Chern characters for matrix factorizations

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Abstract

Chern characters are an important invariant of vector bundles. Three of the main properties of Chern characters for vector bundles are: functoriality, additivity over short exact sequences and multiplicativity over tensor products. The aim of this thesis is to introduce explicitly computable Chern characters for matrix factorizations, which fulfil variants of these three properties of Chern characters for vector bundles. We apply variants of the Chern characters for matrix factorizations also on modules with an eventually periodic free resolution with periodic part given by a matrix factorization and on periodic complexes with periodic part not necessarily given by a matrix factorization.

Zusammenfassung

Chern Charaktere sind eine wichtige Invariante für Vektorbündel. Drei ihrer Haupteigenschaften sind: Funktorialität, Additivität über kurze exakte Sequenzen und Multiplikativität über Tensorprodukte.
Das Ziel dieser Arbeit ist es, explizit berechenbare Chern Charakter für Matrixfaktorisierungen einzuführen, die Varianten dieser drei Eigenschaften von Chern Charakteren für Vektorbündel erfüllen. Wir definieren auch Varianten dieser Chern Charaktere für Matrixfaktorisierungen für Moduln mit letztendlich periodischer freier Auflösung, deren periodischer Teil durch eine Matrixfaktorisierung geben ist und für periodische Komplexe, deren periodischer Teil nicht notwendigerweise durch eine Matrixfaktorisierung gegeben ist.

Contents

Introduction

Let A be a ring and $f \in A$. A *matrix factorization* of f over A is a pair of matrices

$$(\varphi, \psi) \in M_{n \times l}(A) \times M_{l \times n}(A),$$

such that $\varphi\psi = f \operatorname{id}_n$, $\psi\varphi = f \operatorname{id}_l$. Matrix factorizations have been first introduced by David Eisenbud in [Eis80] to characterise resolutions over a hypersurface ring $B = A/(f)$ with A a regular local ring. He showed that the minimal free resolution of any B-module will become 2-periodic and the periodic part is given by a matrix factorization. The modules, which have a 2-periodic, minimal free resolution form the start, are precisely the *maximal Cohen-Macaulay modules*. A module is said to be maximal Cohen-Macaulay if its depth equals the Krull dimension of the ring.

This result of Eisenbud was used to show that the ring $B = \mathbb{C}\{X_1, \ldots, X_n\}/(f)$ has only finitely many different isomorphism classes of indecomposable maximal Cohen-Macaulay modules if and only if f has a *simple singularity*, i.e. from the A-D-E list. Knörrer showed in [Knö87] the backwards implication. Buchweitz, Greuel and Schreyer showed the forward implication in [BGS87]. One says in this case that the ring B has *finite Cohen-Macaulay representation type*. The Cohen-Macaulay representation type is called *discrete* if there are countably but not finitely many different isomorphism classes. In [BGS87] the authors also show, that $\mathbb{C}\{X_1, \ldots, X_n\}/(f)$ has discrete representation type if and only if f is of type A_∞ or D_∞. Furthermore, they give a complete list of all different isomorphism classes in the case of two variables. In [BD08] Burban and Drozd give a list of all different isomorphism classes in the case of three variables for A_∞ and D_∞ singularities. Before it was only known that there are countably but not finitely many different ones. There are two further Cohen-Macaulay representation types for *curve singularities*, namely *tame* and *wild*, see [BG16], Section 4 for the definition. In [DG93] Drozd and Greuel determine all tame curve singularities. There are many more results on the Cohen-Macaulay representation type for more general rings, e.g. [Aus87].

The relation between matrix factorizations and maximal Cohen-Macaulay modules was generalized by Buchweitz in [Buc87]. He showed that for a strongly Gorenstein ring S, the stable category of maximal Cohen-Macaulay S-modules is equivalent to the stable derived category of S. If S is a hypersurface ring as before, then both are equivalent to the homotopy category of matrix factorizations. Orlov generalized this equivalence to the homotopy category of matrix factorizations and the stable derived category of a scheme in [Orl04]. Also matrix factorizations have been used in physics to describe D-branes in Landau-Ginzburg models for example in [KL03] and [CM16]. Furthermore matrix factorizations have applications in mirror symmetry, see for example [BHLW06]. In the light of these applications matrix factorizations have become a topic worth studying for its own sake.

To classify matrix factorizations, one is interested in invariants to tell things apart. One such invariant for matrix factorizations is the so called *Chern character*. Chern characters were first defined for vector bundles over $\mathbb{C}$, for example in [MS74] and [Wei13]. One application for Chern characters for vector bundles is the Hirzebruch-Riemann-Roch theorem. For a Riemann surface S and a line bundle $\mathcal{O}(D)$ to a divisor D this theorem states:

$$\dim(H^0(\mathcal{O}(D))) - \dim(H^1(\mathcal{O}(D))) = \mathrm{ch}_1(\mathcal{O}(D)) + \frac{1}{2}\,\mathrm{ch}_1(T_S).$$

Vector bundles over manifolds have many things in common with finitely generated projective modules, as the first chapter of [Wei13] outlines. An incarnation of this fact is the Serre-Swan theorem, which states that there is an equivalence between the category of continuous vector bundles over a compact complex manifold X and the category of finitely generated projective modules over the ring of continuous functions from X to $\mathbb{C}$. This equivalence is given by taking the global sections of the vector bundle. Therefore, an analogue theory of Chern characters for finitely generated projective modules was developed, see for example [Lod98], chapter 8. Three of the main properties of both Chern characters are:

- Functoriality,
- Additivity over short exact sequences,
- Multiplicativity over tensor products.

In [Shk09] Shklyarov introduces a construction of a Chern character for differential graded categories. These Chern characters live in the *Hochschild homology* of the differential graded category. Hochschild homology was first defined for a (not necessarily commutative) associative algebra A over a commutative ring k. A good source for Hochschild homology of an algebra and many related topics is [Lod98]. Hochschild homology can be thought of as a non commutative generalization of Kähler differentials, because if A is commutative and the rational numbers are a subring of k, then the n-th exterior power $\Omega^n_{A/k}$ of the Kähler differentials is a direct summand of the n-th Hochschild homology $HH_n(A,k)$ of A over k. Furthermore, we show in Section 2.3, that $\Omega^n_{A/k}$ agrees with $HH_n(A,k)$ if A is the polynomial ring over a field k. More generally this holds for any smooth algebras A over a field k (see [Lod98], Theorem 3.4.4). The definition of Hochschild homology of a differential graded category is a generalization of the one for algebras, see [Shk09] 2.3 for a definition. The category of matrix factorizations is differential and $\mathbb{Z}/2\mathbb{Z}$-graded and therefore Shklyarov's theory can be applied to it. While Shklyarov's theory is nice from a theoretical point of view, the Hochschild homology and the Chern characters are hard to compute explicitly. In [Dyc11], Corollary 6.5 Dyckerhoff computed the Hochschild homology of the category of matrix factorizations for the ring $A = \mathbb{C}[X_1, \dots, X_n]$ and $f \in A$ with *isolated critical locus.* In [PV12] Polishchuk and Vaintrob showed that the result from Dyckerhoff also holds for $A = \mathbb{C}[\![X_1, \dots, X_n]\!]$ and found explicit formulas for the Chern characters for this A and f with *isolated singular locus.* Furthermore, they showed a relation between the *Herbrand difference* of two maximal Cohen-Macaulay modules and the Chern characters of the corresponding matrix factorizations. This was conjectured 30 years earlier by Buchweitz and van Straten. The Herbrand difference of two modules can be computed via the nondegenerate residue pairing

$$\Omega^n_{A/\mathbb{C}}/(df \wedge \Omega^{n-1}_{A/\mathbb{C}}) \times \Omega^n_{A/\mathbb{C}}/(df \wedge \Omega^{n-1}_{A/\mathbb{C}}) \to \mathbb{C},$$
$$(gdx_1 \dots dx_n, hdx_1 \dots dx_n) \mapsto \mathrm{Res}_{A/\mathbb{C}} \left(\frac{ghdx_1 \dots dx_n}{\partial_1 f \dots \partial_n f} \right)$$

and plugging in the Chern characters (cf. Section 6.3). This assumes that the number of variables is even. For an odd number of variables the Herbrand difference is identecally zero. This relation between the Herbrand difference and the Chern characters fits into a more general framework, which Shklyarov calls abstract Hirzebruch-Riemann-Roch theorem, see [Shk09], Section 1.2.
Other people have tried to find simpler approaches to Chern characters of matrix factorizations, see for example the dissertation of Xuan Yu [Yu13]. In most cases the n-th Chern character of a matrix factorization (φ, ψ) of f over A is given by a $\mathbb{Q}$-multiple of

$$\mathrm{Tr}((d\varphi d\psi)^n) \in \Omega^{2n}_{A/\mathbb{C}}/(df \wedge \Omega^{2n-1}_{A/\mathbb{C}}),$$

after applying the right isomorphisms. The approaches to get this trace are very different.

The aim of this thesis is to construct easily computable Chern characters for matrix factorizations, which fulfil variants of the three properties listed for Chern characters for vector bundles. Our Chern characters are lifts up to a sign of the ones computed in [PV12] in the cases of the power series ring. In particular we show that the before mentioned trace $\mathrm{Tr}((d\varphi d\psi)^n)$ already as element of $\Omega^{2n}_{A/\mathbb{C}}/(df \wedge d(\Omega^{2n-1}_{A/\mathbb{C}}))$ depends only on the isomorphism class of (φ, ψ). This was predicted by Buchweitz and van Straten in [BvS12], but to our knowledge not proved yet. We also find formulas for Chern characters which are not only invariant under isomorphism but even under homotopy equivalences. This construction works also (to some extent) for periodic complexes. Finally we show that the Chern characters for matrix factorization induce Chern characters for modules which have a eventually periodic resolution with periodic part given by a matrix factorization.

In **Chapter 1** we review the definition of Hochschild, Connes and cyclic homology of an algebra as well as some of their properties. These will be the spaces where some of our invariants will live in. For this we follow parts of [Lod98].
In **Chapter 2** we compute some examples of Hochschild homology and cyclic homology following the unpublished paper [BvS].
In **Chapter 3** introduce the notion of a Chern theory and revise the construction of Chern characters for finitely generated projective modules. In particular we show the additivity over short exact sequences and multiplicativity over tensor products. Most of this is covered in [Lod98], chapter 8. Then we compare the Chern theories for finitely generated projective modules and vector bundles.
In **Chapter 4** we introduce the (homotopy) category of matrix factorizations and some related concepts. Furthermore, we will discuss the structure of the homotopy category of matrix factorizations as triangulated pseudo tensor category. Our main source in this chapter is [Eis80].
In **Chapter 5** we introduce our definition of Chern characters for the (homotopy) category

of matrix factorizations. To the best of our knowledge this approach has not been used before. Then we prove functoriality, additivity over distinguished triangles and analyse the behaviour under tensor products.
In **Chapter 6** we look at the definition of the Chern characters used in [PV12], show that our Chern characters are lifts of those and talk about the relation between Chern characters and the Herbrand difference.
In **Chapter 7** we prove two vanishing results for our Chern characters for matrix factorizations. The first one has to do with the determinant of the matrices, the second one with the singularities of f.
In **Chapter 8** we show that our Chern characters for matrix factorizations induce Chern characters for modules which have an eventually periodic resolution and the periodic part is given by a matrix factorization.
In **Chapter 9** we generalize our Chern characters from matrix factorizations to periodic complexes of arbitrary period length. While this works in general, this theory has still strong limitations.

1 Hochschild, Connes and cyclic homology

In this chapter we introduce the notion of Hochschild, Connes and Cyclic homology and mention some of their properties. We will follow the book of [Lod98], and refer the reader to this book for the proofs and further information about these homology theories.

In this chapter k will always be a commutative ring, A will be an associative, unital (not necessarily commutative) k-algebra, M will be an A-bimodule and $\otimes$ will always mean $\otimes_k$.

1.1 Hochschild homology

1.1 Definition We define $C_n(A, M, k) := M \otimes_k A^{\otimes_k n} := M \otimes_k A \otimes_k A \otimes_k \ldots \otimes_k A$. We write $(a_0, a_1, \ldots, a_n)$ for the element $a_0 \otimes a_1 \otimes \ldots \otimes a_n \in C_n(A, M, k)$, where $a_0 \in M$ and $a_i \in A$ for $i \geq 1$. Furthermore, we define for each $n \geq 0$:

$$
\begin{aligned}
&b\colon C_{n+1}(A, M, k) \to C_n(A, M, k),\ b = \sum_{i=0}^{n+1} (-1)^i d_i,\ \text{where} \\
&d_0(a_0, a_1, a_2 \ldots, a_{n+1}) := (a_0 a_1, a_2, \ldots, a_{n+1}), \\
&d_i(a_0, \ldots, a_{n+1}) := (a_0, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_{n+1}),\ 1 \leq i \leq n, \\
&d_{n+1}(a_0, \ldots, a_{n+1}) := (a_{n+1} a_0, \ldots, a_n).
\end{aligned}
$$

The map b is called the *Hochschild boundary.* This gives a complex, the so called *Hochschild complex*:

$$C_\bullet(A, M, k) := \ldots \xrightarrow{b} M \otimes A \otimes A \xrightarrow{b} M \otimes A \xrightarrow{b} M \to 0 \ldots.$$

Finally, let $H_n(A, M, k)$ be the n-th homology of $C_\bullet(A, M, k)$. It is called the *n-th Hochschild homology of A over k with coefficients in M.* If it is clear from the context over which ring k we work, we omit it from the notation. We write $HH_n(A)$ for $H_n(A, A)$.

1.2 Lemma $H_n(A, M)$ is a module over the center of A, with respect to the action

$$a \cdot (m, a_1, \ldots, a_n) := (am, a_1, \ldots, a_n),\ a \in Z(A).$$

In particular, if A is commutative, $H_n(A, M)$ is an A-module.

1.3 Definition Assume A is commutative.

1. We define an A-linear, surjective map

$$\pi_n\colon H_n(A,M) \to M \otimes_A \Omega^n_{A/k},\ (a_0,\dots,a_n) \mapsto a_0 \otimes_A d(a_1)\dots d(a_n).$$

   Here $\Omega^n_{A/k}$ is the n-th exterior power of the module of Kähler differentials of A over k and d is the associated universal derivation.

2. We define a left action of S_n (the n-th symmetric group) on $C_n(A,M)$ by

$$\sigma(a_0,a_1,\dots,a_n) = (a_0,a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(n)}).$$

   (It depends on the way one is thinking about the composition in S_n for this to be a right or left action.)

3. We define an A-linear map

$$\varepsilon_n\colon M \otimes \Omega^n_{A/k} \to H_n(A,M),\ a_0 d(a_1)\dots d(a_n) \mapsto \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\sigma(a_0,a_1,\dots,a_n).$$

1.4 Proposition If A is commutative, then:

1. π_1 is an isomorphism.

2. ε_n is a right-inverse to π_n up to the factor $n!$, i.e. $\pi\circ\varepsilon_n = n!\ \mathrm{id}$. In particular $M\otimes_A \Omega^n_{A/k}$ is a direct summand of $H_n(A,M)$, if $\mathbb{Q}$ is a subring of k.

1.5 Proposition Set $A^e := A \otimes_k A^{op}$, where A^{op} is the opposite algebra of A. If A is flat as module over k, then we have an isomorphism of $Z(A)$-modules:

$$H_n(A,M) \cong \operatorname{Tor}^{A^e}_n(A,M).$$

1.6 Definition We define $D_n(A,M) \subset C_n(A,M) = M \otimes A^{\otimes n}$ to be the k-module generated by all elements of the form $m \otimes a_1 \otimes \dots \otimes a_n$ with $a_i = 1$ for some i (note that $i = 0$ is excluded). $(D_\bullet(A,M),b)$ is an acyclic subcomplex of $(C_\bullet(A,M),b)$, therefore the homology of $\overline{C}_\bullet(A,M) := (C_\bullet(A,M)/D_\bullet(A,M),b)$ is again $H_n(A,M)$. The complex $\overline{C}_\bullet(A,M)$ is called *normalized Hochschild complex* and sometimes it is more convenient than $C_\bullet(A,M)$.

1.7 Definition If B is a ring or a module, we will write $M_r(B)$, $r\in\mathbb{N}$, for the $r\times r$ square matrices with entries in B and we will write Tr for the trace map

$$\operatorname{Tr}\colon M_r(B) \to B,\ \alpha \mapsto \sum_{i=1}^r \alpha_{ii}.$$

We define for all r the k-linear *generalized trace map*:

$$\begin{aligned}\operatorname{tr}\colon M_r(M)\otimes M_r(A)^{\otimes n} &\to M \otimes A^{\otimes n},\\ (\alpha\otimes\beta\otimes\dots\otimes\gamma) &\mapsto \sum(\alpha_{i_0i_1}\otimes\beta_{i_1i_2}\otimes\dots\otimes\gamma_{i_ni_0}),\end{aligned}$$

where the sum runs over all possible indices with $1 \le i_0,\dots,i_n \le r$.

1.8 Lemma We have isomorphisms of k-modules

$$M_r(A) \cong M_r(k) \otimes A,\ M_r(M) \cong M_r(k) \otimes M.$$

For $u_0, \dots, u_n \in M_r(k)$, $a_0 \in M$ and $a_1, \dots a_n \in A$ we have:

$$\operatorname{tr}(u_0 a_0 \otimes \dots \otimes u_n a_n) = \operatorname{Tr}(u_0 \dots u_n) a_0 \otimes \dots \otimes a_n.$$

1.9 Definition Using Lemma 1.8 one can verify that $\operatorname{tr} \circ d_i = d_i \circ \operatorname{tr}$ for all i and thus we get the following maps on complexes and homologies

$$\operatorname{tr}\colon C_\bullet(M_r(A), M_r(M)) \to C_\bullet(A, M),$$
$$\operatorname{tr}_n\colon H_n(M_r(A), M_r(M)) \to H_n(A, M).$$

1.10 Proposition (Morita invariance) The map tr_n is an isomorphism with inverse inc_n, where

$$\operatorname{inc}\colon C_\bullet(A, M) \to C_\bullet(M_r(A), M_r(M))$$

is defined by writing the elements of A respectively M in the upper left corner and zeros elsewhere.

The next statement is not in the book of Loday and therefore we will give a proof.

1.11 Proposition If A is commutative, the composition

$$H_n(M_r(A), M_r(M)) \xrightarrow{\operatorname{tr}_n} H_n(A, M) \xrightarrow{\pi_n} M \otimes_A \Omega^n_{A/k}$$

is given by the formula

$$\pi_n \circ \operatorname{tr}_n(\alpha_0 \otimes \dots \otimes \alpha_n) = \operatorname{Tr}(\alpha_0 \otimes_A d\alpha_1 \dots d\alpha_n).$$

Here the universal differential d is applied componentwise. If in addition $M = A$, then $M \otimes_A \Omega^n_{A/k}$ is isomorphic to $\Omega^n_{A/k}$ and $\operatorname{Tr}(\alpha_0 \otimes_A d\alpha_1 \dots d\alpha_n)$ is mapped to $\operatorname{Tr}(\alpha_0 d\alpha_1 \dots d\alpha_n)$ under this isomorphism.

Proof. We use Lemma 1.8 and verify this on elements of the form $u_0 a_0 \otimes \dots \otimes u_n a_n$, where $u_0, \dots, u_n \in M_r(k)$, $a_0 \in M$ and $a_1, \dots a_n \in A$:

$$\begin{aligned}
&\pi_n \circ \operatorname{tr}(u_0 a_0 \otimes \dots \otimes u_n a_n) = \pi_n(\operatorname{Tr}(u_0 \dots u_n) a_0 \otimes \dots \otimes a_n) \\
&= \operatorname{Tr}(u_0 \dots u_n) a_0 \otimes_A da_1 \dots da_n = \operatorname{Tr}(u_0 a_0 \otimes_A u_1 da_1 \dots u_n da_n) \\
&= \operatorname{Tr}(u_0 a_0 \otimes_A d(u_1 a_1) \dots d(u_n a_n)).
\end{aligned}$$

□

1.12 Definition (Functoriality) Let A, B be two associative, unital k-algebras, $A \xrightarrow{f} B$ a homomorphism of k-algebras (which need not respect the unit). Then the maps

$$C_n(f)\colon A^{\otimes n+1} \to B^{\otimes n+1},\ (a_0, \dots a_n) \mapsto (f(a_0), \dots, f(a_n))$$

commute with the Hochschild boundary b and therefore give chain maps and maps on Hochschild homology

$$C_\bullet(f)\colon C_\bullet(A, A) \to C_\bullet(B, B),$$
$$H_n(f)\colon H_n(A, A) \to H_n(B, B).$$

1.13 Example If $g \in A$ is an invertible element, then the map

$$A \to A,\ a \mapsto gag^{-1}.$$

is a map of unital k-algebras, we call this map again g.

1.14 Proposition The map

$$H_n(g)\colon H_n(A,A) \to H_n(A,A),\ (a_0,\dots,a_n) \mapsto (ga_0g^{-1},\dots ga_ng^{-1})$$

is the identity on $H_n(A,A)$.

1.15 Definition 1. For all $n \in \mathbb{N}_0$ we define the following map

$$B_n\colon C_n(A,A) = A^{\otimes n+1} \to C_{n+1}(A,A) = A^{\otimes n+2},$$
$$(a_0,\dots a_n) \mapsto \sum_{i=0}^{n}(-1)^{ni}\left((1,a_i,\dots,a_n,a_0,\dots,a_{i-1}) - (a_i,1,a_{i+1},\dots,a_n,a_0,\dots,a_{i-1})\right).$$

2. As $B_nb + bB_{n+1} = 0$, we get induced maps on Hochschild homology, which we denote with B_n again. In Hochschild homology one can work in the normalized setting (Definition 1.6) and use the simpler map

$$\overline{B}_n\colon \overline{C}_n(A,A) = A \otimes (A/k)^{\otimes n} \to \overline{C}_{n+1}(A,A) = A \otimes (A/k)^{\otimes n+1},$$
$$(a_0,\dots a_n) \mapsto \sum_{i=0}^{n}(-1)^{ni}(1,a_i,\dots,a_n,a_0,\dots,a_{i-1}).$$

1.16 Lemma If A is commutative, one has the following commutative diagram

$$\begin{tikzcd}
\Omega^n_{A/k} \arrow[r, "\varepsilon_n"] \arrow[d, "d"] & HH_n(A) \arrow[r, "\pi_n"] \arrow[d, "B_n=\overline{B}_n"] & \Omega^n_{A/k} \arrow[d, "(n+1)d"] \\
\Omega^{n+1}_{A/k} \arrow[r, "\varepsilon_{n+1}"] & HH_{n+1}(A) \arrow[r, "\pi_{n+1}"] & \Omega^{n+1}_{A/k}
\end{tikzcd}$$

in this sense B_n lifts the exterior derivative to the level of Hochschild homology.

1.17 Definition Let $p,q \in \mathbb{N}_0$, $n = p+q$ and S_n the n-th symmetric group.

1. A permutation $\sigma \in S_n$ is called a *p,q-shuffle*, if $\sigma(1) < \sigma(2) < \ldots < \sigma(p)$ and $\sigma(p+1) < \ldots < \sigma(p+q)$.

2. We denote with $S_{p,q} \subset S_n$ the set of all p,q-shuffles.

For the rest of this section A' is another associative, unital k-algebra.

1.18 Definition We define the *p,q-shuffle product map* sh_{pq} by

$$C_p(A,A) \otimes C_q(A',A') \to C_{p+q}(A\otimes A', A\otimes A'),$$
$$(a_0,\dots,a_p)\otimes(a'_0,\dots,a'_q) \mapsto \sum_{\sigma\in S_{p,q}} \mathrm{sgn}(\sigma)\sigma(a_0\otimes a'_0, a_1\otimes 1_{A'},\dots,a_p\otimes 1_{A'}, 1_A\otimes a'_1,\dots,1_A\otimes a'_q).$$

We also write $(a_0,\dots,a_p)\times(a'_0,\dots a'_q)$ instead of $\mathrm{sh}_{p,q}((a_0,\dots,a_p)\otimes(a'_0,\dots a'_q))$.

1.19 Lemma For $x \in C_p(A,A)$ and $y \in C_q(A',A')$ we have

$$b(\mathrm{sh}_{p,q}(x \otimes y)) = \mathrm{sh}_{p-1,q}(b(x) \otimes y) + (-1)^p \,\mathrm{sh}_{p,q-1}(x \otimes b(y)).$$

1.20 Definition We define

$$\mathrm{sh}_n = \bigoplus_{p+q=n} \mathrm{sh}_{p,q} \colon (C_\bullet(A,A) \otimes C_\bullet(A',A'))_n \to C_n(A \otimes A').$$

By Lemma 1.19 this gives a map of complexes sh, which we call *the shuffle product map.*

1.21 Proposition If A' and $HH_n(A')$ are flat over k for all n, then the map

$$\mathrm{sh}_* \colon \bigoplus_{p+q=n} HH_p(A) \otimes HH_q(A') \to HH_n(A \otimes A')$$

induced by sh is an isomorphism for all n.

1.22 Definition For $A = A'$ and A commutative we can compose the maps $\mathrm{sh}_{p,q}$, with the map

$$C_n(A \otimes A, A \otimes A) \to C_n(A,A)$$

induced by the multiplication map (of k-algebras) $A \otimes A \to A$

$$C_p(A,A) \otimes C_q(A,A) \to C_n(A \otimes A, A \otimes A) \to C_n(A,A).$$

By 1.19 this composition induces a map on Hochschild homology, which we call $\mathrm{sh}_{p,q}$ again

$$\begin{aligned} HH_p(A) \otimes HH_q(A) &\to HH_{p+q}(A), \\ (a_0, \dots, a_p) \otimes (a'_0, \dots a'_q) &\mapsto \sum_{\sigma \in S_{p,q}} \mathrm{sgn}(\sigma)\sigma(a_0a'_0, a_1, \dots, a_p, a'_1, \dots, a'_q). \end{aligned}$$

This gives $HH_*(A)$ the structure of a graded commutative algebra.

1.23 Lemma If A is commutative (and $p!$, $q!$ are invertible in k), the following diagram commutes

$$\begin{tikzcd} \Omega^p_{A/k} \times \Omega^q_{A/k} \arrow[r, "\varepsilon_p \times \varepsilon_q"] \arrow[d, "\wedge"] & HH_p(A) \times HH_q(A) \arrow[r, "\pi_p \times \pi_q"] \arrow[d, "\mathrm{sh}_{p,q}"] & \Omega^p_{A/k} \times \Omega^q_{A/k} \arrow[d, "\frac{n!}{p!q!}\wedge"] \\ \Omega^{p+q}_{A/k} \arrow[r, "\varepsilon_{p+q}"] & HH_{p+q}(A) \arrow[r, "\pi_{p+q}"] & \Omega^{p+q}_{A/k} \end{tikzcd}$$

1.2 Connes homology

1.24 Definition The k-linear map

$$\begin{aligned} t = t_n \colon A^{\otimes n+1} &\to A^{\otimes n+1}, \\ (a_0, \dots, a_n) &\mapsto (-1)^n (a_n, a_0, \dots, a_{n-1}) \end{aligned}$$

is called the *cyclic operator.*

1.25 Lemma We have $(1-t)b' = b(1-t)$, where $b' = \sum_{i=0}^{n-1}(-1)^i d_i = b - (-1)^n d_n$ with d_i and b the Hochschild differential as in Definition 1.1.

1.26 Definition Because of Lemma 1.25, the Hochschild differential induces a well defined map $b\colon A^{\otimes n+1}/(1-t) \to A^{\otimes n}/(1-t)$. Therefore, we get a complex

$$C^\lambda_\bullet(A,k) := \ldots \xrightarrow{b} A^{\otimes 3}/(1-t) \xrightarrow{b} A^{\otimes 2}/(1-t) \xrightarrow{b} A \to 0.$$

This complex is called *Connes complex* and its n-th homology group is denoted with $H^\lambda_n(A,k)$. We will omit k from the notation if it is clear from the context.

1.27 Definition We define $D^\lambda_n(A) \subset C^\lambda_n(A) = A^{\otimes n+1}/(1-t)$ to be the subcomplex generated as k-module by the elements $(a_0,\ldots,a_n)$ with $a_i = 1$ for some $i \geq 0$. This gives a subcomplex of $C^\lambda_\bullet(A)$ and hence we get a new complex by dividing out $D^\lambda_\bullet(A)$. We denote the n-th homology of this new complex with $\overline{H}^\lambda_n(A)$. This homology is called *reduced Connes homology*.

1.28 Definition If A is commutative we get well defined k-linear maps

$$\pi_n\colon H^\lambda_n(A) \to \Omega^n_{A/k}/d(\Omega^{n-1}_{A/k}),\ (a_0,\ldots,a_n) \mapsto a_0 d(a_1)\ldots d(a_n).$$

And as $D^\lambda_n(A) \subset \ker(\pi)$ this induces a map

$$\pi_n\colon \overline{H}^\lambda_n(A) \to \Omega^n_{A/k}/d(\Omega^{n-1}_{A/k}),\ (a_0,\ldots,a_n) \mapsto a_0 d(a_1)\ldots d(a_n).$$

Let us remark, that those maps are not well defined, if we do not divide out $d(\Omega^{n-1}_{A/k})$ in the right hand side.

1.29 Lemma We have $\operatorname{tr}\circ t = t \circ \operatorname{tr}$ as maps from $M_r(A)^{\otimes n} \to A^{\otimes n}$.

1.30 Definition Lemma 1.29 shows that the generalized trace map gives a map of complexes $C^\lambda_\bullet(M_r(A)) \to C^\lambda_\bullet(A)$, so we get well-defined maps

$$\operatorname{tr}_n\colon H^\lambda_n(M_r(A)) \to H^\lambda_n(A),\ (\alpha\otimes\beta\otimes\ldots\otimes\gamma) \mapsto \sum(\alpha_{i_0i_1}\otimes\beta_{i_1i_2}\otimes\ldots\otimes\gamma_{i_ni_0}).$$

We have $\operatorname{tr}\colon D^\lambda_n(M_r(A)) \to D^\lambda_n(A)$ and hence we also get a map (see Definition 1.27)

$$\operatorname{tr}_n\colon \overline{H}^\lambda_n(M_r(A)) \to \overline{H}^\lambda_n(A),\ (\alpha\otimes\beta\otimes\ldots\otimes\gamma) \mapsto \sum(\alpha_{i_0i_1}\otimes\beta_{i_1i_2}\otimes\ldots\otimes\gamma_{i_ni_0}).$$

1.31 Proposition (Morita invariance) For $\mathbb{Q} \subset k$ the map $\operatorname{tr}_n\colon H^\lambda_n(M_r(A)) \to H^\lambda_n(A)$ is an isomorphism. If in addition k is a direct summand of A as k-module, then the same is true for reduced Connes homology.

1.32 Proposition If A is commutative, then the composition

$$\pi_n \circ \operatorname{tr}_n\colon H^\lambda_n(M_r(A)) \to \Omega^n_{A/k}/d(\Omega^{n-1}_{A/k})$$

maps $(\alpha_0,\ldots,\alpha_n)$ to $\operatorname{Tr}(\alpha_0 d\alpha_1\ldots d\alpha_n)$ (cf. Proposition 1.11). The same is true for reduced Connes homology.

1.33 Definition (Functoriality) Let A, B be two associative, unital k-algebras, $A \xrightarrow{f} B$ a homomorphism of k-algebras (which does not need to respect the unit). Then we have maps

$$C_n(f)\colon A^{\otimes n+1} \to B^{\otimes n+1},\ (a_0, \dots a_n) \mapsto (f(a_0), \dots, f(a_n)).$$

These maps commute with b and t and therefore give a map of complexes and maps on Connes homology

$$C^\lambda_\bullet(f)\colon C^\lambda_\bullet(A) \to C^\lambda_\bullet(B),$$
$$C^\lambda_n(f)\colon H^\lambda_n(A) \to H^\lambda_n(B).$$

1.34 Proposition If $\mathbb{Q} \subset k$ and $g \in A$ is invertible, then the maps

$$A^{\otimes n+1} \to A^{\otimes n+1},\ a_0 \otimes \dots \otimes a_n \mapsto g a_0 g^{-1} \otimes \dots \otimes g a_n g^{-1}$$

give a chain map from $C^\lambda_n(A)$ to itself. The induced map on the homology $H^\lambda_n(A)$ is the identity for all $n \geq 0$.

1.3 Cyclic homology

1.35 Definition Let N be the map $1 + t + \dots + t^n\colon A^{\otimes n+1} \to A^{\otimes n+1}$, where t is the cyclic operator from Definition 1.24.

1.36 Lemma One can verify the following equations

$$(1-t)N = N(1-t) = 0, \qquad b^2 = 0, \qquad (b')^2 = 0,$$
$$(1-t)b' = b(1-t), \qquad Nb = b'N.$$

Here b is the Hochschild boundary and $b' = \sum_{i=0}^{n-1}(-1)^i d_i = b - (-1)^n d_n$ as in Lemma 1.25 (some of this equations were already mentioned).

1.37 Definition Lemma 1.36 is precisely saying, that we get the following bicomplex

$$\begin{tikzcd}
\vdots \arrow[d, "b"] & \vdots \arrow[d, "-b'"'] & \vdots \arrow[d, "b"] & \vdots \arrow[d, "-b'"'] & \\
A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & \dots \arrow[l, "N"'] \\
A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & \dots \arrow[l, "N"'] \\
A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & A \arrow[l, "1-t"'] & \dots \arrow[l, "N"']
\end{tikzcd}$$

This bicomplex is called the *cyclic bicomplex* and we will denote it by $CC_{\bullet\bullet}(A)$. The leftmost A is supposed to be in bidegree $(0,0)$, so $CC_{pq}(A) = C_p(A) = A^{\otimes p+1}$. Furthermore, we define the n-th *cyclic homology group* $HC_n(A)$ of A over k to be

$$HC_n(A) = H_n(\mathrm{Tot}(CC(A))),$$

where on the right hand side we mean the n-th homology of the total complex.

1.38 Definition There is a natural surjection p from $\mathrm{Tot}(CC(A))$ to $C^{\lambda}_{\bullet}(A)$, which is given by the quotient map $A^{\otimes\bullet} \to A^{\otimes\bullet}/(1-t)$ on the first column of $CC(A)$ and 0 on the other columns.

1.39 Proposition Suppose $\mathbb{Q} \subset k$. Then the induced maps $p_n\colon HC_n(A) \to H^{\lambda}_n(A)$ are isomorphisms for all n.

1.40 Lemma The generalized trace map $\mathrm{tr}\colon M_r(A)^{\otimes n} \to A^{\otimes n}$ gives a map of bicomplexes $CC(M_r(A)) \to CC(A)$.

1.41 Proposition (Morita invariance) The maps on homology induced by tr

$$\mathrm{tr}_n\colon H_n(M_r(A), M_r(M)) \to H_n(A, M)$$

are isomorphisms with inverse inc_n for all n, where

$$\mathrm{inc}\colon CC(A) \to CC(M_r(A))$$

is defined by writing the element of A in the upper left corner and zeros elsewhere.

1.42 Definition (Functoriality) Let A, B be two associative, unital k-algebras, $A \xrightarrow{f} B$ a homomorphism of k-algebras (which does not need to respect the unit). Then we had maps

$$C_n(f)\colon A^{\otimes n+1} \to B^{\otimes n+1},\ (a_0, \dots a_n) \mapsto (f(a_0), \dots, f(a_n)).$$

These maps commute with b, b' and t and therefore give a map of bicomplexes and maps on cyclic homology

$$CC(f)\colon CC(A) \to CC(B),$$
$$HC_n(f)\colon HC_n(A) \to HC_n(B).$$

1.43 Proposition If $g\colon A \to A$ is the map, which conjugates with an invertible element $g \in A$, then

$$H_n(g)\colon HC_n(A, A) \to HC_n(A, A)$$

is the identity on $HC_n(A, A)$.

Furthermore, any map $f\colon A \to B$ induces an isomorphism on Hochschild homology if and only if it induces an isomorphism on cyclic homology.

2 Examples of Hochschild- and cyclic homology groups

In this chapter k will always be a commutative ring, A will be an associative, unital, commutative k-algebra.

2.1 Universal finite differential module

In this section we introduce the universal finite differential module. We need this module, because Kähler differentials for the rings $\mathbb{C}\{X_1, \dots, X_n\}$ or $\mathbb{C}[\![X_1, \dots, X_n]\!]$ are rather unpleasant, e.g. not finitely generated as modules, while the universal finite differentials are.

Recall, that the module of Kähler differentials of A over k is defined to be an A-module $\Omega_{A/k}$ with a k-linear derivation $d\colon A \to \Omega_{A/k}$ with the following universal property:
For any A-module M and any k-linear derivation $D\colon A \to M$, there is a unique A-linear map making the following diagram commute

$$\begin{tikzcd} A \arrow[r, "D"] \arrow[d, "d"] & M \\ \Omega_{A/k} \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

2.1 Definition The universal finite differential module of A over k is a finitely generated A-module $\Omega^{\text{fin}}_{A/k}$ with a k-linear derivation $d\colon A \to \Omega^{\text{fin}}_{A/k}$, which has the same universal property as the module of Kähler differentials of A over k, but we require the universal property only to hold for finitely generated A-modules M. For exterior powers we will write Ω^i_{fin} instead of $(\Omega^{\text{fin}}_{A/k})^i$, if A and k are understood.

Unlike $\Omega_{A/k}$, $\Omega^{\text{fin}}_{A/k}$ does not always exist. However, it does exist in the two cases we are interested in: If $k = \mathbb{C}$ and $A = \mathbb{C}\{X_1, \dots, X_n\}$ or $\mathbb{C}[\![X_1, \dots, X_n]\!]$, then

$$\Omega^{\text{fin}}_{A/k} = \bigoplus_{i=1}^{n} A \cdot dX_i.$$

For the rings $A = \mathbb{C}[X_1, \dots, X_n]$ or $\mathbb{C}[X_1, \dots, X_n]_{(X_1,\dots,X_n)}$ we have changed nothing, i.e.

$$\Omega_{A/k} = \Omega^{\text{fin}}_{A/k} = \bigoplus_{i=1}^{n} A \cdot dX_i.$$

2.2 Definition From the universal property of the Kähler differentials with $M = \Omega^{\text{fin}}_{A/k}$ and D the derivation from the definition of $\Omega^{\text{fin}}_{A/k}$ we get a unique map, which commutes with d:

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "d"] & \Omega^{\text{fin}}_{A/k} \\ \Omega_{A/k}. \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

2.2 The relative de Rham complex

In this section we introduce the relative de Rham complex and some related concepts. We will need this complex to compute examples for cyclic homology. Furthermore, it will be used in the second vanishing result of Chapter 7.

In this section we assume $k = \mathbb{C}$, $A = \mathbb{C}[X_1, \dots, X_n]$, $\mathbb{C}\{X_1, \dots, X_n\}$, $\mathbb{C}[\![X_1, \dots, X_n]\!]$ or $\mathbb{C}[X_1, \dots, X_n]_{(X_1,\dots,X_n)}$ and $f \in A$ is a polynomial. We write f_i for $\partial_i f$. The last three cases for A are local and we will denote the maximal ideal with $\mathfrak{m}$. Sometimes we write complex instead of cochain complex. Ω^i will mean $\Omega^i_{A/k}$ and Ω^i_{fin} will mean $(\Omega^{\text{fin}}_{A/k})^i$.

2.3 Definition

1. We define $J_f = (f_1, \dots, f_n)$ and call it the *Milnor ideal of f*. We define $T_f = (f, f_1, \dots, f_n)$ and call it the *Tjurina ideal of f*.

2. We say f has *isolated critical locus*, if the Krull-dimension of A/J_f is zero. We say f has *isolated singular locus*, if the Krull-dimension of A/T_f is zero. Note, that the Krull dimension is zero if and only if the $\mathbb{C}$-vector space dimension is finite.

3. A/J_f respectively A/T_f are called the *Milnor-* respectively *Tjurina-algebra of f*.

4. If f has isolated critical locus, we define $\mu_f = \dim_{\mathbb{C}}(A/J_f)$ and call it the *Milnor number of f*. If f has isolated singular locus, we define $\tau_f = \dim_{\mathbb{C}}(A/T_f)$ and call it the *Tjurina number of f*.

If f is quasi-homogeneous with total weight w and weight w_i for X_i, then

$$wf = \sum_{i=1}^{n} w_i X_i f_i.$$

Therefore $f \in J_f$, $J_f = T_f$ and we do not need to distinguish between the Tjurina and the Milnor ideal.

The following lemma is often helpful, if A is one of the three local rings.

2.4 Lemma For an ideal $I \subsetneq A$ the following are equivalent:

1. $\dim_{\mathbb{C}}(A/I) < \infty$,

2. $\mathfrak{m}^k \subset I$ for some k,

3. $\mathfrak{m} = \sqrt{I}$,

4. For all i there is a k_i with $X_i^{k_i} \in I$.

2.5 Lemma If $I \subset \mathbb{C}[X_1, \dots, X_n]_{(X_1,\dots,X_n)}$ is an ideal, then the following are equivalent:

1. $\dim_{\mathbb{C}}(\mathbb{C}[X_1, \dots, X_n]_{(X_1,\dots,X_n)}/I) < \infty$,

2. $\dim_{\mathbb{C}}(\mathbb{C}\{X_1, \dots, X_n\}/I.\mathbb{C}\{X_1, \dots, X_n\}) < \infty$,

3. $\dim_{\mathbb{C}}(\mathbb{C}[\![X_1, \dots, X_n]\!]/I.\mathbb{C}[\![X_1, \dots, X_n]\!]) < \infty$.

Proof. The implications from 1. to 2. to 3. follow from the fourth condition in Lemma 2.4. The implication 3. to 1. follows, because condition three from Lemma 2.4 is the same as the exactness of the sequence $0 \to A/\sqrt{I} \to A/m \to 0$ and $\mathbb{C}[\![X_1, \dots, X_n]\!]$ is faithfully flat over $\mathbb{C}[X_1, \dots, X_n]_{(X_1,\dots,X_n)}$. $\square$

The next proposition is a non trivial consequence of the theory of complex spaces.

2.6 Proposition ([GLS07], Lemma 2.3) If $A = \mathbb{C}\{X_1, \dots, X_n\}$, then f has isolated critical locus if and only if it has isolated singular locus.

Because of Lemma 2.5 the previous proposition also holds for $A = \mathbb{C}[\![X_1, \dots, X_n]\!]$ or $\mathbb{C}[X_1, \dots, X_n]_{(X_1,\dots,X_n)}$. For $A = \mathbb{C}[X_1, \dots, X_n]$ this is wrong. As a counter example consider $f(X, Y) = \frac{3}{2}X^2 - Y^2 + (X^3 - 2XY^2 - X^2Y^2)$.

2.7 Definition We get the following three complexes of A-modules:

$$C^\bullet_{\mathrm{dR}} : 0 \to A \xrightarrow{d} \Omega^1_{\mathrm{fin}} \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_{\mathrm{fin}} \to 0,$$
$$C^\bullet_{\mathrm{df}} : 0 \to A \xrightarrow{df\wedge} \Omega^1_{\mathrm{fin}} \xrightarrow{df\wedge} \dots \xrightarrow{df\wedge} \Omega^n_{\mathrm{fin}} \to 0,$$
$$C^\bullet_{f} : 0 \to A \xrightarrow{d} \Omega^1_{f} \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_{f} \to 0,$$

where $\Omega^i_f = \Omega^i_{\mathrm{fin}}/df \wedge \Omega^{i-1}_{\mathrm{fin}}$. We denote the i-th cohomology of this complexes with H^i_{dR}, H^i_{df} and H^i_f respectively. The last complex is called *relative de Rham complex.*

2.8 Definition Let $r_1, \dots, r_i$ be elements of a ring R and $e_j \in R^i$ the element with a 1 as j-th entry and zeros elsewhere.

1. *The Koszul complex* $C_\bullet(r_1, \ldots, r_i)$ is given by

$$C_m = C_m(r_1, \ldots, r_n) = \bigwedge^m R^i \text{ with differential}$$
$$\partial_m \colon C_m \to C_{m-1}, \quad e_{i_1} \wedge \ldots \wedge e_{i_m} \mapsto \sum_{j=1}^m (-1)^{j+1} r_j e_{i_1} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_m}.$$

We denote the m-th homology of this complex with $H_m(r_1, \ldots, r_i)$.

2. *The Koszul cochain complex* $C^\bullet(r_1, \ldots, r_i)$ is given by

$$C^m = C^m(r_1, \ldots, r_n) = \bigwedge^m R^i \text{ with differential}$$
$$\partial^m \colon C^m \to C^{m+1}, \quad e_{i_1} \wedge \ldots \wedge e_{i_m} \mapsto r \wedge e_{i_1} \wedge \ldots \wedge e_{i_m} \text{ with } r = (r_1, \ldots, r_i) \in R^i.$$

This differential is also called exterior multiplication with r. We denote the m-th cohomology of this cochain complex with $H^m(r_1, \ldots, r_i)$.

The Koszul chain complex and complex are dual to each other, in the sense, that $C^\bullet(r_1, \ldots, r_n)$ is isomorphic to $\hom_R(C_\bullet(r_1, \ldots, r_n), R)$ as cochains complex and the other way around (as complexes). For more details on the Koszul complex we refer to Section 1.6 of [BH93].

2.9 Proposition ([BH93], Theorem 1.6.10 and Proposition 1.6.16) Let $r_1, \ldots, r_i$ be elements of a ring R.

1. If the ideal $(r_1, \ldots, r_i)$ contains a weak R-sequence of length m, then

$$H_{i+1-j}(r_1, \ldots, r_i) = 0 \text{ for } 1 \leq j \leq m.$$

2. If the ideal $(r_1, \ldots, r_i)$ contains a weak R-sequence of length m, then

$$H^{j-1}(r_1, \ldots, r_i) = 0 \text{ for } 1 \leq j \leq m.$$

3. $H_j(r_1, \ldots, r_i) \cong H^{i-j}(r_1, \ldots, r_i)$ for $0 \leq j \leq i$.

Because $df = \sum_{i=1}^n f_i dX_i$ the complex C_{df} is precisely the Koszul cochain complex for the elements $f_1, \ldots, f_n$. Therefore, Proposition 2.9 implies the following lemma:

2.10 Lemma If J_f contains a weak A-sequence of length i, then $H^j_{\mathrm{df}} = 0$ for $1 \leq j < i$.

We denote with Kdim the Krull dimension. If $\mathrm{Kdim}(A/J_f) \leq i$, then J_f contains a weak A-sequence of length $n - i$. For A local this follows from the formula

$$\operatorname{grade} J_f \geq \operatorname{depth} A - \mathrm{Kdim}(A/J_f) = n - i.$$

This statement is Exercise 1.2.23 from [BH93] and $\operatorname{grade} J_f$ is by definition the maximal length of an A-sequence in J_f ([BH93], Definition 1.2.6). If A is the polynomial ring, this is also true, one can see this as follows:
By 1.2.10 from [BH93] there is a prime ideal $\mathfrak{p} \supset J_f$ with $\operatorname{grade} J_f = \operatorname{depth} A_{\mathfrak{p}}$. Now pick a maximal ideal $\mathfrak{n} \supset \mathfrak{p}$. Then by 1.2.23 from [BH93] we have

$$\operatorname{grade} \mathfrak{p}.A_{\mathfrak{n}} \geq \operatorname{depth} A_{\mathfrak{n}} - \operatorname{Kdim}(A_{\mathfrak{n}}/\mathfrak{p}.A_{\mathfrak{n}}) \geq n - i.$$

Since the localization of an $A_{\mathfrak{n}}$-sequence is an $A_{\mathfrak{p}}$-sequence ([BH93], Corollary 1.1.3) we get

$$\operatorname{grade} J_f = \operatorname{depth} A_{\mathfrak{p}} = \operatorname{grade} \mathfrak{p}.A_{\mathfrak{p}} \geq \operatorname{grade} \mathfrak{p}.A_{\mathfrak{n}} \geq n - i.$$

For the rest of this section we want to show the vanishing for some of the cohomology groups H^i_f under assumptions on f.

2.11 Definition We denote with $\mathbb{C}\langle t, \partial_t \rangle$ the polynomial ring in two not commuting variables and define $D = \mathbb{C}\langle t, \partial_t \rangle / (t\partial_t - \partial_t t - 1)$.

Notice, that $\omega \in \Omega^i$ defines a class in H^i_f if and only if $d\omega = df \wedge \eta$ for some $\eta \in \Omega^i$. Because $d(f\omega) = df \wedge \omega + f d\omega = df \wedge (\omega + f\eta)$, we see that H^i_f has the structure of a $\mathbb{C}[t]$-module, where the multiplication with t is the multiplication with f. Depending on A this even gives the structure of a $\mathbb{C}[t]_{(t)}$, $\mathbb{C}\{t\}$, or $\mathbb{C}[\![t]\!]$-module. We want to show next, that under some assumptions on f, H^i_f has the structure of a D-module.

2.12 Definition For each $0 \leq i \leq n-1$ with $H^i_{\mathrm{df}} = 0$ we define a $\mathbb{C}$-linear map

$$\partial_t = \partial_t^{(i)} \colon H^i_f \to \Omega^i / (df \wedge \Omega^{i-1} + d(\Omega^{i-1})),$$

which maps (the class of) $\omega \in \Omega^i$ to (the class of) $\eta \in \Omega^i$ if and only if $d\omega = df \wedge \eta$.

For all four possible choices for A we have $H^j_{\mathrm{dR}} = 0$ for $j > 0$, which is important for the next lemma. In particular $d \colon \Omega^{n-1} \to \Omega^n$ is surjective and hence $H^n_f = 0$.

2.13 Lemma

1. If $H^{i+1}_{\mathrm{df}} = 0$, then ∂_t maps to $H^i_f \subset \Omega^i / (df \wedge \Omega^{i-1} + d(\Omega^{i-1}))$.
2. If $H^i_{\mathrm{dR}} = H^i_{\mathrm{dR}}(A/\mathbb{C}) = 0$, then ∂_t is injective.
3. If $H^{i+1}_{\mathrm{dR}} = 0$, then H^i_f is contained in the image of ∂_t.

Proof.

1. If $\partial_t \omega = \eta$, then $d\omega = df \wedge \eta$. By applying d we get $0 = -df \wedge d\eta$. If $H^{i+1}_{\mathrm{df}} = 0$, it follows $d\eta = df \wedge \eta'$, which precisely says $\eta \in H^i_f$.
2. If $\partial_t \omega = 0$, then $d\omega = df \wedge \eta$ for some $\eta \in df \wedge \Omega^{i-1} + d(\Omega^{i-1})$. Write $\eta = df \wedge \eta' + d\eta''$. We get $d\omega = df \wedge d\eta''$, hence $d(\omega + df \wedge \eta'') = 0$. With our assumption $H^i_{\mathrm{dR}} = 0$, we get $\omega = df \wedge \eta'' + d\eta'''$, which exactly says, that the class of ω is zero in H^i_f.

3. Let $\eta \in H_f^i$ be arbitrary. Then $d(df \wedge \eta) = -(df \wedge df \wedge \eta') = 0$. Because $H_{\mathrm{dR}}^{i+1} = 0$, this implies $df \wedge \eta = d\omega$, which says $\omega \in H_f^i$ and $\partial_t \omega = \eta$.

$\square$

2.14 Lemma If $H_{\mathrm{df}}^i = 0$ and $H_{\mathrm{df}}^{i+1} = 0$, then the multiplication with f and the map ∂_t give H_f^i the structure of a D-module, i.e. $\partial_t(f\omega) - f\partial_t\omega = \omega$ for each $\omega \in H_f^i$.

Proof. Since $\omega \in H_f^i$ we have $d\omega = df \wedge \eta$ for some $\eta \in \Omega^i$. This implies $\partial_t \omega = \eta$. Now compute $d(f\omega) = df \wedge \omega + f d\omega = df \wedge (\omega + f\eta)$, hence $\partial_t(f\omega) = \omega + f\eta$. $\square$

2.15 Proposition If f is quasi homogeneous, $H_{\mathrm{df}}^i = 0$, $H_{\mathrm{df}}^{i+1} = 0$ and $i \geq 1$, then $H_f^i = 0$.

Proof. We know that $\partial_t \colon H_f^i \to H_f^i$ is bijective, because $H_{\mathrm{df}}^i = 0$, $H_{\mathrm{df}}^{i+1} = 0$, $H_{\mathrm{dR}}^i = 0$ and $H_{\mathrm{dR}}^{i+1} = 0$. Now since f is quasi homogeneous, there are weights $w_i \in \mathbb{Q}_+$, such that every occurring monomial of f has degree $w = 1$ for $\deg(X_i) = w_i$. Then we get a grading on Ω^i by setting $\deg(X_i) = \deg(dX_i) = w_i$. Because $\deg(df) = \deg(f) = 1$, the subspaces $df \wedge \Omega^{i-1}$ and $d(\Omega^{i-1})$ are generated as $\mathbb{C}$-vector spaces by homogeneous elements. Therefore, we get a (non negative) grading on H_f^i. Now pick $0 \neq \omega \in H_f^i$ homogeneous of minimal degree. Then $d\omega = df \wedge \eta$ for some η and hence $\eta = \partial_t(\omega) \in H_f^i$. Because $d\omega$ has either same degree as ω or $d\omega = 0$, we see that η has either degree $\deg(\omega) - 1$ or $\eta = 0$. This implies, that ∂_t can not be injective, unless $H_f^i = 0$. $\square$

2.16 Corollary If f is quasi homogeneous and $\mathrm{Kdim}(A/T_f) \leq i$, then $H_f^j = 0$ for $1 \leq j < n - i - 1$.

Proof. Because f is quasi homogeneous, we have $A/T_f = A/J_f$ and we have seen, that $H_{\mathrm{df}}^j = 0$ for $1 \leq j < n - i$ under the assumption $\mathrm{Kdim}(A/J_f) \leq i$. $\square$

The assumption of f being quasi homogeneous is not needed, if $\mathrm{Kdim}(A/T_f) = 0$ and A is the ring of (convergent) power series. The following proposition was proven by Brieskorn in [Bri70], Proposition 1.7. Brieskorn showed it only for the $\mathbb{C}\{t\}$ torsion free part of H_f^n and later Sebastiani ([Seb70], Corollaire 1) showed, that H_f^n is torsion free.

2.17 Proposition ([Bri70], Proposition 1.7) If $A = \mathbb{C}\{X_1, \ldots, X_n\}$ and $\dim_{\mathbb{C}} A/T_f < \infty$. Then

1. If $n \geq 2$, then $H_f^0 = \mathbb{C}\{f\}$.
2. $H_f^i = 0$ for $i \neq 0, n-1$.
3. H_f^i is a free $\mathbb{C}\{t\}$-module of rank μ_f.

Then in Proposition 3.2 of [Bri70] the author shows, that Proposition 2.17 also holds for the ring $\mathbb{C}[\![X_1, \ldots, X_n]\!]$. He proves, that the homology for the completion $\mathbb{C}[\![X_1, \ldots, X_n]\!]$ of $\mathbb{C}\{X_1, \ldots, X_n\}$ at the maximal ideal $(X_1, \ldots, X_n)$ is the completion of the homology over $\mathbb{C}\{X_1, \ldots, X_n\}$ as $\mathbb{C}\{t\}$-module at the maximal ideal (t).

2.3 Examples for Hochschild homology

In this section k is a field, $A = k[X_1, \ldots, X_n]$, $f \in A \backslash k$, $K = k[t]$, $B = A/(f)$ and we see A as a K-module via $t \cdot a = fa$. We write f_i for $\partial_i f = \partial_{X_i} f$ and Ω^i for $\Omega^i_{A/k}$

We want to compute the following three Hochschild homology groups:

$$H_m(A, A, k),\ H_m(A, A, K) \text{ and } H_m(B, B, k) \text{ for all } m.$$

Clearly A and B are flat over k, because k is a field. Also A is flat over K, because A is torsion free and torsion free modules over a principal ideal domain are flat. Therefore, by Proposition 1.5 we have

$$H_m(A, A, k) \cong \mathrm{Tor}_m^{A^e}(A, A) \text{ with } A^e = A \otimes_k A^{op} = A \otimes_k A$$

and similarly for our other two cases. Hence to compute $H_m(A, A, k)$, we are interested in a free resolution of A as A^e-module. We will get this resolutions from a Koszul complex.

By Proposition 2.9 the Koszul complex corresponding to a regular sequence $r_1, \ldots, r_i$ is a free resolution of $H_0(r_1, \ldots, r_i) = R/(r_1, \ldots, r_i)$. We can use this in our specific cases as follows:
$A^e = A \otimes_k A \cong k[Y_1, \ldots, Y_n, Z_1, \ldots, Z_n]$, where $Y_i = X_i \otimes 1$ and $Z_i = 1 \otimes X_i$. There is a surjective map

$$\Theta\colon A^e \cong k[Y_1, \ldots, Y_n, Z_1, \ldots, Z_n] \to A = k[X_1, \ldots, X_n],\ Y_i \mapsto X_i,\ Z_i \mapsto X_i.$$

The kernel is the ideal generated by the regular sequence $Y_1 - Z_1, \ldots, Y_n - Z_n \in A^e$. In conclusion $C_\bullet(Y_1 - Z_1, \ldots, Y_n - Z_n)$ is a free resolution of $A^e/(Y_1 - Z_1, \ldots, Y_n - Z_n) \cong A$ as A^e-module. Now to compute $H_m(A, A, k) \cong \mathrm{Tor}_m^{A^e}(A, A)$ we have to tensor over A^e with A. Note that $A^e \otimes_{A^e} A \cong A$ and the isomorphism is mapping both $Y_i, Z_i \in A^e$ to X_i. Therefore, after tensoring we have the complex with $\bigwedge^m A^n$ in degree m and zero differential, because the Koszul differential is given by multiplication with $Y_i - Z_i$, which becomes $X_i - X_i = 0$ after tensoring. Putting everything together we proved:

2.18 Proposition $H_m(A, A, k)$ is a free A-module of rank $\binom{n}{m}$. In particular if $A = k$, then $H_0(k, k, k) \cong k$ and $H_m(k, k, k) = 0$ for $m > 0$.

Now we continue with the computation of

$$H_m(A, A, K) \cong \mathrm{Tor}_m^U(A, A) \text{ with } U = A \otimes_K A^{op} = A \otimes_K A.$$

Note that $U \cong A^e/f(X) \otimes 1 - 1 \otimes f(X) \cong k[Y,Z]/(F(Y,Z))$ with $F(Y,Z) = f(Y) - f(Z)$. Here we wrote X instead of $X_1, \ldots, X_n$ and the same for Y, Z. We need now a free resolution of A as U-module. Because $\Theta(F(Y,Z)) = F(X,X) = f(X) - f(X) = 0$, there are elements $\alpha_1, \ldots, \alpha_n \in k[Y,Z]$ with $F(X,Y) = \sum_{j=1}^n \alpha_j (Y_j - Z_j)$. In other words, F is the image under the first Koszul differential

$$\partial_1 \colon \bigwedge^1 (k[Y,Z])^n \cong C_1(Y_1 - Z_1, \ldots, Y_n - Z_n) \to C_0(Y_1 - Z_1, \ldots, Y_n - Z_n) \cong k[Y,Z]$$

of $\alpha := (\alpha_1, \ldots, \alpha_n) \in (k[Y,Z])^n \cong \bigwedge^1 (k[Y,Z])^n$.

2.19 Definition Fix $\alpha \in (A^e)^n$ with $\partial_1(\alpha) = F$ and set $C_i := C_i(Y_1 - Z_1, \ldots, Y_n - Z_n)$.

1. We define $\sigma = \sigma_\alpha^{(i)} \colon \bigwedge^i (A^e)^n \to \bigwedge^{i+1} (A^e)^n$, $e \mapsto \alpha \wedge e$, the *exterior multiplication* with α.

2. We define a bigraded diagram $C_{\bullet\bullet}$

$$\begin{tikzcd}
\vdots \arrow[d, "\partial_4"] & \vdots \arrow[d, "\partial_3"] & \vdots \arrow[d, "\partial_2"] & \vdots \arrow[d, "\partial_1"] & & \cdot^{\cdot^{\cdot}} \\
C_3 \arrow[d, "\partial_3"] & C_2 \arrow[l, "\sigma"'] \arrow[d, "\partial_2"] & C_1 \arrow[l, "\sigma"'] \arrow[d, "\partial_1"] & C_0 \arrow[l, "\sigma"'] & & \\
C_2 \arrow[d, "\partial_2"] & C_1 \arrow[l, "\sigma"'] \arrow[d, "\partial_1"] & C_0 \arrow[l, "\sigma"'] & & & \\
C_1 \arrow[d, "\partial_1"] & C_0 \arrow[l, "\sigma"'] & & & & \\
C_0 & & & & &
\end{tikzcd}$$

   where the bottom left C_0 is supposed to be in bidegree $(0,0)$ and we increase the degree in the opposite directions of the maps ∂, σ. Note the rows and columns are complexes, but $C_{\bullet\bullet}$ is not a double complex, because the squares do not anticommute.

3. Define the module $C_{\text{odd}} = \bigoplus_{i=1}^\infty C_{2i-1}$ and $C_{\text{even}} = \bigoplus_{i=0}^\infty C_{2i}$.

2.20 Lemma The map $\partial_{m+1}\sigma + \sigma\partial_m \colon C_m \to C_m$ is the multiplication with F for all $0 \le m \le n-1$. In other words, we get a double complex $C_{\bullet\bullet} \otimes_{A^e} U$, because the squares anticommute modulo F.

Proof. We compute

$$(\partial_{m+1}\sigma)(e) = \partial_{m+1}(\alpha \wedge e) = \partial_1(\alpha)e - \alpha \wedge \partial_m(e) = Fe - (\sigma\partial_m)(e) \text{ for all } e \in C_m.$$

$\square$

2.21 Definition We define the double complex $\overline{C}_{\bullet\bullet}$ to be $C_{\bullet\bullet} \otimes_{A^e} U$. The corresponding total complex will be denoted with $\operatorname{Tot}(\overline{C}_{\bullet\bullet})$. The total complex is explicitly given by

$$\begin{aligned} &\ldots \xrightarrow{\partial+\sigma} \overline{C}_{\text{odd}} \xrightarrow{\partial+\sigma} \overline{C}_{\text{even}} \xrightarrow{\partial+\sigma} \overline{C}_{\text{odd}} \xrightarrow{\partial+\sigma} \ldots \\ &\xrightarrow{\partial+\sigma} \overline{C}_4 \oplus \overline{C}_2 \oplus \overline{C}_0 \xrightarrow{\partial+\sigma} \overline{C}_3 \oplus \overline{C}_1 \xrightarrow{\partial+\sigma} \overline{C}_2 \oplus \overline{C}_0 \xrightarrow{\partial+\sigma} \overline{C}_1 \xrightarrow{\partial} \overline{C}_0, \end{aligned}$$

where $\overline{C}_i = C_i \otimes_{A^e} U$ and we denote the induced mappings from ∂ and σ again with the same symbol.

2.22 Proposition The complex $\operatorname{Tot}(\overline{C}_{\bullet\bullet})$ is a free resolution of A as U-module.

Proof. This is the easiest case of [Eis80], Theorem 7.2. We will give a proof of this easy case for convenience. The proof here is a slight modification of a proof of a similar statement presented to us by Manfred Lehn. Let us look at the spectral sequence to the double complex $\overline{C}_{\bullet\bullet}$ with E^2 terms given by first taking vertical homology and then horizontal homology. We start by computing the vertical homology. The vertical complexes were given by the complex $\overline{C}_\bullet := C_\bullet(Y_1 - Z_1, \ldots, Y_n - Z_n) \otimes_{A^e} U$. We have the following exact sequence of complexes.

$$0 \to C_\bullet(Y_1 - Z_1, \ldots, Y_n - Z_n) \xrightarrow{\cdot F} C_\bullet(Y_1 - Z_1, \ldots, Y_n - Z_n) \to \overline{C}_\bullet \to 0.$$

This induces a long exact sequences of homologies. Because $H_i(Y_1 - Z_1, \ldots, Y_n - Z_n) = 0$ for $i > 0$, we have $H_i(\overline{C}_\bullet) = 0$ for $i > 1$. Because $H_0(Y_1 - Z_1, \ldots, Y_n - Z_n) \cong A$, the end of the long exact sequence is given by

$$0 \to H_1(\overline{C}_\bullet) \to A \to A \to H_0(\overline{C}_\bullet) \to 0.$$

The map in the middle is induced by the multiplication with F. Multiplication with F induces the zero map on $H_0(Y_1 - Z_1, \ldots, Y_n - Z_n) \cong A$, because it is null homotopic by Lemma 2.20. Therefore, we get

$$H_1(\overline{C}_\bullet) \cong H_0(Y_1 - Z_1, \ldots, Y_n - Z_n) \cong A, \quad H_0(\overline{C}_\bullet) \cong H_0(Y_1 - Z_1, \ldots, Y_n - Z_n) \cong A.$$

Hence after taking vertical homology we are left with

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d, "0"] & \iddots \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & A \arrow[l] \arrow[d, "0"] & A \arrow[l, "\sigma"'] & \\
0 \arrow[d] & A \arrow[l] \arrow[d, "0"] & A \arrow[l, "\sigma"'] & & \\
A \arrow[d, "0"] & A \arrow[l, "\sigma"'] & & & \\
A. & & & &
\end{tikzcd}$$

Now to take horizontal homology we claim, that σ is the composition

$$A \cong H_0(\overline{C}_\bullet) \overset{h_0}{\cong} H_0(Y_1 - Z_1, \ldots, Y_n - Z_n) \overset{h_1}{\cong} H_1(\overline{C}_\bullet) \cong A$$

and hence an isomorphism. The isomorphism $h_1^{-1}\colon H_1(\overline{C}) \to H_0(Y_1 - Z_1, \ldots, Y_n - Z_n)$ comes from the snake lemma applied to the diagram

$$\begin{tikzcd}
0 \arrow[r] & C_1 \arrow[r, "\cdot F"] \arrow[d, "\partial_1"] & C_1 \arrow[r] \arrow[d, "\partial_1"] & \overline{C}_1 = C_1 \otimes_{A^e} U \arrow[r] \arrow[d, "\partial_1"] & 0 \\
0 \arrow[r] & C_0 \arrow[r, "\cdot F"] & C_0 \arrow[r, "p"] & \overline{C}_0 = C_0 \otimes_{A^e} U \arrow[r] & 0
\end{tikzcd}$$

Then h_1^{-1} is constructed in the following way: For $x \in H_1(\overline{C}_\bullet)$ pick a preimage $y \in C_1$. Now $p(\partial_1(y)) = 0$, hence there is $z \in C_0$ with $Fz = \partial_1(y)$, then $h_1^{-1}(x)$ is the class of z in $H_0(Y_1 - Z_1, \ldots, Y_n - Z_n)$.
If $a \in A^e = C_0$ with $x = \sigma(a) \bmod F \in H_1(\overline{C}_\bullet)$, then we can pick the preimage $y = \sigma(a)$. Furthermore, we have

$$\partial_1(\sigma(a)) = \partial_1(\alpha \wedge a) = \partial_1(\alpha) \cdot a = F \cdot a$$

by definition of σ and α. Hence we can pick $h_1^{-1}(\sigma(a)) = a$ or, in other words, we have $h_1(a) = \sigma(a) \bmod F$ for each $a \in A^e$. The map h_0 is given by $a \bmod F \mapsto a$ for $a \in A^e$. In conclusion the composition $h_1 h_0(a \bmod F) = h_1(a) = \sigma(a) \bmod F$. So indeed σ induces an isomorphism $H_0(\overline{C}_\bullet) \to H_1(\overline{C}_\bullet)$. Hence we showed, that taking horizontal homology results in

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \iddots \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] & \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] & & \\
0 \arrow[d] & 0 \arrow[l] & & & \\
A. & & & &
\end{tikzcd}$$

So all groups E^2 are zero except $E^2_{0,0} = A$. This implies, that the zero homology of the total complex is A and all others are zero. Therefore, the total complex is a free resolution of A as U-module. □

Now to compute $H_m(A, A, K) \cong \mathrm{Tor}_m^U(A, A)$ we have to tensor the complex $\mathrm{Tot}(\overline{C}_{\bullet\bullet})$ with A over U. Then after identifying U^n with $(k[Y,Z]/F(Y,Z))^n$ and $U^n \otimes_U A$ with A^n, then tensoring translates to plugging in X_i for both Y_i and Z_i. Therefore, the Koszul differential ∂ will again be zero after tensoring. σ was given by exterior multiplication with $\alpha = (\alpha_1, \ldots, \alpha_n) \in k[Y,Z]^n$, such that $f(Y) - f(Z) = F = \sum_{j=1}^n \alpha_j(Y,Z)(Y_j - Z_j)$. Now we need the following lemma:

2.23 Lemma If $(\alpha_1, \dots, \alpha_n) \in k[Y, Z]^n$ with $f(Y) - f(Z) = \sum_{j=1}^n \alpha_j(Y, Z)(Y_j - Z_j)$, then $\alpha_i(X, X) = f_i(X)$ for all $i = 1, \dots, n$.

Proof. Fix i and differentiate the equation $f(Y) - f(Z) = \sum_{j=1}^n \alpha_j(Y, Z)(Y_j - Z_j)$ with respect to Y_i to get

$$\partial_{Y_i} f(Y) = \alpha_i(Y, Z) + \sum_{j=1}^n (\partial_{Y_i} \alpha_j(Y, Z))(Y_j - Z_j).$$

Now plug in X_i for both Y_i and Z_i to get the claim from the lemma. $\square$

Therefore, the map σ is the exterior multiplication with $\partial_X f := (f_1, \dots, f_n)$ after tensoring with A over U.

Putting this together, we get the following lemma:

2.24 Lemma The m-th Hochschild homology $H_m(A, A, K)$ is the m-th homology of the total complex of the double complex

$$\begin{tikzcd}
\vdots \arrow[d, "0"] & \vdots \arrow[d, "0"] & \vdots \arrow[d, "0"] & \vdots \arrow[d, "0"] & \iddots \\
\bigwedge^3 A^n \arrow[d, "0"] & \bigwedge^2 A^n \arrow[l, "e(f)"'] \arrow[d, "0"] & \bigwedge^1 A^n \arrow[l, "e(f)"'] \arrow[d, "0"] & A \arrow[l, "e(f)"'] & \\
\bigwedge^2 A^n \arrow[d, "0"] & \bigwedge^1 A^n \arrow[l, "e(f)"'] \arrow[d, "0"] & A \arrow[l, "e(f)"'] & & \\
\bigwedge^1 A^n \arrow[d, "0"] & A \arrow[l, "e(f)"'] & & & \\
A, & & & &
\end{tikzcd}$$

where $e(f)$ is the exterior multiplication with $\partial_X f := (f_1, \dots, f_n)$, i.e. $e(f)(a) = \partial_X f \wedge a$. In other words, the rows are (truncations of) the Koszul cochain complex $C^\bullet(f_1, \dots, f_n)$.

2.25 Definition We define $\Omega^{\text{even}} := \bigoplus_{j=0}^\infty \Omega^{2j}$ and $\Omega^{\text{odd}} = \bigoplus_{j=1}^\infty \Omega^{2j-1}$.

The complex $C^\bullet(f_1, \dots, f_n)$ is the same as the complex $C^\bullet_{\text{df}}$ from Definition 2.7 coming from wedging with df. In this notation the double complex from Lemma 2.24 is given by

$$\begin{tikzcd}
\vdots \arrow[d, "0"] & \vdots \arrow[d, "0"] & \vdots \arrow[d, "0"] & \vdots \arrow[d, "0"] & \iddots \\
\Omega^3 \arrow[d, "0"] & \Omega^2 \arrow[l, "df\wedge"'] \arrow[d, "0"] & \Omega^1 \arrow[l, "df\wedge"'] \arrow[d, "0"] & A \arrow[l, "df\wedge"'] & \\
\Omega^2 \arrow[d, "0"] & \Omega^1 \arrow[l, "df\wedge"'] \arrow[d, "0"] & A \arrow[l, "df\wedge"'] & & \\
\Omega^1 \arrow[d, "0"] & A \arrow[l, "df\wedge"'] & & & \\
A. & & & &
\end{tikzcd}$$

The corresponding total complex is explicitly given by

$$\begin{aligned}
&\dots \xrightarrow{df\wedge} \Omega^{\text{odd}} \xrightarrow{df\wedge} \Omega^{\text{even}} \xrightarrow{df\wedge} \Omega^{\text{odd}} \xrightarrow{df\wedge} \dots \\
&\xrightarrow{df\wedge} \Omega^4 \oplus \Omega^2 \oplus A \xrightarrow{df\wedge} \Omega^3 \oplus \Omega^1 \xrightarrow{df\wedge} \Omega^2 \oplus A \xrightarrow{df\wedge} \Omega^1 \xrightarrow{0} A,
\end{aligned}$$

Now we can express the Hochschild homology groups in the cohomology groups of the complex $C^\bullet_{\text{df}}$ and get the following proposition:

2.26 Proposition The Hochschild homology groups are given by

$$H_m(A,A,K) = \begin{cases} \Omega^m_f \oplus \bigoplus_{j=1}^{\lfloor m/2 \rfloor} H^{m-2j}_{\text{df}}, & m \le n, \\ \bigoplus_{j=0}^{\lfloor n/2 \rfloor} H^{n-2j}_{\text{df}}, & m > n,\ m \equiv n \bmod 2, \\ \bigoplus_{j=0}^{\lfloor (n-1)/2 \rfloor} H^{n-1-2j}_{\text{df}}, & m > n,\ m \equiv n-1 \bmod 2. \end{cases}$$

We saw in Proposition 2.9, that f having an isolated singularity implies

$$H^n_{\text{df}} = \Omega^n_f = \Omega^n / df \wedge \Omega^{n-1} \cong A/(f_1, \dots, f_n), \quad H^m_{\text{df}} = 0 \text{ for } m \ne n.$$

For the case of an isolated singularity Proposition 2.26 specialises to:

2.27 Corollary ([BvS], Theorem 1.3) If f has an isolated singularity, then the Hochschild homology groups are given by

$$H_m(A,A,K) = \begin{cases} \Omega^m_f, & m \le n, \\ \Omega^n_f, & m > n,\ m \equiv n \bmod 2, \\ 0, & m > n,\ m \equiv n-1 \bmod 2. \end{cases}$$

Next we want to compute $H_m(B,B,k)$ under the assumption, that f has an isolated singularity.

2.28 Lemma We have $H_m(B,B,k) \cong H_m(B,B,K) \cong H_m(A,B,K)$ for all $m \geq 0$.

Proof. The Hochschild complexes computing the above three Hochschild homologies have in degree i the following A-modules:

$$C_i(B,B,k) = B \otimes_k B^{\otimes_k i}, \quad C_i(B,B,K) = B \otimes_K B^{\otimes_K i}, \quad C_i(A,B,K) = B \otimes_K A^{\otimes_K i}.$$

But these three A-modules are isomorphic. The first two by the isomorphisms

$$B \otimes_k B^{\otimes_k i} \to B \otimes_K B^{\otimes_K i}, \quad b_0 \otimes_k \ldots \otimes_k b_i \mapsto b_0 \otimes_K \ldots \otimes_K b_i.$$

The second two by the isomorphism

$$B \otimes_K A^{\otimes_K i} \to B \otimes_K B^{\otimes_K i}, \quad b \otimes_K a_1 \otimes_K \ldots \otimes_K a_i \mapsto b \otimes_K [a_1] \otimes_K \ldots \otimes_K [a_i],$$

where $[a_j]$ denotes the class of $a_j \in A$ as element of B. To verify the bijectivity one can check, that the componentwise defined inverses are indeed well defined (also it is enough to show $B \otimes_K A \cong B \otimes_K B \cong B \otimes_k B$ by associativity of the tensor product). Because these isomorphisms are defined componentwise they commute with the Hochschild boundary and give therefore the stated isomorphisms of Hochschild homology groups. $\square$

2.29 Lemma There is a long exact sequence of K-modules

$$\ldots \to H_{i+1}(A,B,K) \to H_i(A,A,K) \xrightarrow{\cdot f} H_i(A,A,K) \to H_i(A,B,K) \to \ldots$$

Proof. We have the following exact sequence of A-modules

$$0 \to A \xrightarrow{\cdot f} A \to B \to 0.$$

Now set $U = A \otimes_K A$, to get a long exact sequence

$$\ldots \to \mathrm{Tor}_{i+1}^U(A,B) \to \mathrm{Tor}_i^U(A,A) \xrightarrow{\cdot f} \mathrm{Tor}_i^U(A,A) \to \mathrm{Tor}_i^U(A,B) \to \ldots$$

Because A is flat as K-module, this long exact sequence is the same as

$$\ldots \to H_{i+1}(A,B,K) \to H_i(A,A,K) \xrightarrow{\cdot f} H_i(A,A,K) \to H_i(A,B,K) \to \ldots$$

$\square$

2.30 Lemma If f has an isolated singularity, then the multiplication with f is injective as a map $\Omega_f^m \to \Omega_f^m$ for all $m < n$.

Proof. Because f has isolated singularity the map of A-modules $df \wedge \colon \Omega_f^m \to \Omega^{m+1}$ is injective for $m < n$. Hence Ω_f^m is a torsion-free A-module as submodule of the free module Ω^{m+1}. $\square$

2.31 Definition Define M_f to be the kernel of the map $(\cdot f)\colon \Omega_f^n \to \Omega_f^n$. If f is quasi homogeneous and the characteristic of k is zero, then f is contained in the Milnor ideal of f and hence $M_f = \Omega_f^n$.

2.32 Corollary If f has an isolated singularity, then the Hochschild homology groups are given by

$$H_m(B,B,k) = \begin{cases} \Omega^m_{B/k}, & m \leq n, \\ \Omega^n_{B/k}, & m > n,\ m \equiv n \bmod 2, \\ M_f, & m > n,\ m \equiv n-1 \bmod 2. \end{cases}$$

Proof. We know $H_i(A,B,K) \cong H_i(B,B,k)$ (Lemma 2.28). Therefore, we use the long exact sequence from Lemma 2.29. If $m \leq n$, then the relevant part of the long exact sequence looks like this:

$$H_m(A,A,K) \xrightarrow{\cdot f} H_m(A,A,K) \to H_m(A,B,K) \to H_{m-1}(A,A,K) \xrightarrow{\cdot f} H_{m-1}(A,A,K).$$

plugging in our result for $H_i(A,A,K)$ gives

$$\Omega^m_f \xrightarrow{\cdot f} \Omega^m_f \to H_m(A,B,K) \to \Omega^{m-1}_f \xhookrightarrow{\cdot f} \Omega^{m-1}_f.$$

As the rightmost map is injective we have

$$H_m(A,B,K) \cong \Omega^m_f / f\Omega^m_f = \Omega^m/(df \wedge \Omega^{m-1} + f\Omega^m) \cong \Omega^m_{B/k}.$$

If $m > n$ with $m \equiv n \bmod 2$, then the relevant sequence looks like

$$0 \to H_{m+1}(A,B,K) \to \Omega^n_f \xrightarrow{\cdot f} \Omega^n_f \to H_m(A,B,K) \to 0.$$

This implies $H_{m+1}(A,B,K) = M_f$ and $H_m(A,B,K) \cong \Omega^n_{B/k}$. (For $H_{n+1}(A,B,K)$ we get the same sequence without the rightmost zero). $\square$

2.4 Examples for cyclic homology

In this section $A = \mathbb{C}[X_1,\ldots,X_n]$, $f \in A \backslash \mathbb{C}$, $K = \mathbb{C}[t]$, $B = A/(f)$ and we see A as a K-module via $t \cdot a = fa$. We write f_i for $\partial_i f = \partial_{X_i} f$ and Ω^i for $\Omega^i_{A/\mathbb{C}}$

We want to compute the following three cyclic homology groups:

$$HC_m(A,\mathbb{C}),\ HC_m(A,K) \text{ and } HC_m(B,\mathbb{C}).$$

For the last two cases, we will only be able to compute this groups for $m \leq n$ and only under the assumption, that f has an isolated singularity.

Let us start with $HC_m(A, \mathbb{C})$. We look at the following double complex $B_{\bullet\bullet}(A, \mathbb{C})$:

$$\begin{tikzcd}
\vdots \arrow[d, "b"] & \vdots \arrow[d, "b"] & \vdots \arrow[d, "b"] & \vdots \arrow[d, "b"] & \iddots \\
A^3 \arrow[d, "b"] & A^2 \arrow[l, "\mathcal{B}"'] \arrow[d, "b"] & A^1 \arrow[l, "\mathcal{B}"'] \arrow[d, "b"] & A \arrow[l, "\mathcal{B}"'] & \\
A^2 \arrow[d, "b"] & A^1 \arrow[l, "\mathcal{B}"'] \arrow[d, "b"] & A \arrow[l, "\mathcal{B}"'] & & \\
A^1 \arrow[d, "b"] & A \arrow[l, "\mathcal{B}"'] & & & \\
A, & & & &
\end{tikzcd}$$

where $A^i = A^{\otimes_{\mathbb{C}} i}$ and $\mathcal{B}$ is the map from Definition 1.15. We had to change the name to $\mathcal{B}$, because the letter B is in use. By [Lod98] 2.1.8 the total complex of this double complex computes the cyclic homology $HC_m(A, \mathbb{C})$ and similar for $HC_m(A, K)$, $HC_m(B, \mathbb{C})$. We want to change the columns of this double complex to a quasi isomorphic complex better suited for our computation.

Recall the maps $\pi_i \colon A^i \to \Omega^i$ from Definition 1.3. In characteristic zero π_i realized Ω^i as a direct summand of $HH^i(A, \mathbb{C})$. We have computed $HH^i(A, \mathbb{C}) \cong \Omega^i$, which implies π_i induces an isomorphism $HH^i(A, \mathbb{C}) \to \Omega^i$ by Lemma 2.33. Furthermore, $\pi_i \circ b = 0$, in particular we have the following quasi-isomorphism of complexes

$$\begin{tikzcd}
\dots \arrow[r, "b"] & A^{n+3} \arrow[r, "b"] \arrow[d] & A^{n+2} \arrow[r, "b"] \arrow[d] & A^{n+1} \arrow[r, "b"] \arrow[d, "\pi"] & \dots \arrow[r, "b"] & A^2 \arrow[r, "b"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\pi"] & 0 \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \Omega^n \arrow[r, "0"] & \dots \arrow[r, "0"] & \Omega^1 \arrow[r, "0"] & A \arrow[r] & 0
\end{tikzcd}$$

2.33 Lemma Let R be a noetherian, commutative ring and M, N two R-modules with M finitely generated. If M is isomorphic to $M \oplus N$ as R-module, then $N = 0$.

Proof. N is finitely generated as submodule of a finitely generated module over a noetherian ring. Because N is zero if and only if its localization at every prime ideal of R is zero, we can assume that R is local. Now tensor with the residue field E, to get an isomorphism $M \otimes E \cong (M \otimes E) \oplus (N \otimes E)$ of finitely generated vector spaces. Counting the dimensions gives $N \otimes E = 0$, which implies $N = 0$ by the Nakayama lemma. □

By Lemma 1.16 we know, that we have the following commutative diagram for all m:

$$\begin{tikzcd}
HH_m(A) \arrow[r, "\pi_m"] \arrow[d, "\mathcal{B}"] & \Omega^m_{A/\mathbb{C}} \arrow[d, "(m+1)d"] \\
HH_{m+1}(A) \arrow[r, "\pi_{m+1}"] & \Omega^{m+1}_{A/\mathbb{C}}.
\end{tikzcd}$$

Therefore, the maps π_m give a map of double complexes from $B_{\bullet\bullet}(A,\mathbb{C})$ to the following double complex $D_{\bullet\bullet}(A,\mathbb{C})$:

$$\begin{tikzcd}
\vdots \arrow[d, "0"] & \vdots \arrow[d, "0"] & \vdots \arrow[d, "0"] & \vdots \arrow[d, "0"] & \iddots \\
\Omega^3 & \Omega^2 \arrow[l, "3d"'] \arrow[d, "0"] & \Omega^1 \arrow[l, "2d"'] \arrow[d, "0"] & A \arrow[l, "d"'] & \\
\Omega^2 \arrow[from=2-1, d, "0"] & \Omega^1 \arrow[l, "2d"'] \arrow[d, "0"] & A \arrow[l, "d"'] & & \\
\Omega^1 \arrow[d, "0"] & A \arrow[l, "d"'] & & & \\
A. & & & &
\end{tikzcd}$$

This map of double complexes induces quasi-isomorphisms in each column and therefore a quasi-isomorphism of the two total complexes ([Lod98], Proposition 1.0.12). In conclusion we can compute the cyclic homology groups $HC_m(A,\mathbb{C})$ by computing the m-th homology of the total complex of $D_{\bullet\bullet}(A,\mathbb{C})$. Computing this total homology is easy, because of the zero vertical differential, hence we get the following proposition:

2.34 Proposition

$$HC_m(A,\mathbb{C}) = \begin{Bmatrix} A, & m=0 \\ \Omega^m/d(\Omega^{m-1}), & 0<m<n \\ 0, & m \geq n \end{Bmatrix} \oplus \begin{Bmatrix} \mathbb{C}, & m \text{ even, } m>0 \\ 0, & m \text{ odd} \end{Bmatrix}$$

The summand $\mathbb{C}$ comes from the kernel $d\colon A \to \Omega^1$.

For the rest of this section we assume, that f has an isolated singularity. Under this assumption the computation of $HC_m(A,K)$ and $HC_m(B,\mathbb{C})$ is pretty similar. We computed:

$$H_m(A,A,K) = \Omega^m/df \wedge \Omega^{m-1} \cong \Omega^m_{A/K} \text{ for } m \leq n.$$

Thus as before we get a quasi-isomorphism, but only for the first $n+1$ terms

$$\begin{tikzcd}
\dots \arrow[r, "b"] & A^{n+3} \arrow[r, "b"] & A^{n+2} \arrow[r, "b"] & A^{n+1} \arrow[r, "b"] \arrow[d, "\pi"] & \dots \arrow[r, "b"] & A^2 \arrow[r, "b"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\pi"] & 0 \\
\dots \arrow[r] & \Omega^n_{A/K} \arrow[r] & 0 \arrow[r] & \Omega^n_{A/K} \arrow[r, "0"] & \dots \arrow[r, "0"] & \Omega^1_{A/K} \arrow[r, "0"] & A \arrow[r] & 0
\end{tikzcd}$$

Here A^i means $A^{\otimes_K i}$. This means, that we can compute the cyclic homology groups $HC_m(A,K)$ for $m \leq n$ as the first homology groups of the total complex of the following double complex

$$\begin{tikzcd}
\Omega^n_{A/K} \arrow[d, "0"] & \Omega^{n-1}_{A/K} \arrow[l, "nd"'] \arrow[d, "0"] & \Omega^{n-2}_{A/K} \arrow[l, "(n-1)d"'] \arrow[d, "0"] & \Omega^{n-3}_{A/K} \arrow[l, "(n-2)d"'] \arrow[d, "0"] & \cdots \\
\vdots \arrow[d, "0"] & \vdots \arrow[d, "0"] & \vdots \arrow[d, "0"] & \vdots \arrow[d, "0"] & \iddots \\
\Omega^3_{A/K} \arrow[d, "0"] & \Omega^2_{A/K} \arrow[l, "3d"'] \arrow[d, "0"] & \Omega^1_{A/K} \arrow[l, "2d"'] \arrow[d, "0"] & A \arrow[l, "d"'] & \\
\Omega^2_{A/K} \arrow[d, "0"] & \Omega^1_{A/K} \arrow[l, "2d"'] \arrow[d, "0"] & A \arrow[l, "d"'] & & \\
\Omega^1_{A/K} \arrow[d, "0"] & A \arrow[l, "d"'] & & & \\
A. & & & &
\end{tikzcd}$$

The rows are truncations of the relative de Rham complex. Thus we can express the cyclic homology groups $HC_m(A, K)$ in the cohomology groups of the relative de Rham complex $C_f^\bullet$ and get the following proposition:

2.35 Proposition If $m \leq n$ and f has an isolated singularity, then the cyclic homology groups are given by

$$HC_m(A, K) = \Omega^m/(df \wedge \Omega^{m-1} + d(\Omega^{m-1})) \oplus \bigoplus_{j=1}^{\lfloor m/2 \rfloor} H_f^{m-2j}.$$

If the number of variables n is at least 2, then $H_f^0 \cong K$. If f is quasi homogeneous, we have showed $H_f^i = 0$ for $i \neq 0, n-1$. Under both of this conditions we therefore get:

$$HC_m(A, K) = \Omega^m/(df \wedge \Omega^{m-1} + d(\Omega^{m-1})) \oplus \begin{cases} K, & m \text{ even}, m > 0, \\ 0, & m \text{ odd}. \end{cases}$$

To compute $HC_m(B, \mathbb{C})$ one can do the same with K replaced by $\mathbb{C}$ and $\Omega^i_{A/K}$ replaced with $\Omega^i_{B/\mathbb{C}}$. Thus we can express the cyclic homology groups in term of $H^i_{\mathrm{dR}}(B/\mathbb{C})$ the de Rham cohomology of B over $\mathbb{C}$ and get the following proposition:

2.36 Proposition If $m \leq n$ and f has an isolated singularity, then the cyclic homology groups are given by

$$HC_m(B, \mathbb{C}) = \Omega^m_{B/\mathbb{C}}/d(\Omega^{m-1}_{B/\mathbb{C}}) \oplus \bigoplus_{j=1}^{\lfloor m/2 \rfloor} H^{m-2j}_{\mathrm{dR}}(B/\mathbb{C}).$$

If f is quasi homogeneous and the number of variables n is at least 2, then $H^i_{\mathrm{dR}}(B/\mathbb{C}) = 0$ for $i \neq 0, n-1$ and $H^0_{\mathrm{dR}}(B/\mathbb{C}) \cong \mathbb{C}$. This is Proposition 8.19 in [Loo84]. There it is stated for the ring of convergent power series instead of polynomials (without the quasi homogeneous assumption). From that it follows for the polynomial ring in the quasi homogeneous case, because $H^i_{\mathrm{dR}}(B/\mathbb{C})$ is graded. Under both conditions Proposition 2.36 simplifies to:

$$HC_m(A,K) = \Omega^m_{B/\mathbb{C}}/d(\Omega^{m-1}_{B/\mathbb{C}}) \oplus \begin{cases} \mathbb{C}, & m \text{ even}, m > 0, \\ 0, & m \text{ odd}. \end{cases}$$

Furthermore, $\Omega^{n-1}_{B/\mathbb{C}}/d(\Omega^{n-2}_{B/\mathbb{C}})$ is a $\mathbb{C}$-vector space of dimension μ_f. This is showed in [Loo84] for convergent power series. This implies the same for the polynomial ring in the quasi homogeneous case.

In the unpublished paper [BvS] the cyclic homology groups for $HC_m(A,K)$ and $HC_m(B,\mathbb{C})$ for $m > n$ are also computed. Unfortunately the argument given appears to be wrong. The authors claim that the Hochschild complex $C_\bullet(A,A,K)$ is quasi isomorphic to the complex $L_\bullet$ given by

$$\begin{aligned} &\ldots \xrightarrow{df\wedge} \Omega^{\mathrm{odd}} \xrightarrow{df\wedge} \Omega^{\mathrm{even}} \xrightarrow{df\wedge} \Omega^{\mathrm{odd}} \xrightarrow{df\wedge} \ldots \\ &\xrightarrow{df\wedge} \Omega^4 \oplus \Omega^2 \oplus A \xrightarrow{df\wedge} \Omega^3 \oplus \Omega^1 \xrightarrow{df\wedge} \Omega^2 \oplus A \xrightarrow{df\wedge} \Omega^1 \xrightarrow{0} A, \end{aligned}$$

which we used to compute the Hochschild homology groups $H_m(A,A,K)$. But $L_\bullet$ is quasi isomorphic to its cohomology, i.e. the complex

$$HL_\bullet : \ \ldots \to 0 \to \Omega^n_f \to 0 \to \Omega^n_f \xrightarrow{0} \Omega^{n-1}_f \xrightarrow{0} \ldots \xrightarrow{0} \Omega^1_f \xrightarrow{0} \Omega^0.$$

The quasi-isomorphism is given by the projection of $\Omega^m \to \Omega^m_f = \Omega^m/df \wedge \Omega^{m-1}$. In particular we could replace the columns of the double complex computing the cyclic homology groups $HC_m(A,K)$ which were given by the Hochschild complex $C_\bullet(A,A,K)$ by $HL_\bullet$, without changing the homology of the total complex. The resulting double complex will in high enough degrees look like the following:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \iddots \\
0 \arrow[d] & \Omega^n_f \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \Omega^n_f \arrow[l] \arrow[d] & \ldots \arrow[l] \\
\Omega^n_f \arrow[d] & 0 \arrow[l] \arrow[d] & \Omega^n_f \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \ldots \arrow[l] \\
0 \arrow[d] & \Omega^n_f \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \Omega^n_f \arrow[l] \arrow[d] & \ldots \arrow[l] \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}$$

Now it follows, that we get more and more summands Ω^n_f in the higher homology groups of the total complex. This contradicts the result of the paper [BvS] itself, as well as a result from Burghelea and Vigué Poirrier in [BVP88], Theorem 4.3. This theorem computes the cyclic homology groups $HC_m(B,\mathbb{C})$ under the assumption, that f is homogeneous. We will compute $HC_m(B,\mathbb{C})$ in the quasi homogeneous case with a similar approach.

By Theorem 2.2.1 of [Lod98], there is the *Connes' periodicity exact sequence*

$$\ldots \to H_m(A,A,K) \to HC_m(A,K) \xrightarrow{S} HC_{m-2}(A,K) \to H_{m-1}(A,A,K) \to \ldots$$

and similar with B instead of A and $\mathbb{C}$ instead of K. Since $H_m(A,A,K) = 0$ for $m > n$ and $m \equiv n-1 \mod 2$, we get the following exact sequence for each $l > 1$

$$0 \to HC_{n+2l+1}(A,K) \to HC_{n+2l-1}(A,K) \to \Omega^n_f \to HC_{n+2l}(A,K) \to HC_{n+2l-2}(A,K) \to 0.$$

This implies that $HC_{n+2l+1}(A,K)$ injects into $HC_{n+2l-1}(A,K)$ and $HC_{n+2l}(A,K)$ surjects onto $HC_{n+2l-2}(A,K)$. Unfortunately we are not able to determine the maps and can not say more.

If f is quasi homogeneous one can say more about the map $S\colon HC_m(B,\mathbb{C}) \to HC_{m-2}(B,\mathbb{C})$. In this case the algebra B is graded with $\mathbb{C}$ in degree 0. By [Lod98], 2.2.13 and 4.1.12 we get for each m a decomposition

$$HC_m(B,\mathbb{C}) = \overline{HC}_m(B,\mathbb{C}) \oplus HC_m(\mathbb{C},\mathbb{C})$$

and the map S is zero on $\overline{HC}_m(B,\mathbb{C})$. By 4.1.13 in [Lod98], the Connes' periodicity exact sequence reduces to

$$0 \to \overline{HC}_{m-1}(B,\mathbb{C}) \to H_m(B,B,\mathbb{C}) \to \overline{HC}_m(B,\mathbb{C}) \to 0, \text{ for } m > 0.$$

Plugging in $m = n+1$ and our previous results

$$\overline{HC}_n(B,\mathbb{C}) = 0, \quad H_{n+1}(B,B,\mathbb{C}) \cong \Omega^n_f \cong \mathbb{C}^{\mu_f}$$

show, that $\overline{HC}_{n+1}(B,\mathbb{C}) \cong H_{n+1}(B,B,\mathbb{C}) \cong \Omega^n_f \cong \mathbb{C}^{\mu_f}$. Plugging in $m = n+2$ gives

$$0 \to \overline{HC}_{n+1}(B,\mathbb{C}) \to H_{n+2}(B,B,\mathbb{C}) \to \overline{HC}_{n+2}(B,\mathbb{C}) \to 0.$$

The groups $\overline{HC}_{n+1}(B,\mathbb{C})$, $H_{n+2}(B,B,\mathbb{C})$ are both isomorphic to $\mathbb{C}^{\mu_f}$, $\mu_f < \infty$ and the first map is injective and hence surjective. Therefore, we can conclude $\overline{HC}_{n+2}(B,\mathbb{C}) = 0$. Continuing in this manner gives for all $l > 1$:

$$\overline{HC}_{n+2l-1}(B,\mathbb{C}) \cong \mathbb{C}^{\mu_f}, \quad \overline{HC}_{n+2l}(B,\mathbb{C}) = 0.$$

Since

$$HC_m(\mathbb{C},\mathbb{C}) = \begin{cases} \mathbb{C}, & m \text{ even}, \\ 0, & m \text{ odd}, \end{cases}$$

we have showed the following proposition:

2.37 Proposition If f has an isolated singularity, the number of variables n is at least 2 and f is quasi homogeneous, then:

$$HC_m(B,\mathbb{C}) \cong \left\{\begin{array}{ll} B, & m=0 \\ \Omega^m_{B/\mathbb{C}}/d(\Omega^{m-1}_{B/\mathbb{C}}), & 0<m<n \\ 0, & m \geq n, m \equiv n \bmod 2 \\ \mathbb{C}^{\mu_f}, & m \geq n, m \equiv n-1 \bmod 2 \end{array}\right\} \oplus \left\{\begin{array}{ll} \mathbb{C}, & m \text{ even},\ m>0 \\ 0, & m \text{ odd} \end{array}\right\}$$

3 Chern theories for projective modules

This chapter will introduce the Chern characters associated to a finitely generated, projective module, which take values in the Connes, cyclic homology or in the de Rham cohomology. We will follow chapter eight of [Lod98].

In this chapter k will always be a commutative ring, A will be an associative, unital, not necessarily commutative k-algebra and $\otimes$ will always mean $\otimes_k$. The term A-module will mean left A-module. All modules in this chapter will be assumed to be finitely generated.

3.1 Finitely generated projective modules as images of idempotent matrices

3.1 Definition Let us recall, that one possible definition for *projective* A-modules is the following. An A-module P is projective if and only if for any two A-modules M, N and A-linear maps $P \to M$, $N \twoheadrightarrow M$ the following commutative diagram can be completed

$$\begin{tikzcd} & N \arrow[d, two heads] \\ P \arrow[r] \arrow[ur, dotted, "\exists"] & M. \end{tikzcd}$$

3.2 Construction Assume P is a projective A-module and generated by r elements, so we can find a surjection $e\colon A^r \twoheadrightarrow P$. If $Q := \ker(e)$, then we get an exact sequence

$$0 \to Q \to A^r \xrightarrow{e} P \to 0.$$

Because P is projective, there is an A-linear map $i_P\colon P \to A^r$ with $\mathrm{id}_P = e \circ i_P$. The map $e_{A^r} := i_P \circ e = \mathrm{id}_P \oplus 0$ is an endomorphism of $A^r \cong P \oplus Q$ and can therefore be given by a matrix $e_{A^r} \in M_r(A)$. We have $e_{A^r}^2 = e_{A^r}$ and P is isomorphic to the image of e_{A^r}. We will often omit the subscript of e_{A^r}, in particular with $e \in M_r(A)$ we will always mean the matrix associated to e_{A^r}. Furthermore this construction can be inverted, i.e. any image of an idempotent $e \in M_r(A)$ is a projective A-module. In particular Q is the image of $1 - e$ and hence projective.

3.3 Example ([Lod98] Example 8.3.6) For $A = \mathbb{C}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ we define

$$p = \begin{pmatrix} x & y + iz \\ y - iz & -x \end{pmatrix}, \; e = \frac{\mathrm{id} + p}{2}.$$

Then $p^2 = \mathrm{id}$, which implies $e \in M_2(A)$ is an idempotent and hence the image of e is a projective A-module. In this example we could replace $\mathbb{C}[x, y, z]$ by any commutative ring R and $x, y+iz, y-iz$ by arbitrary elements r_1, r_2, r_3 of R and set $A = R/(r_1^2 + r_2 \cdot r_3 - 1)$.

If P and P' are two projective A-modules, then we can carry out Construction 3.2 for both P and P'. If $r < r'$ we can add superfluous copies of A to A^r and Q to get $r = r'$, which adds zero rows respectively columns to the matrix e and the resulting matrix is still an idempotent with image isomorphic to P. In conclusion we can always achieve that e and e' have the same size. In this situation we get the following statements.

3.4 Lemma Let P, P' be projective modules, which are isomorphic to the images of idempotents $e, e' \in M_r(A)$. Then:

1. $P \cong P'$ and $Q \cong Q'$ if and only if there is a invertible $g \in M_r(A)$, s. t. $e' = geg^{-1}$.

2. If P and P' are isomorphic, then Q and Q' do not have to be isomorphic and hence e and e' are not necessarily conjugate.

3. If P and P' are isomorphic, then there is an invertible $g \in M_{2r}(A)$, s.t.
$$\begin{pmatrix} e' & 0 \\ 0 & 0 \end{pmatrix} = g \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} g^{-1}.$$

Proof. Note that $e = \mathrm{id}_P \oplus 0 \colon A^r \to A^r$ and similar for e'.

1. For the forward implication take $g = \varphi \oplus \psi$, where φ, ψ are the isomorphisms. The other implication follows because the images of conjugate maps are isomorphic.

2. See the two examples below.

3. If $\varphi \colon P \to P'$ is an isomorphism, we set
$$\alpha := \varphi \oplus 0 \colon A^r \cong P \oplus Q \to P' \oplus Q' \cong A^r,$$
$$\beta := \varphi^{-1} \oplus 0 \colon A^r \cong P' \oplus Q' \to P \oplus Q \cong A^r.$$
Then we can take
$$g = \begin{pmatrix} \alpha & 1 - e' \\ 1 - e & \beta \end{pmatrix}, \; g^{-1} = \begin{pmatrix} \beta & 1 - e \\ 1 - e' & \alpha \end{pmatrix}.$$

□

3.5 Example 1. For non-commutative rings A it can happen that $A \cong A^2$ as A-modules. Then A, A^2 are isomorphic to the images of the (non-conjugate) idempotents
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(A).$$
An example of such a ring is $A = \mathrm{End}_{\mathbb{Z}}(\bigoplus_{\mathbb{N}} \mathbb{Z})$.

2. If $M \oplus A \cong A^3$, but $M \ncong A^2$, then we get two idempotents $e, e' \in M_3(A)$ for the projective module A, coming from the exact sequences

$$0 \to M \to A^3 \to A \to 0, \quad 0 \to A^2 \to A^3 \to A \to 0.$$

By the first part of Lemma 3.4 the two idempotents we get are not conjugate. More general, we can take M to be any stably free, but not free module. One example is $A = \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$, $M = \{(a,b,c) \in A^3 \mid xa+yb+zc = 0\}$. The two idempotents are

$$\begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This example comes from the tangential bundle on the 2-sphere.

3.2 Definition of a Chern theory

There are theories of Chern characters for vector bundles and for finitely generated projective modules, both of these theories are additive over short exact sequences and multiplicative over tensor products. In this section we put these properties in a more general framework and define what the notion of a Chern theory for a k-linear category $\mathfrak{C}$ is. Finding such Chern theories is one of the main goals in this thesis.

3.6 Definition 1. A *Chern theory for a k-linear category* $\mathfrak{C}$ consists of the following:

- A target $H = \prod_{n=n_0}^{\infty} H_n$, where all H_n are k-modules. For us n_0 will be either 0 or 1.
- Maps $\mathrm{ch}_n \colon \mathrm{Obj}(\mathfrak{C}) \to H_n$, which are constant on isomorphism classes in $\mathfrak{C}$ and the product of these maps $\mathrm{ch} \colon \mathrm{Obj}(\mathfrak{C}) \to H$.

The map ch_n is called the *n-th Chern map*, ch the *total chern map*, for $c \in \mathfrak{C}$ we call $\mathrm{ch}_n(c)$, $\mathrm{ch}(c)$ the *n-th* respectively *total Chern character of* c. Although the class $\mathrm{Obj}(\mathfrak{C})$ does not need to be a set, we will still use the word map, which usually have sets as sources.

2. A Chern theory is said to be *additive* or *exact additive*, if the category $\mathfrak{C}$ is exact and ch_n are additive over exact sequences, i.e.

$$\mathrm{ch}_n(C_1) + \mathrm{ch}_n(C_3) = \mathrm{ch}_n(C_2)$$

for all exact sequences $C_1 \to C_2 \to C_3$ in $\mathfrak{C}$.

3. A Chern theory is said to be *triangulated additive*, if the category $\mathfrak{C}$ is triangulated and ch_n are additive over distinguished triangles, i.e.

$$\mathrm{ch}_n(C_1) + \mathrm{ch}_n(C_3) = \mathrm{ch}_n(C_2)$$

for all distinguished triangles $C_1 \to C_2 \to C_3 \to \Sigma(C_1)$ in $\mathfrak{C}$.

4. A Chern theory is said to be *multiplicative with weights* $k(n) \in k^*$, if $\mathfrak{C}$ is a pseudo tensor category, H has the structure of an associative, graded k-algebra and for all $C_1, C_2 \in \mathrm{Obj}(\mathfrak{C})$ we have

$$\mathrm{ch}_n(C_1 \otimes C_2) = k(n) \sum_{i=n_0}^{n-n_0} \mathrm{ch}_i(C_1)\,\mathrm{ch}_{n-i}(C_2).$$

It is called *multiplicative*, if $k(n) = 1$ for all n, i.e.

$$\mathrm{ch}_n(C_1 \otimes C_2) = \sum_{i=n_0}^{n-n_0} \mathrm{ch}_i(C_1)\,\mathrm{ch}_{n-i}(C_2), \text{ which means for total Chern characters}$$
$$\mathrm{ch}(C_1 \otimes C_2) = \mathrm{ch}(C_1)\,\mathrm{ch}(C_2).$$

If necessary we will add superscripts to ch_n, etc. to describe its target further or add phrases like "with values in the Connes homology". The definition of triangulated additivity seems like an obvious replacement for the exact additivity, if one works with triangulated categories instead of exact categories. We generalized the notion of being multiplicative, because the Chern theories for matrix factorizations, which we will define, are multiplicative with weights, but not multiplicative. Furthermore, the Chern theories we define satisfy some functoriality. This is the content of Section 5.3, but we are not sure how we should phrase the general definition.

3.3 Chern theory with values in the Connes homology

3.7 Lemma If $e \in M_r(A)$ with $e^2 = e$ then:

1. $b(e^{\otimes 2n}) = 0, \quad b(e^{\otimes 2n+1}) = e^{\otimes 2n}$.

2. $t(b(e^{\otimes 2n+1})) = t(e^{\otimes 2n}) = -e^{\otimes 2n} = -b(e^{\otimes 2n+1})$.

The first equations show, that $e^{\otimes 2n}$ represents the zero class in $HH_{2n-1}(M_r(A))$, $H^{\lambda}_{2n-1}(M_r(A))$. The second shows, if 2 is not a zero divisor in k, then $e^{\otimes 2n+1}$ represents a class in $H^{\lambda}_{2n}(M_r(A))$ (but not in $HH_{2n-1}(M_r(A))$).

3.8 Definition We define

$$\alpha^{(l)} \colon M_r(A) \to M_{r+l}(A)$$

to be the map which adds l zero rows and columns to a matrix. Then by functoriality we get maps

$$\alpha^{(l)}_\bullet \colon C^{\lambda}_\bullet(M_r(A)) \to C^{\lambda}_\bullet(M_{r+l}(A)),$$
$$\alpha^{(l)}_n \colon H^{\lambda}_n(M_r(A)) \to H^{\lambda}_n(M_{r+l}(A)).$$

From the definition of tr it follows that

$$\mathrm{tr}^{(r+l)} \circ \alpha_\bullet^{(l)} = \mathrm{tr}^{(r)} \colon C_\bullet^\lambda(M_r(A)) \to C_\bullet^\lambda(A),$$

where we indicated the source with the superscript at tr. In particular the same is true for maps on the Connes homology.

3.9 Proposition Suppose the rational numbers are a subring of k. Let P, P' be isomorphic, projective A modules, which are the images of idempotents $e \in M_r(A)$, $e' \in M_{r'}(A)$. Then for all $n \geq 0$

$$\mathrm{tr}_{2n}(e^{\otimes 2n+1}) = \mathrm{tr}_{2n}((e')^{\otimes 2n+1}) \in H_{2n}^\lambda(A).$$

In particular the element $\mathrm{tr}_{2n}(e^{\otimes 2n+1}) \in H_{2n}^\lambda(A)$ depends only on (the isomorphism class of) P and not on the choice of e.

Proof. Without loss of generality we have $r \leq r'$. By replacing e by $\alpha_{r'-r}(e)$, which is still an idempotent with image isomorphic to P and both have the same image under tr_{2n} , we can assume that $r = r'$. By the third part of Lemma 3.4 the matrices $\alpha_r(e)$ and $\alpha_r(e')$ are conjugate. With Proposition 1.34 follows that

$$\mathrm{tr}_{2n+1}^{(r)}(e^{\otimes 2n+1}) = \mathrm{tr}_{2n+1}^{(2r)}(\alpha_r(e)^{\otimes 2n+1}) = \mathrm{tr}_{2n+1}^{(2r)}(\alpha_r(e')^{\otimes 2n+1}) = \mathrm{tr}_{2n+1}^{(r)}((e')^{\otimes 2n+1}).$$

□

Note that we can weaken the assumptions of Proposition 3.9 to $\frac{(2n)!}{n!}$ is not a zero divisor in k instead of $\mathbb{Q} \subset k$. This will be explained after Equation 3.2.

3.10 Definition Let $P(A)$ be the (k-linear, exact, tensor) category of finitely generated projective A-modules.

3.11 Definition Let $\mathbb{Q} \subset k$. We get the following Chern theory for $P(A)$:

- The target is $H_{\mathrm{ev}}^\lambda = \prod_{n=0}^{\infty} H_{2n}^\lambda(A)$.
- $\mathrm{ch}_n^\lambda\colon \mathrm{Obj}(P(A)) \to H_{2n}^\lambda(A)$, $P \mapsto \mathrm{tr}_{2n}(e^{\otimes 2n+1})$, where $e \in M_r(A)$ is a projector with image isomorphic to P.

Lemma 3.12 will show, that this Chern theory is additive, which means that ch_n and ch induce well defined maps on the Grothendieck group $K_0(A)$.

3.12 Lemma If $0 \to P_1 \to E \to P_2 \to 0$ is a short exact sequence of A-modules with P_1, P_2 projective, then E is projective and for all $n \geq 0$ we have

$$\mathrm{ch}_n^\lambda(E) = \mathrm{ch}_n^\lambda(P_1) + \mathrm{ch}_n^\lambda(P_2).$$

Proof. Let $e_i \in M_{r_i}(A)$ be the idempotent to P_i, we can assume $r := r_1 = r_2$. As P_2 is projective the short exact sequence splits and $E \cong P_1 \oplus P_2$. Then E is projective, because it is the image of the idempotent

$$e := \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix}.$$

It remains to show that $\mathrm{tr}_{2n}(e^{\otimes 2n+1}) = \mathrm{tr}_{2n}(e_1^{\otimes 2n+1}) + \mathrm{tr}_{2n}(e_2^{\otimes 2n+1})$. Note that $e^{\otimes 2n+1}$ is the sum of the 2^{2n+1} summands

$$\begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix},$$
$$\ldots, \quad \begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix}.$$

Furthermore, tr_{2n} vanishes on all of those except the first and the last one. Now with the k-linearity of tr_{2n} it follows that

$$\begin{aligned} \mathrm{tr}_{2n}(e^{\otimes 2n+1}) &= \mathrm{tr}_{2n}\left(\begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}^{\otimes 2n+1}\right) + \mathrm{tr}_{2n}\left(\begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix}^{\otimes 2n+1}\right) \\ &= \mathrm{tr}_{2n}(e_1^{\otimes 2n+1}) + \mathrm{tr}_{2n}(e_2^{\otimes 2n+1}). \end{aligned}$$

(here the map tr_{2n} is used with different sources). □

3.4 Chern theory with values in the de Rham cohomology

In this section we assume $\mathbb{Q} \subset k$ to be a subfield, the k-algebra A is commutative, P is a finitely generated, projective A-module and $e \in M_r(A)$ an idempotent with $P \cong \mathrm{im}(e)$.

Recall the k-linear map from Definition 1.28

$$\begin{aligned} \pi_n \colon H_n^\lambda(A) &\to \Omega^n_{A/k}/d(\Omega^{n-1}_{A/k}), \\ a_0 \otimes \ldots \otimes a_n &\mapsto a_0 d(a_1) \ldots d(a_n). \end{aligned}$$

By composing we get maps

$$\mathrm{ch}_n^{dR} = \pi_{2n-1} \circ \mathrm{ch}_n^\lambda \colon \mathrm{Obj}(P(A)) \to H^{2n}_{\mathrm{dR}}(A/k),$$

which we will use to define a Chern theory. But first we want to show that the image of ch_n^{dR} lies in the de Rham cohomology. To proof this we need some properties of traces of matrices with entries in the graded commutative ring $\Omega^*_{A/k} = \bigoplus_{i=0}^\infty \Omega^i_{A/k}$.

3.13 Lemma Let W be a square matrix with entries in $\Omega^*_{A/k}$, then $d(\mathrm{Tr}(W)) = \mathrm{Tr}(d(W))$.

3.14 Lemma For matrices $W_1 \in M_{l\times m}(\Omega^{n_1}_{A/k})$, $W_2 \in M_{m\times l}(\Omega^{n_2}_{A/k})$ we have

$$\mathrm{Tr}(W_1 W_2) = (-1)^{n_1 n_2}\,\mathrm{Tr}(W_2 W_1).$$

Proof. Write down a proof for the ordinary invariance of the trace map under cyclic permutations over commutative rings and add signs for the anticommutativity of differential forms. □

3.15 Lemma Let $e \in M_r(A)$ be an idempotent matrix, then we have the following equations:

$$\begin{aligned} e^2 &= e, & ede + (de)e &= de, \\ e(de)e &= 0, & edede &= de(de)e. \end{aligned}$$

Proof. $e^2 = e$ holds by definition. Apply d to get $ede + (de)e = de$. Multiply with e from either site to get $e(de)e = 0$. Apply d to get $edede = de(de)e$. □

3.16 Lemma For any idempotent matrix $e \in M_r(A)$ we have $d\,\mathrm{Tr}(e(de)^{2n}) = 0$.

Proof.

$$\begin{aligned} d\,\mathrm{Tr}(e(de^{2n}) &\overset{3.13}{=} \mathrm{Tr}(de(de)^{2n}) \overset{3.15}{=} \mathrm{Tr}((ede + (de)e)(de)^{2n}) \overset{3.14}{=} 2\,\mathrm{Tr}(e(de)^{2n+1}) \\ &\overset{3.15}{=} 2\,\mathrm{Tr}(e^2(de)^{2n+1}) \overset{3.14}{=} 2\,\mathrm{Tr}(e(de)^{2n+1}e) \overset{3.15}{=} 2\,\mathrm{Tr}(e(de)e(de)^{2n}) \overset{3.15}{=} 0. \end{aligned}$$

□

3.17 Definition We define the following additive Chern theory for $P(A)$:

- The target is $H^{\mathrm{ev}}_{\mathrm{dR}}(A/k) = \prod_{n=0}^{\infty} H^{2n}_{\mathrm{dR}}(A/k)$.
- $\mathrm{ch}_n^{dR}\colon \mathrm{Obj}(P(A)) \to H^{2n}_{\mathrm{dR}}(A/k)$, $P \mapsto \frac{1}{n!}\pi_{2n}(\mathrm{tr}_{2n}(e^{\otimes 2n+1})) = \frac{1}{n!}\mathrm{Tr}(e(de)^{2n})$, where $e \in M_r(A)$ is a projector with image isomorphic to P.

Proposition 3.22 will show that this Chern theory is multiplicative.
One can construct these invariants directly without using Connes homology by using so called connections (this in done in [Lod98] at the beginning of chapter 8).

3.18 Example

1. If P is not only projective but even free, we can choose the identity matrix for e and hence $\mathrm{ch}_n^{dR}(P)$ will be 0 for $n > 0$, as $de = 0$. Note that in this situation $\mathrm{ch}_n^{\lambda}(P)$ does not need to be zero, for example for $k = A = P = \mathbb{C}$ a choice for e is the 1×1 matrix 1 and then $\mathrm{ch}_n^{\lambda}(P)$ is the class of $1^{\otimes 2n+1}$, which is the generator of $H^{\lambda}_{2n} \cong k$.
2. Let us compute this invariants for our Example 3.3. In this example we had $A = \mathbb{C}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ and

$$p = \begin{pmatrix} x & y + iz \\ y - iz & -x \end{pmatrix}, \; e = \frac{1 + p}{2}$$

we get

$$\begin{aligned}
\mathrm{ch}_0^{dR}(\mathrm{im}(e)) &= \mathrm{Tr}(e) = \frac{1}{2}(\mathrm{Tr}(\mathrm{id}_2) + \mathrm{Tr}(p)) = 1,\\
\mathrm{ch}_1^{dR}(\mathrm{im}(e)) &= \mathrm{Tr}(edede) = \frac{i}{2}(xdydz + ydzdx + zdxdy),\\
\mathrm{ch}_n^{dR}(\mathrm{im}(e)) &= 0 \text{ for } n > 1.
\end{aligned}$$

For the rest of this section we want to show the multiplicativity of the Chern theory defined in Definition 3.17. The multiplicativity is also the reason for the factorials appearing in this definition.

3.19 Lemma Let $M, M' \in M_r(\Omega^*_{A/k})$ and $N, N' \in M_s(\Omega^*_{A/k})$ be matrices. Then we get

$$\begin{aligned}
d(M \otimes N) &= dM \otimes N + (-1)^{|M|} M \otimes dN,\\
\mathrm{tr}(M \otimes N) &= \mathrm{tr}(M)\,\mathrm{tr}(N),\\
(M \otimes N) \cdot (M' \otimes N') &= (-1)^{|N|\cdot|M'|} M \cdot M' \otimes N \cdot N'.
\end{aligned}$$

Here $|M|$ denotes the natural number, s.t. the entries of M belong to $\Omega^{|M|}_{A/k} \subset \Omega^*_{A/k}$.

Proof. All of those equations can be shown by a straightforward computation using the definitions. □

3.20 Lemma Let P_1, P_2 be projective A-modules, which are isomorphic to the images of the idempotents e_1, e_2. Then $e := e_1 \otimes e_2$ is an idempotent with image isomorphic to $P := P_1 \otimes_A P_2$. In particular P is projective.

Proof. Let r_1, r_2 be the sizes of the matrices e_1, e_2 and Q_1, Q_2 their kernels. With the third equation from Lemma 3.19 we get $e^2 = (e_1 \otimes e_2)^2 = e_1^2 \otimes e_2^2 = e_1 \otimes e_2 = e$, so $e \in M_{r_1 r_2}(A)$ is an idempotent. Furthermore, we have

$$A^{r_1} = P_1 \oplus Q_1, \quad A^{r_2} = P_2 \oplus Q_2,$$

which implies that

$$A^{r_1 r_2} = A^{r_1} \otimes_A A^{r_2} = (P_1 \otimes_A P_2) \oplus (P_1 \otimes_A Q_2) \oplus (Q_1 \otimes_A P_2) \oplus (Q_1 \otimes_A Q_2).$$

If one interprets elements of P_i, Q_i as certain column vectors from A^{r_i}, one gets (again with the third equation from Lemma 3.19), that $e_1 \otimes e_2$ is the identity on $P_1 \otimes_A P_2$ and zero on the other summands. □

3.21 Lemma Let e_1, e_2 be idempotent square matrices and $e := e_1 \otimes e_2$ then

$$\mathrm{Tr}(e(de)^{2n}) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{Tr}(e_1(de_1)^{2i})\,\mathrm{Tr}(e_2(de_2)^{2(n-i)}).$$

Proof. We have $de = d(e_1 \otimes e_2) = de_1 \otimes e_2 + e_1 \otimes de_2$. So $e(de)^{2n}$ is the sum of tensors of matrices of the form $e_1E_1 \otimes e_2E_2$, where E_i is a product of $2n$ matrices and each factor is either e_i or de_i and for each i either the i-th factor of E_1 or E_2 contains a d. Now using the third and fourth equality from Lemma 3.15 we see that all each block of de_1 in E_1 has to have an even length or $e_1E_1 = 0$. With the same argument for e_2E_2 we see that each block of e_i and de_i in E_i must have an even length or $e_1E_1 \otimes e_2E_2 = 0$. In the remaining matrices $e_1E_1 \otimes e_2E_2$ we can swap all occurring e_i in front (again with the last equation from Lemma 3.15) and since the total number of occurring d is $2n$, we will have $e_1E_1 \otimes e_2E_2 = e_1(de_1)^{2i} \otimes e_2(de_2)^{2(n-i)}$ for some $i = 0 \dots n$. Taking the trace gives

$$\operatorname{Tr}(e_1(de_1)^{2i} \otimes e_2(de_2)^{2(n-i)}) = \operatorname{Tr}(e_1(de_1)^{2i}) \operatorname{Tr}(e_2(de_2)^{2(n-i)}).$$

Finally we are left to count how often the summand $e_1(de_1)^{2i} \otimes e_2(de_2)^{2(n-i)}$ occurs, which is $\binom{n}{i}$ times. $\square$

Lemma 3.20 and 3.21 imply Proposition 3.22 (mind the factor $\frac{1}{n!}$ in the definition of ch_n^{dR}).

3.22 Proposition If P, P' are two projective modules over A, then

$$\operatorname{ch}_n^{dR}(P \otimes_A P') = \sum_{i=0}^{n} \operatorname{ch}_i^{dR}(P) \operatorname{ch}_{n-i}^{dR}(P').$$

For the total Chern Character this translates to the nice formula

$$\operatorname{ch}^{dR}(P \otimes_A P') = \operatorname{ch}^{dR}(P) \operatorname{ch}^{dR}(P').$$

3.5 Chern theory with values in the cyclic homology

In this section P is a finitely generated, projective A-module and $e \in M_r(A)$ an idempotent with $P \cong \operatorname{im}(e)$.

In Definition 1.38 and Proposition 1.39 we have given maps $p_n \colon HC_n(A) \to H_n^\lambda(A)$ for all n, which are isomorphisms for $\mathbb{Q} \subset k$. We will now try to lift our construction for Chern characters with values in the Connes homology along this maps. So we want to construct invariants $\operatorname{ch}_n^{\mathrm{cy}}(P) \in HC_{2n}(A)$, s.t.

$$p_{2n}(\operatorname{ch}_n^{\mathrm{cy}}(P)) = \operatorname{ch}_n^\lambda(P) \in H_{2n}^\lambda(A). \tag{3.1}$$

Unfortunately we will in general not be able to achieve 3.1 and we will have to settle for

$$p_{2n}(\operatorname{ch}_n^{\mathrm{cy}}(P)) = (-1)^n \frac{(2n)!}{n!} \operatorname{ch}_n^\lambda(P) \in H_{2n}^\lambda(A). \tag{3.2}$$

If $\mathbb{Q}$ was a subring of k we could divide by the integer coefficient and get (3.1), but in this case cyclic homology agrees with Connes homology, thus the construction of $\operatorname{ch}_n^{\mathrm{cy}}(P)$ would give nothing new. Note that we did only prove the invariance of $\operatorname{ch}_n^\lambda(P)$ in the case $\mathbb{Q} \subset k$. Once the invariance of $\operatorname{ch}_n^{\mathrm{cy}}(P)$ is shown, one can weaken the assumption for the invariance of $\operatorname{ch}_n^\lambda(P)$ to $\frac{(2n)!}{n!}$ is not a zero divisor in k by using (3.2).

3.23 Construction An element of $HC_{2n}(M_r(A)) = H_{2n}(\mathrm{Tot}(CC(A)))$ is the equivalence class of a tuple

$$(A_1, A_2, \dots, A_{2n+1}) \in M_r(A) \oplus M_r(A)^{\otimes 2} \oplus \dots \oplus M_r(A)^{\otimes 2n+1},$$

lying in the kernel of the differential of the total complex. This is translated to the following equations

$$\begin{aligned} b(A_{2n+1}) &= -(1-t)(A_{2n}), \\ b'(A_{2n}) &= N(A_{2n-1}), \\ b(A_{2n-1}) &= -(1-t)(A_{2n-2}), \\ &\vdots \\ b'(A_2) &= N(A_1). \end{aligned} \tag{3.3}$$

The map p_{2n} is given by $(A_1, A_2, \dots, A_{2n+1}) \mapsto A_{2n+1}$ and $\mathrm{ch}_n^\lambda(P) = \mathrm{tr}_{2n}(e^{\otimes 2n+1})$. So in order to get (3.1) we want to find $A_i \in M_r(A)^{\otimes i}$ s.t. $A_1, \dots, A_{2n}, e^{\otimes 2n+1}$ satisfy the equations (3.3). As

$$2b(e^{\otimes 2n+1}) = 2e^{\otimes 2n} = (1-t)(e^{\otimes 2n})$$

a good candidate for A_{2n} is $-e^{\otimes 2n}$, but we have to accept the factor 2 in front of $e^{\otimes 2n+1}$. Similar we have

$$(2n-1)b'(e^{\otimes 2n}) = (2n-1)e^{\otimes 2n-1} = N(e^{\otimes 2n-1})$$

so a good candidate for A_{2n-1} is $e^{\otimes 2n-1}$, but we have to accept the factor $2n-1$ in front of $e^{\otimes 2n}$ and hence $2(2n-1)$ in front of $e^{\otimes 2n+1}$. Continuing until we reach $A_1 = e$, we will get that

$$\begin{aligned} c(e)_n &= (y_n, z_n, y_{n-1}, z_{n-1}, \dots, y_0), \text{ where} \\ y_i &= (-1)^i (2^i \prod_{j=1}^{i-1} (2j+1)) e^{\otimes 2i+1} = (-1)^i \frac{(2i)!}{i!} e^{\otimes 2i+1}, \ i \geq 0, \\ z_i &= (-1)^{i-1} (2^{i-1} \prod_{j=1}^{i-1} (2j+1)) e^{\otimes 2i} = (-1)^{i-1} \frac{(2i)!}{2(i!)} e^{\otimes 2i}, \ i \geq 1, \end{aligned}$$

defines an element of $HC_{2n}(M_r(A))$. Note that the rational numbers appearing in the definition of y_i and z_i are indeed integers. By construction $\mathrm{tr}_{2n}(c(e)_n)$ is mapped by p_{2n} to $y_n = (-1)^n \frac{(2n)!}{2(n!)} \mathrm{ch}_n^\lambda(P)$, so fulfils (3.2).

3.24 Proposition The value $\mathrm{tr}_{2n}(c(e)_n) \in HC_{2n}(A)$ does not depend on the choice of e, but only on (the isomorphism class of) P.

Proof. The proof can be done in the same way as for Connes homology (see Proposition 3.9). We can drop the assumption $\mathbb{Q} \subset k$, because we use Proposition 1.43 instead of Proposition 1.34, which was the reason for this assumption. $\square$

3.25 Definition We get the following additive Chern theory for $P(A)$:

- The target is $HC_{\text{ev}} = \prod_{n=0}^{\infty} HC_{2n}(A)$.

- $\mathrm{ch}_n^{\mathrm{cy}} \colon \mathrm{Obj}(P(A)) \to HC_{2n}(A)$, $P \mapsto \mathrm{tr}_{2n}(c(e)_n)$, where $e \in M_r(A)$ is a projector with image isomorphic to P.

The additivity over short exact sequences is shown componentwise in the same way as for Connes homology (cf. Lemma 3.12).

3.6 Summary

For $\mathbb{Q} \subset k$ we have defined three different additive Chern theories $\mathrm{ch}^{\mathrm{cy}}$, ch^{λ} and $\mathrm{ch}^{\mathrm{dR}}$ for $P(A)$. $\mathrm{ch}^{\mathrm{dR}}$ is also multiplicative and required our ring A to be commutative. For each $n \in \mathbb{N}_0$ we have the following commutative diagram

$$\begin{tikzcd}
 & HC_{2n}(A) \arrow[d, "(-1)^n \frac{(2n)!}{2(n!)} p_{2n}"] \\
K_0(A) \arrow[ur, "\mathrm{ch}_n^{\mathrm{cy}}"] \arrow[r, "\mathrm{ch}_n^{\lambda}"] \arrow[dr, "\mathrm{ch}_n^{\mathrm{dR}}"'] & H_{2n}^{\lambda}(A) \arrow[d, "\frac{1}{n!}\pi_{2n}"] \\
 & H_{dR}^{2n}(A).
\end{tikzcd}$$

For fixed n it is sufficient that $2(n!)$ is invertible in k for the existence of the maps $\mathrm{ch}_n^{\mathrm{cy}}$, ch_n^{λ} and $\mathrm{ch}_n^{\mathrm{dR}}$.

3.7 Vector bundles and projective modules

In this section we want to discuss the relation between the Chern theory for finitely generated modules and topological vector bundles. We will only look at the case where the topological space is a complex manifold.

In this section X will be a n-dimensional complex manifold, which is C^∞ as manifold over $\mathbb{R}$ and E will be a C^∞-vector bundle over X. From now on we will say smooth instead of C^∞. We will denote the smooth global sections of E with $\Gamma(E)$ and the smooth functions from X to $\mathbb{C}$ with $C^\infty(X)$.

Let us first recall the Serre-Swan theorem ([Swa62]). This theorem states, that if Y is a compact hausdorff topological space, then taking continuous global sections gives an equivalence between the category of real (respectively complex) vector bundles over Y and the category of finitely generated projective modules over the ring of continuous functions from Y to $\mathbb{R}$ (respectively $\mathbb{C}$).

If X is a smooth manifold (not necessarily compact), then the smooth global sections of a smooth vector bundle are a finitely generated module over the smooth functions on X. If X is connected, we get again a equivalence of categories ([Nes03]). This is sometimes called the smooth Serre-Swan theorem.

Now to compare the Chern theories for vector bundles and finitely generated projective modules over $C^\infty(X)$, we would like to have $\mathrm{ch}_n(E) = \mathrm{ch}_n^{\mathrm{dR}}(\Gamma(E))$ for every smooth vector bundle E. One problem in verifying this equality is, that there are many different ways to define Chern characters (in both cases). Depending on the way one uses, the two Chern characters will live in different cohomology theories. One way for vector bundles is using Chern classes, the splitting principle and Chern roots. This is done for example in (cf. [Wei13], chapter 2). With this method $\mathrm{ch}_n(E)$ is an element of $2n$-th singular cohomology $H^{2n}(X, \mathbb{C})$. On the other hand $\mathrm{ch}_n^{\mathrm{dR}}(\Gamma(E))$ is an element of $H^{2n}_{\mathrm{dR}}(C^\infty(X)/\mathbb{C})$. So we will first have to find suitable maps to compare these Chern characters.

Let us look at the following complex

$$C^\bullet_{\mathrm{dR}}(X) : 0 \to C^\infty(X) \xrightarrow{d} A^1(X) \xrightarrow{d} \dots \xrightarrow{d} A^{2n}(X) \to 0,$$

where $A^i(X) = \Gamma(\bigwedge^i \hom_{\mathbb{R}}(\tau, \mathbb{C}))$, i.e. the global sections of the i-th exterior power of the complexification of the cotangent bundle τ. If $U \subset X$ is an open subset and x_i local complex coordinates for U, then the sections $A^i(U)$ defined over U can be described as

$$A^1(U) = \bigoplus_{j=1}^n C^\infty(U) dx_j \oplus \bigoplus_{j=1}^n C^\infty(U) d\overline{x}_j, \quad A^i(U) = \bigwedge^i A^1(U).$$

Locally the map d is then given by

$$a dx_I \mapsto \sum_{j=1}^n \partial_{x_j}(a) dx_j \wedge dx_I + \partial_{\overline{x}_j}(a) d\overline{x}_j \wedge dx_I.$$

The i-th cohomology of this complex is usually denoted with $H^i_{\mathrm{dR}}(X)$ and is called the *de Rham cohomology of* X. We will now outline a construction for $\mathrm{ch}_n(E) \in H^{2n}_{\mathrm{dR}}(X)$. This construction works with connections for vector bundles and is done in [MS74], appendix C. There is an isomorphism $H^i_{\mathrm{sing}}(X, \mathbb{C}) \cong H^i_{\mathrm{dR}}(X)$ ([Voi07], Corollary 4.37) and the two different constructions of $\mathrm{ch}_n(E)$ are the same after applying this isomorphism.

We have described the Chern characters for finitely generated projective modules purely with traces of idempotents, but it can also be constructed with connections for modules. A connection for a module M over a k-algebra A is a k-linear map

$$\nabla \colon M \to M \otimes_A \Omega_{A/k} \text{ with } \nabla(am) = a\nabla(m) + m \otimes d(a)$$

for all $a \in A$, $m \in M$. The construction of $\mathrm{ch}_n(M)$ from a connection is done in [Lod98], chapter 8. Loday arrives at the same formula for the Chern characters as we do via traces. A connection for a vector bundle E is a $\mathbb{C}$-linear map

$$\nabla \colon \Gamma(E) \to \Gamma(E \otimes \tau^*) \text{ with } \nabla(fs) = f\nabla(s) + s \otimes df$$

for all $s \in \Gamma(E)$, $f \in C^\infty(X)$. Here $\tau^* = \hom_{\mathbb{R}}(\tau, \mathbb{C})$ is the complexification of the cotangent bundle.

Because $\Gamma(E \otimes \tau^*) \cong \Gamma(E) \otimes \Gamma(\tau^*)$, we see that this definition of a connection for the vector bundle E is almost the same as a connection for the module $\Gamma(E)$ over the $\mathbb{C}$-algebra $C^\infty(X)$, except one needs to replace $\Omega_{C^\infty(X)/\mathbb{C}}$ with $A^1(X) = \Gamma(\tau^*)$. Now both theories continue along the same lines:

- ∇ determines a unique map $\nabla^{(1)} \colon \Gamma(E) \otimes A^1(X) \to \Gamma(E) \otimes A^2(X)$ with
$$\nabla^{(1)}(s \otimes \omega) = s \otimes d\omega - \nabla s \wedge \omega \text{ for } \omega \in \Gamma(\tau^*) \text{ and } s \in \Gamma(E).$$
And the same for modules with $A^i(X)$ replaced by $\Omega^i_{C^\infty(X)/\mathbb{C}}$.

- The composition $K := \nabla^{(1)} \circ \nabla \colon \Gamma(E) \to \Gamma(E) \otimes A^2(X)$ can be locally given by a matrix with entries in $A^2(U)$. This matrix is unique, up to conjugation with an invertible matrix. The map K is usually called the *curvature of* ∇. The same holds for modules with $\Omega^2_{C^\infty(X)/\mathbb{C}}$ and without the word locally.

- Because these matrices are unique up to conjugation, it makes sense to evaluate these matrix on any invariant polynomial P in the entries of the matrix, i.e. a polynomial satisfying $P(M) = P(GMG^{-1})$, or more general on any power series, which is the sum of invariant homogeneous polynomials. In the case for vector bundles, this indeed gives a global object.

- The n-th Chern character is then the element of $A^{2n}(X)$ (respectively $\Omega^{2n}_{C^\infty(X)/\mathbb{C}}$) one gets by evaluating the matrix on the degree n part of the power series $\mathrm{Tr}(e^{K/2\pi i})$ (respectively $\mathrm{Tr}(e^K)$).

- Then one verifies, that this defines indeed an object in the cohomology, i.e. lies in the kernel of d, does not depend on the choice of connection and every vector bundle respectively projective module has a connection.

Now by the universal property of $\Omega_{C^\infty(X)/\mathbb{C}}$ there is a unique map making the following diagram of $C^\infty(X)$-modules commute:

$$\begin{tikzcd}
C^\infty(X) \arrow[r, "d"] \arrow[d, "d"] & A^1(X) \\
\Omega_{C^\infty(X)/\mathbb{C}}. \arrow[ur, dotted, "\lambda", "\exists!"'] &
\end{tikzcd}$$

Then taking exterior powers gives also maps $\wedge^j\lambda \colon \Omega^j_{C^\infty(X)/\mathbb{C}} \to A^j(X)$ for all j. Since these maps commute with d, we get a map of complexes and hence a map between cohomologies $H^j(\lambda) \colon H^j_{\mathrm{dR}}(C^\infty(X)/\mathbb{C}) \to H^j_{\mathrm{dR}}(X)$. By our discussion above the image of the n-th Chern character of the module $\Gamma(E)$ under $H^{2n}(\lambda)$ is $(2\pi i)^n$ times the Chern character of the vector bundle E.

3.26 Example Let γ^n be the tautological line bundle on $\mathbb{P}^n_{\mathbb{C}}$ given by

$$\{(l,x) \in \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1} \mid l \in \mathbb{P}^n_{\mathbb{C}}, x \in l\} \to \mathbb{P}^n_{\mathbb{C}}, \quad (l,x) \mapsto l.$$

The global sections of γ^n are the same as smooth maps

$$h\colon S^n_{\mathbb{C}} \to \mathbb{C} \text{ with } h(ux) = u^{-1}h(x) \text{ for } u \in \mathbb{C}, |u| = 1, x \in S^n_{\mathbb{C}}.$$

If h is such a function and $[x] \in \mathbb{P}^n_{\mathbb{C}}$ with $|x| = 1$, then we get a global section by the assignment $[x] \mapsto ([x], h(x)x)$. The right hand side is independent of the choice of x and each global section is of this form. The functions $\overline{x}_0, \dots, \overline{x}_n\colon S^n_{\mathbb{C}} \to \mathbb{C}$ (projection to the i-th $\mathbb{C}$-coordinate and complex conjugation) are such functions. These global sections generate $\Gamma(\gamma^n)$, because for each point of $\mathbb{P}^n_{\mathbb{C}}$ one coordinate function does not vanish and γ^n is a line bundle. We get then the following maps of $C^\infty(\mathbb{P}^n_{\mathbb{C}})$ modules:

$$C^\infty(\mathbb{P}^n_{\mathbb{C}})^{n+1} \xrightarrow{\alpha} \Gamma(\gamma^n), \qquad \Gamma(\gamma^n) \xrightarrow{\beta} C^\infty(\mathbb{P}^n_{\mathbb{C}})^{n+1},$$

$$e_i \mapsto \overline{x}_i, \qquad h \mapsto \sum_{i=0}^{n} x_i h e_i.$$

Note that $x_i h$ is indeed a well defined function on $\mathbb{P}^n_{\mathbb{C}}$. Furthermore, α is surjective and $\alpha \circ \beta = \mathrm{id}$, so $\beta \circ \alpha$ is a idempotent with image isomorphic to $\Gamma(\gamma^n)$. The matrix to this idempotent is given by

$$e := \begin{pmatrix} \overline{x}_0 x_0 & \overline{x}_1 x_0 & \dots & \overline{x}_n x_0 \\ \overline{x}_0 x_1 & \overline{x}_1 x_1 & \dots & \overline{x}_n x_1 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{x}_0 x_n & \overline{x}_1 x_n & \dots & \overline{x}_n x_n \end{pmatrix}.$$

Then we get a connection ∇ of the projective $R := C^\infty(\mathbb{P}^n_{\mathbb{C}})$-module $\Gamma(\gamma^n)$ by the following composition:

$$\Gamma(\gamma^n) \hookrightarrow \Gamma(\gamma^n) \oplus \ker(e) \cong R^{n+1} \xrightarrow{d^{n+1}} (\Omega_{R/\mathbb{C}})^{\oplus(n+1)} \cong R^{n+1} \otimes_R \Omega_{R/\mathbb{C}} \xrightarrow{e\otimes\mathrm{id}} \Gamma(\gamma^n) \otimes \Omega_{R/\mathbb{C}}$$

this connection is called the Levi-Civita connection (and can be defined similar for any finitely generated projective module over any ring). By composing ∇ with the map $\mathrm{id} \otimes \lambda\colon \Gamma(\gamma^n) \otimes \Omega_{R/\mathbb{C}} \to \Gamma(\gamma^n) \otimes A^1(\mathbb{P}^n_{\mathbb{C}})$ we get a connection for the vector bundle γ^n. The first Chern character given by this connection can be computed by the formula

$$\mathrm{Tr}(edede) \in H^2_{\mathrm{dR}}(R/\mathbb{C}) \text{ respectively } \frac{1}{2\pi i} H^2(\lambda)(\mathrm{Tr}(edede)) \in H^2_{\mathrm{dR}}(\mathbb{P}^n_{\mathbb{C}}),$$

where d is the universal differential $R \to \Omega_{R/\mathbb{C}}$.

4 The (homotopy) category of matrix factorizations

In this chapter A is a commutative ring and $f \in A$. All modules in this chapter will be assumed to be finitely generated.

4.1 Matrix factorizations and homotopy

4.1 Definition A matrix factorization of f over A is a pair of matrices

$$(\varphi, \psi) \in M_{n\times l}(A) \times M_{l\times n}(A),$$

s.t. $\varphi\psi = f\,\mathrm{id}_n$, $\psi\varphi = f\,\mathrm{id}_l$.

We will see examples of matrix factorizations and corresponding resolutions in Section 4.4 and Example 5.23. For the rest of this chapter we assume, that $0 \neq f$ is not a zero divisor. In this case we get the following lemma.

4.2 Lemma If $0 \neq f \in A$ is a non zero divisor and (φ, ψ) is a matrix factorization of f over A, then φ, ψ are square matrices of the same size, injective as mappings and either matrix determines the other uniquely.

4.3 Definition Let (φ, ψ), (φ', ψ') two matrix factorizations.

1. A *morphism of matrix factorizations* is a pair of two A-linear maps (α, β), which make the following diagram commute:

$$\begin{tikzcd} A^m \arrow[r, "\psi"] \arrow[d, "\alpha"] & A^m \arrow[r, "\varphi"] \arrow[d, "\beta"] & A^m \arrow[d, "\alpha"] \\ A^{m'} \arrow[r, "\psi'"] & A^{m'} \arrow[r, "\varphi'"] & A^{m'} \end{tikzcd}$$

   We denote the set of all morphisms with $MF((\varphi, \psi), (\varphi', \psi'))$. A morphism is an *isomorphism of matrix factorizations* if α and β are isomorphisms.

2. Two morphisms (α,β), (α',β') are said to be *homotopic* if there are two A-linear maps s_0, s_1 fitting in the following diagram

$$\begin{tikzcd} A^m \arrow[r, "\psi"] \arrow[d, "\alpha-\alpha'"'] & A^m \arrow[r, "\varphi"] \arrow[d, "\beta-\beta'"] \arrow[dl, "s_1"] & A^m \arrow[d, "\alpha-\alpha'"] \arrow[dl, "s_0"] \\ A^{m'} \arrow[r, "\psi'"] & A^{m'} \arrow[r, "\varphi'"] & A^{m'}, \end{tikzcd}$$

such that $\alpha-\alpha'=\varphi'\circ s_0+s_1\circ\psi$ and $\beta-\beta'=\psi'\circ s_1+s_0\circ\varphi$.

3. A morphism (α,β) is called a *homotopy equivalence* if there is a morphism (α',β') in the other direction, such that both compositions are homotopic to the identity.

4. We say (φ,ψ), (φ',ψ') are *homotopy equivalent* if there is a homotopy equivalence between them. We refer to all matrix factorizations homotopy equivalent to a given one (φ,ψ) as the *homotopy class of* (φ,ψ) and denote it with $[\varphi,\psi]$.

4.4 Lemma In the first part of Definition 4.3 either of the two equations

$$\alpha\circ\varphi=\varphi'\circ\beta,\quad \beta\circ\psi=\psi'\circ\alpha$$

implies the other. The same is true for the two equations

$$\alpha-\alpha'=\varphi'\circ s_0+s_1\circ\psi,\quad \beta-\beta'=\psi'\circ s_1+s_0\circ\varphi$$

from the second part.

Proof. We show that $\alpha-\alpha'=\varphi'\circ s_0+s_1\circ\psi$ implies $\beta-\beta'=\psi'\circ s_1+s_0\circ\varphi$. The other implications can be proven in a similar way. By composing with ψ' from the left and φ from the right we get

$$\psi'\circ(\alpha-\alpha')\circ\varphi=\psi'\circ\varphi'\circ s_0\circ\varphi+\psi'\circ s_1\circ\psi\circ\varphi,$$

which simplifies to

$$f(\beta-\beta')=f(\psi'\circ s_1+s_0\circ\varphi).$$

Now cancel f, which is possible because f is a non zero divisor. $\square$

There is a different $\mathbb{Z}/2\mathbb{Z}$-graded approach to matrix factorizations, which is often used in the literature (for example in the cited paper [PV12]). We work with Definition 4.1 of a matrix factorizations, unless stated otherwise.

4.5 Definition (Matrix factorization, $\mathbb{Z}/2\mathbb{Z}$-graded version)

1. A matrix factorization is a pair (E,δ), where $E=E_0\oplus E_1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded module with E_0, E_1 free and $\delta\colon E\to E$ is an odd endomorphism, satisfying $\delta^2=f\,\mathrm{id}_E$. We will write δ as block matrix $\left(\begin{smallmatrix}0&\varphi\\\psi&0\end{smallmatrix}\right)$ with $\varphi\colon E_0\to E_1$ and $\psi\colon E_1\to E_0$.

2. The homomorphism complex of (E, δ) and (E', δ') is the $\mathbb{Z}/2\mathbb{Z}$-graded complex

$$MF_{\mathbb{Z}/2\mathbb{Z}}((E, \delta), (E', \delta')) = MF_0((E, \delta), (E', \delta')) \oplus MF_1((E, \delta), (E', \delta'))$$

of graded maps from E to E' of degree 0 respectively 1, together with the differential D given by $\alpha \mapsto \delta' \circ \alpha - (-1)^{|\alpha|}\alpha \circ \delta$ for homogeneous α of degree $|\alpha|$.

Morphisms in the sense of Definition 4.3 are elements of the kernel $D\colon MF_0 \to MF_1$ and two morphisms in this sense are homotopic if and only if their difference is in the image of $D\colon MF_1 \to MF_0$.

4.6 Definition We denote with $MF(A, f)$ the category of all matrix factorizations of f over A and with $[MF(A, f)]$ the corresponding homotopy category. This means $[MF(A, f)]$ has the same objects as $MF(A, f)$, but homotopy classes of morphisms in $MF(A, f)$ as morphisms (which is the same as applying H_0 to the morphism complexes in the language of Definition 4.5). Therefore, the isomorphisms in $[MF(A, f)]$ are precisely the homotopy equivalences.

4.2 Minimal free resolutions and matrix factorizations

In this section we recap basic results about free resolutions of modules, which can be found (for example) in [Eis95]. Then we state some of the properties of minimal free resolutions, in particular we are interested in their connections to matrix factorizations. This connection was the main reason for the introduction of matrix factorizations by Eisenbud in [Eis80].

All rings in this section are noetherian and all modules are finitely generated.

4.7 Definition 1. A *free resolution of* a module M, is a (possibly infinite) complex

$$F_\bullet := \ldots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to 0,$$

where each F_i is a free module and sits in degree i. Furthermore, $(F_\bullet, \partial_\bullet)$ is exact, except at F_0, where the homology is M. The exact complex

$$\ldots \to F_2 \to F_1 \to F_0 \xrightarrow{\varepsilon} M \to 0$$

is called an *augmented free resolution of* M.

2. If A is local with maximal ideal $\mathfrak{m}$, then a *minimal free resolution of* M is a free resolution with one of the following equivalent conditions:

   a) $\operatorname{im}(\partial_i) \subset \mathfrak{m}F_{i-1}$ for $i \geq 1$.

   b) The rank of F_0 is the minimal number of generators of M (which is equals $\dim_{A/\mathfrak{m}}(M/\mathfrak{m}M)$) and the rank of F_i equals the minimal number of generators of $\ker(\partial_{i-1})$ for all $i \geq 1$ (with $\partial_0 = \varepsilon$).

3. If A is local and $F_\bullet$ is a minimal free resolution of M, then the i-th *Betti number* of M is defined to be the rank of F_i.

4. A complex $F_\bullet$ is called *periodic of period length* n if there is an isomorphism of complexes $s\colon F_\bullet[n] \to F_\bullet$. If $F_i = 0$ for all $i < j$, then we require $s_i\colon F_{n+i} \to F_i$ to be isomorphism only for $i \geq j$.

5. A complex is called *periodic* if it is periodic of period length n for some n.

6. A complex $F_\bullet$ with $F_i = 0$ for all $i < j$ is called *eventually periodic* if it is periodic after cutting off the first k non zero terms for some k.

Now we state two important facts about free resolutions, which will be used throughout this thesis.

4.8 Lemma ([Wei94], 2.2.5, 2.2.6) Free resolutions exist and are unique up to homotopy equivalence.

4.9 Lemma ([Eis95], 20.1, 20.2) If A is local, then minimal free resolutions exist and are unique up to isomorphisms of complexes. In particular for a given module M either all resolutions are (eventually) periodic or none are.

4.10 Proposition ([Eis80], 4.1) Let A be a regular local ring and $I \subset A$ an ideal generated by an A-sequence. Set $B = A/I$, and let

$$F_\bullet : \ldots \to F_1 \to F_0$$

be a minimal B-free resolution. If $\{\operatorname{rank} F_i\}$ is bounded, then $F_\bullet$ becomes periodic of period 2 after at most $1 + \operatorname{Kdim} B$ steps.

This proposition was later generalized for example in [GP90].

4.11 Proposition ([GP90], 1.2) Let A be a Cohen-Macaulay local ring of multiplicity ≤ 7 or a Gorenstein local ring of multiplicity ≤ 11. If M is a finitely generated A-module with bounded Betti numbers, then M is eventually periodic of period 2.

Here M being periodic means every minimal free resolution is so. The authors of [GP90] construct minimal (i.e. Cohen-Macaulay of multiplicity 8 or Gorenstein of multiplicity 12) examples of minimal free resolutions with bounded Betti numbers and arbitrary period length respectively not periodic ([GP90], 3.1 and 3.4). Their multiplicity 8 example is given here.

4.12 Example 1. If P is a projective A-module, which is isomorphic to the image of the idempotent $e \in M_n(A)$ (see Construction 3.2), then

$$\ldots \xrightarrow{e} A^n \xrightarrow{1-e} A^n \xrightarrow{e} A^n \xrightarrow{1-e} A^n$$

is a 2-periodic free resolution of P. If A is local this resolution is not minimal.

2. The next example shows, that (even over local rings) periodic, minimal free resolutions of arbitrary length exist. This example is Proposition (3.4) in [GP90]. Let $0 \neq \alpha \in \mathbb{C}$. We define

$$\begin{aligned} R &= R_\alpha = \mathbb{C}[W,X,Y,Z]/I, \text{ where} \\ I &= I_\alpha = (\alpha WY + XY, WZ + XZ, YZ, W^2, X^2, Y^2, Z^2). \end{aligned}$$

R is local with maximal Ideal $\mathfrak{m} = (W,X,Y,Z) = \sqrt{I}$. Now we define

$$d_n := \begin{pmatrix} W & \alpha^n Y + Z \\ 0 & X \end{pmatrix} \text{ for } n \in \mathbb{Z}.$$

As $d_n d_{n+1} = 0$ (modulo I) for all $n \in \mathbb{Z}$ we get a complex

$$\dots \overset{d_{n+1}}{\to} R^2 \overset{d_n}{\to} R^2 \overset{d_{n-1}}{\to} \dots \tag{4.1}$$

We will show, that this complex is exact. Note that $\dim_{\mathbb{C}} R = 8$, because

$$1, W, X, Y, Z, WX, WY, WZ$$

is a basis. We have $\dim \operatorname{im}(d_n) \geq 8$ for all n, because

$$d_n\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}W\\0\end{pmatrix}, \quad d_n\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}\alpha^n Y + Z\\X\end{pmatrix}, \quad d_n\left(\begin{pmatrix}0\\W\end{pmatrix}\right) = \begin{pmatrix}\alpha^n WY + WZ\\WX\end{pmatrix},$$
$$d_n\left(\begin{pmatrix}X\\0\end{pmatrix}\right) = \begin{pmatrix}WX\\0\end{pmatrix}, \quad d_n\left(\begin{pmatrix}Y\\0\end{pmatrix}\right) = \begin{pmatrix}WY\\0\end{pmatrix}, \quad d_n\left(\begin{pmatrix}0\\Y\end{pmatrix}\right) = \begin{pmatrix}0\\-\alpha WY\end{pmatrix},$$
$$d_n\left(\begin{pmatrix}Z\\0\end{pmatrix}\right) = \begin{pmatrix}WZ\\0\end{pmatrix}, \quad d_n\left(\begin{pmatrix}0\\Z\end{pmatrix}\right) = \begin{pmatrix}0\\-WZ\end{pmatrix},$$

are $\mathbb{C}$-independent elements of image of d_n. We can conclude

$$8 \leq \dim \operatorname{im}(d_n) = \dim R^2 - \dim \ker(d_n) \leq 16 - \dim \operatorname{im}(d_{n+1}) \leq 16 - 8 = 8.$$

This implies $\dim \operatorname{im}(d_n) = \dim \ker(d_n) = 8$ for all n and hence $\operatorname{im}(d_{n+1}) = \ker(d_n)$. By cutting off the exact sequence in (4.1), we get a free resolution of the R-module $\operatorname{coker}(d_n)$

$$\dots \overset{d_{n+2}}{\to} R^2 \overset{d_{n+1}}{\to} R^2 \overset{d_n}{\to} R^2 \to 0.$$

This resolution is minimal, as the entries of d_k are in $\mathfrak{m}$.
If α does not have a finite order, then this complex will not be eventually periodic. If the order of α is l, then this complex is periodic with minimal periodic length l. To show this statement about the minimality of the period length, it is enough to show that unless l divides $t - s$ the matrices d_t and d_s are not equal up to base change, i.e. there are no invertible matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in M_2(R)$, s.t.

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) d_t = d_s \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right). \tag{4.2}$$

Since d_t and d_s are homogeneous of degree 1 and R is the quotient of a polynomial ring by an ideal homogeneously generated in degree 2, equation (4.2) is also true in the polynomial ring if we replace the (classes of the) polynomials $a, \ldots, d, a', \ldots, d'$ with their constant parts, which we will denote with an overline.

$$\left(\begin{smallmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{smallmatrix}\right) d_t = d_s \left(\begin{smallmatrix} \overline{a'} & \overline{b'} \\ \overline{c'} & \overline{d'} \end{smallmatrix}\right).$$

Looking at the top right entry of this matrix equation we find

$$\overline{a}\alpha^t Y + \overline{a}Z + \overline{b}X = \overline{b'}W + \overline{d'}\alpha^s Y + \overline{d'}Z.$$

Hence we get

$$\overline{b'} = 0,\ \overline{b} = 0,\ \overline{a}\alpha^t = \overline{d'}\alpha^s,\ \overline{a} = \overline{d'},$$

which gives $\overline{a} = \overline{d'} = 0$, unless $\alpha^t = \alpha^s$. If the constant part of a, b, b', d' are zero, then the determinant of the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ is no unit. This completes the proof.

4.13 Proposition ([Eis80], 5.1) For any matrix factorization (φ, ψ) of x over A if $(x)/(x^2)$ is free over $B = A/(x)$, we get a periodic free resolution of the B-module $\operatorname{coker}(\varphi)$ as follows

$$\mathbf{F}(\varphi, \psi) : \ldots \xrightarrow{\overline{\psi}} A^l \xrightarrow{\overline{\varphi}} A^n \xrightarrow{\overline{\psi}} A^l \xrightarrow{\overline{\varphi}} A^n,$$

where the matrices $\overline{\varphi}, \overline{\psi}$ are given by reducing the entries modulo x.

4.14 Proposition ([Eis80], 5.2) Let A be a regular local ring with infinite residue class field, and let $I \subset A$ be an ideal generated by an A-sequence. Set $B = A/I$. If $\mathbf{F}$ is a periodic minimal B-free resolution, then there exist a local ring B_1, a non zero divisor $x \in B_1$, and a matrix factorization (φ, ψ) of x over B_1, such that $\mathbf{F} = \mathbf{F}(\varphi, \psi)$.

4.15 Proposition ([Eis80], 5.3) Let A be a noetherian ring, and let $\ldots \xrightarrow{\varphi} G \xrightarrow{\psi} F \xrightarrow{\varphi} G$ be a free resolution of a finitely generated R-module which is periodic of period 2. Then $\operatorname{rank} F = \operatorname{rank} G$.

4.16 Proposition ([Eis80], 6.1) Let A be a regular local ring, $x \in A$, and let $B = A/x$. Let $d = \dim A$. If

$$\mathbf{F} : \ldots \to F_1 \to F_0$$

is the minimal B-free resolution of a finitely generated B-module N, then:

1. $\mathbf{F}$ becomes periodic of period 2 after $d + 1$ steps,

2. $\mathbf{F}$ is periodic (necessarily of period 2) if and only if N is a maximal Cohen-Macaulay B-module with no free summand.

3. Every periodic minimal resolution over B comes from a matrix factorization of x over A (in the sense of Proposition 4.13).

4.17 Definition 1. We call the two matrix factorization $(1, x)$ and $(x, 1)$ of x over A *trivial*.

2. The *direct sum* of two matrix factorizations (φ, ψ) and (φ', ψ') of x over A is defined to be
$$\left(\begin{pmatrix}\varphi & 0\\ 0 & \varphi'\end{pmatrix}, \begin{pmatrix}\psi & 0\\ 0 & \psi'\end{pmatrix}\right).$$

3. A matrix factorization is called *reduced*, if it is not isomorphic to any matrix factorization having a trivial matrix factorizations as direct summand.

If A is local with maximal ideal $\mathfrak{m}$ and $x \in A$ is a non zero divisor, a matrix factorization of x over A is reduced if and only if all entries are in $\mathfrak{m}$, i.e. the induced free resolution is minimal.

4.18 Proposition ([Eis80], 6.3) Let A be a regular local ring and let $B = A/(x)$ be a proper factor ring. Write $\overline{(-)}$ for reduction modulo x. The associations

$$(\varphi, \psi) \mapsto \mathbf{F}_{(\varphi,\psi)} : \ldots \xrightarrow{\overline{\varphi}} \overline{G} \xrightarrow{\overline{\psi}} \overline{F} \xrightarrow{\overline{\varphi}} \overline{G}$$

and

$$(\varphi, \psi) \mapsto M_{(\varphi,\psi)} = \operatorname{coker} \varphi$$

induces bijections between the sets of

1. Equivalence classes of reduced matrix factorizations of x over A,

2. Isomorphism classes of nontrivial periodic minimal free resolutions over B,

3. Maximal Cohen-Macaulay B-modules without free summands.

This was later generalized by Buchweitz in [Buc87] to the equivalence of categories stated in Proposition 4.21. Buchweitz also works with not necessarily commutative strongly Gorenstein rings. A proper quotient of a regular local ring is strongly Gorenstein. For the rest of this section we will denote with S a not necessarily commutative ring, which is supposed to be noetherian on both sides.

4.19 Definition ([Buc87], 4.1, 4.2.1) 1. S is said to be *strongly Gorenstein* if it is of finite injective dimension as both a left or right module over itself.

2. A S-module M is called *maximal Cohen-Macaulay* if it is $\hom_S(-, S)$-acyclic, i.e. $\operatorname{Ext}^i_S(M, S) = 0$ for $i > 0$.

4.20 Definition ([Buc87]) 1. We define $\operatorname{mod}(S)$ to be the category of finitely generated left S-modules.

2. We define $\underline{\operatorname{mod}}(S)$ to be the so called *stabilization of* $\operatorname{mod}(S)$, i.e. the same objects, but factoring out morphisms, which factor through a finitely generated projective module.

3. We define the category $MCM(S)$ to be the full subcategory of $\mathrm{mod}(S)$ consisting of the maximal Cohen-Macaulay modules.

4. We define $\underline{MCM}(S)$ to be the so called *stabilization of* $MCM(S)$, i.e. the same objects, but factoring out morphisms, which factor through a finitely generated projective S-module.

5. We define $\underline{APC}(S)$ to be the homotopy category of (unbounded) acyclic complexes of finitely generated projective S-modules.

6. $D(S)$ denotes the *derived category* of S.

7. A complex of S-modules is called *perfect*, if it is isomorphic in $D(S)$ to a finite complex of finitely generated projective S-modules.

8. We define $D^b_{\mathrm{perf}}(S)$ to be the subcategory of $D^b(S)$ consisting of all perfect complexes.

9. $D^b_{\mathrm{perf}}(S)$ is a thick subcategory of $D^b(S)$. Therefore, we can define the (triangulated) quotient $\underline{D^b}(S) := D^b(S)/D^b_{\mathrm{perf}}(S)$ and call it the *stabilized derived category of* S.

The following theorem shortened version of [Buc87], 4.4.1. Buchweitz gives the equivalences of categories explicitly.

4.21 Proposition If S is strongly Gorenstein, then the three categories $\underline{MCM}(S)$, $\underline{APC}(S)$ and $\underline{D^b}(S)$ are equivalent.

4.22 Corollary If A is a regular local ring and $B = A/(x)$ is a proper quotient, then B is strongly Gorenstein and the homotopy category of matrix factorizations of x over A $[MF(A,x)]$ is equivalent to either of the categories $\underline{MCM}(B)$, $\underline{APC}(B)$ and $\underline{D^b}(B)$.

4.3 Triangulated structure on the homotopy category

Some of the categories, which are equivalent to $[MF(A,f)]$ have the structure of a triangulated category. In this section we introduce the induced triangulated structure on $[MF(A,f)]$.

All matrix factorizations in this section are assumed to be of f over A.

4.23 Definition Let $(\alpha,\beta)\colon (\varphi,\psi) \to (\varphi',\psi')$ be a morphism of matrix factorizations.

1. We define *the suspension of* (φ,ψ) as the matrix factorization $\Sigma((\varphi,\psi)) := (-\psi,-\varphi)$. *The suspension of* (α,β) is the morphism of matrix factorizations $\Sigma(\alpha,\beta) = (\beta,\alpha)$. Σ defines a functor from $[MF(A,f)]$ to itself with $\Sigma^2 = \mathrm{id}$.

2. The *mapping cone of* (α, β) is defined to be the matrix factorization

$$C_{(\alpha,\beta)} := \left(\begin{pmatrix} \varphi' & \alpha \\ 0 & -\psi \end{pmatrix}, \begin{pmatrix} \psi' & \beta \\ 0 & -\varphi \end{pmatrix} \right).$$

3. There are two obvious morphisms of matrix factorizations

$$(\varphi', \psi') \xrightarrow{(\mathrm{id}\times 0, \mathrm{id}\times 0)} C_{(\alpha,\beta)} \xrightarrow{(0\oplus \mathrm{id}, 0\oplus \mathrm{id})} \Sigma((\varphi, \psi)).$$

4. A *triangle* is a sequence of matrix factorizations and morphisms between them

$$(\varphi, \psi) \to (\varphi', \psi') \to (\varphi'', \psi'') \to \Sigma(\varphi, \psi).$$

5. A *morphism of two triangles* is given by three morphisms of matrix factorizations, making the following diagram commute

$$\begin{tikzcd}
(\varphi, \psi) \arrow[r] \arrow[d, "(\alpha_1,\beta_1)"] & (\varphi', \psi') \arrow[r] \arrow[d, "(\alpha_2,\beta_2)"] & (\varphi'', \psi'') \arrow[r] \arrow[d, "(\alpha_3,\beta_3)"] & \Sigma(\varphi, \psi) \arrow[d, "\Sigma(\alpha_1,\beta_1)"] \\
(\Phi, \Psi) \arrow[r] & (\Phi', \Psi') \arrow[r] & (\Phi'', \Psi'') \arrow[r] & \Sigma(\Phi, \Psi)
\end{tikzcd}.$$

   A morphism of triangles is an *isomorphism of triangles*, if all vertical maps are isomorphisms in $[Mf(A, f)]$, i.e. homotopy equivalences.

6. We call a triangle *distinguished*, if it is isomorphic to a triangle coming from a mapping cone, i.e.

$$(\varphi, \psi) \xrightarrow{(\alpha,\beta)} (\varphi', \psi') \xrightarrow{(\mathrm{id}\times 0, \mathrm{id}\times 0)} C_{(\alpha,\beta)} \xrightarrow{(0\oplus \mathrm{id}, 0\oplus \mathrm{id})} \Sigma((\varphi, \psi)).$$

   We denote the set of all distinguished triangles with Δ.

4.24 Proposition The category $[MF(A, f)]$ together with the suspension functor Σ and Δ as set of distinguished triangles defines a triangulated category.

Proof. This is Proposition 3.3 of [Orl04] or Theorem 5.1.1 in the master thesis [Lan16], which takes a more elementary approach to verify it. ☐

4.4 Tensor structure on the homotopy category

In this section we introduce the pseudo tensor triangulated structure of $[MF(A, f)]$. We mainly follow, what has been done in [Yu13]. One can also compare with [Yos98].

In this section k is a commutative ring and A, A' are (unital) commutative k-algebras and $f' \in A'$ (and still $f \in A$). $\otimes$ will mean $\otimes_k$.

4.25 Definition Assume we have two matrix factorizations $(\varphi,\psi) \in M_n(A)^2$ of f over A and $(\varphi',\psi') \in M_{n'}(A')^2$ of f' over A'. We define the *tensor product* to be the following matrix factorization of $f \otimes 1 + 1 \otimes f'$ over $A \otimes A'$

$$(\varphi,\psi) \otimes_{\mathrm{MF}} (\varphi',\psi') = \left(\begin{pmatrix} \varphi \otimes \mathrm{id}_{n'} & \mathrm{id}_n \otimes \varphi' \\ -\mathrm{id}_n \otimes \psi' & \psi \otimes \mathrm{id}_{n'} \end{pmatrix}, \begin{pmatrix} \psi \otimes \mathrm{id}_{n'} & -\mathrm{id}_n \otimes \varphi' \\ \mathrm{id}_n \otimes \psi' & \varphi \otimes \mathrm{id}_{n'} \end{pmatrix} \right).$$

If $A = A'$ we map $A \otimes A$ to A via the multiplication map, then $(\varphi,\psi) \otimes_{\mathrm{MF}} (\varphi',\psi')$ is a matrix factorization of $f + f'$ over A.

The formula for $\otimes_{\mathrm{MF}}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded version of the tensor product for complexes.

4.26 Definition Let $a = (a_1,\dots,a_m), b = (b_1,\dots,b_m) \in A^m$. Then we define the matrix factorization $\{a,b\} := (a_1,b_1) \otimes_{\mathrm{MF}} \dots \otimes_{\mathrm{MF}} (a_m,b_m)$ of $f = a \cdot b = a_1b_1 + \dots + a_mb_m$ and call it *the Koszul matrix factorization corresponding to the pair* (a,b).

If a is a regular A-sequence and $f = a_1b_1 + \dots + a_mb_m \in I = (a_1,\dots a_m)$, then $\{a,b\}$ gives the periodic part of a B-resolution of the B-module R/I ([PV12], 2.3.1).

4.27 Example Let us look examples for Koszul matrix factorization for $m = 1,2$.

1. Set $A := \mathbb{C}[\![X,Y]\!]$, $f := XY$, $B = A/(f)$ and consider the 1×1 matrix factorization $\{(X),(Y)\} = (X,Y)$. This matrix factorization is the periodic part of the minimal free resolution of the B-module $\mathbb{C}[\![Y]\!] = B/(X) = \mathrm{coker}(X)$.

2. Set $A = \mathbb{C}[\![W,X,Y,Z]\!]$, $f = WX - YZ$ and $B = A/(f)$ and consider the 2×2 matrix factorization

$$\{(W,Y),(X,-Z)\} = (W,X) \otimes_{\mathrm{MF}} (Y,-Z) = \left(\begin{pmatrix} W & Y \\ Z & X \end{pmatrix}, \begin{pmatrix} X & -Y \\ -Z & W \end{pmatrix} \right).$$

   This matrix factorization can either be seen as the 2-periodic part of the minimal free resolution of the B-module $\mathrm{coker}(\varphi)$ or the eventually 2-periodic part in the minimal free resolution of $M = \mathbb{C}[\![X,Z]\!] = A/(W,Y)$. M is not maximal Cohen-Macaulay, because the minimal free resolution will have an non periodic start.

For the rest of this section we will assume, that 2 is invertible in k. Let $(\varphi,\psi) \in M_n(A)$ a matrix factorization of f over A and $(\varphi',\psi') \in M_{n'}(A')$ of f' over A'. If $A = A'$ and $f = f'$, then $(\varphi,\psi) \otimes_{\mathrm{MF}} (\varphi',\psi')$ is a matrix factorization of $2f$. We are interested in a tensor product $[MF(A,f)] \times [MF(A,f)] \to [MF(A,f)]$. Therefore, we will introduce the factor $\frac{1}{2}$ to the second matrix.

4.28 Definition We define a tensor product $[MF(A,f)] \times [MF(A,f)] \to [MF(A,f)]$ given by first applying $\otimes_{\mathrm{MF}}$ and then multiplying the second matrix with $\frac{1}{2}$. We denote this tensor product with $(\varphi,\psi) \otimes^{\frac{1}{2}} (\varphi',\psi')$ or $(\varphi,\psi) \otimes^{\frac{1}{2}}_{\mathrm{MF}} (\varphi',\psi')$.

4.29 Definition ([Yu13], 4.1.1) 1. A *pseudo tensor category* is a category $\mathfrak{C}$ equipped with a tensor product $\otimes\colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$, s.t. for any $a, b, c \in \mathfrak{C}$, there is a natural isomorphism, called the *associator*

$$\alpha_{a,b,c}\colon (a \otimes b) \otimes c \cong a \otimes (b \otimes c)$$

and a natural isomorphism, called the *braiding*

$$B_{a,b}\colon a \otimes b \cong b \otimes a.$$

We require the associator to satisfy the *pentagon identity*, i.e. the following diagram commutes

$$\begin{tikzcd}
 & & (a \otimes b) \otimes (c \otimes d) \arrow[drr, "\alpha_{a,b,c\otimes d}"] & & \\
((a \otimes b) \otimes c) \otimes d \arrow[urr, "\alpha_{a\otimes b,c,d}"] \arrow[dr, "\alpha_{a,b,c}\otimes \mathrm{id}_d"'] & & & & a \otimes (b \otimes (c \otimes d)) \\
 & (a \otimes (b \otimes c)) \otimes d \arrow[rr, "\alpha_{a,b\otimes c,d}"] & & a \otimes ((b \otimes c) \otimes d) \arrow[ur, "\mathrm{id}_a \otimes \alpha_{b,c,d}"] &
\end{tikzcd}$$

We also require the associator and the braiding to satisfy the *hexagon identity*, i.e. the following diagram commutes

$$\begin{tikzcd}
(a \otimes b) \otimes c \arrow[r, "\alpha_{a,b,c}"] \arrow[d, "B_{a,b}\otimes \mathrm{id}_c"] & a \otimes (b \otimes c) \arrow[r, "B_{a,b\otimes c}"] & (b \otimes c) \otimes a \arrow[d, "\alpha_{b,c,a}"] \\
(b \otimes a) \otimes c \arrow[r, "\alpha_{b,a,c}"] & b \otimes (a \otimes c) \arrow[r, "\mathrm{id}_b \otimes B_{a,c}"] & b \otimes (c \otimes a)
\end{tikzcd}$$

2. If $\mathfrak{C}$ is triangulated, we will say $\mathfrak{C}$ is a *pseudo tensor triangulated category* if it is pseudo tensor and in addition $- \otimes a$, $a \otimes -$ are triangulated functors.

4.30 Proposition ([Yu13], 4.1.22) The triangulated category $[MF(A, f)]$ together with the tensor product $\otimes^{\frac{1}{2}}$ is a pseudo tensor triangulated category.

5 Chern theories for matrix factorizations

In this chapter we will define Chern theories for $[MF(A,f)]$, $MF(A,f)$ and exhibit their functoriality with respect to k-algebra maps, their behaviour under cones of morphisms of matrix factorizations and the tensor product of matrix factorizations.

In this chapter k is a commutative ring and A is an associative, unital, commutative k-algebra, $0 \neq f \in A$ is supposed to be a non zero divisor, $B = A/(f)$ a proper quotient, $(\varphi, \psi) \in M_r(A)^2$ a matrix factorization of f over A and Ω^n will mean $\Omega^n_{A/k}$.

5.1 Chern theories for $[MF(A,f)]$

In this section we will construct four Chern theories for the category $[MF(A,f)]$.

Recall the maps $M_r(A) \to M_{r+1}(A)$ defined by boxing with zeros, that give maps $HH_n(M_r(A)) \to HH_n(M_{r+1}(A))$, which commute with the generalized trace map.

5.1 Definition We define $M(A)$ as the direct limit $\lim_r M_r(A)$. Then we get also generalized trace maps $\mathrm{tr}_n\colon HH_n(M(A)) = \lim_r HH_n(M_r(A)) \to HH_n(A)$, which are still isomorphisms. For details on this we refer to [Lod98].

We can define a k-algebra structure on B via the quotient map $A \to B$ (this is well defined as $k \cdot (f) \subset (f)$). Then we also get k-algebra structures on $M_l(B)$ and $M(B)$.

5.2 Proposition Let (φ, ψ) be a matrix factorization. Then the element

$$(\varphi \otimes_k \psi)^{\otimes_k n} := \varphi \otimes_k \psi \otimes_k \ldots \otimes_k \varphi \otimes_k \psi \in HH_{2n-1}(M(B))$$

depends only on the homotopy class of (φ, ψ). Here we interpret φ, ψ as matrices over B by reducing modulo f componentwise.

Proof. This follows from the next more general Lemma 5.3. □

5.3 Lemma Let T be an associative (not necessarily commutative) k-algebra and $a_0, \ldots, a_n, b_0, \ldots, b_n \in T$ satisfying

$$a_i a_{i+1} = 0 = b_i b_{i+1} \text{ for all } i \in \mathbb{Z}/(n+1)\mathbb{Z}.$$

Furthermore, let $f_0, \dots, f_n, g_0, \dots, g_n \in A$, s.t.

$$f_{i-1}a_i = b_i f_i,\ g_{i-1}b_i = a_i g_i \text{ for all } i \in \mathbb{Z}/(n+1)\mathbb{Z}.$$

Finally let $s_0, \dots, s_n, t_0, \dots, t_n \in A$ with

$$\begin{aligned} f_i g_i - 1_i^{(b)} &= b_{i+1}t_i + t_{i-1}b_i \text{ for all } i \in \mathbb{Z}/(n+1)\mathbb{Z}, \\ g_i f_i - 1_i^{(a)} &= a_{i+1}s_i + s_{i-1}a_i \text{ for all } i \in \mathbb{Z}/(n+1)\mathbb{Z}, \end{aligned}$$

where $1_i^{(b)}, 1_i^{(a)} \in T$, such that $1_i^{(a)}a_{i+1} = a_{i+1}$ and $a_i 1_i^{(a)} = a_i$ and similar for b instead of a. Then we have the following equation in $HH_n(T)$:

$$a_0 \otimes \dots \otimes a_n = b_0 \otimes \dots \otimes b_n.$$

If a_i, b_i are the differentials in two different periodic complexes (or matrix factorizations), then these equations are precisely saying, that those two complexes (or matrix factorizations) are homotopy equivalent (via the maps f, g and everything has the same period length). The elements $1_i^{(b)}, 1_i^{(a)}$ are the identity matrices of various sizes (for matrix factorizations we have $1_i^{(a)} = 1_{i+1}^{(a)}$, $1_i^{(b)} = 1_{i+1}^{(b)}$ for all i). Non square matrices are then seen as square matrices (and hence elements of T) by filling up with zeros.

Proof. In this proof we will often omit the subscripts. Clearly

$$b(a_0 \otimes \dots \otimes a_n) = 0 = b(b_0 \otimes \dots \otimes b_n),$$

so these two represent elements of $HH_n(T)$. We want to show the following equations in HH_n:

$$\begin{aligned} (a \otimes \dots \otimes a) &= (asa + a \otimes \dots \otimes asa + a), \\ (b \otimes \dots \otimes b) &= (gbf \otimes \dots \otimes gbf). \end{aligned}$$

Once we have showed these two claims, we get

$$\begin{aligned} (a \otimes \dots \otimes a) &= (asa + a \otimes \dots \otimes asa + a) = (agf \otimes \dots \otimes agf) \\ &= (gbf \otimes \dots \otimes gbf) = (b \otimes \dots \otimes b), \end{aligned}$$

where the second and third equality hold, because

$$asa + a = a(gf - 1^{(a)} - as) + a = agf - a - 0 + a = agf = gbf.$$

To prove the first claim let us define

$$\begin{aligned} A_i &:= a_0 s_n a_0 + a_0 \otimes \dots \otimes a_i s_{i-1} a_i + a_i \otimes a_{i+1} \otimes \dots \otimes a_n, \\ C_i &:= a_0 s_n a_0 + a_0 \otimes \dots \otimes a_i s_{i-1} a_i + a_i \otimes a_{i+1} s_i + 1_i^{(a)} \otimes a_{i+1} \otimes \dots \otimes a_n. \end{aligned}$$

Then $b(C_i) = \pm(A_i - A_{i+1})$ and hence

$$(a \otimes \dots \otimes a) = A_{-1} = A_n = (asa + a \otimes \dots \otimes asa + a).$$

To prove the second claim let us define

$$
\begin{aligned}
B_i &:= b_0 f_0 \otimes g_0 b_1 f_1 \otimes \ldots \otimes g_{i-1} b_i f_i \otimes g_i b_{i+1} \otimes b_{i+2} \otimes \ldots \otimes b_n, \ n \geq i \geq -1, \\
D_i &:= b_0 f_0 \otimes g_0 b_1 f_1 \otimes \ldots \otimes g_{i-1} b_i f_i \otimes g_i \otimes b_{i+1} \otimes b_{i+2} \otimes \ldots \otimes b_n, \ n \geq i \geq 0, \\
B'_i &:= b_0 f_0 \otimes g_0 b_1 f_1 \otimes \ldots \otimes g_{i-1} b_i f_i \otimes g_i b_{i+1} f_{i+1} g_{i+1} \otimes b_{i+2} \otimes \ldots \otimes b_n, \ n \geq i \geq 0, \\
B'_{-1} &:= b_0 f_0 g_0 \otimes b_1 \otimes \ldots \otimes b_n, \\
D'_i &:= b_0 f_0 \otimes g_0 b_1 f_1 \otimes \ldots \otimes g_{i-1} b_i f_i \otimes g_i b_{i+1} \otimes t_i b_{i+1} + 1^{(b)}_{i+1} \otimes b_{i+2} \otimes \ldots \otimes b_n, \ n \geq i \geq 0, \\
D'_{-1} &:= b_0 \otimes t_n b_0 + 1^{(b)}_0 \otimes b_1 \otimes \ldots \otimes b_n.
\end{aligned}
$$

Then we have $b(D_i) = \pm(B_i - B'_{i-1})$, $b(D'_i) = \pm(B_i - B'_i)$ and hence

$$(b \otimes \ldots \otimes b) = B_{-1} = B_n = (gbf \otimes \ldots \otimes gbf).$$

□

5.4 Definition As direct consequence of Proposition 5.2, we can define the following Chern theory for $[MF(A, f)]$

- The target is $HH_{\text{odd}}(B) := \prod_{n=1}^{\infty} HH_{2n-1}(B)$.
- $[\mathrm{ch}]^{\mathrm{HH,\ odd}}_n\colon \ \mathrm{Obj}([MF(A, f)]) \to HH_{2n-1}(B)$, $(\varphi, \psi) \mapsto \mathrm{tr}_{2n-1}((\varphi \otimes \psi)^{\otimes 2n})$.

In Section 5.4 we will show that this Chern theory is triangulated additive. The compatibility with the tensor product is more complicated and will be dealt with in Section 5.5. We will refer to it as *Chern theory with values in odd Hochschild homology.*

In Lemma 1.16 we had the following commutative square

$$
\begin{array}{ccc}
HH_{2n-1}(B) & \xrightarrow{\pi_{2n-1}} & \Omega^{2n-1}_{B/k} \\
\Big\downarrow{\scriptstyle \overline{B}_{2n-1}} & & \Big\downarrow{\scriptstyle (2n)d} \\
HH_{2n}(B) & \xrightarrow{\pi_{2n}} & \Omega^{2n}_{B/k}
\end{array}
$$

5.5 Definition By composing with the three maps $\pi_{2n-1}, \overline{B}_{2n-1}, d$ we can define three new triangulated additive Chern theories (assuming the denominators are invertible):

$$[\mathrm{ch}]_n^{\text{HH, even}} := \frac{1}{n}\overline{B}_{2n-1} \circ [\mathrm{ch}]_n^{\text{HH, odd}} \colon \operatorname{Obj}([MF(A,f)]) \to HH_{2n}(B),$$
$$(\varphi,\psi) \mapsto \frac{1}{n}B(\mathrm{tr}_{2n-1}((\varphi\otimes\psi)^{\otimes 2n})) = \frac{1}{n}\,\mathrm{tr}_{2n-1}(B((\varphi\otimes\psi)^{\otimes 2n}))$$
$$= \mathrm{tr}_{2n-1}(1\otimes(\varphi\otimes\psi)^{\otimes 2n}) - \mathrm{tr}_{2n-1}(1\otimes(\psi\otimes\varphi)^{\otimes 2n})$$

$$[\mathrm{ch}]_n^{\text{dR, odd}} := \frac{1}{(2n-1)!}\pi_{2n-1} \circ [\mathrm{ch}]_n^{\text{HH, odd}} \colon \operatorname{Obj}([MF(A,f)]) \to \Omega^{2n-1}_{B/k},$$
$$(\varphi,\psi) \mapsto \frac{1}{(2n-1)!}\pi_{2n-1}(\mathrm{tr}_{2n-1}((\varphi\otimes\psi)^{\otimes 2n})) = \frac{1}{(2n-1)!}\,\mathrm{Tr}(\varphi d\psi(d\varphi d\psi)^{n-1})$$

$$[\mathrm{ch}]_n^{\text{dR, even}} := \frac{1}{n}d \circ [\mathrm{ch}]_n^{\text{dR, odd}} = \frac{1}{(2n)!}\pi_{2n} \circ [\mathrm{ch}]_n^{\text{HH, even}} \colon \operatorname{Obj}([MF(A,f)]) \to d(\Omega^{2n-1}_{B/k}),$$
$$(\varphi,\psi) \mapsto \frac{2}{(2n)!}d(\mathrm{Tr}(\varphi d\psi(d\varphi d\psi)^{n-1})) = \frac{2}{(2n)!}\,\mathrm{Tr}((d\varphi d\psi)^n)$$

The coefficients are (only) important for the behaviour under tensor products (this is the content of Section 5.5).

Of course each of these three new Chern theories carries possibly less information than $[\mathrm{ch}]_n^{\text{HH, odd}}$, therefore one might ask why we introduce them. We introduce $[\mathrm{ch}]_n^{\text{dR, odd}}$, because it is easy computable and in most cases one can check if this invariant is zero or not, which we can not do for $[\mathrm{ch}]_n^{\text{HH, odd}}$ in general. We introduce $[\mathrm{ch}]_n^{\text{HH, even}}$ and $[\mathrm{ch}]_n^{\text{dR, even}}$, because they behave better with tensor products and will appear in the description of the behaviour of the odd Chern characters under tensor products.

We will give examples in the next section, because the Chern theories for $MF(A,f)$ are very similar and hence we can give examples for both at once.

5.2 Chern theories for $MF(A, f)$

In this section we will construct two Chern theories for the category $MF(A,f)$.

In this section we denote with $\mathrm{GL}_r(A)$ the multiplicative group of invertible $r\times r$ matrices with entries in A.

We consider the value

$$\mathrm{Tr}_n(\varphi,\psi) := \mathrm{Tr}(\varphi d\psi(d\varphi d\psi)^{n-1}) \in \Omega^{2n-1}/df \wedge \Omega^{2n-2} + fd(\Omega^{2n-2}).$$

We want to show that $\mathrm{Tr}_n(\varphi,\psi)$ depends only on the isomorphism class (in $MF(A,f)$) of (φ,ψ). To do this we need to assume, that $H^{2n-1}_{\mathrm{dR}}(A/k) = 0$.

Let us remind that the isomorphism class of (φ, ψ) consists of all matrix factorizations of the form $(H\varphi G^{-1}, G\psi H^{-1})$, where $H, G \in \mathrm{GL}_r(A)$. Therefore, we need to show, that for all $H, G \in \mathrm{GL}_r(A)$ and for all matrix factorizations (φ, ψ) the following equation holds:

$$\mathrm{Tr}_n(\varphi, \psi) = \mathrm{Tr}_n(H\varphi G^{-1}, G\psi H^{-1}) \in \Omega^{2n-1}/(df \wedge \Omega^{2n-2} + fd(\Omega^{2n-2})). \tag{5.1}$$

Lemma 5.8 will show that it is enough to show, that for all $G \in \mathrm{GL}_r(A)$ and for all matrix factorizations (φ, ψ) the following equation holds:

$$\mathrm{Tr}_n(\varphi, \psi) = \mathrm{Tr}_n(\varphi G^{-1}, G\psi) \in \Omega^{2n-1}/(df \wedge \Omega^{2n-2} + fd(\Omega^{2n-2})). \tag{5.2}$$

Then we show that Equation (5.2) holds. The proof is divided into several steps.

5.6 Lemma 1. For $H \in \mathrm{GL}_r(A)$ we have

$$Hd(H^{-1}) = -d(H)H^{-1}, \quad H^{-1}d(H) = -d(H^{-1})H.$$

2. For any matrix factorization (φ, ψ) we have

$$\varphi d\psi = -d\varphi\psi \mod df, \quad \psi d\varphi = -d\psi\varphi \mod df.$$

Proof. Apply d to the equation $HH^{-1} = \mathrm{id} = H^{-1}H$ respectively $\varphi\psi = f\,\mathrm{id} = \psi\varphi$. □

5.7 Lemma For any matrix factorization (φ, ψ) we have

$$\mathrm{Tr}_n(\varphi, \psi) = -\mathrm{Tr}_n(\psi, \varphi) \in \Omega^{2n-1}/df \wedge \Omega^{2n-2}.$$

Proof. We compute modulo df:

$$\begin{aligned}\mathrm{Tr}_n(\varphi, \psi) &= \mathrm{Tr}(\varphi d\psi(d\varphi d\psi)^{n-1}) \overset{5.6}{=} -\mathrm{Tr}(d\varphi\psi(d\varphi d\psi)^{n-1}) \overset{3.14}{=} -\mathrm{Tr}(\psi(d\varphi d\psi)^{n-1}d\varphi) \\ &= -\mathrm{Tr}(\psi d\varphi(d\psi d\varphi)^{n-1}) = -\mathrm{Tr}_n(\psi, \varphi).\end{aligned}$$

□

5.8 Lemma Equation (5.1) follows from Equation (5.2).

Proof. We compute in $\Omega^{2n-1}/(df \wedge \Omega^{2n-2} + fd(\Omega^{2n-2}))$:

$$\begin{aligned}\mathrm{Tr}_n(\varphi, \psi) &\overset{(5.2)}{=} \mathrm{Tr}_n(\varphi G^{-1}, G\psi) \overset{5.7}{=} -\mathrm{Tr}_n(G\psi, \varphi G^{-1}) \overset{(5.2)}{=} -\mathrm{Tr}_n(G\psi H^{-1}, H\varphi G^{-1}) \\ &\overset{5.7}{=} \mathrm{Tr}_n(H\varphi G^{-1}, G\psi H^{-1}).\end{aligned}$$

□

5.9 Definition 1. We set $\varphi_0 = \varphi d(G^{-1})$, $\varphi_1 = d\varphi G^{-1}$, $\psi_0 = dG\psi$ and $\psi_1 = Gd\psi$. Then we get

$$d(\varphi G^{-1}) = \varphi_0 + \varphi_1, \quad d(G\psi) = \psi_0 + \psi_1.$$

2. For $\alpha = (\alpha_1, \dots, \alpha_{2n-1}) \in \{0,1\}^{2n-1}$ we define

$$(\Phi, \Psi)_\alpha := \mathrm{Tr}(\varphi G^{-1} \psi_{\alpha_1} \varphi_{\alpha_2} \psi_{\alpha_3} \dots \varphi_{\alpha_{2n-2}} \psi_{\alpha_{2n-1}}).$$

   Then we get

$$\mathrm{Tr}_n(\varphi G^{-1}, G\psi) = \mathrm{Tr}(\varphi G^{-1} d(G\psi)(d(\varphi G^{-1})d(G\psi))^{n-1}) = \sum_{\alpha \in \{0,1\}^{2n-1}} (\Phi, \Psi)_\alpha.$$

3. For $\alpha = (\alpha_1, \dots, \alpha_{2n-1}) \in \{0,1\}^{2n-1}$ and $k \in \mathbb{N}_0$ we define

$$\alpha^{(k)} := (\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_{2n-1}, \alpha_1, \dots \alpha_k),$$

   i.e. the entries of α get shifted to the left k times.

5.10 Lemma We have $(\Phi, \Psi)_\alpha = (\Phi, \Psi)_{\alpha^{(k)}}$ modulo df for all k.

Proof. It is enough to proof the case $k = 1$. Let us assume $\alpha = (1, 0, \alpha_3, \dots, \alpha_{2n-1})$, the other cases work similar. Then

$$\begin{aligned}(\Phi, \Psi)_\alpha &= \mathrm{Tr}(\varphi G^{-1}\psi_1 \dots) = \mathrm{Tr}(\varphi G^{-1} G d\psi \dots) = \mathrm{Tr}(\varphi d\psi \dots) = -\mathrm{Tr}(d\varphi \psi \dots) \\ &= -\mathrm{Tr}(d\varphi G^{-1} G\psi \dots).\end{aligned}$$

Repeating this step (swapping the d of a block of two matrices to the block of two matrices to the left (using Lemma 5.6)) we will get

$$(\Phi, \Psi)_\alpha = (-1)^{2n-1} \mathrm{Tr}(d\varphi G^{-1} dG\psi \dots G\psi).$$

Note that one can apply Lemma 5.6 to blocks of the form $G\psi\varphi d(G^{-1})$, because $\psi\varphi = f\,\mathrm{id}$ commutes with every other matrix. Now we compute

$$\begin{aligned}(\Phi, \Psi)_\alpha &= -\mathrm{Tr}(d\varphi G^{-1} dG\psi \dots G\psi) = -\mathrm{Tr}(G^{-1} dG\psi \dots G\psi d\varphi) \\ &= \mathrm{Tr}(G^{-1} dG\psi \dots G d\psi \varphi) = \mathrm{Tr}(\varphi G^{-1} dG\psi \dots G d\psi) = (\Phi, \Psi)_{\alpha^{(1)}}\end{aligned}$$

□

5.11 Definition Let S be a ring.

1. For $1 \leq i, j \leq r$, $i \neq j$ and $s \in S$ we write $E_{ij}(s)$ for the $r \times r$ matrix with r ones on the diagonal and only one further non zero entry s in place i, j. We call these matrices *elementary matrices.*

2. We write $E_r(S)$ for the multiplicative subgroup of $\mathrm{GL}_r(S)$ generated by

$$\{E_{ij}(s) \mid 1 \leq i, j \leq r,\ i \neq j,\ s \in S\}$$

   and call it *the elementary subgroup.*

5.12 Lemma Let $W \in M_r(\Omega^*_A/k)$ be arbitrary:

1. We have $d((E_{ij}(a))^{\varepsilon_1})Wd((E_{ij}(a))^{\varepsilon_2}) = 0$ for any choice of $(\varepsilon_1, \varepsilon_2) \in \{1, -1\}^2$.

2. For $G = E_{ij}(a)$ we have $(\Phi, \Psi)_\alpha = 0$ if $\alpha \in \{0,1\}^{2n-1}$ contains at least two zeros.

Proof. 1. Because $(E_{ij}(a))^{-1} = E_{ij}(-a)$, the only non zero entry of $d(E_{ij}(a))^{\varepsilon_i}$ is da or $-da$. Any product of two elements from the set $\{da, -da\}$ is zero. This implies $d((E_{ij}(a))^{\varepsilon_1})Wd((E_{ij}(a))^{\varepsilon_2}) = 0$.

2. If α contains at least two zeros, $(\Phi, \Psi)_\alpha$ is the trace of a product of matrices containing $d((E_{ij}(s))^{\varepsilon})$ at least two times. Now use the first part.

□

5.13 Lemma Equation (5.2) holds, if $G = E_{ij}(a)$.

Proof.

$$\begin{aligned}
\mathrm{Tr}_n(\varphi G^{-1}, G\psi) &= \sum_{\alpha \in \{0,1\}^{2n-1}} (\Phi, \Psi)_\alpha \overset{5.12}{=} \sum_{\substack{\alpha \in \{0,1\}^{2n-1}, \\ \alpha \text{ contains at most one } 0}} (\Phi, \Psi)_\alpha \\
&\overset{5.10}{=} (\Phi, \Psi)_{(1,\dots,1)} + (2n-1)(\Phi, \Psi)_{(0,1,\dots,1)} \\
&= \mathrm{Tr}_n(\varphi, \psi) + (2n-1)(\Phi, \Psi)_{(0,1,\dots,1)}.
\end{aligned}$$

We want to show now $(\Phi, \Psi)_{(0,1,\dots,1)} \in df \wedge \Omega^{2n-2} + fd(\Omega^{2n-2})$, which will finish the proof. For $n = 1$ we have

$$(\Phi, \Psi)_{(0,1,\dots,1)} = (\Phi, \Psi)_{(0)} = \mathrm{Tr}(\varphi G^{-1} dG \psi) = f\,\mathrm{Tr}(G^{-1} dG) = 0.$$

The last equation is, because $G^{-1}dG$ has only one non zero entry da, which is not on the main diagonal.
Now let us look at the case $n \geq 2$, there we have

$$\begin{aligned}
&fd(\mathrm{Tr}(\varphi d(G^{-1})Gd\psi(d\varphi d\psi)^{n-2})) \\
&= f\,\mathrm{Tr}(d\varphi d(G^{-1})Gd\psi(d\varphi d\psi)^{n-2}) - f\,\mathrm{Tr}(\varphi d(G^{-1})dGd\psi(d\varphi d\psi)^{n-2}) \\
&\overset{5.12}{=} \mathrm{Tr}(fd\varphi d(G^{-1})Gd\psi(d\varphi d\psi)^{n-2}) - 0 = \mathrm{Tr}(\varphi\psi d\varphi d(G^{-1})Gd\psi(d\varphi d\psi)^{n-2}) \\
&\overset{5.6}{=} -\mathrm{Tr}(\varphi d\psi\varphi d(G^{-1})Gd\psi(d\varphi d\psi)^{n-2}) = -(\Phi, \Psi)_{(1,0,1,\dots,1)} \overset{5.10}{=} -(\Phi, \Psi)_{(0,1,\dots,1)}.
\end{aligned}$$

This shows $(\Phi, \Psi)_{(0,1,\dots,1)} \in df \wedge \Omega^{2n-2} + fd(\Omega^{2n-2})$. □

5.14 Lemma Equation (5.2) holds, if $G \in E_r(A)$.

Proof. $E_r(A)$ is generated by $M := \{E_{ij}(s) \mid 1 \leq i, j \leq r,\ i \neq j,\ a \in A\}$. Note that inverses of elements of M are again in M. Clearly if Equation (5.2) holds for $G_1, G_2 \in \mathrm{GL}_r$ (for all matrix factorizations), then it also holds for G_1G_2. Now use Lemma 5.13. □

5.15 Lemma If $G \in \mathrm{GL}_r(A)$, then $\begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \in E_{2r}(A)$.

Proof. Multiplying a matrix G from the right with $E_{ij}(a)$ (i.e. $GE_{ij}(a)$) translates to adding a times the i-th column of G to the j-th column of G. Similar for left multiplication and rows. Therefore, we see that any upper triangular and any lower triangular matrix with ones on the diagonal is in $E_{2r}(A)$. Now compute

$$\begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_r & 0 \\ G^{-1} & \mathrm{id}_r \end{pmatrix} \begin{pmatrix} \mathrm{id}_r & \mathrm{id}_r - G \\ 0 & \mathrm{id}_r \end{pmatrix} \begin{pmatrix} \mathrm{id}_r & 0 \\ -\mathrm{id}_r & \mathrm{id}_r \end{pmatrix} \begin{pmatrix} \mathrm{id}_r & \mathrm{id}_r - G^{-1} \\ 0 & \mathrm{id}_r \end{pmatrix} \in E_{2r}(A).$$

□

5.16 Lemma Let $M \in M_r(\Omega^k)$. Then $\mathrm{Tr}(M^n) = 0$, if k is odd and n is even.

Proof. If the characteristic of k is not 2, then this follows from Lemma 3.14 (graded cyclic permutation in the trace). We proof now the characteristic 2 case (the proof works in any characteristic). If $M = (m_{ij})_{1 \le i,j \le r}$, then

$$\mathrm{Tr}(M^n) = \sum_{1 \le i_1, \dots, i_n \le r} m_{i_1 i_2} m_{i_2 i_3} \dots m_{i_n i_1}.$$

For $\beta = (i_1, \dots, i_n) \in \{1, \dots r\}^n$ we define $m_\beta := m_{i_1 i_2} m_{i_2 i_3} \dots m_{i_n i_1}$. Since the m_{ij} anti-commute we have $m_\beta = -m_{\beta^{(1)}}$, where $\beta^{(1)} := (i_2, \dots, i_n, i_1)$. Now either $\beta, \beta^{(1)}, \dots, \beta^{(n-1)}$ are pairwise distinct, then m_β appears in the sum $\frac{n}{2}$ times with a plus and with a minus sign. Or $\beta = \beta^{(k)}$ for some $k < n$, then β is periodic, therefore some m_{ij} appears two times in m_β, hence $m_\beta = 0$.

□

5.17 Lemma If $(\varphi, \psi) = (\mathrm{id}_r, f\,\mathrm{id}_r)$ and $H^{2n-1}_{\mathrm{dR}}(A/k) = 0$, then for all $G \in \mathrm{GL}_r(A)$ we have

$$\mathrm{Tr}_n(\varphi G^{-1}, G\psi) = 0 \in \Omega^{2n-1}/df \wedge \Omega^{2n-2} + fd(\Omega^{2n-2}).$$

In particular Equation (5.2) holds for this trivial matrix factorization.

Proof. We have $d\varphi = 0$ and $d\psi = 0 \mod df$. Therefore, $(\Phi, \Psi)_\alpha = 0 \mod df$ if α contains at least one 1. Thus we get

$$\mathrm{Tr}_n(\varphi G^{-1}, G\psi) = \sum_{\alpha \in \{0,1\}^{2n-1}} (\Phi, \Psi)_\alpha \overset{df}{=} (\Phi, \Psi)_{(0,\dots,0)} = f^n\, \mathrm{Tr}(G^{-1} dG (dG^{-1} dG)^{n-1}).$$

Note that $dG^{-1}G = -G^{-1}dG$, which implies $dG^{-1} = -G^{-1}dGG^{-1}$. Therefore

$$\begin{aligned} d(\mathrm{Tr}(G^{-1}dG(dG^{-1}dG)^{n-1})) &\overset{3.13}{=} \mathrm{Tr}((dG^{-1}dG)^n) = \mathrm{Tr}((-G^{-1}dGG^{-1}dG)^n) \\ &= (-1)^n\, \mathrm{Tr}((G^{-1}dG)^{2n}) \overset{5.16}{=} 0. \end{aligned}$$

With the assumption $H^{2n-1}_{\mathrm{dR}}(A/k) = 0$ this implies $\mathrm{Tr}(G^{-1}dG(dG^{-1}dG)^{n-1}) = d\eta$ for some $\eta \in \Omega^{2n-2}$. So we get

$$f^n\, \mathrm{Tr}(G^{-1}dG(dG^{-1}dG)^{n-1}) = fd(f^{n-1}\eta) - (n-1)df f^{n-1}\eta \in df \wedge \Omega^{2n-2} + fd(\Omega^{2n-2}).$$

□

5.18 Lemma Equation 5.2 holds, if $H^{2n-1}_{\mathrm{dR}}(A/k) = 0$.

Proof. Let (φ, ψ) be an arbitrary matrix factorization and $G \in \mathrm{GL}_r(A)$. Define a new matrix factorization and a new invertible matrix

$$(\Phi, \Psi) := \left(\begin{pmatrix} \varphi & 0 \\ 0 & \mathrm{id}_r \end{pmatrix}, \begin{pmatrix} \psi & 0 \\ 0 & f\,\mathrm{id}_r \end{pmatrix} \right), \quad H = \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix}$$

Note that $H \in E_{2r}(A)$ by Lemma 5.15. Now we compute

$$\begin{aligned} \mathrm{Tr}_n(\varphi, \psi) &\overset{5.17}{=} \mathrm{Tr}_n(\varphi, \psi) + \mathrm{Tr}_n(\mathrm{id}_r, f\,\mathrm{id}_r) = \mathrm{Tr}_n(\Phi, \Psi) \overset{5.14}{=} \mathrm{Tr}_n(\Phi H^{-1}, H\Psi) \\ &= \mathrm{Tr}_n(\varphi G^{-1}, G\psi) + \mathrm{Tr}_n(\mathrm{id}_r\, G, G^{-1} f\,\mathrm{id}_r) \overset{5.17}{=} \mathrm{Tr}_n(\varphi G^{-1}, G\psi). \end{aligned}$$

□

Lemma 5.18 and 5.8 imply the following proposition:

5.19 Proposition Equation 5.1 holds if $H^{2n-1}_{\mathrm{dR}}(A/k) = 0$. In particular

$$\mathrm{Tr}_n(\varphi, \psi) := \mathrm{Tr}(\varphi d\psi (d\varphi d\psi)^{n-1}) \in \Omega^{2n-1}/df \wedge \Omega^{2n-2} + f d(\Omega^{2n-2})$$

depends only on the isomorphism class of (φ, ψ) if $H^{2n-1}_{\mathrm{dR}}(A/k) = 0$.

Note that the assumption $H^{2n-1}_{\mathrm{dR}}(A/k) = 0$ is not needed for $n > 1$ if A is local or euclidean. This is because the invertible matrices over these rings have nice generating sets. For these rings the matrices $E_{ij}(a)$ and $D_i(u)$, where $D_i(u)$ is the diagonal matrix with ones on the diagonal except a unit u at the i-th place, generate $\mathrm{GL}_r(A)$ (we prove this for local rings below, for euclidean rings the proof is similar). Now one can prove Lemma 5.12 and 5.13 with $E_{ij}(a)$ replaced with $D_i(u)$. Both proofs are very similar to the proofs for $E_{ij}(a)$, except the case $n = 1$ in the proof of Lemma 5.13. There one will need $H^1_{\mathrm{dR}}(A/k) = 0$ to show that $f\,\mathrm{Tr}(D_i(u)^{-1} d(D_i(u))) \in df \wedge \Omega^0 + f d(\Omega^0)$. One can do this in the following way: $d(u^{-1}du) = d(u^{-1})du = -u^{-2}dudu = 0$, then by assumption $u^{-1}du = d\eta$ for some $\eta \in \Omega^0$. Then

$$f\,\mathrm{Tr}(D_i(u)^{-1} d(D_i(u))) = f u^{-1} du = f d\eta \in df \wedge \Omega^0 + f d(\Omega^0).$$

5.20 Lemma For any commutative local ring S the group $\mathrm{GL}_r(S)$ is generated by matrices $E_{ij}(s)$, $D_i(u)$, $1 \le i, j \le r$, $i \ne j$, $s \in S$, $u \in S^*$.

Proof. We use induction for r. The case $r = 1$ is clear. Let $G = (g_{ij}) \in \mathrm{GL}_r(S)$ be given. We want to show, that we can transform G to the identity matrix by multiplying G from the right and the left by matrices $E_{ij}(s)$ and $D_i(u)$. These multiplications translate to adding arbitrary multiples of columns respectively rows to other columns respectively rows and multiplying rows respectively columns with a unit.
Case 1: g_{11} is a unit, then we can transform G to a matrix with a one as first entry and zeros elsewhere in the first row and column. Then we are done by induction.

Case 2: g_{11} is not a unit, then another entry, say g_{1j_0}, of the first row is a unit, because the determinant is a unit and contained in the ideal generated by the entries of the first row. Then by adding the j_0-th column to the first one we get a matrix with first entry a unit. Then we are done by case 1. $\square$

5.21 Definition Let $H^{2n-1}_{\mathrm{dR}}(A/k) = 0$ for all $n \geq 0$. As consequence of Proposition 5.19 we get the following Chern theory for $MF(A, f)$

- The target is $\Omega^{\mathrm{odd}}/df \wedge \Omega^{\mathrm{even}} + fd(\Omega^{\mathrm{even}}) := \prod_{n=1}^{\infty} \Omega^{2n-1}/df \wedge \Omega^{2n-2} + fd(\Omega^{2n-2})$.
- $\mathrm{ch}_n^{\mathrm{dR,\ odd}}\colon \mathrm{Obj}(MF(A, f)) \to \Omega^{2n-1}/df \wedge \Omega^{2n-2} + fd(\Omega^{2n-2})$,
  $(\varphi, \psi) \mapsto \frac{1}{(2n-1)!} \mathrm{Tr}_{2n-1}((\varphi d\psi(d\varphi d\psi)^{n-1})$.

We will refer to it as *Chern theory with values in odd forms.*

5.22 Definition By applying the map $\frac{1}{n}d$ we get the following Chern theory for $MF(A, f)$

- The target is $d(\Omega^{\mathrm{odd}})/df \wedge d(\Omega^{\mathrm{even}}) = \prod_{n=1}^{\infty} d(\Omega^{2n-1})/df \wedge d(\Omega^{2n-2})$.
- $\mathrm{ch}_n^{\mathrm{dR,\ even}}\colon \mathrm{Obj}(MF(A, f)) \to d(\Omega^{2n-1})/df \wedge d(\Omega^{2n-2})$, $(\varphi, \psi) \mapsto \frac{2}{(2n)!} \mathrm{Tr}((d\varphi d\psi)^n)$.

We will refer to it as *Chern theory with values in even forms.*

Note, that $\mathrm{ch}_n^{\mathrm{dR,\ even}}$ can be defined without the assumption $H^{2n-1}_{\mathrm{dR}}(A/k) = 0$. This is because we did need this assumption for $\mathrm{ch}_n^{\mathrm{dR,\ odd}}$ to get rid of terms of the form $\mathrm{Tr}(G^{-1}dG(dG^{-1}dG)^{n-1})$, $G \in \mathrm{GL}_r(A)$ and those are mapped to zero by the map d.

As we will show in Section 5.4, that the Chern theories $\mathrm{ch}^{\mathrm{dR,\ odd}}$, $\mathrm{ch}^{\mathrm{dR,\ even}}$ are additive over cones, but cones do not give $MF(A, f)$ the structure of an exact or triangulated category, therefore we can not speak about additivity of these Chern theories. We will talk about compatibility with the tensor product in Section 5.5. As for the category $[MF(A, f)]$ we introduce $\mathrm{ch}^{\mathrm{dR,\ even}}$, because of the better behaviour under tensor products. Furthermore, $\mathrm{ch}^{\mathrm{dR,\ even}}$ is the closest to the Chern characters computed in [PV12]. The comparison of these two Chern characters will be done in Chapter 6.

Let us now compute $\mathrm{ch}^{\mathrm{dR,\ odd}}$ and $\mathrm{ch}^{\mathrm{dR,\ even}}$ in some examples. All statements made are equally true for $[\mathrm{ch}]^{\mathrm{dR,\ odd}}$ and $[\mathrm{ch}]^{\mathrm{dR,\ even}}$.

5.23 Example 1. Let $A = k[X_1, \ldots, X_{2n}]$ (or $A = k[\![X_1, \ldots, X_{2n}]\!]$) and $f = X_1X_2 + \ldots + X_{2n-1}X_{2n}$. Then the Koszul matrix factorization

$$K := \{(X_1, X_3, \ldots, X_{2n-1}), (X_2, X_4, \ldots, X_{2n})\} = (X_1, X_2) \otimes_{\mathrm{MF}} \ldots \otimes_{\mathrm{MF}} (X_{2n-1}, X_{2n})$$

is a matrix factorization of size 2^{n-1} of f over A. Clearly

$$\mathrm{ch}_1^{\mathrm{dR,\ odd}}(X_i, X_j) = X_i dX_j,\ \mathrm{ch}_m^{\mathrm{dR,\ odd}}(X_i, X_j) = 0,\ m > 1,$$
$$\mathrm{ch}_1^{\mathrm{dR,\ even}}(X_i, X_j) = dX_i dX_j,\ \mathrm{ch}_m^{\mathrm{dR,\ even}}(X_i, X_j) = 0,\ m > 1.$$

By applying the formulas for tensor products from Section 5.5 Corollary 5.29 and 5.27 several times we get

$$\begin{aligned}
\mathrm{ch}_n^{\text{dR, even}}(K) &= dX_1 \dots dX_{2n} \neq 0,\\
\mathrm{ch}_m^{\text{dR, even}}(K) &= 0, \text{ for } m < n,\\
\mathrm{ch}_n^{\text{dR, odd}}(K) &= \sum_{i=1}^{n} X_{2i-1} dX_1 \dots \widehat{dX_{2i-1}} \dots dX_{2n} \neq 0,\\
\mathrm{ch}_m^{\text{dR, odd}}(K) &= 0, \text{ for } m < n.
\end{aligned}$$

2. Let $A = k[X_1, \dots, X_4]$ (or $A = k[\![X_1, \dots, X_4]\!]$). Let us look at the matrices

$$\varphi := \begin{pmatrix} X_1 & X_2 & X_3 \\ X_2 & X_3 & X_2 \\ X_3 & X_4 & X_1 \end{pmatrix}, \quad \psi = \mathrm{adj}(\varphi).$$

These two give a matrix factorization of the (reducible) polynomial $f := \det(\varphi)$. One can check

$$\mathrm{ch}_1^{\text{dR, even}}(\varphi, \psi) = 0, \quad \mathrm{ch}_2^{\text{dR, even}}(\varphi, \psi) = 6(X_1X_3 - X_3^2) dX_1 \dots dX_4 \neq 0.$$

3. Let $A = k[X_1, \dots, X_9]$ (or $A = k[\![X_1, \dots, X_9]\!]$). Let us look at the matrices

$$\varphi := \begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{pmatrix}, \quad \psi = \mathrm{adj}(\varphi).$$

These two give a matrix factorization of the irreducible polynomial $f := \det(\varphi)$. One can check (using a computer and for example Singular)

$$\begin{aligned}
&\mathrm{ch}_2^{\text{dR, even}}(\varphi, \psi) \neq 0, \ \mathrm{ch}_3^{\text{dR, even}}(\varphi, \psi) \neq 0,\\
&\mathrm{ch}_1^{\text{dR, even}}(\varphi, \psi) = 0, \ \mathrm{ch}_m^{\text{dR, even}}(\varphi, \psi) = 0 \text{ for } m \geq 4.
\end{aligned}$$

4. Let $g \in \mathbb{C}[X, Y, Z]$ a homogeneous polynomial of degree 3, then after a linear isomorphism we get $g = \lambda(X^3 + Y^3 + Z^3) + \mu XYZ$. For the Hessematrix H_g of g we have

$$H_g = \begin{pmatrix} 6\lambda X & \mu Z & \mu Y \\ \mu Z & 6\lambda Y & \mu X \\ \mu Y & \mu X & 6\lambda Z \end{pmatrix}, \quad \det(H_g) = (216\lambda^3 + 2\mu^3) XYZ - 6\lambda\mu^2(X^3 + Y^3 + Z^3),$$

so the determinant is a different homogeneous polynomial of degree 3 in the same normal form. For general $f = \lambda'(X^3+Y^3+Z^3)+\mu' XYZ$ we find 3 polynomials g with $\det(H_g) = f$. Then $(H_g, \mathrm{adj}(H_g))$ gives a matrix factorization of f. A straightforward computation shows, that $\mathrm{ch}_n^{\text{dR, odd}}$, $\mathrm{ch}_n^{\text{dR, even}}$ will be zero for all n and all of these matrix factorizations.

Unfortunately examples, in which not all Chern characters are zero, go beyond something one can or wants to compute by hand very quickly.

5.3 Functoriality

In this section we want to show, that all our six Chern theories for $MF(A,f)$ respectively $[MF(A,f)]$ have a certain functoriality.

In this section A' is a further associative, unital, commutative k-algebra, $\lambda\colon A \to A'$ is a map of k-algebras, $f' = \lambda(f)$ a non zero divisor, $B' = A'/(f')$ a proper quotient and $(\Omega')^n$ will mean $\Omega^n_{A'/k}$.

The map λ induces a functor $MF(A,f) \to MF(A',f')$ by applying λ to matrices entrywise. It also induces a functor from $[MF(A,f)] \to [MF(A',f')]$, because the images of homotopic morphisms are homotopic. Furthermore, λ induces a map of k-algebras $B \to B'$, which induces a map $HH_n(B) \to HH_n(B')$ for all n, as we have seen in Definition 1.12. Let us call all the induced maps λ_*, then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Obj}([MF(A,f)]) \arrow[r, "{[\mathrm{ch}]_n^{\text{HH, odd}}}"] \arrow[d, "\lambda_*"] & HH_{2n-1}(B) \arrow[d, "\lambda_*"] \\
\mathrm{Obj}([MF(A',f')]) \arrow[r, "{[\mathrm{ch}]_n^{\text{HH, odd}}}"] & HH_{2n-1}(B'),
\end{tikzcd}$$

because both compositions apply λ to every occurring element of A. A similar result holds for $[\mathrm{ch}]_n^{\text{HH, even}}$. For the Chern theories in forms we consider the well defined map

$$\lambda_*\colon \Omega^n \to (\Omega')^n, \quad a da_1 \dots da_n \mapsto \lambda(a) d\lambda(a_1) \dots d\lambda(a_n).$$

Our Chern theories had values in quotients of Ω^n and all of the k-submodules, which we divided out are mapped by λ_* to the corresponding submodule with primes. Therefore, we get similar commutative diagrams for those four Chern theories. Let us write down the diagram for $\mathrm{ch}_n^{\text{dR, odd}}$:

$$\begin{tikzcd}
\mathrm{Obj}(MF(A,f)) \arrow[r, "{\mathrm{ch}_n^{\text{dR, odd}}}"] \arrow[d, "\lambda_*"] & \Omega^{2n-1}/df \wedge \Omega^{2n-2} + fd(\Omega^{2n-2}) \arrow[d, "\lambda_*"] \\
\mathrm{Obj}(MF(A',f')) \arrow[r, "{\mathrm{ch}_n^{\text{dR, odd}}}"] & (\Omega')^{2n-1}/df' \wedge (\Omega')^{2n-2} + f'd((\Omega')^{2n-2}).
\end{tikzcd}$$

5.4 Additivity over cones

In this section we want to show that all of our Chern theories are additive over cones. We will denote with $(\varphi,\psi) \in M_r(A)^2$, $(\varphi',\psi') \in M_{r'}(A)^2$ two matrix factorizations of f and $(\alpha,\beta)\colon (\varphi,\psi) \to (\varphi',\psi')$ will be a morphism of matrix factorizations.

We will show the additivity over cones for the Chern theory $[\mathrm{ch}]_n^{\text{HH, odd}}$. From that it also follows for the Chern theories $[\mathrm{ch}]_n^{\text{HH, even}}$, $[\mathrm{ch}]_n^{\text{dR, odd}}$ and $[\mathrm{ch}]_n^{\text{dR, even}}$, as these are the compositions of $[\mathrm{ch}]_n^{\text{HH, odd}}$ with k-linear maps.

5.24 Lemma We have the following equality of elements of $HH_{2n-1}(B)$

$$[\mathrm{ch}]_n^{\mathrm{HH,\ odd}}(\varphi,\psi) = -[\mathrm{ch}]_n^{\mathrm{HH,\ odd}}(\psi,\varphi).$$

Proof. We have the following equality in $M_r(B)^{\otimes 2n}$

$$b(1,(\varphi,\psi)^n) = \sum_{i=0}^{2n}(-1)^i d_i(1,(\varphi,\psi)^n) = (\varphi,\psi)^n - 0 + \ldots - 0 + (\psi,\varphi)^n$$

and therefore we get the following equality in $HH_{2n-1}(B)$

$$[\mathrm{ch}]_n^{\mathrm{HH,\ odd}}(\varphi,\psi) = \mathrm{tr}((\varphi,\psi)^n) = -\,\mathrm{tr}((\psi,\varphi)^n) = -[\mathrm{ch}]_n^{\mathrm{HH,\ odd}}(\psi,\varphi).$$

□

5.25 Proposition We have the following equality of elements of $HH_{2n-1}(B)$

$$[\mathrm{ch}]_n^{\mathrm{HH,\ odd}}(\varphi',\psi') = [\mathrm{ch}]_n^{\mathrm{HH,\ odd}}(\varphi,\psi) + [\mathrm{ch}]_n^{\mathrm{HH,\ odd}}(C_{(\alpha,\beta)}).$$

Proof. Let us define $(\varphi'',\psi'') := C_{(\alpha,\beta)}$. By the definition of the cone we have

$$\varphi'' = \begin{pmatrix}\varphi' & \alpha\\ 0 & -\psi\end{pmatrix} = \begin{pmatrix}\varphi' & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & \alpha\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & -\psi\end{pmatrix},$$
$$\psi'' = \begin{pmatrix}\psi' & \beta\\ 0 & -\varphi\end{pmatrix} = \begin{pmatrix}\psi' & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & \beta\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & -\varphi\end{pmatrix}.$$

Hence, $(\varphi''\otimes\psi'')^{\otimes n}$ is the sum of tensor products of the form $F_1\otimes\ldots\otimes F_{2n}$, where in the odd places we have either of the three matrices $\left(\begin{smallmatrix}\varphi' & 0\\ 0 & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0 & \alpha\\ 0 & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0 & 0\\ 0 & -\psi\end{smallmatrix}\right)$ and similar for the even places. Note that tr_{2n-1} vanishes on all summands except $(\left(\begin{smallmatrix}\varphi' & 0\\ 0 & 0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}\psi' & 0\\ 0 & 0\end{smallmatrix}\right))^{\otimes n}$ or $(\left(\begin{smallmatrix}0 & 0\\ 0 & -\psi\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}0 & 0\\ 0 & -\varphi\end{smallmatrix}\right))^{\otimes n}$. This is because F_i needs to have a non zero column with the same index as a non zero row of F_{i+1} for all $i\in\mathbb{Z}/2n\mathbb{Z}$ or $\mathrm{tr}(F_1\otimes\ldots\otimes F_{2n}) = 0$. Therefore, we get

$$\begin{aligned}
&[\mathrm{ch}]_n^{\mathrm{HH,\ odd}}(\varphi'',\psi'') = \mathrm{tr}_{2n-1}\left((\varphi''\otimes\psi'')^{\otimes n}\right)\\
&= \mathrm{tr}_{2n-1}((\left(\begin{smallmatrix}\varphi' & 0\\ 0 & 0\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}\psi' & 0\\ 0 & 0\end{smallmatrix}\right))^{\otimes n}) + \mathrm{tr}_{2n-1}((\left(\begin{smallmatrix}0 & 0\\ 0 & -\psi\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}0 & 0\\ 0 & -\varphi\end{smallmatrix}\right))^{\otimes n})\\
&= \mathrm{tr}_{2n-1}\left((\varphi'\otimes\psi')^{\otimes n}\right) + \mathrm{tr}_{2n-1}\left(((-\psi)\otimes(-\varphi))^{\otimes n}\right)\\
&= \mathrm{tr}_{2n-1}\left((\varphi'\otimes\psi')^{\otimes n}\right) + \mathrm{tr}_{2n-1}\left((\psi\otimes\varphi)^{\otimes n}\right)\\
&= [\mathrm{ch}]_n^{\mathrm{HH,\ odd}}(\varphi',\psi') + [\mathrm{ch}]_n^{\mathrm{HH,\ odd}}(\psi,\varphi)\\
&= [\mathrm{ch}]_n^{\mathrm{HH,\ odd}}(\varphi',\psi') - [\mathrm{ch}]_n^{\mathrm{HH,\ odd}}(\varphi,\psi).
\end{aligned}$$

□

As the category $[MF(A,f)]$ is a triangulated category, with distinguished triangles being triangles isomorphic to ones coming from cones, Proposition 5.25 precisely says that our four Chern theories for the category $[MF(A,f)]$ are triangulated additive.

We have already seen the statement from Lemma 5.24 for $\mathrm{ch}_n^{\mathrm{dR,\ odd}}$ in Lemma 5.7, which already holds in $\Omega^{2n-1}_{A/k}/df \wedge \Omega^{2n-2}_{A/k}$ instead of $\Omega^{2n-1}_{A/k}/df \wedge \Omega^{2n-2}_{A/k} + fd(\Omega^{2n-1}_{A/k})$. Now one can prove the additivity over cones for $\mathrm{ch}_n^{\mathrm{dR,\ odd}}$ as in Lemma 5.25 by replacing the generalized trace map with the regular trace map and the tensor product with the matrix product. From that the additivity over cones for $\mathrm{ch}_n^{\mathrm{dR,\ even}}$ follows or can be proven similarly. The proof will already work in $\Omega^{2n}_{A/k}$ instead of $\Omega^{2n}_{A/k}/df \wedge d(\Omega^{2n-2}_{A/k})$.

5.5 Compatibility with tensor products

In this section we will exhibit, how our Chern theories behave under tensor products of matrix factorizations. We will denote with $(\varphi, \psi) \in M_r(A)^2$, $(\varphi', \psi') \in M_{r'}(A)^2$ two matrix factorizations of f and $(\Phi, \Psi) := (\varphi, \psi) \otimes_{\mathrm{MF}} (\varphi', \psi') \in M_{2rr'}(A)^2$, which is a matrix factorization of $2f$. Furthermore, the symbol $\otimes$ between 2 matrices will always mean the Kronecker product and $\mathrm{id} = \mathrm{id}_r$ or $\mathrm{id}_{r'}$ depending on the context.

Let us start with our four Chern theories with values in forms, i.e.

$$\begin{aligned}
[\mathrm{ch}]_n^{\mathrm{dR,\ odd}}(\varphi, \psi) &= \frac{1}{(2n-1)!} \operatorname{Tr}(\varphi d\psi (d\varphi d\psi)^{n-1}) \in \Omega^{2n-1}_{B/k} = \Omega^{2n-1}/(df \wedge \Omega^{2n-2} + f\Omega^{2n-1}),\\
[\mathrm{ch}]_n^{\mathrm{dR,\ even}}(\varphi, \psi) &= \frac{2}{(2n)!} \operatorname{Tr}((d\varphi d\psi)^{n}) \in d(\Omega^{2n-1}_{B/k}) = d(\Omega^{2n-1})/(df \wedge d\Omega^{2n-2} + d(f\Omega^{2n-1})),\\
\mathrm{ch}_n^{\mathrm{dR,\ odd}}(\varphi, \psi) &= \frac{2}{(2n-1)!} \operatorname{Tr}(\varphi d\psi (d\varphi d\psi)^{n-1}) \in \Omega^{2n-1}/(df \wedge \Omega^{2n-2} + fd(\Omega^{2n-1})),\\
\mathrm{ch}_n^{\mathrm{dR,\ even}}(\varphi, \psi) &= \frac{2}{(2n)!} \operatorname{Tr}((d\varphi d\psi)^{n-1}) \in d(\Omega^{2n-1})/df \wedge d\Omega^{2n-2}.
\end{aligned}$$

Note that the target

$$\bigoplus_{n=1}^{\infty} d(\Omega^{2n-1})/df \wedge d(\Omega^{2n-2})$$

of the total Chern character $\mathrm{ch}^{\mathrm{dR,\ even}}$ has a graded product given by wedging of forms, while the target

$$\bigoplus_{n=1}^{\infty} \Omega^{2n-1}/df \wedge \Omega^{2n-2} + fd(\Omega^{2n-1})$$

of $\mathrm{ch}^{\mathrm{dR,\ odd}}$ is a module over the target of $\mathrm{ch}^{\mathrm{dR,\ even}}$ with respect to wedging of forms. The same is true for $[\mathrm{ch}]^{\mathrm{dR,\ odd}}$ and $[\mathrm{ch}]^{\mathrm{dR,\ even}}$. We will prove formulas for $\otimes_{\mathrm{MF}}$, which will hold in $\Omega^{2n-1}/df \wedge \Omega^{2n-2}$ (odd case) respectively Ω^{2n} (even case). In particular these formulas will hold for all four of the above Chern theories.

5.26 Lemma For $(\Phi, \Psi) = (\varphi, \psi) \otimes_{\mathrm{MF}} (\varphi', \psi')$ the matrix $(d\Phi d\Psi)^n$ is given by the following matrix, which we will denote with $\Lambda(n)$:

$$\begin{pmatrix} \sum_{i=0}^{n} \binom{2n}{2i} (d\varphi d\psi)^i \otimes (d\varphi' d\psi')^{n-i} & \sum_{i=0}^{n-1} -\binom{2n}{2i+1} d\varphi (d\psi d\varphi)^i \otimes d\varphi' (d\psi' d\varphi')^{n-1-i} \\ \sum_{i=0}^{n-1} \binom{2n}{2i+1} d\psi (d\varphi d\psi)^i \otimes d\psi' (d\varphi' d\psi')^{n-1-i} & \sum_{i=0}^{n} \binom{2n}{2i} (d\psi d\varphi)^i \otimes (d\psi' d\varphi')^{n-i} \end{pmatrix}$$

Proof. We use induction for n. For $n = 1$ we have

$$\begin{aligned}
d\Phi d\Psi &= d\left(\begin{pmatrix} \varphi \otimes \mathrm{id} & \mathrm{id} \otimes \varphi' \\ -\mathrm{id} \otimes \psi' & \psi \otimes \mathrm{id} \end{pmatrix}\right) d\left(\begin{pmatrix} \psi \otimes \mathrm{id} & -\mathrm{id} \otimes \varphi' \\ \mathrm{id} \otimes \psi' & \varphi \otimes \mathrm{id} \end{pmatrix}\right) \\
&= \begin{pmatrix} d\varphi \otimes \mathrm{id} & \mathrm{id} \otimes d\varphi' \\ -\mathrm{id} \otimes d\psi' & d\psi \otimes \mathrm{id} \end{pmatrix} \begin{pmatrix} d\psi \otimes \mathrm{id} & -\mathrm{id} \otimes d\varphi' \\ \mathrm{id} \otimes d\psi' & d\varphi \otimes \mathrm{id} \end{pmatrix} \\
&= \begin{pmatrix} d\varphi d\psi \otimes \mathrm{id} + \mathrm{id} \otimes d\varphi' d\psi' & -2 d\varphi \otimes d\varphi' \\ 2 d\psi \otimes d\psi' & d\psi d\varphi \otimes \mathrm{id} + \mathrm{id} \otimes d\psi' d\varphi' \end{pmatrix}.
\end{aligned}$$

For the induction step we have to verify, that $\Lambda(n+1) = \Lambda(n)\Lambda(1)$. We will only verify, that the upper left entry is the same. The other entries can be checked in a similar way. The coefficient in front of $(d\varphi d\psi)^i \otimes (d\varphi' d\psi')^{n+1-i}$ in $\Lambda(n+1)$ is $\binom{2(n+1)}{2i}$. There are exactly three ways to get $(d\varphi d\psi)^i \otimes (d\varphi' d\psi')^{n+1-i}$ as product in $\Lambda(n)\Lambda(1)$:

$$\begin{aligned}
&\left(\binom{2n}{2(i-1)} (d\varphi d\psi)^{i-1} \otimes (d\varphi' d\psi')^{n-(i-1)}\right) \cdot (d\varphi d\psi \otimes \mathrm{id}), \\
&\left(\binom{2n}{2i} (d\varphi d\psi)^{i} \otimes (d\varphi' d\psi')^{n-i}\right) \cdot (\mathrm{id} \otimes d\varphi' d\psi'), \\
&\left(-\binom{2n}{2(i-1)+1} d\varphi (d\psi d\varphi)^{i-1} \otimes d\varphi' (d\psi' d\varphi')^{n-1-i}\right) \cdot (2 d\psi \otimes d\psi').
\end{aligned}$$

Note that the minus sign in the last product will become a plus sign after multiplying the tensor products of matrices. So it remains to check

$$\binom{2(n+1)}{2i} = \binom{2n+1}{2i} + \binom{2n+1}{2i-1} = \binom{2n}{2i} + 2\binom{2n}{2i-1} + \binom{2n}{2(i-1)}.$$

$\square$

5.27 Corollary For $(\Phi, \Psi) = (\varphi, \psi) \otimes_{\mathrm{MF}} (\varphi', \psi')$ we have

$$\mathrm{Tr}((d\Phi d\Psi)^n) = \sum_{i=1}^{n-1} 2\binom{2n}{2i} \mathrm{Tr}((d\varphi d\psi)^i) \, \mathrm{Tr}((d\varphi' d\psi')^{n-i}).$$

For the n-th even Chern character this translates to

$$\mathrm{ch}_n^{\mathrm{dR,even}}(\Phi, \Psi) = \sum_{i=1}^{n-1} \mathrm{ch}_i^{\mathrm{dR,even}}(\varphi, \psi) \, \mathrm{ch}_{n-i}^{\mathrm{dR,even}}(\varphi', \psi')$$

and for the even total Chern character we get

$$\mathrm{ch}^{\mathrm{dR,even}}(\Phi, \Psi) = \mathrm{ch}^{\mathrm{dR,even}}(\varphi, \psi) \, \mathrm{ch}^{\mathrm{dR,even}}(\varphi', \psi').$$

Proof. By Lemma 5.26 and 3.19 we get

$$\begin{aligned}
&\operatorname{Tr}((d\Phi d\Psi)^n) \\
&= \operatorname{Tr}\left(\sum_{i=0}^{n}\binom{2n}{2i}(d\varphi d\psi)^i \otimes (d\varphi' d\psi')^{n-i}\right) + \operatorname{Tr}\left(\sum_{i=0}^{n}\binom{2n}{2i}(d\psi d\varphi)^i \otimes (d\psi' d\varphi')^{n-i}\right) \\
&= \sum_{i=0}^{n}\binom{2n}{2i}\operatorname{Tr}((d\varphi d\psi)^i)\operatorname{Tr}((d\varphi' d\psi')^{n-i}) + \sum_{i=0}^{n}\binom{2n}{2i}\operatorname{Tr}((d\psi d\varphi)^i)\operatorname{Tr}((d\psi' d\varphi')^{n-i}) \\
&= \sum_{i=1}^{n-1} 2\binom{2n}{2i}\operatorname{Tr}((d\varphi d\psi)^i)\operatorname{Tr}((d\varphi' d\psi')^{n-i}).
\end{aligned}$$

The last equality holds as $\operatorname{Tr}((d\varphi d\psi)^i) = -\operatorname{Tr}((d\psi d\varphi)^i)$ for all $i \geq 1$. Therefore:

$$\begin{aligned}
\operatorname{Tr}((d\varphi d\psi)^i)\operatorname{Tr}((d\varphi' d\psi')^{n-i}) &= \operatorname{Tr}((d\psi d\varphi)^i)\operatorname{Tr}((d\psi' d\varphi')^{n-i}) \text{ for } 0 < i < n, \\
\operatorname{Tr}((d\varphi d\psi)^i)\operatorname{Tr}((d\varphi' d\psi')^{n-i}) &= -\operatorname{Tr}((d\psi d\varphi)^i)\operatorname{Tr}((d\psi' d\varphi')^{n-i}) \text{ for } i = 0, n.
\end{aligned}$$

The statement for the Chern characters follows, since the factor $2\binom{2n}{2i}$ cancels with the factor $\frac{2}{(2i)!}$ in the definition of $\operatorname{ch}_i^{\mathrm{dR,even}}$. □

5.28 Corollary The top left $r \times r$ block respectively the bottom right of the matrix $\Phi d\Psi(d\Phi d\Psi)^{n-1}$ are given by:

$$\begin{gathered}
\sum_{i=1}^{n-1}\binom{2n-1}{2i-1}\left[\varphi d\psi(d\varphi d\psi)^{i-1} \otimes (d\varphi' d\psi')^{n-i} + (d\varphi d\psi)^{n-i} \otimes \varphi' d\psi'(d\varphi' d\psi')^{i-1}\right], \\
\sum_{i=1}^{n-1}\binom{2n-1}{2i-1}\left[\psi d\varphi(d\psi d\varphi)^{i-1} \otimes (d\psi' d\varphi')^{n-i} + (d\psi d\varphi)^{n-i} \otimes \psi' d\varphi'(d\psi' d\varphi')^{i-1}\right]
\end{gathered}$$

Proof. Multiply

$$\Phi d\Psi = \begin{pmatrix} \varphi d\psi \otimes \mathrm{id} + \mathrm{id}\,\varphi' d\psi' & d\varphi \otimes \varphi' - \varphi \otimes d\varphi' \\ \psi \otimes d\psi' - d\psi \otimes \psi' & \psi d\varphi \otimes \mathrm{id} + \mathrm{id} \otimes \psi' d\varphi' \end{pmatrix}$$

with the explicit form of $(d\Phi d\Psi)^{n-1}$ from Lemma 5.26. □

5.29 Corollary For the odd Chern characters we get the following formula of elements in $\Omega^{2n-1}/df \wedge \Omega^{2n-2}$:

$$\begin{aligned}
\operatorname{Tr}(\Phi d\Psi(d\Phi d\Psi)^{n-1}) = \sum_{i=1}^{n-1} 2\binom{2n-1}{2i-1}\big[&\operatorname{Tr}(\varphi d\psi(d\varphi d\psi)^{i-1})\operatorname{Tr}((d\varphi' d\psi')^{n-i}) \\
&+ \operatorname{Tr}((d\varphi d\psi)^{n-i})\operatorname{Tr}(\varphi' d\psi'(d\varphi' d\psi')^{i-1})\big].
\end{aligned}$$

For the n-th odd Chern character this translates to

$$\operatorname{ch}_n^{\mathrm{dR,odd}}(\Phi, \Psi) = \sum_{i=1}^{n-1}\operatorname{ch}_i^{\mathrm{dR,odd}}(\varphi, \psi)\operatorname{ch}_{n-i}^{\mathrm{dR,even}}(\varphi', \psi') + \operatorname{ch}_{n-i}^{\mathrm{dR,even}}(\varphi, \psi)\operatorname{ch}_i^{\mathrm{dR,odd}}(\varphi', \psi')$$

and for the odd total Chern character we get

$$\operatorname{ch}^{\mathrm{dR,odd}}(\Phi, \Psi) = \operatorname{ch}^{\mathrm{dR,odd}}(\varphi, \psi)\operatorname{ch}^{\mathrm{dR,even}}(\varphi', \psi') + \operatorname{ch}^{\mathrm{dR,even}}(\varphi, \psi)\operatorname{ch}^{\mathrm{dR,odd}}(\varphi', \psi').$$

Proof. The proof is analogues to the even case. Except we will need to compute modulo df to get (Lemma 5.7)

$$\operatorname{Tr}(\varphi d\psi(d\varphi d\psi)^{i-1}) = -\operatorname{Tr}(\psi d\varphi(d\psi d\varphi)^{i-1}).$$

□

Recall, that $[MF(A,f)]$ is a pseudo tensor category with tensor product $\otimes_{\mathrm{MF}}^{\frac{1}{2}}$ and not with $\otimes_{\mathrm{MF}}$. The formulas for $\otimes_{\mathrm{MF}}^{\frac{1}{2}}$ follow directly from the ones for $\otimes_{\mathrm{MF}}$ by multiplying the right hand side with $\frac{1}{2^n}$, i.e.

$$\begin{aligned}
\mathrm{ch}_n^{\mathrm{dR,odd}}((\varphi,\psi)\otimes_{\mathrm{MF}}^{\frac{1}{2}}(\varphi,\psi)) &= \frac{1}{2^n}\sum_{i=1}^{n-1}\big[\mathrm{ch}_i^{\mathrm{dR,odd}}(\varphi,\psi)\,\mathrm{ch}_{n-i}^{\mathrm{dR,even}}(\varphi',\psi') \\
&\qquad + \mathrm{ch}_{n-i}^{\mathrm{dR,even}}(\varphi,\psi)\,\mathrm{ch}_i^{\mathrm{dR,odd}}(\varphi',\psi')\big], \\
\mathrm{ch}_n^{\mathrm{dR,even}}((\varphi,\psi)\otimes_{\mathrm{MF}}^{\frac{1}{2}}(\varphi,\psi)) &= \frac{1}{2^n}\sum_{i=1}^{n-1}\mathrm{ch}_i^{\mathrm{dR,even}}(\varphi,\psi)\,\mathrm{ch}_{n-i}^{\mathrm{dR,even}}(\varphi',\psi')
\end{aligned}$$

and the same for $[\mathrm{ch}]_n^{\mathrm{dR,odd}}$, $[\mathrm{ch}]_n^{\mathrm{dR,even}}$, which means, that $[\mathrm{ch}]_n^{\mathrm{dR,even}}$ is multiplicative with weights $\frac{1}{2^n}$. As one can check it is not possible to get rid of the factor $\frac{1}{2^n}$ by introducing new factors to the definition of $[\mathrm{ch}]_n^{\mathrm{dR,even}}$. These new factors $l(n)$ would have to satisfy $l(1)^2 = \frac{1}{4}l(2)$, $l(1)l(2) = \frac{1}{8}l(3)$, $l(1)l(3) = l(2)l(2) = \frac{1}{16}l(4)$ and these equations imply $16l(1)^4 = 32l(1)^4$. $\mathrm{ch}_n^{\mathrm{dR,even}}$ satisfies the same equations as $[\mathrm{ch}]_n^{\mathrm{dR,even}}$, but we can not speak of multiplicativity in this case, because $MF(A,f)$ is not a pseudo tensor category with either of the two tensor products $\otimes_{\mathrm{MF}}$ or $\otimes_{\mathrm{MF}}^{\frac{1}{2}}$.

Now we want to show that we get similar formulas for our Chern theories with values in the Hochschild homology, i.e.

$$\begin{aligned}
&[\mathrm{ch}]_n^{\mathrm{HH,\ odd}}(\varphi,\psi) = (\varphi,\psi)^{\otimes_k n} \in HH_{2n-1}(B), \\
&[\mathrm{ch}]_n^{\mathrm{HH,\ even}}(\varphi,\psi) = (1,(\varphi,\psi)^{\otimes_k n}) - (1,(\psi,\varphi)^{\otimes_k n}) \in HH_{2n}(B).
\end{aligned}$$

Let us first describe the product on the target of the even total Chern character

$$HH_{\mathrm{even}}(B) = \bigoplus_{n=1}^{\infty} HH_{2n}(B).$$

For each n,m we have the map $\mathrm{sh}_{2n,2m}\colon HH_{2n}(B)\otimes_k HH_{2m}(B) \to HH_{2(m+n)}(B)$. These maps define a product on $HH_{\mathrm{even}}(B)$ in the obvious way. Note that this product is associative, this can be verified on graded pieces. For $a := (a_0,a_1,\dots a_l) \in HH_l(B)$, $b := (b_0,b_1,\dots b_m) \in HH_m(B)$ and $c := (c_0,c_1,\dots c_n) \in HH_n(B)$ we compute

$$\begin{aligned}
\mathrm{sh}_{l+m,n}(\mathrm{sh}_{l,m}(a,b),c) &= \sum_{\sigma\in S_{l,m,n}} \mathrm{sgn}(\sigma)\sigma(a_0b_0c_0,a_1,\dots,a_l,b_1,\dots,b_m,c_1,\dots c_n) \\
&= \mathrm{sh}_{l,m+n}(a,\mathrm{sh}_{m,n}(b,c)),
\end{aligned}$$

where $S_{l,m,n} \subset S_{l+m+n}$ is the set of all permutations, which do not change the order of the $a_1, \ldots, a_l$ relative to each other and the same with b_i and c_i. Similarly the target of the odd total Chern character $HH_{\text{odd}}(B) = \bigoplus_{n=1}^{\infty} HH_{2n-1}(B)$ has a module structure over $HH_{\text{even}}(B)$. We want to show now, that the behaviour of the Chern characters under tensor products can be described with this product. The proofs for these formulas are non commutative versions of the proofs for the Chern theories with values in forms. In particular the formulas for tensor products in the Hochschild homology imply the formulas for tensor products in the quotients of Kähler differentials, but we wanted to cover the easier case first.

To show the formulas for the Chern characters with values in the Hochschild homology we can assume $r = r'$, i.e. our matrix factorizations (φ, ψ) and (φ', ψ') have the same size. This is no loss of generality, since by filling up the smaller one with f respectively 1 on the diagonal of the first respectively second matrix, we get a matrix factorization that is homotopy equivalent.

5.30 Lemma Let $A_1, \ldots, A_m, B_1, \ldots, B_n, \mathrm{id} = \mathrm{id}_r \in M_r(A)$. Then:

1. $\mathrm{tr}(A_1 \otimes \mathrm{id}, \ldots, A_m \otimes \mathrm{id}, \mathrm{id} \otimes B_1, \ldots, \mathrm{id} \otimes B_n) = \mathrm{tr}(A_1, \ldots, A_m) \otimes_k \mathrm{tr}(B_1, \ldots, B_n)$ as elements of $A^{\otimes_k m+n}$.

2. $\mathrm{tr}(A_1 \otimes \mathrm{id}, \ldots, A_m \otimes \mathrm{id}) = r \cdot \mathrm{tr}(A_1, \ldots, A_m)$ and $\mathrm{tr}(\mathrm{id} \otimes B_1, \ldots, \mathrm{id} \otimes B_n) = r \cdot \mathrm{tr}(B_1, \ldots, B_n)$.

3. For all $\sigma \in S_{m+n}$ we have

   $$\mathrm{tr}(\sigma(A_1 \otimes \mathrm{id}, \ldots, A_m \otimes \mathrm{id}, \mathrm{id} \otimes B_1, \ldots, \mathrm{id} \otimes B_n)) = \sigma(\mathrm{tr}(A_1, \ldots, A_m) \otimes_k \mathrm{tr}(B_1, \ldots, B_n))$$

   as elements of $A^{\otimes_k m+n}$.

Proof. 1. We remind that for $M = (m_{ij}), N \in M_r(A)$ the Kronecker product is the following $r^2 \times r^2$ matrix:

$$M \otimes N = \begin{pmatrix} m_{11}N & m_{12}N & \cdots \\ m_{21}N & m_{22}N & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

We express every number $1 \leq l \leq r^2$ as $l = (q(l) - 1) \cdot r + r(l)$, $1 \leq q(l) \leq r$ and $0 \leq r(l) \leq r - 1$ (division with remainder). Furthermore, we write $a_{ij}^{(l)}$ respectively $\alpha_{ij}^{(l)}$ for the entries of the matrix A_l respectively $A_l \otimes \mathrm{id}$ and similar with b, β, B_l, $\mathrm{id} \otimes B_l$. Then we have

$$\alpha_{ij}^{(l)} = \begin{cases} a_{q(i)q(j)}^{(l)}, & \text{if } r(i) = r(j) \\ 0, & \text{else,} \end{cases} \qquad \beta_{ij}^{(l)} = \begin{cases} b_{r(i)r(j)}^{(l)}, & \text{if } q(i) = q(j) \\ 0, & \text{else.} \end{cases}$$

Now by definition we have

$$\begin{aligned}
&\operatorname{tr}(A_1\otimes \mathrm{id},\dots,A_m\otimes \mathrm{id},\mathrm{id}\otimes B_1,\dots,\mathrm{id}\otimes B_n)\\
&=\sum_{1\le i_1,\dots,i_{m+n}\le r^2}(\alpha^{(1)}_{i_1i_2},\dots,\alpha^{(m)}_{i_mi_{m+1}},\beta^{(1)}_{i_{m+1}i_{m+2}},\dots\beta^{(n)}_{i_{m+n}i_1})\\
&=\sum_{\substack{1\le q_1,\dots,q_{m+n}\le r\\ 1\le r_1,\dots,r_{m+n}\le r}}(\alpha^{(1)}_{(q_1-1)\cdot r+r_1,(q_2-1)\cdot r+r_2},\dots,\beta^{(n)}_{(q_{m+n}-1)\cdot r+r_{n+m},(q_1-1)\cdot r+r_1})
\end{aligned}$$

Here we have written q_l instead of $q(i_l)$ and r_l instead of $r(i_l)$. Now we know that $r_1=\ldots=r_{m+1}$ or one of the alphas is zero and likewise $q_{m+1}=\ldots q_{m+n}=q_1$ or one of the betas will be zero. Hence we can sum over a smaller index set and replace $r_1,\dots,r_m$ by r_{m+1} respectively $q_{m+1},\dots,q_{m+n}$ by q_1. Continuing the computation we get

$$\begin{aligned}
&=\sum_{\substack{1\le q_1,\dots,q_m\le r\\ 1\le r_{m+1},\dots,r_{m+n}\le r}}(\alpha^{(1)}_{(q_1-1)\cdot r+r_{m+1},(q_2-1)\cdot r+r_{m+1}},\dots,\beta^{(n)}_{(q_1-1)\cdot r+r_{n+m},(q_1-1)\cdot r+r_{m+1}})\\
&=\sum_{\substack{1\le q_1,\dots,q_m\le r\\ 1\le r_{m+1},\dots,r_{m+n}\le r}}(a^{(1)}_{q_1q_2},\dots,a^{(m)}_{q_mq_1},b^{(1)}_{r_{m+1}r_{m+2}},\dots,b^{(n)}_{r_{m+n}r_{m+1}})\\
&=\left[\sum_{1\le q_1,\dots,q_m\le r}(a^{(1)}_{q_1q_2},\dots,a^{(m)}_{q_mq_1})\right]\otimes_k\left[\sum_{1\le r_{n+1},\dots,r_{m+n}\le r}(b^{(1)}_{r_{m+1}r_{m+2}},\dots,b^{(n)}_{r_{m+n}r_{m+1}})\right]\\
&=\operatorname{tr}(A_1,\dots,A_m)\otimes_k\operatorname{tr}(B_1,\dots,B_n)
\end{aligned}$$

2. As in the first part we get

$$\begin{aligned}
&\operatorname{tr}(A_1\otimes \mathrm{id},\dots,A_m\otimes \mathrm{id})=\sum_{1\le i_1,\dots,i_m\le r^2}(\alpha^{(1)}_{i_1i_2},\dots,\alpha^{(m)}_{i_mi_1})\\
&=\sum_{\substack{1\le q_1,\dots,q_m\le r\\ 1\le r_1,\dots,r_m\le r}}(\alpha^{(1)}_{(q_1-1)\cdot r+r_1),(q_2-1)\cdot r+r_2},\dots,\alpha^{(m)}_{(q_m-1)\cdot r+r_m),(q_1-1)\cdot r+r_1})\\
&=\sum_{\substack{1\le q_1,\dots,q_m\le r\\ 1\le r_1\le r}}(\alpha^{(1)}_{(q_1-1)\cdot r+r_1),(q_2-1)\cdot r+r_1},\dots,\alpha^{(m)}_{(q_m-1)\cdot r+r_1),(q_1-1)\cdot r+r_1})\\
&=\sum_{\substack{1\le q_1,\dots,q_m\le r\\ 1\le r_1\le r}}(a^{(1)}_{q_1q_2},\dots,a^{(m)}_{q_mq_1})=r\cdot\operatorname{tr}(A_1,\dots,A_m)
\end{aligned}$$

3. The third statement is equivalent to, the first part of this lemma with more than one block of matrices with the identity in first respectively second place in the Kronecker product. Let us show that

$$\begin{aligned}
&\operatorname{tr}(A_1\otimes \mathrm{id},\mathrm{id}\otimes B_1,\dots,A_{m-1}\otimes \mathrm{id},\mathrm{id}\otimes B_{m-1},A_m\otimes \mathrm{id})\\
&=\sum_{\substack{1\le i_1,\dots,i_m\le r\\ 1\le j_1,\dots,j_{m-1}\le r}}(a^{(1)}_{i_1i_2},b^{(1)}_{j_1j_2},\dots,a^{(m-1)}_{i_{m-1}i_m},b^{(m-1)}_{j_{m-1}j_1},a^{(m)}_{i_mi_1}).
\end{aligned}$$

This assumes that each block of $-\otimes\mathrm{id}$ respectively $\mathrm{id}\otimes-$ consists of exactly one matrix, the general case can be proven in the same way (just replace $A_1\otimes\mathrm{id}$ with $A_{11}\otimes\mathrm{id},\dots,A_{1m_1}\otimes\mathrm{id}$ etc. and use more indices). Now calculate

$$\begin{aligned}
&\mathrm{tr}(A_1\otimes\mathrm{id},\mathrm{id}\otimes B_1,\dots,A_{m-1}\otimes\mathrm{id},\mathrm{id}\otimes B_{m-1},A_m\otimes\mathrm{id})\\
=&\sum_{1\le i_1,\dots,i_{2m-1}\le r^2}(\alpha^{(1)}_{i_1i_2},\beta^{(1)}_{i_2i_3},\dots,\alpha^{(m-1)}_{i_{2m-3}i_{2m-2}},\beta^{(m-1)}_{i_{2m-2}i_{2m-1}},\alpha^{(m)}_{i_{2m-1}i_1})\\
=&\sum_{\substack{1\le q_1,\dots,q_{2m-1}\le r\\1\le r_1,\dots,r_{2m-1}\le r}}(\alpha^{(1)}_{(q_1-1)r+r_1,(q_2-1)r+r_2},\dots,a^{(m)}_{(q_{2m-1}-1)r+r_{2m-1},(q_1-1)r+r_1})\\
=&\sum_{\substack{1\le q_1,q_3,\dots,q_{2m-3},q_{2m-1}\le r\\1\le r_2,r_4,\dots,r_{2m-4},r_{2m-2}\le r}}(a^{(1)}_{q_1q_3},b^{(1)}_{r_2r_4},a^{(2)}_{q_3q_5},b^{(2)}_{r_4r_6},\dots,a^{(m-1)}_{q_{2m-3}q_{2m-1}},b^{(m-1)}_{r_{2m-2}r_2},a^{(m)}_{q_{2m-1}q_1})
\end{aligned}$$

□

Let us recall the group $S_{p,q}\subset S_n$, $n=p+q$ of p,q-shuffles, which was important for the product maps sh_{pq} (Definition 1.17 and 1.22).

5.31 Lemma We have the following equation of elements of $B^{\otimes 2n}$:

$$\begin{aligned}
\mathrm{tr}((\Phi,\Psi)^{\otimes_k n})=\mathrm{tr}\Big[\sum_{p=1}^{n}\sum_{\sigma\in S_{2p-1,2(n-p)}}&\mathrm{sgn}(\sigma)\Big(\sigma((\varphi\otimes\mathrm{id},\psi\otimes\mathrm{id})^{\otimes_k p},(\mathrm{id}\otimes\varphi',\mathrm{id}\otimes\psi')^{\otimes_k n-p})\\
+\sigma((\psi\otimes\mathrm{id},\varphi\otimes\mathrm{id})^{\otimes_k p},(\mathrm{id}\otimes\psi',\mathrm{id}\otimes\varphi')^{\otimes_k n-p})&+\sigma((\mathrm{id}\otimes\varphi',\mathrm{id}\otimes\psi')^{\otimes_k p},(\varphi\otimes\mathrm{id},\psi\otimes\mathrm{id})^{\otimes_k n-p})\\
+\sigma((\mathrm{id}\otimes\psi',\mathrm{id}\otimes\varphi')^{\otimes_k p},(\psi\otimes\mathrm{id},\varphi\otimes\mathrm{id})^{\otimes_k n-p})\Big)\Big].
\end{aligned}$$

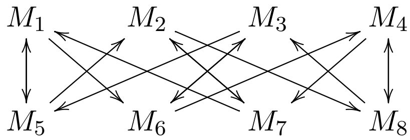
Proof. We have

$$\begin{aligned}
\Phi&=\begin{pmatrix}\varphi\otimes\mathrm{id}&\mathrm{id}\otimes\varphi'\\-\mathrm{id}\otimes\psi'&\psi\otimes\mathrm{id}\end{pmatrix}=M_1+M_2+M_3+M_4\\
&:=\begin{pmatrix}\varphi\otimes\mathrm{id}&0\\0&0\end{pmatrix}+\begin{pmatrix}0&\mathrm{id}\otimes\varphi'\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\-\mathrm{id}\otimes\psi'&0\end{pmatrix}+\begin{pmatrix}0&0\\0&\psi\otimes\mathrm{id}\end{pmatrix},\\
\Psi&=\begin{pmatrix}\psi\otimes\mathrm{id}&-\mathrm{id}\otimes\varphi'\\\mathrm{id}\otimes\psi'&\varphi\otimes\mathrm{id}\end{pmatrix}=M_5+M_6+M_7+M_8\\
&:=\begin{pmatrix}\psi\otimes\mathrm{id}&0\\0&0\end{pmatrix}+\begin{pmatrix}0&-\mathrm{id}\otimes\varphi'\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\\mathrm{id}\otimes\psi'&0\end{pmatrix}+\begin{pmatrix}0&0\\0&\varphi\otimes\mathrm{id}\end{pmatrix}.
\end{aligned}$$

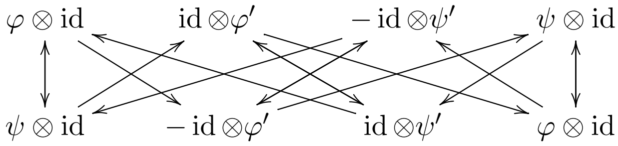
We want to show, that

$$\begin{aligned}
&\mathrm{tr}(M_1,\Psi,(\Phi,\Psi)^{n-1})\\
&=\mathrm{tr}\Big[\sum_{p=1}^{n}\sum_{\sigma\in S_{2p-1,2(n-p)}}\mathrm{sgn}(\sigma)\sigma((\varphi\otimes\mathrm{id},\psi\otimes\mathrm{id})^{\otimes_k p},(\mathrm{id}\otimes\varphi',\mathrm{id}\otimes\psi')^{\otimes_k n-p})\Big],
\end{aligned}$$

which is the first summand at the right hand side in the claim of this lemma. Now $\operatorname{tr}(M_1, \Psi, (\Phi, \Psi)^{n-1})$ is the sum with summands of the form $\operatorname{tr}(M_1, M_{i_2}, M_{i_3}, \dots, M_{i_{2n}})$ with $i_{2l} \in \{5, 6, 7, 8\}$ and $i_{2l-1} \in \{1, 2, 3, 4\}$. Many of those traces will be zero, e. g. any trace with a M_1 or M_3 in front of M_7 or M_8. Let us summarize this in the following directed graph



where an arrow from M_i to M_j means, that M_j may appear after M_i in the traces without the trace being zero (in general). Now a non zero summand $\operatorname{tr}(M_1, M_{i_2}, M_{i_3}, \dots, M_{i_{2n}})$ corresponds to a path of length $2n$ from M_1 to M_1 in the above graph. If $\operatorname{tr}(M_1, \dots, M_{i_{2n}})$ corresponds to such a path, then the matrices M_i can be exchanged with their only non zero $r \times r$ block, without changing the result of the trace. Let us rewrite the graph with M_i replaced its corresponding non zero $r \times r$ block



Now looking at the possible paths of length $2n$ from the top left $\varphi \otimes \mathrm{id}$ to itself, one will see, that one gets exactly all sequences in $\varphi \otimes \mathrm{id}, \psi \otimes \mathrm{id}, \mathrm{id} \otimes \varphi'$ and $\mathrm{id} \otimes \psi'$, starting with $\varphi \otimes \mathrm{id}$, in which the subsequence consisting of $\varphi \otimes \mathrm{id}$ and $\psi \otimes \mathrm{id}$ is of the form $(\varphi \otimes \mathrm{id}, \psi \otimes \mathrm{id})^p$, the same with $'$ and these are two alternating sequences are shuffled into each other arbitrarily. In other words, exactly all summands of

$$\operatorname{tr}\Big[\sum_{p=1}^{n} \sum_{\sigma \in S_{2p-1,2(n-p)}} \operatorname{sgn}(\sigma)\sigma((\varphi \otimes \mathrm{id}, \psi \otimes \mathrm{id})^{\otimes_k p}, (\mathrm{id} \otimes \varphi', \mathrm{id} \otimes \psi')^{\otimes_k n-p})\Big].$$

It remains to check, that the sign is correct, but this is true, because one gets a factor (-1) exactly, when one has an odd amount of $\mathrm{id} \otimes \varphi', \mathrm{id} \otimes \psi'$ after $\varphi \otimes \mathrm{id}$, which gives $\operatorname{sgn}(\sigma)$. One can show this rigorously in the following way: Each path of length $2n$ from M_1 to itself is the concatenation of a path from M_1 to M_1, M_2, M_3 or M_4 of length 2 with one of length $2n-2$ in the opposite direction. Now analyse all such path precisely by doing induction on the path length. In a similar way one can show that we get the other three summands from the claim of this lemma starting with one of the other three possible matrices M_4, M_2, M_3 (here we listed the matrices in the order of the corresponding summands). □

5.32 Lemma We have the following equation of elements of $HH_{2(n-p)}(B)$:

$$(1, \operatorname{tr}((\varphi', \psi')^{\otimes_k n-p})) - (1, \operatorname{tr}((\psi', \varphi')^{\otimes_k n-p}) = \frac{1}{n-p} \overline{B}_{2(n-p)-1}\big(\operatorname{tr}(\varphi', \psi')^{\otimes_k n-p}\big)$$

Proof. By Lemma 1.29 we have $\operatorname{tr} \circ t = t \circ \operatorname{tr}$. In particular

$$(t \circ \operatorname{tr})((\varphi', \psi')^{\otimes_k n-p}) = \operatorname{tr}(t((\varphi', \psi')^{\otimes_k n-p})) = -\operatorname{tr}((\psi', \varphi')^{\otimes_k n-p}),$$
$$(t^2 \circ \operatorname{tr})((\varphi', \psi')^{\otimes_k n-p}) = \operatorname{tr}(t^2((\varphi', \psi')^{\otimes_k n-p})) = \operatorname{tr}((\varphi', \psi')^{\otimes_k n-p}).$$

Now we have by definition of $\overline{B}$ and t:

$$\begin{aligned}\overline{B}_{2(n-p)-1}(b_0,\dots,b_{2(n-p)-1}) &= \sum_{j=0}^{2(n-p)-1} (-1)^{(2(n-p)-1)j}(1,b_j,\dots,b_{2(n-p)-1},b_0,\dots,b_{j-1})\\ &= \sum_{j=0}^{2(n-p)-1} (1,t^j(b_0,\dots,b_{2(n-p)-1})).\end{aligned}$$

Plugging in $\mathrm{tr}((\varphi',\psi')^{\otimes_k n-p})$ for the b_j gives the result. $\square$

5.33 Proposition We have

$$\begin{aligned}[\mathrm{ch}]_n^{\mathrm{HH,odd}}(\Phi,\Psi) = \sum_{p=1}^{n-1} \mathrm{sh}_{2p-1,2(n-p)}\big([\mathrm{ch}]_p^{\mathrm{HH,odd}}(\varphi,\psi),[\mathrm{ch}]_{n-p}^{\mathrm{HH,even}}(\varphi',\psi'))\\ + [\mathrm{ch}]_p^{\mathrm{HH,odd}}(\varphi',\psi'),[\mathrm{ch}]_{n-p}^{\mathrm{HH,even}}(\varphi,\psi))\big)\end{aligned}$$

Proof. We compute

$$\begin{aligned}&[\mathrm{ch}]_n^{\mathrm{HH,odd}}(\Phi,\Psi) = \mathrm{tr}((\Phi,\Psi)^{\otimes_k n}) \overset{5.31}{=} \mathrm{tr}\Big[\sum_{p=1}^{n}\sum_{\sigma\in S_{2p-1,2(n-p)}} \mathrm{sgn}(\sigma)\\ &\Big(\sigma((\varphi\otimes\mathrm{id},\psi\otimes\mathrm{id})^{\otimes_k p},(\mathrm{id}\otimes\varphi',\mathrm{id}\otimes\psi')^{\otimes_k n-p}) + \sigma((\psi\otimes\mathrm{id},\varphi\otimes\mathrm{id})^{\otimes_k p},(\mathrm{id}\otimes\psi',\mathrm{id}\otimes\varphi')^{\otimes_k n-p})\\ &+\sigma((\mathrm{id}\otimes\varphi',\mathrm{id}\otimes\psi')^{\otimes_k p},(\varphi\otimes\mathrm{id},\psi\otimes\mathrm{id})^{\otimes_k n-p}) + \sigma((\mathrm{id}\otimes\psi',\mathrm{id}\otimes\varphi')^{\otimes_k p},(\psi\otimes\mathrm{id},\varphi\otimes\mathrm{id})^{\otimes_k n-p})\Big)\Big]\end{aligned}$$

By Lemma 5.24 the $p=n$ term of the first summand will cancel with the $p=n$ term of the second summand and similar for third and fourth summand. Now apply Lemma 5.30 to simplify

$$\begin{aligned}&[\mathrm{ch}]_n^{\mathrm{HH,odd}}(\Phi,\Psi) = \sum_{p=1}^{n-1}\sum_{\sigma\in S_{2p-1,2(n-p)}} \mathrm{sgn}(\sigma)\\ &\Big(\sigma(\mathrm{tr}((\varphi,\psi)^{\otimes_k p}),\mathrm{tr}((\varphi',\psi')^{\otimes_k n-p})) + \sigma(\mathrm{tr}((\psi,\varphi)^{\otimes_k p}),\mathrm{tr}((\psi',\varphi')^{\otimes_k n-p}))\\ &+\sigma(\mathrm{tr}((\varphi',\psi')^{\otimes_k p}),\mathrm{tr}((\varphi,\psi)^{\otimes_k n-p})) + \sigma(\mathrm{tr}((\psi',\varphi')^{\otimes_k p}),\mathrm{tr}((\psi,\varphi)^{\otimes_k n-p}))\Big).\end{aligned} \tag{5.3}$$

By definition of the shuffle product we have

$$\begin{aligned}&\sum_{\sigma\in S_{2p-1,2(n-p)}} \mathrm{sgn}(\sigma)\sigma(\mathrm{tr}((\varphi,\psi)^{\otimes_k p}),\mathrm{tr}((\varphi',\psi')^{\otimes_k n-p}))\\ &=\mathrm{sh}_{2p-1,2(n-p)}(\mathrm{tr}((\varphi,\psi)^{\otimes_k p}),(1,\mathrm{tr}((\varphi',\psi')^{\otimes_k n-p})))\end{aligned}$$

and similarly with the other summands. We can further simplify the sum of first and second summand of (5.3):

$$
\begin{aligned}
&\sum_{\sigma \in S_{2p-1,2(n-p)}} \operatorname{sgn}(\sigma)\big(\sigma(\operatorname{tr}((\varphi,\psi)^{\otimes_k p}),\operatorname{tr}((\varphi',\psi')^{\otimes_k n-p})) + \sigma(\operatorname{tr}((\psi,\varphi)^{\otimes_k p}),\operatorname{tr}((\psi',\varphi')^{\otimes_k n-p}))\big) \\
&= \operatorname{sh}_{2p-1,2(n-p)}\big(\operatorname{tr}((\varphi,\psi)^{\otimes_k p}),(1,\operatorname{tr}((\varphi',\psi')^{\otimes_k n-p}))\big) + \big(\operatorname{tr}((\psi,\varphi)^{\otimes_k p}),(1,\operatorname{tr}((\psi',\varphi')^{\otimes_k n-p}))\big) \\
&\overset{5.24}{=} \operatorname{sh}_{2p-1,2(n-p)}\big(\operatorname{tr}((\varphi,\psi)^{\otimes_k p}),(1,\operatorname{tr}((\varphi',\psi')^{\otimes_k n-p}))\big) - \big(\operatorname{tr}((\varphi,\psi)^{\otimes_k p}),(1,\operatorname{tr}((\psi',\varphi')^{\otimes_k n-p}))\big) \\
&= \operatorname{sh}_{2p-1,2(n-p)}\big(\operatorname{tr}((\varphi,\psi)^{\otimes_k p}),(1,\operatorname{tr}((\varphi',\psi')^{\otimes_k n-p})) - (1,\operatorname{tr}((\psi',\varphi')^{\otimes_k n-p}))\big) \\
&\overset{5.32}{=} \operatorname{sh}_{2p-1,2(n-p)}\big(\operatorname{tr}(\varphi,\psi)^{\otimes_k p},\frac{1}{n-p}\overline{B}_{2(n-p)-1}(\operatorname{tr}(\varphi',\psi')^{\otimes_k n-p})\big) \\
&= \operatorname{sh}_{2p-1,2(n-p)}\big([\mathrm{ch}]_p^{\mathrm{HH,odd}}(\varphi,\psi),[\mathrm{ch}]_{n-p}^{\mathrm{HH,even}}(\varphi',\psi')\big).
\end{aligned}
$$

Doing the same with the third and fourth summand of (5.3) will give

$$
\operatorname{sh}_{2p-1,2(n-p)}\big([\mathrm{ch}]_p^{\mathrm{HH,odd}}(\varphi',\psi'),[\mathrm{ch}]_{n-p}^{\mathrm{HH,even}}(\varphi,\psi)\big).
$$

□

5.34 Corollary For the even Chern characters this translates to the simpler formula

$$
[\mathrm{ch}]_n^{\mathrm{HH,even}}(\Phi,\Psi) = \sum_{p=1}^{n-1} \operatorname{sh}_{2p,2(n-p)}\big([\mathrm{ch}]_p^{\mathrm{HH,even}}(\varphi,\psi),[\mathrm{ch}]_{n-p}^{\mathrm{HH,even}}(\varphi',\psi')\big).
$$

Proof. Apply $\frac{1}{n}\overline{B}_{2n-1}$ to the equation for the odd Chern characters and use the following identity of elements in the normalized Hochschild homology:

$$
\overline{B}_{i+j}\operatorname{sh}_{ij}(x,\overline{B}_j y) = \operatorname{sh}_{i+1,j}(\overline{B}_i x,\overline{B}_j y) = (-1)^{|x|+1}\overline{B}_{i+j}\operatorname{sh}_{i+1,j-1}(\overline{B}_i x, y).
$$

This follows from [Lod98], Corollary 4.3.4 and the fact that $\overline{B}^2$ is zero in the normalized setting. □

As for the Chern characters with values in forms, this is saying, that $[\mathrm{ch}]_n^{\mathrm{HH,even}}$ is multiplicative with weights $\frac{1}{2^n}$ in the pseudo tensor category $([MF(A,f)],\otimes_{\mathrm{MF}}^{\frac{1}{2}})$. An immediate consequence of the formulas for the tensor products of matrix factorizations is the following corollary.

5.35 Corollary If the matrix factorization (φ,ψ) is a tensor product of n other matrix factorizations, then $\mathrm{ch}_m(\varphi,\psi) = 0$ for $m < n$. Here ch can be any of our six Chern theories.

6 Comparison with the Chern characters from Alexander Polishchuck and Arkady Vaintrob

In this chapter we will compare our Chern characters with the ones computed in [PV12] by Alexander Polishchuck and Arkady Vaintrob. We will see that ours are lifts of theirs up to a sign. We will also discuss the correlation between the Herbrand difference and Chern characters for matrix factorizations.

In this chapter $k = \mathbb{C}$, $A = \mathbb{C}[\![X_1, \dots, X_n]\!]$, $f \in A$, $B = A/(f)$ a proper quotient, we will write ∂_i for ∂_{X_i}, $f_i := \partial_i f$, $J_f = (f_1, \dots, f_n)$, f is assumed to have an isolated singularity (i.e. $\dim_{\mathbb{C}}(A/J_f) < \infty$), $\Omega^m = \Omega^m_{A/\mathbb{C}}$ and $dX := dX_1 \wedge \dots \wedge dX_n$. The authors of [PV12] work with matrix factorizations in the $\mathbb{Z}/2\mathbb{Z}$-graded version, therefore we will do the same in this chapter. We denote with $\overline{E} = (E = E_0 \oplus E_1, \delta = \left(\begin{smallmatrix} 0 & \varphi \\ \psi & 0 \end{smallmatrix}\right))$ a matrix factorization of f over A in the sense of the $\mathbb{Z}/2\mathbb{Z}$-graded version of Definition 4.5.

6.1 The definition of the Chern characters from Alexander Polishchuck and Arkady Vaintrob

In this section we will write DG for differential graded, which is always meant to be defined over k.

There is a definition of Hochschild homology for a DG category (see for example [Shk09] 2.3 or [Kel06] 5.3, we use the terminology from [Shk09]). Any k-algebra T can be seen as DG algebra sitting in degree 0 and zero differential. A DG algebra can be seen as DG category $\mathfrak{C}(T)$ consisting of one object with the given DG algebra as homomorphisms. In this case the definition of Hochschild homology for $\mathfrak{C}(T)$ and our definition from Chapter 1 agree, up to a regrading, i.e. $HH_{-l}(\mathfrak{C}(T)) = HH_l(T)$ for all $l \geq 0$.

In [PV12] the authors compute the Hochschild homology of the differential $\mathbb{Z}/2\mathbb{Z}$-graded category $MF(A, f)$ (using results from [Dyc11]). They prove that (Equation (2.28))

$$HH_\bullet(MF(A, f)) \xrightarrow{\sim} (A/J_f) \otimes \Omega^n[n], \tag{6.1}$$

where $[n]$ means a shift of degrees, i.e. only the $(-n)$-th Hochschild homology is not zero. Since they work over $\mathbb{Z}/2\mathbb{Z}$ there are only two groups HH_0 and HH_1.

For any DG category $\mathfrak{C}$ there is a theory for Chern characters, i.e. a function

$$\mathrm{ch}_{\mathfrak{C}}^{HH}\colon \mathrm{Obj}(\mathfrak{C}) \to HH_\bullet(\mathfrak{C})$$

(see [Shk09] 1.2). In [PV12] (Theorem 3.2.3) the authors prove that

$$\mathrm{ch}_{n,MF(A,f)}^{HH}(\overline{E}) = \mathrm{str}_A(\partial_n\delta \circ \ldots \circ \partial_1\delta)dX \in (A/J_f)\otimes\Omega^n = HH_n(MF(A,f)),$$

after applying their isomorphism from Equation (6.1). The map str is the so called *super trace*. The matrices δ and $\partial_i\delta$ have the structure of 2×2 block matrices. In this setting the super trace is given by

$$\mathrm{str}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{Tr}(a) - \mathrm{Tr}(d).$$

Furthermore, there is a k-bilinear pairing

$$HH_{-n}(\mathfrak{C}) \times HH_n(\mathfrak{C}^{\mathrm{op}}) \to k$$

(see [Shk09], 1.2). This pairing is nondegenerate for a proper smooth DG algebra (see [Shk09], 6.2). It is also nondegenerate for the category $MF(A,f)$ and

$$MF(A,f)^{\mathrm{op}} \cong MF(A,-f)$$

(see [PV12] Equation (2.14)), therefore

$$HH_\bullet(MF(A,f)) = HH_\bullet(MF(A,f)^{\mathrm{op}}).$$

Hence we get a nondegenerate pairing

$$HH_n(MF(A,f)) \times HH_n(MF(A,f)) \to k$$

($\mathbb{Z}/2\mathbb{Z}$-graded, so $HH_n = HH_{-n}$). Now Corollary 4.1.3 of [PV12] says, that this pairing is given by

$$((A/J_f)\otimes\Omega^n) \times ((A/J_f)\otimes\Omega^n) \to k, \quad (g\otimes dX, h\otimes dX) \mapsto (-1)^{\binom{n}{2}}\,\mathrm{Res}(g\cdot h),$$

where $\mathrm{Res}(g) = \mathrm{Res}_{\mathbb{C}[X]/\mathbb{C}}\left(\frac{gdX}{f_1,\ldots,f_n}\right)$ is the Grothendieck residue (see [Lip84] or [Har66]). Let us explain how to compute $\mathrm{Res}_{\mathbb{C}[X]/\mathbb{C}}\left(\frac{gdX}{f_1,\ldots,f_n}\right)$.

With $\frac{gdX}{f_1,\ldots,f_n}$ we mean the element $\frac{gdX}{f_1\cdots f_n}$ of the n-th local cohomology of Ω^n. The local cohomology can be computed with the cocomplex

$$C^\bullet_{f_1,\ldots,f_n} : 0 \to \Omega^n \to \bigoplus_{i=1}^n \Omega^n_{f_i} \to \bigoplus_{1\le i<j\le n} \Omega^n_{f_i,f_j} \to \ldots \to \Omega^n_{f_1,\ldots,f_n} \to 0,$$

where $\Omega^n_{f_i,f_j}$ is the localization with respect to the multiplicative subset $\{f_i^l\cdot f_j^m \mid l,m\in\mathbb{N}_0\}$ and similarly with one or more f_i as index. The differential is given by the sum of natural embeddings with signs

$$\Omega_{f_{i_1},\ldots,f_{i_l}} \xrightarrow{(-1)^{s-1}} \Omega_{f_{j_1},\ldots,f_{j_{l+1}}}, \text{ if } (i_1,\ldots,i_l) = (j_1,\ldots,\widehat{j_s},\ldots,j_{l+1}).$$

The f_i can be replaced with any system of parameters, i.e. any n elements of A, which generate an ideal containing a power of the maximal ideal. For more details on local cohomology see [BH93] (it is defined in Definition 3.5.2 and Theorem 3.5.6 is the statement that it can be computed with the above cocomplex).

Now if $t_1, \dots, t_n$ is a system of parameters, then we find natural numbers $i_1, \dots, i_n$ and a matrix $W = (w_{ij}) \in M_n(A)$ with

$$\begin{pmatrix} X_1^{i_1} \\ \vdots \\ X_n^{i_n} \end{pmatrix} = \begin{pmatrix} w_{11} & \dots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \dots & w_{nn} \end{pmatrix} \cdot \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}.$$

Then $\frac{gdX}{t_1,\dots,t_n} = \frac{\det(W)gdX}{X_1^{i_1},\dots,X_n^{i_n}}$ as elements in the n-th local cohomology (see [Lip84], Lemma 7.2) and $\operatorname{Res}_{\mathbb{C}[X]/\mathbb{C}}\left(\frac{\det(W)gdX}{X_1^{i_1},\dots,X_n^{i_n}}\right)$ is the coefficient in front of $X_1^{i_1-1}\cdots X_n^{i_n-1}$ in the power series $\det(W)g$. Although the products in the denominator commute, the notation with the commata in the denominator makes sense, because we have to keep track from which cocomplex $C^\bullet_{t_1,\dots,t_n}$ they came, to compare them as elements in the n-th local cohomology.

6.2 Comparison of the two Chern theories

In this section we want to show that our Chern characters $\mathrm{ch}^{\mathrm{dR,\ even}}$ from Definition 5.22 are lifts up to a sign of the Chern characters computed in [PV12].

Let us first compute

$$\begin{aligned}
\operatorname{str}_A(\partial_n\delta\dots\partial_1\delta) &= \operatorname{str}_A\left(\begin{pmatrix} 0 & \partial_n\varphi \\ \partial_n\psi & 0 \end{pmatrix}\cdots\begin{pmatrix} 0 & \partial_1\varphi \\ \partial_1\psi & 0 \end{pmatrix}\right) \\
&= \begin{cases} \operatorname{str}_A\begin{pmatrix} \partial_n\varphi\partial_{n-1}\psi\dots\partial_2\varphi\partial_1\psi & 0 \\ 0 & \partial_n\psi\partial_{n-1}\varphi\dots\partial_2\psi\partial_1\varphi \end{pmatrix}, & n \text{ even}, \\ \operatorname{str}_A\begin{pmatrix} 0 & \partial_n\varphi\partial_{n-1}\psi\dots\partial_2\varphi\partial_1\psi \\ \partial_n\psi\partial_{n-1}\varphi\dots\partial_2\psi\partial_1\varphi & 0 \end{pmatrix}, & n \text{ odd}, \end{cases} \\
&= \begin{cases} \operatorname{Tr}(\partial_n\varphi\partial_{n-1}\psi\dots\partial_2\varphi\partial_1\psi) - \operatorname{Tr}(\partial_n\psi\partial_{n-1}\varphi\dots\partial_2\psi\partial_1\varphi), & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}
\end{aligned}$$

Therefore, we will assume from now on, that n is even.

Now let us look at the targets of the two Chern character maps. The one in [PV12] maps to

$$(A/J_f) \otimes \Omega^n = \Omega^n / df \wedge \Omega^{n-1}$$

and our top Chern character $\mathrm{ch}^{\mathrm{dR,\ even}}_{\frac{n}{2}}$ maps to

$$\Omega^n / df \wedge d(\Omega^{n-2}).$$

Hence the target of the Chern characters from [PV12] is a quotient of our target.

6.1 Lemma We have the following equality in Ω^n:

$$\mathrm{ch}^{\mathrm{dR,\ even}}_{\frac{n}{2}}(\overline{E}) = \frac{2}{n!}\operatorname{Tr}((d\varphi d\psi)^{\frac{n}{2}}) = \frac{1}{n!}\operatorname{str}_A((d\delta)^n).$$

Proof. We calculate

$$\begin{aligned}\operatorname{str}_A((d\delta)^n) &= \operatorname{str}_A\left(\begin{pmatrix} 0 & d\varphi \\ d\psi & 0\end{pmatrix}^n\right) = \operatorname{str}_A\begin{pmatrix}(d\varphi d\psi)^{\frac{n}{2}} & 0 \\ 0 & (d\psi d\varphi)^{\frac{n}{2}}\end{pmatrix} \\ &= \operatorname{Tr}((d\varphi d\psi)^{\frac{n}{2}}) - \operatorname{Tr}((d\psi d\varphi)^{\frac{n}{2}}) = 2\operatorname{Tr}((d\varphi d\psi)^{\frac{n}{2}}) = n!\,\mathrm{ch}^{\mathrm{dR,\ even}}_{\frac{n}{2}}(\overline{E}).\end{aligned}$$

□

6.2 Lemma For a permutation $\sigma \in S_n$ we have the following equality in A/J_f:

$$\operatorname{str}_A(\partial_n\delta \dots \partial_1\delta) = \operatorname{sgn}(\sigma)\operatorname{str}_A(\partial_{\sigma(n)}\delta \dots \partial_{\sigma(1)}\delta).$$

Proof. In this proof we write δ_i for $\partial_i\delta$. We will show, that

$$\operatorname{str}_A(\delta_n \dots \delta_1) = -\operatorname{str}_A(\delta_{n-1}\delta_n\delta_{n-2}\dots\delta_1)$$

as elements of $A_{n-1} := A/(f_1, \dots, f_{n-2}, f_n)$. Because f has an isolated singularity (any permutation of) $f_1, \dots, f_n$ is a regular sequence, in particular f_{n-1} is a non zero divisor in A_{n-1}. Therefore it is enough to show $f_{n-1}\operatorname{str}_A(\delta_n \dots \delta_1) = -f_{n-1}\operatorname{str}_A(\delta_{n-1}\delta_n\delta_{n-2}\dots\delta_1)$. Since $\delta^2 = f$ we have $\delta_i\delta + \delta\delta_i = f_i$ for all i, in particular we can commute δ_i with δ with a sign in str_A for $i \neq n-1$, while computing in A_{n-1}. We have

$$\begin{aligned}f_{n-1}\operatorname{str}_A(\delta_n\dots\delta_1) &= \operatorname{str}_A(f_{n-1}\delta_n\dots\delta_1) = \operatorname{str}_A((\delta_{n-1}\delta + \delta\delta_{n-1})\delta_n\dots\delta_1) \\ &= \operatorname{str}_A(\delta_{n-1}\delta\delta_n\dots\delta_1) + \operatorname{str}_A(\delta\delta_{n-1}\delta_n\dots\delta_1) =: s_1 + s_2.\end{aligned}$$

Let us look at the first summand s_1:

$$\begin{aligned}&\operatorname{str}_A(\delta_{n-1}\delta\delta_n\dots\delta_1) = -\operatorname{str}_A(\delta_{n-1}\delta_n\delta\delta_{n-1}\dots\delta_1) = -\operatorname{str}_A(\delta_{n-1}\delta_n(f_{n-1} - \delta_{n-1}\delta)\delta_{n-2}\dots\delta_1) \\ &= -\operatorname{str}_A(\delta_{n-1}\delta_n f_{n-1}\delta_{n-2}\dots\delta_1) + \operatorname{str}_A(\delta_{n-1}\delta_n\delta_{n-1}\delta\delta_{n-2}\dots\delta_1) \\ &= -f_{n-1}\operatorname{str}_A(\delta_{n-1}\delta_n\delta_{n-2}\dots\delta_1) + (-1)^{n-2}\operatorname{str}_A(\delta_{n-1}\delta_n\delta_{n-1}\delta_{n-2}\dots\delta_1\delta).\end{aligned}$$

Now using the graded cyclic invariance of str_A we have

$$(-1)^{n-2}\operatorname{str}_A(\delta_{n-1}\delta_n\delta_{n-1}\dots\delta_1\delta) = (-1)^{n-2+n+1}\operatorname{str}_A(\delta\delta_{n-1}\delta_n\delta_{n-1}\dots\delta_1) = -s_2.$$

Putting everything together we get the lemma for the transposition $((n-1)\ n)$ in A_{n-1}. Other transpositions can be checked similarly. □

6.3 Proposition Writing again $\mathrm{ch}^{\mathrm{dR,\ even}}_{\frac{n}{2}}(\overline{E})$ for its image under the quotient map $\Omega^n/(df \wedge d(\Omega^{n-2})) \to \Omega^n/(df \wedge \Omega^{n-1})$ we get

$$\mathrm{ch}^{\mathrm{dR,\ even}}_{\frac{n}{2}}(\overline{E}) = (-1)^{\frac{n}{2}}\,\mathrm{ch}^{HH}_{n,MF(A,f)}(\overline{E}).$$

Proof. In this proof we write δ_i for $\partial_i\delta$. Using the previous two lemmas we compute

$$\begin{aligned}
\mathrm{ch}^{\mathrm{dR,\ even}}_{\frac{n}{2}}(\overline{E}) &\overset{6.1}{=} \frac{1}{n!}\,\mathrm{str}_A((d\delta)^n) = \frac{1}{n!}\,\mathrm{str}_A\left(\left(\sum_{i=1}^{n}\delta_i dX_i\right)^n\right)\\
&= \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{str}_A(\delta_{\sigma(1)}\dots\delta_{\sigma(n)})dX_{\sigma(1)}\dots dX_{\sigma(n)}\\
&= \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\,\mathrm{str}_A(\delta_{\sigma(1)}\dots\delta_{\sigma(n)})dX \overset{6.2}{=} \mathrm{str}_A(\delta_1\dots\delta_n)dX\\
&\overset{6.2}{=} (-1)^{\frac{n}{2}}\,\mathrm{str}_A(\delta_n\dots\delta_1)dX = (-1)^{\frac{n}{2}}\,\mathrm{ch}^{HH}_{n,MF(A,f)}(\overline{E}).
\end{aligned}$$

□

We will see in Corollary 7.6, that $\mathrm{ch}^{\mathrm{dR,\ even}}_m(\overline{E}) = 0$ for $m < \frac{n}{2}$ under the assumptions of this chapter on k, A and f. To be more precise we will only show this for their images under the map $\Omega^{2m}/df \wedge d(\Omega^{2m-2}) \to \Omega^{2m}_{\mathrm{fin}}/df \wedge d(\Omega^{2m-2}_{\mathrm{fin}})$, i.e for universal finite differentials instead of Kähler differentials. This difference will only be important if f or an entry of φ, ψ is a power series, which is not contained in the localisation of the polynomial ring at the maximal ideal $(X_1, \dots, X_n)$.

6.3 Chern characters for matrix factorizations and Herbrand difference

In this section we want to exhibit the relation between Herbrand difference and Chern characters of matrix factorizations.

We have seen in Proposition 4.16, that every resolution over $B = \mathbb{C}[\![X_1, \dots, X_n]\!]/(f)$ becomes periodic and the periodic part is given by a matrix factorization.

6.4 Lemma If M, M' are two B-modules, then $\mathrm{Ext}^l_B(M, M')$ and $\mathrm{Tor}^B_l(M, M')$ are finitely generated B/J_f-modules for $l > n+1$. In particular $\mathrm{Ext}^l_B(M, M')$ and $\mathrm{Tor}^B_l(M, M')$ have finite length, because B/J_f is a finite $\mathbb{C}$-vector space by our assumption on f.

Proof. We have to show, that f_i annihilates $\mathrm{Ext}^l_B(M, M')$ and $\mathrm{Tor}^B_l(M, M')$ for all i and $l > n+1$. This follows if the multiplication with f_i is homotopic to zero in a complex computing Ext respectively Tor. For any matrix factorization (φ, ψ) of f over A we have, that the endomorphism (f_i, f_i) is homotopic to zero, which one can see in the following diagram

$$\begin{tikzcd}
A^r \arrow[r, "\psi"] \arrow[d, "f_i"'] & A^r \arrow[r, "\varphi"] \arrow[d, "f_i"] \arrow[dl, "\partial_i\varphi"] & A^r \arrow[d, "f_i"] \arrow[dl, "\partial_i\psi"] \\
A^r \arrow[r, "\psi"'] & A^r \arrow[r, "\varphi"'] & A^r,
\end{tikzcd}$$

because $\partial_i\varphi\psi + \varphi\partial_i\psi = f_i = \partial_i\psi\varphi + \psi\partial_i\varphi$. Now for $l \geq n+1$ the resolution of M is 2-periodic and given by the reduction module f of a matrix factorization. Reducing the above homotopy modulo f gives a homotopy on the 2-periodic part of the resolution. Now apply $\hom_B(-, M')$ respectively $- \otimes_B M'$ to this complex to get the result. $\square$

We can now make the following definition, because the lengths appearing are 2-periodic and finite.

6.5 Definition For two B-modules M, M' we define the *Herbrand difference*

$$\mathrm{h}(M, M') = \operatorname{length} \operatorname{Ext}_B^{2l}(M, M') - \operatorname{length} \operatorname{Ext}_B^{2l-1}(M, M'), \quad l >> 0.$$

Furthermore, we define the *Hochster-Theta pairing*

$$\Theta(M, M') = \operatorname{length} \operatorname{Tor}^B_{2l}(M, M') - \operatorname{length} \operatorname{Tor}^B_{2l-1}(M, M'), \quad l >> 0.$$

This definition implies $h(M, M') = \Theta(M^*, M')$, where $M^* = \hom_B(M, B)$ is the dual.

6.6 Example Let us look at $A = \mathbb{C}[\![X, Y]\!]$, $f = XY$, $B = A/(f)$, $M = A/(X) = B/(X)$ and $M' = A/(Y) = B/(Y)$. Then a minimal free resolution of M is given by the reduction of the matrix factorization (X, Y) (and likewise for M'):

$$\ldots \stackrel{\cdot X}{\to} B \stackrel{\cdot Y}{\to} B \stackrel{\cdot X}{\to} B \to 0.$$

Applying $\hom_B(-, M')$ to this resolution gives the following complex:

$$0 \to M' \stackrel{\cdot X}{\to} M' \stackrel{\cdot Y}{\to} M' \stackrel{\cdot X}{\to} \ldots$$

The group $\operatorname{Ext}^l_B(M, M')$ is the l-th homology of this complex, hence

$$\operatorname{Ext}_B^{2l-1}(M, M') \cong \mathbb{C}, \quad \operatorname{Ext}_B^{2l}(M, M') \cong 0, \quad \mathrm{h}(M, M') = 0 - 1 = -1.$$

Similar one computes $\mathrm{h}(M, M) = 1$, $\mathrm{h}(M', M') = 1$, $\mathrm{h}(M', M) = -1$.

For more properties of the Hochster-Theta pairing and the Herbrand difference one can look for example in [BvS12]. We get the following correlation between $\mathrm{ch}^{HH}_{MF(A,f)}$ and the Herbrand difference.

6.7 Proposition ([PV12], Theorem 4.1.4) If M, M' are two B-modules with $M = \operatorname{coker}(\varphi)$, $M' = \operatorname{coker}(\varphi')$ for two matrix factorizations (φ, ψ), (φ', ψ') of f and

$$\mathrm{ch}^{HH}_{n,MF(A,f)}(\varphi, \psi) = g dX \in \Omega^n / df \wedge \Omega^{n-1}$$

and the same for (φ', ψ') and g', then

$$\mathrm{h}(M, M') = (-1)^{\binom{n}{2}} \operatorname{Res}(g \cdot g') = (-1)^{\binom{n}{2}} \operatorname{Res}_{\mathbb{C}[X]/\mathbb{C}} \left(\frac{g g' dX}{f_1, \ldots, f_n} \right).$$

In particular h is constant zero for an odd number of variables.

The authors of [PV12] formulate this theorem with the homologies of the $\mathbb{Z}/2\mathbb{Z}$-graded morphism complex of matrix factorizations (in the sense of Definition 4.5). These homologies can be shown to be isomorphic to the Ext_B^{2l} respectively Ext_B^{2l-1}, $l \geq 1$.
As our Chern character $\mathrm{ch}_{\frac{n}{2}}^{\text{dR, even}}$ differs only by a sign from $\mathrm{ch}_{n,MF(A,f)}^{HH}$ this proposition remains true, if we formulate it with our Chern character.

6.8 Example In the situation of our previous Example 6.6 we have

$$\mathrm{ch}_{\frac{n}{2}}^{\text{dR, even}}(X,Y) = dXdY, \quad \mathrm{ch}_{\frac{n}{2}}^{\text{dR, even}}(Y,X) = dYdX = -dXdY,$$

i.e. $g = 1$, $g' = 1$ and indeed

$$\begin{aligned}
(-1)^{\binom{n}{2}} \mathrm{Res}(g \cdot g') &= (-1)^{\binom{2}{2}} \mathrm{Res}(-1) = -\mathrm{Res}_{\mathbb{C}[X,Y]/\mathbb{C}} \left(\frac{-dXdY}{Y, X} \right) \\
&= \mathrm{Res}_{\mathbb{C}[X,Y]/\mathbb{C}} \left(\frac{-dXdY}{X, Y} \right) = -1 = \mathrm{h}(M, M'), \\
(-1)^{\binom{n}{2}} \mathrm{Res}(g \cdot g) &= (-1)^{\binom{n}{2}} \mathrm{Res}(g' \cdot g') = 1 = \mathrm{h}(M, M) = \mathrm{h}(M', M').
\end{aligned}$$

7 Two vanishing results

In this chapter we will prove two vanishing results for our Chern theories with values in forms. The first one will hold for all four, while the second one only holds for the even ones.

In this chapter k will always be a commutative ring, A will be an associative, unital, commutative k-algebra, $f \in A$ is a non zero divisor, $B = A/(f)$ a proper quotient and $(\varphi, \psi) \in M_r(A)^2$ will be a matrix factorization of f over A and Ω^n will mean $\Omega^n_{A/k}$.

7.1 Implications from the determinant

In this section we want to show, that all first Chern characters with values in forms $(\mathrm{ch}_1^{\mathrm{dR,\ odd}}, \mathrm{ch}_1^{\mathrm{dR,\ even}}, [\mathrm{ch}]_1^{\mathrm{dR,\ odd}}, [\mathrm{ch}]_1^{\mathrm{dR,\ even}})$ vanish, if the determinant of φ is of the form uf^l for $0 \leq l \leq r$ with a unit $u \in A$. In particular this is the case for f irreducible.

7.1 Lemma For $W \in M_r(A)$ let us denote with $\mathrm{adj}(W)$ the adjugate matrix of W. Then we have:

1. $\mathrm{Tr}(dW \,\mathrm{adj}(W)) = d(\det(W))$.
2. $\mathrm{Tr}(W d(\mathrm{adj}(W))) = (r-1)d(\det(W))$.
3. $\mathrm{Tr}(dW d(\mathrm{adj}(W))) = 0$.

Proof. We begin by proving the first part. Let us write W_{ij} for the entry in the i-th row and j-th column of W and similarly with $\mathrm{adj}(W)$ and dW. We ignore the signs in this proof, because d introduces no new signs and the signs are correct, because of the known formula $W \,\mathrm{adj}(W) = \det(W)\,\mathrm{id}_r$. By definition $\mathrm{adj}(W)_{ji}$ is the determinant of the submatrix of W one gets by deleting the i-th row and j-th column, hence $W_{ij}\,\mathrm{adj}(W)_{ji}$ consists of exactly the summands of $\det(W)$, which contain the factor W_{ij}. So $dW_{ij}\,\mathrm{adj}(W)_{ji}$ consists exactly of the summands of $d(\det W)$, which have the d in front of W_{ij}. Therefore, $(dW\,\mathrm{adj}(W))_{ii}$ consists exactly of those summands of $d(\det(W))$, which have the d in front of W_{ij} for some $j = 1, \dots r$. Summing over i we get

$$\mathrm{Tr}(dW\,\mathrm{adj}(W)) = d(\det W).$$

The second part follows from the first, because

$$\begin{aligned}&\mathrm{Tr}(dW\,\mathrm{adj}(W)) + \mathrm{Tr}(Wd(\mathrm{adj}(W))) = \mathrm{Tr}(dW\,\mathrm{adj}(W) + Wd(\mathrm{adj}(W)))\\ &= \mathrm{Tr}(d(W\,\mathrm{adj}(W))) = d\,\mathrm{Tr}(\det(W)\,\mathrm{id}) = rd(\det(W)).\end{aligned}$$

The third part follows from either the first or the second part by applying d. □

7.2 Lemma If $\det(\varphi) = f^m$ for $0 \le m \le r$, then

$$\mathrm{Tr}(\varphi d\psi) = (r-m)df, \quad \mathrm{Tr}(d\varphi d\psi) = 0.$$

Proof. The second formula follows from the first by applying d. Now prove the first equality. Let first be $m \ge 1$. Since f is a non zero-divisor, we have $f^{m-1}\psi = \mathrm{adj}(\varphi)$. Now we can compute

$$\begin{aligned}(r-1)mf^{m-1}df &= (r-1)d(\det(\varphi)) = \mathrm{Tr}(\varphi d(\mathrm{adj}(\varphi))) = \mathrm{Tr}(\varphi d(f^{m-1}\psi))\\ &= (m-1)f^{m-2}df\,\mathrm{Tr}(\varphi\psi) + f^{m-1}\,\mathrm{Tr}(\varphi d\psi)\\ &= r(m-1)f^{m-1}df + f^{m-1}\,\mathrm{Tr}(\varphi d\psi).\end{aligned}$$

Hence we get

$$\mathrm{Tr}(\varphi d\psi) = ((r-1)m - r(m-1))df = (r-m)df.$$

For $m = 0$ it follows, that $\det(\psi) = f^r/\det(\varphi) = f^r$, therefore

$$\begin{aligned}\mathrm{Tr}(d\varphi\psi) = \mathrm{Tr}(\psi d\varphi) = (r-r)df = 0,\\ \mathrm{Tr}(d\varphi\psi) + \mathrm{Tr}(\varphi d\psi) = \mathrm{Tr}(d(\varphi\psi)) = rdf.\end{aligned}$$

This implies $\mathrm{Tr}(\varphi d\psi) = rdf$. □

7.3 Proposition If $\det(\varphi) = uf^m$ for a unit $u \in A$ and $0 \le m \le r$, then

$$\mathrm{ch}_1^{\mathrm{dR,\ odd}}(\varphi,\psi) = \frac{1}{(2n-1)!}\,\mathrm{Tr}(\varphi d\psi) = 0 \in \Omega^1/(df \wedge \Omega^0 + fd(\Omega^0)).$$

In particular, if f is irreducible, then $\det(\varphi)$ has this form and hence

$$\mathrm{ch}_1^{\mathrm{dR,\ odd}}(\varphi,\psi) = 0 \in \Omega^1/(df \wedge \Omega^0 + fd(\Omega^0)).$$

Proof. As multiplying the first row of φ with u^{-1} and simultaneously the first column of ψ with u is an isomorphism of matrix factorizations, which does not change the value of $\mathrm{Tr}(\varphi d\psi)$ as element of $\Omega^1/(df \wedge \Omega^0 + fd(\Omega^0))$, we may assume that $u = 1$. Then $\mathrm{Tr}(\varphi d\psi) = (r-m)df = 0 \in \Omega^1/df \wedge \Omega^0$. □

The vanishing of $\mathrm{ch}_1^{\mathrm{dR,\ odd}}(\varphi,\psi)$ follows, since we divide out

$$(df \wedge \Omega^0 + f\Omega^1) \supset (df \wedge \Omega^0 + fd(\Omega^0))$$

in their target. The vanishing of the two even Chern characters follows by applying d.

7.2 Implications from the singularities of f

In this section we will show the vanishing of $\mathrm{ch}_i^{\mathrm{dR,\,even}}$ under some assumptions on the singularities of f and A. We use notations and results from Section 2.2.

7.4 Lemma For each $i \geq 1$ we have the following equation in $\Omega^{2i}_{A/k}$

$$2idf \wedge \operatorname{Tr}(\varphi d\psi(d\varphi d\psi)^{i-1}) = 2f \operatorname{Tr}((d\varphi d\psi)^i) = 2fd(\operatorname{Tr}(\varphi d\psi(d\varphi d\psi)^{i-1})).$$

Proof. The second equality is clear, therefore we show the first one. As $\varphi\psi = f\,\mathrm{id} = \psi\varphi$ we get $\varphi d\psi + d\varphi\psi = df = \psi d\varphi + d\psi\varphi$. Let us now show with induction for i

$$(\varphi d\psi)^2(d\varphi d\psi)^{i-1} = (1-2i)df\varphi d\psi(d\varphi d\psi)^{i-1} - \varphi(d\psi d\varphi)^i\psi.$$

For $i = 1$ we have $(\varphi d\psi)^2 = \varphi d\psi(\varphi d\psi) = \varphi d\psi(df - d\varphi\psi) = -df\varphi d\psi - \varphi d\psi d\varphi\psi$. For the induction step we get

$$\begin{aligned}
(\varphi d\psi)^2(d\varphi d\psi)^i &= ((1-2i)df\varphi d\psi(d\varphi d\psi)^{i-1} - \varphi(d\psi d\varphi)^i\psi)d\varphi d\psi \\
&= (1-2i)df\varphi d\psi(d\varphi d\psi)^i - \varphi(d\psi d\varphi)^i(df - d\psi\varphi)d\psi \\
&= (-2i)df\varphi d\psi(d\varphi d\psi)^i + \varphi(d\psi d\varphi)^i d\psi(\varphi d\psi) \\
&= (-2i)df\varphi d\psi(d\varphi d\psi)^i + \varphi(d\psi d\varphi)^i d\psi(df - d\varphi\psi) \\
&= (1-2(i+1))df\varphi d\psi(d\varphi d\psi)^i - \varphi(d\psi d\varphi)^{i+1}\psi.
\end{aligned}$$

Now we compute

$$\begin{aligned}
&df \wedge \operatorname{Tr}(\varphi d\psi(d\varphi d\psi)^{i-1}) \\
&= (\varphi d\psi + d\varphi\psi)\operatorname{Tr}(\varphi d\psi(d\varphi d\psi)^{i-1}) = \operatorname{Tr}((\varphi d\psi + d\varphi\psi)\varphi d\psi(d\varphi d\psi)^{i-1}) \\
&= \operatorname{Tr}((\varphi d\psi)^2(d\varphi d\psi)^{i-1}) + \operatorname{Tr}(d\varphi\psi\varphi d\psi(d\varphi d\psi)^{i-1}) \\
&= (1-2i)df \wedge \operatorname{Tr}(\varphi d\psi(d\varphi d\psi)^{i-1}) - \operatorname{Tr}(\varphi(d\psi d\varphi)^i\psi) + f\operatorname{Tr}((d\varphi d\psi)^i) \\
&= (1-2i)df \wedge \operatorname{Tr}(\varphi d\psi(d\varphi d\psi)^{i-1}) + 2f\operatorname{Tr}((d\varphi d\psi)^i). \qquad \square
\end{aligned}$$

For the rest of this section we will write ω_i for $\operatorname{Tr}(\varphi d\psi(d\varphi d\psi)^{i-1})$ and assume, that $k = \mathbb{C}$, $A = \mathbb{C}[X_1, \dots, X_n]$ or $\mathbb{C}\{X_1, \dots, X_n\}$ or $\mathbb{C}[\![X_1, \dots, X_n]\!]$ or $\mathbb{C}[X_1, \dots, X_n]_{(X_1,\dots,X_n)}$, $f \in A$ is a polynomial. For the (formal) power series ring we will work over universal finite differentials instead of Kähler differentials, by applying the unique map from Definition 2.2 to everything.

7.5 Corollary

1. $f\omega_i$ represents an element of H_f^{2i-1}.
2. If $H_{\mathrm{df}}^{2i-1} = 0$, then $\partial_t(f\omega_i) = (i+1)\omega_i$.
3. If H_{df}^{2i-1} and H_{df}^{2i} are zero, then $\omega_i \in H_f^{2i-1}$.

Proof. We have $d(f\omega_i) = df \wedge \omega_i + f d\omega_i \overset{7.4}{=} df \wedge \omega_i + i df \wedge \omega_i = (i+1) df \wedge \omega_i$. This shows the first and second claim as soon as ∂_t (cf. Definition 2.12) is well defined, which requires $H^{2i-1}_{\mathrm{df}} = 0$. The third claim follows from the second as soon as $H^{2i}_{\mathrm{df}} = 0$, because ∂_t maps to H^i_f in this case by Lemma 2.13. □

For the next corollary we recall, that

$$\mathrm{ch}_i^{\text{dR, even}}(\varphi, \psi) = \frac{2}{(2n)!} \operatorname{Tr}((d\varphi d\psi)^i) = \frac{2}{(2n)!} d\omega_i \in \Omega^{2i}/df \wedge d(\Omega^{2i-2}).$$

7.6 Corollary 1. If H^{2i-1}_{df} and H^{2i}_{df} are zero, then $\operatorname{Tr}((d\varphi d\psi)^i) = 0 \in \Omega^{2i}/df \wedge \Omega^{2i-1}$.

2. If H^{2i-1}_{df}, H^{2i}_{df} and H^{2i-1}_f are zero, then $\mathrm{ch}_i^{\text{dR, even}}(\varphi, \psi) = 0$.

Proof. 1. We have $\omega_i \in H^{2i-1}_f$, hence $d\omega_i = \operatorname{Tr}((d\varphi d\psi)^i)$ is a multiple of df.

2. We have $\omega_i \in H^{2i-1}_f = 0$, hence $\omega_i = df \wedge \eta + d(\eta')$. Now apply d to get

$$\mathrm{ch}_i^{\text{dR, even}}(\varphi, \psi) = -\frac{2}{(2n)!} df \wedge d\eta = 0.$$

□

In Section 2.2 we described some criteria for the vanishing of H^j_{df} and H^j_f. In particular the assumptions for the second claim in the previous Corollary are satisfied if either of following conditions hold:

- $2i < (n - \operatorname{Kdim}(A/T_f))$ and f is quasi homogeneous.
- $A = \mathbb{C}\{X_1, \ldots, X_n\}$ or $\mathbb{C}[\![X_1, \ldots, X_n]\!]$, f has isolated singularity and $2i < n$.

8 Chern theory for modules from matrix factorizations

In this chapter we define Chern theories for modules, which have an eventually 2-periodic resolution and the periodic part is given by a matrix factorization (in the sense of Proposition 4.13). The Chern characters will be the ones from the matrix factorization.

In this chapter k is a commutative ring and A is an associative, unital, commutative, noetherian k-algebra, $0 \neq f \in A$ is supposed to be an non zero divisor, $B = A/(f)$ a proper quotient, $(\varphi, \psi) \in M_r(A)^2$, $(\varphi', \psi') \in M_{r'}(A)^2$ two matrix factorizations of f over A and M, M' are finitely generated B-modules.

8.1 The case of $[MF(A, f)]$

In Corollary 4.22 we have seen that the categories $[MF(A, f)]$ and $\underline{MCM}(B)$ are equivalent for A regular local and $B = A/f$ a proper quotient. The functor, which realises this equivalence was taking the cokernel. This functor can be defined for any ring A.

8.1 Definition We define the functor

$$\begin{aligned} [\mathfrak{coker}]\colon [MF(A, f)] &\to \underline{\mathrm{mod}}(B), \\ (\varphi, \psi) &\mapsto [\mathfrak{coker}](\varphi, \psi) = \mathrm{coker}(\varphi), \\ (\alpha, \beta)\colon (\varphi, \psi) \to (\varphi', \psi') &\mapsto [\mathfrak{coker}]([\alpha, \beta]) = \big[\, \mathrm{coker}(\varphi) \to \mathrm{coker}(\varphi'), [a] \mapsto [\alpha(a)]\big]. \end{aligned}$$

Note that we have to take $\underline{\mathrm{mod}}(B)$ as target instead of $\mathrm{mod}(B)$ or $[\mathfrak{coker}]([\alpha, \beta])$ will depend on the representative of $[\alpha, \beta]$.

8.2 Definition Two B-modules M,M' are said to be *stably isomorphic*, if they are isomorphic in $\underline{\mathrm{mod}}(B)$. We will denote it with $M \overset{s}{\cong} M'$. By [Hel60], Theorem 2.2, this is equivalent to the existence of two projective B-modules P,P' with $M \oplus P$ isomorphic to $M' \oplus P'$ as B-modules.

8.3 Definition We denote the essential image of $[\mathfrak{coker}]$ by $[\mathfrak{coker}](A, f)$. It can be described as the set of B-modules, which are stably isomorphic to a B-module, which has a 2-periodic, free resolution given by a matrix factorization.

8.4 Proposition The functor $[\mathfrak{coker}]$ is full and faithful.

Proof. For a morphism $[\alpha,\beta]\colon(\varphi,\psi)\to(\varphi',\psi')$ in $[MF(A,f)]$ and a morphism $[\gamma]\colon \operatorname{coker}(\varphi)\to\operatorname{coker}(\varphi')$ in $\underline{\mathrm{mod}}(B)$, we have: $[\mathfrak{coker}]([\alpha,\beta])=[\gamma]$, if and only if the following diagram commutes

$$\begin{tikzcd} \dots \arrow[r] & A^r \arrow[r, "\psi"] \arrow[d, "\alpha"] & A^r \arrow[r, "\varphi"] \arrow[d, "\beta"] & A^r \arrow[r, "\varepsilon"] \arrow[d, "\alpha"] & \operatorname{coker}(\varphi) \arrow[r] \arrow[d, "\gamma"] & 0 \\ \dots \arrow[r] & A^{r'} \arrow[r, "\psi'"] & A^{r'} \arrow[r, "\varphi'"] & A^{r'} \arrow[r, "\varepsilon'"] & \operatorname{coker}(\varphi') \arrow[r] & 0, \end{tikzcd} \tag{8.1}$$

for some representative of $[\alpha,\beta]$ respectively $[\gamma]$. Furthermore, it is enough to check that the first two right squares commute, then the other squares also commute by Lemma 4.4. Now to show that the functor is full, i.e. surjective on morphisms, it is enough to show that for a given map γ we can find two maps α and β, such that the first two squares in the diagram (8.1) commute. This can be done in the following way:
There exists a map α making the first square commutative, because A^r is projective and ε' is surjective. Then $\varepsilon'\circ\alpha\circ\varphi=\gamma\circ\varepsilon\circ\varphi=0$ follows, so $\alpha\circ\varphi$ maps to the kernel of ε', which is equal to the image of φ'. Hence there is a map β making the second square commute by the projectivity of A^r.
To show that it is faithful, we have to show: For every map $\gamma\colon \operatorname{coker}(\varphi)\to\operatorname{coker}(\varphi')$, which factors over a projective B-module (i.e. represents the class of the zero homomorphism in $\underline{\mathrm{mod}}(B)$) and every choice of (α,β) making the diagram (8.1) commute, the morphisms of matrix factorizations (α,β) and $(0,0)$ are homotopic. By reducing modulo f we get the following diagram of B-modules

$$\begin{tikzcd} \dots \arrow[r] & B^r \arrow[r, "\psi"] \arrow[d, "\alpha"] & B^r \arrow[r, "\varphi"] \arrow[d, "\beta"] & B^r \arrow[r, "\varepsilon"] \arrow[d, "\alpha"] & \operatorname{coker}(\varphi) \arrow[r] \arrow[d, "\gamma"] & 0 \\ \dots \arrow[r] & B^{r'} \arrow[r, "\psi'"] & B^{r'} \arrow[r, "\varphi'"] & B^{r'} \arrow[r, "\varepsilon'"] & \operatorname{coker}(\varphi') \arrow[r] & 0. \end{tikzcd}$$

Both rows are free B-resolutions, in this situation one calls the chain map consisting of α, β a lift of γ. By our assumption on γ we have

$$\operatorname{coker}(\varphi)\overset{\gamma}{\to}\operatorname{coker}(\varphi')=\operatorname{coker}(\varphi)\overset{\gamma_1}{\to}P\overset{\gamma_2}{\to}\operatorname{coker}(\varphi')$$

for some projective module P and we can insert the projective resolution

$$\dots\to 0\to 0\to P\to P$$

of P as third row in the diagram. Now we will use the fact that one can lift any map of B-modules to a chain map of corresponding projective resolutions (for example [Eis95], Lemma 20.3). Lifting γ_1 and γ_2 we get another lift of γ, which consists only of zero maps, except for the first map. Moreover any two lifts of γ are homotopic as complexes over B (also [Eis95], Lemma 20.3). In particular there are maps s,t fitting into a diagram

$$\begin{tikzcd} & B^r \arrow[r, "\varphi"] \arrow[dl, "s"'] \arrow[d, "\beta"] & B^r \arrow[dl, "t"] \\ B^{r'} \arrow[r, "\psi'"] & B^{r'} & \end{tikzcd}$$

with $\beta = \beta - 0 = t\varphi + \psi' s$. The maps s, t are given by matrices with entries in B. Chose representatives arbitrarily to view s, t as maps of $A^r \to A^{r'}$. Then for each $a \in A^r$ we get the following equation in $A^{r'}$:

$$(\beta - t\varphi + \psi' t)(a) = f\rho(a)$$

for some map $\rho\colon A^r \to A^{r'}$, but ρ is A-linear, since $\beta - t\varphi + \psi' t$ is and f is not a zero divisor. Moreover $f\rho = \psi'(\varphi'\rho)$, hence we get $\beta = t\varphi + \psi'(s + \varphi'\rho)$. From Lemma 4.4 it follows that

$$\alpha = \alpha - 0 = \varphi' t + (s + \varphi'\rho)\psi.$$

So we have verified that (α, β) is homotopic to $(0, 0)$. □

8.5 Corollary The functor $[\mathfrak{coker}]\colon [MF(A, f)] \to [\mathfrak{coker}](A, f)$ is an equivalence of categories.

Proof. A functor is an equivalence of categories if and only if it is full, faithful and essentially surjective. □

Now we could define four additive Chern theories for the category $[\mathfrak{coker}](A, f)$ by composing the four we have defined for $[MF(A, f)]$ with any inverse to $[\mathfrak{coker}]$ (all inverses will give the same Chern theories). But we want to generalize this a bit further. Therefore, we need some more preparation.

8.6 Definition Let M be a B-module and $\ldots \to F_3 \overset{d_3}{\to} F_2 \overset{d_2}{\to} F_1 \overset{d_1}{\to} M \overset{d_0}{\to} 0$ an augmented free resolution of M. Then we define $[\mathrm{syz}]_i(M) := \ker(d_i)$ and call it an i-th *syzygy of* M.

8.7 Lemma The stable isomorphism class of $[\mathrm{syz}]_i(M)$ depends only on the stable isomorphism class of M.

Proof. It is enough to show the case $i = 1$. Furthermore, it is enough to show, that the stable isomorphism class of $[\mathrm{syz}]_1(M)$ depends only on the isomorphism class of M, because the sum of two resolutions gives a resolution for the direct sum of modules and $[\mathrm{syz}]_1(F)$ is stable isomorphic to a free module for F free. This implies

$$[\mathrm{syz}]_1(M \oplus F) \overset{s}{\cong} [\mathrm{syz}]_1(M) \oplus [\mathrm{syz}]_1(F) \overset{s}{\cong} [\mathrm{syz}]_1(M).$$

The remaining case is Schanuel's Lemma, for example [Eis95], Exercise A3.13. □

8.8 Proposition Let M be a B-module with $[\mathrm{syz}]_i(M) \overset{s}{\cong} [\mathfrak{coker}]([\varphi, \psi])$, then

$$(-1)^i \operatorname{ch}_n(\varphi, \psi)$$

depends only on the stable isomorphism class of M (in particular not on i). Here ch can be any of our four Chern theories for the category $[MF(A, f)]$.

Proof. By Lemma 8.7 the stable isomorphism class of $[\mathrm{syz}]_i(M)$ depends only on the stable isomorphism class of M. Furthermore, $[\mathfrak{coker}]$ is a equivalence of categories and therefore preimages of isomorphic objects are isomorphic. Hence the homotopy class of (φ, ψ) is determined by the stable isomorphism class of M for a fixed i. More over

$$[\mathrm{syz}]_{i+1}(M) \overset{s}{\cong} [\mathrm{syz}]_1([\mathrm{syz}]_i(M)) \overset{s}{\cong} [\mathfrak{coker}]([\psi, \varphi])$$

and we have seen flipping the two matrices only introduces a factor (-1) to all of our four Chern theories for $[MF(A, f)]$. $\square$

8.9 Definition Let us define the category $\underline{\mathrm{evMF}}(A, f)$ to be the full subcategory of $\underline{\mathrm{mod}}(B)$, which consists of the modules M with $[\mathrm{syz}]_i(M) \in [\mathfrak{coker}](A, f)$ for some i. In particular $\underline{\mathrm{evMF}}(A, f)$ contains all B-modules, which have a free resolution, which is eventually periodic and the periodic part is given by a matrix factorization.

8.10 Definition As a direct consequence of Proposition 8.8 any of our four Chern theories for $[MF(A, f)]$ gives us a Chern theory for $\underline{\mathrm{evMF}}(A, f)$ by setting

$$\mathrm{ch}_n(M) = (-1)^i \,\mathrm{ch}_n(\varphi, \psi),$$

where (φ, ψ) is a matrix factorization with $[\mathrm{syz}]_i(M) \overset{s}{\cong} [\mathfrak{coker}]([\varphi, \psi])$.

8.2 The case of $MF(A, f)$

In this section we will see, that the Chern theories for $MF(A, f)$ induce Chern theories for a full subcategory of $\mathrm{mod}(B)$, similarly to the case of $[MF(A, f)]$. For this to work we require our ring A to be local with maximal ideal $\mathfrak{m}$.

8.11 Definition A functor F is called *conservative*, if it reflects isomorphisms, i.e. if $F(\alpha)$ is an isomorphism, then so is α. If F is a full, conservative functor and a, b are two objects, then $F(a)$ is isomorphic to $F(b)$ if and only if a is isomorphic to b.

Similarly to the case for the category $[MF(A, f)]$ we have the functor

$$\begin{aligned}
\mathfrak{coker}\colon MF(A, f) &\to \mathrm{mod}(B),\\
(\varphi, \psi) &\mapsto \mathfrak{coker}(\varphi, \psi) = \mathrm{coker}(\varphi),\\
(\alpha, \beta)\colon (\varphi, \psi) \to (\varphi', \psi') &\mapsto \mathfrak{coker}(\alpha, \beta) = \big(\mathrm{coker}(\varphi) \to \mathrm{coker}(\varphi'), [a] \mapsto [\alpha(a)]\big).
\end{aligned}$$

However, $\mathfrak{coker}$ is full, but it is not even conservative, which implies not faithful. For example it maps the non-zero object $(1, f)$ to the zero object. Therefore, we restrict the functor $\mathfrak{coker}$ to the full subcategory $MF(A, f)_{\mathrm{red}}$ of reduced matrix factorizations (see Definition 4.17)

$$\mathfrak{coker}_{\mathrm{red}}\colon MF(A, f)_{\mathrm{red}} \to \mathrm{mod}(B).$$

Then $\mathfrak{coker}_{\mathrm{red}}$ is full and conservative (but not necessarily faithful).

8.12 Example Let $A = \mathbb{C}[\![X]\!]$, $f = x^2$, $B = A/f$ and look at the (reduced) matrix factorization (x, x) of f over A. Then $(0, 0)$, (x, x) both give morphisms of the matrix factorization (x, x) to itself and both are mapped to the zero map by $\mathfrak{coker}$. So $\mathfrak{coker}$ is not faithful in this example.

8.13 Lemma The functor $\mathfrak{coker}_{\mathrm{red}}$ is full and conservative. The functor $\mathfrak{coker}$ is full.

Proof. For $(\alpha, \beta)\colon (\varphi, \psi) \to (\varphi', \psi')$ and $\gamma\colon\ \mathrm{coker}(\varphi) \to \mathrm{coker}(\varphi')$, we have:
$\mathfrak{coker}(\alpha, \beta) = \gamma$, if and only if the following diagram commutes

$$\begin{tikzcd} \ldots \arrow[r] & A^r \arrow[r, "\psi"] \arrow[d, "\alpha"] & A^r \arrow[r, "\varphi"] \arrow[d, "\beta"] & A^r \arrow[r, "\varepsilon"] \arrow[d, "\alpha"] & \mathrm{coker}(\varphi) \arrow[r] \arrow[d, "\gamma"] & 0 \\ \ldots \arrow[r] & A^{r'} \arrow[r, "\psi'"] & A^{r'} \arrow[r, "\varphi'"] & A^{r'} \arrow[r, "\varepsilon'"] & \mathrm{coker}(\varphi') \arrow[r] & 0. \end{tikzcd} \tag{8.2}$$

Again it is enough to verify, that the first two right squares commute. Now one can show that $\mathfrak{coker}$ and $\mathfrak{coker}_{\mathrm{red}}$ are full in the same way as for [$\mathfrak{coker}$].
To show that $\mathfrak{coker}_{\mathrm{red}}$ is conservative we have to show: for an isomorphism γ and any choice of α, β making the diagram 8.2 commute, α, β are also isomorphisms. We can reduce the square modulo f, then the rows become minimal free B-resolutions of the cokernels. Then the same argument as in the proof for the uniqueness of minimal free resolutions up to isomorphisms of complexes shows, that α, β are isomorphisms as maps from $B^r \to B^{r'}$. Let us sketch this: Reduce the first square modulo $\mathfrak{m}$. By the Nakayama Lemma $\varepsilon, \varepsilon'$ become isomorphisms of $A/\mathfrak{m} = B/\mathfrak{m}$ vector spaces. γ is still an isomorphism, because its determinant is not in $\mathfrak{m}$. Then α is an isomorphism modulo $\mathfrak{m}$. Then α is also an isomorphism as map of B modules, because its determinant is not in $\mathfrak{m}$. Now repeat this argument for $\ker(\varepsilon)$, $\ker(\varepsilon')$ instead of $\mathrm{coker}(\varphi)$, $\mathrm{coker}(\varphi)$ to get that β is an isomorphism. α and β are still isomorphisms over A, since the determinant will only change by a multiple of f and $f \in \mathfrak{m}$. $\square$

8.14 Definition We denote the essential image of $\mathfrak{coker}$ by $\mathfrak{coker}(A, f)$. It can be described as the full subcategory of modules in $\mathrm{mod}(B)$ which have a 2-periodic, free resolution given by a matrix factorization.

8.15 Definition Let M be a B-module and $\ldots \to F_3 \overset{d_3}{\to} F_2 \overset{d_2}{\to} F_1 \overset{d_1}{\to} M \overset{d_0}{\to} 0$ an augmented minimal free resolution of M. Then we define $\mathrm{syz}_i(M) := \ker(d_i)$ and call it the i-th *syzygy of* M.

8.16 Lemma The isomorphism class of $\mathrm{syz}_i(M)$ depends only on the isomorphism class of M.

8.17 Lemma ([Eva73], Proposition 1) For noetherian local rings the cancellation property holds for finitely generated modules, i.e.

$$M \oplus M' \cong M \oplus M'' \text{ implies } M' \cong M''$$

for all finitely generated modules M, M' and M''.

8.18 Lemma If M is a B-module (or over a ring with cancellation property) and we write $M \cong M' \oplus B^k$ where M' has no free summand, then the isomorphism class of M' is determined by (the isomorphism class of) M.

Proof. Assume $M \cong M'' \oplus B^l$ is another such decomposition. Then $M' \oplus B^k \cong M'' \oplus B^l$ and by the cancellation property we can cancel $B^{\min(k,l)}$. Assume $k \geq l$, then we get $M' \oplus B^{k-l} \cong M''$ and as M'' has no free summand $k - l = 0$. $\square$

8.19 Lemma If M is a B-module and $[\mathrm{syz}]_i(M) \overset{s}{\cong} \mathfrak{coker}(\varphi, \psi)$ for a fixed choice of $[\mathrm{syz}]_i(M)$, then we can find a matrix factorization (φ', ψ') with $\mathrm{ch}(\varphi, \psi) = \mathrm{ch}(\varphi', \psi')$ and $[\mathrm{syz}]_i(M) \cong \mathfrak{coker}(\varphi', \psi')$. Here ch can be any of our six Chern theories for matrix factorizations.

Proof. $[\mathrm{syz}]_i(M) \overset{s}{\cong} \mathfrak{coker}(\varphi, \psi)$ means, that there are projective B-modules P, P', such that $[\mathrm{syz}]_i(M) \oplus P \cong \mathfrak{coker}(\varphi, \psi) \oplus P'$, but projective modules are free over local rings. By the cancellation property we can cancel the module of smaller rank and hence we are left with

$$[\mathrm{syz}]_i(M) \oplus B^l \cong \mathfrak{coker}(\varphi, \psi) \text{ or } [\mathrm{syz}]_i(M) \cong \mathfrak{coker}(\varphi, \psi) \oplus B^l.$$

In the second case define $(\varphi', \psi') := (\varphi, \psi) \oplus (f, 1)^l$. Then $[\mathrm{syz}]_i(M) \cong \mathfrak{coker}(\varphi', \psi')$ and we did not change the Chern character, because $\mathrm{ch}(f, 1) = 0$ for all of our six Chern theories. In the first case (φ, ψ) is isomorphic as matrix factorization to a sum

$$(\varphi'', \psi'') \oplus (1, f)^k \oplus (f, 1)^m \text{ with } (\varphi'', \psi'') \text{ reduced.}$$

Then we must have $m \geq l$, because the corkernels of the other two matrix factorizations have no free summands. Then we can pick $(\varphi', \psi') = (\varphi'', \psi'') \oplus (1, f)^k \oplus (f, 1)^{(m-l)}$ (we could also alter the exponent of $(1, f)$ arbitrarily). $\square$

8.20 Proposition Let M be a B-module with $[\mathrm{syz}]_i(M) \overset{s}{\cong} \mathfrak{coker}(\varphi, \psi)$, then

$$(-1)^i \, \mathrm{ch}_n(\varphi, \psi)$$

depends only on the isomorphism class of M. Here ch can be any of our two Chern theories for the category $MF(A, f)$.

Proof. One possible choice for $[\mathrm{syz}]_i(M)$ is $\mathrm{syz}_i(M)$ and all choices for $[\mathrm{syz}]_i(M)$ are stably isomorphic. By Lemma 8.19 we can replace $\overset{s}{\cong}$ with $\cong$. Now (φ, ψ) is isomorphic to a sum $(\varphi', \psi') \oplus (1, f)^k \oplus (f, 1)^l$ with (φ', ψ') reduced. Hence we get

$$\mathrm{syz}_i(M) \cong \mathrm{coker}(\varphi') \oplus \mathrm{coker}(1)^k \oplus \mathrm{coker}(f)^l = \mathrm{coker}(\varphi') \oplus B^l$$

and $\mathrm{coker}(\varphi')$ has no free summand. By Lemma 8.18 the isomorphism class of $\mathrm{coker}(\varphi')$ is uniquely determined by the isomorphism class of $\mathrm{syz}_i(M)$. Moreover, for both of our Chern theories for $MF(A, f)$ we have $\mathrm{ch}_n(\varphi, \psi) = \mathrm{ch}_n(\varphi', \psi')$. The right hand side depends only on the isomorphism class of (φ', ψ'), which is determined by the isomorphism class of $\mathrm{coker}(\varphi')$, which is determined by the isomorphism class of $\mathrm{syz}_i(M)$, which is determined by the isomorphism class of M. Now changing i gives precisely the sign $(-1)^i$. This is proved as in Proposition 8.8. $\square$

8.21 Definition Let us define the category $\mathrm{evMF}(A, f)$ to be the full subcategory of $\mathrm{mod}(B)$, which consists of the modules M with $[\mathrm{syz}]_i(M) \in \mathfrak{coker}(A, f)$ for some i. In other words, $\mathrm{evMF}(A, f)$ contains exactly all B-modules, which have a free resolution, which is eventually periodic and the periodic part is given by a matrix factorization.

Note that by Lemma 8.19 the following three conditions are equivalent:

- $[\mathrm{syz}]_i(M) \in \mathfrak{coker}(A, f)$,
- $\mathrm{syz}_i(M) \in \mathfrak{coker}(A, f)$,
- $[\mathrm{syz}]_i(M) \overset{s}{\cong} \mathfrak{coker}(\varphi, \psi)$ for some (φ, ψ).

8.22 Definition As a direct consequence of Proposition 8.20, any of our two Chern theories for $MF(A, f)$ gives us a Chern theory for $\mathrm{evMF}(A, f)$ by setting

$$\mathrm{ch}_n(M) = (-1)^i \, \mathrm{ch}_n(\varphi, \psi),$$

where (φ, ψ) is a factorization with $[\mathrm{syz}]_i(M) \overset{s}{\cong} \mathfrak{coker}(\varphi, \psi)$. We will show in the next section that these two Chern theories are exact additive. In particular the category $\mathrm{evMF}(A, f)$ is exact.

8.3 Additivity over short exact sequences

8.23 Lemma For any morphism $(\alpha, \beta)\colon (\varphi'', \psi'') \to (\varphi', \psi')$ we have the following sequence of matrix factorizations

$$(\varphi', \psi') \xrightarrow{(\mathrm{id} \times 0, \mathrm{id} \times 0)} C_{(\alpha,\beta)} \xrightarrow{(0 \oplus \mathrm{id}, 0 \oplus \mathrm{id})} \Sigma(\varphi'', \psi'').$$

The image of this sequence under $\mathfrak{coker}$ is a short exact sequence of B-modules:

$$0 \to \mathfrak{coker}(\varphi', \psi') \to \mathfrak{coker}(C_{(\alpha,\beta)}) \to \mathfrak{coker}(\Sigma(\varphi'', \psi'')) \to 0.$$

Proof. The three involved matrix factorizations give B-resolutions of their cokernels and lifting the maps $\mathfrak{coker}(\mathrm{id} \times 0, \mathrm{id} \times 0)$ and $\mathfrak{coker}(0 \oplus \mathrm{id}, 0 \oplus \mathrm{id})$ to this resolutions results in a short exact sequence of complexes. The short exact sequence

$$0 \to \mathfrak{coker}(\varphi', \psi') \to \mathfrak{coker}(C_{(\alpha,\beta)}) \to \mathfrak{coker}(\Sigma(\varphi'', \psi'')) \to 0$$

is the end of the corresponding long exact homology sequence and hence exact. □

The converse is also true:

8.24 Lemma For a short exact sequence of B-modules

$$0 \to M' \to M \to M'' \to 0$$

with $M' = \operatorname{coker}(\varphi') = \mathfrak{coker}(\varphi', \psi')$ and $M = \operatorname{coker}(-\psi'') = \mathfrak{coker}(\Sigma(\varphi'', \psi''))$, there is a morphism of matrix factorizations $(\alpha, \beta)\colon (\varphi'', \psi'') \to (\varphi', \psi')$, such that

$$M = \operatorname{coker}\begin{pmatrix} \varphi' & \alpha \\ 0 & -\psi'' \end{pmatrix} = \mathfrak{coker}\left(\begin{pmatrix} \varphi' & \alpha \\ 0 & -\psi'' \end{pmatrix}, \begin{pmatrix} \psi' & \beta \\ 0 & -\varphi'' \end{pmatrix}\right) = \mathfrak{coker}(C_{(\alpha,\beta)}).$$

Proof. The proof is the proof for the horseshoe Lemma, adapted to our two periodic context of matrix factorizations. Let us look at the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & M' \arrow[u] \arrow[r, "f"] & M \arrow[u] \arrow[r, "g"] & M'' \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & A^r \arrow[u, "\varepsilon'"] \arrow[r, "i_0"] \arrow[ur, "\varepsilon_0'"] & A^r \oplus A^{r'} \arrow[u, "\varepsilon"] \arrow[r, "p_0"] & A^{r'} \arrow[u, "\varepsilon''"'] \arrow[ul, "\varepsilon_0''"'] \arrow[r] & 0 \\
0 \arrow[r] & A^r \arrow[u, "\varepsilon'" phantom, ""] \arrow[u, "\varphi'"] \arrow[r, "i_1"] & A^r \oplus A^{r'} \arrow[u, dotted, "\varphi"] \arrow[r, "p_1"] & A^{r'} \arrow[u, "-\psi''"'] \arrow[r] & 0 \\
0 \arrow[r] & A^r \arrow[u, "\psi'"] \arrow[r, "i_2"] & A^r \oplus A^{r'} \arrow[u, dotted, "\psi"] \arrow[r, "p_2"] & A^{r'} \arrow[u, "-\varphi''"'] \arrow[r] & 0
\end{tikzcd}$$

The first and third column are exact at M' and the first A^r respectively M'' and the first $A^{r'}$. i_k and p_k are the canonical injections respectively projections. $\varepsilon_0' = f \circ \varepsilon'$, ε_0'' exists because of the projectivity of $A^{r'}$ and $\varepsilon = \varepsilon_0' \oplus \varepsilon_0''$ (then ε is surjective by the snake-lemma). The squares involving the middle column commute if and only if

$$\varphi = \begin{pmatrix} \varphi' & \alpha \\ 0 & -\psi'' \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi' & \beta \\ 0 & -\varphi'' \end{pmatrix}$$

for arbitrary matrices α and β. Furthermore, (φ, ψ) is a matrix factorization with $\operatorname{coker}(\varphi)$ isomorphic to M if and only if the following three equations hold:

$$\operatorname{im}(\varphi) = \ker(\varepsilon), \quad \varphi\psi = f \operatorname{id} = \psi\varphi.$$

The first is equivalent (again with the snake-lemma) to $\operatorname{im}(\varphi) \subset \ker(\varepsilon)$, which is the same as $f \circ \varepsilon' \circ \alpha = \varepsilon_0'' \circ \psi''$. There is an α satisfying this condition by the projectivity of $A^{r'}$, since $f \circ \varepsilon'$ maps surjectively onto $\ker(g) \supset \operatorname{im}(\varepsilon''_0 \circ \psi'')$.

$$\begin{tikzcd}
0 & & \\
M' \arrow[u] \arrow[r, "f"] & M \arrow[r, "g"] & M'' \\
A^r \arrow[u, "\varepsilon'"] \arrow[ur, "\varepsilon'_0"] & & A^{r'} \arrow[u, "\varepsilon''"'] \arrow[ul, "\varepsilon''_0"'] \\
& & A^{r'} \arrow[u, "-\psi''"] \arrow[ull, dotted, "\alpha"]
\end{tikzcd}$$

The second and third equation translate to (by looking at the top right entry)

$$\varphi' \circ \beta + \alpha \circ (-\varphi'') = 0 = \psi' \circ \alpha + \beta \circ (-\psi''),$$

which are precisely the conditions for a morphism of matrix factorizations. Now again by the projectivity of $A^{r'}$ and since φ' maps surjectively onto $\ker(\varepsilon') \supset \operatorname{im}(\alpha \circ (-\varphi''))$, there exists a β satisfying the left equality.

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \\
& A^r \arrow[u, "\varepsilon'"] \arrow[ur, "\varepsilon'_0"] & & A^{r'} \arrow[u, "\varepsilon''"'] \arrow[ul, "\varepsilon''_0"'] \\
& A^r \arrow[u, "\varphi'"] & & A^{r'} \arrow[u, "-\psi''"] \arrow[ull, "\alpha"] \\
& & & A^{r'} \arrow[u, "-\varphi''"] \arrow[ull, dotted, "-\beta"]
\end{tikzcd}$$

The right equality follows from the left, by Lemma 4.4. $\square$

8.25 Lemma For $i \geq 1$ and a short exact sequence of B-modules

$$0 \to M' \to M \to M'' \to 0$$

and any choice of $[\mathrm{syz}]_i(M')$ and $[\mathrm{syz}]_i(M'')$ it is possible to choose $[\mathrm{syz}]_i(M)$, such that the sequence

$$0 \to [\mathrm{syz}]_i(M') \to [\mathrm{syz}]_i(M) \to [\mathrm{syz}]_i(M'') \to 0$$

is still exact (the maps come from the lifts of the map $M' \to M$ respectively $M \to M''$ to the resolution, in which $[\mathrm{syz}]_i(M)$, etc. appear as kernels).

Proof. It is enough to prove the case $i = 1$. A choice of $[\mathrm{syz}]_1(M')$, $[\mathrm{syz}]_1(M'')$ is the choice of a surjection of a free B-module $F' \to M'$ and $F'' \to M''$ and as in the proof before (or as in the horseshoe lemma) we get a fitting surjection $F = F' \oplus F'' \to M$ and a commutative diagram with all columns and the first two rows exact:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & M' \arrow[u] \arrow[r] & M \arrow[u] \arrow[r] & M'' \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & F' \arrow[u, "\varepsilon'"] \arrow[r, "\iota"] & F \arrow[u, "\varepsilon"] \arrow[r, "\pi"] & F'' \arrow[u, "\varepsilon''"'] \arrow[r] & 0 \\
0 \arrow[r] & {[\mathrm{syz}]_1(M')} \arrow[u, "i'"] \arrow[r] & \ker(\varepsilon) \arrow[u, "i"] \arrow[r] & {[\mathrm{syz}]_1(M'')} \arrow[u, "i''"] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Now the last row is exact by the snake Lemma and $\ker(\varepsilon)$ is a possible choice for $[\mathrm{syz}]_1(M)$. $\square$

8.26 Example Set $A = \mathbb{C}[\![X, Y]\!]$, $f = XY$, $B = A/f$ and look at the two reduced matrix factorization (X, Y) and $\Sigma(X, Y) = (-Y, -X)$. We have the following exact sequence of B-modules:

$$0 \to B/X = \mathfrak{coker}(X, Y) \xrightarrow{\cdot Y} B \to B/Y = \mathfrak{coker}(-Y, -X) \to 0.$$

Then by the construction in Lemma 8.24 we get $\left(\begin{pmatrix} X & -1 \\ 0 & -Y \end{pmatrix}, \begin{pmatrix} Y & -1 \\ 0 & -X \end{pmatrix}\right)$ as the matrix factorization in the middle, which is isomorphic to $\left(\begin{pmatrix} 1 & 0 \\ 0 & XY \end{pmatrix}, \begin{pmatrix} XY & 0 \\ 0 & 1 \end{pmatrix}\right)$. In particular the matrix factorization in the middle does not need to be reduced, even if the other two are. Furthermore, the sequence $0 \to \mathrm{syz}_i(B/X) \to \mathrm{syz}_i(B) \to \mathrm{syz}_i(B/Y) \to 0$ is not exact for any $i > 0$, because $\mathrm{syz}_i(B)$ is zero and the other two are not, i.e. in Lemma 8.25 does not work with syz instead of [syz].

In the next Corollary ch can be any of the six Chern theories defined in Section 8.1 and 8.2. For the two from Section 8.2 the ring A (and hence B) has to be local. There is a functor $F\colon \mathrm{mod}(B) \to \underline{\mathrm{mod}}(B)$, which is the identity on objects and sends a morphism to its class. The objects from $\mathrm{evMF}(A, f)$ are mapped to objects of $\underline{\mathrm{evMF}}(A, f)$. For $M \in \mathrm{evMF}(A, f)$ and ch a Chern theory from Section 8.1 we denote $\mathrm{ch}(F(M))$ by $\mathrm{ch}(M)$.

8.27 Corollary If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of B-modules with $M', M'' \in \mathrm{evMF}(A, f)$, then $M \in \mathrm{evMF}(A, f)$ and we have $\mathrm{ch}(M) = \mathrm{ch}(M') + \mathrm{ch}(M'')$.

Proof. Pick i big enough and $[\mathrm{syz}]_i(M')$, $[\mathrm{syz}]_i(M'')$, such that

$$[\mathrm{syz}]_i(M') \cong \mathfrak{coker}(\varphi', \psi'), \quad [\mathrm{syz}]_i(M'') \cong \mathfrak{coker}(-\psi'', -\varphi'') = \mathfrak{coker}(\Sigma(\varphi'', \psi'')).$$

Now use Lemma 8.25 to pick $[\mathrm{syz}]_i$ fitting in an exact sequence. Then apply Lemma 8.24 to get $[\mathrm{syz}]_i \cong \mathfrak{coker}(C_{(\alpha,\beta)})$ for some morphism (α, β). Then $M \in \mathrm{evMF}(A, f)$ and

$$\mathrm{ch}(M) = (-1)^i \,\mathrm{ch}(C_{(\alpha,\beta)}) = (-1)^i \,\mathrm{ch}(\varphi', \psi') + (-1)^i \,\mathrm{ch}(\Sigma(\varphi'', \psi'')) = \mathrm{ch}(M') + \mathrm{ch}(M'')$$

by additivity of the Chern theory for matrix factorizations over cones. $\square$

8.28 Definition A subcategory is called *strictly full*, if it is full and closed under isomorphisms.

8.29 Corollary Assume A is local. The category $\mathrm{evMF}(A, f)$ is exact and the two Chern theories from Definition 8.22 are exact additive.

Proof. A strictly full additive subcategory of abelian category, which is closed under extensions, is exact, with the exact structure coming from the exact sequences in the abelian category. The category $\mathrm{evMF}(A, f)$ is a strictly full additive subcategory of the abelian category $\mathrm{mod}(B)$. Furthermore, it is closed under extensions by Corollary 8.27. Therefore, $\mathrm{evMF}(A, f)$ is exact and our two Chern theories are exact additive again by Corollary 8.27. $\square$

8.4 Compatibility with the tensor product

In this section we want to exhibit the properties under tensor products. Unfortunately we rather have to give negative results in the sense, that we do not know much about it.

The category $[MF(A, f)]$ is a triangulated, pseudo tensor category and equivalent to $[\mathfrak{coker}](A, f)$, therefore the later is also a triangulated, pseudo tensor category with the induced structures from $[MF(A, f)]$. Unfortunately, this induced tensor product on $[\mathfrak{coker}](A, f)$ has (to our knowledge) nothing to do with known tensor product for modules and we are not sure how to describe it directly. The tensor product for modules does not make sense in the stable category, since B^l is stably isomorphic to 0, but $M \otimes_B B^l \cong M^l$ is not stably isomorphic to $0 = M \otimes_B 0$ in general.

8.30 Example Let $F = B^l$ be a free module, then $F \cong [\mathfrak{coker}]((f, 1)^{\oplus l})$. Furthermore, let $M = \mathfrak{coker}(\varphi, \psi)$ another B-module. Then for modules we have

$$\mathrm{ch}(M \otimes_B F) = \mathrm{ch}(M^{\oplus l}) = l\,\mathrm{ch}(M).$$

On the other hand one can verify, that $(\varphi, \psi) \otimes_{\mathrm{MF}}^{\frac{1}{2}} (f, 1)^{\oplus l}$ is isomorphic as matrix factorization to $(\mathrm{id}_{rl}, f\,\mathrm{id}_{rl}), (f\,\mathrm{id}_{rl}, \mathrm{id}_{rl})$, which is mapped to $B^{rl} \overset{s}{\cong} 0$ by $[\mathfrak{coker}]$.

9 Chern theory for periodic complexes

In this chapter k is a commutative ring and A is an associative, unital, commutative k-algebra and Ω^n will mean $\Omega^n_{A/k}$.

The main ingredient in the proof that the Chern theory $[\mathrm{ch}]_n^{\text{HH, odd}}$ from Definition 5.4 is constant on homotopy equivalence classes was Lemma 5.3. In this Lemma we worked with matrix factorizations as 2-periodic complexes. The proof worked for any period length and did not require the complex to be exact. Hence we get a Chern theory for periodic complexes (of fixed period length), which is invariant under periodic homotopy equivalences of the same period length. Let us make this more precise.

9.1 Definition Let us denote with $\mathrm{per}_m(A)$ the category, which has as objects m-periodic complexes of finitely generated free A-modules and as morphisms m-periodic chain maps. Furthermore, we denote with $[\mathrm{per}]_m(A)$ the corresponding homotopy category.

For the rest of this section we will write complex instead of m-periodic complex of finitely generated, free A-modules and the same for resolutions. Lemma 5.3 implies:

9.2 Lemma For a complex $C_\bullet$ with differentials $\delta_m, \ldots, \delta_1$ the value of

$$(\delta_m, \ldots, \delta_1)^n \in HH_{mn-1}(M(A))$$

depends only on the isomorphism class of $C_\bullet$ in $[\mathrm{per}]_m(A)$ or, in other words, is invariant under m-periodic homotopy equivalences.

Note this value is zero if m is odd, n is even and characteristic not 2, because

$$b(\mathrm{id}, (\delta_m, \ldots, \delta_1)^n) = (\delta_m, \ldots, \delta_1)^n + (-1)^{nm}(\delta_1, \delta_m, \ldots, \delta_2)^n.$$

Now we can define 4 different Chern theories for the category $[\mathrm{per}]_m(A)$ as for the category $[MF(A, f)]$. The n-th Chern character will then be given by

$$\begin{aligned}
&\mathrm{tr}((\delta_m, \ldots, \delta_1)^n) \in HH_{mn-1}(A),\\
&\overline{B}_{nm-1}(\mathrm{tr}((\delta_m, \ldots, \delta_1)^n)) = n\sum_{i=1}^{m}(-1)^{i-1}\,\mathrm{tr}(1, (\delta_{i-1}, \ldots, \delta_1, \delta_m, \ldots, \delta_i) \in HH_{mn}(A),\\
&\pi_{nm-1}(\mathrm{tr}((\delta_m, \ldots, \delta_1)^n)) = \mathrm{Tr}((\delta_m d\delta_{m-1} \ldots d\delta_1(d\delta_m d\delta_{m-1} \ldots d\delta_1)^{n-1}) \in \Omega^{mn-1},\\
&d(\mathrm{Tr}((\delta_m d\delta_{m-1} \ldots d\delta_1(d\delta_m d\delta_{m-1} \ldots d\delta_1)^{n-1})) = \mathrm{Tr}((d\delta_m d\delta_{m-1} \ldots d\delta_1)^n) \in \Omega^{mn}.
\end{aligned}$$

Here we have left out any normalization factors, because those were needed for the behaviour under tensor products and we will not talk about this behaviour for these invariants. These Chern theories will be additive over mapping cones of m-periodic chain maps, with the same proof as for the 2-periodic case.

Unfortunately, we do not know any interesting example, which does not come from a matrix factorization, where those invariants are not all zero. An interesting example could be a minimal free resolution with period length > 2. In the second part of Example 4.12 we had a minimal free resolution of arbitrary period length m, but all invariants in Ω^{nm-1} respectively Ω^{nm} will be zero, unless $m = n = 1$. We do not know if the invariants in the Hochschild homology are zero or not.

Let us finish with three examples which show that the condition for the chain maps to have the same period length as the complex is necessary, even when working with minimal resolutions of the same module. In particular we do not get a useful Chern theory for modules with eventually periodic minimal resolutions (and the periodic part does not come from a matrix factorization).

9.3 Example 1. If $\delta_m, \ldots, \delta_1$ are the differentials of a minimal resolution, then we get another resolution by multiplying the differentials with arbitrary invertible $a_m, \ldots a_1 \in A$. We get the following isomorphisms of complexes:

$$\begin{tikzcd}
C_0 = C_m \arrow[r, "\delta_m"] \arrow[d, "\frac{1}{a}"] & C_{m-1} \arrow[r, "\delta_{m-1}"] \arrow[d, "\mathrm{id}"] & \ldots \arrow[r, "\delta_2"] & C_1 \arrow[r, "\delta_1"] \arrow[d, "\mathrm{id}"] & C_0 \arrow[d, "\mathrm{id}"] \\
C_0 = C_m \arrow[r, "a\delta_m"] \arrow[d, "a"] & C_{m-1} \arrow[r, "\delta_{m-1}"] \arrow[d, "a_m"] & \ldots \arrow[r, "\delta_2"] & C_1 \arrow[r, "\delta_1"] \arrow[d, "\frac{a}{a_1}"] & C_0 \arrow[d, "a"] \\
C_0 = C_m \arrow[r, "a_m\delta_m"] & C_{m-1} \arrow[r, "a_{m-1}\delta_{m-1}"] & \ldots \arrow[r, "a_2\delta_2"] & C_1 \arrow[r, "a_1\delta_1"] & C_0,
\end{tikzcd}$$

where $a = a_m \ldots a_1$ and the first chain isomorphism is not periodic, but the second is. Furthermore, there is no periodic chain homotopy equivalence between the first two rows in general, because their Chern characters will vary by a power of a. One could call such a map periodic up to a factor, because we can continue it in this manner with increasing powers of $\frac{1}{a}$. The next example shows, that more can happen, than just invertible factors.

2. $A = \mathbb{C}[\![X,Y]\!]/(X^2 - Y^2)$ and define

$$\varphi := \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}.$$

We can complete φ with either of the following matrices to a minimal two periodic free resolution of $\operatorname{coker}(\varphi)$

$$\psi_1 = \begin{pmatrix} X & -Y \\ -Y & X \end{pmatrix}, \ \psi_2 = \begin{pmatrix} -Y & X \\ X & -Y \end{pmatrix}.$$

(φ, ψ_1) gives a matrix factorization, while (φ, ψ_2) does not. We compute

$$\mathrm{Tr}(\varphi d\psi_1) = 2(XdX - YdY) = 0 \in \Omega^1_{A/k},$$
$$\mathrm{Tr}(\varphi d\psi_2) = 2(YdX - XdY) \neq 0 \in \Omega^1_{A/k}.$$

This shows, that in general these traces can differ by more than a unit constant. There is a 4-periodic isomorphism of complexes between these two different 2-periodic complexes

$$\begin{tikzcd}
A^2 \arrow[r, "\left(\begin{smallmatrix} X & Y \\ Y & X \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)"] & A^2 \arrow[r, "\left(\begin{smallmatrix} X & -Y \\ -Y & X \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)"] & A^2 \arrow[r, "\left(\begin{smallmatrix} X & Y \\ Y & X \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] & A^2 \arrow[r, "\left(\begin{smallmatrix} X & -Y \\ -Y & X \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] & A^2 \arrow[r, "\left(\begin{smallmatrix} X & Y \\ Y & X \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)"] & A^2 \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)"] \\
A^2 \arrow[r, "\left(\begin{smallmatrix} X & Y \\ Y & X \end{smallmatrix}\right)"'] & A^2 \arrow[r, "\left(\begin{smallmatrix} -Y & X \\ X & -Y \end{smallmatrix}\right)"'] & A^2 \arrow[r, "\left(\begin{smallmatrix} X & Y \\ Y & X \end{smallmatrix}\right)"'] & A^2 \arrow[r, "\left(\begin{smallmatrix} -Y & X \\ X & -Y \end{smallmatrix}\right)"'] & A^2 \arrow[r, "\left(\begin{smallmatrix} X & Y \\ Y & X \end{smallmatrix}\right)"'] & A^2,
\end{tikzcd}$$

but the inequality of the two traces above implies there is no 2-periodic homotopy equivalence between them.

3. Let $A, \varphi, \psi_1, \psi_2$ as in the previous example. Furthermore, we define

$$M = \mathrm{coker}(\varphi) = A^2 / \langle \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right), \left(\begin{smallmatrix} Y \\ X \end{smallmatrix}\right) \rangle_A.$$

Then we have the following short exact sequence of A-modules

$$0 \to M \xrightarrow{g} A^2 \xrightarrow{h} M \to 0,$$

where h is the canonical projection onto quotient and g is defined by

$$g(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)) = \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right), \quad g(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)) = \left(\begin{smallmatrix} -Y \\ X \end{smallmatrix}\right).$$

Now we pick the two periodic resolution given by (φ, ψ_1) for the first copy of M and (φ, ψ_2) for the second. Then the 4-periodic free resolution of A^2 with differentials given by

$$\begin{pmatrix} \varphi & \alpha \\ 0 & \varphi \end{pmatrix}, \begin{pmatrix} \psi_1 & \beta \\ 0 & \psi_2 \end{pmatrix}, \begin{pmatrix} \varphi & \gamma \\ 0 & \varphi \end{pmatrix}, \begin{pmatrix} \psi_1 & \delta \\ 0 & \psi_2 \end{pmatrix}, \text{ where}$$
$$\alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \delta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

gives a short exact sequence of resolutions. These are all possible choices in degree 0 for $\alpha, \beta, \gamma, \delta$ to get 4-periodicity. In particular there is no possible choice to make it 2-periodic.

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