$

where

$$\mathrm{Wr}(y_1, \dots, y_n) := \det \begin{pmatrix} y_1 & \cdots & y_n \\ \frac{d}{dz} y_1 & \cdots & \frac{d}{dz} y_n \\ \vdots & \vdots & \vdots \\ \left(\frac{d}{dz}\right)^{n-1} y_1 & \cdots & \left(\frac{d}{dz}\right)^{n-1} y_n \end{pmatrix}.$$

(iii) *the solution space of $\mathrm{Sym}^n L$ is spanned by $\{y_1 \cdot \dots \cdot y_n \mid y_i \in \mathrm{Sol}_L \text{ for all } i = 1, \dots, n\}$.*

We end the general discussion of differential modules and differential operators by a consequence of Theorem 1.2.5, see [PS02, Corollary 2.35], which relates factors of a monic differential operator to submodules of its corresponding marked differential module.

1.3.11 Corollary *For each monic differential operator $L \in k[\partial]$ of degree $\deg(L) \geq 1$ with corresponding marked differential module (M, e), Picard-Vessiot ring PV and differential Galois group G, there are natural bijections between*

(i) *the G-invariant subspaces of $\ker(\partial, \mathrm{PV} \otimes_k M)$,*

(ii) *the submodules of M and*

(iii) *the monic right factors of L.*

1.4 Local and rational differential modules

In this section, we investigate differential operators and differential modules over the rings $\mathbb{C}[z]$ of polynomials and $\mathbb{C}[\![z]\!]$ of formal power series and their quotient fields $\mathbb{C}(z)$ and $\mathbb{C}((z))$. We equip the ring $\mathbb{C}[\![z]\!]$ with the valuation

$$\nu\left(\sum_{m=0}^{\infty} A_m z^m\right) := \min\{i \mid A_i \neq 0\}.$$

Differential objects over $\mathbb{C}[\![z]\!]$ are called *local* and differential objects over $\mathbb{C}[z]$ are called *rational*. On each of those rings, we consider the usual derivation $\partial_z := \frac{d}{dz}$ and the logarithmic derivation $\vartheta_z := z\partial_z$. We denote the ring of differential operators with respect to (R, ∂_z) by $R[\partial]$ and the ring of differential operators with respect to (R, ϑ_z) by $R[\vartheta]$ for each $R \in \{\mathbb{C}[z], \mathbb{C}[\![z]\!], \mathbb{C}(z), \mathbb{C}((z))\}$. Note, that for each differential operator $L \in k[\partial]$, there is an element $g \in R$ such that $gL = \sum_{i=0}^{n} a_i \partial^i \in R[\partial]$ is *reduced*, i.e. $\gcd(a_0, \ldots, a_n) = 1$. For simplicity, we identify gL with L and can hence also write $L \in R[\partial]$. We use the same notational convention if we replace ∂ by ϑ.

Moreover, we recently write

$$\mathbb{C}[z, \vartheta] := \mathbb{C}[z][\vartheta]$$

and denote elements $L \in \mathbb{C}[z, \vartheta]$ in the form $L = \sum_{i=0}^{m} z^i P_i$, with $P_0, \ldots, P_m \in \mathbb{C}[\vartheta]$.

Consequently, we denote a differential module over (R, ∂_z) by (M, ∂) and a differential module over (R, ϑ_z) by (M, ϑ). For $k = \mathbb{C}(z)$ or $\mathbb{C}((z))$, it is easy to see that $k[\partial]$ and $k[\vartheta]$ are isomorphic to each other.

In the sequel, we will freely switch between these two rings of differential operators, since each derivation has its own advantages. In particular, we can regard a differential k-module (M, ∂) as differential k-module (M, ϑ) and vice versa.

We investigate the induced action of $\mathrm{End}_{\mathbb{C}}(k)$ on the ring of differential operators over k. We only study those endomorphisms which are completely determined by their image of z. In particular, we can define such an endomorphism on $\mathbb{C}(z)$ for each $f \in \mathbb{C}(z) \setminus \mathbb{C}$ and on $\mathbb{C}((z))$ for each $f \in z\mathbb{C}[\![z]\!]$. We extend the action of each such an endomorphism to the ring $k[\partial]$ in the following way:

1.4.1 Definition The linear map

$$f^*\colon k[\partial] \to k[\partial], z \mapsto f(z),\ \partial \mapsto \frac{1}{f'}\partial$$

is called the *twist* by f. If f^* is invertible, we furthermore write

$$f^\vee := (f^*)^{-1}(z).$$

1.4.2 Proposition *The twist f^* is an endomorphism of $k[\partial]$. Furthermore, f^* is an isomorphism if and only if the map $z \mapsto f(z)$ is an isomorphism on k.*

1.4.3 Remark. (i) As the rings $k[\partial]$ and $k[\vartheta]$ are isomorphic, twists of the rings $f^*(k[\vartheta])$ are given by

$$f^*\vartheta = \frac{f}{\vartheta_z(f)}\vartheta.$$

(ii) For each two elements $f, g \in \mathbb{C}(z)$ or $z\mathbb{C}[\![z]\!]$ we have the transformation rules $(f \circ g)^* = g^* f^*$ and $(f \circ g)^\vee = g^\vee \circ f^\vee$, if both elements induce isomorphisms.

Differential objects over $\mathbb{C}(z)$ and $\mathbb{C}((z))$ are related via natural localization maps:

1.4.4 Definition For each $p \in \mathbb{C}$, we consider the translation isomorphism

$$\tau_p \colon \mathbb{C}(z)[\partial] \to \mathbb{C}(z)[\partial],\ z \mapsto z+p,\ \partial \mapsto \partial$$

and the morphism of differential rings $\delta \colon \mathbb{C}(z)[\partial] \to \mathbb{C}((z))[\partial]$ which sends each $f \in \mathbb{C}(z)$ to its Laurent-expansion at 0. We call

$$\iota_p \colon \mathbb{C}(z) \to \mathbb{C}((z)),\ \iota_p := \delta \circ \tau_p$$

the *localization map* at $z = p$ and $\iota_p(\mathbb{C}(z)[\partial])$ the *localization at* p. For a rational differential module (M, ∂), we define its *localization at* p $(\iota_p(M), \partial)$ by the local differential module $\iota_p(M) := \mathbb{C}((z)) \otimes_{\mathbb{C}(z)} M$ with respect to ι_p.

We get a completely similar notion of the localization at ∞ by putting $\tau_\infty(z) = 1/z$ and $\tau_\infty(\partial) = -z^2\partial$. Note, that $\iota_p(\vartheta) = \vartheta(1 + p/z)$ for $p \in \mathbb{C}$ while $\iota_\infty(\vartheta) = -\vartheta$.

1.5 Singularities of local differential modules

We state the explicit description of the universal Picard-Vessiot ring of $(\mathbb{C}((z)), \vartheta_z)$ due to P.A. Hendriks and M. van der Put. The construction of this ring can be reviewed in [PS02, Section 3.2].

1.5.1 Proposition *The universal Picard-Vessiot ring of* $(\mathbb{C}((z)), \vartheta_z)$ *is given by*

$$\mathcal{F} = \mathbb{C}((z))\left[\{z^a\}_{a\in\mathbb{C}}, \{e(q)\}_{q\in\mathcal{Q}}, \ell\right],$$

where

(i) $\mathcal{Q} = \bigcup_{m\geq 1} \frac{1}{z^m}\mathbb{C}\left[\frac{1}{z}\right]$

(ii) *the only relations between the occurring symbols are* $z^{a+b} = z^a z^b$ *for all* $a, b \in \mathbb{C}$, $z^k \in \mathbb{C}((z))$ *for each* $k \in \mathbb{Z}$, $e(q_1 + q_2) = e(q_1)e(q_2)$ *for all* $q_1, q_2 \in \mathcal{Q}$ *and* $e(0) = 1$.

(iii) *the extension of* ϑ_z *to* $\mathcal{F}$ *is given by* $\vartheta_z(z^a) = az^a$, $\vartheta_z(e(q)) = qe(q)$ *and* $\vartheta_z(\ell) = 1$.

As remarked in [PS02], for the localization $(\iota_p(M), \vartheta)$ of a differential $\mathbb{C}(z)$-module (M, ϑ) at a point $p \in \mathbb{C}$, one can interpret the additional symbols occurring in the Picard-Vessiot ring of $\iota_p(M)$ intuitively as the functions $\ell = \ln(z)$, $z^a = \exp(\ln(a)z)$ and $e(q) = \exp\left(\int q\frac{dz}{z}\right)$ on a proper sector S centered at $z = 0$.
With this explicit description of the universal Picard-Vessiot ring at hand, we introduce the following notions of singularities of local differential modules.

1.5.2 Definition We call a local differential module M with Picard-Vessiot ring PV

(i) *regular*, if $\mathrm{PV} \subset \mathbb{C}((z))$.

(ii) *regular singular*, if it is not regular and $\mathrm{PV} \subset \mathbb{C}((z))[\{z^a\}_{a\in\mathbb{C}}, \ell]$.

(iii) *irregular* else.

We call an operator $L \in \mathbb{C}((z))[\vartheta]$ regular, resp. regular singular, resp. irregular, if its associated differential module $\mathbb{C}((z))[\vartheta]/\mathbb{C}((z))[\vartheta]L$ is. For a rational differential module N, we call $p \in \mathbb{P}^1$ a *singularity*, if its localization $\iota_p(N)$ is not regular. Further, we call $p \in \mathbb{P}^1$ a *regular singularity*, if $\iota_p(N)$ is regular singular and an *irregular singularity*, if $\iota_p(N)$ is irregular. If N has no irregular singularities, we call it a *fuchsian* differential module.

We are mainly interested in fuchsian differential modules. Therefore, we first state criteria to check if a local differential module is not irregular, which can be reproduced from [PS02, Definition 3.9, Definition 3.11 and Proposition 3.16]

1.5.3 Proposition *For each marked differential $\mathbb{C}((z))$-module (M, e) with corresponding minimal operator $L = \vartheta^n + \sum_{i=0}^{n-1} a_i\vartheta^i$ the following statements are equivalent*

(i) *M is not irregular.*

(ii) *M contains a $\mathbb{C}[\![z]\!]$-lattice $\Sigma \subset M$, which is invariant under the action of ϑ.*

(iii) *The coefficients of L satisfy $\nu(a_i) \geq 0$ for all $0 \leq i \leq n-1$.*

The differential Galois group of a non-irregular differential module is generated by the action of one single element of the absolute differential Galois group of $(\mathbb{C}((z)), \vartheta_z)$ on the corresponding covariant solution space, the so-called formal monodromy.

1.5.4 Definition Consider the universal Picard-Vessiot ring $\mathcal{F}$ of $\mathbb{C}((z))$.
The *formal monodromy* $\gamma \in \mathrm{Aut}_{\mathbb{C}((z))[\vartheta]}(\mathcal{F})$ is given by

(i) $\gamma(z^a) := \exp(2\pi i a)z^a$ for all $a \in \mathbb{C}$.

(ii) $\gamma(\ell) := \ell + 2\pi i$.

(iii) $\gamma(e(q)) := e(\gamma(q))$ for all $q \in \mathcal{Q}$.

For a local differential module M, we call the natural action of γ on its covariant solution space the *formal monodromy* of M. For a differential operator $L \in \mathbb{C}((z))[\partial]$, we call the natural action of γ on Sol_L the *formal monodromy* of L.

We want to study and describe the solution spaces of non-irregular differential modules and the action of the formal monodromy on them in more detail. For non-regular differential operators, the best known and most successful method to compute the solution space is a classical one named after Frobenius, although probably already used by Euler. An outline of this method is presented in [Inc56, Chapter XVI] and [CC87]. We do not reproduce the whole procedure here but only state some consequences of it which mainly concern the eigenvalues of the formal monodromy. To determine eigenvalues of the action of the formal monodromy and to make statements on the shape of its Jordan form, it is appropriate to introduce the classical notion of the indicial equation of an operator.

1.5.5 Definition For

$$0 \neq L = \sum_{i=0}^{n} a_i \vartheta^i = \sum_{i=0}^{n} \left(\sum_{j=-k}^{\infty} a_{i,j} z^j \right) \vartheta^i \in \mathbb{C}((z))[\vartheta]$$

and $v = \min\{\nu(a_i) \mid 0 \leq i \leq n\}$, we call

$$\operatorname{Ind}(L) := \sum_{i=0}^{n} a_{i,v} T^i \in \mathbb{C}[T]$$

the *indicial equation* of L. The roots of $\operatorname{Ind}(L)$ are called the *exponents* of L. The *multiplicity* of an exponent is its multiplicity as a root of $\operatorname{Ind}(L)$. For $L \in \mathbb{C}(z)[\vartheta]$ and a point $p \in \mathbb{P}^1$, we set

$$\operatorname{Ind}_p(L) := \operatorname{Ind}(\iota_p(L)).$$

We have the following useful:

1.5.6 Lemma *A differential operator $0 \neq L \in \mathbb{C}((z))[\vartheta]$ is not irregular if and only if* $\deg(\operatorname{Ind}(L)) = \deg(L)$.

Proof As left multiplication with elements in $\mathbb{C}((z))$ does not change the indicial equation, we may assume that L is monic. If L is additionally regular singular, then we have $L = \vartheta^n + \sum_{j=0}^{n-1} \left(\sum_{i=0}^{\infty} a_i z^i \right) \vartheta^j$ and thus $\deg(\operatorname{Ind}(L)) = n = \deg(L)$. If $\deg(\operatorname{Ind}(L)) = n = \deg(L)$, we have $\min\{r \in \mathbb{Z} \mid a_{r,j} \neq 0 \text{ for some } j\} = 0$, since L is monic and so $L = \vartheta^n + \sum_{i=0}^{n-1} b_i \vartheta^i$ with $\deg(b_i) \geq 0$ for all $i = 0, \dots, n-1$. By Proposition 1.5.3, L is not irregular. □

The relation between the exponents of an operator and the shape of its solutions is as follows:

1.5.7 Proposition *Consider a regular singular differential operator $0 \neq L \in \mathbb{C}((z))[\vartheta]$ with set of exponents $E = \{\lambda_1, \dots, \lambda_r\}$ and for each exponent λ_i the set $\lambda_i + \mathbb{Z}$ equipped with the natural order $\leq$ on it.*

(i) *If λ_i biggest exponent of L which lies in $\lambda_i + \mathbb{Z}$, then* $\operatorname{Sol}(L)$ *contains an element of* $z^{\lambda_i} \mathbb{C}[\![z]\!]^*$.

(ii) *If λ_i has multiplicity m, there are elements $f_0, \dots, f_{m-1} \in z^{\lambda_i} \mathbb{C}[\![z]\!]$ such that* $\operatorname{Sol}(L)$ *contains $\sum_{j=0}^{k} \frac{1}{j!} \ell^j f_{k-j}$ for each $0 \leq k \leq m-1$.*

(iii) *If* $\operatorname{Sol}(L) \cap z^a \mathbb{C}[\![z]\!]^* \neq \{0\}$, *then L has an exponent in $a + \mathbb{Z}_{\geq 0}$.*

As a direct consequence concerning the formal monodromy, we get:

1.5.8 Corollary *Consider a local regular singular differential module M, its formal monodromy γ and a cyclic vector $e \in M$ with minimal operator L. Furthermore, denote the exponents of L by $E = \{\lambda_1, \dots, \lambda_r\}$.*

(i) *The set of eigenvalues of γ is precisely given by* $\{\exp(2\pi i \lambda_1), \dots, \exp(2\pi i \lambda_r)\}$.

(ii) *If λ_i has multiplicity k, the Jordan form of γ has a block of size k attached to the eigenvalue* $\exp(2\pi i \lambda_i)$.

Note, that the exponents of a local differential operator determine the Jordan form of its formal monodromy completely, if each two exponents do not differ by a non-zero integer. Otherwise, there are also criteria to determine the Jordan form which we do not consider as they are unfortunately quite unhandy to use for our purposes.

For operations on local differential modules, it is clear how the formal monodromy acts on the resulting solution space by Theorem 1.2.5. The corresponding operations for differential operators also induce operations on its indicial equations. We present some special cases, which we will use frequently in the sequel and can be proven directly.

1.5.9 Lemma *For each* $0 \neq L \in \mathbb{C}(z)[\vartheta]$ *and* $p \in \mathbb{C}$, *we have*

(i)
$$\mathrm{Ind}_p(L \otimes R)(T) = \mathrm{Ind}_p(L)\,(T + \mathrm{Ind}_p(R)(0))$$
for each differential operator $R \in \mathbb{C}(z)[\vartheta]$ *of degree one such that* $\iota_p(R)$ *is not irregular.*

(ii)
$$\mathrm{Ind}(f^*(\iota_p(L))) = \mathrm{Ind}_p\left(\frac{1}{\nu(f)}L\right)$$
for each $f \in z\mathbb{C}[\![z]\!]$.

1.6 Fuchsian differential modules

We discuss fuchsian differential modules and their relations to the classical notion of a fuchsian differential operator. One of the main differences to the local case is that we have a very natural notion of a singularity for operators $L \in \mathbb{C}(z)[\partial]$ which is slightly different to the one for the associated differential module $M = \mathbb{C}(z)[\partial]/\mathbb{C}(z)[\partial]L$.

1.6.1 Definition The *singular locus* $S \subset \mathbb{P}^1$ of a monic differential operator $L = \sum_{i=0}^n a_i\partial^i \in \mathbb{C}(z)[\partial]$ is the union of the singularities of the a_i together with the point ∞. The points $p \in \mathbb{P}^1 \setminus S$ are the *regular points* of L. Furthermore, we call $s \in S$

(i) an *apparent singularity* of L, if $\iota_s(L)$ is regular.

(ii) a *regular singularity* of L, if $\iota_s(L)$ is regular singular.

(iii) an *irregular singularity* of L, if $\iota_s(L)$ is irregular.

We call L *fuchsian*, if it has no irregular singularities.

1.6.2 Remark. For a monic operator $L = \sum_{i=0}^n a_i\partial^i \in \mathbb{C}(z)[\partial]$ and $p \in \mathbb{C}$, we have an alternative formula for the indicial equation $\mathrm{Ind}_p(L)$, namely

$$\mathrm{Ind}_p(L) = \sum_{i=0}^{n} \mathrm{res}_{z=p}\left((z-p)^{n-i}a_i\right)\prod_{j=0}^{i-1}(T-j) \in \mathbb{C}[T].$$

In particular, if $p \notin S$, the set of exponents of $\iota_p(L)$ reads $\{0, 1, \ldots, n-1\}$ but the converse is not true.

By [PS02, Lemma 6.11], the coefficients of a fuchsian differential operator are of the following shape.

1.6.3 Lemma *A monic differential operator $L = \sum_{i=0}^{n} a_i \partial^i \in \mathbb{C}(z)[\partial]$ of degree n with singular locus $\{s_1, \ldots, s_r\} \cup \{\infty\} \subset \mathbb{C} \cup \{\infty\}$ is fuchsian if and only if*

$$a_k = \frac{b_k}{(z - s_1)^{n-k} \ldots (z - s_r)^{n-k}},$$

where b_k is a polynomial of degree less or equal than $(n-k)(r-1)$.

As already observed by L. Fuchs the exponents of a fuchsian differential operator satisfy the so-called *Fuchs relation.*

1.6.4 Proposition *For each fuchsian operator L with singular locus $S = \{s_1, \ldots, s_r, \infty\}$, the roots $\lambda_{1,j}, \ldots, \lambda_{n,j}$ of $\mathrm{Ind}_{s_j}(L)$ and $\lambda_{1,r+1}, \ldots, \lambda_{n,r+1}$ of $\mathrm{Ind}_{\infty}(L)$ - which are all counted with multiplicity - fulfill the equation*

$$\sum_{i,j} \lambda_{i,j} = (r-1)\binom{n}{2}.$$

By [PS02, Proposition 3.16], the link to fuchsian differential modules is as follows.

1.6.5 Proposition *A differential $\mathbb{C}(z)$-module M is fuchsian if and only if it has a cyclic vector $e \in M$ whose minimal operator L is fuchsian. Furthermore, the singularities of M are contained in the singular locus of L.*

By [PS02, Lemma 3.10], fuchsian differential modules form a subcategory in $\mathrm{Diff}_{\mathbb{C}(z)}$ which is stable under tensor operations:

1.6.6 Lemma *Submodules, direct sums, duals, tensor products, exterior powers and symmetric powers of fuchsian differential modules are again fuchsian. The same holds for factors, duals, tensor products, exterior powers and symmetric powers of fuchsian differential operators.*

We collect the local data, i.e. the singularities and the exponents, of a fuchsian operator in its so-called *Riemann scheme.*

1.6.7 Definition Given a fuchsian operator $L \in \mathbb{C}(z)[\partial]$ of degree n with singular locus $\{s_1, \ldots, s_m\}$ and real exponents $\lambda_{1,j} \leq \lambda_{2,j} \leq \cdots \leq \lambda_{n,j}$ at each s_i, we call

$$\mathcal{R}(L) := \left\{ \begin{array}{cccc} s_1 & s_2 & \ldots & s_m \\ \hline \lambda_{1,1} & \lambda_{1,2} & \ldots & \lambda_{1,m} \\ \vdots & \vdots & \ldots & \vdots \\ \lambda_{n,1} & \lambda_{n,2} & \ldots & \lambda_{n,m} \end{array} \right\}$$

the *Riemann scheme* of L.

Chapter 2

Differential operators and geometry

In complex algebraic geometry, differential operators arise naturally from regular connections on $\mathbb{P}^1$ minus a finite set of points. More precisely, the local system of horizontal sections of these connections can be seen as variation of cohomologies of complex algebraic manifolds. We briefly recall properties of local systems and regular connections in the first section of this chapter, which are taken from Deligne's standard value [Del70]. Each regular connection we consider admits an extension to a regular singular one on $\mathbb{P}^1$. It is convenient to describe them from an algebraic point of view, as this provides a link to fuchsian differential $\mathbb{C}(z)$-modules. Those relations are stated in section two. In section three, we introduce the notion of Picard-Fuchs operators and discuss some of their properties. In the fourth section, we introduce differential Calabi-Yau operators as Picard-Fuchs operators of special families of Calabi-Yau manifolds. In the whole section, we use several standard notations and concepts of complex algebraic geometry without any further explanation.

2.1 Local systems

Throughout this section, we fix a finite non-empty subset $S \subset X$ and put $X := \mathbb{P}^1 \setminus S$. Furthermore, we fix an orientation on X. By a *local system* $\mathbb{L}$ on X, we mean a locally constant sheaf of $\mathbb{C}$-vector spaces. Its rank is denoted by $\mathrm{rk}(\mathbb{L})$. The category of local systems on X is denoted by $\mathrm{LocSys}_{\mathbb{C}}(X)$. We point out the relation between local systems and representations of the fundamental group, regular connections and differential operators. Via the choice of a base point $x_0 \in X$, each local system $\mathbb{L}$ on X induces a representation of the fundamental group with respect to this base point in the following way:
For any closed path $\gamma\colon [0,1] \to X$ with $\gamma(0) = \gamma(1)$, we have a sequence of isomorphisms

$$\mathbb{L}_{x_0} \to (\gamma^*\mathbb{L})_0 \to \Gamma([0,1], \gamma^*\mathbb{L}) \to (\gamma^*\mathbb{L})_1 \to \mathbb{L}_{x_0},$$

as $[0,1]$ is simply connected. The composition of these isomorphisms yields an element $\rho_{\mathbb{L}}(\gamma) \in \mathrm{GL}\left(\mathbb{L}_{x_0}\right)$ which only depends on the homotopy class of γ.

2.1.1 Definition For a local system $\mathbb{L}$ on X and a base point $x_0 \in X$, we call the $\mathbb{C}$-vector space $\mathrm{GL}(\mathbb{L}_{x_0})$ together with the map

$$\rho_{\mathbb{L}}\colon \pi_1(X, x_0) \to \mathrm{GL}(\mathbb{L}_{x_0}),\ \gamma \mapsto \rho_{\mathbb{L}}(\gamma)$$

its induced *monodromy representation*. Furthermore, for each point $s \in S$, the image $\rho_{\mathbb{L}}(\gamma_s)$ of a loop $\gamma_s \in \pi_1(X, x_0)$ which encircles s in positive direction and no other points of S is called the *local monodromy* of $\mathbb{L}$ at s.

For a fixed base point $x_0 \in X$, we denote the category of finite dimensional representations of $\mathbb{C}$-vector spaces of $\pi_1(X, x_0)$ by $\mathrm{Repr}_{\pi_1(X,x_0)}$. By [Del70, Theorem 1.3] we have:

2.1.2 Proposition *For each choice of a base point $x_0 \in X$, the attachment $\mathbb{L} \to \rho_{\mathbb{L}}$ induces an equivalence of categories* $\mathrm{LocSys}_{\mathbb{C}}(X) \cong \mathrm{Repr}_{\pi_1(X,x_0)}$.

A more detailed description of local systems via their local monodromies is done in Chapter 5.1.
Local systems on X also have a natural relation to regular connections.

2.1.3 Definition A *regular connection* $(\mathcal{V}, \nabla)$ on X consists of a holomorphic vector bundle $\mathcal{V}$ together with an additive, $\mathbb{C}$-linear map

$$\nabla \colon \mathcal{V} \to \Omega^1_X \otimes \mathcal{V}$$

which satisfies the *Leibniz rule*

$$\nabla(f \otimes v) = df \otimes v + f\nabla(v)$$

for all sections f of $\mathcal{O}_X$ and v of $\mathcal{V}$. Regular connections on X form a category which we denote by $\mathrm{Reg}(X)$. A section v of $\mathcal{V}$ is called *horizontal* if $\nabla(v) = 0$ holds.

As a special case of [Del70, Theorem 2.17], we get:

2.1.4 Proposition *The attachment*

$$\mathbb{V} \mapsto (\mathbb{V} \otimes \mathcal{O}_X, 1 \otimes d)$$

induces an equivalence of categories $\mathrm{LocSys}_{\mathbb{C}}(X) \cong \mathrm{Reg}(X)$.

The inverse of the functor stated above is given by sending a regular connection to its local system of horizontal sections.
Let us have a short look at the local situation:
Choose an open set $U \subset X$ and a coordinate z on U. Then the map

$$\nabla_{\frac{d}{dz}} \colon \mathcal{V}(U) \to \mathcal{V}(U), \nabla_{\frac{d}{dz}} = \left(\frac{d}{dz} \otimes \mathrm{id}\right) \circ \nabla$$

turns $\mathcal{V}(U)$ into a differential $\mathcal{O}_X(U)$ module. In particular, each stalk $\mathcal{V}_z$ inherits the structure of a differential module over the ring of convergent power series $\mathbb{C}\{z\}$. Writing $(\mathcal{V}, \nabla) = (\mathbb{V} \otimes \mathcal{O}_X, \nabla)$, the stalk $\mathbb{V}_z$ of the underlying local system coincides with the covariant solution space of the differential $\mathbb{C}\{z\}$-module $\mathcal{V}_z$.
Given a differential operator $L \in \mathbb{C}(z)[\partial]$ with singular locus S, we can regard its action on sections of the sheaf $\mathcal{O}_X$. A classical result of Cauchy, see e.g. [Del70, Théorème I.4.5], assures, that the solutions form a local system.

2.1.5 Proposition *For each differential operator $L \in \mathbb{C}(z)[\partial]$ of degree n with singular locus S, the sheaf $\mathbb{L}$ whose sections are given by*

$$\mathbb{L}(U) = \{y \in \mathcal{O}_X(U) \mid Ly = 0\}$$

is a local system of rank n on X.

In particular, if L appears in the local situation as minimal operator of a section of $\mathcal{V}(U)$, each stalk $\mathbb{L}_z$ is dual to the stalk $\mathbb{V}_z$.
If L is fuchsian, the method of Frobenius even yields a statement concerning the solutions of L in a neighbourhood of a singular point $s \in S$:
The discussion done in [Inc56, Section 16.2-16.4] reveals that for each choice of a branch of the logarithm $\ln(z)$, each solution of L lies in $\exp(\mu \ln(z))\mathbb{C}\{z\}[\ln(z)]$, where $\mu \in \mathbb{C}$. These solutions form a $\mathbb{C}$-vector space which we denote by Sol_L^{an}. Moreover, if U is chosen small enough, each choice of a branch of $\ln(z)$ yields an isomorphism between Sol_L and Sol_L^{an}, by replacing ℓ by $\ln(z)$ and z^μ by $\exp(\mu \ln(z))$. The discussion done in [Del70, Section I.6] assures that the monodromy representation attached to $\mathbb{L}$ is related to the formal monodromy of L in the following way:

2.1.6 Lemma *Consider a fuchsian differential operator $L \in \mathbb{C}(z)[\partial]$ of degree n with associated local system $\mathbb{L}$ on $\mathbb{P}^1 \setminus S$. Then for each choice of a base point $x_0 \in \mathbb{P}^1 \setminus S$ and $s \in S$, the local monodromy of $\mathbb{L}$ at s is conjugated to the action of the formal monodromy on $\mathrm{Sol}_{\iota_s(L)}$ under the identification $\mathbb{L}_{x_0} \cong \mathbb{C}^n \cong \mathrm{Sol}_{\iota_s(L)}$.*

Each section of $\mathbb{L}$ is uniquely determined by its germ in $\mathbb{L}_{x_0}$. Therefore, we can apply each of the local monodromies $T_s \in \mathrm{GL}(\mathbb{L}_{x_0})$ to sections of $\mathbb{L}$. Similar to [PS02, Theorem 5.8], this gives the following relation to the differential Galois group of L.

2.1.7 Proposition *For each choice of a base point $x_0 \in X$, the differential Galois group of a fuchsian differential operator L is isomorphic to the Zariski closure of the monodromy group $\rho_{\mathbb{L}}(\pi_1(X, x_0)) \subset \mathrm{GL}(\mathbb{L}_{x_0})$.*

2.2 Regular singular connections

In the geometric situation, we also study connections on $\mathbb{P}^1$ which may acquire poles of order one at a finite number of points. These connections appear as extensions of regular connections on $\mathbb{P}^1 \setminus S$ for a finite non-empty subset $S \subset \mathbb{P}^1$. We identify S with the divisor $\sum_{s \in S} s$ on $\mathbb{P}^1$. The sheaf of one-forms on $\mathbb{P}^1$ which admit a pole of order less or equal than one at the points of S is as usual denoted by $\Omega^1_{\mathbb{P}^1}(\log S)$.

2.2.1 Definition A *regular singular connection* $(\mathcal{V}, \nabla)$ on $\mathbb{P}^1$ with singular locus S consists of a holomorphic vector bundle $\mathcal{V}$ on $\mathbb{P}^1$ together with an additive, $\mathbb{C}$-linear map

$$\nabla \colon \mathcal{V} \to \Omega^1_{\mathbb{P}^1}(\log S) \otimes \mathcal{V}$$

which satisfies the Leibniz rule. Regular singular connections on $\mathbb{P}^1$ with singular locus S form a category which we denote by $\mathrm{RegSing}(\mathbb{P}^1, S)$.

Using the GAGA principle established by Serre in [Ser56], we can compare holomorphic vector bundles on $\mathbb{P}^1$ to algebraic vector bundles on the projective space of the affine line. We denote $\mathbb{P}^1$ considered as $\mathbb{C}$-scheme by $\mathbf{P^1}$. In particular, its underlying topological space carries the Zariski-topology.
Completely similar to the analytic point of view, we set:

2.2.2 Definition A *regular singular connection* on $\mathbf{P^1}$ with singular locus S is a pair $(\mathcal{M}, \nabla)$ consisting of an algebraic vector bundle $\mathcal{M}$ on $\mathbf{P^1}$ together with an additive, $\mathbb{C}$-linear map

$$\nabla \colon \mathcal{M} \to \Omega_{\mathbf{P^1}}(\log S) \otimes \mathcal{M}$$

which satisfies the Leibniz rule. The category of regular singular connections on $\mathbf{P^1}$ with singular locus S is denoted by $\mathrm{RegSing}(\mathbf{P^1}, S)$.

The GAGA principle assures that the algebraic and the analytic setting are equivalent, see also [PS02, Section 6.2]

2.2.3 Proposition *The categories* $\mathrm{RegSing}(\mathbb{P}^1, S)$ *and* $\mathrm{RegSing}(\mathbf{P^1}, S)$ *are equivalent.*

Moreover, regular singular connections on $\mathbf{P^1}$ are linked to differential $\mathbb{C}(z)$-modules in the following way:
Given a regular singular connection $(\mathcal{M}, \nabla)$ on $\mathbf{P^1}$, its generic fiber $\mathcal{M}_\eta$ is a finite dimensional $\mathbb{C}(z)$-vector space. The generic fiber $\Omega_{\mathbf{P^1},\eta}(\log S)$ of $\Omega_{\mathbf{P^1}}(\log S)$ coincides with the universal differential $\mathbb{C}(z)dz$ of $\mathbb{C}(z)$ over $\mathbb{C}$. The map ∇ induces a map

$$\nabla_\eta \colon \mathcal{M}_\eta \to \mathbb{C}(z)dz \otimes \mathcal{M}_\eta$$

which remains $\mathbb{C}$-linear, additive and satisfies the Leibniz rule for $\mathbb{C}(z)$-vector spaces. Therefore, the map

$$\partial := \nabla_\eta \otimes \left(\frac{d}{dz} \otimes \mathrm{id} \right)$$

turns $(\mathcal{M}_\eta, \partial)$ into a differential $\mathbb{C}(z)$-module.
In more general terms, see e.g. [PS02, Chapter 6.4 and 6.5], this procedure yields:

2.2.4 Proposition *The attachment* $(\mathcal{M}, \nabla) \mapsto (\mathcal{M}_\eta, \partial)$ *yields a functor from* $\mathrm{RegSing}\left(\mathbf{P^1}, S\right)$ *to the category of fuchsian differential* $\mathbb{C}(z)$*-modules with singular locus* S*.*

2.3 Picard-Fuchs operators

We briefly describe the geometric setting we want to consider. Take a complex algebraic $(n+1)$-dimensional manifold Y together with a proper morphism $\pi \colon Y \to \mathbb{P}^1$ whose generic fiber is smooth. In particular, there is a finite subset $S \subset \mathbb{P}^1$ such that the fiber $Y_z := \pi^{-1}(z)$ is a compact complex manifold if $z \in \mathbb{P}^1 \setminus S$. In this situation, we speak of a *family* of compact complex manifolds. The n-th complexified cohomology group - which is in fact a finite dimensional $\mathbb{C}$-vector space - of the fiber Y_z is denoted by $H^n(Y_z)$. We fix an integer $1 \leq k \leq n$ and investigate the push forward $\mathbb{H} := R^k \pi_* \underline{\mathbb{C}}_Y$ of the constant sheaf $\underline{\mathbb{C}}_Y$ to $\mathbb{P}^1 \setminus S$. This sheaf is in fact a local system on $\mathbb{P}^1 \setminus S$ with the property that $\mathbb{H}_z = H^k(Y_z)$ holds. As discussed in Proposition 2.1.4, it gives rise to a regular connection $(\mathcal{H}, \nabla)$ on $\mathbb{P}^1 \setminus S$, a so-called *Gauß-Manin connection*.
For a non-constant family, we get restrictions on the local monodromy of $\mathbb{H}$, see e.g. [Kat70].

2.3.1 Theorem (Monodromy Theorem) *For each choice of a base point $x_0 \in \mathbb{P}^1 \setminus S$ and for each $s \in S$, the local monodromy $T_s \in \mathrm{GL}\left(H^k(Y_{x_0})\right)$ is quasi-unipotent, i.e. there is an $N \in \mathbb{N}$ such that $T_s^N - \mathrm{id}$ is nilpotent. Moreover, we have $(T_s^N - \mathrm{id})^{k+1} = 0$.*

By taking the minimal operator of a section, we can relate $(\mathcal{H}, \nabla)$ to a differential operator L locally. To study differential operators of this type globally, we extend the connection $(\mathcal{H}, \nabla)$ to a regular singular connection on $\mathbb{P}^1$. According to results of P. Griffiths [Gri70] and P. Deligne [Del70, Théorème II.7.9], this is indeed possible.

2.3.2 Theorem (Regularity Theorem) *The connection $(\mathcal{H}, \nabla)$ admits an extension to a regular singular connection $\left(\overline{\mathcal{H}}, \overline{\nabla}\right)$ with singular locus S on $\mathbb{P}^1$.*

Locally, say over the point $z = 0$, this extension can be done in such a way that the eigenvalues of $\overline{\nabla}_{z\frac{d}{dz}}$ on the $\mathbb{C}$-vector space $\overline{\mathcal{H}}_0 / z\overline{\mathcal{H}}_0$ lie in $(-1, 0]$. If this holds at any point $s \in S$, the extension is called *canonical*.
As stated in Proposition 2.2.3, we can attach to the canonical extension $\left(\overline{\mathcal{H}}, \overline{\nabla}\right)$ a regular singular connection on $\mathbf{P^1}$ and - by passing to the generic point - a fuchsian differential $\mathbb{C}(z)$-module $(M_{\mathcal{H}}, \partial)$ with singular locus in S. The differential operator L considered in the local situation can by construction be seen as the restriction of a minimal operator of an element in $M_{\mathcal{H}}$. Therefore, we state:

2.3.3 Definition With respect to the terminology introduced above, the minimal operator of a non-zero element $e \in M_{\mathcal{H}}$ is called a *Picard-Fuchs operator*.

The relation to the classical notion of Picard-Fuchs equations can be seen as follows:
Choose $k = n$ and assume that each cohomology group $H^n(Y_z)$ underlies Poincaré duality, i.e. the cup product induces a non-degenerate bilinear $(-1)^n$-symmetric pairing on $H^n(Y_z)$. This pairing can be extended to the corresponding vector bundle $\overline{\mathcal{H}}$ such that it is additionally compatible with $\overline{\nabla}$, i.e. we have

$$\frac{d}{dz}\left(\langle v, w\rangle\right) = \langle \overline{\nabla}_{\frac{d}{dz}} v, w\rangle + \langle v, \overline{\nabla}_{\frac{d}{dz}} w\rangle$$

for each choice of a coordinate z and sections v, w of $\overline{\mathcal{H}}$. In particular, if v is a section of $\mathbb{H}$ and L a differential operator which annihilates w, the expression $\langle v, w\rangle$ is a solution of L. Identifying $\mathbb{H}$ with its dual by means of $\langle \cdot, \cdot \rangle$, the expression $\langle v, w\rangle$ can be seen as integrating the n-form w over a locally constant n-cycle and hence coincides with the classical notion of a period.
We state some well-known examples of Picard-Fuchs operators.

2.3.4 Example (i) Consider a differential operator $L \in \mathbb{Q}[z, \vartheta]$ of degree one which has an algebraic solution f with $f^n \in \mathbb{Q}(z)$ for some $n \in \mathbb{N}$. Then L is the Picard-Fuchs operator for the family of points $y^n = f$.

(ii) Consider the polynomial equation

$$y^2 = x(x-1)(x-z).$$

For each value $z \in \mathbb{C} \setminus \{0, 1\}$, the roots $C(z)$ of the resulting equation have the structure of a Riemann surface of genus one. All those surfaces fit to the so-called *Legendre family*. Furthermore, the vector space of holomorphic one-forms on $C(z)$ is spanned by

$$\omega(z) = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-z)}}.$$

We are thus looking for a differential equation with coefficients in $\mathbb{C}(z)$ which annihilates the integral of $\omega(z)$ over a locally constant cycle. As e.g. explained in [AMSSZ09, Section 1.1], this operator is given by

$$L = \vartheta^2 - z\left(\vartheta + \frac{1}{2}\right)^2.$$

2.4 Differential Calabi-Yau operators

From now on, we consider special classes of families $\pi\colon Y \to \mathbb{P}^1$, namely those, for which each smooth fiber is a Calabi-Yau manifold of complex dimension n. Moreover, we only study the middle cohomology of such a family, i.e. the local system $\mathbb{H} = R^n\pi_*\underline{\mathbb{C}}_Y$. We briefly state additional properties of the corresponding Gauß-Manin connection and Picard-Fuchs operators and formulate further restrictions on our family. More details and missing definitions can e.g. be found in [CK00] and [Kul98].

As Calabi-Yau manifolds are compact Kähler manifolds, the n-th cohomology of each smooth fiber of our family carries a *pure Hodge-structure* of weight n, which is given by the decreasing Hodge filtration

$$F^k(Y_t) := \bigoplus_{r \geq k} H^{r,n-r}(Y_t)$$

such that

$$H^n(Y_t) = F^p(Y_t) \oplus \overline{F^{n-p+1}(Y_t)}$$

holds for each $1 \leq p \leq n$. We write $h^{p,q} := \dim_{\mathbb{C}} H^{p,q}(Y_t)$ for the *Hodge-numbers* of Y_t.

As Y_t is Calabi-Yau, we particularly have $h^{n,0} = 1$. We assume further that $h^{n-1,1} = 1$ if $n \geq 3$, as this implies by a theorem by Bogomolov, Tian and Todorov, see [Bog78], [Tia87], [Tod89] that the full complex moduli space of each smooth fiber is a one-dimensional complex manifold.

Consider the Gauß-Manin connection $(\mathcal{H}, \nabla)$ on $\mathbb{P}^1 \setminus S$ related to $\mathbb{H} := R^n\pi_*\underline{\mathbb{C}}_Y$. Via the map π, the k-th parts of the Hodge filtrations $F^\bullet(Y_t)$ on the smooth fibers fit to vector bundles $\mathcal{F}^k$ and give rise to a filtration

$$\mathcal{F}^n \hookrightarrow \ldots \hookrightarrow \mathcal{F}^0 = \mathcal{H}$$

of $\mathcal{H}$ by locally free subsheaves. With respect to this filtration, the Gauß-Manin connection satisfies the *Griffiths transversality property*, i.e. we have

$$\nabla\left(\mathcal{F}^i\right) \subset \Omega^1_{\mathbb{P}^1 \setminus S} \otimes \mathcal{F}^{i-1}$$

for all $i = 0, \ldots, n$. The data $(\mathcal{H}, \mathbb{H}, \mathcal{F}^\bullet)$ is a complex variation of Hodge structure of weight n. Moreover, its primitive part is polarized by the extension of the Poincaré-pairing to the bundle $\mathcal{H}$. Via Proposition 2.2.3, we can attach a fuchsian differential $\mathbb{C}(z)$-module $(M_{\mathcal{H}}, \partial)$ to the canonical extension $(\overline{\mathcal{H}}, \overline{\nabla})$ on $\mathbb{P}^1$. Moreover, the work of W. Schmid [Sch73] assures that the bundles $\mathcal{F}^i$ can be extended to subbundles $\overline{\mathcal{F}^i}$ of the canonical extension $(\overline{\mathcal{H}}, \overline{\nabla})$ of $(\mathcal{H}, \nabla)$ such that the Griffiths transversality property still holds. We are interested in Picard-Fuchs equations which are induced by sections of the rank one bundle $\overline{\mathcal{F}^n}$.

2.4.1 Definition We call a minimal operator of an element $e \in M_{\mathcal{H}}$ which is induced by a global section of $\overline{\mathcal{F}^n}$ a *differential Calabi-Yau operator*.

In the sequel, we only investigate families of Calabi-Yau manifolds with the following further properties:

2.4.2 Assumptions

- We assume that each global section e of $\overline{\mathcal{F}^n}$ induces together with its derivatives $\overline{\nabla}_{\frac{d}{dz}} e, \ldots, \overline{\nabla}^n_{\frac{d}{dz}} e$ with respect to each choice of a coordinate an irreducible subvariation of Hodge structure outside the singular locus of $\overline{\nabla}$. We denote the corresponding connection on $\mathbb{P}^1 \setminus S$ by $(\mathcal{V}, \nabla) = (\mathbb{V} \otimes \mathcal{O}_{\mathbb{P}^1 \setminus S}, 1 \otimes d)$ and its canonical extension by $(\overline{\mathcal{V}}, \overline{\nabla})$. Note, that $\mathbb{V}$ is a local system of rank n. To keep the notations simpler, we denote the induced decreasing filtration of subbundles on $\mathcal{V}$ also by $\mathcal{F}^\bullet$.

- We assume that $0 \in S$ and that for any choice of a base point $x_0 \in \mathbb{P}^1 \setminus S$ the monodromy $T_0 \in \mathrm{GL}(\mathbb{V}_{x_0})$ of $\mathbb{V}$ at $z = 0$ is *maximally unipotent*, i.e. we have that $T_0 - \mathrm{id}$ is nilpotent with $(T_0 - \mathrm{id})^n \neq 0$. In this case, we call $z = 0$ a *MUM point* of the family. There are several reasons to assume that the family admits a MUM point: For families of Calabi-Yau threefolds, the first examples of families without a MUM point were found very recently, see e.g. [GG10]. Moreover, the existence of a MUM point allows us to apply results borrowed from mirror symmetry.

- We assume that the fibers Y_t over $t \in \mathbb{Q}$ are defined over $\mathbb{Q}(z)$. This imposes additional arithmetic properties on differential Calabi-Yau operators, see Section 3.2.

Chapter 3

Algebraic properties of differential Calabi-Yau operators

In this chapter, we collect algebraic properties of differential Calabi-Yau operators related to families which fulfill Assumptions 2.4.2. From a purely algebraic point of view, this leads to the notion of CY-type differential operators. Several algebraic properties of those of degree four are e.g. discussed in [Yu08], [AESZ05] and [AZ06]. Keeping the underlying geometric situation in mind, we use the notations which were introduced in the preceding chapter. In particular, we investigate regular connections $(\mathcal{V}, \mathcal{F}^{\bullet}, \nabla)$ which are locally generated by a section of $\mathcal{F}^n$ together with its derivatives. We are not able to touch all properties that are induced from Hodge theory. In particular, we do not discuss any of the properties that rely on the holomorphic structure of the fibers.

Throughout this section, let $L = \partial^{n+1} + \sum_{i=0}^{n} a_i \partial^i \in \mathbb{Q}(z)[\partial]$ be an irreducible monic differential operator and

$$(M_L, e) = (\mathbb{C}(z)[\partial]/\mathbb{C}(z)[\partial]L\ , [1])$$

its corresponding marked differential $\mathbb{C}(z)$-module. The elements of the set $\mathcal{E} := \{e, \ldots, \partial^n e\}$ form a basis of M_L and e should correspond to a global section of $\overline{\mathcal{F}^n}$ in the geometric situation. We fix the filtration $E^{\bullet}$ on M_L via

$$E^i := span\{e, \ldots, \partial^{n-i} e\}$$

and call it the *cyclic filtration* associated to e. Similar to the Griffiths transversality property, we have $\partial E^i \subset E^{i-1}$.

In the first section, we discuss the algebraic counterpart of the Poicaré pairing. This gives a special, non-degenerate $(-1)^n$-symmetric pairing on M_L, which yields the operator L to be self-dual. In the second section, we discuss arithmetic properties of holomorphic solutions of L in general and especially at the MUM point. In the third section, we briefly discuss the limiting mixed Hodge structure of the family at the MUM point, which forces the exponents of L at this point to be equal. Together with the $(-1)^n$-symmetric pairing, this gives a formal normal form at the MUM point which is constructed in the fourth section. In fact, we can choose a special coordinate which controls possible local transformations between two operators of this type. Claiming further integrality properties on the special coordinate - which is mainly motivated by mirror symmetry - we end up with the notion of an operator of CY-type in section five.

3.1 The Poincaré pairing

We first discuss effects of the Poincaré pairing on the operator L and its associated marked differential module (M_L, e). As this pairing extends to the regular singular connection $(\overline{\mathcal{V}}, \overline{\nabla})$, we are looking for a non-degenerate $(-1)^n$-symmetric form

$$\langle \cdot, \cdot \rangle \colon M_L \times M_L \to \mathbb{C}(z)$$

such that

$$\partial \langle m_1, m_2 \rangle = \langle \partial m_1, m_2 \rangle + \langle m_1, \partial m_2 \rangle$$

holds. This can be rephrased by requesting that $\langle \cdot, \cdot \rangle \in \mathrm{Hom}_{\mathbb{C}(z)[\partial]} \left(\mathrm{Sym}^2 M_L, \mathbb{C}(z)\right)$ for n odd and $\langle \cdot, \cdot \rangle \in \mathrm{Hom}_{\mathbb{C}(z)[\partial]} \left(\bigwedge^2 M_L, \mathbb{C}(z)\right)$ for n even. Furthermore, as e represents a section of $(n, 0)$-forms, we claim that $\langle e, \partial^i e \rangle = 0$ holds for all $i = 0, \ldots, n-1$ because of the Griffiths transversality property.

3.1.1 Definition We say that L satisfies property (P) if there is a non-degenerate form $\langle \cdot, \cdot \rangle \colon M_L \times M_L \to \mathbb{C}(z)$ such that

(i) $\langle \cdot, \cdot \rangle \in \mathrm{Hom}_{\mathbb{C}(z)[\partial]} \left(\mathrm{Sym}^2 M_L, \mathbb{C}(z)\right)$ for n even and $\langle \cdot, \cdot \rangle \in \mathrm{Hom}_{\mathbb{C}(z)[\partial]} \left(\bigwedge^2 M_L, \mathbb{C}(z)\right)$ for n odd.

(ii) $\langle e, \partial^i e \rangle = 0$ for $i = 0, \ldots, n-1$.

This condition induces relations on the coefficients of L in the following way.

3.1.2 Proposition *The operator L satisfies property (P) if and only if L is self-dual in the sense that there is a $0 \neq \alpha \in \mathbb{C}(z)$ such that $L\alpha = \alpha L^\vee$ holds. In particular, we have $\alpha' = -2a_n/(n+1)\alpha$.*

Proof Suppose that L satisfies (P). As $\langle \cdot, \cdot \rangle$ is non-degenerate, we have

$$\alpha := \langle e, \partial^n e \rangle \neq 0.$$

Furthermore, the map

$$\varphi \colon M_L \to M_L^\vee, \ \varphi(m) := \langle m, \cdot \rangle$$

is an isomorphism of differential modules. With respect to the basis $\{e^\vee, \ldots, (\partial^n e)^\vee\}$ dual to $\mathcal{E}$, the element $\varphi(e)$ is exactly $\alpha\, (\partial^n e)^\vee$. By Proposition 1.3.3, this implies $L\alpha = \alpha L^\vee$. Conversely, if $L\alpha = \alpha L^\vee$ for $\alpha \neq 0$, we have a corresponding differential isomorphism

$$\psi \colon M_L \to M_L^\vee, \ \psi(e) := \alpha\, (\partial^n e)^\vee .$$

Thus the form

$$\langle \cdot, \cdot \rangle \colon M_L \times M_L \to \mathbb{C}(z), \ \langle m_1, m_2 \rangle := \left(\psi(m_1)\right)(m_2)$$

is non-degenerate and satisfies $\langle e, \partial^i e \rangle = 0$ for all $i = 0, \ldots, n-1$. As ψ is a differential morphism, we have

$$\langle m_1, m_2 \rangle' = \langle \partial m_1, m_2 \rangle + \langle m_1, \partial m_2 \rangle$$

for each two elements $m_1, m_2 \in M_L$. It remains to prove that $\langle \cdot, \cdot \rangle$ is $(-1)^n$-symmetric. By construction of $\langle \cdot, \cdot \rangle$, we have $\langle e, \partial^i e \rangle = 0$ for $0 \leq i < n$ and $\langle e, \partial^n e \rangle = \alpha$. Compatibility with the derivation gives $\partial^i\langle e,\cdot\rangle = \langle\partial^i e,\cdot\rangle$. By the formulae stated in Proposition 1.3.5, we conclude that $\langle\partial^i e, e\rangle = 0$ for $0 \leq i < n$ and $\langle\partial^n e, e\rangle = (-1)^n\alpha$. Fix $N \leq n$ and suppose that we have shown the identity

$$\langle\partial^m e, \partial^k e\rangle = (-1)^n\langle\partial^k e, \partial^m e\rangle$$

for every $m \leq N$ and every $0 \leq k \leq n$. By compatibility with the derivation we get

$$\begin{aligned}\langle\partial^{N+1}e, \partial^k e\rangle &= \partial\langle\partial^N e, \partial^k e\rangle - \langle\partial^N e, \partial^{k+1}e\rangle \\ &= (-1)^n\partial\langle\partial^k e, \partial^N e\rangle + (-1)^{n+1}\langle\partial^{k+1}e, \partial^N e\rangle = (-1)^n\langle\partial^k e, \partial^{N+1}e\rangle.\end{aligned}$$

Thus the desired result follows by induction.
Finally, comparing the coefficients of L and $L^\vee$, one readily sees that $\alpha' = -2a_n\alpha/n+1$ holds. □

We derive several corollaries and related facts to property (P).
As a direct consequence concerning the differential Galois group, we get:

3.1.3 Corollary *If L satisfies (P), the differential Galois group of M_L lies in $\mathrm{Sp}_{n+1}(\mathbb{C})$ for $n+1$ even and in $\mathrm{SO}_{n+1}(\mathbb{C})$ for $n+1$ odd.*

Proof Write PV for the Picard-Vessiot ring of M_L, G for the differential Galois-group of M_L and $\langle\cdot,\cdot\rangle$ for the form associated to property (P). We can extend $\langle\cdot,\cdot\rangle$ in a natural way to $\mathrm{PV}\otimes M_L$. Denote by $W \subset \mathrm{PV}\otimes M_L$ the covariant solution space of M_L. As for each two elements $w_1, w_2 \in W$ we have

$$\langle w_1, w_2\rangle' = \langle\partial w_1, w_2\rangle + \langle w_1, \partial w_2\rangle = 0,$$

the restriction of $\langle\cdot,\cdot\rangle$ to W maps to $\mathbb{C}$. As $\langle\cdot,\cdot\rangle$ is non-degenerate, this restriction is non-trivial by Remark 1.2.3. Take $\sigma \in G$ and $r_1\otimes m_1, r_2\otimes m_2 \in \mathrm{PV}\otimes M_L\cap W$. As $\langle m_1, m_2\rangle \in \mathbb{C}(z)$ and $\langle r_1\otimes m_1, r_2\otimes m_2\rangle \in \mathbb{C}$ are invariant under the action of σ, we get

$$\begin{aligned}\langle\sigma(r_1\otimes m_1), \sigma(r_2\otimes m_2)\rangle &= \langle\sigma(r_1)\otimes m_1, \sigma(r_2)\otimes m_2\rangle = \sigma(r_1)\sigma(r_2)\langle m_1, m_2\rangle \\ &= \sigma(r_1r_2)\sigma(\langle m_1, m_2\rangle) = \sigma\left(\langle r_1\otimes m_1, r_2\otimes m_2\rangle\right) \\ &= \langle r_1\otimes m_1, r_2\otimes m_2\rangle.\end{aligned}$$

Hence the elements of G preserve the restriction of $\langle\cdot,\cdot\rangle$ to W. This yields the result. □
By Theorem 1.2.5, the form induces submodules of $\bigwedge^2 M_L$ and $\mathrm{Sym}^2 M_L$.

3.1.4 Corollary *Suppose that L satisfies property (P).*

(i) *If $\deg(L) > 2$ is even, $\bigwedge^2 M_L$ has an one dimensional differential submodule W, which is not contained in the differential submodule N generated by $e\wedge\partial e$.*

(ii) *If $\deg(L) > 1$ is odd, $\mathrm{Sym}^2 M_L$ has an one dimensional differential submodule W, which is not contained in the differential submodule N generated by $e\cdot e$.*

Proof We only state the proof for $\deg(L) > 2$ even, as the one for the odd case is completely similar. The $\mathbb{C}$-span of $\langle\cdot,\cdot\rangle$ is contained in the contravariant solution space of $\bigwedge^2 M_L$ and is fixed by the differential Galois group of $\bigwedge^2 M_L$. As $\dim_{\mathbb{C}(z)}\left(\bigwedge^2 M_L\right) > 1$, the form $\langle\cdot,\cdot\rangle$ induces a one dimensional differential submodule W of $\bigwedge^2 M_L$ by Corollary 1.3.11. In particular, $\mathrm{Hom}_{\mathbb{C}(z)[\partial]}(W, \mathbb{C}(z))$ is by construction spanned by $\langle\cdot,\cdot\rangle$. Let N be the differential submodule of $\bigwedge^2 M_L$ which is generated by $e \wedge \partial e$ and assume that $W \subset N$. By Remark 1.2.3, the natural map

$$\mathrm{Hom}_{\mathbb{C}(z)[\partial]}(N, \mathbb{C}(z)) \to \mathrm{Hom}_{\mathbb{C}(z)[\partial]}(W, \mathbb{C}(z))$$

is surjective. As the restriction of $\langle\cdot,\cdot\rangle$ to N is identically zero, this is impossible. □

If L is additionally fuchsian, property (P) induces a symmetry of its exponents, as soon as they are real numbers.

3.1.5 Proposition *For each fuchsian operator L with singular locus S that satisfies property (P) and has real exponents $\lambda_{s,1} \leq \cdots \leq \lambda_{s,n+1}$ at each $s \in S$, we have*

(i)

$$\frac{2}{n+1}\sum_{i=1}^{n+1} \lambda_{s,n+1} \in \mathbb{Z}$$

and

(ii)

$$\lambda_{s,i} + \lambda_{s,n+1-i} = \lambda_{s,j} + \lambda_{s,n+1-j}$$

for all $1 \leq i, j \leq n+1$.

Proof As $L = \partial^{n+1} + \sum_{i=0}^{n} a_i \partial^i$ satisfies property (P), we have $\alpha^{-1} L \alpha = L^\vee$ for an $0 \neq \alpha \in \mathbb{C}(z)$ with $\alpha' = -2a_n\alpha/(n+1)$. Solving this equation locally reveals that

$$\frac{2}{n+1}\mathrm{res}_{z=s}(a_n) \in \mathbb{Z}.$$

Furthermore, by Remark 1.6.2 we have the relation

$$\binom{n+1}{2} - \sum_{i=0}^{n+1} \lambda_{s,i} = \mathrm{res}_{z=s}(a_n),$$

which yields the first result. For the second statement, assume without loss of generality that $s = 0$. As L is fuchsian, there is a polynomial $g \in \mathbb{C}(z)$ such that we can write $gL = \sum_{i=0}^{m} z^i P_i(\vartheta) \in \mathbb{C}[z, \vartheta]$. By the rules $(PQ)^\vee = Q^\vee P^\vee$, $\vartheta^\vee = -\vartheta - 1$ and $\vartheta z^i = z^i(\vartheta + i)$, we get that

$$(gL)^\vee = L^\vee g = \sum_{i=0}^{m} z^i P_i(-1 - \vartheta - i).$$

In particular, there is an integer $r \in \mathbb{Z}$ such that the exponents of $L^\vee$ at $z = 0$ are given by $-r - \lambda_{0,n+1} \leq \cdots \leq -r - \lambda_{0,1}$. By Lemma 1.5.9, there is an $a \in \mathbb{Q}$ such that

$$\lambda_{p,i} + a = -r - \lambda_{p,n+1-i}$$

holds for each $1 \leq i \leq n+1$. As this implies $\lambda_{p,i} + \lambda_{p,n+1-i} = -r - a$, we get the result. □

Furthermore, one computes directly that the cyclic filtration on M_L behaves under the action of $\langle\cdot,\cdot\rangle$ as follows:

3.1.6 Corollary *If L satisfies (P), we have*

$$\left(E^i\right)^\perp = E^{n-1-i}$$

with respect to $\langle \cdot, \cdot \rangle$ for every $i = 0, \dots, n-1$.

3.2 Solutions at the MUM point

As we assume that the families we consider can be realized over the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$, the solutions of related Picard-Fuchs operators inherit arithmetic properties. More generally, by results of L. Siegel, B. Dwork, Y. André and many others, the coefficients of each local solution $y \in \overline{\mathbb{Q}}[\![z]\!]$ grow moderately in a p-adic sense and are in fact so-called G-functions.

3.2.1 Definition A formal power series $f = \sum_{m=0}^{\infty} A_m z^m \in \overline{\mathbb{Q}}[\![z]\!]$ is called a *G-function*, if

(i) it has a positive radius of convergence in $\mathbb{C}$.

(ii) there is a differential operator $P \in \overline{\mathbb{Q}}[z, \vartheta]$, such that f is a solution of $\iota_0(P)$.

(iii) there is a sequence of positive integers $(c_n)_{n\in\mathbb{N}}$ and an algebraic number field $K \subset \overline{\mathbb{Q}}$ such that
$$\sup_n \left(\frac{1}{n}\ln(c_n)\right) < \infty$$
and $c_n A_j \in \mathcal{O}_K$ for all $j \leq n$.

If the solution space $\iota_t(L)$ of a given differential operator $L \in \overline{\mathbb{Q}}(z)[\partial]$ at an arbitrary point $t \in \overline{\mathbb{Q}}$ contains a non-zero G-function, we call L a *G-operator*.

By [And89, Appendix V] we have:

3.2.2 Theorem (André) *If $L \in \overline{\mathbb{Q}}(z)[\partial]$ is a Picard-Fuchs operator, then each element in* $\mathrm{Sol}_{\iota_t(L)} \cap \overline{\mathbb{Q}}[\![z]\!]$ *is a G-function for all $t \in \overline{\mathbb{Q}}$.*

We are hence looking for operators L with this property. As e.g. discussed in [DGS94, Chapter VIII], a remarkable theorem of G.V. and D.V. Chudnovsky shows that it holds for irreducible G-operators.

3.2.3 Theorem *If $L \in \overline{\mathbb{Q}}(z)[\partial]$ is an irreducible G-operator, each element of* $\mathrm{Sol}_{\iota_t(L)} \cap \overline{\mathbb{Q}}[\![z]\!]$ *is a G-function for all $t \in \overline{\mathbb{Q}}$.*

Combining this statement with a theorem of N. Katz stated in [DGS94, Chapter III.6] yields:

3.2.4 Proposition *Each irreducible G-operator is fuchsian and has only rational exponents.*

Note, that by the Monodromy Theorem 2.3.1 and the Regularity Theorem 2.3.2 this statement holds for Picard-Fuchs operators.
As the operator L we consider is irreducible, it suffices by Theorem 3.2.3 to prove that a holomorphic solution $y = z^r + \sum_{m=r+1}^{\infty} A_m z^m$ of $\iota_0(L)$ is a G-function. Furthermore, as $z = 0$

is a MUM point, the holomorphic solutions of $\iota_0(L)$ form a one-dimensional $\mathbb{C}$-vector space. If $\iota_0(L)$ is regular singular, each holomorphic solution of $\iota_0(L)$ satisfies conditions (i) and (ii). Even with a closed formula for the coefficients of such a solution at hand, it can become a challenging task to check the third condition of Definition 3.2.1. A result of Y. André assures that we get solutions of the following shape.

3.2.5 Definition A formal power series $f = \sum_{m=0}^{\infty} A_m z^m \in \mathbb{Q}[\![z]\!]$ is *N-integral* if there is an $N \in \mathbb{N}$ such that $N^m A_m \in \mathbb{Z}$ for all $m \geq 0$.

The result [And89, Theorem IX.4.2] of Y. André implies the following statement concerning solutions near a MUM point.

3.2.6 Theorem *Consider a family $\pi\colon Y \to \mathbb{P}^1$ whose smooth fibers over $t \in \mathbb{Q}$ are defined over $\mathbb{Q}(z)$ and such that $z = 0$ is a MUM point. Then each Picard-Fuchs operator of the family has an N-integral solution at $z = 0$.*

We take this property into account for our description.

3.2.7 Definition We say that L satisfies property (N) if $\iota_0(L)$ has an N-integral solution.

The for our purposes most important examples of N-integral power series are the following ones:

3.2.8 Proposition (i) *The Taylor expansion around $z = 0$ of each algebraic function $\psi \in \mathbb{Q}(z)^{alg}$ is N-integral up to multiplication with an element $c \in \overline{\mathbb{Q}} \setminus \{0\}$.*

(ii) *For each choice of rational numbers $a_1, \dots, a_n \in \mathbb{Q}$, the generalized hypergeometric function*
$$ {}_nF_{n-1}\left(\begin{matrix} a_1, \dots, a_n \\ 1, \dots, 1 \end{matrix} \,\middle|\, z\right) = \sum_{m=0}^{\infty} \frac{(a_1)_m \dots (a_n)_m}{(m!)^n} z^m, $$
where we write $(a_i)_0 = 1$ and $(a_i)_m = a_i(a_i+1)\dots(a_i+m-1)$ for each $1 \leq i \leq n$, is N-integral.

Proof The first result is a classical theorem of G. Eisenstein, see e.g. [Eis52]. For the second statement, it suffices to do the proof for $n = 1$. As we have ${}_1F_0(a|z) = (1-z)^{-a}$ for each $a \in \mathbb{Q}$, this case is already covered by the first statement. □

One can check directly, that the class of N-integral power series is preserved under the following operations.

3.2.9 Lemma *The class of N-integral power series is stable under taking products, multiplicative inversion, twists by N-integral power series and inversion of twists. Furthermore, each N-integral power series $f = z^r + \sum_{m=r+1}^{\infty} A_m z^m$ admits an N-integral m-th root.*

3.3 Exponents at the MUM point

We discuss some properties of the canonical extension $(\overline{\mathcal{V}}, \overline{\nabla})$ at the MUM point. Therefore, we pass to the local situation and regard $(\mathcal{V}, \nabla, \mathcal{F}^\bullet)$ as a variation of Hodge structure over the punctured disc Δ^* centered at $z = 0$.

In general, the extensions of the subbundles $\overline{\mathcal{F}^{\bullet}}$ do not give rise to a pure Hodge structure on the fiber $\overline{\mathcal{V}}_0$. According to the work [Sch73] of W. Schmid, we can put an additional filtration on $\overline{\mathcal{V}}_0$ such that it carries a *mixed Hodge structure*. We briefly recall some parts of this procedure.
The fiber $\overline{\mathcal{V}}_0$ can be described in two different ways: As the canonical extension is regular singular, there is a frame $F \subset \overline{\mathcal{V}}(\Delta)$ with associated lattice $\Sigma = \mathbb{C}\{z\}F$ such that $\overline{\nabla}_{z\frac{d}{dz}}(\Sigma) \subset \Sigma$ holds and the real parts of the eigenvalues of the *Euler operator*

$$E_\Sigma \in \operatorname{End}_{\mathbb{C}}(\Sigma/z\Sigma), \; [\sigma] \mapsto \left[\overline{\nabla}_{z\frac{d}{dz}}(\sigma)\right]$$

lie in $(-1, 0]$, see e.g. [Kul98, Section II.6.3]. In particular, we get $\overline{\mathcal{V}}_0 = \Sigma$.
Alternatively, we consider the upper half plane $\mathfrak{h} \subset \mathbb{C}$ and the universal covering

$$e\colon \mathfrak{h} \to \Delta^*, \; u \mapsto \exp(2\pi i u)$$

of Δ^*. The pullback $e^*\mathbb{V}$ is a constant local system on $\mathfrak{h}$ and we have an isomorphism

$$\mathbb{V}_z = \mathbb{V}_{e(u)} \cong (e^*\mathbb{V})_u \cong \Gamma(\mathfrak{h}, e^*\mathbb{V})$$

for each $z \in \Delta^*$. Therefore, the $\mathbb{C}$-vector space

$$V_\infty := \Gamma(\mathfrak{h}, e^*\mathbb{V})$$

is called the *canonical fiber* of $\mathbb{V}$. We have an isomorphism of $\mathbb{C}$-vector spaces

$$\psi_z\colon V_\infty \to \Sigma/t\Sigma,$$

as described in [Kul98, Section II.6.4], which depends on the choice of the coordinate z.
Using those identifications, we equip $\overline{\mathcal{V}}_0$ with two filtrations: An increasing one defined on V_∞ and a decreasing one defined on $\Sigma/z\Sigma$. By Nakayama's Lemma, each filtration U on these vector spaces can be lifted to a filtration on $\overline{\mathcal{V}}_0$ uniquely, which we also denote by U. To define the increasing filtration, we recall the following general procedure.

3.3.1 Lemma *For each finite dimensional vector space V together with a nilpotent map $N \in \operatorname{End}(V)$ such that $N^q \neq 0$ and $N^{q+1} = 0$ hold, there is a uniquely determined increasing filtration $W_\bullet$ with the following properties:*

(i) *$N(W_i) \subset W_{i-2}$ for each $i \in \mathbb{Z}$.*

(ii) *For $k \geq 1$, the map N^k induces an isomorphism $\operatorname{Gr}^W_{q+k} V \cong \operatorname{Gr}^W_{q-k} V$, where $\operatorname{Gr}^W_i V = W_i/W_{i-1}$ is the i-th graded part of the filtration.*

As each fiber $\mathbb{V}_z$ is isomorphic to V_∞, we can lift the monodromy T_0 to a map $T_\infty \in \operatorname{GL}(V_\infty)$, which remains maximally unipotent. Therefore,

$$N = \log(T_\infty) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(T_\infty - \mathrm{id})^k \in \operatorname{GL}(V_\infty)$$

is a nilpotent map and we consider the by means of Lemma 3.3.1 associated filtration on V_∞.

3.3.2 Definition For $N = \log(T_\infty) \in \mathrm{GL}(V_\infty)$, the increasing filtration $W(N)_\bullet$ with properties given in Lemma 3.3.1 is called the *monodromy weight filtration.*

The defining properties of $W(N)_\bullet$ stated in Lemma 3.3.1 provide a way to compute each $W(N)_k$ iteratively. In our case, one can check directly that this filtration is of the following shape.

3.3.3 Lemma *Suppose that the Jordan form of $N = \log(T_\infty) \in \mathrm{GL}(V_\infty)$ consists of one Jordan block of size $n+1$. Then the monodromy weight filtration $W(N)_\bullet$ on V_∞ satisfies*

$$W_{2k}(N) = W_{2k+1}(N) = \ker\left(N^{k+1}\right)$$

for each $k = 0, \dots, n$. In particular, we have

$$\dim \mathrm{Gr}_l^{W(N)} V_\infty = \begin{cases} 1, & l \text{ even} \\ 0, & l \text{ odd.} \end{cases}$$

It is possible to put a decreasing filtration $F_\infty^\bullet$ on $\Sigma/t\Sigma \cong V_\infty$ such that the induced filtration on $\overline{\mathcal{V}}_0$ gives the requested extension of the the bundles $\mathcal{F}^\bullet$. Moreover, $(V_\infty, W(N)_\bullet, F_\infty^\bullet)$ is a mixed Hodge structure, the so-called *limiting mixed Hodge structure*, i.e. $F_\infty^\bullet$ induces pure Hodge structures on the graded pieces $\mathrm{Gr}_k^{W(N)} V_\infty$ of the filtration $W_\bullet(N)$.
As the even graded pieces $\mathrm{Gr}_{2k}^{W(N)} V_\infty$ of the monodromy weight filtration are one-dimensional, they carry Hodge structures of Tate type, while the odd graded pieces are zero dimensional. Therefore, this mixed Hodge-structure is called of *Hodge-Tate type.* As remarked in [Del97, Section 6], this implies that $V_\infty = F_\infty^p \oplus W(N)_{2p-2}$ holds for all $0 \le p \le n$ and that $\dim \mathrm{Gr}_{F_\infty}^p V_\infty = 1$.
Hodge-Tate structures induce important consequences for the exponents of the corresponding Picard-Fuchs operator at $z = 0$.

3.3.4 Lemma *In our situation, there is a section $e(z)$ of $\overline{\mathcal{F}^n}$ such that*

$$F_\infty^p = \left\{e(0), \overline{\nabla}_{z\frac{d}{dz}} e(0), \dots, \overline{\nabla}^{n-p}_{z\frac{d}{dz}} e(0)\right\}.$$

Proof As by definition $\overline{\mathcal{F}^n}_0 = F_\infty^n$ and $\overline{\mathcal{F}^n}$ has rank one, there is a section $e(z)$ of $\overline{\mathcal{F}^n}$ whose restriction $e(0)$ lies in the frame F with $\Sigma = \mathbb{C}\{z\}F$ and spans F_∞^n. Denote for $v \in \Sigma$ its class in $\Sigma/z\Sigma$ by $[v]$. As $W(N)_{2n} = \ker(N^n)$ and $F_\infty^n \oplus W(N)_{2n} = V_\infty$, we have $N^n[e(0)] \neq 0$. Since N is nilpotent and $\dim_{\mathbb{C}}(V_\infty) = n+1$, the elements $[e(0)], N[e(0)], \dots, N^n[e(0)]$ form a basis of V_∞. By [Kul98, Section II.6.4], the identification of $\Sigma/z\Sigma$ and V_∞ yields

$$E_\Sigma = -\frac{1}{2\pi i} N.$$

Hence the elements $[e(0)], -2\pi i E_\Sigma[e(0)], \dots, (-2\pi i)^n E_\Sigma^n[e(0)]$ form a basis of V_∞. As the Griffiths transversality property of the limit Hodge filtration yields $E_\Sigma\left(F_\infty^p\right) \subset F_\infty^{p-1}$ and we have $\dim \mathrm{Gr}_{F_\infty}^p V_\infty = 1$, we conclude the result. □

3.3.5 Corollary *The exponents of a differential Calabi-Yau operator $L \in \mathbb{C}[z, \vartheta]$ at $z = 0$ are all equal.*

Proof By Lemma 3.3.4, we can choose a section $e(z)$ of $\overline{\mathcal{F}^n}$ such that the restrictions $B = \left\{[e(0)], \left[\overline{\nabla}_{z\frac{d}{dz}} e(0)\right], \ldots, \left[\overline{\nabla}^n_{z\frac{d}{dz}} e(0)\right]\right\}$ form a basis of V_∞. As $\overline{\mathcal{F}^n}$ has rank one, it suffices to show the result for the minimal operator P of $e(z)$. Considering the action of the Euler operator E_Σ on V_∞ with respect to B, one readily sees that its characteristic polynomial equals $\mathrm{Ind}_0(P)$. As by our assumptions the eigenvalues of E_Σ are all equal to zero, this yields the result. □

We take the result of Corollary 3.3.5 into account for our algebraic description. As the monodromy at $z = 0$ is maximally unipotent, it is clear that the exponents of L at $z = 0$ are integers.

3.3.6 Definition We say that L satisfies property (M) if $\mathrm{Ind}_0(L) = (T - r)^{n+1} \in \mathbb{C}[T]$ for an $r \in \mathbb{Z}$.

According to Lemma 1.5.6, property (M) implies that $z = 0$ is a regular singularity of L and we can write

$$L = (\vartheta - r)^n + \sum_{i=1}^{m} z^i P_i$$

according to the notational convention explained in Section 1.4. Furthermore, $\iota_0(L)$ has a unique solution of the form $y = z^r + \sum_{m=r+1}^{\infty} A_m z^m \in \mathbb{Q}[\![z]\!]$. If $r \geq 0$ and L additionally satisfies property (N), this solution is N-integral. Moreover, by Proposition 1.5.7, there are formal power series $f_0 \in z^r\mathbb{Q}[\![z]\!]^*$ and $f_1, \ldots, f_n \in z^r\mathbb{Q}[\![z]\!]$, such that the elements $y_i = \sum_{j=0}^{i} \ell^j f_{i-j}$ form a basis of $\mathrm{Sol}_{\iota_0(L)}$. We call such a basis a *Frobenius basis*.

3.4 The local normal form at the MUM point

For differential operators which satisfy properties (M) and (P), we mimic results concerning the limiting mixed Hodge structure $(V_\infty, W(N)_\bullet, F_\infty^\bullet)$ on the localized differential module $\iota_0(M_L)$. In the geometric situation, the subspaces $V^p = F_\infty^p \cap W_{2p}$ considered in [Del97, Section 6] induce a decomposition $V_\infty = \oplus_{p\geq 0} V^p$, which extends to the fiber $\overline{\mathcal{V}}_0$. This gives a local normal form of the operator L at $z = 0$ which we regain on a purely algebraic level.

Throughout this subsection, let L satisfy properties (M) and (P). We denote by $\langle \cdot, \cdot \rangle$ the form on M_L induced by property (P) and by PV the Picard-Vessiot ring of $\iota_0(M_L)$. Further, we assume without loss of generality that the exponents of L at $z = 0$ are all equal to zero.

The counterpart of the canonical fiber V_∞ on the level of differential modules is the covariant solution space

$$V := \ker(\partial, \mathrm{PV} \otimes \iota_0(M_L))$$

of $\iota_0(M_L)$. By our assumptions, the action T of the formal monodromy on V is maximally unipotent. We consider its logarithm $N = \log(T)$ to define the monodromy weight filtration.

3.4.1 Definition The *monodromy weight filtration* $W_\bullet$ on V is the ascending filtration induced by N as described in Lemma 3.3.1. Via the isomorphism $\mathrm{PV} \otimes V \cong \mathrm{PV} \otimes \iota_0(M_L)$, this filtration induces an ascending filtration on $\mathrm{PV} \otimes \iota_0(M_L)$. We call it the monodromy weight filtration on $\mathrm{PV} \otimes \iota_0(M_L)$ and denote it by $W_\bullet$ as well.

As stated in Lemma 3.3.3, we have $W_{2k}(N) = W_{2k+1}(N) = \ker\left(N^{k+1}\right)$. A direct computation reveals the following:

3.4.2 Lemma *We have*

$$W_i^{\perp} = W_{2n-1-i}$$

with respect to $\langle \cdot, \cdot \rangle$ *for each* $i \geq 0$.

By property (M) and Proposition 1.5.7, the solution space $\mathrm{Sol}_{\iota_0(L)}$ is a subset of $\mathbb{C}[\![z]\!][\ell]$ and therefore inherits a natural ascending filtration by the powers of ℓ. Expressing the cyclic vector e with respect to a horizontal base of $\mathrm{PV} \otimes M_L$, one directly checks the following:

3.4.3 Lemma *For each* $w_k \in V \cap W_{2k}$, *we have*

$$\langle w_k, e \rangle \in \left(\bigoplus_{i=0}^{k} \mathbb{C}[\![z]\!]\ell^i \right) \cap \mathrm{Sol}_L \, .$$

We now construct a special basis of $\iota_0(M_L)$.

3.4.4 Proposition *There is a basis* $D = \{d_0, \ldots, d_n\} \subset \iota_0(M_L)$ *which has the following properties:*

(i) *We have* $d_i \in E^i \cap W_{2i}$ *for each* $i = 0, \ldots, n$.

(ii) *Setting* $d_{n+1} := 0$ *there are* $\alpha_i \in \mathbb{C}[\![z]\!]^*$ *with* $\alpha_i(0) = 1$ *such that* $\vartheta(\alpha_i d_i) = d_{i+1}$ *for each* $i = 0, \ldots, n$.

Proof We construct the requested data recursively. Choose a basis $\{h_0, \ldots, h_n\} \subset V$ such that $W_{2k} = \mathit{span}\{h_0, \ldots, h_k\}$. This basis is in fact a horizontal basis of $\mathrm{PV} \otimes \iota_0(M_L)$ and we can write $e = \sum_{i=0}^{n} \nu_i h_i$ with

$$\nu_{n-k} \in \left(\bigoplus_{j=0}^{k} \mathbb{C}[\![z]\!]\ell^j \right) \cap \mathrm{Sol}_{\iota_0(L)}$$

by Lemma 3.4.2 and Lemma 3.4.3. Choose a Frobenius basis $\{\sum_{j=0}^{k} \frac{1}{j!}\ell^j f_{k-j}\}_{0 \leq k \leq n}$ of $\mathrm{Sol}_{\iota_0(L)}$ with $f_0 \in \mathbb{C}[\![z]\!]^*$ and $f_1, \ldots, f_n \in \mathbb{C}[\![z]\!]$. Then we have

$$\nu_{n-k} = \sum_{j=0}^{k} \frac{1}{j!}\ell^j \left(\sum_{i=j}^{k} c_{i,k,j} f_{k-i} \right)$$

for certain $c_{i,k,j} \in \mathbb{C}$. In particular, $c_{k,k,k} \neq 0$ for each $0 \leq k \leq n$.
It is clear that $d_0 := e \in E^n \cap W_{2n}$. In order to keep the following iteration process a bit more transparent, we put

$$u_0(i) := \nu_{n-i}$$

and

$$u_k(i) := \left(\frac{u_{k-1}(i)}{u_{k-1}(k-1)} \right)'$$

for all $k = 0, \ldots, n$ and all $i \geq k$, where $(\cdot)'$ denotes the action of $z\frac{d}{dz}$. In particular, we have

$$d_0 = e = \sum_{i=0}^{n} u_0(n-i) h_i.$$

We also put

$$g_j^{(0)} := g_j := \sum_{i=j}^{k} c_{i,k,j} f_{k-i}$$

for all $j = 0, \ldots, n$. By definition, we have

$$u_0(0) = c_{n,n,n} f_0 \in \mathbb{C}[\![z]\!]^*$$

and set

$$\alpha_0 := u_0(0)^{-1}.$$

As $\vartheta E^i \subset E^{i-1}$ and $\vartheta h_i = 0$ for each $0 \le i \le n$, we get

$$d_1 := \vartheta(\alpha_0 d_0) = \sum_{i=0}^{n-2} u_1(n-i) h_i \in E^{n-1} \cap W_{2(n-1)}.$$

Furthermore, we have

$$u_1(k) = \left(\frac{\nu_{n-k}}{\nu_n}\right)' = \sum_{j=0}^{k} \left(\frac{g_j}{j! g_n} \ell^j\right)' =$$
$$\sum_{j=0}^{k-1} \frac{1}{j!} \ell^j \left(\frac{g_{j+1}}{g_n} + \left(\frac{g_j}{g_n}\right)'\right) := \sum_{j=0}^{k-1} \frac{1}{j!} \ell^j g_j^{(1)}.$$

In particular, we get

$$u_1(1) = g_{n-1}^{(1)} = 1 + \left(\frac{g_{n-1}}{g_{n-2}}\right)' \in \mathbb{C}[\![z]\!]^*,$$

as $z\frac{d}{dz}\left(\mathbb{C}[\![z]\!]\right) \subset z\mathbb{C}[\![z]\!]$. We put $\alpha_1 := u_1(1)^{-1}$ and get

$$\begin{aligned} d_2 := \vartheta\left(\alpha_1 d_1\right) &= \sum_{i=0}^{n-1} \left(\frac{u_1(n-i)}{u_1(1)}\right)' h_i \\ &= \sum_{i=0}^{n-2} u_2(n-i) h_i \in E^{n-2} \cap W_{2(n-2)}. \end{aligned}$$

An iteration of this process yields the result. □

3.4.5 Definition We call a set $D = \{d_0, \ldots, d_n\}$ which satisfies the properties stated in Proposition 3.4.4 a *Deligne frame* induced by L and the elements $\alpha_0, \ldots, \alpha_n \in \mathbb{C}[\![z]\!]^*$ the associated *structure series*.

3.4.6 Remark. The construction of a Deligne frame $D = \{d_0, \ldots, d_n\}$ presented in the proof of Proposition 3.4.4 depends on the choice of a Frobenius basis $\{\sum_{j=0}^{k} \frac{1}{j!} \ell^j f_{k-j}\}_{0 \le k \le n}$ of $\mathrm{Sol}_{\iota_0(L)}$. If we choose another Frobenius basis, the resulting Deligne frame $D' = \{d'_0, \ldots, d'_n\}$ differs from D only by $d_i = c d'_i$ for a $c \in \mathbb{C}^*$. The structure series $\alpha_0, \ldots, \alpha_{n-1}$ of both Deligne frames are the same and given by $\alpha_i = u_i(i)^{-1}$, where u_i is the function introduced in the proof of Proposition 3.4.4. Therefore, we may also speak of the structure series of L.

Having constructed a Deligne-frame induced by L, we can write the differential operator $\iota_0(L)$ in the following way.

3.4.7 Corollary *Let $D = \{d_0, \dots, d_n\}$ be a Deligne frame induced by L with associated structure series $\alpha_0, \dots, \alpha_n$. Then*

(i) *the image of $\iota_0(L)$ in $\mathbb{C}[\![z]\!][\vartheta]$ is given by*

$$\vartheta\alpha_n\vartheta\alpha_{n-1}\cdot\dots\cdot\vartheta\alpha_1\vartheta\alpha_0.$$

(ii) *we have $\alpha_i = c_i\alpha_{n+1-i}$ with $c_i \in \mathbb{C}^*$ for all $1 \leq i \leq n$.*

Proof The first statement follows directly from the construction of D, as L is the minimal operator of d_0. To prove the second statement, we first observe that $\langle d_i, d_j\rangle = 0$ if $j \neq n-i$ and $\langle \alpha_i d_i, \alpha_{n-i}d_{n-i}\rangle \in \mathbb{C}^*$ for all $0 \leq i \leq \frac{n-1}{2}$ by Corollary 3.1.6 and Lemma 3.4.2. Writing $(\cdot)' = z\frac{d}{dz}$ we get

$$\begin{aligned}
0 &= (\langle \alpha_i\alpha_{i-1}d_{i-1}, \alpha_{n+1-i}\alpha_{n-i}d_{n-i}\rangle)' \\
&= \langle \alpha_i'\alpha_{i-1}d_{i-1}, \alpha_{n+1-i}\alpha_{n-i}d_{n-i}\rangle + \langle \alpha_i d_i, \alpha_{n+1-i}\alpha_{n-i}d_{n-i}\rangle \\
&\quad + \langle \alpha_i\alpha_{i-1}d_{i-1}, \alpha_{n+1-i}'\alpha_{n-i}d_{n-i}\rangle + \langle \alpha_i\alpha_{i-1}d_{i-1}, \alpha_{n+1-i}d_{n+1-i}\rangle \\
&= c_1\alpha_{n+1-i} + c_2\alpha_i
\end{aligned}$$

for certain $c_1, c_2 \in \mathbb{C}^*$ and hence the result. □

We can use a local coordinate to rewrite $\iota_0(L)$ further:

3.4.8 Definition The solution $q \in z\mathbb{C}[\![z]\!]^*$ of the differential equation

$$\vartheta_z q = \alpha_1^{-1}q$$

with $q'(0) = 1$ is called *special coordinate* or *q-coordinate* of L.

In more explicit terms, for each two solutions $y_0 = f_0 \in \mathbb{C}[\![z]\!]$ and $y_1 = \ell f_0 + f_1$ with $f_1 \in \mathbb{C}[\![z]\!]$ of $\iota_0(L)$ there is a $c \in \mathbb{C}^*$ such that the q-coordinate of L is given by

$$q = c\exp\left(\frac{y_1}{y_0}\right) \in z\mathbb{C}[\![z]\!].$$

As $q \in z\mathbb{C}[\![z]\!]^*$, we get that

$$\vartheta_q := q^*\vartheta = \frac{q}{\vartheta_z q}\vartheta = \alpha_1\vartheta.$$

Therefore, we have

$$\vartheta\alpha_n\vartheta\alpha_{n-1}\cdot\dots\cdot\vartheta\alpha_1\vartheta\alpha_0 = \frac{1}{\alpha_1}\vartheta_q\frac{\alpha_n}{\alpha_1}\vartheta_q\frac{\alpha_{n-1}}{\alpha_1}\dots\vartheta_q\frac{\alpha_2}{\alpha_1}\vartheta_q\alpha_0.$$

Possible local transformations between operators that satisfy (M) and (P) are encoded in their q-coordinates in the following way.

3.4.9 Proposition *Consider two operators L_1 and L_2 which both satisfy (M) and (P) and have q-coordinates $q_1, q_2 \in z\mathbb{C}[\![z]\!]^*$. Then L_1 and L_2 can be locally transformed into each other by twisting with an element $\psi \in z\mathbb{C}[\![z]\!]^*$ with $\psi'(0) = 1$ if and only if*

$$\left(q_1^\vee\right)^* (\iota_0(L_1)) = \left(q_2^\vee\right)^* (\iota_0(L_2)).$$

Proof Assume first that $\iota_0(L_2) = \psi^* \iota_0(L_1)$. As $\psi'(0) = 1$, one checks directly that $q_2 = q_1 \circ \psi$ holds. Hence the transformation rules stated in Remark 1.4.3 yield

$$\begin{aligned}\left(q_2^\vee\right)^* \iota_0(L_2) &= \left((q_1 \circ \psi)^\vee\right)^* \psi^* \iota_0(L_1) = \left(\psi \circ (q_1 \circ \psi)^\vee\right)^* \iota_0(L_1) \\ &= \left(\psi \circ \psi^\vee \circ q_1^\vee\right)^* \iota_0(L_1) = \left(q_1^\vee\right)^* \iota_0(L_1).\end{aligned}$$

On the other hand, if $(q_1^\vee)^* (\iota_0(L_1)) = (q_2^\vee)^* \iota_0(L_2)$ we get $\iota_0(L_2) = (q_1^\vee \circ q_2)^* \iota_0(L_1)$. As $q_1'(0) = q_2'(0) = 1$ the same holds for $(q_1^\vee \circ q_2)'(0)$. □

The preceding proposition gives obstructions against a transformation of two such operators in terms of their Deligne frames:

3.4.10 Corollary *Consider two operators L and $\tilde{L}$ with $\deg(L) = \deg(\tilde{L})$ which both satisfy (M) and (P), have structure series $\alpha_0, \dots, \alpha_n$ and $\tilde{\alpha}_0, \dots, \tilde{\alpha}_n$ and q-coordinates $q, \tilde{q} \in z\mathbb{C}[\![z]\!]^*$. There is a $\psi \in z\mathbb{C}[\![z]\!]^*$ with $\psi'(0)$ such that $\tilde{L} = \psi^*(L)$ holds if and only if $\alpha_0 \circ q^\vee = \tilde{\alpha}_0 \circ \tilde{q}^\vee$ and*

$$\frac{\alpha_i}{\alpha_1} \circ q^\vee = \frac{\tilde{\alpha}_i}{\tilde{\alpha}_1} \circ \tilde{q}^\vee$$

for all $1 \leq i \leq n$.

From the viewpoint of mirror symmetry, the transformation $z \mapsto q$ coincides with the so-called *mirror map*, see e.g. [CK00, Section 2.3 or Section 6.3]. There are various reasons which lead to the conjecture that $q \in z\mathbb{Q}[\![z]\!]$ should be N-integral. For instance, the q-coordinate of many families of elliptic curves has a modular interpretation, see e.g. [Yos87]. According to the observations in [COGP92], the expression $\alpha_1/\alpha_2 \circ q^\vee$ seems to be related to other series whose coefficients count curves on the mirror of the family in the case of threefolds. For further related literature, see e.g. [COGP92], [CK00], [LY96] and the references therein.

3.4.11 Definition We say that a differential operator $L \in \mathbb{C}[z, \vartheta]$ satisfies property (Q) if its q-coordinate is N-integral. If additionally all quotients α_i/α_1 of its structure series are N-integral, we say that it satisfies property (Q+).

It was conjectured in [Alm09] that (Q) implies (Q+). A good candidate for a counterexample is stated in Section A.2.6.

3.5 Differential operators of CY-type

We have all ingredients to define a class of operators whose members have algebraic properties similar to those of differential Calabi-Yau operators.

3.5.1 Definition An irreducible differential operator $L \in \mathbb{Q}[z, \vartheta]$ of degree $n+1$ is *of CY-type* if

(i) it satisfies property (M), i.e. there is an $r \in \mathbb{Z}$ such that $\mathrm{Ind}_0(L) = (T-r)^{n+1}$ holds,

(ii) it satisfies property (N), i.e. $\iota_0(L)$ has an N-integral solution,

(iii) it satisfies property (P), i.e. there is a $0 \neq \alpha \in \mathbb{C}(z)$ such that $L\alpha = \alpha L^\vee$ holds,

(iv) it satisfies property (Q), i.e. it satisfies properties (M) and (P) and the q-coordinate of $\iota_0(L)$ is N-integral and

(v) it satisfies property (Q+), i.e. it satisfies properties (M), (P) and (Q) and α_i/α_1 is N-integral for all $0 \leq i \leq n$, where $\alpha_0, \dots \alpha_n$ are the structure series of L.

As evidence for this description, we prove in Corollary 4.2.2 that differential Calabi-Yau operators related to families of relatively minimal elliptic curves with section over $\mathbb{Q}$ which have a fiber of type I_n at $z = 0$ are of CY-type. Note, that condition (Q+) is empty for operators of degree less or equal than three. In the case of differential Calabi-Yau operators of degree four, it is proven in [Vol07] that property (Q+) holds. The proof is very involved and uses as well p-adic Hodge theory as 1-motives.
Under the transformations introduced before, differential operators of CY-type behave as follows:

3.5.2 Lemma *Consider a CY-type operator L of degree $n+1$. Then*

(i) *$L \otimes R$ is of CY-type for each fuchsian differential operator $R \in \mathbb{Q}[z][\partial]$ of degree one whose exponents lie in $\mathbb{Z}$ at $z = 0$ and in $\frac{1}{n+1}\mathbb{Z}$ at each other singularity.*

(ii) *ψ^*L is of CY-type for each $\psi \in \mathbb{Q}(z)^{alg}$ such that $\iota_0(\psi) \in z\mathbb{Q}[\![z]\!]$ and $\psi^*L \in \mathbb{Q}[z, \vartheta]$.*

Proof For $L \otimes R$, the result is clear. For ψ^*L, properties (M), (N) and (P) are also clear. Denote by y_0 a holomorphic N-integral solution of $\iota_0(L)$ and by $y_1 = \ln(z)y_0 + r$ with $r \in \mathbb{Q}[\![z]\!]$ another solution of $\iota_0(L)$. As we can write $\psi = z^h e$ with $e \in \mathbb{Q}[\![z]\!]^*$, we get

$$\ln(\psi(z)) = h\ln(z) + \ln(e).$$

Hence the q-coordinate $\tilde{q}$ of ψ^*L and the q-coordinate of L are related by

$$\tilde{q}^h = \frac{q \circ \psi}{e(0)} \in z\mathbb{Q}[\![z]\!].$$

According to Lemma 3.2.9, the latter series has an N-integral h-th root in $z\mathbb{Q}[\![z]\!]$. As each two roots of a power series only differ by multiplication of the whole series with a root of unity, we get property (Q). Finally, the construction done in Proposition 3.4.4 reveals that if

$\alpha_0, \ldots, \alpha_n$ are the structure series of L , then $\psi^*\alpha_0, \ldots, \psi^*\alpha_n$ are the structure series of ψ^*L. Hence property (Q+) also follows from Lemma 3.2.9. □

In the geometric situation, a twist by $\psi \in \mathbb{Q}(z)^{alg}$ corresponds to a covering, where the tensor product with g changes the cyclic vector by a multiple of it. As both operations do not change the geometry of the family significantly, we get a natural notion of equivalent operators of CY-type.

3.5.3 Definition We call two operators L_1 and L_2 of CY-type to be *equivalent* - written $L_1 \sim L_2$ - if $L_1 = \psi^* L_2 \otimes g$ for a $g \in \mathbb{Q}(z)^{\mathrm{alg}}$ with $\nu(g)_0 \in \mathbb{Z}$ and a $\psi \in \mathbb{Q}(z)^{\mathrm{alg}}$ with $\psi(0) = 0$.

We try to collect as many operators of CY-type as possible up to equivalence. Some specialties for operators of CY-type which are at most of degree five are stated in the next chapter.

Chapter 4

CY-type operators of low degree

In this chapter, we state some special properties of CY-type operators of degree less or equal than five and relations amongst them. For those of degree one, we give a complete classification which assures that operators of this type correspond to square roots of rational functions. In particular, the differential Galois group of such an operator has at most two elements and each two of them are equivalent. For those of degree two, we have no complete classification at hand. All known examples can be realized as an algebraic pullback of a hypergeometric differential operator. The differential Galois group of those operators is $\mathrm{SL}_2(\mathbb{C})$. We further prove that arbitrary symmetric powers of CY-type operators of degree two again are of CY-type. In fact, each CY-type operator of degree three can be written as second symmetric power of a CY-type operator of degree two. For CY-type operators of degree four, there are unless third symmetric powers of ones of degree two those whose differential Galois group is $\mathrm{Sp}_4(\mathbb{C})$. It is not clear, whether there are up to equivalence finitely or infinitely many operators of this type. Finally, we state relations between CY-type operators of degree four and five which lead to the conjecture that each CY-type operator of degree five can up to equivalence be written as an exterior square of one of degree four.

4.1 Degree one

We give a complete classification of CY-type operators of degree one. For those operators, properties (Q) and (Q+) are empty and we just have to deal with properties (M), (N) and (P). The classification reveals that each CY-type operator of degree one is in fact a Picard-Fuchs operator.

The solutions of a fuchsian differential operator of degree one with rational exponents are algebraic functions. Moreover, they are completely determined by the Riemann scheme of the operator. A direct computation shows:

4.1.1 Lemma *For each choice of distinct points $s_1, \ldots, s_r \in \mathbb{C}$ and rationals $e_1, \ldots, e_n \in \mathbb{Q}$, the solution space of $\partial - \sum_{i=1}^n a_i/(z - s_i)$ is spanned by the algebraic function*

$$y = \prod_{i=1}^{n} (z - s_i)^{a_i}.$$

We classify CY-type operators of degree one via their solutions.

4.1.2 Proposition *A differential operator $L \in \mathbb{Q}[z,\vartheta]$ of degree one is of CY-type if and only if its solution space is spanned by $y = \sqrt{P/Q}$ with $P, Q \in \mathbb{Q}[z]$ such that y is holomorphic near $z = 0$.*

Proof If the solution space of L is spanned by y, it is clear that $L = \partial - y'/y$. By Proposition 3.2.8, L satisfies property (N). As y is holomorphic at $z = 0$, the exponent of $\iota_0(L)$ is an integer and L satisfies property (M). Moreover, L is related to its dual via

$$Ly^2 = \partial y^2 - y'y = y^2(\partial + y'/y) = y^2 L^\vee.$$

As y^2 is rational, L also fulfills property (P) and therefore is of CY-type. Suppose now that L is of CY-type. As it is irreducible and has by properties (M) and (N) at $z = 0$ a solution which is a G-function, it follows from Proposition 3.2.4 that L is fuchsian with rational coefficients. Because of property (P) and Proposition 3.1.5, its exponents lie in $\frac{1}{2} + \mathbb{Z}$ and the result follows from Lemma 4.1.1. □

4.1.3 Corollary (i) *Each CY-type operator of degree one is a Picard-Fuchs operator.*

(ii) *Each two CY-type operators of degree one are equivalent.*

(iii) *The differential Galois group of each CY-type operator of degree one is either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$*

Proof The first statement is covered by Example 2.3.4, while the second statement is a direct consequence of the notion of equivalence. The solution space of L is spanned by an algebraic function g and therefore the differential Galois group of L coincides with the Galois group of the extension $\mathbb{Q}(z)(g) \supset \mathbb{Q}(z)$. As this extension has by Proposition 4.1.2 at most degree two, we conclude the third statement. □

4.2 Degree two and three

As opposed to CY-type operators of degree one, we have no complete classification for CY-type operators of degree greater or equal than two. For those of degree two, property (Q+) is empty. As the differential Galois group of such an operator contains a maximally unipotent element, the discussion done in [PS02, Section 4.3.4] directly reveals that it coincides with $\mathrm{SL}_2(\mathbb{C})$.

Prototypes of these operators are Picard-Fuchs operators related to families of relatively minimal elliptic curves over $\mathbb{Q}$ with section. Those with three and four singular fibers were completely classified in [SH85] and [Her91]. The idea for this classification goes back to the following theorem by Fricke and Klein, see e.g. [FK90, pp. 30-34 and 60-62].

4.2.1 Theorem *Let $y^2 = 4x^3 - g_2(t)x - g_3(t)$ be a family of elliptic curves. Then the normalized period*

$$\Omega(t) = \sqrt{\frac{g_3(t)}{g_2(t)}} \int_{\gamma(t)} \frac{dx}{y}$$

is a solution of the differential equation

$$J(t)^*(E) := J(t)^* \left(\partial^2 + \frac{1}{z}\partial + \frac{31/144z - 1/36}{z^2(z-1)^2}\right),$$

where

$$J(t) = \frac{g_2(t)^3}{g_2(t)^3 - 27g_3(t)^2}.$$

In particular, each Picard-Fuchs operator related to a family of elliptic curves is a rational pullback of the differential operator E. We conclude the following result.

4.2.2 Corollary *Each differential Calabi-Yau operator of a family of elliptic curves which has a MUM point at $z = 0$ is equivalent to a CY-type operator of degree two.*

Proof By a direct computation, we first observe that

$$\frac{1}{z}^*(E) \otimes (\vartheta + z(\vartheta + 1/4)) = 144\vartheta^2 - z(288\vartheta^2 - 31) + 4z^2(6\vartheta + 1)(6\vartheta - 1) =: L.$$

By Lemma 3.5.2, it suffices to prove that L is of CY-type. Since $\mathrm{Ind}_0(L) = T^2$ holds, it fulfills property (M). Property (P) can be computed directly. With respect to the notation introduced in [BH89, Formula (2.5)] we have

$$L \otimes \left(\vartheta - z\left(\vartheta + \frac{1}{4}\right)\right) = D\left(\frac{1}{12}, \frac{5}{12}, 1, 1\right).$$

Therefore, the function ${}_2F_1\left(\frac{1}{12}, \frac{5}{12}, 1; z\right)(1-z)^{\frac{1}{4}}$ is a solution of L at $z = 0$ and property (N) is fulfilled by Proposition 3.2.8. Let $q \in z\mathbb{Q}[\![z]\!]$ be the q-coordinate of L. As $1/(q^\vee)$ is up to the transformation $z \mapsto 1728z$ the Fourier expansion of the well-known modular form

$$j = q^{-1} + \sum_{n\geq 0}^{\infty} c(n)q^n = q^{-1} + 744q + 196884q^2 + \dots,$$

see e.g. [Sil99, Theorem V.1.1], the q-coordinate of L is N-integral. Hence L also fulfills property (Q). □

Using CY-type operators of degree two, we can produce CY-type operators of arbitrary degree with differential Galois group $\mathrm{SL}_2(\mathbb{C})$ via symmetric powers. This result can be seen as a reflection of the representation theory of $\mathrm{SL}_2(\mathbb{C})$, see e.g. [FH04, Chapter 11].

4.2.3 Lemma (i) $\mathrm{Sp}_2(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C})$

(ii) *The action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^{n+1}$ via Sym^n is irreducible. For $n = 2$, it is furthermore isomorphic to the natural action of $\mathrm{SO}_3(\mathbb{C})$ on $\mathbb{C}^3$.*

The result on the level of operators relies on the following general properties of differential operators of degree two which are taken from [PS02, Proposition 4.26].

4.2.4 Lemma *For each differential operator $L = \partial^2 + a\partial + b \in \mathbb{C}(z)[\partial]$ we have that*

(i) $\deg(\mathrm{Sym}^n L) = n + 1$.

(ii) $\mathrm{Sym}^n L = L_{n+1}$, *where L_i is recursively given by $L_0 = 1$, $L_1 = \partial$ and*

$$L_{i+1} = \partial L_i + iaL_i + i(n - i + 1)bL_{i-1}.$$

(iii) *each solution of* $\mathrm{Sym}^n L$ *is a product of* n *solutions of* L.

4.2.5 Proposition *Consider a CY-type operator* L *of degree two. Then its* n*-th symmetric power* $\mathrm{Sym}^n L$ *is a CY-type operator of degree* $n+1$.

Proof By the first statement of Lemma 4.2.4, the representation theory of $\mathrm{SL}_2(\mathbb{C})$ and Corollary 1.3.11, each of the operators $\mathrm{Sym}^n L$ is irreducible and has degree $n+1$. Denote by (M_L, e) the marked differential module corresponding to L and by $(M_{\mathrm{Sym}^n L}, e^n)$ the marked differential module corresponding to $\mathrm{Sym}^n(L)$, regarded as a differential submodule of $\mathrm{Sym}^n(M_L)$. Since by Proposition 1.3.10 the n-th power y_0^n of each holomorphic solution y_0 of $\iota_0(L)$ is a solution of $\mathrm{Sym}^n(L)$, this operator satisfies property (N) by Lemma 3.2.9. As L fulfills property (P), there is a non-degenerate, anti-symmetric pairing $\langle\cdot,\cdot\rangle$ on M_L. Setting

$$(v_1\cdots v_n, w_1\cdots w_n) = \langle v_1, w_1\rangle\cdots\langle v_n, w_n\rangle,$$

we get a pairing $(\cdot,\cdot)$ on $M_{\mathrm{Sym}^n(L)}$. By a direct computation, this implies that $\mathrm{Sym}^n(L)$ satisfies property (P). To show property (M), suppose without loss of generality that $\mathrm{Ind}_0(L) = T^2$. The recursion formula for $\mathrm{Sym}^n(L)$ given in Lemma 4.2.4 inductively yields that the sum of the exponents of $\iota_0\,(\mathrm{Sym}^n L)$ equals zero. Furthermore, each holomorphic solution of $\iota_0(\mathrm{Sym}^n L)$ is a $\mathbb{C}$-multiple of y_0^n. Hence, the exponent of y_0^n at $z = 0$ - which is zero - is the biggest exponent of $\iota_0(\mathrm{Sym}^n L)$. As L satisfies property (P), the relation between the exponents given in Proposition 3.1.5 gives property (M). It remains to check properties (Q) and (Q+). Therefore, we first observe that if y_0, y_1 form a Frobenius basis of $\iota_0(L)$, the elements $w_0,\ldots,w_n$ with $w_i = y_0^{n-i}y_1^i$ form a Frobenius basis of $\iota_0(\mathrm{Sym}^n(L))$ by the third statement of Lemma 4.2.4. The q-coordinate of $\mathrm{Sym}^n(L)$ fulfills the differential equation

$$q' = \frac{{w_1}'}{w_0}q = \frac{y_1}{y_0}q$$

and thus coincides with the q-coordinate of L. This yields property (Q). Setting $u_0(i) := w_i$ and

$$u_{k+1}(i) = \left(\frac{u_k(i)}{u_k(k)}\right)',$$

we know by Remark 3.4.6 that the structure series of L are $u_0^{-1}(0), u_1^{-1}(1),\ldots,u_n^{-1}(n) \in \mathbb{C}[\![z]\!]^*$. One checks inductively that

$$u_k^{-1}(i) = (i-k)\cdots i\left(\left(\frac{w_1}{w_0}\right)^{i-k+1}\right)'$$

holds. In particular, we have $u_1(1)/u_k(k) \in \mathbb{Q}$ for each $1 \le k \le n$ which yields property (Q+). □

As we have seen in the proof of the proposition above, the Deligne frames of those CY-type operators which are symmetric powers of operators of degree two are completely determined by their q-coordinate. By Theorem 1.2.5, the differential Galois group of these operators is $\mathrm{SL}_2(\mathbb{C})$. We suppose that the converse also holds:

4.2.6 Conjecture The differential Galois group of a CY-type operator of degree $n+1$ is $\mathrm{SL}_2(\mathbb{C})$ if and only if it can be written as n-th symmetric power of a CY-type operator of degree two.

As evidence for Conjecture 4.2.6, we present the relation between CY-type operators of degree two and three. This can be seen as a generalization of a classical theorem by G. Fano, see [Fan00].

4.2.7 Proposition *Each CY-type operator of degree three can be written as the second symmetric square of a CY-type operator of degree two.*

Proof Consider a CY-type operator $P = \partial^3 + a_2\partial^2 + a_1\partial + a_0$. Property (P) induces the relation

$$a_0 = \frac{1}{3}a_1a_2 - \frac{1}{3}a_2a_2' - \frac{2}{27}a_2^3 + \frac{1}{2}a_1' - \frac{1}{6}a_2''$$

amongst the coefficients of P. By the recursion formula stated in Lemma 4.2.4, we get $\mathrm{Sym}^2(L) = P$ for

$$L = \partial^2 + \frac{1}{3}a_2\partial + \frac{1}{4}a_1 - \frac{1}{12}a_2' - \frac{1}{18}a_2^2.$$

This operator also satisfies property (P). We can assume without loss of generality that the exponents of $\iota_0(P)$ are all equal to zero. A direct computation shows, that the sum of the exponents of $\iota_0(L)$ is zero as well. As the square y_0^2 of each solution y_0 of $\iota_0(L)$ is a solution of $\iota_0(P)$, the biggest exponent of $\iota_0(L)$ is zero. Therefore, property (M) follows from Proposition 3.1.5. As $\iota_0(P)$ admits an N-integral solution, L also does by Lemma 3.2.9 and hence fulfills property (N). Finally, as we have seen in the proof of Proposition 4.2.5 that the q-coordinates of L and P coincide, L satisfies property (Q). □

This statement can also be seen as reflection of the isomorphism mentioned in Lemma 4.2.3 on the level of differential operators. On the geometric side, prototypes for CY-type operators of degree three are differential Calabi-Yau operators for families of K3 surfaces with Picard number nineteen.

4.3 Degree four and five

As a consequence of the classification done in [Hes01], we prove that CY-type operators of degree four fall in two main classes: Those, whose differential Galois group is isomorphic to $\mathrm{SL}_2(\mathbb{C})$ - which acts in its natural Sym^3 representation on the solution space of the operator - and those whose differential Galois group is isomorphic to $\mathrm{Sp}_4(\mathbb{C})$. Assuming Conjecture 4.2.6, we have already discussed the first class of such operators and are mainly interested in the second one.

For each potential CY-type operator L of degree four which is not a symmetric power, we have to check both properties (Q) and (Q+). Even if a closed formula for the coefficients of an N-integral solution of $\iota_0(L)$ is known, it is almost impossible to check property (Q) in the general case. For hypergeometric equations, there p-adic methods established by B. Dwork, see [Dwo73]. Those techniques were applied to the case of generalized hypergeometric operators by C. Krattenthaler and T. Rivoal in [KR10] as well as to specializations of systems of generalized hypergeometric equations by the same authors in [KR08]. Very recently, E. Delagyue has established a nice criterion which covers the N-integrality of the q-coordinate for many known operators, see [Del11]. There is at the moment no chance to prove property (Q+) in a purely algebraic manner.

Those criteria which we can not prove rigorously are checked numerically up to a point of conviction. Recall, that both the holomorphic solution of $\iota_0(L)$ we consider and the q-coordinate

are solutions of differential equations of degree one with coefficients in $\mathbb{Q}[\![z]\!]$. A straightforward Ansatz reveals that the m-th coefficient of such a solution is a linear combination of rational numbers divided by m. Thus, we see that for a general operator more and more different prime numbers enter the factorization of the denominator of the coefficient. In practice, we hence do not believe that property (N), (Q) or (Q+) holds, if more than five different prime numbers enter the factorization of the denominator for the m-th coefficient of the corresponding power series. For arbitrary differential operators, this typically happens if $m \geq 20$.
As we have already seen in Corollary 3.1.4, the second exterior power of a CY-type operator of degree four has degree five and its differential Galois group is contained in $\mathrm{SO}_5(\mathbb{C})$. We show that most of the CY-properties are transported by this procedure and that it seems to be invertible. These relations were already observed in [AZ06] and can in fact be seen as reflection of the following group theoretic statement on the level of differential operators, see e.g. [FH04, Chapter 18].

4.3.1 Lemma *The action of* $\mathrm{Sp}_4(\mathbb{C})$ *on* $\mathbb{C}^6$ *via* $\bigwedge^2$ *splits into a one-dimensional and an irreducible five-dimensional representation. The latter one is isomorphic to the natural action of* $\mathrm{SO}_5(\mathbb{C})$ *on* $\mathbb{C}^5$.

We first state the following useful results of explicit computations.

4.3.2 Proposition *For each irreducible differential operator* $L = \partial^4 + \sum_{i=0}^{3} a_i \partial^i \in \mathbb{C}(z)[\partial]$ *which satisfies property (P) we have:*

(i)
$$a_1 = \frac{1}{2}a_2a_3 - \frac{1}{8}a_3^3 + a_2' - \frac{3}{4}a_3a_3' - \frac{1}{2}a_3''.$$

(ii) $\bigwedge^2 M = W \oplus N$, *where* (M, e) *is the marked differential module corresponding to* L, N *is the five dimensional differential submodule generated by* $e \wedge \partial e$ *and* W *is the one dimensional differential submodule generated by*
$$w = \left(a_2 - \frac{1}{4}a_3^2 - \frac{1}{2}a_3'\right) e \wedge \partial e + \frac{1}{2}a_3 e \wedge \partial^2 e + e \wedge \partial^3 e - \partial e \wedge \partial^2 e.$$

(iii) $\bigwedge^2 L$ *satisfies property (P).*

Proof The first result follows directly by writing down the pairing $\langle \cdot, \cdot \rangle$ on M_L with respect to the cyclic basis generated by e. The second result can be achieved by writing down
$$\wedge^2\psi\colon \bigwedge^2 M \to \bigwedge^2 M^\vee,\ m_1 \wedge m_2 \mapsto \psi(m_1) \wedge \psi(m_2)$$
with respect to the basis $\{\partial^i e \wedge \partial^j e\}_{i<j}$ induced by E. For the third statement, we observe that
$$\wedge^2\psi(e \wedge \partial e) = \alpha^2 \left(\partial^2 e\right)^\vee \wedge \left(\partial^3 e\right)^\vee$$
holds. With respect to the cyclic basis E' generated by $e \wedge \partial e$, one shows that
$$\wedge^2\psi(e \wedge \partial e) = \alpha^2 \left(\partial^4 (e \wedge \partial e)\right)^\vee$$

holds. As this yields $\bigwedge^2 L\alpha^2 = \alpha^2(\bigwedge^2 L)^\vee$, property (P) is satisfied by Proposition 3.1.2. $\square$

With those results at hand, we can prove that the second exterior power of a CY-type operator of degree four inherits properties of CY-type operators.

4.3.3 Proposition *Let $L = \partial^4 + \sum_{i=0}^3 a_i\partial^i \in \mathbb{Q}(z)[\partial]$ be a CY-type operator of degree four. Then $\bigwedge^2 L$ is irreducible and satisfies properties (M), (N) and (P).*

Proof We have already shown in Proposition 4.3.2 that $\bigwedge^2 L$ satisfies property (P). Writing $\bigwedge^2 L = \partial^5 + \sum_{i=0}^4 b_i\partial^i$ we get

$$-\frac{2}{5}b_4\alpha^2 = \left(\alpha^2\right)' = 2\alpha\alpha' = -a_3\alpha^2$$

and thus $b_4 = 5a_3/2$. Assume without loss of generality that $\mathrm{Ind}_0(L) = T^4$. Then the sum of the exponents of $\iota_0(\bigwedge^2 L)$ is -5. Choose a Frobenius basis $\{y_0, \dots, y_3\}$ of the solution space of $\iota_0(L)$. By Proposition 1.3.10, the solution space of $\iota_0(\bigwedge^2 L)$ is spanned by the Wronskian $\mathrm{Wr}(y_i, y_j)$ for $0 \leq i < j \leq 3$. Therefore, all exponents of $\iota_0\left(\bigwedge^2 L\right)$ are integers and the space holomorphic solutions of $\iota_0\left(\bigwedge^2 L\right)$ is spanned by $\mathrm{Wr}(y_0, y_1)$. Moreover, the exponent of $\mathrm{Wr}(y_0, y_1)$ - which equals -1 - is the biggest one of $\iota_0(\bigwedge^2 L)$. As $\bigwedge^2 L$ fulfills property (P), property (M) follows from Proposition 3.1.5. To check property (N), observe that we have $\mathrm{Wr}(y_0, y_1) = y_0y_1' - y_0'y_1 = -y_0^2\,(y_1/y_0)'$. Property (Q) assures that the logarithmic derivative $q'/q = (y_1/y_0)'$ of the q-coordinate of L is N-integral. As L satisfies property (N), the function y_0 is N-integral and hence $\bigwedge^2 L$ also satisfies property (N). Finally, we show that $\bigwedge^2 L$ is irreducible. Denote by (M, e) the marked differential module associated to L and by N the differential submodule of $\bigwedge^2 M$ which is spanned by $e \wedge \partial e$. Since M is irreducible, the module $\bigwedge^2 M$ is semi-simple, i.e. each differential submodule has a complement which is also a differential a submodule, by [PS02, Exercise 2.38]. Therefore, if $\bigwedge^2 L$ and hence N were reducible, the representation matrix of ϑ on N would up to base change be of the form

$$C := \begin{pmatrix} C_1 & 0 & \dots & 0 \\ 0 & C_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_t \end{pmatrix} \in \mathrm{Mat}_5(\mathbb{C})$$

with $t > 1$. As $\bigwedge^2 L$ satisfies property (M), this is impossible. $\square$

Note, that the logarithmic derivative of q-coordinate $\tilde{q}$ of $\bigwedge^2(L)$ is given by

$$\tilde{q}'/\tilde{q} = (\mathrm{Wr}(y_2, y_0)/\mathrm{Wr}(y_1, y_0))' = \left((y_2/y_0)'/(y_1/y_0)'\right)'.$$

By Remark 3.4.6, the latter expression coincides with the quotient α_1/α_2, where $\alpha_0, \dots, \alpha_3$ denotes the structure series of L. As L satisfies property (Q+), we conclude that $\tilde{q}'/\tilde{q}$ is N-integral. We furthermore suppose - but are not able to prove - that $\bigwedge^2(L)$ even fulfills properties (Q) and (Q+).

As the formal monodromy of a CY-type operator is by definition maximally unipotent at $z = 0$, its differential Galois group is not finite. By the classification achieved in [Hes01], Proposition 4.3.3 leaves two possibilities.

4.3.4 Corollary *The differential Galois group of a CY-type operator of degree four is either* $\mathrm{SL}_2(\mathbb{C})$ *or* $\mathrm{Sp}_4(\mathbb{C})$.

Unless the case of degree two and three, the local normal form of a CY-type operator derived in Corollary 3.4.7 is not trivial. Expressed in the special coordinate, it depends on one single power series. The following terminology is borrowed from physics.

4.3.5 Definition For a CY-type operator $L \in \mathbb{Q}[z, \vartheta]$ of degree four with associated marked differential module (M_L, e) and symplectic form $\langle \cdot, \cdot \rangle$ we call $Y := \langle e, \vartheta^3 e \rangle \in \mathbb{C}(z)$ the *Yukawa coupling* of L.

The Yukawa coupling determines the local normal form of L in the following way.

4.3.6 Proposition *Consider a CY-type operator* $L \in \mathbb{Q}[z, \vartheta]$ *of degree four with structure series* $\alpha_0, \dots, \alpha_3$*, special coordinate* $q \in z\mathbb{Q}[\![z]\!]$ *and associated marked differential module* (M_L, e)*. Then there is an* $c \in \mathbb{C}^*$ *such that*

$$\langle \alpha_0 e, \vartheta_q^3(\alpha_0 e) \rangle = c \frac{\alpha_1}{\alpha_2}$$

Proof Consider a Deligne frame $\{d_0, \dots, d_3\}$ induced by L. As $\vartheta_q = \alpha_1 \vartheta$ holds we get

$$\vartheta_q^3 \alpha_0 e = \vartheta_q^2 \alpha_1 d_1 = \vartheta_q \alpha_1 d_2 = \alpha_1 \left(\left(\frac{\alpha_1}{\alpha_2} \right)' \alpha_2 d_2 + \frac{\alpha_1}{\alpha_2} d_3 \right),$$

where $(\cdot)' = z\frac{d}{dz}$. As $\alpha_1 = c\alpha_3$ for a constant $c \in \mathbb{C}^*$, $\langle e, d_2 \rangle = 0$ and $\langle \alpha_0 e, \alpha_3 d_3 \rangle \in \mathbb{C}^*$, this gives the result. □

The expression $\langle \alpha_0 e, \vartheta_q^3(\alpha_0 e) \rangle$ is often called *normalized Yukawa coupling*. In particular, Conjecture 4.2.6 induces the following:

4.3.7 Conjecture The differential Galois group of a CY-type operator of degree four is $\mathrm{SL}_2(\mathbb{C})$ if and only if its normalized Yukawa-coupling is constant.

We now discuss a construction which is inverse to the one done in Proposition 4.3.3, i.e. how to get an operator of degree four which satisfies (M), (N) and (P) from a CY-type operator of degree five. Again, this relation is similar to a classical statement of G. Fano [Fan00] and is studied in [AZ06].

4.3.8 Lemma *Consider an irreducible differential operator* $P = \partial^5 + \sum_{i=0}^4 b_i \partial^i \in \mathbb{C}(z)[\partial]$ *of degree five which fulfills (P). Then there is a uniquely determined differential operator* $L = \partial^4 + \sum_{i=0}^3 a_i \partial^i \in \mathbb{C}(z)[\partial]$ *which fulfills (P) such that* $\bigwedge^2 L = P$ *holds. In this situation, we write* $\bigvee_2(P) := L$.

Proof We prove the statement by a direct computation. First, assume that $b_4 = 0$. By property (P), we get that

$$P = \partial^5 + b_3 \partial^3 + \frac{3}{2} b_3' \partial^2 + b_1 \partial + \frac{1}{2} b_1' - \frac{1}{4} b_3'''.$$

By [AZ06, Proposition 3], each operator $L = \partial^4 + \sum_{i=0}^{3} a_i\partial^i$ of degree four whose second exterior power has degree five fulfills property (P). Furthermore, as $b_4 = 0$, one computes that $a_3 = 0$. Hence we make the Ansatz

$$L = \partial^4 + a_2\partial^2 + a_2'\partial + a_0.$$

As this operator fulfills (P), its second exterior power also does by Proposition 4.3.2 and a direct computation yields

$$\bigwedge^2 L = \partial^5 + 2a_2\partial^3 + 3a_2'\partial^2 + (a_2^2 + 3a_2'' - 4a_0)\partial + a_2''' + a_2'a_2 - 2a_0'.$$

Comparing the coefficients of $\bigwedge^2 L$ and P reveals that our Ansatz was successful. If $b_4 \neq 0$, there is a unique $\beta \in \mathbb{C}(z)^{alg}$ such that $\beta P\beta^{-1} = \partial^5 + \sum_{i=0}^{3} \tilde{b}_i\partial^i$. If L satisfies (P) and $\bigwedge^2 L = \beta P\beta^{-1}$ one readily checks that for $\gamma^2 = \beta$ we get $\bigwedge^2 \gamma L\gamma^{-1} = P = \bigwedge^2(-\gamma)L(-\gamma)^{-1}$. This completes the proof. □

We also have a relation between $\mathrm{Sym}^2\left(\bigvee_2(P)\right)$ and $\bigwedge^2(P)$ which once again is inspired by representations of $\mathrm{Sp}_4(\mathbb{C})$ and follows directly from the computations done in [Alm06, Section 2.2].

4.3.9 Lemma *Let $L = \partial^4 + \sum_{i=0}^{3} a_i\partial^i \in \mathbb{C}(z)[\partial]$ satisfy (P) with $L\alpha = \alpha L^\vee$ for $0 \neq \alpha \in \mathbb{C}(z)$. Then*

$$\bigwedge^2\left(\bigwedge^2 L\right) = \mathrm{Sym}^2 L \otimes z^2\alpha.$$

With this relation at hand, we are able to check the following result.

4.3.10 Proposition *For each CY-type operator $P \in \mathbb{Q}(z)[\partial]$ of degree five which has an odd exponent at $z = 0$, the operator $L = \bigvee_2(P)$ fulfills properties (M), (N) and (P).*

Proof We have already seen in Lemma 4.3.8 that L fulfills property (P). To check property (M), assume without loss of generality that the exponents of $\iota_0(P)$ are all equal to -1. A direct computation shows that the sum of the exponents of $\iota_0\left(\bigvee_2(P)\right)$ is zero. The solution space of $\iota_0(P)$ admits a Frobenius basis and is spanned by the wronskians of solutions of $\iota_0(L)$. Therefore, the solution space of L also admits a Frobenius basis $y_0, \ldots, y_3$. As $\mathrm{Wr}(y_0, y_1) \in z^{-1}\mathbb{Q}[\![z]\!]$ is a solution of P, its exponent is -1 and the exponent of y_0 is zero. Hence zero is the biggest exponent of $\iota_0\left(\bigvee_2(P)\right)$ and property (M) is fulfilled by Proposition 3.1.5. By an argument completely similar to the one in the proof of Proposition 4.3.3, we get that $\bigwedge^2(P)$ has an N-integral holomorphic solution g at $z = 0$. By Lemma 4.3.9, we have $g = z^2\alpha y_0^2$ for y_0 as before and $0 \neq \alpha \in \mathbb{C}(z)$ such that $\bigvee_2(P)\alpha = \alpha\bigvee_2(P)^\vee$ holds. As $z^2\alpha \in \mathbb{C}(z)$, we get that y_0^2 is N-integral as well. Hence, $\bigvee_2(P)$ also fulfills property (N) by Lemma 3.2.9. □

Propositions 4.3.3 and 4.3.10 lead together with our computational experience to the following:

4.3.11 Conjecture Each CY-type operator of degree five is equivalent to the second exterior power of a CY-type operator of degree four.

Chapter 5

Monodromy tuples and their constructions

Consider a finite non-empty set $S \subset \mathbb{P}^1$ with $r+1$ elements and $X := \mathbb{P}^1 \setminus S$. By additional choices on the underlying topological data, each local system of rank n on X is determined by a tuple of matrices $(T_1, \dots, T_{r+1}) \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ such that $\prod_{i=1}^{r+1} T_i = \mathbb{1}_n$ holds. Such a tuple of matrices is called an m-tuple. We discuss elementary properties of m-tuples in the first section of this chapter. Here, the so-called multiplicative Deligne-Simpson Problem (MDSP) - i.e. the question, if there is for a given tuple of Jordan matrices a corresponding m-tuple - is of our particular interest. In the second section, we discuss the notion of rigid local systems and m-tuples. Those local systems are completely determined by their local data, i.e. the Jordan forms of the matrices of a corresponding m-tuple. By the work [Kat96] of N. Katz, rigid local systems of arbitrary rank can be constructed from local systems of rank one via tensor products and middle convolutions. We review the middle convolution and the middle Hadamard product on the level of m-tuples in the third section. In section four, we state a way to attack the MDSP constructively and the main theorem of Katz concerning the construction of linearly rigid m-tuples.

Let throughout the whole chapter S be a finite non-empty subset of $\mathbb{P}^1$. We also fix an orientation on $\mathbb{P}^1$. Moreover, we denote by G a reductive linear algebraic group together with an embedding $G \to \mathrm{GL}_n(\mathbb{C})$ and do not distinguish between G and its image under this embedding. A Jordan-block matrix of size m whose diagonal entries are all equal to α is denoted by $\alpha J(m)$ and $J(m)$ if $\alpha = 1$. The Jordan form $\mathbf{J}(A)$ of an arbitrary matrix $A \in \mathrm{GL}_n(\mathbb{C})$ is written as direct sum of the corresponding Jordan-block matrices. In the sequel, we call such a direct sum of Jordan-block matrices simply a *Jordan matrix* or a *Jordan form*. If a block matrix $\alpha J(m)$ appears with multiplicity k, we denote that by $\alpha J(m)^k$.

5.1 Local systems and monodromy tuples

We introduce the notion of m-tuples.

5.1.1 Definition A tuple of matrices $T := (T_1, \dots, T_{r+1}) \in G^{r+1} \subset \mathrm{GL}_n(\mathbb{C})^{r+1}$ is called an *m-tuple* if

$$\prod_{i=1}^{r+1} T_i = \mathbb{1}_n.$$

We call $\mathrm{rk}(T) := n$ the *rank* of the m-tuple. If especially G is a symplectic or an orthogonal group, we call the m-tuple *symplectic*, resp. *orthogonal*. We call

$$\mathbf{J}(T) = (\mathbf{J}(T_1), \ldots, \mathbf{J}(T_{r+1})) \in \mathrm{GL}_n(\mathbb{C})$$

its associated *tuple of Jordan forms*. The action of the matrices of an m-tuple on $\mathbb{C}^n$ induce a representation of the group generated by this matrices. We denote this representation by $< T >$. We call two m-tuples $T, T' \in G^{r+1}$ to be *equivalent*, written $T \sim T'$, if there is an $H \in \mathrm{GL}_n(\mathbb{C})$ such that $H^{-1} T_i H = T_i'$ holds for each $1 \leq i \leq r+1$. In the sequel, we do not distinguish between m-tuples which are equivalent to each other.

The category of m-tuples with $r+1$ elements and the obvious notion of morphisms, i.e. $AT_i = T_i' A$ for a matrix A and each $1 \leq i \leq r+1$, is denoted by Tup_{r+1}. In this category, we can take tensor products, symmetric powers and exterior powers in the usual way.
We state a link between local systems on $\mathbb{P}^1 \setminus S$ and m-tuples depending on the following choice of topological data.

5.1.2 Definition A *base star* on $\mathbb{P}^1 \setminus S$ is a tuple

$$\Gamma\left(\underline{S}\right) := (x_0; (s_1, \ldots, s_{r+1}); (\gamma_{s_1}, \ldots, \gamma_{s_{r+1}}))$$

consisting of

(i) the elements of the set $S = \{s_1, \ldots, s_{r+1}\}$ written as a tuple

$$\underline{S} = (s_1, \ldots, s_{r+1}) \in \left(\mathbb{P}^1\right)^{r+1}$$

with pairwise distinct entries.

(ii) a base point $x_0 \in \mathbb{P}^1 \setminus S$.

(iii) a set of generators $\gamma_{s_1}, \ldots, \gamma_{s_{r+1}} \in \pi_1(\mathbb{P}^1 \setminus S, x_0)$ such that γ_{s_i} encircles s_i in positive direction and no other points of S and moreover the composition $\gamma_{s_1} \cdots \gamma_{s_{r+1}}$ is the trivial path.

We call the tuple $\underline{S}$ the *points* of $\Gamma(\underline{S})$.

5.1.3 Proposition *For each base star $\Gamma(\underline{S})$, the functor*

$$\mathrm{Repr}_{\pi_1(\mathbb{P}^1 \setminus S, x_0)}(\mathbb{C}) \to \mathrm{Tup}_{r+1},\ \rho \mapsto (\rho(\gamma_{s_1}), \ldots, \rho(\gamma_{s_{r+1}}))$$

is an equivalence of categories which is compatible with tensor operations. In particular, we get an equivalence of categories

$$\tau_{\Gamma(\underline{S})} \colon \mathrm{LocSys}(\mathbb{P}^1 \setminus S) \to \mathrm{Tup}_{r+1}, \mathbb{L} \to \left(\rho_{\mathbb{L}}(\gamma_{s_1}), \ldots, \rho_{\mathbb{L}}(\gamma_{s_{r+1}})\right)$$

which is compatible with tensor operations.

This relation leads to the following terminology.

5.1.4 Definition Consider a local system $\mathbb{L}$ on $\mathbb{P}^1 \setminus S$ and a base star $\Gamma(\underline{S})$. Then we call the m-tuple $\tau_{\Gamma(\underline{S})}(\mathbb{L})$ a *monodromy tuple* of $\mathbb{L}$. If the local system $\mathbb{L}$ is induced by a fuchsian differential operator L, we also speak of a *monodromy tuple* of L.

We introduce further terminology for m-tuples which reflects properties of their induced representations.

5.1.5 Definition We call an m-tuple T

(i) *irreducible* if its natural representation $< T >$ is irreducible.

(ii) *orthogonal*, resp. *symplectic*, if its matrices lie in an orthogonal, resp. symplectic, group.

(iii) *quasi-unipotent* if all its matrices are quasi-unipotent.

It is natural to classify m-tuples via the conjugacy classes of the matrices of their irreducible parts. This leads directly to the following problem.

Problem (The multiplicative Deligne-Simpson Problem (MDSP))
Let $J = (J_1, \ldots, J_{r+1}) \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ be a tuple of Jordan matrices. Decide, whether there is an irreducible m-tuple $T = (T_1, \ldots, T_{r+1}) \in G^{r+1}$ such that the Jordan form of T_i is J_i. This problem is called the *multiplicative Deligne-Simpson Problem*, or MDSP in short, for J. We say that the MDSP for J is *solvable* in G, if there is an irreducible m-tuple $T \in G^{r+1}$ with the requested properties. Such an m-tuple T is called a *solution* of the related MDSP. If a solution can be realized as monodromy tuple of a fuchsian differential operator $L \in \mathbb{C}[z, \vartheta]$, we call L a solution of the MDSP as well.

The study of the MDSP was pioneered by P. Deligne and C. Simpson, see e.g. [Sim91]. An overview concerning solutions of this problem is given in [Kos02]. Explicit solutions of the MDSP for an arbitrary tuple of Jordan matrices are in general hard to find for the following two reasons:
First, the product relation of the matrices of a potential solution $T = (T_1, \ldots, T_{r+1})$ yields that T_{r+1} is determined by the matrices $T_1, \ldots, T_r$. However, if we only know the Jordan forms $\mathbf{J}(T_1), \ldots, \mathbf{J}(T_r)$, we do in general not know $\mathbf{J}(T_{r+1})$. The second difficulty lies in the irreducibility of the m-tuple T. If we drop this claim on the solutions, we would e.g. for each choice of Jordan matrices $J_1, \ldots, J_r$ get that the MDSP for $(J_1, \ldots, J_r, \mathbf{J}\left(J_r^{-1} \ldots J_1^{-1}\right))$ is solvable by this tuple of Jordan forms itself.
We briefly discuss the MDSP in the simple but very important case of rank one.

5.1.6 Example Consider a tuple of non-zero numbers $J = (J_1, \ldots, J_{r+1}) \in \mathrm{GL}_1(\mathbb{C})^{r+1}$. Then the MDSP for J is solvable if and only if J is an m-tuple, i.e. we have $\prod_{i=1}^{r+1} J_i = 1$. In this case, J itself is a solution.

For an m-tuple T of rank bigger than one, it is a non-trivial problem to detect whether it is irreducible just by looking at its tuple of Jordan matrices. To state a criterion, we introduce the following notation.

5.1.7 Definition For $A \in \mathrm{GL}_n(\mathbb{C})$ we write

$$\gamma(A) := \mathrm{rk}(A - \mathbb{1}_n).$$

The criterion given in [Sco77, Theorem 1] reads:

5.1.8 Lemma *Each irreducible m-tuple $T = (T_i)_{1 \le i \le r+1} \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ fulfills*

$$\sum_{i=1}^{r+1} \gamma(T_i) \ge 2n.$$

This inequality can actually be used to show that a given tuple of Jordan matrices can not be induced by an irreducible m-tuple.

5.2 Rigidity

In this section, we study m-tuples which are completely determined by the Jordan forms of their matrices, so-called linearly rigid ones. We first introduce the notion of rigidity as it is done in [Kat96, Section 1.0].

5.2.1 Definition An m-tuple $T = (T_1, \ldots, T_{r+1}) \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ is called *linearly rigid* if we have $T \sim T'$ for each other m-tuple $T' = (T'_1, \ldots, T'_{r+1}) \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ with $\mathbf{J}(T) = \mathbf{J}(T')$. A local system on $\mathbb{P}^1 \setminus S$ is called *linearly rigid* if an associated monodromy tuple has this property.

5.2.2 Example We have the following prominent examples of linearly rigid local systems.

(i) Each local system of rank one is linearly rigid.

(ii) As revealed by B. Riemann in [Rie57], each local system on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ which is induced by a hypergeometric differential operator

$$D(\alpha_1, \alpha_2, \beta_1, \beta_2) = (\vartheta + \beta_1 - 1)(\vartheta + \beta_2 - 1) - z(\vartheta + \alpha_1)(\vartheta + \alpha_2)$$

is linearly rigid. By Levelt's thesis [Lev61], this result also holds for generalized hypergeometric differential operators $D(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)$.

We can read off whether an irreducible m-tuple is linearly rigid from the Jordan forms of its matrices. For simplicity, we introduce the following terminology.

5.2.3 Definition For $A \in G \subset \mathrm{GL}_n(\mathbb{C})$ we set

$$\delta_G(A) := \mathrm{codim}(C_G(A)),$$

where

$$C_G(A) = \{H \in G \mid H^{-1}AH = A\}$$

denotes the centralizer of A in G and the co-dimension is taken with respect to the dimension of G as an algebraic group over $\mathbb{C}$. If $G = \mathrm{GL}_n(\mathbb{C})$, we just write $\delta(A)$.

5.2.4 Remark. By [Car85, Chapter 13] and [Sim91, Lemma 7], the dimension of the centralizer $C_{\mathrm{GL}_n(\mathbb{C})}(A)$ of a given matrix $A \in \mathrm{GL}_n(\mathbb{C})$ can be computed as follows:
If A is unipotent, its Jordan form reads $\mathbf{J}(A) = \bigoplus_{i=1}^{k} J(v_i)$. The sizes $v_1 \ge \cdots \ge v_k$ of the distinct Jordan blocks of $\mathbf{J}(A)$ give a partition $V(A) = (v_1, \ldots, v_k)$ of n. We pass to its dual partition $W(T) = (w_1, \ldots, w_l)$ by setting

$$w_i := \#\{j \mid v_j \ge i\}$$

and get

$$\dim C_{\mathrm{GL}_n(\mathbb{C})}(A) = \sum_{i=1}^{l} w_i^2.$$

If A is not unipotent with set of eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$, its Jordan form reads

$$\mathbf{J}(A) = \bigoplus_{i=1}^{m} \lambda_i \bigoplus_{k=1}^{j_i} J(u_{i,k}).$$

For each λ_i, the sizes of the Jordan blocks $u_{i,1} \geq \cdots \geq u_{i,j_i}$ give partitions $U_i = (u_{i,1}, \ldots, u_{i,j_i})$. Then we define the partition $V(A) = (v_1, \ldots, v_N)$ with $N = \max_{i=1,\ldots,m}\{j_i\}$ via

$$v_r := \sum_{k=0}^{m} u_{k,r}$$

with the convention $u_{i,r} = 0$ if $r > j_i$, pass to its dual partition $W(A)$ and again get $\dim C_{\mathrm{GL}_n(\mathbb{C})}(A) = \sum_{i=1}^{l} w_i^2$.

We have the following criterion to check whether an irreducible local system is linearly rigid, see [Kat96, Theorem 1.1.2].

5.2.5 Theorem *An irreducible local system $\mathbb{L}$ on $\mathbb{P}^1 \setminus S$ is linearly rigid if and only if*

$$\chi(j_* \operatorname{End}(\mathbb{L})) = 2 \dim H^0\left(\mathbb{P}^1, j_* \operatorname{End}(\mathbb{L})\right) - \dim H^1\left(\mathbb{P}^1, j_* \operatorname{End}(\mathbb{L})\right) = 2,$$

where $j\colon \mathbb{P}^1 \setminus S \to \mathbb{P}^1$ denotes the natural inclusion.

As revealed in the proof of this theorem and in [SV99, Theorem 2.3], this gives a criterion for rigidity on the level of m-tuples.

5.2.6 Proposition *Each irreducible m-tuple $T = (T_1, \ldots, T_{r+1}) \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ fulfills*

$$\sum_{i=1}^{r+1} \delta(T_i) - 2(n^2 - 1) \geq 0.$$

Moreover, we have

$$\sum_{i=1}^{r+1} \delta(T_i) - 2(n^2 - 1) = 0$$

if and only if T is linearly rigid.

The notion of rigidity can be extended from $\mathrm{GL}_n(\mathbb{C})$ to arbitrary reductive algebraic groups G as it was e.g. done in [SV99, Definition 3.1]. We do not have a criterion to prove G-rigidity as stated in Proposition 5.2.6 and thus give a reasonable generalization of the expression $\sum_{i=1}^{r+1} \delta(T_i) - 2(n^2 - 1)$ to an arbitrary tuple of matrices $T \in G^{r+1}$ as stated in [SV99, Corollary 3.2].

5.2.7 Definition For a tuple of matrices $T = (T_1, \dots, T_{r+1}) \in G^{r+1}$ we call

$$i_G(T) := \sum_{i=1}^{r+1} \delta_G(T_i) - 2(\dim(G) - \dim(Z(G))$$

the *rigidity-index* of T with respect to G, where $Z(G)$ denotes the center of G. If $G = \mathrm{GL}_n(\mathbb{C})$, we just write $i(T)$ instead of $i_G(T)$.

Note, that we have defined the rigidity index for arbitrary tuples of matrices. Moreover, we have $i_G(T) = i_G(\mathbf{J}(T))$ for each m-tuple T. As $\dim \mathrm{GL}_n(\mathbb{C}) = n^2$ and $\dim Z(\mathrm{GL}_n(\mathbb{C})) = 1$, Proposition 5.2.6 can be rephrased by saying that an irreducible m-tuple T is linearly rigid if and only if $i(T) = i(\mathbf{J}(T)) = 0$ holds.
One of the striking results of N. Katz concerning the solvability of the MDSP in the linearly rigid case was established in [Kat96, Sections 6.3 and 6.4]:
He gives an algorithmic procedure to decide if for a given tuple of Jordan matrices $J \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ with $i(J) = 0$ the MDSP for J is solvable or not. Moreover, if the MDSP for J is solvable, this procedure provides a construction of the solution T starting with an m-tuple of rank one by application of a finite sequence of operations with m-tuples of rank one. The key fact concerning these operations is that they preserve irreducibility and rigidity of m-tuples, but may change the rank of them.
We state a more precise statement in Theorem 5.4.3. In fact, we need two operations for the construction of solutions of the MDSP for linearly rigid tuples, namely tensor products and middle convolutions with special m-tuples of rank one.
Tensor products with m-tuples of rank one have the following properties.

5.2.8 Lemma *Consider an m-tuple $T \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ and an m-tuple $P \in \mathrm{GL}_1(\mathbb{C})^{r+1}$ of rank one. Then*

(i) *T is irreducible if and only if $T \otimes P$ is.*

(ii) *there is an m-tuple $P' \in \mathrm{GL}_1(\mathbb{C})^{r+1}$ such that $(T \otimes P) \otimes P' = T$. In this case, we write $P^{-1} := P'$.*

(iii) *$i(T) = i(T \otimes P)$.*

Proof The first and second statement are clear. For an arbitrary matrix $A \in \mathrm{GL}_n(\mathbb{C})$, the procedure described in Remark 5.2.4 shows that $W(A) = W(\alpha A)$ holds for each $\alpha \in \mathbb{C}^*$. Hence $\delta(A) = \delta(\alpha A)$, which gives the third statement. □

In particular, we can interpret the tensor product with m-tuples of rank one as an invertible operation which preserves irreducibility and the rigidity index. Concerning solutions of the MDSP, this yields:

5.2.9 Corollary *Consider a tuple of Jordan matrices $J \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ and an arbitrary m-tuple P of rank one. Then the MDSP for J is solvable if and only if the MDSP for $J \otimes P$ is.*

5.3 The middle convolution and the middle Hadamard product

The original definition of the middle convolution by N. Katz given in [Kat96, Chapter 4.3] is very general, but in our context quite unhandy to use. The translation to the level of local systems is e.g. given in [Det05, Section 2.3]. We rather describe the middle convolution on the level of m-tuples which was established in [DR00, Definition 2.5] and state some of its properties taken from [Det05], [DR00] and [DR07].

5.3.1 Definition For an m-tuple $T = (T_i)_{1\le i\le r+1} \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ and $\alpha \in \mathbb{C}^*$ we set

(i)

$$B_k := \begin{pmatrix}
\mathbb{1}_n & 0 & & \cdots & & 0 & \\
& \ddots & & & & & \\
& & \mathbb{1}_n & & & & \\
\alpha(T_1-\mathbb{1}_n) & \cdots & \alpha(T_{k-1}-\mathbb{1}_n) & \alpha T_k & (T_{k+1}-\mathbb{1}_n) & \cdots & (T_r-\mathbb{1}_n) \\
& & & & \mathbb{1}_n & & \\
& & & & & \ddots & \\
0 & & & \cdots & & 0 & \mathbb{1}_n
\end{pmatrix}$$

for $k = 1, \ldots, r$ and $B_{r+1} = (B_1 \cdots B_r)^{-1}$.

(ii) $K(T) := \bigoplus_{i=1}^{r} \mathrm{Kern}(T_i - \mathbb{1}_n)$.

(iii) $L(T) := \bigcap_{i=1}^{r} \mathrm{Kern}(B_i - \mathbb{1}_{nr})$.

(iv)

$$\mathrm{MC}_\alpha(T) := (\mathrm{MC}_\alpha(T_i))_{1\le i\le r+1} = (\pi(B_i))_{1\le i\le r+1} \in \mathrm{GL}(\mathbb{C}^{nr}/(K(T)+L(T)))^{r+1}.$$

The m-tuple $\mathrm{MC}_\alpha(T)$ is called the *middle convolution* of T with α, where $\pi\colon \mathbb{C}^{nr} \to \mathbb{C}^{nr}/(K(T)+L(T))$ denotes the canonical projection.

As stated in [DR00, Lemma 2.7], the rank of $\mathrm{MC}_\alpha(T)$ turns out to depend on the tuple of Jordan forms $\mathbf{J}(T)$ only.

5.3.2 Lemma *For each m-tuple $T = (T_1, \ldots, T_{r+1}) \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ and each $\alpha \in \mathbb{C}^* \setminus \{1\}$, the rank of $\mathrm{MC}_\alpha(T)$ is given by*

$$\mathrm{rk}(\mathrm{MC}_\alpha(T)) = \sum_{i=1}^{r} \gamma(T_i) + \gamma\left(\alpha^{-1}T_{r+1}\right) - n.$$

Moreover, for an irreducible m-tuple $T \in \mathrm{GL}_n(\mathbb{C})$, the tuple of Jordan forms $\mathbf{J}(\mathrm{MC}_\alpha(T))$ is completely determined by the tuple of Jordan forms $\mathbf{J}(T)$. Therefore, we introduce the operation MC_α for tuples of Jordan forms as well.

5.3.3 Definition For each tuple of Jordan matrices $J = (J_1, \dots, J_{r+1}) \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ with

$$J_i = \bigoplus_{\rho \in \mathbb{C}} \rho J(j)^{v(i,\rho,j)}$$

and each $\alpha \in \mathbb{C}^* \setminus \{1\}$, we set $C_\alpha(J) := \sum_{i=1}^r \gamma(J_i) + \gamma(\alpha^{-1} J_{r+1}) - n$,

$$c_{\alpha,i}(J) := \gamma(J_i) + n - \gamma(\alpha J_i)$$

and

$$c_{\alpha,r+1}(J) := \gamma(\alpha^{-1} J_{r+1}) + n - \gamma(J_{r+1}).$$

If $c_{\alpha,i}(J) \leq C_\alpha(J)$ holds for all $1 \leq j \leq r+1$, the tuple of Jordan matrices

$$\mathrm{MC}_\alpha(J) := (\mathrm{MC}_\alpha(J_1), \dots, \mathrm{MC}_\alpha(J_{r+1}))$$

of rank $C_\alpha(J)$ is defined via

$$\mathrm{MC}_\alpha(J_i) = \bigoplus_{\rho \in \mathbb{C} \setminus \{1, \alpha^{-1}\}} \bigoplus_j \alpha\rho J(j)^{v(i,\rho,j)} \bigoplus_{j \geq 2} \alpha J(j-1)^{v(i,1,j)} \\ \bigoplus J(j+1)^{v(i,\alpha^{-1},j)} \bigoplus J(1)^{C_\alpha(J) - c_{\alpha,i}(J)}$$

and

$$\mathrm{MC}_\alpha(J_{r+1}) = \bigoplus_{\rho \in \mathbb{C} \setminus \{1, \alpha\}} \bigoplus_j \alpha^{-1}\rho J(j)^{v(r+1,\rho,j)} \bigoplus J(j-1)^{v(r+1,\alpha,j)} \\ \bigoplus \alpha^{-1} J(j+1)^{v(r+1,1,j)} \bigoplus \alpha^{-1} J(1)^{C_\alpha(J) - c_{\alpha,r+1}(J)}.$$

By [Det05, Lemma 3.4.2], this operation is compatible with the middle convolution for irreducible m-tuples.

5.3.4 Proposition *For each irreducible m-tuple $T = (T_1, \dots, T_{r+1}) \in \mathrm{GL}_n(\mathbb{C})$ and each $\alpha \in \mathbb{C}^* \setminus \{1\}$, we have*

$$\mathbf{J}(\mathrm{MC}_\alpha(T)) = \mathrm{MC}_\alpha(\mathbf{J}(T)).$$

The middle convolution $\mathrm{MC}_\alpha(T)$ with $\alpha \in \mathbb{C}^* \setminus \{1\}$ is an invertible operation which preserves irreducibility and rigidity of T, see [DR00, Remark 3.1, Corollary 3.6 and Corollary 4.4].

5.3.5 Lemma *Consider an irreducible m-tuple $T = (T_1, \dots, T_{r+1}) \in \mathrm{GL}_n(\mathbb{C})$ such that at least two matrices T_i, T_j with $1 \leq i, j \leq r$ are not the identity if $n = 1$. Then for each $\alpha \in \mathbb{C}^* \setminus \{1\}$ we have that*

(i) $\mathrm{MC}_\alpha(T)$ *is irreducible.*

(ii) $\mathrm{MC}_{\alpha^{-1}}(\mathrm{MC}_\alpha(T)) = T$.

(iii) $i(T) = i(\mathrm{MC}_\alpha(T))$.

The second and third statement also hold if we replace T by $\mathbf{J}(T)$.
Concerning the MDSP this yields:

5.3.6 Corollary *Consider an $\alpha \in \mathbb{C}^* \setminus \{1\}$ and a tuple of Jordan matrices $J \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ such that at least two matrices of J_i, J_j with $1 \leq i, j \leq r$ are not the identity if $n = 1$. Then the MDSP for J is solvable if and only if the MDSP for $\mathrm{MC}_\alpha(J)$ is.*

We also introduce the middle Hadamard product MH_α with $\alpha \in \mathbb{C}^*$ on the level of m-tuples and tuples of Jordan forms. This operation is closely related to the middle convolution with MC_α.

5.3.7 Definition Let $T = (T_1, \dots, T_{r+1}) \in \mathrm{GL}_n(\mathbb{C})$ be an m-tuple and $\alpha \in \mathbb{C}^*$. We call

$$\mathrm{MH}_\alpha(T) := \mathrm{MC}_{\alpha^{-1}}\left(T \otimes (\alpha, 1, \dots, 1, \alpha^{-1})\right)$$

the *middle Hadamard product* of T with α. If T is irreducible, the middle Hadamard product for the tuple of Jordan forms $\mathbf{J}(T)$ is defined similarly.

Again, we are also able to compute the Jordan forms of the matrices of $\mathrm{MH}_\alpha(T)$ explicitly:

5.3.8 Proposition *For each irreducible m-tuple $T \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ and each $\alpha \in \mathbb{C}^* \setminus \{1\}$, the Jordan forms of the matrices of $\mathrm{MH}_\alpha(T)$ are given by*

$$\begin{aligned}
\mathbf{J}(\mathrm{MH}_\alpha(T_i)) &= \bigoplus_{\rho \in \mathbb{C}\setminus\{1,\alpha\}} \alpha^{-1}\rho J(j)^{v(i,\rho,j)} \bigoplus J(j+1)^{v(i,\alpha,j)} \\
(2 \leq i \leq r) &\qquad \bigoplus_{j \geq 2} \alpha^{-1} J(j-1)^{v(i,1,j)} \bigoplus J(1)^{l_i} \\
\mathbf{J}(\mathrm{MH}_\alpha(T_1)) &= \bigoplus_{\rho \in \mathbb{C}\setminus\{1,\alpha^{-1}\}} \rho J(j)^{v(i,\rho,1)} \bigoplus J(j+1)^{v(i,1,1)} \\
&\qquad \bigoplus_{j \geq 2} \alpha^{-1} J(j-1)^{v(i,\alpha^{-1},1)} \bigoplus J(1)^{l_1} \\
\mathbf{J}(\mathrm{MH}_\alpha(T_{r+1})) &= \bigoplus_{\rho \in \mathbb{C}\setminus\{1,\alpha\}} \rho J(j)^{v(r+1,\rho,j)} \bigoplus_{j \geq 2} J(j-1)^{v(r+1,1,j)} \\
&\qquad \bigoplus \alpha J(j+1)^{v(r+1,\alpha,j)} \bigoplus \alpha J(1)^{l_{r+1}}
\end{aligned}$$

where l_i is determined by $\mathrm{rk}(\mathrm{MH}_\alpha(T))$.

Similar to the statements for the middle convolution, the middle Hadamard product has the following properties.

5.3.9 Proposition *Consider an irreducible m-tuple $T = (T_i)_{1 \leq i \leq r+1} \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ such that at least two matrices of T are not the identity if $n = 1$ and $\alpha \in \mathbb{C}^*$.*

(i) *If $\alpha \neq 1$, the rank of $\mathrm{MH}_\alpha(T)$ is precisely*

$$\sum_{i=2}^{r+1} \gamma(T_i) + \gamma(\alpha T_1) - n.$$

(ii) *$\mathrm{MH}_\alpha(T)$ is irreducible.*

(iii) *If $i(T) = 0$, then $i(\mathrm{MH}_\alpha(T)) = 0$.*

5.4 Constructive solutions of the MDSP

A constructive way to attack the MDSP for a tuple of Jordan matrices J is to construct another tuple of Jordan matrices J' for which the MDSP is solvable. If all of the operations along the way are invertible and preserve the irreducibility of m-tuples, we also get a solution of the MDSP for J. As we have seen in Corollaries 5.2.9 and 5.3.6, this is true for taking tensor products with m-tuples of rank one and middle convolutions with $\alpha \in \mathbb{C}^* \setminus \{1\}$.

5.4.1 Proposition *Let $J \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ be a tuple of Jordan matrices, $P \in \mathrm{GL}_1(\mathbb{C})^{r+1}$ be an m-tuple of rank one and $\alpha \in \mathbb{C}^* \setminus \{1\}$. Then the MDSP for J is solvable if and only if the MDSP for $\mathrm{MC}_\alpha(J \otimes P)$ is solvable. In particular, T is a solution of the MDSP for J if and only if $\mathrm{MC}_\alpha(T \otimes P)$ is a solution of the MDSP for $\mathrm{MC}_\alpha(J \otimes P)$.*

Solving the MDSP for an arbitrary tuple of Jordan matrices $J = (J_1, \ldots, J_{r+1}) \in \mathrm{GL}_n(\mathbb{C})$ in explicit terms is almost impossible if the rank of the tuple is large. We try to apply tensor products with m-tuples of rank one and middle convolutions to J in such a way that the resulting tuple J' has lower rank than J.
By Lemma 5.1.8 and Lemma 5.3.2, we hence look for an m-tuple $(\beta_1, \ldots, \beta_{r+1}) \in \mathrm{GL}_1(\mathbb{C})^{r+1}$ and an $\alpha \in \mathbb{C}^* \setminus \{1\}$ such that

$$\sum_{i=1}^{r} \gamma(\beta_i J_i) + \gamma(\alpha^{-1}\beta_{r+1} J_{r+1}) \leq 2n \leq \sum_{i=1}^{r+1} \gamma(\beta_i J_i)$$

holds. It is appropriate to choose the m-tuple $(\beta_1, \ldots, \beta_{r+1})$ in such a way that the right hand side of the inequality above is minimal. If it is less or equal than $2n-1$, the MDSP for J has no solution. If it is bigger or equal than $2n$, we look at J_{r+1} to decide whether we can choose an $\alpha \in \mathbb{C}^* \setminus \{1\}$ such that $\mathrm{rk}(\mathrm{MC}_\alpha(J)) < n$. Iterating this procedure as often as possible, we end in the optimal case up with a rank one tuple J' for which the MDSP is completely discussed in Example 5.1.6. By Proposition 5.4.1, we can invert all steps along the way and regain a solution of the MDSP for J.
We illustrate this in some examples of rank two.

5.4.2 Example

(i) Consider the tuple $J = (J(2), J(2), \alpha J(1) \oplus \alpha^{-1} J(1)) \in \mathrm{GL}_2(\mathbb{C})^3$ with $\alpha \in \mathbb{C}^* \setminus \{1\}$. By Proposition 5.3.4, we have

$$\mathrm{MC}_\alpha(J) = \left(\alpha, \alpha, \alpha^{-2}\right).$$

As this is an m-tuple of rank one, the MDSP for J is solvable. Using the formulae stated in Definition 5.3.1, one computes a solution

$$T = \left(\begin{pmatrix} 1 & \alpha^{-1} - 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 - \alpha & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 - \alpha^{-1} \\ \alpha - 1 & \alpha + \alpha^{-1} - 1 \end{pmatrix} \right).$$

Alternatively, this solution can be written $T = \mathrm{MH}_\alpha(1, \alpha, \alpha^{-1})$ by Definition 5.3.7.

(ii) Consider the tuple $J = (J(2), \alpha J(1) \oplus \alpha^{-1} J(1), \alpha J(1) \oplus \alpha^{-1} J(1)) \in \mathrm{GL}_2(\mathbb{C})^3$ with $\alpha \in \mathbb{C}^* \setminus \{1, -1\}$. We get

$$J \otimes (1, \alpha^{-1}, \alpha) = \left(J(2), \alpha^{-2} J(1) \oplus J(1), \alpha^2 J(1) \oplus J(1)\right).$$

By Lemma 5.1.8, each m-tuple with these Jordan forms is reducible. Consequently, the MDSP for J is not solvable. Moreover, computing $\mathrm{MC}_{\alpha^2}(J \otimes (1, \alpha^{-1}, \alpha))$ with the formula stated in Definition 5.3.3 is impossible, as the resulting tuple would be of rank one, while its second matrix would have a Jordan block of size two.

(iii) Consider the tuple $(J(2), J(2), J(2), J(2)) \in \mathrm{GL}_2(\mathbb{C})^4$. It is not possible to reduce the rank of this tuple via tensor products with m-tuples of rank one and middle convolutions with $\alpha \in \mathbb{C}^*$.

The striking result of N. Katz concerning linearly rigid m-tuples assures that the procedure described above yields an m-tuple of rank one.

5.4.3 Theorem *For each tuple of Jordan matrices $J \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ with $i(J) = 0$, the MDSP for J is solvable if and only if there are m-tuples $B_1, \ldots, B_k \in (\mathrm{GL}_1(\mathbb{C}))^{r+1}$ of rank one and elements $\alpha_1, \ldots, \alpha_k \in \mathbb{C}^*$ such that*

$$J' = \mathrm{MC}_{\alpha_k}(\ldots \mathrm{MC}_{\alpha_1}(J \otimes B_1) \cdots \otimes B_k)$$

is an m-tuple of rank one. In particular, the up to equivalence unique solution of the MDSP $T \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ is given by

$$T = \mathrm{MC}_{\alpha_1^{-1}}\left(\ldots\left(\mathrm{MC}_{\alpha_k^{-1}}(J') \otimes B_{k-1}^{-1}\right)\ldots\right) \otimes B_1^{-1},$$

where the $B_1^{-1}, \ldots B_k^{-1}$ are as explained in Lemma 5.2.8.

In particular, we are not able to reduce the rank of the third tuple discussed in Example 5.4.2 as this tuple is not linearly rigid. Also note, that for a given linearly rigid m-tuple T the minimal number of middle convolutions in its construction is given by the size of the largest Jordan block of each of its matrices.

Although the operations on m-tuples allow us to construct each solution of the MDSP for linearly rigid m-tuples explicitly, doing the computation by hand may get tedious, as one has to detect the subspaces $K(T)$ and $L(T)$ introduced in Definition 5.3.1 before each application of the middle convolution.

Chapter 6

From monodromy tuples to differential operators

We translate the constructions for monodromy tuples introduced in the last chapter to the level of differential operators in an appropriate way. In the first section, we relate tensor constructions of differential operators as considered in Definition 1.3.4 to their associated local systems. The translation of the middle convolution to the level of fuchsian systems was done by M. Dettweiler and S. Reiter in [DR07]. By the choice of a cyclic vector, this leads to a version on the level of fuchsian differential operators. The disadvantage of this approach is that there usually is no canonical choice of a cyclic vector and the corresponding minimal operator acquires apparent singularities. For our purposes - as we would be looking for very special cyclic vectors - we rather consider the middle convolution on the level of differential operators directly. In the second section of this chapter, we introduce the middle convolution on the level of solutions of a fuchsian differential operator. This is done similarly to the approaches taken in [IKSY91] and [DR07]. In the third section, we construct a fuchsian differential operator whose solution space is spanned by the middle convolutions of the solutions of a given fuchsian differential operator with z^a. Therefore, we impose some mild technical positivity assumptions on the exponents of the operators we consider. This induces the notion of the middle convolution on the level of fuchsian differential operators. The construction is not completely explicit, as the middle convolution appears as a uniquely determined right factor of known degree of a given fuchsian differential operator. However, in most of the cases we consider, this factor can be computed directly. We also derive the notion of the middle Hadamard product on the level of fuchsian differential operators in the fourth section. In the fifth section, we briefly discuss the construction of explicit solutions of the MDSP via fuchsian differential operators.
Several results of this chapter are also achieved in [BR12].

6.1 Tensor products

Tensor products of differential operators and the characterization of their solution spaces were already considered in Definition 1.3.4 and Proposition 1.3.10. Comparing the tensor product of two monic differential operators L_1 and L_2 with the tensor product of their corresponding marked differential modules (M_1, e_1) and (M_2, e_2) typically reveals two phenomena:

The vector $e_1 \otimes e_2 \in M_1 \otimes M_2$ is not cyclic and the singular locus of the operator $L_1 \otimes L_2$ contains additional apparent singularities. Those facts yield the following:

6.1.1 Proposition *Consider two irreducible, fuchsian differential operators $L, L' \in \mathbb{C}[z, \vartheta]$ with associated local systems $\mathbb{L}$ on $\mathbb{P}^1 \setminus S$ and $\mathbb{L}'$ on $\mathbb{P}^1 \setminus S'$. Then*

(i) *there is a finite set $S'' \supset S \cup S'$ such that the local system induced by $L \otimes L'$ is a subsheaf of $j^*(\mathbb{L} \boxtimes \mathbb{L}')$, where $j\colon \mathbb{C} \setminus S'' \to \mathbb{C} \setminus (S \cup S')$ denotes the canonical inclusion.*

(ii) *the local system induced by $\mathrm{Sym}^k(L)$ is a subsheaf in $\mathrm{Sym}^k(\mathbb{L})$.*

(iii) *the local system induced by $\bigwedge^k(L)$ is a subsheaf $\bigwedge^k(\mathbb{L})$.*

As we are especially interested in tensor products with operators of degree one, we introduce the following suggestive notation.

6.1.2 Definition Consider a fuchsian differential operator $L \in \mathbb{C}[z, \vartheta]$ and a fuchsian differential operator $P \in \mathbb{C}[z, \vartheta]$ of degree one whose solution space is spanned by an algebraic function $g \in \mathbb{C}(z)^{alg}$. Then we write $L \otimes g := L \otimes P$.

In analogy to Lemma 5.2.8, we immediately get the following properties.

6.1.3 Lemma *Consider an irreducible fuchsian operator $L \in \mathbb{C}(z)[\partial]$ with associated local system $\mathbb{L}$ on $\mathbb{P}^1 \setminus S$ and a fuchsian operator $O \in \mathbb{C}(z)[\partial]$ of degree one with associated local system $\mathbb{O}$ on $\mathbb{P}^1 \setminus S'$. Then*

(i) *The local system induced by $L \otimes O$ is precisely given by $\mathbb{L} \boxtimes \mathbb{O}$.*

(ii) *There is a fuchsian differential operator $O^{-1} \in \mathbb{C}(z)[\partial]$ of degree one such that $(L \otimes O) \otimes O^{-1} = L$.*

6.2 The middle convolution of solutions

In the upcoming sections, we translate the notion of the middle convolution from the level of m-tuples to the level of differential operators. More generally, the operation $\mathrm{MC}_\alpha(T)$ can be interpreted as the middle convolution of T with the m-tuple (α, α^{-1}). We first introduce a differential operator whose monodromy tuple is of this shape.

6.2.1 Definition Let $a \in \mathbb{Q} \setminus \mathbb{Z}$ and $\alpha = \exp(2\pi i a)$. We set

$$O_a := \vartheta - a \in \mathbb{C}[z, \vartheta].$$

The solution space of O_a is spanned by the algebraic function z^a. Therefore, the monodromy tuple induced by O_a is $(\exp(2\pi i a), \exp(-2\pi i a))$ with respect to each choice of a base star $\Gamma((0, \infty))$. By this choice, two operators O_a and O_b induce the same monodromy tuple if and only if $(a - b) \in \mathbb{Z}$.

In the literature, there are various notions of convolution. The approach we follow here is classical, see e.g. [Inc56, Section 18.45], and uses integrals over double loops.

To state those integrals properly, we fix the following topological data:

Convention For a given a fuchsian differential operator L with singular locus S, we fix a coordinate on $\mathbb{P}^1$ and a base star

$$\Gamma(\underline{S}) := (x_0; (s_1, \ldots, s_r, \infty); (\gamma_{s_1}, \ldots, \gamma_{s_r}, \gamma_\infty))$$

on $\mathbb{P}^1 \setminus S$ such that $s_{r+1} = \infty$ holds. The monodromy tuple of L is always taken with respect to this base star. For each point $z \in \mathbb{P}^1 \setminus S$, we further consider the base star

$$\Gamma(\underline{S}(z)) := (x_0; (s_1, \ldots, s_r, z, s_{r+1}); (\gamma_{s_1}, \ldots, \gamma_{s_r}, \gamma_z, \gamma_\infty))$$

on $\mathbb{P}^1 \setminus (S \cup \{z\})$.
For each function f which is a local solution of a fuchsian differential operator with singular locus in $S' \subset \mathbb{P}^1$, the effect of analytic continuation along a loop $\gamma \in \pi_1(\mathbb{P}^1 \setminus S', x_0)$ is denoted by $\gamma \cdot f$.

6.2.2 Definition Let $L \in \mathbb{C}[z, \vartheta]$ be fuchsian with singular locus S, $f \in \mathrm{Sol}_L$ and $a \in \mathbb{Q} \setminus \mathbb{Z}$. For $s \in S$, the expression

$$C_a^s(f) := \int_{[\gamma_s, \gamma_z]} f(x)(z-x)^a \frac{dx}{z-x}$$

is called the *convolution* of f and z^a with respect to the *Pochhammer contour*

$$[\gamma_s, \gamma_z] := \gamma_s^{-1} \gamma_z^{-1} \gamma_s \gamma_z.$$

A priori, it is not clear for which functions f and for which values $a \in \mathbb{Q} \setminus \mathbb{Z}$ the expression $C_a^s(f)$ is meaningful. The relation $C_a^s(f)$ to an integral over a line in $\mathbb{C}$ discussed in [IKSY91, Chapter 2, Lemma 3.3.1, Proposition 3.3.2 and Proposition 3.3.7] gives the following result.

6.2.3 Proposition *Consider a function of the form $f(z) = (z-s)^\mu g(z)$ which is a local solution of an arbitrary fuchsian differential operator with singular locus S, such that $g(z)$ is holomorphic in a neighborhood of $z = s$, $g(s) \neq 0$ and $\mu \notin \mathbb{Z}_{<0}$. Then for each $a \in \mathbb{Q} \setminus \mathbb{Z}$, we get*

$$C_a^s(f) = \left(1 - e^{2\pi i a}\right)\left(1 - e^{2\pi i \mu}\right) \int_s^z f(x)(z-x)^{a-1} dx,$$

where $\int_s^z$ denotes the integral over a path from s to z which neither meets nor encircles any points of $S \setminus \{s\}$. The right hand side of the equation above gives rise to a holomorphic function in a neighbourhood of $z = p$.

We have a similar result for $C_a^\infty(f)$, assuming that $\mu \notin a + \mathbb{Z}_{<0}$.
The relation of the convolution to line-integrals enables us to deduce explicit formulae for some local solutions of a fuchsian differential operator. We denote the *Beta function* by

$$\mathrm{B}(p, q) := \int_0^1 x^{p-1}(1-x)^{q-1} dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

considered as analytic continuation of the expression on the very right.

6.2.4 Proposition *Let $L \in \mathbb{C}[z,\vartheta]$ be a fuchsian differential operator with singular locus S.*

(i) *If $f(z+s) = z^{\mu}\sum_{m=0}^{\infty} A_m z^m$ is a solution of $\iota_s(L)$ for $s \in S$ and μ is not a negative integer, we get*

$$C_a^s(f)(z) = \left(1 - e^{2\pi i\mu}\right)\left(1 - e^{2\pi i a}\right)(z-s)^{\mu+a}\sum_{m=0}^{\infty} \mathrm{B}(\mu+1+m,a)A_m(z-s)^m$$

and

$$C_a^s\left(\ln(z-s)f\right)(z) = 2\pi i\left(e^{2\pi i a} - 1\right)(z-s)^{\mu+a}\sum_{m=0}^{\infty} \mathrm{B}(\mu+1+m,a)A_m(z-s)^m,$$

if $\mu \in \mathbb{N}$.

(ii) *If $t = \frac{1}{z}$, $f = t^{\mu}\sum_{m=0}^{\infty} A_m t^m$ is a solution of $\iota_\infty(L)$ and μ is not contained in $a + \mathbb{Z}_{<0}$, we have*

$$C_a^\infty(f) = \left(1 - e^{2\pi i(\mu-a)}\right)\left(1 - e^{-2\pi i a}\right)t^{\mu-a}\sum_{m=0}^{\infty} \mathrm{B}(\mu-a+m,a)A_m t^m$$

and

$$C_a^\infty\left(\ln(t)f\right) = 2\pi i\left(e^{-2\pi i a} - 1\right)t^{\mu-a}\sum_{m=0}^{\infty} \mathrm{B}(\mu-a+m,a)A_m t^m,$$

if $\mu \in a + \mathbb{N}$.

Proof By Proposition 6.2.3 we get

$$C_a^s(f)(z) = \left(1 - e^{2\pi i a}\right)\left(1 - e^{2\pi i\mu}\right)\int_s^z f(x)(z-x)^{a-1}dx.$$

As

$$\begin{aligned}
\int_s^z f(x)(z-x)^{a-1}dx &= \int_0^{z-s} x^{\mu}\sum_{m=0}^{\infty} A_m x^m (z-s-x)^{a-1}dx \\
&= \sum_{m=0}^{\infty}\int_0^{z-s} A_m(z-s-x)^{a-1}x^{\mu+m}dx \\
&= \sum_{m=0}^{\infty} A_m(z-s)^{\mu+a+m}\int_0^{z-s}\left(1-\frac{x}{z-s}\right)^{a-1}\left(\frac{x}{z-s}\right)^{\mu+m}dx \\
&= \sum_{m=0}^{\infty} A_m(z-p)^{\mu+a+m}\int_0^1 (1-s)^{a-1}s^{\mu+m}ds
\end{aligned}$$

holds if the absolute value of $z-s$ is sufficiently small, we obtain the first part of the result. By a direct computation, we have

$$\begin{aligned}
C_a^s\left(\ln(z-s)f\right) &= (1-\exp(2\pi i a))\int_s^z (\mathrm{id}-\gamma_s)\left(\ln(x-s)f(x)(z-x)^{a-1}\right)dx \\
&= -2\pi i(1-\exp(2\pi i a))\int_p^z f(x)(z-x)^{a-1}dx
\end{aligned}$$

and hence the second statement.
The statements for $t = \frac{1}{z}$ can be shown completely similar after having observed that

$$C_a^\infty(f) = (1 - \exp(2\pi i(\mu - a))(1 - \exp(-2\pi i a)) \int_0^t x^{-1-a}(1 - xz)^{a-1} f\left(\frac{1}{x}\right) dx$$

holds, since the action of γ_∞ and γ_z on the integrand commute and $z^{-1-a} f\left(\frac{1}{z}\right)$ has no residue at $z = 0$. □

Immediate consequences are:

6.2.5 Corollary (i) *If f is holomorphic at $z = s$, we have $C_a^s(f) = 0$.*

(ii) *If $z^a f$ is holomorphic at $z = \infty$, we have $C_a^\infty(f) = 0$.*

(iii) *If $(z - s)^a f$ is N-integral, $C_a^s(f)$ also is.*

6.3 The middle convolution of fuchsian differential operators

Having discussed the middle convolution of solutions of a given fuchsian differential operator L with singular locus S with z^a, we are now looking for a fuchsian differential operator whose solution space is spanned by the set

$$C_a(\mathrm{Sol}_L) := \bigcup_{s \in S} \{C_a^s(f) \mid L(f) = 0\}.$$

By direct computations, we first construct a fuchsian differential operator $\mathcal{C}_a(L)$ with singular locus S whose solution space contains $C_a(\mathrm{Sol}_L)$. Hence, the operator of smallest degree whose solution space contains $C_a(\mathrm{Sol}_L)$ is a right factor of $\mathcal{C}_a(L)$. By Lemma 1.6.6, this operator is fuchsian as well. Next, we show that $C_a(\mathrm{Sol}_L)$ is as a set invariant under the action of the formal monodromy γ_s for each $s \in S$. The restrictions of those elements to the Picard-Vessiot ring of $\mathcal{C}_a(L)$ generate the differential Galois group of $\mathcal{C}_a(L)$ by Proposition 2.1.7. Consequently, Lemma 1.3.9 provides that there is a fuchsian differential operator whose solution space is spanned by $C_a(\mathrm{Sol}_L)$.
For simplicity, we always want to be able to connect the expressions $C_a^s(f)$ to expressions established in Proposition 6.2.4. By the restrictions on the exponents of f stated there, we do the construction we are after for the following class of operators.

6.3.1 Definition For each $a \in \mathbb{Q} \setminus \mathbb{Z}$, we call an operator $L \in \mathbb{C}[z, \vartheta]$ *a-positive*, if it is fuchsian, has no exponents in $\mathbb{Z}_{<0}$ at each point $p \in \mathbb{C}$ and no exponents in $a + \mathbb{Z}_{<0}$ at $z = \infty$.

6.3.2 Remark. Suppose that L has exponents in $\mathbb{Z}_{<0}$ at $p \in \mathbb{C}$. By Lemma 1.5.9, there is an $N_p \in \mathbb{N}$ such that the operator $L \otimes (z - p)^{N_p}$ has no exponent in $\mathbb{Z}_{<0}$ at $z = p$. Therefore, for each $a \in \mathbb{Q} \setminus \mathbb{Z}$, we can iteratively find an operator R of degree one and an $\tilde{a} \in a + \mathbb{Z}$ such that the monodromy tuples of L and $L \otimes R$ with respect to $\Gamma(\underline{S})$ coincide and the latter operator is $\tilde{a}$-positive. Hence, the notion of being a-positive is just a technical one and causes no restrictions on the monodromy tuples we want to study.

Note, that for an a-positive operator L with singular locus S, the expression $C_a^s(f)$ is for each solution $f \in \mathrm{Sol}_L$ and each $s \in S$ meaningful in the sense of Proposition 6.2.3.
We construct the operator $\mathcal{C}_a(L)$ directly by using the following:

6.3.3 Lemma *Consider $a \in \mathbb{Q} \setminus \mathbb{Z}$, an a-positive differential operator $L \in \mathbb{C}[z, \vartheta]$ and a solution f of L. We have the relations*

(i)
$$\frac{d}{dz} C_a^s(f) = C_a^s\left(\frac{d}{dz} f\right) = (a-1) C_{a-1}^s(f).$$

(ii)
$$C_a^s\left(z^i f\right) = \prod_{j=0}^{i-1} \left(\frac{z\frac{d}{dz}}{a+j} - 1\right) C_{a+i}^s(f).$$

(iii)
$$C_a^s\left(z\frac{d}{dz} f\right) = \left(z\frac{d}{dz} - a\right) C_a^s(f).$$

Proof Using Leibniz's rule for differentiating under the integral sign, we get

$$\begin{aligned}\frac{d}{dz} \int_{[\gamma_s,\gamma_z]} f(x)(z-x)^{a-1} dx &= \int_{[\gamma_s,\gamma_z]} f(x) \frac{d}{dz}(z-x)^{a-1} dx \\ &= -\int_{[\gamma_s,\gamma_z]} f(x) \frac{d}{dx}(z-x)^{a-1} dx.\end{aligned}$$

As the monodromy of $f(x)(z-x)^{a-1}$ along the path $[\gamma_s, \gamma_z]$ is trivial, integration by parts yields

$$-\int_{[\gamma_s,\gamma_z]} f(x) \frac{d}{dx}(z-x)^{a-1} dx = \int_{[\gamma_s,\gamma_z]} \left(\frac{d}{dx} f(x)\right)(z-x)^{a-1} dx$$

and hence the first result. The other statements can be obtained by direct computation and the results established before. □

6.3.4 Proposition *Let $a \in \mathbb{Q} \setminus \mathbb{Z}$, $L = \sum_{i=0}^m z^i P_i(\vartheta) \in \mathbb{C}[z, \vartheta]$ be a-positive and f a solution of L. Then $C_a^s(f)$ is a solution of*

$$\mathcal{C}_a(L) := \sum_{i=0}^m z^i \prod_{j=1}^i (\vartheta - a + j) \prod_{k=0}^{m-i-1} (\vartheta - k) P_i(\vartheta - a)$$

for each $s \in S$.

Proof For $0 \le i \le m$ and $b \in \mathbb{Q} \setminus \mathbb{Z}$, we have

$$\begin{aligned}C_{b+i}^s(g) &= \frac{1}{\prod_{l=1}^{m-i}(b+m-l)} \left(\frac{d}{dz}\right)^{m-i} C_{b+m}^s(g) \\ &= z^{i-m} \frac{z^{m-i}}{\prod_{l=1}^{m-i}(b+m-l)} \left(\frac{d}{dz}\right)^{m-i} C_{b+m}^s(g) \\ &= z^{i-m} \frac{\prod_{k=0}^{m-i-1}(\vartheta - k)}{\prod_{l=1}^{m-i}(b+m-l)} \left(C_{b+m}^s(g)\right)\end{aligned}$$

for each g which is a solution of some $R \in \mathbb{C}[z,\vartheta]$ by Lemma 6.3.3. Thus

$$\begin{aligned}
z^m C_b^s\left(L(f)\right) &= z^m \sum_{i=0}^{m} C_b^s\left(z^i P_i(\vartheta)(f)\right) = z^m \sum_{i=0}^{m} \prod_{j=0}^{i-1}\left(\frac{\vartheta}{b+j}-1\right)\left(C_{b+i}^s\left(P_i(\vartheta)(f)\right)\right) \\
&= \sum_{i=0}^{m} \prod_{j=0}^{i-1}\left(\frac{\vartheta}{b+j}-1\right) z^i \frac{\prod_{k=0}^{m-i-1}(\vartheta-k)}{\prod_{l=1}^{m-i}(b+m-l)}\left(C_{b+m}^s\left(P_i(\vartheta)(f)\right)\right) \\
&= \sum_{i=0}^{m} z^i \prod_{j=0}^{i-1}\left(\frac{\vartheta+i-m}{b+j}-1\right)\frac{\prod_{k=0}^{m-i-1}(\vartheta-k)}{\prod_{l=1}^{m-i}(b+m-l)} P_i(\vartheta-(b+m))\left(C_{b+m}^s(f)\right) \\
&= \frac{\sum_{i=0}^{m} z^i \prod_{j=1}^{i}(\vartheta-m-b+j)\prod_{k=0}^{m-i-1}(\vartheta-k)P_i(\vartheta-(b+m))}{\prod_{i=0}^{m-1}(b+i)}\left(C_{b+m}^s(f)\right).
\end{aligned}$$

Putting $b = a - m$, we get the desired result. □

By the definition of $\mathcal{C}_a(L)$, we directly get:

6.3.5 Corollary *Let $a \in \mathbb{Q}\setminus\mathbb{Z}$ and L be a-positive with singular locus S. Then $\mathcal{C}_a(L)$ is also fuchsian with singular locus $S\cup\{0\}$.*

Proof The formula for $\mathcal{C}_a(L)$ given in the proposition above assures that $\mathcal{C}_a(L)$ is regular singular at $z=0$ and $z=\infty$ and that its singular locus is $S\cup\{0\}$. For $\tau_s = z+s$ with $s \in S$, one checks that $C_a^{s'}(\tau_s^*(f)) = \tau_s^*\left(C_a^{s+s'}(f)\right)$. In particular, $\tau_s^* C_a(L)$ is a right factor of $C_a(\tau_s^* L)$ for all $s \in S$. As the latter operator is by construction regular singular at $z=0$, the first one also is by Lemma 1.6.6. Hence, $\mathcal{C}_a(L)$ is regular singular at each $s \in S$ which gives the result. □

We briefly indicate the effect of analytic continuation of elements in $C_a(\mathrm{Sol}_L)$ along a loop $\delta \in \pi_1(\mathbb{P}^1\setminus S, z)$. As it is e.g. done in [DR00, Chapter 4], this can be studied from a topological point of view. We do not reproduce the precise arguments here, as they are quite technical and rely amongst other things on representations of the Artin braid group. The idea is to describe the analytic continuation of $C_a^s(f)$ by integrals with the same integrand as before, but deformed Pochhammer loops. Therefore, one chooses for $z \in \mathbb{P}^1\setminus S$ a base star

$$\Delta(\underline{S}) = (z; (s_1,\dots,s_r,\infty), (\delta_{s_1},\dots,\delta_{s_r},\delta_\infty))$$

on $\mathbb{P}^1\setminus S$ whose paths are conjugated to those of $\Gamma(\underline{S})$ by a path connecting z and x_0. In particular, given a differential operator L with singular locus S, its monodromy tuples with respect to $\Gamma(\underline{S})$ and $\Delta(\underline{S})$ are equivalent. Then, one investigates how the base star $\Gamma(\underline{S}(z))$ is deformed by continuation of each of its paths along δ_{s_i}. A nice visualization of this classical idea which was used to describe the monodromy of Jordan-Pochhammer differential equations can be found in [Inc56, Section 18.45].

All in all, one can check that the action of the path $\delta_{s_k} \circlearrowleft$ on the loops $(\gamma_{s_1}, \ldots, \gamma_{s_r}, \gamma_z)$ is given by

$$\delta_{s_k} \circlearrowleft (\gamma_{s_1}, \ldots, \gamma_{s_r}, \gamma_z) := \left(\gamma_{s_1}, \ldots, \gamma_{s_{k-1}}, \gamma_{s_k}^{\gamma_z}, \gamma_{s_{k+1}}^{[\gamma_{s_k}, \gamma_z]}, \ldots, \gamma_{s_r}^{[\gamma_{s_k}, \gamma_z]}, \gamma_z^{\gamma_{s_k}\gamma_z}\right),$$

where

$$\gamma^\delta = \delta^{-1}\gamma\delta$$

for each two paths $\gamma, \delta \in \pi_1(\mathbb{P}^1 \setminus (S \cup \{z\}), x_0)$.
With respect to this action, we now have

$$\delta_{s_k} \cdot C_a^{s_i}(f) = \int_{[\delta_{s_k} \circlearrowleft \gamma_{s_i}, \delta_{s_k} \circlearrowleft \gamma_z]} f(x)(z-x)^{a-1} dx.$$

A direct computation shows the following rules.

6.3.6 Proposition *Consider $a \in \mathbb{Q} \setminus \mathbb{Z}$, an irreducible, a-positive operator $L \in \mathbb{C}[z, \vartheta]$ and put $\alpha = \exp(2\pi i a)$. Then for each solution f of L we have*

(i) $\delta_{s_k} \cdot C_a^{s_i}(f) = C_a^{s_i}(f) + C_a^{s_k}(\alpha(\gamma_{s_i} - 1) \cdot f)$ *if $i < k$,*

(ii) $\delta_{s_k} \cdot C_a^{s_k}(f) = C_a^{s_k}(\alpha\gamma_{s_k} \cdot f)$,

(iii) $\delta_{s_k} \cdot C_a^{s_i}(f) = C_a^{s_i}(f) + C_a^{s_k}((\gamma_{s_i} - 1) \cdot f)$ *if $i > k$,*

where 1 denotes the trivial loop.

Those computations yield the differential operator we want to construct.

6.3.7 Proposition *If $a \in \mathbb{Q} \setminus \mathbb{Z}$ and L is a-positive with singular locus S, there is a fuchsian operator in $\mathbb{C}[z, \vartheta]$ whose solution space is spanned by*

$$C_a(\mathrm{Sol}_L) = \bigcup_{s \in S} \{C_a^s(f) \mid L(f) = 0\}.$$

Proof By Proposition 6.3.6, the actions of the loops $\delta_{s_1}, \ldots, \delta_{s_r}$ described above leave $C_a(\mathrm{Sol}_L)$ as a set invariant. Hence, $C_a(\mathrm{Sol}_L)$ is as a set invariant under the action of the monodromy group of $\mathcal{C}_a(L)$. As this operator is fuchsian, the result follows from Lemma 1.3.9, Corollary 1.3.11 and Lemma 1.6.6. □

6.3.8 Definition With respect to the notations in Proposition 6.3.7, we denote the differential operator whose solution space is spanned by $C_a(\mathrm{Sol}_L)$ by $L \star_C O_a$. It is called the *middle convolution* of L and O_a.

Finally, we show that the monodromy tuple induced by $L \star_C O_a$ coincides with the middle convolution of the monodromy tuples of L and O_a, if L is irreducible and a-positive.

6.3.9 Theorem *Let $a \in \mathbb{Q} \setminus \mathbb{Z}$, $\alpha = \exp(2\pi i a)$ and $L \in \mathbb{C}[z, \vartheta]$ be irreducible and a-positive. Let furthermore T be the monodromy tuple of L with respect to the base star $\Gamma(\underline{S})$. Then the monodromy tuple of $L \star_C O_a$ with respect to the same base star is equivalent to $\mathrm{MC}_\alpha(T)$. In particular, $L \star_C O_a$ is an irreducible right factor of $\mathcal{C}_a(L)$ of degree*

$$\deg(L \star_C O_a) = \sum_{i=1}^{r} \gamma(T_i) + \gamma(\alpha^{-1} T_{r+1}) - \deg(L).$$

Proof We put $n = \deg(L)$, fix a basis $f_1, \ldots, f_n$ of Sol_L and identify $\mathrm{Sol}_L \cong \mathbb{C}^n$ via this choice. We also set $F := (f_1, \ldots, f_n)$ and consider for each $w = (w_1, \ldots, w_n)$ the product $F \cdot w = \sum_{i=1}^n f_i w_i$. In particular, we have

$$\gamma_{s_i} \cdot (F \cdot w) = F \cdot T_i w$$

for each $s_i \in S$. We write

$$\begin{aligned} C_a(F) &= (C_a(F)_1, \ldots, C_a(F)_{nr}) \\ &:= (C_a^{s_1}(f_1), \ldots, C_a^{s_1}(f_n), C_a^{s_2}(f_1), \ldots, C_a^{s_2}(f_n), \ldots, C_a^{s_r}(f_1), \ldots, C_a^{s_r}(f_n)) \,. \end{aligned}$$

Consider furthermore B_k, $K(T), L(T)$ and π as introduced in Definition 5.3.1. We have shown in Proposition 6.3.6 that the map

$$\varphi \colon \mathbb{C}^{nr} \to C_a(\mathrm{Sol}_L),\ (v_1, \ldots, v_{nr}) \mapsto \sum_{i=1}^{nr} C_a(F)_i v_i$$

is surjective and invariant under the action of the monodromy on both spaces, i.e. we have

$$\varphi(B_k v) = \delta_{s_k}(\varphi(v))$$

for each $v \in \mathbb{C}^{nr}$. For simplicity, we write $v = (w_1, \ldots, w_r)$ with $w_k = (w_{k,1}, \ldots, w_{k,n}) \in \mathbb{C}^n$ and identify w_k with $(0, \ldots, 0, w_k, 0, \ldots, 0)$ in the obvious way. We show that $K(T) \subset \ker(\varphi)$ and $L(T) \subset \ker(\varphi)$. For $w_k \in \mathrm{Kern}(T_k - \mathrm{id})$, we have

$$\varphi(w_k) = \sum_{i=1}^n C_a^{s_k}(f_i) w_{k,i} = C_a^{s_k}\left(\sum_{i=1}^n F \cdot w\right).$$

As $F \cdot w \in \mathrm{Kern}(\gamma_{s_k} - \mathrm{id})$ and L is a-positive, we get that $F \cdot w$ is holomorphic at s_k. Therefore, we have $C_a^{s_k}\left(\sum_{i=1}^n F \cdot w\right) = 0$ by Corollary 6.2.5, yielding $K(T) \subset \ker(\varphi)$. To prove $L(T) \subset \ker(\varphi)$, we first observe that the product relation $\gamma_{s_1} \cdots \gamma_{s_r} \gamma_z \gamma_\infty = 1$ yields

$$\begin{aligned} C_a^\infty(f) &= \int_{[\gamma_z^{-1}\gamma_{s_r}^{-1} \ldots \gamma_{s_1}^{-1}, \gamma_z]} f(x)(z-x)^{a-1} dx \\ &= \sum_{i=1}^r \int_{[\gamma_{s_i}^{-1}, \gamma_z]} \gamma_{s_{i-1}}^{-1} \ldots \gamma_{s_1}^{-1} f(x)(z-x)^{a-1} dx \\ &= -\sum_{i=1}^r C_a^{s_i}\left(\gamma_{s_i}^{-1}\gamma_{s_{i-1}}^{-1} \ldots \gamma_{s_1}^{-1} f\right) \end{aligned}$$

for each $f \in \mathrm{Sol}_L$. It follows from Corollary 6.2.5 that $C_a^\infty(f) = 0$ if $f \in \ker(\alpha^{-1}\gamma_\infty - \mathrm{id}) = \ker(\alpha\gamma_\infty^{-1} - \mathrm{id})$, which precisely means that $f \in \ker(\alpha\gamma_{s_1} \cdots \gamma_{s_r} - \mathrm{id})$, as $\gamma_z \cdot f = f$. Thus we get

$$0 = \alpha \sum_{i=1}^r C_a^{s_i}\left(\gamma_{s_{i+1}} \ldots \gamma_{s_r} f\right).$$

By [DR00, Lemma 2.7], we can find for each $v \in L(T)$ an $w \in \ker(\alpha T_1 \ldots T_r - \mathbb{1}_n)$ such that

$$v = (T_2 \cdots T_r w, T_3 \cdots T_r w, \ldots, w)$$

holds, as $\alpha \neq 1$. Therefore we get

$$\varphi(v) = \alpha \sum_{i=1}^{r} C_a^{s_i}\left(F \cdot (T_{i+1} \cdots T_r w)\right)) = 0$$

and hence $L(T) \subset \ker(\varphi)$. In particular, we have just shown that the map φ factors over $\mathbb{C}^{nr}/(K(T) + L(T))$. Consider the map

$$\overline{\varphi}\colon \mathbb{C}^{nr}/(K(T) + L(T)) \to C_a(\mathrm{Sol}_L),\ (\overline{v_1}, \ldots, \overline{v_{nr}}) \mapsto \sum_{i=1}^{nr} C_a(F)_i \overline{v_i}.$$

This map is surjective and invariant under the action of the monodromy. By [DR00, Corollary 3.6] the action of the monodromy on $\mathbb{C}^{nr}/(K(T)+L(T))$ is irreducible. Therefore, the kernel of $\overline{\varphi}$ is trivial. The operator L has at least one non-apparent singularity $s \in \mathbb{C}$ and hence at least one solution f such that $C_a^s(f) \neq 0$ holds, by Proposition 6.2.4. Thus, $C_a(\mathrm{Sol}_L) \neq \{0\}$ and $\overline{\varphi}$ is an isomorphism. □

We have reached our goal and constructed a differential operator whose induced monodromy tuple is $\mathrm{MC}_\alpha(T)$ as right factor of known degree of the operator $\mathcal{C}_a(L)$. Nevertheless, computing factors of differential operators is a non-trivial problem. One can e.g. use the command `DFactor` in the package `DEtools` in `MAPLE`. However, if this procedure does not find a factorization of a given differential operator L, it is not clear that L is irreducible, see e.g. [Hoe97]. We indicate that for a special class of differential operators, the right factors we are looking for can be obtained easily. Therefore, we recall a classical result by L. Heffter, see [Hef90], which relates the number of holomorphic solutions of a given operator reduced $L \in \mathbb{C}[z, \vartheta]$ at its singularities outside $\{0, \infty\}$ with its degree in z.

6.3.10 Lemma *Consider a reduced fuchsian differential operator $L = \sum_{i=0}^{m} z^i P_i(\vartheta) \in \mathbb{C}[z, \vartheta]$ such that $P_m(\vartheta) \neq 0$ and its monodromy tuple $T = (T_1, \ldots, T_{r+1})$ with respect to a base star $\Gamma((0, s_2, \ldots, s_r, \infty))$. Then $m \geq \sum_{i=2}^{r} \gamma(T_i)$.*

This inequality justifies the following terminology:

6.3.11 Definition If $m = \sum_{i=2}^{r} \gamma(T_i)$ holds in the situation of Lemma 6.3.10, we call L *small.*

If we restrict ourselves to small a-positive operators, we can make the following predictions on the left factors of $\mathcal{C}_a(L)$.

6.3.12 Lemma *Consider $a \in \mathbb{Q} \setminus \mathbb{Z}$ and an irreducible, a-positive small differential operator $L = \sum_{i=0}^{m} z^i P_i(\vartheta) \in \mathbb{C}[z, \vartheta]$ which has a non-apparent singularity at $z = 0$. Then*

(i) *$\mathcal{C}_a(L)$ is small.*

(ii) *$\mathcal{C}_a(L)$ admits a factorization $R \cdot (L \star_C O_a)$, where R is a product of operators of degree one, whose non-apparent singularities lie in $\{0, \infty\}$.*

Proof We choose a base star $\Gamma((0, s_2, \ldots, s_r, \infty))$, denote the monodromy tuple of L by T and the monodromy tuple of $\mathcal{C}_a(L)$ by T'. By Proposition 5.3.4, we have $\gamma(T_i) = \gamma(\mathrm{MC}_\alpha(T_i))$ for $\alpha = \exp(2\pi i a)$ and each $1 \leq i \leq r+1$. As $L \star_C O_a$ is a right factor of $\mathcal{C}_a(L)$, the matrix

$\mathrm{MC}_\alpha(T_i)$ appears up to conjugation as a block inside T_i'. In particular, $\gamma(\mathrm{MC}_\alpha(T_i)) \leq \gamma(T_i')$. As

$$\mathcal{C}_a(L) = \sum_{i=0}^{m} z^i \prod_{j=1}^{i} (\vartheta - a + j) \prod_{k=0}^{m-i-1} (\vartheta - k) P_i(\vartheta - a)$$

and this operator is reduced, the inequality given in Lemma 6.3.10 together with the assumption that L is small leads to

$$m = \sum_{i=2}^{r} \gamma(T_i) = \sum_{i=2}^{r} \gamma(\mathrm{MC}_\alpha(T_i)) \leq \sum_{i=2}^{r} \gamma(T_i') \leq m.$$

This gives $\gamma(T_i) = \gamma(\mathrm{MC}_\alpha(T_i)) = \gamma(T_i')$ for all $1 \leq i \neq j \leq r$ and hence both results. □

6.3.13 Remark. If L is small, a result which seems not yet published and was told us by V. Levandovskyy in a personal communication even implies that all factors of R are of the form $\vartheta + c$ with $c \in \mathbb{C}$. Hence the desired factorization of $\mathcal{C}_a(L)$ can be computed directly.

6.4 The middle Hadamard product of fuchsian differential operators

We also translate the middle Hadamard product to the level of differential operators and briefly summarize all properties which are transferred from the middle convolution. As before, we interpret the middle Hadamard product as an operation involving an m-tuple of rank one.

6.4.1 Definition For $a \in \mathbb{Q} \setminus \mathbb{Z}$, we set

$$I_a := \vartheta - z(\vartheta + a) \in \mathbb{C}[z, \vartheta].$$

The solution space of I_a is spanned by the algebraic function $(1-z)^{-a}$. Therefore, for each choice of a base star $\Gamma((1,\infty))$, the monodromy tuple induced by I_a is $(\exp(-2\pi a), \exp(2\pi i a))$ Two operators I_a and I_b induce the same monodromy tuple if and only if $(a-b) \in \mathbb{Z}$ holds. Similar to the procedure done before, we consider the Hadamard product of solutions of a fuchsian differential operator with $(1-z)^{-a}$.

6.4.2 Definition Let $L \in \mathbb{C}[z, \vartheta]$ be fuchsian with singular locus S, $f \in \mathrm{Sol}_L$ and $a \in \mathbb{Q} \setminus \mathbb{Z}$. For $s \in S$, the expression

$$H_a^s(f) := \int_{[\gamma_s, \gamma_z]} f(x) \left(1 - \frac{z}{x}\right)^{-a} \frac{dx}{x}$$

is called the *Hadamard product* of f and $(1-z)^{-a}$ with respect to the Pochhammer contour $[\gamma_s, \gamma_z]$.

By a direct computation, we have the following relation between the convolution and the Hadamard product.

6.4.3 Lemma *Let $L \in \mathbb{C}[z, \vartheta]$ be fuchsian, $f \in \mathrm{Sol}_L$ and $a \in \mathbb{Q} \setminus \mathbb{Z}$. There are elements $c_1, c_2 \in \mathbb{C}^*$ such that*

$$C_a^p(f) = c_1 H_{1-a}^p(z^a f)$$

and

$$H_a^p(f) = c_2 C_{1-a}^p(z^{a-1}f)$$

hold.

With these relations at hand, we transfer properties of the middle convolution to the middle Hadamard product. On the level of solutions we have:

6.4.4 Proposition *Consider a fuchsian differential operator $L \in \mathbb{C}[z,\vartheta]$ with singular locus S.*

(i) *For each $s \in S$ and each solution f of $\iota_s(L)$ with $(z+s)^{a-1}f(z+s) = z^\mu \sum_{m=0}^{\infty} A_m z^m$ and $\mu \notin \mathbb{Z}_{<0}$, we have*

$$H_a^s(f) = \left(1 - e^{2\pi i\mu}\right)(1-\alpha)(z-p)^{\mu+1-a} \sum_{m=0}^{\infty} \mathrm{B}(\mu+1+m, 1-a) A_m (z-s)^m.$$

Further, if $\mu \in \mathbb{N}_0$, we have

$$H_a^s(\ln(z-s)f) = -2\pi i(1-\alpha)(z-s)^{\mu+1-a} \sum_{m=0}^{\infty} \mathrm{B}(\mu+1+m, 1-a) A_m (z-s)^m.$$

(ii) *For $t = \frac{1}{z}$ and each solution $f = t^\mu \sum_{m=0}^{\infty} A_m t^m$ of $\iota_\infty(L)$ with $\mu \notin \mathbb{Z}_{<0}$, we have*

$$H_a^\infty(f) = \left(1 - e^{2\pi i\mu}\right)(1-\alpha^{-1}) t^\mu \sum_{m=0}^{\infty} \mathrm{B}(\mu+m, 1-a) A_m t^m.$$

Further, if $\mu \in \mathbb{N}_0$, we have

$$H_a^\infty(\ln(t)f) = -2\pi i(1 - \exp(2\pi i\mu)) \sum_{m=0}^{\infty} \mathrm{B}(\mu+m, 1-a) A_m t^m.$$

6.4.5 Remark. Classically, one defines the Hadamard product of two formal power series $f(z) = \sum_{m=0}^{\infty} A_m z^m$ and $g(z) = \sum_{m=0}^{\infty} B_m z^m$ via

$$\sum_{m=0}^{\infty} A_m z^m \star_H \sum_{m=0}^{\infty} B_m z^m = \sum_{m=0}^{\infty} A_m B_m z^m.$$

Under this point of view, if f is a solution of L which is holomorphic at $z = 0$, we have

$$H_a^0(f) = f(z) \star_H (1-z)^{-a}.$$

As already seen for the middle convolution in Corollary 6.2.5, the middle Hadamard product preserves N-integrality of solutions as well.

For the middle Hadamard product on the level of fuchsian differential operators, the claim that the considered operators are a-positive is replaced by the assumption that $L \otimes O_{a-1}$ is $(1-a)$-positive. In particular, the exponents of $\iota_0(L)$ do not lie in $1 - a + \mathbb{Z}_{<0}$ and the exponents of $\iota_s(L)$ do not lie in $\mathbb{Z}_{<0}$ for each $s \in \mathbb{P}^1 \setminus \{0\}$.

6.4.6 Definition For each $a \in \mathbb{Q} \setminus \mathbb{Z}$ and each fuchsian differential operator $L = \sum_{i=0}^{m} z^i P_i \in \mathbb{C}[z, \vartheta]$ such that $L \otimes O_{a-1}$ is $(1-a)$-positive, we put

$$\mathcal{H}_a(L) := \sum_{i=0}^{m} z^i \prod_{j=0}^{i-1} (\vartheta + a + j) \prod_{k=0}^{m-i-1} (\vartheta - k) P_i.$$

As direct consequence of Lemma 6.4.3, we have:

6.4.7 Proposition *Consider $a \in \mathbb{Q} \setminus \mathbb{Z}$ and a differential operator L such that $L \otimes O_{a-1}$ is $(1-a)$-positive. Then for each $f \in \mathrm{Sol}_L$ and each $s \in S$, the Hadamard-product $H_a^s(f)$ is a solution of $\mathcal{H}_a(L)$.*

As in the case of the middle convolution, the middle Hadamard product can be realized as monodromy tuple of a right factor of $\mathcal{H}_a(L)$.

6.4.8 Definition For each $a \in \mathbb{Q} \setminus \mathbb{Z}$ and each differential operator $L = \sum_{i=0}^{m} z^i P_i \in \mathbb{C}[z, \vartheta]$ such that $L \otimes O_{a-1}$ is $(1-a)$-positive, there is a fuchsian differential operator whose solution space is spanned by the set

$$H_a(\mathrm{Sol}_L) := \bigcup_{s \in S} \{H_a^s(f) \mid L(f) = 0\}.$$

We call this operator the *middle Hadamard product* $L \star_H I_a$ of L with I_a.

Similar to Theorem 6.3.9, we have:

6.4.9 Theorem *Consider $a \in \mathbb{Q} \setminus \mathbb{Z}$ and $\alpha = \exp(2\pi i a)$. If L is irreducible such that $L \otimes O_{a-1}$ is $(1-a)$-positive and induces a monodromy tuple T, then $L \star_H I_a$ is an irreducible right factor of $\mathcal{H}_a(L)$. Furthermore, for each choice of a base star $\Gamma(0, s_2, \ldots, s_r, \infty))$, its induced monodromy tuple is equivalent to $\mathrm{MH}_\alpha(T)$.*

6.4.10 Remark. For computational issues, one can instead of computing the operator $\mathcal{H}_a(L)$ also compute the tensor product of the recurrence equations associated to L and I_a and convert the resulting recurrence equation into a differential operator. By Theorem 6.4.9, the middle Hadamard product $L \star_H I_a$ is a right factor of the latter operator. This can e.g. be done by using the command `hadamardproduct` of the package `gfun` in `MAPLE`.

6.5 Constructive solutions of the MDSP

The translations of the operations we consider to find constructive solutions of the MDSP for a given of tuple of Jordan matrices to the level of differential operators enable us to realize solutions of the MDSP as monodromy tuples of differential operators.

6.5.1 Proposition *Let $J \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ be a tuple of Jordan matrices, $S = \{s_1, \ldots, s_r, \infty\} \subset \mathbb{P}^1$, $P \in \mathrm{GL}_1(\mathbb{C})^{r+1}$ be an m-tuple of rank one and $\alpha \in \mathbb{C}^* \setminus \{1\}$. Further, fix a base star $\Gamma(\underline{S})$ on $\mathbb{P}^1 \setminus S$. Then the monodromy tuple T of $L \in \mathbb{C}[z, \vartheta]$ is a solution of the MDSP for J if and only if there is a fuchsian differential operator $R \in \mathbb{C}[z, \vartheta]$ and an $a \in \mathbb{Q} \setminus \mathbb{Z}$ such that the monodromy tuple of $(L \otimes R) \star_C O_a$ is a solution of the MDSP for $\mathbf{J}(\mathrm{MC}_\alpha(T \otimes P))$.*

Proof Follows directly from Lemma 6.1.3, Remark 6.3.2 and Theorem 6.3.9. □

We investigate Example 5.4.2 from this point of view.

6.5.2 Example (i) According to Example 5.4.2, a solution T of the MDSP for the tuple $J = (J(2), J(2), \alpha J(1) \oplus \alpha^{-1} J(1)) \in \mathrm{GL}_2(\mathbb{C})^3$ is given by $T = \mathrm{MH}_\alpha(1, \alpha, \alpha^{-1})$. By Theorem 6.4.9, for each choice of a base star $\Gamma((0, 1, \infty))$ and $a \in \mathbb{Q} \setminus \mathbb{Z}$ the monodromy tuple of

$$L = I_{1-a} \star_H I_a = \vartheta^2 - z(\vartheta + a)(\vartheta + 1 - a)$$

is equivalent to T. Note, that we even have $\mathcal{H}_{1-a}(I_a) = I_{1-a} \star_H I_a$ in this case. It is easy to see that this operator satisfies properties (M), (N) and (P). For $a = \frac{1}{2}$, the operator L coincides with the Picard-Fuchs operator of the Legendre family stated in Example 2.3.4.

(ii) For $J = (J(2), J(2), J(2), J(2)) \in \mathrm{GL}_2(\mathbb{C})^4$, it is not possible to reduce the rank of this tuple via tensor products with m-tuples of rank one and middle convolutions with $\alpha \in \mathbb{C}^*$. However, we will see in Section 9.4 that for each choice of $s \in \mathbb{C}^* \setminus \{1\}$, $c \in \mathbb{C}$ and a base star $\Gamma((0, 1, s, \infty))$ the monodromy tuple of

$$L(c, s) = s\vartheta^2 - z(\vartheta^2(s + 1) + \vartheta(1 + s + c) + z^2(\vartheta + 1)^2$$

is a solution of the MDSP for J. We only know very few values $s \in \mathbb{C}^*$ and $c \in \mathbb{C}$ such that $L(c, s)$ is has an N-integral solution at $z = 0$.

Chapter 7

Linearly rigid operators of CY-type

In this chapter, we construct operators of CY-type of arbitrary degree which have a linearly rigid monodromy tuple. Therefore, we introduce the notion of m-tuples of CY-type in the first section by recalling the consequences of properties (M), (N) and (P) for differential operators on their corresponding monodromy tuples. As one of the matrices of such a tuple has to be maximally unipotent, they fall into a class of m-tuples for which C. Simpson described the linearly rigid ones in [Sim91, Theorem 4]. These m-tuples split into four families for each of which we discuss the MDSP for the CY-tuples inside. For the solvable ones, we translate the construction of a solution to the level of differential operators via the results established in Chapter 6. The translation will be done in such a way that each of the corresponding operators fulfills properties (M) and (N). Also property (P) seems to be fulfilled for each solution. However, properties (Q) and (Q+) only seem to hold for a finite number of those operators. In particular, we conjecture that the number of CY-tuples J of given rank n with $i(J) = 0$ for which a solution of the MDSP can be realized as monodromy tuple of an operator of CY-type is finite. Furthermore, each solution of CY-type seems to be unique up to equivalence.
Several results of this chapter are also achieved in [BR12].

7.1 Tuples of CY-type

We first recall common properties of monodromy tuples of CY-type differential operators for an arbitrary choice of topological data. Because of properties (N) and (M), the matrices of such a monodromy tuple are quasi-unipotent and one of them is maximally unipotent. Property (P) implies that the monodromy tuple is symplectic for n even and orthogonal for n odd. At first sight properties (Q) and (Q+) do not lead to any obvious restrictions. This gives the notion of a CY-type tuple of matrices.

7.1.1 Definition A tuple $T \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ is said to be of *CY-type* if

(i) T is symplectic for n even and orthogonal for n odd.

(ii) T is quasi-unipotent.

(iii) one of the matrices T_i is maximally unipotent.

As stated in Remark 5.2.4, for each element $A \in \mathrm{GL}_n(\mathbb{C})$, the codimension $\delta(A)$ of its centralizer in $\mathrm{GL}_n(\mathbb{C})$ is given by

$$\delta(A) = n^2 - \sum_{i=1}^{l} w_i^2,$$

where $W(A) = (w_1, \dots, w_l)$ is the partition constructed there. Therefore, it is convenient to describe linearly rigid tuples of rank n in terms of tuples of partitions of n. As we are interested in tuples for which one matrix is maximally unipotent, one of the corresponding partitions has to be $(1, \dots, 1)$. A characterization of linearly rigid tuples of this type was done by C. Simpson in [Sim91].

7.1.2 Theorem (c.f. [Sim91, Theorem 4]) *Let $J \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ be a tuple of Jordan matrices with $i(J) = 0$ and one maximally unipotent matrix. If the MDSP for J is solvable, it consists of three matrices and the associated tuple of partitions $(W(J_1), W(J_2), W(J_3))$ is up to permutation one the following:*

(i) $((1, \dots, 1), (n-1, 1), (1, \dots, 1))$, *the so-called hypergeometric case.*

(ii) $((1, \dots, 1), (k, k-1, 1), (k, k))$ *for $n = 2k \geq 4$ even, the so-called even case.*

(iii) $((1, \dots, 1), (k+1, k), (k, k, 1))$ *for $n = 2k + 1 \geq 5$ odd, the so-called odd case.*

(iv) $((1, \dots, 1), (2, 2, 2), (4, 2))$, *the so-called extra case.*

For the cases mentioned in Theorem 7.1.2, we state all CY-tuples of Jordan matrices $J = (J_1, J_2, J_3)$ for which the MDSP is solvable in the following sections. As the solvability of the MDSP is independent of the order of the matrices J_i, we assume without loss of generality that J_1 is maximally unipotent and that $\delta(J_2) \leq \delta(J_3)$. Next, we construct fuchsian differential operators L, whose monodromy tuples T are solutions of the MDSP for J with respect to each choice of a base star $\Gamma((0, 1, \infty))$. The explicit computation of them can be carried out in `MAPLE`. The involved choices of parameters in $\mathbb{Q} \setminus \mathbb{Z}$ are made in such a way that the operators fulfill the positivity assumptions in each step of the construction and the solutions seem to fulfill properties (M), (N) and (P). Thereafter, we investigate which of the solutions additionally fulfill properties (Q) and (Q+) numerically. Using the formulae stated in Proposition 6.4.4, we are furthermore able to compute several local solutions of those operators explicitly and state them together with the Riemann schemes of the operators in Section A.2.
To state Jordan matrices which correspond to the tuples of partitions explicitly, we introduce some notational conventions.

Convention When we state a Jordan matrix as sum of Jordan-block matrices, we write $\alpha := \alpha J(1)$. If several blocks $\alpha_1 J(m_1) \oplus \dots \oplus \alpha_j J(m_j)$ all appear with multiplicity n, we denote this by $(\alpha_1 J(m_1) \oplus \dots \oplus \alpha_j J(m_j))^n$. For numbers $\alpha_1, \dots, \alpha_k \in \mathbb{C}^*$ we mean by $[\alpha_1 \oplus \dots \alpha_k]$ that if $\alpha_i = \alpha_j$, their corresponding Jordan blocks in the bracket glue to one big block. To abbreviate notation further, we put

$$[\alpha] := \left[\alpha \oplus \alpha^{-1}\right].$$

Furthermore, we always assume that $\alpha_i \in \mathbb{C}^*$ for each $i \in \mathbb{N}$.

In those cases that appear in Theorem 7.1.2, the Jordan matrices corresponding to the tuples of partitions can be read off from the Tables in Section A.1, where we use the notational conventions introduced above.
We furthermore introduce a notational convention concerning tensor products with m-tuples of rank one, which is omnipresent during our constructions.

7.1.3 Definition Consider a tuple of matrices $T \in \mathrm{GL}_n(\mathbb{C})^{r+1}$ and $\alpha \in \mathbb{C}^*$. We put

$$T \otimes K_i^j(\alpha) := (T_1, \ldots, T_{i-1}, \alpha T_i, T_{i+1}, \ldots, T_{j-1}, \alpha^{-1} T_j, T_{j+1}, \ldots, T_{r+1}) \in G^{r+1}$$

with the convention $T \otimes K_i^i(\alpha) = T$.

Note, that this operation fulfills the obvious rule $T \otimes K_i^j(\alpha) \otimes K_j^k(\alpha) = T \otimes K_i^k(\alpha)$.
To guarantee that the solutions of the MDSP we consider are symplectic or orthogonal tuples, we state the following properties of the convolution and the middle Hadamard product which are direct consequences of [DR00, Corollary 5.10 and Theorem 5.14].

7.1.4 Proposition *For each irreducible m-tuple $T \in \mathrm{GL}_n(\mathbb{C})^{r+1}$, we have that*

(i) $\mathrm{MC}_{-1}(T)$ *is orthogonal if T is symplectic.*

(ii) $\mathrm{MC}_{-1}(T)$ *is symplectic if T is orthogonal.*

(iii) *for each $\alpha \in \mathbb{C}^*$, the m-tuple $\mathrm{MH}_{\alpha^{-1}}(\mathrm{MH}_\alpha(T))$ is orthogonal or symplectic if T is orthogonal or symplectic.*

We furthermore make use of the following general group theoretical fact, see e.g. [SS97, Theorem B].

7.1.5 Lemma *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a simple self dual group which contains a maximally unipotent element. If G is not $\mathrm{SL}_2(\mathbb{C})$ acting as Sym^{n-1}, it is symplectic if n is even and orthogonal if n is odd.*

7.2 The hypergeometric case

We start with the hypergeometric case. If $n = 2k > 2$ is even, the only CY-tuples of Jordan matrices J in this case are up to the order of the matrices and tensor products with m-tuples of rank one of the form $J_2 = J(2) \oplus J(1)^{n-2}$ and $J_3 = [[\alpha_1] \oplus \cdots \oplus [\alpha_k]]$. By Lemma 5.1.8, the corresponding MDSP has no solution if $\alpha_i = 1$ for any $1 \leq i \leq k$. In each of these cases, the matrices of the tuple

$$J' = \mathrm{MC}_{\alpha_k}\left(\mathrm{MC}_{\alpha_k^{-1}}(J) \otimes K_1^3(\alpha_k)\right) \otimes K_3^1(\alpha_k)$$

of rank $n-2$ read

$$J' = (J(n-2), J(2) \oplus J(1)^{n-4}, [[\alpha_1] \oplus \cdots \oplus [\alpha_{k-1}]]).$$

By induction on the rank of the tuple - where the step from rank two to rank one is completely similar to the first of the constructions in Example 5.4.2 - we get

$$\mathrm{MC}_{\alpha_1^{-1}}\left(\ldots \mathrm{MC}_{\alpha_k}\left(\mathrm{MC}_{\alpha_k^{-1}}(J) \otimes K_1^3(\alpha_k)\right) \otimes K_3^1(\alpha_k) \ldots\right) \otimes K_1^3(\alpha_1) = (1, \alpha_1^{-1}, \alpha_1).$$

Hence the inversion formulae given in Lemma 5.3.5 yield a solution

$$T = \mathrm{MH}_{\alpha_k^{-1}}\left(\mathrm{MH}_{\alpha_k}\left(\dots \mathrm{MH}_{\alpha_1^{-1}}\left((1,\alpha_1^{-1},\alpha_1)\right)\dots\right)\right)$$

of the MDSP for J. By Proposition 7.1.4 and Lemma 7.1.5, the matrices of this solution are indeed elements of $\mathrm{Sp}_n(\mathbb{C})$.

If $n = 2k+1$ is odd with $k > 0$, the only extended CY-tuples of Jordan matrices J in this case are up to order of the matrices tensor products with m-tuples of rank one of the form

$$J_2 \in \{J(2) \oplus J(1)^{n-2}, J(1) \oplus -J(1)^{n-1}\}$$

and $J_3 = [[\alpha_1] \oplus \dots \oplus [\alpha_k]] \oplus J(1)$. By Lemma 5.1.8, the MDSP of the corresponding tuple has no solution if $J_2 = J(2) \oplus J(1)^{n-2}$. In the other case, we have $\alpha_i \neq -1$ for each $1 \leq i \leq k$ by Lemma 5.1.8. Then

$$\mathrm{MC}_{-1}\left(J \otimes K_2^3(-1)\right) \otimes K_1^3(-1)$$

is as in the hypergeometric case for rank $n-1$ and hence covered by the construction done before. By Proposition 7.1.4, the matrices of this solution are indeed elements of $\mathrm{SO}_n(\mathbb{C})$.

It remains to discuss the case $n = 2$. Here, the extended CY-tuples are up to the order of their matrices given by $J_2 = [\alpha]$ and $J_3 = [\beta]$. By Lemma 5.1.8, the MDSP has no solution if $\alpha = \beta^{-1}$. Otherwise, we get

$$\mathrm{MC}_{\alpha\beta}\left(J \otimes K_3^2(\alpha)\right) = \left(\alpha\beta, \alpha^{-1}\beta, \beta^{-2}\right).$$

Therefore, a solution of the MDSP for this tuple is given by

$$T = \mathrm{MH}_{\alpha\beta}\left((1, \alpha^{-1}\beta, \alpha\beta^{-1})\right) \otimes K_2^3(\alpha).$$

With those results at hand, the translation of the constructions to differential operators gives the following:

7.2.1 Proposition (The hypergeometric case)

(i) *In the hypergeometric case, the CY-tuples of Jordan matrices of rank n for which the MDSP is solvable are up to permutation and tensor products with m-tuples of rank one given by*

$$R^{hyp}(\alpha,\beta,2) = (J(2), [\alpha], [\beta])$$

with $\alpha \neq \beta^{-1}$ for $n = 2$,

$$R^{hyp}(\alpha_1,\dots,\alpha_k,2k) = (J(2k), J(2) \oplus J(1)^{2(k-1)}, [[\alpha_1] \oplus \dots \oplus [\alpha_k]])$$

for $n = 2k > 2$ even and

$$R^{hyp}(\alpha_1,\dots,\alpha_k,2k+1) = (J(2k+1), J(1) \oplus -J(1)^{2k}, [[\alpha_1] \oplus \dots [\alpha_k]] \oplus J(1))$$

for $n = 2k+1$ odd, where $\alpha_j \neq 1$ for all $1 \leq j \leq k$.

(ii) *For each choice of a base star $\Gamma((0,1,\infty))$ and $a, b \in \mathbb{Q} \setminus \mathbb{Z}$ such that $\alpha = \exp(2\pi i a)$, $\beta = \exp(2\pi i b)$ and $a = b$ if $(a-b) \in \mathbb{Z}$ hold, a solution of the MDSP for $R^{hyp}(\alpha,\beta,2)$ is given by the monodromy tuple of*

$$\begin{aligned} P^{hyp}(a,b,2) &= (I_{a-b} \star_H I_{a+b}) \otimes I_{-a} \\ &= \vartheta^2 - z(2\vartheta^2 + a^2 - b^2 - a) + z^2(\vartheta - b)(\vartheta + b). \end{aligned}$$

(iii) *If $n = 2k$ is even, for each choice of a base star $\Gamma((0,1,\infty))$ and $a_1,\ldots,a_k \in \mathbb{Q}\setminus\mathbb{Z}$ such that $\alpha_j = \exp(2\pi i a_j)$ hold, a solution of the MDSP for $R^{hyp}(\alpha_1,\ldots,\alpha_k,n)$ is given by the monodromy tuple of*

$$\begin{aligned} P^{hyp}(a_1,\ldots,a_k,2k) &= I_{a_1} \star_H I_{1-a_1} \star_H \ldots I_{a_k} \star_H I_{1-a_k} \\ &= \vartheta^n - z(\vartheta+a_1)(\vartheta+1-a_1)\cdots(\vartheta+a_k)(\vartheta+1-a_k). \end{aligned}$$

(iv) *If $n = 2k+1$ is odd, for each choice of a base star $\Gamma((0,1,\infty))$ and $a_1,\ldots,a_k \in \mathbb{Q}\setminus\mathbb{Z}$ such that $\alpha_j = \exp(2\pi i a_j)$ hold, a solution of the MDSP for $R^{hyp}(\alpha_1,\ldots,\alpha_k,n)$ is given by the monodromy tuple of $P^{hyp}(a_1,\ldots,a_k,2k+1)\otimes K_1^{-1}$, where*

$$\begin{aligned} &P^{hyp}(a_1,\ldots,a_k,2k+1) = \left(I_{a_1} \star_H I_{1-a_1} \star_H \ldots I_{a_k} \star_H I_{1-a_k} \star_H I_{\frac{1}{2}}\right) \\ &= \vartheta^n - z(\vartheta+a_1)(\vartheta+1-a_1)\cdots(\vartheta+a_k)(\vartheta+1-a_k)\,(\vartheta+1/2)\,. \end{aligned}$$

(v) *Each of the operators $P^{hyp}(a,b,2)$ and $P^{hyp}(a_1,\ldots,a_k,n)$ satisfies properties (M), (N) and (P).*

Proof We only state a proof of the fifth part of the proposition. Note, that property (M) can be read off directly and property (N) is a consequence of Proposition 3.2.8. Furthermore, we have

$$P_1(a_1,\ldots,a_k,n)^\vee = (\vartheta+1)^n - z(\vartheta+2-a_1)(\vartheta+1+a_1)\cdots(\vartheta+2-a_k)(\vartheta+1+a_k)$$

for n even. Therefore $P_1(a_1,\ldots,a_k,n)z = zP_1(a_1,\ldots,a_k,n)^\vee$ which yields property (P). □

Although we find infinitely many operators which fulfill properties (M), (N) and (P), our computational approach to check (Q) and (Q+) suggests that those properties only hold for finitely many ones.

7.2.2 Conjecture The operator $P^{hyp}(a,b,2)$ is of CY-type if and only if $0 < b \le a < 1$ and $a,b \in \{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}\}$. For $n > 2$, $a_1,\ldots,a_k \in \mathbb{Q}\setminus\mathbb{Z}$ and $\alpha_j = \exp(2\pi i a_j)$, the operator $P^{hyp}(a_1,\ldots,a_k,n)$ is of CY-type if and only if $0 \le a_j < 1$, and $\{\alpha_1,\ldots,\alpha_k\}$ is the full set of roots of a product of cyclotomic polynomials.

7.2.3 Remark. (i) The operators mentioned in Conjecture 7.2.2 are up to a transformation $z \mapsto \lambda z$ exactly those, for which property (Q) is proven in [KR10, Theorem 1].

(ii) As a consequence of [BH89, Corollary 3.6], the monodromy group for the operators mentioned in Conjecture 7.2.2 can be realized over $\mathbb{Z}$.

7.3 The even case

We continue our constructions with the even family mentioned in Theorem 7.1.2. For $n = 4$, the only extended CY-tuples of Jordan matrices J in this case are up to the order of the matrices of the form $J_2 = [\alpha] \oplus J(1)^2$ with $\alpha \neq 1$ and $J_3 = [\beta]^2$. Then we get

$$\mathrm{MC}_\beta\left(\mathrm{MC}_{\beta^{-1}}(J)\otimes K_1^3(\beta)\right)\otimes K_3^1(\beta) = (J(2),[\alpha],J(2))$$

and thus

$$\mathrm{MC}_\alpha\left(\mathrm{MC}_\beta\left(\mathrm{MC}_{\beta^{-1}}(J)\otimes K_1^3(\beta)\right)\otimes K_3^1(\beta)\otimes K_3^2(\alpha)\right)\otimes K_3^1(\alpha)=(1,\alpha^{-1},\alpha).$$

The inversion formulae given in Lemma 5.3.5 yield a solution

$$T=\mathrm{MH}_{\beta^{-1}}\left(\mathrm{MH}_\beta\left(\mathrm{MH}_\alpha((1,\alpha^{-1},\alpha))\otimes K_2^3(\alpha)\right)\right)$$

of the MDSP for J.
For $n=2k>4$, all possibilities with induced partition (k,k) are of the form $J_3=[\beta]^k$. The possibilities with induced partition $(k,k-1,1)$ are up multiplication with scalars

$$J_2=-J(2)\oplus J(1)^k\oplus -J(1)^{k-2}$$

if k is even and

$$J_2=J(2)\oplus J(1)^{k-1}\oplus -J(1)^{k-1}$$

if k is odd. By Lemma 5.1.8, the MDSP has no solution if $\beta=1$. We first discuss the special case $\beta=-1$. If k is even, we get

$$\begin{aligned}J'&=\mathrm{MC}_{-1}\left(\mathrm{MC}_{-1}(J)\otimes K_1^3(-1)\right)\otimes K_1^2(-1)\\&=\Big(J(2k-2),J(2)\oplus J(1)^{k-2}\oplus -J(1)^{k-2},-J(2)^{k-1}\Big).\end{aligned}$$

The latter tuple is in the even series for $\beta=-1$ again. Similarly, if k is odd, we get

$$\begin{aligned}J'&=\mathrm{MC}_{-1}\left(\mathrm{MC}_{-1}(J)\otimes K_1^3(-1)\right)\otimes K_1^2(-1)\\&=\Big(J(2k-2),-J(2)\oplus J(1)^{k-1}\oplus -J(1)^{k-3},-J(2)^{k-1}\Big),\end{aligned}$$

which also is in the even series for $\beta=-1$. Hence, we inductively get that

$$\mathrm{MC}_{-1}(\ldots\mathrm{MC}_{-1}\left(\mathrm{MC}_{-1}(J)\otimes K_1^3(-1)\right)\otimes K_1^2(-1)\ldots)=(1,-1,-1).$$

As the operation $\cdot\otimes K_1^2(-1)$ coincides with $\cdot\otimes K_1^3(-1)\otimes K_2^3(-1)$, the inversion formulae given in Lemma 5.3.5 yield a solution

$$T=\mathrm{MH}_{-1}\left(\mathrm{MH}_{-1}\left(\ldots\left(\mathrm{MH}_{-1}((1,-1,-1))\otimes K_2^3(-1)\right)\cdots\otimes K_2^3(-1)\right)\right)$$

of the MDSP for J. If $\beta\neq 1$ one checks that the tuple

$$\mathrm{MC}_\beta(\mathrm{MC}_{\beta^{-1}}(J)\otimes K_1^3(\beta))\otimes K_3^1(\beta)\otimes K_2^3(-1)$$

lies in the even family for the special case considered before. Thus we get the general solution

$$T=\mathrm{MH}_{\beta^{-1}}\left(\mathrm{MH}_\beta\left(\ldots\left(\mathrm{MH}_{-1}((1,-1,-1))\otimes K_2^3(-1)\right)\cdots\otimes K_2^3(-1)\right)\right)$$

of the MDSP for J. By Proposition 7.1.4 and Lemma 7.1.5, the matrices of this solution are indeed elements of $\mathrm{Sp}_n(\mathbb{C})$.
The translation of these constructions to the level of differential operators yields:

7.3.1 Proposition (The even case)

(i) *In the even case, the CY-tuples of Jordan matrices of even rank n for which the MDSP is solvable are up to permutation given by*

$$R^{even}(\alpha, \beta, 4) = \left(J(4), [\alpha] \oplus J(1)^2, [\beta]^2\right)$$

for $n = 4$,

$$R^{even}(\beta, 2k) = \left(J(2k), -J(2) \oplus J(1)^k \oplus -J(1)^{k-2}, [\beta]^k\right)$$

if $n = 2k \geq 4$ is divisible by four and

$$R^{even}(\beta, 2k) = \left(J(2k), J(2) \oplus J(1)^{k-1} \oplus -J(1)^{k-1}, [\beta]^k\right)$$

else, where in each of the cases $\alpha, \beta \neq 1$.

(ii) *For each choice of a base star $\Gamma((0, 1, \infty))$ and $a, b \in \mathbb{Q} \setminus \mathbb{Z}$ such that $\alpha = \exp(2\pi i a)$, $\beta = \exp(2\pi i b)$, $a = b$ if $(a - b) \in \mathbb{Z}$ and $a = 1 - b$ if $(1 - b - a) \in \mathbb{Z}$ hold, a solution of the MDSP for $R^{even}(\alpha, \beta, 4)$ is given by the monodromy tuple of*

$$\begin{aligned} P^{even}(a, b, 4) &= ((I_a \star_H I_a) \otimes I_{1-a}) \star_H I_b \star_H I_{1-b} \\ &= \vartheta^4 - z(\vartheta + b)(\vartheta + 1 - b)(2\vartheta^2 + 2\vartheta + a^2 - a + 1) \\ &\quad + z^2(\vartheta + b)(\vartheta + 1 - b)(\vartheta + b + 1)(\vartheta + 2 - b). \end{aligned}$$

(iii) *For each choice of a base star $\Gamma((0, 1, \infty))$ and $b \in \mathbb{Q} \setminus \mathbb{Z}$ such that $\beta = \exp(2\pi i b)$ holds, a solution of the MDSP for $R^{even}(\beta, 2k)$ is given by the monodromy tuple of*

$$P^{even}(b, 2k) = R\,(k - 1) \star_H I_b \star_H I_{1-b},$$

where

$$\begin{aligned} R(1) &= \left(I_{\frac{1}{2}} \star_H I_{\frac{1}{2}}\right) \otimes I_{\frac{1}{2}} \\ &= 4\vartheta^2 - z\left(8\vartheta^2 + 8\vartheta + 3\right) + 4z^2(\vartheta + 1)^2 \end{aligned}$$

and

$$R(m + 1) = \left(R(m) \star_H I_{\frac{1}{2}} \star_H I_{\frac{1}{2}}\right) \otimes I_{\frac{1}{2}}.$$

(iv) *Each operator $P^{even}(a, b, 4)$ satisfies properties (M), (N) and (P) and each operator $P^{even}(b, 2k)$ satisfies (M) and (N).*

Although we are not able to give a proof, it seems that each operator $P^{even}(b, 2k)$ satisfies property (P) as well.
Similar investigations as before lead to the following conjecture.

7.3.2 Conjecture The operators $P^{even}(a, b, 4)$ and $P^{even}(b, n)$ are of CY-type if and only if $a, b \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$.

7.4 The odd case

Next, we investigate the odd case for $n = 2k+1$ mentioned in Theorem 7.1.2. The only CY-tuples of Jordan matrices J are up to order of the matrices and tensor products with m-tuples of rank one of the form

$$J_2 \in \{J(1)^k \oplus -J(1)^{k+1}, J(2)^k \oplus J(1)\}$$

if k is odd,

$$J_2 \in \{J(1)^{k+1} \oplus -J(1)^k, J(2)^k \oplus J(1)\}$$

if k is even and $J_3 = [[\alpha] \oplus J(1)] \oplus [\alpha]^{k-1}$. Suppose that $J_2 = J(2)^k \oplus J(1)$. By Lemma 5.1.8, the MDSP has no solution if and $\alpha = 1$. For $\alpha \neq 1$, we get

$$\mathrm{MC}_{\alpha^{-1}}(J) \otimes K_1^3(\alpha) \otimes K_2^3(\alpha) = \left(J(2k), \alpha J(1)^k \oplus J(1)^k, \alpha^{-1}J(2) \oplus \alpha^{-1}J(1)^{k-2} \oplus J(1)^k\right).$$

By Lemma 5.1.8, the MDSP for this tuple and consequently also the MDSP for the original one has no solution. We turn to the remaining possibilities for J_2. By Lemma 5.1.8, the MDSP has no solution if k is odd and $\alpha = -1$ or if k is even and $\alpha = 1$. If k is odd and $\alpha = 1$, we get

$$\begin{aligned} J' &= \mathrm{MC}_{-1}\left(\mathrm{MC}_{-1}\left(J \otimes K_2^3(-1)\right) \otimes K_1^3(-1)\right) \otimes K_1^2(-1) \\ &= \left(J(2k-1), J(1)^k \oplus -J(1)^{k-1}, J(1) \oplus -J(2)^{k-1}\right), \end{aligned}$$

which is precisely the special case where $k-1$ is even and $\alpha = -1$. On the other hand, if k is even and $\alpha = -1$, we obtain

$$\begin{aligned} J' &= \mathrm{MC}_{-1}\left(\mathrm{MC}_{-1}(J) \otimes K_1^3(-1)\right) \otimes K_1^3(-1) \\ &= \left(J(2k-1), J(1)^{k-1} \oplus -J(1)^k, J(3) \oplus J(2)^{k-2}\right), \end{aligned}$$

which is precisely the special case where $k-1$ is odd and $\alpha = 1$.
In both of these special cases, iterating this process gives the rank three tuple

$$Q = (J(3), -J(1)^2 \oplus J(1), J(3)).$$

As this tuple is covered by the hypergeometric case, the MDSP is solvable for both of the special cases. Finally, if k is odd and $\alpha \neq \pm 1$, we get that

$$J' = \mathrm{MC}_{-\alpha}\left(\mathrm{MC}_{-\alpha^{-1}}\left(J \otimes K_2^3(-1)\right) \otimes K_1^3(-\alpha)\right) \otimes K_3^1(-\alpha) \otimes K_2^3(-1)$$

is covered by the case where $k-1$ is even and $\alpha = -1$. Similarly, if k is even and $\alpha \neq \pm 1$, we get that

$$J' = \mathrm{MC}_{\alpha}\left(\mathrm{MC}_{\alpha^{-1}}(J) \otimes K_1^3(\alpha)\right) \otimes K_3^1(\alpha)$$

is covered by the case where $k-1$ is odd and $\alpha = 1$. Hence the MDSP is solvable for those tuples J and a solution reads

$$T = \mathrm{MH}_{\alpha^{-1}}\left(\mathrm{MH}_{\alpha}\left(\dots\left(\mathrm{MH}_{-1}\left(\mathrm{MH}_{-1}(Q)\right) \otimes K_2^3(-1)\right) \cdots \otimes K_2^3(-1)\right)\right) \otimes K_2^3(-1)$$

if $n > 3$ and k is odd and

$$T = \mathrm{MH}_{\alpha^{-1}}\left(\mathrm{MH}_{\alpha}\left(\dots\left(\mathrm{MH}_{-1}\left(\mathrm{MH}_{-1}(Q)\right)\otimes K_2^3(-1)\right)\dots\otimes K_2^3(-1)\right)\right)$$

if k is even.
By Proposition 7.1.4 and Lemma 7.1.5, the matrices of the induced solution are elements of $\mathrm{SO}_n(\mathbb{C})$.

7.4.1 Proposition (The odd case)

(i) *In the odd case, the CY-tuples of Jordan matrices of rank $n = 2k+1$ for which the MDSP is solvable are up to permutation and tensor products with m-tuples of rank one given by*

$$R^{odd}(\alpha, 2k+1) = \Big(J(2k+1), J(1)^k \oplus -J(1)^{k+1}, [[\alpha] \oplus J(1)] \oplus [\alpha]^{k-1}\Big)$$

if k is odd and $\alpha \neq -1$ and by

$$R^{odd}(\alpha, 2k+1) = \Big(J(2k+1), J(1)^{k+1} \oplus -J(1)^{k}, [[\alpha] \oplus J(1)] \oplus [\alpha]^{k-1}\Big)$$

if k is even and $\alpha \neq 1$.

(ii) *For each choice of a base star $\Gamma((0,1,\infty))$ and $a \in \mathbb{Q}\setminus\mathbb{Z}$ such that $\alpha = \exp(2\pi i a)$ holds, a solution of the MDSP for $R^{odd}(\alpha, 2k+1)$ is given by the monodromy tuple of*

$$P^{odd}(a, 2k+1) \otimes I_{\frac{1}{2}} = \left(R\,(k-1) \star_H I_a \star_H I_{1-a}\right) \otimes I_{\frac{1}{2}}$$

if k is odd and

$$P^{odd}(a, 2k+1) = R\,(k-1) \star_H I_a \star_H I_{1-a}$$

if k is even, where

$$\begin{aligned} R(1) &= \Big(I_{\frac{1}{2}} \star_H I_{\frac{1}{2}} \star_H I_{\frac{1}{2}}\Big) \otimes I_{\frac{1}{2}} \\ &= 8\vartheta^3 - z(2\vartheta+1)(8\vartheta^2+8\vartheta+5) + 8z^2(\vartheta+1)^3 \end{aligned}$$

and

$$R(m+1) = \Big(R(m) \star_H I_{\frac{1}{2}} \star_H I_{\frac{1}{2}}\Big) \otimes I_{\frac{1}{2}}.$$

(iii) *Each operator $P^{odd}(a, 2k+1)$ satisfies properties (M) and (N).*

As in the even case, property (P) seems to hold for each operator $P^{odd}(a,n)$, although we are not able to prove it.
Similar investigations as before lead to the following conjecture.

7.4.2 Conjecture For each odd number $n \geq 3$ the operator $P^{odd}(a, 2k+1)$ is of CY-type if and only if $a \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$.

7.5 The extra case

Finally, we turn to the extra case. Here, the only possibilities for an extended CY-tuple of Jordan matrices are up to tensor products with m-tuples of rank one and the order of the matrices of the form

$$J_2 \in \{-J(1)^2 \oplus J(1)^4, J(2)^2 \oplus J(1)^2\}$$

and

$$J_3 = [[\alpha] \oplus \pm J(1)]^2.$$

The combinations for which $J_3 = [[\alpha] \oplus J(1)]^2$ holds are ruled out by Lemma 5.1.8. Therefore, consider $J_3 = [[\alpha] \oplus -J(1)]^2$. If $J_2 = -J(1)^2 \oplus -J(1)^2$, Proposition 7.1.4 assures that the tuple

$$\mathrm{MC}_{-1}(J) \otimes K_1^3(-1) = (J(5), J(2)^2 \oplus J(1), [\alpha]^2 \oplus J(1)$$

lies in the odd family for $n = 5$, but its associated MDSP has no solution by Proposition 7.4.1. This leaves the possibility

$$J = \left(J(6), J(2)^2 \oplus J(1)^2, [[\alpha] \oplus -J(1)]^2\right).$$

In this case we get that

$$J' = \mathrm{MC}_{-1}(J) \otimes K_1^3(-1) = \left(J(5), J(1)^3 \oplus -J(1)^2, [\alpha]^2 \oplus J(1)\right).$$

As we have constructed a solution of the MDSP for this tuple in Proposition 7.4.1, the MDSP for J is solvable. By Proposition 7.1.4 and Lemma 7.1.5, each solution lies in $\mathrm{Sp}_6(\mathbb{C})^3$.

7.5.1 Proposition (The extra case)

(i) *In the extra case, the CY-tuples of Jordan matrices for which the MDSP is solvable are up to permutation and tensor products with m-tuples of rank one given by*

$$R^{extra}(\alpha) = \left(J(6), J(2) \oplus J(1)^2, [[\alpha] \oplus -J(1)]^2\right)$$

for $\alpha \neq 1$.

(ii) *For each choice of a base star* $\Gamma((0, 1, \infty))$ *and* $a \in \mathbb{Q} \setminus \mathbb{Z}$ *such that* $\alpha = \exp(2\pi i a)$ *holds, a solution of the MDSP for* $R^{extra}(\alpha)$ *given by the monodromy tuple of*

$$\begin{aligned} P^{extra}(a) &= \left(\left(I_{\frac{1}{2}} \star_H I_{\frac{1}{2}} \star_H I_{\frac{1}{2}}\right) \otimes I_{\frac{1}{2}}\right) \star_H I_a \star_H I_{1-a} \star_H I_{\frac{1}{2}} \\ &= 16\,\vartheta^6 - z\,(2\,\vartheta + 1)^2 \left(8\,\vartheta^2 + 8\,\vartheta + 5\right)(\vartheta + a)\,(\vartheta + 1 - a) \\ &\quad + 4\,z^2\,(2\,\vartheta + 3)\,(2\,\vartheta + 1)\,(\vartheta + 2 - a)\,(\vartheta + 1 - a)\,(\vartheta + 1 + a)\,(\vartheta + a). \end{aligned}$$

(iii) *Each operator* $P^{extra}(a)$ *satisfies properties (M), (N) and (P).*

As before, further computations lead to the following conjecture.

7.5.2 Conjecture The operator $P^{extra}(a)$ is of CY-type if and only if $a \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$.

7.5.3 Example We discuss some specialties which seem to hold for CY-type operators of degree six examplarily for those of the extra case. Unless the case of CY-type operators of degree up to five, the local normal form of such a CY-type operator depends on the two series α_1/α_2 and α_1/α_3, where $\alpha_0, \ldots, \alpha_5$ is the corresponding structure series. We state the first terms of the q-coordinate q and the expressions $Y_1 := \alpha_1/\alpha_2 \circ q^\vee$ and $Y_2 := \alpha_1/\alpha_3 \circ q^\vee$ of each of the CY-type operators in the extra case. The latter two series are invariant under local transformations $\psi \in z\mathbb{Q}[\![z]\!]^*$ with $\psi'(0) = 1$.

$$a = \frac{1}{2},\ q = z + 1152\,z^2 + 2142144\,z^3 + 4940537856\,z^4 + \ldots$$
$$Y_1 = 1 + 576\,z + 1124928\,z^2 + 2363940864\,z^3 + 5117880638016\,z^4 + \ldots$$
$$Y_2 = 1 + 640\,z + 766336\,z^2 + 1364156416\,z^3 + 2588791110016\,z^4 + \ldots$$

$$a = \frac{1}{3},\ q = z + 1968\,z^2 + 6170184\,z^3 + 23955371648\,z^4 + \ldots$$
$$Y_1 = 1 + 1008\,z + 3312144\,z^2 + 11698683264\,z^3 + 42584436803088\,z^4 + \ldots$$
$$Y_2 = 1 + 1008\,z + 2131920\,z^2 + 6383299968\,z^3 + 20312552381904\,z^4 + \ldots$$

$$a = \frac{1}{4},\ q = z + 4736\,z^2 + 35158976\,z^3 + 322519506944\,z^4 + \ldots$$
$$Y_1 = 1 + 2496\,z + 19446336\,z^2 + 161921114112\,z^3 + 1391283515923008\,z^4 + \ldots$$
$$Y_2 = 1 + 2176\,z + 11626880\,z^2 + 81837481984\,z^3 + 613286180895104\,z^4 + \ldots$$

$$a = \frac{1}{6},\ q = z + 32640\,z^2 + 1633116096\,z^3 + 100632924938240\,z^4 + \ldots$$
$$Y_1 = 1 + 17856\,z + 945580608\,z^2 + 52710537240576\,z^3 + 3042075955330751040\,z^4 + \ldots$$
$$Y_2 = 1 + 12672\,z + 507482496\,z^2 + 23635655368704\,z^3 + 1188567502482340224\,z^4 + \ldots$$

In the corresponding situation of degree four, the series $Y := Y_1$ is the normalized Yukawa coupling. As observed by P. Candelas et al. in [COGP92], the numbers n_d of the expansion

$$Y = 1 + \sum_{d=1}^{\infty} n_d d^3 \frac{z^d}{1 - z^d}$$

are in the known examples integral and supposed to count geometric objects. Proceeding similarly, we can write Y_1 and Y_2 as a Lambert series

$$Y_i = 1 + \sum_{d=1}^{\infty} m_{i,d} d^{k_i} \frac{z^d}{1 - z^d}$$

and check for which choice of k_i the corresponding numbers $m_{i,d}$ are integers. In the examples of degree six we constructed in this section, this seems to hold for $k_1 = 2$ and $k_2 = 1$. We state the first four numbers $m_{i,d}$ for all CY-type operators in the extra case.

$$a = \frac{1}{2}, \quad \begin{aligned} m_1 &: 576, 281088, 262660032, 319867469568, \ldots \\ m_2 &: 640, 382848, 454718592, 647197585920, \ldots \end{aligned}$$

$$a = \frac{1}{3}, \quad \begin{aligned} m_1 &: 1008, 827784, 1299853584, 2661527093184, \ldots \\ m_2 &: 1008, 1065456, 2127766320, 5078137562496, \ldots \end{aligned}$$

$$a = \frac{1}{4}, \quad \begin{aligned} m_1 &: 2496, 4860960, 17991234624, 86955218529792, \ldots \\ m_2 &: 2176, 5812352, 27279159936, 153321542317056, \ldots \end{aligned}$$

$$a = \frac{1}{6}, \quad \begin{aligned} m_1 &: 17856, 236390688, 5856726358080, 190129747149073152, \ldots \\ m_2 &: 12672, 253734912, 7878551785344, 297141875493714432, \ldots \end{aligned}$$

Chapter 8

Fourth order symplectically rigid operators of CY-type

In this chapter, we discuss the MDSP for CY-tuples $J \in \mathrm{Sp}_4(\mathbb{C})^{r+1}$ with rigidity index $i_{\mathrm{Sp}_4(\mathbb{C})}(J) = 0$. This holds for each rank four CY-tuple of the hypergeometric and the even series introduced in Theorem 7.1.2. We thus first concentrate on the remaining cases and classify those for which the MDSP is solvable in the first section. By the general classification of linearly rigid m-tuples, it is not possible to reduce a non-linearly rigid m-tuple to a tuple of rank one via tensor products and middle convolutions with m-tuples of rank one. However, it turns out that the requested tuples can be constructed starting with a rank one tuple if we also consider symmetric- and exterior powers of m-tuples. The construction of solutions of the MDSP via fuchsian differential operators is done in the second section. Again, the involved parameters are chosen in such a way that each solution satisfies properties (M), (N) and (P). Properties (Q) and (Q+) - which are checked numerically - only seem to hold for a finite number of operators. This leads to a conjecture concerning the solvability of the MDSP by operators of CY-type similar to the one in the linearly rigid case. In the third section of this chapter, we compare all potential CY-type operators with $\mathrm{Sp}_4(\mathbb{C})$-rigid monodromy tuple we constructed with the ones tabulated in [AESZ05]. In particular, all known CY-type operators of this type are covered by our approach.
Several results of this chapter are also achieved in [BR12].

8.1 Characterization

We characterize those CY-tuples of Jordan matrices of rank four with $i_{\mathrm{Sp}_4(\mathbb{C})}(J) = 0$ for which the MDSP is solvable. Therefore, we again use the notational conventions introduced in Section 7.1. Unlike for $\mathrm{GL}_n(\mathbb{C})$, for a given element $A \in \mathrm{Sp}_4(\mathbb{C})$ the number $\delta_{\mathrm{Sp}_4(\mathbb{C})}(A)$ is not completely determined by the associated partition $W(A)$ but can e.g. be read off from the tables given in [Car85, Chapter 13]. We have tabulated each of the possibilities in Table A.2. By means of Lemma 4.3.1, we may further identify $\mathrm{Sp}_4(\mathbb{C})$ with $\mathrm{SO}_5(\mathbb{C})$. To each Jordan matrix in $\mathrm{Sp}_4(\mathbb{C})$, the corresponding Jordan matrices in $\mathrm{SO}_5(\mathbb{C})$ are also stated in Table A.2. We first investigate for which CY-tuples of Jordan matrices the MDSP is solvable.

8.1.1 Proposition *Let $G = \mathrm{Sp}_4(\mathbb{C})$ and J be a CY-tuple of Jordan matrices of rank four such that $i(J) \neq 0$ and $i_G(J) = 0$ hold. If the MDSP for J is solvable in G, the tuple consists* *of three matrices and coincides up to permutation and tensor products with m-tuple of rank one with one of the following possibilities:*

(i) $S_1(\alpha,\beta) = (J(4), J(1)^2 \oplus -J(1)^2, [[\alpha] \oplus [\beta]])$ *where* $(\alpha,\beta) \neq \pm(1,1)$.

(ii) $S_2(\alpha,\beta) = (J(4), [\alpha] \oplus J(1)^2, [\beta] \oplus -J(1)^2)$ *where* $\alpha \neq 1$ *and* $\beta \neq -1$.

Proof Since $\dim G = 10$ and $\dim Z(G) = 0$, we have

$$i_G(J) = \sum_{i=1}^{r+1} \delta_G(J_i) - 20$$

by Definition 5.2.7. As one of the matrices of J, say J_1, is maximally unipotent, this implies

$$\sum_{i=2}^{r+1} \delta_G(J_i) - 12 = 0.$$

We also assume without loss of generality that $\delta_G(J_i) \leq \delta_G(J_k)$ if $i < k$.
As stated in Table A.2, this leaves the following possibilities for $\delta_G(J_i)$:

(i) $\delta_G(J_2) = \delta_G(J_3) = \delta_G(J_4) = 4$.

(ii) $\delta_G(J_2) = 4$ and $\delta_G(J_3) = 8$.

(iii) $\delta_G(J_2) = \delta_G(J_3) = 6$.

In the first of these cases, the combinations $J_2 = J_3 = J_4 = J(2) \oplus J(1)^2$, $\{J_2, J_3, J_4\} = \{J(2) \oplus J(1)^2, -(J(2) \oplus J(1))\}$ and $J_2 = J(1)^2 \oplus -J(1)^2$, $J_3 \neq J_2$, $J_4 \neq J_2$ are ruled out by Lemma 5.1.8. Consider $J_2 = J_3 = J(1)^2 \oplus -J(1)^2$. As we are looking for solutions in $\mathrm{Sp}_4(\mathbb{C})$, the second exterior power of J then has for each possible choice of J_4 a reducible submodule in $\mathrm{SO}_5(\mathbb{C})$. In particular, J is reducible. Hence, in the first of the cases stated above, the MDSP for J is never solvable.
In the second of these cases, we get $i(J) = 0$ if $J_2 = J(2) \oplus J(1)^2$. This leaves tuples of the form $S_1(\alpha,\beta)$.
In the third case, we get $i(J) = 0$ if $(J_2, J_3) = \left([\alpha] \oplus J(1)^2, [\beta]^2\right)$. By Lemma 5.1.8, the MDSP has for the possibilities

$$(J_2, J_3) = \left([\alpha] \oplus J(1)^2, [\beta] \oplus J(1)^2\right)$$

and $J_2 = J_3 = J(2) \oplus -J(1)^2$ no solution. If $(J_2, J_3) = \left([\alpha]^2, [\beta]^2\right)$, the MDSP has no solution by considering the tuple itself or its second exterior power. Therefore, the only remaining cases are like $S_2(\alpha,\beta)$, which completes the proof. □

8.2 Constructions

We construct solutions of the MDSP for $S_1(\alpha,\beta)$ and $S_2(\alpha,\beta)$ introduced in Proposition 8.1.1 as monodromy tuples of fuchsian differential operators. Parts of the computations were done using commands of the package `DEtools` in `MAPLE`.

8.2.1 Proposition *Consider $\alpha, \beta \in \mathbb{C}^*$, such that neither $\alpha \neq \pm i\beta$ nor $\alpha, \beta \in \{1, -1\}$. Then for each choice of a base star $\Gamma((0,1,\infty))$ and $a \in \mathbb{Q}\backslash\left(\frac{1}{4}+\mathbb{Z}\cup\frac{3}{4}+\mathbb{Z}\right)$, $b \in \mathbb{Q}\backslash\left(\frac{1}{4}+\mathbb{Z}\cup\frac{3}{4}+\mathbb{Z}\right)$ such that $\alpha = \exp(2\pi i(a+b))$, $\beta = \exp(2\pi i(a-b))$, $a = b$ if $(a-b) \in \mathbb{Z}$ and $a = 1-b$ if $(1-b-a) \in \mathbb{Z}$ hold, a solution of the MDSP for*

$$S_1(\alpha, \beta) = (J(4), J(1)^2 \oplus -J(1)^2, [[\alpha] \oplus [\beta]])$$

is given by the monodromy tuple of the operator

$$\begin{aligned}
P_3(a,b) &:= \left(\bigwedge^2 \left(\left(I_{\frac{3}{4}+a} \star_H I_{\frac{3}{4}-a} \otimes z^{\frac{1}{2}}\right) \star_H I_{\frac{1}{4}+b} \star_H I_{\frac{1}{4}-b}\right) \otimes z^{\frac{1}{2}}\right) \star_H I_{\frac{3}{2}} \\
&= 64\,\vartheta^4 - z\left(128\,\vartheta^4 + 256\,\vartheta^3 + \vartheta^2(304 - 128(a^2+b^2))\right) \\
&\quad - z\left(\vartheta\,(176 - 128(a^2+b^2)) + 39 - 48(a^2+b^2) - 256\,a^2b^2\right) \\
&\quad + 64z^2\,(\vartheta+1-a-b)\,(\vartheta+1+a-b) \\
&\qquad\qquad (\vartheta+1-a+b)\,(\vartheta+1+a+b)\,.
\end{aligned}$$

Further, each operator $P_3(a,b)$ satisfies properties (M), (N) and (P).

Proof Up to a tensor product with an m-tuple of rank one, we may assume that

$$\gamma(-[[\alpha] \oplus [\beta]]) = 4$$

holds. We get

$$\begin{aligned}
J' &= \mathrm{MC}_{-1}(S_1(\alpha,\beta)) \otimes K_1^3(-1) \\
&= \left(J(3) \oplus -J(1)^2, J(2)^2 \oplus J(1), -[[\alpha] \oplus [\beta]] \oplus J(1)\right).
\end{aligned}$$

By Proposition 7.1.4, J' is orthogonal. Therefore, it can by Lemma 4.2.3 and Table A.2 be written as the tuple of Jordan matrices of a direct summand of $\bigwedge^2(\tilde{J})$ which lies in $\mathrm{SO}_5(\mathbb{C})$, where

$$\tilde{J} = (iJ(2) \oplus -iJ(2), J(2) \oplus J(1)^2, [[\zeta] \oplus [\chi]])$$

with $\zeta\chi = \alpha$ and $\zeta\chi^{-1} = \beta$. If $\zeta, \chi \in \{i, -i\}$, the MDSP for $\tilde{J}$ has no solution by Lemma 5.1.8 and Lemma 6.1.3. In the remaining cases, we get

$$\mathrm{MC}_{-i\zeta}\left(\mathrm{MC}_{i\chi}\left(\mathrm{MC}_{i\chi^{-1}}(J \otimes K_1^3(-i)) \otimes K_1^3(-i\chi)\right) \otimes K_3^1(-i\chi)\right) = (-i\zeta, i\zeta, -\zeta^{-2}).$$

Hence, the MDSP is solvable and a solution is given by

$$T = \mathrm{MH}_{-1}\left(\bigwedge^2\left(\mathrm{MH}_{i\chi^{-1}}\left(\mathrm{MH}_{i\chi}\left(\mathrm{MH}_{-i\zeta}\left((1, i\zeta, -i\zeta^{-1})\right) \otimes K_1^3(-1)\right)\right)\right) \otimes K_1^3(-1)\right).$$

According to Proposition 7.1.4 and Lemma 4.2.3, the matrices of such a solution indeed lie in $\mathrm{Sp}_4(\mathbb{C})$. Note, that for each a, b as stated in the result, the operator

$$\begin{aligned}
Q &= \left(\left(I_{\frac{3}{4}+a} \star_H I_{\frac{3}{4}-a}\right) \otimes z^{\frac{1}{2}}\right) \star_H I_{\frac{1}{4}+b} \star_H I_{\frac{1}{4}-b} \\
&= 64\vartheta^2(2\vartheta-1)^2 - z(4\vartheta+1+4b)(4\vartheta+1-4b)(4\vartheta+1+4a)(4\vartheta+1-4a)
\end{aligned}$$

is irreducible and fulfills property (P). Therefore, we get $\deg(\bigwedge^2 Q) = 5$ by Corollary 3.1.4. By Lemma 6.1.3 and Theorem 6.4.9, the monodromy tuple of $P_3(a,b)$ is a solution of the MDSP for $S_1(\alpha,\beta)$. As all operations involved preserve N-integrality of solutions, it fulfills property (N). Properties (M) and (P) can be checked by direct computation. □

8.2.2 Remark. Alternatively, one checks that the direct summand of the tuple $\bigwedge^2(J)$ which lies in $\mathrm{SO}_5(\mathbb{C})$ is in the hypergeometric series and that its MDSP is solvable if neither $\alpha \neq \pm i\beta$ nor $\alpha, \beta \in \{1, -1\}$. In particular, we have

$$P_3\left(\frac{1}{2}a - \frac{1}{4}, \frac{1}{2}b - \frac{1}{4}\right) = \bigvee\nolimits_2 \left(P^{hyp}(a, b, 5)\right) \otimes z^{-1/2}(1-z)^{-3/4}.$$

8.2.3 Proposition *Consider $\alpha, \beta \in \mathbb{C}^*$ such that $\alpha \neq 1$ and $\beta \neq -1$. Then for each choice of a base star $\Gamma((0, 1, \infty))$ and $a, b \in \mathbb{Q} \setminus \mathbb{Z}$ such that $\alpha = \exp(\pi i(a + b))$, $\beta = \exp(2\pi i(a - b))$ $a = b$ if $(a - b) \in \mathbb{Z}$ and $a = 1 - b$ if $(1 - b - a) \in \mathbb{Z}$ hold, a solution of the MDSP for*

$$S_2(\alpha, \beta) = (J(4), [\alpha] \oplus J(1)^2, [\beta] \oplus -J(1)^2])$$

is given by the monodromy tuple of

$$\begin{aligned} P_4(a, b) &:= \mathrm{Sym}^2 \left(I_a \star_H I_b \otimes I_{\frac{1-a-b}{2}}\right) \star_H I_{\frac{1}{2}} \\ &= 4\,\vartheta^4 - 2\,z\,(2\,\vartheta + 1)^2 \left(\vartheta^2 + \vartheta + 2\,ab - a + 1 - b\right) \\ &\quad + z^2\,(2\,\vartheta + 1)\,(2\,\vartheta + 3)\,(\vartheta + 1 + a - b)\,(\vartheta + 1 + b - a)\,. \end{aligned}$$

Further, each operator $P_4(a, b)$ satisfies properties (M), (N) and (P).

Proof We get

$$J' = \mathrm{MC}_{-1}(J) \otimes K_1^3(-1) = (J(3), [[-\alpha] \oplus J(1)], [[\beta] \oplus J(1)]).$$

By Proposition 7.1.4, J' is orthogonal and can according to Lemma 4.2.3 be written as tuple of Jordan matrices of $\mathrm{Sym}^2(\tilde{J})$, where

$$\tilde{J} = (J(2), [i\zeta], [\chi])$$

with $\zeta^2 = \alpha$ and $\chi^2 = \beta$. Then we get

$$\mathrm{MC}_{i\zeta\chi}\left(\tilde{J} \otimes K_3^2(i\zeta)\right) = (i\zeta\chi, -i\zeta^{-1}\chi, \chi^{-2}).$$

Therefore, the MDSP is solvable and a solution is given by

$$T = \mathrm{MH}_{-1}\left(\mathrm{Sym}^2\left(\mathrm{MH}_{i\zeta\chi}\left(1, -i\zeta^{-1}\chi, i\zeta\chi^{-1}\right) \otimes K_2^3(i\zeta)\right)\right).$$

By Proposition 7.1.4 and Lemma 4.2.3, the matrices of such a solution indeed lie in $\mathrm{Sp}_4(\mathbb{C})$. Lemma 6.1.3 and Theorem 6.4.9 assure that the monodromy tuple of $P_4(a, b)$ is a solution of the MDSP for J. As all operations involved preserve the N-integrality of solutions, each operator $P_4(a, b)$ fulfills property (N). Properties (M) and (P) can be checked by direct computation. $\square$

8.3 Comparison to earlier results

For each allowed choice of $a, b \in \mathbb{Q} \setminus \mathbb{Z}$, the operators $P^{hyp}(a,b,4), P^{even}(a,b,4), P_3(a,b)$ and $P_4(a,b)$ fulfill properties (M), (N) and (P). We investigate for all operators $P_3(a,b)$ and $P_4(a,b)$ whether (Q) and (Q+) hold numerically. Similar to the cases for $P^{hyp}(a,b,4)$ and $P^{even}(a,b,4)$, this seems to lead to a finite number of operators of CY-type. It turns out that all operators of CY-type we find via this method appear in the table [AESZ05, Appendix A] up to a transformation $z \mapsto \lambda z$. In each of the appearing cases, this transformation yields the holomorphic solution $y_0 = 1 + \sum_{m=1}^{\infty} A_m z^m$ to be minimal over $\mathbb{Z}$, in the sense that there is no $1 \neq n \in \mathbb{N}$ such that $A_m/(n^m) \in \mathbb{Z}$ for all $m \in \mathbb{N}$. The q-coordinate $q = z + \sum_{m=2}^{\infty} B_m z^m$ of those transformed operators turns also out to be minimal over $\mathbb{Z}$. To make this transformation explicit, we consider for each $a = r/s \in \mathbb{Q}$ with $r \in \mathbb{Z}$, $s \in \mathbb{N}$ and $\gcd(r,s) = 1$ its image under the map

$$\beta\colon \mathbb{Q} \setminus \{0\} \to \overline{\mathbb{Z}},\ a \mapsto s \prod_{i=1}^{n} s_i^{\frac{1}{s_i - 1}},$$

where $s_1, \ldots, s_n$ denote the distinct prime divisors of s.
In the upcoming tables, the entry *Nr.* refers to the labeling of the resulting CY-type operator in [AESZ05].

(i) As already stated, it seems that $P^{hyp}(a,b,4)$ is an operator of CY-type if and only if $a, b \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$ or $(a,b) \in \left\{\left(\frac{1}{5}, \frac{2}{5}\right), \left(\frac{1}{8}, \frac{3}{8}\right), \left(\frac{1}{10}, \frac{3}{10}\right), \left(\frac{1}{12}, \frac{5}{12}\right)\right\}$. After the transformation $z \mapsto \beta(a)^2\beta(b)^2 z$, we obtain the following CY-type operators stated in [AESZ05]

a	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{1}{10}$	$\frac{1}{12}$
b	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{5}$	$\frac{3}{8}$	$\frac{3}{10}$	$\frac{5}{12}$
Nr.	3	5	6	14	4	11	8	10	12	13	1	7	2	9

(ii) As already stated, it seems that $P^{even}(a,b)$ is an operator of CY-type if and only if $a, b \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$. After the transformation $z \mapsto \beta(a)^2\beta(b)^2 z$, we obtain the following CY-type operators stated in [AESZ05]

a	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
b	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$
Number	111	110	30	112	141	142	196	143

a	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
b	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$
Number	189	194	197	199	190	195	198	61

(iii) To make our observations more transparent, we substitute $c = 2a + \frac{1}{2}$ and $d = 2b + \frac{1}{2}$. Then it seems that $P_3(a,b)$ is an operator of CY-type if and only if $c, d \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$ or $(c,d) \in \left\{\left(\frac{1}{5}, \frac{2}{5}\right), \left(\frac{1}{8}, \frac{3}{8}\right), \left(\frac{1}{10}, \frac{3}{10}\right), \left(\frac{1}{12}, \frac{5}{12}\right)\right\}$. After the transformation $z \mapsto 4\beta(c)^2\beta(d)^2 z$, we get

c	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{1}{10}$	$\frac{1}{12}$
d	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{5}$	$\frac{3}{8}$	$\frac{3}{10}$	$\frac{5}{12}$
Nr.	$\tilde{3}$	$\tilde{5}$	$\tilde{6}$	$\widetilde{14}$	$\tilde{4}$	$\widetilde{11}$	$\tilde{8}$	$\widetilde{10}$	$\widetilde{12}$	$\widetilde{13}$	$\tilde{1}$	$\tilde{7}$	$\tilde{2}$	$\tilde{9}$

where the number $\widetilde{i}$ refers to the operators defined in [Alm06]. As shown there, these operators are equivalent to $206-219$ in [AESZ05]. The alternative construction mentioned in Remark 8.2.2 together with Conjecture 4.3.11 yields the conjecture that $P_3(a,b)$ is of CY-type if and only if $\bigwedge^2 P_3(a,b)$ is.

(iv) It seems that $P_4(a,b)$ is of CY-type if and only if $(2a,2b) \in \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6}\right\}$ and additionally $2(a+b) \in \mathbb{Z}$ or $2(a-b) \in \mathbb{Z}$ holds. After the transformation $z \mapsto 4\beta(a)\beta(b)z$, we get the following CY-type operators stated in [AESZ05].

a	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
b	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{5}{6}$
Nr.	3^*	6^*	14^*	4^*	4^{**}	8^*	8^{**}

a	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{12}$
b	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{5}{12}$	$\frac{7}{12}$
Nr.	10^*	10^{**}	13^*	13^{**}	7^*	7^{**}	9^*	9^{**}

Chapter 9

CY-type operators of degree four and index two

In this chapter, we investigate the MDSP for CY-tuples $J \in \mathrm{Sp}_4(\mathbb{C})^{r+1}$ with $i_{\mathrm{Sp}_4(\mathbb{C})}(J) = 2$. In the second section, we show that in the solvable case r has to be two or three. Moreover, we show that the solvability of the MDSP for tuples with $r = 3$ yields the solvability of the MDSP for certain tuples of rank two, if we also consider squares and roots of m-tuples. These operations are briefly introduced in the first section of this chapter. In the third section, we realize solutions of the MDSP of the appearing tuples of rank two by Heun operators. We further show that this allows us to construct solutions of the MDSP for the CY-type tuples of rank four we started with. The constructions are done in such a way that each of the resulting differential operators satisfies properties (M) and (P). As both the Heun operators we consider and the solutions we get admit an accessory parameter, it is very difficult to detect those which satisfy properties (N), (Q) and (Q+). However, it seems that each of the defining properties of a CY-type operator is preserved by the constructions. Therefore, we also determine appropriate Heun operators of CY-type as pullbacks of hypergeometric differential operators, which is done in the fourth section of this chapter. The resulting CY-type operators of degree four are stated in Section A.2.6. Some of them were previously unknown.

9.1 Additional operations

We briefly introduce additional operations we consider for tuples of Jordan matrices and differential operators. First, we consider the natural action of permutations on tuples of Jordan matrices.

9.1.1 Definition Consider a tuple of Jordan matrices $J = (J_1, \ldots, J_{r+1}) \in \mathrm{GL}_n(\mathbb{C})$ and a permutation $\pi \in S_{r+1}$. We put

$$J_\pi := (J_{\pi(1)}, \ldots, J_{\pi(r+1)}) \in \mathrm{GL}_n(\mathbb{C}).$$

It is easy to see that the action of a permutation on J is compatible with the middle convolution in the following sense:

9.1.2 Lemma *Consider a tuple of Jordan forms $J = (J_1, \ldots, J_{r+1}) \in \mathrm{GL}_n(\mathbb{C})$ such that $\sum_{i=1}^{r+1} \gamma(J_i) \geq 2n$ holds and a permutation $\pi \in S_{r+1}$. Then we have*

$$\mathrm{MC}_\alpha(J_\pi) = \left(\mathrm{MC}_\alpha\left(J \otimes K_{r+1}^{\pi(r+1)}(\alpha)\right) \otimes K_{\pi(r+1)}^{r+1}(\alpha)\right)_\pi$$

for each $\alpha \in \mathbb{C}^* \setminus \{1\}$.

We briefly introduce the notion of squares and roots on the level of tuples of Jordan forms. On the topological point of view, this corresponds to the change of the local data of a local system on $\mathbb{P}^1 \setminus S$ by a $2:1$-covering.

9.1.3 Definition For a tuple of Jordan forms $J = (J_1, \ldots, J_{r+1}) \in \mathrm{GL}_n(\mathbb{C})^{r+1}$, we put

$$J^2 := \left(\mathbf{J}\left(J_1^2\right), J_2, \ldots, J_r, J_2, \ldots, J_r, \mathbf{J}\left(J_{r+1}^2\right)\right) \in \mathrm{GL}_n(\mathbb{C})^{2r}.$$

Further, given a tuple of Jordan forms $J = (J_1, \ldots, J_{2r}) \in \mathrm{GL}_n(\mathbb{C})^{2r}$ such that $J_i = J_{r-1+i}$ holds for each $2 \leq i \leq r$, we put

$$\sqrt{J} = \left(\mathbf{J}\left(\sqrt{J_1}\right), J_2, \ldots, J_r, \mathbf{J}\left(\sqrt{J_{r+1}}\right)\right) \in \mathrm{GL}_n(\mathbb{C})^{r+1}$$

with $\mathbf{J}\left(\sqrt{J_1}^2\right) = J_1$ and $\mathbf{J}\left(\sqrt{J_{r+1}}^2\right) = J_{r+1}$.

Note, that $\sqrt{J}$ is not unique.

We investigate similar operations on the level of differential operators. The twist by $z \mapsto z^2$ was already discussed in Section 1.4. We now introduce an operation corresponding to $z \mapsto \sqrt{z}$.

9.1.4 Definition The differential subring $\mathbb{C}[z^2, \vartheta] \subset \mathbb{C}[z, \vartheta]$ is - as a set -given by

$$\mathbb{C}[z^2, \vartheta] := \left\{\sum_{i=0}^{m} z^{2i} P_i \mid P_i \in \mathbb{C}[\vartheta]\right\}.$$

Furthermore, we consider the ring homomorphism

$$\sqrt{z}^* : \mathbb{C}[z^2, \vartheta] \to \mathbb{C}[z, \vartheta],\ z^2 \mapsto z,\ \vartheta \mapsto 2\vartheta.$$

Looking at local solutions of L, one directly gets the following useful criterion to check whether an operator lies in $\mathbb{C}[z^2, \vartheta]$.

9.1.5 Lemma *An irreducible fuchsian differential operator $L \in \mathbb{C}[z, \vartheta]$ lies in $\mathbb{C}[z^2, \vartheta]$ if and only if $\iota_0(L)$ has a solution $f \in z^\mu \mathbb{C}[\![z^2]\!]^*$.*

9.1.6 Corollary *For each irreducible fuchsian differential operator $L \in \mathbb{C}[z, \vartheta]$, we have*

$$L \otimes (-z)^*(L) \in \mathbb{C}[z^2, \vartheta].$$

Proof As $\iota_0(L)$ has a solution $f = z^\mu \sum_{m=0}^{\infty} A_m z^m$, we get that

$$g = z^\mu \sum_{m=0}^{\infty} A_m z^m z^\mu \sum_{m=0}^{\infty} A_m (-z)^m = z^{2\mu} \sum_{m=0}^{\infty} \sum_{k=0}^{m} (-1)^k A_k A_{m-k} z^m \in z^{2\mu} \mathbb{C}[\![z^2]\!]$$

is a solution of $\iota_0(L \otimes (-z)^*(L))$. Therefore, the result follows from Lemma 9.1.5. □

The preceding Corollary justifies:

9.1.7 Definition For each irreducible fuchsian operator $L \in \mathbb{C}[z, \vartheta]$, we put

$$\Delta(L) := \sqrt{z}^* \left(L \otimes (-z)^*(L)\right) \in \mathbb{C}[z, \vartheta].$$

9.2 Characterization

We give a characterization of all m-tuples $T \in \mathrm{Sp}_4(\mathbb{C})^4$ of CY-type with $i_{\mathrm{Sp}_4(\mathbb{C})}(T) = 2$ via their Jordan forms.

9.2.1 Proposition *Consider $G = \mathrm{Sp}_4(\mathbb{C})$ and a CY-tuple of Jordan forms $J \in G^{r+1}$ with $i_G(J) = 2$. If the MDSP for J is solvable in G, we have $r \leq 3$. For $r = 3$, the tuple J coincides up to tensor product with m-tuples of rank one and permutation of its elements with one of the following:*

(i) $M_1(\alpha) = \left(J(4), J(2) \oplus J(1)^2, J(2) \oplus J(1)^2, [\alpha]^2\right)$ *for* $\alpha \neq 1$.

(ii) $M_2(\alpha) = \left(J(4), J(2) \oplus J(1)^2, J(2) \oplus J(1)^2, [\alpha] \oplus -J(1)^2\right)$ *for* $\alpha \neq 1$.

(iii) $M_3(\alpha) = \left(J(4), J(2) \oplus J(1)^2, J(1)^2 \oplus -J(1)^2, [\alpha] \oplus J(1)^2\right)$ *for* $\alpha \neq 1$.

(iv) $M_4(\alpha) = \left(J(4), J(2) \oplus J(1)^2, J(1)^2 \oplus -J(1)^2, [\alpha]^2\right)$.

(v) $M_5(\alpha) = \left(J(4), J(1)^2 \oplus -J(1)^2, J(1)^2 \oplus -J(1)^2, [\alpha] \oplus -J(1)^2\right)$.

Proof We assume without loss of generality that $J_1 = J(4)$ and put the other matrices into any order. As $i_G(J) = 2$ and $\delta_G(J(4)) = 8$, we have

$$\sum_{i=2}^{r+1} \delta_G(J_i) - 14 = 0.$$

By Table A.2, this yields $r \leq 3$ and $\delta_G(J_2) = \delta_G(J_3) = 4$ and $\delta_G(J_4) = 6$ if $r = 3$. In particular, we get that

$$J_2, J_3 \in \{\pm\left(J(2) \oplus J(1)^2\right), -J(1)^2 \oplus J(1)^2\}$$

and

$$J_4 \in \left\{[\alpha]^2, \pm\left([\alpha]_{\alpha\neq 1} \oplus J(1)^2\right)\right\}.$$

If $J_2 = J_3 = J(2) \oplus J(1)^2$, the cases for which $\gamma(J_4) = 2$ are excluded by Lemma 5.1.8. This leaves the possibilities $J_4 = [\alpha]^2$ and $J_4 = [\alpha] \oplus -J(1)^2$, where $\alpha \neq 1$.
If $J_2 = J(1)^2 \oplus -J(1)^2$ and $J_3 = J(2) \oplus J(1)^2$, the cases for which $\gamma(J_4) = 1$ are excluded.
If $J_2 = J_3 = J(1)^2 \oplus -J(1)^2$, we get that $\gamma\left(\bigwedge^2 J_1\right) + \gamma\left(-\bigwedge^2 J_2\right) + \gamma\left(-\bigwedge^2 J_3\right) = 6$ as we are looking for solutions in G. Therefore, the possibilities for which $\gamma\left(\bigwedge^2 J_4\right) \leq 3$ are ruled out by Lemma 5.1.8. This discussion leaves the possible tuples $M_1(\alpha) - M_5(\alpha)$ up to permutation of their matrices and tensor products with m-tuples of rank one. $\square$

Next, we show that the solvability of the MDSP for each of the tuples $M_1(\alpha) - M_5(\alpha)$ yields the solvability of the MDSP for a tuple of rank two.

9.2.2 Lemma *For each tuple $M_i(\alpha)$ introduced in Proposition 9.2.1, there is a tuple of Jordan matrices $\tilde{M}_i(\alpha)$ of rank two such that the MDSP for $\tilde{M}_i(\alpha)$ is solvable if the MDSP for $M_i(\alpha)$ is.*

Proof We discuss each of the families of CY-tuples separately.
For $M_1(\alpha)$ with $\alpha \neq 1$, we find that

$$\begin{aligned}\tilde{M}_1(\alpha) &= \mathrm{MC}_\alpha\left(\mathrm{MC}_{\alpha^{-1}}(M_1(\alpha)) \otimes K_1^4(\alpha)\right) \otimes K_4^1(\alpha)\\ &= (J(2), J(2), J(2), J(2))\end{aligned}$$

and therefore

$$M_1(\alpha) = \mathrm{MH}_{\alpha^{-1}}\left(\mathrm{MH}_\alpha\left(\tilde{M}_1(\alpha)\right)\right).$$

For $M_2(\alpha)$ with $\alpha \neq 1$, we find that

$$\mathrm{MC}_{-1}(M_2(\alpha)) \otimes K_1^4(-1) = \mathrm{Sym}^2(\tilde{M}_2(\alpha))$$

for

$$\tilde{M}_2(\alpha) = (J(2), J(1) \oplus -J(1), J(1) \oplus -J(1), [\beta]),$$

where $\beta^2 = \alpha$. Therefore, we get

$$M_2(\alpha) = \mathrm{MH}_{-1}\left(\mathrm{Sym}^2(\tilde{M}_2(\alpha))\right).$$

For $M_3(\alpha)$, we find

$$\begin{aligned}J' &= \mathrm{MC}_{-1}(M_3(\alpha) \otimes K_1^4(-1)) \otimes K_1^4(-1)\\ &= (J(3) \oplus -J(1), -J(1) \oplus J(1)^3, J(2)^2, [[-\alpha] \oplus J(1)] \oplus J(1)).\end{aligned}$$

Therefore, we get

$$\begin{aligned}J'' = \left(J'_{(2,4)}\right)^2 = (J(3) \oplus J(1), [[-\alpha] \oplus J(1)] \oplus J(1), J(2)^2,&\\ [[-\alpha] \oplus J(1)] \oplus J(1), J(2)^2, J(1)^4).&\end{aligned}$$

By Proposition 7.1.4, J'' lies in $\mathrm{PSO}_4(\mathbb{C})$ which is isomorphic to $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$ by taking tensor products, see e.g. [FH04, Chapter 18.2]. Therefore, we can write

$$J'' = \left(J(2) \otimes J(2), [\beta] \otimes [\beta], J(2) \otimes J(1)^2, [\beta] \otimes [\beta], J(2) \otimes J(1)^2, J(1)^2 \otimes J(1)^2\right)$$

where $\beta^2 = -\alpha$. In particular, the solvability of the MDSP for $M_3(\alpha)$ implies the solvability of the MDSP for $\tilde{M}_3(\alpha) = (J(2), [\beta], [\beta], J(2))$.
For $M_4(\alpha)$, we first obtain

$$\begin{aligned}\tilde{J} &= \mathrm{MC}_{-1}\left(\mathrm{MC}_\alpha\left(\mathrm{MC}_{\alpha^{-1}}(M_4(\alpha)) \otimes K_1^4(\alpha)\right) \otimes K_4^1(\alpha) \otimes K_3^4(-1)\right) \otimes K_1^4(-1)\\ &= (J(1) \oplus -J(1) \oplus \alpha J(1) \oplus \alpha^{-1}J(1), J(2)^2, -J(1) \oplus J(1)^3, -J(1)^2 \oplus J(1)^2).\end{aligned}$$

Again, by the isomorphism mentioned in [FH04, Chapter 18.2], we find

$$\left(\tilde{J}_{(3,4)}\right)^2 = \left([\alpha] \otimes [\alpha], J(2) \otimes J(1)^2, [i] \otimes [i], J(2) \otimes J(1)^2, [i] \otimes [i], J(1)^2 \otimes J(1)^2\right).$$

Hence the solvability of the MDSP for $M_4(\alpha)$ implies the solvability of the MDSP for $\tilde{M}_4(\alpha) = (J(2), [i], [i], [\alpha])$.

For $M_5(\alpha)$, we find

$$\begin{aligned}\tilde{J} &= \mathrm{MC}_{-1}\left(M_5(\alpha)\right) \otimes K_1^4(-1)\\ &= (J(3) \oplus -J(1)^2, J(2)^2 \oplus J(1), J(2)^2 \oplus J(1), [\alpha \oplus J(1)] \oplus J(1)^2).\end{aligned}$$

By Proposition 7.1.4, $\tilde{J}$ lies in $\mathrm{SO}_5(\mathbb{C})$ and hence can be written as the direct summand of $\bigwedge^2(J'')$ in $\mathrm{SO}_5(\mathbb{C})$, where

$$J'' = \left(iJ(2) \oplus -iJ(2), J(2) \oplus J(1)^2, J(2) \oplus J(1)^2, \beta J(1)^2 \oplus \beta^{-1} J(1)^2\right)$$

with $\beta^2 = \alpha$. Thus, the solvability of the MDSP for $M_5(\alpha)$ induces the solvability of the MDSP for

$$\begin{aligned}\tilde{M}_5(\alpha) &= \mathrm{MC}_{-i\beta}\left(\mathrm{MC}_{i\beta^{-1}}\left(J'' \otimes K_1^4(i)\right) \otimes K_1^4(-i\beta)\right) \otimes K_4^1(-i\beta)\\ &= (-J(2), -J(1) \oplus J(1), -J(1) \oplus J(1), J(2)).\end{aligned}$$

This discussion completes the proof of the Theorem. □

Note, that not all constructions we used in the proof above are invertible and hence it is not clear if the solvability of the MDSP for $\tilde{M}_i(\alpha)$ implies the solvability of the MDSP for $M_i(\alpha)$. In the next section, we prove that this is indeed true by choosing special solutions of the MDSP for the tuples $\tilde{M}_i(\alpha)$ in terms if differential operators.

9.3 Constructions

We first introduce some simplifying conventions and notations concerning differential operators of degree two.

9.3.1 Definition For a fuchsian differential operator $L \in \mathbb{C}[z, \vartheta]$ of degree two which has rational exponents $e_{1,s} \leq e_{2,s}$ at each singularity $s \in S$, we call

$$\lambda_s := e_{2,s} - e_{1,s}$$

the *signature* of the singularity s. With respect to an order on S, the tuple of signatures of all points in S is denoted by $\mathrm{sign}(L)$ and called the *signature* of L.

9.3.2 Example For a Picard-Fuchs operator related to a family of elliptic curves, a fiber of type

- I_b, $b > 0$, has signature 0.
- II has signature $\frac{1}{3}$.
- III has signature $\frac{1}{2}$.
- IV has signature $\frac{2}{3}$.

Furthermore, if $s \in \mathbb{C}$, we assume that the smallest exponent of $\iota_s(L)$ is zero. As by the Fuchs relation the equality

$$e_{1,\infty} + e_{2,\infty} = |S| - 2 - \sum_{s \in S \setminus \{\infty\}} \lambda_s$$

holds, all exponents of L then are determined by its signature. As we only want to consider operators for which the exponents at each of their singularities fix the Jordan form of the corresponding local monodromy, we assume that all entries in $\mathrm{sign}(L)$ are strictly smaller than one. A class of such operators are the following ones:

9.3.3 Definition For $h := (t,u,v,w,c,s) \in (\mathbb{C} \cap [0,1))^4\, \mathbb{C} \times \mathbb{C} \setminus \{0,1\}$, we call

$$\begin{aligned} R(h) &= 4s\vartheta(\vartheta - t) - 4z(\vartheta^2(s+1) + \vartheta(1 - v - t + s(1-u-t)) + c) \\ &\quad + z^2(2\vartheta + 2 - t - u - v + w)(2\vartheta + 2 - t - u - v - w) \end{aligned}$$

the associated *Heun operator*.

9.3.4 Remark. The Riemann scheme of $R(h)$ is given by

$$\mathcal{R}\left(R(h)\right) = \left\{ \begin{array}{cccc} 0 & 1 & s & \infty \\ \hline 0 & 0 & 0 & 1 - \frac{1}{2}(t+u+v+w) \\ t & u & v & 1 - \frac{1}{2}(t+u+v-w) \end{array} \right\}.$$

Its signature reads $\mathrm{sign}(R(h)) = (t,u,v,w)$.

We are now able to construct fuchsian differential operators whose monodromy tuples are solutions of the MDSP for $M_1(\alpha) - M_5(\alpha)$. We state these operators for each tuple in the following sequence of propositions separately. Again, most of the computations involved were carried out using the package `DEtools` in `MAPLE`. The construction for $M_1(\alpha)$ was already discussed in [AZ06, Section 7].

9.3.5 Proposition *For each choice of a point $s \in \mathbb{C} \setminus \{0,1\}$, a parameter $c \in \mathbb{C}$, a base star $\Gamma((0,1,s,\infty))$ and $a \in \mathbb{Q} \setminus \mathbb{Z}$ such that $\exp(2\pi i a) = \alpha$ holds, a solution of the MDSP for*

$$M_1(\alpha) = \left(J(4), J(2) \oplus J(1)^2, J(2) \oplus J(1)^2, [\alpha]^2\right)$$

is given by the monodromy tuple of

$$\begin{aligned} Q_1(c,s,a) &= \vartheta^4 s - z\,(\vartheta + a)\,(\vartheta + 1 - a)\left(\vartheta^2(1+s) + \vartheta(1+s) + c\right) \\ &\quad + z^2\,(\vartheta + 2 - a)\,(\vartheta + 1 - a)\,(\vartheta + 1 + a)\,(\vartheta + a)\,. \end{aligned}$$

Proof As we have seen in the proof of Proposition 9.2.1, we have

$$M_1(\alpha) = \mathrm{MH}_{\alpha^{-1}}\left(\mathrm{MH}_\alpha\left(\tilde{M}_1(\alpha)\right)\right),$$

where $\tilde{M}_1(\alpha) = (J(2), J(2), J(2), J(2))$. As a solution of the MDSP for $\tilde{M}_1(\alpha)$ is given by the monodromy tuple of $R(0,0,0,0,c,s)$, the monodromy tuple of the operator

$$Q_1(c,s,a) = R(0,0,0,0,c,s) \star_H I_a \star_H I_{1-a}$$

is a solution of the MDSP for $M_1(\alpha)$. Computing this operator explicitly gives the result. □

9.3.6 Proposition *For each choice of a point* $s \in \mathbb{C} \setminus \{0,1\}$, *a parameter* $c \in \mathbb{C}$, *a base star* $\Gamma((0,1,s,\infty))$ *and* $\lambda \in \mathbb{Q} \setminus \mathbb{Z}$ *such that* $\exp(2\pi i \lambda) = \alpha$ *holds, a solution of the MDSP for*

$$M_2(\alpha) = \left(J(4), J(2) \oplus J(1)^2, J(2) \oplus J(1)^2, [\alpha] \oplus -J(1)^2\right)$$

is given by the monodromy tuple of

$$\begin{aligned} Q_2(c,s,\lambda) = {} & 4\,\vartheta^4 s - z\,(2\,\vartheta+1)^2\left(\vartheta^2 + \vartheta^2 s + \vartheta + \vartheta\, s + 4\,c\right) \\ & + z^2\,(2\,\vartheta+3)\,(2\,\vartheta+1)\,(\vartheta+1+\lambda)\,(\vartheta+1-\lambda) \end{aligned}$$

Proof As we have seen in the proof of Proposition 9.2.1, we have

$$(M_2(\alpha), S) = \mathrm{MH}_{-1}\left(\mathrm{Sym}^2(\tilde{M}_2(\alpha))\right),$$

where

$$\tilde{M}_2(\alpha) = (J(2), J(1) \oplus -J(1), J(1) \oplus -J(1), [\beta])$$

with $\beta^2 = \alpha$. As a solution of the MDSP for $\tilde{M}_2(\alpha)$ is given by the monodromy tuple of the operator $R\left(0, \frac{1}{2}, \frac{1}{2}, \lambda, c, s\right)$, the monodromy tuple of the operator

$$Q_2(c,s,\lambda) = \mathrm{Sym}^2\left(R\left(0, \frac{1}{2}, \frac{1}{2}, \lambda, c, s\right)\right) \star_H I_{\frac{1}{2}}$$

is a solution of the MDSP for $M_2(\alpha)$. Computing this operator explicitly gives the result. □

9.3.7 Proposition *For each choice of a point* $s \in \mathbb{C} \setminus \{0, \pm 1\}$, *a parameter* $c \in \mathbb{C}$, *a base star* $\Gamma\left((0,1,-(s-1)^2/4s,\infty)\right)$ *and* $\lambda \in \mathbb{Q} \setminus \mathbb{Z}$ *such that* $\exp(2\pi i\lambda) = \alpha$ *holds, a solution of the MDSP for*

$$M_3(\alpha) = (J(4), J(2) \oplus J(1)^2, J(1)^2 \oplus -J(1)^2, [\alpha] \oplus J(1)^2)$$

is given by

$$\begin{aligned} Q_3(c,s,\lambda) = {} & \vartheta^4(s-1)^4 - (s-1)^2 z(\vartheta^4(s^2-10s+1) + 2\vartheta^3(s^2-10s+1)) \\ & - (s-1)^2 z(\vartheta^2(s^2(\lambda-\lambda^2+1) + 2s(\lambda^2+\lambda+c-12) + 1 + \lambda - \lambda^2 + 2c)) \\ & - (s-1)^2 z(\vartheta(s^2\lambda(1-\lambda) + 2s(\lambda^2+\lambda+c-7) + \lambda(1-\lambda) + 2c)) \\ & - (s-1)^2 z(c^2 - 3s + \lambda(cs + c + s\lambda)) \\ & - 4sz^2(\vartheta+1)^2(\vartheta^2(2s^2-8s+2) + 2\vartheta(2s^2-8s+2)) \\ & - 4sz^2(\vartheta+1)^2(s^2(3+\lambda-2\lambda^2) + s(4\lambda^2+2\lambda+2c-13) + 3 + \lambda + 2c - 2\lambda^2) \\ & - 4s^2z^3(\vartheta+1)(\vartheta+2)(2\vartheta-2\lambda+3)(2\vartheta+2\lambda+3) \end{aligned}$$

Proof We mimic a construction inverse to the one done in Proposition 9.2.1 on the level of differential operators. Consider the operator

$$P_1 = \left(\left(\frac{2zs}{z+1+s(z-1)}\right)^* (R(0,\lambda,\lambda,0))\right) \otimes (z^2-1)^{-\lambda/2}((s+1)z+1-s)^{\lambda-1}.$$

Its Riemann scheme reads

$$\mathcal{R}(P_1) = \left\{ \begin{array}{ccccc} 0 & -1 & 1 & (s-1)/(s+1) & \infty \\ \hline 0 & -\frac{\lambda}{2} & -\frac{\lambda}{2} & 0 & 1 \\ 0 & \frac{\lambda}{2} & \frac{\lambda}{2} & 0 & 2 \end{array} \right\}.$$

Moreover, for each choice of a base star $\Gamma(0,-1,1,(s-1)/(s+1),\infty)$, its monodromy tuple is a solution of the MDSP for $(J(2),[\beta],[\beta],J(2),J(1)^2)$. The latter tuple is precisely $\tilde{M}_3(\alpha)$ extended by the identity matrix. A direct computation shows that the operator $\Delta(P_1)$ has Riemann scheme

$$\mathcal{R}(\Delta(P_1)) = \left\{ \begin{array}{ccccc} 0 & 1 & (s-1)^2/(s+1)^2 & \tilde{s} & \infty \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & \frac{3}{2} \\ 0 & -\lambda & 1 & 2 & 2 \\ \frac{1}{2} & \lambda & 1 & 4 & 3 \end{array} \right\},$$

where $\tilde{s} \in \mathbb{C}$ is an apparent singularity. Furthermore, the Jordan forms of the monodromy tuple of $\Delta(P_1)$ are given by

$$(J(3) \oplus -J(1), [[-\alpha] \oplus J(1)] \oplus J(1), J(2)^2, J(1)^4, -J(1) \oplus J(1)^3)$$

for each choice of a base star $\Gamma((0,1,(s-1)^2/(s+1)^2,\tilde{s},\infty))$. A direct computation further shows that $\Delta(P_1) \star_H I_{\frac{1}{2}}$ admits a right factor P_2 with singular locus $\{0,1,(s-1)^2/(s+1)^2,\infty\}$ whose monodromy tuple for each choice of a base star $\Gamma((0,1,(s-1)^2/(s+1)^2,\infty))$ is a solution of the MDSP for

$$(J(4), [\alpha]^2 \oplus J(1)^2, J(1)^2 \oplus -J(1)^2, J(2) \oplus J(1)^2) = M_3(\alpha)_{(2,4)}.$$

Hence, the desired differential operator is given by

$$Q_3(\lambda, c, s) = \left(\frac{z}{z-1}\right)^* (P_2).$$

Carrying out all computations explicitly gives the result. □

9.3.8 Proposition *For each choice of a point $s \in \mathbb{C} \setminus \{0, \pm 1\}$, a parameter $c \in \mathbb{C}$, a base star $\Gamma\left(0,1,-(s-1)^2/4s,\infty\right)$ and $\lambda \in \mathbb{Q} \setminus \mathbb{Z}$ such that $\exp(\pi i \lambda) = \alpha$ holds, a solution of the MDSP for*

$$M_4(\alpha) = \left(J(4), J(2) \oplus J(1)^2, J(1)^2 \oplus -J(1)^2, [\alpha]^2\right)$$

is given by the monodromy tuple of

$$\begin{aligned}
Q_4(c,s,\lambda) &= 4\,\vartheta^4\,(s-1)^4 - z(s-1)^2(4\vartheta^4(s^2-10s+1))\\
&\quad - z(s-1)^2(8\vartheta^3(s^2-10s+1)) + \vartheta^2(5s^2+s(8c+8\lambda^2-90)+8c+5))\\
&\quad - z(s-1)^2(\vartheta(s^2+s(8c+8\lambda^2-50)+8c+1))\\
&\quad - z(s-1)^2(s(3\lambda^2+2c-11)+2c+4c^2)\\
&\quad - 4sz^2(2\vartheta+2+\lambda)(2\vartheta+2-\lambda)(\vartheta^2(2s^2-8s+2))\\
&\quad - 4sz^2(2\vartheta+2+\lambda)(2\vartheta+2-\lambda)(\vartheta(4s^2-16s+4))\\
&\quad - 4sz^2(2\vartheta+2+\lambda)(2\vartheta+2-\lambda)(3s^2+s(\lambda^2-11+2c)+3+2c))\\
&\quad - 4s^2z^3(2\vartheta+2-\lambda)(2\vartheta+2+\lambda)(2\vartheta+4-\lambda)(2\vartheta+4+\lambda).
\end{aligned}$$

Proof We mimic a construction inverse to the one done in Proposition 9.2.1 on the level of differential operators. Consider the operator

$$P_1 = \left(\left(\frac{(s+1)z+1-s}{2z}\right)^*\left(R\left(0,\frac{1}{2},\frac{1}{2},\lambda\right)\right)\right)\otimes z^{-1/2}(z^2-1)^{1/4}.$$

Its Riemann scheme reads

$$\mathcal{R}(P_1) = \left\{\begin{array}{cccc} 0 & -1 & 1 & (s-1)/(s+1)\\ \hline -\frac{\lambda}{2} & \frac{1}{4} & \frac{1}{4} & 0\\ \frac{\lambda}{2} & \frac{3}{4} & \frac{3}{4} & 0 \end{array}\right\}.$$

Moreover, for each choice of a base star $\Gamma((s-1)/(s+1),-1,1,0,\infty)$, its monodromy tuple is a solution of the MDSP for $(J(2),[i],[i],[\alpha],J(1)^2)$. The latter tuple is precisely $\tilde{M}_4(\alpha)$ extended by the identity matrix. A direct computation shows that the operator $\Delta(P_1)$ has Riemann scheme

$$\mathcal{R}(\Delta(P_1)) = \left\{\begin{array}{ccccc} 0 & 1 & (s-1)/(s+1)^2 & \tilde{s} & \infty\\ \hline 0 & \frac{1}{2} & 0 & 0 & 0\\ \frac{1}{2} & 1 & 0 & 1 & \frac{1}{2}\\ -\frac{\lambda}{2} & \frac{3}{2} & 1 & 2 & 1\\ \frac{\lambda}{2} & 2 & 1 & 4 & 2 \end{array}\right\},$$

where $\tilde{s}\in\mathbb{C}$ is an apparent singularity. The operator $\left(\frac{z}{z-1}\right)^*(\Delta(P_1))\star_H I_{\frac{1}{2}}$ has an irreducible right factor P_2 whose Riemann scheme reads

$$\mathcal{R}(P_2) = \left\{\begin{array}{cccc} 0 & 1 & -(s-1)^2/4s & \infty\\ \hline 0 & 0 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{1}{2} & \frac{1}{2}\\ -\frac{\lambda}{2} & 1 & \frac{3}{2} & \frac{3}{2}\\ \frac{\lambda}{2} & 2 & 2 & \frac{3}{2} \end{array}\right\}.$$

Hence the desired differential operator reads

$$\left(P_2 \otimes \left(4sz + (s-1)^2\right)^{-1/2}\right) \star_H I_{1+\frac{\lambda}{2}} \star_H I_{1-\frac{\lambda}{2}} =: Q_4(c, s, \lambda).$$

Carrying out all computation explicitly gives the result. □

9.3.9 Proposition *For each choice of a point $s \in \mathbb{C} \setminus \{0,1\}$, a parameter $c \in \mathbb{C}$, a base star $\Gamma((0,1,s,\infty))$ and $a \in \mathbb{Q} \setminus \mathbb{Z}$ such that $\alpha = -\exp(2\pi i a)$ holds, a solution of the MDSP for*

$$M_5(\alpha) = \left(J(4), J(1)^2 \oplus -J(1)^2, J(1)^2 \oplus -J(1)^2, [\alpha] \oplus -J(1)^2\right)$$

is given by

$$\begin{aligned}
Q_5(c,s,a) &= 16s^2\vartheta^4 - 4sz(8\vartheta^4(s+1) + 16\vartheta^3(s+1)) \\
&- 4sz(2\vartheta^2(s(9 + a(1-a)) + 9 + a(1-a) + 4c)) \\
&- 4sz(2\vartheta(s(5 + a(1-a)) + 5 + a(1-a) + 4c)) \\
&- 4sz(s(2 - a^2 + a) + 4c(a^2 - a + 1) + a + 2 - a^2) \\
&+ z^2(16\vartheta^4(s^2 + 4s + 1) + 64\vartheta^3(s^2 + 4s + 1)) \\
&+ z^2(\vartheta^2(4s^2(2a - 2a^2 + 23) + 32s(c + a(1-a) + 15) + 4(2a(1-a) + 8c + 23))) \\
&+ z^2(16\vartheta(s^2(a(1-a) + 4) + 64s(c + a(1-a) + 7) + 8(2a(1-a) + 7 + 8c))) \\
&+ z^2(s^2(1+a)(2+a)(2-a)(3-a)) \\
&- 2sz^2(a^4 - 2a^3 + a^2(21 - 4c) + a(4c - 20) - 84 - 16c)) \\
&+ z^2((a^2 + 3a + 2 + 4c)(a^2 - 5a + 6 + 4c)) \\
&- 2z^3(2\vartheta + 3)^2(4\vartheta^2(s+1) + 12\vartheta(s+1)) \\
&- 2z^3(2\vartheta + 3)^2(s(11 + 3a(1-a)) + 3a(1-a) + 11 + 4c) \\
&+ z^4(2\vartheta + 3)(2\vartheta + 5)(2\vartheta + 5 - 2a)(2\vartheta + 3 + 2a).
\end{aligned}$$

Proof As we have seen in the proof of Proposition 9.2.1, we have

$$M_5(\alpha) = \mathrm{MH}_{-1}\left(\bigwedge^2 \left(\mathrm{MH}_{i\beta^{-1}}\left(\mathrm{MH}_{-i\beta}(\tilde{M}_5(\alpha))\right) \otimes K_4^1(i)\right)|_{\mathrm{SO}_5(\mathbb{C})}\right),$$

with

$$\tilde{M}_5(\alpha) = ((-J(2), -J(1) \oplus J(1), -J(1) \oplus J(1), J(2))$$

and $\beta^2 = -\alpha$. For each choice of a base star $\Gamma((0,1,s,\infty))$, the monodromy tuple of the operator $R(0, \frac{1}{2}, \frac{1}{2}, 0, c, s) \otimes z^{-1/2}$ is a solution of the MDSP for $\tilde{M}_5(\alpha)$. Therefore, a solution of the MDSP for $M_5(\alpha)$ is given by the monodromy tuple of the operator

$$\bigwedge^2 \left(\left(\left(R\left(0, \frac{1}{2}, \frac{1}{2}, 0, c, s\right) \otimes z^{-\frac{1}{2}}\right) \star_H I_{\frac{5}{4} - \left(\frac{1}{4} + \frac{a}{2}\right)} \star_H I_{\frac{1}{4} + \left(\frac{1}{4} + \frac{a}{2}\right)}\right) \otimes z^{\frac{3}{4}}\right) \star_H I_{\frac{3}{2}},$$

which we call $Q_5(c, s, a)$. □

We summarize all the propositions of this section.

9.3.10 Theorem *Let $G = \mathrm{Sp}_4(\mathbb{C})$ and $J \in \mathrm{Sp}_4(\mathbb{C})^4$ be a CY-tuple of Jordan matrices with $i_G(J) = 2$. The MDSP for J is solvable if and only if J coincides up to permutation of its elements and tensor products with m-tuples of rank one with one of the tuples $M_1(\alpha) - M_5(\alpha)$ introduced in Proposition 9.2.1.*

9.4 Comparison to earlier results

We try to find those operators in families $Q_1 - Q_5$ for which additionally properties (N), (Q) and (Q+) are fulfilled. First observe that all of these properties depend on the choice of s and c and that none of them seems to hold for an arbitrary choice. As the space of possible (s, c)-values is $\mathbb{C} \setminus \{0, \pm 1\} \times \mathbb{C}$, a straight forward search would be very elaborate. Another option is to guarantee that properties (N) and (Q) hold for the Heun operator $R(h)$ we start with. Indeed, by Lemma 3.2.9 and Corollary 6.2.5, we see that property (N) is kept by all constructions we use. We try to find as much Heun operators as possible which are algebraic pullbacks of operators of CY-type, as for those property (Q) holds by Lemma 3.5.2. A complete list of possible pullbacks is stated in [VF12].

In the sequel, we give a list of differential operators of degree two which are pullbacks of hypergeometric ones and have an appropriate signature. The coefficients of their q-coordinates are minimal over $\mathbb{Z}$, which we have achieved after a transformation $z \mapsto cz$. We also state corresponding parameters s, c, which are not unique. Note, that the cross-ratio of the singularities is invariant under the transformation $z \mapsto \lambda z$. For those which are Picard-Fuchs operators of one-parameter families of relatively minimal elliptic curves over $\mathbb{Q}$ with section, we also state the related families in their Weierstraß- form $Y^2 = 4X^3 - g_2(z)X - g_3(z)$ which we have computed using the results of [Her91].

For $\mathrm{sign}(R(h)) = (0, 0, 0, 0)$, we find the following six operators.

Number	**Operator**	**(s,c)**
1	$\vartheta^2 - z\left(7\,\vartheta^2 + 7\,\vartheta + 2\right) - 8\,z^2\,(\vartheta+1)^2$	$\left(-\frac{1}{8}, \frac{1}{4}\right)$
2	$\vartheta^2 - z\left(11\,\vartheta^2 + 11\,\vartheta + 3\right) - z^2\,(\vartheta+1)^2$	$\left(\frac{55\sqrt{5}-123}{2}, \frac{15\sqrt{5}-33}{2}\right)$
3	$\vartheta^2 - z\left(10\,\vartheta^2 + 10\,\vartheta + 3\right) + 9\,z^2\,(\vartheta+1)^2$	$\left(\frac{1}{9}, \frac{1}{3}\right)$
4	$\vartheta^2 - 4z\left(3\,\vartheta^2 + 3\,\vartheta + 1\right) + 32\,z^2\,(\vartheta+1)^2$	$(2, 1)$
5	$\vartheta^2 - 3z\left(3\,\vartheta^2 + 3\,\vartheta + 1\right) + 27\,z^2\,(\vartheta+1)^2$	$\left(\frac{1+i\sqrt{3}}{2}, \frac{1+i\sqrt{3}}{6}\right)$
6	$\vartheta^2 - z\left(17\,\vartheta^2 + 17\,\vartheta + 6\right) + 72\,z^2\,(\vartheta+1)^2$	$\left(\frac{9}{8}, \frac{3}{4}\right)$

Up to a tensor product with an operator of degree one, all of them are Picard-Fuchs operators for the following families of elliptic curves:

Number		**Families**
1	g_2	$3\,(4\,z+1)\left(64\,z^3 + 48\,z^2 - 12\,z + 1\right)$
	g_3	$\left(8\,z^2 + 4\,z - 1\right)\left(512\,z^4 + 512\,z^3 + 8\,z - 1\right)$
2	g_2	$3(z^4 - 12z^3 + 14z^2 + 12z + 1)$
	g_3	$z^6 - 18\,z^5 + 75\,z^4 + 75\,z^2 + 18\,z + 1$
3	g_2	$3\,(3\,z+1)\left(243\,z^3 + 243\,z^2 + 9\,z + 1\right)$
	g_3	$\left(27\,z^2 + 18\,z - 1\right)\left(729\,z^4 + 972\,z^3 + 270\,z^2 + 36\,z + 1\right)$
4	g_2	$3(65536\,z^4 - 32768\,z^3 + 5120\,z^2 - 256\,z + 1)$
	g_3	$\left(128\,z^2 - 32\,z + 1\right)\left(131072\,z^4 - 65536\,z^3 + 10240\,z^2 - 512\,z - 1\right)$
5	g_2	$3\,(9\,z-1)\left(6561\,z^3 - 2187\,z^2 + 243\,z - 1\right)$
	g_3	$14348907\,z^6 - 9565938\,z^5 + 2657205\,z^4 - 367416\,z^3 + 24057\,z^2 - 486\,z - 1$

6	g_2	$108\,(12\,z-1)\left(15552\,z^3-3888\,z^2+252\,z-1\right)$
	g_3	$216\,\left(216\,z^2-36\,z+1\right)\left(373248\,z^4-124416\,z^3+13824\,z^2-504\,z-1\right)$

Hence all operators $1-6$ are up to a tensor product rational pullbacks of the hypergeometric differential operator $I_{\frac{1}{12}} \star_H I_{\frac{5}{12}}$ by Theorem 4.2.1.

For $\mathrm{sign}(L) = (0, \lambda, \lambda, 0)$ we only found suitable operators if $\lambda \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$.

Amongst those, the operators with signature $(0, \frac{1}{2}, \frac{1}{2}, 0)$ can be constructed as follows: Consider an operator L with $\mathrm{sign}(L) = (0,0,0,0)$ whose singularities are given by $\{0, \alpha_1, \alpha_2, \infty\}$, where α_1, α_2 are roots of a polynomial $p(X) = aX^2 + bX + 1 \in \mathbb{Z}[X]$. Then

$$L' = \left((-z/p(z))^\vee\right)^* L \in \mathbb{Q}[z, \vartheta]$$

has four singularities $\{0, \beta_1, \beta_2, \infty\}$ and $\mathrm{sign}(L') = (0, \frac{1}{2}, \frac{1}{2}, 0)$ holds. Moreover, L' is of CY-type, if L is.

For $\lambda \in \{\frac{1}{3}, \frac{1}{4}\}$, we also obtain two operators of signature $(0, \lambda, \lambda, 0)$. All in all we get:

Nr., λ	**Operator**	**(s,c)**
$1'$, $\frac{1}{2}$	$4\,\vartheta^2 - 2z\left(28\,\vartheta^2+14\,\vartheta+3\right)+81\,z^2\,(2\,\vartheta+1)^2$	$\left(\frac{8-7i\sqrt{2}}{16}, \frac{4i\sqrt{2}+7}{54}\right)$
$2'$, $\frac{1}{2}$	$4\,\vartheta^2 - 2z\left(44\,\vartheta^2+22\,\vartheta+5\right)+125\,z^2\,(2\,\vartheta+1)^2$	$\left(\frac{117+44i}{125}, \frac{11+2i}{50}\right)$
$3'$, $\frac{1}{2}$	$\vartheta^2 - 2z\left(10\,\vartheta^2+5\,\vartheta+1\right)+16\,z^2\,(2\,\vartheta+1)^2$	$\left(4, \frac{1}{2}\right)$
$4'$, $\frac{1}{2}$	$\vartheta^2 - 2z(12\vartheta^2+6\vartheta+1)+4z^2(2\vartheta+1)^2$	$\left(17-12\sqrt{12}, \frac{3}{2}-\sqrt{2}\right)$
$5'$, $\frac{1}{2}$	$4\,\vartheta^2 + 6z\left(12\,\vartheta^2+6\,\vartheta+1\right)-27\,z^2\,(2\,\vartheta+1)^2$	$\left(-7+4\sqrt{3}, \frac{2\sqrt{3}-3}{6}\right)$
$6'$, $\frac{1}{2}$	$4\,\vartheta^2 - 2z\left(68\,\vartheta^2+34\,\vartheta+5\right)+z^2\,(2\,\vartheta+1)^2$	$\left(577-408\sqrt{2}, \frac{85}{2}-30\sqrt{2}\right)$
7, $\frac{1}{3}$	$9\,\vartheta^2 - 3z\left(39\,\vartheta^2+26\,\vartheta+7\right)+49\,z^2\,(3\,\vartheta+2)^2$	$\left(\frac{71}{98}+\frac{39}{98}i\sqrt{3}, \frac{13+3i\sqrt{3}}{42}\right)$
8, $\frac{1}{3}$	$9\,\vartheta^2 + 12z\left(15\,\vartheta^2+10\,\vartheta+2\right)-8\,z^2\,(3\,\vartheta+2)^2$	$\left(-26-15\sqrt{3}, -\frac{10}{3}-2\sqrt{3}\right)$
9, $\frac{1}{4}$	$16\vartheta^2 - 4z(24\vartheta^2+18\vartheta+5)+25\,z^2\,(4\,\vartheta+3)^2$	$\left(\frac{24i-7}{25}i, \frac{4i+3}{20}\right)$
10, $\frac{1}{4}$	$16\vartheta^2 - 4z(56\vartheta^2+42\vartheta+9)+\,z^2\,(4\,\vartheta+3)^2$	$\left(97+56\sqrt{3}, \frac{63}{4}+9\sqrt{3}\right)$

Up to a tensor product with an operator of degree one, we get the following Picard-Fuchs operators for families of elliptic curves.

Number		**Families**
7	g_2	$147\,\left(196\,z^2-26\,z+1\right)\left(9604\,z^2-490\,z+1\right)$
	g_3	$343\,\left(196\,z^2-26\,z+1\right)\left(13176688\,z^4-1882384\,z^3+86436\,z^2-980\,z-1\right)$
8	g_2	$3(64z^4-192z^3+64z^2+24z+1)$
	g_3	$512\,z^6-2304\,z^5+2496\,z^4+312\,z^2+36\,z+1$
$2'$	g_2	$3\,\left(500\,z^2+44\,z+1\right)\left(20\,z^2+20\,z+1\right)$
	g_3	$-\left(500\,z^2+44\,z+1\right)^2\left(4\,z^2-8\,z-1\right)$
$5'$	g_2	$\left(81\,z^2-54\,z-3\right)(9\,z+1)(3\,z-1)$
	g_3	$-\left(729\,z^4-972\,z^3+270\,z^2+36\,z+1\right)\left(27\,z^2+1\right)$

By Theorem 4.2.1 and the discussion done before this yields a realization of the operators $1' - 6'$, 7 and 8 as twists of the hypergeometric operator $I_{\frac{1}{12}} \star_H I_{\frac{5}{12}}$ by an algebraic function followed by a tensor product with an operator of degree one. Up to a tensor product with an operator of degree one, we get operator number 9 as twist of $I_{\frac{3}{8}} \star I_{\frac{1}{8}}$ by

$$\varphi(z) = -\frac{256z^5}{(z-1)^4(25z^2-6z+1)}$$

and operator number 10 as twist of $I_{\frac{3}{8}} \star I_{\frac{1}{8}}$ by

$$\psi(z) = -\frac{6912z^3}{(3z-1)^4(9z^2-42z+1)}.$$

In the case $\operatorname{sign}(L) = \left(0, \frac{1}{2}, \frac{1}{2}, \lambda\right)$, we only found suitable operators if $\lambda \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$. All operators of this type can be constructed from those having signature $(0, \lambda, \lambda, 0)$ completely similar to the construction of those with signature $(0, \frac{1}{2}, \frac{1}{2}, 0)$ from those with signature $(0, 0, 0, 0)$. By this approach, we find:

Nr., λ	**Operator**	**(s,c)**
$1''$, $\frac{1}{2}$	$\vartheta^2 + 2z\left(14\vartheta^2 + 7\vartheta + 1\right) - 8z^2(4\vartheta+1)(4\vartheta+3)$	$\left(-8, -\frac{1}{2}\right)$
$2''$, $\frac{1}{2}$	$\vartheta^2 + z\left(44\vartheta^2 + 22\vartheta + 3\right) - z^2(4\vartheta+1)(4\vartheta+3)$	$\left(\frac{55\sqrt{5}-123}{2}, -\frac{33+15\sqrt{5}}{8}\right)$
$3''$, $\frac{1}{2}$	$\vartheta^2 - z\left(40\vartheta^2 + 20\vartheta + 3\right) + 9z^2(4\vartheta+1)(4\vartheta+3)$	$\left(9, \frac{3}{4}\right)$
$4''$, $\frac{1}{2}$	$\vartheta^2 - 4z\left(12\vartheta^2 + 6\vartheta + 1\right) + 32z^2(4\vartheta+1)(4\vartheta+3)$	$\left(2, \frac{1}{4}\right)$
$5''$, $\frac{1}{2}$	$\vartheta^2 + 3z\left(12\vartheta^2 + 6\vartheta + 1\right) + 27z^2(4\vartheta+1)(4\vartheta+3)$	$\left(\frac{1-i\sqrt{3}}{2}, \frac{3-i\sqrt{3}}{24}\right)$
$6''$, $\frac{1}{2}$	$\vartheta^2 - 2z\left(34\vartheta^2 + 17\vartheta + 3\right) + 72z^2(4\vartheta+1)(4\vartheta+3)$	$\left(\frac{9}{8}, \frac{3}{16}\right)$
$7'$, $\frac{1}{3}$	$\vartheta^2 + z\left(26\vartheta^2 + 13\vartheta + 2\right) - 3z^2(3\vartheta+1)(3\vartheta+2)$	$\left(-\frac{1}{27}, \frac{2}{27}\right)$
$8'$, $\frac{1}{3}$	$\vartheta^2 - 4z\left(10\vartheta^2 + 5\vartheta + 1\right) + 48z^2(3\vartheta+1)(3\vartheta+2)$	$\left(\frac{23+10i\sqrt{2}}{27}, \frac{5+i\sqrt{2}}{27}\right)$
$9'$, $\frac{1}{4}$	$\vartheta^2 - z\left(12\vartheta^2 + 6\vartheta + 1\right) - z^2(8\vartheta+3)(8\vartheta+5)$	$\left(-4, -\frac{1}{4}\right)$
$10'$, $\frac{1}{4}$	$\vartheta^2 - z\left(28\vartheta^2 + 3\vartheta + 3\right) + 12z^2(8\vartheta+3)(8\vartheta+5)$	$\left(\frac{4}{3}, \frac{1}{4}\right)$

Up to a tensor product with an operator of degree one, only operator $7'$ is a Picard-Fuchs operator for a family of elliptic curves. This family is determined by

$$\begin{aligned} g_2 &= -3\left(432z^2 - 40z + 1\right)(16z-1) \\ g_3 &= \left(432z^2 - 40z + 1\right)^2(4z-1). \end{aligned}$$

By the preceding discussion, each operator $1'' - 6''$ and $7' - 10'$ can be realized as twist of a hypergeometric operator by an algebraic function followed by a tensor product with an operator of degree one.

It seems that the operators of CY-type in families $Q_1 - Q_5$ are exactly those operators we get if we start with one of the Heun operators mentioned above. The resulting operators are stated in Section A.2.6.

Appendix A

Tables

A.1 Matrices

Partition of n	**Jordan matrices**
$(1,\dots,1)$	$[\alpha_1 \bigoplus \cdots \bigoplus \alpha_n]$
$(n-1,1)$	$[\alpha_1 \bigoplus \alpha_2] \bigoplus \alpha_1 J(1)^{n-2}$
(k,k)	$[\alpha_1 \bigoplus \alpha_2]^k$
$(k,k-1,1)$	$[\alpha_1 \bigoplus \alpha_2 \bigoplus \alpha_3] \bigoplus [\alpha_1 \bigoplus \alpha_2]^{k-2} \bigoplus \alpha_1$
$(k+1,k)$	$[\alpha_1 \bigoplus \alpha_2]^k \bigoplus \alpha_1$
$(k,k,1)$	$[\alpha_1 \bigoplus \alpha_2 \bigoplus \alpha_3] \bigoplus [\alpha_1 \bigoplus \alpha_2]^{k-1}$
$(4,2)$	$[\alpha_1 \bigoplus \alpha_2]^2 \bigoplus \alpha_1 J(1)^2$
$(2,2,2)$	$[\alpha_1 \bigoplus \alpha_2 \bigoplus \alpha_3]^2$

Table A.1: Tuples of partitions and Jordan forms.

$\delta_{\mathrm{Sp}_4(\mathbb{C})}$	**Jordan form in** $\mathrm{Sp}_4(\mathbb{C})$	**Jordan form in** $\mathrm{SO}_5(\mathbb{C})$
4	$\pm(J(2) \oplus J(1)^2)$	$J(2) \oplus J(2) \oplus J(1)$
	$-J(1)^2 \oplus J(1)^2$	$-J(1)^4 \oplus J(1)$
6	$[\alpha]^2$	$\big[[\alpha^2] \oplus J(1)\big] \oplus J(1)^2$
	$\pm([\alpha] \oplus J(1)^2)_{\alpha\neq 1}$	$[\alpha]^2 \oplus J(1)$
8	$[[\alpha] \oplus [\beta]]_{\alpha\neq\beta^{\pm 1}}$	$[[\alpha\beta] \oplus [\alpha\beta^{-1}] \oplus J(1)]$
	$[[\alpha] \oplus [\alpha]]$	$J(3) \oplus \alpha^2 \oplus \alpha^{-2}$

Table A.2: Jordan forms in $\mathrm{Sp}_4(\mathbb{C})$ and $\mathrm{SO}_5(\mathbb{C})$.

A.2 Operators

This section of the Appendix provides additional data for the families of differential operators we constructed in the last three chapters. For the families of differential operators which have a linearly rigid or a symplectically rigid monodromy tuple, we state their coefficients if known, their Riemann scheme, an N-integral holomorphic solution at $z = 0$ and further solutions at $z = 1$ if possible. For the families of differential operators which have a monodromy tuple of rigidity index two in $\mathrm{Sp}_4(\mathbb{C})$, we state their coefficients, their Riemann scheme and a table of resulting operators of CY-type, which are transformed in such a way that the coefficients of their q-coordinates are minimal over $\mathbb{Z}$. In these tables, the entry *number* refers to the number of the corresponding Heun operator stated in Section 9.4. If the corresponding CY-type operator of degree four appears in [AESZ05], the entry *source* provides its number given there. If it is equivalent to an operator stated in [AESZ05], we indicate that by the symbol $\sim$. If it doesn't appear in the original version of [AESZ05], we write $-$. Nevertheless, these operators appear as numbers $386 - 393$ in the updated version.

A.2.1 Hypergeometric

$$P^{hyp}(a_1,\ldots,a_k,2k) = \vartheta^n - z(\vartheta+a_1)(\vartheta+1-a_1)\cdots(\vartheta+a_k)(\vartheta+1-a_k)$$

- Riemann scheme:

$$\left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & a_1 \\ 0 & 1 & 1-a_1 \\ \vdots & \vdots & \vdots \\ 0 & 2(k-1) & a_k \\ 0 & k-1 & 1-a_k \end{array}\right\}$$

- Special solution at $z = 0$: $f(z) = \sum_{m=0}^{\infty} A_m z^m$ with

$$A_m = \binom{a_1+m-1}{m}\binom{m-a_1}{m}\cdots\binom{a_k+m-1}{m}\binom{m-a_k}{m}$$

- Special solution at $z = 1$: $g(z+1) = z^{k-1}\sum_{m=0}^{\infty} B_m^{(k)} z^m$ with

$$B_m^{(k)} = \sum_{i\le l\le m} (-1)^{m-l} B_i^{(k-1)} \mathrm{B}(l-i+1-a_k,a_k)\mathrm{B}(k-1+l,1-a_k)$$
$$\mathrm{B}(m-l+a_k,1-a_k)\mathrm{B}(k-a_k+m,a_k)$$

starting with

$$B_m^{(1)} = (-1)^m \mathrm{B}(m+a_1,1-a_1)\mathrm{B}(m+1-a_1,a_1).$$

$$P^{hyp}(a_1,\dots,a_k,2k+1)=\vartheta^n - z(\vartheta+a_1)(\vartheta+1-a_1)\cdots(\vartheta+a_k)(\vartheta+1-a_k)\left(\vartheta+\frac{1}{2}\right)$$

- Riemann scheme:

$$\left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & a_1 \\ 0 & 1 & 1-a_1 \\ \vdots & \vdots & \vdots \\ 0 & 2k-2 & a_k \\ 0 & 2k-1 & 1-a_k \\ 0 & \frac{2k-1}{2} & \frac{1}{2} \end{array}\right\}$$

- Special solution at $z=0$: $f(z)=\sum_{m=0}^{\infty} A_m z^m$ with

$$A_m=\binom{2n}{n}\binom{a_1+m-1}{m}\binom{m-a_1}{m}\cdots\binom{a_k+m-1}{m}\binom{m-a_k}{m}$$

- Special solution at $z=1$:

$$g(z+1)=z^{\frac{2k-1}{2}}\sum_{m=0}^{\infty}\sum_{l=0}^{m}(-1)^{m-l}\mathrm{B}\left(k+m,\frac{1}{2}\right)\mathrm{B}\left(m-l+\frac{1}{2},-\frac{1}{2}\right)B_l^{(k)}z^m$$

with

$$B_m^{(k)}=\sum_{i\le l\le m}(-1)^{m-l}B_i^{(k-1)}\mathrm{B}(l-i+1-a_k,a_k)\mathrm{B}(k-1+l,1-a_k)$$
$$\mathrm{B}(m-l+a_k,1-a_k)\mathrm{B}(k-a_k+m,a_k)$$

starting with

$$B_m^{(1)}=(-1)^m\mathrm{B}(m+a_1,1-a_1)\mathrm{B}(m+1-a_1,a_1).$$

A.2.2 The even case

$$P^{even}(a,b,4) = \vartheta^4 - z(\vartheta+b)(\vartheta+1-b)(2\vartheta^2+2\vartheta+a^2-a+1) \\ + z^2(\vartheta+b)(\vartheta+1-b)(\vartheta+b+1)(\vartheta+2-b):$$

- Riemann scheme:

$$\left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & b \\ 0 & 1 & 1-b \\ 0 & a & 1+b \\ 0 & 1-a & 2-b \end{array}\right\}$$

- Special solution at $z=0$: $f(z) = \sum_{m=0}^{\infty} A_m z^m$ with

$$A_m = \binom{b+m-1}{m}\binom{m-b}{m}\sum_{k=0}^{m}\binom{a+m-k-1}{m-k}^2\binom{k-a}{k}$$

- Special solutions at $z=1$: $g(z+1) = z^\gamma \sum_{m=0}^{\infty} B_m^{(\gamma)} z^m$ at $z=1$ where

$$B_m^{(\gamma)} = \mathrm{B}(1+\gamma-b+m,b)\sum_{l=0}^{m}(-1)^l\alpha(l)\mathrm{B}(1-b+l,-\gamma)\binom{-b}{m-l}\binom{l-1+\gamma}{\gamma-1},$$

with $\gamma \in \{a, 1-a\}$ and

$$\alpha(l) = {}_3F_2\left(\left.\begin{matrix} -l,\ \gamma,\ \gamma \\ 1+\gamma,\ b-l \end{matrix}\right| 1\right)$$

$P^{even}(b, 2k)$

- Riemann scheme:

$$\left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & b \\ 0 & \frac{1}{2} & 1-b \\ \vdots & \vdots & \vdots \\ 0 & k-\frac{3}{2} & k-1-b \\ 0 & k-1 & k-1+b \\ 0 & \frac{k-1}{2} & k-b \end{array}\right\}$$

- Special solution at $z = 0$:

$$f(z) = \sum_{m=0}^{\infty} A_m^{(k-1)} \binom{b+m-1}{m} \binom{m-b}{m} z^m,$$

where $A_m^{(0)} = 1$ for all $m \in \mathbb{N}_0$ and

$$A_m^{(i+1)} = \sum_{j=0}^{m} A_{m-j}^{(i)} \binom{-\frac{1}{2}+m-j}{m-j}^2 \binom{j-\frac{1}{2}}{j}.$$

- Special solution at $z = 1$:

$$g(z+1) = z^{\frac{k-1}{2}} \sum_{m=0}^{\infty} B_m z^m$$

with

$$B_m = \sum_{i \le l \le m} (-1)^{m-l} B_i^{(k-1)} \mathrm{B}(l-i+1-b, b) \mathrm{B}(k-1+l, 1-b)$$
$$\mathrm{B}(m-l+b, 1-b) \mathrm{B}(k-b+m, b),$$

where

$$B_m^{(0)} = (-1)^m \mathrm{B}\left(\frac{1}{2}+m, \frac{1}{2}\right)^2$$

and

$$B_m^{(r+1)} = \sum_{i \le l \le m} (-1)^{m-l} B_i^{(r)} \mathrm{B}\left(\frac{1}{2}+l-i, \frac{1}{2}\right) \mathrm{B}\left(r+\frac{1}{2}+l, \frac{1}{2}\right)$$
$$\mathrm{B}\left(\frac{1}{2}+m-l, \frac{1}{2}\right) \mathrm{B}\left(r+1+m, \frac{1}{2}\right)$$

A.2.3 Odd case $P^{odd}(a, 2k+1)$

- Riemann scheme:

$$\left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & a \\ 0 & \frac{1}{2} & 1-a \\ \vdots & \vdots & \vdots \\ 0 & k-1 & k-1+a \\ 0 & k-\frac{1}{2} & k-a \\ 0 & k & \frac{k}{2} \end{array}\right\}$$

- Special solution at $z = 0$: $f(z) = \sum_{m=0}^{\infty} A_m z^m$ with

$$A_m = A_m^{(k)} \binom{a+m-1}{m} \binom{m-a}{m},$$

where

$$A_m^{(1)} = \sum_{j=0}^{m} \binom{m-j-\frac{1}{2}}{m-j}^3 \binom{j-\frac{1}{2}}{j}$$

and

$$A_m^{(r+1)} = \sum_{j=0}^{m} A_{m-j}^{(r)} \binom{m-j-\frac{1}{2}}{m-j}^2 \binom{j-\frac{1}{2}}{j}.$$

- Special solution at $z = 1$: $g(z+1) = z^k \sum_{m=0}^{\infty} B_m z^m$ at $z = 1$ with

$$B_m = \sum_{i \le l \le m} (-1)^{m-l} B_i^{(k)} \mathrm{B}(l-i+1-b, b) \mathrm{B}(k-1+l, 1-b)$$
$$\mathrm{B}(m-l+b, 1-b) \mathrm{B}(k-b+m, b),$$

where

$$B_m^{(0)} = \sum_{l \le m} (-1)^m - l\mathrm{B}\left(\frac{1}{2}+l, \frac{1}{2}\right)^2 \mathrm{B}(m-l+\frac{1}{2}, \frac{1}{2})\mathrm{B}(m+1, \frac{1}{2})$$

and

$$B_m^{(r+1)} = \sum_{i \le l \le m} (-1)^{m-l} B_i^{(r)} \mathrm{B}\left(\frac{1}{2}+l-i, \frac{1}{2}\right) \mathrm{B}\left(r+l, \frac{1}{2}\right)$$
$$\mathrm{B}\left(\frac{1}{2}+m-l, \frac{1}{2}\right) \mathrm{B}\left(r+\frac{1}{2}+m, \frac{1}{2}\right)$$

A.2.4 Extra case

$$P^{extra}(a) = 16\,\vartheta^6 - z\,(2\,\vartheta+1)^2\left(8\,\vartheta^2+8\,\vartheta+5\right)(\vartheta+a)\,(\vartheta+1-a)$$
$$+\,4\,z^2\,(2\,\vartheta+3)\,(2\,\vartheta+1)\,(\vartheta+2-a)\,(\vartheta+1-a)\,(\vartheta+1+a)\,(\vartheta+a)\,.$$

- Riemann scheme:

$$\left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 1 & a \\ 0 & 2 & 1-a \\ 0 & 2 & 1+a \\ 0 & 3 & 2-a \end{array}\right\}.$$

- Special solution at $z=0$: $\quad f(z)=\sum_{m=0}^{\infty} A_m z^m$ with

$$A_m = \binom{m-\frac{1}{2}}{m}\binom{a+m-1}{m}\binom{m-a}{m}\sum_{j=0}^{m}\binom{m-j-\frac{1}{2}}{m-j}^3\binom{j-\frac{1}{2}}{j}.$$

A.2.5 Index zero in $\mathrm{Sp}_4(\mathbb{C})$

$$\begin{aligned}
P_3(a,b) :&= \left(\bigwedge^2 \left(\left(I_{\frac{3}{4}+a} \star_H I_{\frac{3}{4}-a} \otimes z^{\frac{1}{2}}\right) \star_H I_{\frac{1}{4}+b} \star_H I_{\frac{1}{4}-b}\right) \otimes z^{\frac{1}{2}}\right) \star_H I_{\frac{3}{2}} \\
&= 64\,\vartheta^4 - z\left(128\,\vartheta^4 + 256\,\vartheta^3 + \vartheta^2(304 - 128(a^2+b^2))\right) \\
&\quad - z\left(\vartheta\,(176 - 128(a^2+b^2)) + 39 - 48(a^2+b^2) - 256\,a^2b^2\right) \\
&\quad + 64z^2\,(\vartheta+1-a-b)\,(\vartheta+1+a-b) \\
&\qquad\qquad (\vartheta+1-a+b)\,(\vartheta+1+a+b)
\end{aligned}$$

- Riemann scheme:

$$\left\{\begin{array}{ccc}
0 & 1 & \infty \\
\hline
0 & -\frac{1}{2} & 1-a-b \\
0 & 0 & 1+a-b \\
0 & 1 & 1-a+b \\
0 & \frac{3}{2} & 1+a+b
\end{array}\right\}.$$

- Special solution at $z=0$: $f(z) = \sum_{m=0}^{\infty} A_m z^m$ at $z=0$ with

$$A_m = \binom{\frac{1}{2}+m}{m} \sum_{k=0}^{m} \left(2k - \frac{1}{2} - m\right) \alpha\left(-\frac{1}{2}, k\right) \alpha(0, m-k),$$

where

$$\begin{aligned}
\alpha(\nu,m) :=& \mathrm{B}\left(\frac{3}{4}+a+\nu+m, \frac{1}{4}-a\right) \mathrm{B}\left(\frac{3}{4}-a+\nu+m, \frac{1}{4}+a\right) \\
& \mathrm{B}\left(\frac{3}{4}+b+\nu+m, \frac{3}{4}-b\right) \mathrm{B}\left(\frac{3}{4}-b+\nu+m, \frac{3}{4}+b\right)
\end{aligned}$$

$$\begin{aligned}P_4(a,b) :&= \mathrm{Sym}^2\left(I_a \star_H I_b \otimes I_{\frac{1-a-b}{2}}\right) \star_H I_{\frac{1}{2}}\\&= 4\,\vartheta^4 - 2\,z\,(2\,\vartheta+1)^2\left(\vartheta^2+\vartheta+2\,ab-a+1-b\right)\\&\quad + z^2\,(2\,\vartheta+1)\,(2\,\vartheta+3)\,(\vartheta+1+a-b)\,(\vartheta+1+b-a)\end{aligned}$$

- Riemann scheme:

$$\left\{\begin{array}{ccc}0 & 1 & \infty\\ \hline 0 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{3}{2}\\ 0 & -\frac{1}{2}+a+b & 1+a-b\\ 0 & \frac{3}{2}-a-b & 1-a+b\end{array}\right\}.$$

- Special solution at $z=0$: $f(z)=\sum_{m=0}^{\infty} A_m z^m$ with

$$A_m = \binom{m-\frac{1}{2}}{m}\sum_{k=0}^{m}\binom{a+k-1}{k}\binom{b+k-1}{k}\binom{m-k-a}{m-k}\binom{m-k-b}{m-k}.$$

- Special solutions at $z=1$: $g_{(a,b)}(z+1)$ and $g_{(1-a,1-b)}(z+1)$ with

$$g_{(a,b)}(z+1) = z^{\frac{3}{2}-a-b}\sum_{m=0}^{\infty} B_m^{(a,b)} z^m$$

where

$$B_m^{(a,b)} = \mathrm{B}\left(2-a-b+m,\frac{1}{2}\right)\sum_{l=0}^{m}\mathrm{B}\,(1-b+l,1-a)\,\alpha(l)\binom{-\frac{1}{2}}{m-l}\binom{a-1}{l}$$

and

$$\alpha(l) = {}_4F_3\left(\left.\begin{matrix}-l,\ 1-b,\ 1-a,\ a-1-l+b\\ b-l,\ a-l,\ 2-a-b\end{matrix}\,\right|\,1\right).$$

A.2.6 Index two in $\mathrm{Sp}_4(\mathbb{C})$

$$Q_1(c,s,a) = \vartheta^4 s - z\,(\vartheta+a)\,(\vartheta+1-a)\,\big(\vartheta^2(1+s)+\vartheta(1+s)+c\big)$$
$$+\, z^2\,(\vartheta+2-a)\,(\vartheta+1-a)\,(\vartheta+1+a)\,(\vartheta+a)$$

Riemann scheme:

$$\left\{\begin{array}{cccc} 0 & 1 & s & \infty \\ \hline 0 & 0 & 0 & a \\ 0 & 1 & 1 & 1-a \\ 0 & 1 & 1 & 1+a \\ 0 & 2 & 2 & 2-a \end{array}\right\}.$$

CY-operators: We seem to get operators of CY-type if $a \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$. In particular, we find

Number	Operator	Source
1	$\vartheta^4 - z\,(7\,\vartheta^2+7\,\vartheta+2)\,(\vartheta+a)\,(\vartheta+1-a)$ $-8\,z^2\,(\vartheta+2-a)\,(\vartheta+1-a)\,(\vartheta+1+a)\,(\vartheta+a)$	$a=\frac{1}{2}$: 45 $a=\frac{1}{3}$: 15 $a=\frac{1}{4}$: 68 $a=\frac{1}{6}$: 62
2	$\vartheta^4 - z\,(11\,\vartheta^2+11\,\vartheta+3)\,(\vartheta+a)\,(\vartheta+1-a)$ $-z^2\,(\vartheta+2-a)\,(\vartheta+1-a)\,(\vartheta+1+a)\,(\vartheta+a)$	$a=\frac{1}{2}$: 25 $a=\frac{1}{3}$: 24 $a=\frac{1}{4}$: 51 $a=\frac{1}{6}$: 63
3	$\vartheta^4 - z\,(10\,\vartheta^2+10\,\vartheta+3)\,(\vartheta+a)\,(\vartheta+1-a)$ $+9\,z^2\,(\vartheta+2-a)\,(\vartheta+1-a)\,(\vartheta+1+a)\,(\vartheta+a)$	$a=\frac{1}{2}$: 58 $a=\frac{1}{3}$: 70 $a=\frac{1}{4}$: 69 $a=\frac{1}{6}$: 64
4	$\vartheta^4 - 4\,z\,(3\,\vartheta^2+3\,\vartheta+1)\,(\vartheta+a)\,(\vartheta+1-a)$ $+32\,z^2\,(\vartheta+2-a)\,(\vartheta+1-a)\,(\vartheta+1+a)\,(\vartheta+a)$	$a=\frac{1}{2}$: 36 $a=\frac{1}{3}$: 48 $a=\frac{1}{4}$: 38 $a=\frac{1}{6}$: 65
5	$\vartheta^4 - 3\,z\,(3\,\vartheta^2+3\,\vartheta+1)\,(\vartheta+a)\,(\vartheta+1-a)$ $+27\,z^2\,(\vartheta+2-a)\,(\vartheta+1-a)\,(\vartheta+1+a)\,(\vartheta+a)$	$a=\frac{1}{2}$: 133 $a=\frac{1}{3}$: 134 $a=\frac{1}{4}$: 135 $a=\frac{1}{6}$: 136
6	$\vartheta^4 - z\,(17\,\vartheta^2+17\,\vartheta+6)\,(\vartheta+a)\,(\vartheta+1-a)$ $+72\,z^2\,(\vartheta+2-a)\,(\vartheta+1-a)\,(\vartheta+1+a)\,(\vartheta+a)$	$a=\frac{1}{2}$: 137 $a=\frac{1}{3}$: 138 $a=\frac{1}{4}$: 139 $a=\frac{1}{6}$: 140

$$Q_2(c,s,\lambda) = 4\,\vartheta^4 s - z\,(2\,\vartheta+1)^2\,\left(\vartheta^2+\vartheta^2 s+\vartheta+\vartheta\,s+4\,c\right)$$
$$+\,z^2\,(2\,\vartheta+3)\,(2\,\vartheta+1)\,(\vartheta+1+\lambda)\,(\vartheta+1-\lambda)\,,$$

Riemann scheme:

$$\left\{\begin{array}{cccc} 0 & 1 & s & \infty \\ \hline 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & 1-\lambda \\ 0 & 1 & 1 & 1+\lambda \\ 0 & 2 & 2 & \frac{3}{2} \end{array}\right\}.$$

CY-operators: For $\lambda = \frac{1}{2}$ we get operators of CY-type, which we have already constructed before and thus omit them in the following table. Despites that we find

Number	**Operator**	**Source**
$1'$	$4\,\vartheta^4 - 2\,z\,(2\,\vartheta+1)^2\,\left(7\,\vartheta^2+7\,\vartheta+3\right)$ $+81\,z^2\,(2\,\vartheta+1)\,(\vartheta+1)^2\,(2\,\vartheta+3)$	41
$2'$	$4\,\vartheta^4 - 2\,z\,(2\,\vartheta+1)^2\,\left(11\,\vartheta^2+11\,\vartheta+5\right)$ $+125\,z^2\,(2\,\vartheta+1)\,(\vartheta+1)^2\,(2\,\vartheta+3)$	187
$3'$	$\vartheta^4 - z\,(2\,\vartheta+1)^2\,\left(5\,\vartheta^2+5\,\vartheta+2\right)$ $+16\,z^2\,(2\,\vartheta+1)\,(\vartheta+1)^2\,(2\,\vartheta+3)$	16
$4'$	$\vartheta^4 - 2\,z\,(2\,\vartheta+1)^2\,\left(3\,\vartheta^2+3\,\vartheta+1\right)$ $+4\,z^2\,(2\,\vartheta+1)\,(\vartheta+1)^2\,(2\,\vartheta+3)$	42
$5'$	$4\,\vartheta^4 + 6\,z\,(2\,\vartheta+1)^2\,\left(3\,\vartheta^2+3\,\vartheta+1\right)$ $-27\,z^2\,(2\,\vartheta+1)\,(\vartheta+1)^2\,(2\,\vartheta+3)$	184
$6'$	$4\,\vartheta^4 - 2\,z\,(2\,\vartheta+1)^2\,\left(17\,\vartheta^2+17\,\vartheta+5\right)$ $+z^2\,(2\,\vartheta+1)\,(\vartheta+1)^2\,(2\,\vartheta+3)$	29
$7'$	$4\,\vartheta^4 + 2\,z\,(2\,\vartheta+1)^2\,\left(13\,\vartheta^2+13\,\vartheta+4\right)$ $-3\,z^2\,(2\,\vartheta+1)\,(3\,\vartheta+2)\,(3\,\vartheta+4)\,(2\,\vartheta+3)$	26
$8'$	$\vartheta^4 - 8\,z\,(2\,\vartheta+1)^2\,\left(5\,\vartheta^2+5\,\vartheta+2\right)$ $+192\,z^2\,(2\,\vartheta+1)\,(3\,\vartheta+2)\,(3\,\vartheta+4)\,(2\,\vartheta+3)$	strange

$9'$	$\vartheta^4 - 4\,z\,(2\,\vartheta+1)^2\left(3\,\vartheta^2+3\,\vartheta+1\right)$ $-16\,z^2\,(2\,\vartheta+1)\,(4\,\vartheta+3)\,(4\,\vartheta+5)\,(2\,\vartheta+3)$	18
$10'$	$\vartheta^4 - 4\,z\,(2\,\vartheta+1)^2\left(7\,\vartheta^2+7\,\vartheta+3\right)$ $+48\,z^2\,(2\,\vartheta+1)\,(4\,\vartheta+3)\,(4\,\vartheta+5)\,(2\,\vartheta+3)$	183

Note that if we start with $8'$, we end up with an operator which seems to fulfill (Q) but not (Q+). Actually, this is the only known example of this type.

$$
\begin{aligned}
Q_3(c,s,\lambda) = {} & \vartheta^4(s-1)^4 - (s-1)^2 z(\vartheta^4(s^2-10s+1) + 2\vartheta^3(s^2-10s+1)) \\
& - (s-1)^2 z(\vartheta^2(s^2(\lambda-\lambda^2+1) + 2s(\lambda^2+\lambda+c-12) + 1 + \lambda - \lambda^2 + 2c)) \\
& - (s-1)^2 z(\vartheta(s^2\lambda(1-\lambda) + 2s(\lambda^2+\lambda+c-7) + \lambda(1-\lambda) + 2c)) \\
& - (s-1)^2 z(c^2 - 3s + \lambda(cs + c + s\lambda)) \\
& - 4sz^2(\vartheta+1)^2(\vartheta^2(2s^2-8s+2) + 2\vartheta(2s^2-8s+2)) \\
& - 4sz^2(\vartheta+1)^2(s^2(3+\lambda-2\lambda^2) + s(4\lambda^2+2\lambda+2c-13) + 3 + \lambda + 2c - 2\lambda^2) \\
& - 4s^2z^3(\vartheta+1)(\vartheta+2)(2\vartheta-2\lambda+3)(2\vartheta+2\lambda+3)
\end{aligned}
$$

Riemann scheme:

$$
\left\{
\begin{array}{cccc}
0 & 1 & -\frac{(s-1)^2}{4s} & \infty \\
\hline
0 & 0 & -\frac{1}{2} & 1 \\
0 & 1 & 0 & \frac{3}{2}-\lambda \\
0 & 1 & 1 & \frac{3}{2}+\lambda \\
0 & 2 & \frac{3}{2} & 2
\end{array}
\right\}
$$

CY-operators:

Number	Operator	Source
1	$\vartheta^4 - z\left(28 + 145\,\vartheta^4 + 290\,\vartheta^3 + 285\,\vartheta^2 + 140\,\vartheta\right) + 32\,z^2\left(194\,\vartheta^2 + 388\,\vartheta + 327\right)(\vartheta+1)^2 - 20736\,z^3(\vartheta+2)(\vartheta+1)(2\,\vartheta+3)^2$	~ 100
2	$\vartheta^4 - z\left(19\,\vartheta^2 + 19\,\vartheta + 6\right)\left(7\,\vartheta^2 + 7\,\vartheta + 2\right) + 8\,z^2\left(127\,\vartheta^2 + 254\,\vartheta + 224\right)(\vartheta+1)^2 - 500\,z^3(\vartheta+2)(\vartheta+1)(2\,\vartheta+3)^2$	~ 101
3	$\vartheta^4 + 2z\left(4\,\vartheta^4 + 8\,\vartheta^3 + 37\,\vartheta^2 + 33\,\vartheta + 9\right) - 36\,z^2\left(92\,\vartheta^2 + 184\,\vartheta + 189\right)(\vartheta+1)^2 - 20736\,z^3(\vartheta+2)(\vartheta+1)(2\,\vartheta+3)^2$	~ 103
4	$\vartheta^4 + z\left(80 + 240\,\vartheta^4 + 480\,\vartheta^3 + 592\,\vartheta^2 + 352\,\vartheta\right) + 2048\,z^2\left(6\,\vartheta^2 + 12\,\vartheta + 5\right)(\vartheta+1)^2 - 65536\,z^3(\vartheta+2)(\vartheta+1)(2\,\vartheta+3)^2$	~ 107
5	$\vartheta^4 + z\left(72 + 243\,\vartheta^4 + 486\,\vartheta^3 + 567\,\vartheta^2 + 324\,\vartheta\right) + 5832\,z^2\left(3\,\vartheta^2 + 6\,\vartheta + 4\right)(\vartheta+1)^2 + 78732\,z^3(\vartheta+2)(\vartheta+1)(2\,\vartheta+3)^2$	~ 165

6	$\vartheta^4 + z\left(180 + 575\,\vartheta^4 + 1150\,\vartheta^3 + 1379\,\vartheta^2 + 804\,\vartheta\right)$ $+3168\,z^2\left(26\,\vartheta^2 + 52\,\vartheta + 27\right)(\vartheta+1)^2$ $-20736\,z^3(\vartheta+2)(\vartheta+1)(2\,\vartheta+3)^2$	~ 144
$1'$	$\vartheta^4 + z\left(210 + 776\,\vartheta^4 + 1552\,\vartheta^3 + 1738\,\vartheta^2 + 962\,\vartheta\right)$ $+324\,z^2\left(580\,\vartheta^2 + 1160\,\vartheta + 747\right)(\vartheta+1)^2$ $+13436928\,z^3(\vartheta+1)^2(\vartheta+2)^2$	–
$2'$	$\vartheta^4 + z\left(310 + 1016\,\vartheta^4 + 2032\,\vartheta^3 + 2410\,\vartheta^2 + 1394\,\vartheta\right)$ $+500\,z^2\left(532\,\vartheta^2 + 1064\,\vartheta + 563\right)(\vartheta+1)^2$ $+4000000\,z^3(\vartheta+1)^2(\vartheta+2)^2$	–
$3'$	$\vartheta^4 + z\left(152 + 368\,\vartheta^4 + 736\,\vartheta^3 + 1020\,\vartheta^2 + 652\,\vartheta\right)$ $-8192\,z^2\left(\vartheta^2 + 2\,\vartheta + 6\right)(\vartheta+1)^2$ $-9437184\,z^3(\vartheta+1)^2(\vartheta+2)^2$	–
$4'$	$\vartheta^4 - 16\,z\left(6\,\vartheta^2 + 6\,\vartheta - 1\right)(2\,\vartheta+1)^2$ $-1024\,z^2\left(60\,\vartheta^2 + 120\,\vartheta + 97\right)(\vartheta+1)^2$ $-2097152\,z^3(\vartheta+1)^2(\vartheta+2)^2$	~ 291
$5'$	$\vartheta^4 - 18\,z\left(12\,\vartheta^2 + 12\,\vartheta + 5\right)\left(3\,\vartheta^2 + 3\,\vartheta + 1\right)$ $+2916\,z^2\left(36\,\vartheta^2 + 72\,\vartheta + 55\right)(\vartheta+1)^2$ $-5038848\,z^3(\vartheta+1)^2(\vartheta+2)^2$	~ 73
$6'$	$\vartheta^4 + z\left(-46 - 1144\,\vartheta^4 - 2288\,\vartheta^3 - 1590\,\vartheta^2 - 446\,\vartheta\right)$ $-4\,z^2\left(2300\,\vartheta^2 + 4600\,\vartheta + 3621\right)(\vartheta+1)^2$ $-18432\,z^3(\vartheta+1)^2(\vartheta+2)^2$	–
7	$\vartheta^4 + z\left(126 + 419\,\vartheta^4 + 838\,\vartheta^3 + 985\,\vartheta^2 + 566\,\vartheta\right)$ $+196\,z^2\left(250\,\vartheta^2 + 500\,\vartheta + 301\right)(\vartheta+1)^2$ $+28812\,z^3(6\,\vartheta+11)(6\,\vartheta+7)(\vartheta+2)(\vartheta+1)$	–
8	$\vartheta^4 + z\left(-24 - 248\,\vartheta^4 - 496\,\vartheta^3 - 400\,\vartheta^2 - 152\,\vartheta\right)$ $+64\,z^2\left(112\,\vartheta^2 + 224\,\vartheta + 187\right)(\vartheta+1)^2$ $-1536\,z^3(6\,\vartheta+11)(6\,\vartheta+7)(\vartheta+2)(\vartheta+1)$	–

9	$\vartheta^4 + z\left(70 + 264\,\vartheta^4 + 528\,\vartheta^3 + 586\,\vartheta^2 + 322\,\vartheta\right)$ $+100\,z^2\left(228\,\vartheta^2 + 456\,\vartheta + 335\right)(\vartheta+1)^2$ $+40000\,z^3(4\,\vartheta+5)(4\,\vartheta+7)(\vartheta+2)(\vartheta+1)$	–
10	$\vartheta^4 - 2\,z\left(23\,\vartheta^2 + 23\,\vartheta + 5\right)(2\,\vartheta+1)^2$ $-4\,z^2\left(380\,\vartheta^2 + 760\,\vartheta + 657\right)(\vartheta+1)^2$ $-192\,z^3(4\,\vartheta+5)(4\,\vartheta+7)(\vartheta+2)(\vartheta+1)$	~ 266

$$\begin{aligned}
Q_4(c,s,\lambda) &= 4\,\vartheta^4\,(s-1)^4 - z(s-1)^2(4\vartheta^4(s^2-10s+1)) \\
&- z(s-1)^2(8\vartheta^3(s^2-10s+1)) + \vartheta^2(5s^2+s(8c+8\lambda^2-90)+8c+5)) \\
&- z(s-1)^2(\vartheta(s^2+s(8c+8\lambda^2-50)+8c+1)) \\
&- z(s-1)^2(s(3\lambda^2+2c-11)+2c+4c^2) \\
&- 4sz^2(2\vartheta+2+\lambda)(2\vartheta+2-\lambda)(\vartheta^2(2s^2-8s+2)) \\
&- 4sz^2(2\vartheta+2+\lambda)(2\vartheta+2-\lambda)(\vartheta(4s^2-16s+4)) \\
&- 4sz^2(2\vartheta+2+\lambda)(2\vartheta+2-\lambda)(3s^2+s(\lambda^2-11+2c)+3+2c)) \\
&- 4s^2z^3(2\vartheta+2-\lambda)(2\vartheta+2+\lambda)(2\vartheta+4-\lambda)(2\vartheta+4+\lambda)
\end{aligned}$$

Riemann scheme:

$$\left\{\begin{array}{cccc}
0 & 1 & -\frac{(s-1)^2}{4s} & \infty \\
\hline
0 & 0 & -\frac{1}{2} & 1-\frac{\lambda}{2} \\
0 & 1 & 0 & 1+\frac{\lambda}{2} \\
0 & 1 & 1 & 2-\frac{\lambda}{2} \\
0 & 2 & \frac{3}{2} & 2+\frac{\lambda}{2}
\end{array}\right\}.$$

CY-operators:

Number	**Operator**	**Source**
$1''$	$\vartheta^4 - z\left(2320\,\vartheta^4+4640\,\vartheta^3+4228\,\vartheta^2+1908\,\vartheta+360\right)$ $+1024\,z^2\,(4\,\vartheta+5)\,(4\,\vartheta+3)\left(97\,\vartheta^2+194\,\vartheta+144\right)$ $-1327104\,z^3\,(4\,\vartheta+5)\,(4\,\vartheta+7)\,(4\,\vartheta+3)\,(4\,\vartheta+9)$	~ 33
$2''$	$\vartheta^4 - z\left(2128\,\vartheta^4+4256\,\vartheta^3+3076\,\vartheta^2+948\,\vartheta+116\right)$ $+16\,z^2\,(4\,\vartheta+5)\,(4\,\vartheta+3)\left(1016\,\vartheta^2+2032\,\vartheta+1585\right)$ $-32000\,z^3\,(4\,\vartheta+5)\,(4\,\vartheta+7)\,(4\,\vartheta+3)\,(4\,\vartheta+9)$	~ 302
$3''$	$\vartheta^4 + z\left(128\,\vartheta^4+256\,\vartheta^3+1288\,\vartheta^2+1160\,\vartheta+300\right)$ $-144\,z^2\,(4\,\vartheta+5)\,(4\,\vartheta+3)\left(368\,\vartheta^2+736\,\vartheta+657\right)$ $-1327104\,z^3\,(4\,\vartheta+5)\,(4\,\vartheta+7)\,(4\,\vartheta+3)\,(4\,\vartheta+9)$	~ 293
$4''$	$\vartheta^4 + z\left(3840\,\vartheta^4+7680\,\vartheta^3+9280\,\vartheta^2+5440\,\vartheta+1200\right)$ $+32768\,z^2\,(4\,\vartheta+3)\,(4\,\vartheta+5)\left(6\,\vartheta^2+12\,\vartheta+5\right)$ $-4194304\,z^3\,(4\,\vartheta+5)\,(4\,\vartheta+7)\,(4\,\vartheta+3)\,(4\,\vartheta+9)$	–

$5''$	$\vartheta^4 - z\left(3888\,\vartheta^4 + 7776\,\vartheta^3 + 8748\,\vartheta^2 + 4860\,\vartheta + 1044\right)$ $+11664\,z^2\,(4\,\vartheta+3)\,(4\,\vartheta+5)\left(24\,\vartheta^2 + 48\,\vartheta + 29\right)$ $-5038848\,z^3\,(4\,\vartheta+5)\,(4\,\vartheta+7)\,(4\,\vartheta+3)\,(4\,\vartheta+9)$	~ 154
$6''$	$\vartheta^4 + z\left(9200\,\vartheta^4 + 18400\,\vartheta^3 + 21628\,\vartheta^2 + 12428\,\vartheta + 2712\right)$ $+9216\,z^2\,(4\,\vartheta+5)\,(4\,\vartheta+3)\left(143\,\vartheta^2 + 286\,\vartheta + 144\right)$ $-1327104\,z^3\,(4\,\vartheta+5)\,(4\,\vartheta+7)\,(4\,\vartheta+3)\,(4\,\vartheta+9)$	–
$7'$	$\vartheta^4 - z\left(1000\,\vartheta^4 + 2000\,\vartheta^3 + 1618\,\vartheta^2 + 618\,\vartheta + 102\right)$ $+12\,z^2\,(6\,\vartheta+5)\,(6\,\vartheta+7)\left(419\,\vartheta^2 + 838\,\vartheta + 647\right)$ $-7056\,z^3\,(6\,\vartheta+5)\,(6\,\vartheta+13)\,(6\,\vartheta+7)\,(6\,\vartheta+11)$	–
$8'$	$\vartheta^4 - z\left(1792\,\vartheta^4 + 3584\,\vartheta^3 + 4192\,\vartheta^2 + 2400\,\vartheta + 528\right)$ $+768\,z^2\,(6\,\vartheta+5)\,(6\,\vartheta+7)\left(31\,\vartheta^2 + 62\,\vartheta + 34\right)$ $-36864\,z^3\,(6\,\vartheta+5)\,(6\,\vartheta+13)\,(6\,\vartheta+7)\,(6\,\vartheta+11)$	–
$9'$	$\vartheta^4 - z\left(912\,\vartheta^4 + 1824\,\vartheta^3 + 1796\,\vartheta^2 + 884\,\vartheta + 180\right)$ $+176\,z^2\,(8\,\vartheta+7)\,(8\,\vartheta+9)\left(24\,\vartheta^2 + 48\,\vartheta + 35\right)$ $-6400\,z^3\,(8\,\vartheta+17)\,(8\,\vartheta+9)\,(8\,\vartheta+15)\,(8\,\vartheta+7)$	–
$10'$	$\vartheta^4 + z\left(1520\,\vartheta^4 + 3040\,\vartheta^3 + 3628\,\vartheta^2 + 2108\,\vartheta + 468\right)$ $+48\,z^2\,(8\,\vartheta+9)\,(8\,\vartheta+7)\left(184\,\vartheta^2 + 368\,\vartheta + 183\right)$ $-2304\,z^3\,(8\,\vartheta+17)\,(8\,\vartheta+9)\,(8\,\vartheta+15)\,(8\,\vartheta+7)$	–

$$
\begin{aligned}
Q_5(c,s,a) = {} & 16s^2\vartheta^4 - 4sz(8\vartheta^4(s+1) + 16\vartheta^3(s+1)) \\
& - 4sz(2\vartheta^2(s(9+a(1-a)) + 9 + a(1-a) + 4c)) \\
& - 4sz(2\vartheta(s(5+a(1-a)) + 5 + a(1-a) + 4c)) \\
& - 4sz(s(2-a^2+a) + 4c(a^2-a+1) + a + 2 - a^2) \\
& + z^2(16\vartheta^4(s^2+4s+1) + 64\vartheta^3(s^2+4s+1)) \\
& + z^2(\vartheta^2(4s^2(2a-2a^2+23) + 32s(c+a(1-a)+15) + 4(2a(1-a)+8c+23))) \\
& + z^2(16\vartheta(s^2(a(1-a)+4) + 64s(c+a(1-a)+7) + 8(2a(1-a)+7+8c))) \\
& + z^2(s^2(1+a)(2+a)(2-a)(3-a)) \\
& - 2sz^2(a^4 - 2a^3 + a^2(21-4c) + a(4c-20) - 84 - 16c)) \\
& + z^2((a^2+3a+2+4c)(a^2-5a+6+4c)) \\
& - 2z^3(2\vartheta+3)^2(4\vartheta^2(s+1) + 12\vartheta(s+1)) \\
& - 2z^3(2\vartheta+3)^2(s(11+3a(1-a)) + 3a(1-a) + 11 + 4c) \\
& + z^4(2\vartheta+3)(2\vartheta+5)(2\vartheta+5-2a)(2\vartheta+3+2a)
\end{aligned}
$$

Riemann scheme:

$$
\left\{
\begin{array}{cccc}
0 & 1 & s & \infty \\
\hline
0 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\
0 & 0 & 0 & \frac{5}{2} \\
0 & 1 & 1 & \frac{3}{2}+a \\
0 & \frac{3}{2} & \frac{3}{2} & \frac{5}{2}-a
\end{array}
\right\}
$$

CY-operators: We seem to get CY-operators if $a \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$. To be more precise, we find

Number	Operator	Source
$1'$	$16\vartheta^4 - 8z\left(56\vartheta^4 + 112\vartheta^3 + (132+14a(1-a))\vartheta^2)\right)$ $-8z\left((76+14a(1-a))\vartheta + 17 + 4a(1-a)\right)$ $+4z^2\left(1432\vartheta^4 + 5728\vartheta^3 + (10670+716a(1-a))\vartheta^2\right)$ $+4z^2\left((9884+1432a(1-a))\vartheta\right)$ $+4z^2\left(64a^3 - 32a^4 - 868a^2 + 836a + 3681\right)$ $-324\,z^3\,(2\,\vartheta+3)^2\left(28\,\vartheta^2 + 84\,\vartheta + 21\,a(1-a) + 80\right)$ $+6561\,z^4\,(2\,\vartheta+5)\,(2\,\vartheta+3)\,(2\,\vartheta+3+2\,a)\,(2\,\vartheta+5-2\,a)$	$a=\frac{1}{2}$: 150 $a=\frac{1}{3}$: 151 $a=\frac{1}{4}$: 152 $a=\frac{1}{6}$: 153

	$16\,\vartheta^4-8z\left(88\,\vartheta^4+176\,\vartheta^3+2(104+11a(1-a))\vartheta^2\right)$	
	$-8z\,(2(60+11a(1-a))\vartheta+27+6a(1-a))$	$a=\frac{1}{2}:\ 121$
	$+4z^2\left(2936\vartheta^4+11744\vartheta^3+(20822+1468a(1-a))\vartheta^2\right)$	$a=\frac{1}{3}:\sim 168$
$2'$	$+4z^2\,((18156+2936a(1-a))\vartheta)$	$a=\frac{1}{4}:\sim 229$
	$+4z^2\left(8a^3-4a^4-1612a^2+1608a+6417\right)$	$a=\frac{1}{6}:\sim 230$
	$-500\,z^3\,(2\,\vartheta+3)^2\left(44\,\vartheta^2+132\,\vartheta+33a(1-a)+126\right)$	
	$+15625\,z^4\,(2\,\vartheta+5)\,(2\,\vartheta+3)\,(2\,\vartheta+3+2\,a)\,(2\,\vartheta+5-2\,a)$	
	$\vartheta^4-z(40\vartheta^4+80\vartheta^3+(94+10a(1-a))\vartheta^2)$	
	$-z((54+10a(1-a))\vartheta+12+3a(1-a))$	$a=\frac{1}{2}:\ 39$
	$+3z^2\left(176\vartheta^4+704\vartheta^3+(1188+88a(1-a))\vartheta^2\right)$	$a=\frac{1}{3}:\ 50$
$3'$	$+3z^2\,((968+176a(1-a))\vartheta)$	$a=\frac{1}{4}:\ 37$
	$+3z^2(3a^4-6a^3-89a^2+92a+320)$	$a=\frac{1}{6}:\ 66$
	$-32\,z^3\,(2\,\vartheta+3)^2\left(20\,\vartheta^2+60\,\vartheta+15\,a(1-a)+57\right)$	
	$+256\,z^4\,(2\,\vartheta+5)\,(2\,\vartheta+3)\,(2\,\vartheta+3+2\,a)\,(2\,\vartheta+5-2\,a)$	
	$\vartheta^4-2z\left(24\vartheta^4+48\vartheta^3+(56+6a(1-a))\vartheta^2\right)$	
	$-2z\,((32+6a(1-a))\vartheta+7+2a(1-a))$	$a=\frac{1}{2}:\sim 122$
	$+4z^2\left(152\vartheta^4+608\vartheta^3+(926+76a(1-a))\vartheta^2\right)$	$a=\frac{1}{3}:\sim 170$
$4'$	$+4z^2\,((636+152a(1-a))\vartheta)$	$a=\frac{1}{4}:\sim 231$
	$+4z^2\left(8a^4-16a^3-64a^2+72a+169\right)$	$a=\frac{1}{6}:\sim 232$
	$-16\,z^3\,(2\,\vartheta+3)^2\left(12\,\vartheta^2+36\,\vartheta-9\,a^2+34+9\,a\right)$	
	$+16\,z^4\,(2\,\vartheta+5)\,(2\,\vartheta+3)\,(2\,\vartheta+3+2\,a)\,(2\,\vartheta+5-2\,a)$	
	$16\vartheta^4+24z\left(24\vartheta^4+48\vartheta^3+(56+6a(1-a))\vartheta)\right)$	
	$+24z\,((32+6a(1-a))\vartheta+7+2a(1-a))$	$a=\frac{1}{2}:\sim 164$
	$+4z^2\left(1080\vartheta^4+4320\vartheta^3+(5670+540a(1-a))\vartheta^2\right)$	$a=\frac{1}{3}:\sim 172$
$5'$	$+4z^2\,((2700+1080a(1-a))\vartheta)$	$a=\frac{1}{4}:\sim 227$
	$+4z^2\left(108a^4-216a^3-324a^2+432+225\right)$	$a=\frac{1}{6}:\sim 228$
	$-324\,z^3\,(2\,\vartheta+3)^2\left(12\,\vartheta^2+36\,\vartheta+9a(1-a)+34\right)$	
	$+729\,z^4\,(2\,\vartheta+5)\,(2\,\vartheta+3)\,(2\,\vartheta+3+2\,a)\,(2\,\vartheta+5-2\,a)$	
	$16\vartheta^4-8z\left(136\vartheta^4+272\vartheta^3+(316+34a(1-a))\vartheta^2\right)$	
	$-8z\,((180+34a(1-a))\vartheta+39+12a(1-a))$	$a=\frac{1}{2}:\ 44$
	$+4z^2\left(4632\vartheta^4+18528\vartheta^3+(27342+2316a(1-a))\vartheta^2\right)$	$a=\frac{1}{3}:\ 53$
$6'$	$+4z^2\,((17628+4632a(1-a))\vartheta)$	$a=\frac{1}{4}:\ 52$
	$+4z^2\left(288a^4-576a^3-1860a^2+2148a+4209\right)$	$a=\frac{1}{6}:\ 149$
	$-4\,z^3\,(2\,\vartheta+3)^2\left(68\,\vartheta^2+204\,\vartheta+51a(1-a)+192\right)$	
	$+z^4\,(2\,\vartheta+5)\,(2\,\vartheta+3)\,(2\,\vartheta+3+2\,a)\,(2\,\vartheta+5-2\,a)$	

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Abstract

This thesis is devoted to the study of Picard-Fuchs operators associated to one-parameter families of n-dimensional Calabi-Yau manifolds whose solutions are integrals of $(n, 0)$-forms over locally constant n-cycles. Assuming additional conditions on these families, we describe algebraic properties of these operators which leads to the purely algebraic notion of operators of CY-type. Moreover, we present an explicit way to construct CY-type operators which have a linearly rigid monodromy tuple. Therefore, we first use the translation of the existence algorithm by N. Katz for rigid local systems to the level of tuples of matrices which was established by M. Dettweiler and S. Reiter. An appropriate translation to the level of differential operators yields families which contain operators of CY-type. Considering additional operations, we are also able to construct special CY-type operators of degree four which have a non-linearly rigid monodromy tuple. This provides both previously known and new examples.

Zusammenfassung

In der vorliegenden Arbeit beschäftige ich mich mit Picard-Fuchs Operatoren für Familien von Calabi-Yau Mannigfaltigkeiten der Dimension n mit einem Deformationsparameter, deren Lösungen durch Integrale einer $(n, 0)$-Form über lokal konstante n-Zykel gegeben sind. Ich beschreibe algebraische Eigenschaften dieser Differenzialoperatoren für Familien, die weiteren Bedingungen unterliegen. Dies führt zum rein algebraischen Begriff des Differenzialoperators vom CY-Typ. Für diejenigen, deren Monodromietupel linear starr ist, stelle ich außerdem eine explizite Konstruktionsmethode vor. Hierzu verwende ich den Existenzalgorithmus für linear starre lokale Systeme von N. Katz und die von M. Dettweiler und S. Reiter erarbeitete Version desselben für Matrixtupel. Daraufhin entwickele ich eine geeignete Version für Differenzialoperatoren, was zu Familien führt, die Differenzialoperatoren vom CY-Typ enthalten. Des Weiteren bin ich unter Verwendung zusätzlicher Operationen in der Lage spezielle Differenzialoperatoren vierter Ordnung vom CY-Typ zu konstruieren, deren Monodromietupel nicht linear starr ist. Dies führt sowohl zu bereits bekannten als auch zu neuen Beispielen für Operatoren dieser Klasse.

Lebenslauf

Michael Bogner

Staatsangehörigkeit: deutsch

Ich wurde am 23.06.1983 als jüngster Sohn von Wolfgang und Monika Bogner in Mainz geboren. Von 1989 bis 1993 besuchte ich die Heinrich-Mumbächer Grundschule in Mainz-Bretzenheim. Danach besuchte ich von 1993 bis 2002 das Bischöfliche Willigis Gymnasium in Mainz, an dem ich im März 2002 mein Abitur (Note *sehr gut*, 1,3) ablegte. Von Juni 2002 bis Februar 2003 absolvierte ich meinen Zivildienst beim Kreisverband Mainz-Bingen des Deutschen Roten Kreuzes. Ich begann im April 2003 mein Diplomstudium in Mathematik mit Nebenfach Phyik an der Johannes Gutenberg-Universität in Mainz, das ich im Oktober 2008 mit der Note *mit Auszeichnung* abschloss. Seit November 2008 bin ich als wissenschaftlicher Mitarbeiter an der Johannes Gutenberg-Universität in Mainz beschäftigt.