Hypersurfaces with Many Singularities

History – Constructions – Algorithms – Visualization

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Abstract

Part 1: Known Constructions. About 40 pages of historical survey of the classical subject of hypersurfaces with many singularities constitute the first part of the present work. The maximum number $\mu^n(d)$ of singularities on a hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$ is known in very few cases only, e.g. in $\mathbb{P}^3(\mathbb{C})$ for $d \leq 6$. Apart from such exceptions, there only exist upper and lower bounds.

We hope that this overview will not only serve as an introductory text and a guide to the literature, but that it will also give the reader some new ideas and references to interesting articles which might serve as a starting point for further research. To make this easier, we do in fact not only summarize known results, but we also give some direct generalizations and concrete examples which have not been considered so far.

Part 2: New Constructions and Algorithms. The main part of this thesis is devoted to new constructions. First, we prove the existence of hypersurfaces of any given degree d in $\mathbb{P}^n$ with many A_j-singularities based on the theory of dessins d'enfants (chapter 5). This yields new asymptotic lower bounds for the maximum number of such singularities in most cases. Our construction is a variant of the well-known construction of Chmutov from 1992. In the real case, we are able to prove an upper bound which shows that a real variant of Chmutov's construction is in some sense asymptotically the best possible one.

In low degree, it is usually possible to obtain better results than those given by the general constructions and upper bounds. As described in the historical survey, all known constructions use nice geometrical arguments and symmetry to reduce the problem at hand to a solvable one. In this thesis, we give several algorithmic approaches which do either work without such an intuition or use experiments over prime fields which replace the intuition. Our method which uses the geometry of prime field experiments allows us to construct a septic in $\mathbb{P}^3$ with 99 real nodes in chapter 8 which improves Chmutov's record, 93. But this method still involves human interaction.

We then describe an algorithm which reduces the construction of surfaces of degree $d \leq 7$ with the greatest known number of nodes to a short computer algebra computation. We can even apply it to higher degree: For $d = 9$, we obtain a surface with 226 nodes which also improves Chmutov's current record, 216. This algorithm can certainly be applied to many other concrete problems in algebraic geometry.

Part 3: Visualization. Many interesting examples of the subject are defined over the real numbers. Thus, we are quite often in a position that allows us to use visualization of singular surfaces. For several years there already exists software which produces nice images, e.g. Endraß's SURF. Based on these existing programs we developped some tools allowing a dynamical experience of algebraic curves and surfaces: SPICY, SURFEX, and SURFEX.LIB. We demonstrate their usefulness in the last part of this work. Our example is the construction of nice equations for all 45 topological types of real cubic surfaces in projective three-space which is one of the most classical subjects in algebraic geometry.

Acknowledgements

First of all, I thank my advisor without whose motivation and inspiration the present work would never have existed. In fact, this already applies to the time when I was still writing my diploma thesis: His ability to communicate the fascination of algebraic geometry and in particular of the subject of hypersurfaces with many singularities was one of the main reasons why I decided not to leave university. He always had an open ear for my problems and some good advice during many valuable discussions.

I also thank the authors of the great computer programs SINGULAR and SURF; I use them every day! There are many more people I have to thank. Often an invitation for a talk at a seminar or a conference, or a short chat during a coffee break gave me the motivation and inspiration to work on a problem for several months.

Last, but not least, I thank all the (ex-)members of the algebraic geometry group in Mainz. The great ambiance caused by all these nice and creative people was the basis for the work presented here.

Ein ganz großer Dank geht an meine Eltern und Geschwister. Ohne ihren Rückhalt und ihre Unterstützung während der letzten dreißig Jahre für alles, was ich mir in den Kopf gesetzt habe, würde die vorliegende Arbeit nicht existieren.

Schließlich danke ich meiner Freundin. Für das Aufmuntern, wenn etwas gerade mal nicht so klappen wollte, wie ich es mir gedacht hatte. Und auch für das Mitfreuen, wenn etwas gerade mal besonders gut funktioniert hat. Ohne ihre Motivation und ihren positiven Einfluss stünde in dieser Arbeit sicher nur die Hälfte.

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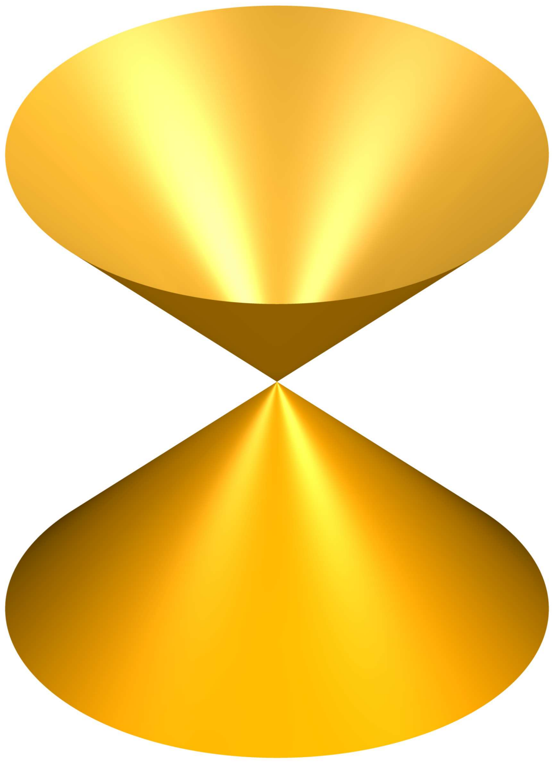


A cone, a quadric surface with the simplest type of singularity: a node, also called ordinary double point or A_1-singularity. How many nodes can a surface of degree d in $\mathbb{P}^3$ have?

Introduction

The Problem

The Most General Question. A generic hypersurface of degree $d \in \mathbb{N}$ in $\mathbb{P}^n := \mathbb{P}^n(\mathbb{C})$ is smooth. Thus, it is natural to ask:

QUESTION 0.1. *Which combinations of singularities can occur on a hypersurface in $\mathbb{P}^n$ of given degree d?*

It is easy to answer this question for $d = 1, 2$. It is obvious that a hyperplane ($d = 1$) cannot have any singularity. It is also easy to classify quadrics ($d = 2$) w.r.t. the singularities occurring on them. E.g., a quadric in $\mathbb{P}^n$ can contain at most one isolated singularity. This can only be an ordinary double point.

In $\mathbb{P}^n$, $n \leq 3$, it is also possible to treat the cases $d = 3, 4$: For the *cubic surfaces* in $\mathbb{P}^3$ all possible combinations of singularities are known since Schläfli's work in 1863 (see section 1.1 on page 13). All possible combinations of singularities on quartic surfaces ($d = 4$) in $\mathbb{P}^3$ are also known; the last remaining open questions have been answered in 1997 using computers (see section 4.8 on page 53).

The Question on the Maximum Number. At the moment, the answer to the previous question seems unreachable if $d \geq 5$ or $n \geq 4$. In the present work, we thus consider the slightly simpler problem:

QUESTION 0.2. *What is the maximum number $\mu^n(d)$ of isolated singularities on a hypersurface of degree d in $\mathbb{P}^n$?*

We have already seen that this is easy if $d = 1, 2$: $\mu^n(1) = 0$ and that $\mu^n(2) = 1$ for all n. On the other hand, the maximum number $\mu^2(d)$ of isolated singularities on a plane curve in $\mathbb{P}^2$ is $\binom{d}{2}$, established by d general lines.

In higher dimensions, there is no such result known yet. In fact, a direct analogue cannot exist in $\mathbb{P}^n$, $n \geq 3$, because in this case a hypersurface with only isolated singularities has to be irreducible. It is also well-known that an irreducible plane curve of degree d with k nodes exists if and only if $0 \leq k \leq \frac{1}{2}(d-1)(d-2)$ (see [**Sev21**, p. 329] for a classical exposition). However, in higher dimensions this question turned out to be a hard one: Despite many efforts, $\mu^3(d)$ is only known for $d \leq 6$ until now. If we ask for the maximum number of singularities of some given type (different from nodes, e.g. cusps), the question is still open in general, even in the case of plane curves we only know the answer for low degrees.

The aim of the present work is to improve the knowledge around the questions above. Our focus is on the geometry and equations of the hypersurfaces and methods for constructing interesting examples. Note that in principal, for each d there is an algorithm which computes the surfaces of degree d with the maximum number of nodes. But this involves very large systems of non-linear equations and can only be performed in special cases. We work out such an example in chapter 7. In more complicated cases, we need other ideas.

Some Notation

Singularities. A point $p \in \mathbb{C}^n$ is called a *singular point* (or *singularity*) of the hypersurface $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ if $f(p) = 0$ and $\frac{\partial f}{\partial x_i}(p) = 0$ for all $i = 1, 2, \ldots, n$. It is called *isolated* if there exists an open neighborhood of p which does not contain any other singular point. This is equivalent to $\dim(\mathbb{C}[x_1, x_2, \ldots, x_n]/(f, J_f)) < \infty$, where $J_f := (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n})$ denotes the *Jacobian ideal*.

Most of the time, we will only deal with a special kind of isolated singularities, so-called *double points*: Let $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ define an isolated hypersurface singularity (also called f) at the origin of $\mathbb{C}^n$. If the *tangent cone* $\operatorname{tc}(f)$ — i.e. the homogeneous part of f of the lowest degree — has degree two then the singularity is called a *double point*.

An *ordinary j-tuple point* in $\mathbb{C}^n$ is an isolated singularity in $\mathbb{C}^n$ which is locally a cone over a smooth hypersurface of degree j in $\mathbb{C}^{n-1}$. An *ordinary double point* is also called *(ordinary) node* or *A_1-singularity.* This is equivalent to the property that the *hessian* — i.e. the determinant of the matrix of second order derivatives of f — does not vanish at the singular point. It is also equivalent to the property that f can be written in the form $x_1^2 + x_2^2 + \cdots + x_n^2$ in some local coordinates at the origin. More generally, f is an *A_j-singularity* if it can be written in the form $x_1^{j+1} + x_2^2 + \cdots + x_n^2$ in some local coordinates at the origin. We call an A_2-singularity an *(ordinary) cusp* and an A_3-singularity a *tacnode*. See, e.g., [**AGZV85a, AGZV85b, Dim87**] for more information on singularities.

The Maximum Numbers of Singularities. The maximum number of isolated singularities of some given type on hypersurfaces in some projective space will appear throughout this work in different situations. To clarify which maximum number we mean, we will use different notations for each of these: Let $d \in \mathbb{N}$, $n \in \mathbb{N}$. Let T be a type of an isolated hypersurface singularity in $\mathbb{C}^n$ (e.g., $T = A_1, A_2, D_4$). Then:

- $\mu^n(d)$ denotes the maximum number of singularities a hypersurface of degree d in $\mathbb{P}^n$ can have.
- $\mu^n_T(d)$ denotes the maximum possible number of singularities of type T on a hypersurface of degree d in $\mathbb{P}^n$ which has only singularities of type T. E.g., $\mu^3_{A_1}(d)$ is the maximum number of nodes which a nodal surface in $\mathbb{P}^3$ can have.
- Many results hold for hypersurfaces of degree d in $\mathbb{P}^n$ with only rational double points as singularities. We thus introduce the notation: $\mu^n_{Dp}(d)$.
- We will need similar notations for other classes of singularities, e.g. $\mu^n_A(d)$ for the maximum possible number of A_j-singularities.
- By $\mu^n_j(d)$ we denote the maximum number of ordinary j-tuple points a hypersurface of degree d in $\mathbb{P}^n$ can have. E.g., $\mu^5_3(d)$ is the maximum number of ordinary 5-tuple points a surface in $\mathbb{P}^3$ can have.

Our main object of study are hypersurfaces in $\mathbb{P}^3$, so we write $\mu(d) := \mu^3(d)$, $\mu_T(d) := \mu^3_T(d)$, etc. for short. For a given hypersurface f in $\mathbb{P}^n$ which has only isolated singularities we use similar notations. E.g., $\mu(f)$ denotes the number of singularities and $\mu_{A_1}(f)$ the number of nodes of f.

As already mentioned, these maximal numbers are only known in very few cases. Thus, upper and lower bounds for them will occur frequently in the main text. There are some obvious inequalities for $n, d \in \mathbb{N}$: $\mu^n(d) \geq \mu^n_A(d) \geq \mu^n_{A_1}(d)$. But notice that it is not known if $\mu^n_{A_1}(d) = \mu^n(d)$. I.e., it is not known if the maximum number of singularities can be achieved with only ordinary double points.

Symmetry. Most of the examples which we will encounter are *symmetric* in the following sense: If a group G acts on $\mathbb{P}^n(\mathbb{C})$ then a hypersurface in $\mathbb{P}^n(\mathbb{C})$ which is given by a homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ is called *G-symmetric* if f is G-invariant, i.e. if $f \in \mathbb{C}[x_0, \ldots, x_n]^G$.

Notice that it is not known if the maximum number of singularities $\mu^n(d)$ on a hypersurface of degree d in $\mathbb{P}^n$ can always be realized by an example which is G-symmetric, where G is not the trivial group. Nevertheless, for studying hypersurfaces with many singularities, we will often have to restrict ourselves to hypersurfaces which are G-symmetric for some non-trivial finite group G.

Main Results

Most of the results presented in this Ph.D. thesis have already appeared as preprints on arXiv.org [**Lab04, Lab05a, Lab05b, BLvS**], some others are already published or accepted for publication [**LvS03, HL05**]. The present work places them in a bigger framework and gives some additional information and results.

The previously unpublished content includes in particular a large historical survey on known constructions and a new algorithm. This algorithm is certainly the most important result of this thesis: It reduces all known constructions of nodal surfaces of degree $d \leq 8$ with the maximum known number of nodes to a computer algebra calculation (see part2, chapter 9), and also yields the new results $\mu(7) \geq 99$, $\mu(9) \geq 226$.

Part 1: Known Constructions. The subject of hypersurfaces with many singularities has a long and rich history which started with the classification of the singular cubic surfaces by Schläfli in 1963. In our opinion, it is necessary to know these developments if one really wants to understand the ideas behind our new constructions which form the main part of our work.

We thus start with a historical overview of the subject. In fact, we go slightly beyond this and give some obvious generalizations and detailed studies in cases in which it seems appropriate to us. E.g., equation (2.9) which follows from Gallarati's generalization of B. Segre's ideas shows that the maximum number $\mu_{A_2}(6)$ of cusps on a sextic is greater or equal to 36 which is a fact that has been overlooked for some time. Another example is our concrete computation of Varchenko's spectral bound in the case of A_j-singularities (section 3.7). This leads to an interpretation of this bound as so-called octahedral numbers in the case $j \geq 2d-1$ (section 4.13).

Part 2: New Constructions and Algorithms. Our main results are contained in the second part of this thesis. Therein, we present some new constructions of hypersurfaces with many singularities which lead to new lower bounds for the maximum number $\mu^n_T(d)$ of singularities of type T on a hypersurface of degree d in $\mathbb{P}^n$ in many cases. In our opinion, the methods used for these constructions are of independent interest themselves because they can certainly be applied in many other situations.

At first sight, our most important result is certainly the construction of a surface in $\mathbb{P}^3$ with 99 nodes (chapter 8) which shows:

$$99 \leq \mu(7) \leq 104.$$

This is the first construction of odd degree $d > 5$ which exceeds the general lower bound given by Chmutov in 1992. After Chmutov's discovery there appeared surfaces with more nodes for $d = 6, 8, 10, 12$. These were found by taking a family of surfaces which depends on some parameters and each of whose members was invariant under some large symmetry group. The symmetry reduced the number of free parameters drastically, and it was possible to determine these using other geometrical arguments.

In large odd degree the only useful symmetry one can impose seems to be dihedral symmetry, i.e. the symmetry of the d-gon. But this kind of symmetry is essentially two-dimensional and thus leaves us with many parameters. The best way to solve this problem seems to guess some additional geometric properties of the hopefully existing surface with many singularities — but how? Our idea is to use experiments over prime fields to get these ideas. Based on these additional geometrical properties, it is then not very difficult to use computer algebra to eliminate all free parameters.

In some cases, it is even possible to solve the problem completely algorithmically. Either by directly working in characteristic zero and using elimination and primary decomposition (chapter 7), or by lifting the prime field parameters to characteristic zero using the chinese remainder theorem together with a rational recovery algorithm (chapter 9). Indeed, we implemented the latter algorithm as a SINGULAR library called SEARCHINFAMILIES.LIB. Using this, it is a triviality to reproduce the constructions of all known records for $\mu_{A_1}(d)$ for $d \leq 7$, even our own one for septics. When applying it to the next interesting case which is $d = 9$ we obtain a nonic with 226 nodes which shows:

$$226 \leq \mu(9) \leq 246.$$

Our algorithm is very general so that it can certainly be applied to many other concrete problems in algebraic geometry. In our opinion, all this makes the development of this algorithm the most important result of this thesis.

But we do not only describe algorithmic ways to construct some special examples. We also give a general construction of hypersurfaces in $\mathbb{P}^n$ with many A_j-singularities which does not use computers at all (chapter 5). It is based on Chmutov's well-known construction of nodal hypersurfaces. Our proof uses the so-called Dessins d'Enfants. The numbers of A_j-singularities of our examples exceed the known lower bounds in most cases. E.g. in $\mathbb{P}^3$, we get:

$$\mu_{A_j}(d) \gtrapprox \frac{3j+2}{6j(j+1)} d^3, \quad j \geq 2.$$

In $\mathbb{P}^n, n \geq 5$, our examples even improve the lower bounds in the nodal case slightly.

We then make a short excursus to the world of real algebraic geometry (chapter 6). We use a relation to the theory of real line arrangements to show that the numbers of nodes of Breske's real variants of Chmutov's surfaces are in some sense asymptotically the largest possible ones. This confirms a conjecture of Chmutov in the special case of real line arrangements.

Summarizing, we get table 0.1 on the facing page which gives the best known lower and upper bounds for the maximum number $\mu_{A_j}(d)$ of A_j-singularities on a surface of degree d for $j = 1, 2, 3, 4$.

We mark those cases in bold in which our constructions improve (to our knowledge) the previously known lower bounds. For $j \geq 2$ and $d \geq 5$, all best known lower bounds are either attained by our examples from chapter 5 or by Gallarati's generalization of B. Segre's idea which we work out in detail in section 2.5. The

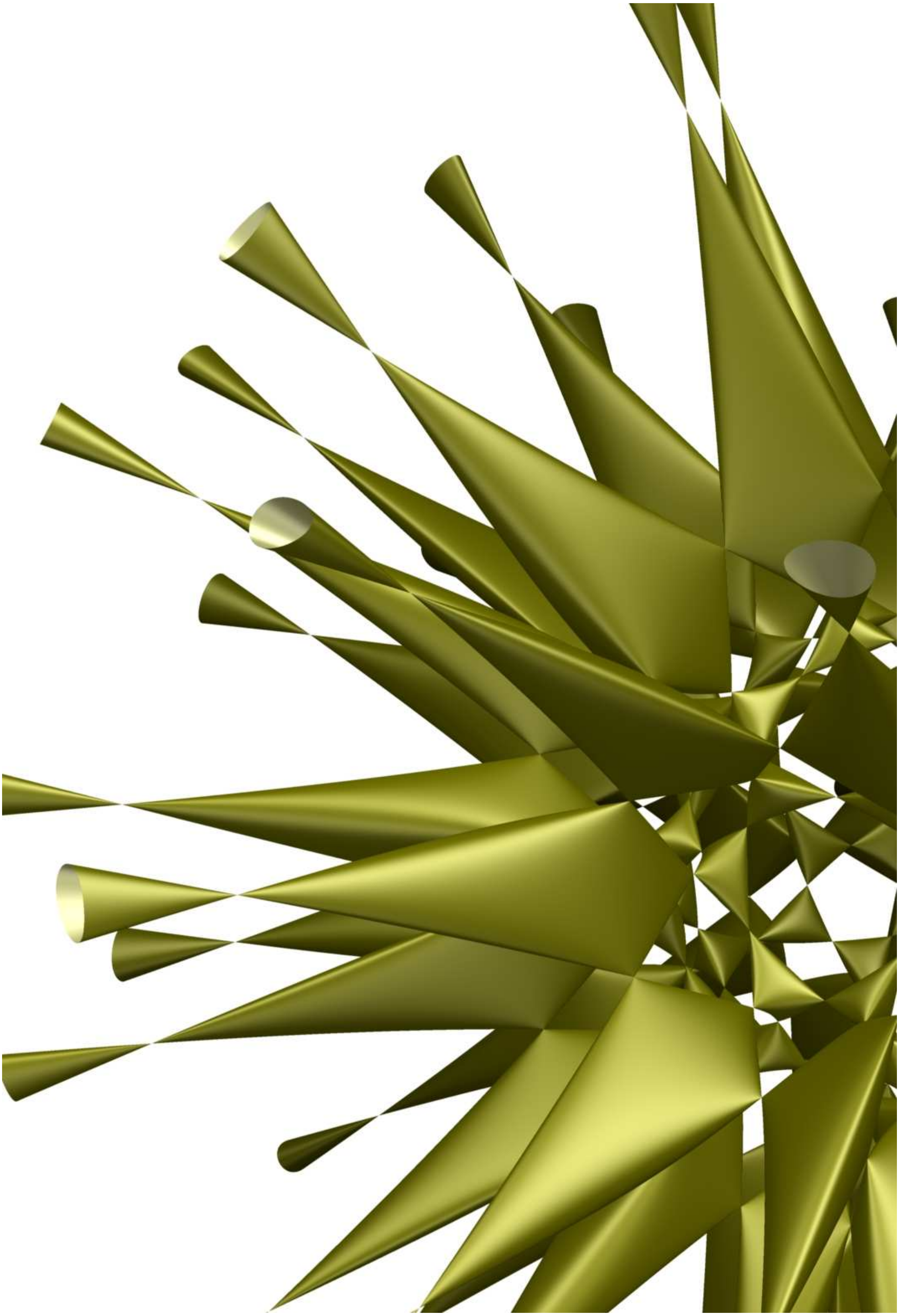
j \ d	3	4	5	6	7	8	9	10	11	12	d
1	4 / 4	16 / 16	31 / 31	65 / 65	**99** / 104	168 / 174	**226** / 246	345 / 360	425 / 480	600 / 645	$\approx$ 5/12 / 4/9 $\cdot d^3$
2	3 / 3	8 / 8	15 / 20	36 / 37	**52** / 62	**70** / 98	**126** / 144	**159** / 202	**225** / 275	**300** / 363	$\approx$ **2/9** / 1/4 $\cdot d^3$
3	1 / 1	6 / 6	10 / 13	15 / 26	**31** / 44	64 / 69	**72** / 102	**114** / 144	**140** / 195	**198** / 258	$\approx$ **11/72** / 8/45 $\cdot d^3$
4	1 / 1	4 / 4	10 / 11	15 / 20	21 / 35	**32** / 54	**54** / 80	100 / 112	**110** / 152	**132** / 201	$\approx$ **7/60** / 5/36 $\cdot d^3$

TABLE 0.1. An overview of our main results. The bold numbers indicate the cases in which the present work improves the previously best known lower bounds.

constructions for the other surfaces reaching the best known lower bounds in the nodal case (i.e., $j = 1$) are briefly described in our historical survey (part 1).

Part 3: Visualization. If a surface with many singularities is defined over the reals then it is sometimes nice to have a picture of it. But this is not the only reason why one would like to have good visualizations of singular surfaces: In the last part of this thesis we show how to use our visualization tools SPICY and SURFEX to construct good equations for all 45 topological types of real cubic surfaces with only rational double points. Furthermore, in many cases visualization is a very good tool to understand the geometry of some constructions in an intuitive way. And this can help to construct new intesting examples based on these known ones.

All pictures of algebraic surfaces in this thesis were produced using our SINGULAR library SURFEX.LIB. This is a SINGULAR interface for our tool SURFEX which also adds some features, e.g. the ability to draw one-dimensional real parts of surfaces which are not contained in the real two-dimensional component.



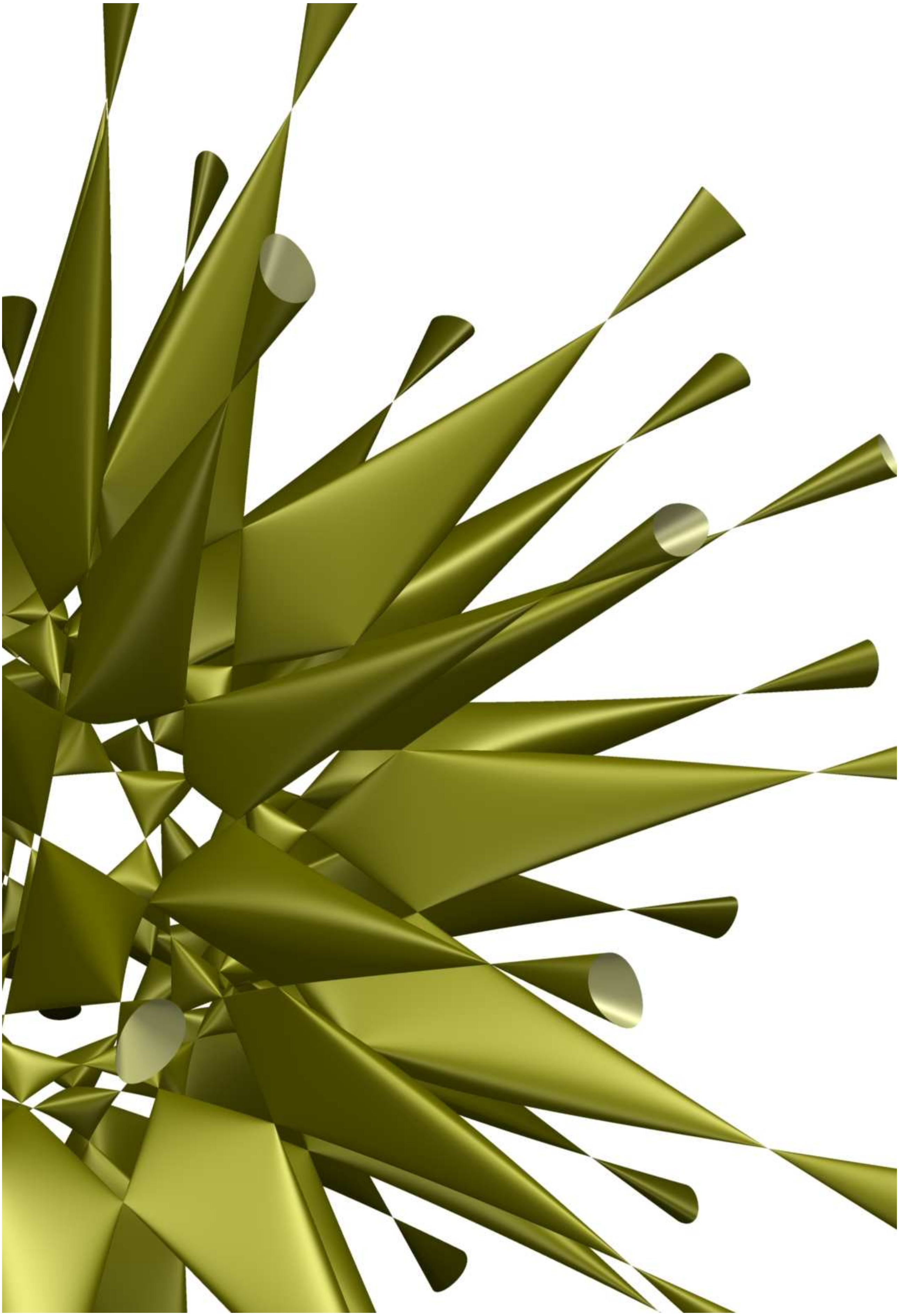


Figure on the preceding pages: Barth's 345-nodal icosahedral-symmetric dectic from 1996. Like his famous 65-nodal sextic, Barth constructed it by studying a one-parameter family of symmetric surfaces. See [**Lab03a**] for more images and movies of algebraic surfaces.

Part 1

Known Constructions

Introduction

In this historical overview, we present the work on the question on the maximum number of singularities on a hypersurface of degree d in $\mathbb{P}^n := \mathbb{P}^n(\mathbb{C})$ which has been done before the appearance of the present work. We try to mention all major results on the subject. It is clear that we cannot go into the details at many places. In view of our main results contained in the other parts of this thesis, our focus will be on the geometry and equations of the hypersurfaces.

Some very brief survey articles have already appeared on surfaces with many singularities (e.g., [**Tog50**], [**Gal84**], [**End95**]). Ours aims to be a bit more exhaustive in two senses: First, we do not only mention very few important results; second, we do not only summarize the ideas, but we also give some natural generalizations and concrete examples. An example is our concrete computation of Varchenko's spectral bound in the case of A_j-singularities (section 3.7). This leads to an interpretation of this bound as so-called octahedral numbers in the case $j \geq 2d-1$ (section 4.13).

Another aim of this survey is to give geometers who want to construct new examples of hypersurfaces with many isolated singularities a kind of encyclopedia at hand which one can use to get new ideas or to combine and improve old ones. At the same time, it can serve as a guide to the literature which tries to be as complete as possible. Beside this, we want to point out some of the interesting historical developments by presenting this overview in (more or less) chronical order and by indicating the relations between the constructions as often as possible.

Our summary is devided into four parts each of which starts with a short introduction. This might be particularly helpful for an impatient reader who just wants to get a very short overview. Finally, we want to mention that the large number of papers on the subject in van Straten's library and one of his unpublished notes have proven to be quite useful as a starting point for our work.

A 16-nodal Kummer surface. In 1864, Kummer noticed that Fresnel's wave surface had 16 nodes and that this was indeed the maximum possible number of nodes on a quartic surface in $\mathbb{P}^3$.

CHAPTER 1

The Important First Steps (until 1915)

After the trivial cases of degree $d \leq 2$, the first interesting case is the one of surfaces of degree three, the so-called *cubic surfaces.* These were already classified with respect to the singularities occuring on them in 1863 by Schläfli. Only one year later, Kummer noticed that the maximum number of isolated singularities on a quartic was 16.

In the following years, several interesting constructions and upper bounds appeared including Rohn's construction of surfaces of degree d with $\approx \frac{1}{4}d^3$ nodes and Basset's upper bound $\mu_{Dp}(d) \lessapprox \frac{1}{2}d^3$ for the maximum number of double points on a surface of degree d. Also, the first nodal hypersurfaces in higher dimensions showed up, but mainly as a tool for understanding surfaces in $\mathbb{P}^3$ in a better way.

1.1. Cubic Surfaces

One of the first major achievements on algebraic surfaces was Cayley's and Salmon's observation in 1849 that a smooth cubic surface contains lines exactly 27 lines [**Cay49**]. In fact, they also noticed [**Sal49b**] that there are still 27 lines when certain singularities occur if the lines are counted with the correct multiplicity. The automorphism group of the configuration of the 27 lines contains the simple group of order 25920 as an index two normal subgroup. This configuration and the group played an important role in the development of group theory until the end of the 19^{th} century. See, e.g., Dickson's book [**Dic01**, chapter XIV, p. 292-298].

1.1.1. Schläfli's Classification. Shortly after this discovery, Schläfli presented the classification with respect to the singularities and the reality of the lines [**Sch63**] (see also [**Sch58**] and [**Cay69**]). This very explicit article also contains many (projective) equations, e.g. of the four-nodal cubic surface

$$\text{Cay}_3 := \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 0 \tag{1.1}$$

which is nowadays often called *Cayley Cubic* (fig. 1.1 on the next page). To our knowledge, it is not clear who first discovered its existence, but Cayley was certainly one of the first to know it. Any four-nodal cubic is projectively equivalent to this one. Another nice equation of this cubic is the following (compare also (1.5)):

$$\text{Cay}_3 : \quad x_0^3 + x_1^3 + x_2^3 + x_3^3 + \frac{1}{4}x_4^3 = 0, \quad x_0 + x_1 + x_2 + x_3 + x_4 = 0. \tag{1.2}$$

In chapter 12 on page 143 we give explicit affine equations and images for all real topological types of cubic surfaces.

The *class* $d^*(f)$ of a surface f of degree d is the number of tangency points f has with a generic pencil of hyperplanes (see e.g., [**BW79**, section 3]). This number is also the degree of the dual surface f^* of f. A smooth surface of degree d has class $d(d-1)^2$. In the times of Schläfli's work mentioned above, it was well-known

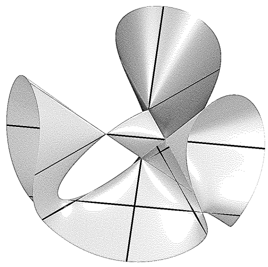


FIGURE 1.1. The four-nodal Cayley Cubic with the affine equation: $4(x^3+3x^2-3xy^2+3y^2+\frac{1}{2})+3(x^2+y^2)(z-6)-z(3+4z+7z^2)$. It contains exactly three lines of multiplicity one and six lines of multiplicity four.

(apparently due to Salmon [**Sal47**], [**Sal49a**], using results of Poncelet [**Pon29**, §93], see also [**Sal80**]) that each singularity of type A_j of f diminishes the class by $j+1 \geq 2$ which gives: $d^*(f) \leq d(d-1)^2 - 2\mu_A(f)$, where $\mu_A(f)$ denotes the number of A_j-singularities of f. It was also well-known that for a surface of degree $d \geq 3$ we have $d^*(f) \geq 3$. This yields:

$$\mu_A(d) \leq \frac{1}{2}\left(d(d-1)^2 - 3\right). \tag{1.3}$$

d	1	2	3	4	5	6	7	8	9	10	11	12	d
$\mu_A(d) \leq$	0	1	4	16	38	73	124	194	286	403	548	724	$\approx \frac{1}{2}d^3$

Together with the existence of the four-nodal cubic (1.1) we get:

$$\mu(3) = \mu_A(3) = \mu_{A_1}(3) = \mathbf{4}. \tag{1.4}$$

Knowing that a cusp (i.e., an A_2-singularity) reduces the class by 3, the preceding bound can be used to show that the maximum number of cusps is 3. For higher singularities this technique is not sufficient. E.g., it does not give any reason for the non-existence of a cubic with an A_8-singularity. In [**Sch63**], Schläfli presents a more detailed study of the geometry of A_j-singularities to show that they only exist on cubic surfaces for $j \leq 5$.

1.1.2. Further Results. There are several other important works on cubic surfaces which also influenced the theory of hypersurfaces with many singularities. E.g., Clebsch's article [**Cle71**] which contains the description of his famous *Diagonal Cubic Surface* in $\mathbb{P}^3$ with 27 real lines, see fig. 1.2 on the next page. It is given by cutting a Σ_5-symmetric hypercubic with a Σ_5-symmetric hyperplane in $\mathbb{P}^4$:

$$\text{Cle}_3: \quad x_0^3+x_1^3+x_2^3+x_3^3+x_4^3 = 0, \quad x_0+x_1+x_2+x_3+x_4 = 0. \tag{1.5}$$

Although this surface is smooth it will appear again in subsequent sections.

Klein's article [**Kle73**] was probably one of the first applications of deformation theory to algebraic surfaces: Starting from a cubic surface with four nodes he constructed the other topological types of cubic surfaces by deformations. For a recent detailed classification of real cubic surfaces with singularities, see [**KM87**].

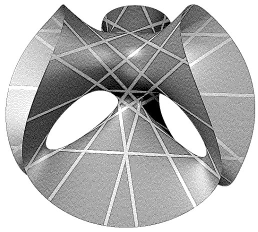


FIGURE 1.2. The Clebsch Diagonal Cubic. We copied the equation for this affine view from a SURF script of S. Endraß. In (1.5) he replaced the x_i by the tetrahedral coordinates $y_0 = 1 - x_2 - \sqrt{2}x_0$, $y_1 = 1 - x_2 + \sqrt{2}x_0$, $y_2 = 1 + x_2 + \sqrt{2}x_1$, $y_3 = 1 + x_2 - \sqrt{2}x_1$.

1.1.3. Models of Surfaces. The algebraic geometers of the 19^{th} century did not only describe abstract properties of cubic surfaces. They were also interested in the intuitive understanding of their geometry. Clebsch was probably the first who suggested to construct a real world (plaster) model of a cubic surface. At his suggestion, Wiener produced such a model of the Clebsch Diagonal Cubic in 1869. Together with some other models, it was presented at several exhibitions in the world. Other well-known series of models were produced by Klein and Rodenberg (see [**Rod04**] and [**Rod79**]). For more recent works concerning real-world models, see [**Fis86, Kae99**] and chapter 11.

1.2. Kummer Quartics

Only one year after Schläfli's classification of the cubic surfaces, Kummer studied quartics with the maximum number of singularities systematically. In [**Kum75a**] he remarked in 1864 that Fresnel's Wave Surface was an algebraic surface of degree 4 containing 16 nodes. This classical surface was discovered in 1819 during Fresnel's studies on crystal optics and his ideas of a wave theory of light (see [**OR**] for more bibliographical information). The equation of Fresnel's Wave Surface presented in [**Sal80**] as an example of a quartic derived from an ellipsoid (see fig. 1.3 on the following page) is:

$$\begin{aligned}\mathrm{Fres}_{a,b,c} \;:=\; & \big((x^2(R_2-b^2)(R_2-c^2)\big) + \big(y^2(R_2-a^2)(R_2-c^2)\big) \\ & +\big(z^2(R_2-a^2)(R_2-b^2)\big) - (R_2-a^2)(R_2-b^2)(R_2-c^2),\end{aligned}$$

where $R_2 := x^2 + y^2 + z^2$ and the constants $a, b, c \in \mathbb{C}$ can be chosen arbitrarily.

Kummer also noticed in [**Kum75a**] that 16 was the maximum possible number of singularities on a quartic — using the reasoning (1.3) based on the upper bound for the class. This showed:

$$\mu(4) = \mu_A(4) = \mu_{A_1}(4) = \mathbf{16}. \tag{1.6}$$

His systematic treatment of all 16-nodal quartics in [**Kum75a**] and [**Kum75b**] is the reason why such surfaces are nowadays called *Kummer Surfaces*. We copied a slightly adapted version of Kummer's equation given in [**Kum75b**] from a SINGULAR script of S. Endraß, see fig. 1.3 for a picture:

$$\mathrm{Ku}_\mu := \left(x^2+y^2+z^2-\mu^2\right)^2 - \lambda\, y_0\, y_1\, y_2\, y_3, \qquad (1.7)$$
$$\lambda = \tfrac{3\mu^2-1}{3-\mu^2},\ \mu \in \mathbb{C},$$

where the y_i are the tetrahedral coordinates already used for Clebsch's Diagonal Cubic in fig. 1.2 on the preceding page. A very nice book on the Kummer quartic was written by Hudson [**Hud90**] a few years later. Another famous monograph on singular quartic surfaces is [**Jes16**].

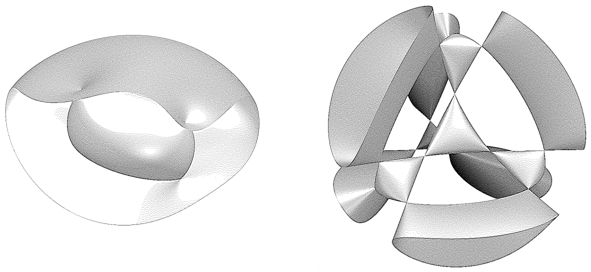


FIGURE 1.3. Fresnel's Wave Surface $\mathrm{Fres}_{1,\frac{3}{10},\frac{1}{2}}$ of 1819 has 16 nodes only four of which are real. Kummer's tetrahedral-symmetric Surface $\mathrm{Ku}_{1.3}$ of 1864, instead, has 16 real nodes.

1.3. Rohn's Construction of Quartics with 8–16 Nodes

In [**Roh86**], Rohn studied quartics with 8–16 nodes in a systematic way by examining the sextic plane curve obtained as the branch locus of the projection of the quartic to a plane. The 16-nodal Kummer quartic corresponds to the case in which the sextics factors into six straight lines.

One of his equations of the 12-nodal quartic is still one of the most important methods for finding surfaces with many singularities as we will see later. Fig. 1.4 on the next page illustrates the idea for curves with A_2-singularities. Rohn's case of a quartic with 12 nodes involved four planes and a smooth quadric instead of lines and circles, see fig. 1.5 on the facing page.

I am indebted to Viat. Kharlamov who informed me of the fact that this idea is contained in Rohn's article (and is probably even older, compare e.g. Schläfli's article [**Sch63**]). All articles of the second half of the 20th century known to us attribute this construction to B. Segre, see section 2.2. A reason for this might be that in his famous book on singular quartic surfaces [**Jes16**], Jessop attributes the systematic treatment of quartics with many nodes to Rohn, but he neither gives a reference to Rohn's article [**Roh86**], nor explains this construction explicitly.

For the more general case of degree d, Rohn's construction (he only discusses the special case $d = 4$) can be described as follows: d general hyperplanes l_i in $\mathbb{P}^3$ intersect in $\binom{d}{2}$ lines which meet a general surface q of degree $\lfloor \frac{d}{2} \rfloor$ in $\binom{d}{2}\lfloor \frac{d}{2} \rfloor \approx \frac{1}{4}d^3$ points. Thus, the surfaces

$$\mathrm{Ro}_d : \prod_{i=1}^{d} l_i - q^2 = 0 \qquad (1.8)$$

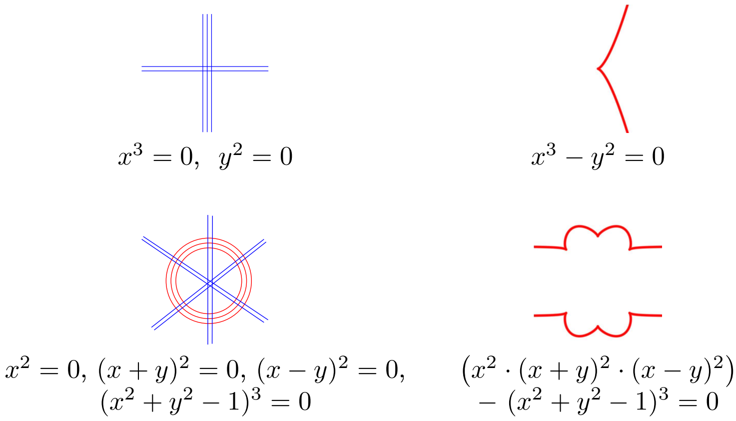


FIGURE 1.4. Globalizing the local equation of a singularity.

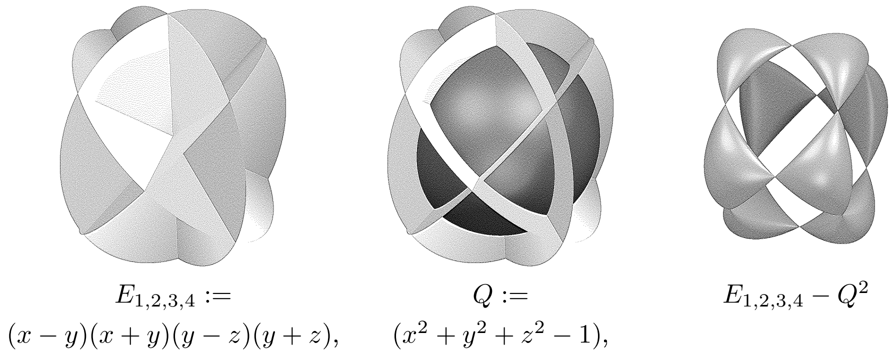


FIGURE 1.5. Rohn's 12-nodal surface [**Roh86**, p. 33] constructed by globalizing the local equation of a node.

have $\binom{d}{2}\lfloor\frac{d}{2}\rfloor = \begin{cases} \frac{1}{4}d^3 - \frac{1}{4}d^2, & d \text{ even} \\ \frac{1}{4}d^3 - \frac{1}{2}d^2 + \frac{1}{4}d, & d \text{ odd} \end{cases}$ nodes.

1.4. Basset's Upper Bound for Surfaces

Besides these constructions there also appeared new upper bounds. In 1906, Basset [**Bas06a**], [**Bas06b**] improved the bound (1.3) for the maximum number of isolated double points on a surface f of degree d in $\mathbb{P}^3$. But the approximate behaviour did not change: $\mu_{Dp}(d) \lessapprox \frac{1}{2}d^3$. Basset's idea was to project a nodal surface f of degree d and class d^* in $\mathbb{P}^3$ from a general point. This yields a $(d-2)$-fold covering $f \to \mathbb{P}^2$ ramified along a plane curve C of degree $d(d-1)$ and class d^*. Applying the Plücker Formulas to C yields:

$$\mu_{Dp}(d) \leq \frac{1}{2}\Big(d(d-1)^2 - 5 - \sqrt{d(d-1)(3d-14)+25}\Big). \tag{1.9}$$

Many years later, Stagnaro remarked that Basset's article was not rigorous enough and gave a modern proof [**Sta83**]. Furthermore, his proof yielded a generalization of Basset's bound to ordinary q-fold points, see section 3.1.3.

The following table gives the knowledge on $\mu_{Dp}(d)$ at this point. Although Rohn used the method described above only for constructing quartics, we list the number of nodes on surfaces of higher degree that one obtains in this way. The bold numbers indicate the cases in which Basset's upper or Rohn's lower bound improve the previously known bounds:

d	1	2	3	4	5	6	7	8	9	10	11	12	d
$\mu_{Dp}(d) \leq$	0	1	4	16	**34**	**66**	**114**	**181**	**270**	**383**	**524**	**696**	$\approx \frac{1}{2}d^3$
$\mu_{A_1}(d) \geq$	0	1	4	16	**20**	**45**	**63**	**112**	**144**	**225**	**275**	**396**	$\approx \frac{1}{4}\mathbf{d^3}$

We only want to mention in passing that in the years after Basset's discovery, several other people tried to improve his bound. E.g., Lefschetz [**Lef13**] and Hollcroft [**Hol23**], [**Hol28**], [**Hol29**] succeeded, but only under a certain assumption which Lefschetz calls the "postulate of singularities". As Lefschetz mentioned, for plane curves this postulate is equivalent to the "almost obvious" admission that when a curve can have κ_1 cusps it can also have any number of cusps smaller than κ_1. Nowadays, it is known that such properties can be shown by proving the non-obstructedness of a certain deformation functor.

1.5. Some Hypersurfaces in Higher Dimensions

1.5.1. C. Segre's 10-nodal Cubic in $\mathbb{P}^4$. Already at the end of the 19^{th} century, the first hypersurfaces in higher dimensions with many singularities were constructed: In 1887, C. Segre described a 10-nodal cubic in $\mathbb{P}^4$ [**Seg87**] (see also [**Seg88**], [**Cas88**]) which is the maximum possible number by an argument similar to (1.3). It can be shown that there is in fact only one such cubic up to projective equivalence, see e.g. [**Kal86**]. When denoting by $\sigma_j(x_0, x_1, \ldots, x_5)$, $j \in \mathbb{N}$, the j-th elementary symmetric polynomial in $\mathbb{P}^5$, C. Segre's cubic has the following nice equation:

$$\text{Seg}_3 : \quad \sigma_1(x_0, x_1, \ldots, x_5) = \sigma_3(x_0, x_1, \ldots, x_5) = 0. \tag{1.10}$$

The 10 nodes are the elements of the Σ_6-orbit of the point $(1 : 1 : 1 : -1 : -1 : -1)$. Another equally nice equation is (compare Clebsch's Diagonal Cubic (1.5) in section 1.1 on page 13 and Kalker's cubics in section 3.11 on page 41):

$$\text{Seg}_3 : \quad \sum_{i=0}^{5} x_i = \sum_{i=0}^{5} x_i^3 = 0. \tag{1.11}$$

C. Segre also noticed (see [**Seg87**] and also [**Tog50**, p. 53]) that it is possible to construct a Kummer quartic with the help of his 10-nodal Cubic in $\mathbb{P}^4$ (in fact, this seems to be his major motivation for constructing the cubic in $\mathbb{P}^4$). To understand this construction of the Kummer quartic, let us start with a cubic hypersurface H in $\mathbb{P}^4$ and a general point P on it. We may assume that P has the coordinates $(1 : 0 : 0 : 0 : 0)$ s.t. H has the form

$$H = F_3 + 2x_0F_2 + x_0^2F_1,$$

where the $F_i \in \mathbb{C}[x_1, x_2, x_3, x_4]$ have degree $i = 1, 2, 3$. The projection of H from P to $\mathbb{P}^3$ is a 2-fold ramified covering with branch locus

$$\Phi_4 := \det \begin{pmatrix} F_1 & F_2 \\ F_2 & F_3 \end{pmatrix}. \tag{1.12}$$

In general, Φ_4 has $1{\cdot}2{\cdot}3 = 6$ singularities at the points in which all the F_i, $i = 1, 2, 3$, vanish. But if H is the 10-nodal Segre cubic Seg_3 then one expects to get 10 additional nodes on Φ_4. Indeed, in this way we get the 16-nodal Kummer quartic.

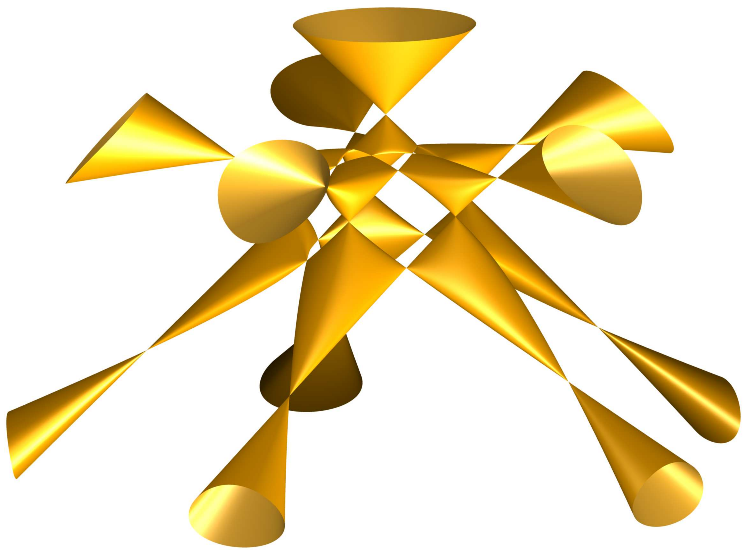
1.5.2. Burkhardt's 45-nodal Quartic in $\mathbb{P}^4$. In 1891, Burkhardt constructed a quartic in $\mathbb{P}^4$ with 45 nodes and showed this to be the maximum possible number [**Bur91**]. Its beautiful geometrical and combinatorial properties connected to the group of the 27 lines of the cubic surfaces were worked out in [**Bak46**] and [**Tod47**]. The fact that the Burkhardt quartic is also unique up to projective equivalence is a much more recent result [**dJSBvdV90**]. Similar to C. Segre's 10-nodal cubic, this unique 45-nodal quartic can be given by elementary symmetric polynomials:

$$\mathrm{Bu}_4: \quad \sigma_1(x_0, x_1, \dots, x_5) = \sigma_4(x_0, x_1, \dots, x_5) = 0. \tag{1.13}$$

Nowadays, we know that the Burkhardt quartic is unobstructed which shows the existence of quartics with exactly 0 up to 45 nodes. Determinantal equations for such quartics were given only recently [**Pet98**].

1.5.3. Some Cubics with Many Singularities. The first cubic in $\mathbb{P}^5$ with the maximum number of ordinary double points was Veneroni's 15-nodal hypersurface [**Ven14**]. The description of the space of all such hypercubics was the main subject of Kalker's Ph.D. thesis 70 years later [**Kal86**].

Lefschetz considered higher-dimensional hypersurfaces with many higher singularities. E.g., he constructed a cubic hypersurface in $\mathbb{P}^4$ with 5 cusps which is the maximum possible number, see [**Lef12**].



Barth's D_5-symmetric Togliatti quintic from 1993. Togliatti already showed the existence of 31-nodal quintics in 1940, but he did not give concrete equations.

CHAPTER 2

The Problem is Difficult (1915–1959)

After the first fruitful years the development of the area of hypersurfaces with many singularities slowed down a bit. In fact, the first striking result after Basset's upper bound of 1906 was the construction of Togliatti's 31-nodal quintic surface in 1940 (section 2.1). It seems that it was only after this discovery that the geometers realized the difficulty and relevance of the problem (see e.g., [**Tog50**]).

In the following years, several papers appeared on the subject. The major results in this direction were probably B. Segre's counterexamples to Severi's claimed upper bound (section 2.2), and B. Segre's observation that pull-back under a branched covering is a good way to produce many singularities (section 2.4).

2.1. Togliatti's Cubics in $\mathbb{P}^5$ and Quintic in $\mathbb{P}^3$

More than 20 years after Veneroni, Togliatti also constructed 15-nodal cubics in $\mathbb{P}^5$ [**Tog36**] ([**Tog37**] contains a simplified version) and he also proved that this was the maximum possible number of nodes on such a hypersurface. As Togliatti remarked on the last page of [**Tog37**], his family contains Veneroni's as a special case. His three-parameter family of 15-nodal cubics is:

$$\text{Tog}_3: \quad x_3x_4x_5 + x_3A + x_4B + x_5C = 0, \tag{2.1}$$

where $A, B, C \in \mathbb{C}[x_0, x_1, x_2]$ are defined as follows:

$$A := -x_0^2 + lx_1^2 + \frac{1}{k}x_2^2, \quad B := \frac{1}{l}x_0^2 - x_1^2 + hx_2^2, \quad C := kx_0^2 + \frac{1}{h}x_1^2 - x_2^2,$$

and where the three parameters $0 \neq h, k, l \in \mathbb{C}$ satisfy the condition $hkl + 1 \neq 0$. A particularly nice equation of a 15-nodal cubic in $\mathbb{P}^5$ arises for $h = k = l = 1$.

Togliatti's cubics are much better known than Veneroni's because Togliatti used them to show the existence of a 31-nodal quintic surface Tog_{31} in $\mathbb{P}^3$ [**Tog40**] which was the first new lower bound for the maximum number $\mu(d)$ of singularities on a surface of degree d in $\mathbb{P}^3$ since 50 years:

$$\mu_{A_1}(5) \geq \mathbf{31}. \tag{2.2}$$

Togliatti's construction is a variant of C. Segre's construction of the Kummer quartic (1.12). Togliatti started with a smooth hypercubic H in $\mathbb{P}^5$. As there are four conditions on a line to be contained in such a cubic and as the Grassmanian of lines has dimension 8, we get a four-dimensional family of lines on a generic hypercubic H. Assuming that the line l is given by $x_2 = x_3 = x_4 = x_5 = 0$, the cubic can be written in the from

$$H = A + 2x_1B + 2x_1C + x_0^2D + 2x_0x_1E + x_1^2F,$$

where $A, B, C, D, E, F \in \mathbb{C}[x_2, \ldots, x_5]$. When intersecting $H \subset \mathbb{P}^4$ with the $\mathbb{P}^3$ of $\mathbb{P}^2$'s containing l we get a cubic consisting of the line l and a residual conic which will be a pair of lines if

$$\Phi_5 := \det \begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix} = 0. \tag{2.3}$$

Φ_5 is a quintic surface in $\mathbb{P}^3$ with coordinates $x_2, \ldots, x_5$. This surface has 16 singular points corresponding to the points in which all the 2×2 minors of the matrix vanish. Now, if the hypercubic C in $\mathbb{P}^5$ has some nodes one expects the quintic surface Φ to have the same number of extra nodes. Using a 15-nodal cubic we get the desired $16 + 15 = 31$ nodes on Φ_5 which we denote by Tog_{31} in that case. Nowadays, all 31-nodal quintics in $\mathbb{P}^3$ all called Togliatti quintics because Beauville showed in [**Bea79b**] using a result of Catanese [**Cat81**] that all 31-nodal quintics in $\mathbb{P}^3$ can actually be obtained with Togliatti's construction.

Other more explicit constructions of 31-nodal quintics were given later: In 1983, Stagnaro constructed a 31-nodal quintic in $\mathbb{P}^3$, and a real dihedral-symmetric such quintic was found by Barth in the 90's. The latter was described in Endraß's Ph.D. thesis [**End96**] (see also section 4.2 on page 47).

2.2. Severi's Wrong Assumption and B. Segre's First Construction

In 1946, Severi wrote an article [**Sev46**] on an upper bound of $\binom{d+2}{3} - 4 \approx \frac{1}{6}d^3$ singularities which was shown to be wrong by B. Segre only shortly afterwards [**Seg47**].

In fact, Severi considered the following property as being intuitively clear: μ ordinary double points diminish the number of moduli of the surface at least by μ. As Lefschetz already noticed (see end of section 1.4), we have to be very careful with such arguments. Burns and Wahl [**BW74**] analyzed this problem in 1974: They showed that the minimal resolution $X \to f$ of a μ-nodal surface f of degree d is unobstructed if and only if the set of nodes Σ is d-independent, i.e. for any partition $\Sigma = \Sigma' \cup \Sigma''$, one may find a hypersurface of degree d containing Σ' and missing Σ''. To obtain an example of a surface of the lowest possible degree with obstructed minimal resolution, they considered the variants

$$\mathrm{BW}_d := \prod_{i=1}^{d} l_i(x_0, x_1, x_2) - x_3^d = 0 \tag{2.4}$$

of Rohn's construction (1.8) with $\binom{d}{2}$ singularities of type A_{d-1} (the l_i are general linear forms in x_0, x_1, x_2). Indeed, for $d = 5$ this is a quintic with ten A_4-singularities which is an example of an obstructed minimal resolution of a surface of the lowest degree.

Burns and Wahl [**BW74**] also mentioned that B. Segre's counterexamples [**Seg47**] to Severi's claim lead to unobstructed minimal resolutions. These can be constructed as follows. Consider the form

$$\Phi := \det \begin{pmatrix} f_{11} & \cdots & f_{1r} \\ \vdots & \ddots & \vdots \\ f_{r1} & \cdots & f_{rr} \end{pmatrix},$$

where $f_{ij} = f_{ji}$ are forms of degree k in four variables. Such a surface Φ of degree $r \cdot k$ in $\mathbb{P}^3$ has in general nodes at the $\delta := \binom{r+1}{3} \cdot k^3$ points in which the $(r-1) \times (r-1)$ minors of the matrix vanish.

In section 9 of his article, B. Segre specialized the f_{ij} and got surfaces of degree $r{\cdot}k$ with exactly $\delta_1 := \delta + \frac{r}{2}k^2(k-1) = \frac{rk}{6}(r^2k^2 + 2k^2 - 3k)$ nodes. For $r = 2$, these are surfaces of even degree $d = 2k$ with exactly $\frac{1}{4}d^3 - \frac{1}{4}d^2$ nodes which disproved Severi's claim.

As already mentioned in section 1.3 on page 16, Viat. Kharlamov informed me of the fact that for $r = 2$, these surfaces had already been found by Rohn [**Roh86**] 60 years earlier in the case of quartics.

Togliatti [**Tog50**] gave an overview of the results on hypersurfaces with many singularities known until 1950 and pointed out the difficulty of the subject. His survey article turned out to have some influence on the development of the subject: In fact, several authors cited this article as a motivation for working in this field in the following years.

2.3. Gallarati's General Constructions

In [**Gal51b**], Gallarati remarked that the special case of $r = 2$ of B. Segre's construction also worked for odd degree in order to show that Severi's claim fails for all $d \geq 12$. Again, this was basically a rediscovery of Rohn's construction from section 1.3 on page 16.

Gallarati [**Gal51a**] also gave another construction of nodal surfaces of degree d in $\mathbb{P}^3$ with approximately $\approx \frac{1}{4}d^3$ nodes. His construction improved the old bound 1.8 on page 16 in the lower order terms:

$$\mu_{A_1}(d) \geq \begin{cases} \frac{1}{4}d^3 + \mathbf{\frac{1}{4}d^2} - d, & d \text{ even} \\ \frac{1}{4}d^3 - \mathbf{\frac{1}{4}d^2} - \frac{1}{4}d + 1, & d \text{ odd.} \end{cases} \tag{2.5}$$

It is interesting to note that he also gave a construction of surfaces of odd degree d with exactly one triple point and many additional nodes whose number of nodes $\delta(d)$ exceeded the previously mentioned ones:

$$\delta(d) = \frac{1}{4}d^3 + \mathbf{\frac{1}{4}d^2} - \frac{9}{4}d - \frac{9}{4}. \tag{2.6}$$

The following table lists the bounds known up to this point. Again, the bold numbers indiciate the cases in which Gallarati's construction improved the previously known bounds. In the cases in which $\mu_{A_1}(d)$ differs from $\mu(d)$, we give both numbers:

d	5	6	7	8	9	10	11	12	d
$\mu_{Dp}(d) \leq$	34	66	114	181	270	383	524	696	$\approx \frac{1}{2}d^3$
$\mu_{A_1}(d)$ $(\mu(d)) \geq$	31	**57**	**72(81)**	**136**	**160(181)**	**265**	**300(337)**	**456**	$\approx \frac{1}{4}d^3$

2.4. B. Segre's Second Construction

In [**Seg52**], B. Segre introduced another nice construction of surfaces with many singularities using pull-back under a branched covering. He considered the map

$$\Omega : \mathbb{P}^3 \to \mathbb{P}^3, \ (x_0 : x_1 : x_2 : x_3) \mapsto (x_0^2 : x_1^2 : x_2^2 : x_3^2).$$

This map has degree eight and the pull-back of a form $f(x_0, x_1, x_2, x_3)$ of degree d under this map is a form $f(x_0^2, x_1^2, x_2^2, x_3^2)$ of degree $2d$. A node of f outside the coordinate tetrahedron corresponds to eight nodes of the transformed surface. Taking f to be tangent to the tetrahedron, one gets additional nodes. In this way, B. Segre constructed surfaces of degree $4, 6, 8$ with $16, 63, 153$ nodes, respectively.

E.g., for the Kummer quartic he took a tetrahedron each of whose four planes touch a smooth quadric in generic points.

B. Segre explained that his construction can also be applied to any nodal surface F_0 of degree d_0 with k_0 nodes. The resulting surface has degree $2d_0$ and $8k_0+4$ nodes. Applied successively to his 153-nodal octic and so on, this yields surfaces F_i of degree $d = 2^i \cdot 8$ with

$$\mu_{A_1}(d) > \frac{\mathbf{153}}{\mathbf{8^3}}\mathbf{d^3} \tag{2.7}$$

nodes. This was the first time that an asymptotic lower bound of more than $\frac{1}{4}d^3$ singularities on a surface of degree d appeared. We get the following table (Basset's upper bound was still the best one which was valid without additional assumptions):

d	5	6	7	8	9	10	11	12	d
$\mu_{Dp}(d) \leq$	34	66	114	181	270	383	524	696	$\approx \frac{1}{2}d^3$
$\mu_{A_1}(d)\ (\mu(d)) \geq$	31	**63**	80(81)	**153**	180(181)	265	336(337)	**508**	$\approx \frac{\mathbf{153}}{\mathbf{512}}\mathbf{d^3}$

In his paper, B. Segre also remarked that it might be possible to adapt his construction of the 153-nodal octic to get a 160-nodal one. This would improve this lower bound to $\approx \frac{160}{8^3}d^3 = \frac{5}{16}d^3$.

In the same paper, B. Segre also tried to improve the upper bounds for a surface f with only isolated double points. He did not succeed in general, but under the assumption that f does not possess an infinite number of *tritangent planes* (i.e. planes which are tangent to the surface in three points) and that its *parabolic curve* (the intersection of the surface with its hessian) and its *flecnodal curve* (the points at which there is a line having at least 4-point contact with the surface) do not contain any common component. E.g., he showed that — under these assumptions — a quintic cannot have more than 31 singularities and a sextic cannot have more than 63 ones. Of course, this did not prove that there were no surfaces with more nodes, but it gave an idea where to search for such examples.

2.5. Gallarati's Generalization of B. Segre's Second Construction

Shortly after B. Segre's discovery, Gallarati [**Gal52a**] generalized the map Ω to higher dimensions and higher singularities:

$$\Omega_j^n : \mathbb{P}^n \to \mathbb{P}^n, \ (x_0 : x_1 : \cdots : x_n) \mapsto (x_0^j : x_1^j : \cdots : x_n^j). \tag{2.8}$$

As an example analogous to B. Segre's construction of the Kummer quartic, Gallarati took a smooth quadric in $\mathbb{P}^n$ touching the $n+1$ hyperplanes of the coordinate $(n+1)$-hedron in generic points. Via Ω_2^n this gives a hyperquartic in $\mathbb{P}^n$ with $(n+1)\cdot 2^{n-1}$ nodes. E.g., Gallarati obtained a 40-nodal hyperquartic V_{40} in $\mathbb{P}^4$.

2.5.1. A Formula.

Gallarati did not give a general formula for the number and type of singularities one obtains in this way. But it is easy to derive a formula for hypersurfaces with A_j-singularities similar to B. Segre's case of nodal surfaces in $\mathbb{P}^3$: Let F_0 be a hypersurface in $\mathbb{P}^n$ of degree d_0 with k_0 singularities of type A_j. Take $n+1$ general hyperplanes tangent to F_0 as the coordinate $(n+1)$-hedron. The degree of the map Ω_{j+1}^n is $(j+1)^n$ away from the coordinate hyperplanes. It is $(j+1)^{n-1}$ on a general intersection point of two of the coordinate hyperplanes, and $(j+1)^{n-i}$, $i = 2, 3, \ldots, n$, for even more special points on the coordinate hyperplanes.

For our generic choice of coordinate hyperplanes tangent to F_0 the pull-back under Ω^n_{j+1} thus gives a hypersurface F_1 in $\mathbb{P}^n$ of degree $d_1 := (j+1)\cdot d_0$ with

$$\mu_{A_j}(F_1) = (j+1)^n \cdot k_0 + (n+1)\cdot(j+1)^{n-1} \tag{2.9}$$

singularities of type A_j. E.g., applied to a smooth quadric in $\mathbb{P}^3$ this gives:

COROLLARY 2.1. *Let $j \in \mathbb{N}$. There exist surfaces in $\mathbb{P}^3$ of degree $d = 2\cdot(j+1)$ with $4\cdot(j+1)^2$ singularities of type A_j.*

Specializing even further to $n = 3, j = 2$, we obtain:

$$\mu_{A_2}(6) \geq \mathbf{36}. \tag{2.10}$$

Notice that this is quite interesting because we know nowadays from Miyaoka's bound (section 3.10 on page 40) that $\mu_{A_2}(6) \leq 37$ holds.

Applying the preceding construction to F_1, we obtain a hypersurface F_2 in $\mathbb{P}^n$ of degree $d_2 := (j+1)^2\cdot d_0$ with

$$\mu_{A_j}(F_2) = (j+1)^n\big((j+1)^n\cdot k_0 + (n+1)\cdot(j+1)^{n-1}\big) + (n+1)(j+1)^{n-1}$$

singularities of type A_j. Iterating this, we get a hypersurface F_i of degree $d_i := (j+1)^i\cdot d_0$ with

$$\mu_{A_j}(F_i) = (j+1)^{ni}\cdot k_0 + \frac{n+1}{j+1}\cdot\Big(\frac{(j+1)^{n(i+1)}-1}{(j+1)^n-1} - 1\Big) \tag{2.11}$$

singularities of type A_j. Approximately, we thus have:

$$\mu_{A_j}(F_i) \approx \frac{1}{d_0^n}\cdot\left(k_0 + \frac{(n+1)\cdot(j+1)^{n-1}}{(j+1)^n-1}\right)\cdot d_i^n \quad \text{for } i \text{ large.} \tag{2.12}$$

Notice that it is easy to compute how many singularities we need to improve the best known lower bounds using the formula (2.12). E.g., let us look at nodal surfaces: To improve Chmutov's lower bound $\approx \frac{5}{12}d^3$ for the maximum number of nodes on a surface of degree d (section 4.1 on page 45) it suffices to construct a surface of degree d_0 with k_0 nodes, s.t. $k_0 > \frac{5}{12}d_0^3 - \frac{16}{7}$. Comparing this with the best known upper bound (section 3.10 on page 40), we find, e.g., that a 13652-nodal surface of degree 32 or a 109225-nodal surface of degree 64 would be sufficient.

We also want to mention that B. Segre's idea was rediscovered and worked out in detail in the case of plane curves with A_j-singularities by Hirano in 1992 [**Hir92**]. E.g., he found the lower bound of $\approx \frac{9}{32}d^2$ cusps on a plane curve of degree d in the way described above by starting from a smooth conic. To our knowledge, the currently best known lower bound for the maximum number of cusps on a plane curve is Vik.S. Kulikov's [**Kul03**]. He was able to choose at every other step of the iteration one of the coordinate axes to be bitangent to the curve which gives $\mu^2_{A_2}(d) \gtrapprox \frac{283}{60\cdot16}d^2$ when starting from a three-cuspidal quartic. D. Paccagnan (a student of Stagnaro) announced in an abstract of a talk at the ICM 1998 a slightly better lower bound, but this was never published. The currently best known upper bound is: $\mu^2_{A_2}(d) \leq \frac{5}{16}d^2 - \frac{3}{8}d$. This result is probably due to Ivinskis [**Ivi85**], see also: [**Hir86**], [**Sak93**]. To our knowledge, the maximum number of cusps on a plane curve of degree d is still unknown for $d > 12$.

2.5.2. Gallarati's Applications of the Construction. In his article, Gallarati performed the computation presented in the previous section in the special case of nodal surfaces in $\mathbb{P}^3$, i.e. $j = 2$ and $n = 3$. This gave a slight improvement to B. Segre's lower bound because B. Segre only considered one coordinate hyperplane (instead of four) to be tangent to the surface. For the maximum number of nodes on a surface of degree $d = 2^k \cdot 8$ Gallarati thus obtained:

$$\mu_{A_1}(d) \gtrapprox \left(\frac{\mathbf{153}}{\mathbf{512}} + \frac{\mathbf{1}}{\mathbf{224}}\right)\mathbf{d^3}. \tag{2.13}$$

But one cannot only obtain hypersurfaces with many A_j-singularities using this construction as Gallarati's example of a surface of degree 9 in $\mathbb{P}^3$ with 36 ordinary triple points showed. This surface is nicely connected to cubic surfaces with Eckardt points: He started with a smooth cubic surface with four Eckardt points, i.e. points in which three of the 27 lines meet. Taking as the coordinate tetrahedron the four planes tangent to these Eckardt points, Ω_3^3 yields a nonic with $4 \cdot 3^2 = 36$ triple points (recently, Stagnaro used similar ideas to get 39 triple points [**Sta04**]).

Gallarati then used the 40-nodal quartic V_{40} in $\mathbb{P}^4$ obtained above to construct a sextic in $\mathbb{P}^3$ in a way similar to the construction of the Kummer quartic (1.12) and the Togliatti quintic (2.3): Taking one of the nodes of V_{40} as the origin $P := (1 : 0 : \cdots : 0)$ of the coordinate system, V_{40} has the form:

$$V_{40} := x_0^2 F_2 + 2x_0 F_3 + F_4 = 0, \tag{2.14}$$

where $F_i \in \mathbb{C}[x_1, x_2, x_3, x_4]$ are of degree $i,\ i = 2, 3, 4$. The projection from P to the $\mathbb{P}^3$ given by $x_0 = 0$ is a 2-fold ramified covering with branch locus

$$\mathrm{Ga}_{63} := \det \begin{pmatrix} F_2 & F_3 \\ F_3 & F_4 \end{pmatrix}. \tag{2.15}$$

Ga_{63} has $2{\cdot}3{\cdot}4 + (40 - 1) = 63$ double points which is the same number of nodes as B. Segre's sextic. Gallarati also remarked that a similar construction could not work if we started with a 45-nodal quartic in $\mathbb{P}^4$ because it would give a 68-nodal sextic which is not possible because of Basset's bound. But it is interesting to note that van Straten's suggestion to try to start with the 3-parameter family of 42-nodal quartics yields a 3-parameter family of 65-nodal sextics as shown in [**Pet98**], see also section 4.5 on page 50.

2.6. Kreiss's Construction

In 1955, Kreiss described a construction of some surfaces of even degree $d = 2k$ with many nodes [**Kre55**]. Similar to a construction of Castelnuovo [**Cas91**], they have the form

$$f = Q(f_1, f_2, f_3),$$

where $Q(u, v, w)$ is a conic in $\mathbb{P}^2$ and the f_i are forms of degree k which are assumed to define k^3 simple points. A generic surface f has these k^3 points as nodes. We now take hyperplanes $E_{ij},\ i = 1, 2, 3,\ j = 1, 2, \ldots, n$, and put $f_i := \prod_{j=1}^{k} E_{ij}$.

The fibre of the rational map $\mathbb{P}^3 \to \mathbb{P}^2,\ x \mapsto (f_1(x) : f_2(x) : f_3(x))$ over a generic point of the form $(0 : \alpha : \beta)$ will have $k\binom{k}{2}$ singular points corresponding to the intersection points of the $\binom{k}{2}$ lines $E_{1i} \cap E_{1j}$ with the surface $\beta f_2 - \alpha f_3 = 0$. If one chooses the conic Q to be tangent to $u = 0, v = 0, w = 0$ in $\mathbb{P}^2$ one obtains a surface of degree $d = 2k$ with

$$\mu_{A_1}(d) \geq k^3 + 3k\binom{k}{2} = \mathbf{\frac{5}{16}d^3 - \frac{3}{8}d^2}, \quad d = 2k, k \in \mathbb{N}, \tag{2.16}$$

singularities which are all nodes in general.

Then Kreiss assumed that in the net spanned by f_1, f_2, f_3 there was a fourth surface $f_4 = af_1 + bf_2 + cf_3$ which decomposed as a product of k linear forms. Then by making Q also tangent to the line $au + bv + cw = 0$ in $\mathbb{P}^2$ we would get a surface with $k^3 + 4k\binom{k}{2} = 3k^3 - 2k^2$ singular points. To show this, Kreiss argued as follows: To have a syzygy of the form $\sum_{i=1}^{4} E_{1i} \cdots E_{ki} = 0$ between four k-tuples of linear forms we have $16k$ coefficients at our disposal which are subject to $\binom{k+3}{3}$ algebraic equations. As the inequality $16k \leq \binom{k+3}{3}$ holds exactly for $k \leq 7$, Kreiss claimed to have constructed surfaces of degree $d = 2k, 2 \leq k \leq 7$, with

$$3k^3 - 2k^2 = \frac{3}{8}d^3 - \frac{1}{4}d^2, \quad d = 2k, \quad 2 \leq k \leq 7, \tag{2.17}$$

nodes.

Van Straten remarked that such a construction is indeed possible for $k = 2$ if one takes the three pairs of parallel planes of a cube, but that the problem with Kreiss's argument for other k is the fact that one has to remove degenerate solutions of the above set of equations and that this might leave us with the empty set.

Nevertheless, Kreiss's work is often cited, and it took a long time until constructions giving at least the number of nodes that Kreiss's construction would give. Because of this influence, we list the lower bounds that Kreiss claimed to have found despite van Straten's previously mentioned remark:

d	4	6	8	10	12	14
$\mu_{A_1}(d) \geq$	16	63	**160**	**325**	**576**	**931**

Taking into account Gallarati's improvement (2.13) of B. Segre's lower bound based on an existing nodal surface, we get with the 576-nodal dodectic the existence of surfaces of degree $d = 2^k \cdot 12$ with

$$\mu(d) \gtrapprox \mathbf{\frac{253}{756}d^3 \approx 0.3347d^3} \tag{2.18}$$

nodes.

2.7. Gallarati's 160-nodal Construction

Despite Kreiss's 160-nodal octic in $\mathbb{P}^3$, Gallarati wrote an article on another such surface because of its interesting construction [**Gal57**]. He started with the form

$$V^9 = x_1x_2x_3x_4x_5 - y_1y_2y_3y_4y_5$$

in $\mathbb{P}^9$. V is singular along the 100 $\mathbb{P}^5$'s obtained by equating two of the x_i and two of the y_i to zero. So, a general linear section gives a family of 100-nodal quintics in $\mathbb{P}^4$. Gallarati then argued that one could choose this section so that it acquired a triple point P and that the lines joining P and the 100 nodes of the quintic were not contained in the tangent cone at P. Thus the ramification locus of the form

$$\mathrm{Ga}_{160} := \det\begin{pmatrix} F_3 & F_4 \\ F_4 & F_5 \end{pmatrix} = 0 \tag{2.19}$$

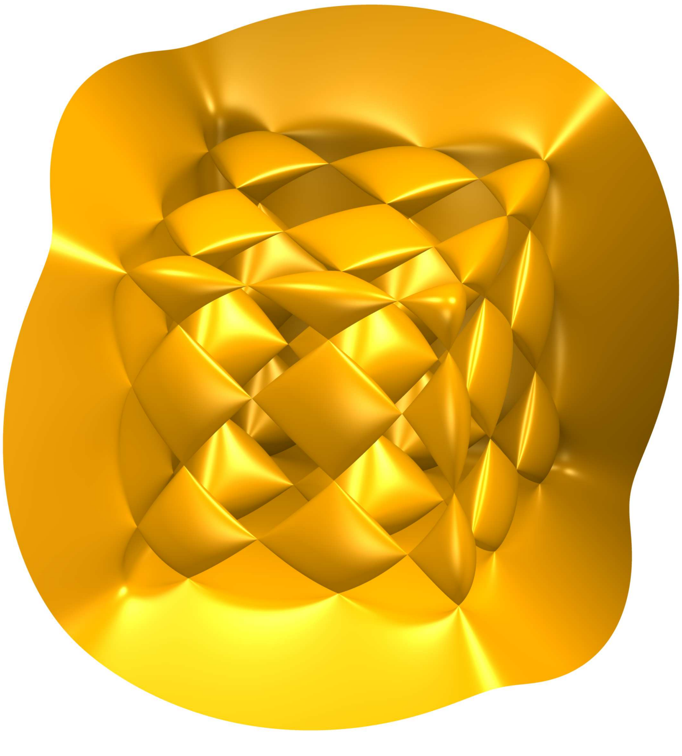
has $3 \cdot 4 \cdot 5 + 100 = 160$ nodes.

Moreover, Gallarati remarked that it might be possible to specialize further and to obtain an octic with more than 160 nodes in this way. To our knowledge, this is still unknown.

Van Straten mentioned that it might be possible to go to still higher dimensions: E.g., a general linear section of

$$V^{13} = x_1x_2x_3x_4x_5x_6x_7 - y_1y_2y_3y_4y_5y_6y_7$$

in $\mathbb{P}^3$ has 225 nodes. Again, one might hope to be able to choose this section so that it acquires a quadruple point which could then give a surface of degree 10 with $4 \cdot 5 \cdot 6 + 225 = 345$ nodes. This would be the same number of nodes as Barth's dectic (see section 4.5 on page 50). Some questions arising from this observation are the following: Does this construction work? If it does, is the surface different from Barth's (section 4.5)? Can we go on?



Chmutov's septic TChm_7^3, constructed around 1982 using Tchebychev polynomials. Variants of this basic idea are still the best ones for constructing hypersurfaces with many nodes of high degree.

CHAPTER 3

Modern Methods (1960–1990)

From the 1960's on, a systematic theory of singularities (see e.g. [**Mil68**], [**AGZV85a, AGZV85b**]) and their deformations was developped. These new methods allowed significant improvements of the known bounds around 1980.

Highlights of the period between 1960 and 1990 were Beauville's proof for $\mu_{A_1}(5) = 31$ in 1979 (section 3.3) as well as Varchenko's (1983, section 3.7) and Miyaoka's (1984, section 3.10) upper bounds. These were the first upper bounds for the maximum number of nodes on a surface of degree d which had a better asymptotic behaviour than the 100 year-old upper bound $\mu_{Dp}(d) \lessapprox \frac{1}{2}d^3$ based on the class of the surface. In fact, Miyaoka's bound $\mu_{Dp}(d) \lessapprox \frac{4}{9}d^3$ is still the best known bound for surfaces and Varchenko's spectral bound is still the best known one for hypersurfaces in higher dimensions. The strength of Varchenko's bound can be illustrated by the fact that it is exact for cubic hypersurfaces in $\mathbb{P}^n$ as Kalker's examples from 1986 showed (section 3.11). Another important contribution was Chmutov's idea to use Tchebychev polynomials for constructing hypersurfaces with many singularities (section 3.8).

3.1. Stagnaro's Results on Surfaces with Many Singularities

3.1.1. Surfaces with a j-tuple Point. In [**Sta68**], Stagnaro considered surfaces in $\mathbb{P}^3$ of the form

$$\mathcal{F}_{2m+j} : x_0^{2m} F_j + 2x_0^m F_{m+j} + F_{2m+j} = 0, \tag{3.1}$$

where $F_i \in \mathbb{K}[x_1, x_2, x_3]$ are forms of degree i, $i \in \{j, m+j, 2m+j\}$ and $\mathbb{K}$ is an algebraicly closed field of a characteristic which is not a $2m(2m+1)$ divisor. The surfaces $\mathcal{F}_{2m+j}$ have a j-tuple point in $(1:0:0:0)$.

For $j = 1$ and $m = 2q-1$ he then chose the F_i in a special way s.t. $\mathcal{F}_{2m+j} = \mathcal{F}_{4q-1}$ was a surface of degree $4q-1$ with $4q(2q-1)^2$ nodes and $12q-9$ singularities of type $A_{2(q-1)}$. His reasoning still contained an arbitrary form $\Theta_{2(q-2)}$ of degree $2(q-2)$. For a particular example, this can be chosen in a particular way to obtain even more singularities. E.g., this allowed him to show the existence of a septic $\mathcal{F}_7$ with 72 nodes and 16 additional cusps. Notice that the previously best lower bound for $\mu(d)$, 81, was also given by a construction of surfaces with singularities different from nodes, namely Gallarati's surfaces with a triple point and additional nodes (2.6):

$$\mu(7), \mu_{Dp}(7) \geq \mathbf{88}, \text{ although still, we have only: } \mu_{A_1}(7) \geq 72. \tag{3.2}$$

With the help of the 28 bitangents to a quartic plane curve, Stagnaro then used the above technique to show the existence of surfaces $\mathcal{F}_{2m+4}$ of degree $2m+4$ with $m\binom{2m+8}{2}$ isolated double points and an ordinary quadruple point in $(1:0:0:0)$. E.g., for $m = 2$ this is an octic with a quadruple point and 132 additional nodes.

3.1.2. A Sextic with 64 Nodes and a Septic with 90 Singularities. 10 years later [**Sta78**], Stagnaro constructed a surface of degree 6 with 64 singularities which showed:

$$\mu_{A_1}(6) \geq \mathbf{64}. \tag{3.3}$$

Notice that until this point, three sextics with 63 nodes had been known (see sections 2.4, 2.6, 2.5). According to B. Segre's upper bound mentioned at the end of section 2.4, a 64-nodal sextic cannot verify B. Segre's assumptions. And indeed, Stagnaro showed that his sextic St_{64} had an infinite number of tritangent planes. Its construction is based on a very special configuration of lines and conics in the plane. [1]

With an analogous method he constructed a surface of degree 7 with 72 nodes and 18 additional cusps. These are two more cusps than those of the example of the previous section. We have:

$$\mu_A(7) \geq \mathbf{90}, \text{ although still, we have only: } \mu_{A_1}(7) \geq 72. \tag{3.4}$$

Under certain assumptions, Stagnaro also gave a slight improvement of Basset's upper bound which computes to 65 for the case of degree 6. Nowadays, we know that 65 is the correct bound for sextics (see section 4.5 on page 50).

3.1.3. Stagnaro's Upper Bound for Ordinary q-fold Points. In [**Sta83**], Stagnaro gave a modern proof of Basset's bound and generalized it to ordinary q-fold points. Denoting by $\mu_q(d)$ the maximum number of q-fold points on a surface of degree d, he showed:

$$\mu_q(d) \leq \frac{4d(d-1)(d-2)}{q(q-1)(4q-5)} \tag{3.5}$$

and also:

$$\begin{aligned}\mu_q(d) \quad \leq \quad & \tfrac{1}{2q(q-1)^3} \cdot \Big(2d(d-1)^2(q-1) - 13q + 16 \\ & -\sqrt{4d(d-1)(3d-11q+8)(q-1) + (13q-16)^2}\Big).\end{aligned} \tag{3.6}$$

The exactness of (3.5) for $d = 5$ was already known [**Gal52b**]. An interesting remark of Stagnaro was that this bound is exact in several other cases, too (although (3.6) is better for d large). To prove this, he took the following generalization of Castelnuovo's construction [**Cas91**] (see also section 2.6): He considered surfaces A_s, B_s, C_s of degree s meeting in s^3 distinct points. If F_q is a generic form of degree q then

$$\mathrm{Stag}_{s,q} := F_q(A_s, B_s, C_s) \tag{3.7}$$

is a surface of degree $s{\cdot}q$ in $\mathbb{P}^3$ with s^3 ordinary q-fold points. Playing this against (3.5), he showed that s^3 was the maximum number of q-fold points on a surface of degree $s{\cdot}q$ if

$$q \geq \frac{1}{8}\Big(3(3s^3 - 4s^2 + 3) + \sqrt{9(3s^3 - 4s^2 + 3)^2 - 16(5s^3 - 8s + 5)}\Big). \tag{3.8}$$

This yielded an infinite number of cases in which the exact value of $\mu_q(d)$ was known. E.g., for $s = 2$, (3.8) is equivalent to $q \geq 8$, so for a surface of degree $2{\cdot}q$,

[1] Van Straten checked Stagnaro's equation of St_{64} using computer algebra and found it to be wrong. Its construction consists of several pages of geometrical arguments, so maybe the construction is basically correct, but only contains some typos. Because of the lengthy argument we were not able to figure this out.

$q \geq 8$, the maximum number of q-fold points is $2^3 = 8$. So, $\mu_8(16) = 8$, $\mu_9(18) = 8$, etc. In general, the criterion gives exactness only for cases in which the multiplicity of the ordinary singularities are large compared to the degree.

Beside this, Stagnaro summarized the best known constructions until this point in a table. There, he cited his septic from [**Sta78**] (section 3.1.2) with 72 nodes and 18 cusps as a surface with 90 ordinary double points which yielded some confusion in the literature of the following years.

Only shortly afterwards, Stagnaro wrote a preprint [**Sta84**] (cited in [**Gal84**] and [**Wer87**]) in which he claimed to construct a sextic with 66 nodes which would be the maximum possible number of double points according to Basset's upper bound. But shortly afterwards, (in his own MathSciNet review of Gallarati's historical overview), Stagnaro noticed that his construction was false. Since [**JR97**], we know that such a sextic cannot exist (see also section 4.5).

3.2. Teissier's and Piene's Formulas for the Class

The geometers of the 1970's realized that the old formulas for the class of a singular hypersurface (see for example the one preceding equation (1.3)) were either not general enough or not proven in a rigorous way. This (and generalizations of such results) were the motivation for Teissier [**Tei75**] and Piene [**Pie78**, p. 266] to show that if a hypersurface f of degree d in $\mathbb{P}^n$ has only isolated singularities $s_1, \ldots, s_k$ then its class d^* can be computed:

$$d^* = d(d-1)^{n-1} - \sum_{i=1}^{k} e_{s_i}, \tag{3.9}$$

where e_{s_i} denotes the multiplicity of the Jacobian Ideal at a singular point s_i. This number e_{s_i} can also be expressed as follows (see [**Bru81**]): $e_{s_i} = \mu(s_i) + \mu_1(s_i)$, where $\mu(s_i)$ is the Milnor number of the singularity of f at s_i and $\mu_1(x)$ is the Milnor number of a generic hyperplane section of f through s_i. Since $d^* \geq 0$ and $\mu(s_i) + \mu_1(s_i) \geq 2$, this gives:

$$\mu(d) \leq \frac{1}{2} d(d-1)^{n-1}. \tag{3.10}$$

This was the first upper bound for the maximum number of singular points of a hypersurface f with only isolated singularities which held in this generality. For surfaces in $\mathbb{P}^3$ with only double points, this bound was of course not as good as Basset's bound (section 1.4), because it was a generalization of the bound which had been known before the appearance of Basset's results.

3.3. Beauville's Proof of $\mu_{A_1}(5) = 31$ Using Coding Theory

The first major breakthrough after the results of the 19^{th} century was Beauville's proof for

$$\mu_{A_1}(5) = \mathbf{31}. \tag{3.11}$$

He called a set of isolated ordinary double points s_i, $i \in I$, on a quintic f in $\mathbb{P}^3$ *even* if the sum of their exceptional divisors E_i on the blown up surface F was divisible by two in $\operatorname{Pic}(F)$ or equivalently that the sum of the E_i was zero in $H^2(F, \mathbb{Z}/2)$. Beauville showed that even sets of nodes containing 16 and 20 elements were the only non-empty ones on a nodal quintic (these actually occur, see [**Bea79a**] and [**Cat81**]). Supposing that the quintic f had at least 32 nodes $s_1, \ldots, s_{32}$, he associated to the E_i a homomorphism $\phi : \mathbb{F}_2^{32} \to H^2(F, \mathbb{Z}/2)$. This ϕ has a kernel

K of dimension $\dim(K) \geq 6$. Looking at $K \subset \mathbb{F}_2^{32}$ as a code over the field with two elements, its only weights are 16 and 20 according to the remark above. But this contradicts the following fact from coding theory: If the weights of K are greater or equal to $\frac{m}{2}$ then $m \geq 2^{\dim(K)-1}$; in case of equality, K is isomorphic to a code which has $\frac{1}{2}\dim(K)$ as its only weight.

3.4. Bruce's Upper Bounds

In [**Bru81**], Bruce improved the general upper bound (3.9) for the number of singular points on a hypersurface of degree d in $\mathbb{P}^n$ with only isolated singularities. For surfaces in $\mathbb{P}^3$ of odd degree d, his bound is also better than Basset's bound, although it still stayed $\approx \frac{1}{2}d^3$. For the maximum number $\mu(d)$ of isolated singularities on a hypersurface of degree d in $\mathbb{P}^n$, he showed:

$$\begin{array}{rcll} \mu(d) & \leq & \frac{1}{2d}\left((d-1)^n(d+1)+(d-1)\right), & n \text{ even}, \\ \mu(d) & \leq & \frac{1}{2}(d-1)^n, & n \text{ odd}, d \text{ odd}, \\ \mu(d) & \leq & \frac{1}{2d}\left((d-1)^n(d+1)+1\right), & n \text{ odd}, d \text{ even}. \end{array} \tag{3.12}$$

His proof was based on a deformation theoretical result of Siersma [**Sie74**] and the computation of the rank of the intersection matrix of $x_1^d + x_2^d + \cdots x_n^d$ for n even and of $x_1^d + x_2^d + \cdots x_n^d + x_{n+1}^2$ for n odd using [**Mil68**]. For the maximum number of singularities on a surface in $\mathbb{P}^3$, the following bounds were known up to this point:

d	5	6	7	8	9	10	11	d
$\mu(d)(\mu_{A_1}(d)) \leq$	**32** (31)	**73** (66)	**108**	**193** (181)	**256**	**401**(383)	**500**	$\approx \frac{1}{2}d^3$
$\mu_{A_1}(d)(\mu(d)) \geq$	31	64	72 (90)	160	160(181)	325	300(337)	$\approx \frac{1}{3}d^3$

Notice that Bruce's list [**Bru81**, p. 50] does not show Gallarati's surfaces of degree $d = 9, 11$ with a triple point and 180 and 336 additional nodes, respectively (see section 2.3 on page 23).

3.5. Catanese's and Ceresa's Sextics with up to 64 Nodes

Clemens's work on double covers of $\mathbb{P}^3$ [**Cle83**] (see also section 3.13) and his notion of defect raised new interest on the problem of the existence and the construction of surfaces f of degree d in $\mathbb{P}^3$ having a given number μ_0 of nodes as its only singularites.

Such questions were motivations for Catanese and Ceresa to construct sextics in $\mathbb{P}^3$ with any given number $\mu_0 = 1, 2, \ldots, 64$ of nodes [**CC82**]. They applied B. Segre's idea to use pull-back under a branched covering, see section 2.4 on page 23. B. Segre had only obtained a 63-nodal sextic in this way. For the construction of a 64-nodal one the authors thus had to use different specializations of the coordinate tetrahedron.

Catanese and Ceresa also claimed to have shown that 64 is the maximum number of nodes possible on a sextic constructed in this way. Barth's 65-nodal sextic [**Bar96**] disproved this, see section 4.5 on page 50.

3.6. Givental's Upper Bound

Only a few years after Beauville's proof that the maximum number of nodes on a quintic in $\mathbb{P}^3$ was exactly 31, Givental established a general upper bound for the maximum number of isolated singularities on a hypersurface of degree d in $\mathbb{P}^n$ which computes to 31 in the case of quintic surfaces.

It is worth noting that the proof of his bound is much simpler than the one of Varchenko's spectral bound (see section 3.7). There is a drawback to this: The approximate behaviour of Givental's bound is still $\mu(d) \lessapprox \frac{1}{2}d^n$. But for low degree it is much better than the previously known bounds:

d	5	6	7	8	9	10	11	12	d
$\mu(d)(\mu_{Dp}(d)) \leq$	**31**	**68** (66)	**104**	**180**	**247**	**376**	**484**	**680**	$\approx \frac{1}{2}d^3$
$\mu_{A_1}(d)(\mu(d)) \geq$	31	64	72 (90)	160	160(181)	325	300(337)	576	$\approx \frac{1}{3}d^3$

Givental's bound can be computed as follows: Let I be the set of multiindices m, lying strictly inside the n-dimensional cube with side d:

$$I := \{m \in \mathbb{Z}^n \mid 0 < m_i < d\}. \tag{3.13}$$

We give names to the number of elements in the following subsets of I:

$$\begin{aligned} M &:= \# \left\{ |m| = \left(\tfrac{n}{2} + 2k\right) d \right\}, \\ K &:= \# \left\{ |m| = \left(\tfrac{n}{2} + 2k - 1\right) d \right\}, \\ R &:= \# \left\{ |m| - \left(\tfrac{n}{2} + 2k - 1\right) d = \pm 1 \text{ or } \pm \tfrac{1}{2} \right\}, \end{aligned}$$

where $|m| := m_1 + \cdots + m_n$ as usual and $k \in \mathbb{Z}$. With these notations, Givental's upper bound on the maximum number $\mu^n(d)$ of isolated singularities on a hypersurface of degree d in $\mathbb{P}^n$ is:

$$\mu^n(d) \leq \frac{1}{2}\left((d-1)^n + M - K - R\right). \tag{3.14}$$

As the major motivation for the research on the subject, Givental mentioned the following conjecture on the number of singular points on a hypersurface. It was formulated by Arnold in 1981 in a discussion of Bruce's article [**Bru81**] as Varchenko said in the introduction of [**Var83**]: Arnold suggested the bound $\mu^n(d) \leq A_n(d)$, where

$$A_n(d) := \# \left\{ (k_1, \ldots, k_n) \in I \;\Big|\; \frac{1}{2}(n-2)d + 1 < \sum k_i \leq \frac{1}{2}nd \right\}. \tag{3.15}$$

$A_n(d)$ is thus a certain number of integer points within an n-dimensional cube. Givental's bound is slightly greater than $A_n(d)$ for most degrees. Nowadays, the numbers $A_n(d)$ are called Arnold numbers. The correctness of Arnold's conjecture was shown only shortly afterwards by Varchenko (see next section).

3.7. Varchenko's Spectral Bound

Not long after Givental's new upper bound, Varchenko was able to prove the conjecture of Arnold (see equation (3.15)) by showing a theorem on the spectrum of a singularity [**Var83**]. Basically, the spectrum consists of the eigenvalues of the monodromy operator of the singularity, see e.g. [**Kul98**], [**AGZV85b**, ch. 14] for details on the spectrum. Varchenko's nowadays called Spectral Bound was the first upper bound for the maximum number of singularities on a surface of degree d which had an approximative behaviour of less than $\frac{1}{2}d^3$. In fact, he showed:

$$\mu^n(d) \leq \mathbf{A_n(d)}, \tag{3.16}$$

where $A_n(d)$ is the Arnold number defined in (3.15). For surfaces in $\mathbb{P}^3$, this computes to:

$$(3.17) \qquad \mu(d) \leq \begin{cases} \frac{23}{48}d^3 - \frac{9}{8}d^2 + \frac{5}{6}d, & d \equiv 0 \mod 2, \\ \frac{23}{48}d^3 - \frac{23}{16}d^2 + \frac{73}{48}d - \frac{9}{16}, & d \equiv 1 \mod 2. \end{cases}$$

This leads to the following table:

d	5	6	7	8	9	10	11	12	d
$\mu(d)(\mu_A(d)) \leq$	31	68 (66)	104	180	**246**	**375**	**480**	**676**	$\approx \mathbf{\frac{23}{48}d^3}$
$\mu_{A_1}(d)(\mu(d)) \geq$	31	64	72 (90)	160	160(181)	325	300(337)	576	$\approx \frac{1}{3}d^3$

Arnold and Givental computed the approximate behaviour of $A_n(d)$:

$$(3.18) \qquad A_n(d) \approx \sqrt{\frac{6}{\pi n}} d^n + O(d^{n-1}) \text{ for large } n.$$

As already mentioned, Varchenko's previous bound is based on a property of the spectrum of a singularity, more precisely the so-called semicontinuity of the spectrum (see [**Var83**] and also [**Kul98**], [**AGZV85b**, ch. 14]). It cannot only be applied to ordinary double points, but to any type of isolated singularity in n-dimensional space for which it is possible to compute the spectrum. For many cases, this computation has already been performed. The spectrum can even be calculated using Endraß's SINGULAR library `spectrum.lib` or Schulze's library `gaussman.lib`. These libraries also contain procedures for computing the bound for the maximum number of singularities of a given type on a hypersurface of degree d in $\mathbb{P}^n$ based on the semicontinuity property:

```
LIB "gaussman.lib"; LIB "spectrum.lib";
proc varchenko_bound_general(int n, int d, poly sing) {
    poly p = 0;
    for(int i=1; i<=n; i=i+1) { p = p + var(i)^d; }
    list s = spectrumnd(p);
    list ss = spectrumnd(sing);
    return(spsemicont(s,list(ss),1)[1]); }
```

E.g., with this procedure the SINGULAR code

```
ring r = 0,(x,y,z),ds;
varchenko_bound_general(3, 7, x^2+y^2+z^2);
```

gives `104` which is the Varchenko's bound for the maximum number of nodes on a septic surface in $\mathbb{P}^3$.

To explain how to compute formulas for the bound in more general cases, let us look at A_j-singularities on surfaces of degree d in $\mathbb{P}^3$. It is known that Varchenko's spectral bound can be described by a polynomial of degree 3 in d, but we could not find explicit statements for $j > 1$ in the literature. In the following we explain briefly how to proceed in order to compute these polynomials.

For even degree $d \geq 4$ the spectrum $sp(d)$ of the singularity $x^d + y^d + z^d = 0$ in $\mathbb{C}^3$ consists of the spectral numbers $s_d(i) = \frac{i+2}{d}, i = 1, 2, \ldots, 3(d-1) - 2$, with multiplicities $m_d(i)$, where

- $m_d(1) = 1$,
- $m_d(i+1) = m_d(i) + 1 + i, \quad i < d - 1$,
- $m_d(i+1) = m_d(i) + 2(i_{mid} - i) + 1, \quad d - 1 \leq i < i_{mid} := \frac{3d}{2} - 2$,
- $m_d(3(d-1) - 1 - i) = m_d(i), \quad 1 \leq i \leq i_{mid}$ (symmetry of the spectrum).

The spectrum of an A_j-singularity is also well-known (see e.g. [**AGZV85b**, p. 389]). Its spectral numbers are $\frac{j+2}{j+1}, \frac{j+3}{j+1}, \ldots, \frac{2j+1}{j+1}$, all with multiplicity 1.

EXAMPLE 3.1. *The spectrum $sp(6)$ of the singularity $x^6+y^6+z^6$ is:*

i	1	2	3	4	5	6	7	8	9	10	11	12	13
spectral number s_i	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	$\frac{8}{6}$	$\frac{9}{6}$	$\frac{10}{6}$	$\frac{11}{6}$	$\frac{12}{6}$	$\frac{13}{6}$	$\frac{14}{6}$	$\frac{15}{6}$
multiplicity m_i	1	3	6	10	15	18	19	18	15	10	6	3	1

The spectral numbers of the A_2-singularity are: $\frac{8}{6}, \frac{10}{6}$, both with multiplicity 1. □

To compute Varchenko's bound we have to choose an open interval of length 1, say $I = (\frac{i_r+2-d}{d}, \frac{i_r+2}{d})$, of the spectrum $sp(d)$ which contains all spectral numbers of the A_j-singularity and such that the sum of the multiplicities of the spectral numbers in the interval is minimal. Then we have to sum up all the multiplicities in this interval and divide by j.

Let us write $d = k\cdot(j+1)+l$. Then we may choose $I := (\frac{i_r+2-d}{d}, \frac{i_r+2}{d})$, where $i_r := k\cdot(2j+1) + \left\lfloor \frac{l\cdot(2j+1)}{j+1} \right\rfloor - 1$. We introduce some notations: $n_l := i_{mid} - (d-1)$, $n_r := i_r - i_{mid} - 1$, $n_{ll} := d-1-n_l-n_r$, $m_{mid} = \sum_{i=1}^{d-1} i + (\frac{d}{2}-1)^2$. Using these we can compute Varchenko's bound $\mathrm{Var}_{A_j}(d)$ for the maximum number of A_j-singularities on a surface of degree d in $\mathbb{P}^3$ for the case $d, j \in \mathbb{N}$ with $d \geq 4$:

$$\mathrm{Var}_{A_j}(d) = \frac{1}{j}\cdot\Big(\frac{1}{2}\cdot\big(\sum_{i=1}^{d-1} i + \sum_{i=1}^{d-1} i^2 - \sum_{i=1}^{d-1-n_{ll}} i - \sum_{i=1}^{d-1-n_{ll}} i^2\big) + (n_r+n_{ll})\cdot m_{mid} - \sum_{i=1}^{n_r} i^2 - \sum_{i=1}^{n_l-1} i^2\Big) \quad (3.19)$$ □

EXAMPLE 3.2. *Let us look at the case $d=6, j=2$ as in example 3.1. In this case, the constants used above have the following values: $k=2$, $l=0$, $i_r=9$, $i_{mid}=7$, $n_l=2$, $n_r=1$, $n_{ll}=2$, $m_{mid}=19$. We can now easily compute the bound $\mathrm{Var}_{A_2}(d)$ in (3.19) for $d=6$ (compare the table in example 3.1):*

$$\mathrm{Var}_{A_2}(6) = \frac{1}{2}\cdot\Big(\underbrace{\frac{15+55-6-14}{2}}_{=10+15} + \underbrace{3\cdot 19 - 1 - 1}_{=18+19+18}\Big) = 40.$$ □

Using some summation formulas we find the following bounds for $d \geq 4$:

- $\mu_{A_1}(d) \leq \mathrm{Var}_{A_1}(d) = \begin{cases} \frac{23}{48}d^3 - \frac{9}{8}d^2 + \frac{5}{6}d, & d \equiv 0 \mod 2, \\ \frac{23}{48}d^3 - \frac{23}{16}d^2 + \frac{73}{48}d - \frac{9}{16}, & d \equiv 1 \mod 2. \end{cases}$
- $\mu_{A_2}(d) \leq \mathrm{Var}_{A_2}(d) = \begin{cases} \frac{31}{108}d^3 - \frac{25}{36}d^2 + \frac{1}{2}d, & d \equiv 0 \mod 3, \\ \frac{31}{108}d^3 - \frac{31}{36}d^2 + \frac{17}{18}d - \frac{10}{27}, & d \equiv 1 \mod 3, \\ \frac{31}{108}d^3 - \frac{7}{9}d^2 + \frac{3}{4}d - \frac{5}{27}, & d \equiv 2 \mod 3. \end{cases}$
- $\mu_{A_3}(d) \leq \mathrm{Var}_{A_3}(d) = \begin{cases} \frac{235}{1152}d^3 - \frac{49}{96}d^2 + \frac{13}{36}d, & d \equiv 0 \mod 4, \\ \frac{235}{1152}d^3 - \frac{235}{384}d^2 + \frac{785}{1152}d - \frac{35}{128}, & d \equiv 1 \mod 4, \\ \frac{235}{1152}d^3 - \frac{37}{64}d^2 + \frac{173}{288}d - \frac{3}{16}, & d \equiv 2 \mod 4, \\ \frac{235}{1152}d^3 - \frac{209}{384}d^2 + \frac{569}{1152}d - \frac{35}{384}, & d \equiv 3 \mod 4. \end{cases}$

The formulas are not correct for $d=3$ for some j because the spectrum of the $x^3+y^3+z^3=0$ singularity does not have enough spectral numbers to fit into the description above.

3.8. Tchebychev Polynomials and Hypersurfaces with Many Nodes

Chmutov suggested (see [**AGZV85b**, p. 419] or Varchenko's overview article [**Var84**, p. 2782]) to consider the hypersurface TChm_d^n of degree d in $\mathbb{P}^n$ with affine equation:

$$\mathrm{TChm}_d^n: \quad \sum_{j=0}^{n-1} T_d(x_j) = \begin{cases} 0, & n \text{ even}, \\ -1, & n \text{ odd}, \end{cases} \tag{3.20}$$

where

$$T_d(z) := \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} (-1)^i \binom{n}{2i} z^{n-2i}(1-z^2)^i \tag{3.21}$$

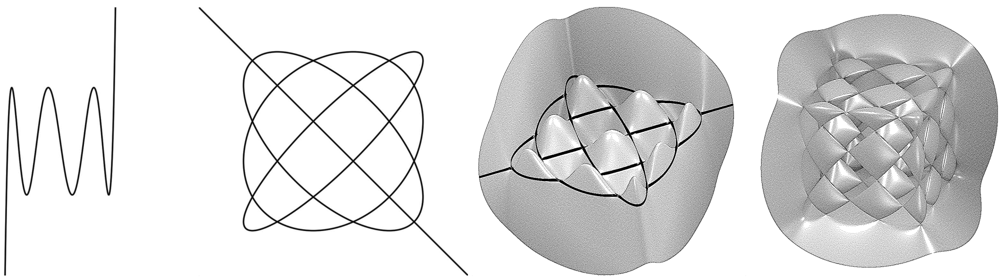
denotes the Tchebychev polynomial of degree d having two critical values ± 1 (see [**Riv74**]). T_d can be recursively defined as follows:

$$T_0(z) := 1, \quad T_1(z) := z, \quad T_d(z) := 2z \cdot T_{d-1}(z) - T_{d-2}(z). \tag{3.22}$$

These polynomials have many other nice properties. We only mention two more of them. First, the $T_d(z)$ satisfy the equation:

$$T_d(\cos(\alpha)) = \cos(d\alpha), \ \alpha \in [0, \pi], \tag{3.23}$$

and its derivative $T_d'(z)$ vanishes at $\alpha_k := \cos\left(\frac{k\pi}{d}\right), 1 \leq k \leq d-1$ which gives rise to a maximum (resp. minimum) if k is even (resp. odd). Second, the plane curves $C_1 := T_d(x) + T_d(y)$ (resp. $C_2 := T_d(x) - T_d(y)$) factor into $\frac{d}{2}$ irreducible conics (resp. $\frac{d-2}{2}$ irreducible conics and two lines) if d is even and they both factor into $\frac{d-1}{2}$ irreducible conics and a line if d is odd (see [**Wer87**, p. 34]).



$y = T_7(x)$ $\qquad$ $\mathrm{TChm}_7^2 = 0$ $\qquad$ $\mathrm{TChm}_7^2 = 5z$ $\qquad$ $\mathrm{TChm}_7^3 = 0$

FIGURE 3.1. The Geometry of Chmutov's Hypersurfaces.

It is easy to see that the hypersurfaces TChm_d^n are singular exactly at the points $(\alpha_{k_1}, \ldots, \alpha_{k_n})$, $1 \leq k_i \leq n-1$, where $\lfloor \frac{n}{2} \rfloor$ of the indices k_i are odd and the other are even (see fig. 3.1 for an illustration of the case $n = 2$). All singularities are nodes and their number is

$$\mu(\mathrm{TChm}_d^n) = c_n d^n + O(d^{n-1}),$$

where $c_3 = \frac{3}{8}$ and more precisely $\mu(\mathrm{TChm}_d^3) = \frac{3}{8}d^2(d-2)$ if d is even and $\mu(\mathrm{TChm}_d^3) = \frac{3}{8}(d-1)^3$ if d is odd. This showed:

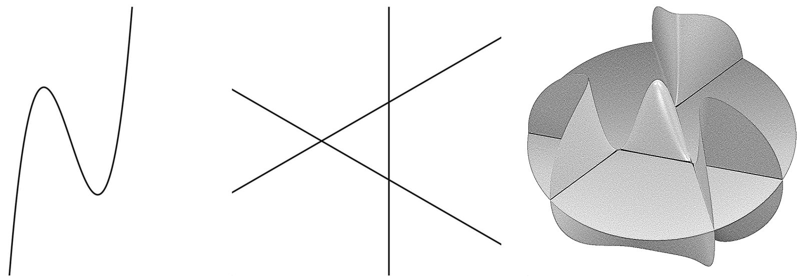
$$\mu_{A_1}(d) \gtrapprox \mathbf{\frac{3}{8}d^3} \tag{3.24}$$

which was the best approximate behaviour for $n = 3$ known up to this point. It is easy to compute the exact number also in higher dimensions, e.g. for d odd and n even we get $\mu(\mathrm{TChm}_d^n) = \left(\frac{d-1}{2}\right)^n \cdot \binom{n}{n/2}$ nodes. A computation of Givental concerning the approximate behaviour with respect to n showed: $c_n \approx \sqrt{\frac{2}{\pi n}}$ for large n, see [**Var84**, p. 2782] (compare (3.18)). When assuming the correctness of Kreiss's construction, it improves the bounds for low degree d only in a few cases:

d	5	6	7	8	9	10	11	12	d
$\mu(d)(\mu_A(d)) \leq$	31	68 (66)	104	174	246	360	480	645	$\approx \frac{4}{9}d^3$
$\mu_{A_1}(d)(\mu(d)) \geq$	31	64	**81** (90)	160	**192**	325	**375**	576	$\approx \mathbf{\frac{3}{8}d^3}$

3.9. Givental's Cubics in $\mathbb{P}^n$

Givental (see [**AGZV85b**, p. 419], [**Var84**, p. 2782]) used Chmutov's idea to construct cubics in $\mathbb{P}^n$ with a number of nodes that almost reached Varchenko's spectral bound. Instead of Tchebychev Polynomials which are polynomials with few critical values in one variable, he used a polynomial with few critical values in two variables: To understand the construction, let us start with a regular triangle $R_3(x, y)$ whose non-zero critical point has critical value $+1$, see fig. 3.2.



$y = T_3(x)$ $\qquad$ $R_3(x, y) = 0$ $\qquad$ $z - R_3(x, y) = 0$

FIGURE 3.2. The Tchebychev Polynomial $T_3(x)$ of degree three and a regular triangle, once seen in the plane, once in space.

Then the number of singular points (all are nodes) of the cubic hypersurface in $\mathbb{P}^n$ with affine equation

$$(3.25)\quad \mathrm{Giv}_3^n : \sum_{j=0}^{\frac{n}{2}-1} (-1)^{j\cdot(1+(n \bmod 2))} R_3(x_{2j}, x_{2j+1}) = -(n \bmod 2)\frac{T_3(x_{n-1}) - 1}{2},$$

is $g_n \approx 2^n \sqrt{\frac{16}{3\pi n}}$ for n large. Givental also noticed that $A_n(3) \approx 2^n \sqrt{\frac{8}{\pi n}}$ for n large which showed: $\frac{g_n}{A_n(3)} \approx \sqrt{\frac{2}{3}} \approx 0.8165$. In fact, Varchenko's spectral bound is exact for cubics in $\mathbb{P}^n$ as Kalker showed only shortly afterwards, see section 3.11.

In both cited texts [**AGZV85b**, p. 419], [**Var84**, p. 2782], the equations for Giv_3^n are only given for $n \equiv 0 \mod 4$, but they list the numbers of nodes that can be obtained using Givental's construction. These numbers can be realized using the equations given above:

n	1	2	3	4	5	6	7	8	9	n
$\mu^n(3) = \mu^n_{A_1}(3) =$	1	3	4	10	15	35	56	126	210	$\binom{n+1}{\lfloor \frac{n}{2} \rfloor}$
$\mu^n(3) \geq$	1	3	4	10	15	**33**	**54**	**118**	**189**	$\approx \mathbf{2^n \sqrt{\frac{16}{3\pi n}}}$

3.10. Miyaoka's Bound for Surfaces with Rational Double Points

Parallel to Varchenko's spectral bound, there appeared another very important upper bound due to Miyaoka. In [**Miy84**], Miyaoka proved an inequality that he could apply to a normal surface f_d of degree d in $\mathbb{P}^3$ with only rational double points as singularities to prove:

$$\mu_{Dp}(d) \leq \mathbf{\frac{4}{9} d(d-1)^2}. \tag{3.26}$$

This is still the best known upper bound for the maximum number of rational double points on a surface in $\mathbb{P}^3$ for large degree. Only for odd low degree, Varchenko's bound is better in some cases. The following table gives the bounds known up to this point:

d	5	6	7	8	9	10	11	12	d
$\mu(d)(\mu_{Dp}(d)) \leq$	31	68 (66)	104	174	246	**360**	480	**645**	$\approx \mathbf{\frac{4}{9} d^3}$
$\mu_{A_1}(d)(\mu(d)) \geq$	31	64	81 (90)	160	192	325	375	576	$\approx \frac{1}{3} d^3$

Miyaoka's bound can also be applied to compute the maximum number of some particular type of rational double points: Let X_p be the germ of a quotient singularity $(\mathbb{C}/G_p)_0$, where G_p is a finite subgroup of $\mathrm{GL}(2, \mathbb{C})$ having the origin as its unique fixed point. Let $\widetilde{X}_E$ be the minimal resolution of X_p and E the exceptional divisor. The Euler numbers $e(\widetilde{X}_E)$ and $e(E)$ coincide. Put $\nu(p) := e(E) - \frac{1}{|G_p|}$. The non-negative rational number $\nu(p)$ is an analytic invariant of the quotient singularity X_p which is not difficult to compute in the cases we are interested in:

$$\begin{aligned} &\nu(p) &&= 0 && \text{if } X_p \text{ is smooth,} \\ &\nu(p) &&= 2 - \tfrac{1}{j} && \text{if } X_p \text{ is an ordinary } j\text{-tuple point,} \\ &\nu(p) &&= j + 1 - \tfrac{1}{j+1} = \tfrac{j(j+2)}{j+1} && \text{if } X_p \text{ is an } A_j\text{-singularity.} \end{aligned} \tag{3.27}$$

Miyaoka showed: If X is a projective surface with only rational double points $p_1, \ldots, p_k$ and whose dualizing line bundle $\mathcal{O}_X(K_x)$ is numerically effective, then

$$\sum_{i=1}^{k} \nu(p_i) + e(D) \leq c_2(\widetilde{X}) - \frac{1}{3}(K_X + D)^2 = 12\chi(\mathcal{O}_X) - \frac{1}{3}(4K_X^2 + 2K_X D + D^2) \tag{3.28}$$

for any effective normal crossing divisor D away from the singularity. Now, let $f_d \subset \mathbb{P}^3$ be a normal surface of degree d with only rational double points $p_1, \ldots, p_k$. Then (3.28) implies:

$$\sum_{i=1}^{k} \nu(p_i) \leq \frac{2}{3} d(d-1)^2. \tag{3.29}$$

E.g., as $\nu(p) \geq \frac{3}{2}$ for singular points, we get (3.26). For any fixed $j \in \mathbb{N}$, (3.27) yields for the maximum number $\mu_{A_j}(d)$ of A_j-singularities on a surface of degree d

in $\mathbb{P}^3$:

$$\mu_{A_j}(d) \le \text{Miy}_{A_j}(d) := \frac{\mathbf{2}}{\mathbf{3}}\frac{\mathbf{j+1}}{\mathbf{j(j+2)}}\mathbf{d(d-1)^2}. \tag{3.30}$$

There also exists a generalization of Miyaoka's result to more general singularities by J. Wahl, see section 4.10 and [**Wah94**].

3.11. Kalker's Cubics in $\mathbb{P}^n$

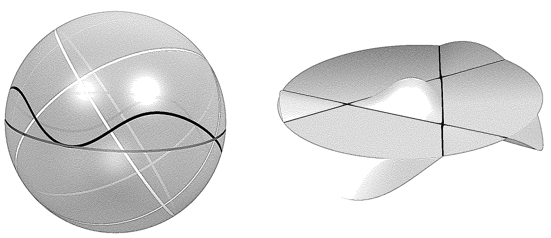
In his Ph.D. thesis [**Kal86**], Kalker gave examples of cubics in $\mathbb{P}^n$ that showed that Varchenko's spectral bound for the maximum number $\mu^n(3)$ of singularities on cubic hypersurfaces in $\mathbb{P}^n$ was sharp for all $n \in \mathbb{N}$:

$$\mu^n(3) = A_n(3) = \binom{\mathbf{n+1}}{\lfloor \frac{\mathbf{n}}{\mathbf{2}} \rfloor}. \tag{3.31}$$

He defined cubics Kal_3^n as generalizations of the equations of the four-nodal Cayley cubic in $\mathbb{P}^3$ (1.2) and C. Segre's cubic in $\mathbb{P}^4$ (1.11) (we chose a slightly different notation for Kalker's equations for odd n in order to underline the similarity to the Cayley cubic):

$$\text{Kal}_3^n : \begin{array}{lll} \sum_{i=0}^{n+1} x_i^3 = 0, & \sum_{i=0}^{n+1} x_i = 0, & \text{for } n \text{ even}, \\ \sum_{i=0}^{n} x_i^3 + \frac{1}{4}x_{n+1}^3 = 0, & \sum_{i=0}^{n+1} x_i = 0, & \text{for } n \text{ odd}. \end{array} \tag{3.32}$$

The singularities of these cubics in $\mathbb{P}^n$ are exactly the points in which the two hypersurfaces in $\mathbb{P}^{n+1}$ are tangent to each other (fig. 3.3).



$\text{Kal}_3^1 \subset \mathbb{P}^1$ $\text{Kal}_3^2 \subset \mathbb{P}^2$

FIGURE 3.3. The Geometry of Kalker's Hypersurfaces. $\text{Kal}_3^1 \subset \mathbb{P}^1$ consists of two points, one doubled: In the projective view one can see the (black) fermat cubic $x_0^3 + x_1^3 + \frac{1}{4}x_2^3$ touching the hyperplane $x_0 + x_1 + x_2 \simeq \mathbb{P}^1$ in one point and meeting it transversally in another one. To illustrate the construction of the Kalker cubic $\text{Kal}_3^2 \subset \mathbb{P}^2$ which takes place in $\mathbb{P}^3$, we take the affine chart $x_3 = 1$. Here one sees the three points in which the hyperplane $x_0 + x_1 + x_2 + 1 \simeq \mathbb{P}^2$ touches the cubic $x_0^3 + x_1^3 + x_2^3 + 1^3$. Kal_3^2 consists of three lines.

The following table lists the numbers of nodes on Kalker's hypercubics. This is one of the very rare cases in which we know the maximum number of singularities:

n	0	1	2	3	4	5	6	7	8	9	10	n
$\mu^n(3) = \mu^n_{A_1}(3) =$	0	1	3	4	10	15	**35**	**56**	**126**	**210**	**462**	$\binom{\mathbf{n+1}}{\lfloor \frac{\mathbf{n}}{\mathbf{2}} \rfloor}$

Of course, this construction can also be generalized to higher degrees, although Kalker did not mention it because he was only interested in cubics. Moreover, this construction does not seem to be very good for $d > 3$.

3.12. Two Nodal Quintics in $\mathbb{P}^4$

Since the end of the 19th century, the maximum numbers of nodes on a threefold in $\mathbb{P}^4$ of degree ≤ 4 are known, see 1.5 on page 18. We have just seen that Varchenko's bound is exact for cubics in any dimension. In 1985 and 1986, Hirzebruch [**Hir87**] and Schoen [**Sch86**] constructed the first examples that came close to Varchenko's upper bound (section 3.7 on page 35) for quintics in $\mathbb{P}^4$: $\mu^4(5) \leq 135$.

Schoen's quintic

$$\text{Sch}_5^4: \sum_{i=0}^{4} x_i^5 - 5\prod_{i=0}^{4} x_i = 0 \tag{3.33}$$

has exactly 125 nodes (see also [**Wer87**, p. 84]).

He was only interested in threefolds, but it is obvious how his construction can be generalized to higher dimensions. But the variants $\widetilde{\text{Sch}}_d^{d-1}$ in $\mathbb{P}^{d-1}$ given by $\widetilde{\text{Sch}}_d^{d-1}: \sum_{i=0}^{d-1} x_i^d - d\prod_{i=0}^{d-1} x_i = 0$ have only $d^{d-2} = d^{n-1}$ nodes.

Hirzebruch's quintic can also be generalized to higher dimensions and other degrees. In fact, the construction is exactly Givental's (3.25), but instead of a triangle $R_3(x,y)$, he took a five-gon $R_5(x,y)$ (see fig. 3.4) and Hirzebruch only applied it in four-dimensional space because he was only interested in quintic threefolds. The $(d-1)^2 = 16$ distinct critical points of $R_5(x,y)$ lie on three different critical levels: $\binom{5}{2} = 10$ have critical value 0 (the intersections of the five lines), 5 with critical value v_1 (within each triangle) and 1 with critical value $v_0 \neq 1$ (the center).

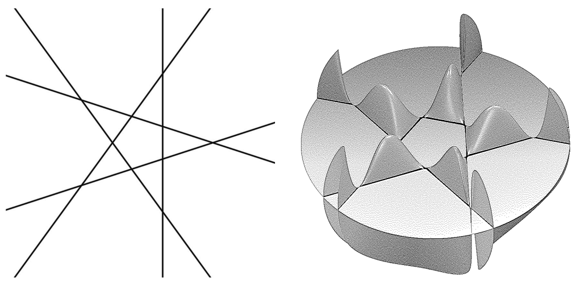


FIGURE 3.4. The regular five-gon, once seen in the plane, once in space.

It is now clear that the quintic in $\mathbb{P}^4$ given by

$$\text{Hirz}_5^4: R_5(x_0,x_1) - R_5(x_2,x_3) = 0 \tag{3.34}$$

has exactly 126 ordinary nodes, 100 coming from the intersection of two lines, and 25 + 1 others. Thus:

$$\mu_{A_1}^4(5) \geq \mathbf{126}. \tag{3.35}$$

In view of Givental's idea (3.25), this construction can also be generalized to higher dimensions and degrees. Although we could not find this generalization in the literature, its basic idea was certainly known to those working on the subject at that time. We denote by $R_5(x,y)$ the regular five-gon normalized s.t. the critical value over the origin is $+1$. Notice that then the other non-zero critical point is -1. We can translate Givental's equations (3.25) for cubics word by word:

$$\text{(3.36)}\quad \mathrm{GH}_5^n: \quad \sum_{j=0}^{\frac{n}{2}-1} (-1)^{j\cdot(1+(n \bmod 2))} R_5(x_{2j}, x_{2j+1}) = -(n \bmod 2)\frac{T_5(x_{n-1})-1}{2}.$$

For low n, we get the following numbers of nodes on quintics in $\mathbb{P}^n$:

n	1	2	3	4	5	6	7	8	9
$\mu^n(5) \leq$	2	10	31	135	456	1918	6728	27876	100110
$\mu^n_{A_1}(5) \geq$	2	10	31	**126**	**420**	**1620**	**5750**	**23126**	**78300**

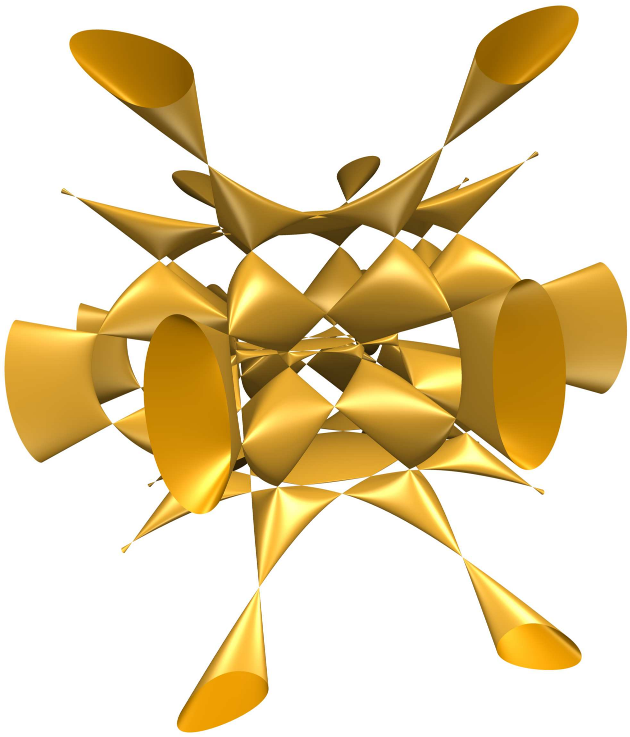
If we replace in GH_5^n the polynomial $R_5(x,y)$ by another polynomial of degree d which has exactly three critical values of the form $0, 1, -1$ then the formula can be used verbatim. Instead, if it is not possible to bring the critical values of a polynomial into this form then another formula is better, see e.g. 4.1 on page 45.

3.13. The Defect and the Existence of Certain Nodal Hypersurfaces

In **[Cle83]**, Clemens introduced the notion of defect $\delta(X) := b_4(X) - b_2(X)$ of a nodal hypersurface X in $\mathbb{P}^n$, where b_i denotes the i^{th} Betti number. If X deforms, but maintains its number of nodes, the defect remains constant.

Based on this article, Borcea **[Bor90]** considered the special case of nodal threefolds with trivial dualizing sheaf and was able to interpret the defect as follows: Let X be a quintic threefold with $\mu(X)$ nodes and defect $\delta(X)$. Then there exist quintics with $0 \leq k \leq \mu(X)$ nodes with at most $\delta(X)$ exceptions. The same holds for double solids ramified over a nodal octic surface. This result can be seen as a generalization of a result of Greuel/Karras in **[GK89]** which states that hypersurfaces with all lower numbers of nodes exist if X is unobstructed, i.e. if a certain cohomology group vanishes. Using the defect, their condition can be written as $\delta(X) = 0$.

Several people computed the defect in some special cases. E.g., Werner **[Wer87]** treated the case of several variants of Chmutov's octics which are of the form presented in section 3.8 modulo sign changes. Among these, there is a 108-nodal example with defect $\delta = 0$. Using Borcea's previous result this shows the existence of octics with $0, 1, \ldots, 108$ nodes. Another of the octics has 144 nodes and defect $\delta = 9$. Thus between 108 and 144 at most nine gaps may appear. Schoen's quintic in $\mathbb{P}^4$ (section 3.12 on the facing page) has 125 nodes and defect $\delta = 24$. Thus, up to 125 at most 24 gaps may exist, but most of them have to occur for large numbers of nodes because Borcea gives an 100-nodal example with defect $\delta = 3$. To our knowledge it has not been checked yet which numbers of nodes actually occur on octic surfaces or quintic threefolds. It might be possible to apply our methods presented in part 2 of this thesis for this purpose because most of the examples can certainly be found in some obvious families.



Endraß's 168-nodal octic from 1996. He located it within a five-parameter family of $D_8 \times \mathbb{Z}_2$-symmetric 112-nodal surfaces of degree eight.

CHAPTER 4

Recent Results (1991 until now)

Since Miyaoka's and Varchenko's upper bounds for the maximum possible number of nodes from the early 1980's there has not yet appeared any essentially new idea for producing new upper bounds. But since the early 1990's several new lower bounds have been found.

First of all, Chmutov improved his own general construction in the case of nodal surfaces and threefolds. Both families are still the best known ones for general degree.

Apart from that, several special cases have been improved. E.g., van Straten constructed a quintic in $\mathbb{P}^4$ with 130 nodes, and Barth constructed his famous sextic in $\mathbb{P}^3$ with 65 nodes. Using methods similar to Beauville's proof of $\mu_{A_1}(5) = 31$, Jaffe and Ruberman were then able to show that $\mu_{A_1}(6) = 65$. So, $\mu_{A_1}(d)$ is known for $d \leq 6$.

In the cases of degree $8, 10, 12$ there also appeared constructions exceeding Chmutov's general lower bound. But for odd degree $d > 5$ no such surface was found.

Parallely, people started to consider also other singularities of small degree. E.g., Barth constructed a quartic with the maximum number of 8 cusps and a quintic with 15 cusps. Based on results of Nikulin and Urabe on K3 lattice theory, Yang completed the enumeration of all combinations of singularities on quartics in $\mathbb{P}^3$ using computers.

4.1. Chmutov's Hypersurfaces using Folding Polynomials

When trying to generalize Givental's cubics from section 3.9 on page 39 in the case of surfaces to higher degree d one realizes the following. For the number of nodes on the resulting surfaces only the critical points with two different critical values on a plane curve are relevant. This immediately leads to the question what the maximum number of critical points on two critical levels of a plane curve of degree d is. Of course, a trivial upper bound is $2 \cdot \binom{d}{2} \approx d^2$. Chmutov succeeded in proving a stronger result [**Chm84**] similar to Varchenko's spectral bound. In [**Chm95**] he mentioned the special case of non-degenerate critical points for which this leads to an upper bound of $\approx \frac{7}{8}d^2$ critical points on two levels. As he remarked in [**Chm92**], this bound immediately implies a bound for the maximum number $\mu^3_{\text{sep}}(d)$ of nodes on a surface of degree d of the form of Givental's cubics $p(x, y) + q(z) = 0$ (he called them *surfaces in separated variables*):

$$\mu^3_{\text{sep}}(d) \lessapprox \frac{7}{16} d^3. \tag{4.1}$$

This is less than Miyaoka's upper bound: $\frac{7}{16}d^3 = \frac{63}{144}d^3 < \frac{64}{144}d^3 = \frac{4}{9}d^3$. Thus, it is not possible to reach Miyaoka's upper bound with surfaces in separated variables.

In view of this upper bound, it is natural to ask how close one can get. We are thus looking for plane curves with very few different critical values (in fact, the minimum is three for plane curves of degree $d \geq 4$). The first remark is that regular d-gons have exactly $\binom{d}{2}$ critical values with critical value 0 and only d critical values on the other critical levels.

Chmutov [**Chm92**] realized that the so-called folding polynomials $F_d^{A_2}(x,y)$ associated to the root system A_2 (see [**Wit88**] and also [**HW88, EL82**]) are very well-suited for this purpose. In fact, they give $\approx \frac{5}{6}d^2$ critical points on two levels. In [**Chm95**], Chmutov even conjectured that this is the maximum number of critical points on two critical levels. The folding polynomials $F_d^{A_2}(x,y)$ can be defined as follows:

$$F_d^{A_2}(x,y) := 2 + \det \begin{pmatrix} x & 1 & 0 & \cdots\cdots & 0 \\ 2y & x & \ddots & \ddots & \vdots \\ 3 & y & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots\cdots & 0 & 1 & y & x \end{pmatrix} + \det \begin{pmatrix} y & 1 & 0 & \cdots\cdots & 0 \\ 2x & y & \ddots & \ddots & \vdots \\ 3 & x & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots\cdots & 0 & 1 & x & y \end{pmatrix}. \tag{4.2}$$

These polynomials are generalizations of the Tchebychev polynomials in many senses (see [**Wit88, HW88, EL82**]). The property which is important for Chmutov is the fact that they have very few (in fact, three) different critical values. He showed that such a polynomial $F_d^{A_2}$ had $\binom{d}{2}$ critical points with critical value 0 and

$$\begin{array}{ll} \frac{1}{3}d(d-3) & \text{if } d \equiv 0 \mod 3, \\ \frac{1}{3}(d(d-3)+2) & \text{otherwise} \end{array} \tag{4.3}$$

critical points with critical value -1. The other critical points have critical value 8. As one might guess from the number $\binom{d}{2}$ of critical points on the level 0, the polynomial $F_d^{A_2}$ consists in fact of d lines.

The number of nodes of Chmutov's surfaces defined by the affine equation

$$\mathrm{Chm}_d^3: \quad F_d^{A_2}(x_0,x_1) + \frac{1}{2}(T_d(x_2)+1) = 0 \tag{4.4}$$

can easily be computed using (4.3):

$$\begin{array}{ll} \frac{1}{12}\left(5d^3 - 13d^2 + 12d\right) & \text{if } d \equiv 0 \mod 6, \\ \frac{1}{12}\left(5d^3 - 13d^2 + 16d - 8\right) & \text{if } d \equiv 2,4 \mod 6, \\ \frac{1}{12}\left(5d^3 - 14d^2 + 13d - 4\right) & \text{if } d \equiv 1,5 \mod 6, \\ \frac{1}{12}\left(5d^3 - 14d^2 + 9d\right) & \text{if } d \equiv 3 \mod 6. \end{array} \tag{4.5}$$

Thus:

$$\mu_{A_1}(d) \gtrapprox \frac{\mathbf{5}}{\mathbf{12}}\mathbf{d^3}. \tag{4.6}$$

For low even degree, Kreiss's construction (if correct) is at least as good as Chmutov's: $\mu(\mathrm{Chm}_8^3) = 321 < 325$ and for $\mu(\mathrm{Chm}_{12}^3) = 576$.

d	5	6	7	8	9	10	11	12	d
$\mu(d)(\mu_{D_p}(d)) \leq$	31	68 (66)	104	174	246	360	480	645	$\approx \frac{4}{9}d^3$
$\mu_{A_1}(d)(\mu(d)) \geq$	31	64	**93**	160	**216**	325	**425**	576	$\approx \frac{\mathbf{5}}{\mathbf{12}}\mathbf{d^3}$

Of course, the folding polynomials $F_d^{A_2}(x,y)$ can also be used in higher dimensions (compare Hirzebruch's quintic (3.34) and Givental's cubics (3.25)). In **[Chm92]** Chmutov only mentioned the case $\mathbb{P}^4$ in which $\mathrm{Chm}_d^4 := F_d^{A_2}(x_0,x_1) - F_d^{A_2}(x_2,x_3)$ gives threefolds with approximately $\frac{7}{18}d^4$ nodes. With Varchenko's spectral bound, we get:

d	3	4	5	6	7	8	9	10	11	12
$\mu^4(d) \leq$	10	45	135	320	651	1190	2010	3195	4840	7051
$\mu^4_{A_1}(d) \geq$	10	45	126	**277**	**566**	**1029**	**1720**	**2745**	**4150**	**6013**

In view of the equations of Givental's cubics in $\mathbb{P}^n$, we can easily write down equations for hypersurfaces with many nodes in $\mathbb{P}^n$ using the folding polynomials. As we could not find them in the literature, we give them here explicitly:

$$\mathrm{Chm}_d^n : \quad \sum_{j=0}^{\lfloor\frac{n}{2}\rfloor-1} (-1)^j F_d^{A_2}(x_{2j}, x_{2j+1}) = (n \bmod 2)\cdot\frac{1}{2}(T_d(x_{n-1})+1). \tag{4.7}$$

In some cases, we get more nodes if we replace the sign $(-1)^j$ by 1, e.g. if $n=5$. Furthermore, for small degree, it is easy to find out which are the best plane curves for this purpose. E.g., in degree 4, there is a better choice than $F_4^{A_2}$: The union of four lines, s.t. two of non-singular critical points have critical value -1 and the remaining one has critical value $+1$. Of course, Chmutov's upper bound for surfaces in separated variables also generalizes to higher dimensions.

But notice that Chmutov's older construction (section 3.8 on page 38) is asymptotically better than his new construction for any fixed $n \geq 5$ and large d. Intuitively, the reason for this is that the two non-zero critical values of the polynomials $T_d(x)+T_d(y)$ also sum up to zero; this is not the case for $F_d^{A_2}$. In $\mathbb{P}^5$, we get the following table for low degree:

d	3	4	5	6	7	8	9
$\mu^5(d) \leq$	15	126	456	1506	3431	7872	14412
$\mu^5_{A_1}(d) \geq$	15	**104**	420	1080	**2583**	5760	**10368**

4.2. Barth's 31-nodal Quintic in $\mathbb{P}^3$

Although Barth only published his construction of a 31-nodal quintic in $\mathbb{P}^3$ as a preprint, it is quite interesting and we thus describe it here shortly (a longer exposition can be found in **[End96]**).

As a starting point, Barth took a family of quintics $\prod_{i=0}^4 P_i(x,y) - az\cdot Q^2$ coming from Rohn's construction 1.3 on page 16. In order to be able to reduce the problem in three-space to a planar one, he took planes P_i and a quadric Q which admit the symmetry D_5 of a five-gon: $F_{a,b,d} := P - az\cdot Q^2$, where

$$\begin{aligned} P &:= \prod_{j=0}^4 \left(\cos\left(\tfrac{\pi j}{4}\right)x + \sin\left(\tfrac{\pi j}{4}\right)y - w\right) \\ &= \tfrac{1}{16}\Big(x^5 - 5x^4w - 10x^3y^2 - 10x^2y^2w + 20x^2w^3 \\ &\qquad +5xy^4 - 5y^4w + 20y^2w^3 - 16w^5\Big), \\ Q &:= x^2+y^2+bz^2+zw+dw^2, \end{aligned} \tag{4.8}$$

and where $a,b,d \in \mathbb{C}$ are still to be determined.

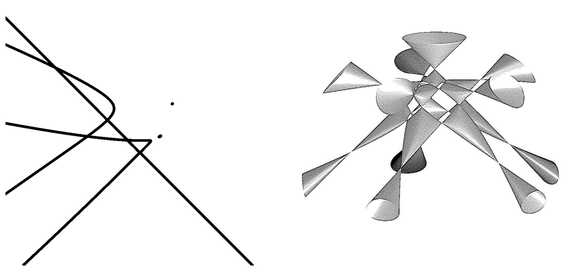


FIGURE 4.1. Barth's Togliatti quintic with 31 nodes and its restriction to the plane $y = 0$. Notice that this plane quintic consists of a line and an irreducible quartic. This latter has three nodes two of which are solitary (also called A_1^+) points.

A generic member of this family has 20 nodes. Because of the symmetry, four of these are in each of the symmetry planes

$$E_j := \left\{ \sin(\frac{\pi j}{n})x = \cos(\frac{\pi j}{n})y \right\}, j = 0, 1, \ldots, 4.$$

In fact, the symmetry allowed Barth to restrict his attention to one of these planes, say E_0. Every node N of the plane curve $C_{a,b,d} := E_0 \cap F_{a,b,d}$ induces an orbit of length five if N does not lie on the axes $x = y = 0$. In order to get 31 ordinary double points, we thus have to find parameters a, b, d, s.t. the plane quintic $\mathrm{Bar}_{31} := C_{a,b,d}$ has one additional node on the axes $x = y = 0$ and two additional nodes away from this axes.

But an irreducible quintic can have at most six nodes, s.t. our curve $C_{a,b,d}$ has to be reducible in order to have the seven nodes that we need. In fact, a lengthy analysis shows that the curve is a union of a line and a three-nodal quartic with the parameters

$$a = -\frac{5}{32}, \quad b = -\frac{5 - \sqrt{5}}{20}, \quad d = -(1 + \sqrt{5}).$$

The only node on the axes $x = y = 0$ comes from the intersection of the line with the quartic. The surface $F_{a,b,d}$ corresponding to these parameters has therefore $20 + 2 \cdot 5 + 1 = 31$ nodes, see fig. 4.1.

4.3. Van Straten's 130-nodal Quintic in $\mathbb{P}^4$

As we have seen in section 1.5 on page 18, C. Segre's 10-nodal cubic and the 45-nodal Burkhardt Quartic in $\mathbb{P}^4$ have nice Σ_6-symmetric equations. In **[vS93]**, van Straten analyzed all singular examples in the space of all Σ_6-symmetric quintics in the $\mathbb{P}^4$ given by cutting the $\mathbb{P}^5$ by the hyperplane $\sigma_1(x_0, \ldots, x_5) = 0$. It is spanned by $\sigma_5(x_0, \ldots, x_5)$ and $\sigma_2(x_0, \ldots, x_5)\sigma_3(x_0, \ldots, x_5)$. Besides several other interesting quintics, this pencil

$$\mathrm{vS}_{(\alpha:\beta)} := \alpha \cdot \sigma_5(x_0, \ldots, x_5) + \beta \cdot \sigma_2(x_0, \ldots, x_5) \cdot \sigma_3(x_0, \ldots, x_5) \tag{4.9}$$

contains the 130-nodal example $\mathrm{vS}_{(1:1)}$ showing (compare section 3.12):

$$\mu^4_{A_1}(d) \geq \mathbf{130}. \tag{4.10}$$

The nodes of this quintic form three orbits under the operation of the Σ_6 on the coordinates:

$$\begin{array}{ll} (1:1:1:-1:-1:-1) & 10 \text{ nodes,} \\ (1:1:-1:-1:\sqrt{-3}:\sqrt{-3}) & 90 \text{ nodes,} \\ (1:1:1:1:\sqrt{-3}-2:\sqrt{-3}-2) & 30 \text{ nodes.} \end{array} \tag{4.11}$$

As one might expect, this 130-nodal quintic has some nice properties similar to those of C. Segre's cubic and the Burkhardt quartic. But in contrast to these two varieties in $\mathbb{P}^4$, it is not invariant under the simple group of order 25920.

4.4. Goryunov's Symmetric Quartics in $\mathbb{P}^n$

Inspired by the construction of the 130-nodal van Straten quintic, Goryunov **[Gor94]** looked at all nodal quartics and cubics in $\mathbb{P}^n$ which are invariant under the reflection groups A_n or B_n.

Of course, in the case of cubics in $\mathbb{P}^n$, his examples could not give more nodes than Kalker's examples (section 3.11 on page 41) because those already reached Varchenko's upper bound. In fact, Goryunov found isomorphic cubics using his method.

But his B_n-symmetric quartics gave rise to new lower bounds. His construction is based on his observation that one can reformulate the condition that a hypersurface in $\mathbb{P}^n$ has a singularity in the case of hypersurfaces symmetric under the considered reflection groups: It turned out to be equivalent to the condition that the corresponding hypersurface in the orbit space is nontransversal to the discriminant of the group. He thus constructed his examples with many nodes by finding a hypersurface in the orbit space that is nontransversal to the strata of very long orbits.

Using this method he showed that the B_{n+1}-symmetric hypersurface

$$\text{Gory}_4^n(a): \quad 2\cdot(a+1)\cdot\Big(\sum_{0\le i<j\le n} x_i^2x_j^2\Big) - a\cdot\Big(\sum_{0\le j\le n} x_j^2\Big)^2 = 0 \tag{4.12}$$

has exactly $2^a\binom{n+1}{a+1}$ nodes. This number is maximal for $a = \lfloor\frac{2n}{3}\rfloor$ and $a = \lfloor\frac{2n+1}{3}\rfloor$ which both yield:

$$\mu_{A_1}^n(4) \geq \mathbf{2}^{\lfloor\frac{\mathbf{2n}}{\mathbf{3}}\rfloor}\binom{\mathbf{n+1}}{\lfloor\frac{\mathbf{2n}}{\mathbf{3}}\rfloor+\mathbf{1}}. \tag{4.13}$$

His A_{n+1}-symmetric quartics exceed this number only for $n = 4$ (this gives the 45-nodal Burkhardt Quartic) and $n = 7$. We obtain the following table (with Varchenko's upper bound):

n	2	3	4	5	6	7	8	9	10	n
$\mu^n(4) \leq$	6	16	45	126	357	1016	2907	8350	24068	$\approx \frac{\sqrt{3}}{2}\frac{3^{n+1}}{\sqrt{\pi n}}$
$\mu^n(4) \geq$	6	16	45	**120**	**336**	**938**	**2688**	**7680**	**21120**	$\approx \frac{\mathbf{3}}{\mathbf{4}}\frac{\mathbf{3^{n+1}}}{\sqrt{\pi \mathbf{n}}}$

In table **[Gor94**, p. 148**]**, Goryunov listed Chmutov's old hypersurfaces (section 3.8 on page 38) as the previously known best lower bounds although the generalization of Chmutov's new construction (section 4.1 on page 45) leads to greater numbers of nodes for small n. But also in comparison to these, Goryunov's examples are better for all n.

4.5. Barth's Icosahedral-Symmetric Surfaces and $\mu_{A_1}(6) = 65$

Similar to his construction of the 31-nodal quintic in $\mathbb{P}^3$, Barth also used the idea to analyze a pencil of symmetric surfaces to treat the case of degree six and ten [**Bar96**]. The main advantage of these two cases is the fact that one can use an even larger symmetry group than in the case of the five-gon-symmetry for the quintic: Barth's surfaces of degree 6 and 10 are invariant under the symmetry group of the icosahedron in euclidean three-space $\mathbb{R}^3$ which contains the dihedral group D_5 as a subgroup.

Let $\tau := \frac{1}{2}(1+\sqrt{5})$. The six planes through the origin which are orthogonal to the six diagonals of the regular icosahedron are given by the affine equation $P := (\tau^2x^2 - y^2)(\tau^2y^2 - z^2)(\tau^2z^2 - x^2)$. Consider the family

$$F_\alpha := P - \alpha{\cdot}Q^2,$$

where $Q := x^2 + y^2 + z^2 - 1$ is a sphere and $\alpha \in \mathbb{C}$ is a parameter still to be determined (compare Rohn's construction in section 1.3 on page 16).

For generic values of $\alpha \neq 0$, the surface F_α has 45 singularities. 30 of these come from the intersection of P and Q as in Rohn's construction, 15 are at infinity. Barth then enforced a third orbit of 20 singularities on the ten lines joining two opposit centers of faces of the icosahedron. Because of the symmetry he could restrict the computations to one of these lines which led to $\alpha = \frac{1}{4}(2\tau + 1)$. Altogether, he obtained a surface

$$\mathrm{Bar}_{65} := F_{\frac{1}{4}(2\tau+1)}$$

with $30 + 15 + 20 = 65$ nodes (see fig. 4.2).

A similar construction gave a surface Bar_{345} of degree 10 with 345 nodes (see also fig. 4.2). Its equation is as follows:

$$\begin{aligned} 8(x^2 - \tau^4y^2)(y^2 - \tau^4z^2)(z^2 - \tau^4x^2)\Big(x^4 + y^4 + z^4 - 2(x^2y^2 + y^2z^2 + z^2x^2)\Big) \\ +(3+5\tau)(x^2+y^2+z^2-1)^2\Big(x^2+y^2+z^2-(2-\tau)\Big)^2 = 0. \end{aligned} \tag{4.14}$$

Taking into account both surfaces we have the new lower bounds:

$$\mu_{A_1}(6) \geq \mathbf{65}, \quad \mu_{A_1}(10) \geq \mathbf{345}. \tag{4.15}$$

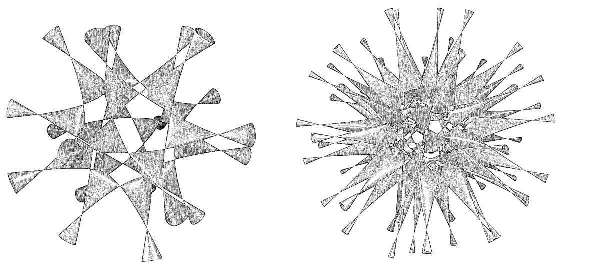


FIGURE 4.2. Barth's 65-nodal sextic and 345-nodal dectic.

As already mentioned in section 3.5 on page 34, the existence of the 65-nodal sextic was even more astonishing in view of Catanese's and Ceresa's claimed upper bound for surfaces constructed using B. Segre's 8-fold covering method. In fact, as one can see from the equations, Barth's sextic is of this type, but exceeds this bound. And indeed, by analyzing his construction carefully, Barth was able to trace down the error in Catanese's and Ceresa's reasoning.

An interesting fact was computed by van Straten using deformation theory and computer algebra (see next section): The Barth sextic is contained in a three-parameter family of 65-nodal sextics. Furthermore, van Straten suggested to try to construct this family explicitly using Gallarati's construction (section 2.5 on page 24). This was done by Pettersen in his Ph.D. thesis [**Pet98**].

Only very shortly after Barth's discovery of the 65-nodal sextic, Jaffe and Ruberman [**JR97**] were able to show $\mu_{A_1}(6) \leq 65$ using arguments similar to Beauville's (section 3.3 on page 33) which finally showed:

$$\mu_{A_1}(6) = \mathbf{65}. \tag{4.16}$$

After the appearance of this result in degree six, Endraß considered even sets of nodes and their codes related to it in more generalality, see [**End98, End99**]. But this did not allow him to deduce new upper bounds for higher degrees.

4.6. Deformations of Nodal Hypersurfaces

Van Straten's computation of the fact that Barth's 65-nodal sextic varies in a three-parameter family is an application of his deformation theory for nodal hypersurfaces in $\mathbb{P}^n$ which he developped in a still unfinished paper [**vS94**]. His theory is based on the deformation theory for non-isolated singularities which he developped in [**dJvS90**] together with de Jong.

Van Straten considered the affine cone over the singular locus $\Sigma := \Sigma(X)$ of the nodal hypersurface X which is given by a homogenous polynomial $X \in P := \mathbb{C}[x_0, \ldots, x_n]$ of degree d. This allowed him to apply the above deformation theory of non-isolated singularities. The deformation functor $\mathrm{Def}(X, \Sigma)$ consists of deformations of the projective hypersurface X which induce analytically trivial deformations of the multigerm of X around Σ. $T^1(X, \Sigma)$ is the space of infinitesimal deformations and $T^2(X, \Sigma)$ the obstruction space.

We get for the infinitesimal embedded deformations $T^1(X) = (P/J)_d$, where $J := \mathrm{Jac}(X) := (\frac{\partial X}{\partial x_0}, \ldots, \frac{\partial X}{\partial x_n})$ denotes the jacobian ideal of X. For the infinitesimal deformations of the multigerm (X, Σ), we have: $T^1(\mathcal{O}_{(X,\Sigma)}) = \oplus_{x \in \Sigma} T^1(\mathcal{O}_{X,x})$. As Σ is reduced in our case, van Straten could apply some vanishing results which make a long exact sequence from [**dJvS90**] collaps to:

$$0 \to T^1(X, \Sigma) \to T^1(X) \to T^1(\mathcal{O}_{(X,\Sigma)}) \to T^2(X, \Sigma) \to 0.$$

We denote the saturation of J w.r.t. $\mathfrak{m} := (x_0, \ldots, x_n)$ by $I := J : \mathfrak{m}^\infty$. Van Straten argued that the above sequence is isomorphic to the degree d part of the sequence of graded P-modules:

$$0 \to H^0_{\mathfrak{m}}(P/J) \to P/J \to P/I \to H^1_{\mathfrak{m}}(P/J) \to 0.$$

This immediately yields:

$$\begin{aligned} \dim T^1(X, \Sigma) - \dim T^2(X, \Sigma) &= \dim(P/J)_d - \sum_{x \in \Sigma} \tau(X, x) \\ &= \binom{n+d}{d} - (n+1)^2 - \#(\mathrm{nodes}(X)). \end{aligned} \tag{4.17}$$

As $H^0_{\mathfrak{m}}(P/J) = I/J$, we get from the above exact sequences:

$$T^1(X,\Sigma) = (I/J)_d. \tag{4.18}$$

We can thus compute $\dim T^1(X,\Sigma)$ using computer algebra and — via (4.17) — also $\dim T^2(X,\Sigma)$ which represents the number of independent conditions imposed on a polynomial of degree d to pass through the nodes Σ.

As an example, let us consider the case of nodal sextics f_6. It is known that $\#(\text{nodes}(f_6)) \le 65$ as we have seen in the previous section. As $\dim T^2(f_6,\Sigma) \ge 0$, we obtain: $\dim T^1(f_6,\Sigma) \ge 68 - 65 = 3$. Indeed, using SINGULAR we can compute via (4.18) that $\dim T^1(\text{Bar}_{65},\Sigma) = 3$ and thus $\dim T^2(\text{Bar}_{65},\Sigma) = 0$ for Barth's 65-nodal sextic Bar_{65} (section 4.5 on page 50).

Let us mention some other immediate consequences of the linear relation (4.17) between $\dim T^1(X,\Sigma)$ and $\dim T^2(X,\Sigma)$:

COROLLARY 4.1. *With the notations above, we have:*

(1) *If there exists a rigid nodal septic f_7 in $\mathbb{P}^3$ (i.e. $\dim T^1(f_7,\Sigma(f_7)) = 0$) then it has exactly 104 nodes and $\dim T^2(f_7,\Sigma(f_7)) = 0$.*
(2) *If we have $\dim T^2(f_d,\Sigma(f_d)) \le c$ for some nodal hypersurface f_d of degree d in $\mathbb{P}^n$ for some $c \in \mathbb{N}_0$ then*

$$\#(\text{nodes}(f_d)) \le \dim(P/J)_d + c = \binom{n+d}{d} - (n+1)^2 + c.$$

(3) *In particular, the number of nodes of an unobstructed nodal hypersurface f_d of degree d in $\mathbb{P}^n$ is bounded by:*

$$\#(\text{nodes}(f)) \le \binom{n+d}{d} - (n+1)^2.$$

In chapter 10 on page 121, we list $\dim T^1(f_d,\Sigma(f_d))$ and $\dim T^2(f_d,\Sigma(f_d))$ for many nodal hypersurfaces f_d of degree d in $\mathbb{P}^n$. From these computations, we can observe many interesting things. E.g., the 168-nodal octic presented in the following section is the only known rigid octic although the restriction from the corollary allows the existence of 149-nodal rigid octics. Why?

If $\dim T^2(X,\Sigma) = 0$ for some nodal hypersurface X then any local deformation can be globalized to X. Thus, the existence of a nodal hypersurface X implies the existence of hypersurfaces with any non-negative number $\le \#(\text{nodes}(X))$ of nodes.

4.7. Endraß's 168-nodal Octics

Barth's construction of the 65-nodal sextic and the 345-nodal dectic showed that Rohn's and B. Segre's constructions were even more powerful than the geometers had thought before. So, Barth's Ph.D. student Endraß considered surfaces of degree 8 which arise in the same way. The main result of his thesis **[End97, End96]** was the construction of on octic with 168 nodes.

He started with a D_8-invariant 9-parameter family $F := P - Q$ of surfaces of degree 8, where

$$\begin{aligned} P &:= \prod_{j=0}^{7}\left(\cos\left(\tfrac{\pi j}{4}\right)x + \sin\left(\tfrac{\pi j}{4}\right)y - w\right) \\ &= \tfrac{1}{4}(x^2-w^2)(y^2-w^2)\left((x+y)^2-2w^2\right)\left((x-y)^2-2w^2\right), \\ Q &:= a(x^2+y^2)^2 + (x^2+y^2)(bz^2+czw+dw^2) \\ &\quad +ez^4+fz^3w+gz^2w^2+hzw^3+iw^4. \end{aligned} \tag{4.19}$$

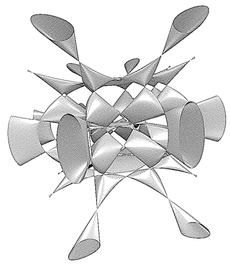


FIGURE 4.3. One of the two 168-nodal Endraß Octics.

For a generic choice of the parameters $a, b, \ldots, i \in \mathbb{C}$, this has $\binom{8}{2} \cdot 4 = 112$ nodes (compare Rohn's construction, section 1.3). For the analysis of the family, Endraß could restrict to two planes because of the symmetry of the construction. After a careful analysis of the curves in these planes, he finally found the parameters $a = -4(1+\sqrt{2})$, $b = 8(2+\sqrt{2})$, $c = 0$, $d = 2(2+7\sqrt{2})$, $e = -16$, $f = 0$, $g = 8(1-2\sqrt{2})$, $h = 0$, $i = -(1+12\sqrt{2})$ which lead to on octic Endr_{168} with 168 nodes:

$$\mu_{A_1}(8) \geq \mathbf{168}. \tag{4.20}$$

In fact, by replacing every $\sqrt{2}$ by $-\sqrt{2}$ Endraß got another 168-nodal octic Endr'_{168} which is not projectively isomorphic to the first one.

Van Straten computed, again using deformation theory (section 4.6), that this octic is rigid. In fact, this is still the only rigid nodal octic and also the rigid nodal surface of the smallest degree known up to now. Van Straten also found an octic with many nodes within the above family; his example has 165 nodes.

4.8. Yang's List of Rational Double Points on Quartics

The classification of all cubic surfaces with respect to the singularities occurring on them was already found in the 19$^{\text{th}}$ century (see section 1.1 on page 13 and [**BW79**] for a modern treatment). Although the greatest number of singularities on a quartic surface was also already determined at that time by Kummer, the classification of all quartic surfaces with respect to their singularities was only completed in 1997 by Yang [**Yan97**].

In a series of papers in the 1980's, Urabe had started to try to classify all quartic surfaces in $\mathbb{P}^3$. He had succeeded in the case of non-normal quartic surfaces [**Ura86a**]. Urabe had also performed the major steps for quartic surfaces with at least one singularity which is not a rational double point, see [**Ura85, Ura86b, Deg90**]. For the only remaining case of quartics with only rational double points, Urabe had managed to reduce the problem to a purely lattice-theoretic problem [**Ura87, Ura90**].

Using Urabe's results together with some K3 lattice theory due to Nikulin [**Nik80**], Yang was finally able to determine all possible combinations of rational double points on a quartic surfaces mainly by applying Nikulin's method systematically using a computer [**Yan97**]. This completed the classification of all quartic surfaces with respect to the singularities occurring on them more than one-hundred years after the same had been done for cubic surfaces.

To give some examples, the result shows that $\mu^3_{A_2}(4) = 8$, i.e. the maximum number of cusps on a quartic is eight. $\mu^3_{A_2}(4) \leq 8$ already follows from Varchenko's bound, but to our knowledge, the other inequality $\mu^3_{A_2}(4) \geq 8$ had not been known previously. Shortly afterwards, Barth obtained the same result by another method [**Bar00b**] – he also gave explicit equations for 8-cuspidal quartics in $\mathbb{P}^3$.

Another interesting extremal case is the highest A_j-singularity that can occur on a quartic. Either from Varchenko's spectral bound or via Nikulin's K3 lattice theory it follows that there cannot be such a singularity for $j > 19$. From Nikulin's results, one can show in an abstract way that a quartic with an A_{19} exists (e.g., by looking at Yang's list). But it is even possible to write down an explicit formula. This was already done in 1982 by Kato and Naruki [**KN82**], basically using explicit methods similar to those which had allowed Schläfli to construct a cubic surface with an A_5-singularity which we mentioned at the end of section 1.1.1 together with results of [**BW79**]:

$$\begin{aligned}(4.21)\quad & 16(x^2+y^2)+32xz^2-16y^3+16z^4-32yz^3+8(2x^2-2xy+5y^2)z^2 \\ & +8(2x^3-5x^2y-6xy^2-7y^3)z+20x^4+44x^3y+65x^2y^2+40xy^3+41y^4=0.\end{aligned}$$

4.9. Sarti's 600-nodal Dodectic

The main result of Sarti's thesis (she is another Ph.D. student of Barth) was the construction of a surface of degree 12 with 600 nodes, see [**Sar01**]. This surface is also invariant under a large symmetry group, namely the reflection group of the regular four-dimensional 600-cell. In fact, Goryunov had already announced the existence of a 600-nodal surface invariant under this group in 1996. But he had not been able to give explicit equations because the equations of the invariant S_{12} of degree 12 had not been known at that time.

This and the other invariants of the group of the 600-cell were found by Sarti in her Ph.D. thesis. We refer to [**Sar01**, p. 438] for the very lengthy equation of S_{12}. Given this, she studied the pencil

$$\mathrm{Sa}_{12}(\lambda): \ S_{12}(x,y,z,w)+\lambda(x^2+y^2+z^2+w^2)^6$$

and found the parameters $\lambda \in \mathbb{C}$, s.t. $\mathrm{Sa}_{12}(\lambda)$ admits singularities. It turned out that $\mathrm{Sa}_{12}(\lambda)$ has orbits of nodes of lengths 300, 600, 360, 60 for $\lambda = -\frac{3}{32}, -\frac{22}{243}, -\frac{2}{25}, 0$, respectively and no other singularities. Thus, $\mathrm{Sa}_{12}(-\frac{22}{243})$ is a surface in $\mathbb{P}^3$ with 600 nodes, see fig. 4.4 on the next page:

$$(4.22)\qquad \mu_{A_1}(12) \geq \mathbf{600}.$$

In an unpublished preprint, Stagnaro [**Sta01**] constructed a surface of degree 12 with 584 nodes only very shortly before the publication of Sarti's 600-nodal example. Stagnaro's construction was therefore never published.

Until this point, the following was known on $\mu(d)$:

d	5	6	7	8	9	10	11	12	d
$\mu(d)(\mu_{A_1}(d)) \leq$	31	68 (65)	104	174	246	360	480	645	$\approx \frac{4}{9}d^3$
$\mu_{A_1}(d) \geq$	31	65	93	168	216	345	425	600	$\approx \frac{5}{12}d^3$

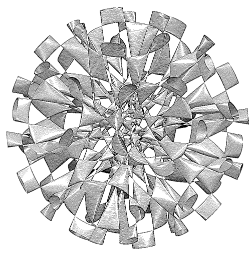


FIGURE 4.4. The 600-nodal Sarti Dodectic.

4.10. Surfaces in $\mathbb{P}^3$ with Triple Points

We already mentioned in section 3.1.3 on page 32 that the maximum number of ordinary triple points on a quintic in $\mathbb{P}^3$ was already shown to be 5 by Gallarati [**Gal52b**]. But for higher degree not much was known until 2000. In particular, for the next smallest degree, six, it was unknown if there existed a sextic with 11 triple points.

Recall that we denote by $\mu_3(d)$ the maximum number of ordinary triple points on a surface of degree d in $\mathbb{P}^3$. $\mu_3(6) \leq 11$ can easily be computed using Varchenko's bound. The authors of [**EPS03**] also found another bound, the so-called polar bound which is bad for high degree d, but for $d = 6$ it gives $\mu_3(6) \leq 10$. This polar bound is based on the fact that the position of the triple points of a surface in $\mathbb{P}^3$ cannot be too special. In fact, the authors of [**EPS03**] showed that if F_d is a surface of degree d with many triple points and V_δ is another surface of degree δ then V_δ cannot contain more than $\frac{1}{6}\delta d(d-1)$ of the triple points of F_d, counted with multiplicities. As they were also able to show the existence of a sextic with 10 triple points, they could conclude that the maximum number of triple points on such a surface was known:

$$\mu_3(6) = \mathbf{10}.$$

The authors were not able to classify all sextics with 10 triple points; this classification was completed by one of them in [**Ste03**].

For degree $d \geq 8$, the best known upper bound for $\mu_3(d)$ is Wahl's generalization [**Wah94**] of Miyaoka's bound (section 3.10 on page 40) which the authors of [**EPS03**] also computed:

$$\mu_3(d) \leq \frac{2}{27} d(d-1)^2, \quad d \geq 7.$$

For $d = 7$, Varchenko's bound is still better and computes to 17. The authors were not able to reach this bound, but they gave a one-parameter family of septics with 16 triple points, see fig. 4.5 on the following page.

In a recent article [**Sta04**], Stagnaro used again a variant of B. Segre's second construction (section 2.4) to get a surface of degree 9 with 39 triple points (upper bound: 42).

4.11. Barth's Surfaces with many Cusps

After having studied surfaces with many nodes in the early 1990's, Barth started to look at surfaces with many cusps (i.e. A_2-singularities) in the late 1990's. His aim was not only to find lower bounds for the maximum number of cusps, but also

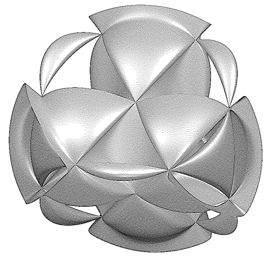


FIGURE 4.5. A Septic with 16 ordinary triple points.

to describe the codes connected to them: Similar to the code over $\mathbb{F}_2$ associated to an even set of nodes, one can construct a code over $\mathbb{F}_3$ to a three-divisible set of cusps.

Barth showed in [**Bar00b**] that a quartic in $\mathbb{P}^3$ cannot contain more than 8 cusps and he also gave explicit examples which proved the existence . As already mentioned, this also follows in an abstract way from Yang's computations, see section 4.8 on page 53. Barth's 8-cuspidal quartics were constructed as projections of 9-cuspidal sextic surfaces in $\mathbb{P}^4$ from one of their nine cusps. The one-parameter family $\mathrm{Bar}_4(k), k \neq \pm 1$, of quartics with 8 cusps is given by:

$$\begin{aligned}(4.23)\qquad &(1+k)^3 x_0^2 x_1^2 + 2k(1-k^2) x_0 x_1 x_2 x_3 - (1-k)^3 x_2^2 x_3^2 \\ &+(1-k^2)(x_0+x_1+x_2+x_3)\Big((1-k)x_2x_3(x_0+x_1) - (1+k)x_0x_1(x_2+x_3)\Big)\end{aligned}$$

One of these quartics is shown in fig. 4.6.

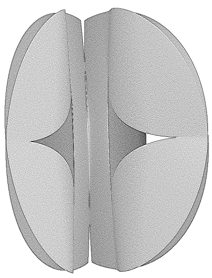


FIGURE 4.6. Barth's quartic with eight cusps for $k = 2$.

In [**Bar00a**], Barth constructed another surface with many cusps: a quintic which is nicely connected to the Clebsch Diagonal Cubic (equation (1.5)). Similar to this surface, it is Σ_5-symmetric and given by a hyperplane section of a hypersurface in $\mathbb{P}^4$:

$$(4.24)\qquad \mathrm{Bar}_{15}: \quad 5s_2s_3 - 12s_5 = 0, \quad s_1 = 0,$$

where $s_k := \sum_{i=0}^{4} x_i^k$. Bar_{15} has A_2-singularities at the 15 points in the Σ_5-orbit of $(1:1:-1:-1:0)$ which shows: $\mu_{A_2}(5) \geq \mathbf{15}$. But this surface has many other interesting geometrical properties. E.g., its intersection with the Clebsch Diagonal Surface Cle_3 consists exactly of 15 lines joining the 15 singularities in pairs, see fig. 4.7 on the next page.

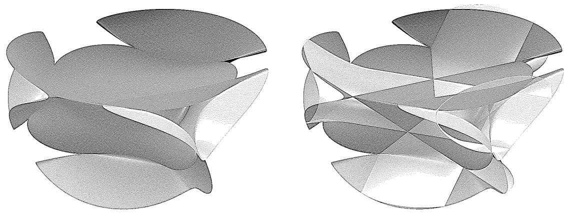


FIGURE 4.7. Barth's quintic with 15 cusps and the same surface together with the Clebsch Diagonal Cubic.

For further work on cuspidal surfaces of low degree and their codes, we refer to [**Bar98, Tan03, BR01, BR04**] . For a short overview on the results on curves, see section 2.5 on page 24.

4.12. Patchworking Singular Varieties

A topic that we did not touch so far for $d \geq 5$ is the general question which combinations of possibly different types of singularities can occur on a hypersurface of degree d in $\mathbb{P}^n$.

One of the first results in this direction is contained in Greuel's, Lossen's and Shustin's joint article [**GLS98**]. They prove the existence of an $\alpha > 0$, s.t. a plane curve with prescribed singularities of topological types $S_1, \dots, S_k$ exists if $\mu(S_1) + \cdots + \mu(S_k) \leq \alpha d^2$. The coefficient α which occurred in this sufficient criterion was later improved by one of the authors [**Los99**].

Most known results on the existence of hypersurfaces in higher dimensions with possibly different prescribed topological types are mainly based on patchworking theorems by Shustin [**Shu98, Shu00**]. Some works using these methods are [**Wes03, SW04**]. Notice that these results are the best known ones in this generality, but when restricting to hypersurfaces with only one particular type of singularity then the other constructions presented in this survey give more singularities.

4.13. Hypersurfaces with high A_j-Singularities ($j > d$)

Most constructions for hypersurfaces with many higher singularities only work for degree d large enough. Most asymptotic behaviours considered in the previous sections were of the type: fix a type of singularity and ask how many of them can occur on hypersurfaces of degree d for $d \to \infty$. It is also natural to ask the other question: Which singularities can occur on a hypersurface in $\mathbb{P}^n$ of a fixed degree d? As already mentioned, this has been answered completely for $d = 3$ (section 1.1) and $d = 4$ (section 4.8). To our knowledge, not much is known for degree $d \geq 5$.

In this section we want to give those few constructions known to us which give singularities f with high Milnor numbers $\mu(f)$, i.e., $\mu(f) > d$. As we could not find any reference in the literature to the corresponding upper bounds we also compute them here.

One Single Isolated Singularity. Consider the polynomials

$$f_{k,l}(x_1, \dots, x_n) := (x_2 - x_1^k)^l + \cdots + (x_n - x_{n-1}^k)^l + s \cdot x_n^{lk} \tag{4.25}$$

d	4	5	6	7	8	9	10	11	12
$j\cdot\mathrm{Var}_{A_j}(d)$, $j \geq 2d-1$	19	44	85	146	231	344	489	670	891

TABLE 4.1. Varchenko's upper bound for the maximum number of A_j-singularities on a surface in $\mathbb{P}^3$ of degree d as a function of j for fixed d, $j \geq 2d-1$. They are all of the form $\frac{o_{d-1}}{j}$, where $o_d = \frac{1}{3}d(2d^2+1)$ are the so-called octahedral numbers.

of degree $l{\cdot}k$ with $s \in \mathbb{C}$ generic. Then it is easy to show that the hypersurface $f_{k,2}$ in $\mathbb{P}^n$ has an A_{2k^n-1}-singularity at the origin. This is a well-known trick variants of which also exist for D_j-singularities (see e.g., [**Wes03**, p. 350]).

We already remarked that a quartic in $\mathbb{P}^3$ with an A_{19}-singularity exists, and that this is the highest possible A_j-singularity according to Varchenko's upper bound (see section 4.8 on Yang's list). Varchenko's bound is also exact for $d=3$: the highest A_j-singularity is an A_5-singularity in this case. So we may ask: Is Varchenko's bound exact for the maximum index j s.t. there exists an A_j-singularity on a surface of fixed degree d?

Globalizing the Equation. It is easy to globalize the local trick from the preceding section to to get surfaces with few A_j-singularities (i.e., $j > d$) in $\mathbb{P}^3$. Of course, there are natural generalizations to higher dimensions and several variants which produce other numbers and types of singularities:

Let $k, l \in \mathbb{N}$, $k \geq l$. The surfaces

$$g_{k,l} := \big((y^l-1)-(x^k-1)\big)^2 + \Big(z-(y^l-1)^{(\lfloor \frac{k}{l} \rfloor)}\Big)^2 + z^{2k} \tag{4.26}$$

of degree $d = 2k$ have $k{\cdot}l$ singularities of type $A_{\frac{2k^2}{l}-1}$.

Interpretation of Varchenko's Bound $\mathrm{Var}_{A_j}(d)$ as Octahedral Numbers if $j \geq 2d-1$. In the case $j > d$ Varchenko's bound (section 3.7) is usually better than Miyaoka's (section 3.10). For fixed d, these upper bounds cannot be described by a polynomial, but by a rational function. We could not find a reference of Varchenko's bound for the number of A_j-singularities for fixed degree in the literature. So, let us compute it here using the concrete expression (3.19) which we gave on page 37. It turns out that equation (3.19) simplifies drastically if we reduce to the case $j \geq 2d-1$. Indeed, if we write $d = k(j+1)+l)$ as on page 37 then $k=0$ and $d=l$. In this way, (3.19) reduces after some easy computations to:

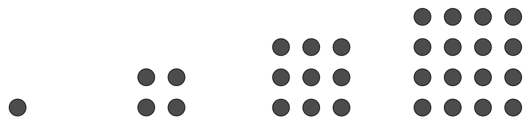
$$\mathrm{Var}_{A_j}(d) = \frac{1}{2j}\Big((2-4d^2)C + (4d-1)C^2 - C^3\Big) + \frac{2d}{3j}(d^2-1), \text{ if } j \geq 2d-1, \tag{4.27}$$

where $C := \left\lfloor \frac{2\cdot d\cdot j}{j+1} \right\rfloor$. But $C = \left\lfloor \frac{2\cdot d\cdot j}{j+1} \right\rfloor = \left\lfloor 2d - \frac{2\cdot d}{j+1} \right\rfloor = 2{\cdot}d-1$ if $j \geq 2{\cdot}d-1$. The previous formula thus collapses to:

$$\mathrm{Var}_{A_j}(d) = \frac{1}{3j}\cdot\big((d-1)\cdot(2(d-1)^2+1)\big), \qquad \text{if } j \geq 2d-1. \tag{4.28}$$

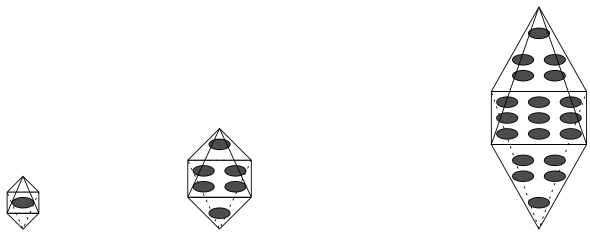
The first values of this bound are listed in table 4.1. When entering the numbers $j{\cdot}\mathrm{Var}_{A_j}(d)$ in the Sloane's on-line encyclopedia of integer sequences [**Slo03**, id: A005900] we learn that these are the so-called *octahedral numbers*. These numbers are a three-dimensional variant of square numbers $sq_i := i^2$ (see fig. 4.8): just fill up the octahedron by layers of squares (see fig. 4.9). This computes to:

$$o_d = \sum_{i=1}^{d} sq_i + \sum_{i=1}^{d-1} sq_i = \frac{1}{3}d(2d^2+1). \tag{4.29}$$



$sq_1 = 1$ $sq_2 = 2^2$ $sq_3 = 3^2$ $sq_4 = 4^2$

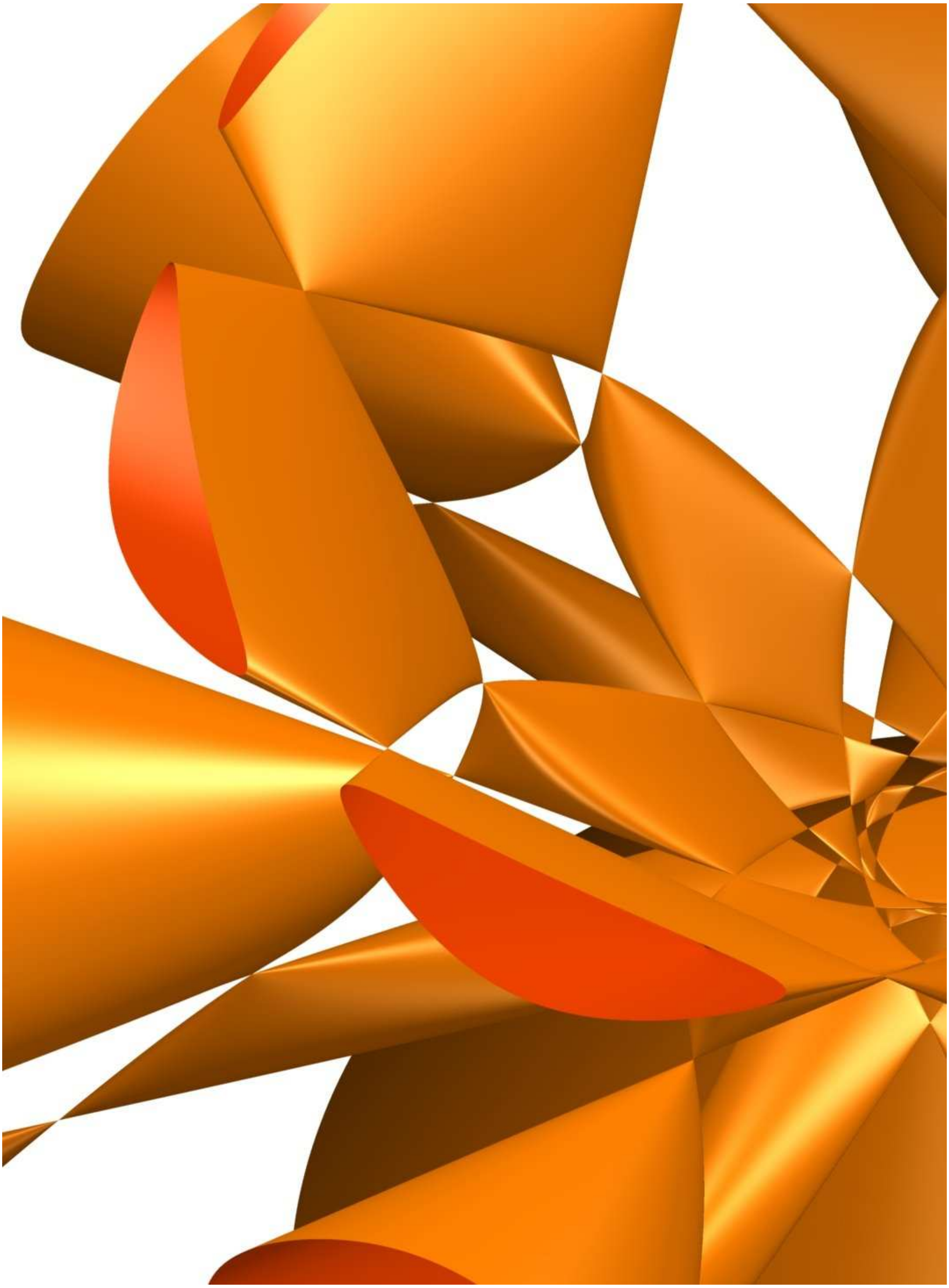
FIGURE 4.8. The square numbers $sq_i = i^2$.



$o_1 = 1$ $o_2 = 1+4+1 = 6$ $o_3 = 1+4+9+4+1 = 19$

FIGURE 4.9. The octahedral numbers $o_d = \sum_{k=1}^{d} k^2 + \sum_{k=1}^{d-1} k^2$.

Do surfaces of degree $d \geq 5$ with an $A_{o_{d-1}}$-singularity exist?



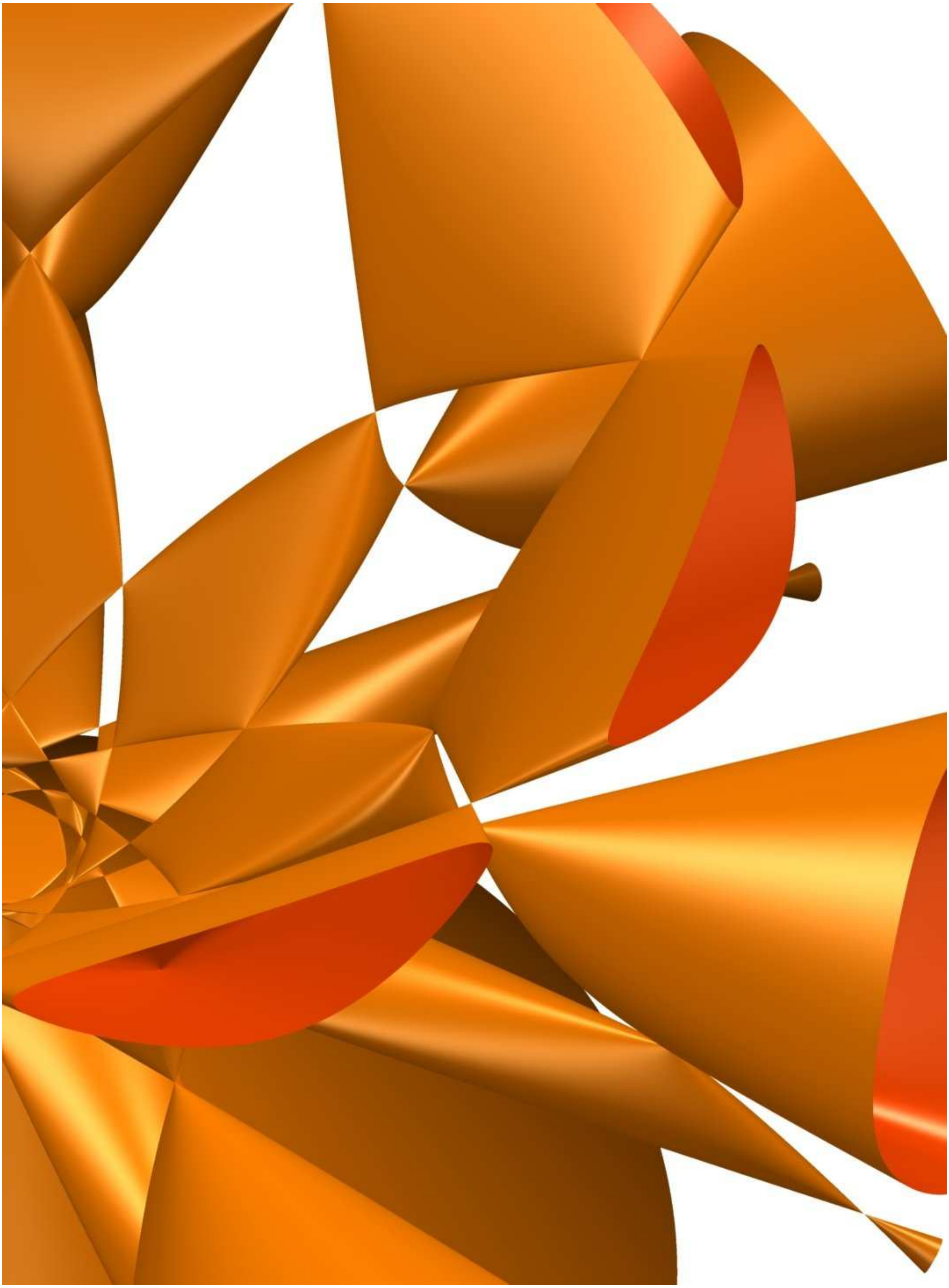


Figure on the preceding pages: A 99-nodal septic, located within a four-parameter family of 63-nodal surfaces using the geometry of prime field experiments. See [**Lab03a**] for more images and movies of algebraic surfaces.

Part 2

New Constructions and Algorithms

Introduction

We give new results related to two types of constructions: Chmutov's construction of nodal surfaces given by polynomials in separated variables (see section 4.1 on page 45) and dihedral-symmetric constructions based on Rohn's idea (see section 1.3 on page 16).

On Variants of Chmutov's Constructions. Our first new result (chapter 5 on page 67) is a variant of Chmutov's construction which gives many A_j-singularities instead of nodes. Our proof of the existence of the hypersurfaces with many A_j-singularities is based on the theory of dessins d'enfants. In most cases, our construction leads to new lower bounds:

$$\mu_{A_j}(d) \gtrapprox \frac{\mathbf{3j+2}}{\mathbf{6j(j+1)}} d^3, \quad j \geq 2.$$

Important ingredients of this construction are certain line arrangements in the plane which have many critical points with the same non-zero critical value. Using a relation to the theory of two-colorings of real line arrangements we are able to show that these arrangements are asymptotically the best possible ones (chapter 6 on page 79).

Using Computers to Study Special Cases. For special cases of low degree, it is usually possible to improve general results such as those presented in the two chapters mentioned in the previous section. The most recent new lower bounds for the maximum number of nodes on surfaces of a given degree were produced by starting with some k-parameter family of such surfaces and then using geometrical arguments to determine the parameters such that a new record was found.

But guessing such geometrical arguments requires a large amount of geometric intuition. And it turned out that the cases of odd degree $d = 7, 9, 11, \ldots$ are quite difficult to treat in that way. So, our idea was to find algorithms to locate interesting examples within the families. We present two essentially different methods: elimination and primary decomposition in characteristic zero (chapter 7), or experiments over prime fields and then lifting to characteristic zero (chapters 8 and 9). The latter allows us to construct a surface of degree 7 in $\mathbb{P}^3$ with 99 nodes (chapter 8) which is the first case of odd degree greater than five which exceeds Chmutov's general lower bound:

$$\mathbf{99} \leq \mu(7) \leq 104.$$

We then describe an algorithm which can be performed automatically by a computer (chapter 9). Our implementation as a SINGULAR library called SEARCHINFAMILIES.LIB reduces the construction of a 99-nodal septic to a 10-minute-long computer algebra computation. Similarly, all records for smaller degrees $d \leq 6$ can be reproduced. When applying the algorithm to the case $d = 9$ we obtain a nonic with 226 nodes which is also a new lower bound:

$$\mathbf{226} \leq \mu(9) \leq 246.$$

Our algorithm is very general so that it can certainly be applied to many other concrete problems in algebraic geometry.

A quintic with 15 cusps which shows the idea of how to construct surfaces with many A_j-singularities using Dessins d'Enfants.

CHAPTER 5

Dessins d'Enfants and Surfaces with Many A_j-Singularities

The best known lower bounds for surfaces of large degree d with A_1-singularities are given by Chmutov's construction (section 4.1). For higher A_j-singularities the best known constructions are still given by a direct generalization of Rohn's construction (section 1.3), $\mu_{A_j}(d) \geq \frac{1}{2}d(d-1)\lfloor\frac{d}{j+1}\rfloor$. For many degrees, one can also use Gallarati's already mentioned generalization of B. Segre's construction (section 2.5) which is usually better than Rohn's if it can be applied.

For singularities different from nodes, there exist only very few special constructions which exceed these general ones. The best known lower bounds in particular cases of low degree are given by Barth, see section 4.11: $\mu_{A_2}(4) = 8$, $15 \leq \mu_{A_2}(5) \leq 20$. In this chapter (see also [**Lab05b**]) we describe a variant of Chmutov's construction which leads to the lower bound (corollary 5.7 on page 73):

$$\mu_{A_j}(d) \gtrapprox \frac{\mathbf{3j+2}}{\mathbf{6j(j+1)}}d^3. \tag{5.1}$$

To our knowledge, this gives asymptotically the best known bounds for any $j \geq 2$. The construction reaches more than $\approx 75\%$ of the theoretical upper bound (see section 3.10 on page 40) in all cases. For quintics in $\mathbb{P}^3$, we also get an example with 15 cusps, so the gap of 5 more possible cusps remains.

Table 5.1 on the following page gives an overview of our results for low j, see also corollaries 5.7 and 5.8. We describe a generalization of our construction to higher dimensions in section 5.6 on page 74. This leads to new lower bounds even in the case of nodal hypersurfaces.

5.1. Chmutov's Idea

We start with some notation: A point $z_0 \in \mathbb{C}$ is a *critical point of multiplicity* $j \in \mathbb{N}$ of a polynomial $g \in \mathbb{C}[z]$ in one variable if the first j derivatives of g vanish at z_0: $g^{(1)}(z_0) = \cdots = g^{(j)}(z_0) = 0$. The number $g(z_0)$ is called the *critical value* of z_0. A critical point of multiplicity $j, j > 1$, is called a *degenerate* critical point. Recall from section 4.1 on page 45 that Chmutov used the following idea to construct surfaces in $\mathbb{P}^3$ with many nodes:

- Let $P_d(x, y) \in \mathbb{C}[x, y]$ be a polynomial of degree d with few different critical values, all of which are non-degenerate. By a coordinate change, we may assume that the two critical values which occur most often are 0 and -1. We assume that they occur $\nu(0)$ and $\nu(-1)$ times, and that $\nu(0) > \nu(-1)$.
- Let $T_d(z) \in \mathbb{R}[z]$ be the Tchebychev polynomial of degree d with critical values -1 and $+1$, where -1 occurs $\lfloor\frac{d}{2}\rfloor$ times and $+1$ occurs $\lfloor\frac{d-1}{2}\rfloor$ times.

j \ d	3	4	5	6	7	8	9	10	11	12	d
1	4 / 4	16 / 16	31 / 31	65 / 65	93 / 104	168 / 174	216 / 246	345 / 360	425 / 480	600 / 645	$\approx$ 5/12 / 4/9 $\cdot d^3$
2	3 / 3	8 / 8	15 / 20	36 / 37	**52** / 62	**70** / 98	**126** / 144	**159** / 202	**225** / 275	**300** / 363	$\approx$ **2/9** / 1/4 $\cdot d^3$
3	1 / 1	6 / 6	10 / 13	15 / 26	**31** / 44	64 / 69	**72** / 102	**114** / 144	**140** / 195	**198** / 258	$\approx$ **11/72** / 8/45 $\cdot d^3$
4	1 / 1	4 / 4	10 / 11	15 / 20	21 / 35	**32** / 54	**54** / 80	100 / 112	**110** / 152	**132** / 201	$\approx$ **7/60** / 5/36 $\cdot d^3$

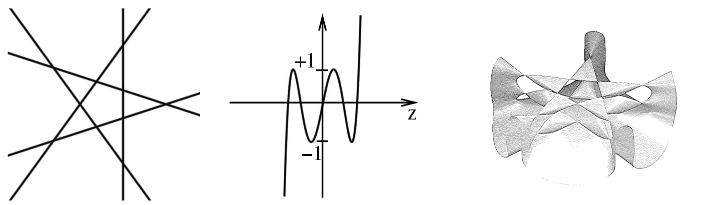
TABLE 5.1. Known upper and lower bounds for the maximum number $\mu_{A_j}(d)$ of singularities of type A_j, $j = 1, 2, 3, 4$, on a surface of degree d in $\mathbb{P}^3$. For $j \geq 2$ and $d \geq 5$, the lower bounds are attained by our examples or by Gallarati's generalization of B. Segre's idea (section 2.5 on page 24).

- It is easy to see that the projective surface given by the affine equation

$$P_d(x,y) + \frac{1}{2}(T_d(z) + 1) = 0 \qquad (5.2)$$

has $\nu(0) \cdot \lfloor \frac{d}{2} \rfloor + \nu(-1) \cdot \lfloor \frac{d-1}{2} \rfloor$ nodes.

Chmutov uses for $P_d(x,y)$ the folding polynomials $F_d^{A_2}$ associated to the root system A_2 defined in (4.2). In the case of degree 5 the best polynomial for this purpose is a regular fivegon $R_5(x,y) \in \mathbb{R}[x,y]$ with the critical value 1 at the origin and with the critical value -1 at the other non-singular critical points. The construction above then gives 30 nodes, see fig. 5.1.



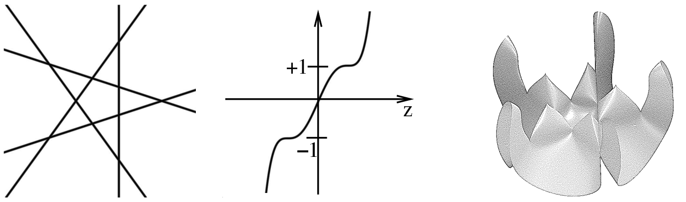
$R_5(x,y)$ $\qquad$ $T_5(z)$ $\qquad$ $R_5(x,y) + \frac{1}{2}(T_5(z)+1)$

FIGURE 5.1. A variant of Givental's and Chmutov's construction: A regular 5-gon $R_5(x,y)$, the Tchebychev polynomial $T_5(z)$ and the surface $R_5(x,y) + \frac{1}{2}(T_5(z)+1)$ with $10 \cdot 2 + 5 \cdot 2 = 30$ nodes.

5.2. Adaption to Higher Singularities

To adapt Chmutov's construction (5.2) to higher singularities of type A_j, we replace the polynomials $T_d(z)$ by polynomials with degenerate critical points.

For the construction of a quintic surface with many cusps, we thus take again the regular 5-gon $R_5(x,y) \in \mathbb{R}[x,y]$ together with a polynomial $T_5^2(z) \in \mathbb{R}[z]$ of degree 5 with the maximum number of critical points of multiplicity two. As the derivative of such a polynomial has degree 4, the maximum number of such critical



$R_5(x,y)$ $T_5^2(z)$ $R_5(x,y)+\frac{1}{2}(T_5^2(z)+1)$

FIGURE 5.2. The construction of a quintic with 15 cusps.

points is $\frac{4}{2}=2$, see fig. 5.2. The critical values of these two critical points have to be different because a horizontal line through both critical points would intersect the curve in six points counted with multiplicities. Similar to the situation for nodes in (5.2) the surface $R_5(x,y)+\frac{1}{2}(T_5^2(z)+1)$ has $10{\cdot}1+5{\cdot}1=15$ singularities of type A_2.

We take surfaces in separated variables defined by polynomials of the form:

$$\mathrm{Chm}(G_d^j) := F_d^{A_2} + G_d^j, \tag{5.3}$$

where $F_d^{A_2}(x,y) \in \mathbb{R}[x,y]$ is the folding polynomial defined in (4.2) and where $G_d^j(z) \in \mathbb{C}[z]$ is a polynomial of degree d with many critical points of multiplicity j with critical values -1 and $+1$. E.g., for $j=1$, the ordinary Tchebychev polynomials $G_d^1(z) := T_d(z)$ yield Chmutov's surfaces with many nodes. In the following sections, we discuss two generalizations of the ordinary Tchebychev polynomials to polynomials with critical points of higher multiplicity which give surfaces of degree d with many A_j-singularities, $j<d$.

5.3. j-Belyi Polynomials via Dessins d'Enfants

The existence of polynomials in one variable with only two different critical values with prescribed multiplicities of the critical points can be established using ideas of Hurwitz [**Hur91**] based on Riemann's Existence Theorem. The interest in this subject was renewed by Grothendieck's *Esquisse d'un programme* (see [**SL97a, SL97b**]). Nowadays, it is commonly known under the name of *Dessins d'Enfants*. We will use the following proposition / definition which is basically taken from [**AZ98**]:

PROPOSITION/DEFINITION 5.1.

(1) *A tree (i.e. a graph without cycles) with a prescribed cyclic order of the edges adjacent to each vertex is called a* **plane tree**. *A plane tree has a natural bicoloring of the vertices (black/white). If we fix the color of one vertex, then this bicoloring is unique.*
(2) *A polynomial in one variable with not more than two different critical values is called a* **Belyi polynomial**.
(3) *For a given Belyi polynomial $p:\mathbb{C}\to\mathbb{C}$ with critical values c_1 and c_2, we define the* **plane tree** $PT(p)$ **associated to** p *to be the inverse image $p^{-1}([c_1,c_2])$ of the interval $[c_1,c_2]$, where $p^{-1}(c_1)$ are the black vertices, and $p^{-1}(c_2)$ are the white vertices of the tree (see fig. 5.3 on the following page).*

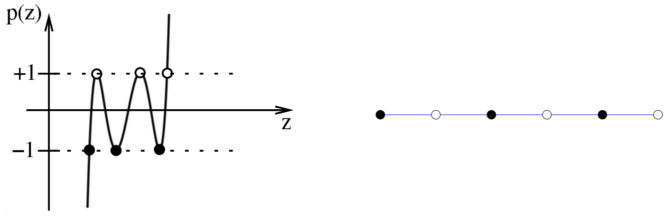


FIGURE 5.3. The ordinary Tchebychev polynomial T_5 with two critical points with critical value -1 and two with critical value $+1$. The right picture shows its plane tree $PT(T_5)$. A vertex with two adjacent edges corresponds to a critical point with multiplicity 1, a vertex with one adjacent edge corresponds to a non-critical point.

(4) *For any plane tree, there exists a Belyi polynomial whose critical points have the multiplicities given by the number of edges adjacent to the vertices minus one and vice verca.*

We will need the following two trivial bounds concerning critical points:

LEMMA 5.2. *Let $d, j \in \mathbb{N}$. Let $g \in \mathbb{C}[z]$ be a polynomial of degree d in one variable with only isolated critical points. Then:*

(1) *The total number of different critical points of g of multiplicity j does not exceed $\lfloor \frac{d-1}{j} \rfloor$.*
(2) *The number of different critical points of g of multiplicity j with the same critical value does not exceed $\lfloor \frac{d}{j+1} \rfloor$.* □

We give a special name to polynomials reaching the first of these bounds:

DEFINITION 5.1. *Let $d, j \in \mathbb{N}$ and let p be a Belyi polynomial of degree d. We call p a j-**Belyi polynomial** if p has the maximum possible number $\lfloor \frac{d-1}{j} \rfloor$ of critical points of multiplicity j.*

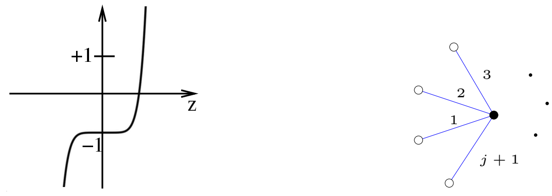
EXAMPLE 5.1. *The ordinary Tchebychev polynomials $T_d^1(z) := T_d(z)$ are 1-Belyi polynomials. $T_5^2(z)$ in fig. 5.2 on the page before is a 2-Belyi Polynomial.* □

A special type of j-Belyi polynomials are those of degree $j+1$. We will join several plane trees corresponding to such j-Belyi polynomials of degree $j+1$ to form larger plane trees in the following sections:

DEFINITION 5.2. *We call the plane tree corresponding to a j-Belyi polynomial of degree $j+1$ a j-**star**. If the center of this tree is a black (resp. white) vertex we call it a $\bullet$- (resp. $\circ$-) **centered** j-**star** (see fig. 5.4 on the facing page).*

5.4. The Polynomials $T_d^j(z)$

A natural generalization of the ordinary Tchebychev polynomials to polynomials $G_d^j(z)$ with degenerate critical points that can be used in the construction of equation (5.3) on page 69 comes from the following intuitive idea: Take polynomials which look similar to the ordinary Tchebychev polynomials (fig. 5.3), but which have higher vanishing derivatives such that they are j-Belyi polynomials.



$T_{j+1}^j(z) = z^{j+1} - 1$ *for* $j = 6$.

FIGURE 5.4. The polynomial $T_{j+1}^j(z)$ with exactly one critical point $z_0 = 0$ of multiplicity j and critical value -1 together with the corresponding $\bullet$-centered j-star.

EXAMPLE 5.2. *A* 3*-Belyi polynomial of degree* 13 *has* $\lfloor \frac{13-1}{3} \rfloor = 4$ *critical points of multiplicity* 3*. The polynomial* T_{13}^3 *has two critical points with critical value* -1 *and two with critical value* $+1$*. The plane tree showing the existence of such a polynomial consists of four connected* 3*-stars. To show the similarity to the ordinary Tchebychev polynomials we draw them in fig. 5.5 as four bouquets of* 1*-stars attached to the plane tree in fig. 5.3 on the preceding page. A straightforward* SINGULAR **[GPS01]** *script to compute the equation of* $T_{13}^3(z)$ *can be found on the website* **[Lab03a]**. □

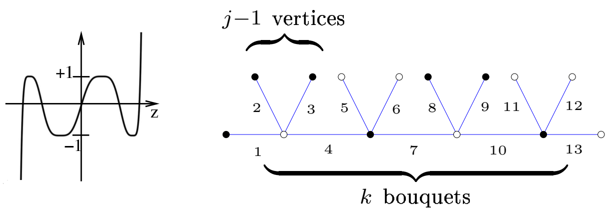


FIGURE 5.5. The bicolored plane tree $PT(T_d^j)$ for the polynomial $T_d^j(z)$ for $j = 3$, $d = 13$, $k := \frac{d-1}{j} = 4$. It consists of k connected j-stars. Here, we line them up to show the similarity to the ordinary Tchebychev polynomials in fig. 5.3 on the facing page. See **[Lab03a]** for a SINGULAR **[GPS01]** script to compute the equation of $T_{13}^3(z)$.

THEOREM/DEFINITION 5.3. *Let* $d, j \in \mathbb{N}$ *with* $d > j$*. There exists a polynomial* $\mathbf{T_d^j(z)}$ *of degree* d *with* $\lceil \frac{1}{2} \lfloor \frac{d-1}{j} \rfloor \rceil$ *critical points of multiplicity* j *with critical value* -1 *and* $\lfloor \frac{1}{2} \lfloor \frac{d-1}{j} \rfloor \rfloor$ *such critical points with critical value* $+1$.

PROOF. The corresponding plane tree $PT(T_d^j)$ can be defined as follows (compare fig. 5.5). For $d = k \cdot (j+1)$, $k \in \mathbb{N}$, we take k connected j-stars. Fixing the center of the first j-star to be white, the plane tree has a unique bicoloring. If $d = l + k \cdot (j+1)$ for some $1 \leq l \leq j$, we attach another l-star to get a polynomial of degree d. □

Although there is an explicit recursive construction of ordinary Tchebychev polynomials and their generalizations to higher dimensions (the folding polynomials, see **[Wit88]**), we do not know a similar explicit construction of the polynomials

$T_d^j(z)$ for $j \geq 2$. To our knowledge, they can only be computed for low degree d until now, e.g. using Groebner Basis. When plugged into the construction (5.3) on page 69 the existence of the polynomials T_d^j immediately implies:

COROLLARY 5.4. *Let $d, j \in \mathbb{N}$ with $d > j$. There exist surfaces*

$$\mathrm{Chm}(T_d^j) := F_d^{A_2} + \frac{1}{2}(T_d^j + 1)$$

of degree d with the following number of singularities of type A_j:

$$\begin{array}{ll} \frac{1}{2}d(d-1)\cdot\lceil\frac{1}{2}\lfloor\frac{d-1}{j}\rfloor\rceil + \frac{1}{3}d(d-3)\cdot\lfloor\frac{1}{2}\lfloor\frac{d-1}{j}\rfloor\rfloor, & \text{if } d \equiv 0 \mod 3, \\ \frac{1}{2}d(d-1)\cdot\lceil\frac{1}{2}\lfloor\frac{d-1}{j}\rfloor\rceil + \frac{1}{3}(d(d-3)+2)\cdot\lfloor\frac{1}{2}\lfloor\frac{d-1}{j}\rfloor\rfloor, & \text{otherwise.} \end{array}$$

□

5.5. The Polynomials $M_d^j(z)$

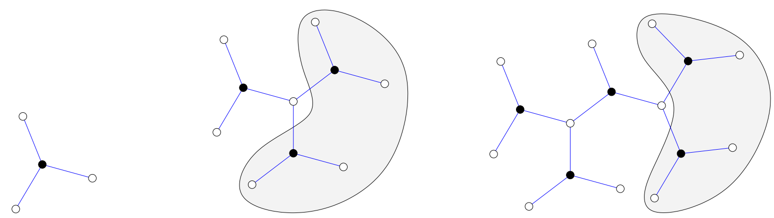
The j-Belyi polynomials $T_d^j(z)$ described in the previous section reach the first bound of lemma 5.2 on page 70. The j-Belyi polynomials $M_d^j(z)$ whose existence will be shown in this section also achieve the second bound of this lemma. We start with two examples:

EXAMPLE 5.3. *The 2-Belyi polynomial $T_9^2(z)$ is the example of the smallest degree from the previous section that does not reach the second bound of lemma 5.2. The plane tree $PT(M_9^2(z))$ in fig. 5.6 shows the existence of a 2-Belyi polynomial of degree 9 that achieves this bound.*

As in the case of the polynomials $T_d^j(z)$, it is possible to compute the polynomials $M_d^j(z)$ explicitly for low j and d. For our case $j = 2, d = 9$ we denote by u the unique critical point with critical value $+1$ and by b_0, b_1, b_2 the three critical points with critical value -1. When requiring $b_2 = 0$ (i.e., $M_9^2(0) = -1$), $M_9^2(z)$ has the derivative

$$\frac{\partial M_9^2}{\partial z}(z) = (z - b_0)^2 \cdot (z - b_1)^2 \cdot z^2 \cdot (z - u)^2.$$

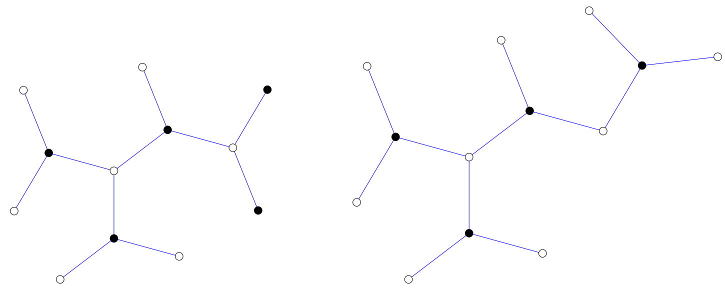
Using SINGULAR **[GPS01]**, *we find: $u^9 = 18$ and b_0 and b_1 are the two distinct roots of $z^2 - 3uz + 3u^2 = 0$. Notice that $b_0, b_1 \notin \mathbb{R}$ even if we take $u \in \mathbb{R}$.* □



(a) $PT(M_3^2)$ *(b)* $PT(M_9^2)$ *(c)* $PT(M_{15}^2)$

FIGURE 5.6. To obtain $PT(M_9^2)$ from the 2-star $PT(M_3^2) = PT(T_3^2)$, we attach two •-centered 2-stars to one of the ○-vertices (marked by the grey background). The corresponding polynomial $M_9^2(z)$ has thus 3 critical points of multiplicity 2 with critical value -1 (the 3 •-centered 2-stars) and 1 such point with critical value $+1$ (the only ○-centered 2-star). M_{15}^2 has five and two, respectively.

EXAMPLE 5.4. *If $d \neq k\cdot(j+1)$ for some $k \in \mathbb{N}$, the construction of a plane tree corresponding to a polynomial reaching both bounds of lemma 5.2 is a little more delicate than in the previous example. The cases $PT(M_{11}^2)$ and $PT(M_{12}^2)$ in fig. 5.7 illustrate this.* □



(a) $PT(M_{11}^2)$ (b) $PT(M_{12}^2)$

FIGURE 5.7. M_{11}^2 and M_{12}^2 have the same number of critical points of multiplicity j. M_{12}^2 has five ones with critical value -1 and only one critical point with critical value $+1$. M_{11}^2 has three critical points with critical value -1 and two with critical value $+1$.

THEOREM/DEFINITION 5.5. *Let $d, j \in \mathbb{N}$ with $d > j$. There exists a polynomial $\mathbf{M_d^j(z)}$ of degree d with $\left\lfloor \frac{d}{j+1} \right\rfloor$ critical points of multiplicity j with critical value -1 and $\left(\left\lfloor \frac{d-1}{j} \right\rfloor - \left\lfloor \frac{d}{j+1} \right\rfloor\right)$ such critical points with critical value $+1$.*

PROOF. The existence of a corresponding plane tree $PT(M_d^j)$ can be shown as follows (compare fig. 5.6 on the preceding page). For $d = j+1$ we define $PT(M_d^j)$ as a •-centered j-star. For $d = (j+1) + k \cdot j \cdot (j+1)$, $k \in \mathbb{N}$, we attach successively sets of j •-centered j-stars as illustrated in figure 5.6. If $d \neq (j+1) + k \cdot j \cdot (j+1)$ for some $k \in \mathbb{N}$ the existence of plane trees $PT(M_d^j)$ can be shown similarly (see fig. 5.7). □

The existence of the polynomials $M_d^j(z)$ has two immediate consequences:

COROLLARY 5.6. *The bounds in lemma 5.2 on page 70 are sharp.* □

It is clear that the polynomials M_d^j cannot have only real coefficients and only real critical points for d large enough. So, the same holds for the singularities of the surfaces of the following corollary:

COROLLARY 5.7. *Let $d, j \in \mathbb{N}$ with $d > j$. There exist surfaces*

$$\mathrm{Chm}(M_d^j) := F_d^{A_2} + M_d^j$$

of degree d with the following number of singularities of type A_j:

$$\begin{cases} \frac{1}{2}d(d-1)\cdot\left\lfloor \frac{d}{j+1} \right\rfloor + \frac{1}{3}d(d-3)\cdot\left(\left\lfloor \frac{d-1}{j} \right\rfloor - \left\lfloor \frac{d}{j+1} \right\rfloor\right), & \text{if } d \equiv 0 \mod 3, \\ \frac{1}{2}d(d-1)\cdot\left\lfloor \frac{d}{j+1} \right\rfloor + \frac{1}{3}(d(d-3)+2)\cdot\left(\left\lfloor \frac{d-1}{j} \right\rfloor - \left\lfloor \frac{d}{j+1} \right\rfloor\right), & \text{otherwise.} \end{cases}$$ □

To get an idea of the quality of our best lower bounds given by our examples $\mathrm{Chm}(M_d^j)$ from corollary 5.7 on the preceding page we compare them with the best known upper bounds: Varchenko's spectral bound (section 3.7 on page 35) and Miyaoka's bound (section 3.10 on page 40). It is well-known that the latter is better for large d. Together with the previous corollary we get:

COROLLARY 5.8. *Let $j \in \mathbb{N}$. For large degree d, the quotient of the number of A_j-singularities on our surfaces $\mathrm{Chm}(M_d^j)$ and the best known upper bound $\mathrm{Miy}_{A_j}(d)$ is:*

$$\frac{\mu_{A_j}(\mathrm{Chm}(M_d^j))}{\mathrm{Miy}_{A_j}(d)} \approx \frac{(j+2)(3j+2)}{4(j+1)^2}.$$

This quotient is greater than $\frac{3}{4}$ for all $j \geq 1$, the limit for $j \to \infty$ is also $\frac{3}{4}$. □

5.6. Generalization to Higher Dimensions

It is possible to generalize the construction of surfaces with many A_j-singularities described in the previous sections to $\mathbb{P}^n$, $n \geq 4$. It turns out that for $n \geq 5$, the folding polynomials $F_d^{A_2}(x, y)$ are no longer the best choice: Even for nodal hypersurfaces, the folding polynomials $F_d^{B_2}(x, y)$ lead to better lower bounds.

5.6.1. Nodal Hypersurfaces in $\mathbb{P}^n$, $n \geq 4$. In section 4.1 on page 45 we explained how Chmutov idea to use the folding polynomials $F_d^{A_2}(x, y)$ associated to the root system A_2 can be generalized to obtain nodal hypersurfaces in higher dimensions. We can improve the asymptotic behaviour of the lower bound slightly by using a folding polynomial associated to another root system. Such polynomials were described in [**Wit88**], and their critical points were studied in [**Bre05**] analogous to the case of A_2 treated by Chmutov in [**Chm92**] (see also section 4.1). It turns out that the folding polynomials $F_d^{B_2}(x, y)$ associated to the root system B_2 are best suited for our purposes. They can be defined recursively as follows:

$$F_0^{B_2} := 1, \qquad F_1^{B_2} := \frac{1}{4}y, \qquad F_2^{B_2} := \frac{1}{4}y^2 - \frac{1}{2}(x^2 - 2y - 4) - 1,$$
$$F_3^{B_2} := \frac{1}{4}y^3 - \frac{3}{4}y(x^2 - 2y - 4) - \frac{3}{4}y,$$

$$F_d^{B_2} := y\left(F_{d-1}^{B_2} + F_{d-3}^{B_2}\right) - \left(2 + (x^2 - 2y - 4)\right) F_{d-2}^{B_2} - F_{d-4}^{B_2}. \tag{5.4}$$

These polynomials have exactly three different critical values: -1, 0, $+1$. The numbers of critical points of $F_d^{B_2}$ are: $\binom{d}{2}$ with critical value 0, $\lfloor\frac{(d-1)}{2}\rfloor\lfloor\frac{d}{2}\rfloor$ with critical value -1. The use of these polynomials improves the asymptotic behaviour (for d large) of the best known lower bound for the maximum number of nodes only slightly. This is given by Chmutov's surfaces TChm_d^n which are defined as a sum of Tchebychev polynomials (see section 3.8 on page 38). In fact, the coefficient of the highest order term of the polynomial describing its number of nodes does not change (see table 5.2 on the next page). Nevertheless, we want to mention:

PROPOSITION 5.9. *Let $n \geq 2$, $d \geq 3$. Then: $\mu(\mathrm{Chm}^n(F_d^{B_2})) > \mu(\mathrm{TChm}_d^n)$.*

It is not true that the folding polynomials $F_d^{A_2}$ and $F_d^{B_2}$ are the best possible choices in all cases. Indeed, for $d = 5$, a regular five-gon leads to more nodes. For $d = 3, 4$ there are better constructions for nodal hypersurfaces in $\mathbb{P}^n$ known. In fact, Kalker (section 3.11) already noticed that Varchenko's upper bound is exact

n	3	4	5	6	7	8	9	10
$\frac{1}{d^n}\cdot\mu(\mathrm{Chm}^n(F_d^{A_2})) \approx$	$\frac{5}{12}$	$\frac{7}{18}$	$\frac{7}{24}$	$\frac{19}{72}$	$\frac{35}{144}$	$\frac{49}{216}$	$\frac{79}{432}$	$\frac{25}{144}$
$\frac{1}{d^n}\cdot\mu(\mathrm{Chm}^n(F_d^{B_2})) \approx \frac{1}{d^n}\cdot\mu(\mathrm{TChm}_d^n) \approx$	$\frac{3}{2^3}$	$\frac{3}{2^3}$	$\frac{5}{2^4}$	$\frac{5}{2^4}$	$\frac{35}{2^7}$	$\frac{35}{2^7}$	$\frac{63}{2^8}$	$\frac{63}{2^8}$

TABLE 5.2. The asymptotic behaviour of the number of nodes on variants of Chmutov's hypersurfaces in $\mathbb{P}^n$. As Chmutov already realized in **[Chm92]**, the $\mathrm{Chm}^n(F_d^{A_2})$ are only better for $n = 3, 4$. For $n \geq 5$, the best lower bounds are given by our variant $\mathrm{Chm}^n(F_d^{B_2})$ which improves Chmutov's oldest examples TChm_d^n slightly.

for $d = 3$. Goryunov rediscovered the same cubics by another method and also found quartics with many nodes in $\mathbb{P}^n$ (see section 4.4).

5.6.2. Hypersurfaces in $\mathbb{P}^n$ with A_j-Singularities, $j \geq 2, n \geq 4$. Similar to the case of surfaces which we discussed in the preceding sections, we can adapt the equations for the nodal hypersurfaces to get hypersurfaces $\mathrm{Chm}^{j,n}(F_d^{B_2})$ (or $\mathrm{Chm}^{j,n}(F_d^{A_2})$, $\mathrm{TChm}_d^{j,n}$) with many A_j-singularities:

$$(5.5) \qquad \mathrm{Chm}^{j,n}(F_d^{B_2}) : \sum_{i=0}^{\lfloor \frac{n-3}{2} \rfloor} F_d^{B_2}(x_{2i}, x_{2i+1}) = \begin{cases} T_d(x_{n-2}) + M_d^j(x_{n-1}), & n \,\text{even} \\ -\frac{1}{2}(M_d^j(x_{n-1}) + 1), & n \,\text{odd}. \end{cases}$$

This leads to the asymptotic behaviour given in table 5.3.

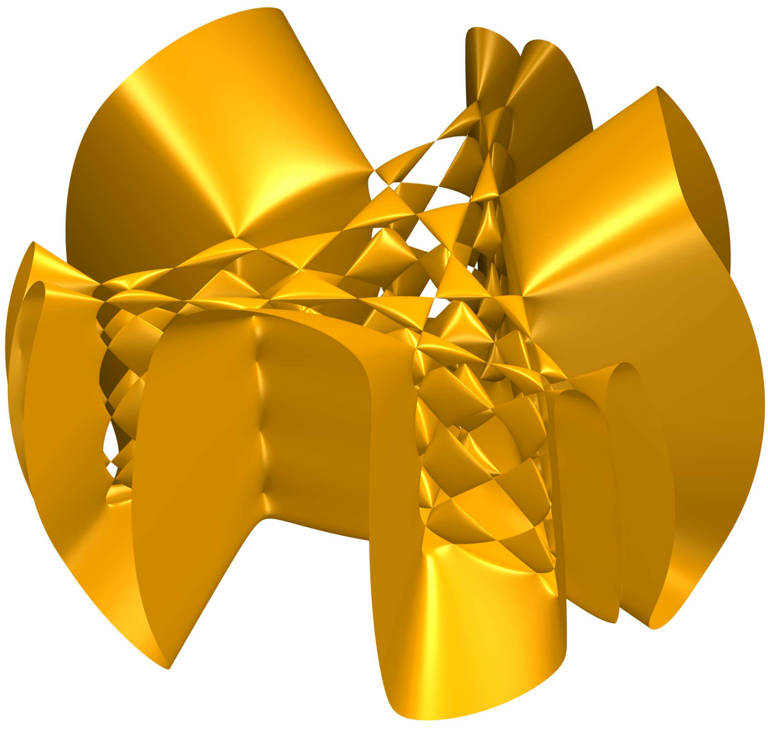
n	3	4	5	6	7	8
$\frac{1}{d^n}\cdot\mu_{A_2}^n(d) \gtrapprox$	$\frac{2}{9}$	$\frac{13}{72}$	$\frac{1}{6}$	$\frac{13}{96}$	$\frac{55}{384}$	$\frac{15}{128}$
$\frac{1}{d^n}\cdot\mu_{A_3}^n(d) \gtrapprox$	$\frac{11}{72}$	$\frac{1}{8}$	$\frac{11}{96}$	$\frac{3}{32}$	$\frac{25}{256}$	$\frac{125}{1536}$
$\frac{1}{d^n}\cdot\mu_{A_4}^n(d) \gtrapprox$	$\frac{7}{60}$	$\frac{23}{240}$	$\frac{7}{80}$	$\frac{23}{320}$	$\frac{19}{256}$	$\frac{1}{16}$
$\frac{1}{d^n}\cdot\mu(\mathrm{Chm}^{j,n}(F_d^{A_2})) \approx$	$\frac{3j+2}{6j(j+1)}$	$\frac{5j+3}{12j(j+1)}$	$\frac{7j+3}{18j(j+1)}$	$\frac{7j+4}{24j(j+1)}$	$\frac{19j+16}{72j(j+1)}$	$\frac{35j+19}{144j(j+1)}$
$\frac{1}{d^n}\cdot\mu(\mathrm{Chm}^{j,n}(F_d^{B_2})) \approx$	$\frac{2j+1}{4j(j+1)}$	$\frac{3j+2}{8j(j+1)}$	$\frac{3j+2}{8j(j+1)}$	$\frac{5j+3}{16j(j+1)}$	$\frac{20j+15}{64j(j+1)}$	$\frac{35j+20}{128j(j+1)}$

TABLE 5.3. The asymptotic behaviour of the number of A_j-singularities on a hypersurface of degree d in $\mathbb{P}^n$. $\mathrm{Chm}^{j,n}(F_d^{B_2})$ is better than $\mathrm{Chm}^{j,n}(F_d^{A_2})$ for $n \geq 6$.

Notice that we usually get fewer singularities if we add a sign $(-1)^i$ in the sum in contrast to equation (4.7) where the alternating sign is often better because the folding polynomial $F_d^{A_2}$ has other critical values than $F_d^{B_2}$.

Of course, for small d, n, j, it is often easy to write down better lower bounds. E.g., if n is even and d is small, it is often better to replace $T_d(x_{n-2}) + M_d^j(x_{n-1})$ by a plane curve with the maximum known number of cusps. For some specific values of d, $j \geq 2$, $n \geq 4$ there are even better lower bounds known. E.g., we already mentioned in section 1.5.3 that Lefschetz constructed a cubic hypersurface in $\mathbb{P}^4$ with 5 cusps which is the maximum possible number.

For $n = 2$, our construction presented in subsection 5.6.2 on the page before only leads to plane curves of degree d with $\approx \frac{1}{4} \cdot d^2$ cusps whereas the generalization of B. Segre's construction (equation (2.12) gives $\approx \frac{9}{32} \cdot d^2$ such singularities when starting with a smooth conic.



Breske's 216-nodal real variant of Chmutov's nonic.

CHAPTER 6

Real Line Arrangements and Surfaces with Many Real Nodes

We make a short excursus to the world of real algebraic geometry (see also [**BLvS**]). More precisely, we consider the relationship between the maximum possible number $\mu_{A_1}(d)$ of nodes on a surface of degree d and the maximum possible number $\mu^{\mathbb{R}}_{A_1}(d)$ of real nodes on a real surface in $\mathbb{P}^3(\mathbb{R})$. Obviously, $\mu^{\mathbb{R}}_{A_1}(d) \leq \mu_{A_1}(d)$, but do we even have $\mu^{\mathbb{R}}_{A_1}(d) = \mu_{A_1}(d)$? In other words: Can the maximum number of nodes be achieved with real surfaces with real singularities?

The previous question arises naturally because all results in low degree $d \leq 12$ suggest that it could be true (see chapter 4 on page 45 and table 6.1). In contrast to this, until very recently, the best known asymptotic lower bound, $\mu_{A_1}(d) \gtrapprox \frac{5}{12}d^3$, was only reached by Chmutov's construction (section 4.1 on page 45) which yields singularities with non-real coordinates. But during the writing of Breske's diploma thesis [**Bre05**] under the direction of van Straten it turned out that the folding polynomials used by Chmutov can be adapted to have real critical points. Of course, these give rise to variants of Chmutov's surfaces with only real nodes. In this chapter, we briefly explain how this can be done. See table 6.1 on the following page for the lower bounds for $\mu_{A_1}(d)$ resulting from this. In the real case we can distinguish between two types of A_1-singularities, *conical nodes* ($x^2 + y^2 - z^2 = 0$) and *solitary points* ($x^2 + y^2 + z^2 = 0$): Breske's construction produces only conical nodes.

Notice that in general there are no better real upper bounds for $\mu^{\mathbb{R}}_{A_1}(d)$ known than the well-known complex ones of Miyaoka (section 3.10) and Varchenko (section 3.7). But for solitary points there exist better bounds via the relation to the zero$^{\text{th}}$ Betti number (see e.g., [**Kha96**]). E.g., Rohn showed in 1913 that a real quartic surface in $\mathbb{P}^3(\mathbb{R})$ cannot have more than 10 solitary points although it can have 16 conical nodes. We show a real upper bound of $\approx \frac{5}{6}d^2$ for the maximum number of critical points on two levels of real simple line arrangements consisting of d lines. In [**Chm95**], Chmutov conjectured this to be the maximum number for all complex plane curves of degree d. He also noticed [**Chm92**] that such a bound directly implies an upper bound for the number of real nodes of certain surfaces. Our upper bound shows that Breske's folding polynomials are asymptotically the best possible real line arrangements for this purpose.

6.1. Variants of Chmutov's Surfaces with Many Real Nodes

In this section, we briefly describe how Breske adapted Chmutov's construction to get surfaces with many real nodes. Recall that the $F^{A_2}_d(x, y)$ have critical points with only three different critical values: 0, -1, and 8 (see section 4.1 on page 45). Thus, the surface $\text{Chm}^{A_2}_d(x, y, z)$ is singular exactly at those points at which the

d	1	2	3	4	5	6	7	8	9	10	11	12	13	d
$\mu^{\mathbb{R}}_{A_1}(d) \leq$	0	1	4	16	31	65	104	174	246	360	480	645	832	$\frac{4}{9}d(d-1)^2$
$\mu^{\mathbb{R}}_{A_1}(d) \geq$	0	1	4	16	31	65	99	168	**216**	345	**425**	600	**732**	$\approx \mathbf{\frac{5}{12}d^3}$

TABLE 6.1. Except for $d = 9$, the currently known bounds for the maximum number $\mu_{A_1}(d)$ (resp. $\mu^{\mathbb{R}}_{A_1}(d)$) of nodes on a surface of degree d in $\mathbb{P}^3(\mathbb{C})$ (resp. $\mathbb{P}^3(\mathbb{R})$) are equal. The bold numbers indicate in which cases Breske's variants of Chmutov's surfaces improve the previously known lower bound for $\mu^{\mathbb{R}}_{A_1}(d)$.

critical values of $F^{A_2}_d(x,y)$ and $\frac{1}{2}(T_d(z)+1)$ sum up to zero (i.e., either both are 0 or the first is -1 and the second is $+1$).

Notice that the plane curve defined by $F^{A_2}_d(x,y)$ consists in fact of d lines. But these are not real lines and the critical points of this folding polynomial also have non-real coordinates. It is natural to ask whether there is a real line arrangement which leads to the same number of critical points. In her Diploma thesis, Breske computed the critical points of the other folding polynomials. Among these, there are the following examples which are the real line arrangements we have been looking for (see [**Bre05**, p. 87–89]):

We define the real folding polynomial $F^{A_2}_{\mathbb{R},d}(x,y) \in \mathbb{R}[x,y]$ associated to the root system A_2 as $F^{A_2}_{\mathbb{R},d}(x,y) := F^{A_2}_d(x+iy,\ x-iy)$, where i is the imaginary number. It is easy to see that the $F^{A_2}_{\mathbb{R},d}(x,y)$ have indeed real coefficients. The numbers of critical points are the same as those of $F^{A_2}_d(x,y)$; but now they have real coordinates as the following lemma shows:

LEMMA 6.1 (see [**Bre05**]). *The real folding polynomial $F^{A_2}_{\mathbb{R},d}(x,y)$ associated to the root system A_2 has $\binom{d}{2}$ real critical points with critical value* 0 *and*

$$\frac{1}{3}d^2 - d, \quad \text{if } d \equiv 0 \mod 3, \qquad \frac{1}{3}d^2 - d + \frac{2}{3}, \quad \text{otherwise} \tag{6.1}$$

real critical points with critical value -1. *The other critical points also have real coordinates and have critical value* 8.

PROOF. We proceed similar to the case discussed by Chmutov, see [**Bre05**, p. 87–95] for details. To calculate the critical points of the real folding polynomial $F^{A_2}_{\mathbb{R},d}$, we use the map $h^1 : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$(u,v) \mapsto \big(\cos(2\pi(u+v))+\cos(2\pi u)+\cos(2\pi v),\ \ \sin(2\pi(u+v))-\sin(2\pi u)-\sin(2\pi v)\big).$$

This is in fact just the real and imaginary part of the first component of the generalized cosine h considered by Withers [**Wit88**] and Chmutov [**Chm92**]. It is easy to see that h^1 is a coordinate change if $u - v > 0$, $u + 2v > 0$, and $2u + v < 1$. It transforms the polynomial $F^{A_2}_{\mathbb{R},d}$ into the function $G^{A_2}_d : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$G^{A_2}_d(u,v) := F^{A_2}_{\mathbb{R},d}(h^1(u,v)) = 2\cos(2\pi du) + 2\cos(2\pi dv) + 2\cos(2\pi d(u+v)) + 2.$$

The calculation of the critical points of $G^{A_2}_d$ is exactly the same as the one performed in [**Chm92**]. As the function $G^{A_2}_d$ has $(d-1)^2$ distinct real critical points in the region defined by $u - v > 0$, $u + 2v > 0$, and $2u + v < 1$, the images of these points under the map h^1 are all the critical points of the real folding polynomial $F^{A_2}_{\mathbb{R},d}$ of degree d. In contrast to Chmutov, we here get real critical points because h^1 is a map from $\mathbb{R}^2$ into itself. □

None of the other root systems yield more critical points on two levels. But as mentioned in section 5.6 on page 74, the real folding polynomials associated to the root system B_2 give hypersurfaces in $\mathbb{P}^n$, $n \geq 5$, which improve the previously known lower bounds for the maximum number of nodes in higher dimensions slightly ([**Bre05**] gives a detailed discussion of all these folding polynomials and their critical points).

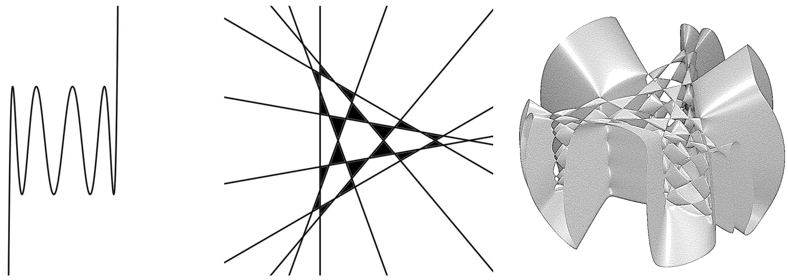


FIGURE 6.1. For degree $d = 9$ we show the Tchebychev polynomial $T_9(z)$, the real folding polynomial $F^{A_2}_{\mathbb{R},9}(x, y)$ associated to the root system A_2, and the surface $\mathrm{Chm}^{A_2}_{\mathbb{R},9}(x, y, z)$. The bounded regions in which $F^{A_2}_{\mathbb{R},9}(x, y)$ takes negative values are marked in black.

The lemma immediately gives the following variant of Chmutov's surfaces:

THEOREM 6.2 (see [**Bre05**]). *Let $d \in \mathbb{N}$. The real projective surface of degree d defined by*

$$\mathrm{Chm}^{A_2}_{\mathbb{R},d}(x, y, z) := F^{A_2}_{\mathbb{R},d}(x, y) + \frac{1}{2}(T_d(z) + 1) \in \mathbb{R}[x, y, z] \tag{6.2}$$

has the following number of real nodes:

$$\begin{cases} \frac{1}{12}\left(5d^3 - 13d^2 + 12d\right), & \text{if } d \equiv 0 \mod 6, \\ \frac{1}{12}\left(5d^3 - 13d^2 + 16d - 8\right), & \text{if } d \equiv 2, 4 \mod 6, \\ \frac{1}{12}\left(5d^3 - 14d^2 + 13d - 4\right), & \text{if } d \equiv 1, 5 \mod 6, \\ \frac{1}{12}\left(5d^3 - 14d^2 + 9d\right), & \text{if } d \equiv 3 \mod 6. \end{cases} \tag{6.3}$$

These numbers are the same as the numbers of complex nodes of Chmutov's surfaces $\mathrm{Chm}^{A_2}_d(x, y, z)$. To our knowledge, the result gives new lower bounds for the maximum number $\mu^{\mathbb{R}}_{A_1}(d)$ of real singularities on a surface of degree d in $\mathbb{P}^3(\mathbb{R})$ for $d = 9, 11$ and $d \geq 13$, see table 6.1 on the preceding page. Notice that all best known lower bounds for $\mu^{\mathbb{R}}_{A_1}(d)$ are attained by surfaces with only conical nodes which is not astonishing in view of the upper bounds for solitary points mentioned in the introduction.

6.2. On Two-Colorings of Real Simple Line Arrangements

The real folding polynomials $F^{A_2}_{\mathbb{R},d}(x, y)$ used in the previous section are in fact *real simple (straight) line arrangements* in $\mathbb{R}^2$, i.e., lines no three of which meet in a point. Such arrangements can be 2-colored in a natural way (see fig. 6.1): We label in black those connected components (*cells*) of $\mathbb{R}^2 \setminus \{F^{A_2}_{\mathbb{R},d}(x,y)=0\}$ in which $F^{A_2}_{\mathbb{R},d}(x,y)$ takes negative values, the others in white. The bounded black regions in fig. 6.1 contain exactly one critical point with critical value -1 each.

Harborth has shown in [**Har81**] that the maximum number $M_b(d)$ of black cells in such real simple line arrangements of d lines satisfies:

$$M_b(d) \leq \begin{cases} \frac{1}{3}d^2 + \frac{1}{3}d, & d \text{ odd}, \\ \frac{1}{3}d^2 + \frac{1}{6}d, & d \text{ even}. \end{cases} \tag{6.4}$$

d of these cells are unbounded. This is a purely combinatorial result which is strongly related to the problem of determining the maximum number of triangles in such arrangements which has a long and rich history (see [**GO04**]). Notice that this bound is better than the one obtained by Kharlamov using Hodge theory [**Kha05**]. It is known that the bound (6.4) is exact for infinitely many values of d. The real folding polynomials $F^{A_2}_{\mathbb{R},d}(x,y)$ almost achieve this bound. Moreover, these arrangements have the very special property that all critical points with a negative (resp. positive) critical value have the same critical value -1 (resp. $+8$).

To translate the upper bound on the number of black cells into an upper bound on critical points we use the following lemma:

LEMMA 6.3 (see Lemme 10, 11 in [**OR03**]). *Let f be a real simple line arrangement consisting of $d \geq 3$ lines. f has exactly $\binom{d-1}{2}$ bounded open cells each of which contains exactly one critical point. All the critical points of f are non-degenerate.*

It is easy to prove the lemma, e.g. by counting the number of bounded cells and by observing that each such cell contains at least one critical point. Comparing this with the number $(d-1)^2 - \binom{d}{2} = \binom{d-1}{2}$ of all non-zero critical points gives the result. Now we can show that our real line arrangements are asymptotically the best possible ones for constructing surfaces with many singularities:

THEOREM 6.4. *The maximum number of critical points with the same non-zero critical value $0 \neq v \in \mathbb{R}$ of a real simple line arrangement is bounded by $M_b(d) - d$, where d is the number of lines. In particular, the maximum number of critical points on two levels of such an arrangement does not exceed $\binom{d}{2} + M_b(d) - d \approx \frac{5}{6}d^2$.*

PROOF. In view of the upper bound (6.4) for the maximum number $M_b(d)$ of black cells of a real simple line arrangement we only have to verify that any bounded cell contains only one critical point. But this follows from the preceding lemma. □

Chmutov showed a much more general result ([**Chm84**], see [**Chm95**] for the case of non-degenerate critical points): For a plane curve of degree d the maximum number of critical points on two levels does not exceed $\approx \frac{7}{8}d^2$. In [**Chm95**], he conjectured $\approx \frac{5}{6}d^2$ to be the actual maximum which is attained by the complex line arrangements $F^{A_2}_d(x,y)$ he used for his construction (and also by the real line arrangements $F^{A_2}_{\mathbb{R},d}(x,y)$). Thus, our theorem 6.4 is the verification of Chmutov's conjecture in the particular case of real simple line arrangements. As Chmutov remarked in [**Chm92**], such an upper bound immediately implies an upper bound on the maximum number of nodes on a surface in separated variables:

COROLLARY 6.5. *A surface of the form $p(x,y) + q(z) = 0$ cannot have more than $\approx \frac{1}{2}d^2 \cdot \frac{1}{2}d + \frac{1}{3}d^2 \cdot \frac{1}{2}d = \frac{5}{12}d^3$ nodes if $p(x,y)$ is a real simple line arrangement. This number is attained by the surfaces $\mathrm{Chm}^{A_2}_{\mathbb{R},d}(x,y,z)$ defined in theorem 6.2.*

Comparing this number with the upper bound $\approx \frac{5}{12}d^3$ on the zero$^{\text{th}}$ Betti number (see e.g., [**Kha96**, p. 533]) one is tempted to ask if it is possible to deform our singular surfaces to get examples with many real connected components. But our surfaces $\text{Chm}^{A_2}_{\mathbb{R},d}(x,y,z)$ only contain A_1^- singularities which locally look like a cone $(x^2+y^2-z^2=0)$. When removing the singularities from the zero-set of the surface every connected component contains at least three of the singularities. Thus, the zero$^{\text{th}}$ Betti number of a small deformation of our surfaces are not larger than $\approx \frac{5}{3\cdot 12}d^3$ which is far below the number $\approx \frac{13}{36}d^3$ resulting from Bihan's construction [**Bih03**] which is based on Viro's patchworking method.

Conversely, we may ask if it is always possible to move the lines of a simple real line arrangement in such a way that all critical points which have a critical value of the same sign can be chosen to have the same critical value. If this were true then it would be possible to improve our lower bound for the maximum number $\mu^{\mathbb{R}}_{A_1}(d)$ of real nodes on a real surface of degree d slightly because it is known that the upper bounds for the maximum number $M_b(d)$ of black cells are in fact exact for infinitely many d. E.g., in the already cited article [**Har81**], Harborth gave an explicit arrangement of 13 straight lines which has $\frac{1}{3}\cdot 13^2 + \frac{1}{3}\cdot 13 - 13 = 47$ bounded black regions. When regarding this arrangement as a polynomial of degree $d = 13$ it has exactly one critical point with a negative critical value within each of the black regions. Such a polynomial would lead to a surface with $\binom{13}{2}\cdot\lceil\frac{13-1}{2}\rceil + 47\cdot\lfloor\frac{13-1}{2}\rfloor = 750 > 732$ nodes. Similarly, such a surface of degree 9 would have $228 > 216$ nodes. In the case of degree 7 the construction would only yield 96 nodes which is less than the number 99 found in [**Lab04**].

6.3. Concluding Remarks

Notice that it is not clear that line arrangements are the best plane curves for our purpose, and we may ask: Is it possible to exceed the number of critical points on two levels of the line arrangements $F^{A_2}_{\mathbb{R},d}(x,y)$ using irreducible curves of higher degrees? Either in the real or in the complex case? This is not true for the real folding polynomials. E.g., those associated to the root system B_2 consist of many ellipses and yield surfaces with fewer singularities (see [**Bre05**]).

We can also ask for the maximum number $\mu^{\mathbb{R}}_{A}(d)$ of real A_j-singularities. It is clear that constructions similar to those in chapter 5 on page 67 cannot give the same number of real nodes because of the intermediate value theorem (Zwischenwertsatz). It would be nice to use real dessins d'enfants (see e.g., [**Bru04**]) to check which numbers are actually possible to obtain.

A sextic with 30 real cusps and 10 real nodes at infinity, constructed using an algorithm in characteristic zero.

CHAPTER 7

An Algorithm in Characteristic Zero

We give an algorithm (see also [**Lab05a**]) that can be used to find hypersurfaces with many singularities within families of hypersurfaces. As we will see, it is based on very recent features of the computer algebra system SINGULAR. The idea to such an algorithm is not so new. In fact, our main observation was to notice that we can use features of the most recent versions of this computer algebra system to perform the algorithm on a computer in our particular case.

We describe this algorithm using the example of the construction of a sextic surface in $\mathbb{P}^3$ with 35 cusps. From this, it is easy to figure out how to proceed in general. When we uploaded the preprint [**Lab05a**] to arXiv.org we believed that this 35-cuspidal example was the one with the maximum known number of cusps. Only recently we realized that Gallarati's variant of B. Segre's construction (see section 2.5 on page 24) leads to a sextic with 36 cusps. We present the algorithm here because it can certainly be applied in many similar situations.

In the works mentioned in part 1, the authors used geometric arguments to reduce a problem depending on several parameters to polynomials each depending only on one parameter. The roots of these polynomials could then easily be found by hand or by computer algebra. But what can we do when there are no geometric arguments available to reduce the problem to equations in one variable each? In this case, we can still use a similar approach by replacing root-finding of a polynomial in one variable by primary decomposition.

7.1. The Family of 30-cuspidal Sextics

As our starting point, we take the 4-parameter family $f_{s,t,u,v} \subset \mathbb{P}^3$ with dihedral symmetry D_5 defined by:

$$\begin{array}{rcl} p & := & z \cdot \Pi_{j=0}^{4} \left[\cos\left(\frac{2\pi j}{7}\right) x + \sin\left(\frac{2\pi j}{7}\right) y - z\right] \\ & = & \frac{z}{16}\Big[x\left(x^4 - 2{\cdot}5{\cdot}x^2y^2 + 5{\cdot}y^4\right) \\ & & \qquad -5{\cdot}z{\cdot}\left(x^2+y^2\right)^2 + 4{\cdot}5{\cdot}z^3{\cdot}\left(x^2+y^2\right) - 16{\cdot}z^5\Big], \\ q_{s,t,u,v} & := & s{\cdot}(x^2+y^2) + t{\cdot}z^2 + u{\cdot}zw + v{\cdot}w^2, \\ f_{s,t,u,v} & := & p - q_{s,t,u,v}^3. \end{array} \tag{7.1}$$

p is the product of z and 5 planes in $\mathbb{P}^3(\mathbb{C})$ meeting in the point $(0:0:0:1)$ with the symmetry D_5 of the 5-gon with rotation axes $\{x = y = 0\}$. $q_{s,t,u,v}$ is also D_5-symmetric, because x and y only appear as $x^2 + y^2$.

The generic surface $f_{s,t,u,v}$ has $15{\cdot}2 = 30$ singularities of type A_2 at the intersections of the tripled quadric $q_{s,t,u,v}$ with the $\binom{6}{2}$ pairwise intersection lines of the 6 planes p. $2{\cdot}5 = 10$ of the singularities lie in the $\{z = 0\}$ plane, the other $4{\cdot}5 = 20$ not. The coordinates of the latter 20 can be obtained from the 4 singularities in the $\{y = 0\}$ plane using the symmetry of the family. To see that the $\{y = 0\}$ plane contains 4 cusps, note that $p|_{y=0} = z \cdot (z - x) \cdot (x^2 - 2xz - 4z^2)^2$: For generic values of the parameters, this doubled quadric factor meets the tripled quadric $q_{s,t,u,v}$ in $2 \cdot 2$ points.

Note that

$$f_{s,t,u,v}(x,y,z,\lambda w) = f_{s,t,\lambda u,\lambda^2 v}(x,y,z,w) \ \forall \lambda \in \mathbb{C}^*, \tag{7.2}$$

s.t. we can choose $v := 1$ (it is easy to see that $v = 0$ corresponds to a degenerate case). Therefore, we write:

$$f_{s,t,u} := f_{s,t,u,1} \text{ and } q_{s,t,u} := q_{s,t,u,1}.$$

7.2. The Sextics with 35 Cusps

To find surfaces in this 3-parameter family with more singularities, we compute the discriminant $Disc_{f_{s,t,u}} \in \mathbb{C}[s,t,u]$ of the family $f_{s,t,u}$ by first dividing out the base locus (the intersections of the double lines of p with the quadric q) from the singular locus (we use saturation, because we have to divide out the base locus six times):

$$\begin{aligned} sl &:= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w}\right), \\ bl &:= \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}, \frac{\partial p}{\partial w}, q\right), \\ I &:= sl \ : \ bl^{\infty}. \end{aligned}$$

Then we eliminate the variables x, y, z from this quotient. In fact, because of the symmetry we restrict our attention to the $\{y = 0\}$ plane, which speeds up the computations: Every singularity in the plane $\{y = 0\}$ which is not on the rotation axes $\{x = y = 0\}$ generates an orbit of length 5 of singularities of the same type.

A short SINGULAR computation then gives the discriminant $Disc_{f_{s,t,u}} \in \mathbb{Q}[s,t,u]$, which factorizes into $Disc_{f_{s,t,u}} = D_{f,1} \cdot D_{f,2} \cdot D_{f,3}$, where:

$$\begin{aligned} D_{f,1} = \ & 2^{20}{\cdot}3^6 \cdot s^5 \cdot \left(2^4{\cdot}s^2 + 2^2{\cdot}3{\cdot}st + t^2\right) \cdot (s+t)^2 \\ & + \left(-2^{19}{\cdot}3^6\right) \cdot s^5 \cdot \left(2{\cdot}11{\cdot}s^2 + 19{\cdot}st + 2{\cdot}t^2\right) \cdot (s+t) \cdot u^2 \\ & + 2^{16}{\cdot}3^6 \cdot s^5 \cdot \left(41{\cdot}s^2 + 2{\cdot}3{\cdot}7{\cdot}st + 2{\cdot}3{\cdot}t^2\right) \cdot u^4 \\ & + \left(-2^{14}{\cdot}3^3\right) \cdot s^3 \\ & \quad \cdot \left(2{\cdot}3^3{\cdot}7{\cdot}s^3u^6 + 2^2{\cdot}3^3{\cdot}s^2tu^6 + 2^6{\cdot}5^2{\cdot}s^3 - 2^5{\cdot}5^2{\cdot}s^2t - 5^2{\cdot}61{\cdot}st^2 - 5^3{\cdot}t^3\right) \\ & + 2^{12}{\cdot}3^3 \cdot s^3 \cdot \left(3^3{\cdot}s^2u^6 - 2^5{\cdot}5^2{\cdot}s^2 - 2{\cdot}5^2{\cdot}61{\cdot}st - 3{\cdot}5^3{\cdot}t^2\right) \cdot u^2 \\ & + 2^{10}{\cdot}3^3{\cdot}5^2 \cdot s^3 \cdot (61{\cdot}s + 3{\cdot}5{\cdot}t) \cdot u^4 \\ & + \left(-2^6{\cdot}5^3\right) \cdot \left(2^2{\cdot}3^3{\cdot}s^3u^6 + 2^6{\cdot}5{\cdot}23{\cdot}s^3 + 2^5{\cdot}3{\cdot}5{\cdot}s^2t + 2^2{\cdot}3{\cdot}5^2{\cdot}st^2 + 5^2{\cdot}t^3\right) \\ & + 2^4{\cdot}3{\cdot}5^4 \cdot \left(2^5{\cdot}s^2 + 2^3{\cdot}5{\cdot}st + 5{\cdot}t^2\right) \cdot u^2 \\ & + \left(-2^2{\cdot}3{\cdot}5^5\right) \cdot \left(2^2{\cdot}s + t\right) \cdot u^4 \\ & + 5^5 \cdot \left(u^4 - 2^2{\cdot}u^2 + 2^4\right) \cdot \left(u^2 + 2^2\right), \\ D_{f,2} = \ & \left(-2^4\right) \cdot t^2 + 2^3 \cdot t \cdot \left(u^2 + 2\right) + \left(2{\cdot}u - \left(u^2 + 2^2\right)\right) \cdot \left(2{\cdot}u + \left(u^2 + 2^2\right)\right), \\ D_{f,3} = \ & 2^2 \cdot t + (2 - u) \cdot (2 + u). \end{aligned}$$

We hope that some singularities of the discriminant correspond to examples of surfaces $f_{s,t,u}$ with more A_2-singularities. Note that only $D_{f,1}$ depends on the parameter s. Using computer algebra, it is easy to verify that the intersections of two of the 3 components $D_{f,1}, D_{f,2}, D_{f,3}$ of $Disc_{f_{s,t,u}}$ do not yield to surfaces with many additional singularities.

So, we use Singular againto compute the primary decomposition of the singular locus of $D_{f,1}$ over $\mathbb{Q}$: $sl(D_{f,1}) = \mathfrak{sl}_{f,1} \cap \mathfrak{sl}_{f,2} \cap \mathfrak{sl}_{f,3} \cap \mathfrak{sl}_{f,4}$, where

$$\begin{aligned}
\mathfrak{sl}_{f,1} &= \Big(2^2 \cdot (2^2 \cdot s - t) + u^2, \quad 2^6 \cdot 3^3 \cdot s^3 - 5\Big) \\
\mathfrak{sl}_{f,2} &= \Big(-2^2 \cdot (2^2 \cdot 3 \cdot s + 5 \cdot t) + 5 \cdot u^2, \quad 2^4 \cdot 3^2 \cdot s^2 + 2^2 \cdot 3 \cdot 5 \cdot s + 5^2\Big) \\
\mathfrak{sl}_{f,3} &= \Big(2^{15} \cdot 3^3 \cdot t^6 - 2^{14} \cdot 3^4 \cdot t^5 \cdot u^2 + 2^{11} \cdot 3^4 \cdot 5 \cdot t^4 \cdot u^4 - 2^6 \cdot 3^3 \cdot 5 \cdot t^3 \cdot (2^5 \cdot u^6 - 11 \cdot 31) \\
&\qquad +2^4 \cdot 3^4 \cdot 5 \cdot t^2 \cdot (2^3 \cdot u^6 - 11 \cdot 31) \cdot u^2 - 2^2 \cdot 3^4 \cdot t \cdot (2^4 \cdot u^6 - 5 \cdot 11 \cdot 31) \cdot u^4 \\
&\qquad + (2^3 \cdot 3^3 \cdot u^{12} - 3^3 \cdot 5 \cdot 11 \cdot 31 \cdot u^6 + 2^6 \cdot 5^2 \cdot 19^3), \\
&\quad 2^{11} \cdot 3^2 \cdot t^4 - 2^{11} \cdot 3^2 \cdot t^3 \cdot u^2 + 2^8 \cdot 3^3 \cdot t^2 \cdot u^4 \\
&\qquad -2^2 \cdot (2^5 \cdot 3^2 \cdot tu^6 - 2^2 \cdot 5 \cdot 7 \cdot 19 \cdot 211 \cdot s - 5 \cdot 73 \cdot 193 \cdot t) \\
&\qquad +u^2 \cdot (2^3 \cdot 3^2 \cdot u^6 - 5 \cdot 73 \cdot 193) \Big) \\
\mathfrak{sl}_{f,4} &= \Big(2^2 \cdot 3 \cdot s - 5, \quad -4 \cdot (t+1) + u^2\Big).
\end{aligned}$$

All these prime ideals define smooth curves in the 3-dimensional parameter space. When projecting the curve C_3 defined by $\mathfrak{sl}_{f,3}$ to the s,t- or the s,u-plane, we get in both cases six straight lines defined by the equation

$$2^{15} \cdot 3^3 \cdot s^6 - 2^6 \cdot 3^3 \cdot 5 \cdot s^3 + 5^2 = 0. \tag{7.3}$$

This shows that C_3 consists in fact of the union of six plane curves. Over the algebraic extension $\mathbb{Q}(s)$, it is easy to compute the equation of these:

$$C_{3,s} = 5 \cdot u^2 - 2^2 \cdot 5 \cdot t - 2^{11} \cdot 3^2 \cdot s^4 - 2^4 \cdot 5 \cdot s \in \mathbb{Q}(s)[t,u]. \tag{7.4}$$

To show that there is a surface with 35 A_2-singularities, we take the most simple point of this curve, the one with $u = 0$:

THEOREM 7.1 (35-cuspidal Sextic). *Let $s_0 \in \mathbb{C}$ be one of the six roots of (7.3). Let $(t_0, 0)$ be the point on C_{3,s_0} with $u = 0$. Then the sextic $S_{35} := f_{s_0,t_0,0} \subset \mathbb{P}^3$ has exactly 35 singularities of type A_2 and no other singularities.*

PROOF. We use computer algebra. The SINGULAR script and its output can be downloaded from the webpage [**Lab03a**]. Here, we give the basic ideas. With $u = 0$ in C_{3,s_0}, we find: $t_0 = -4 \cdot s_0 \left(\frac{2^7 \cdot 3^2}{5} \cdot s_0^3 + 1\right)$. For the corresponding surface

$$S_{35} := f_{s_0, -4 \cdot s_0 \left(\frac{2^7 \cdot 3^2}{5} \cdot s_0^3 + 1\right), 0} \tag{7.5}$$

we first check that the total milnor number is 70. Then we verify that the surface has 35 singularities of type A_2: For each orbit of singularities, we compute the ideal of one of the singularities and check explicitly that it is a cusp. To show this it suffices to verify that its milnor number is exactly two. E.g., for the orbit of the five non-generic singularities, we take the cusp S_{yw} that lies in the $\{y = 0\}$ plane:

$$S_{yw} = \left(-\frac{2^7 \cdot 3^2}{5} s_0^3 + 8 : 0 : 1 : 0\right).$$

□

Note that the coefficients of the surface S_{35} are not real. In fact, the ideal $\mathfrak{sl}_{f,3}$ does not contain any real point, because equation (7.3) does not have any real root. In particular, it is not possible to use the software `surf` [**End03**] to draw an image of this sextic. This also holds for the more general family $f_{s,t,u,v}$ because of equation (7.2). The curves defined by the ideals $\mathfrak{sl}_{f,2}$ and $\mathfrak{sl}_{f,4}$ lead to only one additional higher singularity, and we are not interested in such examples.

But in the case of the prime ideal $\mathfrak{sl}_{f,1}$, we get surfaces with 30 real A_2-singularities and 10 real A_1-singularities (see also fig. 7.1). Again, we choose a point in the parameter-space with $u = 0$:

THEOREM 7.2. *The sextic* $f_{s_0,t_0,0} \subset \mathbb{P}^3$, *where* $s_0 := \frac{1}{3\cdot 2^2}\sqrt[3]{5} \in \mathbb{R}$, $t_0 = 2^2 \cdot s_0 \in \mathbb{R}$, *has exactly* 30 *singularities of type* A_2, 10 *singularities of type* A_1, *and no other singularities. Furthermore, all the singularities are real.*

PROOF. Similar to the preceding one. □

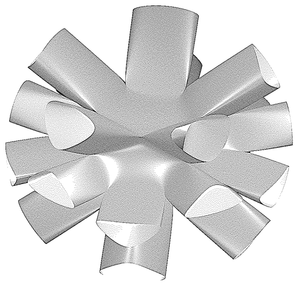


FIGURE 7.1. A sextic with 30 cusps and 10 nodes at infinity. Some movies illustrating this are available from [**Lab03a**].

7.3. Concluding Remarks

In our application, we could restrict our attention to a plane because of the symmetry of the family, so that the number of variables decreased. This speeded up the computations. But the case of septics with many nodes was too time-consuming to be treated in this way: Our construction of a 99-nodal surface of degree 7 (see next chapter) involves computations in positive characteristics and then liftings to characteristic zero using the geometry of the examples.

In other applications, it might be easy to divide out the base locus and to compute the discriminant, e.g. by using the geometry of the family. Then it only remaines to study the discriminant for finding examples which have more singularities than the generic member of the family.

7.4. The SINGULAR Code

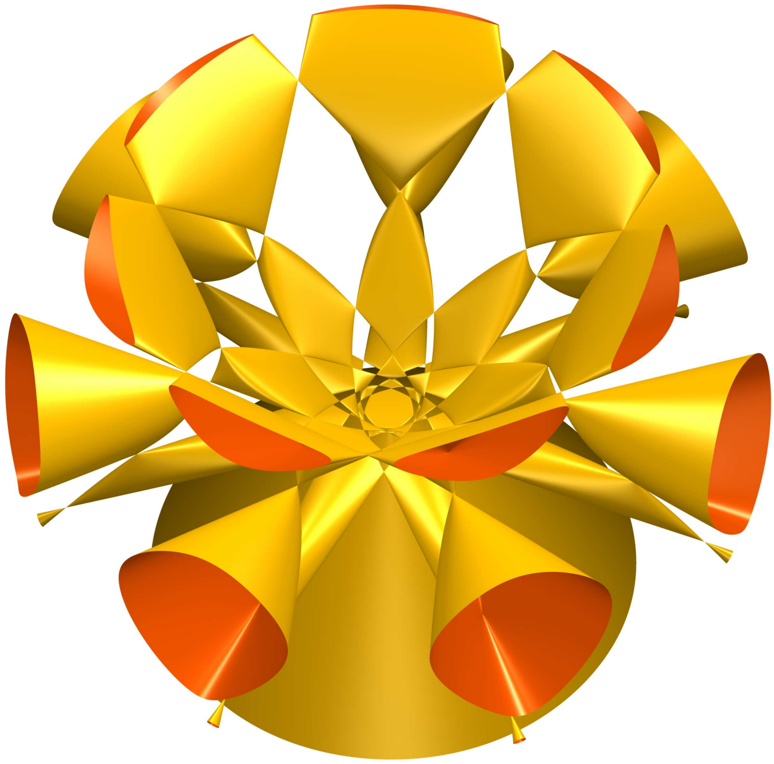
```
"::::::: :::::";
"A Sextic with $35$ Cusps";
"(Oliver Labs)";
"";
"This Singular script computes the parameters s,t,u,v,";
"s.t. the surface f_{s,t,u,v} of the article has $35$ cusps.";
"";
"This script also contains the proof that this surface has ";
"$35$ such singularities and no other singularities.";
"::::::: :::::";
"";

LIB "primdec.lib";
LIB "sing.lib";
LIB "classify.lib";
LIB "zeroset.lib";

proc mycodim(ideal stdi)
"
ASSUME: stdi is already in standard bases form!
"
{
    return(nvars(basering)-dim(stdi));
}

proc std_primdecGTZ(ideal I)
"
RETURN: A list, similar to the one returned by primdecGTZ, but with
        some extra information.
Calls primdecGTZ and then calls std() for each of the prime ideals
replace the prime ideals by their standard-basis.
The third sub-item of each item of the list is
the dimension of the prime ideal,
the fourth sub-item is its multiplicity.
"
{
    list pd = primdecGTZ(I);
    list pd_neu;
    int i;
    list coords;
    ideal stdtmp;
    for(i=1; i<=size(pd); i++) {
        stdtmp = std(pd[i][2]);
        pd_neu[i] = list(pd[i][1], stdtmp, dim(stdtmp), mult(stdtmp));
    }
    return(pd_neu);
}


/////////////////////////////
int pr = 0;
```

```
//////////////
// The ring in which the algebraic number t is defined:
//
ring r = pr, (x,y,z,w,s,t,u,v), dp;

// The 6 planes p:
poly p = z*(16*x^5-160*x^3*y^2+80*x*y^4
 -80*x^4*z-160*x^2*y^2*z+320*x^2*z^3
 -80*y^4*z+320*y^2*z^3-256*z^5)/256;

// The quadric q:
poly q = (s*(x^2+y^2) +t*z^2 +u*w*z +v*w^2);

// The family of sextics with 30 cusps:
poly f = p - q^3;

ideal jf = diff(f,w), diff(f,y), diff(f,z), diff(f,x);
ideal jfy = substitute(jf, y,0);

ideal bl = diff(p,x), diff(p,z), diff(p,w), diff(p,y), q;
ideal bly = substitute(bl, y,0);
"";"";"sl:";"";
jfy;

"";"";"bl:";"";
bly;

"";"";"Compute I and eliminate x and z:";"";
poly discr;
"";"sat...";
ideal I = sat(jfy,bly)[1];
"";"std...";
I = std(I);

"";"eliminate x and z...";
ideal el = eliminate(I,xz);
el;
discr = el[1];

"";"";"From now on we choose v=1.";"";

//map mp = r, x,y,z,w,s,t,1,v;
map mp = r, x,y,z,w,s,t,u,1;
discr = mp(discr);
"";"";"Factorize Disc_f:";"";
factorize(discr);
"";

poly mpf = mp(f);
//"discr for u=1:";discr;

// the conditions on the parameters that yield
// additional singularities on the x=y=0 axes
// (precomputed)
```

```
poly conduv(1), conduv(2);
conduv(1) = 4*v*(t+1)-u^2;
conduv(2) = (u^2-4tv)^2 + 4*v*(u^2+4*v*(1-t));
// for the discriminant, we do not want the
// conditions conduv(i) that describe the cases
// that give a singularity on the x=y=0 axes:
"";"";"Notice that the largest component is exactly the one that describes";
"the cases that do not give a singularity on the x=y=0 axes:";"";
discr = quotient(discr,mp(conduv(1)))[1];
discr = quotient(discr,mp(conduv(2)))[1];
discr;

"";"";"Primary decomposition of sl(D_{f,1}) (takes a few seconds):";"";
if(0==1) {
    // The following takes a few seconds.
    // So, by default, we do not execute this part of the script.
    // Change 0==1 to 1==1 in the preceding if-statement
    // if you want this part to be executed.
    list sl_f = std_primdecGTZ(slocus(discr));
    sl_f;
} else {
    "skipped (precomputed).";
}

ideal sl_f3 = u4-68su2-8tu2+1216s2+272st+16t2,
    48s2u2-2496s3-192s2t+5,
    18432s4-5u2+80s+20t;

poly els = eliminate(sl_f3, tu)[1];
"";"The six values for s (equation (3)):";"";
els;

"-----";"";"Switch to the extension Q(s):";"";
string els_str = string(els);
"els:",els_str;
ring rs = (0,s),(t,u,x,y,z,w),dp;
execute("minpoly = "+els_str+";");
ideal sl_f3 = imap(r,sl_f3);
sl_f3 = std(sl_f3);
"";"";"equation (4):";"";
sl_f3;

"";"";"The value t_0(s) in the proof the $35$-cuspidal sextic theorem:";"";
poly p_t = subst(sl_f3[1], u, 0);
number n_t = leadcoef(- ((p_t / leadcoef(p_t)) - t));
n_t;

"";"";"The equation of S_{35}:";"";
poly f = imap(r,mpf);
f = substitute(f, u,0, t,n_t);
f;

"-----";"";"The total milnor number:";"";
ideal jf = diff(f,x), diff(f,y), diff(f,z), diff(f,w);
```

```
jf = std(jf);
"codim:", mycodim(jf), ", milnor:", mult(jf);

"";"";"The total milnor number on w=1:";"";
poly fw = substitute(f, w,1);
ideal jfw = fw, diff(fw,x), diff(fw,y), diff(fw,z);
jfw = std(jfw);
"codim:", mycodim(jfw), ", milnor:", mult(jfw);

"";"";"The total milnor number on y=0, w=0, z=1:";"";
poly fyw = substitute(f, y,0, w,0, z,1);
ideal jfyw = fyw, diff(fyw,x), diff(fyw,z);
// first throw away the non-existent point (0:0:0:0):
jfyw = sat(jfyw,xyzw)[1];
// then compute the total milnor number:
jfyw = std(jfyw);
"codim:", mycodim(jfyw), ", milnor:", mult(jfyw);

"";"Check that this is exactly one point by computing the radical:";"";
ideal radjfyw = radical(jfyw);
radjfyw = std(radjfyw);
"codim:", mycodim(radjfyw), ", milnor:", mult(radjfyw);
"";"This shows that S_{yw} is an A_2 singularity.";"";

"";"";"The ideal describing the point S_{yw} in the affine z=1 chart:";"";
list lSyw = primdecGTZ(jfyw);
ideal ptSyw = y,w,subst(lSyw[1][2],z,1);
ptSyw;

"-----";"";"Check that all the 30 other singularities are non-nodes:";"";
fw = substitute(f, w,1);
jfw = fw, diff(fw,x), diff(fw,y), diff(fw,z);
// then compute the total milnor number:
jfw = std(jfw);
"codim:", mycodim(jfw), ", milnor:", mult(jfw);
ideal nonnodes = fw, jfw, det(jacob(jacob(fw)));
nonnodes = std(nonnodes);
"codim(nonnodes):", mycodim(nonnodes), ", milnor(nonnodes):", mult(nonnodes);

"-----";"";"Check that there is no singularity on y=0, z=0 and w=1:";"";
ideal jfyz = fw, diff(fw,x), diff(fw,y), diff(fw,z);
jfyz = substitute(jfyz, y,0, z,0);
jfyz = std(jfyz);
"dim:",dim(jfyz), ", milnor:", mult(jfyz);

"-----";"";"Check that all the 10 singularities on z=0, w=1 are A_2s:";"";
"Compute the total milnor number:";
ideal jfz = subst(jfw,z,0);
jfz = std(jfz);
"codim:", mycodim(jfz), ", milnor:", mult(jfz);
"";"Check that there are exactly 10 singularities on z=0:";
"radical...";
ideal radjfz = radical(jfz);
"std...";
```

```
radjfz = std(radjfz);
"codim:", mycodim(radjfz), ", milnor:", mult(radjfz);
"";"As all the 10 are non-nodes, they all have milnor number 2";
"and are thus A_2-singularities.";

"-----";"";"Check that all the 4 singularities on y=0, w=1 are A_2s:";"";
"Compute the total milnor number:";
ideal jfy = subst(jfw,y,0);
jfy = std(jfy);
jfy;
"codim:", mycodim(jfy), ", milnor:", mult(jfy);
"";"Check that there are exactly 4 singularities on y=0:";
"Compute the primary decomposition of jfy...";
list ljfy = std_primdecGTZ(jfy,1);
ljfy;
"The 3rd and 4th entry are dimension and multiplicity";
"of the prime component:";
"codim:", ljfy[1][3], ", milnor:", ljfy[1][4];
"";"As all the 4 are non-nodes, they all have milnor number 2";
"and are thus A_2-singularities.";
"From the symmetry of the construction we thus know that";
"all the 20=4*5 singularities";
"which are in the D_5-orbits of these four singularities";
"are A_2-singularities.";

"-----";"";"Thus the surface S_{35} of degree 6 has exactly 35 cusps";
"and no other singularities.";
"";"This completes the proof the theorem.";"";
$;
```



A septic with 99 nodes, constructed using the geometry over prime fields.

CHAPTER 8

Using the Geometry over Prime Fields

We have already seen that the restrictions on the maximum number $\mu_{A_1}(d)$ of nodes on a nodal surface of degree d known so far are as follows:

degree	2	3	4	5	6	7	8	9	10	11	12	d
$\mu_{A_1}(d) \geq$	1	4	16	31	65	93	168	216	345	425	600	$\frac{5}{12}d^3$
$\mu_{A_1}(d) \leq$	1	4	16	31	65	104	174	246	360	480	645	$\frac{4}{9}d^3$

In this chapter we show (see also [**Lab04**]):

$$\mu_{A_1}(7) \geq \mu_{A_1}^{\mathbb{R}}(7) \geq \mathbf{99}. \tag{8.1}$$

The upper bound $\mu(7) \leq 104$ is given by Varchenko's spectrum bound (section 3.7). Notice that for $d = 7$ Miyaoka's bound (section 3.10) is 112, but Givental's bound (section 3.6) also computes to 104.

The previously known septic with the greatest number of nodes was the example of Chmutov with 93 nodes (see section 4.1 on page 45). For $d \leq 5$ and the even degrees $d = 6, 8, 10, 12$ there are examples exceeding Chmutov's lower bound: sections 4.5, 4.7, 4.9. These had been obtained by using some beautiful geometric arguments based on Rohn's (section 1.3) and B. Segre's idea (section 2.4).

Here, we explain how to use the geometry of computer algebra experiments over prime fields to treat the case $d = 7$ and to find the first surface of odd degree greater than 5 that exceeds Chmutov's general lower bound. Given an explicit equation of a family of hypersurfaces, there are some other approaches for finding those examples with the greatest number of nodes. We were not able to apply the techniques which do not involve computer algebra and which were used for degree $d = 6, 8, 10, 12$ because for these one needs a priori some good idea on the geometry of the surface. We neither succeeded using the computer algebra techniques from chapter 7 in the present case because of computer performance restrictions.

Instead, we choose a more geometric and experimental approach to study the family. The idea to use experiments over prime fields was already used by other people, e.g. Schreyer and Tonoli [**ST02**]. But in their case they were able to use deformation theoretical arguments to show that their examples lift to some special Calabi-Yau threefolds in characteristic zero. In our case, we lift the modular examples explicitly to characteristic zero using their geometry.

8.1. The Family

Inspired by many authors (see in particular sections 1.3, 4.2, 4.5, 4.7), we look for septics with many nodes in $\mathbb{P}^3(\mathbb{C})$ within a 7-parameter family of surfaces

$$S_{a_1,a_2,\dots,a_7} := P - U_{a_1,a_2,\dots,a_7}$$

of degree 7 admitting the dihedral symmetry D_7 of a 7-gon:

$$\begin{aligned}
P &:= 2^6 \cdot \Pi_{j=0}^{6} \left[\cos\left(\frac{2\pi j}{7}\right) x + \sin\left(\frac{2\pi j}{7}\right) y - z\right] \\
&= x \cdot \left[x^6 - 3{\cdot}7{\cdot}x^4y^2 + 5{\cdot}7{\cdot}x^2y^4 - 7{\cdot}y^6\right] \\
&\quad +7{\cdot}z{\cdot}\left[\left(x^2+y^2\right)^3 - 2^3{\cdot}z^2{\cdot}\left(x^2+y^2\right)^2 + 2^4{\cdot}z^4{\cdot}\left(x^2+y^2\right)\right] - 2^6{\cdot}z^7, \\
U_{a_1,\ldots,a_7} &:= (z+a_5w)\left(a_1z^3 + a_2z^2w + a_3zw^2 + a_4w^3 + (a_6z+a_7w)(x^2+y^2)\right)^2.
\end{aligned}$$

P is the product of 7 planes in $\mathbb{P}^3(\mathbb{C})$ meeting in the point $(0:0:0:1)$ and admitting D_7-symmetry with rotation axes $\{x = y = 0\}$: In fact, P is invariant under the map $y \mapsto -y$ and $P \cap \{z = z_0\}$ is a regular 7-gon for $z_0 \neq 0$. U is also D_7-symmetric, because x and y only appear as $x^2 + y^2$.

As we have already seen in section 1.3 on Rohn's construction of nodal quartics, such a surface S has generically nodes at the $3 \cdot 21 = 63$ intersections of the $\binom{7}{2} = 21$ doubled lines of P with the doubled cubic of U. We look for parameters $a_1, a_2, \ldots, a_7$, s.t. the corresponding surface has 99 nodes.

As $S_{a_1,a_2,\ldots,a_7}(x,y,z,\lambda w) = S_{a_1,\lambda a_2,\lambda^2 a_3,\lambda^3 a_4,\lambda a_5,a_6,\lambda a_7}(x,y,z,w)$ $\forall \lambda \in \mathbb{C}^*$, we choose $a_7 := 1$. Moreover, experiments over prime fields suggest that the maximum number of nodes on such surfaces is 99 and that such examples exist for $a_6 = 1$. As we are mainly interested in finding an example with 99 nodes, we restrict ourselves to the sub-family:

$$S := S_{a_1,a_2,a_3,a_4,a_5,1,1} = P - U_{a_1,a_2,a_3,a_4,a_5,1,1}.$$

Some other cases, e.g. $a_6 = 0$, also lead to 99-nodal septics (see e.g. chapter 9).

8.2. Reduction to the Case of Plane Curves

To simplify the problem of locating examples with 99 nodes within our family S, we restrict our attention to the $\{y = 0\}$ -plane and search for plane curves $S|_{y=0}$ (we write S_y for short) with many nodes. This is motivated by the symmetry of the construction:

LEMMA 8.1 (see [**End96**]). *A member $S = S_{a_1,a_2,a_3,a_4,a_5,1,1}$ of our family of surfaces has only ordinary double points as singularities, if $(1 : i : 0 : 0) \notin S$ and the surface does only contain ordinary double points as singularities in the plane $\{y = 0\}$. If the plane septic S_y has exactly n nodes and if exactly n_{xy} of these nodes are on the axes $\{x = y = 0\}$ then the surface S has exactly $n_{xy} + 7 \cdot (n - n_{xy})$ nodes and no other singularities. Each singularity of S_y which is not on $\{x = y = 0\}$ gives an orbit of 7 singularities of S under the action of the dihedral group D_7.*

PROOF. Because of the D_7-symmetry of the construction, we only have to show that there are no other singularities than the claimed ones. It is easy to prove (see [**End96**, p. 18, cor. 2.3.10] for details) that any isolated singularity of S which is not contained in one of the orbits of the nodes of S_y would yield a non-isolated singularity which intersects the plane $\{y = 0\}$. But this contradicts the assumption that the surface S does only contain ordinary double points on $\{y = 0\}$. □

So, we first look for septic plane curves of the form S_y with many nodes, then we verify that these singularities are indeed also nodes of the surface. Via the lemma, we are then able to conclude that the surface has only ordinary double points. In order to understand the geometry of the plane septic S_y better we look at the singularities that occur for generic values of the parameters. First, we compute:

$$\begin{aligned} P|_{y=0} &= x^7 + 7\cdot x^6 z - 7\cdot 2^3\cdot x^4 z^3 + 7\cdot 2^4\cdot x^2 z^5 - 2^6\cdot z^7 \\ &= \frac{(x-z)}{2^4}\cdot\big(\underbrace{x+(-\rho)z}_{=:L_1}\big)^2\cdot\big(\underbrace{2x+(\rho^2+4\rho)z}_{=:L_2}\big)^2\cdot\big(\underbrace{2x+(-\rho^2-2\rho+8)z}_{=:L_3}\big)^2, \\ U|_{y=0} &= (z+a_5 w)\big(\underbrace{(z+w)x^2 + a_1 z^3 + a_2 z^2 w + a_3 z w^2 + a_4 w^3}_{=:C}\big)^2, \end{aligned}$$

where ρ satisfies:

$$\rho^3 + 2^2\rho^2 - 2^2\rho - 2^3 = 0. \tag{8.2}$$

The three points G_{ij} of intersection of C with the line L_i are ordinary double points of the plane septic $S_y = P|_{y=0} - U|_{y=0}$ for generic values of the parameters, s.t. we have $3\cdot 3 = 9$ *generic* singularities (see fig. 8.1).

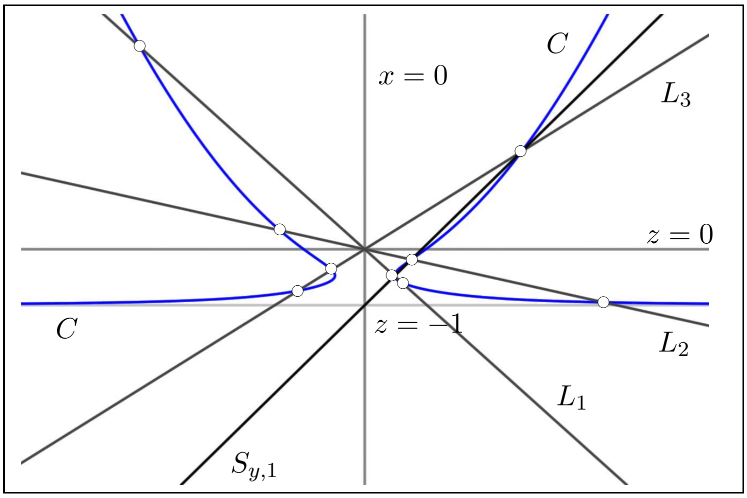


FIGURE 8.1. The three doubled lines L_i and the doubled cubic C intersect in $3\cdot 3 = 9$ points G_{ij}. These are the *generic* singularities of the plane septic S_y.

8.3. Finding Solutions over some Prime Fields

In the early times of computer algebra, the software was only able to work over finite prime fields. It is well-known that the reduction modulo a prime p of a hypersurface has the same number and type of singularities for almost all p. So, the common practice in the early 1990's was to compute this for a hopefully sufficient number of different primes.

We take the other direction. By running over all possible parameter combinations over some small prime fields $\mathbb{F}_p$ using the computer algebra system SINGULAR [**GPS01**], we find some 99-nodal surfaces over these fields: For a given set of parameters $a_1, a_2, \ldots, a_5$, we can easily check the actual number of nodes on the corresponding surface using computer algebra (see [**GP02**, appendix A, p. 487]).

As indicated in the previous section, we work in the plane $\{y = 0\}$ for faster computations. It turns out that the greatest number of nodes on S_y is 15 over the small prime fields $\mathbb{F}_p$, $11 \leq p \leq 53$: See table 8.1 on the facing page. The prime fields $\mathbb{F}_p$, $2 \leq p \leq 7$, are not listed because they are special cases: These primes appear as coefficients or exponents in the equation of our family. In each of the cases we checked, one of the 15 singular points lies on the axes $\{x = 0\}$, such that the corresponding surface has exactly $14 \cdot 7 + 1 = 99$ nodes and no other singularities.

8.4. The Geometry of the 15-nodal septic Plane Curve

To find parameters $a_1, a_2, \ldots, a_5$ in characteristic 0 we want to use geometric properties of the 15-nodal septic plane curve S_y. But as we do not know any such property yet, we use our prime field examples to get some good ideas:

OBSERVATION 8.2. *In all our prime field examples of* 15*-nodal plane septics* S_y*, we have:*

(1) *S_y splits into a line $S_{y,1}$ and a sextic $S_{y,6}$: $S_y = S_{y,1} \cdot S_{y,6}$. The plane curve $S_{y,6}$ of degree 6 has $15 - 6 = 9$ singularities. Note that this property is similar to the one of the 31-nodal D_5-symmetric quintic in $\mathbb{P}^3(\mathbb{C})$ constructed by W. Barth (section 4.2 on page 47).*

The line and the sextic have some interesting geometric properties (see fig. 8.1 on the preceding page and fig. 8.3 on page 102):

(2) *$S_{y,1} \cap S_{y,6} = \{R, G_{1j_1}, G_{2j_2}, G_{3j_3}, O_1, O_2\}$, where R is a point on the axes $\{x = 0\}$ and the G_{ij_k} are three of the 9 generic singularities G_{ij} of S_y, one on each line L_i, and O_1, O_2 are some other points that neither lie on $\{x = 0\}$, nor on one of the L_i.*

(3) *The sextic $S_{y,6}$ has the six generic singularities G_{ij}, $(i,j) \in \{1,2,3\}^2 \setminus \{(1,j_1),(2,j_2),(3,j_3)\}$, and three exceptional singularities: E_1, E_2, E_3.*

In many prime field experiments, we have furthermore:

(4) *In the projective x, z, w-plane, the point R has the coordinates $(0 : -1 : 1)$, s.t. the line $S_{y,1}$ has the form $S_{y,1} :\ z + t \cdot x + w = 0$ for some parameter t (see also table 8.1 on the facing page).*

The other cases ($R = (0 : c : 1), c \neq -1$) lead to more complicated equations and will not be discussed here.

Using this observation as a guess for our septic in characteristic 0, we obtain several polynomial conditions on the parameters. Using SINGULAR to eliminate variables, we find the following relation between the parameters a_4 and t:

$$t \cdot \big(\underbrace{a_4 t^3 + t}_{=:\alpha}\big)^2 + t - 1 = 0, \tag{8.3}$$

which can be parametrized by α: $t = -\frac{1}{1+\alpha^2}$, $a_4 = (\alpha(1+\alpha^2)-1)(1+\alpha^2)^2$. Further eliminations allow us to express all the other parameters in terms of α:

- $a_1 = \alpha^7 + 7\alpha^5 - \alpha^4 + 7\alpha^3 - 2\alpha^2 - 7\alpha - 1$,
- $a_2 = (\alpha^2 + 1)(3\alpha^5 + 14\alpha^3 - 3\alpha^2 + 7\alpha - 3)$,
- $a_3 = (\alpha^2 + 1)^2(3\alpha^3 + 7\alpha - 3)$,
- $a_5 = -\frac{\alpha^2}{1+\alpha^2}$.

Field	a_1	a_2	a_3	a_4	a_5	$S_{y,1}$	α
$\mathbb{F}_{11}$	2	3	5	2	-5	$z = x - w$	$\alpha = -3$
$\mathbb{F}_{19}$	-7	-2	7	1	8	$z = 8x - w$	$\alpha = 7$
$\mathbb{F}_{19}$	2	0	1	9	7	$z = 9x - w$	$\alpha = -4$
$\mathbb{F}_{19}$	5	-9	7	-3	-1	$z = 2x - w$	$\alpha = -3$
$\mathbb{F}_{23}$	-5	11	10	1	7	$z = -9x - w$	$\alpha = -2$
$\mathbb{F}_{31}$	-15	-13	-5	13	-10	$z = -2x - w$	$\alpha = -13$
$\mathbb{F}_{31}$	1	-2	14	-9	11	$z = 15x - w$	$\alpha = -11$
$\mathbb{F}_{31}$	14	-10	-13	-14	-11	$z = -13x - w$	$\alpha = -7$
$\mathbb{F}_{43}$	-11	15	0	-13	-6	$z = -6x - w$	$\alpha = 7$
$\mathbb{F}_{43}$	20	16	-1	-14	10	$z = -12x - w$	$\alpha = 14$
$\mathbb{F}_{43}$	-9	3	-3	-11	5	$z = 18x - w$	$\alpha = -21$
$\mathbb{F}_{53}$	-8	20	14	18	11	$z = 25x - w$	$\alpha = 4$
$\mathbb{F}_{53}$	-2	-10	-14	-26	16	$z = -9x - w$	$\alpha = 24$
$\mathbb{F}_{53}$	10	25	-4	22	25	$z = -16x - w$	$\alpha = 25$

TABLE 8.1. A few examples of parameters giving 15-nodal septic plane curves (and 99-nodal surfaces) over prime fields (see [**Lab03a**] for more exhaustive tables).

8.5. The 1-parameter Family of Plane Sextics

Once more we use our explicit examples of 15-nodal septic plane curves over prime fields to finally be able to write down a condition for α in characteristic 0.

First, note that we can now easily obtain the equation of $S_{y,6}$ by dividing the equation of our septic curve S_y by the equation of the line $S_{y,1} = z + tx + w = z - \frac{1}{1+\alpha^2}x + w$. $S_{y,6}$ is a sextic which has 6 nodes for generic α, but should have 9 double points for some special values of α. One idea to determine these particular values is to find a geometric relation between the 6 generic singular points and the 3 exceptional ones.

8.5.1. Three Conics. Looking at the equations describing the singular points of our examples of 9-nodal sextics $S_{y,6}$ over the prime fields, we see the following:

OBSERVATION 8.3. *For all our 9-nodal examples of plane sextics over prime fields, there are three conics through six of these points each (see fig. 8.2 on the next page):*

(1) *one conic C_0 through the 6 generic singularities,*
(2) *one conic C_1 through the 3 exceptional singularities and 3 of the generic ones,*
(3) *one conic C_2 through the 3 exceptional singularities and the other 3 generic ones.*

Moreover, the three conics have the following properties over the prime fields:

(4) *C_1 has the form:*

$$C_1 : \; x^2 + kz^2 + (k+4)zw = 0, \tag{8.4}$$

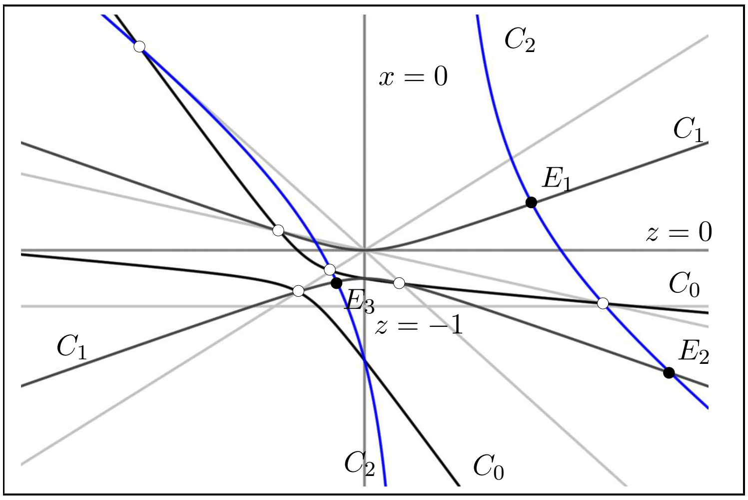


FIGURE 8.2. Three conics relating the 9 double points of the sextic $S_{y,6}$. E_1, E_2, and E_3 (black) are the exceptional singularities (i.e. they do not lie on one of the lines L_i, see fig. 8.1 on page 97). The white points are the generic singularities, coming from the intersection of the doubled cubic C with the three doubled lines L_i.

where k is a still unknown parameter. In particular, C_1 is symmetric with respect to $x \mapsto -x$ and contains the point $(0:0:1)$.

(5) *C_0 intersects the other two conics on the $\{x = 0\}$ -axes (see fig. 8.2):*

$$X_1 := C_0 \cap C_1 \cap \{x = 0\}, \quad X_2 := C_0 \cap C_2 \cap \{x = 0\}. \tag{8.5}$$

To determine the new parameter k in equation (8.4), we will use (8.5). We compute the two points of C_0 on the $\{x = 0\}$ -axes explicitly using SINGULAR: First, the ideal $I^{gen}_{S_{y,6}}$ describing the six generic singularities of $S_{y,6}$ can be computed from the ideal $I^{gen}_{S_y} := (C,\, L_1L_2L_3)$ describing the 9 generic singularities of S_y by calculating the following ideal quotient: $I^{gen}_{S_{y,6}} = I^{gen}_{S_y} : S_{y,1}$. Now, the equation of C_0 can be obtained by taking the degree-2-part of the ideal $I^{gen}_{S_{y,6}}$:

$$C_0: \quad \begin{aligned} &\alpha x^2 + (\alpha^3 + 5\alpha - 1)xz + (\alpha^3 + \alpha - 1)xw \\ (\alpha^5 + 6\alpha^3 - \alpha^2 + \alpha - 1)z^2 &+ (2\alpha^5 + 8\alpha^3 - 2\alpha^2 + 6\alpha - 2)zw \\ &+(\alpha^5 + 2\alpha^3 - \alpha^2 + \alpha - 1)w^2 = 0. \end{aligned} \tag{8.6}$$

Thus, $\{P^+, P^-\} := C_0 \cap \{x = 0\} = \left\{ \left(0 : \frac{-2(\alpha^3+3\alpha-1)(1+\alpha^2) \pm \beta(\alpha)}{2(\alpha^5+6\alpha^3-\alpha^2+\alpha-1)} : 1 \right) \right\}$, where

(8.7)
$$\begin{aligned} \beta(\alpha)^2 \quad &:= \quad (\alpha^3 + 3\alpha - 1)^2(1 + \alpha^2)^2 \\ &\quad\quad -4(\alpha^5 + 6\alpha^3 - \alpha^2 + \alpha - 1)(1 + \alpha^2)(\alpha^3 + \alpha - 1) \\ &= \quad (1 + \alpha^2)(\alpha^8 - 4\alpha^7 + 3\alpha^6 - 22\alpha^5 - 5\alpha^4 + 16\alpha^3 + 6\alpha^2 + 2\alpha - 3). \end{aligned}$$

C_1 intersects the $\{x = 0\}$ -axes in exactly two points: $(0 : 0 : 1)$ and X_1. Hence, we can determine the two possibilities for the parameter $k \in \mathbb{Q}(\alpha, \beta(\alpha))$ in equation (8.4) for C_1: Together with the z and w-coordinates of the points $P^{\pm}$,

$C_1 \cap \{x=0\} = \{kz^2 + kzw + 4zw = 0\}$ leads to the following two possibilities:

$$C_1 : \quad x^2 + \frac{-4P_z^{\pm}}{P_z^{\pm}(P_z^{\pm}+1)} z(z+w) + 4zw = 0. \tag{8.8}$$

8.5.2. The Condition on α. The equations of the conics C_0 and C_1 will allow us to compute the condition on α, s.t. the sextic $S_{y,6}$ has 9 singularities, using the following (see observation 8.3 and fig. 8.2):

- C_0 intersects the three doubled lines L_i exactly in the six generic singularities.
- C_1 intersects the three doubled lines L_i exactly in three of these six generic singularities and the origin (which counts three times).

Thus, the set of z-coordinates of the three points $(C_1 \cap L_1L_2L_3) \setminus \{(0:0:1)\}$ has to be contained in the set of z-coordinates of the six points $C_0 \cap L_1L_2L_3$. This means that the remainder q of the following division (res_x denotes the resultant with respect to x)

$$res_x(C_0,\, L_1L_2L_3) = p(z) \cdot \left(\frac{1}{z^3} \cdot res_x(C_1,\, L_1L_2L_3) \right) + q(z) \tag{8.9}$$

should vanish: $q = 0$.

As the degree of the remainder is $\deg(q) = 2$, this gives 3 conditions on α and $\beta(\alpha)$, coming from the fact that all the 3 coefficients of $q(z)$ have to vanish. It turns out that it suffices to take one of these, the coefficient of z^2, which can be written in the form $c(\alpha) + \beta(\alpha)d(\alpha)$, where $c(\alpha)$ and $d(\alpha)$ are polynomials in $\mathbb{Q}[\alpha]$. As a condition on α only we can take:

$$cond(\alpha) := \big(c(\alpha) + \beta(\alpha)d(\alpha)\big) \cdot \big(c(\alpha) - \beta(\alpha)d(\alpha)\big) \in \mathbb{Q}[\alpha],$$

which is of degree 150.

This condition $cond(\alpha)$ vanishes for those α for which the corresponding surface has 99 nodes and for several other α. To obtain a condition which exactly describes those α we are looking for, we factorize $cond(\alpha) = f_1 \cdot f_2 \cdots f_k$ (e.g., using SINGULAR again). Substituting in each of these factors our solutions over the prime fields, we see that the only factor that vanishes is: $7\alpha^3 + 7\alpha + 1 = 0$.

8.6. The Equation of the 99-nodal Septic

Up to this point, it is still only a guess — verified over some prime fields — that the values α satisfying the condition above give 99-nodal septics in characteristic 0. But it turns out that we have indeed:

THEOREM 8.4 (99-nodal Septic). *Let $\alpha \in \mathbb{C}$ satisfy:*

$$7\alpha^3 + 7\alpha + 1 = 0. \tag{8.10}$$

Then the surface S_α in $\mathbb{P}^3(\mathbb{C})$ of degree 7 with equation $S_{99} := S_\alpha := P - U_\alpha$ has exactly 99 ordinary double points and no other singularities, where

$$\begin{aligned} P &:= x\cdot\left[x^6 - 3\cdot7\cdot x^4y^2 + 5\cdot7\cdot x^2y^4 - 7\cdot y^6\right] \\ &\quad +7\cdot z\cdot\left[\left(x^2+y^2\right)^3 - 2^3\cdot z^2\cdot\left(x^2+y^2\right)^2 + 2^4\cdot z^4\cdot\left(x^2+y^2\right)\right] - 2^6\cdot z^7, \\ U_\alpha &:= (z + a_5w)\left((z+w)(x^2+y^2) + a_1z^3 + a_2z^2w + a_3zw^2 + a_4w^3\right)^2, \end{aligned}$$

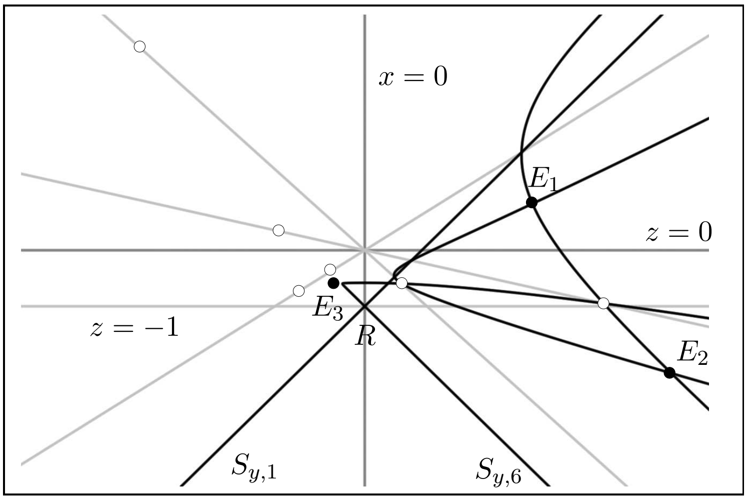


FIGURE 8.3. The 15-nodal plane septic $S_{y_{\alpha_{\mathbb{R}}}} = S_{y,1_{\alpha_{\mathbb{R}}}} \cdot S_{y,6_{\alpha_{\mathbb{R}}}}$ (see (8.11) on page 102); the singularities of the sextic $S_{y,6_{\alpha_{\mathbb{R}}}}$ are marked by large circles: The three exceptional singularities E_1, E_2, E_3 are marked in black, the generic singularities in white. The five left-most nodes are real isolated ones. Only five of the six intersections of the line $S_{y,1_{\alpha_{\mathbb{R}}}}$ and the sextic $S_{y,6_{\alpha_{\mathbb{R}}}}$ are visible because we just show a small part of the whole (x, z)-plane.

$$\begin{aligned} a_1 &:= -\tfrac{12}{7}\alpha^2 - \tfrac{384}{49}\alpha - \tfrac{8}{7}, & a_2 &:= -\tfrac{32}{7}\alpha^2 + \tfrac{24}{49}\alpha - 4, \\ a_3 &:= -4\alpha^2 + \tfrac{24}{49}\alpha - 4, & a_4 &:= -\tfrac{8}{7}\alpha^2 + \tfrac{8}{49}\alpha - \tfrac{8}{7}, \\ a_5 &:= 49\alpha^2 - 7\alpha + 50. \end{aligned}$$

There is exactly one real solution $\alpha_{\mathbb{R}} \in \mathbb{R}$ to the condition (8.10),

$$\alpha_{\mathbb{R}} \approx -0.14010685, \tag{8.11}$$

and all the singularities of $S_{\alpha_{\mathbb{R}}}$ are also real.

PROOF. The straightforward SINGULAR computation to show that the surface has 99 nodes and no other singularities takes too long on our computer. So, instead, we use computer algebra to verify the assumptions of lemma 8.1 on page 96: The point $(1 : i : 0 : 0) \notin S_\alpha$, and the surface S_α contains exactly 15 nodes on $\{y = 0\}$ one of which lies on the axes $\{x = y = 0\}$. Our SINGULAR code for checking this is listed in the appendix and can also be obtained from [**Lab03a**]. In this way, we are able to conlude that S_α has exactly $14{\cdot}7 + 1 = 99$ nodes as singularities.

Using the fact that $\beta(\alpha)^2 = \left(\frac{12}{7}\right)^2$ together with the geometric description of the singularities of the plane septic given in the previous sections, it is also straightforward to verify the reality assertion. □

8.7. Further Remarks

The existence of the real $\alpha_{\mathbb{R}}$ allows us to use our tool SURFEX [**HLM05**] to compute an image of the 99-nodal septic $S_{\alpha_{\mathbb{R}}}$ (fig. 8.4 on the next page). When denoting the maximum number of real singularities a septic in $\mathbb{P}^3(\mathbb{R})$ can have by $\mu^{\mathbb{R}}(7)$, we get, with the remarks mentioned in the introduction:

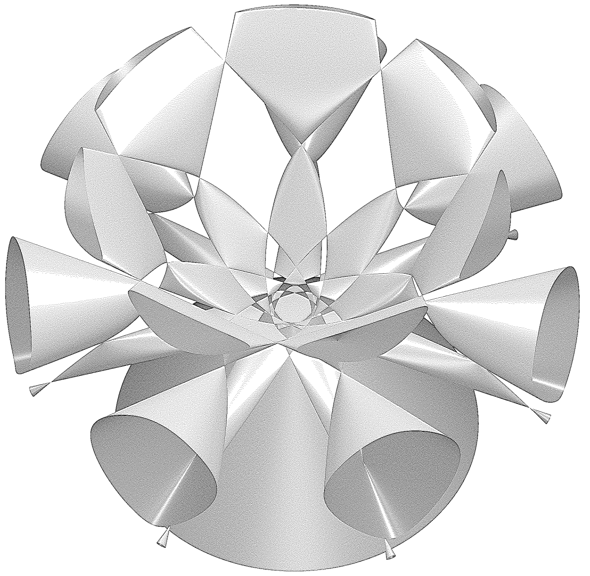


FIGURE 8.4. A part of the affine chart $w = 1$ of the real septic with 99 nodes, see [**Lab03a**] for more images and movies.

COROLLARY 8.5.

$$99 \leq \mu^{\mathbb{R}}(7) \leq \mu(7) \leq 104.$$

Note that the previously known lower bound, 93, was reached by S. V. Chmutov's surface (section 4.1). It can be computed using deformation theory and SINGULAR (see section 4.6 on page 51) that the space of obstructions for globalizing all local deformations is zero. We thus obtain:

COROLLARY 8.6. *There exist surfaces of degree* 7 *in* $\mathbb{P}^3(\mathbb{R})$ *with exactly* k *real nodes and no other singularities for* $k = 0, 1, 2, \ldots, 99$.

Recently, there has been some interest in surfaces that do exist over some finite fields, but which are not liftable to characteristic 0. The reduction of our 99-nodal septic S_α modulo 5 (note: $1 \in \mathbb{F}_5$ satisfies (8.10): $7{\cdot}1^3 + 7{\cdot}1 + 1 \equiv 0$ modulo 5) neither gives a 99-nodal surface nor a highly degenerated one as one might expect because the exponent 5 appears several times in the defining equation. Instead, we can easily verify the following using computer algebra:

COROLLARY 8.7. *For* $\alpha_5 := 1 \in \mathbb{F}_5$ *the surface* $S_{\alpha_5} \subset \mathbb{P}^3(\mathbb{F}_5)$ *defined as in the above theorem has* 100 *nodes and no other singularities.*

Of course, not all the coordinates of its singularities are in $\mathbb{F}_5$, but in some algebraic extension. The septic has similar geometric properties as our 99-nodal surface; in addition it has one node at the intersection of the $\{x = y = 0\}$ axes and $\{w = 0\}$. Until now, we were not able to determine if this 100-nodal septic defined over $\mathbb{F}_5$ can be lifted to characteristic zero.

8.8. A Conjecture

We hope to be able to apply our technique for finding surfaces with many nodes within families of surfaces to similar problems. E.g., it should be possible to construct surfaces with dihedral symmetry of degree 9 and 11 with many ordinary double points. In fact, our experiments over prime fields suggest the following conjecture which is already established for $d = 3, 5, 7$ (see figure 8.5 which illustrates the geometry of the plane curve S_y):

CONJECTURE 8.8. *For any odd $d \geq 3$, there exists a surface S of degree d with $\frac{1}{8} \cdot \left(3d^3 - 4d^2 - 7d + 8\right)$ nodes with the following geometric properties:*

(1) *S has dihedral symmetry D_d and is constructed based on Rohn's idea (section 1.3): $S = P - (z + a_0 w) \cdot (S_{\frac{d-1}{2}})^2$, where P is a product of d planes*

$$P = \Pi_{j=0}^{d-1} \left[\cos\left(\frac{2\pi j}{d}\right) x + \sin\left(\frac{2\pi j}{d}\right) y - z \right]$$

and $S_{\frac{d-1}{2}}$ is a surface of degree $\frac{d-1}{2}$.

(2) *The plane curve $S_y := S \cap \{y = 0\}$ factors into a line and a curve of degree $d-1$: $S_y = S_{y,1} \cdot S_{y,d-1}$.*

(3) *S_y has $\frac{d-1}{2} + \left(\frac{d-1}{2}\right)^2 + \frac{1}{2}\frac{d-1}{2}\left(\frac{d-1}{2} - 1\right)$ nodes.*

(4) *Exactly one of the nodes of S_y, say R, lies on the rotation axes $\{x = y = 0\}$ of the dihedral operation. In fact, R is the intersection of the line $S_{y,1}$ with the rotation axes $\{x = y = 0\}$.*

(5) *The generic surface from Rohn's construction has nodes at the intersection of the $\binom{d}{2} = d \cdot \left(\frac{d-1}{2}\right)$ intersection lines of the d planes defined by P with the surface $S_{\frac{d-1}{2}}$ of degree $\frac{d-1}{2}$. Because of the dihedral symmetry of the construction $\frac{1}{d} \cdot d \cdot \left(\frac{d-1}{2}\right)^2$ of the nodes of the plane septic S_y come from this general construction.*

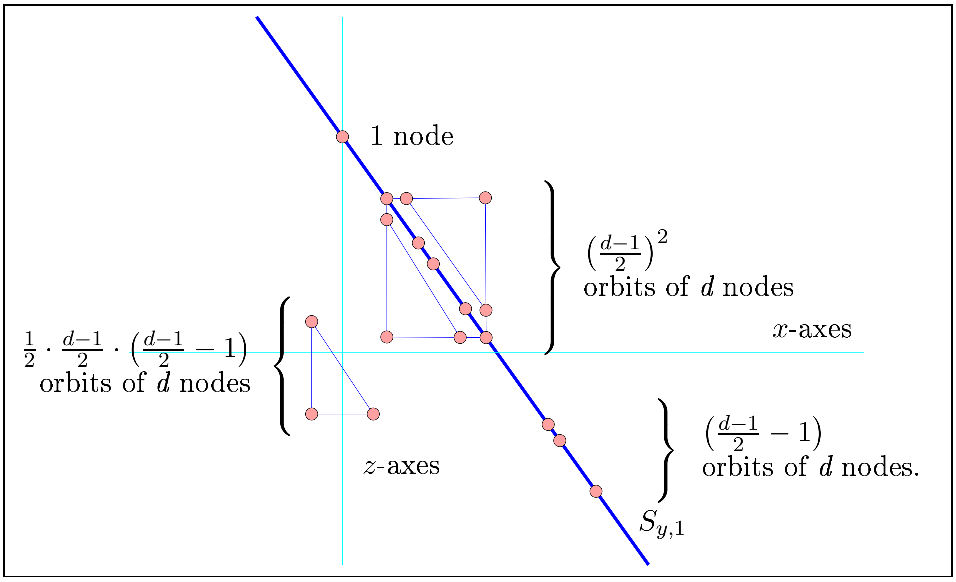


FIGURE 8.5. The geometry of the conjectured plane curve S_y.

For $d \leq 11$, the number of nodes conjectured above exceeds Chmutov's lower bound for the maximum number of nodes on a surface of degree d (section 4.1). But for $d \geq 13$, Chmutov's examples have more nodes. Thus, if the conjecture cannot be improved then it does only yield very few new lower bounds: $\mu_{A_1}(9) \geq 226$ and $\mu_{A_1}(11) \geq 430$.

8.9. Appendix: The SINGULAR Code

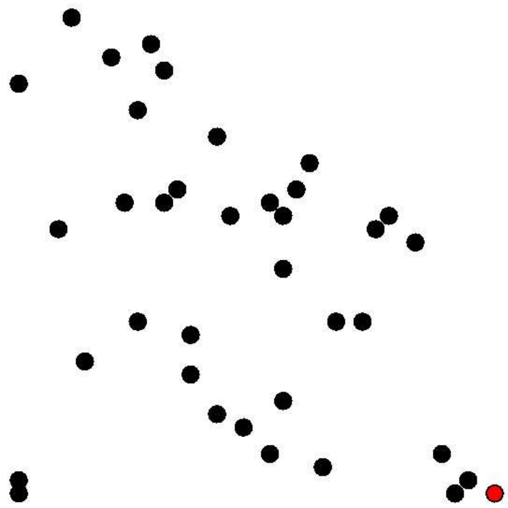
8.9.1. The SINGULAR Library manySings.lib.

```
LIB "sing.lib";
LIB "primdec.lib";
LIB "primitiv.lib";

proc mapPtTo0(ideal pt) {
 def oring=basering; poly allvars=1; list ll=list(); poly ptmp;
 for(int i=1; i<=nvars(oring); i++) { allvars = allvars * var(i); }
  for(i=1; i<=nvars(oring); i++) { ptmp=eliminate(pt, allvars/var(i))[1];
   ll[i]=1/leadcoef(ptmp)*var(i)-((ptmp/leadcoef(ptmp))-var(i));}
 return(list(ll)); } // end proc mapPtTo0(ideal pt)

proc checkOnlyNodes(list l, string strf, int p) {
 def oring=basering; ideal itmp; string imp; string strmp;
 for(int i=1; i<=size(l); i++) { "";"-----";"i =",i;"";
  execute("itmp="+l[i]+";"); mult(std(itmp)),"point(s)";
  "eliminate..."; imp=string(minpoly)+","+string(eliminate(itmp,x));
  ring rel(i)=(p,gamma),(al,z),dp; execute("ideal mp="+imp+";");
   strmp=string(subst(cleardenom(primitive(mp)[1]),z,gamma));
  ring rga(i)=(p,gamma), (x,y,z), dp; execute("minpoly="+ strmp+";");
   list lal=primdecGTZ(7*x^3 + 7*x + 1); number al;
   if(mult(std(lal[2][2]))==1) {
    al=-leadcoef(lal[2][2][1]/leadcoef(lal[2][2][1])-x);
   } else { al=-leadcoef(lal[1][2][1]/leadcoef(lal[1][2][1])-x); }
   execute("ideal jf=y,"+l[i]+";"); "primdec..."; list lf=primdecGTZ(jf);
   list ml=mapPtTo0(lf[1][2]); map m=rga(i),ml[1][1],ml[1][2],ml[1][3];
   execute("poly Saff="+strf+";"); poly lSaff=m(Saff);
  ring rloc(i)=(p,gamma), (x,y,z), ds; execute("minpoly="+ strmp+";");
   poly lSaff=imap(rga(i),lSaff);
   "milnor..."; isTrue(milnor(lSaff)==1);
   setring oring; } } // end proc checkOnlyNodes(list l, string strf, int p)
```

8.9.2. The SINGULAR Code to Prove Theorem 8.4.

```
// The execution of the whole program takes a few minutes.
LIB "manySings.lib"; // also available from www.AlgebraicSurface.net
proc isTrue(int c) { if(c==0) { return("FALSE"); } else { return("TRUE");}}

int p=0; ring r=(p,al), (x,y,z,w), dp; minpoly=7*al^3 + 7*al + 1;
number a(1)=-12/7*al^2-384/49*al-8/7; number a(2)=-32/7*al^2+24/49*al-4;
number a(3)=-4*al^2+24/49*al-4; number a(4)=-8/7*al^2+8/49*al-8/7;
number a(5)=49*al^2-7*al+50;
poly P=x*(x^6-3*7*x^4*y^2+5*7*x^2*y^4-7*y^6)
       +7*z*((x^2+y^2)^3-2^3*z^2*(x^2+y^2)^2+2^4*z^4*(x^2+y^2)) -2^6*z^7;
poly U=(z+a(5)*w)*(a(1)*z^3+a(2)*z^2*w+a(3)*z*w^2+a(4)*w^3+(z+w)*(x^2+y^2))^2;
poly S=P-U; // the 99-nodal surface

poly Si=substitute(S, x,1 ,z,0 ,w,0); ideal yi=y^2+1; yi=std(yi);
"";"Check that (1:i:0:0) is not on S:", isTrue(reduce(Si,yi)!=0);
```

```
ideal jSxy=x,y,jacob(S); jSxy=std(jSxy);
"";"There is only one node on x=y=0:",isTrue(mult(jSxy)==1&&dim(jSxy)==1);
ideal jSwy=w,y,jacob(S); jSwy=std(jSwy);
"";"There is no singularity at infinity:",isTrue(dim(jSwy)==0);

poly Saff=subst(S, w,1); string strSaff=string(Saff);
poly Syaff=substitute(S, y,0, w,1);
poly Sy1=subst(z-1/(1+al^2)*x+w,w,1); poly Sy6=Syaff/Sy1;
"";"Check the factorization Sy=Sy1*Sy6:", isTrue(Sy1*Sy6 - Syaff == 0);
ideal j16=Sy1,Sy6; list l16=primdecSY(j16); j16=std(j16);
"Sy1 & Sy6 meet in 6 pts (with mult.):",isTrue(dim(j16)==2&&mult(j16)==6);
ideal j6=Sy6,diff(Sy6,x),diff(Sy6,z); j6=std(j6);
"Sy6 has 9 singularities (with mult.):",isTrue(dim(j6)==2&&mult(j6)==9);
ideal iSy16=Sy1,j6;
"None of these 9 singularities lies on Sy1:",isTrue(dim(std(iSy16))==-1);
poly L1L2L3=x^3+4*x^2*z-4*x*z^2-8*z^3;
poly C0=(al)*x^2+(al^3+5*al-1)*x*z+(al^5+6*al^3-al^2+al-1)*z^2
  +(al^3+al-1)*x+(2*al^5+8*al^3-2*al^2+6*al-2)*z+(al^5+2*al^3-al^2+al-1);
ideal jG6=L1L2L3, subst(C0,w,1);
"The 6 generic sing. of Sy6..."; list lG6=primdecSY(jG6);
"The 3 special sing. of Sy6..."; list lS6=primdecSY(quotient(j6,jG6));

// The prime ideals defining the 15 singularities on {y=0}:
list lsings=list(string(l16[1][2]),string(l16[2][2]),string(l16[3][2]),
                 string(lG6[1][2]),string(lG6[2][2]),string(lS6[1][2]));
"";"Verify that these 15 singularities on {y=0} are nodes of the surface";
"by mapping each point to zero and checking locally:";
checkOnlyNodes(lsings,strSaff,p);$;
```



The $\{y=0\}$ plane section of the affine chart $w=1$ of a 226-nodal nonic over $\mathbb{F}_{61}$. It can be found and lifted to characteristic zero using an algorithm for locating interesting examples within families of algebraic varieties.

CHAPTER 9

Locating Interesting Examples within Families

Suppose we have a k-parameter family of algebraic varieties $V_{a_1,a_2,\dots,a_k}$ defined over some algebraic extension $\mathbb{K} := \mathbb{Q}(\alpha)$ of $\mathbb{Q}$ in which we hope to exist a particularly *interesting* example.

Suppose furthermore that there exists an algorithm which allows us to detect using computer algebra if a variety $V_{a_1,a_2,\dots,a_k}$ is *interesting* for given values of the parameters $a_1, a_2, \dots, a_k$. Then the algorithm which we describe in this chapter and which we implemented as the SINGULAR [**GPS01**] library SEARCHINFAMILIES.LIB allows us to locate these examples in many cases.

As we have seen in the preceding chapters, all surfaces of degree $d \leq 8$ with the greatest known number of nodes can be constructed by locating them within families with dihedral symmetry. Furthermore, we have already seen that for a given surface it is easy to compute its number of nodes using computer algebra. Thus, the problem of finding surfaces with many nodes is exactly of the type described in the previous paragraph.

And indeed, we will see in the following sections that the construction of all known surfaces of degree $d \leq 7$ which lead to the best known lower bounds for the maximum number $\mu_{A_1}(d)$ of nodes can be reduced to a computer algebra calculation. Moreover, we apply the method to the case of degree $d = 9$ which leads to a new lower bound:

$$\mu_{A_1}(9) \geq \mathbf{226}. \tag{9.1}$$

Recall that we have seen in the first part of this Ph.D. thesis and in chapter 8 that the restrictions on $\mu_{A_1}(d)$ known before the present chapter are as follows:

degree	2	3	4	5	6	7	8	9	10	11	12	d
$\mu_{A_1}(d) \geq$	1	4	16	31	65	99	168	216	345	425	600	$\frac{5}{12}d^3$
$\mu_{A_1}(d) \leq$	1	4	16	31	65	104	174	246	360	480	645	$\frac{4}{9}d^3$

Thus, in degree $d = 9$ there remains a gap of 20 nodes between our construction which leads to 226 nodes and the best known upper bound 246.

9.1. Some Introductory Examples

In the previous chapter, we used the geometry of the prime field examples to obtain a conjecture for some restrictions on the parameters. We could then verify them by simply computing the number of nodes of the resulting surface.

The process of figuring out the needed geometric properties of the prime field examples involved creative human interaction. Here, we use a purely arithmetic way to lift the prime field examples which can be performed automatically. Nevertheless, geometric insight can speed up the algorithm significantly.

field	β	$\mu(\mathrm{vS}_{1:\beta})$
$\mathbb{F}_2$	—	—
$\mathbb{F}_3$	—	—
$\mathbb{F}_5$	1	205
$\mathbb{F}_5$	-1	205
$\mathbb{F}_7$	1	130
$\mathbb{F}_{11}$	1	130
$\mathbb{F}_{13}$	1	130
$\mathbb{F}_{17}$	1	130

TABLE 9.1. Examples of $\mathrm{vS}_{(1:\beta)}$ over some prime fields which have at least 130 ordinary double points.

The examples which we present in this section illustrate the basic ideas behind our method.

9.1.1. Van Straten's 130-nodal Quintic in $\mathbb{P}^4$. Let us look for examples with many isolated singularities within van Straten's two-parameter family (see section 4.3 on page 48)

$$\mathrm{vS}_{(\alpha:\beta)} := \alpha{\cdot}\sigma_5(x_0,\dots,x_5) + \beta{\cdot}\sigma_2(x_0,\dots,x_5){\cdot}\sigma_3(x_0,\dots,x_5)$$

of hypersurfaces of degree 5 in the $\mathbb{P}^4$ cut out by $x_5 = -(x_1 + x_2 + \cdots + x_4)$. It is clear that the corresponding hypersurface has a non-isolated singularity if $\alpha = 0$, so let us normalize to $\alpha := 1$. This leaves us with a one-parameter family $\mathrm{vS}_{(1:\beta)}$.

Running through all possible parameters β over the prime fields $\mathbb{F}_2, \mathbb{F}_3, \dots, \mathbb{F}_{19}$, we find the examples with at least 130 ordinary double points listed in table 9.1. The whole computation takes approximately two minutes on our computer.

From this table, it is easy to guess that $\mathrm{vS}_{(1:1)}$ is indeed a 130-nodal quintic in $\mathbb{P}^4$ in characteristic zero. This guess can now be verified, again using computer algebra. Notice that the reduction modulo five gives a 205-nodal quintic in $\mathbb{P}^4$ which cannot exist in characteristic zero because of Varchenko's upper bound which is 135 nodes (see section 3.7).

9.1.2. Barth's 65-nodal Sextic in $\mathbb{P}^3$. Let us compute the parameters for which Barth's one-parameter family of 45-nodal sextics $F_\alpha = P - \alpha{\cdot}Q^2$ has exactly 65 nodes (see section 4.5 on page 50). The SINGULAR script which computes table 9.2 on the facing page only runs for a few seconds.

It is easy to guess from the table that α has to satisfy some quadratic condition. For each prime for which there exist exactly two solutions we compute the monic quadratic polynomial with the two values of α as roots. These monic quadratic polynomials over the prime fields are not difficult to lift by lifting each coefficient to some rational number. This can be done using Wang's rational recovery algorithm or one of its variants (see e.g., **[CE95]**):

ALGORITHM 1 (Wang's algorithm).

Input: *A modulus* $M \in \mathbb{Z}$ *and a residue* $U \in \mathbb{Z}/(M)$.

field	α	polynomial	$\mu(F_\alpha)$
$\mathbb{F}_2$	—	—	—
$\mathbb{F}_3$	—	—	—
$\mathbb{F}_5$	-2	$\alpha+2$	65
$\mathbb{F}_7$	—	—	—
$\mathbb{F}_{11}$	$5,\ -4$	$\alpha^2-\alpha+2$	65
$\mathbb{F}_{13}$	—	—	—
$\mathbb{F}_{17}$	—	—	—
$\mathbb{F}_{19}$	$3,\ -2$	$\alpha^2-\alpha-6$	65
$\mathbb{F}_{23}$	—	—	—
$\mathbb{F}_{29}$	$5,\ -4$	$\alpha^2-\alpha+9$	65
$\mathbb{F}_{31}$	$-1,\ 2$	$\alpha^2-\alpha-2$	65
$\mathbb{F}_{37}$	—	—	—
$\mathbb{F}_{41}$	$-13,\ 14$	$\alpha^2-\alpha-18$	65
$\mathbb{F}_{43}$	—	—	—

TABLE 9.2. Examples of F_α over some prime fields which have at least 65 ordinary double points.

Output: *A pair (A, B) of integers s.t. $A \equiv BU \mod M$ and $|A|, B < \sqrt{\frac{1}{2}M}$ with $B > 0$ if such a pair exists. Otherwise, return NIL.*

```
(A_1, A_2) := (M, U);    (V_1, V_2) := (0, 1);
loop
    if |V_2| >= sqrt(1/2 M) then return NIL;
    if A_2 < sqrt(1/2 M) then return (sign(V_2)A_2, |V_2|);
    Q := floor(A_1 / A_2);    (A_1, V_1) := (A_1, V_1) - Q(A_2, V_2);
    swap(A_1, A_2);    swap(V_1, V_2);
```

Of course, Wang's algorithm only works fine if the modulus M is big enough. Thus, in our situation we first have to use the chinese remainder theorem on all our prime field examples to be able to apply the rational recovery algorithm. This immediately yields:

$$\alpha^2 - \alpha - \frac{1}{16} = 0.$$

Again, it is easy to verify using computer algebra that this is indeed the correct parameter.

9.1.3. A Reducible Case. To illustrate a problem which may occur because of different algebraic numbers we consider the ideal $I = ((x^2-2)\cdot(x^2-3),\ y-1) \subset \mathbb{Q}[x, y]$. Table 9.3 on the next page lists all $\mathbb{F}_p$-rational points of I (i.e. points of I with coordinates in $\mathbb{F}_p$) over some small prime fields $\mathbb{F}_p$. Of course, the existence of such points is related to the existence of square roots of two and three in these fields.

It may happen that the prime field experiments take too much time, so that we do not have enough primes p for which the maximum number of $\mathbb{F}_p$-rational points exists. E.g., in our example we found only one prime, namely 23, for which all four points are $\mathbb{F}_p$-rational. Such a problem does not exist in the case in which we are in the comfortable position to be able to produce as many examples as we wish. These cases are easy to solve, in particular if we already know in advance which primes have good reduction and which not (see e.g., [**ABKR00**]): just take a reduced Groebner basis of the ideal defining the points and lift the coefficients

p	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{7}$	ideal	#
11	—, —	2, −2	−3	$y+3,\ xy+3x+2y-5,\ x^2-4$	2
13	—, —	3, −3	2	$y-2,\ xy-2x+3y-6,\ x^2+4$	2
17	3, −3	—, —	5	$y-5,\ xy-5x+3y+2,\ x^2+8$	2
19	—, —	—, —	−8	1	0
23	9, −9	−10, 10	10	$y-10,\ xy-10x-10y+8,\ x^4+3x^2+4$	4
29	—, —	—, —	−4	1	0
31	4, −4	—, —	9	$y-9,\ xy-9x+4y-5,\ x^2+15$	2
37	—, —	−5, 5	16	$y-16,\ xy-16x-5y+6,\ x^2+12$	2
41	12, −12	—, —	6	$y-6,\ xy-6x-12y-10,\ x^2+20$	2
43	—, —	—, —	−6	1	0

TABLE 9.3. For some prime fields $\mathbb{F}_p$ we show all $\mathbb{F}_p$-rational points of the ideal $I = \big((x^2-\frac{1}{2})\cdot(x^2-\frac{1}{3}),\ y-\frac{1}{7}\big)$. The column "#" lists the number of these points and the column "ideal" shows a reduced Groebner basis of the ideal describing them.

using Wang's rational recovery algorithm. But in our applications, the bottle neck of the algorithm are the experiments and we usually cannot produce many more prime field examples in short time.

To solve the problem we simply choose only subsets of all primes which lead to the second most number of $\mathbb{F}_p$-rational points. In our example, the maximum number of $\mathbb{F}_p$-rational points is 4 and the second most is 2. As $\frac{4}{2}=2$, at least half of the cases in which there are exactly two $\mathbb{F}_p$-rational points has to come from the same factor ($(x^2-\frac{1}{2})$ or $(x^2-\frac{1}{3})$) of the reducible polynomial of I.

There are six primes, $11, 13, 17, 31, 37, 41$, with exactly two $\mathbb{F}_p$-rational points. Thus, for all $\binom{6}{6/2} = \binom{6}{3} = 20$ combinations of three of these primes we try to lift their ideals in the same way as for the 65-nodal sextic. E.g., for the set of primes $\{11, 13, 37\}$ Wang's rational recovery algorithm already produces the guess $x^2-\frac{1}{3}$, $y-\frac{1}{7}$. This guess can then be verified over the rational numbers using computer algebra.

9.2. The Algorithm

We now describe the algorithm in the general situation. All main ideas are already contained in the examples presented in the previous section. The purpose of the algorithm can be formulated as follows:

ALGORITHM 2.

Input: - *An ideal* $F \subset \mathbb{K}[a_1, \ldots, a_k, x_0, x_1, \ldots, x_n]$, *where the* a_i *are considered as parameters. For each choice* $\overline{a_i}$ *of the* a_i, *this yields an ideal* $F_{\overline{a_1},\overline{a_2},\ldots,\overline{a_k}} \in \mathbb{K}[x_0, x_1, \ldots, x_n]$.

- *A procedure* `checkInterest(ideal I)` *which checks if a given ideal* $I := F_{\overline{a_1},\overline{a_2},\ldots,\overline{a_k}}$ *is* interesting.

Output: *An ideal in* $\mathbb{K}[a_1, \ldots, a_k]$ *defining parameters* $\overline{a_1}, \overline{a_2}, \ldots, \overline{a_k}$ *s.t.* $F_{\overline{a_1},\overline{a_2},\ldots,\overline{a_k}}$ *is* interesting *in the sense defined by the specified procedure.*

In our cases, each ideal $I = F_{\overline{a_1},\overline{a_2},\ldots,\overline{a_k}} \in \mathbb{K}[x_0, x_1, \ldots, x_n]$ is just one polynomial describing a hypersurface in $\mathbb{P}^n$ for which `checkInterest(ideal I)` verifies that its number of nodes is high. The algorithm consists of several steps:

Step 1: Prime Field Experiments. We run through all possible parameter combinations over some small prime fields $\mathbb{F}_{p_1}, \mathbb{F}_{p_2}, \ldots, \mathbb{F}_{p_m}$ and use the procedure `checkInterest` to pick the *interesting* parameter vectors. These possible combinations may be restricted by giving a list of conditions.

If the original equation of the surface is defined over some algebraic extension $\mathbb{K} := \mathbb{Q}(\alpha)$ of $\mathbb{Q}$ then we simply add α to the list of parameters and add its minimal polynomial to the list of conditions on the parameters.

EXAMPLE 9.1. *In our application, the prodeduce* `checkInterest` *will simply compute the number of singularities of the given surface and return* `true` *if it is the number we have been looking for or* `false` *if not.*

Step 2: The Ideals over the Prime Fields. For each prime $p \in \{p_1, \ldots, p_m\}$ we view the *interesting* parameter vectors as points in the parameter space and compute the ideal I_p describing all points by intersecting the point ideals. We then compute a reduced Groebner basis of the I_p to make the occurring monomials unique.

In order to be able to lift the ideals to characteristic zero we first have to figure out which of the modular ideals come from the same ideal in characteristic zero. To do this, we sort the ideals I_{p_i} first w.r.t. the number of *interesting* parameter points they define and second w.r.t. the monomials which occur in the ideal. We pick the set S_I with the greatest number of prime field ideals with the same monomials.

Step 3: Lifting the Ideal. Then we lift each coefficient occurring in the reduced Groebner basis of the ideals in S_I using Wang's rational recovery algorithm. As indicated in the example of section 9.1.3 on page 111 this might lead to some problems and require some more computations if different algebraic numbers are involved. Such a situation can only occur if the variety in the parameter space is reducible.

If all the coefficients occurring in the ideal can be lifted to characteristic zero then we proceed with the next step.

If not then we go back to the first step and perform some new experiments. If we have already obtained partial results then we may use these in order to speed up the computations.

Step 4: Checking the Guess. Using the procedure `checkInterest` again we now verify the guess which the lifting process has produced. If it is not yet the correct one then we go back to the first step and perform some more experiments.

9.3. Dihedral-symmetric Surfaces of Degree $d \leq 6$ with Many Nodes

As indicated in the introduction to this chapter, the algorithm described in the previous sections reduces the construction of surfaces of degree $d \leq 6$ in $\mathbb{P}^3$ which have the maximum possible number of nodes to a triviality once we had the idea to look for dihedral-symmetric examples:

Implementing a procedure `checkInterest(ideal I)` which checks if the surface given by I has the correct number of nodes is easy. Then it only remains to write down the D_d-symmetric (resp. D_{d-1}-symmetric) families of surfaces based on Rohn's construction (see section 1.3).

For the concrete results we do not use any further geometric intuition, although this might lead to much nicer results: In this section, we are only interested in a proof of concept, i.e. in showing that our algorithm produces the correct results even if we apply it in a very naive way. The computations were performed on a 1 MHz Mobile Centrino Laptop with 512 MB memory. In all examples, almost all the time was used for the experiments. Although the equations are easy to compute, we copied most of them together with the projectivities from [**End96**].

9.3.1. A D_3-symmetric 4-nodal Cubic. In degree $d = 3$, the family of dihedral-symmetric surfaces based on Rohn's construction is

$$f_3^{a_1,a_2} := p - q^{a_1,a_2},$$

where

$$\begin{aligned} p &:= x^3 - 3xy^2 + 3x^2w + 3y^2w - 4w^3, \\ q^{a_1,a_2} &:= a_1 \cdot (z - a_2 w)) \cdot z^2. \end{aligned}$$

As we are only interested in projectively different surfaces, we may choose $a_2 := 1$ because $f_3^{a_1,a_2}(x,y,\lambda z,w) = f_3^{a_1\lambda^3, \frac{a_2}{\lambda}}(x,y,z,w)\ \forall\ \lambda \in \mathbb{C}^*$ (see [**End96**, p. 22]). This leaves us with a one-parameter family $f_3^{a_1,1}$ of three-nodal cubics.

It suffices to perform the experiments for all primes $p \in \{5, 7, 11, \ldots, 29\}$. The whole algorithm runs two seconds, including experiments, lifting and verification of the result in characteristic zero. It finds the 4-nodal cubic $f_3^{\frac{27}{4},1}$.

9.3.2. A D_3-symmetric 16-nodal Kummer Quartic. In degree $d = 4$, the family is

$$f_4^{a_1,a_2,a_3,a_4} := p - q^{a_1,a_2,a_3,a_4},$$

where

$$\begin{aligned} p &:= \frac{1}{4} \cdot z \cdot \left(x^3 - 3xy^2 + 3x^2w + 3y^2w - 4w^3\right), \\ q^{a_1,a_2,a_3,a_4} &:= \left(a_1(x^2+y^2) + a_2 z^2 + a_3 zw + a_4 w^2\right)^2. \end{aligned}$$

Again, we are only interested in projectively different surfaces. Thus, because of $f_4^{a_1,a_2,a_3,a_4}(x,y,\lambda^2 z,w) = \lambda^2 f_4^{\frac{a_1}{\lambda},\lambda^3 a_2,\lambda a_3,\frac{a_4}{\lambda}}(x,y,z,w)\ \forall\ \lambda \in \mathbb{C}^*$ we may choose $a_1 = 1$. In order to obtain only finitely many solutions we choose furthermore $a_4 := 1$. This leaves us with a two-parameter family $f_4^{1,a_2,a_3,1}$ of 12-nodal quartics.

It suffices to perform the experiments for all primes $p \in \{5, 7, 11, \ldots, 29\}$. The whole algorithm runs nine seconds, including experiments, lifting and verification of the result in characteristic zero. It finds the 16-nodal quartic $f_4^{1,\left(\frac{5}{4}\right)^3,\frac{-5}{32},1}$.

9.3.3. A D_5-symmetric 31-nodal Togliatti Quintic. In degree $d = 5$, the family is

$$f_5^{a_1,a_2,a_3,a_4,a_5} := p - q^{a_1,a_2,a_3,a_4,a_5},$$

where

$$\begin{aligned} p &:= x^5 - 5(x^4+y^4)w - 10x^2y^2(x+w) + 20(x^2+y^2)w^3 + 5xy^4 - 16w^5, \\ q^{a_1,\dots,a_5} &:= a_1{\cdot}z{\cdot}\left(a_2(x^2+y^2) + a_3z^2 + a_4zw + a_5w^2\right)^2. \end{aligned}$$

Again, we may choose $a_2 := 1$, $a_4 := 1$. This leaves us with a three-parameter family $f_5^{a_1,1,a_3,1,a_5}$ of 20-nodal quintics.

It suffices to perform the experiments for all primes $p \in \{11, 13, 17, \dots, 31\}$. The whole algorithm runs six minutes, including experiments, lifting and verification of the result in characteristic zero. It finds the 31-nodal quintic $f_5^{a_1,1,a_3,1,a_5}$ where the a_i are given by the ideal $(2a_1+5,\ 20a_3+a_5+6,\ a_5^2+2a_5-4)$.

9.3.4. A D_5-symmetric 65-nodal Sextic. In degree $d = 6$, we take the D_5-symmetric family

$$f_6^{a_1,a_2,a_3,a_4,a_5,a_6} := p - q^{a_1,a_2,a_3,a_4,a_5,a_6},$$

where

$$\begin{aligned} p &:= w{\cdot}\left(x^5 - 5(x^4+y^4)w - 10x^2y^2(x+w) + 20(x^2+y^2)w^3 + 5xy^4 - 16w^5\right), \\ q^{a_1,\dots,a_6} &:= a_1{\cdot}\left((z-a_2w){\cdot}(a_3(x^2+y^2) + a_4z^2 + a_5zw + a_6w^2)\right)^2. \end{aligned}$$

In order to obtain only finitely many solutions and as we are only interested in projectively equivalent surfaces we may choose $a_3 := 1$, $a_4 := -1$, $a_5 = 0$. This leaves us with a three-parameter family $f_6^{a_1,a_2,1,-1,0,a_6}$ of 45-nodal sextics.

It suffices to perform the experiments for all primes $p \in \{7, 11, 13, \dots, 39\}$. The whole algorithm runs 22 minutes, including experiments, lifting and verification of the result in characteristic zero. It finds the 65-nodal sextic $f_6^{\frac{-5}{16},0,1,-1,0,4}$.

9.4. Another D_7-symmetric Septic with 99 Nodes

Without using any creative ideas, but just by following our algorithm, we wish to recover the result $\mu_{A_1}(7) \geq 99$ which we found in chapter 8 on page 95. We start again with the 7-parameter family of all D_7-symmetric septics

$$f_7^{a_1,a_2,\dots,a_7} := p - q^{a_1,\dots,a_7},$$

where

$$\begin{aligned} p &:= 2^6 \cdot \Pi_{j=0}^6 \left[\cos\left(\frac{2\pi j}{7}\right)x + \sin\left(\frac{2\pi j}{7}\right)y - z\right] \\ &= x{\cdot}\left[x^6 - 3{\cdot}7{\cdot}x^4y^2 + 5{\cdot}7{\cdot}x^2y^4 - 7{\cdot}y^6\right] \\ &\quad +7{\cdot}z{\cdot}\left[(x^2+y^2)^3 - 2^3{\cdot}z^2{\cdot}(x^2+y^2)^2 + 2^4{\cdot}z^4{\cdot}(x^2+y^2)\right] - 2^6{\cdot}z^7, \\ q^{a_1,\dots,a_7} &:= (z+a_5w)\left(a_1z^3 + a_2z^2w + a_3zw^2 + a_4w^3 + (a_6z+a_7w)(x^2+y^2)\right)^2. \end{aligned}$$

Although we may choose $a_7 := 1$, these are too many parameters to perform a prime field search over the whole family in short time. So, we have to impose some additional conditions. Either by looking at the examples in smaller degree or by checking the geometry of some experiments over very small prime fields, it is natural to expect that there should exist a 99-nodal septic S s.t. the plane curve $S|_{y=0}$ factors into a line $S_{y,1}$ and a sextic $S_{y,6}$ with the property that $S_{y,1}$ passes through three of the generic singularities of the construction (see section 8.3 on page 97).

From some prime experiments we see immediately that there is in fact a one-parameter family of such 99-nodal surfaces. Thus we may speed up our search by requiring the line $S_{y,1}$ to be a special one: $S_{y,1} = x + c$. This suffices to produce a 99-nodal septic surface using our algorithm as we will see below.

9.4.1. Computing the conditions. It is easy to translate the restrictions above into algebra (in the following, we use the affine chart $w = 1$):

(1) The plane curve $S|_{y=0} \in \mathbb{Q}[x, z]$ is zero on the whole line $S_{y,1} = x + c$, i.e. $(S|_{y=0})|_{x=-c} \equiv 0$. As $(S|_{y=0})|_{x=-c}$ is a polynomial of degree 7 in one variable z this gives $7 + 1$ conditions on the parameters $a_1, \dots, a_7$ because each of the coefficients has to vanish.

(2) The generic singularities are given by the intersection of the doubled cubic $C|_{y=0}$ and the three lines $L_1L_2L_3|_{y=0}$ (i.e., this is also a cubic plane curve). The fact that the line $S_{y,1} = x + c$ passes through three of these generic singularities can be translated by simply substituting x by $-c$ in the two cubics. When dividing each of the two cubic polynomials $(C|_{y=0})|_{x=-c} \in \mathbb{Q}[z]$ and $(L_1L_2L_3|_{y=0})|_{x=-c} \in \mathbb{Q}[z]$ by its leading coefficient we get two polynomials in one variable which should be equal. Thus, we get three conditions on the parameters $a_1, \dots, a_7$.

By taking all these conditions in one ideal I_{conds} we see that we are left with essentially two unknown parameters because the dimension of I_{conds} can easily be computed to be two. It turns out that we can ideed express all parameters as functions of two of them, namely c and a_6, by computing a lexicographical Groebner basis of I_{conds}:

$$a_1^2 = -64, \qquad a_2 = -\frac{1}{2}a_1 c, \qquad a_3 = -a_6 c^2 - \frac{1}{2}a_1 c^2,$$
$$a_4 = -\frac{1}{8}a_1 c^3 - c^2, \qquad a_5 = c, \qquad a_7 = 1.$$

9.4.2. Experimental Result. When performing our algorithm on this two-parameter family we find after 10 minutes:

$$c^2 = -\left(\frac{1}{7}\right)^2, \quad a_6 = 0.$$

This simplifies the expressions for the other parameters:

$$a_1 = 56c, \quad a_2 = \frac{4}{7}, \quad a_3 = \frac{4c}{7}, \quad a_4 = \left(\frac{2}{7}\right)^3, \quad a_5 = c, \quad a_6 = 0, \quad a_7 = 1$$

We denote the ideal defining these parameters by I_{sol}.

9.4.3. Verification.

THEOREM 9.1. *The surface $S_{a_1,\dots,a_7}$ of degree 7 has exactly 99 nodes and no other singularities if the $a_i \in \mathbb{Q}(c)$ are as specified by the ideal I_{sol} in section 9.4.2.*

PROOF. By computer algebra. In order not to have to compute in an extension of $\mathbb{Q}$ (which is usually quite time-consuming), we first notice that I_{sol} defines exactly two points in the parameter space. Thus, dividing the multiplicity of the singular locus of the surface $S := S_{a_1,\dots,a_7}$ by two gives its total milnor number. The following sequence of SINGULAR commands computes this:

```
ideal sl = diff(S,x),diff(S,y),diff(S,z),diff(S,w);
I_sol = groebner(I_sol);
sl = reduce(sl, I_sol);
"milnor:", (mult(std(sl)) div mult(I_sol));
```

In a similar way, we can verify that these 99 singularities are indeed isolated points and moreover have multiplicity one, i.e. they are all nodes. □

9.5. A D_9-symmetric Nonic with 226 Nodes

In exactly the same way as we constructed the 99-nodal septic in section 9.4 on page 115, we can proceed to find a nonic with many nodes. We start with the family $f_9^{a_0,\ldots,a_9} := p - q^{a_0,\ldots,a_9}$, where

$$\begin{aligned} p &:= 2^6 \cdot \Pi_{j=0}^{8} \left[\cos\left(\frac{2\pi j}{9}\right) x + \sin\left(\frac{2\pi j}{9}\right) y - z\right] \\ &= x^9 - 36x^7y^2 + 126x^5y^4 - 84x^3y^6 + 9xy^8 - 9x^8z - 36x^6y^2z - 54x^4y^4z \\ &\quad - 36x^2y^6z - 9y^8z + 120x^6z^3 + 360x^4y^2z^3 + 360x^2y^4z^3 + 120y^6z^3 \\ &\quad - 432x^4z^5 - 864x^2y^2z^5 - 432y^4z^5 + 576x^2z^7 + 576y^2z^7 - 256z^9, \\ q^{a_0,\ldots,a_9} &:= (z + a_0w) \cdot \big(a_1z^4 + a_2z^3w + a_3z^2w^2 + a_4zw^3 + a_5w^4 \\ &\qquad + (a_6z^2 + a_7zw + a_8w^2)(x^2 + y^2) + a_9(x^2 + y^2)^2\big)^2. \end{aligned}$$

We may choose $a_5 := 1$. After the use of the same geometrical assumptions as in the case of the 99-nodal septic in the preceding section, we are left with a four-parameter family. For this family the experiments take quite a lot of time, so we try to guess another parameter. It turns out that the maximum number of nodes which we find in $\mathbb{F}_{13}$ using our algorithm is 226. From these experiments we guess that there are 226-nodal nonics for $a_3 = 0$ (similar to the result $a_6 = 0$ for the 99-nodal septic). This reduces our problem to a search over a three-parameter family. Nevertheless, the experiments take several hours. Finally, we get:

THEOREM 9.2. *The eight surfaces $S_{226} := f_9^{a_0,\ldots,a_9}$ of degree 9 with*

$$a_1^2 = -256, \quad a_8^2 = -\frac{9a_1}{8}, \quad c^4 = \frac{a_1}{128}, \quad a_0 = c \quad a_2 = -\frac{a_1c}{2},$$
$$a_3 = 0, \quad a_6 = -\frac{3a_1}{4}, \quad a_4 = \frac{1}{c} \quad a_7 = \frac{a_1c}{4} + \frac{1}{c^3}, \quad a_9 = \frac{21a_1}{16}$$

and $a_8 = \frac{3}{2c^2}$ have exactly 226 nodes and no other singularities.

PROOF. By computer algebra. See proof of theorem 9.1. □

The previously known maximum number of nodes on a nonic was 216, attained by the surface of degree 9 from Chmutov's series (section 4.1 on page 45). We now have:

COROLLARY 9.3.

$$\mu_{A_1}(9) \geq \mathbf{226}.$$

Of course, in view of the fact that the 99 nodes even exist over the real numbers it is natural to ask for the existence of a nonic with 226 real nodes. In analogy to chapter 8 on page 95 we can write down a promissing family, but it has one more parameter and we did not have enough time to perform the computation yet.

Notice that our 226-nodal nonic has one additional node on the rotation axes $\{x = y = 0\}$ in $\mathbb{F}_{29}$. This is similar to the case of septics where there exists one additional node on the rotation axes in $\mathbb{F}_5$.

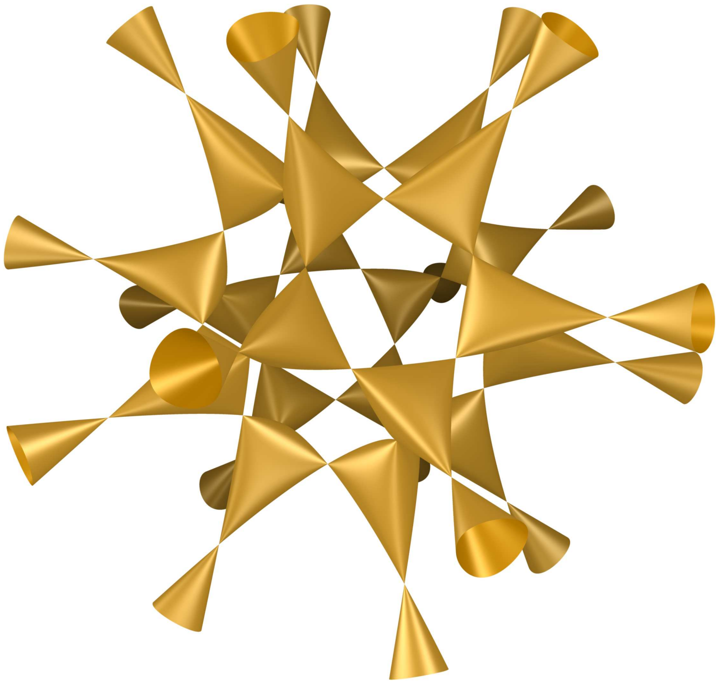
9.6. Discussion

Unfortunately, we cannot predict a priori how long the algorithm will run for a given family, but it is clear that it has to terminate some time if we neglect hardware and software restrictions. It neither gives a proof of the non-existence of examples which had not been found. Nevertheless, our algorithm has several advantages:

- The algorithm is highly parallelizable. Indeed, the bottle neck of the method are the prime field experiments, and it is easy to distribute these experiments over several machines.
- It produces partial results which can be used as guesses to speed up the computations significantly.

Our method has certainly many applications in other areas of algebraic geometry. We only mention a few cases connected to singularities in which it might be useful:

- Dihedral-symmetric surfaces of degree $d = 11, 13, \dots$ with many nodes.
- Dihedral-symmetric surfaces with non-maximal numbers of nodes; e.g., it is not clear which numbers of nodes may occur on octics (see section 3.13 on page 43 on the defect).
- (Real) line arrangements of degree $9, 11, \dots$ with many critical points on two levels (see chapter 6 on page 79).
- Surfaces with many cusps, in particular quintics because the gap between the maximum known 15 and the best known upper bound 20 is very large (see section 4.11 and chapter 5).



Barth's icosahedral-symmetric 65-nodal sextic. Shortly after its discovery in 1996, Jaffe and Ruberman showed that 65 is indeed the maximum possible number of nodes on a sextic surface in $\mathbb{P}^3$.

CHAPTER 10

Tables Showing the Current State of Knowledge

This chapter gives a tabular overview on the current state of knowledge on the subject of hypersurfaces with many singularities. In all tables, bold numbers indicate the cases in which the present thesis improves the previously best known bounds.

At some places, there appear question marks. These are sometimes caused by running time restrictions because the computation of the dimension of the tangent space of the deformation functor of the nodal hypersurfaces can take a lot of time.

Another reason might be that we have simply not yet implemented the equation of the hypersurface in SINGULAR. Sometimes, this task is not trivial or at least a huge amount of work because some constructions are only given by vague or lengthy arguments. In some cases (e.g. Kreiss's construction, section 2.6 on page 26), it is even not clear if the construction really works.

Once we have computed more numbers, we will place updated tables on our webpage [**Lab03a**].

10.1. Nodal Hypersurfaces

In $\mathbb{P}^3$ and $\mathbb{P}^4$, the best known constructions for large degree d are still given by Chmutov's construction from 1992, see section 4.1. For $n \geq 5$ and large d, the best known construction is our variant of Chmutov's construction based on Breske's folding polynomials associated to the root system B_2, see section 5.6.1.

In the following tables, we give an overview on the currently best known bounds for the maximum number of nodes for small n or d. The tables do not only show the names of the persons who discovered the hypersurfaces. We also give the references to the sections of this Ph.D. thesis in which we introduced the hypersurface and the year in which it was discovered.

Furthermore, we give the dimensions of the space of infinitesimal deformations and the obstruction space of van Straten's deformation functor $\mathrm{Def}(X, \Sigma)$ (see section 4.6 on page 51). For shortness, we write t^i for $\dim T^1(X, \Sigma(X))$, $i = 1, 2$, throughout.

10.1.1. Nodal Surfaces in $\mathbb{P}^3$. We start with the most important table: Nodal hypersurfaces in $\mathbb{P}^3$, table 10.1 on the next page. As explained in the historical part of this work, this subject has a very long and rich history. The two bold numbers, 99 and 226, indicate the cases in which the present thesis improves the previously known bounds.

d	$\mu^3_{A_1}(d) \le$	#	name, section, and year	t^1	t^2
3	4, Schläfli: s. 1.1.1, 1863	3	Chmutov: s. 4.1, 1992	1	0
		4	Schläfli: s. 1.1.1, 1863	0	0
4	16, Kummer: s. 1.2, 1864	14	Chmutov: s. 4.1, 1992	5	0
		16	Fresnel, Kummer: s. 1.2, 1819/64	3	0
5	31, Beauville: s. 3.3, 1979	28	Chmutov: s. 4.1, 1992	12	0
		31	Togliatti: s. 2.1, 1940	9	0
6	65, Jaffe/Ruberman: s. 4.5, 1997	57	Chmutov: s. 4.1, 1992	11	0
		63	Gallarati: s. 2.5, 1952	5	0
		64	Stagnaro: s. 3.1.2, 1978	?	?
		64	Catanese-Ceresa: s. 3.5, 1982	4	0
		65	Barth: s. 4.5, 1996	3	0
7	104, Varchenko: s. 3.7, 1983	81	Chmutov: s. 3.8, 1982	23	0
		93	Chmutov: s. 4.1, 1992	11	0
		99	L.: s. 8.6, 2004	5	0
8	174, Miyaoka: s. 3.10, 1984	128	Endraß: s. 4.7, 1996	28	7
		153	B. Segre: s. 2.4, 1952	?	?
		154	Chmutov: s. 4.1, 1992	5	10
		160	Gallarati: s. 2.5, 1952	6	17
		160	Kreiss: s. 2.6, 1955	?	?
		165	van Straten: unpublished, 1997	1	17
		168	Endraß: s. 4.7, 1996	0	19
9	246, Varchenko: s. 3.7, 1983	192	Chmutov: s. 3.8, 1982	23	11
		216	Chmutov: s. 4.1, 1992	7	19
		226	L.: s. 9.5, 2005	?	?
10	360, Miyaoka: s. 3.10, 1984	321	Chmutov: s. 4.1, 1992	2	53
		325	Kreiss: s. 2.6, 1955	?	?
		345	Barth: s. 4.5, 1996	0	75
11	480, Varchenko: s. 3.7, 1983	425	Chmutov: s. 4.1, 1992	3	80
		430 ?	L.(conjecture): s. 8.8, 2004	?	?
12	645, Miyaoka: s. 3.10, 1984	576	Kreiss: s. 2.6, 1955	?	?
		576	Chmutov: s. 4.1, 1992	2	139
		600	Sarti: s. 4.9, 2001	0	161
13	829, Varchenko: s. 3.7, 1983	729 ?	L.(conjecture): s. 8.8, 2004	?	?
		732	Chmutov: s. 4.1, 1992	?	?
14	1051, Miyaoka: s. 3.10, 1984	931	Kreiss: s. 2.6, 1955	?	?
		949	Chmutov: s. 4.1, 1992	?	?
d	$\approx 4/9d^3$, Miyaoka: s. 3.10, 1984	$\approx \frac{5}{12}d^3$	Chmutov: s. 4.1, 1992	?	?

TABLE 10.1. Nodal Hypersurfaces in $\mathbb{P}^3$

10.1.2. Surfaces in $\mathbb{P}^3$ with Many Real Nodes. Except for $d = 9$, the currently known bounds for the maximum number $\mu_{A_1}(d)$ (resp. $\mu^{\mathbb{R}}_{A_1}(d)$) of nodes on a surface of degree d in $\mathbb{P}^3(\mathbb{C})$ (resp. $\mathbb{P}^3(\mathbb{R})$) are equal. The upper bounds are the same as the complex ones listed in the previous table: Varchenko's (section 3.7) and Miyaoka's (section 3.10).

d	1	2	3	4	5	6	7	8	9	10	11	12	13	d
$\mu^{\mathbb{R}}_{A_1}(d) \leq$	0	1	4	16	31	65	104	174	246	360	480	645	832	$\frac{4}{9}d(d-1)^2$
$\mu^{\mathbb{R}}_{A_1}(d) \geq$	0	1	4	16	31	65	99	168	216	345	425	600	732	$\approx \frac{5}{12}d^3$

TABLE 10.2. The currently best known bounds on the maximum number of real nodes.

10.1.3. Nodal Hypersurfaces in $\mathbb{P}^4$. Not many people have worked on nodal hypersurfaces in $\mathbb{P}^4$ of large degree. To our knowledge, the general constructions described in the historical part of this thesis are the only available results for $d \geq 6$. Therefore, table 10.3 is quite short.

For $d = 6, 7, 8$, it would certainly be possible to apply constructions similar to the one of van Straten's 130-nodal quintic, e.g. by using our algorithm from chapter 9.

d	$\mu^4_{A_1}(d) \leq$	#	name, ref., and year	t^1	t^2
3	10	10	Segre: s. 1.5.1, 1887	0	0
4	45	41	Chmutov: s. 4.1, 1992	4	0
		45	Burkardt: s. 1.5.2, 1891	0	0
		45	Goryunov: s. 4.4, 1994	0	0
5	135	120	Chmutov: s. 4.1, 1992	1	20
		125	Schön: s. 3.12, 1986	0	24
		126	Hirzebruch: s. 3.12, 1987	0	25
		130	van Straten: s. 4.3, 1993	0	29
6	320	277	Chmutov: s. 4.1, 1992	0	92
7	651	566	Chmutov: s. 4.1, 1992	?	?
8	1190	1029	Chmutov: s. 4.1, 1992	?	?
9	2010	1720	Chmutov: s. 4.1, 1992	?	?
10	3195	2745	Chmutov: s. 4.1, 1992	?	?
d	?	$\frac{7}{18}d^4$	Chmutov: s. 4.1, 1992	?	?

TABLE 10.3. Nodal Hypersurfaces in $\mathbb{P}^4$. The upper bounds are given by Varchenko's spectral bound (section 3.7).

10.1.4. Nodal Hypersurfaces in $\mathbb{P}^5$. For nodal hypersurfaces in $\mathbb{P}^5$ the situation is similar to the one in $\mathbb{P}^4$: there are very few (or even no) results for $d \geq 6$. But as our variants of Chmutov's construction lead to new lower bounds, table 10.4 shows three of these variants.

Note that although our construction leads to a new lower bound it does not improve the highest order term $\frac{5}{16}d^5$ of the polynomial describing the number of nodes. Thus, this number in the bottom row of the table is not marked in bold. But as one can see from the table, our construction improves the previously known lower bounds quite a bit in small degree.

d	$\mu^5_{A_1}(d) \leq$	#	name, ref., and year	t^1	t^2
3	15	15	Veneroni: s. 1.5.3, 1914	5	0
		15	Togliatti: s. 2.1, 1936	5	0
4	126	40	Chmutov: s. 3.8	50	0
		80	Chmutov: s. 3.8	20	10
		104	Chmutov/L.: s. 4.1, 2005	?	?
		120	Goryunov: s. 4.4, 1994	0	30
5	456	320	Chmutov: s. 3.8	15	119
		392	L.: s. 5.6.1, 2005	?	?
		420	Hirzebruch/L.: s. 3.12, 2005	?	?
6	1506	810	Chmutov: s. 3.8	25	409
		1035	Chmutov/L.: s. 4.1, 2005	?	?
		1179	L.: s. 5.6.1, 2005	?	?
7	3431	2430	Chmutov: s. 3.8	?	?
		2583	Chmutov/L.: s. 4.1, 2005	?	?
		2781	L.: s. 5.6.1, 2005	?	?
8	7872	4320	Chmutov: s. 3.8	?	?
		5488	Chmutov/L.: s. 4.1, 2005	?	?
		6016	L.: s. 5.6.1, 2005	?	?
9	14412	10240	Chmutov: s. 3.8	?	?
		10368	Chmutov/L.: s. 4.1, 2005	?	?
		11328	L.: s. 5.6.1, 2005	?	?
10	27237	12500	Chmutov: s. 3.8	?	?
		16000	Chmutov: s. 3.8	?	?
		20525	L.: s. 5.6.1, 2005	?	?
d	?	$\frac{5}{16}d^5$	L.: s. 5.6.1, 2005	?	?

TABLE 10.4. Nodal Hypersurfaces in $\mathbb{P}^5$. The upper bounds are given by Varchenko's spectral bound (section 3.7).

10.1.5. Nodal Cubic Hypersurfaces in $\mathbb{P}^n$. The nodal cubic hypersurfaces are one of the very rare cases in which $\mu^n_{A_1}(d)$ (and even $\mu^n(d)$) is known.

The first who showed this was Kalker in his Ph.D. thesis in 1986. As explained in section 3.11 on page 41, he simply wrote down equations which realize the upper bound provided by Varchenko's spectral bound (section 3.7). Later, Goryunov obtained the same number of nodes on cubics by a different method (section 4.4).

n	$\mu^n_{A_1}(3) \leq$	#	name, section, and year	t^1	t^2
3	4	4	Schläfli: s. 1.1.1, 1863	0	0
4	10	10	Segre: s. 1.5.1, 1887	0	0
5	15	15	Veneroni: s. 1.5.3, 1914	5	0
		15	Togliatti: s. 2.1, 1936	5	0
6	35	33	Givental: s. 3.9, $\approx$ 1982	2	0
		35	Kalker: s. 3.11, 1986	0	0
		35	Goryunov: s. 4.4, 1994	0	0
7	56	54	Givental: s. 3.9, $\approx$ 1982	2	0
		56	Kalker: s. 3.11, 1986	0	0
		56	Goryunov: s. 4.4, 1994	0	0
8	126	118	Givental: s. 3.9, $\approx$ 1982	0	34
		126	Kalker: s. 3.11, 1986	0	42
		126	Goryunov: s. 4.4, 1994	0	42
9	210	189	Givental: s. 3.9, $\approx$ 1982	3	72
		210	Kalker: s. 3.11, 1986	0	90
		210	Goryunov: s. 4.4, 1994	0	90
10	462	414	Givental: s. 3.9, $\approx$ 1982	0	249
		462	Kalker: s. 3.11, 1986	0	297
		462	Goryunov: s. 4.4, 1994	0	297
n odd	$\binom{n+1}{[(n+1)/2]}$	$\binom{n+1}{[(n+1)/2]}$	Kalker: s. 3.11, 1986	?	?
n even	$\binom{n+1}{[n/2]}$	$\binom{n+1}{[n/2]}$	Kalker: s. 3.11, 1986	?	?

TABLE 10.5. Cubics in $\mathbb{P}^n$. The upper bounds are given by Varchenko's spectral bound, see section 3.7 on page 35.

10.1.6. Nodal Quartic Hypersurfaces in $\mathbb{P}^n$. Although Goryunov (section 4.4) used the same method for constructing his quartics as for his cubics, the quartics do not reach Varchenko's upper bound (section 3.7).

Is Goryunov's construction already the best possible or is it possible to produce more nodes? It would be interesting to try to answer to this question, at least for small n. As table 10.6 shows, already for $n = 5$ there is a gap of 6 between Goryunov's lower bound and Varchenko's upper bound.

d	$\mu^n_{A_1}(4) \leq$	#	name, section, and year	t^1	t^2
3	16	16	Fresnel, Kummer: s. 1.2, 1819/64	3	0
4	45	24	Chmutov: s. 3.8, 1982	0	0
		45	Burkardt: s. 1.5.2, 1891	0	0
5	126	40	Chmutov: s. 3.8, 1982	50	0
		80	Chmutov: s. 3.8, 1982	20	10
		104	L.: s. 5.6.1, 2005	?	?
		120	Goryunov: s. 4.4, 1994	0	30
6	357	160	Chmutov: s. 3.8, 1982	36	35
		300	L.: s. 5.6.1, 2005	?	?
		336	Goryunov: s. 4.4, 1994	0	175
7	1016	280	Chmutov: s. 3.8, 1982	63	77
		560	Chmutov: s. 3.8, 1982	28	322
		804	L.: s. 5.6.1, 2005	?	?
		896	Goryunov: s. 4.4, 1994	0	630
		938	Goryunov: s. 4.4, 1994	?	?
8	2907	1120	Chmutov: s. 3.8, 1982	36	742
		2337	L.: s. 5.6.1, 2005	?	?
		2688	Goryunov: s. 4.4, 1994	0	2274
n	$\approx \frac{\sqrt{3}}{2} \cdot \frac{3^{n+1}}{\sqrt{\pi n}}$	$2^{2n/3}\binom{n+1}{[2n/3]+1}$	Goryunov: s. 4.4, 1994	?	?

TABLE 10.6. Quartics in $\mathbb{P}^n$. The upper bounds are given by Varchenko's spectral bound, see section 3.7 on page 35.

10.2. Higher Singularities

Not much is known on the maximum number of higher singularities on hypersurfaces of degree d in $\mathbb{P}^n$. Even in $\mathbb{P}^3$, there are only few results such as Barth's surfaces with many cusps (section 4.11 on page 55).

Our general constructions from chapter 5 on page 67 improve most known lower bounds for the maximum number of A_j-singularities on a hypersurface of degree d in $\mathbb{P}^n$, $n \geq 3$, significantly.

To our knowledge, there are almost no results for other singularities except very general ones like those based on Viro's patchworking method (section 4.12 on page 57).

10.2.1. Hypersurfaces with A_j-Singularities in $\mathbb{P}^3$. The projective three-space is still the field of most active research in the subject of hypersurfaces with many singularities. Our results from chapter 5 on page 67 improve most previously known bounds as table 10.7 shows.

Note that even the cases of $j \geq 2$, $d \geq 5$ which are not marked in bold have been overlooked for some time. These lower bounds come from Gallarati's generalization of B. Segre's construction which we have been working out in section 2.5 on page 24.

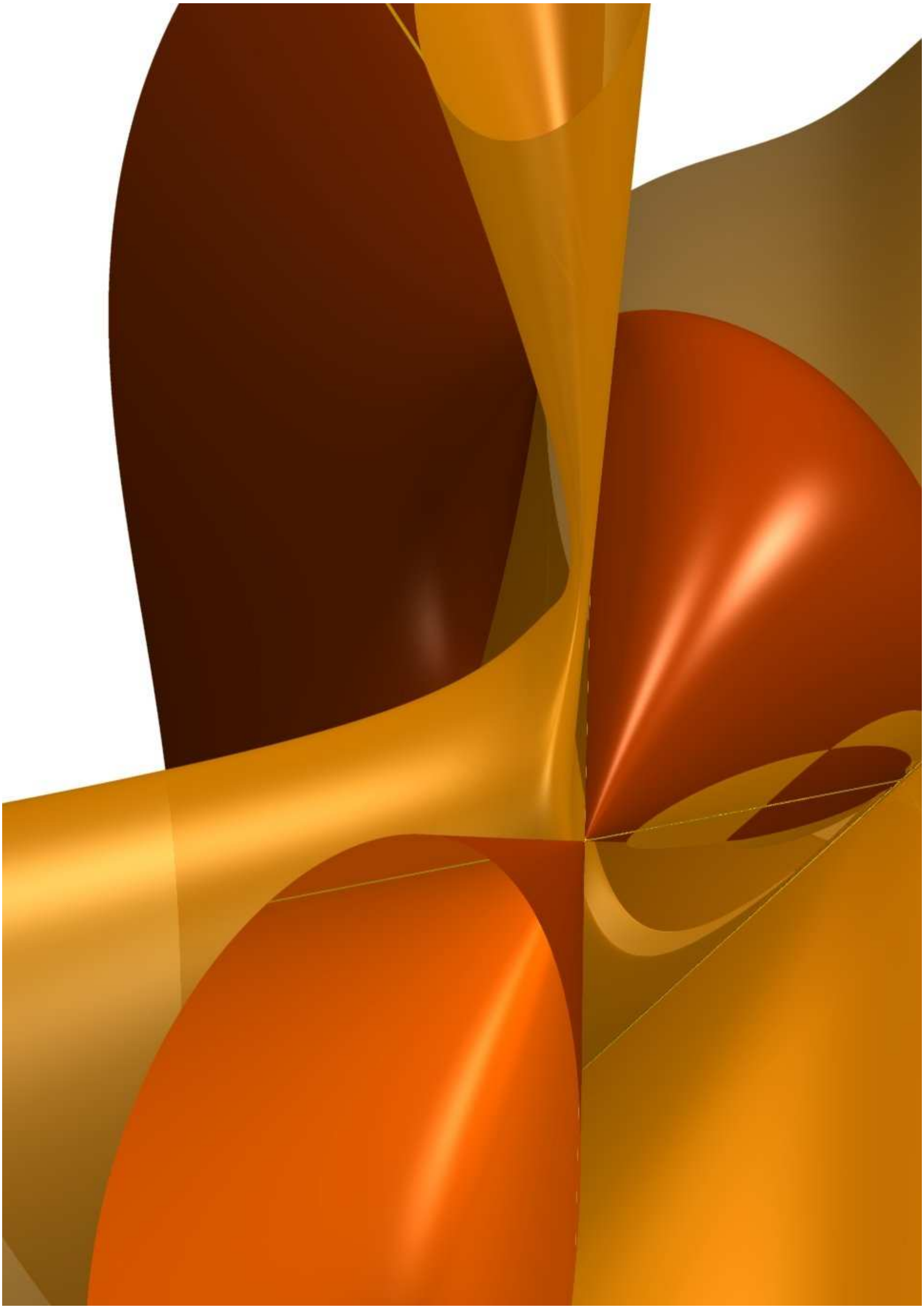
j \ d	3	4	5	6	7	8	9	10	11	12	d
1	4 / 4	16 / 16	31 / 31	65 / 65	**99** / 104	168 / 174	**226** / 246	345 / 360	425 / 480	600 / 645	$\approx$ 5/12 / 4/9 $\cdot d^3$
2	3 / 3	8 / 8	15 / 20	36 / 37	**52** / 62	**70** / 98	**126** / 144	**159** / 202	**225** / 275	**300** / 363	$\approx$ **2/9** / 1/4 $\cdot d^3$
3	1 / 1	6 / 6	10 / 13	15 / 26	**31** / 44	64 / 69	**72** / 102	**114** / 144	**140** / 195	**198** / 258	$\approx$ **11/72** / 8/45 $\cdot d^3$
4	1 / 1	4 / 4	10 / 11	15 / 20	21 / 35	**32** / 54	**54** / 80	100 / 112	**110** / 152	**132** / 201	$\approx$ **7/60** / 5/36 $\cdot d^3$

TABLE 10.7. Known upper and lower bounds for the maximum number $\mu_{A_j}(d)$ of singularities of type A_j, $j = 1, 2, 3, 4$, on a surface of degree d in $\mathbb{P}^3$.

10.2.2. Hypersurfaces in $\mathbb{P}^n$ with A_j-Singularities, $j \geq 2, n \geq 4$. Our results from chapter 5 on page 67 also improve most previously known bounds in higher dimensions. In the lower two rows, table 10.8 shows the asymptotic behaviour of our two variants of the construction with many A_j-singularities: One uses Breske's folding polynomials associated to the root system B_2, the other uses those associated to the root system A_2 which Chmutov already used in the nodal case. Notice that for high degree d, $\text{Chm}^{j,n}(F_d^{B_2})$ is better than $\text{Chm}^{j,n}(F_d^{A_2})$ for $n \geq 5$ if $j \geq 2$.

n	3	4	5	6	7	8
$\frac{1}{d^n} \cdot \mu^n_{A_2}(d) \gtrapprox$	$\mathbf{\frac{2}{9}}$	$\mathbf{\frac{13}{72}}$	$\mathbf{\frac{1}{6}}$	$\mathbf{\frac{13}{96}}$	$\mathbf{\frac{55}{384}}$	$\mathbf{\frac{15}{128}}$
$\frac{1}{d^n} \cdot \mu^n_{A_3}(d) \gtrapprox$	$\mathbf{\frac{11}{72}}$	$\mathbf{\frac{1}{8}}$	$\mathbf{\frac{11}{96}}$	$\mathbf{\frac{3}{32}}$	$\mathbf{\frac{25}{256}}$	$\mathbf{\frac{125}{1536}}$
$\frac{1}{d^n} \cdot \mu^n_{A_4}(d) \gtrapprox$	$\mathbf{\frac{7}{60}}$	$\mathbf{\frac{23}{240}}$	$\mathbf{\frac{7}{80}}$	$\mathbf{\frac{23}{320}}$	$\mathbf{\frac{19}{256}}$	$\mathbf{\frac{1}{16}}$
$\frac{1}{d^n} \cdot \mu(\text{Chm}^{j,n}(F_d^{A_2})) \approx$	$\mathbf{\frac{3j+2}{6j(j+1)}}$	$\mathbf{\frac{5j+3}{12j(j+1)}}$	$\frac{7j+3}{18j(j+1)}$	$\frac{7j+4}{24j(j+1)}$	$\frac{19j+16}{72j(j+1)}$	$\frac{35j+19}{144j(j+1)}$
$\frac{1}{d^n} \cdot \mu(\text{Chm}^{j,n}(F_d^{B_2})) \approx$	$\frac{2j+1}{4j(j+1)}$	$\frac{3j+2}{8j(j+1)}$	$\mathbf{\frac{3j+2}{8j(j+1)}}$	$\mathbf{\frac{5j+3}{16j(j+1)}}$	$\mathbf{\frac{20j+15}{64j(j+1)}}$	$\mathbf{\frac{35j+20}{128j(j+1)}}$

TABLE 10.8. The asymptotic behaviour of the number of A_j-singularities on a hypersurface of degree d in $\mathbb{P}^n$. $\text{Chm}^{j,n}(F_d^{B_2})$ is better than $\text{Chm}^{j,n}(F_d^{A_2})$ for $n \geq 6$.



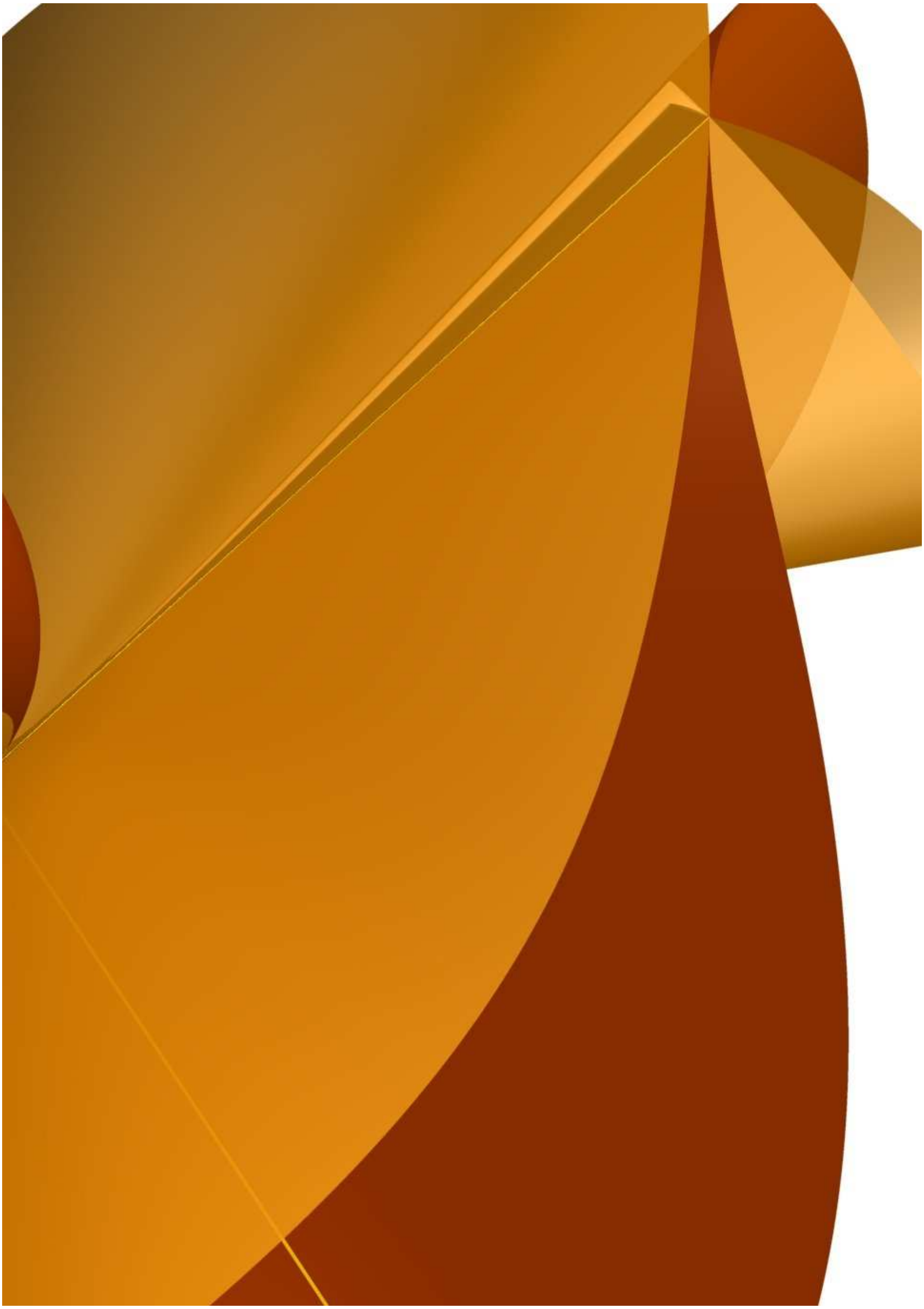


Figure on the preceding pages: A cubic surface (dark) with one A_3-singularity and two nodes. The brighter surface is its covariant of degree 9 which cuts out its lines. See [**LvS00**] for more images and movies of cubic surfaces.

Part 3

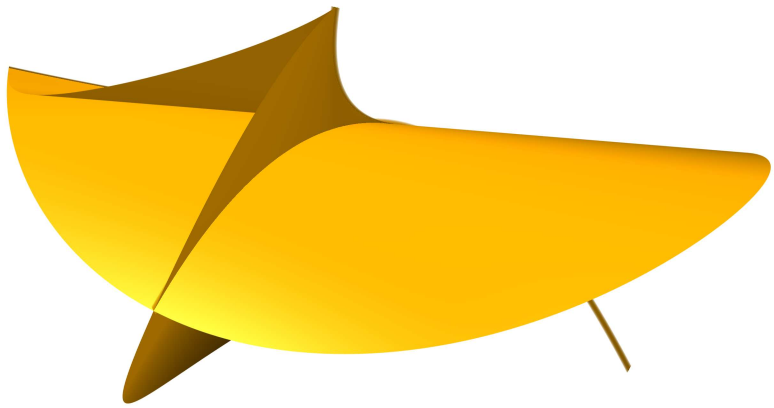
Visualization

Introduction

If a surface with many singularities is defined over the reals then it is sometimes nice to have an image of it. But this is not the only reason why one would like to have good visualizations of singular surfaces.

In chapter 12 we show how to use our visualization tools SPICY and SURFEX to construct good equations for all 45 topological types of real cubic surfaces with only rational double points. Furthermore, in many cases visualization is a very good tool to understand the geometry of some constructions in an intuitive way. And this can help to construct new intesting examples based on these known ones.

Before that, we give a short overview of different methods for visualizing algebraic geometry ranging from classical approaches to modern interactive computer software.



The swallowtail. Our SINGULAR library SURFEX.LIB is able to visualize this famous surface correctly. It contains a real curve which is not contained in the real two-dimensional part of the surface.

CHAPTER 11

Methods for Visualizing Algebraic Geometry

11.1. Classical Approaches

Since the early days of algebraic geometry, mathematicians visualize their objects of study. Drawings by hand are easy for curves and surfaces of degree $d \leq 2$. It is even not difficult to draw curves of higher degree when computing many points and other important data like the coordinates of their singularities and inflexion points. Drawing images of cubic surfaces is already much more involved. Nevertheless, the literature of the 19^{th} century contains some very good visualizations. Some people (e.g., Clebsch, Wiener, Rodenberg, and Klein) even produced real-world models of algebraic surfaces of low degree as we already mentioned in section 1.1.3 on page 15. These were mostly made out of plaster or wood. Of course, the production of interesting surfaces of higher degree ($d \geq 5$) was almost impossible because of their complexity. Models of algebraic surfaces were even produced and sold for high prices (see [**Sch11**]). But from the 1930's on visualization of mathematics was frowned upon for many years.

11.2. The First Visualization Software

Visualization entered back into the world of algebraic geometry in the mid-1980's. E.g., Fischer's book on mathematical models appeared at that time; in connection with this, some of the old plaster models were reproduced.

Shortly afterwards, the first software visualization tools have been developped. Until now the one that produces the best images of singular algebraic surfaces is still Endraß's SURF [**End03**] the first version of which he implemented during the writing of his diploma thesis. SURF is based on the raytracing method similar to POVRAY. The latter is a much more general program which allows raytracing of any real-world scene. But besides the fact that we personally prefer the images produced by SURF, Endraß's software has the advantage of being quicker. This is important for our application as we will see later. SURF was even used to construct a model of the Clebsch Diagonal Cubic at Fischer's university at Düsseldorf which is a few meters tall (see [**Kae99**]): The constructors used the software for drawing many plane sections of the surface which served as the basis for the modelling process.

Another promissing approach to the visualization of algebraic surfaces is triangulation. In the smooth case, it is not difficult to implement a good algorithm for this purpose. In the singular case, the best existing software is still Morris's software ASURF from the LSMP package [**Mor03**] for which he implemented a web-frontend using JAVAVIEW (see [**Pol01**]). His program is based on heuristics and does not produce satisfactory results in many cases. To our knowledge, recent ideas on the triangulation of singular surfaces, e.g. by Mourrain's group in Nice, have not been implemented yet.

Together with so-called $3d$-printers the triangulation software allows the machine-production of real-world models. To our knowledge, this technique was first used by mathematicians-sculpturs like Helaman Ferguson, Bathsheba Grossman, Georg Hart. Recently, the architect Jonathan Chertok reproduced the whole Rodenberg series by this method based on the equations communicated to him by several mathematicians including the author. In order to make the production of such models easier, we implemented an extension for SURF based on Johannes Beigel's version of the program which uses the triangulation library GTS. Unfortunately, this software is not in a publishable state yet, but it already allowed us to produce several examples, e.g. the first model of a 30-cuspidal sextic surface with the symmetry of an icosahedron and a reproduction of a Clebsch diagonal cubic (fig. 11.1).

FIGURE 11.1. A reproduction of Clebsch's diagonal cubic surface using a $3d$-printer based on the data produced using our extension of SURF.

11.3. Interactive Software

With our interactive visualization software SPICY and SURFEX we aim to go one step further: The user can include the coordinates of points of a plane geometry construction into the equations of algebraic plane curves and surfaces. If the user then moves the points then the images of the algebraic varieties change accordingly. This makes the interactive visualization of deformations and other processes possible.

11.3.1. SPICY — **Interactive Constructive and Algebraic Geometry.** The core of the computer software SPICY (up to now only available as a pre-version from [**Lab03b**]) is a constructive geometry program designed both for visualizing geometrical facts interactively on a computer and for including them in publications. Its main features are:

- Connection to external software like the computer algebra system SINGULAR ([**GPS01**]) and the visualization software SURF ([**End03**]) which enables the user to include algebraic curves and surfaces in dynamic constructions.
- Comfortable graphical user-interface (cf. fig. 11.2) for interactive constructions using the computer-mouse including macro-recording, animation, etc.
- High quality export to `.fig`-format (and in combination with external software like XFIG or FIG2DEV export to many other formats, like `.eps`, `.pstex`, etc.).

We implemented the first particular example of such a tool (called XCSPRG, downloadable from [**LvS00**]) during the writing of our diploma thesis under the direction of D. van Straten. Van Straten had the idea that SURF should be fast enough to be able to recompute two or three images of cubic surfaces per second. In this way, he wanted to be able to manipulate six points in the plane and see the changing surfaces at the same time. This is exactly the purpose of XCSPRG.

After having received my diploma I developped SPICY as a much more general and powerful tool. Let us illustrate its usefulness again with the example of cubic surfaces:

EXAMPLE 11.1. *We take three pairs of two points in the plane each pair connected by a straight line (see fig. 11.2). It is well-known that the blowup of the plane*

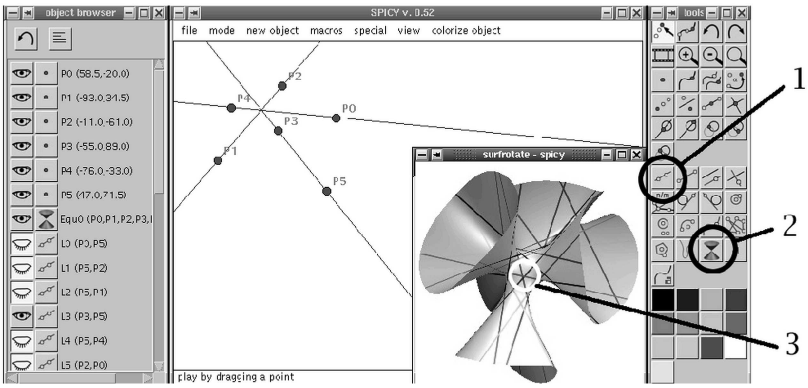


FIGURE 11.2. A screen shot of the SPICY user interface showing three lines, that meet in a point and the corresponding cubic surface, which contains an Eckardt Point (3). Buttons 1 and 2 are used to draw the lines and the surface, respectively.

in the six points yields a smooth cubic surface if neither three of the points are on a common line nor six of them are on a common conic. Furthermore, the blowup is bijective outside the six base points, and straight lines connecting the base points are mapped to straight lines on the cubic surface. Thus, in order to construct a cubic surface with an Eckardt point (i.e. a point in which three lines meet) we only have to manipulate the six base points until the three lines in the plane meet in a point (see [**LvS03**] *for details).*

11.3.2. SURFEX **— Intuitive Visualization, even in the Internet.** We often simply need a good and easy way to visualize one or more surfaces and/or curves on them. Basically, Schmidt's new version 1.0.3 of Endraß's program SURF can already produce the required images, but it has some major deficiencies concerning the usage. First, one needs to know SURF's programming language. Second, rotation within SURF is far from intuitive. The purpose of our tool SURFEX [**HLM05**] is exactly to fill in this blank. Thus, SURFEX is basically an easy-to-use frontend for SURF which allows intuitive rotation, scaling, and usage in general, even in the internet. We demonstrate its usefulness at a concrete example in the next chapter.

11.3.3. SURFEX.LIB **— a** SINGULAR **Interface for** SURFEX**.** The current version of SURFEX has the problem that it uses the raytracer SURF for visualizing algebraic surfaces. And the raytracing technique is not able to visualize real one-dimensional parts of a surface such as the handle of the Whitney umbrella if it is not specified as the intersection of surfaces.

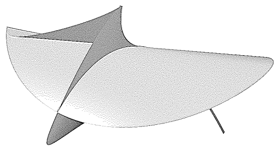
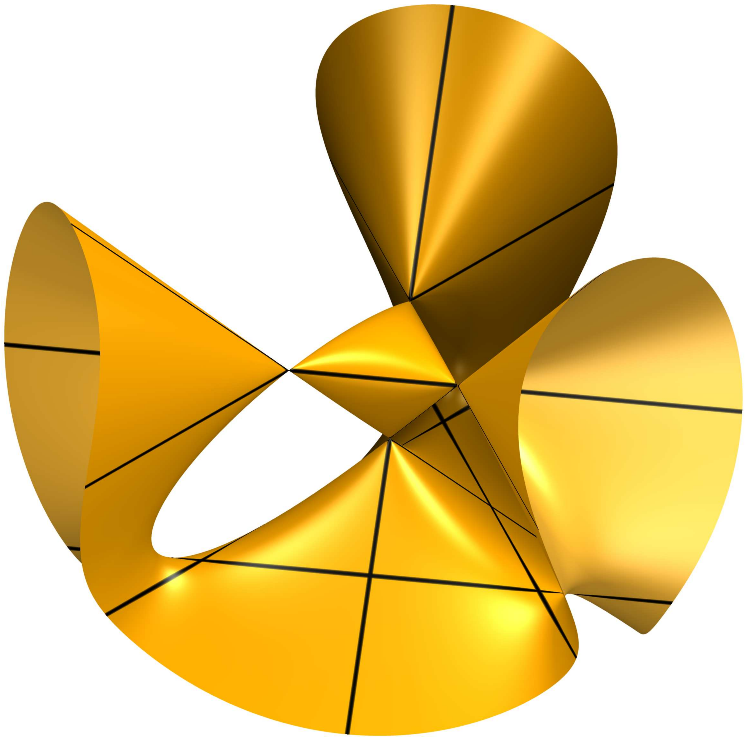


FIGURE 11.3. SURFEX.LIB can also visualize surfaces with real curves which are not contained in the real two-dimensional part of the surface such as the swallowtail.

In combination with SINGULAR, this problem can be solved. SINGULAR can compute the singular locus of a given surface and can then pass those surfaces which cut out the singular curves to SURFEX. E.g., the following code produces a correct image of the swallowtail (see fig. 11.3):

```
LIB "surfex.lib";
ring r = 0,(x,y,z),dp;
poly swallowtail = -4*y^2*z^3-16*x*z^4+27*y^4
                   +144*x*y^2*z+128*x^2*z^2-256*x^3;
plotRotated(swallowtail, list(x,y,z),2);
```



The four-nodal Cayley cubic and its nine lines. The facts that a cubic cannot contain more than four singularities and that any four-nodal cubic contains exactly nine different lines was already known to the geometers of the 19^{th} century.

CHAPTER 12

Illustrating the Classification of Real Cubic Surfaces

In this chapter we demonstrate the usefulness of our visualization tools SPICY and SURFEX for working with algebraic surfaces. Our example is the very classical subject of real cubic surfaces. We will see that the use of our software does not only allow us to visualize existing surfaces, but also helps to produce equations of surfaces (see also [**LvS03**], [**HL05**], [**LvS00**]).

In 1987, Knörrer and Miller [**KM87**] classified all real cubic surfaces in $\mathbb{P}^3$ with respect to their topological type. Roughly, the authors say that two cubic surfaces have the same topological type if they can be transformed continuously into each other without changing the shape. A similar classification had already been given by Schläfli in the 19^{th} century [**Sch63**], but Knörrer and Miller obtained more precise and more complete results. Some of these are based on ideas of Bruce and Wall [**BW79**] who gave a modern treatment of the complex case.

Here, we restrict ourselves to cubic surfaces with only rational double points which is the most interesting part of the classification. We give an explicit real affine equation for each class in their list (see table 12.2 on page 147). These allow us to draw images for each class showing all singularities and lines (see fig. 12.3, 12.4, 12.5) using our software SURFEX [**HLM05**].

In the already cited article, Schläfli also gave equations for each of his types and described their construction in a very geometric way. In many cases, it is easy to find real affine equations from these with the help of our tool SURFEX. But in the other cases, there are too many free parameters and we have to use other methods such as the deformation techniques described by Klein [**Kle73**].

To perform these deformations explicitly, it is useful to have a visualization software at hand. We explain how to use our software SURFEX for such purposes. SURFEX can be used directly on our webpage [**Lab03a**]. It can produce high quality raytraced images for publications in color or in black/white. Indeed, all the images in this chapter are produced using SURFEX in connection with SINGULAR [**GPS01**]. This computer algebra program has been used to compute a primary decomposition of the ideal (f, F_9) describing the 27 lines of f with multiplicities which allowed us to draw the lines on the surfaces using SURFEX. Here, F_9 denotes Clebsch's covariant of degree 9 (see, e.g., [**LvS03**, appendix 4.1] for a determinental formula for this covariant).

The webpage `www.CubicSurface.net` [**LvS00**] contains some movies and more images. SURFEX [**HLM05**] uses S. Endraß's SURF [**End03**] to produce the high quality raytraced images of the surfaces and R. Morris's LSMP [**Mor03**] and K. Polthier's JAVAVIEW [**Pol01**] to allow rotation and scaling of a triangulated preview.

Several mathematicians have already given real affine equations for particularly interesting cubic surfaces such as the Clebsch Diagonal Surface or the four-nodal cubic surface. For some examples of Rodenberg's series there also exist affine equations. But this series is restricted to only a few types of cubic surfaces, and several of Rodenberg's models do not show all the projective real lines because some are at infinity. In fact, this was Rodenberg's intention: His aim was to give an overview of the possible singularities on cubic surfaces and the possible affine views of the projective surfaces.

Here, instead, we do not show different affine views of the same surface. We choose real affine equations that allow us to show all singularities and lines in a single image (or a single real-world model if we use $3d$-printers).

12.1. Knörrer/Miller's 45 Types of Real Cubic Surfaces

To state Knörrer/Miller's classification of real cubic surfaces with only rational double points as singularities we need the following definition. For details and additional results we refer to their article **[KM87]**.

DEFINITION 12.1 (p. 54/55 in **[KM87]**).

(1) *$\mu_{\mathbb{R}}$ denotes the number of (-2)-curves defined over $\mathbb{R}$ in the dual resolution graph of a rational double point that is defined over $\mathbb{R}$. ν denotes the number of pairs of non-intersecting complex conjugate (-2)-curves in this graph.*

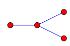
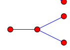
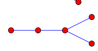
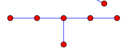
Name	*Old Name*	*Normal Form*	*Coxeter Diagram*	$\mu_{\mathbb{R}}$	ν	
A_{2k}^-	B_{2k+1}	$x^{2k+1}+y^2-z^2$		$2k$	0	$k=1,2$
A_{2k}^+	B_{2k+1}	$x^{2k+1}+y^2+z^2$		0	$k-1$	$k=1$
A_{2k-1}^-	B_{2k}	$x^{2k}+y^2-z^2$		$2k-1$	0	$k=2,3$
A_{2k-1}^+	B_{2k}	$x^{2k}-y^2-z^2$		1	$k-1$	$k=2$
A_1^-	C_2	$x^2+y^2-z^2$		1	0	
$A_1^{\cdot}$	C_2	$x^2+y^2+z^2$		1	0	
D_4^-	U_6	$x^2y-y^3-z^2$		4	0	
D_4^+	U_6	$x^2y+y^3+z^2$		2	1	
D_5^-	U_7	$x^2y+y^4-z^2$		5	0	
E_6^-	U_8	$x^3+y^4-z^2$		6	0	

TABLE 12.1. The types of singularities occuring on real cubic surfaces, their normal forms, their Coxeter diagrams, and the numbers $\mu_{\mathbb{R}}$ and ν.

(2) *Let Σ be a sequence of six points defined over $\mathbb{R}$ in* almost general position *in $\mathbb{P}^2(\mathbb{C})$ in the sense of* [**Dem80**, p. 39]. *Then there exists $r(\Sigma) \in \mathbb{N}_0$, s.t. Σ consists of $2r$ points that are invariant under complex conjugation and $6-2r$ pairwise complex conjugate points. We call $r(\Sigma)$ the* reality index *of Σ.*

(3) *Let X be a cubic surface in $\mathbb{P}(\mathbb{C})$ defined over $\mathbb{R}$ with only rational double points. The* reality index *$r(X)$* of *X is defined as follows: Let $\widetilde{X}$ denote the desingularization of X and $\overline{X}(\Sigma)$ the blowup of $\mathbb{P}^2(\mathbb{C})$ along Σ. Then, $r(X)=r(\Sigma)$, if $\widetilde{X} \cong \overline{X}(\Sigma)$ for a sequence Σ of six points in almost general position in $\mathbb{P}^2(\mathbb{C})$. Otherwise, $r(X)=-1$.*

Using this notion it is possible to compute the number of lines on a real cubic surface:

THEOREM 12.1 (Satz 2.8 in [**KM87**]). *Let $X \subset \mathbb{P}^3(\mathbb{C})$ be a cubic surface defined over $\mathbb{R}$ with only rational double points as singularities. Suppose that the real part $X_{\mathbb{R}} \subset \mathbb{P}^3(\mathbb{R})$ of X has k singular points. Denote by $\mu_{\mathbb{R}}(X)$ the sum of the $\mu_{\mathbb{R}}$ for these singular points and by $\nu(X)$ the sum of the ν of all singularities on X. Then the real part $X_{\mathbb{R}}$ contains exactly $l(X_{\mathbb{R}})$ lines, where*

$$(12.1)\quad l(X_{\mathbb{R}}) = \frac{(2+2r(X)-\mu_{\mathbb{R}}(X))(1+2r(X)-\mu_{\mathbb{R}}(X))}{2} - (r(X)-2)+k-\nu(X).$$

For a cubic surface $X \subset \mathbb{P}^3(\mathbb{C})$ we can read the topology of its real part $X_{\mathbb{R}} \subset \mathbb{P}^3(\mathbb{R})$ from the reality index. E.g., the five smooth cubic surfaces, classically denoted by $F_1, F_2, \ldots, F_5$ (see [**Seg42**]), are classified by the reality index, e.g., $r(F_5) = -1$.

EXAMPLE 12.1. *We illustrate the previous theorem using our software* SPICY*: We construct five points on a circle and another point. Furthermore, we write a* SINGULAR *procedure which computes the equation of the cubic surface and the lines on them (this can be done by only computing 3×3 determinants, see e.g.* [**LvS03**]*). We can now tell* SPICY *to recompute the equation and then* SURF *to draw the corresponding image each time one of the six points has been moved (see figure 12.1, for details we refer to* [**LvS03**]*). Using Knörrer/Miller's formula (12.1), it is easy to compute the number of lines for the surface X in the leftmost figure. This one is smooth, i.e. $k = \mu_{\mathbb{R}}(X) = \nu(X) = 0$, and all the six points are real, i.e. $r(X) = 3$. By the formula, X contains $l(X) = 27$ real lines (which is also easy to see by other means).*

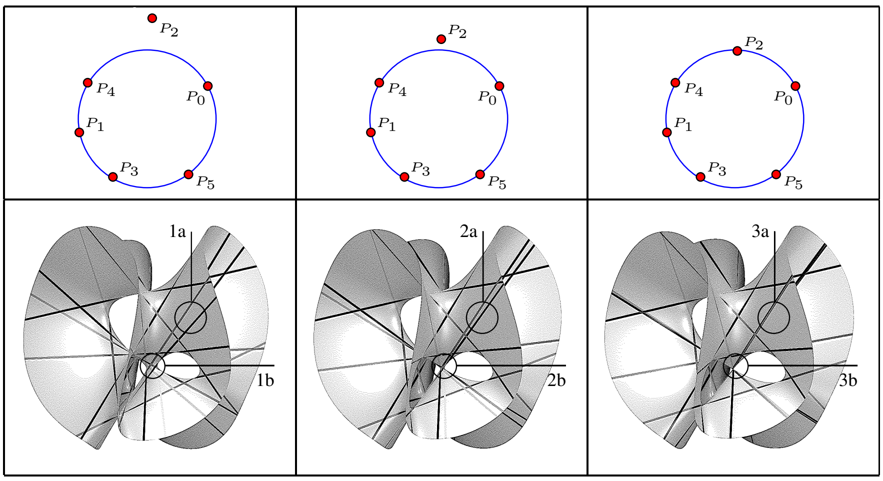


FIGURE 12.1. The blowing-up of the projective plane in six points, such that all six are on a common conic, is a cubic surface with an ordinary double point. Note the changing of the lines, when we drag the point P_2. When P_2 lies on the conic through the other five points, $2 \cdot 6$ lines meet in the double point (1b – 3b) and six pairs of two lines coincide (1a – 3a).

Now let the sixth point also be on the circle as in the rightmost figure. Then it is well-known that the corresponding cubic surface develops an A_1^--singularity and that twelve of the 27 *lines pass through this ordinary double point and coincide in pairs. This development can be visualized interactively using* SPICY *by moving the sixth point slowly. According to table 12.1 on page 144* $\mu_{\mathbb{R}}(Y) = 1$, $\nu(X) = 0$ *for the* 1*-nodal cubic surface* Y *and of course* $k = 1$*. Formula (12.1) thus gives* 21 *as required.*

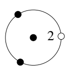
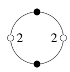
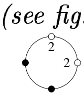
We can now state Knörrer/Miller's main result on cubic surfaces with only rational double points:

THEOREM 12.2 (Classification, Liste 4 in [**KM87**]). *Let* $X \subset \mathbb{P}^3(\mathbb{C})$ *be a cubic surface defined over* $\mathbb{R}$ *with only rational double points and let* $X_{\mathbb{R}} = X \cap \mathbb{P}^3(\mathbb{R})$ *be its real part. Then the topological type of* $X_{\mathbb{R}}$ *is one of the* 45 *types given in table 12.2 on the facing page. If* X *has exactly* $3A_1^-$ *singularities and* X *contains exactly* 12 *lines (no.* 18/19 *in the table) then its topological type can be determined by prop. 12.3 below. Otherwise, the topological type of* X *is determined by its singularities, its number of lines, and the reality index* $r(X)$.

To explain how to distinguish between the topological types 18 and 19, we need Knörrer/Miller's notion of a *configuration type of an* A_1^- *singularity.* We only give a sloppy definition and illustrate it using SURFEX, see [**KM87**, p. 63] for details. For this local study we have to work in affine space:

For an A_1^- singularity, the tangent cone is of the form $x^2 + y^2 - z^2$. This cone intersects the cubic surface X in a curve of degree $2{\cdot}3 = 6$, which consists in fact of six lines, counted with multiplicities. Knörrer/Miller describe such a configuration by a small circle together with six points (counted with multiplicities) because a small real sphere around the singularity intersects X in two small real "circles" (fig. 12.2 on page 148). On each of these circles there lies one point of each of the real lines. Therefore, Knörrer/Miller denote a pair of complex conjugated lines by a point in the center of the circle, the real points are drawn on the circle in the correct order. Different such configurations correspond to cubic surfaces of different topological types.

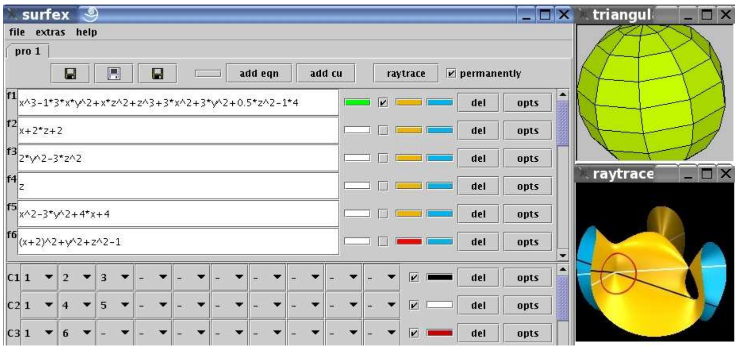
EXAMPLE 12.2. *Example (a) is a configuration with one real point of multiplicity* 2*, two real ones of multiplicity* 1*, and two complex conjugated ones. The other two examples show two doubled and two simple points (see fig. 12.2):*

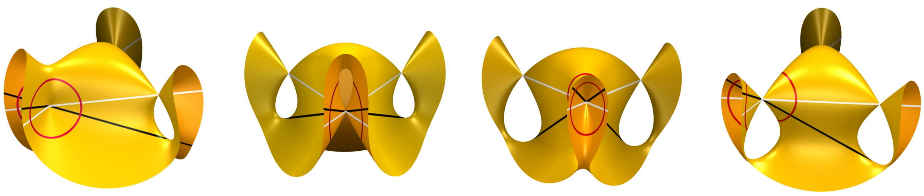
(a)  , *(b)*  *(*KM$_{18}$*),* *(c)*  *(*KM$_{19}$*).* □

PROPOSITION 12.3 (Topological Types 18/19, p. 63 in [**KM87**]). *If a cubic surface* X *has exactly* $3A_1^-$ *singularities and contains* 12 *lines then* X *has the topological type* 18 *if the singular points have a configuration of type*  *(example 12.2 (b)). Otherwise, the* A_1^- *singularities of* X *have a configuration of type*  *(example 12.2 (c)) and* X *has the topological type* 19.

Name	*Sp.*	*Cl.*	*Sing.*	r	l	Equation
KM_1	I	12	$\emptyset$	3	27	$\mathrm{KM}_{27}+\frac{3}{2}(x^2+y^2-z^3)$
KM_2	I	12	$\emptyset$	2	15	$\mathrm{KM}_{27}+\frac{8}{5}((z+1)^2-z^2)$
KM_3	I	12	$\emptyset$	1	7	$\mathrm{KM}_{27}+\frac{2}{3}((z+1)^2+(x-1)^2)-4y^2$
KM_4	I	12	$\emptyset$	0	3	KM_2-4
KM_5	I	12	$\emptyset$	-1	3	$\mathrm{KM}_{27}-\frac{2}{3}((z+1)^2+z^2)$
KM_6	II	10	A_1^-	3	21	$\mathrm{KM}_{27}+2(x^2+y^2)$
KM_7	II	10	A_1^-	2	11	$\mathrm{KM}_{27}+z^3+y^2$
KM_8	II	10	A_1^-	1	5	KM_6-4y^2
KM_9	II	10	A_1^-	0	3	$\mathrm{KM}_6-3(x^2+y^2)$
KM_{10}	II	10	$A_1^{\bullet}$	0	3	$pc+(z+1)\cdot z^2$
KM_{11}	IV	8	$2A_1^-$	3	16	$\mathrm{KM}_{27}+y^2$
KM_{12}	IV	8	$2A_1^-$	2	8	$\mathrm{KM}_{27}+z^2-\frac{1}{5}(x+\frac{1}{2})^2$
KM_{13}	IV	8	$2A_1^-$	1	4	$\mathrm{KM}_{27}-y^2$
KM_{14}	III	9	A_2^-	3	15	$\mathrm{KM}_{21}+\frac{1}{10}(y-1)^2$
KM_{15}	III	9	A_2^-	2	7	$pl+z^3-z^2(x-1)-\frac{1}{5}(x-y)^2$
KM_{16}	III	9	A_2^-	1	3	$\mathrm{KM}_{43}-y^2$
KM_{17}	III	9	A_2^+	0	3	$pc+z^3$
KM_{18}	VIII	6	$3A_1^-$	3	12	$\mathrm{KM}_{43}+z^2(x+\frac{1}{2})$
KM_{19}	VIII	6	$3A_1^-$	3	12	$\mathrm{KM}_{43}+2z^2$
KM_{20}	VIII	6	$3A_1^-$	2	6	$\mathrm{KM}_{27}-z^2$
KM_{21}	VI	7	$A_2^-A_1^-$	3	11	$pl+z^3+z^2(x+y-2)+\frac{1}{10}(x-1)^2$
KM_{22}	VI	7	$A_2^-A_1^-$	2	5	$pl+z^3+z^2(x+y)+\frac{1}{5}(x-1)^2$
KM_{23}	V	8	A_3^-	3	10	$wxy+(x+z)(y^2-(\frac{2}{3}x)^2-(\frac{3}{5}z)^2),\ w=1-x$
KM_{24}	V	8	A_3^-	2	4	$\mathrm{KM}_{32}-\frac{1}{100}z^2(x-z)$
KM_{25}	V	8	A_3^-	1	2	$\mathrm{KM}_{32}+\frac{1}{100}z^2(x-z)$
KM_{26}	V	8	A_3^+	1	4	$2(x^2+y^2)w+2x(z^2-2x^2-4y^2),\ w=1-y$
KM_{27}	XVI	4	$4A_1^-$	3	9	$4(pc+\frac{1}{2})+3(x^2+y^2)(z-6)-z(3+4z+7z^2)$
KM_{28}	XIII	5	$A_2^-2A_1^-$	3	8	$\mathrm{KM}_{43}+z^2(x+2)$
KM_{29}	IX	6	$2A_2^-$	3	7	$\mathrm{KM}_{43}+(x-1)z$
KM_{30}	IX	6	$2A_2^-$	2	3	$\mathrm{KM}_{43}-\frac{3}{10}(x-1)^2$
KM_{31}	X	6	$A_3^-A_1^-$	3	7	$wxz-(x+z)(x^2-y^2),\ w=1-z$
KM_{32}	X	6	$A_3^-A_1^-$	2	3	$wxy-(x+z)(x^2+y^2),\ w=1-z$
KM_{33}	VII	7	A_4^-	3	6	$wxy+y^2z+yx^2-z^3,\ w=1-x-y-z$
KM_{34}	VII	7	A_4^-	2	2	$wxy-y^2z+yx^2-z^3,\ w=1-x-y-z$
KM_{35}	XII	6	D_4^-	3	6	$(x+y+z)^2w+xyz,\ w=\frac{1}{2}(1-x-y-z)$
KM_{36}	XII	6	D_4^+	1	2	$(x+y+z)^2w+(x^2+y^2)z,\ w=\frac{1}{2}(1-x-y-z)$
KM_{37}	XVII	4	$2A_2^-A_1^-$	3	5	$\mathrm{KM}_{43}+(x-1)z^2$
KM_{38}	XVIII	4	$A_3^-2A_1^-$	3	5	$wxz+y^2(x+z),\ w=2(1+x-y+z)$
KM_{39}	XIV	5	$A_4^-A_1^-$	3	4	$wxz-y^2z+\frac{1}{2}x^2y,\ w=\frac{1}{8}(1-y-z)$
KM_{40}	XI	6	A_5^-	3	3	$wxz+y^2z+x^3-z^3,\ w=1-x$
KM_{41}	XI	6	A_5^-	2	1	$wxz+y^2z+x^3+z^3,\ w=1$
KM_{42}	XV	5	D_5^-	3	3	$wx^2+y^2z+xz^2,\ w=1+x$
KM_{43}	XXI	3	$3A_2^-$	3	3	$tl+z^3$
KM_{44}	XIX	4	$A_5^-A_1^-$	3	2	$wxz-y^2z-x^3,\ w=1-z$
KM_{45}	XX	4	E_6^-	3	1	$x^2w-xz^2+y^3,\ w=1-x-y$

TABLE 12.2. Our nice real affine equations for Knörrer/Miller's 45 topological types. The abreviation *Sp.* denotes Schläfli's *species* of the surface, *Cl.* its class, *Sing.* its singularities. r denotes the reality index and l the number of real lines on the surface.





KM_{18} (a) KM_{18} (b) KM_{19} (a) KM_{19} (b)

FIGURE 12.2. The configuration of the lines cut out by the tangent cone at one of the three A_1^- singularities of our surfaces with topological types no. 18 and 19. For each of the surfaces, we show two views (a), (b) from different angles. The white lines have multiplicity two, the black ones have multiplicity one. The figure above illustrates how SURFEX can draw curves on surfaces using the corresponding feature of SURF. To draw the two doubled white lines, we computed the equations `f4`, `f5` cutting these out on the surface using SINGULAR. Then we chose the numbers of the equations from the drop down menu in the row called `C2` and selected the color white.

12.2. Constructing Nice Real Affine Equations

12.2.1. Nice Equations. By a *nice* real affine equation f for a given topological type t we mean an equation, s.t. its projective closure $\overline{f}$ has the required topological type and s.t. the plane at infinity neither contains a singularity nor a line of $\overline{f}$. It has also to be possible to see all its singularities and lines in a single picture (modulo guessing using symmetries). This is not a precise definition. Nevertheless, we formulate our main result in the form of a theorem:

THEOREM 12.4. *For each topological type* $t \in \{1, 2, \ldots, 45\}$ *of real cubic surfaces with only rational double points there is a nice affine equation* KM_t *in the sense of the preceding paragraph. The equations* KM_t *are given in table 12.2 on page 147 and the corresponding pictures are shown in the figures 12.3, 12.4, 12.5. The colors of the lines indicate their multiplicities.*

REMARK 12.5. *For a nice equation for a given topological we do not require the greatest possible symmetry because we want the equations to be generic in the sense that the configuration of the lines on the surface should not be too special. E.g., the Clebsch Cubic Surface has* 10 *so-called* Eckardt Points *in which three of its* 27 *real lines meet, but a generic cubic surface with* 27 *lines does not have any such point.*

1–3:
–
$(r{:}3, 27l.)$
–
$(r{:}2, 15l.)$
–
$(r{:}1, 7l.)$

4–6:
–
$(r{:}0, 3l.)$
–
$(r{:}\text{-}1, 3l.)$
A_1^-
$(r{:}3, 21l.)$

7–9:
A_1^-
$(r{:}2, 11l.)$
A_1^-
$(r{:}1, 5l.)$
A_1^-
$(r{:}0, 3l.)$

10–12:
$A_1^{\cdot}$
$(r{:}0, 3l.)$
$2A_1^-$
$(r{:}3, 16l.)$
$2A_1^-$
$(r{:}2, 8l.)$

13–15:
$2A_1^-$
$(r{:}1, 4l.)$
A_2^-
$(r{:}3, 15l.)$
A_2^-
$(r{:}2, 7l.)$

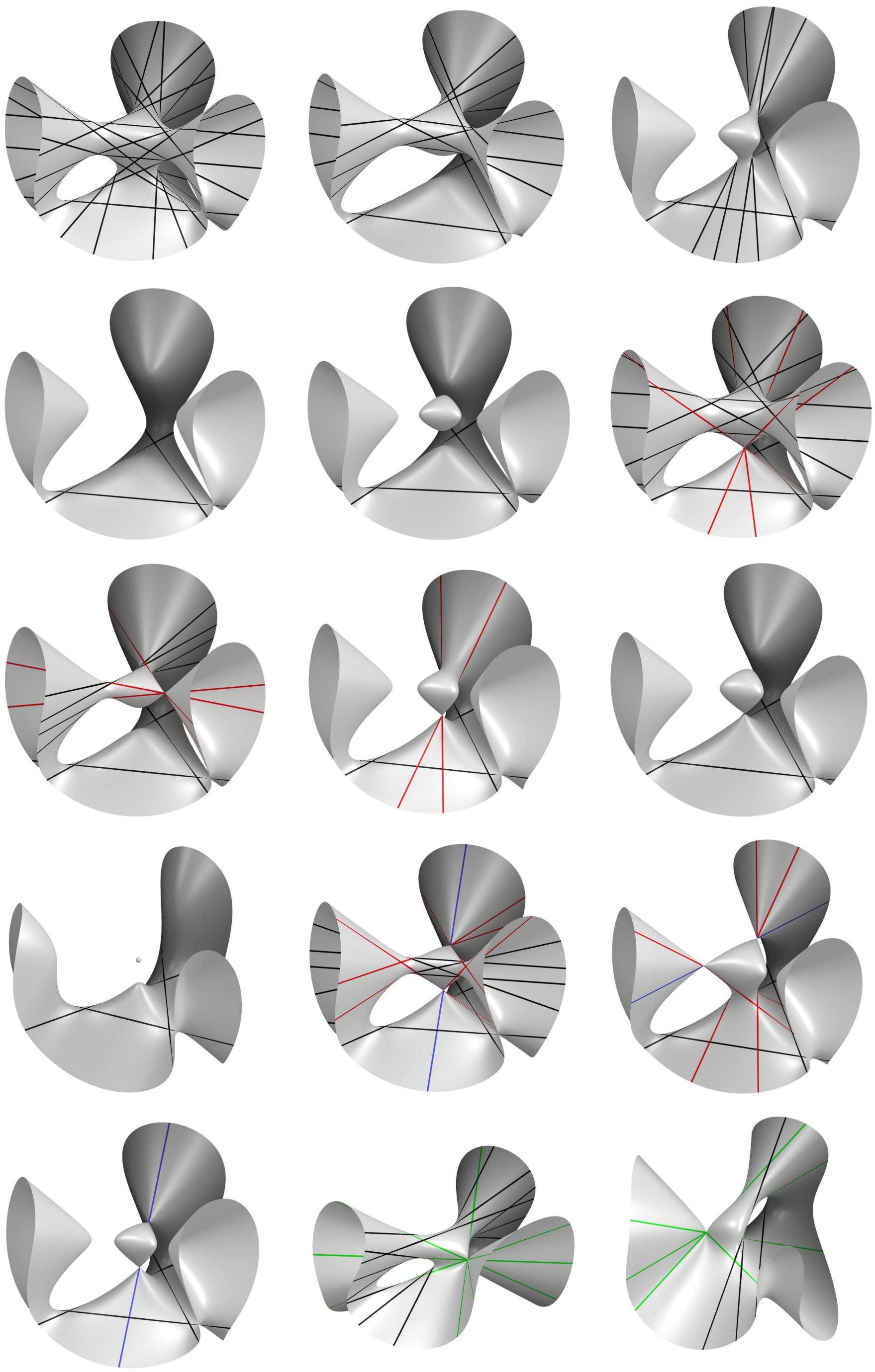


FIGURE 12.3. The surfaces $\mathrm{KM}_1, \ldots, \mathrm{KM}_{15}$. The colors of the lines indicate their multiplicities: ■ 1, ■ 2, ■ 3, ■ 4, ■ 5, ■ 6, ■ 8, ■ 9, ■ 10, ■ 12, ■ 15, ■ 16, ■ 27.

16–18:
A_2^- (r:1, 3l.)
A_2^+ (r:0, 3l.)
$3A_1^-$ (r:3, 12l.)

19–21:
$3A_1^-$ (r:3, 12l.)
$3A_1^-$ (r:2, 6l.)
A_2^-, A_1^- (r:3, 11l.)

22–24:
A_2^-, A_1^- (r:2, 6l.)
A_3^- (r:3, 10l.)
A_3^- (r:2, 4l.)

25–27:
A_3^- (r:1, 2l.)
A_3^+ (r:1, 4l.)
$4A_1^-$ (r:3, 9l.)

28–30:
$A_2^-, 2A_1^-$ (r:3, 8l.)
$2A_2^-$ (r:2, 7l.)
$2A_2^-$ (r:2, 3l.)

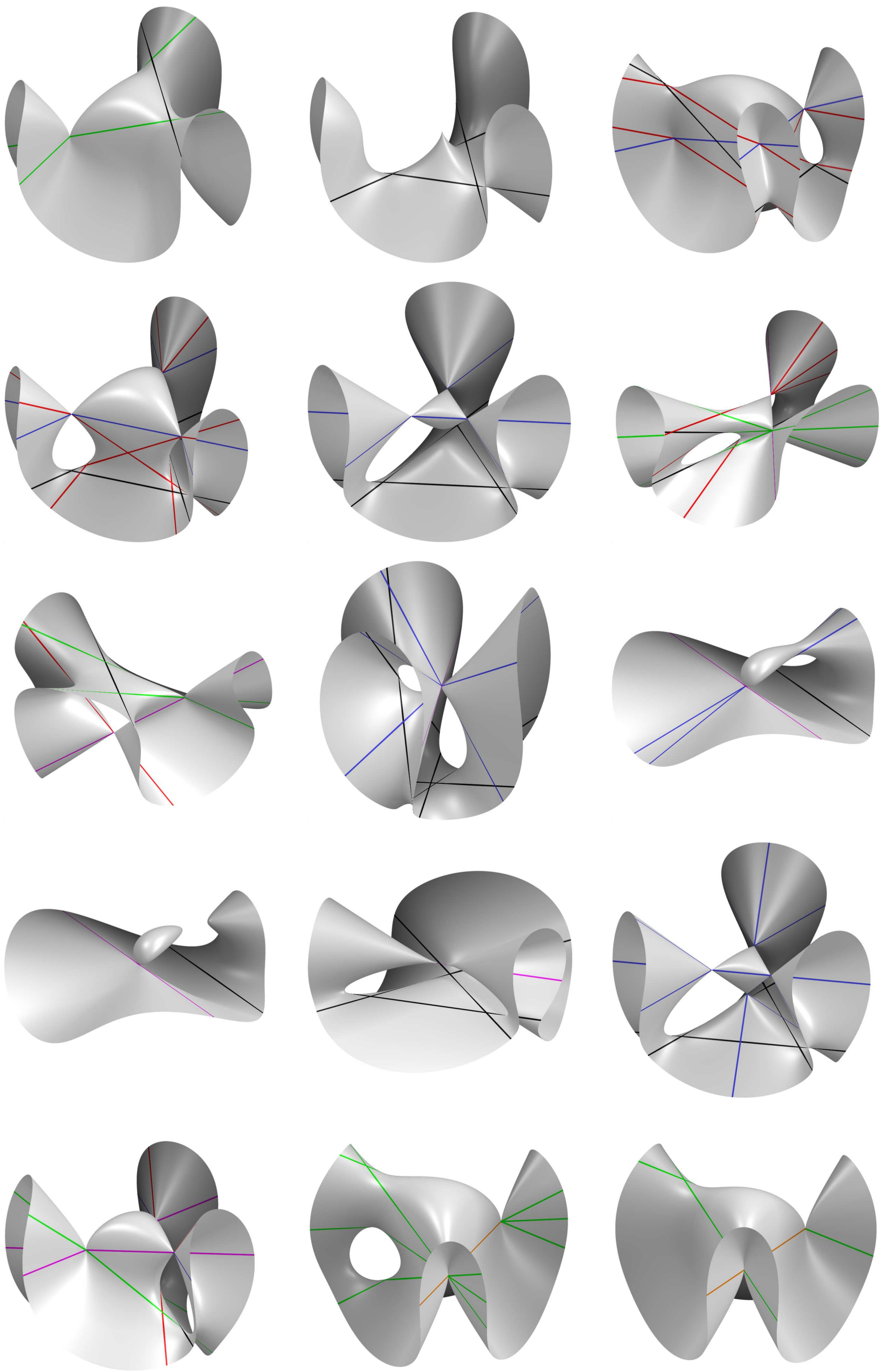


FIGURE 12.4. The surfaces $\mathrm{KM}_{16}, \ldots, \mathrm{KM}_{30}$. The colors of the lines indicate their multiplicities: ■ 1, ■ 2, ■ 3, ■ 4, ■ 5, ■ 6, ■ 8, ■ 9, ■ 10, ■ 12, ■ 15, ■ 16, ■ 27.

31–33:
A_3^-, A_1^- $(r{:}3, 7l.)$
A_3^-, A_1^- $(r{:}2, 3l.)$
A_4^- $(r{:}3, 6l.)$

34–36:
A_4^- $(r{:}2, 2l.)$
D_4^- $(r{:}3, 6l.)$
D_4^+ $(r{:}1, 2l.)$

37–39:
$2A_2^-, A_1^-$ $(r{:}3, 5l.)$
$A_3^-, 2A_1^-$ $(r{:}3, 5l.)$
A_4^-, A_1^- $(r{:}3, 4l.)$

40–42:
A_5^- $(r{:}3, 3l.)$
A_5^- $(r{:}2, 1l.)$
D_5^- $(r{:}3, 3l.)$

43–45:
$3A_2^-$ $(r{:}3, 3l.)$
A_5^-, A_1^- $(r{:}3, 2l.)$
E_6^- $(r{:}2, 1l.)$

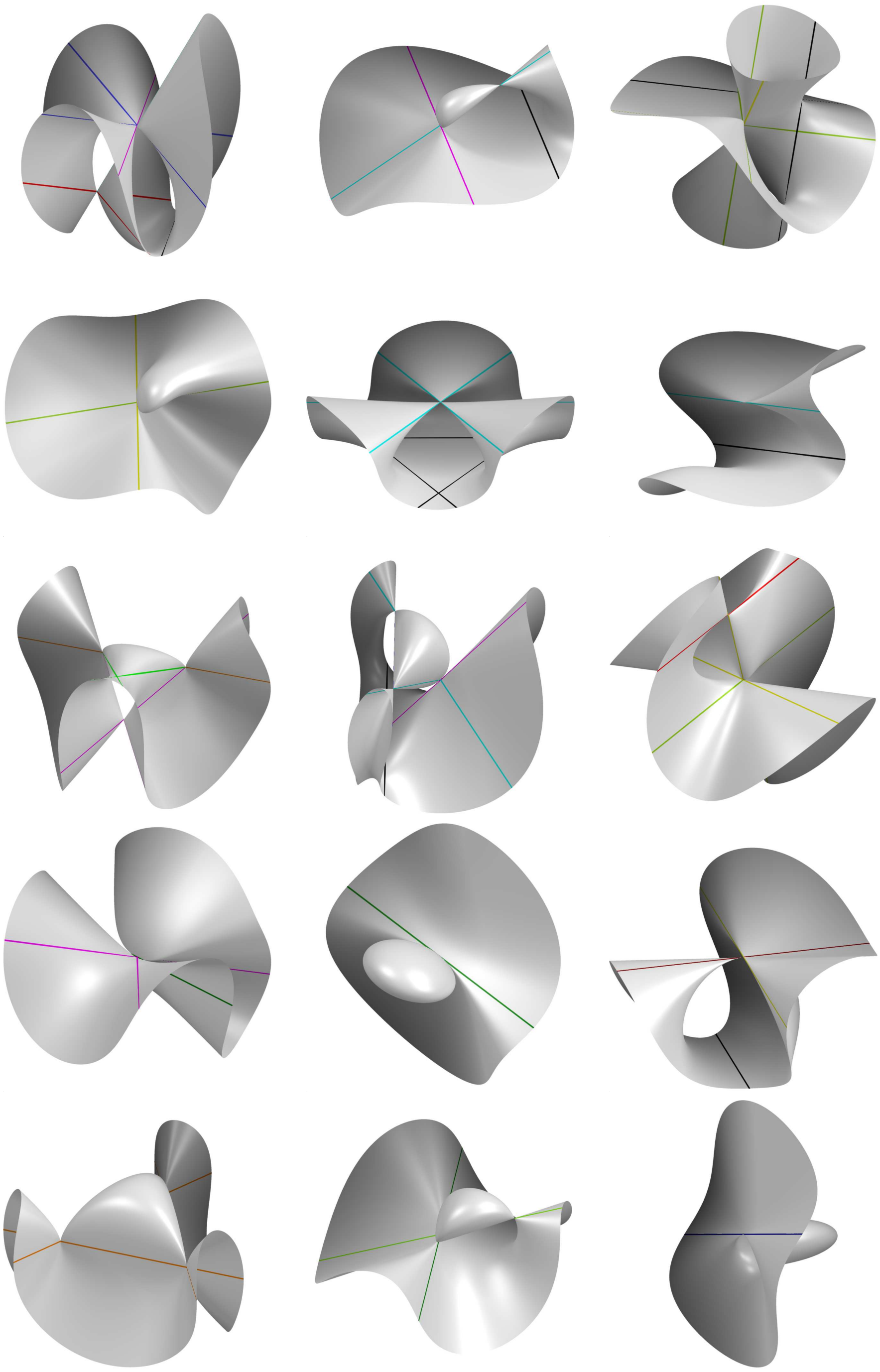


FIGURE 12.5. The surfaces $\mathrm{KM}_{31}, \ldots, \mathrm{KM}_{45}$. The colors of the lines indicate their multiplicities: ■ 1, ■ 2, ■ 3, ■ 4, ■ 5, ■ 6, ■ 8, ■ 9, ■ 10, ■ 12, ■ 15, ■ 16, ■ 27.

REMARK 12.6. *Schläfli orders the cubic surfaces first by their class and then by the worst singularity occuring. This differs from Knörrer/Miller's order which is first by the sum of the Milnor numbers of the singularities and then by the worst singularity occuring.*

In the following subsections we describe how to construct such surfaces.

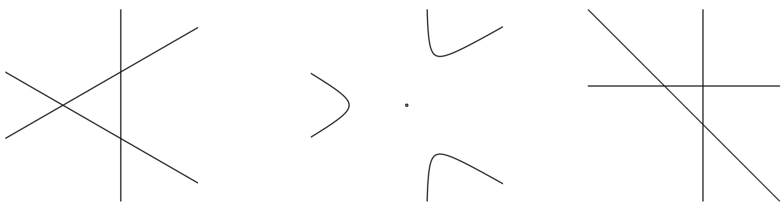
12.2.2. Via Projective Equations. For the projective case, Schläfli already gave equations in [**Sch63**]. He describes in a very geometric way how to construct them. In [**Cay69**], Cayley gives the same equations again and computes a lot of additional data connected to the surfaces.[1]

To obtain a nice real affine equation from one of Schläfli's equations is an easy task for most topological types with higher singularities (A_3 or higher): We just have to choose a good hyperplane at infinity and maybe some constants which is not difficult using our tool SURFEX:

EXAMPLE 12.3. *Let us take the equation* $wxz + y^2z + x^3 = 0$ *given by Schläfli* [**Sch63**, p. 357] *for a projective cubic surface with an* A_1 *and* A_5 *singularity. The choice* $w = 1 - z$ *gives our affine equation* KM_{44}.

For those surfaces with only A_1 and A_2 singularities, this method does not work well because of the great number of free parameters. In this case, we can either write down the equation directly (section 12.2.3), or we can use a deformation process (section 12.2.4) already described by F. Klein in [**Kle73**].

12.2.3. Direct Construction. In some cases, it is easy to write down a nice real affine equation for a topological type directly using symmetry. For this purpose, we will use the three plane curves shown in figure 12.6.



$tl := x^3 + 3x^2 - 3xy^2 + 3y^2 - 4$ $pc := tl + 4$ $pl := (x-1)(y-1)(x+y)$

FIGURE 12.6. Three plane curves, useful for constructing nice equations for cubic surfaces.

EXAMPLE 12.4 (Constructing KM_{43} with three A_2^- Singularities). *We take the polynomial* tl *defining three triangle-symmetric lines (fig. 12.6) in the* x, y*-plane and add the term* z^3*:* $\mathrm{KM}_{43} = tl + z^3$. *At each intersection point of the lines* tl*, this gives a singularity of type* A_2^- *with* z*-coordinate* 0*, see fig. 12.8*(*a*).

The four-nodal surface KM_{27} can be constructed in a similar way. This and a lot more information on nodal surfaces with dihedral symmetry can be found in S. Endraß's Ph.D. thesis [**End96**]. The following example uses a plane curve with a solitary point. In the same way we obtain the surface KM_{26} with an A_3^+ singularity.

[1]Attention, Cayley's list on p. 321 contains some typos.

EXAMPLE 12.5 (Constructing KM_{10} with an $A_1^{\bullet}$ Singularity). *To construct a surface with an $A_1^{\bullet}$ Singularity which has the normal form $x^2+y^2+z^2$ we start with the triangle-symmetric plane cubic pc (fig. 12.6 on the preceding page). The origin is a solitary point (i.e., a singularity with normal form x^2+y^2). Thus the surface $pc+z^2$ has an $A_1^{\bullet}$ singularity with normal form $x^2+y^2+z^2$ and is triangle-symmetric. To obtain the desired affine topology we require a third root on the $\{x=y=0\}$ axes at $z=-1$:* $\mathrm{KM}_{10}=pc+(z+1)\cdot z^2$.

12.2.4. The Deformation Process. Klein's strategy for obtaining surfaces with fewer singularities from surfaces with many singularities is based on the fact that any singularity on a cubic surface can be deformed separately.

By the definition of a singularity, the origin can only be a singularity of an affine surface f if the tangent cone of f has degree at least 2. Thus, in order to smooth an isolated singularity at the origin, we can simply add a term of degree 1 or 0. But which terms can we add to the equation of f without changing the type of a singularity at the origin? For A_1 singularities, this is very easy: These singularities are characterized by the fact that their tangent cone also defines an A_1 singularity.[2] So, we can add any term of degree greater than two and any term of degree two whose coefficient is small enough. E.g. $x^2+y^2-z^2+\frac{1}{10}z^2+\frac{1}{13}xy+x^3$ has a singularity of type A_1^- at the origin.

Using the preceding facts we can deform a cubic surface with four singularities of type A_1^- into one with only three such singularities:

EXAMPLE 12.6 (Smoothing one of four A_1 Singularities). *Let* KM_{27} *be the cubic surface with four A_1^--singularities (see table 12.2 on page 147). Three of its singularities lie in the plane $\{z=0\}$. Using* SURFEX, *it is easy to find an ε, s.t. the surface* $\mathrm{KM}_{27}+\varepsilon z^2$ *has the desired topology (see fig. 12.7):*

Go to the SURFEX *web-page* **[HLM05]**, *start the* SURFEX *program, and enter the equation of* KM_{27}. *Then add a term* `+0.1*z^2` *and check the* `permanently` *checkbox – this will premanently recompute raytraced images of your surface. Drag the computer mouse over the green ball to rotate the surface until you see all singularities. You can scale the image by pressing* s *on your keyboard while dragging. Now your* SURFEX *screen should look similar to fig. 12.7 on the next page. The singularity in the middle has been smoothed in such a way that the neighborhood of the singularity looks like a hyperboloid of one sheet. Adding* `-0.1*z^2` *leads to a neighborhood which looks like a hyperboloid of two sheets.* □

It is a little more subtle to keep singularities of type A_j^- or $A_j^+, j>1$, while deforming others. Forgetting about the sign for a moment, these singularities have the equation $x^{j+1}+y^2+z^2$ in a suitable coordinate system. $A_j, j>1$, singularities are characterized by the property that their tangent cone is of degree two and consists of the union of two different planes.[3]

Let f be a polynomial in three variables x,y,z defining a singularity of type $A_j, j\geq 2$, at the origin. By the finite determinacy theorem (see, e.g., **[Dim87]**), we can add an element of the ideal $I := \mathfrak{m}^2\cdot J_f$ to f without changing the type

[2] This is also the reason why the geometers of the 19^{th} century called the A_1 singularities conical singularities or singularities of type C_2. Other names are *proper node*, *ordinary double point*.

[3] This is the reason why the classical geometers called a singularity of type A_j a biplanar node B_{j+1}. A singularity whose tangent cone consists of a single multiple plane was called a uniplanar node.

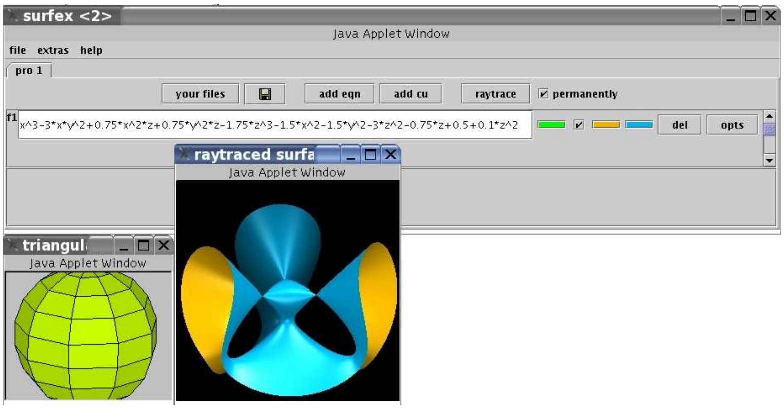


FIGURE 12.7. Smoothing one of the four singularities of the cubic surface KM_{27}.

of the singularity. Here, $\mathfrak{m}$ denotes the maximal ideal (x, y, z) of the origin and thus $\mathfrak{m}^2 = (x^2, xy, xz, y^2, yz, z^2)$. $J_f := (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ is the so-called jacobian ideal generated by the partial derivatives of f.

EXAMPLE 12.7. *We take the singularity of type A_3^- at the origin, defined by $f := x^4+y^2-z^2 = 0$. Its jacobian ideal is $J_f = (x^3, y, z)$. If we choose $g_1 := xy \in \mathfrak{m}^2$ and $g_2 := y \in J_f$ we get $g := g_1g_2 = xy^2$. Then $f + g$ still defines a singularity of type A_3 at the origin. Furthermore, $f + \varepsilon g$ is an A_3^- singularity for ε small enough.*

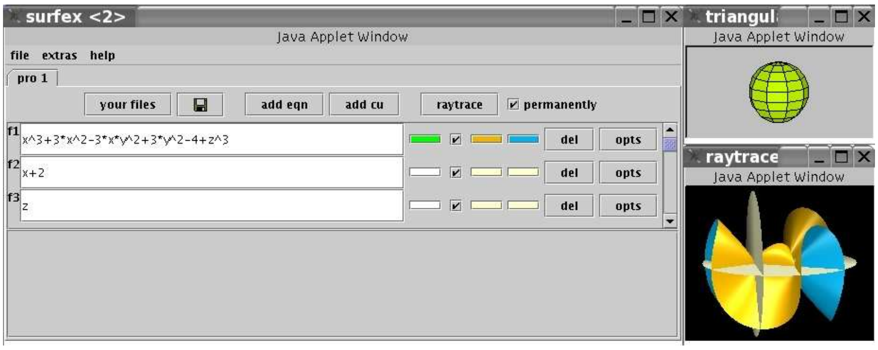
We now come to the global situation of a cubic surface f with only isolated singularities of type $A_j, j \geq 1$. The following example describes how to use the techniques above to deform some of its singularities while keeping others:

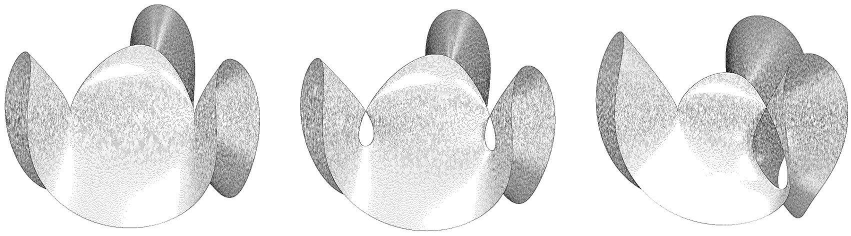
EXAMPLE 12.8 (Deforming two of three A_2^- Singularities to A_1^- Singularities). *We start with the surface KM_{43} which has exactly three singularities of type A_2^- (fig. 12.8(a)). The surface $tl + z^3 + z^2$ (fig. 12.8(b)) has three singularities of type A_1^- at the same coordinates, because the tangent cone is a cone of the form $x^2 - y^2 + z^2$ locally at each of these points. One of these singularities has the coordinates $Q := (-2, 0, 0)$. To get a surface with a singularity of type A_2^- at Q and two singularities of type A_1^-, we need to adjust the construction slightly.*

Our general remarks from the beginning of this subsection tell us that we have to look at the jacobian ideal $J_{\mathrm{KM}_{43}}$ at Q. Over the rational numbers, SINGULAR *gives the following primary decomposition: $J_{\mathrm{KM}_{43}} = (x, y, z^2) \cap (x - 1, y^2 - 3, z^2) \cap (x + 2, y, z^2)$. Locally at Q, the relevant primary component is $(x + 2, y, z^2)$. We choose $E := x + 2 \in (x + 2, y, z^2)$. As $z^2 \in \mathfrak{m}^2$, we then know that $\mathrm{KM}_{43} + z^2 \cdot E$ has a singularity of type A_2 at Q.*

Locally at the other two singularities (which both have x-coordinate 1), E takes the value $1 + 2 = 3$. Thus, at these singularities, $\mathrm{KM}_{43} + z^2 \cdot E$ behaves like $\mathrm{KM}_{43} + z^2 \cdot 3$, which has A_1^- singularities at these points as already seen above.

To check that our choices of planes and constants were reasonable and to understand the construction a little better, we can again use SURFEX. *We type the equation of* KM_{43} *into* SURFEX *as* `f1`. *Then we add another two equations using the* **`add eqn`** *button and choose* `f2` *to be $x+2$ and* `f3` *to be z. If the* **`permanently`** *check-box is activated we already see the three surfaces in one picture. When adjusting the*

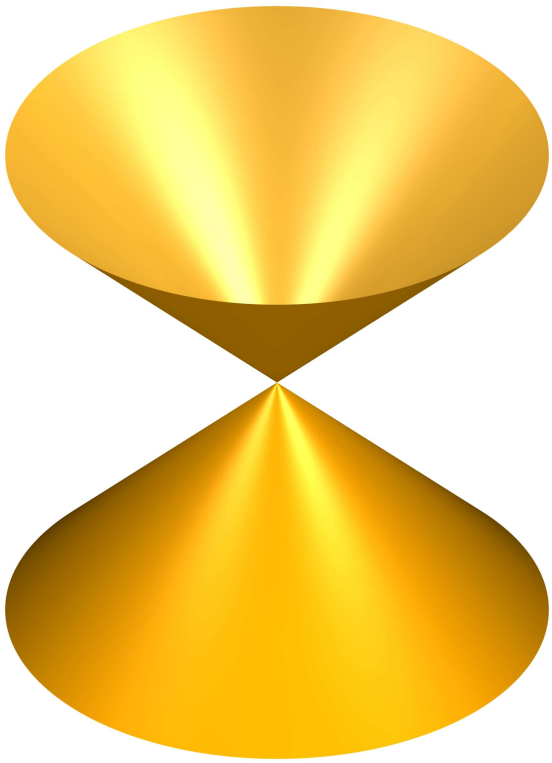




(a) (b) (c)

FIGURE 12.8. Deforming the surface KM_{43} (image (a)) with three singularities of type A_2^- into KM_{28} (image (c)) with one such singularity and two A_1^- singularities.

colors by clicking at the right of the equations, we get a result similar to fig. 12.8. We can hide some of the surfaces by deselecting the checkbox at the right of the equations. When typing into `f1` *the changes described above, we obtain successively the three lower images shown in the figure. We can produce the black/white images used for the present publication in the following way: We press the button showing the small disk, select the* `dithered` *checkbox, choose an appropriate resolution, and then click on* `save`*. A small dialog shows up, where we can give some filename. The high-resolution image is then computed on the webserver. From there, it can then be downloaded using the* `your files` *button in the* SURFEX *window.* □



A cone, a quadric surface with a node. How many nodes can a surface of degree d in $\mathbb{P}^3$ have?

Finally

It is natural to try to apply the methods and algorithms presented in the second part of this work to similar cases. In particular, it would be interesting to construct a surface in $\mathbb{P}^3$ of degree 11 with 430 nodes and to find out if our conjecture on the number of nodes on dihedral-symmetric surfaces (chapter 8) can be improved. If such surfaces exist, will their numbers of nodes be realizable with only real nodes?

Families of varieties within which one searches for some particularly interesting examples also occur in other branches of algebraic geometry. Variants of the algorithm that we presented in chapter 9 can thus also be applied to such problems.

Another wide field with a lot of potential for extensions is the visualization of real hypersurfaces with (many) singularities. First, our visualization tools which we presented in part 3 can be optimized and extended in many aspects. But also the triangulation of real singular varieties which has still not been developped in a satisfactory way would be an interesting achievement. E.g., in combination with (maybe three-dimensional) dynamic constructive geometry software (similar to our tool SPICY) this would open the way to make visualization even more interactive and intuitive.

When browsing through our historical survey (part 1) and our new constructions (part 2), one can see that there are still lots of interesting open questions in the field of hypersurfaces with many singularities and related areas. We hope that the present work encourages many other people to work on this fascinating subject.

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