The Massless Two-loop Two-point Function and Zeta Functions in Counterterms of Feynman Diagrams

Dissertation

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Contents

Chapter 1

Introduction

The necessity to improve theoretical predictions for scattering processes includes the demand for extending calculations in perturbative quantum field theories (pQFTs) to higher loop orders in the perturbation series. Using dimensional regularization, this implies the need to expand the analytic expressions of various Feynman diagrams to higher orders in the regularization parameter ε, with $D = 4 - 2\varepsilon$, as we will explain in the following chapters. Much effort is invested into these higher-order calculations, including attempts to find new ways for addressing this problem. The main part of this thesis provides the description of a new method for expanding the massless two-loop two-point function. Unlike earlier calculations, this method will enable us to expand the integral in principle up to an arbitrary order in ε.

The massless two-loop two-point function is interesting in different respects. On the one hand it is needed for instance in calculations for the process $e^+e^- \to$ hadrons in orders of α_s [GKL 1991, SuSa 1991]. On the other hand there is a number theoretical question associated with this integral: It was shown by different authors that the low-order ε expansion of this integral involves rational numbers and multiple zeta values. A discussion of this issue can be found in Chapter 4. However, it was not clear whether multiple zeta values are sufficient for the expansion of this two-loop function to all orders in ε [Broa 2003]. We will solve this problem and give an answer to this question in Chapter 6.

These number theoretical considerations are related to investigations into mathematical structures underlying perturbative quantum field theories. Although the results of calculations using Feynman diagrams in pQFTs are in good correspondence with experimental results there are a lot of mathematical problems concerning the perturbation series itself that are not yet solved. The high predictive power nevertheless suggests that the mathematical problems might be due to a lack of understanding of the series and that we did not find the correct mathematical formulations so far. A step towards a better understanding of the mathematical structures was done by Dirk Kreimer when he found the *Hopf algebra of renormalization* in renormalizable QFTs [Krei 1998b, CoKr 1998]. Since then, the search for further mathematical structures in Feynman diagrams brought up connections to several mathematical fields. The hope is that improving the knowledge about the structures underlying the perturbation series, one will also be able to understand the results of calculations better. We will make use of such mathematical structures in the following chapters and emphasize their appearance in our calculations.

This thesis is organized as follows: Chapters 2 and 3 provide the basic definitions necessary for subsequent chapters. We start in Chapter 2 with a short introduction to perturbative quantum field theories, only mentioning the very basics. Chapter 3 then defines the Hopf algebra and Lie algebra of rooted trees and Feynman diagrams, emphasizing on the one hand the relation between the antipode of the Hopf algebra and the counterterms of a given Feynman diagram and on the other hand the insertion operation of Feynman graphs that allows us to build Feynman diagrams out of certain building blocks [Krei 1998b, CoKr 1998, CoKr 2000]. However, both chapters 2 and 3 are not intended to give a full review of renormalization of perturbative quantum field theories and its Hopf algebra structure, nor will they fully explain the Hopf algebra and Lie algebra occurring in this context in all of their aspects. We will neglect everything that would distract us from the main path, which necessarily implies that we will omit many aspects that would be interesting in their own right.

Chapters 4 to 6 are dedicated to the calculation of the massless two-loop two-point function, starting in Chapter 4 with a short introduction to some earlier works on the expansion of this function. In Chapter 5 we will then take a closer look at the functions that typically occur in the calculation of Feynman diagrams, and provide an overview of polylogarithms, multiple zeta values, multiple polylogarithms, and related functions. Investigations into these functions and their occurrence in analytic expressions of Feynman diagrams were done in [MUW 2002], where the authors describe a way to expand sums and double sums of fractions of gamma functions in an expansion parameter. An implementation of this work as a computer program is provided by S. Weinzierl's C++ library *nestedsums* [Wein 2002]. Since the sums mentioned before are typical for analytical results of Feynman diagrams, *nestedsums* is a very useful tool in this context and it is this library which enables us to expand the massless two-loop two-point function. In Chapter 6 we will then describe in detail our calculation of the massless two-loop two-point function. Contrary to former work, it will allow us to expand this function up to an arbitrary order in the dimensional regularization parameter ε, as long as three conditions for the set of exponents of its momenta are fulfilled.

In the last two chapters we apply the expansion of the massless two-loop two-point function to the calculation of the non-planar two-loop vertex correction in a massless Yukawa theory and in massless quantum electrodynamics (QED). We then examine in these theories counterterms of graphs that are built by inserting the three divergent one-loop diagrams and the non-planar vertex correction into each other, calculating the vertex corrections with one zero-momentum-transfer vertex. Additionally, we consider the non-planar vertex correction with subdivergences only in one line. The general idea that underlies this construction of graphs has already been used before [Bier 2000] and will therefore only briefly be explained in the first part of Chapter 7, including a presentation of the results for the one-loop diagrams. In the second half of this chapter, we will describe the calculation of the non-planar vertex corrections for the two theories, closing with the description of their implementation as computer programs. Chapter 8 is dedicated to four programs, two for the massless Yukawa theory and two for massless QED, that calculate the counterterm for an input Feynman diagram built from the set of graphs mentioned above. To determine these counterterms the programs use the rooted tree structure of Feynman diagrams and calculate their antipode. Using the set of counterterms that can be built in this way, we investigate connections between the topology of Feynman diagrams and the appearance of Riemann's zeta function in their counterterms. We will end this chapter with the discussion of a list of results we have produced.

Finally, Chapter 9 gives a short summary and outlook.

Chapter 2

Renormalization

Our intention here is not to give a detailed overview of perturbation theory and renormalization but to sketch the general ideas and provide the necessary vocabulary needed in the following. For an extensive introduction we refer the reader to one of the many textbooks such as [PeSch 1995, ItZu 1980, Coll 1984] where these subjects are explained at great length with explicit calculations. Our main interest is to provide examples that illustrate general behavior and correspondences between different expressions, rather than repeating proofs that can be found elsewhere. This especially applies to the next two chapters that are related in many respects, and we will put our emphasis on examples showing these relations.

2.1 General introduction

Consider a Feynman diagram that contains a closed loop. The momentum space Feynman rules tell us that we have to integrate over the momentum of the particles traveling through that loop, taking into account all possible values of this momentum from zero up to infinity. In coordinate space, which is related to the momentum space via Fourier transformation, this integration of the momentum up to infinity corresponds to particles that get infinitesimally close and can lead to so-called *short-distance singularities*. In momentum space we find these singularities in the form of *ultra-violet divergences* (UV-divergences). These divergences would eventually lead to diverging parameters of the theory, which would thus not have any sensible physical meaning. For UV-divergences one can solve this problem by a re-normalization of the parameters of the theory. More precisely, one multiplies the parameters with so-called *Z-factors*. These Z-factors are momentum-free series in $\hbar$ or the coupling constant respectively, and absorb the divergences of the parameters. Note that we set $\hbar = 1$ as usual in the following.

Consider as a general example a generic Lagrangian for a scalar field theory, given by the following formula:

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2} m_0^2 \Phi^2 + \frac{g_0}{n!}\Phi^n, \tag{2.1}$$

Φ denotes a field of a scalar particle and g_0 a coupling constant, $n \in \mathbb{N}$. For $n = 3$ we would obtain a renormalizable theory in 6 dimensions, for $n = 4$ in 4 dimensions. This

Lagrangian consists of different terms: The first monomial is the *free* Lagrangian leading to a freely moving, massless particle. The second one is a *mass term*, which gives a mass to the particle Φ; the third term is the *interaction term*. The parameters g_0 and m_0 are so-called *bare parameters* and have no physical meaning yet. They are related to physically measurable parameters by applying suitable *renormalization conditions*.

The Lagrangian (2.1) generates diagrams with UV-divergent loop integrals. We therefore multiply the field itself, the masses and the couplings by Z-factors which are of the general form:

$$Z_i = 1 + \delta Z_i \tag{2.2}$$

where the contributions to δZ_i have to be determined by the calculation of the counterterms. We find three Z-factors here: Z_Φ for the field, Z_g for the coupling, and Z_m for the mass. Substituting

$$\Phi \to Z_\Phi^{1/2}\Phi, \quad g_0 \to Z_g Z_\Phi^{-n/2} g, \quad m_0^2 \to Z_m Z_\Phi^{-1} m^2 \tag{2.3}$$

we obtain for the Lagrangian:

$$\begin{aligned}\mathcal{L} =& Z_\Phi \frac{1}{2}(\partial_\mu \Phi)^2 - Z_m \frac{1}{2} m^2 \Phi^2 + Z_g \frac{g}{n!}\Phi^n \\ =& \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2} m^2 \Phi^2 + \frac{g}{n!}\Phi^n \\ &+ \delta Z_\Phi \frac{1}{2}(\partial_\mu \Phi)^2 + \delta Z_m \frac{1}{2} m^2 \Phi^2 + \delta Z_g \frac{g}{n!}\Phi^n \\ =& \mathcal{L}_0 + \mathcal{L}_{CT}.\end{aligned} \tag{2.4}$$

$\mathcal{L}_{CT}$ is called the *counterterm Lagrangian*, as this part of the Lagrangian produces the *counterterms*. These are diagrams that correspond to the original Feynman diagrams in such a way that each counterterm provides a contribution to a Z-factor and cancels the divergence of one particular Feynman diagram, rendering it finite. More precisely, the counterterms correspond to Feynman diagrams in which due to (2.2) and (2.3) some vertices and edges are replaced by a contribution of the Z-factor. The Feynman diagrams can be expanded in their external momenta leading to a polynomial in these momenta where the highest degree is limited by the degree of divergence of a diagram. The counterterms are thus also polynomial in masses and momenta.

The Z-factors are determined by different vertex functions. The Z-factor for the coupling, Z_g, is calculated by vertex corrections, Z_m and Z_Φ by self-energies. However, the Z-factors are not unique as there are several ways how to subtract the divergences, corresponding to different *renormalization schemes* and renormalization conditions. These are conditions for the subtraction terms that have to be fulfilled by the *Green's functions*, expectation values of time-ordered products of the fields occurring in the Lagrangian. There are some schemes like for example the BPHZ scheme (Bogoliubov-Parasiuk-Hepp-Zimmermann) that act directly on the integrand making it finite before one integrates over the corresponding momentum. In other schemes one first introduces a *regulator* to identify the divergences. The most obvious idea is to use a *cut-off parameter* Λ as the upper limit of the integration. For $\Lambda \to \infty$, we again obtain an infinite value for the integral. One can then subtract the divergent expression parametrized by Λ in a suitable way and render the integral finite. However, such a parameter violates translational invariance with respect to the momentum and is also not gauge invariant.

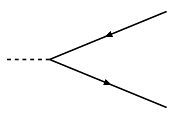
A different regularization method that fulfills the requirements for translational and gauge invariance is the *dimensional regularization* [t'HoVe 1972], which we will use in the following. Applying this regularization scheme one calculates Feynman diagrams in $D = 4 - 2\varepsilon$ dimension. In this one translates the Feynman diagram into a Laurent series in the parameter ε that encodes the divergence, which itself becomes visible for $\varepsilon \to 0$. This term can then be removed for example simply by projecting onto the pole part of the Laurent series and afterwards subtracting this part, which corresponds to an application of the *minimal subtraction* (MS) *scheme*. Of course there is the possibility to subtract finite terms in addition. But the MS scheme most purely shows the mathematics of the divergence structure of graphs. Since this is what we are interested in, the MS scheme is the scheme we will use in the following.

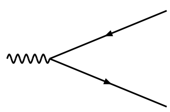
2.2 Feynman diagrams and power counting

After these more general remarks, we want to get into more detail: Feynman diagrams consist of vertices and lines, or edges, of several types.[1] These types are given by the theory under consideration with its various sorts of particles and interactions. The *Feynman rules* assign to each vertex and edge the corresponding analytic expression. The different kinds of propagators, corresponding to the different particles, define the type of an edge and are indicated in the Feynman diagrams in the way we draw them. The next table shows examples for some lines and the propagators corresponding to these lines:

fermion:		solid line,	propagator:	$\frac{1}{k\!\!\!/}$
photon:		curly line,	propagator:	$\frac{1}{k^2}(g_{\mu\nu} + \xi \frac{k_\mu k_\nu}{k^2})$
scalar boson:	- - - -	dashed line,	propagator:	$\frac{1}{k^2}$

Similarly, we obtain for instance the following two vertices for the interactions in Yukawa theory and QED, where we only have three-point vertices between two fermions and one boson:

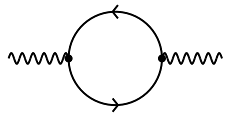
Yukawa vertex:  $-ig$

QED vertex:  $-ie\gamma^\mu$

The variables e and g denote the coupling constants and γ^μ is a *Dirac matrix*. Note that the arrows in diagrams here and in the following indicate the direction of momentum flow.

[1] Please note that we use the terms *Feynman graph* and *Feynman diagram* synonymously.

Following [Krei 2001] we call an edge *internal* or *inner line* if it connects two vertices, and *external* if it connects to only one vertex. Note that we do not consider tadpole graphs. The photon lines in the following picture are external lines, the fermion lines are inner lines.



The *external structure* of a graph is then provided by its external lines. The *type of a vertex* is defined by the set f_v of its incoming and outgoing lines. We call two sets of edges I_1, I_2 *compatible*, $I_1 \sim I_2$, if and only if they contain the same number of edges, of the same type. Two vertices v_1, v_2 are of the same type if f_{v_1} is compatible with f_{v_2}.

We are mainly interested in Yukawa theory and QED, which we will use in Chapter 7 and the programs of Chapter 8. Hence we will base our examples and definitions on these two theories, focusing on Yukawa theory. Changes in formulas or definitions for other theories can be found in the textbooks cited at the beginning of this chapter and will not be taken into account here.

In the process of renormalization one first has to identify the UV-divergent parts of a diagram, that is, the parts of a Feynman diagram whose corresponding analytic expressions diverge, when the momenta tend to infinity. We have already mentioned that the UV-divergences are related to loops of Feynman diagrams. To decide whether there is an UV-divergence emerging from a loop integration or not, one has to consider the large-momentum behavior of the integrand. Consider a general scalar integral

$$I(p_1, ..., p_N) := \int d^4k_1...d^4k_L f(p_1, ..., p_N, k_1, ..., k_L), \tag{2.5}$$

where the p_i, $i \in \{1, ..., N\}$, correspond to the external momenta of the diagram, the k_i, $i \in \{1, ..., L\}$ are the momenta running through the loops of the diagram, and the function f denotes the integrand consisting of fermion and boson propagators. Considering the momentum behavior, each fermion propagator contributes a factor $\frac{1}{|k|}$, each boson propagator a $\frac{1}{|k|^2}$. Roughly speaking, the diagram diverges if there are more powers of momentum in the numerator than in the denominator. This fact is measured and expressed by the *superficial degree of divergence* of a diagram. In 4 dimensions it is defined as the number:

$$\begin{aligned} \omega &= \text{powers of } k \text{ in the numerator - powers of } k \text{ in the denominator} \\ &= 4L - I_F - 2I_B, \end{aligned} \tag{2.6}$$

where we expressed ω by the number I_F of inner fermion lines, I_B of inner boson lines, and the number of loops L of the diagram. In theories other than QED and Yukawa theory one may find an additional term to (2.6) due to couplings that involve derivatives of fields.

This process of determining ω by considering the contribution of the momenta of the propagators and the loop integrations to the divergence of the loop is called *power counting*.

Depending on the value of ω, we find:

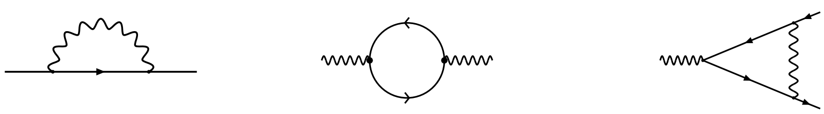
- $\omega < 0$: the diagram is convergent
- $\omega \geq 0$: the diagram can be divergent, more precisely:
  - $\omega = 0$: the corresponding integral tends to infinity $\propto$ *log* Λ and is hence called *logarithmically divergent.*
  - $\omega = 1$: the diagram diverges $\propto \Lambda$ and is called *linear divergent*
  - $\omega = 2$: it is called *quadratically divergent*, etc.

$\omega \geq 0$ is a necessary but not a sufficient condition that an integral diverges. The integral can fulfill $\omega \geq 0$ and still diverge with a "smaller" degree of divergence than ω would indicate. This might happen due to symmetries of a theory. For example, the vacuum polarization in QED has $\omega = 2$ and diverges logarithmically [PeSch 1995]. Additionally, a superficially convergent or divergent diagram can have subdivergences, meaning that not the complete diagram but only certain constituent parts of it correspond to UV-diverging integrals.

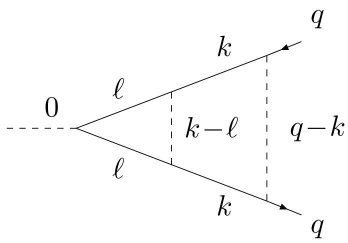
For Yukawa theory and QED we want to express ω by the number of external lines E_F and E_B of a diagram. Knowing that at each vertex two fermion lines and one boson line meet, we can express the number of vertices V as $V = 2I_B + E_B = \frac{1}{2}(2I_F + E_F)$. Taking into account the delta functions related to the vertices and the overall delta function that enforces momentum conservation, we can further express the number of loop integrations as $L = I_F + I_B - V + 1$. Putting all these equations together we obtain:

$$\omega = 4 - \frac{3}{2}E_F - E_B. \tag{2.7}$$

Note that this formula only depends on the number of external lines and not on the number of vertices V or any inner lines I_F and I_B. Hence ω is not changed when we increase the loop order of the graphs! Additionally, we see that increasing the number of external lines, the integral becomes more and more convergent. This leads to three divergent one-loop diagrams for the two theories: the fermion self energy, the vacuum polarization and the vertex correction:



As an example for a divergent diagram we consider a two-loop contribution to the vertex correction in massless Yukawa theory at zero momentum transfer. Please note that we will frequently omit factors $(-ig)^x$, $(i\pi^{D/2})$ etc. when they are not necessary for the understanding and we simply want to illustrate the general behavior.



$$= \int d^4k \int d^4\ell \frac{1}{\not k}\frac{1}{\not \ell}\frac{1}{\not \ell}\frac{1}{\not k}\frac{1}{(q-k)^2}\frac{1}{(k-\ell)^2}$$
$$= \int d^4k \int d^4\ell \frac{1}{\ell^2}\frac{1}{k^2}\frac{1}{(q-k)^2}\frac{1}{(k-\ell)^2}$$

Power counting leads to $\omega = 0$ when we send ℓ and k to infinity jointly, so the integral in total is logarithmically divergent. This divergence is called the *overall divergence* of the graph. Taking into account only the loop integration with respect to ℓ, keeping k fixed, we find again a logarithmic divergence:

$$\int d^4\ell \frac{1}{\ell^2}\frac{1}{(k-\ell)^2} \quad \rightarrow \quad \omega = 0.$$

This corresponds to the expression of a one-loop diagram with external momentum k. Therefore the "inner" loop, with loop momentum ℓ, is a *subdivergence* of this graph.

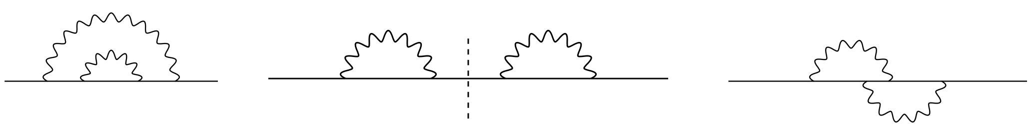
On the other hand for ℓ fixed and $k \to \infty$ one finds:

$$\int d^4k \frac{1}{k^2}\frac{1}{(q-k)^2}\frac{1}{(k-\ell)^2} \quad \rightarrow \quad \omega = -2,$$

which is convergent.

So we have precisely two divergences in this integral: the *superficial* or *overall divergence*, when l and k run to infinity jointly, and one subdivergence, when k is fixed and only l tends to infinity. Graphs like the two self-energy graphs and the vertex correction given above, which are superficially divergent but have no subdivergences, are called *primitive graphs*.

The UV-divergences of a diagram are related to its loops. An important definition in this context is that of *one-particle irreducible* (1PI) graphs. These are graphs that can not be made disconnected by cutting a single inner line. A 1PI primitive graph with two external lines is called a *self-energy*, a graph with three external lines an *interaction* or *vertex graph*. The graph on the left hand side in the following picture is one-particle irreducible, the one in the middle is not. It would split into two disjoint graphs, when one would perform the indicated cut. Such graphs are called *reducible*.



In the 1PI graph on the left hand side we have an example for a graph whose subdivergences are *nested*: one subdivergence lies *inside* another loop or divergence, respectively. For the graph in the middle, the divergences are *disjoint*: The external lines of one of the divergences are not inner lines of another graph. A third possibility is that divergences can *overlap*, like in the third graph of the picture above.

Since the analytic expressions corresponding to reducible diagrams are simple products of the analytic expressions for their irreducible parts, it is sufficient to restrict ourselves to 1PI graphs.

2.3 Dimensional regularization

So far we have identified the divergent parts of a diagram. In a first step towards renormalization we will now regularize these divergences for a massless Yukawa theory by dimensional regularization [t'HoVe 1972, Coll 1984] and afterwards subtract them using the minimal subtraction scheme. Everything which we will only briefly sketch here can be found in all detail with proofs and calculations in [Coll 1984].

In dimensional regularization the integral is analytically continued in the complex plane from 4 to D dimensions, usually defined as $D = 4 - 2\varepsilon$. The D-dimensional integral has the following properties:

- Linearity: $$\int d^D x\, [\, a\, f(\mathbf{x}) + b\, g(\mathbf{x})\,] = a \int d^D x\, f(\mathbf{x}) + b \int d^D x\, g(\mathbf{x}) \tag{2.8}$$

- Scaling behavior: $$\int d^D x\, f(s\mathbf{x}) = s^{-D} \int d^D x\, f(\mathbf{x}) \tag{2.9}$$

- Translational invariance: $$\int d^D x\, f(\mathbf{x}+\mathbf{y}) = \int d^D x\, f(\mathbf{x}) \tag{2.10}$$

In these formulas, $\mathbf{x}$ and $\mathbf{y}$ are vectors in an infinite-dimensional vector space, f and g are functions of vectors $\mathbf{x}$ and $\mathbf{y}$ and a, b, s are scalars.

Using dimensional regularization, the result of calculations of Feynman diagrams depends on the parameter ε. The graphs evaluate to Laurent series in ε, which can diverge for $D \to 4$ or, equivalently, $\varepsilon \to 0$.

For a general D-dimensional integral of momentum k in Euclidean space we get:

$$\int d^D k \frac{[k^2]^a}{[k^2 - m^2]^b} = \pi^{D/2} [-m^2]^{D/2+a-b} \frac{\Gamma(a + D/2)\Gamma(b - a - D/2)}{\Gamma(D/2)\Gamma(b)}. \tag{2.11}$$

Hence in the calculation of this integral we naturally find *Euler's gamma function.* This gamma function can be expanded using:

$$\begin{aligned} \Gamma(1+\varepsilon) &= \exp(-\gamma\varepsilon)\, \exp\left(\sum_{n=2}^{\infty} \frac{\zeta(n)}{n}(-\varepsilon)^n\right), \qquad |\varepsilon| < 1 \\ &= 1 - \gamma\varepsilon + \frac{1}{2}(\gamma^2 + \zeta(2))\varepsilon^2 + O(\varepsilon^3) \end{aligned} \tag{2.12}$$

Note that the integral (2.11) vanishes if there is no external scale, which is provided by the mass in this case or can, for example, be an external momentum in other cases. Hence the integral

$$\int d^D k \frac{1}{[k^2]^a} = 0. \tag{2.13}$$

In the following, we will calculate massless integrals. These integrals are mostly of the form (2.14), which is given in Euclidean space and for $D = 4 - 2\varepsilon$ by:

$$\begin{aligned} I(q;\nu_1,\nu_2) &\equiv \frac{1}{(\mu^2)^{-\varepsilon}} \int \frac{d^D k}{\pi^{D/2}} \frac{1}{[(q-k)^2]^{\nu_1}[k^2]^{\nu_2}} \\ &= \left(\frac{q^2}{\mu^2}\right)^{-\varepsilon} \frac{1}{[q^2]^{(\nu_1+\nu_2-2)}} \frac{\Gamma(2-\nu_1-\varepsilon)\,\Gamma(2-\nu_2-\varepsilon)\,\Gamma(\nu_1+\nu_2-2+\varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(4-\nu_1-\nu_2-2\varepsilon)} \\ &= \frac{1}{(\mu^2)^{-\varepsilon}} \frac{1}{[q^2]^{(\nu_1+\nu_2-2+\varepsilon)}} \frac{\Gamma(2-\nu_1-\varepsilon)\,\Gamma(2-\nu_2-\varepsilon)\,\Gamma(\nu_1+\nu_2-2+\varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(4-\nu_1-\nu_2-2\varepsilon)} \\ &= \frac{1}{(\mu^2)^{-\varepsilon}} [q^2]^{(2-(\nu_1+\nu_2)-\varepsilon)} F_{\nu_1,\nu_2}(\varepsilon) \end{aligned} \tag{2.14}$$

with

$$F_{\nu_1,\nu_2}(\varepsilon) := \frac{\Gamma(2-\nu_1-\varepsilon)\,\Gamma(2-\nu_2-\varepsilon)\,\Gamma(\nu_1+\nu_2-2+\varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(4-\nu_1-\nu_2-2\varepsilon)}. \tag{2.15}$$

The explicit calculation of this integral can be found in the Appendix C. Like the integral (2.11), it evaluates to gamma functions that can be expanded in ε via (2.12).

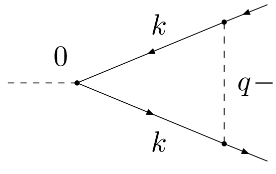
The scale μ^2 is introduced in order to obtain a dimensionless integral. Starting with the Lagrangian one includes this scale in the interaction term to absorb the additional dimension of the coupling constant when we switch to D dimensions, $g \to \mu^{2-D/2} g$ and to keep the coupling constant dimensionless. In the following, we will consider all integrals in units of μ^2, setting $\mu^2 = 1$.

Our aim in later chapters is to investigate the appearance of zeta functions in counterterms rather than to compare our results with experiments. Therefore we will often omit any multiplicative factors like the coupling constant g or multiples of π.

2.4 Counterterms

2.4.1 One-loop diagrams

Let us consider the one-loop vertex function in massless Yukawa theory. It diverges logarithmically in 4 dimensions. In $D = 4 - 2\varepsilon$ dimensions we get:



$$= \int d^D k \frac{1}{\not k} \frac{1}{\not k} \frac{1}{(q-k)^2} = \int d^D k \frac{1}{k^2} \frac{1}{(q-k)^2}.$$

We use (2.14) and expand on the one hand the gamma functions via (2.12), and on the other hand the term $[q^2]^{-\varepsilon}$ into a power series in ε, where we assume for convenience that $q^2 > 0$ (cf. Chapter 8):

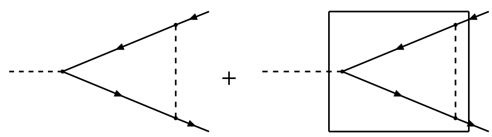
$$[q^2]^{-i\varepsilon} = \exp(\ln([q^2]^{-i\varepsilon})) = \exp(-i\varepsilon \ln[q^2]) = 1 - i\varepsilon \ln[q^2] + O(\varepsilon^2). \tag{2.16}$$

This expansion yields:

$$\int d^D k \frac{1}{(q-k)^2 k^2} = [q^2]^{-\varepsilon} F_{1,1} = [q^2]^{(-\varepsilon)} \frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(1)\Gamma(1)\Gamma(2-2\varepsilon)}$$
$$= \frac{1}{\varepsilon} + 2 - \gamma - \ln[q^2] + O(\varepsilon). \quad (2.17)$$

One can see that by using dimensional regularization, we are now able to isolate the divergence in the term $\frac{1}{\varepsilon}$, which tends to infinity for $\varepsilon \to 0$. Note the appearance of the $\ln[q^2]$-term in the finite part of the expression. This will cause problems when we start to increase the loop order.

Now that the infinity is isolated in the term $\frac{1}{\varepsilon}$, the graph can easily be made finite by simply removing, that means subtracting, this pole part. This is done by adding a counterterm to the graph, which is indented in the following by a box drawn around the graph or a cross at the vertex or line the divergence sits in. The box indicates that one projects onto the pole part of the corresponding Laurent series and subtracts the result:



$$\equiv \frac{1}{\varepsilon} + 2 - \gamma - \ln[q^2] + O(\varepsilon) \;-\; \frac{1}{\varepsilon} = 2 - \gamma - \ln[q^2] + O(\varepsilon) \quad (2.18)$$

In this way we have calculated the first contribution to the Z-factor Z_g. Recall that we multiply g by Z_g: $g \to Z_g g$ to renormalize the expressions. In dimensional regularization, Z_g has the general form:

$$Z_g = 1 - \sum_{j=1}^{\infty} \sum_{k=-j}^{\infty} (g^2)^j c_{j,k} \varepsilon^k \quad (2.19)$$

and is also a Laurent series in ε. Hence we just calculated the first contribution $c_{1,-1} = 1$. We denote this term by $\delta Z_1 := -\frac{c_{1,-1}}{\varepsilon}(-ig)^2$ and get $Z_g = 1 + \delta Z_1$. Substituting the coupling constant g by the "new" value $g \to Z_g g$ in the Feynman diagrams, one immediately obtains the renormalized graph from the Lagrangian.

2.4.2 Multi-loop diagrams

Before we continue investigating the renormalization of UV-divergent expressions of multi-loop diagrams, we want to make a short remark concerning infrared divergences, IR-divergences. These are divergences that come from long-range forces in a massless theory and appear in momentum-space as divergences when some momenta go to zero. We will in our calculations only encounter analytic expressions whose expansions in ε give rise to UV-divergences but not IR-divergences. This is an outcome of the fact that we calculate in $D = 4 - 2\varepsilon$ dimensions: the divergent terms of our expansion in ε stem from certain gamma functions in the numerator of (2.14). If we calculate and express the integral (2.14) in D dimensions and take D different from $D = 4 - 2\varepsilon$, we find that for some values of D other gamma functions provide poles and in this give rise to IR-divergences. Since we restrict ourselves to $D = 4 - 2\varepsilon$, ε small, the different sorts of divergences, UV and IR, do not mix [Coll 1984].

What happens now if we increase the loop order? In dimensional regularization, an n-loop diagram corresponds to a Laurent series that starts with lowest order ε^{-n}. Starting with the two-loop level, we find in the negative orders of the Laurent series terms of the form $\ln^i[q^2]/\varepsilon^j$. Projecting onto the pole part of this series analogously to the one-loop case would lead to a Z-factor containing terms proportional to $\ln[q^2]$. In coordinate space this would give rise to logarithms of differential operators, as by Fourier transformation we obtain the replacement $q_\mu \to i\partial_\mu$. Hence an expansion analogous to (2.16) would give rise to a series involving logarithms of differential operators which eventually leads to an effective action that is no longer local.

A solution to this problem was given by Bogoliubov, Parasiuk and Hepp, who developed a procedure how to renormalize a graph with subdivergences recursively. It is based on the idea that one first renormalizes subdivergences, which will eliminate the non-local terms, leading to a graph $\bar{R}(\Gamma)$. The remaining divergence can then be subtracted by a local counterterm, the counterterm for the overall divergence of the graph. Zimmermann then gave a recipe how to solve this recursion by applying the *forest formula*.
Define:

Γ	=	divergent loop diagram
$\bar{R}(\Gamma)$	=	the graph Γ, in which all subdivergences have been rendered finite, leaving only the overall divergence
$Z(\Gamma)$	=	counterterm for the overall divergence of Γ
Γ_R	=	finite, renormalized graph

The index R of Γ_R denotes the renormalization map, like in our example the MS scheme, $R \equiv R_{MS}$. For each application of the renormalization map R (for each box respectively) one has to multiply the expression with a minus sign. For a primitive one-loop diagram Γ, for example, this means that:

$$\Gamma_R = \Gamma - R(\Gamma) =: \Gamma + Z[\Gamma] \tag{2.20}$$

since the one-loop graph is subdivergence-free.

Zimmermann's forest formula (ZFF) now states the different steps how to construct $Z(\Gamma)$ and Γ_R [Coll 1984], [ItZu 1980] for a general graph:

1. find all the divergences of a graph, denoting them by γ_i, $i \in \mathbb{N}$
2. build all possible sets out of these divergences, the so-called "forests", where the graphs of one forest that correspond to divergent subgraphs have to be either nested or disjoint.

Hence the forests are sets consisting of the superficially divergent subdiagrams γ and, in case the graph is overall divergent, the graph Γ itself. The empty set is also a forest. Forests that do not contain Γ are called *normal* forests, forests including Γ, *full* forests. Since the full forests emerge from the normal forests by simply adding Γ, there are always equally many full and normal forests. The requirement that the subdivergences building a forest may only be disjoint or nested is not a limitation for the set of graphs. One automatically includes overlapping divergences in this way [Coll 1984].

Let γ_X now be the subdivergence that corresponds to a forest X (*e.g.*: $X = \{\gamma_1, \gamma_2\} \Rightarrow \gamma_X = \gamma_1 \cup \gamma_2$). Zimmermann's forest formula then states that $\bar{R}(\Gamma)$, $Z(\Gamma)$ and Γ_R are given by the formulas:

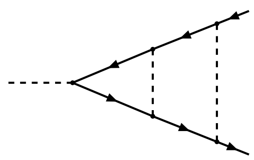
$$\bar{R}(\Gamma) = \sum_{U \in \mathcal{F}_\mathcal{U}} (-)^{n_U} R(\gamma_U)\Gamma/\gamma_U \tag{2.21}$$

$$Z(\Gamma) = \sum_{V \in \mathcal{F}_\mathcal{V}} (-)^{n_V} R(\gamma_V)\Gamma/\gamma_V \tag{2.22}$$

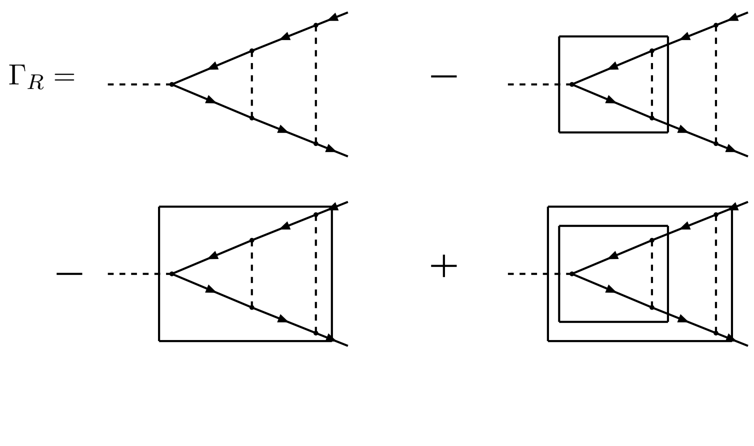
$$\Gamma_R = \sum_{W \in \mathcal{F}_\mathcal{W}} (-)^{n_W} R(\gamma_W)\Gamma/\gamma_W \tag{2.23}$$

where the $\mathcal{F}_\mathcal{U}$ denotes all normal forests U of $\mathcal{F}$, $\mathcal{F}_\mathcal{V}$ all full forests V of $\mathcal{F}$, and $\mathcal{F}_\mathcal{W}$ all possible normal and full forests W. The numbers n_U, n_V, n_W denote the number of elements in the respective forests.[2] Γ/γ_U denotes the graph Γ, in which the graph(s) γ_U have been shrunk to a point. Note that we find: $R(\gamma_1 \cup \gamma_2) = R(\gamma_1)R(\gamma_2)$.

As an example to illustrate Zimmermann's forest formula, consider again the contribution to the two-loop vertex correction

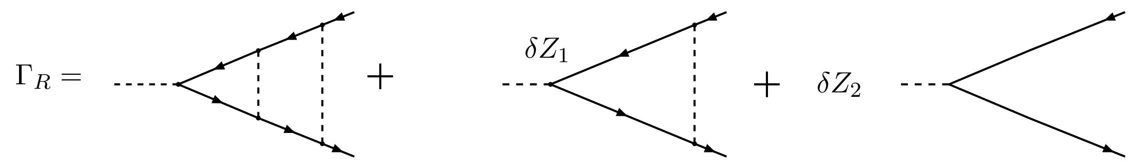


We have seen in Section 2.2 that this graph has one subdivergence given by the inner loop and is overall divergent. We call the subdivergence given by the inner loop $\{\gamma\}$ and the superficial divergence $\{\Gamma\}$. The possible forests are therefore: $\emptyset$, $\{\gamma\}$, $\{\Gamma\}$, $\{\gamma, \Gamma\}$. Applying (2.23) to this graph meaning that we take these forests, apply the renormalization map R and build the sum appropriately, we obtain:



[2]The empty set is a forest, but does not count for the different n's.

The interpretation of these graphs is the following: The second graph contains the counterterm for the subdivergence γ. It is generated by the one-loop counterterm δZ_1 calculated in (2.17) and (2.18). Hence the one-loop renormalization already made the two-loop graph subdivergence-free and what remains is the overall divergence. This, on the other hand, is subtracted by the last two graphs, which bring us the next contribution δZ_2 to the Z-factor: $\delta Z_2 := -(\frac{c_{2,-2}}{\varepsilon^2} + \frac{c_{2,-1}}{\varepsilon})(-ig)^4$. Hence we can represent the renormalized graph Γ_R as



where again δZ_1 is the counterterm for the overall divergence of the one-loop diagram and multiplies the coupling to the external boson line like indicated in the picture. Multiplying δZ_1 also with one or both couplings to the inner boson line would give a contribution of order g^7, g^9 respectively, which is not needed until we consider graphs of this order in g. δZ_2 then is the counterterm for the overall divergence of the two-loop diagram. A short calculation for example in the massless Yukawa theory that we used before shows that in this way one obtains Z-factors that are indeed $\ln[q^2]^i/\varepsilon^j$-free and a renormalized graph of order g^5 in the coupling constant.

Chapter 3

Hopf and Lie algebra

In the last chapter we have demonstrated that there is a certain combinatorial structure underlying the renormalization process of multi-loop diagrams in pQFTs, provided by Zimmermann's forest formula. It was shown by D. Kreimer in 1997 that this structure is encoded in a commutative, but not cocommutative, *Hopf algebra of rooted trees* H_R [Krei 1998b, CoKr 1998]. This Hopf algebra can also be directly formulated on Feynman diagrams, yielding the Hopf algebra of Feynman diagrams H_{FG}. The Feynman rules then are maps, more precisely *characters*, on the Hopf algebra into a suitable target space. The definition of a Hopf algebra and other definitions needed in this context can be found in the Appendix A. It is the antipode of the Hopf algebra of rooted trees which is implemented in the programs given in Chapter 8 and we will therefore introduce this Hopf algebra H_R in detail.

Dually to this associative commutative non-cocommutative Hopf algebra exists a cocommutative Hopf algebra. This latter Hopf algebra is isomorphic to a *universal enveloping algebra* $U(L)$ of a Lie algebra L due to the *Milnor-Moore theorem* [MiMo 1965]. We will also briefly introduce this Lie algebra in the following where we will focus on the Lie algebra of Feynman graphs, since it is the idea of inserting graphs into each other which is one step in the calculation of the massless two-loop two-point function in Chapter 6.

However, we want to emphasize that this chapter will only provide a short sketch of the basic properties of the Hopf and Lie algebras, and we will focus on the structures which we will need in the chapters to come. Similarly, we will not prove the statements which we will make but rather illustrate them with examples. The proofs can be found in [CoKr 1998, CoKr 2000, Krei 2003b, Krei 2003a, Krei 2001, CoKr 2002] which also include reviews on these subjects and related topics.

3.1 The Hopf algebras H_R and H_{FG}

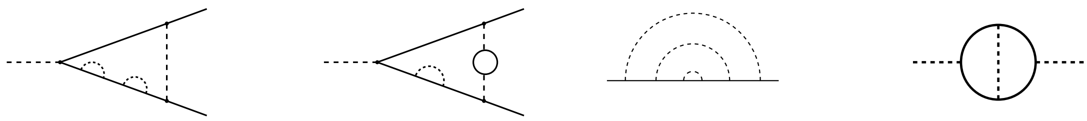
We start with the definition of the Hopf algebra of rooted trees H_R. We will then introduce the Hopf algebra of Feynman diagrams H_{FG} and finish this section by showing the correspondence between H_R, H_{FG}, and Zimmermann's forest formula.

3.1.1 The Hopf algebra of rooted trees H_R

A rooted tree is a simply connected set of vertices and edges, meaning that any two chosen vertices are connected by not more than *one* edge. Each tree contains *one* special vertex which is always drawn at the top of the tree and is called the *root*. We assign to the trees an orientation by defining the edges connected to the root to be *outgoing*, meaning that they are directed away from the root. In general, a vertex has *incoming* and *outgoing* edges where the root is the only vertex with no incoming edge. The next picture shows some examples. The root is marked here by an unfilled vertex.

The last tree in that row is an example for a tree with sidebranchings: there are two edges originating from the root. The other trees have no sidebranchings. The number of edges originating from a vertex v is called the *fertility*, $f(v)$.

Before we continue to define the Hopf algebra, we first want to show how these rooted trees are related to Feynman diagrams. Consider the following four graphs:



The translation from a Feynman diagram to a rooted tree has to be done in the following way: *Set boxes around all subdivergences of a Feynman graph* Γ *and mark their upper horizontal lines with a dot (*$\simeq$ *vertex). Dots of nested boxes, that is boxes where one of them is contained inside the other, are connected with a line (*$\simeq$ *edge).*
For the first three Feynman diagrams of the example above we then get:

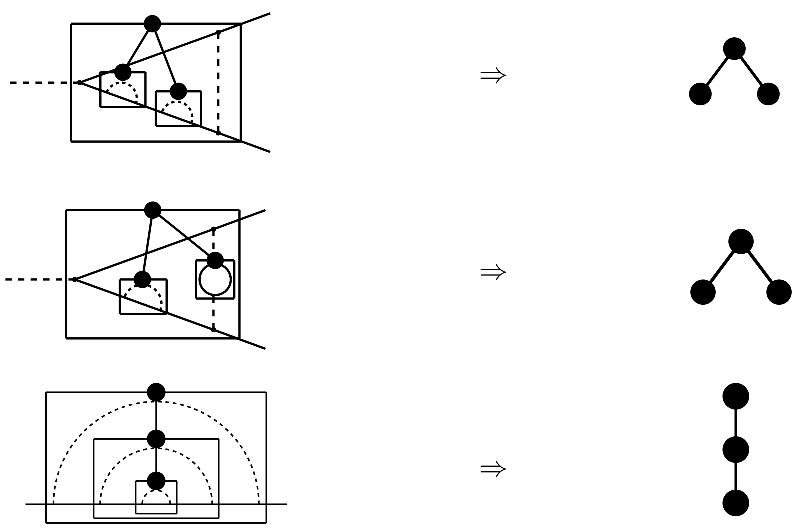
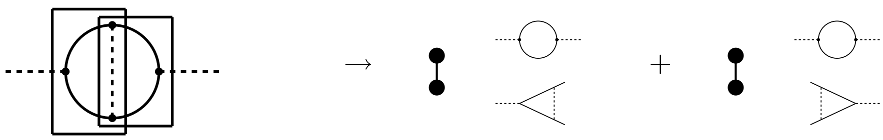


Figure 3.1: Translation of Feynman diagrams to rooted trees.

Hence for each of these Feynman diagrams we obtain a corresponding tree. From the first two pictures one can see that the same tree can belong to different Feynman diagrams. This is because the tree only tells us the relative position of subdivergences, in the sense that it encodes in which way divergent graphs sit inside each other. One can reduce this ambiguity by turning to decorated rooted trees, where one draws the corresponding subdivergences next to the vertices. Still one can easily find examples where the same decorated tree corresponds to different Feynman diagrams. Hence a decorated rooted tree is assigned to a whole set of Feynman diagrams that have the same divergence structure, meaning that they consist of the same divergences in the same nested or disjoint appearance. However, we will soon see that this ambiguity does not cause any problems. Anything valid due to the Hopf algebra structure for the tree is also valid for any element in its corresponding set of graphs.

The situation is different for the fourth graph of our example, which is overall divergent and has two overlapping subdivergences corresponding to the two "halves" of the graph. This graph does not map onto *one* rooted tree, but onto *two* rooted trees in the following way: In a first step, determine the maximal subdivergences of the graph. They are maximal in the sense that there is no other subdivergence that contains them and is not the overall divergence itself. In our example there are two maximal subdivergences, namely the only two subdivergences of the graph that exist. To each maximal forest one then assigns a tree. Decorating these trees with the corresponding diagrams like before we obtain:



The recipe of treating this two-loop graph is in general true for overlapping divergences. Hence overlapping divergences are no big difficulty and simply map onto a sum of graphs instead of one simple tree [Krei 1999].

We now want to define the structure maps of the Hopf algebra on these trees [CoKr 1998, GBVF 2000]. We find a natural grading for rooted trees given by their number of vertices. This is measured by the *grading operator* Y, which is defined by $Yt = nt$, where n = number of vertices of the tree t. The number n is called the *degree* of the tree.

The rooted trees form a vector space, with the linear generators being single trees. One can add these trees and multiply them with coefficients from a field, for example $\mathbb{Q}$. Starting from this vector space we get a *freely generated algebra* A (see Appendix A), whose multiplication is defined as the *disjoint union* of trees. This disjoint union is *commutative* and obviously adds the degrees of the trees. A general element of the algebra therefore is a polynomial in trees. The unit element e is the empty set.

The structure maps of the Hopf algebra are defined by operations on trees. First of all, there are different sorts of *cuts* on their edges. The basic cut is an *elementary cut* which is a cut at a single line of a tree. An *admissible cut* is a set of elementary cuts such that on the way from each vertex up to the root there is at most *one* elementary cut. When we make a cut and think of it as a real cut to a real tree, where we hold the tree by its root, there is always some part of the tree that is pruned and exactly *one* part of the tree that is not. This latter part is the one that we keep in hand, containing the root. Any cut and especially any admissible cut therefore maps a tree onto a monomial of trees, where exactly *one* contains the root and is called $R^C(t)$. The set of trees that are pruned is denoted $P^C(t)$, cf. Fig. 3.2. Additional to these cuts, we have empty cuts and full cuts. The *empty cut* is a cut at *no* edge. Let us figure an additional edge incoming to the root (with no vertex to start from, otherwise one would get a new root). A *full cut* is then a set of cuts that includes a cut at this additional edge. Fig. 3.2 shows some examples. Cuts that are not full cuts are called *normal cuts* and both sets only differ by the one additional cut at the edge which is incoming to the root.

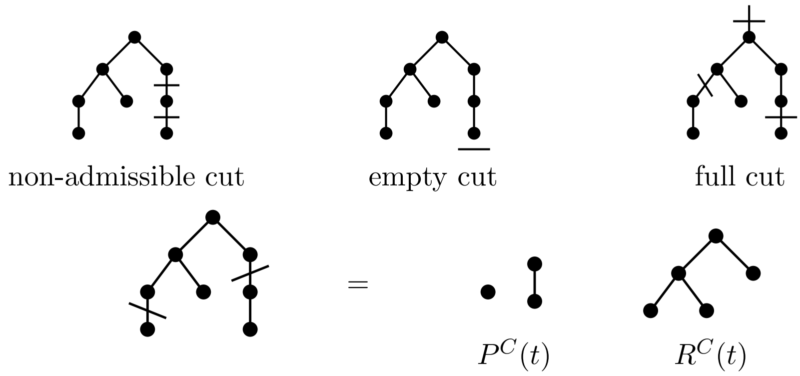


Figure 3.2: Examples for the different sorts of cuts at a tree. Once we apply these cuts to a tree, we obtain pieces of it that are *pruned*, building the set $P^C(t)$, and exactly one part that contains the root, $R^C(t)$.

So far we have an algebra of rooted trees. To obtain a bialgebra, we need to define a counit and a coproduct. The counit ε: $H_R \to \mathbb{Q}$ is defined as:

$$\varepsilon(t) = 0, \quad \forall t \neq e; \qquad \varepsilon(e) = 1. \tag{3.1}$$

The counit maps all trees except the unit $e \in H_R$ to zero. The coproduct Δ is defined by the sum of all admissible cuts:

$$\Delta(e) = e \otimes e \tag{3.2}$$

$$\Delta(t) = e \otimes t + t \otimes e + \sum_{\substack{\textit{admissible} \\ \textit{cuts } C}} P^C(t) \otimes R^C(t). \tag{3.3}$$

Note that the coproduct is non-cocommutative, which may already be seen from the fact that $P^C(t)$ can be a polynomial in trees, while $R^C(t)$ is always one single tree.

Having defined a bialgebra, we need an antipode S to obtain a Hopf algebra. This antipode is defined to be:

$$S(e) = e \tag{3.4}$$

$$S(t) = -t - \sum_{\substack{\textit{admissible} \\ \textit{cuts } C}} S[P^C(t)] R^C(t) \tag{3.5}$$

$$= \sum_{\substack{\textit{all full} \\ \textit{cuts } C}} (-1)^{n_c} P^C(t) R^C(t) \tag{3.6}$$

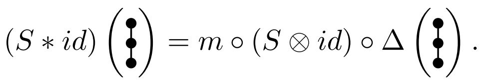
where n_c is the number of elementary cuts at a tree. In the sum over "all full cuts C" we do not restrict ourselves to admissible cuts. Note that one obtains the same trees $P^C(t)$ and $R^C(t)$ by the normal and full cuts. The full cuts mainly serve to provide an additional minus sign in the definition of the antipode (3.6).

In order to renormalize graphs, we need the *convolution product* $*$ of rooted trees and the combination $(S*id)(t) = m\circ(S\otimes id)\circ\Delta(t)$, where m is the multiplication, the disjoint union defined before. Constructing the convolution product with all these definitions, we obtain:

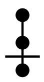
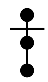
$$\begin{aligned}(S*id)(t) &= m\circ(S\otimes id)\circ\Delta(t)\\ &= t+S(t)+\sum_{\substack{admissible\\ cuts\ C}} S[P^C(t)]R^C(t)\\ &= \varepsilon(t)\\ &= 0, \qquad \forall t\neq e.\end{aligned} \tag{3.7}$$

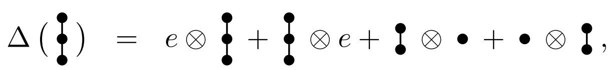
The sum could equally be defined over "all cuts", all normal and full cuts, without restriction to admissible cuts [CoKr 1998]. The result of $(S*id)(t)$ is equal to zero because we can find for each normal cut a corresponding full cut with a relative minus sign. This simply results from the fact that one obtains a full cut from a normal cut by setting one more cut at the incoming line to the root. It therefore shows us that S really is the inverse of id with respect to $*$ and hence the definition for the antipode was correct.

Let us calculate a short example:



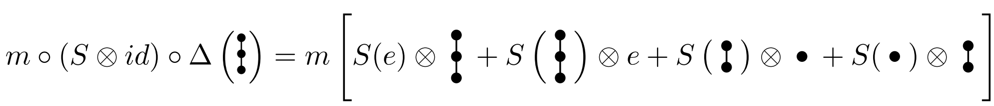
(3.8)

The admissible cuts are  and . According to (3.3) we get:



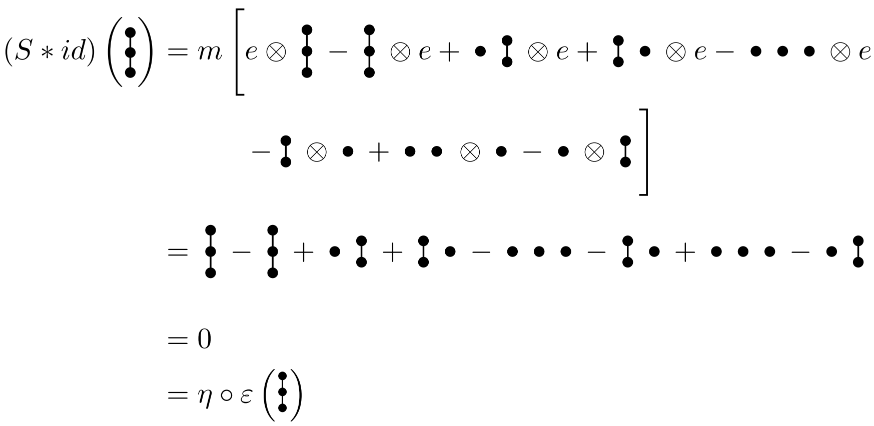
(3.9)

and hence



(3.10)

Calculating the antipode for the trees, we obtain:



(3.11)

This simple example shows again that the structure maps [Kass 1995] were defined correctly and that they indeed define a Hopf algebra on rooted trees.

3.1.2 The Hopf algebra of Feynman graphs H_{FG}

In the previous Section 3.1.1 we saw how one can assign a tree to a Feynman graph by identifying the "1PI primitive" building blocks of a graph, drawing a box around each, and connecting the boxes in such a way that the tree encodes their nested structure. Hence we have the correspondence between trees and 1PI graphs and like the *generators* of H_R are given by single trees, so the corresponding *generators* in this Hopf algebra of Feynman graphs are *1PI graphs*. Their *multiplication* is also given by *disjoint union* in this case.

Let us take a 1PI graph that is overall divergent. Denote the full graph with Γ and subdivergences with γ_i. The *coproduct* is then given by

$$\Delta\Gamma = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma_i \subsetneq \Gamma} \gamma_i \otimes \Gamma/\gamma_i, \tag{3.12}$$

where the sum runs over all subdivergences γ_i. Note that γ_i can be several graphs. Γ/γ_i is the graph Γ in which the subdivergence γ_i has been shrunk to a point. It is called the *cograph* to γ_i.

The antipode in this Hopf algebra is defined as

$$S(\Gamma) = -\Gamma - \sum_{\gamma_i \subset \Gamma} S(\gamma_i)\Gamma/\gamma_i. \tag{3.13}$$

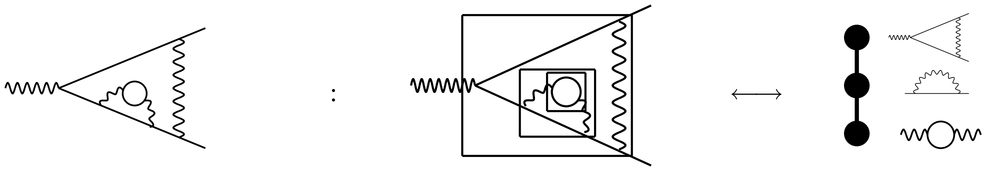
Note the similarity of (3.12) with (3.3) and of (3.13) with (3.5).

Like the grading on H_R given by the number of vertices, there is a natural grading on Feynman graphs given by the loop number of a 1PI graph.

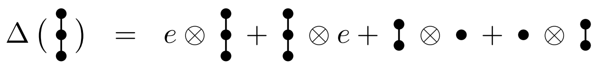
3.1.3 Correspondence between H_R, H_{FG}, and ZFF

We have now defined the Hopf algebras on rooted trees and on Feynman graphs. Although we have shown correspondences between the two, H_R and H_{FG} are not isomorphic as Hopf algebras: While one can uniquely assign to each graph a (decorated) rooted tree, any (decorated) tree is assigned a whole set of Feynman graphs with the same divergence structure. But since the combinatoric of renormalization only depends on the divergence structure of a graph and not on the actual types of divergences, graphs with the same divergence structure have to be renormalized by an application of the structure maps in the same combinatorial way. Hence structure maps applied to a tree provide the correct structure maps on graphs whose divergences occur in such a way that they can be represented by this tree according to the rule given on page 16. For any graph we start with, the two Hopf algebras with their coproducts and antipodes will provide the same combinatorics. Hence we can and will switch between the two representations to express the renormalization of graphs, always using the one that seems to best fit our problem.

So far we have given a combinatorial structure that is encoded in the Hopf algebra of (undecorated) rooted trees and the Hopf algebra of Feynman graphs. The Feynman rules are maps from these Hopf algebras, the Hopf algebra H_R or equivalently the Hopf algebra H_{FG}, into a suitable target space, for example the ring of Laurent series in ε, which we will always use as target space in the following. Let us consider a three-loop graph:

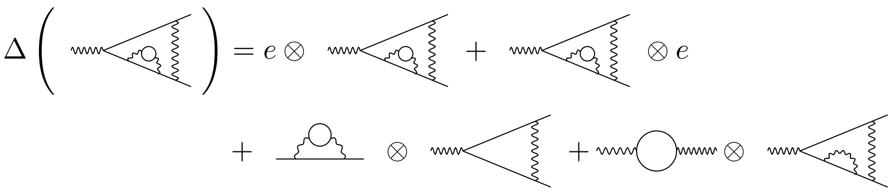


In (3.9) we had for the coproduct of this (undecorated) tree:



(3.14)

Using (3.12) or decorating (3.14) with the corresponding divergences of the graph we obtain:



(3.15)

The Feynman rules now provide a character, an algebra homomorphism, $\Phi : H \to V$ from the Hopf algebra to an analytic expression, an element of the ring of Laurent polynomials in a regularization parameter ε in this case. In our example this Laurent series diverges for $\varepsilon \to 0$. Φ defined in this way is called a *bare character*. Application of the renormalization map R, *e.g.* $R \equiv R_{MS}$ for the MS scheme, means that after mapping into a target space via Φ, one projects this Laurent series onto its pole part:

$$R_{MS}\left(\sum_{i=-n}^{\infty} \frac{c_i}{\varepsilon^i}\right) = \sum_{i=-n}^{-1} \frac{c_i}{\varepsilon^i}, \qquad n \in \mathbb{N}. \tag{3.16}$$

Hence R is a map from $V \to V$.

Define now a new character S_R^Φ. This character provides the counterterm for a graph Γ in the renormalization scheme R and is defined by (cf. for example [Krei 2000a]):

$$S_R^\Phi(\Gamma) = -R[\Phi(\Gamma)] - R\left[\sum_{\gamma\subset\Gamma} S_R^\Phi(\gamma)\Phi(\Gamma/\gamma)\right]. \tag{3.17}$$

To ensure that S_R^Φ really is a character from H to V, *i.e.* that:

$$S_R^\Phi(t_i t_j) = S_R^\Phi(t_i) S_R^\Phi(t_j)\,, \qquad \forall\, t_i, t_j \in H, \tag{3.18}$$

the renormalization map $R : V \to V$ has to fulfill the so-called *multiplicativity constraint* [Krei 2000a]:

$$R(xy) + R(x)R(y) = R(R(x)y) + R(xR(y))\,, \qquad \forall\, x, y \in V. \tag{3.19}$$

In the case of the Laurent series in ε, for instance, it is clear that one can not have the identity $R(xy) = R(x)R(y)$. The pole parts of a product of two series are not the product of the pole terms of the individual series. We only want to mention that (3.19) is the defining equation for a *Baxter algebra*. Hence the target space not only has to be an algebra but a Baxter algebra, and the renormalization map R is a *Rota-Baxter map* [Krei 2000a].

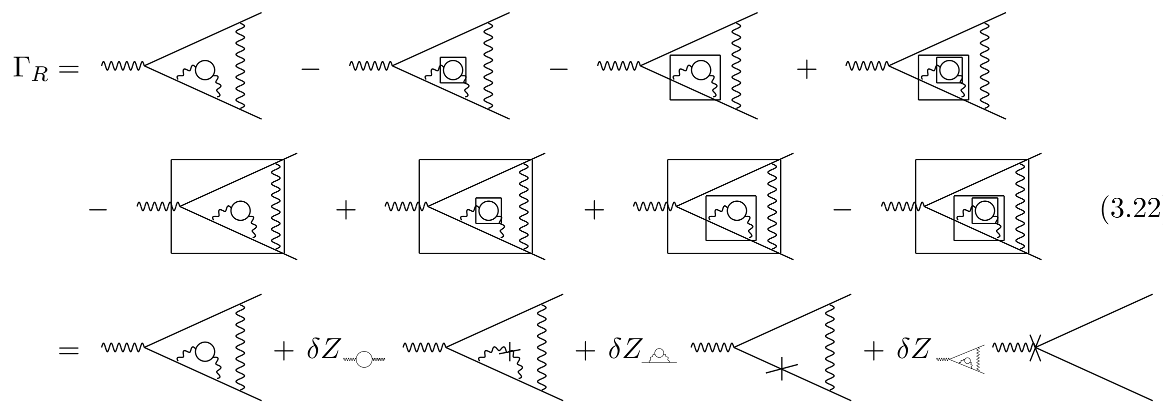
We can now obtain a renormalized graph Γ by applying to it not the *bare character* Φ (*e.g.* assigning to it the corresponding Laurent series) but the *renormalized character* $S_R^\Phi * \Phi(\Gamma)$. The $\bar{R}$ operation (cf. page 12) here is given by:

$$\bar{R}(\Gamma) = \Phi(\Gamma) + \sum_{\gamma\subset\Gamma} S_R^\Phi(\gamma)\Phi(\Gamma/\gamma)\,, \tag{3.20}$$

and we have

$$\Gamma_R = S_R^\Phi * \Phi(\Gamma) = \bar{R}(\Gamma) + S_R^\Phi(\Gamma)\,. \tag{3.21}$$

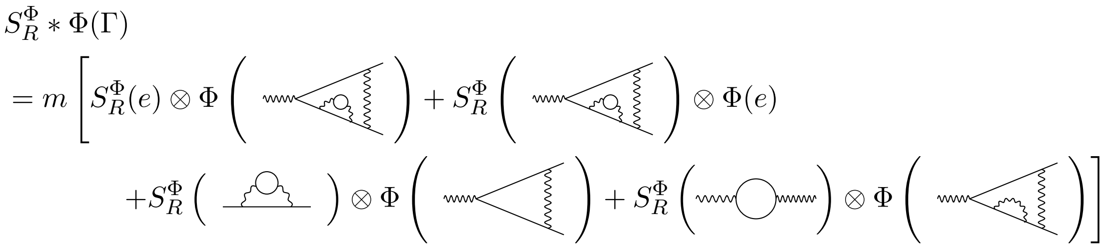
Equation (3.21) yields the same terms as Zimmermann's forest formula. Let us show this by using the three-loop graph that we already introduced at the beginning of this section. We start with Zimmermann's forest formula: Let γ_1 be the boson self-energy, the "innermost primitive" divergence, γ_2 the fermion-self energy that has γ_1 as a subdivergence, and finally Γ the overall divergence. Applying the forest formula we obtain: normal forests: $\emptyset$, $\{\gamma_1\}$, $\{\gamma_2\}$, $\{\gamma_1, \gamma_2\}$; full forests: $\{\Gamma\}$, $\{\gamma_1, \Gamma\}$, $\{\gamma_2, \Gamma\}$, $\{\gamma_1, \gamma_2, \Gamma\}$ and hence:



(3.22)

where the δZ are the counterterms for the overall divergence of their index graph, sitting in the marked line: δZ is given by the second, δZ by the third and fourth and δZ by the four graphs in the last line. Note that the cross that marks a line only states the place where the renormalized subdivergence was sitting and not a counterterm itself, like defined before.

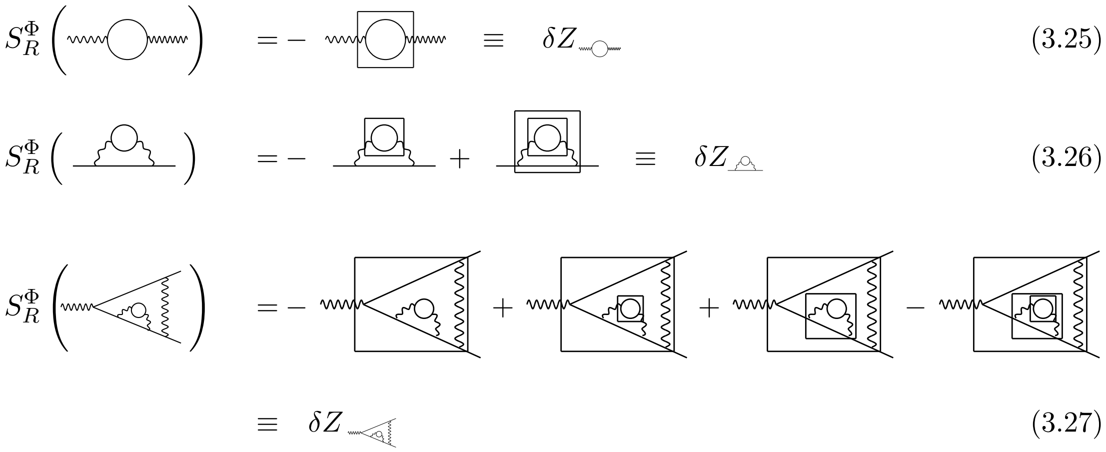
Let now Φ again be the bare map which maps a graph into the ring of Laurent series in ε. Let us then calculate $S_R^\Phi(\Gamma)$. Applying first the coproduct, we obtain:



(3.23)

Using (3.17) we further get:

$$S_R^\Phi(e) \quad = \quad e \tag{3.24}$$



(3.25)

(3.26)

(3.27)

The multiplication map m then simply multiplies $m[S_R^\Phi(x) \otimes \Phi(y)] = S_R^\Phi(x)\Phi(y)$ in the space of Laurent series. We can see here that the character S_R^Φ provides the counterterm for the overall divergence of the graph to which it is applied and hence δZ for the different divergences of a graph, the subdivergences and the overall divergence. These δZ are the contributions to the Z-factor. Inserting these four lines (3.24) – (3.27) back into (3.23) and multiplying the different terms we exactly obtain (3.22).

We have described that the application of the structure maps to trees and to Feynman graphs provide the same terms. Therefore, although the programs of Chapter 8 work on trees and not on graphs, they will produce the terms for the overall divergence of a graph Γ and hence in our example the graphs of $S_R^\Phi(\Gamma)$ given in equation (3.27) for this input graph.

3.2 Lie algebras

Dually to the commutative Hopf algebras H_R and H_{FG} we find for each a cocommutative Hopf algebra. The *Milnor-Moore theorem* [MiMo 1965, CoKr 2000] then states that this latter Hopf algebra is isomorphic to a *universal enveloping algebra* $U(L)$ of a Lie algebra L. We will only define the Lie algebra of Feynman diagrams $\mathcal{L}_{FG}$ here, introducing the *gluing* operation. This operation encodes the insertion of graphs into each other, which is in some sense dual to the splitting of graphs into pieces provided by the coproduct. We will meet this gluing operation again in Chapters 6 to 8. However, we will not make use of the Lie algebra as such in the following. Therefore we will only introduce it briefly here and omit any further representations of this Lie algebra although much more could be said [MeKr 2004a, MeKr 2004b, EFGK 2004a, EFGK 2004b, EFGK 2004c].

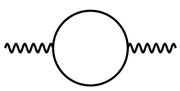
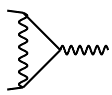
We start by defining the *gluing operation* $*_i$ that maps 1PI graphs to 1PI graphs. One has to provide *gluing data* G_i that indicate where one wants to insert the graph Γ_1 into another graph Γ_2 and, if necessary, in which way (that is which bijection to use). This is the place where we need the definition of *compatible* lines and vertices. For example, take Γ_1 to be a vertex correction graph. If the external lines of Γ_1 are compatible with the type of a vertex v_i in Γ_2 (cf. Chapter 2), $f_{v_i} \sim \Gamma^{[1]}_{1,ext}$, we define[1]

$$\Gamma_2 *_{v_i} \Gamma_1 = \Gamma_2/v_i \cup \Gamma_1/\Gamma^{[1]}_{1,ext}, \tag{3.28}$$

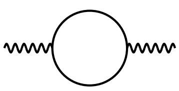
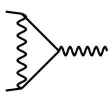
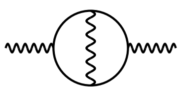
which simply means that we identify the vertex v_i in Γ_2 with Γ_1 and then identify the external lines of Γ_1 with f_{v_i}. Afterwards we sum over all possible bijections between f_{v_i} and $\Gamma^{[1]}_{1,ext}$ and normalize such that topologically different graphs occur only once. This operation can easily be generalized to self-energies and edges. The place where to insert and the bijection build the gluing data. Note, for example, that each term of a coproduct $\tilde{\Delta}(X) = \Delta(X) - 1 \otimes X - X \otimes 1 = \sum_i x'_i \otimes x''_i$ provides unique gluing data, how to insert x' in x'' to obtain x: we know where the subdivergence x' was sitting in x, leading to x''.

As a generalization of the gluing operations, the operation $*$ (without index) is defined such that for two Feynman diagrams Γ_1 and Γ_2, $\Gamma_2 * \Gamma_1$ means that one sums over all possible places for inserting Γ_1 into Γ_2:

$$\Gamma_2 * \Gamma_1 = \sum_{\substack{v \in \Gamma_2^{[0]}, \\ f_v \sim \Gamma^{[1]}_{1,ext}}} \Gamma_2 *_v \Gamma_1 \tag{3.29}$$

For example consider the two graphs:  and 

There are two vertices and therefore two places where one can insert the vertex correction into the vacuum polarization, leading to:

 $*$  $=$ 2 

[1]The upper index [1] at $\Gamma^{[1]}_{1,ext}$ here indicates that we are talking about a set of edges, cf. [Krei 2003a]. An index [0] denotes a set of vertices.

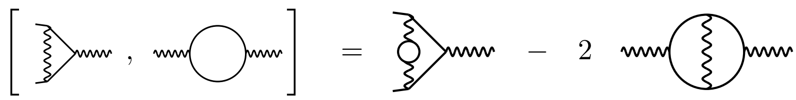
This new composition $\Gamma_2 * \Gamma_1$ is *not associative*, but it fulfills the following relation:

$$\Gamma_3 * (\Gamma_2 * \Gamma_1) - (\Gamma_3 * \Gamma_2) * \Gamma_1 = \Gamma_3 * (\Gamma_1 * \Gamma_2) - (\Gamma_3 * \Gamma_1) * \Gamma_2 \tag{3.30}$$

This is the defining equation for a *pre-Lie algebra structure*. Note that for each side being equal to zero, one would get back the associativity law. Antisymmetrizing this automatically provides a Lie bracket

$$[\Gamma_1, \Gamma_2] = \Gamma_1 * \Gamma_2 - \Gamma_2 * \Gamma_1 \tag{3.31}$$

which fulfills the Jacobi identity. The Lie-bracket for our example would now give:



There are two possible ways to insert the vertex correction into the vacuum polarization at each of the vertices, but only one place to insert the vacuum polarization into the vertex correction.

The only operation which we will use in the following is the gluing operation of inserting singular graphs into each other. We will not make use of the more complex Lie algebra structure of Feynman diagrams, which is discussed in the above references.

Chapter 4

The massless two-loop two-point function — an overview

The next three chapters are dedicated to the expansion of the massless two-loop two-point function. We start in this chapter with a short overview of previous work that has been done to expand this function beginning in 1980. In this context we will first emphasize the *triangle relation* as an *integration-by-parts identity.* This relation provides a standard way to manipulate Feynman diagrams: one can either use it to turn (some of) the Feynman diagrams into other diagrams that are easier to expand, or at least reduce the problem to a set of basic master diagrams, whose expansion is needed. The triangle relation is in general applicable to cases where all exponents of the propagators are positive integers. Unfortunately, the integral can no longer be expanded in ε by the use of standard methods once we allow for arbitrary non-integer exponents ν_i. In this chapter we will describe some ways in which different authors nevertheless succeeded in expanding this integral up to a certain order in ε.

4.1 General remarks

The expansion of the massless two-loop two-point function will be carried out in dimensional regularization, with $D = 4 - 2\varepsilon$. The topology corresponding to this function is drawn in Fig. 4.1.

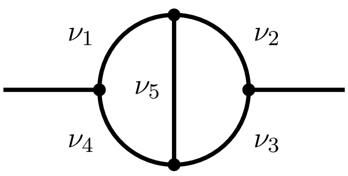


Figure 4.1: The master topology for the two-loop two-point function.

The *indices* ν_i, written next to the lines in Fig. 4.1 are the exponents of the inverse momenta squared that are attached to the corresponding lines in the graph. Analytically, this graph is then given in Minkowski space by

$$\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$$
$$= (-p^2)^{\nu_{12345}-2m+2\varepsilon}\int\frac{d^Dk_1}{i\pi^{\frac{D}{2}}}\int\frac{d^Dk_2}{i\pi^{\frac{D}{2}}}\frac{1}{(-k_1^2)^{\nu_1}(-k_2^2)^{\nu_2}(-k_3^2)^{\nu_3}(-k_4^2)^{\nu_4}(-k_5^2)^{\nu_5}} \tag{4.1}$$

with

$$\nu_{ijk} = \nu_i+\nu_j+\nu_k, \tag{4.2}$$
$$k_3 = k_2-p, \quad k_4 = k_1-p, \quad k_5 = k_2-k_1, \tag{4.3}$$
$$D = 2m-2\varepsilon. \tag{4.4}$$

Note that m is simply a number here and should not be confused with a mass term. The minus signs for the momenta and the factors $\frac{1}{i\pi^{D/2}}$ and $(-p^2)^{\nu_{12345}-2m+2\varepsilon}$ are due to a notation which is used in [BiWe 2003] (cf. Chapter 6) and was introduced for convenience. The factor $(-p^2)^{\nu_{12345}-2m+2\varepsilon}$, for example, makes the integral dimensionless. The indices ν_i are of the general form $\nu_i = n_i + a_i\varepsilon$. This form originates from the fact that subdivergences inside a graph increase the exponents of momenta by terms proportional to $j\varepsilon$, $j\in\mathbb{N}$. We will see this more explicitly in Chapter 7.

The underlying topology for this two-loop diagram of Fig. 4.1 is the master topology for the two-loop two-point case. All other two-loop diagrams can be obtained from this diagram by shrinking one of the internal lines to a point or, equivalently, setting one of the indices ν_i to zero.

The result of the massless one-loop two-point function of Fig. 4.2 in Euclidean space was already given in Chapter 2 and its calculations can be found in Appendix C. For indices ν_1 and ν_2 the result is simply:

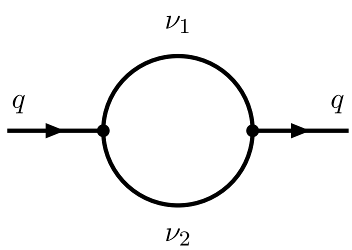


Figure 4.2: The master topology for the one-loop two-point function.

$$\hat{I}^{(1,2)}(m-\varepsilon,\nu_1,\nu_2) = \int\frac{d^Dk}{i\pi^{D/2}}\frac{1}{[k^2]^{\nu_1}[(q-k)^2]^{\nu_2}} =: [q^2]^{(m-\varepsilon-\nu_1-\nu_2)}F_{\nu_1,\nu_2}(\varepsilon) \tag{4.5}$$

with

$$F_{\nu_1,\nu_2}(\varepsilon) = \frac{\Gamma(\nu_{12}-m+\varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)}\frac{\Gamma(m-\varepsilon-\nu_1)\Gamma(m-\varepsilon-\nu_2)}{\Gamma(2m-2\varepsilon-\nu_{12})}. \tag{4.6}$$

The momentum q is the external momentum running through the diagram. We have met this integral already in (2.14). From (4.6) we can see that these integrals can be expressed by a fraction consisting solely of Euler's gamma function. As we have seen in (2.12) these gamma functions can be expanded by the formula

$$\Gamma(1-\varepsilon) = \exp(\gamma\varepsilon)\,\exp\left(\sum_{n=2}^{\infty}\frac{\zeta(n)}{n}(\varepsilon)^n\right), \qquad |\varepsilon| < 1 \tag{4.7}$$

into a Laurent series in ε. The function $\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k}$ in (4.7) is *Riemann's zeta function.* The term $[q^2]^{(m-\varepsilon-\nu_1-\nu_2)}$ can be expanded as in (2.16) by expressing it in the form of an exponential function:

$$[q^2]^x = \exp(\ln([q^2]^x)) = \exp(x\ \ln[q^2]) = 1 + x\ \ln[q^2] + O(x^2). \tag{4.8}$$

A non-integer value for ν_i in this case, with $\nu_i = n_i + a_i\varepsilon$, $n_i, a_i \in \mathbb{Z}$, only changes the argument of a gamma function. Hence it becomes immediately obvious that in the ε expansion of this one-loop integral only Riemann's zeta function appears.

One of the major questions now is whether this is also true for the expansion of the function $\hat{I}^{(2,5)}$, including non-integer exponents.

4.2 Integer exponents ν_i — the triangle relation

The integral $\hat{I}^{(2,5)}$ with $\nu_i \in \mathbb{N}$, $\forall i$, can be expressed in terms of the one-loop functions $\hat{I}^{(1,2)}$ using the *triangle relation.* This relation can be obtained by *integration by parts* [ChTk 1981, DaBo 1990, Davy 1991]. The integration-by-parts identities are based on the translational invariance of the D-dimensional integral (2.10) written in the form:

$$0 = \int d^Dk\,\frac{\partial}{\partial k}(\text{integrand}). \tag{4.9}$$

Applying $\frac{\partial}{\partial k}$ to a general propagator or to a D-dimensional vector k^μ leads to the relations:

$$\begin{aligned}\frac{\partial}{\partial k^\mu}k^\mu &= D\\ \frac{\partial}{\partial k^\mu}\frac{1}{(k+x)^{2\alpha}} &= -2\alpha\frac{(k+x)_\mu}{(k+x)^{\alpha+1}}\\ \frac{\partial}{\partial k^\mu}\frac{1}{(x-k)^{2\alpha}} &= 2\alpha\frac{(x-k)_\mu}{(x-k)^{\alpha+1}}\end{aligned}$$

Note that by doing this, we obtain a vector expression for a derivative of a propagator. Hence we apply $\frac{\partial}{\partial k}\cdot k$ in the integrand instead of $\frac{\partial}{\partial k}$ alone:

$$0 = \int d^Dk\frac{\partial}{\partial k^\mu}\frac{k^\mu}{Denom.}. \tag{4.10}$$

In this way we produce scalar products of momenta in the numerator that themselves can be expressed by linear combinations of the propagators in the denominator of the integrand.

Let us apply $\frac{\partial}{\partial k_{2\,\mu}}(k_2-k_1)^\mu$ to our integral (4.1) to give an example:

$$0 = \int d^Dk_1d^Dk_2\frac{\partial}{\partial k_{2\,\mu}}\frac{(k_2-k_1)^\mu}{[-k_1^2]^{\nu_1}[-k_2^2]^{\nu_2}[-(k_2-p)^2]^{\nu_3}[-(k_1-p)^2]^{\nu_4}[-(k_2-k_1)^2]^{\nu_5}}$$

The derivative acts on each term in the numerator and denominator of the integrand that depends on k_2. Hence we get

$$\begin{aligned}&\int d^Dk_1 d^Dk_2 \frac{\partial}{\partial k_{2\,\mu}} \frac{(k_2-k_1)^\mu}{(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)}\\ &= D\int d^Dk_1 d^Dk_2 \frac{1}{(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)} + 2\nu_2 \int d^Dk_1 d^Dk_2 \frac{k_{2\,\mu}(k_2-k_1)^\mu}{(\nu_1,\nu_2+1,\nu_3,\nu_4,\nu_5)}\\ &+ 2\nu_3 \int d^Dk_1 d^Dk_2 \frac{(k_2-p)_\mu (k_2-k_1)^\mu}{(\nu_1,\nu_2,\nu_3+1,\nu_4,\nu_5)} + 2\nu_5 \int d^Dk_1 d^Dk_2 \frac{(k_2-k_1)_\mu (k_2-k_1)^\mu}{(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5+1)} \qquad (4.11)\end{aligned}$$

where we abbreviated the denominator by giving only its exponents $(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$.

Using the scalar product identities:

$$k_2 \cdot k_1 = \frac{1}{2}\left[-(k_2-k_1)^2 + k_2^2 + k_1^2\right] \qquad (4.12)$$

$$p \cdot k_i = \frac{1}{2}\left[-(p-k_i)^2 + p^2 + k_i^2\right] \qquad (4.13)$$

to cancel expressions in the numerator with propagators in the denominator, we obtain the following relation:

$$0 = \left[(D-2\nu_5-\nu_1-\nu_2) - \nu_2 \mathbf{2}^+(\mathbf{5}^- - \mathbf{1}^-) - \nu_3 \mathbf{3}^+(\mathbf{5}^- - \mathbf{4}^-)\right] \hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) \qquad (4.14)$$

or, solving for $\hat{I}^{(2,5)}$:

$$\begin{aligned}&\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)\\ &= \frac{1}{(D-2\nu_5-\nu_1-\nu_2)} \left[\nu_2 \mathbf{2}^+(\mathbf{5}^- - \mathbf{1}^-) + \nu_3 \mathbf{3}^+(\mathbf{5}^- - \mathbf{4}^-)\right] \hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5). \qquad (4.15)\end{aligned}$$

The operator $\mathbf{i}^\pm$ acts on the integral and increases/decreases the exponent of the propagator i by 1, for example:

$$\mathbf{1}^+ \hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) = \hat{I}^{(2,5)}(m-\varepsilon,\nu_1+1,\nu_2,\nu_3,\nu_4,\nu_5). \qquad (4.16)$$

This is one form of the *triangle relation.* The name for this relation becomes clear when we express (4.14) graphically. The operators that lower the exponents of propagators all act on lines that form a triangle inside the graph. In this case it is “the left” half:

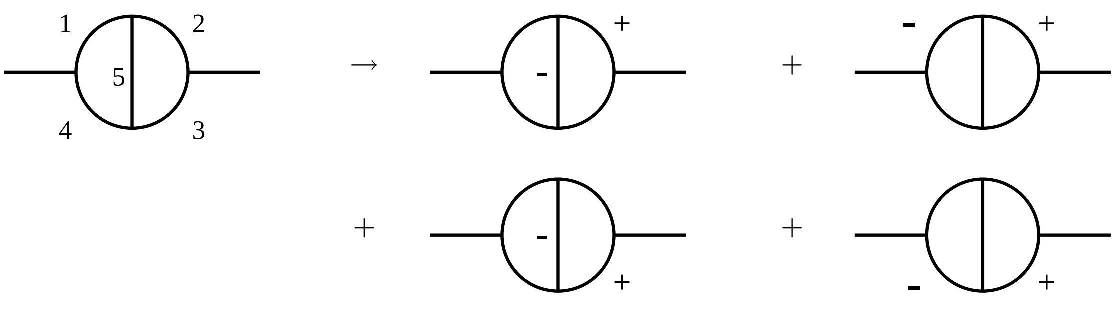


Figure 4.3: The form (4.14) of the triangle relation applied to the two-loop two-point master topology. The + and − signs indicate that the exponent of the momentum corresponding to a line has been increased/decreased by one.

The operators $\mathbf{2}^+$ and $\mathbf{3}^+$ increase the exponent of the propagator attached to lines 2 and 3 by one and $\mathbf{1}^-$, $\mathbf{4}^-$, and $\mathbf{5}^-$ decrease the power of the propagator of the lines 1, 4 and 5 by one.

Due to the symmetry of the graph there are three similar relations for the other lines. Additionally, one obtains relations of this form by taking the derivative with respect to another momentum (other than k_2 in this case) and with other choices of momenta in the numerator. In this way one obtains a whole family of relations between the integral $\hat{I}^{(2,5)}$ and other integrals, where the exponents of the propagators were increased or decreased by one.

Applying (4.14) to $\hat{I}^{(2,5)}(m-\varepsilon,1,1,1,1,1)$ and shrinking lines with index zero to a point, we get for the four topologies on the right hand side of Fig. 4.3, omitting the plus signs:

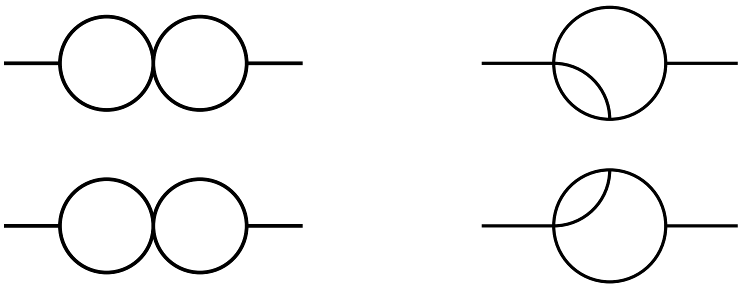


Figure 4.4: The form (4.14) of the triangle relation applied to $\hat{I}^{(2,5)}(m-\varepsilon,1,1,1,1,1)$. Lines with index zero have been shrunk to a point and + signs omitted.

The corresponding integrals read (ordered from left to right in the first and second line):

$$I_1 = \int d^Dk_1 d^Dk_2 \frac{1}{[-k_1^2][-k_2^2]^2[-(k_2-p)^2][-(k_1-p)^2]} \tag{4.17}$$

$$I_2 = \int d^Dk_1 d^Dk_2 \frac{1}{[-k_2^2]^2[-(k_2-p)^2][-(k_1-p)^2][-(k_2-k_1)^2]} \tag{4.18}$$

$$I_3 = \int d^Dk_1 d^Dk_2 \frac{1}{[-k_1^2][-k_2^2][-(k_2-p)^2]^2[-(k_1-p)^2]} \tag{4.19}$$

$$I_4 = \int d^Dk_1 d^Dk_2 \frac{1}{[-k_1^2][-k_2^2][-(k_2-p)^2]^2[-(k_2-k_1)^2]} \tag{4.20}$$

As the topologies of the graphs 1 and 3 already suggest, the corresponding integrals I_1 and I_3 split into two disjoint one-loop integrals of the form (4.5). This leads to a result of the form $F_{1,1}(\varepsilon)^2[q^2]^{-2\varepsilon}$. In the case of the integrals I_2 and I_4, which are just mirror images of each other, we can first do the integration with respect to k_1 corresponding to the one-loop subdivergence and afterwards with respect to k_2. These are again two one-loop integrations with a result $F_{1,1}F_{1,1+\varepsilon}[q^2]^{-2\varepsilon}$ and we get in total the well-known result $\hat{I}^{(2,5)}(m-\varepsilon,1,1,1,1,1) = 6\zeta(3) + O(\varepsilon)$. A more explicit explanation for the fact that the integrals, especially I_2 and I_4 result in the F-functions with indices of the form stated above will be given in Chapter 7.

In case that one of the indices of the two-loop master topology becomes negative, the graph reduces again either to the disjoint or nested integration of one-loop diagrams of Fig. 4.4, or to zero, as one can easily convince oneself.

For general ν_i, with ν_i not necessarily being integers, the triangle relation provides a relation between graphs with different exponents of the propagators. One can use these relations to reduce graphs to a set of "basic" graphs, which then have to be expanded in ε.

Looking at (4.14) or its graphical representation Fig. 4.3 more closely, one can see that it suffices that the exponents $\nu_i \in \mathbb{N}^0$ for $i \in \{1,4,5\}$ in order to reduce the integral to integrals like the ones in Fig. 4.4. A repeated application of relation (4.14) to the integral $\hat{I}^{(2,5)}(m-\varepsilon, n_1, \nu_2, \nu_3, n_4, n_5)$, $n_i \in \mathbb{N}^0, \nu_j \in \mathbb{C}$, would eventually lead to integrals in which one of these n_i is equal to zero, as we have just seen. The resulting integrals can then always be solved using the one-loop result (4.5).

Similar considerations for the other cases of the triangle relation lead to the conclusion that the triangle relation is of particular use whenever we have $\hat{I}^{(2,5)}$ with three integer coefficients at three adjacent lines. Nevertheless, already for $\hat{I}^{(2,5)}(m-\varepsilon, 1, 1, 1, 1, \nu_5)$, $\nu_5 = n_5 + a_5\varepsilon$, with $n_5 \neq 0, a_5 \neq 0$, the integration-by-parts identities are not sufficient anymore to expand the integral. In this case one cannot express the integral purely by functions F_{ν_1,ν_2}. Note again that the one-loop graphs evaluate into a fraction of gamma functions. The fact that $\hat{I}^{(2,5)}(m-\varepsilon, 1, 1, 1, 1, \nu_5)$ is not expressible by these functions is already a hint that a fraction of gamma functions is in general not sufficient.

4.3 Non-integer exponents ν_i

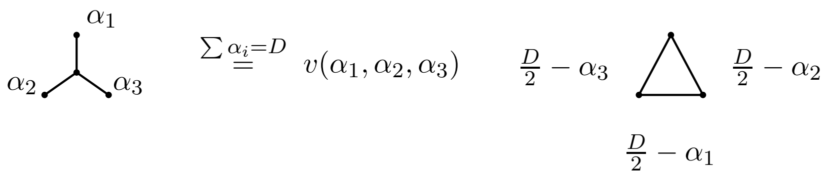
Over the years, several attempts have been undertaken to expand the massless two-loop two-point function for different sets of indices. The investigations into this integral and its Laurent series expansion for non-integer indices started, to the best of our knowledge, in 1980 with a paper of *Chetyrkin*, *Kataev* and *Tkachov* [CKT 1980], applying the *Gegenbauer polynomial x-space technique* to $\hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$. It was followed by the paper [ChTk 1981] where the authors introduced the *integration-by-parts technique* (IBP) which we just described, and combined it with the Gegenbauer polynomial technique. In this way the authors were able to express $\hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ for

a) ν_i integers, $\forall i$

b) $\nu_1 = n + \varepsilon$, and $\nu_2, ..., \nu_5, n$ integers

in the form of a fraction of gamma functions. They also noticed that this is no longer possible once there is a subdivergence sitting in the "middle line" of this graph, which carries the number 5 and index ν_5 in our way of assigning these numbers to the lines, given in Fig. 4.1.

In 1983, D.I.Kazakov [Kaza 1983] calculated an expansion of $\hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ in coordinate space for all indices of the form $\nu_i = 1 + a_i\varepsilon$. He succeeded to expand this integral up to the order ε^3, using the *uniqueness relation.* In its "pure form" this is an identity between a three line vertex and a triangle, stating that one can substitute a three-point vertex, where all the indices of the adjacent lines sum up to D, by a triangle whose indices of the constituent lines sum up to $D/2$, times some product of gamma functions, cf. Fig. 4.5. Starting from this relation, one obtains a set of rules that yield relations between vertices, lines and triangles similar to those obtained by integration-by-parts-identities.



with

$$v(\alpha_1, \alpha_2, \alpha_3) = \pi^{D/2} \prod_{i=1}^{3} \frac{\Gamma(\frac{D}{2} - \alpha_i)}{\Gamma(\alpha_i)}, \quad \alpha_3 = D - \alpha_1 - \alpha_2$$

Figure 4.5: The uniqueness relation. It provides an identity between a three line vertex and a triangle in coordinate space, stating that one can substitute a three-point vertex, where all the indices of the adjacent lines sum up to D, by a triangle, whose indices of the constituent lines sum up to $D/2$, times some product of gamma functions.

In the following two years, Kazakov was able to extend the calculation to the next order in ε, ε^4 [Kaza 1984, Kaza 1985]. In the process of this calculation, he derived a functional equation which is fulfilled by the analytic result of a diagram with indices $\nu_i = 1, i = 1, ..., 4$, ν_5 non-integer, which already indicated that this result might be related to hypergeometric functions of the type ${}_3F_2$ (see Appendix B).

A next major step in the exploration of the massless two-loop two-point function was done by taking into account the symmetry of this diagram. The graph of this function is obviously symmetric under a reflection at the inner vertical line or a reflection with respect to the external lines. This leads to an invariance of the integral under the following two permutations of exponents:

$$(\nu_1, \nu_2, \nu_3, \nu_4) \to (\nu_2, \nu_1, \nu_4, \nu_3) \quad (4.21)$$

$$(\nu_1, \nu_2, \nu_3, \nu_4) \to (\nu_4, \nu_3, \nu_2, \nu_1). \quad (4.22)$$

But there are even more symmetry relations: In [Broa 1986, BaBr 1988] D.J. Broadhurst and D.T. Barfoot exploited the $Z_2 \times S_6$ symmetry group [Broa 1986, BaBr 1988, GoIs 1985], which is of order 1440 and possesses three generators. To be able to investigate this symmetry the authors "closed" the external lines of the diagram, turning it into a three-loop vacuum diagram (cf. Fig. 4.6) and searched for the possible symmetry relations and the transformations that accomplish them.

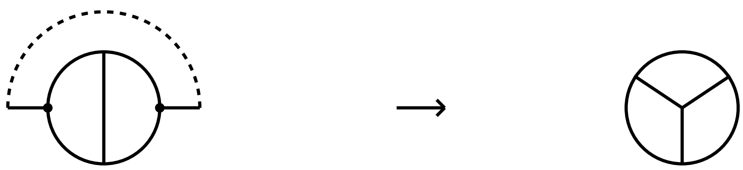


Figure 4.6: In [Broa 1986, BaBr 1988] D.J. Broadhurst and D.T. Barfoot closed the external lines of the two-loop two-point function and obtained a vacuum bubble diagram. They were then able to expand the corresponding function in ε, using symmetry relations for this vacuum diagram.

In this way the integral could first be expanded for indices $\nu_i = 1 + a_i\varepsilon$ up to order ε^5 [Broa 1986] and two years later up to ε^6 [Broa 1986, BaBr 1988]. To carry out the transformations analytically, Barfoot and Broadhurst defined a function $f(\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$, related to the function $\hat{I}^{(2,5)}$, which in our notation is given by:

$$\hat{I}^{(2,5)}(\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5) = \frac{1}{(D-3)\Gamma(D/2-1)^2} \left[\prod_{j=1}^{10} \frac{\Gamma(D/2-\nu_j)}{\Gamma(\nu_j)} \right]^{1/2} \tag{4.23}$$

$$\times f(\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5). \tag{4.24}$$

The variable ν_0 is related to the dimension of the integral and ν_6 to ν_{10} are auxiliary variables associated with the vertices of the tetrahedral vacuum diagram:

$$\begin{aligned} \nu_0 &= D/2 = m - \varepsilon, & \nu_8 &= 2\nu_0 - \nu_{145}, \\ \nu_6 &= 3(D/2) - \nu_{12345}, & \nu_9 &= \nu_{345} - \nu_0, \\ \nu_7 &= 2\nu_0 - \nu_{235}, & \nu_{10} &= \nu_{125} - \nu_0. \end{aligned} \tag{4.25}$$

The function $f(\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ is then invariant under $Z_2 \times S_6$, where the symmetric group S_6 is generated by the six-cycle and the transposition:

$$(\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5) \to (\nu_0, \nu_2, \nu_5, \nu_4, 3\nu_0 - \nu_{12345}, \nu_3), \tag{4.26}$$

$$(\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5) \to (\nu_0, -\nu_0 + \nu_{145}, \nu_2, -\nu_0 + \nu_{345}, \nu_0 - \nu_5, \nu_0 - \nu_4), \tag{4.27}$$

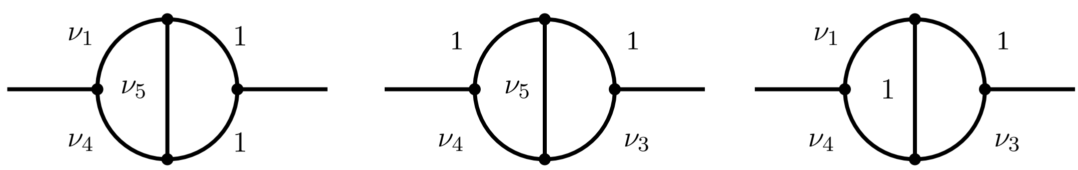
the group Z_2 by the reflection:

$$(\nu_0, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5) \to (\nu_0, \nu_0 - \nu_1, \nu_0 - \nu_2, \nu_0 - \nu_3, \nu_0 - \nu_4, \nu_0 - \nu_5). \tag{4.28}$$

In order to obtain the expansion of the integral for all $\nu_i = 1 + a_i\varepsilon$ up to order ε^6, Broadhurst and Barfoot used the fact that, by applying group theory, one can get the result for the integral with $\nu_i = 1 + a_i\varepsilon$ by the expansion of the same integral, where two adjacent lines have index 1.

D.J. Broadhurst found in the order ε^5-term of the expansion of the function f for the first time that this term not only involved Riemann's zeta functions, but additionally a double sum $U_{6,2} \equiv \sum_{n>m} \frac{(-1)^{n-m}}{n^6 m^2}$, which is not reducible to single zeta functions or a product of them (cf. Chapter 5). This is also the only double sum up to the next order in ε, as no further non-reducible double sum appears in that order [Broa 1986].

In 1996, A.V.Kotikov [Koti 1996] calculated a result for the massless two-loop two-point diagram with three subdivergences, where the two lines with index 1 are adjacent:



and as a special case the graph $\hat{I}^{(2,5)}(m-\varepsilon,1,1,1,1,\nu_5)$. Using an enhanced Gegenbauer-polynomial technique, he was able to express the result of these graphs in terms of ${}_3F_2$ – functions of unit argument. He gave the following formula for the special case $\hat{I}^{(2,5)}(m-\varepsilon,1,1,1,1,\nu_5)$, reformulated in [Groz 2003], to:

$$\begin{aligned}
&\hat{I}^{(2,5)}(m-\varepsilon,1,1,1,1,n)\\
&=2\Gamma(\nu_0-1)\Gamma(\nu_0-n-1)\Gamma(n-2\nu_0+3)\\
&\left[\frac{2\Gamma(\nu_0-1)}{(2\nu_0-2n-4)\Gamma(n+1)\Gamma(3\nu_0-n-4)}\,{}_3F_2\left[\begin{matrix}1,2\nu_0-2,n-\nu_0+2\\ n+1,n-\nu_0+3\end{matrix};1\right]\right]-\frac{\pi cot\pi(2\nu_0-n)}{\Gamma(2\nu_0-2)}
\end{aligned}\tag{4.29}$$

for non-integer n.

In [BGK 1997], D.J.Broadhurst, J.A.Gracey, and D.Kreimer then continued the work of [BaBr 1988] using integration-by-parts identities to construct a recurrence relation

$$\begin{aligned}
&(\nu_5+2-2\nu_0)I_4(\delta,\nu_3,\nu_4,\nu_5)+\frac{(\nu_3+\nu_5+1-2\nu_0)(\nu_4+\nu_5+1-2\nu_0)}{(\nu_5+1-\nu_0)}I_4(\delta-1,\nu_3,\nu_4,\nu_5-1)\\
&=\nu_5(\delta+\nu_5+1-2\nu_0)F_{1,\delta}\left(\frac{F_{\nu_3,\nu_5+1}}{\delta-\nu_4}+\frac{F_{\nu_4,\nu_5+1}}{\delta-\nu_3}\right)
\end{aligned}\tag{4.30}$$

for $I_4(\delta,\nu_3,\nu_4,\nu_5)=\hat{I}^{(2,5)}(2\nu_0-\delta-2,1,1,\nu_3,\nu_4,\nu_5)$. This is again a function for the two-loop two-point graph, where two adjacent lines carry index 1, and $\delta=\nu_{345}-\nu_0$. The function F_{ν_i,ν_j} denotes the function for the one-loop case (4.5). Broadhurst, Gracey and Kreimer then solved the recurrence relation, introducing a function $S(a,b,c,d)$, which itself involves a Saalschützian ${}_3F_2$ – function [Slat 1966]:

$$\begin{aligned}
I_4(\delta,\nu_3,\nu_4,\nu_5)=\nu_5\delta F_{1,\delta+1}&\left(\frac{F_{\nu_3,\nu_5+1}}{2\nu_0-3}S(\nu_0-\nu_3-1,\nu_4-1,\nu_0+\nu_3-\delta-2,\delta-\nu_4)\right.\\
&\left.+\frac{F_{\nu_4,\nu_5+1}}{2\nu_0-3}S(\nu_0-\nu_4-1,\nu_3-1,\nu_0+\nu_4-\delta-2,\delta-\nu_3)\right)
\end{aligned}\tag{4.31}$$

with

$$S(a,b,c,d)=\frac{\pi cot\pi c}{H(a,b,c,d)}-\frac{1}{c}-\frac{b+c}{bc}F(a+c,-b,-c,b+d)\tag{4.32}$$

and

$$H(a,b,c,d)=\frac{\Gamma(1+a)\Gamma(1+b)\Gamma(1+c)\Gamma(1+d)\Gamma(1+a+b+c+d)}{\Gamma(1+a+c)\Gamma(1+a+d)\Gamma(1+b+c)\Gamma(1+b+d)},\tag{4.33}$$

$$F(a,b,c,d)\equiv\sum_{n=1}^{\infty}\frac{(-a)_n(-b)_n}{(1+c)_n(1+d)_n}={}_3F_2\left[\begin{matrix}-a,-b,1\\ 1+c,1+d\end{matrix};1\right]-1.\tag{4.34}$$

The explicit properties fulfilled by $S(a,b,c,d)$ and $F(a,b,c,d)$ can be found in [BGK 1997]. The authors show that $I_4(\delta,\nu_3,\nu_4,\nu_5)$ reduces to Euler's gamma functions when any element of $\{\nu_5,\delta,2\nu_0-\nu_5-2,2\nu_0-\delta-2\}$ is equal to unity.

Taking into account knot theoretic considerations about the different zeta functions that can possibly occur and the group of transformations of ${}_3F_2$ series, Broadhurst, Gracey and Kreimer succeeded in expanding the master two-loop diagram for $\nu_i = 1 + a_i\varepsilon$ up to order ε^8, which corresponds to zeta functions up to the level $\zeta(11)$. In this way they could show that the ε expansion of this two-loop integral not only involves zeta functions of depth one, but starting from order ε^5 also an (alternating) double sum (see Chapter 5). We mentioned the appearance of this function before. The order ε^8 even involves a triple sum. Questions for the different occurring functions were also investigated in [Broa 2003], where the next expansion level ε^9 for the integral defined above was announced.

Chapter 5

Nested sums

In the previous chapter we encountered gamma functions or, more generally, *hypergeometric functions* in the ε-expansion of Feynman graphs. The immediate question arising from this is, how one can expand these functions themselves in the regularization parameter ε. For the one-loop case of (4.5) it was sufficient to apply formula (4.7) for the gamma function near unit argument to provide us with the Taylor expansion of this one-loop diagram and the knowledge that we can only get rational numbers and Riemann's zeta function in this expansion. However, one small step to the more complicated topology of the massless two-loop two-point function already stopped this formula from being sufficient for the expansion. Likewise, we are in this case no longer able to make a statement about the numbers, or functions that could arise: We have not yet excluded the possibility that other (transcendental) functions might occur at orders $\varepsilon^n, n > 9$.

The following Chapter 6 will introduce a new way to expand the massless two-loop two-point function, using the C++ library *nestedsums* [Wein 2002]. This new method will solve both problems. It will enable us to expand the massless two-loop two-point function up to an arbitrary order in ε and will equally enable us to deduce from the calculation process itself which functions can possibly occur. The library *nestedsums*, which we use at the end of these calculations, includes four classes, called `transcendental_fct_type_A()` to `transcendental_fct_type_D()`. These classes are prototypes for sums and double sums of gamma functions of various arguments and can be expanded within *nestedsums* using so-called *Z-sums* and *S-sums*. These objects are generalizations of those functions that can occur in the expansion of the prototype classes. More precisely, the Z- and S-sums are generalizations of *multiple polylogarithms* (MPL), *(multiple) zeta values* (MZV), *harmonic functions*, etc., which typically occur in the expansion of Feynman diagrams. We will refer to this whole family of functions as *multiple nested sums*, a name that will become obvious once we defined them; this will be done in the first section. We will also put some emphasis on the algebraic properties of multiple polylogarithms, introducing the *shuffle* and *stuffle* relations, which we will meet again in another context in Chapter 8. In Section 5.2, we will then present the C++ library *nestedsums*, the theoretical background and the four classes mentioned above. However, we will not describe the employed algorithms in detail and refer the reader to [Wein 2002] instead.

5.1 Nested sums

5.1.1 From polylogarithms to multiple polylogarithms

In this section we want to introduce classical polylogarithms, multiple zeta values and related functions, and also want to provide some of their algebraic properties. We will try to do this in a way that gives an idea of the development of the research on these functions over the years and shows their appearance in different contexts of mathematics and physics. However, this is only a basic sketch and meant as a short introduction. To provide a full review of the work on these functions and their occurrence in physics calculations is out of the scope of this thesis.

Any function which we will introduce here, can either be represented as an infinite sum or as an integral, a fact on which the algebraic properties of these functions will rely. However, we will not always provide both representations in this introductory section, but mostly refer to sums, always having in mind that we will later use *nestedsums*. A table of the different functions with their representation as sums can be found in Section 5.2.

We start our introduction to multiple polylogarithms with Euler [Eule 1771]. He studied the *dilogarithm* function

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-x')}{x'}dx' = \int_0^x \frac{dx'}{x'}\int_0^{x'} \frac{dx''}{1-x''}, \qquad |x| \leq 1. \tag{5.1}$$

The dilogarithm can equally be represented as a sum in the following way:

$$\mathrm{Li}_2(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^2}, \qquad |x| \leq 1. \tag{5.2}$$

For $x = 1$, this sum reduces to a so-called *harmonic sum*, which is generally defined as

$$\mathrm{S}_m(n) = \sum_{i=1}^{n} \frac{1}{i^m}, \tag{5.3}$$

leading to the identity:

$$\mathrm{Li}_2(1) = \sum_{i=1}^{\infty} \frac{1^i}{i^2} = \mathrm{S}_2(\infty). \tag{5.4}$$

A next step with importance for calculations was undertaken by L. Lewin in [Lewi 1981], who gathered the work of different authors and introduced a general standard notation for *classical polylogarithms*

$$\mathrm{Li}_n(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^n}, \tag{5.5}$$

$$\mathrm{Li}_n(x) = \int_0^x \frac{\mathrm{Li}_{n-1}(x)}{x}dx \tag{5.6}$$

and related functions. These functions are now widely used in physics calculations.

In [KMR 1970], Kölbig et al. then revised and corrected a former work of N. Nielsen on a generalization of classical polylogarithms [Niel 1909], rewriting the classical polylogarithm in another form:

$$\mathrm{Li}_n(x) = \frac{(-1)^{n-1}}{(n-2)!} \int_0^1 \log^{n-2} t \, \log(1-xt) \, \frac{dt}{t}, \qquad (n \geq 2). \tag{5.7}$$

N. Nielsen investigated yet more general functions, which Kölbig et al. [KMR 1970] named *Nielsen's generalized polylogarithms*:

$$\mathrm{S}_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \log^{n-1} t \, \log^p(1-xt) \, \frac{dt}{t}. \tag{5.8}$$

These functions reduce to classical polylogarithms for $p = 1$, $\mathrm{S}_{n-1,1}(x) = \mathrm{Li}_n(x)$. However, Kölbig et al. were mainly guided by the desire to find transformations for these functions, rather than to investigate their algebraic properties.

Research in the area of polylogarithms, including their algebraic properties, received a major boost ten years later starting with an article by D. Zagier [Zagi 1994] who extended Riemann's zeta function to the multidimensional case:[1]

$$\zeta(s) = \sum_{n>0}^{\infty} \frac{1}{n^s} \quad \rightarrow \quad \zeta(s_1, ..., s_r) = \sum_{n_1 > ... n_r > 0}^{\infty} \frac{1}{n_1^{s_1} ... n_r^{s_r}} \qquad (s_1 \geq 2, s_i \geq 1). \tag{5.9}$$

The variable r is called the *depth* of the zeta function and $s = s_1 + ... + s_r$ its *weight*. D. Zagier called these multidimensional functions *multiple zeta values* (MZVs).[2] In the same article, he gave a review of connections of zeta functions to various branches in mathematics, to number theory in particular, and referred to their appearance in several different contexts.

Sums of the form (5.9) are in the subsequent literature also called *Euler* or *Euler/Zagier sums* (for example in [BBB 1997]). We will follow [Wein 2002, MUW 2002] here and refer to sums only as Euler-Zagier sums if the upper summation boundary is finite and continue to call them MZVs if the sums are infinite sums. If one also allows the appearance of the number -1 in the numerator of the terms of a sum, these sums are called *alternating* sums [Broa 1996], *e.g.*:

$$\zeta(s_1, ..., s_r; \sigma_1, ..., \sigma_r) = \sum_{n_1 > ... n_r > 0}^{\infty} \frac{\sigma_1^{n_1}}{n_1^{s_1}} ... \frac{\sigma_r^{n_r}}{n_r^{s_r}}, \qquad \sigma_j = \pm 1, \quad \forall j. \tag{5.10}$$

One also finds in the literature the convention that the numbers -1 in the numerator are indicated by a bar over the corresponding variable σ.

[1]Note that Zagier defined the multiple zeta values with a reversed order of the terms of the sum:

$$\zeta(s) = \sum_{n>0}^{\infty} \frac{1}{n^s} \quad \rightarrow \quad \zeta(s_1, ..., s_r) = \sum_{n_r > ... n_1 > 0}^{\infty} \frac{1}{n_1^{s_1} ... n_r^{s_r}} \qquad (s_r \geq 2, s_i \geq 1).$$

The same applies to Goncharov's polylogarithms (see page 41). However, it is more convenient and has become standard by now, to present the MZVs in the way we have done in the text above. It is also the definition that is implemented in *nestedsums* and *GiNaC* (cf. Section 5.2).

[2]Please note that we will often call MZVs simply zeta functions, as long as the meaning is unambiguous.

In the following years, much work on MZVs has been performed by both mathematicians and physicists, either emphasizing the more mathematical point of view or increasingly investigating functions that occur in the calculation of massive or massless Feynman diagrams. Most of the developments in this area, which we will refer to in subsequent paragraphs, took place in parallel and were often inspired by each other. The way in which we present the functions here, is guided by the idea to go from the “simplest” to the “most complex” and is not necessarily meant to be chronologically correct.

We want to start here with *harmonic sums*. J.A.M. Vermaseren reopened the work on harmonic sums [Verm 1999] in physics. He showed by explicit calculations that harmonic sums

$$\mathrm{S}_m(n) = \sum_{i=1}^{n} \frac{1}{i^m} \tag{5.11}$$

$$\mathrm{S}_{-m}(n) = \sum_{i=1}^{n} \frac{(-1)^i}{i^m} \tag{5.12}$$

with $m > 0$, and “higher harmonic sums” or *multiple harmonic sums* that are recursively defined by

$$\mathrm{S}_{m,j_1,\ldots,j_p}(n) = \sum_{i=1}^{n} \frac{1}{i^m} \mathrm{S}_{j_1,\ldots,j_p}(i), \tag{5.13}$$

occurred in the Mellin transforms of functions, which one typically encounters in calculations of Feynman diagrams. Note that a negative m does not mean that the corresponding m in the denominator has a relative minus sign, but that we have an alternating sum with a corresponding $(-1)^i$ in the numerator. One should also be aware that the nested “inner” sums $\mathrm{S}_{j_1,\ldots,j_p}(i)$, in the harmonic sums of (5.13), which are recursive forms of (5.11), all possess a summation limit that runs up to i and not $(i-1)$ as it was, for example, the case for the multiple zeta values. These upper summation limits of nested sums will become more important in the following.

Referring to this work, Remiddi and Vermaseren [ReVe 2000] generalized Nielsen’s polylogarithm and defined the *harmonic polylogarithms* (HPL), $\mathrm{H}(\vec{m}_w; x)$. For the integral representation of this harmonic polylogarithms see [ReVe 2000]. We only want to state the power series expansion for a general harmonic polylogarithm at this point:

$$\mathrm{H}(\vec{m}_w; x) = \sum_{i_1 > \ldots > i_w}^{\infty} \frac{x^{i_1}}{{i_1}^{a_1}} \frac{1}{{i_2}^{a_2}} \cdots \frac{1}{{i_w}^{a_w}}, \tag{5.14}$$

with $\vec{m}_w = a_1(\vec{m}_{w-1}) = (a_1, \ldots, a_w)$. Also in this case one can find an alternating form with numbers -1 in the numerator of the terms in the sum.

We claimed that the harmonic polylogarithms are a generalization of Nielsen’s polylogarithms and indeed one finds from the sum representation of both functions (cf. Table 5.1):

$$\mathrm{S}_{n,p}(x) = \mathrm{H}(n+1, \underbrace{1, \ldots, 1}_{p-1}; x) \tag{5.15}$$

Another major step in examining these multiple nested sums was the definition of *multiple polylogarithms* (MPL) by A.B. Goncharov [Gon 1998]. These are on the one hand multidimensional generalizations of the classical polylogarithm (5.5) and on the other hand generalizations of MZVs to (arbitrary) variables x_i:

$$\mathrm{Li}_{n_1,...,n_m}(x_1,...,x_m) = \sum_{k_1>...>k_m>0}^{\infty} \frac{x_1^{k_1}...x_m^{k_m}}{k_1^{n_1}...k_m^{n_m}}. \tag{5.16}$$

Starting from the middle of the 1990s and following Zagier's work, much progress on the subject of MZVs and related sums could be achieved by D.J. Broadhurst, J.M. Borwein, D.M. Bradley and co-workers. Their work included both a numerical part which consisted of tests of relations between zeta values using computers, and a more algebraic part in which they investigated the underlying mathematical structures. This work led to much progress concerning the algebraic properties of these functions.

We especially want to emphasize the paper [BBBL 2001] of Borwein, Bradley, Broadhurst and Lisoněk, who worked on Goncharov's multiple polylogarithms defining it in a different form by introducing functions λ

$$\lambda \begin{pmatrix} s_1, ..., s_k \\ b_1, ..., b_k \end{pmatrix} := \sum_{\nu_1,...,\nu_k=1}^{\infty} \prod_{j=1}^{k} b_j^{-\nu_j} \left(\sum_{i=j}^{k} \nu_i \right)^{-s_j} \tag{5.17}$$

which equal Gonachrov's MPLs:

$$\mathrm{Li}_{s_k,...,s_1}(x_k,...,x_1) = \lambda \begin{pmatrix} s_1, ..., s_k \\ y_1, ..., y_k \end{pmatrix}, \qquad y_j := \prod_{i=1}^{j} x_i^{-1}. \tag{5.18}$$

Using this function λ, Borwein et al. compiled in [BBBL 2001] an excellent compendium about MPLs and related functions, such as the MZVs. They calculated and proved different algebraic relations fulfilled by MPLs and MZVs, particularly elaborating on the problem how to express MZVs of depth k in terms of MZVs of lower depth. In doing this they investigated the "shuffle" and "stuffle" (quasi-shuffle) relations for these functions, which we will introduce in the next section.

5.1.2 Algebraic relations: shuffle and quasi-shuffle

The interest in relations between different zeta functions already reaches back to Euler who tried to establish some of these relations. He showed for instance that the following relation holds:

$$\zeta(2,1) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n-1} \frac{1}{k} = \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3). \tag{5.19}$$

This is nowadays known to be a special case of the more general *duality relation for MZVs* [BBBL 2001]:

$$\zeta(s_1+2, \underbrace{1,...,1}_{r_1}, ..., s_m+2, \underbrace{1,...,1}_{r_m}) = \zeta(r_m+2, \underbrace{1,...,1}_{s_m}, ..., r_1+2, \underbrace{1,...,1}_{s_1}). \tag{5.20}$$

Zagier, too, tried to find algebraic relations like (5.20). He conjectured for example the identity

$$\zeta(\underbrace{3,1,3,1,...,3,1}_{2n}) = \frac{2\,\pi^{4n}}{(4n+2)!} = \frac{1}{4^n}\zeta(\underbrace{4,4,...,4}_{n}), \tag{5.21}$$

which was later proved to be correct in [BBBL 2001]. In Zagier's article one can also find the following formula [Zagi 1994]:

$$\zeta(k-1,1) = \frac{k-1}{2}\zeta(k) - \frac{1}{2}\sum_{r=2}^{k-2}\zeta(r)\zeta(k-r). \tag{5.22}$$

In this latter relation, a zeta function of depth two is expressed via zeta functions, or a product of zeta functions, of lower depth. To find such relations for zeta functions is one of the big aims in this context. In the attempt to investigate these relations in more detail, it was found that, whenever one multiplies two zeta functions, there are always two different relations that can be obtained. Citing [BBBL 2001] one gets for example:

$$\zeta(2,1)\zeta(2) = 6\zeta(3,1,1) + 3\zeta(2,2,1) + \zeta(2,1,2) \tag{5.23}$$

and equally

$$\zeta(2,1)\zeta(2) = 2\zeta(2,2,1) + \zeta(4,1) + \zeta(2,3) + \zeta(2,1,2). \tag{5.24}$$

Note that if we define the weight w of a product of zeta functions as the sum of the corresponding weights:

$$w(\prod_i \zeta(j_i)) = \sum_i j_i,$$

we observe that the weight is preserved in both cases, (5.23) and (5.24). If we equally define the depth of a product of zeta functions as the sum of the individual depths, it is only preserved in (5.23), but not in (5.24). Equations (5.23) and (5.24) indicate that there are two algebra structures on MZVs (on MPLs). These two algebra relations are the *shuffle product* and the *quasi-shuffle product* [Hoff 2000]. We want to define them in a general way, before we show their connection to zeta functions.

A shuffle relation can be defined in an abstract way on words consisting of letters: Let X be a finite set, called an *alphabet*, whose elements are *letters*. A *word* on the alphabet X is a finite sequence of elements of X:

$$x = x_1...x_k. \tag{5.25}$$

Concatenation of words defines a multiplication on the set X

$$(x_{i_1}...x_{i_k})(x_{i_{k+1}}...x_{i_n}) = x_{i_1}...x_{i_k}x_{i_{k+1}}...x_{i_n} \tag{5.26}$$

and gives rise to a *free monoid* X^* (see Appendix A) on the set of words. The unit element is the empty word. Each word is a product of letters. The length k of the word x defines a natural grading on words. The *shuffle product* on these words is denoted by the symbol $\sqcup\!\sqcup$ and defined via the recursive formula:

$$1 \sqcup\!\sqcup\, \omega = \omega \sqcup\!\sqcup\, 1 = \omega, \tag{5.27}$$

$$xu \sqcup\!\sqcup\, yv = x(u \sqcup\!\sqcup\, yv) + y(xu \sqcup\!\sqcup\, v) \tag{5.28}$$

for words ω, u, v and letters x, y. The shuffle product provides a commutative $\mathbb{K}$-algebra.

For a, b, x, y letters, (5.28) for example leads to:

$$ab \sqcup\!\sqcup\, xy = a(b \sqcup\!\sqcup\, xy) + x(ab \sqcup\!\sqcup\, y) = ... = abxy + axby + axyb + xaby + xayb + xyab. \tag{5.29}$$

One can see that the relative order of the two sets a, b and x, y in the result of the shuffle product is preserved.

Consider now another operation on the letters of X denoted by the bracket $[\cdot,\cdot]$, which is commutative, associative, and adds degrees. A *quasi-shuffle algebra* is then defined as the commutative, associative $\mathbb{K}$-algebra with the product given by:

$$a\omega_1 * b\omega_2 = a(\omega_1 * b\omega_2) + b(a\omega_1 * \omega_2) + [a, b](\omega_1 * \omega_2). \tag{5.30}$$

The zeta functions fulfill this latter quasi-shuffle product via their representation as sums. We will see this explicitly in the next section and ask the reader for some patience at this point.

The shuffle product on the other hand is fulfilled by the zeta functions via their representation as iterated integrals. An iterated integral is generally defined in the following way [Kass 1995]: Let $\omega_1, ..., \omega_n$ be complex-valued 1-forms on a real interval $[a, b]$, with $\omega_i = f_i(s_i)ds_i$. An iterated integral is defined as:

$$\begin{aligned} \int_a^b \omega_1...\omega_n &= \int_a^b f_1(s_1) \int_a^{s_1} f_2(s_2) \int_a^{s_2} \omega_3...\omega_n ds_2 ds_1 \\ &= \prod_{j=1}^{n} \int_a^{y_{j-1}} f_j(y_j) dy_j, \quad y_0 = c. \end{aligned} \tag{5.31}$$

Since the upper integration limit of each integral is the integration variable of the integral "it sits in", it is justified to call these integrals *nested integrals.*

Iterated integrals have the following properties:

$$\begin{aligned} \int_a^b \omega_1...\omega_n &= (-1)^n \int_b^a \omega_n...\omega_1, \\ \int_a^c \omega_1...\omega_n &= \int_a^b \omega_1...\omega_n + \sum_{k=1}^{n-1} \int_a^b \omega_1...\omega_k \int_b^c \omega_{k+1}...\omega_n + \int_b^c \omega_1...\omega_n, \end{aligned}$$

for $a < b < c$, and finally fulfill the important equation:

$$\int_a^b \omega_1...\omega_n \int_a^b \omega_{n+1}...\omega_{n+m} = \sum_\sigma \int_a^b \omega_{\sigma(1)}...\omega_{\sigma(n+m)} \tag{5.32}$$

where σ runs over all $(n,m)-$*shuffles*, *i.e.* over all $\begin{pmatrix} n+m \\ n \end{pmatrix}$ permutations σ of the set $\{1, 2, ..., n+m\}$ such that $\sigma^{-1}(i) < \sigma^{-1}(j)$ for $1 \leq i < j \leq n$ and $n+1 \leq i < j \leq n+m$.

The connection between zeta functions and iterated integrals was given by M.Kontsevich, who found a realization of MZVs in the form of these integrals, that goes back to Drinfel'd [Zagi 1994] :[3]

$$\zeta(i_1, ..., i_k) = \int_0^1 \left(\frac{ds}{s}\right)^{i_1-1} \frac{ds}{1-s} ... \left(\frac{ds}{s}\right)^{i_k-1} \frac{ds}{1-s}. \tag{5.33}$$

Hence we find for example:

$$\zeta(3, 2) = \int_{1>s_1>...>s_5>0} \frac{ds_1}{s_1} \frac{ds_2}{s_2} \frac{ds_3}{1-s_3} \frac{ds_4}{s_4} \frac{ds_5}{1-s_5}. \tag{5.34}$$

Since the iterated integrals fulfill the shuffle relation (5.32), we obtain a corresponding shuffle relation for MZVs. This relation leads to (5.23) for the product of $\zeta(2,1)\zeta(2)$ [BBBL 2001]; the second relation (5.24) can thus be found by multiplication of sums.[4]

The connection of the iterated integral representation to the more abstract words on which we defined the two relations was investigated by Michael E. Hoffman, [Hoff 2000, Hoff 1997]. Let us just give the basic idea of this approach by stating that one can define a map which assigns to each form in the iterated integral representation of a zeta function a letter x or y:

$$\frac{ds}{s} \to x, \quad \frac{ds}{1-s} \to y, \tag{5.35}$$

and hence define for each zeta function a map into the set of words in these two indeterminates by:

$$\zeta(i_1, ..., i_k) \to x^{i_1-1}y...x^{i_k-1}y\,. \tag{5.36}$$

There is the obvious restriction for the words to begin with x and to end with y coming from (5.33). Hoffman then defines the shuffle and quasi-shuffle product on the $\mathbb{Q}$-vector space of these words. We will only show using these words that the equation (5.23) really holds: Using (5.36), we can write $\zeta(2,1)$ and $\zeta(2)$ as:

$$\zeta(2, 1) \equiv xyy, \quad \zeta(2) \equiv xy. \tag{5.37}$$

Building the shuffle relation for $\zeta(2,1)\zeta(2)$ by shuffling these words leads to:

$$xyy ⧢ xy = xyyxy + 3xyxyy + 6xxyyy. \tag{5.38}$$

A translation of this result back into zeta functions via (5.36) gives (5.23).

In [Hoff 2000, Hoff 1997] one can also find relations between the words defined above and so-called *Lyndon words* [Reut 1993, MiPe 2000]. These are words fulfilling a certain order

[3] To work with the more general MPLs, one only has to substitute $\frac{1}{1-s}$ by $\frac{1}{x-s}$. Again, the MZVs are just MPLs for $x_i = 1$.

[4] Note that the shuffle relation is also called *weight-length shuffle* and the quasi-shuffle-relation *depth-length shuffle* or *stuffle* in [BBBL 2001].

relation: Let L be a *totally ordered set*, which means that for any two elements $x, y \in L$ either $x \leq y$ or $y \leq x$. Provide L^*, the free monoid of L, with the *alphabetical or lexicographical order*, that is:

$$\forall\, u, w \in L^*, \quad w \neq e \Rightarrow u < uw, \tag{5.39}$$

$$\forall\, u, v_1, v_2 \in L^*,\ \forall\, x, y \in L,\ x < y \Rightarrow uxv_1 < uyv_2. \tag{5.40}$$

A *Lyndon word* on L^* is a nonempty word which is smaller than all its nontrivial proper right factors; in other words, w is a Lyndon word if $w \neq 1$ and if for each factorization $w = uv$ with $u, v \neq 1$, one has $w < v$. Let $X = \{x_0, x_1\}$ with $x_0 < x_1$, the Lyndon words of length 5 or less on X^* given in alphabetical order are then [MiPe 2000]:

$$\{x_0, x_0^4x_1, x_0^3x_1, x_0^3x_1^2, x_0^2x_1, x_0^2x_1x_0x_1, x_0^2x_1^2, x_0^2x_1^3, x_0x_1, x_0x_1x_0x_1^2, x_0x_1^2, x_0x_1^3, x_0x_1^4, x_1\}.$$

The shuffle and the quasi-shuffle algebra are both freely generated by these Lyndon words. In [MiPe 2000], Minh and Petitot used the Lyndon words to calculate all possible relations between zeta functions up to weight 10, by using a *Gröbner basis* representation. We will use these relations to reduce zeta functions of a certain depth to zeta functions of lower depths in Chapter 8.

5.2 *nestedsums* — the theory and some of its functions

So far we have introduced different sorts of functions that can all be expressed via integrals and (nested) sums. One of the reasons why they were investigated is their appearance in high energy physics calculations. We mentioned in Chapter 4 that there is a relation between knot theory and high energy physics, which was found by D. Kreimer. He could show that there is a close connection between the appearance of zeta functions in the counterterms of Feynman diagrams and its underlying topology [Krei 1997, Krei 2000b]. Kreimer invented a way to translate Feynman diagrams into knots, and ultimately into *braids* (see [Kass 1995]), and found a rule how to relate a certain *braid word*, and hence a certain Feynman diagram, to zeta functions. Investigations into these questions will be the subject of Chapter 7.

In Chapter 4 we also remarked that Kreimer's work could be used to expand the massless two-loop two-point function in the dimensional regularization parameter ε. In this context only zeta functions and alternating zeta functions occurred, for some cases a simple result of the fact that an integral could be expressed solely by a fraction of Euler's gamma function. The more complicated cases then involved a hypergeometric series, which ultimately is also a sum of a fraction of gamma functions.

In the introduction to harmonic sums in the last chapter we also referred to the work of Remiddi and Vermaseren who found harmonic sums in the expansion of Mellin moments of functions occurring in calculations of Feynman diagrams [Verm 1999, ReVe 2000]. This is one example for the idea of finding typical "basis elements" in which one might be able to expand the result of a Feynman diagram.

Generally speaking, the more abstract work on multiple nested sums went along with the idea that one could find functions typical for Feynman diagrams, and that one should be able to express general results in terms of these functions. A step in this direction was done with the creation of the C++ library *nestedsums*, which we will introduce now and which provides a systematic approach to this question.

5.2.1 *nestedsums* — the general idea

The library *nestedsums* is a computer library written in C++ [Wein 2002]. Itself uses another C++ library written for symbolic computations, called *GiNaC*, which was and still is developed at Mainz University [GiNaC].

The basic idea behind *nestedsums* [MUW 2002,Wein 2002] is to introduce newly defined sums that interpolate between the different multiple nested sums, MZVs, Nielsen's polylogarithms, harmonic polylogarithms, etc. for differently chosen sets of values of their parameters. These basic sums are called *Z-sums* and *S-sums*. They are defined in the following way:

$$Z(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{5.41}$$

$$Z(n; m_1, ..., m_k; x_1, ..., x_k) = \sum_{i=1}^{n} \frac{x_1^i}{i^{m_1}} Z(i-1; m_2, ..., m_k; x_2, ..., x_k) \tag{5.42}$$

$$= \sum_{n \geq i_1 > i_2 > ... > i_k > 0} \frac{x_1^{i_1}}{i_1^{m_1}} ... \frac{x_k^{i_k}}{i_k^{m_k}}. \tag{5.43}$$

$$S(n) = \begin{cases} 1, & n > 0 \\ 0, & n \leq 0; \end{cases} \tag{5.44}$$

$$S(n; m_1, ..., m_k; x_1, ..., x_k) = \sum_{i=1}^{n} \frac{x_1^i}{i^{m_1}} S(i; m_2, ..., m_k; x_2, ..., x_k) \tag{5.45}$$

$$= \sum_{n \geq i_1 \geq i_2 \geq ... \geq i_k \geq 1} \frac{x_1^{i_1}}{i_1^{m_1}} ... \frac{x_k^{i_k}}{i_k^{m_k}} \tag{5.46}$$

Hence the Z- and S-sums are nested sums. The variable k is called the *depth*, $w = m_1 + ... + m_k$ the *weight*, as with the zeta functions. Note that the difference between Z-sums and S-sums lies in the summation index of the iterated sum. They can be transformed recursively into each other [MUW 2002]:

$$S(n; m_1, ...; x_1, ...) = \sum_{i_1=1}^{n} \frac{x_1^{i_1}}{i_1^{m_1}} \sum_{i_2=1}^{i_1-1} \frac{x_2^{i_2}}{i_2^{m_2}} S(i_2; m_3, ...; x_3, ...)$$

$$+ S(n; m_1 + m_2, m_3, ...; x_1 x_2, x_3, ...),$$

$$Z(n; m_1, ...; x_1, ...) = \sum_{i_1=1}^{n} \frac{x_1^{i_1}}{i_1^{m_1}} \sum_{i_2=1}^{i_1} \frac{x_2^{i_2}}{i_2^{m_2}} Z(i_2 - 1; m_3, ...; x_3, ...)$$

$$- Z(n; m_1 + m_2, m_3, ...; x_1 x_2, x_3, ...).$$

where the first formula allows to convert from a S-sum to a Z-sum and the second formula vice versa. Table 5.1 shows how the Z- and S-sums interpolate between the different functions.

S-sums

$$S(n; m_1, ..., m_k; x_1, ..., x_k) = \sum_{i_1 \geq ... \geq i_k > 0}^{n} \frac{x_1^{i_1}}{i_1^{m_1}} ... \frac{x_k^{i_k}}{i_k^{m_k}}$$

harmonic sums

$$S(n; m_1, ..., m_k; 1, ..., 1) = \sum_{i_1 \geq ... \geq i_k > 0}^{n} \frac{1}{i_1^{m_1}} ... \frac{1}{i_k^{m_k}}$$

Z-sums

$$Z(n; m_1, ..., m_k; x_1, ..., x_k) = \sum_{i_1 > ... > i_k > 0}^{n} \frac{x_1^{i_1}}{i_1^{m_1}} ... \frac{x_k^{i_k}}{i_k^{m_k}}$$

Euler-Zagier sums

$$Z(n; m_1, ..., m_k; 1, ..., 1) = \sum_{i_1 > ... > i_k > 0}^{n} \frac{1}{i_1^{m_1}} ... \frac{1}{i_k^{m_k}} = Z_{m_1,...,m_k}(n)$$

multiple zeta values

$$Z(\infty; m_1, ..., m_k; 1, ..., 1) = \sum_{i_1 > ... > i_k > 0}^{\infty} \frac{1}{i_1^{m_1}} ... \frac{1}{i_k^{m_k}}$$

classical polylogarithms

$$\mathrm{Li}_n(x) = Z(\infty; n; x) = \sum_{i>0}^{\infty} \frac{x^i}{i^n}$$

Nielsen's polylogarithms

$$\mathrm{S}_{n,p}(x) = Z(\infty; n+1, \underbrace{1, ..., 1}_{p-1}; x, 1, ..., 1) = \mathrm{Li}_{n+1,\underbrace{1,...,1}_{p-1}}(x, \underbrace{1, ..., 1}_{p-1})$$

$$= \sum_{i_1 > ... > i_p > 0}^{\infty} \frac{x^{i_1}}{i_1^{n+1}} \frac{1}{i_2} ... \frac{1}{i_p}$$

harmonic polylogarithms

$$\mathrm{H}_{m_1,...,m_k}(x) = Z(\infty; m_1, ..., m_k; x, 1, ..., 1) = \mathrm{Li}_{m_1,...,m_k}(x, \underbrace{1, ..., 1}_{k-1})$$

$$= \sum_{i_1 > ... > i_k > 0}^{\infty} \frac{x^{i_1}}{i_1^{m_1}} \frac{1}{i_2^{m_2}} ... \frac{1}{i_k^{m_k}}$$

multiple polylogarithms

$$\mathrm{Li}_{m_1,...,m_k}(x_1, ..., x_k) = Z(\infty; m_1, ..., m_k; x_1, ..., x_k) = \sum_{i_1 > ... > i_k > 0}^{\infty} \frac{x_1^{i_1}}{i_1^{m_1}} ... \frac{x_k^{i_k}}{i_k^{m_k}}$$

Table 5.1: The connection between Z-sums, S-sums and multiple nested sums.

We can now define the quasi-shuffle algebra structure. It is fulfilled by the Z- and S-sums and hence also by the MZVs and MPLs. For the product of two sums with the same upper summation limit we get:

$$
\begin{aligned}
&Z(n;m_1,...m_k;x_1...x_k)Z(n;m'_1,...m'_k;x'_1...x'_l)\\
&=\sum_{i_1=1}^{n}\frac{x_1^{i_1}}{i_1^{m_1}}Z(i_1-1;m_2,...m_k;x_2...x_k)Z(i_1-1;m'_1,...m'_k;x'_1...x'_l)\\
&+\sum_{i_2=1}^{n}\frac{x_1'^{i_2}}{i_2^{m'_1}}Z(i_2-1;m_1,...m_k;x_1...x_k)Z(i_2-1;m'_2,...m'_k;x'_2...x'_l)\\
&+\sum_{i=1}^{n}\frac{(x_1x'_1)^i}{i^{m_1+m'_1}}Z(i-1;m_2,...m_k;x_2...x_k)Z(i-1;m'_2,...m'_k;x'_2...x'_l).
\end{aligned}
$$

The concatenation of x_1 with x'_1 in the last line corresponds to the commutative, associative, degree-adding operation $[\cdot,\cdot]$ defined in equation (5.30). Analogously, one obtains for the S-sums:

$$
\begin{aligned}
&S(n;m_1,...m_k;x_1...x_k)S(n;m'_1,...m'_k;x'_1...x'_l)\\
&=\sum_{i_1=1}^{n}\frac{x_1^{i_1}}{i_1^{m_1}}S(i_1;m_2,...m_k;x_2...x_k)S(i_1;m'_1,...m'_k;x'_1...x'_l)\\
&+\sum_{i_2=1}^{n}\frac{x_1'^{i_2}}{i_2^{m'_1}}S(i_2;m_1,...m_k;x_1...x_k)S(i_2;m'_2,...m'_k;x'_2...x'_l)\\
&-\sum_{i=1}^{n}\frac{(x_1x'_1)^i}{i^{m_1+m'_1}}S(i;m_2,...m_k;x_2...x_k)S(i;m'_2,...m'_k;x'_2...x'_l),
\end{aligned}
$$

where we find a minus sign in front of the part belonging to the [,]-operation. The multiplication relation is important because it enables us to write a product of Z-sums (S-sums) with the same summation limit as a sum of single Z-sums (S-sums), *e.g.*:

$$Z_{11}(n)Z_1(n)=Z_{21}(n)+Z_{12}(n)+3Z_{111}(n).$$

We end this short theoretical introduction to *nestedsums* with the remark that, since the Z-sums fulfill a quasi-shuffle algebra relation, they also form a Hopf algebra: M.E.Hoffman showed in [Hoff 2000] that each shuffle algebra is a Hopf algebra and this fact was explicitly formulated for the Z-sums in [MUW 2002].

5.2.2 *nestedsums* — the four classes of functions

There are four types of sums involving gamma functions that are already implemented into *nestedsums*. They are prototypes for functions that often occur in calculations in particle physics. The four classes are:

Type A:

$$\sum_{i=1}^{n} \frac{x^i}{(i+c)^m} \frac{\Gamma(i+a_1+b_1\varepsilon)}{\Gamma(i+c_1+d_1\varepsilon)} ... \frac{\Gamma(i+a_k+b_k\varepsilon)}{\Gamma(i+c_k+d_k\varepsilon)} Z(i+o-1, m_1, ..., m_l, x_1, ..., x_l),$$

Type B:

$$\sum_{i=1}^{n-1} \frac{x^i}{(i+c)^m} \frac{\Gamma(i+a_1+b_1\varepsilon)}{\Gamma(i+c_1+d_1\varepsilon)} ... \frac{\Gamma(i+a_k+b_k\varepsilon)}{\Gamma(i+c_k+d_k\varepsilon)} Z(i+o-1, m_1, ..., m_l, x_1, ..., x_l)$$
$$\times \frac{y^{n-i}}{(n-i+c')^{m'}} \frac{\Gamma(n-i+a'_1+b'_1\varepsilon)}{\Gamma(n-i+c'_1+d'_1\varepsilon)} ... \frac{\Gamma(i+a'_{k'}+b'_{k'}\varepsilon)}{\Gamma(i+c'_{k'}+d'_{k'}\varepsilon)}$$
$$\times Z(n-i+o'-1, m'_1, ..., m'_{l'}, x'_1, ..., x'_{l'}),$$

Type C:

$$-\sum_{i=1}^{n} \binom{n}{i} (-1)^i \frac{x^i}{(i+c)^m} \frac{\Gamma(i+a_1+b_1\varepsilon)}{\Gamma(i+c_1+d_1\varepsilon)} ... \frac{\Gamma(i+a_k+b_k\varepsilon)}{\Gamma(i+c_k+d_k\varepsilon)} S(i+o, m_1, ..., m_l, x_1, ..., x_l),$$

Type D:

$$-\sum_{i=1}^{n-1} \binom{n}{i} (-1)^i \frac{x^i}{(i+c)^m} \frac{\Gamma(i+a_1+b_1\varepsilon)}{\Gamma(i+c_1+d_1\varepsilon)} ... \frac{\Gamma(i+a_k+b_k\varepsilon)}{\Gamma(i+c_k+d_k\varepsilon)} S(i+o, m_1, ..., m_l, x_1, ..., x_l)$$
$$\times \frac{y^{n-i}}{(n-i+c')^{m'}} \frac{\Gamma(n-i+a'_1+b'_1\varepsilon)}{\Gamma(n-i+c'_1+d'_1\varepsilon)} ... \frac{\Gamma(i+a'_{k'}+b'_{k'}\varepsilon)}{\Gamma(i+c'_{k'}+d'_{k'}\varepsilon)}$$
$$\times S(n-i+o', m'_1, ..., m'_{l'}, x'_1, ..., x'_{l'}),$$

where $a_j, a'_j, c_j, c'_j, o, o'$ are integers, c, c' are nonnegative integers and b, b', d, d' are arbitrary numbers. The algorithms for the expansion of these functions are implemented in the classes `transcendental_sum_type_A` to `transcendental_sum_type_D` and a detailed description can be found in [Wein 2002, MUW 2002].

The general idea to achieve the results includes performing the ε expansion for the gamma functions via the formula

$$\Gamma(n+\varepsilon) = \Gamma(1+\varepsilon)\Gamma(n) \left(1 + \varepsilon Z_1(n-1) + \varepsilon^2 Z_{11}(n-1) + \varepsilon^3 Z_{111}(n-1) + ... + \varepsilon^{n-1} Z_{11...1}(n-1)\right) \quad (5.47)$$

for positive n, and

$$\Gamma(-n+1+\varepsilon) = \frac{\Gamma(1+\varepsilon)}{\varepsilon} \frac{(-1)^{n-1}}{\Gamma(n)} \left(1 + \varepsilon S_1(n-1) + \varepsilon^2 S_{11}(n-1) + \varepsilon^3 S_{111}(n-1) + ...\right) \quad (5.48)$$

for negative n.

The functions $\Gamma(1+\varepsilon)$ are then ε-expanded by the *GiNaC* series expansion method. The four classes `transcendental_sum_type_A` to `transcendental_sum_type_D` consist of fractions of gamma functions of different arguments, whose expansion involves Z- and S-sums via (5.47) and (5.48). In the end we multiply these two types of sums and can use their algebra structures to express the result. This is the general idea for the expansion of these four classes of sums, which is implemented in the C++ library *nestedsums*. We will not explicitly describe the entire implementation here and refer the reader again to [Wein 2002].

The four classes `transcendental_sum_type_A` to `transcendental_sum_type_D` are then used by the classes `transcendental_fct_type_A()` to `transcendental_fct_type_D()` modeled on generalizations of hypergeometric functions.

Type A:

$$\frac{\Gamma(d_1)...\Gamma(d_n)}{\Gamma(d_1')...\Gamma(d_{n'}')}\sum_{i=0}^{\infty}\frac{\Gamma(i+a_1)...\Gamma(i+a_k)}{\Gamma(i+a_1')...\Gamma(i+a_{k-1}')}\frac{x^i}{i!},$$

Type B:

$$\frac{\Gamma(d_1)...\Gamma(d_n)}{\Gamma(d_1')...\Gamma(d_{n'}')}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(i+a_1)...\Gamma(i+a_k)}{\Gamma(i+a_1')...\Gamma(i+a_{k-1}')}\frac{\Gamma(j+b_1)...\Gamma(j+b_l)}{\Gamma(j+b_1')...\Gamma(j+b_{l-1}')}$$
$$\times\frac{\Gamma(i+j+c_1)...\Gamma(i+j+c_m)}{\Gamma(i+j+c_1')...\Gamma(i+j+c_m')}\frac{x_1^i}{i!}\frac{x_2^j}{j!},$$

Type C:

$$\frac{\Gamma(d_1)...\Gamma(d_n)}{\Gamma(d_1')...\Gamma(d_{n'}')}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(i+a_1)...\Gamma(i+a_k)}{\Gamma(i+a_1')...\Gamma(i+a_k')}\frac{\Gamma(i+j+c_1)...\Gamma(i+j+c_m)}{\Gamma(i+j+c_1')...\Gamma(i+j+c_{m-1}')}\frac{x_1^i}{i!}\frac{x_2^j}{j!},$$

Type D:

$$\frac{\Gamma(d_1)...\Gamma(d_n)}{\Gamma(d_1')...\Gamma(d_{n'}')}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(i+a_1)...\Gamma(i+a_k)}{\Gamma(i+a_1')...\Gamma(i+a_k')}\frac{\Gamma(j+b_1)...\Gamma(j+b_l)}{\Gamma(j+b_1')...\Gamma(j+b_l')}$$
$$\times\frac{\Gamma(i+j+c_1)...\Gamma(i+j+c_m)}{\Gamma(i+j+c_1')...\Gamma(i+j+c_{m-1}')}\frac{x_1^i}{i!}\frac{x_2^j}{j!}.$$

Consider as an example the generalized hypergeometric function:

$$\begin{aligned}{}_{J+1}F_J(a_1,...,a_{J+1};b_1,...,b_J;x) &= \sum_{i=0}^{\infty}\frac{(a_1)_i...(a_{J+1})_i}{(b_1)_i...(b_J)_i}\frac{x^i}{i!}\\ &= \frac{\Gamma(b_1)...\Gamma(b_J)}{\Gamma(a_1)...\Gamma(a_{J+1})}\sum_{i=0}^{\infty}\frac{x^i}{i!}\frac{\Gamma(i+a_1)}{\Gamma(i+b_1)}...\frac{\Gamma(i+a_J)}{\Gamma(i+b_J)}\Gamma(i+a_{J+1})\\ &= \frac{\Gamma(b_1)...\Gamma(b_J)}{\Gamma(a_1)...\Gamma(a_{J+1})}\sum_{i=0}^{\infty}x^i\frac{\Gamma(i+a_1)}{\Gamma(i+b_1)}...\frac{\Gamma(i+a_J)}{\Gamma(i+b_J)}\frac{\Gamma(i+a_{J+1})}{\Gamma(i+1)}\end{aligned} \tag{5.49}$$

where $(a)_i = \frac{\Gamma(i+a)}{\Gamma(a)}$ is the *Pochhammer symbol*, which is related to the Appell symbol via: $(a)_i \equiv (a,i)$. Going from the second to the third line in (5.49) we used that $\Gamma(i+1) = i!$ for $i \in \mathbb{N}$. Note that there always have to be the same number of gamma functions in the numerator and denominator of such sums, to be able to use *nestedsums*. Taking out the term for $n = 0$, one can rewrite this sum:

$$ {}_{J+1}F_J(a_1,...,a_{J+1};b_1,...,b_J;x) = 1 + \frac{\Gamma(b_1)...\Gamma(b_J)}{\Gamma(a_1)...\Gamma(a_{J+1})} \sum_{i=1}^{\infty} x^i \frac{\Gamma(i+a_1)}{\Gamma(i+b_1)} ... \frac{\Gamma(i+a_J)}{\Gamma(i+b_J)} \frac{\Gamma(i+a_{J+1})}{\Gamma(i+1)}. \tag{5.50} $$

This form of a sum of gamma functions is implemented in `transcendental_fct_type_A()`. Let us calculate an example. Consider the function

$$ {}_2F_1(a\varepsilon, b\varepsilon; 1-c\varepsilon; x) = \sum_{i=0}^{\infty} \frac{(a\varepsilon)_i (b\varepsilon)_i}{(1-c\varepsilon)_i} \frac{x^i}{i!} \tag{5.51} $$

$$ = \frac{\Gamma(1-c\varepsilon)}{\Gamma(a\varepsilon)\Gamma(b\varepsilon)} \sum_{i=0}^{\infty} \frac{x^i}{i!} \frac{\Gamma(i+a\varepsilon)}{\Gamma(i+1-c\varepsilon)} \Gamma(i+b\varepsilon) \tag{5.52} $$

One has to write a program calling `transcendental_fct_type_A()` in the following form:

```
transcendental_fct_type_A(x,
                            lst(a*eps,b*eps),lst(1-c*eps),
                            lst(1-c*eps),lst(a*eps,b*eps),
                            eps,order,expand_status::expansion_required).
```

The rule for the order of the lists in this `transcendental_fct_type_A()` is: first the numerator, then the denominator, first the gamma functions inside the sum, then the gamma functions of the multiplicative factor in front of the sum. The `order` is a parameter up to which the expansion should be performed, `eps` is the expansion parameter and a *GiNaC* symbol here. After converting the result into a standard form using the *nestedsums* function `convert_Zsums_to_standard_form(F21)`, the result for this expansion up to order eps^3 is:

```
F21
=Z(Infinity)+eps^2*b*Li(2,x)*a
+S(1,2,x)*(eps^3*b*a^2+eps^3*b*a*c+eps^3*b^2*a)+eps^3*b*a*Li(3,x)*c
```

`Z(Infinity)` is the unit 1 in *nestedsums*. Hence the result becomes:

$$ {}_2F_1(a\varepsilon, b\varepsilon; 1-c\varepsilon; x) = 1 + \varepsilon^2\, ab\, \mathrm{Li}_2(x) + \varepsilon^3 [\mathrm{S}_{1,2}(x)\, ab\, (a+c+b) + abc\, \mathrm{Li}_3(x)] + O(\varepsilon^4) \tag{5.53} $$

This result agrees with results provided in the literature.

Chapter 6

The massless two-loop two-point function

In Chapter 4 we presented a number of ways to expand the massless two-loop two-point function up to a certain order in ε. Most of them used some sort of relation fulfilled by the integral, with the application of this relation leading to other integrals we can solve. This procedure is a standard way of handling Feynman diagrams: After tensor reduction to scalar integrals or, equivalently, the use of Schwinger parametrization, one applies algebraic relations such as the integration-by-parts identities to reduce the set of integrals to a (hopefully small) set of master integrals, which have to be expanded. However, we will follow the work of S. Weinzierl, P. Uwer, and S. Moch [MUW 2002] here and solve the integrals directly for different exponents of propagators and different space-time dimensions. This can be achieved using the library *nestedsums*, which we introduced in the last chapter. We will therefore transform the analytic expression for the massless two-loop two-point function into the form of the `transcendental_fct_type_A()` and `transcendental_fct_type_B()`, using *Mellin-Barnes integrals*. For a short introduction into the theory of Mellin-Barnes integrals the reader should visit the Appendix B. The idea to express the two-loop two-point function in this form and to use the library *nestedsums* was conceived by Stefan Weinzierl. We then independently performed the calculation and implemented it as a computer program. Afterwards we compared the results of these programs and checked their agreement, also with results in the literature. The calculation and the results have been published in [BiWe 2003].

6.1 The expansion of $\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$

We will stay close to the paper [BiWe 2003] in this chapter and explain the calculation of the integral in the way it is parametrized there. The main idea is to write the integral $\hat{I}^{(2,5)}$ as a double Mellin-Barnes integral. This is done by performing the calculation in a way that follows the Lie algebra gluing operation of Feynman graphs, which we explained in Chapter 3 and which can also be seen in Fig. 6.1. This figure shows that one first calculates the expression for the one-loop three-point graph, which will be a double Mellin-Barnes integral, and combines its result with the analytic expression for the one-loop two-point graph. The exponents of the momenta in the propagators of the one-loop two-point function will be

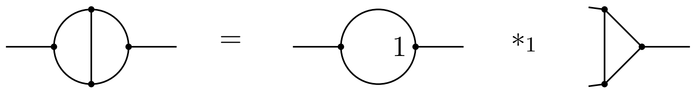


Figure 6.1: In the same way in which one can build the graph for the massless two-loop two-point function by inserting the one-loop three-point graph on the right into the vertex denoted by "1" of the one-loop two-point graph, we will calculate the analytic expression for the massless two-loop two-point function by first calculating the three-point function, transforming it into a double Mellin-Barnes integral, and combining its result afterwards with the analytic expression for the one-loop two-point function.

altered by the insertion of the one-loop three-point subgraph. This will eventually lead us to a double Mellin-Barnes integral representation for the massless two-loop two-point function. In a second step the double Mellin-Barnes integral is then calculated by closing the integration contour and summing up the residues of the integrand in all possible ways. This will give rise to eleven functions that will all be of the form of one of the following two functions G_+ and G_-:

$$
\begin{aligned}
&G_\pm(a_1, a_2, a_3, a_4; b_1, b_2, b_3; c_1, c_2, c_3) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{n+j}}{n!j!} \frac{\Gamma(\mp n - j - a_1)\Gamma(\pm n + j + a_2)\Gamma(\mp n + j + a_3)}{\Gamma(\pm n + j + a_4)} \\
&\qquad \frac{\Gamma(\mp n \mp b_1)\Gamma(n + b_2)}{\Gamma(\mp n \mp b_3)}\Gamma(\mp n \mp b_3)\frac{\Gamma(-j - c_1)\Gamma(j + c_2)}{\Gamma(-j - c_3)},
\end{aligned}
$$

where one or both of the sums can be finite. Functions of this type on the other hand can be expanded in ε with the help of the library *nestedsums*. We will go through all of these steps in detail in the following.

6.1.1 Decomposition of $\hat{I}^{(2,5)}(m - \varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$

We already introduced the integral and all abbreviations in Chapter 4. However, we repeat the conventions here for greater convenience:

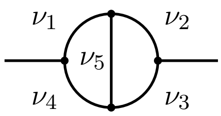
$$
\begin{aligned}
&\hat{I}^{(2,5)}(m - \varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5) \\
&= (-p^2)^{\nu_{12345} - 2m + 2\varepsilon} \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \int \frac{d^D k_2}{i\pi^{\frac{D}{2}}} \frac{1}{(-k_1^2)^{\nu_1}(-k_2^2)^{\nu_2}(-k_3^2)^{\nu_3}(-k_4^2)^{\nu_4}(-k_5^2)^{\nu_5}},
\end{aligned}
\tag{6.1}
$$

with

$$
k_3 = k_2 - p, \quad k_4 = k_1 - p, \quad k_5 = k_2 - k_1, \tag{6.2}
$$

$$
D = 2m - 2\varepsilon. \tag{6.3}
$$

This corresponds to the two-loop two-point graph with the following distribution of momenta:



The factor $(-p^2)^{\nu_{12345}-2m+2\varepsilon}$ makes the integral dimensionless. The exponents ν_i are of the form $\nu_i = n_i + a_i\varepsilon$, where the n_i are positive integers and the a_i nonnegative real numbers. The abbreviation $\nu_{ijk} = \nu_i + \nu_j + \nu_k$ is a short cut for sums of exponents.

We already remarked in Chapter 4 that the form of the exponents originates from the fact that any subdivergence sitting inside a line of a Feynman diagram changes the corresponding exponent of this line from an integer value to the form $\nu_i = n_i + a_i\varepsilon$. We will see this explicitly in Chapter 7. Hence allowing this form of exponents in the general integral will enable us, for instance, to use the massless two-loop two-point function in the ε expansion of the non-planar vertex correction with subdivergences in Section 7.2.

To start with the calculation of (6.1) we group the momenta differently and decompose the integral into two building blocks as follows:

$$\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$$
$$= \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \int \frac{d^D k_2}{i\pi^{\frac{D}{2}}} \frac{(-p^2)^{\nu_{12345}-2m+2\varepsilon}}{[-k_1^2]^{\nu_1}[-k_2^2]^{\nu_2}[-(k_2-p)^2]^{\nu_3}[-(k_1-p)^2]^{\nu_4}[-(k_2-k_1)^2]^{\nu_5}} \tag{6.4}$$

$$= \underbrace{\int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \frac{(-p^2)^{\nu_{14}-m+\varepsilon}}{[-k_1^2]^{\nu_1}[-(k_1-p)^2]^{\nu_4}}}_{=:\,\hat{I}^{(1,2)}(m-\varepsilon,\nu_1,\nu_4)} \underbrace{\int \frac{d^D k_2}{i\pi^{\frac{D}{2}}} \frac{(-p^2)^{\nu_{235}-m+\varepsilon}}{[-k_2^2]^{\nu_2}[-(k_2-p)^2]^{\nu_3}[-(k_2-k_1)^2]^{\nu_5}}}_{=:\,I^{(1,3)}(m-\varepsilon,\nu_2,\nu_3,\nu_5)} \tag{6.5}$$

or again:

$$\hat{I}^{(1,2)}(m-\varepsilon,\nu_1,\nu_4) := (-p^2)^{\nu_{14}-m+\varepsilon} \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \frac{1}{(-k_1^2)^{\nu_1}(-k_4^2)^{\nu_4}} \tag{6.6}$$

$$I^{(1,3)}(m-\varepsilon,\nu_2,\nu_3,\nu_5) := (-p^2)^{\nu_{235}-m+\varepsilon} \int \frac{d^D k_2}{i\pi^{\frac{D}{2}}} \frac{1}{(-k_2^2)^{\nu_2}(-k_3^2)^{\nu_3}(-k_5^2)^{\nu_5}} \tag{6.7}$$

Note that the result of (6.7) depends on $(-k_1^2)$ and $(-k_4^2)$ only by the dimensionless variables $x := \left(\frac{-p^2}{-k_1^2}\right)$ and $y := \left(\frac{-p^2}{-k_4^2}\right)$, which are the propagators of the integral (6.6). In this, we find the idea of inserting the three point graph into the two-point graph mentioned before: After calculating the three-point function (6.7) one obtains a momentum-independent expression times an expression for the momentum that only alters the exponents of the propagators in (6.6).

Apart from the multiplicative factor $(-p^2)^{\nu_{14}-m+\varepsilon}$, the two-point integral (6.6) is just the integral (4.5) leading to the functions $F_{\nu_1,\nu_2}(\varepsilon)$:

$$\begin{aligned} \hat{I}^{(1,2)}(m-\varepsilon,\nu_1,\nu_4) &= \frac{\Gamma(\nu_{14}-m+\varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_4)} \frac{\Gamma(m-\varepsilon-\nu_1)\Gamma(m-\varepsilon-\nu_4)}{\Gamma(2m-2\varepsilon-\nu_{14})} \\ &= F_{\nu_1,\nu_4} \end{aligned} \tag{6.8}$$

6.1.2 Calculation of the Mellin-Barnes integrals for $I^{(1,3)}(m-\varepsilon,\nu_2,\nu_3,\nu_5)$

We start with the calculation of the integral (6.7):

$$I^{(1,3)} = (-p^2)^{\nu_{235}-m+\varepsilon} \int \frac{d^D k_2}{i\pi^{\frac{D}{2}}} \frac{1}{(-k_2^2)^{\nu_2}(-k_3^2)^{\nu_3}(-k_5^2)^{\nu_5}}$$

with $k_3 = k_2 - p$, $k_5 = k_2 - k_1$ and $D = 2m - 2\varepsilon$. Note that in this section we will often omit the function arguments.

Applying the *Feynman parametrization*:

$$\frac{1}{A_1^{m_1} A_2^{m_2} ... A_n^{m_n}} = \int_0^1 dx_1 ... dx_n \, \delta\left(\sum x_i - 1\right) \frac{\Pi x_i^{m_i - 1}}{[\sum x_i A_i]^{\sum m_i}} \frac{\Gamma(m_1 + ... + m_n)}{\Gamma(m_1)...\Gamma(m_n)} \tag{6.9}$$

to the momenta of the integral, we get:

$$I^{(1,3)} = (-p^2)^{\nu_{235} - m + \varepsilon} \Gamma(235)$$
$$\int_0^1 dx_1 dx_2 dx_3 \, \delta\left(\sum x_i - 1\right) \int \frac{d^D k_2}{i\pi^{\frac{D}{2}}} \frac{x_1^{\nu_2 - 1} x_2^{\nu_3 - 1} x_3^{\nu_5 - 1}}{\left(-x_1 k_2^2 - x_2 k_3^2 - x_3 k_5^2\right)^{\nu_2 + \nu_3 + \nu_5}},$$

where we have used the abbreviation

$$\Gamma(235) := \frac{\Gamma(\nu_2 + \nu_3 + \nu_5)}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)}.$$

In a next step, we express the denominator in terms of the momenta k_1, k_2 and p and re-group the expressions:

$$\begin{aligned}
(-x_1 k_2^2 - x_2 k_3^2 - x_3 k_5^2) &= -x_1 k_2^2 - x_2 k_2^2 + 2x_2 k_2 \cdot p - x_2 p^2 - x_3 k_2^2 + 2x_3 k_2 \cdot k_1 - x_3 k_1^2 \\
&= -(x_1 + x_2 + x_3) k_2^2 + 2x_2 k_2 \cdot p - x_2 p^2 + 2x_3 k_2 \cdot k_1 - x_3 k_1^2 \\
&= -k_2^2 + 2x_2 k_2 \cdot p + 2x_3 k_2 \cdot k_1 - x_2 p^2 - x_3 k_1^2,
\end{aligned}$$

where we have used $x_1 + x_2 + x_3 = 1$ in the last step. We now want to isolate the k_2-dependence, knowing that in dimensional regularization we may shift the integration variable k_2 without changing the integral. To do this we add a zero in the form of $\pm(x_2 p + x_3 k_1)^2$. Since

$$(k_2 - x_2 p - x_3 k_1)^2 = k_2^2 - 2x_2 k_2 \cdot p - 2x_3 k_2 \cdot k_1 + (x_2 p + x_3 k_1)^2,$$

we obtain in this way:

$$\begin{aligned}
& -k_2^2 + 2x_2 k_2 \cdot p + 2x_3 k_2 \cdot k_1 - (x_2 p + x_3 k_1)^2 - x_2 p^2 - x_3 k_1^2 + (x_2 p + x_3 k_1)^2 \\
= \; & -(k_2 - x_2 p - x_3 k_1)^2 - x_2 p^2 - x_3 k_1^2 + (x_2 p + x_3 k_1)^2
\end{aligned}$$

Then we shift k_2 via $k_2 \to k_2 - x_2 p - x_3 k_1$ and obtain for the denominator:

$$-(k_2 - x_2 p - x_3 k_1)^2 - x_2 p^2 - x_3 k_1^2 + (x_2 p + x_3 k_1)^2 \to -k_2^2 - x_2 p^2 - x_3 k_1^2 + (x_2 p + x_3 k_1)^2,$$

leading to

$$I^{(1,3)} = (-p^2)^{\nu_{235} - m + \varepsilon} \Gamma(235) \int_0^1 dx_1 dx_2 dx_3 \, \delta\left(\sum x_i - 1\right)$$
$$\times \int \frac{d^D k_2}{i\pi^{\frac{D}{2}}} \frac{x_1^{\nu_2 - 1} x_2^{\nu_3 - 1} x_3^{\nu_5 - 1}}{\left(-k_2^2 + (x_2 p + x_3 k_1)^2 - x_2 p^2 - x_3 k_1^2\right)^{\nu_2 + \nu_3 + \nu_5}}.$$

For the k_2 integration, we use a special case of formula (2.11) in Minkowski space:

$$\int \frac{d^D l}{i\pi^{\frac{D}{2}}} \frac{1}{(-l^2-\Delta)^\alpha} = \frac{\Gamma(\alpha - \frac{D}{2})}{\Gamma(\alpha)} (-\Delta)^{\frac{D}{2}-\alpha}, \tag{6.10}$$

and hence obtain:

$$I^{(1,3)} = (-p^2)^{\nu_{235}-m+\varepsilon}\Gamma(235) \int_0^1 dx_1 dx_2 dx_3\, \delta\left(\sum x_i - 1\right) x_1^{\nu_2-1} x_2^{\nu_3-1} x_3^{\nu_5-1} \frac{\Gamma(\nu_{235}-\frac{D}{2})}{\Gamma(\nu_{235})}$$
$$\times \left(\frac{1}{\left((x_2 p + x_3 k_1)^2 - x_2 p^2 - x_3 k_1^2\right)^{\nu_{235}-\frac{D}{2}}}\right).$$

Making the denominator dimensionless by including the term $(-p^2)^{\nu_{235}-m+\varepsilon}$, it can now be rewritten as:

$$-\frac{(x_2^2 p^2 + 2x_2 x_3 p\cdot k_1 + x_3^2 k_1^2)}{p^2} + x_2 + x_3\frac{k_1^2}{p^2}$$
$$= -x_2(x_2-1) - 2x_2x_3\frac{p\cdot k_1}{p^2} - x_3^2\frac{k_1^2}{p^2} + x_3\frac{k_1^2}{p^2} \quad .$$

Using

$$k_4^2 = (k_1-p)^2 = k_1^2 - 2k_1\cdot p + p^2 \Rightarrow -2k_1\cdot p = k_4^2 - k_1^2 - p^2$$

we obtain

$$\begin{aligned}
& -x_2(x_2-1) - 2x_2x_3\frac{p\cdot k_1}{p^2} - x_3(x_3-1)\frac{k_1^2}{p^2} \\
= \; & -x_2(x_2-1) + x_2x_3\left(\frac{k_4^2}{p^2} - \frac{k_1^2}{p^2} - 1\right) - x_3(x_3-1)\frac{k_1^2}{p^2} \\
= \; & (-x_2^2 + x_2 - x_2x_3) + x_2x_3\frac{k_4^2}{p^2} + (-x_3^2 + x_3 - x_2x_3)\frac{k_1^2}{p^2} \\
= \; & x_2(1-x_2-x_3) + x_2x_3\frac{k_4^2}{p^2} + x_3(1-x_2-x_3)\frac{k_1^2}{p^2} \\
= \; & x_2x_1 + x_2x_3\frac{k_4^2}{p^2} + x_3x_1\frac{k_1^2}{p^2}
\end{aligned}$$

$$\Rightarrow \quad I^{(1,3)} = \Gamma(235) \int_0^1 dx_1 dx_2 dx_3\, \delta\left(\sum x_i - 1\right) x_1^{\nu_2-1} x_2^{\nu_3-1} x_3^{\nu_5-1} \frac{\Gamma(\nu_{235}-\frac{D}{2})}{\Gamma(\nu_{235})}$$
$$\times \left(\frac{1}{\left(x_2x_1 + x_2x_3\frac{k_4^2}{p^2} + x_3x_1\frac{k_1^2}{p^2}\right)^{\nu_{235}-\frac{D}{2}}}\right).$$

We now apply the inverse Mellin-Barnes transformation to the expression in parentheses, according to formula (B.46) of the Appendix:

$$\frac{1}{(A_1+A_2)^\nu} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\sigma A_1^\sigma A_2^{-\nu-\sigma} \frac{\Gamma(-\sigma)\Gamma(\nu+\sigma)}{\Gamma(\nu)} \tag{6.11}$$

The integration contour is a parallel to the imaginary axis at real value c, indented in such a way that it separates the poles of the functions $\Gamma(-\sigma)$ from the ones of $\Gamma(\nu+\sigma)$ (see Appendix B).

Define now A_1 and A_2 to be $A_1 := x_2 x_3 \frac{k_4^2}{p^2}$, $A_2 := (x_3 x_1 \frac{k_1^2}{p^2} + x_1 x_2)$ and $\nu := \nu_{235} - \frac{D}{2}$. We then get for the fraction above:

$$\frac{1}{\left(x_2 x_1 + x_2 x_3 \frac{k_4^2}{p^2} + x_3 x_1 \frac{k_1^2}{p^2}\right)^{\nu_{235}-\frac{D}{2}}}$$
$$= \frac{1}{2\pi i} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} d\sigma \left(x_2 x_3 \frac{k_4^2}{p^2}\right)^{\sigma} \left(x_3 x_1 \frac{k_1^2}{p^2} + x_1 x_2\right)^{-\nu_{235}+\frac{D}{2}-\sigma} \frac{\Gamma(-\sigma)\Gamma(\nu_{235}-\frac{D}{2}+\sigma)}{\Gamma(\nu_{235}-\frac{D}{2})}$$

Applying the inverse Mellin-Barnes transformation to $(x_3 x_1 \frac{k_1^2}{p^2} + x_1 x_2)^{-\nu_{235}+\frac{D}{2}-\sigma}$ again, we obtain:

$$\frac{1}{\left(x_2 x_1 + x_2 x_3 \frac{k_4^2}{p^2} + x_3 x_1 \frac{k_1^2}{p^2}\right)^{\nu_{235}-\frac{D}{2}}}$$
$$= \frac{1}{(2\pi i)^2} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} d\sigma \left(x_2 x_3 \frac{k_4^2}{p^2}\right)^{\sigma} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} d\tau \left(x_3 x_1 \frac{k_1^2}{p^2}\right)^{\tau}$$
$$\times (x_1 x_2)^{-(\nu+\sigma+\tau)} \frac{\Gamma(-\sigma)\Gamma(-\tau)\Gamma(\nu_{235}-\frac{D}{2}+\sigma+\tau)}{\Gamma(\nu_{235}-\frac{D}{2})}.$$

Hence the integral becomes

$$I^{(1,3)} =$$
$$\frac{1}{(2\pi i)^2} \frac{1}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)}$$
$$\times \int_0^1 dx_1 dx_2 dx_3\, \delta(\sum x_i - 1) \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} d\sigma \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} d\tau\; x_1^{\nu_2-\nu-\sigma-1} x_2^{\nu_3-\nu-\tau-1} x_3^{\nu_5+\sigma+\tau-1}$$
$$\times \left(\frac{k_4^2}{p^2}\right)^{\sigma} \left(\frac{k_1^2}{p^2}\right)^{\tau} \Gamma(-\sigma)\Gamma(-\tau)\Gamma(\nu_{235} - \frac{D}{2} + \sigma + \tau).$$

Let us consider in a next step the integrals with respect to x_i. We start with the x_1 integration:

$$\begin{aligned}
& \int_0^1 dx_1 dx_2 dx_3\, \delta\left(\sum x_i - 1\right) x_1^{-\nu_3-\nu_5+\frac{D}{2}-\sigma-1} x_2^{-\nu_2-\nu_5+\frac{D}{2}-\tau-1} x_3^{\nu_5+\sigma+\tau-1} \\
= & \int_0^1 dx_2 \int_0^{1-x_2} dx_3 (1-x_2-x_3)^{-\nu_3-\nu_5+\frac{D}{2}-\sigma-1} x_2^{-\nu_2-\nu_5+\frac{D}{2}-\tau-1} x_3^{\nu_5+\sigma+\tau-1} \\
= & \int_0^1 dx_2 \int_0^{1-x_2} dx_3 (1-x_2-x_3)^{a-1} x_2^{b-1} x_3^{c-1}.
\end{aligned}$$

From the first to the second line the delta function causes not only the "substitution" of x_1 by $1-x_2-x_3$, it also forces the integration with respect to x_3 to go from 0 to $1-x_2$, instead of from 0 to 1. In the last line we introduced $a := -\nu_3 - \nu_5 + \frac{D}{2} - \sigma$, $b := -\nu_2 - \nu_5 + \frac{D}{2} - \tau$

and $c := \nu_5 + \sigma + \tau$ for greater convenience. Compare this result with the representation of the beta function (B.26):

$$\int_0^1 dx x^{a-1}(1-x)^{b-1} = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{6.12}$$

We are using this relation to express the corresponding integral in terms of gamma functions. Set $x_3 = (1-x_2)\,x_3'$, $dx_3 = (1-x_2)\,dx_3'$:

$$\begin{aligned}
&\int_0^1 dx_2 \int_0^{1-x_2} dx_3 (1-x_2-x_3)^{a-1} x_2^{b-1} x_3^{c-1} \\
&= \int_0^1 dx_2 x_2^{b-1} \int_0^1 dx_3' (1-x_2)(1-x_2-(1-x_2)x_3')^{a-1}(1-x_2)^{c-1}(x_3')^{c-1} \\
&= \int_0^1 dx_2 x_2^{b-1} \int_0^1 dx_3' (1-x_2)(1-x_2)^{a-1}(1-x_3')^{a-1}(1-x_2)^{c-1}(x_3')^{c-1} \\
&= \int_0^1 dx_2 x_2^{b-1}(1-x_2)^{a+c-1} \int_0^1 dx_3' (1-x_3')^{a-1}(x_3')^{c-1} \\
&= \frac{\Gamma(b)\Gamma(a+c)}{\Gamma(a+b+c)} \frac{\Gamma(c)\Gamma(a)}{\Gamma(a+c)} \\
&= \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)} \\
&= \frac{\Gamma(\nu_5+\sigma+\tau)\Gamma(m-\varepsilon-\sigma-\nu_{35})\Gamma(m-\varepsilon-\tau-\nu_{25})}{\Gamma(2m-2\varepsilon-\nu_{235})}.
\end{aligned}$$

Gathering all the results and putting them together, we get for the integral:

$$\begin{aligned}
I^{(1,3)} =& \frac{1}{(2\pi i)^2} \frac{1}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)\Gamma(2m-2\varepsilon-\nu_{235})} \\
&\times \int_{\gamma_1-i\infty}^{\gamma_1+i\infty} d\sigma \int_{\gamma_2-i\infty}^{\gamma_2+i\infty} d\tau \left(\frac{k_4^2}{p^2}\right)^{\sigma} \left(\frac{k_1^2}{p^2}\right)^{\tau} \Gamma(-\sigma)\Gamma(-\tau)\Gamma(\sigma+\tau+\nu_{235}-m+\varepsilon) \\
&\qquad \times \Gamma(\sigma+\tau+\nu_5)\Gamma(-\sigma+m-\varepsilon-\nu_{35})\Gamma(-\tau+m-\varepsilon-\nu_{25}).
\end{aligned} \tag{6.13}$$

By inserting this result into (6.6), the terms $\left(\frac{k_4^2}{p^2}\right)^{\sigma}$ and $\left(\frac{k_1^2}{p^2}\right)^{\tau}$ now cause a change in the exponents of the corresponding propagators and we finally obtain the following Mellin-Barnes integral for the integral (6.4):

$$\begin{aligned}
\hat{I}^{(2,5)} =& \frac{1}{(2\pi i)^2} \frac{1}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)\Gamma(2m-2\varepsilon-\nu_{235})} \\
&\times \int_{\gamma_1-i\infty}^{\gamma_1+i\infty} d\sigma \int_{\gamma_2-i\infty}^{\gamma_2+i\infty} d\tau \, \frac{\Gamma(-\sigma)\Gamma(-\sigma+m-\varepsilon-\nu_{35})\Gamma(\sigma+m-\varepsilon-\nu_4)}{\Gamma(-\sigma+\nu_4)} \\
&\qquad \times \frac{\Gamma(-\tau)\Gamma(-\tau+m-\varepsilon-\nu_{25})\Gamma(\tau+m-\varepsilon-\nu_1)}{\Gamma(-\tau+\nu_1)} \\
&\qquad \times \frac{\Gamma(-\sigma-\tau+\nu_{14}-m+\varepsilon)\Gamma(\sigma+\tau-m+\varepsilon+\nu_{235})\Gamma(\sigma+\tau+\nu_5)}{\Gamma(\sigma+\tau+2m-2\varepsilon-\nu_{14})}
\end{aligned} \tag{6.14}$$

6.1.3 Collecting residues — general remarks

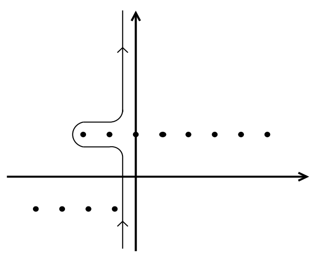
In the last section we transformed the integral $\hat{I}^{(2,5)}$ into a Mellin-Barnes integral (6.14). The aim now is to expand this integral in the dimensional regularization parameter ε. Before we start to perform this expansion we have to discuss some general issues concerning this expansion:

Residues

The basic idea is to close the integration contours of the Mellin-Barnes integrals (6.14) to the left or the right, and to sum up the residues of the gamma functions lying inside the region enclosed by the integration contour. For the residues of the gamma function we find:

$$\text{res}(\Gamma(-x+a), x=a+n) = -\frac{(-1)^n}{n!}\,, \qquad \text{res}(\Gamma(x+a), x=-a-n) = +\frac{(-1)^n}{n!} \tag{6.15}$$

The piece of the integration contour parallel to the imaginary axis lies at a real value such that it separates the poles of the gamma functions of the form $\Gamma(-x+a)$, which lie to the right of it, from those of the functions $\Gamma(x+a)$ placed to its left. If this cannot be achieved by a straight line, this parallel is indented appropriately as shown in the picture on the right.



We choose to close the contour to the right at $+\infty$. Hence only the gamma functions of the form $\Gamma(-x+a)$ contribute when summing up the poles. Since we close the contour to the right, the integration is done clockwise and we get an additional minus sign that cancels the one appearing in the formula for the residues. In total, we get $\frac{(-1)^n}{n!}$ for the gamma function $\Gamma(-x+a)$ at the points $x = a+n$ for $n \in \mathbb{N}$ and an infinite sum $\sum_n \frac{(-1)^n}{n!}$ over all poles.

The general idea

We will always start with the integration with respect to σ and perform the τ integration in a second step. The calculation is done according to the following recipe:

1. Find all gamma functions Γ_σ that can contribute poles for the σ integration

2. Choose one particular gamma function $\Gamma(-\sigma+a) \in \Gamma_\sigma$ and take the whole integral (6.14) at the value $\sigma = a+n$, where this chosen gamma function contributes the poles. Substitute this gamma function by the sum over its residues.

3. Find the gamma functions Γ_τ that contribute poles for the remaining τ integration and continue analogously to step 2) for the σ integration by choosing one particular gamma function out of Γ_τ, taking the integral (6.14) at that certain value of τ for which this chosen gamma function has its poles, and substituting the chosen gamma function by the sum over its residues. Continue in this way with every gamma function in Γ_τ.

4. Take the next gamma function in Γ_σ and apply steps 2 and 3 to the integral for this chosen gamma function.

In this way one obtains all possible combinations of gamma functions with poles inside the integration contour for the σ integration and the τ integration. In a last step one has to sum up all these cases.

We see that there are five gamma functions in the numerator of (6.14) that are of the form $\Gamma(-x+a)$, with x being either σ or τ:

$$\Gamma(-\sigma), \quad \Gamma(-\sigma+m-\varepsilon-\nu_{35}), \quad \Gamma(-\tau), \quad \Gamma(-\tau+m-\varepsilon-\nu_{25}), \quad \Gamma(-\sigma-\tau+\nu_{14}-m+\varepsilon)$$

Note that since one gamma function contains both variables, we have three gamma functions with contributing poles for each integration variable.

We do not have to consider gamma functions $\Gamma(-\sigma+a)$ and $\Gamma(-\tau+b)$ in the denominator of integrals, because they do not contribute poles. However, they can affect the calculation by canceling poles from gamma functions in the numerator of the integral, once we start collecting the residues. Hence they can in our case terminate the sum of the residues which we would obtain and let it end at a finite value. Since the only relevant gamma functions for such a case are $\Gamma(-\sigma+\nu_4)$ and $\Gamma(-\tau+\nu_1)$, this can only happen when ν_1 or ν_4 are integers. The poles in the numerator and denominator then cancel starting at the value ν_1 (ν_4). Consider for example the combination $\Gamma(-\sigma)/\Gamma(-\sigma+\nu_4)$. The function $\Gamma(-\sigma)$ has poles at $\sigma=0,1,2,...$, whereas $\Gamma(-\sigma+\nu_4)$ on the other hand has poles at $\sigma=\nu_4,\nu_4+1,\nu_4+2,....$ Hence, starting from the positive integer ν_4, the poles of the gamma functions cancel and the sum for the residue of the function $\Gamma(-\sigma)$ is restricted to the set $0,...,\nu_4-1$.

The different cases

Let us now have a closer look at the different cases: for each integration we obtain a sum over the residues. Hence the general result will be an infinite double sum. In the following, we will use the convention that the sum obtained from the integral $d\sigma$ will always have the summation index n, and the sum obtained from the $d\tau$ integration will have the summation index j:

$$\int d\sigma \rightarrow \sum_n \qquad\qquad \int d\tau \rightarrow \sum_j$$

Since in some gamma functions both variables occur, the evaluation with respect to σ naturally affects the one over τ: it shifts the poles. Consider for example the following pair of gamma functions:

$$\begin{aligned} &\text{for the } \sigma \text{ integration:} && \Gamma(-\sigma+m-\varepsilon-\nu_{35}),\\ &\text{for the } \tau \text{ integration:} && \Gamma(-\tau-\sigma+\nu_{14}-m+\varepsilon). \end{aligned}$$

Doing the σ integration first, we only get residues if $\sigma=m-\varepsilon-\nu_{35}+n$. Therefore we have to take the remaining gamma functions at this particular value of σ, which also shifts $\Gamma(-\tau-\sigma+\nu_{14}-m+\varepsilon)$ to

$$\Gamma(-\tau-\sigma+\nu_{14}-m+\varepsilon)|_{\sigma=m-\varepsilon-\nu_{35}+n} = \Gamma(-\tau+n+\nu_{1345}-2m+2\varepsilon).$$

Hence the σ integration shifts the poles for τ to the values $\tau=-n-\nu_{1345}+2m-2\varepsilon+j$.

Since we have three gamma functions contributing poles for each integration and we have to take all possible combinations of these functions, we obtain nine terms in total. Additionally, we will have two terms T34 and T35 whose appearance will be explained below:

$$\begin{aligned}
&\text{T11}: && \Gamma(-\sigma) \quad \Gamma(-\tau)\\
&\text{T12}: && \Gamma(-\sigma) \quad \Gamma(-\tau+m-\varepsilon-\nu_{25})\\
&\text{T13}: && \Gamma(-\sigma) \quad \Gamma(-\sigma-\tau+\nu_{14}-m+\varepsilon)|_{\sigma=n}\\[1em]
&\text{T21}: && \Gamma(-\sigma+m-\varepsilon-\nu_{35}) \quad \Gamma(-\tau)\\
&\text{T22}: && \Gamma(-\sigma+m-\varepsilon-\nu_{35}) \quad \Gamma(-\tau+m-\varepsilon-\nu_{25})\\
&\text{T23}: && \Gamma(-\sigma+m-\varepsilon-\nu_{35}) \quad \Gamma(-\sigma-\tau+\nu_{14}-m+\varepsilon)|_{\sigma=m-\varepsilon-\nu_{35}+n}\\[1em]
&\text{T31}: && \Gamma(-\sigma-\tau+\nu_{14}-m+\varepsilon) \quad \Gamma(-\tau)\\
&\text{T32}: && \Gamma(-\sigma-\tau+\nu_{14}-m+\varepsilon) \quad \Gamma(-\tau+m-\varepsilon-\nu_{25})\\
&\text{T33}: && \Gamma(-\sigma-\tau+\nu_{14}-m+\varepsilon) \quad \Gamma(-\sigma-\tau+\nu_{14}-m+\varepsilon)|_{\sigma=-\tau+\nu_{14}-m+\varepsilon+n}\\
&\text{T34}: && \Gamma(-\sigma-\tau+\nu_{14}-m+\varepsilon) \quad \Gamma(-\sigma)|_{\sigma=-\tau+\nu_{14}-m+\varepsilon+n}\\
&\text{T35}: && \Gamma(-\sigma-\tau+\nu_{14}-m+\varepsilon) \quad \Gamma(-\sigma+m-\varepsilon-\nu_{35})|_{\sigma=-\tau+\nu_{14}-m+\varepsilon+n}
\end{aligned} \tag{6.16}$$

The cases Txy are denoted such that the first index always belongs to the gamma function providing the poles for the σ integration and the second one to the function yielding the poles of the τ integration. In the end, we have to sum up all these different cases to obtain the result for (6.14):

$$\hat{I}^{(2,5)} = \text{T11}+\text{T12}+\text{T13}+\text{T21}+\text{T22}+\text{T23}+\text{T31}+\text{T32}+\text{T33}+\text{T34}+\text{T35}. \tag{6.17}$$

Why do we have the cases T34 and T35?

We want to calculate the integral for arbitrary variables m and ν_i. However, to make statements concerning the poles and to be able to decide which poles have to be taken into account one has to do the calculation once for arbitrary but fixed values of the variables m and ν_i. For other values the poles simply "move" in the complex plane, but always just by a finite distance. Therefore, this "moving of poles" does not cause a problem in the general case. One can always deform the contour in such a way that the poles stay inside the integration area. Hence, although the following calculation is done and its result programmed for arbitrary variables, we had to choose some special values for m and ν_i to decide at some point of the calculation whether the poles start to move in or out of the integration area (c.f. T13, T23). We chose $m = 2$, $\nu_i = 1$ and the contour parallel to the imaginary axis at Re$z = -1/3$ as special values. One can convince oneself that this is a possible choice for the real part of the contour parallel to the imaginary axis for the chosen set of m and ν_i.

Some more techniques

Since the functions `transcendental_fct_type_A()` to `transcendental_fct_type_D()` can only handle gamma functions of the form $\Gamma(n+a)$ and not $\Gamma(-n+a)$ inside the terms of an infinite sum, we will transform such gamma functions with a negative variable n or j into ones with a positive n or j via the "reflection formula" (B.16):

$$\Gamma(-n+a) = (-1)^n \frac{\Gamma(a)\Gamma(1-a)}{\Gamma(n+1-a)} \tag{6.18}$$

Note again that, fortunately, this only has to be done for gamma functions appearing inside an infinite sum.

Although the final result of our calculation is finite, we might produce some infinities in intermediate states. Since in the process of our calculation we express the two-loop integral as a combination of several sums with a "1" in the numerator and, on the other hand, the algorithms of *nestedsums* use partial fractioning, one convergent sum might be split into two divergent series. For example, the constituent terms of the convergent sum:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \tag{6.19}$$

would be split into

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)}, \tag{6.20}$$

producing two divergent sums. To avoid this problem, we introduce an additional factor x^n and multiply the term of the sum with it. At the end we then take the limit $x \to 1$. More precisely, we insert $x_1^\sigma x_2^\tau$ into the integrand of (6.14). This leads to convergent sums as long as $0 \leq x_2 < x_1 < 1$. The result for the two-loop integral is then recovered by first sending $x_1 \to 1$ and afterwards $x_2 \to 1$.

Convergence of the integral

To be able to apply the theorem of residues, we have to ensure that the contour at infinity gives a vanishing contribution. We find three conditions for the values ν_i:

$$\begin{aligned} &\nu_1 + \nu_{125} - 2m + 2\varepsilon < 1 \\ &\nu_4 + \nu_{345} - 2m + 2\varepsilon < 1 \\ &\qquad -1 < (\nu_1 + \nu_{125} - 2m + 2\varepsilon) + (\nu_4 + \nu_{345} - 2m + 2\varepsilon). \end{aligned} \tag{6.21}$$

They are obtained in the following way: Since we always start closing the contour with respect to σ and afterwards with respect to τ, we first have to guarantee that the integrand in (6.14) vanishes for $\sigma \to \infty$. In a second step we then look at the integrals corresponding to the cases T1x to T3x. These are the remaining integrals for τ that have to be considered after closing the contour and collecting the residues for σ, and which have to vanish for $\tau \to \infty$.

To derive equations (6.21) we use *Barnes asymptotic expansion* of the gamma function, which is valid for $|x| \to \infty$ and $|\arg x| < \pi$:

$$\ln \Gamma(x+c) \sim (x+c)\ln x - x - \frac{1}{2}\ln\frac{x}{2\pi} - \sum_{\mathrm{n}=1}^{\infty} \frac{\mathrm{B}_{n+1}}{n(n+1)} \left(-\frac{1}{x}\right)^n. \tag{6.22}$$

If we investigate for instance the asymptotic behavior for $\sigma \to \infty$, there are eight gamma functions depending on σ in (6.14) that we have to consider. Six of those appear in the numerator, two in the denominator:

numerator:	denominator:
$\Gamma(-\sigma)$	$\Gamma(-\sigma+\nu_4)$
$\Gamma(-\sigma+m-\varepsilon-\nu_{35})$	$\Gamma(\sigma+\tau+2m-2\varepsilon-\nu_{14})$
$\Gamma(\sigma+m-\varepsilon-\nu_4)$	
$\Gamma(-\sigma-\tau-m+\varepsilon+\nu_{14})$	
$\Gamma(\sigma+\tau-m+\varepsilon+\nu_{235})$	
$\Gamma(\sigma+\tau+\nu_5)$	

These are the contributing functions for $\sigma \to \infty$. Taking the logarithm of the fraction of these gamma functions, we have to add and subtract the expressions for these functions according to (6.22). The terms $\pm\sigma$ immediately cancel when we sum up the eight cases. Considering only the relevant terms, which are the ones proportional to $\ln \sigma$, we find that these consist of $\ln \sigma$ multiplied by a factor originating from the constant c in (6.22) and the term $\frac{ln(\pm\sigma)}{2\pi}$. Considering, for example, the convergence of the first integral with respect to σ, we obtain:

$$(-2m+2\varepsilon+\nu_1+\nu_{125}-2)\ln\sigma. \tag{6.23}$$

Since we are interested in the behavior of the fraction of gamma functions, we now have to exponentiate expression (6.23), leading to

$$\sigma^{(-2\mathrm{m}+2\varepsilon+\nu_1+\nu_{125}-2)}. \tag{6.24}$$

This is the expression that has to vanish for $\sigma \to \infty$, and from this we obtain the first condition in (6.21). The cases T1x and T2x both lead to the second condition, while the third one stems from the case T3x.

6.1.4 Collecting residues — the different cases

T11, T12, T21, and T22

Let us first consider the cases T11, T12, T21, and T22. For all of these cases the σ integration does not affect the poles in τ and they can all be calculated in a similar way. As an example we take T11, and hence the gamma functions $\Gamma(-\sigma)$ and $\Gamma(-\tau)$.

For both gamma functions the poles in the numerator and denominator of the integral might possibly cancel, namely if $\nu_4 = n_4 + a_4\varepsilon$ for σ or $\nu_1 = n_1 + a_1\varepsilon$ for τ are integers. We therefore have to distinguish between the cases:

$$\textit{Case 1}: \quad a_1 = 0, \quad a_4 = 0 \tag{6.25a}$$
$$\textit{Case 2}: \quad a_1 \neq 0, \quad a_4 = 0 \tag{6.25b}$$
$$\textit{Case 3}: \quad a_1 = 0, \quad a_4 \neq 0 \tag{6.25c}$$
$$\textit{Case 4}: \quad a_1 \neq 0, \quad a_4 \neq 0 \tag{6.25d}$$

Case 1:
Both sums are finite: The sum over n ranges from $0, ..., \nu_4 - 1$, the sum over j from $0, ..., \nu_1 - 1$. As both sums are finite, we do not need to apply the "reflection formula" for any of the gamma functions:

$$\begin{aligned} \text{T11} &= \frac{1}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)\Gamma(2m-2\varepsilon-\nu_{235})} \\ &\quad \times \sum_{n=0}^{\nu_4-1}\sum_{j=0}^{\nu_1-1} \frac{(-1)^{(n+j)}x^n y^j}{n!j!} \frac{\Gamma(-n)\Gamma(-n+m-\varepsilon-\nu_{35})\Gamma(n+m-\varepsilon-\nu_4)}{\Gamma(-n+\nu_4)} \\ &\qquad\qquad \times \frac{\Gamma(-j)\Gamma(-j+m-\varepsilon-\nu_{25})\Gamma(j+m-\varepsilon-\nu_1)}{\Gamma(-j+\nu_1)} \\ &\qquad\qquad \times \frac{\Gamma(-n-j+\nu_{14}-m+\varepsilon)\Gamma(n+j-m+\varepsilon+\nu_{235})\Gamma(n+j+\nu_5)}{\Gamma(n+j+2m-2\varepsilon-\nu_{14})} \end{aligned} \tag{6.26}$$

Case 4:
In *case 4*, both sums tend to infinity and one has to apply (6.18) to all places where σ or τ occur with a negative sign inside a gamma function:

$$\begin{aligned} \text{T11} &= \Gamma(-m+\varepsilon+\nu_{14})\Gamma(1+m-\varepsilon-\nu_{14}) \\ &\quad \times \frac{\Gamma(m-\varepsilon-\nu_{35})\Gamma(1-m+\varepsilon+\nu_{35})\Gamma(m-\varepsilon-\nu_{25})\Gamma(1-m+\varepsilon+\nu_{25})}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)\Gamma(\nu_5)\Gamma(2m-2\varepsilon-\nu_{235})\Gamma(1-\nu_1)\Gamma(1-\nu_4)} \\ &\quad \times \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} x^n y^j \frac{\Gamma(n+m-\varepsilon-\nu_4)\Gamma(n+1-\nu_4)\Gamma(j+m-\varepsilon-\nu_1)\Gamma(j+1-\nu_1)}{\Gamma(n+1)\Gamma(n+1-m+\varepsilon+\nu_{35})\Gamma(j+1)\Gamma(j+1-m+\varepsilon+\nu_{25})} \\ &\qquad\qquad \times \frac{\Gamma(n+j-m+\varepsilon+\nu_{235})\Gamma(n+j+\nu_5)}{\Gamma(n+j+2m-2\varepsilon-\nu_{14})\Gamma(n+j+1+m-\varepsilon-\nu_{14})} \end{aligned} \tag{6.27}$$

Cases 2 and *3*:
These are mixed versions of *cases 1* and *4* with one finite and one infinite sum.
Case 2:

$$\begin{aligned} \text{T11} &= \frac{\Gamma(m-\varepsilon-\nu_{25})\Gamma(1-m+\varepsilon+\nu_{25})}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)\Gamma(2m-2\varepsilon-\nu_{235})\Gamma(1-\nu_1)} \\ &\sum_{n=0}^{\nu_4-1}(-x)^n \frac{\Gamma(-n+m-\varepsilon-\nu_{35})\Gamma(n+m-\varepsilon-\nu_4)\Gamma(-n-m+\varepsilon+\nu_{14})\Gamma(1+n+m-\varepsilon-\nu_{14})}{\Gamma(n+1)\Gamma(-n+\nu_4)} \\ &\sum_{j=0}^{\infty} y^j \frac{\Gamma(j+m-\varepsilon-\nu_1)\Gamma(n+j-m+\varepsilon+\nu_{235})\Gamma(n+j+\nu_5)\Gamma(j+1-\nu_1)}{\Gamma(n+j+2m-2\varepsilon-\nu_{14})\Gamma(j+1-m+\varepsilon+\nu_{25})\Gamma(n+j+1+m-\varepsilon-\nu_{14})\Gamma(j+1)} \end{aligned} \tag{6.28}$$

Case 3:

$$
\begin{aligned}
\mathrm{T11} &= \frac{\Gamma(m-\varepsilon-\nu_{35})\Gamma(1-m+\varepsilon+\nu_{35})}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)\Gamma(\nu_5)\Gamma(2m-2\varepsilon-\nu_{235})\Gamma(1-\nu_4)} \\
&\sum_{j=0}^{\nu_1-1}(-y)^j \frac{\Gamma(-j+m-\varepsilon-\nu_{25})\Gamma(j+m-\varepsilon-\nu_1)\Gamma(-j-m+\varepsilon+\nu_{14})\Gamma(1+j+m-\varepsilon-\nu_{14})}{\Gamma(j+1)\Gamma(-j+\nu_1)} \\
&\sum_{n=0}^{\infty} x^n \frac{\Gamma(n+m-\varepsilon-\nu_4)\Gamma(n+j-m+\varepsilon+\nu_{235})\Gamma(n+j+\nu_5)\Gamma(n+1-\nu_4)}{\Gamma(n+j+2m-2\varepsilon-\nu_{14})\Gamma(n+1-m+\varepsilon+\nu_{35})\Gamma(n+j+1+m-\varepsilon-\nu_{14})\Gamma(n+1)}
\end{aligned}
\tag{6.29}
$$

Analogous calculations and statements apply to the combinations of gamma functions leading to T12 (2 cases), T21 (2 cases) and T22 (1 case).

T13 and T23

In this part we need to consider fixed values for m, ν_i and the real part $\mathrm{Re}(\gamma)$ of the integration contour parallel to the imaginary axis. The reason for this is that the poles will move out of the area enclosed by the contour once we start performing the τ integration: After accomplishing the σ integration, a dependence on n from the residue of this first integration enters the relevant gamma function for the τ integration and changes the values of the poles in τ. For T13 we obtain

$$
\Gamma(-\tau-\sigma+\nu_{14}-m+\varepsilon) \;\rightarrow\; \Gamma(-n-\tau+\nu_{14}-m+\varepsilon)\,,
$$

and for T23

$$
\begin{aligned}
\Gamma(-\tau-\sigma+\nu_{14}-m+\varepsilon) &\rightarrow \Gamma(-(m-\varepsilon-\nu_{35}+n)-\tau+\nu_{14}-m+\varepsilon) \\
&= \Gamma(-n-\tau+\nu_{1345}-2m+2\varepsilon).
\end{aligned}
$$

We have to check if, for increasing n, these poles move out of the integration area. Taking the values $m=2$ and $\nu_i=1$, as described before, we obtain for the relevant gamma function yielding the poles for T13: $\Gamma(-n-\tau+\nu_{14}-m+\varepsilon)=\Gamma(-n-\tau+\varepsilon)$. This means that the poles lie at $\tau=-n+\varepsilon+j$ and depend on n. For $n=0$ we obtain poles at $\tau=\varepsilon+j$ leading to poles that are enclosed by the contour and contribute for $j\in\mathbb{N}_0$. For $n=1$ we have $\tau=-1+\varepsilon+j$. Here the pole for $j=0$ is no longer placed inside the integration area! The first pole lying inside is the one for $j=1$ ($\tau=\varepsilon$). It is easy to see by induction that this can be generalized and that the poles for j always contribute starting from $j=n$. Hence the second sum, emerging from the integration over τ, starts from n instead of 0.

Since we are talking about the case T13 and therefore about the function $\Gamma(-\sigma)$, we again have to distinguish between the cases $a_4\neq 0$ and $a_4=0$.

i) $\underline{a_4 \neq 0}$:

$$
\begin{aligned}
\text{T13} \;=\; & \frac{1}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)\Gamma(2m-2\varepsilon-\nu_{235})} \\
& \times \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \sum_{j=n}^{\infty} \frac{(-1)^j}{j!} y^{-n+j+\nu_{14}-m+\varepsilon} \frac{\Gamma(-n+m-\varepsilon-\nu_{35})\Gamma(n+m-\varepsilon-\nu_4)}{\Gamma(-n+\nu_4)} \\
& \times \frac{\Gamma(n-j+m-\varepsilon-\nu_{14})\Gamma(n-j+2m-2\varepsilon-\nu_{1245})\Gamma(-n+j+\nu_4)}{\Gamma(n-j+m-\varepsilon-\nu_4)} \\
& \times \frac{\Gamma(j-2m+2\varepsilon+\nu_{12345})\Gamma(j-m+\varepsilon+\nu_{145})}{\Gamma(j+m-\varepsilon)}
\end{aligned}
\tag{6.30}
$$

In order to be able to use the *nestedsums* functions `transcendental_fct_type_A()` and `transcendental_fct_type_B()` both sums have to start at 0, though. Therefore, the sums have to be rewritten by shifting the indices:

$$
\sum_{n=0}^{\infty}\sum_{j=n}^{\infty} f(n,j) \rightarrow \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} f(n_1,n_2; j=n_1+n_2, n=n_2).
$$

The function f is a replacement character for the fraction of gamma functions obtained in this case. Hence we get

$$
\begin{aligned}
\text{T13} \;=\; & y^{\nu_{14}-m+\varepsilon} \frac{\Gamma(m-\varepsilon-\nu_{35})\Gamma(1-m+\varepsilon+\nu_{35})}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)\Gamma(2m-2\varepsilon-\nu_{235})} \\
& \times \frac{\Gamma(m-\varepsilon-\nu_{14})\Gamma(1-m+\varepsilon+\nu_{14})\Gamma(2m-2\varepsilon-\nu_{1245})\Gamma(1-2m+2\varepsilon+\nu_{1245})}{\Gamma(\nu_4)\Gamma(1-\nu_4)\Gamma(m-\varepsilon-\nu_4)\Gamma(1-m+\varepsilon+\nu_4)} \\
& \times \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{x^{n2} y^{n_1}}{(n_1+n_2)! n_2!} \frac{\Gamma(n_1+\nu_4)\Gamma(n_1+1-m+\varepsilon+\nu_4)}{\Gamma(n_1+1-m+\varepsilon+\nu_{14})\Gamma(n1+1-2m+2\varepsilon+\nu_{1245})} \\
& \times \frac{\Gamma(n_2+1-\nu_4)\Gamma(n_2+m-\varepsilon-\nu_4)}{\Gamma(n_2+1-m+\varepsilon+\nu_{35})} \\
& \times \frac{\Gamma(n_1+n_2-2m+2\varepsilon+\nu_{12345})\Gamma(n_1+n_2-m+\varepsilon+\nu_{145})}{\Gamma(n_1+n_2+m-\varepsilon)}.
\end{aligned}
\tag{6.31}
$$

ii) $\underline{a_4 = 0}$: The sum over n only runs to $\nu_4 - 1$.

$$
\begin{aligned}
\text{T13} \;=\; & \frac{1}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)\Gamma(2m-2\varepsilon-\nu_{235})} \\
& \times \sum_{n=0}^{\nu_4-1} \frac{(-1)^n}{n!} x^n \sum_{j=n}^{\infty} \frac{(-1)^j}{j!} y^{-n+j+\nu_{14}-m+\varepsilon} \frac{\Gamma(-n+m-\varepsilon-\nu_{35})\Gamma(n+m-\varepsilon-\nu_4)}{\Gamma(-n+\nu_4)} \\
& \times \frac{\Gamma(n-j+m-\varepsilon-\nu_{14})\Gamma(n-j+2m-2\varepsilon-\nu_{1245})\Gamma(-n+j+\nu_4)}{\Gamma(n-j+m-\varepsilon-\nu_4)} \\
& \times \frac{\Gamma(j-2m+2\varepsilon+\nu_{12345})\Gamma(j-m+\varepsilon+\nu_{145})}{\Gamma(j+m-\varepsilon)}.
\end{aligned}
\tag{6.32}
$$

In this case it is sufficient to shift the index j in such a way that the sum again starts at $j = 0$:

$$\sum_{j=n}^{\infty} f(n,j) \to \sum_{j'=0}^{\infty} f(n,j';j'=j-n) \tag{6.33}$$

This leads to the result

$$\begin{aligned} \text{T13} \; = \; & \frac{\Gamma(m-\varepsilon-\nu_{14})\Gamma(1-m+\varepsilon+\nu_{14})\Gamma(2m-2\varepsilon-\nu_{1245})\Gamma(1-2m+2\varepsilon+\nu_{1245})}{\Gamma(m-\varepsilon-\nu_4)\Gamma(1-m+\varepsilon+\nu_4)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)\Gamma(2m-2\varepsilon-\nu_{235})} \\ & \times \sum_{n=0}^{\nu_4-1} \frac{(-1)^n}{n!} x^n \frac{\Gamma(-n+m-\varepsilon-\nu_{35})\Gamma(n+m-\varepsilon-\nu_4)}{\Gamma(-n+\nu_4)} y^{\nu_{14}-m+\varepsilon} \\ & \times \sum_{j=0}^{\infty} \frac{(-1)^{n+j}}{(n+j)!} y^j \frac{\Gamma(j+1-m+\varepsilon+\nu_4)\Gamma(j+\nu_4)}{\Gamma(j+1+\nu_{14}-m+\varepsilon)\Gamma(j+1+\nu_{1245}-2m+2\varepsilon)} \\ & \qquad \times \frac{\Gamma(n+j-2m+2\varepsilon+\nu_{12345})\Gamma(n+j-m+\varepsilon+\nu_{145})}{\Gamma(n+j+m-\varepsilon)}. \end{aligned} \tag{6.34}$$

T23 can be calculated analogously to T13, apart from the fact, of course, that there is only one case, similar to the case $a_4 \neq 0$ in T13, since it does not matter whether a_1 and a_4 are real or complex.

T3x - the general case

Consider the case where the σ integration has already been performed and the residues for the function $\Gamma(-\sigma-\tau+\nu_{14}-m+\varepsilon)$ collected, but we still have to do the integration with respect to τ:

$$\begin{aligned} \text{T3}x \; = \; & \frac{1}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)\Gamma(2m-2\varepsilon-\nu_{235})} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d\tau \; x^{\nu_{14}-m+\varepsilon} \; x^{n-\tau} y^{\tau} \\ & \times \frac{\Gamma(n-2m+2\varepsilon+\nu_{12345})\Gamma(n-m+\varepsilon+\nu_{145})}{\Gamma(n+m-\varepsilon)} \\ & \times \frac{\Gamma(\tau-n+m-\varepsilon-\nu_{14})\Gamma(\tau-n+2m-2\varepsilon-\nu_{1345})}{\Gamma(\tau-n+m-\varepsilon-\nu_1)} \\ & \times \Gamma(-\tau)\Gamma(-\tau+m-\varepsilon-\nu_{25})\Gamma(\tau+m-\varepsilon-\nu_1) \frac{\Gamma(-\tau+n+\nu_1)}{\Gamma(-\tau+\nu_1)}. \end{aligned} \tag{6.35}$$

The combination of gamma functions $\frac{\Gamma(-\tau+n+\nu_1)}{\Gamma(-\tau+\nu_1)}$ will soon be of special importance. The different cases T3x are related to the following gamma functions:

T31 : $\Gamma(-\tau)$
T32 : $\Gamma(-\tau+m-\varepsilon-\nu_{25})$
T33 : $\Gamma(-\tau+n+\nu_1)$
T34 : $\Gamma(\tau-n-\nu_{14}+m-\varepsilon)$
T35 : $\Gamma(\tau-n+2m-2\varepsilon-\nu_{1345})$

T31 and T32

In T31, we have to distinguish again between the two cases ν_4 real and ν_4 complex. After closing the contour for the τ integration we obtain

$$\begin{aligned}\mathrm{T31} \; &= \frac{1}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)\Gamma(2m-2\varepsilon-\nu_{235})} \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^n}{n!}\frac{(-1)^j}{j!} x^{\nu_{14}-m+\varepsilon}\, x^{n-j} y^j \\ &\quad \times \frac{\Gamma(n-2m+2\varepsilon+\nu_{12345})\Gamma(n-m+\varepsilon+\nu_{145})}{\Gamma(n+m-\varepsilon)} \\ &\quad \times \frac{\Gamma(j-n+m-\varepsilon-\nu_{14})\Gamma(j-n+2m-2\varepsilon-\nu_{1345})}{\Gamma(j-n+m-\varepsilon-\nu_1)} \\ &\quad \times \Gamma(-j+m-\varepsilon-\nu_{25})\Gamma(j+m-\varepsilon-\nu_1)\frac{\Gamma(-j+n+\nu_1)}{\Gamma(-j+\nu_1)}. \end{aligned} \tag{6.36}$$

i) $\underline{a_4 \neq 0}$:

One problem now arises from the gamma functions that have an argument consisting of a combination of n and j with changing sign, as one cannot achieve a positive sign for both of them simultaneously. The sums will therefore be separated and rewritten in the following way:

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} f(n,j) &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} f(n,j) + \sum_{n=0}^{\infty}\sum_{j=n}^{\infty} f(n,j) - \sum_{j=n=0}^{\infty} f(n,j) \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n} f(n_1,n_2; n=n_1+n_2, j=n_2) \\ &\quad + \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n} f(n_1,n_2; n=n_2, j=n_1+n_2) - \sum_{j=n=0}^{\infty} f(n,j),\end{aligned}$$

and we continue afterwards like in the cases before.

ii) $\underline{a_4 = 0}$:

Let us take a closer look at the fraction $\frac{\Gamma(-j+n+\nu_1)}{\Gamma(-j+\nu_1)}$. This combination does not contribute any poles, since the poles of the denominator start “further to the left” in the complex plane than the ones in the numerator. Nevertheless, this fraction is of course not zero and is in an expanded form equal to

$$\frac{\Gamma(-j+n+\nu_1)}{\Gamma(-j+\nu_1)} = \frac{(-j+n+\nu_1-1)!}{(-j+\nu_1-1)!} = (-j+n+\nu_1-1)\cdot \ldots \cdot(-j+\nu_1). \tag{6.37}$$

For this product to not have any term equal to zero, we need the condition $j < \nu_1$ or $j > n+\nu_1-1$. These are the two cases that have to be calculated:

$$1) \quad \sum_{n=0}^{\infty}\sum_{j=0}^{\nu_1-1} f(n,j) \qquad\qquad 2) \quad \sum_{n=0}^{\infty}\sum_{j=n+\nu_1}^{\infty} f(n,j)$$

Case 1) is completely analogous to cases we have already discussed, with one finite and one infinite sum where no shifting is needed. However, in case 2) the sums have to be shifted again:

$$\sum_{n=0}^{\infty}\sum_{j=\nu_1}^{\infty} f(n,j) = \sum_{n=0}^{\infty}\sum_{j'=n}^{\infty} f(n,j';j'=j-\nu_1) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} f(n_1,n_2;n=n_1+n_2,j'=n_2).$$

Proceeding through the calculation, one also has to apply the identity:

$$\begin{aligned}\frac{\Gamma(-j+n+\nu_1)}{\Gamma(-j+\nu_1)} &= (-j+\nu_1)\cdot\ldots\cdot(-j+n+\nu_1-1)\\ &= (-1)^n(j-\nu_1)\cdot\ldots\cdot(j-\nu_1-n+1)\\ &= (-1)^n\frac{\Gamma(j-\nu_1+1)}{\Gamma(j-\nu_1-n+1)}\end{aligned}$$

to avoid the appearance of terms like

$$\frac{\Gamma(-j+n)}{\Gamma(-j)}.$$

T32 is calculated like T31, except for the fact that there is no case $a_4 = 0$. One only has to do the splitting into three sums as for T31, $a_4 \neq 0$.

T33

The combination of gamma functions

$$\frac{\Gamma(-\tau+n+\nu_1)}{\Gamma(-\tau+\nu_1)}$$

does not contribute any poles and therefore T33 = 0.

T34 and T35

The fact that we have cases T34 and T35 at all has to be explained first. The corresponding gamma functions related to these two are:

$$\begin{aligned}\text{T34}: &\quad \Gamma(\tau-n-\nu_{14}+m-\varepsilon)\\ \text{T35}: &\quad \Gamma(\tau-n+2m-2\varepsilon-\nu_{1345}).\end{aligned}$$

The variables τ and n have different signs again. So for different values of n, poles of this gamma function move into the integration area of the integral over τ and we obtain, similar to the results in T13 and T23, a dependence of the poles of the τ integration on the value of n. We show this by an example: Considering our pre-defined values for m and ν_i, we find for the gamma function of the case T34 that $\Gamma(\tau-n-\nu_{14}+m-\varepsilon) = \Gamma(-\tau-n+\varepsilon)$. Hence the poles are at $\tau = n+\varepsilon-j$ and therefore dependent on n. For $n=0$ we get $\tau=\varepsilon-j$, and only for the case $j=0$ is the pole inside the integration contour. By a similar consideration one can convince oneself that the sum with respect to j for general n always runs up to n.

Analogous statements apply to T35 and this almost suffices to solve this case. The only remaining calculational step is to bring the sums ($n = 0...\infty$, $j = 0...n$) into the form of two infinite sums (with respect to n_1 and n_2). However, what has to be taken into account is that we are dealing with gamma functions of the form $\Gamma(x+a)$ that lead to a residue $+\frac{(-1)^n}{n!}$. Therefore, compared to the other results, T34 and T35 now have a relative minus sign.

Case T34 is slightly more complicated. In this case the two possibilities for a_4, $a_4 \neq 0$ or $a_4 = 0$, again have to be distinguished. Performing this calculation for the general case one obtains a term

$$\frac{\Gamma(\nu_4 - j + n)}{\Gamma(\nu_4 - j)}.$$

This is the analogous form to $\frac{\Gamma(-j+n+\nu_1)}{\Gamma(-j+\nu_1)}$ which appeared is the case T31, $a_1 = 0$. So also in this case we have to consider the two parts $j < \nu_4$ or $j > n + \nu_4 - 1$. Since additionally $j \leq n$ at the same time, the case $j > n + \nu_4 - 1$ does not exist. For $j < \nu_4$ one gets two sums that also have to be transformed by

$$\sum_{j=0}^{\nu_4-1} \sum_{n=j}^{\infty} f(n,j) = \sum_{j=0}^{\nu_4-1} \sum_{n'=0}^{\infty} f(n', j; n' = n - j).$$

Cancellation of terms

Having a closer look at the terms T13, T23, T34 and T35 one finds

$$\mathrm{T13} + \mathrm{T34} = 0$$

$$\mathrm{T23} + \mathrm{T35} = 0$$

This is already the case before the expansion in ε. For both pairs (T13 , T34) and (T23 , T35) the results arise from the same gamma functions just by considering them in a changed order. Hence their cancellation seems to be related to the order in which we perform the integration for σ and τ. This would need some further investigation. To speed up the program, we will omit these four cases from the beginning.

For the integral (6.14) we therefore obtain

$$\hat{I}^{(2,5)} = \mathrm{T11} + \mathrm{T12} + \mathrm{T21} + \mathrm{T22} + \mathrm{T31} + \mathrm{T32}. \tag{6.38}$$

6.2 The expansion of $\hat{I}^{(2,5)}(m - \varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ — the program

In the last section we expressed the result for the integral $\hat{I}^{(2,5)}(m - \varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ in the form of several finite and infinite double sums. These sums are exactly of the form of the four classes `transcendental_fct_type_A()` to `transcendental_fct_type_D()`, which we introduced in Section 5.2. More precisely, we will only need two of these functions, that is `transcendental_fct_type_A()` and `transcendental_fct_type_B()`. Fig. 6.2 shows the tree of the different programs involved. We did not include how the functions themselves call the different functions `transcendental_fct_type_A()` and `transcendental_fct_type_B()`.

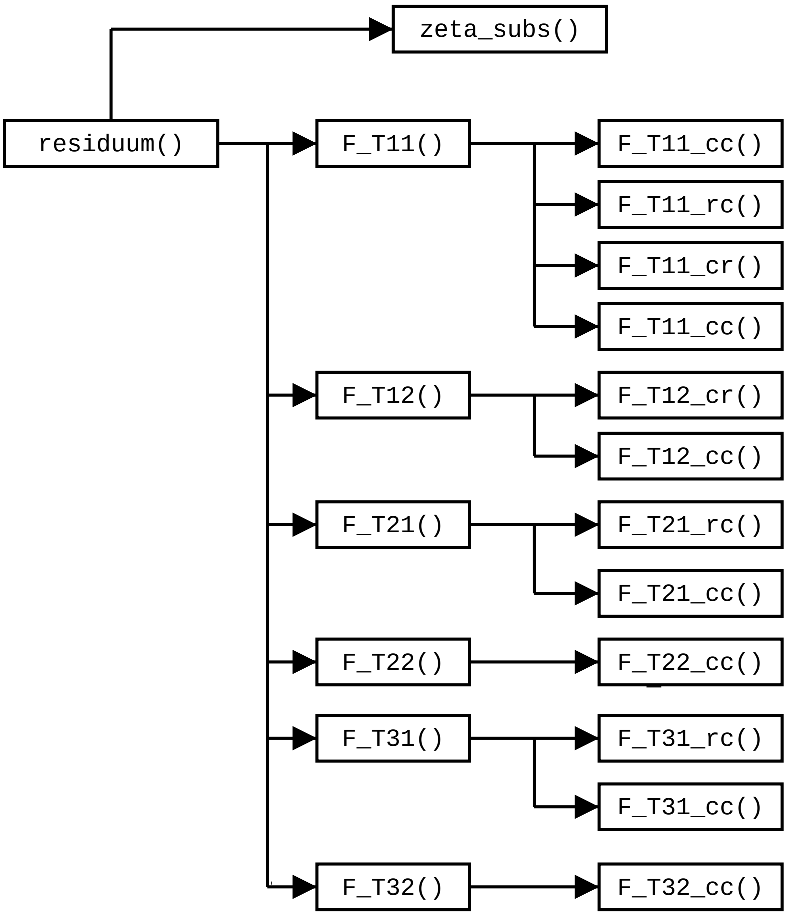


Figure 6.2: The tree of function invocations in the program for expanding the massless two-loop two-point function. We omitted the functions `transcendental_fct_type_A()` and `transcendental_fct_type_B()`, which are called in the end in different combinations (see Section 6.1).

The following functions are implemented:

- `residuum()`: This is the main program in which one can set the *order* up to which the function shall be expanded, the *dimension* of the space $m = \frac{D}{2}$, and the *exponents* of the propagators in the integral, by specifying the integer parts n_i and the integers proportional to ε, a_i of an exponent $\nu_i = n_i + a_i\varepsilon$. It then calls the functions `F_TXY()` with the values set accordingly. The variables X and Y correspond to the names of the functions as defined in Section 6.1.

- `F_T_XY()`: As we have seen in Section 6.1, for some cases we get different sums depending on whether ν_1 and/or ν_4 are real or complex. Therefore the functions `F_TXY()` themselves call functions `F_TXY_ab()`, where a and b can be "r" or "c" according to ν_i being *r*eal or *c*omplex. The first index a corresponds to ν_1, and b to ν_4.

- `F_T_XY_ab()`: The functions `F_TXY_ab()` call the functions `transcendental_fct_type_A()` and `transcendental_fct_type_B()` according to Section 6.1.

- `zeta_subs()`: This function substitutes zeta functions of higher depth by zeta functions of lower depth using the relations given in [MiPe 2000]. Additionally, it substitutes zeta functions of even degree by powers of π, using the relation

$$\zeta(2k) = \frac{(-1)^{k+1}B_{2k}}{2(2k)!}(2\pi)^{2k}, \qquad k \in \mathbb{N} \tag{6.39}$$

with the *Bernoulli numbers*

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad \cdots$$
$$B_n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} B_k, \quad \text{for} \quad n \geq 2. \tag{6.40}$$

6.3 The expansion of $\hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ - the results

Using our program `residuum()`, let us calculate as an arbitrarily chosen example the function $\hat{I}^{(2,5)}$ for the values $\nu_i = 1+\varepsilon$, $\forall i$, $\hat{I}^{(2,5)}(2-\varepsilon, 1+\varepsilon, 1+\varepsilon, 1+\varepsilon, 1+\varepsilon, 1+\varepsilon)$ up to the order ε^4 in the expansion parameter ε:

$$\hat{I}^{(2,5)}(2-\varepsilon, 1+\varepsilon, 1+\varepsilon, 1+\varepsilon, 1+\varepsilon, 1+\varepsilon)$$
$$=$$
$$6\zeta(3)$$
$$+\,\varepsilon[-4\zeta(2,2) - 16\zeta(3,1) + 8/45\pi^4 - 12\gamma_E\zeta(3) + 12\zeta(3)]$$
$$+\,\varepsilon^2[-144\zeta(5) - 8\zeta(2,2) - 16/45\gamma_E\pi^4 + 8\zeta(2,2)\gamma_E + 144\zeta(2,3) - 32\zeta(3,1) + 16/45\pi^4$$
$$+ 45\pi^2\zeta(3) + 204\zeta(3,2) + 216\zeta(2,1,2) - 24\gamma_E\zeta(3) + 12\gamma_E^2\zeta(3) + 408\zeta(3,1,1)$$
$$+ 32\zeta(3,1)\gamma_E - 36\zeta(2,1)\pi^2 + 228\zeta(4,1) + 24\zeta(3) + 228\zeta(2,2,1)]$$
$$+\,\varepsilon^3[-288\zeta(5) - 16\zeta(2,2) - 32/45\gamma_E\pi^4 - 456\gamma_E\zeta(4,1) + 288\zeta(5)\gamma_E - 1656\zeta(2,2,1,1)$$
$$+ 16\zeta(2,2)\gamma_E + 74/3\zeta(2,2)\pi^2 - 408\gamma_E\zeta(3,2) - 1728\zeta(2,1,2,1) + 288\zeta(2,3) - 64\zeta(3,1)$$
$$- 864\zeta(2,1,3) + 884\zeta(3)^2 - 90\gamma_E\pi^2\zeta(3) - 1368\zeta(3,2,1) + 32/45\pi^4 + 90\pi^2\zeta(3)$$
$$- 456\gamma_E\zeta(2,2,1) - 36\zeta(4,1,1) + 408\zeta(3,2) + 288\zeta(2,1,1)\pi^2 + 432\zeta(2,1,2) + 829/945\pi^6$$
$$- 432\gamma_E\zeta(2,1,2) - 1504\zeta(3,3) - 48\gamma_E\zeta(3) - 864\zeta(2,1)\zeta(3) - 8\zeta(2,2)\gamma_E^2 - 1764\zeta(2,2,2)$$
$$+ 24\gamma_E^2\zeta(3) - 1024\zeta(4,2) - 1260\zeta(2,3,1) + 816\zeta(3,1,1) + 64\zeta(3,1)\gamma_E - 3600\zeta(3,1,1,1)$$
$$- 1728\zeta(2,1,1,2) - 288\zeta(2,3)\gamma_E - 892/3\zeta(3,1)\pi^2 + 72\gamma_E\zeta(2,1)\pi^2 - 72\zeta(2,1)\pi^2$$
$$+ 456\zeta(4,1) - 532\zeta(2,4) - 32\zeta(3,1)\gamma_E^2 - 1872\zeta(3,1,2) + 48\zeta(3) + 16/45\gamma_E^2\pi^4$$
$$- 816\gamma_E\zeta(3,1,1) + 456\zeta(2,2,1) - 8\gamma_E^3\zeta(3) + 884\zeta(5,1)]$$
$$+\,\varepsilon^4[-1768\gamma_E\zeta(3)^2 + 25920\zeta(2,1,1,1,2) - 1898/5\zeta(2,1)\pi^4 - 576\zeta(5) - 32\zeta(2,2)$$
$$- 72\gamma_E^2\zeta(2,1)\pi^2 - 4320\pi^2\zeta(2,1,1,1) + 1784/3\zeta(3,1)\gamma_E\pi^2 - 64/45\gamma_E\pi^4 + 1064\gamma_E\zeta(2,4)$$
$$- 912\gamma_E\zeta(4,1) + 576\zeta(5)\gamma_E - 3312\zeta(2,2,1,1) + 32\zeta(2,2)\gamma_E + 8768\zeta(3,3,1)$$
$$+ 6824/3\zeta(3,1)\zeta(3) + 1416\zeta(5,1,1) + 72\gamma_E\zeta(4,1,1) + 3744\gamma_E\zeta(3,1,2) + 816\gamma_E^2\zeta(3,1,1)$$
$$+ 1040\zeta(5)\pi^2 + 148/3\zeta(2,2)\pi^2 - 816\gamma_E\zeta(3,2) - 3456\zeta(2,1,2,1) + 3312\zeta(2,2,1,1)\gamma_E$$

$$
\begin{aligned}
&- 60614\zeta(7) + 576\zeta(2,3) + 1622\pi^2\zeta(4,1) - 1658/945\gamma_E\pi^6 + 1728\gamma_E\zeta(2,1,3)\\
&- 156\zeta(4,1,2) + 2736\gamma_E\zeta(3,2,1) + 4\gamma_E^4\zeta(3) + 10248\zeta(3,2,2) - 1728\zeta(2,1,3)\\
&+ 1768\zeta(3)^2 - 148/3\zeta(2,2)\gamma_E\pi^2 + 16632\zeta(4,1,1,1) - 180\gamma_E\pi^2\zeta(3) + 24336\zeta(3,1,2,1)\\
&- 2736\zeta(3,2,1) + 64/45\pi^4 - 1768\gamma_E\zeta(5,1) + 180\pi^2\zeta(3) + 16/3\zeta(2,2)\gamma_E^3 - 128\zeta(3,1)\\
&- 912\gamma_E\zeta(2,2,1) - 72\zeta(4,1,1) + 816\zeta(3,2) + 7812\zeta(2,4,1) + 576\zeta(2,1,1)\pi^2\\
&+ 1728\gamma_E\zeta(2,1)\zeta(3) + 24048\zeta(3,2,1,1) + 864\zeta(2,1,2) - 864\gamma_E\zeta(2,1,2) - 3008\zeta(3,3)\\
&+ 818\zeta(3,2)\pi^2 - 96\gamma_E\zeta(3) + 334\pi^2\zeta(2,2,1) - 1728\zeta(2,1)\zeta(3) - 16\zeta(2,2)\gamma_E^2\\
&- 288\zeta(5)\gamma_E^2 + 24768\zeta(2,1,3,1) + 90\gamma_E^2\pi^2\zeta(3) - 18020\zeta(5,2) - 13420\zeta(4,3)\\
&- 3528\zeta(2,2,2) - 11372\zeta(3,4) + 3752\zeta(6,1) + 48\gamma_E^2\zeta(3) + 25920\zeta(2,1,1,2,1)\\
&+ 12960\zeta(2,1,1,3) + 4196\zeta(2,2,3) - 2048\zeta(4,2) - 2520\zeta(2,3,1) + 1316/3\zeta(2,3)\pi^2\\
&+ 50976\zeta(3,1,1,1,1) - 36\zeta(2,1,2)\pi^2 + 1632\zeta(3,1,1) + 128\zeta(3,1)\gamma_E - 7200\zeta(3,1,1,1)\\
&- 3456\zeta(2,1,1,2) - 576\zeta(2,1,1)\gamma_E\pi^2 + 24224/3\zeta(2,2)\zeta(3) - 32/135\gamma_E^3\pi^4\\
&+ 8568\zeta(4,2,1) + 2376\zeta(2,1,4) - 576\zeta(2,3)\gamma_E + 25920\zeta(2,1,2,1,1) + 3528\gamma_E\zeta(2,2,2)\\
&+ 456\gamma_E^2\zeta(2,2,1) - 1784/3\zeta(3,1)\pi^2 + 3456\gamma_E\zeta(2,1,1,2) - 6800\zeta(2,5) + 144\gamma_E\zeta(2,1)\pi^2\\
&+ 288\zeta(2,3)\gamma_E^2 - 16\gamma_E^3\zeta(3) - 144\zeta(2,1)\pi^2 + 912\zeta(4,1) + 10180\zeta(2,3,2)\\
&- 1064\zeta(2,4) + 2572\pi^2\zeta(3,1,1) + 26640\zeta(3,1,1,2) - 64\zeta(3,1)\gamma_E^2 - 3744\zeta(3,1,2)\\
&+ 17712\zeta(2,2,2,1) + 26352\zeta(2,2,1,1,1) + 96\zeta(3) + 32/45\gamma_E^2\pi^4 + 2520\gamma_E\zeta(2,3,1)\\
&+ 2048\gamma_E\zeta(4,2) + 456\gamma_E^2\zeta(4,1) + 64/3\zeta(3,1)\gamma_E^3 + 4256\zeta(3,1,3) + 73651/108\pi^4\zeta(3)\\
&+ 1768\zeta(5,1) + 408\gamma_E^2\zeta(3,2) - 1632\gamma_E\zeta(3,1,1) + 912\zeta(2,2,1) + 7200\gamma_E\zeta(3,1,1,1)\\
&+ 3008\gamma_E\zeta(3,3) + 21888\zeta(2,1,2,2) + 1658/945\pi^6 + 20304\zeta(2,2,1,2) + 432\gamma_E^2\zeta(2,1,2)\\
&+ 3456\zeta(2,1,2,1)\gamma_E + 19080\zeta(2,3,1,1) + 12960\zeta(2,1,1)\zeta(3)]
\end{aligned}
$$

We can simplify the zeta functions of depth > 1 by using the Gröbner basis provided by H.N.Minh and M.Petitot in [MiPe 2000]. This reduces the size of the output considerably and we finally obtain:

$$
\begin{aligned}
&\hat{I}^{(2,5)}(2-\varepsilon, 1+\varepsilon, 1+\varepsilon, 1+\varepsilon, 1+\varepsilon, 1+\varepsilon)\\
&= \qquad (6.41)\\
&6\zeta(3)\\
&+ \varepsilon[1/10\pi^4 - 12\gamma_E\zeta(3) + 12\zeta(3)]\\
&+ \varepsilon^2[372\zeta(5) - 1/5\gamma_E\pi^4 + 1/5\pi^4 - \pi^2\zeta(3) - 24\gamma_E\zeta(3) + 12\gamma_E^2\zeta(3) + 24\zeta(3)]\\
&+ \varepsilon^3[744\zeta(5) - 2/5\gamma_E\pi^4 - 744\zeta(5)\gamma_E - 892\zeta(3)^2 + 2\gamma_E\pi^2\zeta(3) + 2/5\pi^4 - 2\pi^2\zeta(3)\\
&\quad + 1199/1260\pi^6 - 48\gamma_E\zeta(3) + 24\gamma_E^2\zeta(3) - 8\gamma_E^3\zeta(3) + 48\zeta(3) + 1/5\gamma_E^2\pi^4]\\
&+ \varepsilon^4[1784\gamma_E\zeta(3)^2 + 1488\zeta(5) - 4/5\gamma_E\pi^4 - 1488\zeta(5)\gamma_E - 62\zeta(5)\pi^2 + 18450\zeta(7)\\
&\quad - 1199/630\gamma_E\pi^6 + 4\gamma_E^4\zeta(3) - 1784\zeta(3)^2 + 4\gamma_E\pi^2\zeta(3) + 4/5\pi^4 - 4\pi^2\zeta(3)\\
&\quad + 1199/630\pi^6 - 96\gamma_E\zeta(3) + 744\zeta(5)\gamma_E^2 - 2\gamma_E^2\pi^2\zeta(3) + 48\gamma_E^2\zeta(3) - 2/15\gamma_E^3\pi^4\\
&\quad - 16\gamma_E^3\zeta(3) + 96\zeta(3) + 2/5\gamma_E^2\pi^4 - 1777/60\pi^4\zeta(3)] \qquad (6.42)
\end{aligned}
$$

In this way one can in principle expand the function $\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$ up to any order in ε, only limited by the available computer memory. In [BiWe 2003] we used a γ_E-free expansion of this integral by multiplying it with the factor

$$c_\Gamma^{-2} = \left(\frac{\Gamma(1-2\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)^2}\right)^2, \tag{6.43}$$

which reasonably lowers the amount of used memory. The result for the function $\hat{I}^{(2,5)}(2-\varepsilon,1+\varepsilon,1+\varepsilon,1+\varepsilon,1+\varepsilon,1+\varepsilon)$ multiplied with this factor becomes:

$$\begin{aligned}
&\hat{I}^{(2,5)}(2-\varepsilon,1+\varepsilon,1+\varepsilon,1+\varepsilon,1+\varepsilon,1+\varepsilon)\\
&= \qquad (6.44)\\
&6\zeta(3)\\
&+\varepsilon[1/10\pi^4+12\zeta(3)]\\
&+\varepsilon^2[372\zeta(5)+1/5\pi^4+24\zeta(3)]\\
&+\varepsilon^3[744\zeta(5)-864\zeta(3)^2+2/5\pi^4+61/63\pi^6+48\zeta(3)]\\
&+\varepsilon^4[1488\zeta(5)+18450\zeta(7)-1728\zeta(3)^2+4/5\pi^4+122/63\pi^6+96\zeta(3)-144/5\pi^4\zeta(3)]
\end{aligned} \tag{6.45}$$

However, using this program we can not only expand $\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$, but we are also able to make statements about the types of functions that can possibly occur in its Taylor expansion. In Chapter 4 we mentioned that D. Broadhurst found a first double sum in the expansion of the function $\hat{I}^{(2,5)}$ [Broa 1986]. This gave rise to the question what sort of functions could in general occur in this expansion [Broa 2003]. We can answer this question now from our calculation: we have seen that we can express all our results in (double) sums of gamma functions in the form of `transcendental_fct_type_A()` and `transcendental_fct_type_B()` that are taken *at unit argument*. Hence the Laurent series expansion of $\hat{I}^{(2,5)}$ only involves rational numbers and multiple zeta values (cf. Table 5.1). We formulated this statement in [BiWe 2003] as a theorem:

Theorem: *Multiple zeta values are sufficient for the Laurent expansion of the two-loop integral $\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$, if all powers of the propagators are of the form $\nu_j = n_j + a_j\varepsilon$, where the n_j are positive integers and the a_j are nonnegative real numbers.*

Apart from the general expansion of the massless two-loop two-point function, this constitutes another important result of our calculation.

Chapter 7

Building massless Feynman diagrams

As one application of the expansion of the function $\hat{I}^{(2,5)}$ we will now calculate the non-planar vertex correction in massless Yukawa theory and massless QED. These two-loop skeleton graphs will be implemented into a program in Chapter 8 with which one can calculate the counterterms for a special class of Feynman diagrams automatically. "A special class" means that we are restricted to Feynman diagrams that can be built from a set P of all primitive one-loop diagrams of the theory, and the non-planar vertex correction as a primitive two-loop diagram:

$P =$ 

By "building" Feynman diagrams from the elements of P we mean that we start with a single graph Γ_1 from this set and insert graphs $\gamma_i \in P$ at places that are compatible with the external structure of the graph γ_i. This again follows the idea of inserting graphs by using gluing data, which we explained in Chapter 3. In this sense, the elements of P are "building blocks" for the graphs we want to investigate.

This idea of using primitive graphs as building blocks of Feynman diagrams and to use them for calculating the counterterms of graphs, was on the one hand exploited by D.Kreimer in [Krei 1998a, Krei 2000b], and was, for the one-loop diagrams of the set P in the above cited massless theories, also subject of my diploma thesis [Bier 2000]. This PhD thesis extends the diploma thesis in the following ways:

- We are calculating the one-loop vertex corrections in Yukawa theory and QED with a different assignment of momenta.
- We are introducing a C++ computer program that calculates the counterterms for an input Feynman diagram automatically.
- We are extending the set P of primitive building blocks by the non-planar vertex correction.

Although the results for the different graphs were calculated by myself, the program implementing these calculations for the massless theories has been written by a former PhD student at Mainz University, Richard Kreckel. It is written in C++ and uses the computer library *GiNaC*. First results obtained from this program, not including the non-planar vertex correction, were published in [BKK 2002]. This program has now been extended to include the non-planar vertex correction. For this, it was necessary to introduce new classes in the program and to modify it to allow insertions into five different places of a graph, instead of only two like for the one-loop case.

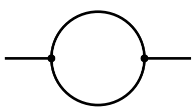
Before we start in the next section with an introduction to the scheme emerging from the insertion of massless graphs into each other, which underlies the computer program, we want to clearly set up the framework in which we are working. Subsequent calculations are all subject to the following conditions:

- We are calculating all graphs in the massless case.
- We are using dimensional regularization with $D = 4 - 2\varepsilon$, hence the parameter m in the function $\hat{I}^{(2,5)}$ is equal to $m = 2$. As the renormalization scheme we are using minimal subtraction.
- The vertex graphs, the one-loop vertex correction, and the non-planar vertex correction are calculated for zero momentum transfer (zmt), which leads to them depending only on *one* external momentum q.
- The vertex graphs are calculated for two different assignments of momenta.
- The non-planar vertex correction is only considered for the case of subdivergences sitting in *one* particular line.

In the first section we will only refer to the one-loop diagrams. Since they have already been covered in my diploma thesis, this section is mainly addressed to readers who do not already know this scheme. Anyone familiar with it may treat the rest of the first section as a recapitulation, or may skip it completely and continue directly with the non-planar vertex correction in Section 7.2.

7.1 One-loop diagrams

As we are calculating dimensionally regularized diagrams, our analytic expressions for the Feynman graphs and their counterterms will be Laurent series in the regularization parameter ε. We start with graphs that are built from primitive one-loop diagrams. These one-loop diagrams will all have the underlying topology of the one-loop two-point function:



This includes the vertex correction, as we only consider it for the case of zero momentum transfer. The topology of the one-loop two-point function is related to the functions F_{ν_1,ν_2}

introduced in (4.5). All integrals corresponding to the one-loop graphs or to graphs built from them will therefore evaluate to sums and/or products of these functions.

We will start by explaining the general scheme for the case of the massless Yukawa theory. We like to stress here again that since we do not calculate processes related to experiments, but rather want to make general statements about the appearance of zeta functions in counterterms, we will omit all coupling constants and other for our purpose "unnecessary" multiplicative factors. These constants would not alter the general results and would just complicate the presentation.

7.1.1 Yukawa theory

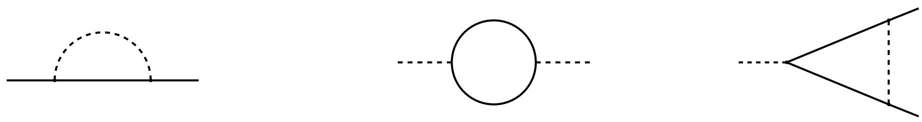
The Lagrangian of the massive Yukawa theory is given by

$$\mathcal{L}_{\text{Yukawa}} = \bar{\Psi}(i\partial\!\!\!/ - m_\Psi)\Psi + \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}m_\Phi^2\Phi^2 - g\bar{\Psi}\Psi\Phi, \tag{7.1}$$

where Ψ is a fermion field, Φ a field for a scalar particle, and g is the coupling constant.

For the massless case one derives from this Lagrangian the following free propagators for particles with momentum q: $\frac{1}{q\!\!\!/}$ for the fermion and $\frac{1}{q^2}$ for the boson. For the vertex correction we find: $-ig\mathbb{1}$, with the unit matrix $\mathbb{1}$. In diagrams, the scalar particle is denoted by a dashed line, the fermion by a solid line.

The three primitive one-loop diagrams in P are (cf. Chapter 2):



fermion self-energy , boson self-energy, vertex correction

We already mentioned that we will calculate the vertex correction for two different assignments of momenta, with *case 1* being the one with the zmt vertex connected to the external boson line, and *case 2* being the one where this vertex is one of the two vertices connected to the external fermion lines:

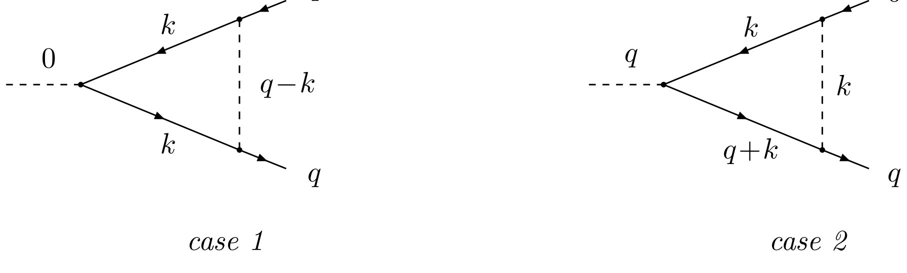
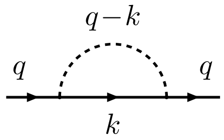


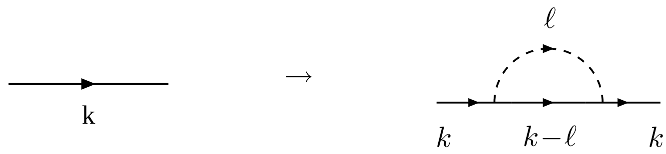
Figure 7.1: The two different assignments of momenta for the one-loop vertex correction.

To explain the general idea underlying the building of graphs, consider the one-loop fermion self-energy for which we get the analytic expression:



$$\equiv \int d^Dk \frac{1}{\not{k}} \frac{1}{(q-k)^2} = \int d^Dk \frac{\not{k}}{k^2(q-k)^2} \overset{(C.5)}{=} [q^2]^{-\varepsilon} \, \frac{1}{2} \, F_{1,1} \, \not{q} \qquad (7.2)$$

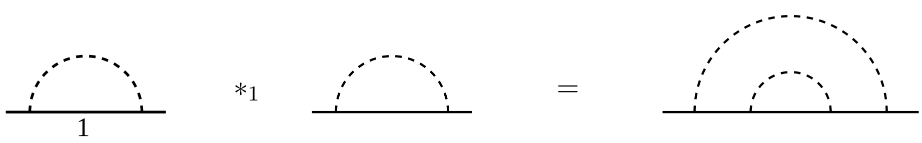
The result for the vector integral in the last step, (C.5), can be found in the Appendix C. Note that the result is proportional to $\not{q}$. If we insert this graph into an inner fermion line of another graph with momentum k, we get for this line a graphical replacement:



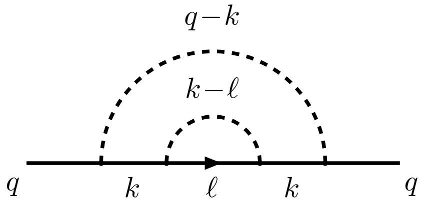
Analytically, this corresponds to the replacement

$$\frac{1}{\not{k}} \quad \rightarrow \quad \frac{1}{\not{k}} \left(\frac{1}{2} F_{1,1} [k^2]^{-\varepsilon} \not{k} \right) \frac{1}{\not{k}} = [k^2]^{-\varepsilon} \frac{1}{2} F_{1,1} \frac{1}{\not{k}} \; . \qquad (7.3)$$

This means that instead of the free propagator $\frac{1}{\not{k}}$, one now has two propagators $\frac{1}{\not{k}}$ surrounding the expression for the subdivergence. Since the subdivergence itself is proportional to the inverse of the free propagator of the line in which it is inserted, it cancels with one part of the "additional" free propagator. What remains is an expression that again has the form of a free fermion propagator $\frac{1}{\not{k}}$, multiplied with a function of ε, $\frac{1}{2}F_{1,1}$, and a momentum-dependent term $[k^2]^{-\varepsilon}$. Since we only insert primitive, *i.e.* in particular, divergent graphs into divergent graphs, this inner fermion line in which we insert is still an inner line of a "bigger", divergent graph Γ, and we have to integrate over the momentum k. If we insert, for example, the one-loop fermion self-energy once into itself:



we obtain:



One can see the insertion at the inner line with momentum k. Analytically this becomes:

$$\begin{aligned}
\int d^D k \frac{1}{\not{k}} \left([k^2]^{-\varepsilon} \frac{1}{2} F_{1,1} \not{k} \right) \frac{1}{\not{k}} \frac{1}{(q-k)^2} &= \frac{1}{2} F_{1,1} \int d^D k \frac{1}{\not{k}} [k^2]^{-\varepsilon} \frac{\not{k}}{\not{k}} \frac{1}{(q-k)^2} \\
&= \frac{1}{2} F_{1,1} \int d^D k \frac{\not{k}}{k^2} [k^2]^{-\varepsilon} \frac{1}{(q-k)^2} \\
&= \frac{1}{2} F_{1,1} \int d^D k \frac{\not{k}}{[k^2]^{1+\varepsilon} [(q-k)^2]}
\end{aligned}$$

After inserting the subdivergence, the integral for this two-loop diagram again has the form of the integral (7.2), with the only difference being the exponent of the fermion momentum in the denominator, which has been changed by $+1 \cdot \varepsilon$. One can easily convince oneself that each additional subdivergence increases this exponent by $+\varepsilon$. Furthermore, this integral is multiplied by the (q-independent) factor $\frac{1}{2} F_{1,1}$ of the one-loop self-energy.

With equation (C.5) one then finds (note from (2.14) that the functions F_{ν_1,ν_2} are symmetric!)

$$\int d^D k \frac{1}{\not{k}} \left([k^2]^{-\varepsilon} \frac{1}{2} F_{1,1} \not{k} \right) \frac{1}{\not{k}} \frac{1}{(q-k)^2} = \frac{1}{2} F_{1,1} \left[F_{\varepsilon,1} + F_{1+\varepsilon,1} \right] [q^2]^{-2\varepsilon} \not{q} \tag{7.4}$$

The important thing here is that the results of the one-loop and also of the two-loop calculation in (7.2) and (7.4) are of the following form: one term of functions that only depends on ε, one term $[q^2]^{-(2)\varepsilon}$, and the inverse of the free propagator $\not{q}$. One generally finds that the result of a graph, built from the primitive one-loop graphs always consists of the following four parts:

1. $(-ig)^{2x}$ with $x =$ number of vertices,
2. a (product of) momentum-independent function(s) of ε,
3. $[q^2]^{-y\varepsilon}$, where $y =$ number of loops in total,
4. the inverse propagator of the external line, with the corresponding momentum, or the inverse unit matrix for the vertex corrections.

These are the ingredients which we also find for all other types of self-energies and vertex corrections, also in QED. Each loop is in particular characterized by its corresponding sum of functions F_{ν_1,ν_2}. Hence we will give these sums special names: fermion self-energies will always be denoted by Σ, vacuum polarizations by Π, and vertex corrections by Γ. We assign to these functions a pair of indices (i, j) that count the number of subdivergences sitting inside the loop and additionally state at which line the subdivergences are located according to the following rules:

$\Sigma_{i,j}$: $i = \#$ subdivergences at the fermion line,
$j = \#$ subdivergences at the boson line.

$\Pi_{i,j}$: $i = \#$ subdivergences at the lower fermion line,
$j = \#$ subdivergences at the upper fermion line.

$\Gamma^1_{i,j}$: $i = \#$ fermion self-energies and vertex corrections
plugged into the zmt vertex and the internal edges connected to it,
$j = \#$ subdivergences at the boson line not connected to this vertex.

$\Gamma^2_{i,j}$: $i = \#$ fermion or boson self-energies and vertex corrections
plugged into the zmt vertex and the internal edges connected to it,
$j = \#$ fermion self-energies at the fermion line not connected to
the vertex of zmt.

We use the same names to denote complete graphs, indicating the difference by placing brackets around the indices of the functions, for instance $\Sigma_{[0,0]}$. However, there are some ambiguities when we name the graphs in this way, as the index does not indicate the type of the subdivergences sitting in one line, but only their total number. Nevertheless, this will not cause too many problems at this point and we will ignore this fact for now.

The result for the one-loop and two-loop fermion self-energies then reads:

$$\Sigma_{[0,0]}(q^2) = \Sigma_{0,0}[q^2]^{-\varepsilon}\not{q} \tag{7.5}$$
$$\Sigma_{[1,0]}(q^2) = \Sigma_{0,0}\Sigma_{1,0}[q^2]^{-2\varepsilon}\not{q} \tag{7.6}$$

One can immediately see that it is possible to reconstruct the corresponding graph from each result. For the two-loop graph, for example, we find that the term $[[q^2]^{-\varepsilon}]^{\mathit{2}} = [q^2]^{-2\varepsilon}$ indicates a graph consisting of *two* loops. The product $\Sigma_{0,0}\Sigma_{1,0}$ makes a statement about the explicit form of the graphs: the part $\Sigma_{0,0}$ for the subdivergence tells us that it is a primitive fermion self-energy. $\Sigma_{1,0}$ however, being the term with the highest number of subdivergences, states the form of the "outer" graph: a fermion self energy with *one* ($\Sigma_{\mathit{1},0}$) subdivergence sitting at the *fermion line* (the *left* index has been increased by one). We will often only provide these functions $\Sigma_{i,j}$, $\Pi_{i,j}$ or $\Gamma_{i,j}$, when talking about subdivergent graphs or corrections. The remaining parts of the result for an integral can be determined uniquely.

Note again that the functions defined above contain all orders in ε including finite orders and not only the divergent parts of the Laurent series.

To complete this section, we give the functions $\Sigma_{i,j}$, $\Pi_{i,j}$, and $\Gamma^{1/2}_{i,j}$ in their general form. They are calculated analogously to the examples above:

$$\Sigma_{i,j} := \frac{1}{2}[F_{i\varepsilon,1+j\varepsilon} + F_{1+i\varepsilon,1+j\varepsilon} - F_{1+i\varepsilon,j\varepsilon}], \tag{7.7}$$
$$\Pi_{i,j}/\mathrm{tr}(\mathbb{1}) := -\frac{1}{2}[F_{i\varepsilon,1+j\varepsilon} - F_{1+i\varepsilon,1+j\varepsilon} + F_{1+i\varepsilon,j\varepsilon}], \tag{7.8}$$
$$\Gamma^1_{i,j} := F_{1+i\varepsilon,1+j\varepsilon}, \tag{7.9}$$
$$\Gamma^2_{i,j} := \frac{1}{2}[F_{2+i\varepsilon,j\varepsilon} - F_{2+i\varepsilon,1+j\varepsilon} + F_{1+i\varepsilon,1+j\varepsilon}]. \tag{7.10}$$

We have divided the boson self-energy by the trace of the unit matrix $\mathrm{tr}(\mathbb{1})$ (trace over spinor indices) for easier comparison of insertions of subgraphs into boson and fermion lines.

7.1.2 QED

The Lagrangian for massive QED is given by

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\ &= \bar{\Psi}(i\not{\partial} - m)\Psi - e\bar{\Psi}\gamma^\mu\Psi A_\mu - \frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2, \end{aligned} \tag{7.11}$$

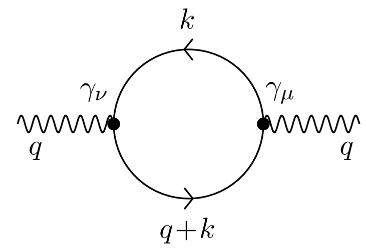
where Ψ is again a fermion field, A_μ is the electromagnetic vector potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, e is the coupling constant, and $D_\mu = \partial_\mu + ieA_\mu$ the gauge-covariant derivative. The term $-1/(2\xi)(\partial_\mu A^\mu)^2$ is the gauge fixing term.

The free propagators in QED for particles with momentum q are: $\frac{1}{\not{q}}$ for the fermion and $\frac{1}{q^2}(g_{\mu\nu} + \xi\frac{q_\mu q_\nu}{q^2})$ for the photon, which is drawn as a wavy line. Note that we have a free parameter ξ, the gauge parameter, entering the calculations. For the vertex we find: $(-ie)\gamma_\mu$, with γ_μ an element of the Dirac algebra, fulfilling the anti-commutation relations $\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$.

The most important differences to the Yukawa theory stem from the gauge parameter in the photon propagator and from the QED vertex $-ie\gamma_\mu$, where the appearance of gamma matrices with their Clifford algebra structure make the calculations more complicated. Nevertheless the scheme introduced in the last chapter that states how graphs are translated into an analytic expression, remains the same. Therefore we will mainly just provide the differences and important changes compared to the Yukawa theory without giving the calculations explicitly. Only the case of the vertex correction is going to be examined in more detail as it will be necessary to introduce matrices for the functions $\Gamma_{i,j}$.

Vacuum polarization

Let us start by calculating the QED vacuum polarization:



It evaluates to the result:

$$\begin{aligned}\Pi^{[0,0]}(q^2) &= -\tfrac{(2-D)}{(1-D)}[q^2]^{-\varepsilon}\frac{1}{2}F_{1,1}\mathrm{tr}(\mathbb{1})\left[g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right]q^2 \\ &= [q^2]^{-\varepsilon}\Pi_{0,0}\left[g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right]q^2 \end{aligned} \tag{7.12}$$

with

$$\Pi_{0,0} = \tfrac{(2-D)}{(1-D)}\frac{1}{2}F_{1,1}\mathrm{tr}(\mathbb{1}) \tag{7.13}$$

Apart from the multiplicative term $\frac{1}{q^2}$, the expression in parentheses in equation (7.12) corresponds to the free propagator for $\xi = -1$ and reflects the transversality of the photon. Later on, we want to insert this vacuum polarization into photon lines of other graphs causing the gauge parameter attached to these lines to be fixed at -1, which one can easily convince oneself by a small calculation. This has to be taken into account in the following.

Since we do not consider a gauge invariant class of Feynman diagrams for higher loop orders, we will only include the one-loop vacuum polarization without subdivergences in this thesis.

Fermion self-energy

For the fermion self-energy we have to consider two cases, as the one-loop vacuum polarization sitting in an inner photon line of a graph as a subdivergence forces the gauge parameter of this line to be $\xi = -1$. Hence we now have to distinguish between the two cases, with and without a subdivergence at the photon line of a graph.

Let j be the number of subdivergences at the photon line. We calculate the result of the fermion self-energy to

$$j = 0: \qquad \tilde{\Sigma}_{i,0} := (2-D)\Sigma_{i,0} + \xi\Sigma'_{i,0}, \tag{7.14}$$

$$\Sigma_{i,0} = \frac{1}{2}\left[F_{1+i\varepsilon,1} + F_{i\varepsilon,1}\right] \text{ and } \Sigma'_{i,0} = \frac{1}{2}\left[F_{-1+i\varepsilon,2} - 2F_{i\varepsilon,2} - F_{i\varepsilon,1} + F_{1+i\varepsilon,2} - F_{1+i\varepsilon,1}\right] \tag{7.15}$$

$$j \neq 0: \qquad \tilde{\Sigma}_{i,j} = \frac{1}{2}\left[(2-D)F_{j\varepsilon,1+i\varepsilon} - (3-D)F_{1+j\varepsilon,i\varepsilon} - (3-D)F_{1+j\varepsilon,1+i\varepsilon}\right.$$
$$\left. + F_{2+j\varepsilon,-1+i\varepsilon} - 2F_{2+j\varepsilon,i\varepsilon} + F_{2+j\varepsilon,1+i\varepsilon}\right] \tag{7.16}$$

The second case could be written as only one function with the help of the Kronecker-Delta:

$$\begin{aligned}\tilde{\Sigma}_{i,j} =& \frac{1}{2}\left[\,(2-D)\,F_{j\varepsilon,1+i\varepsilon} - (3-D)\,F_{1+j\varepsilon,i\varepsilon} - (3-D)\,F_{1+j\varepsilon,1+i\varepsilon}\right.\\ &\left. + F_{2+j\varepsilon,-1+i\varepsilon} - 2\,F_{2+j\varepsilon,i\varepsilon} + F_{2+j\varepsilon,1+i\varepsilon}\,\right](1-\delta_{0,j})\\ &+ \left[\,(2-D)\,\Sigma_{i,0} + \xi\,\Sigma'_{i,0}\,\right]\delta_{0,j},\end{aligned} \tag{7.17}$$

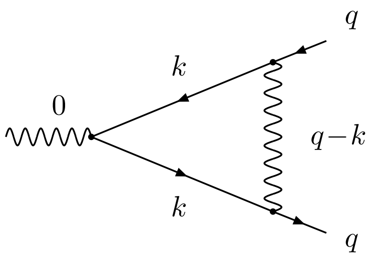
but the programs of Chapter 8 use the first representation.

Note that in our conventions, a gauge parameter of value $\xi = 0$ corresponds to the *Feynman gauge*, $\xi = -1$ to the *Landau gauge* [PeSch 1995].

Vertex corrections

Since the vertex graphs are in our case only dependent on *one* external momentum q, the external structure of the vertex correction consists of two form factors, one for γ_μ and one for $\frac{q_\mu \not{q}}{q^2}$: $\Gamma_\mu(0,q,q) = F_1(q^2)\,\gamma_\mu + F_2(q^2)\,\frac{q_\mu \not{q}}{q^2}$. When we insert this graph into another graph and, correspondingly, this result into an integral for the outer loop of the "bigger" graph, we will get a sum of two integrals: one for the insertion of the part proportional to γ_μ and one for the insertion of the expression proportional to $q_\mu \not{q}/q^2$. This forces us to introduce two-by-two matrices [KrDe 1999] whose four entries are given by four functions $\Delta_{a,b}^{(i,j)}$. The indices (i,j) count the number of internal insertions like before, while the indices a, b have the values 1 or 2, with the case $b = 1$ corresponding to the result that stems from an internal vertex γ_μ, $b = 2$ to the one derived from an internal vertex $q_\mu \not{q}/q^2$. The index a enumerates the two possible form factors in the result.

We again give an example. For the one-loop vertex correction of *case 1* of Fig. 7.1 we obtain:



$$= \left[\Delta_{1,1}^{(0,0)}\gamma_\mu + \Delta_{2,1}^{(0,0)}\frac{q_\mu \not{q}}{q^2}\right][q^2]^{-\varepsilon}.$$

If we insert this graph into itself, we obtain the sum of two integrals:

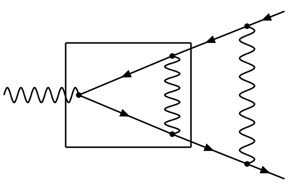
$$\begin{aligned}
&\Delta_{11}^{(0,0)} \int d^D k \gamma^\alpha \frac{1}{\not{k}} (\gamma^\mu) \frac{1}{\not{k}} \gamma^\beta \left[g_{\alpha\beta} + \xi \frac{(q-k)_\alpha (q-k)_\beta}{(q-k)^2}\right] \frac{1}{[(q-k)^2][k^2]^\varepsilon} \\
&+ \Delta_{21}^{(0,0)} \int d^D k \gamma^\alpha \frac{1}{\not{k}} \left(\frac{k^\mu \not{k}}{k^2}\right) \frac{1}{\not{k}} \gamma^\beta \left[g_{\alpha\beta} + \xi \frac{(q-k)_\alpha (q-k)_\beta}{(q-k)^2}\right] \frac{1}{[(q-k)^2][k^2]^\varepsilon} \\
&= \Delta_{11}^{(0,0)} \left[\Delta_{11}^{(1,0)}\gamma^\mu + \Delta_{21}^{(1,0)}\frac{q^\mu \not{q}}{q^2}\right][q^2]^{-2\varepsilon} + \Delta_{21}^{(0,0)} \left[\Delta_{12}^{(1,0)}\gamma^\mu + \Delta_{22}^{(1,0)}\frac{q^\mu \not{q}}{q^2}\right][q^2]^{-2\varepsilon} \\
&= \left[\left(\Delta_{11}^{(0,0)}\Delta_{11}^{(1,0)} + \Delta_{21}^{(0,0)}\Delta_{12}^{(1,0)}\right)\gamma^\mu + \left(\Delta_{11}^{(0,0)}\Delta_{21}^{(1,0)} + \Delta_{21}^{(0,0)}\Delta_{22}^{(1,0)}\right)\frac{q^\mu \not{q}}{q^2}\right][q^2]^{-2\varepsilon}. \qquad (7.18)
\end{aligned}$$

The multiplication of the $\Delta_{a,b}^{(i,j)}$ can be reformulated as a matrix multiplication. Corresponding to each vertex correction we define a matrix

$$M_{i,j}^1 := \begin{pmatrix} \Delta_{11}^{(i,j)} & \Delta_{12}^{(i,j)} \\ \Delta_{21}^{(i,j)} & \Delta_{22}^{(i,j)} \end{pmatrix}, \qquad (7.19)$$

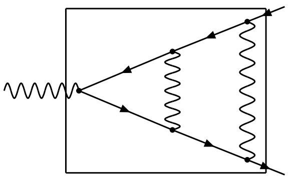
where the upper index refers to the two cases of vertex corrections under consideration. We omit it in the matrix entries for simplicity. For vertex corrections of *case 1*, the index i is the total number of subdivergences at the fermion line, with no difference whether it is of the form Σ or Γ, and j the number of subdivergences at the photon line.

For a two-loop vertex correction, for instance, this means that we begin with the inner vertex correction marked with a box:



$$\equiv M_{0,0}^1 = \begin{pmatrix} \Delta_{11}^{(0,0)} & 0 \\ \Delta_{21}^{(0,0)} & 0 \end{pmatrix}$$

and then multiply this with the matrix for the outer vertex correction, which has one vertex correction as a subdivergence:



$$\equiv M^1_{1,0} = \begin{pmatrix} \Delta^{(1,0)}_{11} & \Delta^{(1,0)}_{12} \\ \Delta^{(1,0)}_{21} & \Delta^{(1,0)}_{22} \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \Delta^{(1,0)}_{11} & \Delta^{(1,0)}_{12} \\ \Delta^{(1,0)}_{21} & \Delta^{(1,0)}_{22} \end{pmatrix} \begin{pmatrix} \Delta^{(0,0)}_{11} & 0 \\ \Delta^{(0,0)}_{21} & 0 \end{pmatrix} = \begin{pmatrix} \Delta^{(1,0)}_{11}\Delta^{(0,0)}_{11} + \Delta^{(1,0)}_{12}\Delta^{(0,0)}_{21} & 0 \\ \Delta^{(1,0)}_{21}\Delta^{(0,0)}_{11} + \Delta^{(1,0)}_{22}\Delta^{(0,0)}_{21} & 0 \end{pmatrix}.$$

Multiplying (7.20) with $(1,0)^T[q^2]^{-2\varepsilon}$, we get the result as the column vector

$$\begin{pmatrix} (\Delta^{(1,0)}_{11}\Delta^{(0,0)}_{11} + \Delta^{(1,0)}_{12}\Delta^{(0,0)}_{21})[q^2]^{-2\varepsilon} \\ (\Delta^{(1,0)}_{21}\Delta^{(0,0)}_{11} + \Delta^{(1,0)}_{22}\Delta^{(0,0)}_{21})[q^2]^{-2\varepsilon} \end{pmatrix}. \tag{7.20}$$

The upper entry of this vector is the form factor $F_1(q^2)$ belonging to γ_μ and the lower one is the form factor $F_2(q^2)$ belonging to $\frac{q^\mu \not{q}}{q^2}$. We can check that this is the same result we obtained in (7.18).

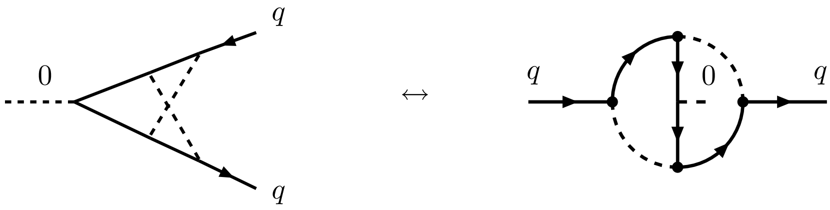
Subdivergences that are *not* vertex corrections, *i.e.* $\Sigma_{i,j}$ and $\Pi_{i,j}$, are multiplied with a unit matrix and inserted in the string of matrices in front of the matrix belonging to the vertex correction which they are a part of. They increase the subscripts i,j accordingly in that matrix. We get two sets of four functions $\Delta^{(i,j)}_{a,b}$ in terms of F_{ν_1,ν_2} functions, one for each assignment of momenta, which we do not want to list here explicitly. The interested reader can find them from our publicly available programs: `http://wwwthep.physik.uni-mainz.de/Publications/theses.html`.

7.2 The non-planar vertex correction

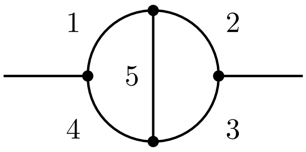
7.2.1 The theory behind the programs

In the last section we have shown how we can build graphs from primitive one-loop diagrams $p_i \in P$ in massless Yukawa theory and QED. We have also calculated or provided all the necessary functions to describe these graphs. In this section we now want to calculate a two-loop building block of P, given by the non-planar two-loop vertex correction. More precisely, we will calculate this non-planar vertex correction for two different assignments of momenta like in the one-loop case.

For the non-planar vertex correction, the integrals (4.5) that sufficed for performing the calculations in the last section, are not sufficient anymore. This is due to the fact that in this primitive two-loop case, we do not find the same master topology as before. Consider a two-loop non-planar vertex correction for a momentum flow similar to the one-loop vertex correction of *case 1*:



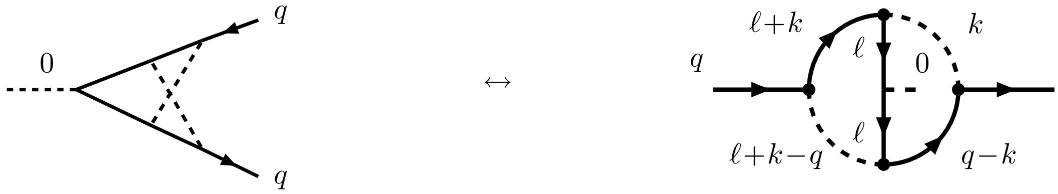
We will again denote the graphs according to the two different momentum flows *case 1* and *case 2* (cf. page 79). Due to the vertex of zero momentum transfer being at the inner vertex, we immediately obtain as the underlying topology the two-loop two-point master topology:



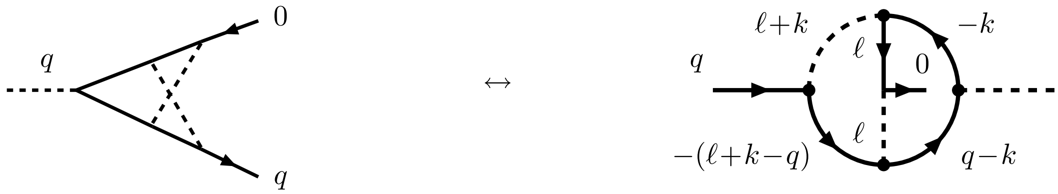
where we have again written the indices of Chapter 6 next to the lines. This topology, on the other hand, immediately leads to the integral $\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$. Since the program includes the possibility for subdivergences occurring inside Feynman diagrams, and since these subdivergences change the exponents of propagators in integrals by an additional part proportional to ε, we need $\hat{I}^{(2,5)}$ also for complex exponents of the momenta.

The following sections will explain in which way we calculated the non-planar vertex correction for massless Yukawa theory and massless QED. These calculations are implemented in the programs of Section 7.2.2.

The four graphs which we will calculate together with the corresponding integrals are, for the massless Yukawa theory in *case 1* and *case 2*:

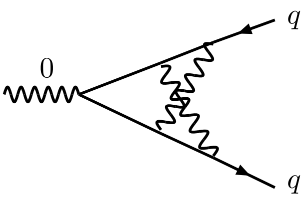


$$\int d^Dkd^D\ell\frac{1}{(\not{q}-\not{k})\,\not{\ell}\,\not{\ell}\,(\not{\ell}+\not{k})}\frac{1}{(\ell+k-q)^2k^2}=\int d^Dkd^D\ell\frac{(\not{q}-\not{k})(\not{\ell}+\not{k})}{(\ell+k-q)^2(q-k)^2(\ell+k)^2\ell^2k^2}$$

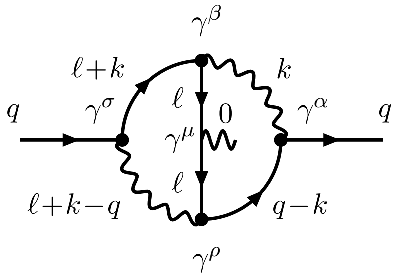


$$\int d^Dkd^D\ell\frac{1}{\not{\ell}\,(-\not{k})\,(\not{q}-\not{k})\,[-(\not{\ell}+\not{k}-\not{q})]}\frac{1}{(\ell+k)^2\ell^2}=\int d^Dkd^D\ell\frac{\not{\ell}\not{k}(\not{q}-\not{k})(\not{\ell}+\not{k}-\not{q})}{(\ell+k-q)^2(q-k)^2(\ell+k)^2\ell^4k^2}$$

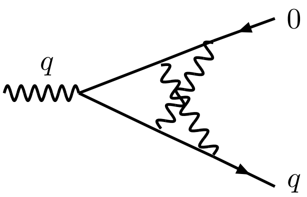
For massless QED in *case 1* and *case 2* we have:



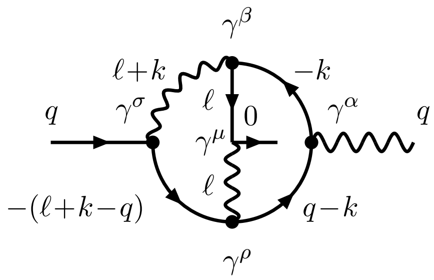
$\leftrightarrow$



$$\int d^Dk d^D\ell\, \gamma^\alpha\, (\not q - \not k)\, \gamma^\rho\, \not\ell\, \gamma^\mu\, \not\ell\, \gamma^\beta\, (\not\ell + \not k)\, \gamma^\sigma$$
$$\times \frac{[g_{\alpha\beta} + \xi(k_\alpha k_\beta/k^2)][g_{\rho\sigma} + \xi((\ell+k-q)_\rho(\ell+k-q)_\sigma/(\ell+k-q)^2)]}{(\ell+k-q)^2(q-k)^2(\ell+k)^2[\ell^2]^2k^2} = A(q^2)\gamma^\mu + B(q^2)\frac{q^\mu \not q}{q^2}$$



$\leftrightarrow$



$$\int d^Dk d^D\ell\, \gamma^\alpha\, \not\ell\, \gamma^\rho\, \not k\, \gamma^\mu\, (\not q - \not k)\, \gamma^\beta\, (\not\ell + \not k - \not q)\, \gamma^\sigma$$
$$\times \frac{[g_{\alpha\beta} + \xi(\ell_\alpha \ell_\beta/\ell^2)][g_{\rho\sigma} + \xi((\ell+k)_\rho(\ell+k)_\sigma/(\ell+k)^2)]}{(\ell+k-q)^2(q-k)^2(\ell+k)^2[\ell^2]^2k^2} = A(q^2)\gamma^\mu + B(q^2)\frac{q^\mu \not q}{q^2}$$

The calculation of the four graphs always follows the same general idea:

- We always take the trace of the integrals, leading to scalar products in the numerators that can be expressed as sums of propagators in the denominators of the integrals via:

  $$q\cdot\ell = \tfrac{1}{2}\left[(q-k)^2 - (\ell+k-q)^2 - k^2 + (\ell+k)^2\right]$$
  $$q\cdot k = \tfrac{1}{2}\left[-(q-k)^2 + q^2 + k^2\right]$$
  $$\ell\cdot k = \tfrac{1}{2}\left[(\ell+k)^2 - k^2 - \ell^2\right]$$

  Products of these scalar products and the cancellation with propagators in the denominator lead to several integrals, either of the form $\hat{I}^{(2,5)}$ or F_{ν_1,ν_2}.

- Whenever possible, we apply the triangle relation. Since we only consider subdivergences at *one* particular line of the non-planar vertex correction, this is sufficient for calculating the cases of subdivergences at the lines with index ν_1 to ν_4.
  For subdivergences at lines ν_1 and ν_4 we use

$$\begin{aligned}
&\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)\\
&\quad= \frac{1}{(D-\nu_1-\nu_4-2\nu_5)}\\
&\quad\Big[\nu_1\left[\hat{I}^{(2,5)}(m-\varepsilon,\nu_1+1,\nu_2,\nu_3,\nu_4,\nu_5-1) - \hat{I}^{(2,5)}(m-\varepsilon,\nu_1+1,\nu_2-1,\nu_3,\nu_5,\nu_4)\right]\\
&\quad + \nu_4\left[\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4+1,\nu_5-1) - \hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3-1,\nu_4+1,\nu_5)\right]\Big],
\end{aligned} \tag{7.21}$$

for subdivergences at lines ν_2 and ν_3

$$\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$$
$$= \frac{1}{(D-\nu_2-\nu_3-2\nu_5)}$$
$$\Big[\nu_2\left[\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2+1,\nu_3,\nu_4,\nu_5-1)-\hat{I}^{(2,5)}(m-\varepsilon,\nu_1-1,\nu_2+1,\nu_3,\nu_5,\nu_4)\right]$$
$$+\nu_3\left[\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3+1,\nu_4,\nu_5-1)-\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3+1,\nu_4-1,\nu_5)\right]\Big]. \tag{7.22}$$

- For subdivergences at line 5 with index ν_5, we transform $\hat{I}^{(2,5)}(2-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$ for different values of the exponents into the form $\hat{I}^{(2,5)}(2-\varepsilon,1,1,1,1+\lambda_\varepsilon\varepsilon,1)$ using relations that are provided in Appendix D.

Let us give an example by calculating the Yukawa *case 1* without subdivergences. Like in the one-loop case, this integral evaluates into a function of the external momentum q and the unit matrix:

$$\int d^Dkd^D\ell\frac{(\not{q}-\not{k})(\not{\ell}+\not{k})}{(\ell+k-q)^2(q-k)^2(\ell+k)^2\ell^2k^2} = F(q^2)\mathbb{1} \tag{7.23}$$

We always divide by $\mathrm{tr}(\mathbb{1})$ so the result obtained from the programs of Chapter 8 will be the function $F(q^2)$, which is a simple function in Yukawa theory or a matrix in QED.

We first take the trace on both sides of (7.23) and obtain for the numerator

$$\mathrm{tr}[(\not{q}-\not{k})(\not{\ell}+\not{k})] = (q\cdot\ell+q\cdot k-k\cdot\ell-k^2)\,\mathrm{tr}(\mathbb{1})\,. \tag{7.24}$$

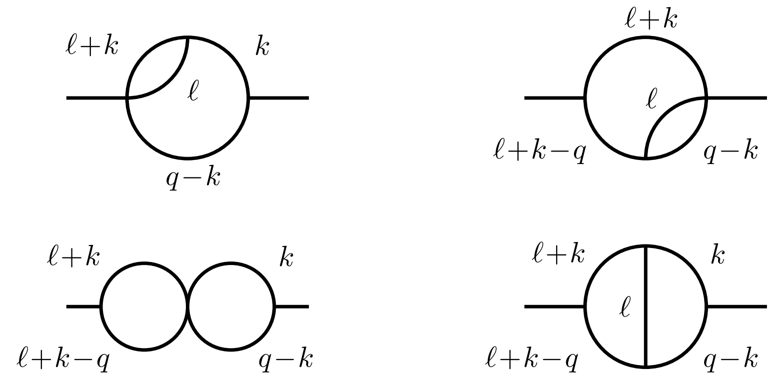
The scalar products can be expressed by a sum of the propagators in the denominator as explained before, and hence

$$\mathrm{tr}[(\not{q}-\not{k})(\not{\ell}+\not{k})] = \frac{1}{2}\left[-(\ell+k-q)^2-k^2+\ell^2+q^2\right]\mathrm{tr}(\mathbb{1})\,. \tag{7.25}$$

Substituting this sum into the numerator of (7.23) leads to four integrals:

$$\int d^Dkd^D\ell\frac{\mathrm{tr}[(\not{q}-\not{k})(\not{\ell}+\not{k})]}{(\ell+k-q)^2(q-k)^2(\ell+k)^2\ell^2k^2}$$
$$=\frac{\mathrm{tr}(\mathbb{1})}{2}$$
$$\times\Big[\int d^Dkd^D\ell\frac{-(\ell+k-q)^2}{(\ell+k-q)^2(q-k)^2(\ell+k)^2\ell^2k^2}+\int d^Dkd^D\ell\frac{-k^2}{(\ell+k-q)^2(q-k)^2(\ell+k)^2\ell^2k^2}$$
$$+\int d^Dkd^D\ell\frac{\ell^2}{(\ell+k-q)^2(q-k)^2(\ell+k)^2\ell^2k^2}+\int d^Dkd^D\ell\frac{q^2}{(\ell+k-q)^2(q-k)^2(\ell+k)^2\ell^2k^2}\Big]$$
$$=\frac{\mathrm{tr}(\mathbb{1})}{2}$$
$$\times\Big[-\int d^Dkd^D\ell\frac{1}{(q-k)^2(\ell+k)^2\ell^2k^2}-\int d^Dkd^D\ell\frac{1}{(\ell+k-q)^2(q-k)^2(\ell+k)^2\ell^2}$$
$$+\int d^Dkd^D\ell\frac{1}{(\ell+k-q)^2(q-k)^2(\ell+k)^2k^2}+q^2\int d^Dkd^D\ell\frac{1}{(\ell+k-q)^2(q-k)^2(\ell+k)^2\ell^2k^2}\Big] \tag{7.26}$$

We have already seen in the context of the triangle relation in Section 4.2 how this cancellation of propagators in numerators and denominators affects integrals and their graphical representation. If we indicate the elimination of a propagator by shrinking the corresponding line in the graph to a point, the integrals in equation (7.26) give rise to the following topologies:



From these pictures, we could immediately read off the results of the calculations, but we want to show them explicitly once: In the first integral, the momentum l only occurs in two propagators. We can therefore group the corresponding propagators and integrate using equation (4.5):

$$\begin{aligned}\int \frac{d^D k d^D \ell}{(q-k)^2(\ell+k)^2\ell^2 k^2} &= \int \frac{d^D k}{(q-k)^2 k^2} \int \frac{d^D \ell}{(\ell+k)^2 \ell^2} = \int \frac{d^D k}{(q-k)^2 k^2} [k^2]^{-\varepsilon} F_{1,1} \\ &= F_{1,1} \int \frac{d^D k}{(q-k)^2 [k^2]^{1+\varepsilon}} = F_{1,1} F_{1,1+\varepsilon} [q^2]^{-2\varepsilon}\end{aligned}$$

In the second integral, both ℓ and k occur in three propagators. Using (2.9) we can shift the integration momenta, for example by setting $k' = \ell + k$:

$$\begin{aligned}\int \frac{d^D k d^D \ell}{(\ell+k-q)^2(q-k)^2(\ell+k)^2\ell^2} &= \int \frac{d^D k' d^D \ell}{(k'-q)^2(q+\ell-k')^2(k')^2\ell^2} \\ &= \int \frac{d^D k'}{(q-k')^2(k')^2} \int \frac{d^D \ell}{((q-k')+\ell)^2\ell^2} = F_{1,1} F_{1,1+\varepsilon} [q^2]^{-2\varepsilon}\end{aligned}$$

The third integral is different, but even easier to solve. Setting $\ell' = \ell + k$, the integral immediately splits into two disjoint integrations:

$$\begin{aligned}\int \frac{d^D k d^D \ell'}{(\ell'-q)^2(q-k)^2(\ell')^2 k^2} &= \int \frac{d^D k}{(q-k)^2 k^2} \int \frac{d^D \ell'}{(\ell'-q)^2(\ell')^2} \\ &= F_{1,1} F_{1,1} [q^2]^{-2\varepsilon}\end{aligned}$$

By shifting the momenta like we just did, we can solve any integral with four propagators. They will always reduce to a product of two functions F_{ν_1,ν_2}, or will be zero when one of the integration variables only occurs in one single propagator. The fourth integral in (7.26) gives rise to the function $\hat{I}^{(2,5)}(m-\varepsilon,1,1,1,1,1)$, which we can now directly expand in ε using the program `residuum` of Chapter 6.[1]

[1] It is immediately clear that a small shift in the momenta transforms the representation of the integral used here into the one of the function $\hat{I}^{(2,5)}$ used in Chapter 6.

Hence we find in these four results the nested structure of the integrations of the first two integrals in (7.26) corresponding to the first two topologies, where we can explicitly see the subintegration over ℓ. The third picture mirrors the disjoint integration of the third integral in (7.26), while the last picture corresponds to the fourth integral that still has five propagators.

In principle, the calculations do not change when we allow for subdivergences in the graphs. One just has to keep in mind that once there are subdivergences, momenta in the numerator will no longer cancel with the corresponding ones in the denominator, but only alter their exponents. We will in these cases try to apply the triangle relation to solve the occurring integrals. Since we restrict ourselves to the case where the non-planar vertex correction has subdivergences only at *one* of its inner lines, the triangle relations are indeed sufficient for subdivergences at lines with the indices ν_1 to ν_4. However, they are not sufficient for subdivergences in the middle line with index ν_5. In this particular case we will need the function $\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$. In the process of calculation, one obtains $\hat{I}^{(2,5)}$ for example with exponents $\hat{I}^{(2,5)}(2-\varepsilon,2,1,1,1,a_5\varepsilon)$. We can expand the function $\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$ for different exponents, as long as the conditions (6.21) are fulfilled. However, we will transform any integral $\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$, whether it fulfills the conditions or not, into the form $\hat{I}^{(2,5)}(m-\varepsilon,1,1,1,1,1+a_5\varepsilon)$, using relations similar to the triangle relation, or combinations of those. One can find these relations and the way in which they were obtained in Appendix D. We then only have to expand the function $\hat{I}^{(2,5)}(2-\varepsilon,1,1,1,1,1+a_5\varepsilon)$ into a Laurent series in ε.

For the QED graphs we again have to take into account that we obtain two form factors:

$$\int d^Dkd^D\ell\,\gamma^\alpha\,(\not{q}-\not{k})\,\gamma^\rho\,\not{\ell}\,\gamma^\mu\,\not{\ell}\,\gamma^\beta\,(\not{\ell}+\not{k})\,\gamma^\sigma$$
$$\times\frac{[g_{\alpha\beta}+\xi(k_\alpha k_\beta/k^2)][g_{\rho\sigma}+\xi((\ell+k-q)_\rho(\ell+k-q)_\sigma/(\ell+k-q)^2)]}{(\ell+k-q)^2(q-k)^2(\ell+k)^2[\ell^2]^2k^2}=A(q^2)\gamma^\mu+B(q^2)\frac{q^\mu\not{q}}{q^2}$$

And thus we again have to calculate the entries for the 2×2 matrices. This is done in the following way: We project onto the form factors by multiplying both sides of the equation once with γ_μ and once with $q_\mu\not{q}/q^2$. We then continue like in the Yukawa theory for the two resulting expressions, which are called zg and zq in the program: we first take the trace of the numerator and substitute scalar products by sums of propagators in the denominator of the integrals. The two form factors can then be built in the following way:

$$\begin{aligned} A &= (zq-zg)/(1-D),\\ B &= (zg-D\,zq)/(1-D). \end{aligned}\tag{7.27}$$

The calculation of the two form factors then follows the calculation in the Yukawa theory and finally expresses these form factors in terms of the functions F_{ν_1,ν_2} and $\hat{I}^{(2,5)}$.

7.2.2 The programs for calculating the non-planar vertex correction

The tree of functions for the calculation of the non-planar vertex correction is shown in Fig. 7.2. Note that the integral $\hat{I}^{(2,5)}$ is called `I_5()`, and the functions F_{ν_1,ν_2} are called `F_ab()` in this program.

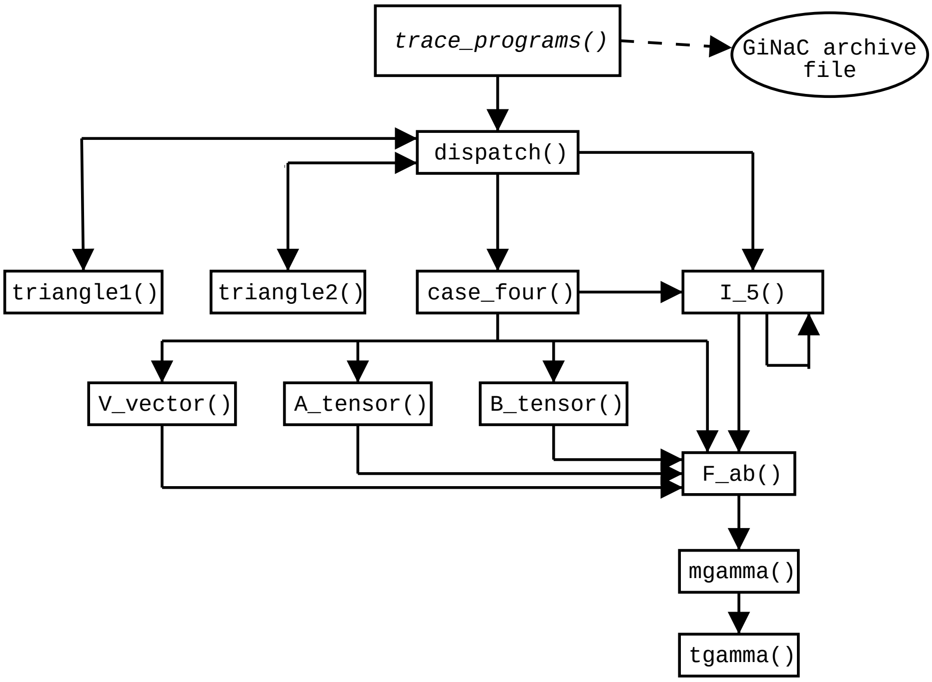


Figure 7.2: The tree of function invocations in the programs for calculating the function or matrix entries for the non-planar vertex correction in massless Yukawa theory and QED.

The main programs

The name `trace_programs()` in Fig. 7.2 is a placeholder for the individual programs

```
yukawa1_gamma2(), yukawa2_gamma2(),
qed1_11_21_gamma2(), qed2_11_21_gamma2(),
qed1_12_22_gamma2(), qed2_12_22_gamma2().
```

These are the main programs for calculating the analytic result of the non-planar vertex correction. For Yukawa theory and QED the numbers 1 and 2 denote the two different cases of momentum flow. In QED the names are extended by the indices of the matrix entries calculated by the respective program.

In these programs, the trace of the numerator of the integrals corresponding to the four graphs is calculated using the *GiNaC* functions `dirac_slash()` and `dirac_trace()`. A D-dimensional Clifford algebra element *e.g.* $\not{q}$, is given by `dirac_slash(q,D)` and `dirac_trace()` then takes the trace of an expression containing such elements. The scalar products that might then occur are substituted by the corresponding sum of momenta of the denominator of the integral, as explained in Section 7.2.1. The exponents of the momenta in the denominators of integrals can be of the form $n_i + a_i\varepsilon$ when subdivergences are inserted (cf. Section 7.1). Hence, in the corresponding programs the exponents consist of two parts, one integer-part and one part proportional to ε.

The functions themselves call the function `dispatch()` from which they receive the result for the different integrals in the form of functions `tgamma()` and `I_5()` (see below). The results for the different integrals are then added and written into an archive file.

The programs in QED calculate the two form factors A proportional to γ_μ and B proportional to $\frac{q_\mu \not{q}}{q^2}$, in the way we explained before.

The `dispatch()` function

This function accepts the expressions for the different integrals encoded by the integer-parts and ε-parts of the exponents of the propagators, produced, for example, in the program `yukawa1_gamma2()`. In a first step, a list `prop_list` of objects of class `propagator` is produced containing the different propagators of an integral encoded by the momentum (p) and its corresponding integer (i) and ε (e) exponent.

Via a scan through that list different variables and counters are set and according to their values the appropriate functions `case_four()`, `triangle1()`, `triangle2()` or `I_5()` for the integrals are called.

`triangle1()` and `triangle2()`

As the name already indicates, these are the functions for the triangle relation of cases 1 and 2, given in (7.21) and (7.22). They map one integral onto four integrals according to the rule (7.21) if there are no subdivergences at the lines with the indices ν_2, ν_3 and ν_5, and to (7.22) if there are no subdivergences at lines with indices ν_1, ν_4 and ν_5. `triangle1()` and `triangle2()` call the function `dispatch()` with the values for the integer and ε parts of the exponents of the propagators set according to the triangle relation.

`case_four()`

This function contains the results for the integrals with different momenta and different numerator structure expressed in terms of the functions `F_ab()`, `V_vector()`, `A_tensor()` and `B_tensor()`. It mainly consists of *if*-clauses that check the different values of special counters and return the results for the integrals accordingly. In the case of integral $\hat{I}^{(2,5)}$ it calls the function `I_5()` .

`V_vector()`

`V_vector()` returns the combination of functions `F_ab()` that corresponds to the result of a vector integral, defined in (C.5).

`A_tensor()` and `B_tensor()`

`A_tensor()` and `B_tensor()` are the combinations of `F_ab()`s that correspond to the A and B part of the result of the tensor integral (C.8).

`F_ab()`

`F_ab()` is the function for the integral (4.5). It returns the fraction of six gamma functions defined there. These gamma functions are called as `mgamma()` functions.

`mgamma()`

`mgamma`(k,l,x) brings the gamma functions into the form constant(k) $*$ `tgamma` $(1 + l\,x)$. This combination proved to be faster in the expansion as Laurent series than an expansion of tgamma $(n + l\,x)$ with $n \neq 1$.

`tgamma()`

This `tgamma()` is the "usual" gamma function and is itself a *GiNaC* function that can easily be expanded in its argument.

`I_5()`

This function is called by `case_four()` whenever we encounter $\hat{I}^{(2,5)}(2-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$, $\nu_i \neq 0$, $\forall i$. We have already explained that in our restricted class of Feynman diagrams, with only one subdivergence, this case can only occur when there is a subdivergence in line "5". `I_5()` uses the equations (D.1) to (D.5) to bring the integral, respectively the function `I_5()`, to the form `I_5(1,0,1,0,1,0,1,0,1,a_5)`, where all the exponents of the propagators are equal to 1 except for the propagator with momentum l, whose exponent is $1 + a_5 \cdot \varepsilon$, $a_5 \in \mathbb{N}$. Hence the output of this function is a sum of functions `F_ab()` and `I_5(1,0,1,0,1,0,1,0,1,a_5)`.

Chapter 8

Counterterms of massless Feynman diagrams

8.1 The programs

In the present chapter we will first describe a set of four programs, called `yukawa1()`, `yukawa2()`, `qed1()`, and `qed2()`, that calculate the counterterm for an input graph. These programs were originally written by Richard Kreckel. He implemented the one-loop scheme of Section 7.1 as a computer program which led to a first publication of results in [BKK 2002]. We refer the reader to this article for the explicit construction of the programs. Here, we want to focus on the way in which one has to use them and the major changes due to the extension of the program by the non-planar vertex correction. The programs together with documentation can be downloaded from `http://wwwthep.physik.uni-mainz.de/Publications/theses.html`.

In the second part of this chapter we will present results obtained with the help of these programs. The questions which we are interested in all concern the connection between the appearance of Riemann's zeta function and the underlying topology of a Feynman diagram.

General remarks

The set of graphs whose counterterms we can calculate is constrained to graphs that are constructed from the building blocks of the set P described in Chapter 7. After mapping such a graph to a tree following Fig. 3.1, the input for the graph into the programs is a decorated rooted tree in list notation. Consider, for instance, a two-loop example and its corresponding rooted tree given in Fig. 8.1. In the third column we have added the tree in list notation (cf. also [Krei 1998b]). Similarly to the rooted trees, these lists encode the relative position of the divergences inside a graph. The symbol $\tilde{\Gamma}$ here and in the following denotes the primitive non-planar vertex correction. All three representations of the graph in Fig. 8.1 provide the same information. The programs then calculate the antipode of this rooted tree by setting all the full cuts defined in equation (3.6) of Chapter 3 and summing up the different terms of the sum — the trees with the different assignments of cuts provided by the antipode. The different terms of the sum, or the different trees, respectively, together build the counterterm for the overall divergence of the graph (cf. Chapter 2 and 3).

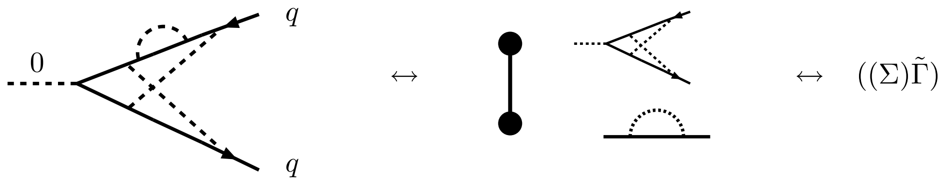


Figure 8.1: Three different representations for a graph: as a Feynman diagram, as its corresponding tree, and in list notation.

In the last chapter we assigned to each divergent loop a function $\Sigma_{i,j}$, $\Pi_{i,j}$, etc (cf. page 82). These functions are related to the vertices of the tree and are implemented in the programs in the form of classes that can be evaluated. The fermion self-energy which corresponds to the function Σ is called `Sigma`, the Vacuum polarization corresponding to Π is called `Vacuum`, the one-loop vertex correction is denoted by `Gamma` (Γ), and the non-planar vertex correction by `Gamma2` ($\tilde{\Gamma}$). Evaluating these classes means expanding the corresponding functions up to an order that is implicitly and correctly set by the program itself. The boolean parameter `cut`, which is also set automatically by the program, then decides whether this Laurent series is to be projected onto its pole part or not, corresponding to the application of the renormalization map R_{MS}.

One remark is necessary at this point. We have seen in Section 7.1 that the general result for a graph mainly consists of two terms: the functions $\Sigma_{i,j}$, $\Pi_{i,j}$, etc., and a term proportional to $[q^2]^{-i\varepsilon}$, where i is the number of loops of the graph. If we now expand the result we have to expand both terms. The expression proportional to $[q^2]$ is, as we have seen in (2.16) and (4.8), expanded via the exponential series:

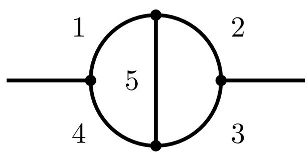
$$[q^2]^{-i\varepsilon} = \exp(\ln([q^2]^{-i\varepsilon})) = \exp(-i\varepsilon \ln[q^2]) = 1 - i\varepsilon \ln[q^2] + O(\varepsilon^2). \tag{8.1}$$

We know that the counterterms of Feynman diagrams do not contain any non-local terms proportional to $\frac{\ln^j[q^2]}{\varepsilon^i}$ (cf. Chapter 2). Although these fractions emerge at different intermediate steps, they cancel once we sum up all contributing terms. Since we know this, we can simply ignore the terms involving $\ln[q^2]$ from the very beginning, and only consider the "1" in (8.1). All other parts proportional to $\frac{\ln^j[q^2]}{\varepsilon^i}$ have to cancel in the sum, which allows us to work solely with the functions $\Sigma_{i,j}$, $\Pi_{i,j}$, etc. On the one hand this makes the calculations as such easier, and on the other hand it lowers the amount of computer memory used. A question that could be asked in this context is, whether this restriction could be a source of mistakes in the program. We know that expressions $\left(\ln[q^2]\right)^i$ always occur in combination with the Euler-Mascheroni constant γ_E, and that the counterterms are also γ_E-free. Indeed we find in the calculation of a counterterm that intermediate terms of the sum in general involve constants γ_E, but that all these constants cancel once we build the sum. This is used as a check inside the program, and any γ_E remaining in the expression for the antipode would cause an error message. Hence we believe that this serves as a check that we were allowed to ignore the $\left(\ln[q^2]\right)^i$ in (8.1) and did not cause any mistakes. A calculation involving the complete series (8.1) is not necessary at this point and would not provide us with further information. The same argument holds for the fact that we restricted ourselves to momenta $q^2 > 0$. Additional imaginary parts due to a momentum $q^2 < 0$ would drop out due to the same reasons as the terms $\ln[q^2]$ or γ_E vanish, and therefore cause no problems.

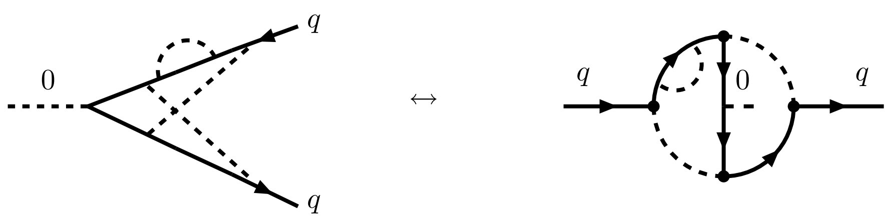
We want to make one technical remark concerning the non-planar vertex correction. Unlike the functions for the one-loop graphs that are defined directly inside the corresponding classes, the result for the two-loop non-planar vertex correction with different subdivergences is stored in *GiNaC* archive files (cf. Fig. 7.2). Hence we have additional functions `yuk1_Gamma2_ev()`, `yuk2_Gamma2_ev()`, `qed1_Gamma2_ev()`, and `qed2_Gamma2_ev()`, that are called inside the classes for the non-planar vertex correction. In these functions we substitute `I_5()` by the function `I_5_ev()` containing the expansion of the function `I_5()` up to the order ε^4 and for $a_5 = 1, ..., 10$. This result has been generated using the program `residuum` of Chapter 6. In the end, `yuk1_Gamma2_ev()` to `qed2_Gamma2_ev()` return the function or the matrix for `Gamma2` in the form of expressions that are properly expanded.

8.2 How to use the programs

Consider again the non-planar vertex correction with its underlying topology. We assigned already in Chapters 4, 6, and 7 a certain number to each line of this topology:



These numbers are now used to indicate in which line the subdivergences are sitting, by adding a subscript with the number of the line to the corresponding divergence. Consider again the graph from our previous example:



The fermion self-energy is sitting at line "1" of the non-planar vertex correction. Therefore the class for this subdivergence is called `Sigma_1`. The same works for the lines "3" and "5", which are the only lines we have to take into account for fermion subdivergences. Note that as we have a vertex of zero momentum transfer, we do not have to distinguish between the two possible pieces of the fermion line above and below the zero momentum transfer vertex which are denoted by the number 5. All the different classes for the four different cases are listed in Appendix E.

In Yukawa theory the input line for this graph would be: `((Sigma_1)Gamma2)`. In QED we additionally have a gauge parameter ξ which is assigned to its corresponding expression in brackets "[ ]", `((Sigma_1[xi])Gamma2[xi])`. The gauge parameter can be a symbol `xi` or any numerical value. Note that the gauge parameter for the graph `Gamma2`, the non-planar vertex correction, is the gauge parameter for both lines.

Here is an example how the programs calculate the result for this graph. We choose QED with the momentum distribution of *case 1*:

```
1 $ ./qed1.out "((Sigma_1[xi])Gamma2[xi])"
2 After decoration the tree has these indices:
3   (Gamma2[1,0,0,0,0][xi](Sigma_1[0,0,0,0,0][xi]))
4 ----+----#----+----#----+----#----+----#----+----#----+----#
5 The antipode of this tree appears to be:
6
7    (2/3+xi+1/4*xi^2-1/12*xi^3)*x^(-2)
8   +(-2/3-5/6*xi-5/24*xi^2-1/24*xi^3)*x^(-1)
```

The graph to be computed is passed in list form as a string on the command line, together with the gauge parameter. Line 3 shows the set of indices attached to the divergences, indicating the place of insertion of the subdivergences. Note how the indices i and j are set up automatically. The next line is a simple progress bar, useful when computations take longer. The result is then printed as a power series in the regularization parameter, here called `x` instead of ε. Since each divergence is primitive and therefore starts with a term $\frac{1}{\varepsilon}$, a general result is a Laurent series whose lowest order equals the number of vertices of a tree, which is two in this example.

8.3 Counterterms and zeta functions

With the help of the programs `yukawa1()` to `qed2()` it is now very easy to investigate the counterterms of Feynman diagrams built from the primitive graphs cited above. We will give some of the results obtained using the program, but before this, we would like to define more explicitly what we mean by the "underlying topology" of a Feynman diagram.

Consider any loop diagram. To obtain the topology of this graph one first ignores the external lines and the types of all inner lines of a graph, drawing them all as a solid line. In a second step one transforms the remaining picture into the form of a circle containing inner lines. If the picture obtained in this way does not immediately show a typical and known topology, one tries to deform the picture in such a way that it is equal to one of the basic topologies like, for example, the ladder topology. Fig. 8.2 shows two examples. In the first row we see a graph with an underlying "swiss cheese topology". The second row shows an example for a graph with a ladder topology. This topology is not immediately obvious, so we deformed the graph from the second to the third picture by pulling the right-most line "into the middle" of the graph, as indicated by the two arrows. This is an allowed operation to identify the topology. To cut a line into two halves, for example, would not be allowed.

In Chapter 4 we have already referred to a connection between the underlying topology of a Feynman diagram and the appearance of Riemann's zeta function in the expansion of the corresponding analytic expression. This connection has been extensively studied by D. Kreimer, who defined a way to translate Feynman diagrams into knots or links and finally into braids [Krei 1997, Krei 2000b]. It could, for example, be shown that in the Laurent series of counterterms of Feynman diagrams with a ladder topology, only rational numbers

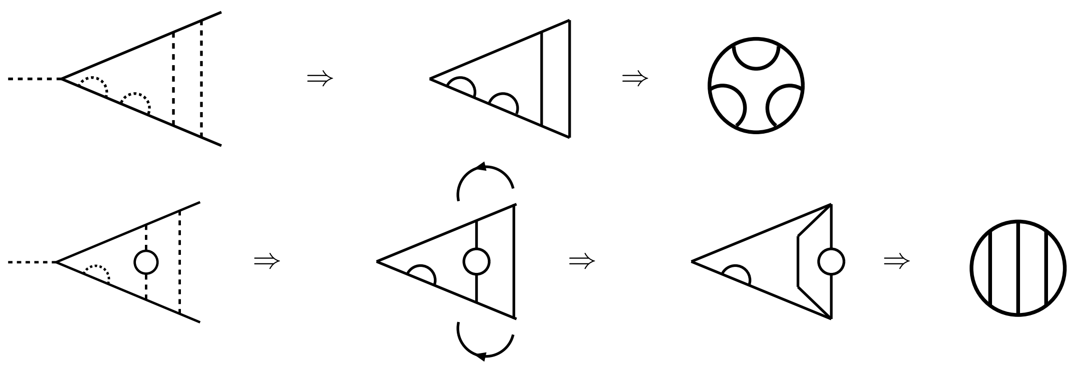


Figure 8.2: Two examples that illustrate how one can find the underlying topology of a Feynman diagram. The first graph leads to a "swiss cheese" topology, the second graph to a ladder topology. For the second graph we have to deform the picture in one step to be able to identify the ladder topology.

occur [DKT 1996, DEMcA 1997, Krei 1997, Krei 2000b, Krei 1998a]. Furthermore, D. Kreimer showed that the first possible non-ladder topology is related to the appearance of $\zeta(3)$, like shown in Fig. 8.3, where we have drawn one possible realization of this topology within our set of graphs.

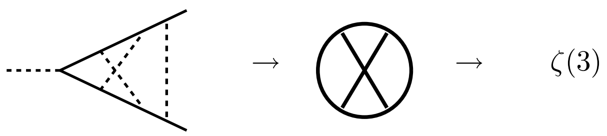
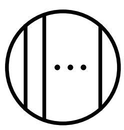


Figure 8.3: The connection between a topology and the appearance of $\zeta(3)$ in its counterterm (see [Krei 1997] for a detailed explanation).

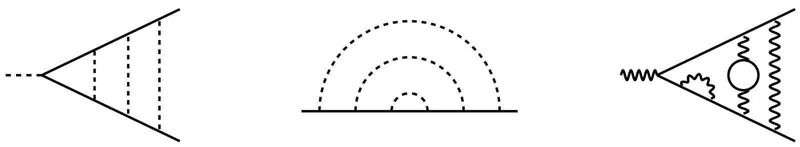
However, to describe the way in which these results were obtained and the interconnections that can be derived from this is out of the scope of this thesis. We restrict ourselves to the presentation of some interesting results that have been obtained using the above mentioned programs. Before that, we have to remark that we will only consider the $\frac{1}{\varepsilon}$- term, the *residue* of the graphs. The reason for this is that it is the only order contributing to the β-function of the renormalization group [Coll 1984].

Ladder diagrams

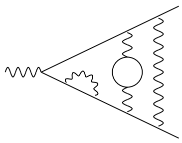
We have already seen an example for a graph with ladder topology in Fig. 8.2. The underlying topology for a graph with ladder topology is in general given by:



The next figure shows graphs that realize this topology, built within our set of graphs:



In Fig. 8.2 we have shown how the third graph indeed corresponds to an underlying ladder topology. For all the graphs with ladder topology which we tested for both theories we could always confirm that the coefficients in the Laurent series of their counterterms only involve rational numbers. As an example we state here the result for the third graph in the above row with a momentum distribution of *case 1* and gauge parameter ξ:

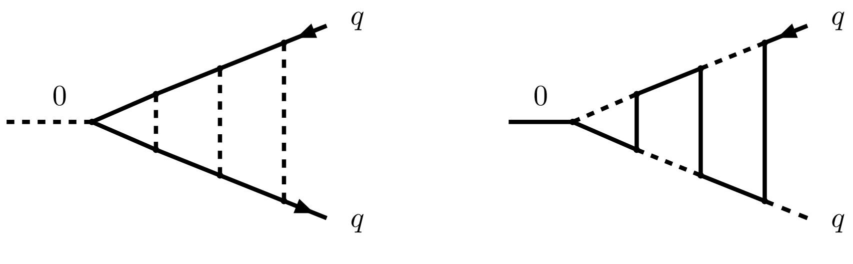


$$= \quad (-\tfrac{1}{3} - \tfrac{2}{3}\xi - \tfrac{1}{3}\xi^2)\varepsilon^{-3} + (\tfrac{17}{18} + \tfrac{11}{9}\xi + \tfrac{5}{18}\xi^2)\varepsilon^{-2} + (\tfrac{7}{36} + \tfrac{1}{3}\xi + \tfrac{5}{36}\xi^2)\varepsilon^{-1}$$

Note that the vacuum polarization in the inner vertex correction causes the gauge parameter of this line to be -1, which has to be inserted manually in the input line of the program. The corresponding vertex correction is convergent for this value of the gauge parameter, so the Laurent series starts with ε^{-3} instead of ε^{-4}.

Symmetries in coefficients of zeta functions - one-loop building blocks

In this section we will only consider the one-loop building blocks of P, excluding the non-planar vertex correction. One question we want to investigate is, whether one can find symmetries that apply to the coefficients of zeta functions, if one changes the flow of momentum in a diagram. Consider, for example, the vertex corrections of *case 1* and *2* in Yukawa theory. We draw the *case 2* here a little different from before, to be able to compare the graphs more easily.



(a) Vertex correction of *case 1*. (b) Vertex correction of *case 2*.

We now insert two fermion self-energies into the innermost vertex correction of both graphs which then represent the same topology but with different momentum flows. Let us denote the graphs according to the names assigned to them in Fig. 8.4. The letter Γ tells us that the outermost divergence is a vertex correction. The upper index indicates the two different cases, while the index $[2, 0, n]$ states the number of subdivergences inside the graph. These subdivergences can be of the form of a fermion self-energy (2), vacuum polarization (0) or vertex correction (n).

$\Gamma^1_{[2,0,n]} =$ $\qquad$ $\Gamma^2_{[2,0,n]} =$

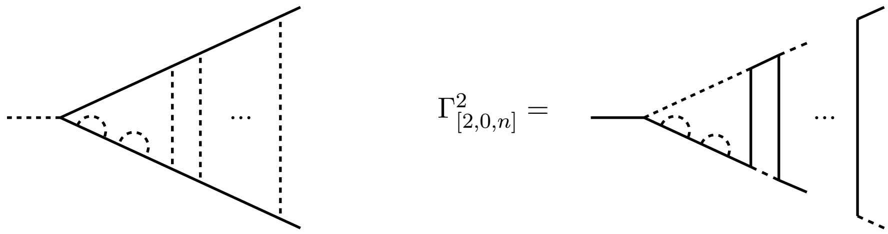


Figure 8.4: The one-loop vertex corrections with their different assignments of momenta. The index $[2,0,n]$ indicates the number of subdivergences inside the graph: two fermion self-energies, no vacuum polarization, and n vertex corrections.

We then obtain for the graphs $\Gamma^1_{[2,0,n]}$ and $\Gamma^2_{[2,0,n]}$, $n \in \{1,...,5\}$:

$$\mathrm{res}(\Gamma^1_{[2,0,1]}) = \frac{5}{48} - \frac{1}{8}\zeta(3) \tag{8.2}$$

$$\mathrm{res}(\Gamma^2_{[2,0,1]}) = \frac{1}{12} - \frac{1}{8}\zeta(3) \tag{8.3}$$

$$\mathrm{res}(\Gamma^1_{[2,0,2]}) = \frac{1}{20} - \frac{9}{40}\zeta(3) - \frac{3}{80}\zeta(4) \tag{8.4}$$

$$\mathrm{res}(\Gamma^2_{[2,0,2]}) = \frac{1}{240} - \frac{1}{20}\zeta(3) - \frac{3}{80}\zeta(4) \tag{8.5}$$

$$\mathrm{res}(\Gamma^1_{[2,0,3]}) = -\left(\frac{23}{90} + \frac{9}{20}\zeta(3) + \frac{7}{80}\zeta(4) + \frac{7}{240}\zeta(5)\right) \tag{8.6}$$

$$\mathrm{res}(\Gamma^2_{[2,0,3]}) = -\left(-\frac{1}{240} + \frac{1}{6}\zeta(3) + \frac{1}{80}\zeta(4) + \frac{7}{240}\zeta(5)\right) \tag{8.7}$$

$$\mathrm{res}(\Gamma^1_{[2,0,4]}) = -\left(\frac{919}{630} + \frac{71}{70}\zeta(3) + \frac{111}{560}\zeta(4) + \frac{1}{12}\zeta(5) + \frac{1}{560}\zeta(3)^2 + \frac{1}{112}\zeta(6)\right) \tag{8.8}$$

$$\mathrm{res}(\Gamma^2_{[2,0,4]}) = -\left(\frac{65}{224} + \frac{11}{140}\zeta(3) + \frac{1}{16}\zeta(4) + \frac{1}{120}\zeta(5) + \frac{1}{560}\zeta(3)^2 + \frac{1}{112}\zeta(6)\right) \tag{8.9}$$

$$\begin{aligned}\mathrm{res}(\Gamma^1_{[2,0,5]}) = -\Bigg(&\frac{6481}{1120} + \frac{33613}{13440}\zeta(3) + \frac{2133}{4480}\zeta(4) + \frac{101}{480}\zeta(5) + \frac{27}{4480}\zeta(3)^2 \\ &+\frac{27}{896}\zeta(6) + \frac{3}{4480}\zeta(3)\zeta(4) + \frac{7}{1920}\zeta(7)\Bigg)\end{aligned} \tag{8.10}$$

$$\begin{aligned}\mathrm{res}(\Gamma^2_{[2,0,5]}) = -\Bigg(&\frac{863}{3360} + \frac{61}{160}\zeta(3) + \frac{27}{1120}\zeta(4) + \frac{7}{120}\zeta(5) + \frac{1}{2240}\zeta(3)^2 \\ &+\frac{1}{448}\zeta(6) + \frac{3}{4480}\zeta(3)\zeta(4) + \frac{7}{1920}\zeta(7)\Bigg)\end{aligned} \tag{8.11}$$

First of all, we see that the first zeta function appearing in Yukawa theory for the graphs $\Gamma^1_{[2,0,1]}$ and $\Gamma^2_{[2,0,1]}$ is $\zeta(3)$, not $\zeta(2)$ as one could expect since $\zeta(2)$ would be the zeta function with the smallest possible integer argument. Note that the underlying topology for these graphs is the "swiss cheese" topology of Fig. 8.2.

The residues are in general a linear combination of terms of varying transcendental weight. Like in Chapter 5 we define the transcendental weight w of a monomial $\prod_i \zeta(j_i)$ as

$$w(\prod_i \zeta(j_i)) = \sum_i j_i.$$

The weight vanishes for a rational number. The above results then show that on the one hand, the highest transcendental weight increases with the number of loops. On the other hand, they confirm that the coefficient of the highest-weight transcendental in the transition from $\mathrm{res}(\Gamma^1_{[2,0,n]})$ to $\mathrm{res}(\Gamma^2_{[2,0,n]})$ is invariant. This is the symmetry we were looking for. A proof of this symmetry behavior of the graphs under a change of momentum-flow can be found in [BKK 2002].

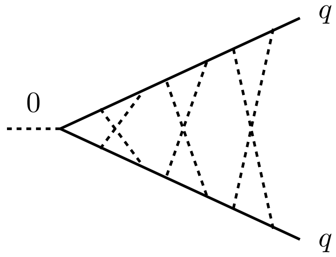
A similar relation holds in QED where we again find the first zeta function to be a $\zeta(3)$ in the graphs $\Gamma^1_{[2,0,1]}$ and $\Gamma^2_{[2,0,1]}$ with "swiss cheese" topology. The symmetry described so far and the general behavior can also be found in QED, independent from the chosen gauge [BKK 2002].

The non-planar vertex correction iterated into itself

Next, we will investigate graphs built from the non-planar vertex correction. For a discussion concerning the relation between the topology of such a graph and the appearance of zeta functions we refer the reader again to [Krei 2000b].

To denote graphs including the non-planar vertex correction, we already called this vertex correction $\tilde{\Gamma}$. The upper index at this $\tilde{\Gamma}$ counts the two possible momentum distributions in the way described above, while we add a fourth index to the set of lower indices to denote the number of non-planar vertex corrections appearing as subdivergences. Hence we find, for example:

$\tilde{\Gamma}^1_{[0,0,0,2]} =$



We again start with the massless Yukawa theory, inserting in a first step the non-planar vertex correction several times into itself. This is only possible at the vertex of zero momentum transfer, leading to the following results for the two cases:

$$\mathrm{res}(\tilde{\Gamma}^1_{[0,0,0,0]}) = \frac{1}{2} \tag{8.12}$$

$$\mathrm{res}(\tilde{\Gamma}^2_{[0,0,0,0]}) = \frac{1}{2} \tag{8.13}$$

$$\mathrm{res}(\tilde{\Gamma}^1_{[0,0,0,1]}) = -\frac{3}{4} + \frac{3}{4}\zeta(3) \tag{8.14}$$

$$\mathrm{res}(\tilde{\Gamma}^2_{[0,0,0,1]}) = -\frac{1}{4} + \frac{3}{4}\zeta(3) \tag{8.15}$$

$$\mathrm{res}(\tilde{\Gamma}^1_{[0,0,0,2]}) = \frac{5}{2} - \frac{7}{2}\zeta(3) + \frac{3}{2}\zeta(3)^2 \tag{8.16}$$

$$\mathrm{res}(\tilde{\Gamma}^2_{[0,0,0,2]}) = \frac{1}{2} - \zeta(3) + \frac{3}{2}\zeta(3)^2 \tag{8.17}$$

$$\mathrm{res}(\tilde{\Gamma}^1_{[0,0,0,3]}) = -\frac{91}{8} + \frac{165}{8}\zeta(3) - \frac{27}{2}\zeta(3)^2 + \frac{27}{8}\zeta(3)^3 + \frac{15}{16}\zeta(5) \tag{8.18}$$

$$\mathrm{res}(\tilde{\Gamma}^2_{[0,0,0,3]}) = -\frac{9}{8} + \frac{27}{8}\zeta(3) - \frac{27}{8}\zeta(3)^2 + \frac{27}{8}\zeta(3)^3 + \frac{15}{16}\zeta(5) \tag{8.19}$$

$$\mathrm{res}(\tilde{\Gamma}^1_{[0,0,0,4]}) = \frac{306}{5} - \frac{273}{2}\zeta(3) + \frac{594}{5}\zeta(3)^2 - \frac{243}{5}\zeta(3)^3 + \frac{81}{10}\zeta(3)^4 - \frac{39}{4}\zeta(5) + 9\zeta(3)\zeta(5) \tag{8.20}$$

$$\mathrm{res}(\tilde{\Gamma}^2_{[0,0,0,4]}) = \frac{16}{5} - \frac{54}{5}\zeta(3) + \frac{81}{5}\zeta(3)^2 - \frac{54}{5}\zeta(3)^3 + \frac{81}{10}\zeta(3)^4 - 3\zeta(5) + 9\zeta(3)\zeta(5) \tag{8.21}$$

$$\begin{aligned}\mathrm{res}(\tilde{\Gamma}^1_{[0,0,0,5]}) = &-\frac{1463}{4} + 969\zeta(3) - 1050\zeta(3)^2 + 585\zeta(3)^3 - \frac{675}{4}\zeta(3)^4 + \frac{81}{4}\zeta(3)^5 \\ &+ 85\zeta(5) - \frac{525}{4}\zeta(3)\zeta(5) + \frac{225}{4}\zeta(3)^2\zeta(5) + \frac{63}{64}\zeta(7)\end{aligned} \tag{8.22}$$

$$\begin{aligned}\mathrm{res}(\tilde{\Gamma}^2_{[0,0,0,5]}) = &-\frac{119}{12} + 40\zeta(3) - \frac{135}{2}\zeta(3)^2 + \frac{135}{2}\zeta(3)^3 - \frac{135}{4}\zeta(3)^4 + \frac{81}{4}\zeta(3)^5 \\ &+ \frac{25}{2}\zeta(5) - \frac{75}{2}\zeta(3)\zeta(5) + \frac{225}{4}\zeta(3)^2\zeta(5) + \frac{63}{64}\zeta(7)\end{aligned} \tag{8.23}$$

One can deduce several things from (8.12) to (8.23):

- Only odd zeta functions occur; there are no zeta functions with an even argument.
- The first zeta function, $\zeta(3)$, appears at the four-loop level, which corresponds to one insertion of the non-planar vertex correction into itself. With each additional non-planar vertex correction as subdivergence we get an additional power of $\zeta(3)$. If i is the number of non-planar vertex correction subdivergences, we get powers of $\zeta(3)$ ranging from $\zeta(3)$ to $\zeta(3)^i$.

- Starting from three subdivergences (8 loops), also $\zeta(5)$ and $\zeta(7)$ appear, first as a singular term and then also multiplied with $\zeta(3)$ and $\zeta(3)^2$.
- Comparing the two different cases the following rational coefficients of zeta functions are found to be equal: For pure powers of $\zeta(3)$, exactly the coefficient of the highest power of zeta functions is the same for both cases; for $\zeta(5)$ it is the coefficient of the product with the highest power of $\zeta(3)$ occurring in it; for $\zeta(7)$ the behavior is similar, where in this case we only have a product with $\zeta(3)^0 = 1$.

The even zeta functions are related to powers of π via the *Bernoulli numbers* B_k:

$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k}}{2(2k)!}(2\pi)^{2k}, \qquad k \in \mathbb{N} \tag{8.24}$$

with

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad \cdots$$
$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k, \quad \text{for} \quad n \geq 2. \tag{8.25}$$

which was also used in the function `zeta_subs()` of Chapter 6 (cf. (6.39) and (6.40)). Hence we can deduce from the first item that there are no powers of π appearing in the counterterms of the non-planar vertex correction.

The situation changes slightly once we turn to QED. Here, the gauge parameter destroys the symmetries we found so far. This needs some explanation. Let us start by noticing that the first three of the observations listed above are still valid for a general gauge parameter ξ. We only have odd zeta functions occurring, with the same power behavior as described above. We have tested this effect up to the orders described for Yukawa theory, but here we only give the counterterm for the primitive graph and one iteration of itself as an illustration:

$$\mathrm{res}(\tilde{\Gamma}^1_{[0,0,0,0]}) = 2 + \xi - \frac{1}{4}\xi^2 \tag{8.26}$$

$$\mathrm{res}(\tilde{\Gamma}^2_{[0,0,0,0]}) = 2 + \xi - \frac{1}{4}\xi^2 \tag{8.27}$$

$$\mathrm{res}(\tilde{\Gamma}^1_{[0,0,0,1]}) = 4 + 12\xi\zeta(3) + \frac{11}{32}\xi^4 - 9\xi - \frac{13}{4}\xi^2 - \frac{1}{2}\xi^3 + 3\xi^2\zeta(3) \tag{8.28}$$

$$\mathrm{res}(\tilde{\Gamma}^2_{[0,0,0,1]}) = -2 + 6\xi\zeta(3) + \frac{3}{32}\xi^4 - 2\xi - \frac{1}{8}\xi^2 + \frac{3}{8}\xi^3 + 6\zeta(3) + \frac{3}{2}\xi^2\zeta(3) \tag{8.29}$$

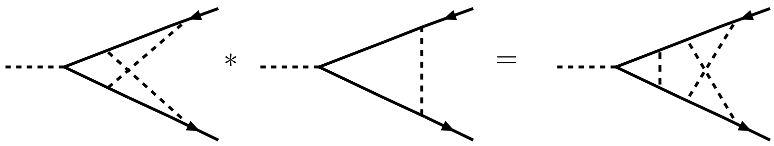
Taking a closer look at the gauge parameter, we can see that for $\xi = 0$ the $\zeta(3)$ in (8.28) even completely vanishes, while we keep a $6\zeta(3)$ in (8.29). On the other hand, one can always find a special gauge in which certain zeta functions agree in their rational coefficient. But still, this is not true for a general gauge parameter ξ. However, since we did not try to consider gauge invariant graphs this should not be regarded as a claim that the symmetry found in

Yukawa theory can not be found in QED. It is only a statement about the results obtained within our set of graphs. The investigation of gauge invariant graphs and their symmetries would need some further work, which has not been carried out so far.

One more remark concerning a test of the program. Before we used the Gröbner basis to simplify relations of zeta functions (cf. Chapter 6), a general result of the form of equations (8.12) to (8.23) contained a term proportional to $q^\mu \not{q}/q^2$ consisting of complicated-looking relations of zeta functions of different depths and weights. They all canceled to zero once we used the relations provided by the Gröbner basis and (8.24). We consider this to be a test for the calculation of the graph `Gamma2`.

Shuffles of the non-planar two-loop and the one-loop vertex correction

In a next step, we build combinations of the one-loop vertex correction Γ and the non-planar vertex correction $\tilde{\Gamma}$. Using the gluing operation of Section 3.2, we define words built out of these letters in the following way: Take one one-loop vertex correction, the letter Γ, and one non-planar vertex correction, the letter $\tilde{\Gamma}$. By gluing Γ into $\tilde{\Gamma}$ or vice versa, we obtain the words $\Gamma\tilde{\Gamma}$ and $\tilde{\Gamma}\Gamma$, respectively:



$$\tilde{\Gamma} * \Gamma = \Gamma\tilde{\Gamma}$$

Note the change of order which is due to the fact that we glue the graph on the right hand side of the $*$ into the graph to the left.

We can then build shuffles of these letters according to Chapter 5, such as:

$$\Gamma \sqcup\!\sqcup \tilde{\Gamma} = \Gamma\tilde{\Gamma} + \tilde{\Gamma}\Gamma \tag{8.30}$$

$$\Gamma\Gamma \sqcup\!\sqcup \tilde{\Gamma} = \Gamma\Gamma\tilde{\Gamma} + \Gamma\tilde{\Gamma}\Gamma + \tilde{\Gamma}\Gamma\Gamma \tag{8.31}$$

$$\Gamma \sqcup\!\sqcup \tilde{\Gamma}\tilde{\Gamma} = \Gamma\tilde{\Gamma}\tilde{\Gamma} + \tilde{\Gamma}\Gamma\tilde{\Gamma} + \tilde{\Gamma}\tilde{\Gamma}\Gamma \tag{8.32}$$

$$\Gamma\Gamma \sqcup\!\sqcup \tilde{\Gamma}\tilde{\Gamma} = \Gamma\Gamma\tilde{\Gamma}\tilde{\Gamma} + \Gamma\tilde{\Gamma}\Gamma\tilde{\Gamma} + \Gamma\tilde{\Gamma}\tilde{\Gamma}\Gamma + \tilde{\Gamma}\Gamma\Gamma\tilde{\Gamma} + \tilde{\Gamma}\Gamma\tilde{\Gamma}\Gamma + \tilde{\Gamma}\tilde{\Gamma}\Gamma\Gamma \tag{8.33}$$

In the following we list the residues for these graphs in Yukawa theory of *case 1*, denoting the graphs by their corresponding words:

$$\Gamma\tilde{\Gamma} = \frac{1}{2} \qquad \tilde{\Gamma}\Gamma = \frac{5}{6} - \zeta(3)$$

$$\Gamma\Gamma\tilde{\Gamma} = \frac{19}{24} \qquad \tilde{\Gamma}\tilde{\Gamma}\Gamma = -\frac{289}{120} - \frac{9}{5}\zeta(3)^2 - \frac{1}{1200}\pi^4 + \frac{37}{10}\zeta(3)$$

$$\Gamma\tilde{\Gamma}\Gamma = \frac{3}{2} + \frac{1}{240}\pi^4 - \frac{5}{4}\zeta(3) \qquad \tilde{\Gamma}\Gamma\tilde{\Gamma} = -\frac{31}{20} + \frac{1}{300}\pi^4 + \frac{11}{10}\zeta(3)$$

$$\tilde{\Gamma}\Gamma\Gamma = \frac{35}{24} - \frac{1}{240}\pi^4 - \zeta(3) \qquad \Gamma\tilde{\Gamma}\tilde{\Gamma} = -\frac{197}{120} - \frac{1}{400}\pi^4 + \frac{6}{5}\zeta(3)$$

$$\Gamma\Gamma\tilde{\Gamma}\tilde{\Gamma} = -\frac{291}{80} + \frac{17}{40}\zeta(5) - \frac{13}{2400}\pi^4 + \frac{49}{20}\zeta(3)$$
$$\Gamma\tilde{\Gamma}\Gamma\tilde{\Gamma} = -\frac{127}{30} + \frac{29}{40}\zeta(5) + \frac{1}{2400}\pi^4 + \frac{93}{40}\zeta(3)$$
$$\Gamma\tilde{\Gamma}\tilde{\Gamma}\Gamma = -\frac{607}{90} + \frac{53}{60}\zeta(5) - \frac{33}{10}\zeta(3)^2 - \frac{1}{90}\pi^4 + \frac{263}{30}\zeta(3) + \frac{1}{100}\pi^4\zeta(3)$$
$$\tilde{\Gamma}\Gamma\Gamma\tilde{\Gamma} = -\frac{61}{18} + \frac{7}{24}\zeta(5) + \frac{1}{144}\pi^4 + \frac{25}{12}\zeta(3)$$
$$\tilde{\Gamma}\Gamma\tilde{\Gamma}\Gamma = -\frac{173}{30} + \frac{21}{40}\zeta(5) - 3\zeta(3)^2 - \frac{1}{2400}\pi^4 + \frac{327}{40}\zeta(3)$$
$$\tilde{\Gamma}\tilde{\Gamma}\Gamma\Gamma = -\frac{299}{48} + \frac{9}{10}\zeta(5) - \frac{27}{10}\zeta(3)^2 + \frac{23}{2400}\pi^4 + \frac{77}{10}\zeta(3) - \frac{1}{100}\pi^4\zeta(3)$$

And hence we obtain:

$$\Gamma \sqcup\!\sqcup \tilde{\Gamma} = \frac{4}{3} - \zeta(3) \tag{8.34}$$
$$\Gamma\Gamma \sqcup\!\sqcup \tilde{\Gamma} = \frac{15}{4} - \frac{9}{4}\zeta(3) \tag{8.35}$$
$$\Gamma \sqcup\!\sqcup \tilde{\Gamma}\tilde{\Gamma} = -\frac{28}{5} + 6\zeta(3) - \frac{9}{5}\zeta(3)^2 \tag{8.36}$$
$$\Gamma\Gamma \sqcup\!\sqcup \tilde{\Gamma}\tilde{\Gamma} = -30 + \frac{63}{2}\zeta(3) - 9\zeta(3)^2 + \frac{15}{4}\zeta(5) \tag{8.37}$$

We can see here that in all cases, the terms involving zeta functions of an even number, expressed as powers of π, vanish! This is true for π^4 alone, and also in the combination with a zeta function of an odd argument in the form of $\pi^4\zeta(3)$. We have tested this to also be true for the shuffle of $\Gamma\Gamma\Gamma \sqcup\!\sqcup \tilde{\Gamma}\tilde{\Gamma}\tilde{\Gamma}$. In the counterterms for this shuffle, zeta functions of the following forms occur: $\zeta(3)$, $\zeta(4)$, $\zeta(5)$, $\zeta(6)$, $\zeta(7)$, $\zeta(8)$, $\zeta(3)^2$, $\zeta(3)^3$, $\zeta(3)\zeta(4)$, $\zeta(3)\zeta(5)$, $\zeta(3)\zeta(6)$, $\zeta(4)\zeta(5)$. If we sum up the counterterms, the coefficients of zeta functions involving $\zeta(4), \zeta(6)$ and $\zeta(8)$ add up to zero. We have checked that the same is true for Yukawa *case 2*.

Also in the case of QED we have checked that we find the same behavior for the shuffles (8.30) to (8.32), this time even for a general gauge parameter ξ!

To summarize, we have shown some results that confirm numerically the connection between the topology of a graph and the appearance of zeta functions in its counterterm. However, we only took a first glance into the underlying symmetries that even seem to involve a shuffle algebra structure.

Chapter 9

Summary and outlook

The main achievement of this thesis is the expansion of the massless two-loop two-point function. Former works on this function only succeeded in expanding it maximally up to order ε^8 in the dimensional regularization parameter ε, and were limited to certain choices of the exponents of its momenta. Our new method allows us to expand this function up to an arbitrary order in ε, where the exponents can have any value as long as the set of exponents fulfills three limiting conditions arising within the calculation.

The calculation of the massless two-loop two-point function included the decomposition of the integral into a one-loop two-point and a one-loop three-point function, where the latter was rewritten into a double Mellin-Barnes integral. Recombining these two we obtained an integral for the two-loop two-point function whose integrand consists of a fraction of Euler's gamma functions. We then closed the integration contour at infinity and collected the residues of the gamma functions. This procedure transformed the integral into sums and double sums of gamma functions that have exactly the form of functions implemented in the C++ library *nestedsums*. Using these functions, we wrote a computer program with which one can expand the massless two-loop two-point function in the parameter ε, with the only limitation on the expansion order being the available computer memory.

Apart from the problem of expanding this integral up to higher loop orders, there is also a number theoretical question concerning the (transcendental) numbers that can occur in this expansion. Before we performed our calculation it was known that up to the order ε^9 only rational numbers and multiple zeta values occur. Taking a closer look at the *nestedsums* functions involved in our calculations, we were able to deduce that rational numbers and multiple zeta values are in fact the *only* numbers that can occur in the expansion of the massless two-loop two-point function. There are no other numbers, transcendental or not, that could appear in any order of ε.

We would like to stress here that the use of Mellin-Barnes integrals and the library *nestedsums* by far not have been fully exploited yet, inviting one to apply it to other Feynman diagrams or other problems occurring in the context of perturbative quantum field theories.

In the second part of this thesis, we applied the expansion of the massless two-loop two-point function to the calculation of the non-planar vertex correction in massless Yukawa theory and massless QED, and implemented the result of these calculations into four programs. These programs use the antipode of the Hopf algebra of rooted trees, which governs the process of renormalization of Feynman diagrams, and calculate this antipode for an input Feynman graph, yielding its counterterm. The set of graphs that can be examined with the help of these programs are graphs in massless Yukawa theory and massless QED built by insertion of the three divergent one-loop diagrams and the non-planar vertex correction into each other, where this insertion of graphs is part of the Lie algebra of Feynman diagrams. The vertex graphs have been calculated with one zero-momentum-transfer vertex and with two different assignments of momenta. Additionally, we restrict ourselves to the case where subdivergences inside the non-planar vertex correction only occur in one chosen line.

Using the programs described, we investigated the connection between the underlying topology of Feynman diagrams and the appearance of zeta functions in the residue of their counterterms. We could confirm that the coefficients of residues in counterterms of graphs with a ladder topology only involve rational numbers. This applies to both theories.

Considering the vertex graphs with the two different assignments of momenta we could show that the coefficients of the corresponding residues are subject to a symmetry relation when we change the momentum flow through the diagrams. This symmetry expresses itself in the fact that the coefficient of the zeta function with the highest transcendental weight remains the same when we change from one assignment of momenta to the other. For Yukawa theory this symmetry is strictly fulfilled. This is not the case in QED, where the gauge parameter partially destroys this symmetry. Since we did not calculate a gauge invariant set of graphs this can not be seen as a contradiction and would have to be clarified by further investigations.

For the non-planar vertex corrections we could show that the counterterms only involve odd zeta functions starting with $\zeta(3)$, but that there are no even zeta functions, *i.e.* powers of π, to any order in ε. This absence of powers of π could also be found in the residues of combinations of graphs given by shuffling the one-loop vertex correction with the non-planar vertex correction. To shuffle graphs means inserting them into each other and summing over the results in a way that is defined by the shuffle relation. This shuffle relation is one of the algebraic relations fulfilled by multiple zeta values, or more generally by multiple polylogarithms. It is not known yet, in which sense the analogy between counterterms and multiple zeta values holds and how deep-reaching it is. Equivalently, it would be interesting to know if there are other relations fulfilled by the counterterms, and whether one could use all these relations and apply them to questions arising from scattering processes. This has to be left open at this point and needs to be investigated in the future.

Appendix A

Algebras

In the following sections, we have compiled some of the different algebraic structures used throughout the previous chapters. Most of the definitions are taken from [Kass 1995, Reut 1993].

A.1 Monoid

A set M with an operation $m : M \times M \to M$ is called a *monoid*, if m is associative and there exists a unit element e with respect to m.

A.2 Free algebra

Let X be a set. Consider the vector space $\mathbb{K}X$ with the basis given by the set of all words $x_{i_1}...x_{i_p}$ in the alphabet X, including the empty word $\emptyset$. Define the degree of the monomial $x_{i_1}...x_{i_p}$ as its length p. *Concatenation* of words defines a multiplication on $\mathbb{K}X$ by

$$(x_{i_1}...x_{i_p})(x_{i_{p+1}}...x_{i_n}) = x_{i_1}...x_{i_p}x_{i_{p+1}}...x_{i_n}. \tag{A.1}$$

Formula (A.1) equips $\mathbb{K}X$ with an algebra structure, called the *free algebra* on the set X. The unit is the empty word: $1 = \emptyset$.

A.3 Hopf algebra

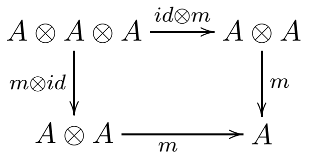
The starting point for the definition of a Hopf algebra is a vector space V over a field $\mathbb{K}$. We will, step by step, add structure maps to this vector space until we obtain the Hopf algebra. This can be visualized by introducing the definitions in the form of *commutative diagrams*.

A $\mathbb{K}$-vector space V is defined by the operation "+" and the *scalar multiplication*. An algebra A has one additional *associative* operation "$\cdot$", called *multiplication*.

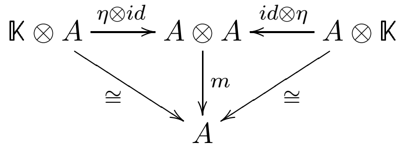
Algebra

A *unital, associative algebra* (A, m, η) is a triple, consisting of a $\mathbb{K}$-vector space A, and linear maps $m : A \otimes A \to A$ (the *multiplication*), and η (the *unit*): $\mathbb{K} \to A$, such that each of the following two diagrams commutes:

associativity:



unit:



Commutativity of a diagram means that following the two possible ways to travel through the diagram, one obtains the same result.

The map η enables us to write the definition of the scalar product via commutative diagrams: scalar multiplication of elements of A with a field element means that to each $a \in A$ there is an element $\lambda a \in A$, with $\lambda \in \mathbb{K}$. This multiplication with a scalar is performed in the following way: the field $\mathbb{K}$ can be embedded as a subalgebra into A, by mapping the unit element $1_{\mathbb{K}}$ onto 1_A. This embedding is done by the function $\eta : \mathbb{K} \to A, \eta : 1_{\mathbb{K}} \mapsto 1_A$. Multiplication of an algebra element $a \in A$ with $\lambda \in \mathbb{K}$ then has to be understood as mapping this element $\lambda 1_{\mathbb{K}}$ of the field to $\lambda 1_A$ in A and multiplying a with λ times 1_A of A.

If we write out the diagrams explicitly, the linear map m acts on the elements of the algebra in the following way:

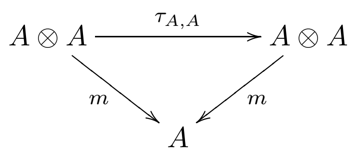
$$\begin{aligned} m : A \otimes A &\to A, \\ a \otimes b &\mapsto m(a \otimes b) \equiv a \cdot b = c \end{aligned}$$

The associativity, for example, can be read off this diagram by following the two possible ways to travel through it:

$$a \otimes b \otimes c \xrightarrow{m \otimes id} a \cdot b \otimes c \xrightarrow{m} (a \cdot b) \cdot c. \qquad \text{(A.2)}$$

$$a \otimes b \otimes c \xrightarrow{id \otimes m} a \otimes b \cdot c \xrightarrow{m} a \cdot (b \cdot c). \qquad \text{(A.3)}$$

The algebra is *commutative*, if in addition the following diagram commutes:



(A.4)

τ is the *flip* that switches the order of the factors: $\tau(a \otimes b) = b \otimes a$.

A *coalgebra* now is a vector space with all the additional structure maps given by the diagrams above, where we simply reverse the arrows and add to each map the prefix “co-”.

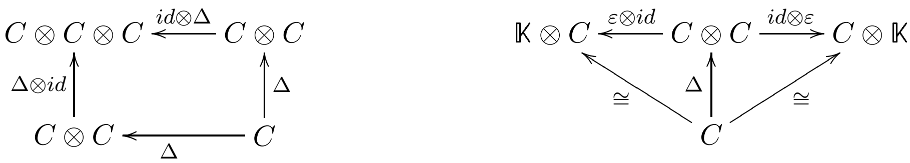
Coalgebra

A *coalgebra* is a triple (C, Δ, ε), where C is a $\mathbb{K}$-vector space, and Δ and ε are two linear maps

$$\begin{array}{lll} \Delta: & C \to C \otimes C & \textit{coproduct} \\ \varepsilon: & C \to \mathbb{K} & \textit{counit} \end{array}$$

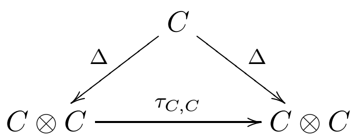
such that the following diagrams commute:

coassociativity: counit:



(A.5)

The coalgebra is *cocommutative* if the corresponding diagram to (A.4) commutes:



(A.6)

A shorthand notation for the coproduct is provided by $Sweedler's$ notation: If x is an element of the coalgebra (C,Δ,ε), the element $\Delta(x)$ of $C \otimes C$ is written in the form

$$\Delta(x) = \sum_i x'_i \otimes x''_i.$$

An element x of a coalgebra C is called *primitive* if:

$$\Delta(x) = 1 \otimes x + x \otimes 1. \tag{A.7}$$

In a next step, we now introduce a $\mathbb{K}$-vector space, which is an algebra and a coalgebra at the same time.

Bialgebra

A *bialgebra* $(B, m, \eta, \Delta, \varepsilon)$ consists of a vector space B, where (B, m, η) is a unital algebra and (B, Δ, ε) a counital coalgebra, such that the maps Δ and ε are unital algebra homomorphisms.

Or, differently speaking: one could say that a bialgebra is a vector space which additionally is an algebra and a coalgebra, with these two structures being compatible, meaning that the product of the coproducts is the coproduct of the products:

$$\Delta(xy) = \Delta(x)\Delta(y), \qquad x, y \in A \tag{A.8}$$

On the left-hand side of this equation we have the product in the tensor product space, which is in Sweedler's notation:

$$\Delta(xy) = \sum_i (xy)'_i \otimes (xy)''_i$$

This is then equal to the right-hand side

$$\Delta(x)\Delta(y) = (\sum_j x'_j \otimes x''_j)(\sum_k y'_k \otimes y''_k) = \sum_{jk} x'_j y'_k \otimes x''_j y''_k$$

to achieve compatibility.

So far we have constructed a vector space that additionally is an algebra and a coalgebra, with both structures being compatible. To get a Hopf algebra we just need one more structure, the *antipode*. It is defined via the *convolution product*:

Convolution product

Let $(C,\Delta_C,\varepsilon_C)$ be a coalgebra and (A,m_A,η_A) an algebra. $Hom(C,A)$ is the vector space of linear maps from C to A. Let $f, g \in Hom(C,A)$. The *convolution product* $*$ of two functions f and g, $f * g$, is defined as

$$\begin{aligned} * : Hom(C,A) \otimes Hom(C,A) &\to Hom(C,A), \\ (f \otimes g) &\mapsto f * g := m_A \circ (f \otimes g) \circ \Delta_C \end{aligned} \tag{A.9}$$

The operation $\circ$ is the usual concatenation of maps and hence $*$ leads to the series of maps

$$C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m_A} A\,. \tag{A.10}$$

In Sweedler's notation this reads:

$$(f * g)(x) = \sum_i f(x'_i) g(x''_i) \tag{A.11}$$

$Hom(C,A)$ is a unital associative algebra. The scalar product, the action of the field on this algebra, is defined as

$$\begin{aligned} \eta_* : \mathbb{K} &\to Hom(C,A), \\ \lambda &\mapsto \eta_*(\lambda) := \lambda(\eta_A \circ \varepsilon_C) \end{aligned}$$

In particular, the unit in this algebra is given by

$$\eta_*(1_{\mathbb{K}}) = 1_{Hom(C,A)} = \eta_A \circ \varepsilon_C.$$

Consider now a bialgebra $(H, m, \eta, \Delta, \varepsilon)$. The functions $f, g \in Hom(C, A)$ become the endomorphisms from H to H, $End(H)$. Hence they also form an associative algebra $(End(H), *, \eta \circ \varepsilon)$, with the convolution product as multiplication:

$$\begin{aligned} * : End(H) \otimes End(H) &\to End(H) \\ (F \otimes G) &\mapsto F * G := m \circ (F \otimes G) \circ \Delta \end{aligned}$$

The unit in $End(H)$ has the form:

$$1_{End(H)} = (\eta \circ \varepsilon), \tag{A.12}$$

$$H \xrightarrow{\varepsilon} \mathbb{K} \xrightarrow{\eta} H \tag{A.13}$$

and therefore maps H onto multiples of 1_H.

Antipode

The identity map id_H is in particular an element in $End(H)$, but, as we have just seen, it is not the unit element with respect to the convolution product. This means that there might be an inverse to $id_H \in End(H)$. If this inverse exists, it is unique. This inverse of id_H with respect to the convolution product is called the *antipode* $S := (id_H)_*^{-1}$. It fulfills

$$S * id_H = id_H * S = \eta \circ \varepsilon = 1_{End(H)}. \tag{A.14}$$

S is an algebra antihomomorphism, *i.e.* $S(a \cdot b) = S(b)S(a)$. If H is commutative or cocommutative, $S^2 = id_H$.

Summing all this up, we get to the definition:

Hopf algebra

A *Hopf algebra* is a tuple $(H, m, \eta, \Delta, \varepsilon, S)$, *i.e.* a bialgebra on which an antipode S exists.

An example: the Hopf algebra of the group algebra G

Let us provide an example to illustrate the Hopf algebra. Consider as an algebra the *group algebra* which is defined as follows. Let G be a group. $\mathbb{K}G$ is the vector space with basis G, *i.e.* $\mathbb{K}G = \{a | a = \sum_{u \in G} a(u)u, u \in G, a(u) \in \mathbb{K}\}$. $\mathbb{K}G$ is commutative if and only if G is an abelian group. The multiplication on $\mathbb{K}G$ is defined by the multiplication on G, and the unit is defined by $\eta(\lambda) := \lambda e$, e being the neutral element in G. Define as the *coproduct* for all elements of $\mathbb{K}G$ the linear map

$$\begin{aligned} \Delta : \mathbb{K}G &\to \mathbb{K}G \otimes \mathbb{K}G, \\ g &\mapsto \Delta(g) := g \otimes g, \end{aligned}$$

and the *counit*

$$\begin{aligned} \varepsilon : \mathbb{K}G &\to \mathbb{K}, \\ g &\mapsto \varepsilon(g) := 1_{\mathbb{K}}. \end{aligned}$$

The identity $id_{\mathbb{K}G}$ maps all g onto g. Therefore we get for the antipode S

$$\begin{aligned} S : \mathbb{K}G &\to \mathbb{K}G, \\ g &\mapsto S(g) := g^{-1}. \end{aligned}$$

This can easily be seen by applying (A.14). On the one hand we get

$$\eta \cdot \varepsilon(g) = \eta(\varepsilon(g)) = e,$$

on the other hand:

$$\begin{aligned} m \circ (S \otimes id_{\mathbb{K}G}) \circ \Delta(g) &= m \circ (S \otimes id_{\mathbb{K}G}) \circ (g \otimes g) \\ &= m \circ (S(g) \otimes g) \\ &= e, \\ \Rightarrow \quad S(g) &= g^{-1}. \end{aligned}$$

An inverse with respect to $*$ for endomorphisms on H

It is now possible to define an inverse for maps $F \in End(H)$ with respect to the convolution product and to define a group, using the antipode. (A.14) can be expressed diagrammatically as:

$$\begin{array}{ccc} H \otimes H & \xrightarrow{S \otimes id} & H \otimes H \\ \uparrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle m} \\ H & \xrightarrow[\eta \circ \varepsilon]{} & H \end{array} \tag{A.15}$$

A map $F \in End(H)$ is an endomorphism in particular an algebra homomorphism, *i.e.* the map F maps the identity 1_H onto 1_H. Hence:

$$(\eta \circ \varepsilon)(c) = F \circ (\eta \circ \varepsilon)(c) = \langle 1_H \rangle, \qquad \forall c \in H, \tag{A.16}$$

where $\langle 1_H \rangle$ is the *linear span* of the unit in H. Since the image of $(\eta \circ \varepsilon)(c)$ is in $\langle 1_H \rangle$, we obtain:

$$\begin{aligned} \Rightarrow \quad (\eta \circ \varepsilon)(c) = F \circ (\eta \circ \varepsilon)(c) \quad &= \quad F \circ [m \circ (S \otimes id) \circ \Delta](c) \\ &= \quad m \circ [((F \circ S) \otimes F)] \circ \Delta(c) \\ &= \quad ((F \circ S) * F)(c), \end{aligned} \tag{A.17}$$

$$\Rightarrow \quad F_*^{-1} = F \circ S, \tag{A.18}$$

which is the usual concatenation of the function F with the antipode S. We express this graphically as

$$\begin{array}{ccccc} H \otimes H & \xrightarrow{S \otimes id} & H \otimes H & & \\ \uparrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle m} & \overset{F}{\searrow} & \\ H & \xrightarrow[E \circ \varepsilon]{} & H & \xrightarrow[F]{} & H \end{array} \tag{A.19}$$

Homomorphisms $\Phi \in Hom(H, V)$

Consider in a next step algebra homomorphisms $\Phi \in Hom(H, V)$, from the Hopf algebra into an algebra V, so-called *characters.* A character in general is an algebra homomorphism from a Hopf algebra into a ring V. As an algebra homomorphism, Φ fulfills

$$\Phi(a\, b) = \Phi(a)\Phi(b). \tag{A.20}$$

We know that we have a convolution product and an antipode on H. This can be used to define a group structure on $Hom(H, V)$ with respect to the convolution product.

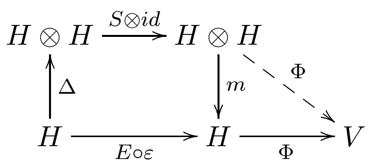
The unit element for the convolution product on $Hom(H, V)$ has the general form $(\eta_V \circ \varepsilon)$.[1] This unit element will now be expressed with the help of the fact that H is a Hopf algebra and as such possesses an antipode that allows defining an inverse.

By the same considerations that led to (A.16), namely that $F \in End(H)$ is an algebra homomorphism and maps 1_H onto 1_H, we find in this case for arbitrary $\Phi \in Hom(H, V)$: $(\eta_V \circ \varepsilon)(c) = \Phi(\eta \circ \varepsilon)(c)$, $\forall c \in H$. Therefore we get analogously to (A.17):

$$\begin{aligned} (\Phi \circ (\eta \circ \varepsilon))(c) &= \Phi \circ (\eta \circ \varepsilon)(c) \\ &= \Phi \circ [m \circ (S \otimes id) \circ \Delta](c) \\ &= m \circ [(\Phi \circ S \otimes \Phi)] \circ \Delta(c) \\ &= ((\Phi \circ S) * \Phi)(c) \end{aligned} \tag{A.21}$$

$$\Rightarrow \quad \Phi_*^{-1} = \Phi \circ S$$

Graphically this can be drawn as:



(A.22)

The map $\Phi \circ S$ is the inverse element of Φ in the group $Hom(H, V)$ of maps from a Hopf algebra H into a target space V.

A.4 Lie algebra

A *Lie algebra* L is a vector space with a bilinear map $[,] : L \times L \to L$, called the *Lie bracket*, satisfying the following two conditions for all $x, y, z \in L$:

1. (antisymmetry) $[x, y] = -[y, x]$

2. (Jacobi identity) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

[1] η_V carries the index V to emphasize the space in which it maps.

A.5 Universal enveloping algebra

Let V be a $\mathbb{K}$-vector space. Define $T^0(V) = \mathbb{K}$, $T^1(V) = V$, $T^n(V) = V^{\otimes n}$ (the tensor product of n copies of V) if $n > 1$. The canonical isomorphisms

$$T^n(V) \otimes T^m(V) \cong T^{n+m}(V) \tag{A.23}$$

induce an associative product on the vector space $T(V) = \bigoplus_{n\geq 0} T^n(V)$. Equipped with this algebra structure, $T(V)$ is called the *tensor algebra* of V. The product on $T(V)$ is explicitly given by

$$(x_1 \otimes ... \otimes x_n)(x_{n+1} \otimes ... \otimes x_{n+m}) = x_1 \otimes ... \otimes x_n \otimes x_{n+1} \otimes ... \otimes x_{n+m}, \tag{A.24}$$

where $x_1, ..., x_n, x_{n+1}, ..., x_{n+m}$ are elements of V. The unit for this product is the image of the unit element 1 in $\mathbb{K} = T^0(V)$. Let i_V be the canonical embedding of $V = T^1(V)$ into $T(V)$. By (A.24) we have

$$x_1 \otimes ... \otimes x_n = i_V(x_1)...i_V(x_n), \tag{A.25}$$

which allows us to set

$$x_1...x_n = x_1 \otimes ... \otimes x_n \tag{A.26}$$

whenever $x_1, ..., x_n$ are elements of V.

To any Lie algebra L we assign an (associative) algebra $U(L)$, called the *enveloping algebra* of L, and a morphism of Lie algebras $i_L : L \to L(U(L))$. We define the enveloping algebra as follows. Let $I(L)$ be the two-sided ideal of the tensor algebra $T(L)$ generated by all elements of the form $xy - yx - [x, y]$, where x, y are elements of L. We then define

$$U(L) = T(L)/I(L). \tag{A.27}$$

Appendix B

The gamma function and related functions

The following definitions and calculations are mainly taken from [PaKa 2001, Apos 1976, Carl 1977].

B.1 The gamma function

Euler's gamma function $\Gamma(x)$ is defined as

$$\Gamma(x) = \int_0^1 (-\log(t))^{x-1} dt \tag{B.1}$$

$$= \int_0^\infty e^{-t} t^{x-1} dt. \tag{B.2}$$

It is analytic in the entire complex plane except for simple poles at $s \in \{0, -1, -2, ...\} \equiv \mathbb{Z}_0^-$.

The second integral is the *Mellin integral* representation of Γ, which we will explain in more detail in section B.2. From equation (B.1) we get $\Gamma(1) = 1$. From equation (B.2) on the other hand, we can derive the *functional equation* for the gamma function:

$$\Gamma(1 + x) = x\Gamma(x). \tag{B.3}$$

For integer values this becomes

$$\Gamma(n + 1) = n\Gamma(n) = n!\,, \tag{B.4}$$

the well-known connection between the function $\Gamma(n)$ and the factorial $n!$.

We need to expand gamma functions that depend on an argument x. This can be done with the help of the following theorem (Weierstrass): For any real number x, except for the negative integers $(0, -1, -2, ..)$, we have the infinite product

$$\Gamma(1 + x) = e^{-\gamma_E x} e^{x/p} \frac{1}{\prod_{p=1}^{\infty} \left(1 + \frac{x}{p}\right)}, \tag{B.5}$$

and for $|x| < 1$

$$\Gamma(1+x) = e^{-\gamma_E x} \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k} x^k\right) \tag{B.6}$$

We use equation (B.6) to expand the gamma function in x. Any gamma function occurring in the calculations can be brought into the form $\Gamma(1+nx)$, $n \in \mathbb{Z}$, with the help of (B.3) and then be expanded in x with (B.6).

The constant γ_E in equations (B.5) and (B.6) is the *Euler-Mascheroni constant*:

$$\gamma_E = \lim_{p\to\infty}\left(1 + \frac{1}{2} + ... + \frac{1}{p} - \log\ p\right) = 0.57721... \tag{B.7}$$

Besides this defining equation, one can also represent γ_E as a sum:

$$\gamma_E = \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k}, \tag{B.8}$$

which one might compare with (B.6).

Substituting the variable t in (B.2) by $(n+a)t'$, one immediately obtains the formula

$$\Gamma(x) = (n+a)^x \int_0^\infty e^{-(n+a)t'} (t')^{x-1} dt' \tag{B.9}$$

$$\Leftrightarrow (n+a)^{-x} = \frac{1}{\Gamma(x)} \int_0^\infty e^{-(n+a)t'} (t')^{x-1} dt' \tag{B.10}$$

which is quite useful in calculations. Another equation that relates an expression similar to the one of (B.10) to a sum over gamma functions leads to *hypergeometric functions*:

$$\begin{aligned}(1-x)^{-a} &= 1 + ax + a(a+1)\frac{x^2}{2!} + ... \\ &= \sum_{m=0}^{\infty} (a,m) \frac{x^m}{m!}, \quad |x| < 1,\end{aligned} \tag{B.11}$$

with $m \in \mathbb{N}$, $a \in \mathbb{C}$. If $a = -n$ is a negative integer, all coefficients with $m > n$ vanish and (B.11) becomes the usual binomial formula $(1-x)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-x)^k$.

The symbol (a,m) is called *Appell's symbol* and is defined as

$$(a,m) = a(a+1)(a+2)...(a+m-1), \tag{B.12}$$

$$(a,0) = 1, \tag{B.13}$$

$$(a,-m) = \frac{1}{(a-1)(a-2)...(a-m)}, \quad a \neq 1, 2, ...m. \tag{B.14}$$

Pochhammer's symbol $(a)_m$ denotes the same quantity: $(a)_m = (a,m)$.

Let $a \in \mathbb{C}$ and the integers $m, n \in \mathbb{N}$ be such that both sides of the following equations are well-defined. We then have:

$$(a, m+n) = (a, m)(a+m, n), \qquad \text{(addition formula)} \qquad \text{(B.15)}$$

$$(a, -n) = (-1)^n/(1-a, n), \qquad \text{(reflection formula)} \qquad \text{(B.16)}$$

$$(2a, 2n) = 2^{2n}(a, n)(a+1/2, n), \qquad \text{(duplication formula)} \qquad \text{(B.17)}$$

Using (B.3) and (B.12) one finds that for $\mathrm{Re}(x) > 0$ and $m \in \mathbb{N}$:

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} = \frac{\Gamma(x+2)}{x(x+1)} = ... = \frac{\Gamma(x+m)}{(x, m)} \qquad \text{(B.18)}$$

$$\Leftrightarrow \quad (x, m) = \frac{\Gamma(x+m)}{\Gamma(x)} \qquad \text{(B.19)}$$

From (B.16) one then immediately obtains the *reflection formula for gamma functions*, which we apply in chapter 6:

$$\frac{\Gamma(1-x+n)}{\Gamma(1-x)} \frac{\Gamma(x-n)}{\Gamma(x)} = (-1)^n \qquad \text{(B.20)}$$

$$\Leftrightarrow \quad \Gamma(-n+x) = (-1)^n \frac{\Gamma(x)\Gamma(1-x)}{\Gamma(n+1-x)}. \qquad \text{(B.21)}$$

Let us consider (B.11) again. This is the first example of a hypergeometric function, namely

$$(1-x)^{-a} = \sum_{m=0}^{\infty} (a, m) \frac{x^m}{m!} = \sum_{m=0}^{\infty} \frac{\Gamma(a+m)}{\Gamma(a)} \frac{x^m}{m!} =: {}_1F_0(a; x), \quad |x| < 1. \qquad \text{(B.22)}$$

A general hypergeometric function ${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; x)$ is defined to be:

$${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; x) := \sum_{m=0}^{\infty} \frac{(a_1, m)...(a_p, m)}{(b_1, m)...(b_q, m)} \frac{x^m}{m!} \qquad \text{(B.23)}$$

$$= \frac{\Gamma(b_1)...\Gamma(b_q)}{\Gamma(a_1)...\Gamma(a_p)} \sum_{m=0}^{\infty} \frac{x^m}{m!} \frac{\Gamma(a_1+m)...\Gamma(a_p+m)}{\Gamma(b_1+m)...\Gamma(b_q+m)}. \qquad \text{(B.24)}$$

We need to expand such hypergeometric functions that depend on a regularization parameter x and even generalizations of them to double sums. This is done with the help of the library *nestedsums* (cf. chapter 5).

Other functions closely related to the gamma function are the *beta function* $B(x, y)$ and the *psi* or *digamma function* $\Psi(x)$:

The beta function is defined as a fraction of gamma functions:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(y, x). \qquad \text{(B.25)}$$

Like the gamma function, the beta function can be represented by integrals:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \qquad \text{(B.26)}$$

$$= \int_0^{\infty} \frac{t^{x-1}}{(t+1)^{x+y}} dt, \quad \mathrm{Re}(x, y) > 0. \qquad \text{(B.27)}$$

The psi or digamma function is defined for any positive integer as the logarithmic derivative of $\Gamma(x)$, that is:

$$\Psi(x) = \frac{d}{dx}(\log(\Gamma(x))) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{B.28}$$

The *polygamma functions* $\Psi_n(x) = \Psi^{(n)}(x)$ are defined by

$$\Psi_n(x) = \frac{d^{n+1}}{dx^{n+1}}(\log(\Gamma(x))), \quad \Psi_0(x) = \Psi(x). \tag{B.29}$$

Recursively we get: $\Psi_n(x+1) = \Psi_n(x) + \frac{(-1)^n n!}{x^{n+1}}$, and explicitly for the digamma function:

$$\Psi(1+x) = -\gamma_E + \sum_{k=2}^{\infty}(-1)^k \zeta(k) x^{k-1}, \quad |x| < 1, \tag{B.30}$$

$$= -\frac{1}{1+x} - (\gamma_E - 1) + \sum_{k=2}^{\infty}(-1)^k(\zeta(k) - 1)x^{k-1}, \quad |x| < 1. \tag{B.31}$$

One can see from (B.30) that $\Psi(1) = \Gamma'(1) = -\gamma_E$.

B.2 Mellin-Barnes integrals and gamma functions

We will now define the Mellin transform of a function and the Mellin-Barnes integral, following very closely the book [PaKa 2001]. The Mellin transform is closely related to the *Laplace transform* of a function. Consider the two-sided (bilateral) Laplace transform of a function $g(x)$:

$$\mathcal{L}[g;s] = \int_{-\infty}^{\infty} e^{-s\tau} g(\tau) d\tau. \tag{B.32}$$

It is holomorphic and converges absolutely in a strip $a < \mathrm{Re}(s) < b$, where a and b are real constants ($a < b$) such that for every (small) positive ε:

$$g(\tau) = \begin{cases} O(e^{(a+\varepsilon)\tau}) & \text{as } \tau \to +\infty \\ O(e^{(b-\varepsilon)\tau}) & \text{as } \tau \to -\infty \end{cases}. \tag{B.33}$$

If we now put $\tau = -\log x$ and $f(x) \equiv g(-\log x)$, we find:

$$\mathcal{L}[g;s] = \int_0^{\infty} x^{s-1} g(-\log\ x) dx = \int_0^{\infty} x^{s-1} f(x) dx \tag{B.34}$$

This is defined to be the *Mellin transform* on $(0,\infty)$ of the function $f(x)$:

$$M[f;s] = F(s) = \int_0^{\infty} x^{s-1} f(x) dx. \tag{B.35}$$

The integral (B.35) defines the Mellin transform in a vertical strip in the s plane. The boundaries are determined by the analytic structure of $f(x)$ for $x \to 0_+$ and $x \to +\infty$. Suppose that

$$f(x) = \begin{cases} O(x^{-a-\varepsilon}) & \text{as } x \to 0_+ \\ O(x^{-b+\varepsilon}) & \text{as } x \to +\infty \end{cases} \tag{B.36}$$

where $\varepsilon > 0$ and $a < b$. Then the integral (B.35) converges absolutely and defines an analytic function in the strip $a < \mathrm{Re}(s) < b$.

We are interested in the inversion formula for $M[f;s]$. This follows directly from the inversion formula for the Laplace transform. For continuous $g(\tau)$ in $\mathcal{L}[g;s]$ as defined above, we have the inversion

$$g(\tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s\tau}\mathcal{L}[g;s]ds, \tag{B.37}$$

where $a < c < b$. With the same variable transformation as in (B.34), we get the result:

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}M[f;s]ds \quad (a < c < b). \tag{B.38}$$

This *inversion formula for the Mellin transform* is valid at all points $x \geq 0$ for which $f(x)$ is continuous.

Let us give some examples. We already denoted equation (B.2), the Mellin integral definition of the gamma function:

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx. \tag{B.39}$$

Comparing this equation (B.39) with (B.35) for $f(x) = e^{-x}$ it is now obvious why this name is justified. The inverse Mellin integral (or *Mellin-Barnes integral*) for $\Gamma(x)$ leads to:

$$e^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma(s)ds, \qquad |\arg x| < \frac{1}{2}\pi;\ x \neq 0, \tag{B.40}$$

where the vertical line Re(s)=c, with $c > 0$ is lying to the right of all poles of $\Gamma(s)$.

We will not get further into questions of convergence of these functions including the range of values on which they are defined, like the restriction to ($|\arg x| < \frac{1}{2}\pi;\ x \neq 0$) in (B.40). For more information see for example [PaKa 2001].

The integral (B.40) can be analytically continued to the entire complex plane by

$$e^{-x} = \frac{1}{2\pi i}\int_C x^{-s}\Gamma(s)ds, \tag{B.41}$$

where C denotes a loop in the complex s plane that encircles the poles of $\Gamma(s)$ (in the positive sense) with endpoints at infinity at $\mathrm{Re}(s) < 0$.

Consider as another example the beta function defined in (B.25) to (B.27):

$$B(x,y) = \int_0^1 \tau^{x-1}(1-\tau)^{y-1}d\tau = \int_0^\infty \frac{\tau^{x-1}}{(1+\tau)^{x+y}}d\tau \tag{B.42}$$

$$= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (\mathrm{Re}(x,y) > 0). \tag{B.43}$$

We have

$$M[(1+x)^{-a};s] = \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)} \tag{B.44}$$

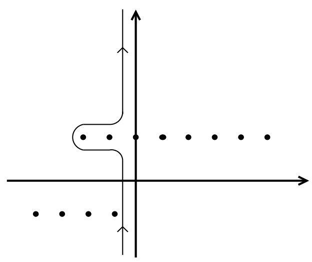
for $0 < \text{Re}(s) < \text{Re}(a)$. So we obtain for the inversion formula:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(a-s)x^{-s}ds = \frac{\Gamma(a)}{(1+x)^a} \tag{B.45}$$

for $0 < c < \text{Re}(a)$ and $|\arg x| < \pi$.
Replacing s by $-s$ in (B.45) we get an equivalent formula:

$$\frac{1}{(x+y)^a} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, x^s y^{-a-s}\frac{\Gamma(-s)\Gamma(a+s)}{\Gamma(a)} \tag{B.46}$$

which we used in the calculations of chapter 6.
The value of c is such that the parallel to the imaginary axis is placed between the poles of the Gamma function $\Gamma(-s)$ and $\Gamma(a+s)$, and is indented appropriately if necessary. This situation is sketched in the picture to the right.



There are many more relations between gamma functions and Mellin-Barnes integrals. In particular, there are relations between hypergeometric functions of different kinds and their Mellin-Barnes integral representations, see [PaKa 2001]. For all of these integrals one always has to be careful as for which values of the parameters they are defined.

Appendix C

Integrals

All scalar, vector and tensor integrals are calculated in Euclidean space. However, we will omit the index E for the momenta. Additionally, we always calculate in $D = 4 - 2\varepsilon$ dimensions.

C.1 Scalar integral

$$\int \frac{d^Dk}{\pi^{D/2}} \frac{1}{[(q+k)^2]^{\nu_1}[k^2]^{\nu_2}} = \int \frac{d^Dk}{\pi^{D/2}} \frac{1}{[(q-k)^2]^{\nu_1}[k^2]^{\nu_2}} = [q^2]^{(2-(\nu_1+\nu_2)-\varepsilon)} F_{\nu_1,\nu_2}(\varepsilon) \tag{C.1}$$

with:

$$F_{\nu_1,\nu_2}(\varepsilon) := \frac{\Gamma(2-\nu_1-\varepsilon)\,\Gamma(2-\nu_2-\varepsilon)\,\Gamma(\nu_1+\nu_2-2+\varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(4-\nu_1-\nu_2-2\varepsilon)} \tag{C.2}$$

Proof:
We start using *Feynman parametrization*:

$$\frac{1}{A_1^{m_1}A_2^{m_2}...A_n^{m_n}} = \int_0^1 dx_1...dx_n\,\delta\left(\sum x_i - 1\right) \frac{\Pi x_i^{m_i-1}}{[\sum x_i A_i]^{\sum m_i}} \frac{\Gamma(m_1+...+m_n)}{\Gamma(m_1)...\Gamma(m_n)} \tag{C.3}$$

and obtain

$$\int \frac{d^Dk}{\pi^{D/2}} \frac{1}{[(q+k)^2]^{\nu_1}[k^2]^{\nu_2}} = \int \frac{d^Dk}{\pi^{D/2}} \int_0^1 dx_1 dx_2 \frac{\delta(x_1+x_2-1)x_1^{\nu_1-1}x_2^{\nu_2-1}}{[x_1(q+k)^2+x_2k^2]^{\nu_1+\nu_2}} \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{C.4}$$

For the momentum-dependent denominator we obtain:

$$x_1(q+k)^2 + x_2k^2 = x_1q^2 + x_1 2q\cdot k + x_1k^2 + x_2k^2.$$

The delta function enforces that $x_2 = 1 - x_1$:

$$\begin{aligned} x_1q^2 + x_1 2q\cdot k + x_1k^2 + x_2k^2 &= x_1q^2 + x_1 2q\cdot k + x_1k^2 + k^2 - x_1k^2 \\ &= x_1q^2 + x_1 2q\cdot k + k^2 \\ &= (k+x_1q)^2 + x_1q^2 - x_1^2q^2 \end{aligned}$$

Shifting the momentum k to $k \to (k + x_1 q)$ we obtain

$$(k + x_1 q)^2 + x_1 q^2 - x_1^2 q^2 = (k + x_1 q)^2 + x_1(1 - x_1) q^2 \longrightarrow k^2 + x_1 x_2 q^2.$$

Inserting this denominator back into (C.4) we get

$$\int \frac{d^D k}{\pi^{D/2}} \frac{1}{[(q+k)^2]^{\nu_1} [k^2]^{\nu_2}} = \frac{\Gamma(\nu_1 + \nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int \frac{d^D k}{\pi^{D/2}} \int_0^1 dx_1 dx_2 \frac{\delta(x_1 + x_2 - 1) x_1^{\nu_1 - 1} x_2^{\nu_2 - 1}}{[k^2 + x_1 x_2 q^2]^{\nu_1 + \nu_2}}.$$

In a next step we use a generalization of formula (2.11) in Euclidean space

$$\int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2 + m^2)^\alpha} = \frac{\Gamma(\alpha - \frac{D}{2})}{\Gamma(\alpha)} \left(m^2\right)^{\frac{D}{2} - \alpha}$$

and obtain

$$\int \frac{d^D k}{\pi^{D/2}} \frac{1}{[(q+k)^2]^{\nu_1} [k^2]^{\nu_2}} = [q^2]^{(D/2 - \nu_1 - \nu_2)} \frac{\Gamma(\nu_1 + \nu_2 - D/2)}{\Gamma(\nu_1)\Gamma(\nu_2)}$$
$$\int_0^1 dx_1 dx_2 \delta(x_1 + x_2 - 1) x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} [x_1 x_2]^{(D/2 - \nu_1 - \nu_2)}.$$

The integrals over the two variables x_1 and x_2 become

$$\begin{aligned} &\int_0^1 dx_1 dx_2 \delta(x_1 + x_2 - 1) x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} [x_1 x_2]^{(D/2 - \nu_1 - \nu_2)} \\ &= \int_0^1 dx_1 x_1^{\nu_1 - 1} (1 - x_1)^{\nu_2 - 1} [x_1 (1 - x_1)]^{(D/2 - \nu_1 - \nu_2)} \\ &= \int_0^1 dx_1 x_1^{D/2 - \nu_2 - 1} (1 - x_1)^{D/2 - \nu_1 - 1}. \end{aligned}$$

With the definition of the beta function (B.26) we obtain:

$$\begin{aligned} \int \frac{d^D k}{\pi^{D/2}} \frac{1}{[(q+k)^2]^{\nu_1} [k^2]^{\nu_2}} &= [q^2]^{(\frac{D}{2} - \nu_1 - \nu_2)} \frac{\Gamma(\frac{D}{2} - \nu_1)\Gamma(\frac{D}{2} - \nu_2)\Gamma(\nu_1 + \nu_2 - \frac{D}{2})}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(D - \nu_1 - \nu_2)} \\ &\overset{D=4-2\varepsilon}{\equiv} [q^2]^{(2 - \nu_1 - \nu_2 - \varepsilon)} \frac{\Gamma(2 - \nu_1 - \varepsilon)\Gamma(2 - \nu_2 - \varepsilon)\Gamma(\nu_1 + \nu_2 - 2 + \varepsilon)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(4 - \nu_1 - \nu_2 - 2\varepsilon)}. \end{aligned}$$

C.2 Vector integral

$$\begin{aligned} \int \frac{d^D k}{\pi^{D/2}} \frac{\not{k}}{[(q+k)^2]^{\nu_1} [k^2]^{\nu_2}} &= -\int \frac{d^D k}{\pi^{D/2}} \frac{\not{k}}{[(q-k)^2]^{\nu_1} [k^2]^{\nu_2}} \\ &= [q^2]^{2 - (\nu_1 + \nu_2) - \varepsilon} \frac{1}{2} \left[F_{\nu_1 - 1, \nu_2} - F_{\nu_1, \nu_2} - F_{\nu_1, \nu_2 - 1}\right] \not{q} \end{aligned} \qquad \text{(C.5)}$$

Proof:

The calculation is based on the Lorentz invariant decomposition of the result: After integration, the integral can only depend on the external scale, the external momentum q in this case. Additionally, due to Lorentz invariance, the result has again to be of the form of a Lorentz vector times a function of q^2 of the correct dimension, forming the only possible Lorentz scalar. Hence the general result will be of the form:

$$\int \frac{d^D k}{\pi^{D/2}} \frac{k_\mu}{[(q+k)^2]^{\nu_1}[k^2]^{\nu_2}} = q_\mu A(q^2)[q^2]^{2-(\nu_1+\nu_2)-\varepsilon} \tag{C.6}$$

For the calculation of $A(q^2)$, we multiply by q^μ:

$$\int \frac{d^D k}{\pi^{D/2}} \frac{k_\mu\, q^\mu}{[(q+k)^2]^{\nu_1}[k^2]^{\nu_2}} = A(q^2)[q^2]^{2-(\nu_1+\nu_2)-\varepsilon} q^2 \tag{C.7}$$

and use the identity

$$k_\mu q^\mu \equiv k \cdot q = \frac{1}{2}\left((q+k)^2 - q^2 - k^2\right)$$

Canceling momenta in the numerator and denominator of the integrand then leads to

$$\begin{aligned} A(q^2)[q^2]^{3-(\nu_1+\nu_2)-\varepsilon} &= \frac{1}{2}\Bigg(\int \frac{d^D k}{\pi^{D/2}} \frac{1}{[(q+k)^2]^{\nu_1-1}[k^2]^{\nu_2}} \\ &\quad - \int \frac{d^D k}{\pi^{D/2}} \frac{q^2}{[(q+k)^2]^{\nu_1}[k^2]^{\nu_2}} - \int \frac{d^D k}{\pi^{D/2}} \frac{1}{[(q+k)^2]^{\nu_1}[k^2]^{\nu_2-1}}\Bigg) \\ &= \frac{1}{2}[q^2]^{3-(\nu_1+\nu_2)-\varepsilon}\left[F_{\nu_1-1,\nu_2} - F_{\nu_1,\nu_2} - F_{\nu_1,\nu_2-1}\right] \end{aligned}$$

$$\Rightarrow \qquad A(q^2) = \frac{1}{2}\left[F_{\nu_1-1,\nu_2} - F_{\nu_1,\nu_2} - F_{\nu_1,\nu_2-1}\right]$$

C.3 Tensor integral

$$\int \frac{d^D k}{\pi^{D/2}} \frac{k_\mu k_\nu}{[(q-k)^2]^{\nu_1}[k^2]^{\nu_2}} = \int \frac{d^D k}{\pi^{D/2}} \frac{k_\mu k_\nu}{[(q+k)^2]^{\nu_1}[k^2]^{\nu_2}} = [q^2]^{3-(\nu_1+\nu_2)-\varepsilon}\left[A\, g_{\mu\nu} + B\, \frac{q_\mu q_\nu}{q^2}\right] \tag{C.8}$$

with

$$\begin{aligned} A &= \tfrac{1}{(1-D)}\left[\frac{1}{4}F_{\nu_1-2,\nu_2} - \frac{1}{2}F_{\nu_1-1,\nu_2} - \frac{1}{2}F_{\nu_1-1,\nu_2-1} + \frac{1}{4}F_{\nu_1,\nu_2} - \frac{1}{2}F_{\nu_1,\nu_2-1} + \frac{1}{4}F_{\nu_1,\nu_2-2}\right] \\ B &= \tfrac{1}{(1-D)}\times \\ &\left[-\frac{D}{4}F_{\nu_1-2,\nu_2} + \frac{D}{2}F_{\nu_1-1,\nu_2} + \frac{D}{2}F_{\nu_1-1,\nu_2-1} - \frac{D}{4}F_{\nu_1,\nu_2} + \frac{1}{2}(2-D)F_{\nu_1,\nu_2-1} - \frac{D}{4}F_{\nu_1,\nu_2-2}\right]. \end{aligned}$$

Proof:

Similar considerations like the ones for the vector integral lead to the conclusion that the result of the tensor integral again has to be a tensor depending solely on q. Hence we have as the most general form for this result $[A(q^2)\, g_{\mu\nu} + B(q^2)\, \frac{q_\mu q_\nu}{q^2}]$. To determine the coefficients A and B, we contract the integral once with $g^{\mu\nu}$ and once with $q_\mu q_\nu/q^2$.

For the first case we get:

$$\int \frac{d^Dk}{\pi^{D/2}} \frac{k_\mu g^{\mu\nu} k_\nu}{[(q+k)^2]^{\nu_1}[k^2]^{\nu_2}} = [A\, g_{\mu\nu} g^{\mu\nu} + B\, \frac{q_\mu g^{\mu\nu} q_\nu}{q^2}]\, [q^2]^{3-(\nu_1+\nu_2)-\varepsilon}$$

$$\Leftrightarrow \int \frac{d^Dk}{\pi^{D/2}} \frac{1}{[(q+k)^2]^{\nu_1}[k^2]^{\nu_2-1}} = [\, A \cdot D + B\,]\, [q^2]^{3-(\nu_1+\nu_2)-\varepsilon}$$

using $g_{\mu\nu} g^{\mu\nu} = D$. Hence we get

$$[\, A \cdot D + B\,] = F_{\nu_1,\nu_2-1} \tag{C.9}$$

Contracting the integral (C.8) with $\frac{q^\mu q^\nu}{q^2}$, we get on the other hand

$$\frac{1}{q^2} \int \frac{d^Dk}{\pi^{D/2}} \frac{k_\mu q^\mu k_\nu q^\nu}{[(q+k)^2]^{\nu_1}[k^2]^{\nu_2}} = \frac{1}{q^2} \int \frac{d^Dk}{\pi^{D/2}} \frac{(k\cdot q)^2}{[(q+k)^2]^{\nu_1}[k^2]^{\nu_2}} \tag{C.10}$$

using

$$(k \cdot q) = \frac{1}{2}[(q+k)^2 - q^2 - k^2]$$

one can again cancel momenta in the numerator with those in the denominator of the integrand like for (C.5), eventually leading to six integrals, where the exponents of the momenta in the denominator have been lowered by 1 or 2:

$$\begin{aligned}(\text{C.10}) = \frac{1}{4}[q^2]^{3-(a+b)-(1+i+j)\varepsilon} \big[F_{(a-2)+i\varepsilon,b+j\varepsilon} - 2F_{(a-1)+i\varepsilon,b+j\varepsilon} - 2F_{(a-1)+i\varepsilon,(b-1)+j\varepsilon} \\ +F_{a+i\varepsilon,b+j\varepsilon} + 2F_{a+i\varepsilon,(b-1)+j\varepsilon} + F_{a+i\varepsilon,(b-2)+j\varepsilon}\big]\end{aligned} \tag{C.11}$$

Using that (C.8) $\cdot \frac{q^\mu q^\nu}{q^2}$ is equal to:

$$[A + B]\, [q^2]^{3-(a+b)-(1+i+j)\varepsilon}$$

and comparing both sides, we get

$$\begin{aligned}A + B = \frac{1}{4}F_{(a-2)+i\varepsilon,b+j\varepsilon} - \frac{1}{2}F_{(a-1)+i\varepsilon,b+j\varepsilon} - \frac{1}{2}F_{(a-1)+i\varepsilon,(b-1)+j\varepsilon} + \frac{1}{4}F_{a+i\varepsilon,b+j\varepsilon} \\ + \frac{1}{2}F_{a+i\varepsilon,(b-1)+j\varepsilon} + \frac{1}{4}F_{a+i\varepsilon,(b-2)+j\varepsilon}\end{aligned} \tag{C.12}$$

Taking (C.9) and (C.12) together and solving for A and B, we get:

$$\begin{aligned}A = \tfrac{1}{(1-D)} \Big[\frac{1}{4}F_{(a-2)+i\varepsilon,\nu_2} - \frac{1}{2}F_{(a-1)+i\varepsilon,\nu_2} - \frac{1}{2}F_{(a-1)+i\varepsilon,(b-1)+j\varepsilon} \\ + \frac{1}{4}F_{\nu_1,\nu_2} - \frac{1}{2}F_{\nu_1,(b-1)+j\varepsilon} + \frac{1}{4}F_{\nu_1,(b-2)+j\varepsilon}\Big] \\ B = \tfrac{1}{(1-D)} \Big[-\frac{D}{4}F_{(a-2)+i\varepsilon,\nu_2} + \frac{D}{2}F_{(a-1)+i\varepsilon,\nu_2} + \frac{D}{2}F_{(a-1)+i\varepsilon,(b-1)+j\varepsilon} \\ - \frac{D}{4}F_{\nu_1,\nu_2} + \frac{1}{2}(2-D)F_{\nu_1,(b-1)+j\varepsilon} - \frac{D}{4}F_{\nu_1,(b-2)+j\varepsilon}\Big]\end{aligned}$$

Appendix D

Relations for $\hat{I}^{(2,5)}$

As we already mentioned in chapter 7 we want to bring integrals $\hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ with $\nu_5 = n_5 + a_5\varepsilon$, $a_5 \neq 0$, into the form $\hat{I}^{(2,5)}(m-\varepsilon, 1, 1, 1, 1, 1+a_5\varepsilon)$, so we will have no problems concerning the convergence of the integral, and only have to expand *one* integral in ε. To achieve this, we use other relations besides the triangle relations (7.21) and (7.22) to lower the exponents ν_1, ν_2, ν_3, and ν_4 by one or to increase the exponent of ν_5 by one.

The relations used to lower one of the exponents except ν_5 can be obtained similar to the triangle relations where we simply shift other momenta of the integral and/or set up the vector integral not by using the momentum l, but the momentum k. Since the calculation is completely analogous to the calculation for the triangle relations, we will simply state the shifted momentum, the derivative, and the momentum added in the numerator of the integral. Furthermore, we will write the relations in the short-cut notation we used before, only stating the exponents.

Momentum: $l' = l + k - q$, $k' = q - k$, numerator: l^μ, derivative: $\frac{\partial}{\partial l^\mu}$

$$\begin{aligned}
&\hat{I}^{(2,5)}(m-\varepsilon, \nu_1+1, \nu_2, \nu_3, \nu_4, \nu_5) \\
&= \Big[-(D-\nu_1-2\nu_4-\nu_5)\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5) \\
&\quad + \nu_1\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1+1, \nu_2, \nu_3, \nu_4-1, \nu_5) \\
&\quad + \nu_5 \Big[\hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4-1, \nu_5+1) - \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3-1, \nu_4, \nu_5+1) \Big] \Big] / \nu_1
\end{aligned} \tag{D.1}$$

Momentum: $l' = l + k - q$, $k' = q - k$, numerator: k^μ, derivative: $\frac{\partial}{\partial k^\mu}$

$$\begin{aligned}
&\hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2+1, \nu_3, \nu_4, \nu_5) \\
&= \Big[-(D-\nu_2-2\nu_3-\nu_5)\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5) \\
&\quad + \nu_2\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2+1, \nu_3-1, \nu_4, \nu_5) \\
&\quad + \nu_5 \Big[\hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3-1, \nu_4, \nu_5+1) - \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4-1, \nu_5+1) \Big] \Big] / \nu_2
\end{aligned} \tag{D.2}$$

Momentum: $l' = l + k$, numerator: k^μ, derivative: $\frac{\partial}{\partial k^\mu}$

$$\begin{aligned}
&\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3+1,\nu_4,\nu_5)\\
&=\Big[-(D-2\nu_2-\nu_3-\nu_5)\,\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)\\
&\quad+\nu_3\,\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2-1,\nu_3+1,\nu_4,\nu_5)\\
&\quad+\nu_5\,\Big[\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2-1,\nu_3,\nu_4,\nu_5+1)\;-\;\hat{I}^{(2,5)}(m-\varepsilon,\nu_1-1,\nu_2,\nu_3,\nu_4,\nu_5+1)\Big]\Big]\,/\nu_3
\end{aligned} \tag{D.3}$$

Momentum: $l' = l + k$, numerator: l^μ, derivative: $\frac{\partial}{\partial l^\mu}$

$$\begin{aligned}
&\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4+1,\nu_5)\\
&=\Big[-(D-2\nu_1-\nu_4-\nu_5)\,\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)\\
&\quad+\nu_4\,\hat{I}^{(2,5)}(m-\varepsilon,\nu_1-1,\nu_2,\nu_3,\nu_4+1,\nu_5,)\\
&\quad+\nu_5\,\Big[\hat{I}^{(2,5)}(m-\varepsilon,\nu_1-1,\nu_2,\nu_3,\nu_4,\nu_5+1)\;-\;\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2-1,\nu_3,\nu_4,\nu_5+1)\Big]\Big]\,/\nu_4
\end{aligned} \tag{D.4}$$

The last two relations are the symmetric counterparts of the first two. They all state how one can decrease one single exponent by one, except for ν_5. It is not possible to find a relation for $\hat{I}^{(2,5)}$ in a similar form that decreases ν_5 by one. This shows that ν_5, corresponding to the momentum at the "center line" of the master topology, is a "special" exponent.

Once we have lowered the exponents ν_1, ν_2, ν_3, and ν_4 to 1 using (D.1) to (D.4), we will then be left with integrals of the form $\hat{I}^{(2,5)}(m-\varepsilon,1,1,1,1,-1+a_5\varepsilon)$. If ν_5 is the only non-integer exponent (the only line with a subdivergence) we can bring these integrals to the form $1+a_5\varepsilon$ plus functions F_{ν_1,ν_2}, by using a combination of the equations (7.21) and (D.1) to (D.4). We will first derive these functions for general exponents and show in a second step that they give rise to the relations we need. Let us start by considering the triangle relation 1:

$$\begin{aligned}
0 &= (D-\nu_1-\nu_4-2\nu_5)\,\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)\\
&\quad+\nu_4\,\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3-1,\nu_4+1,\nu_5)\;-\;\nu_4\,\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3,\nu_4+1,\nu_5-1)\\
&\quad+\nu_1\,\hat{I}^{(2,5)}(m-\varepsilon,\nu_1+1,\nu_2-1,\nu_3,\nu_4,\nu_5)\;-\;\nu_1\,\hat{I}^{(2,5)}(m-\varepsilon,\nu_1+1,\nu_2,\nu_3,\nu_4,\nu_5-1)
\end{aligned}$$

Inserting (D.4) for $\hat{I}^{(2,5)}(m-\varepsilon,\nu_1,\nu_2,\nu_3-1,\nu_4+1,\nu_5)$ and (D.1) for $\hat{I}^{(2,5)}(m-\varepsilon,\nu_1+1,\nu_2-1,$

$\nu_3, \nu_4, \nu_5)$ with ν_5 replaced by $\nu_5 - 1$ we get (the multiplicative factors ν_1 and ν_4 cancel):

$$\begin{aligned}
0 = &(D - \nu_1 - \nu_4 - 2\nu_5)\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5) \\
&+ \nu_4\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3 - 1, \nu_4 + 1, \nu_5) \\
&+ (D - 2\nu_1 - \nu_4 - (\nu_5 - 1))\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5 - 1) \\
&- \nu_4\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1 - 1, \nu_2, \nu_3, \nu_4 + 1, \nu_5 - 1) \\
&- (\nu_5 - 1)\, \Big[\hat{I}^{(2,5)}(m-\varepsilon, \nu_1 - 1, \nu_2, \nu_3, \nu_4, \nu_5) - \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2 - 1, \nu_3, \nu_4, \nu_5) \Big] \\
&+ \nu_1\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1 + 1, \nu_2 - 1, \nu_3, \nu_4, \nu_5) \\
&+ (D - \nu_1 - 2\nu_4 - (\nu_5 - 1))\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5 - 1) \\
&- \nu_1\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1 + 1, \nu_2, \nu_3, \nu_4 - 1, \nu_5 - 1) \\
&- (\nu_5 - 1)\, \Big[\hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4 - 1, \nu_5) - \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3 - 1, \nu_4, \nu_5) \Big] .
\end{aligned}$$

Although it is not too obvious at first sight, this is the equation we need. We have to solve it for $\hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5 - 1)$ and obtain:

$$\begin{aligned}
&(2D - 3\nu_1 - 3\nu_4 - 2(\nu_5 - 1))\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5 - 1) \\
&= \\
&- (D - \nu_1 - \nu_4 - 2\nu_5)\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5) \\
&- \nu_4\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3 - 1, \nu_4 + 1, \nu_5) + \nu_4\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1 - 1, \nu_2, \nu_3, \nu_4 + 1, \nu_5 - 1) \\
&+ (\nu_5 - 1)\, \Big[\hat{I}^{(2,5)}(m-\varepsilon, \nu_1 - 1, \nu_2, \nu_3, \nu_4, \nu_5) - \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2 - 1, \nu_3, \nu_4, \nu_5) \Big] \\
&- \nu_1\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1 + 1, \nu_2 - 1, \nu_3, \nu_4, \nu_5) + \nu_1\, \hat{I}^{(2,5)}(m-\varepsilon, \nu_1 + 1, \nu_2, \nu_3, \nu_4 - 1, \nu_5 - 1) \\
&+ (\nu_5 - 1)\, \Big[\hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3, \nu_4 - 1, \nu_5) - \hat{I}^{(2,5)}(m-\varepsilon, \nu_1, \nu_2, \nu_3 - 1, \nu_4, \nu_5) \Big]
\end{aligned} \qquad (D.5)$$

We can split the integrals on the right-hand side into two groups: For one group, $\nu_5 - 1$ has been increased by one, like we wanted. For the second set of integrals that still depends on $\nu_5 - 1$, there is always an additional exponent ν_1, ν_2, ν_3, or ν_4 lowered by one. If the other exponents are all equal to one, the integrals of the second set will simply factorize into a product of two F_{ν_1,ν_2} functions for which the integer part of the exponent may be negative. If the exponents are not equal to 1 but are natural numbers, a repeated application of this equation will also lead to the factorization into functions F_{ν_1,ν_2}. In any case we will be left with an integral of the form

$$\hat{I}^{(2,5)}(m-\varepsilon, 1, 1, 1, 1, 1 + a_5\varepsilon).$$

Equations (D.1) to (D.5) are implemented in the function `I_5()`.

Appendix E

The classes

This chapter lists the different classes implemented in the programs `yukawa1`, `yukawa2`, `qed1`, and `qed2` described in chapter 8.

yukawa1

`Sigma`	Fermion self-energy, possibly sitting as subdivergence at the fermion line of `Sigma`, the fermion lines of `Gamma`, or the “lower” fermion line of Vacuum. This `Sigma` is not used for `Gamma2`.
`Sigma_flipped`	Fermion self-energy, sitting as subdivergence at the “upper” fermion line of Vacuum.
`Sigma_1`	Fermion self-energy, sitting as subdivergence at line 1 of `Gamma2`.
`Sigma_4`	Fermion self-energy, sitting as subdivergence at line 4 of `Gamma2`.
`Sigma_5`	Fermion self-energy, sitting as subdivergence at line 5 of `Gamma2`.
`Vacuum`	Boson self-energy, possibly sitting as subdivergence in `Sigma` or in `Gamma`.
`Vacuum_2`	Boson self-energy, sitting as subdivergence in line 2 of `Gamma2`.
`Vacuum_3`	Boson self-energy, sitting as subdivergence in line 3 of `Gamma2`.
`Gamma`	One-loop vertex correction, possibly sitting in `Gamma`.
`Gamma_5`	One-loop vertex correction, sitting at entry 5 of `Gamma2`.
`Gamma2`	Non-planar vertex correction, possibly sitting in `Gamma`.
`Gamma2_5`	Non-planar vertex correction, sitting at entry 5 of `Gamma2`.

yukawa2

Sigma	Fermion self-energy, possibly sitting as subdivergence at the fermion line of Sigma, the fermion lines of Gamma connected to the zmt vertex, or the "lower" fermion line of Vacuum. This Sigma is not used for Gamma2.
Sigma_flipped	Fermion self-energy, sitting as subdivergence at the "upper" fermion line of Vacuum or the fermion line of Gamma not connected to the zmt vertex.
Sigma_2	Fermion self-energy, sitting as subdivergence at line 2 of Gamma2.
Sigma_3	Fermion self-energy, sitting as subdivergence at line 3 of Gamma2.
Sigma_4	Fermion self-energy, sitting as subdivergence at line 4 of Gamma2.
Sigma_5	Fermion self-energy, sitting as subdivergence at line 5 of Gamma2.
Vacuum	Boson self-energy, not used as subdivergence.
Vacuum_Sigma	Boson self-energy, sitting as subdivergence in Sigma.
Vacuum_Gamma	Boson self-energy, sitting as subdivergence in Gamma.
Vacuum_1	Boson self-energy, sitting as subdivergence in line 1 of Gamma2.
Vacuum_5	Boson self-energy, sitting as subdivergence in line 5 of Gamma2.
Gamma	One-loop vertex correction, possibly sitting in Gamma.
Gamma_5	One-loop vertex correction, sitting at entry 5 of Gamma2.
Gamma2	Non-planar vertex correction, possibly sitting in Gamma.
Gamma2_5	Non-planar vertex correction, sitting at entry 5 of Gamma2.

qed1

Sigma	Fermion self-energy, possibly sitting as subdivergence at the fermion line of Sigma or the fermion lines of Gamma. This Sigma is not used for Gamma2.
Sigma_1	Fermion self-energy, sitting as subdivergence at line 1 of Gamma2.
Sigma_4	Fermion self-energy, sitting as subdivergence at line 4 of Gamma2.
Sigma_5	Fermion self-energy, sitting as subdivergence at line 5 of Gamma2.
Vacuum	Boson self-energy, possibly sitting as subdivergence in Sigma or in Gamma.
Vacuum_2	Boson self-energy, sitting as subdivergence in line 2 of Gamma2.
Vacuum_3	Boson self-energy, sitting as subdivergence in line 3 of Gamma2.
Gamma	One-loop vertex correction, possibly sitting in Gamma.
Gamma_5	One-loop vertex correction, sitting at entry 5 of Gamma2.
Gamma2	Non-planar vertex correction, possibly sitting in Gamma.
Gamma2_5	Non-planar vertex correction, sitting at entry 5 of Gamma2.

qed2

`Sigma`	Fermion self-energy, possibly sitting as subdivergence at the fermion line of `Sigma` or the fermion lines of `Gamma` connected to the zmt vertex. This `Sigma` is not used for `Gamma2`.
`Sigma_flipped`	Fermion self-energy, sitting at the fermion line of `Gamma` not connected to the zmt vertex.
`Sigma_2`	Fermion self-energy, sitting as subdivergence at line 2 of `Gamma2`.
`Sigma_5`	Fermion self-energy, sitting as subdivergence at line 5 of `Gamma2`.
`Vacuum`	Boson self-energy, not used as subdivergence.
`Vacuum_Sigma`	Boson self-energy, sitting as subdivergence in `Sigma`.
`Vacuum_Gamma`	Boson self-energy, sitting as subdivergence in `Gamma`.
`Vacuum_1`	Boson self-energy, sitting as subdivergence in line 1 of `Gamma2`.
`Vacuum_5`	Boson self-energy, sitting as subdivergence in line 5 of `Gamma2`.
`Gamma`	One-loop vertex correction, possibly sitting in `Gamma`.
`Gamma_5`	One-loop vertex correction, sitting at entry 5 of `Gamma2`.
`Gamma2`	Non-planar vertex correction, possibly sitting in `Gamma`.
`Gamma2_5`	Non-planar vertex correction, sitting at entry 5 of `Gamma2`.

Bibliography

[Apos 1976] T.M. Apostol: *Introduction to Analytic Number Theory*; Springer: Undergraduate Texts in Mathematics, 1976

[BaBr 1988] D.T. Barfoot, D.J. Broadhurst: *$Z_2 \times S_6$ symmetry of the two-loop diagram*; Z. Phys. **C41** (1988) 81–85

[Bier 2000] I. Bierenbaum: *Riemanns Zeta-Funktion in iterierten Einschleifendiagrammen*; Diplomarbeit, Mainz (2000)

[BKK 2002] I. Bierenbaum, R. Kreckel, D. Kreimer: *On the Invariance of Residues of Feynman Graphs*; J. Math. Phys. **43** (2002) 4721–4740 [hep-th/0111192]

[BiWe 2003] I. Bierenbaum, S. Weinzierl: *The massless two-loop two-point function*; Eur. Phys. J. **C32** (2003) 67–78, [hep-ph/0308311]

[BBB 1997] J.M. Borwein, D.M. Bradley, D.J. Broadhurst: *Evaluations of k fold Euler Zagier sums: a compendium of results for arbitrary k*; Electron. J. Combin. Vol. 4 No. 2 (1997) (Wilf Fetschrift) #R5 [hep-th/9611004]

[BBBL 2001] J.M. Borwein, D.M. Bradley, D.J. Broadhurst and P. Lisonek: *Special values of Multidimensional Polylogarithms*; Trans. Amer. Math. Soc., Vol. 353 No. 3, (2001) 907–941

[Broa 1986] D.J. Broadhurst: *Exploiting the 1,440-fold Symmetry of the Master Two-Loop Diagram*; Z. Phys. **C32** (1986) 249–253

[Broa 1996] D.J. Broadhurst: *On the enumeration of irreducible k-fold Euler sums and their roles in knot theory and field theory*; OUT-4102-62, Apr 1996. 34pp. [hep-th/9604128]

[Broa 2003] D.J. Broadhurst: *Where do the tedious products of ζ's come from?*; Nucl. Phys. Proc. Suppl. **116** (2003) 432 [hep-ph/0211194]

[BGK 1997] D.J. Broadhurst, J.A. Gracey, D. Kreimer: *Beyond the triangle and uniqueness relations: non-zeta counterterms at large N from positive knots*; Z. Phys. **C75** (1997) 559–574 [hep-th/9607174]

[Carl 1977] B.C. Carlson: *Special functions of applied Mathematics*; New York: Academic Press 1977

[CKT 1980] K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov: *New approach to the evaluation of multiloop Feynman integrals: the Gegenbauer polynomial x-space technique*; Nucl. Phys. **B174** (1980) 345–377

[ChTk 1981] K.G. Chetyrkin, F.V. Tkachov: *Integration by parts: the algorithm to calculate β-functions in 4 loops*; Nucl. Phys. **B192** (1981) 159–204

[Coll 1984] J.C. Collins: *Renormalization*; Cambridge: Cambridge Univ. Press, 1984

[CoKr 1998] A. Connes, D. Kreimer: *Hopf algebras, Renormalization and Noncommutative geometry*; Commun. Math. Phys. **199** (1998) 203–242 [hep-th/9808042]

[CoKr 2000] A. Connes, D. Kreimer: *Renormalization in quantum field theory and the Riemann-Hilbert problem. II: The beta-function, diffeomorphisms and the renormalization group*; Commun. Math. Phys. **216** (2001) 215–241 [arXiv:hep-th/0003188];
A. Connes, D. Kreimer: *Renormalization in quantum field theory and the Riemann-Hilbert problem. I: The Hopf algebra structure of graphs and the main theorem*; Commun. Math. Phys. **210** (2000) 249–273 [arXiv:hep-th/9912092]

[CoKr 2002] A. Connes, D. Kreimer: *Insertion and Elimination: the doubly infinite Lie algebra of Feynman graphs*; Annales Henri Poincare **3** (2002) 411–433 [arXiv:hep-th/0201157]

[Davy 1991] A.I. Davydychev: *A simple formula for reducing Feynman diagrams to scalar integrals*; Moscow State Univ. preprint INP MSU 91-IO/214

[DaBo 1990] A.I. Davydychev; Boos E.E. : *A method of evaluating massive Feynmand integrals*; Moscow State Univ. preprint INP MSU 90-11/157, (1990)

[DKT 1996] R. Delbourgo, A. Kalloniatis, G. Thompson: *Dimensional renormalization: Ladders and rainbows*; Phys. Rev. **D54** (1996) 5373–5376

[DEMcA 1997] R. Delbourgo, D. Elliott, D.S. McAnally: *Dimensional renormalization in Φ^3 theory: Ladders and rainbows*; Phys. Rev. **D55** (1997) 5230–5233

[EFGK 2004a] K. Ebrahimi-Fard, L. Guo, D. Kreimer: *Integrable Renormalization I: the Ladder Case*; J. Math. Phys. **45** (2004) 3758–3769 [arXiv:hep-th/0402095]

[EFGK 2004b] K. Ebrahimi-Fard, L. Guo, D. Kreimer: *Integrable Renormalization II: the general case*; [arXiv:hep-th/0403118]

[EFGK 2004c] K. Ebrahimi-Fard, L. Guo, D. Kreimer: *Spitzer's Identity and the Algebraic Birkhoff Decomposition in pQFT*; J. Phys. **A37** (2004) 11037–11052 [arXiv:hep-th/hep-th/0407082]

[Eule 1771] L. Euler: *Meditationes Circa Singulare serierum Genus*; Novi Comm. Acad. Sci. Petropol., **20** (1775) 140–186, Reprinted in "Opera omnia", ser. I, **15**, B.G.Teubner, Berlin, 1927, pp. 217–267

[GiNaC] C. Bauer, A. Frink, R. Kreckel : *Introduction to the GiNaC Framework for Symbolic Computation within the C++ Programming Language*; J. Symb. Comput. **33**, (2002) 1; URL: `http://www.ginac.de` [arXiv:cs-sc/0004015]

[Gon 1998] A.B. Goncharov : *Multiple Polylogarithms, Cyclotomy and Modular Complexes*; Math. Res. Lett., **5** (1998) no. 4, 497–516. MR 1 653 320

[GoIs 1985] S.G. Gorishnii, A.P. Isaev : *An approach to the caluculation of many-loop massless Feynman integrals*; Theor. Math. Phys. **62** (1985) 232–240

[GKL 1991] S.G. Gorishnii, A.L. Kataev, S.A. Larin : *The* $O(\alpha_s^3)$ *corrections to* $\sigma_{tot}(e^+e^- \to$ *hadrons*$)$ *and* $\Gamma(\tau^- \to \nu_\tau +$ *hadrons*$)$ *in QCD*; Phys. Lett. **B259** (1991) 144–150

[GBVF 2000] J.M. Gracia-Bondia , J.C. Várilly , H. Figueroa : *Elements of Noncommutative Geometry*; Boston [u.a.]: Birkhäuser 2000. (Birkhäuser Advanced Texts)

[Groz 2003] A.G. Grozin : *Lectures on multiloop calculations*; Int. J. Mod. Phys. **A19** (2004) 473–520 [hep-ph/0307297]

[Hoff 2000] M.E. Hoffman : *Quasi-shuffle products*; J. Algebraic Combin. **11** (2000) 49–68 [math.QA/9907173]

[Hoff 1997] M.E. Hoffman : *The algebra of multiple harmonic series*; J. Algebra **194** (1997) 477–495

[ItZu 1980] C. Itzykson , Zuber, J.B. : *Quantum Field Theory*; New York: McGraw-Hill, 1980

[Kass 1995] C. Kassel : *Quantum Groups*; New York: Springer, 1995. (Graduate Texts in Mathematics 155)

[Kaza 1983] D.I. Kazakov : *The method of uniqueness, a new powerful technique for multiloop calculations*; Phys. Lett. **B133** (1983) 406–410

[Kaza 1984] D.I. Kazakov : *Calculation of Feynman diagrams by the "uniqueness" method*; Theor. Math. Phys. **58** (1984) 223–230

[Kaza 1985] D.I. Kazakov : *Many loop calculations: the uniqueness method and functional equations*; Theor. Math. Phys. **62** (1985) 84–89

[KMR 1970] K.S. Kölbig, J.A. Mignaco , E. Remiddi : *On Nielsens's generalized polylogarithms and their numerical calculations*; B.I.T. **10** (1970) 38;
K.S. Kölbig: *Nielsens's generalized polylogarithms*; SIAM J. Math. Anal. **17** (1986) 1232

[Koti 1996] A.V. Kotikov : *The Gegenbauer Polynomial Technique: the evaluation of a class of Feynman diagrams*; Phys. Lett. **B375** (1996) 240–248 [hep-ph/9512270]

[Krei 1997] D. Kreimer : *Habilitationsschrift: Renormalization and Knot Theory*; J. Knot Theor. Ramifications **6** (1997) 479–581 [q-alg/9607022]

[Krei 1998a] D. Kreimer: *On knots in subdivergent diagrams*; Eur. Phys. J. **C2** (1998) 757 [hep-th/9610128]

[Krei 1998b] D. Kreimer: *On the Hopf algebra structure of perturbative quantum field theories*; Adv. Theor. Math. Phys. **2** (1998) 303 [q-alg/9707029]

[Krei 1999] D. Kreimer: *On overlapping divergences*; Commun. Math. Phys. **204** (1999) 669 [hep-th/9810022]

[Krei 2000a] D. Kreimer: *Chen's iterated integral represents the operator product expansion*; Adv. Theor. Math. Phys. **3** (2000) 3 [hep-th/9901099]

[Krei 2000b] D. Kreimer: *Knots and Feynman Diagrams*; Cambridge: Cambridge Lecture Notes in Physics **13**, 2000

[Krei 2001] D. Kreimer: *Structures in Feynman Graphs -Hopf Algebras and Symmetries*; talk given at the "Dennisfest", "Graphs and Patterns in Mathematics and Theoretical Physics", Stony Brook, June 2001, final version, to appear in the Proceedings [arXiv:hep-th/0202110]

[Krei 2003a] D. Kreimer: *New mathematical structures in renormalizable quantum field theories*; Annals Phys. **303** (2003) 179–202; Erratum-ibid. **305** (2003) 79 [arXiv:hep-th/0211136]

[Krei 2003b] D. Kreimer : *Unique factorization in perturbative QFT*; Talk given at "RadCor 2002 - Loops and Legs 2002", Kloster Banz, Germany, Sep 8-13, 2002, Nucl. Phys. Proc. Suppl. **116** (2003) 392–396 [arXiv:hep-ph/0211188]

[KrDe 1999] D. Kreimer, R. Delbourgo: *Using the Hopf algebra structure of QFT in calculations*; Phys. Rev. **D60** (1999) 105025 [hep-th/9903249]

[Lewi 1981] L. Lewin: *Dilogarithms and Associated Functions*; North Holland 1958; *Polylogarithms and Associated Functions*; North Holland 1981

[MeKr 2004a] I. Mencattini, D. Kreimer: *Insertion and Elimination Lie Algebra: the Ladder case*; Lett. Math. Phys. **67** (2004) 61–74 [arXiv:math.QA/0308042]

[MeKr 2004b] I. Mencattini, D. Kreimer: *The Structure of the Ladder Insertion-Elimination Lie algebra*; [arXiv:math-ph/0408053]

[MiMo 1965] J.W. Milnor , J.C. Moore: *On the structure of Hopf algebras*; Ann. Math. **81** (1965) 211–264

[MiPe 2000] H.N. Minh, M.Petitot: *Lyndon words, polylogarithms and the Riemann ζ function*; Disc. Math. **217** (2000) 273

[MUW 2002] S. Moch, P. Uwer, S. Weinzierl: *Nested sums, expansion of transcendental functions and multiscale multiloop integrals*; J. Math. Phys. **43** (2002) 3363–3386, [hep-ph/0110083]

[Niel 1909] N. Nielsen: *Der Eulersche Dilogarithmus und seine Verallgemeinerungen*; Nove Acta Leopoldina Halle 90 (1909) 123

[PaKa 2001] R.B. Paris , D. Kaminsky : *Asymptotics and Mellin-Barnes integrals*; Cambridge: Cambridge Univ. Press, 2001

[PaTa 1984] P. Pascual , R. Tarrach : *QCD: Renormalization for the Practitioner*; Berlin [u.a.]: Springer, 1984. (Lecture Notes in Physics; **194**)

[PeSch 1995] M.E. Peskin , D.V. Schroeder : *An introduction to quantum field theory*; Reading, Mass.: Perseus Books, 1995

[ReVe 2000] E. Remiddi , J.A.M. Vermaseren : *Harmonic polylogarithsm*; Int. J. Mod. Phys. **A15** (2000) 725 [hep-ph/9905237]

[Reut 1993] C. Reutenauer : *Free Lie Algebras*; Oxford: Clarendon Press, 1993

[Slat 1966] L.J. Slater : *Generalized hypergeometric functions*; Cambridge: Cambridge Univ. Press, 1966

[SuSa 1991] L.R. Surguladze, M.A. Samuel : *Total hadronic cross-section in e+ e- annihilation at the four loop level of perturbative QCD*; Phys. Rev. Lett. **66** (1991) 560–563, Erratum ibid. **66** (1991) 2416

[t'HoVe 1972] t'Hooft, G. , Veltman, M. : *Regularization and renormalization of gauge fields*; Nucl. Phys. **B44** (1972) 189–213

[Tkach 1981] F.V. Tkachov : *A theorem on analytical calculability of four loop renormalization group functions*; Phys. Lett. **B100** (1981) 65–68

[Verm 1999] J.A.M. Vermaseren : *Harmonic sums, Mellin transforms and Integrals*; Int. J. Mod. Phys. **A14** (1999) 2037 [hep-ph/9806280]

[Wein 2002] S. Weinzierl : *Symbolic expansion of transcendental functions*; Comput. Phys. Commun. **145** (2002) 357–370 [math-ph/0201011]

[Wein 2003] S. Weinzierl : *Algebraic Algorithms in Perturbative Calculations*; Proceedings of Les Houches "Frontiers in Number Theory, Physics and Geometry". (2003) [hep-th/0305260]

[Zagi 1994] D. Zagier : *Values of zeta functions and their applications*; in First European Congress of Mathematics Vol.II, Birkhäuser Boston, Boston, 1994, 497–512