“Thermal Relaxation for Particle Systems in Interaction with Several Bosonic Heat Reservoirs”

“Thermische Relaxation von Teilchensystemen in Wechselwirkung mit mehreren bosonischen Wärmebädern”

Dissertation
zur Erlangung des Grades
“Doktor der Naturwissenschaften”

am Fachbereich Mathematik und Informatik
der Johannes Gutenberg-Universität Mainz

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geboren am 5. Juli 1974 in Wiesbaden

Mainz, 1. Juli 2004

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1. Berichterstatter / 1st Referee:	Prof. Dr. VB
2. Berichterstatter / 2nd Referee:	Dr. MM
3. Berichterstatter / 3rd Referee:	Prof. Dr. IMS
Tag der mündlichen Prüfung / Date of the Oral Examination:	16.07.2004

(D77) Dissertation, Johannes Gutenberg-Universität Mainz

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Bibliographic Information Published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the internet at http://dnb.ddb.de.

This dissertation has been published by: Books on Demand GmbH, Norderstedt.

The book is available in bookstores, ISBN 3-8334-1866-4.

Abstract

The present thesis is concerned with the study of a *quantum physical system* composed of a small particle system (such as a spin chain) and several quantized massless boson fields (as photon gasses or phonon fields) at positive temperature. The setup serves as a simplified model for matter in interaction with thermal "radiation" from different sources. Hereby, questions concerning the dynamical and thermodynamic properties of particle-boson configurations *far from thermal equilibrium* are in the center of interest. We study a specific situation where the particle system is brought in contact with the boson systems (occasionally referred to as heat reservoirs) where the reservoirs are prepared close to thermal equilibrium states, each at a different temperature. We analyze the interacting time evolution of such an initial configuration and we show thermal relaxation of the system into a stationary state, i.e., we prove the existence of a time invariant state which is the unique limit state of the considered initial configurations evolving in time. As long as the reservoirs have been prepared at different temperatures, this stationary state features thermodynamic characteristics as stationary *energy fluxes* and a *positive entropy production rate* which distinguishes it from being a thermal equilibrium at any temperature. Therefore, we refer to it as *non-equilibrium stationary state* or simply *NESS*.

The physical setup is phrased mathematically in the language of C^**-algebras*. The thesis gives an extended review of the application of operator algebraic theories to quantum statistical mechanics and introduces in detail the mathematical objects to describe matter in interaction with radiation. The C^*-theory is adapted to the concrete setup. The algebraic description of the system is lifted into a Hilbert space framework. The appropriate Hilbert space representation is given by a *bosonic Fock space* over a suitable L^2-space. The first part of the present work is concluded by the derivation of a spectral theory which connects the dynamical and thermodynamic features with spectral properties of a suitable generator, say K, of the time evolution in this Hilbert space setting. That way, the question about thermal relaxation becomes a spectral problem. The operator K is of *Pauli-Fierz type*.

The spectral analysis of the generator K follows. This task is the core part of the work and it employs various kinds of functional analytic techniques. The operator

K results from a perturbation of an operator L_0 which describes the non-interacting particle-boson system. All spectral considerations are done in a perturbative regime, i.e., we assume that the strength of the coupling is sufficiently small. The extraction of dynamical features of the system from properties of K requires, in particular, the knowledge about the spectrum of K in the nearest vicinity of eigenvalues of the unperturbed operator L_0. Since convergent Neumann series expansions only qualify to study the perturbed spectrum in the neighborhood of the unperturbed one on a scale of order of the coupling strength we need to apply a more refined tool, *the Feshbach map*. This technique allows the analysis of the spectrum on a smaller scale by transferring the analysis to a spectral subspace. The need of spectral information on arbitrary scales requires an iteration of the Feshbach map. This procedure leads to an *operator-theoretic renormalization group*. The reader is introduced to the Feshbach technique and the renormalization procedure based on it is discussed in full detail. Further, it is explained how the spectral information is extracted from the renormalization group flow.

The present dissertation is an extension of two kinds of a recent research contribution by Jakšić and Pillet to a similar physical setup. Firstly, we consider the more delicate situation of bosonic heat reservoirs instead of fermionic ones, and secondly, the system can be studied uniformly for small reservoir temperatures. The adaption of the Feshbach map-based renormalization procedure by Bach, Chen, Fröhlich, and Sigal to concrete spectral problems in quantum statistical mechanics is a further novelty of this work.

Contents

Introduction

Survey of the Problem and Organization of the Thesis

This dissertation is a contribution to mathematical rigorous *non-equilibrium quantum statistical mechanics* considering as example a simplified quantum electrodynamical model for the interaction of matter with radiation (in a wider sense) at positive temperature. In the center of interest is a finite dimensional particle system (such as an N-dimensional spin or a spin chain of finite length) which interacts with several quantized massless boson fields of infinite spatial expansion. These bosonic fields might physically be realized as photon gasses (radiation) or as phonon fields (quantized modes of vibration). Both sorts of bosons are responsible for two different types of heat transfer: while the photons represent a *heat radiation* the propagation of phonons is responsible for *heat conduction* in solids. The boson fields, occasionally also referred to as heat reservoirs, interacting with the particle system need not to be of the same type. A possible scenario covered within this general framework could be a particle exposed to heat radiation through thermal photons and in contact with a solid transferring heat through thermal phonons. It is assumed that the bosonic subsystems are not interacting with each other. For simplicity of notion we henceforth do not differentiate between the two mentioned types of bosons and refer to them as photons. The key problem we are going to investigate in this work is the question about the existence and the nature of stationary states and their dynamical stability. Along with these dynamical issues we also study the thermodynamic properties of the system. Hereby, the setup of the system will be the following. Each of the photon reservoirs will be prepared in or close to a thermal equilibrium state describing a photon ensemble of finite particle density in absence of *Bose-Einstein condensation*. Each photon system for itself, isolated from the other constituents, behaves like a free photon gas and it features the property of *return to equilibrium (RtE)*, i.e., the photon gas will thermally relax under the time evolution into the equilibrium it started close by. The drive of a single infinitely expanded photon system towards equilibrium transfers to the finite particle system once they are coupled to each other. Hence, the composed particle-photon system shows dynamical stability in the sense that states which are close to a thermal equilibrium at a given temperature are attracted by the latter. In this context, proximity has to be understood in the relative entropy sense. Such a setup is said to be *close to equilibrium*. The situation becomes more subtle when several photon systems at possibly different temperatures are coupled to the same particle system, this setup would be *far from equilibrium*. Each reservoir shows a comparable endeavor of thermal relaxation, however, towards different equilibria. This competition among the reservoirs prevents the system from approaching a thermal equilibrium state. Nevertheless, the system presumably will feature thermal relaxation in the sense that it will converge into a stationary state. This limit state will be distinguished from a thermal equilibrium state by the fact that it shows *non-vanishing stationary heat fluxes* between the reservoirs and a *positive entropy production rate*.

We shall mention that the assumption about the finiteness of the particle degrees of freedom is crucial for our analysis. The thermal relaxation of the particle-photon system is controlled by the so called *Fermi golden rule level shift* which is a lower bound to the rate of exponential decay of initial configurations into the stationary state. The Fermi golden rule level shift is computed as the minimal probability of any transition of a particle state from a higher energy level down to a lower one under emission of photons carrying away the energy difference. For infinitely many particle energy levels the infimum of these transition probabilities is typically zero such that the Fermi golden rule level shift does not allow any prediction about decay properties. The arguments applied in this work do not work any more for a vanishing Fermi golden rule level shift. In fact, it is still a non-trivial open problem to prove the thermal relaxation of particle systems with infinitely many energy levels interacting with photons.

In this thesis we put the described physical situation of particle-photon interaction away from equilibrium, and the analysis on it, on a mathematical footing. Hereby, we extend the model proposed in [8] to several bosonic reservoirs and we proceed as follows. The appropriate mathematical concept for treating quantum statistical systems is given by the theory of *C^*-algebras*. In Chapter 1 we thoroughly discuss the mathematical model for the particle and the photon systems. We introduce the Hamiltonian description of each subsystem to define a Heisenberg time evolution on a suitable algebra of observables. Algebra and time evolution form a *C^*- or W^*-dynamical system*. Among the states on this algebra we specify those which describe thermal equilibria. The mathematical setup for the infinite photon systems differs significantly from the considerations on the finite particle system. The thermodynamic concepts, as the notion of a thermal equilibrium state, are easily accessible in the context of a finite system and need an adaption to the infinite situation. We outline the strategy how the photon system can be treated as a *thermodynamic limit* of finite systems of a photon gas confined to increasing but bounded boxes. The preparation of the subsystems in thermal equilibrium states provides us with a representation of the C^*-algebra of observables, the *GNS representation*, which gives rise to a particular *modular structure* of the dynamical system. We launch an excursus about the *Tomita-Takesaki theory*, dealing with this modular structure, which provides algebraical tools in an analytical framework. The coupling between the particle and the photon systems is realized as a local perturbation of the non-interacting setup. This perturbation is obtained by incorporating boson creation and annihilation processes to first order into the Hamiltonian description. We can interpret this sort of perturbation as a simplified model for minimal coupling which, in turn, describes the realistic interaction of electrons with radiation in non-relativistic quantum electrodynamics. The setup belongs to the class of *Pauli-Fierz systems*. The first chapter is concluded by the statement of the main theorem of this work. It rephrases the thermal relaxation properties of the particle system in interaction with several bosonic reservoirs at different temperatures, as discussed above, in a mathematical language. We prove in the case of differing reservoir temperatures the existence of a *non-equilibrium stationary state (NESS)* which is attracting for all physical configurations close to the setup where the subsystems, for itself, are at equilibrium. Hereby, the approach is exponentially fast with a decay rate proportional to the second power of the interaction strength and proportional to the temperature of the reservoirs. Further we show that the NESS features non-vanishing heat fluxes and the entropy production rate in this state is strictly positive. The case where the photon reservoirs started at the same temperature is covered as a limiting case. The dynamical behavior is equivalent to the previous situation, i.e., the thermal relaxation occurs exponentially fast. The thermodynamic characteristic of the limit state, however, is quite different since in the equal temperature situation the relaxation is towards an equilibrium configuration. All our results are perturbative in the strength of the coupling and uniform for small temperatures provided that temperature differences are not too large.

Within Chapter 2 we derive a spectral theory for the thermal relaxation process following [28]. The aim is to connect the dynamical behavior of the interacting system with spectral properties of a suitable generator of the time evolution. The GNS representation together with Tomita-Takesaki's modular theory allow us to transfer the dynamical problems into a more convenient Hilbert space framework. Within this framework the time evolution is generated by a family of so-called *Liouville operators*. We single out one of these operators, the *C-Liouville operator K*, whose null space encodes the information about the NESS while the localization of the continuous spectrum discloses the long time behavior of the evolution. However, before the connection of the dynamical behavior and the operator can be established the C-Liouvillean has to undergo a *spectral deformation*. Since the operator K is in general neither self-adjoint nor normal nor accretive it is the spectral deformation which is necessary to give meaning to the evolution generated by K. In our work we apply a combination of two deformation techniques, the *dilation deformation* as used in [8] and the *translation deformation* applied in [23, 28]. It turns out that the NESS is given as the *zero resonance eigenvectors* of K w.r.t. the chosen deformation, i.e., in terms of the zero eigenvectors of the spectrally deformed operator K_θ and of its adjoint $(K_\theta)^*$. The shift of the spectrum of K_θ into the upper half plane, except for the simple zero eigenvalue, accounts for the exponentially fast decay towards the NESS. The first two chapters build a logical unit addressing the conceptional part of the work which is closely related to contributions of Bach, Fröhlich and Sigal, [8], and of Jakšić and Pillet [24, 27, 28].

The second part consisting of the Chapters 3 - 5 is devoted to the underlying spectral analysis of the problem. We launch Chapter 3 with a complete description of the spectrum of the family of deformed Liouvilleans, the corresponding proof will spread over the whole second part. Different spectral regions require different techniques of analysis. The Liouvilleans of the interacting systems can be seen as perturbations of a deformed free Liouville operator $L_{0,\theta}$ whose spectrum is totally understood. As a rule of thumb we can keep that the closer we get to real eigenvalues of the unperturbed problem the more difficult the analysis becomes and the more sophisticated techniques are applied. As a first step in understanding the spectrum we exclude spectrum in regions far enough from the spectrum of the unperturbed Liouville operator with the means of convergent Neumann series employing relative norm estimates of the perturbation part of the Liouvilleans. This technique will fail when we aim to study the spectral vicinity of the formerly real eigenvalues of $L_{0,\theta}$ due to the divergence of the free resolvent. The *Feshbach technique* is a suitable replacement since it allows a decision about the invertibility of the perturbed resolvent based on an equivalent spectral problem after having projected out the singularity of the free resolvent. This method was introduced as a tool for spectral analysis in quantum field theory by Bach, Fröhlich and Sigal in [6] and was generalized in collaboration with Chen in [4]. The *smooth Feshbach map* generates an operator with *isospectral properties*. This operator is the free operator $L_{0,\theta}$ supplemented, in leading order in the strength of the coupling, by a matrix which is responsible for the shift of the unperturbed eigenvalues away from the real axis. This matrix is therefore called *level shift operator*. Since the shift of eigenvalues directly effects the exponential decay rates of excited configurations of the system the level shift operator is closely related to the *Fermi golden rule*.

The zero eigenvalue plays a special role which is founded by its high degree of degeneracy. Though the level shift operator lifts the degeneracy, a simple eigenvalue stays at the origin of the complex plane, in leading order of the perturbation, and the isolation of this eigenvalue is given by a gap proportional to the temperature of the reservoirs. Hence, for small temperatures (compared to the strength of the coupling) standard perturbation theory fails again to make conclusions about the spectrum of the full operator which we obtained after the application of the smooth

Feshbach map. Since our endeavor is to derive all results uniformly in low temperatures we need to reapply the Feshbach method to tackle the spectral problems in the neighborhood of zero. That way we enter into a process of iterative Feshbach applications building the core part of an *operator-theoretic renormalization group* which goes back to [4, 6]. The outsourcing of the renormalization procedure into Chapter 4 accounts for the difficulties which arise in the study of the spectrum close to zero.

In the last chapter of the main text, Chapter 5, we collect the information which was gained by the renormalization procedure in Chapter 4 to assemble a spectral picture of the Liouville operators in the neighborhood of zero. This is the first time that the renormalization group of [4] is applied to a concrete model in positive temperature quantum electrodynamics in order to draw a quantitative picture of the spectrum in the vicinity of eigenvalues.

In a third part of this thesis we embrace in five appendices the necessary technical tools for the considerations in the main text. Outsourcing the technicalities shall enhance the readability of the main text without holding back the analytical issues from the reader.

Comparison with the Literature

The field of equilibrium and non-equilibrium statistical quantum mechanics has become recently a very active area of research. We range our work among the significant contributions to related problems in this field where a small system is in interaction with its environment. The environment is considered as an infinite part of a system which allows dissipation. This environment can be of various kinds and we mention as examples the quantized bosonic field (as photon or phonon field like in our case), a fermi gas or simply an infinite region of a spin system (as a largely expanded part of a crystal interacting with a small crystalline zone). The small system usually represents a confined particle and its realization may range in the various models from a finite ensemble of spins to an electron in a binding Coulomb potential. During the presentation of models in statistical quantum mechanics we occasionally draw a parallel to the zero temperature situation.

A first rigorous treatment of the dissipative properties of the Pauli-Fierz *spin-boson model* (a single spin coupled to a bosonic field) was undertaken by Jakšić and Pillet in the series of papers [22, 23, 24], a review of these results is given in [25]. The main achievement of that work is the development of a spectral theory for the RtE property connecting the spectrum of a so-called *standard Liouville operator*, an appropriate generator of the time evolution in the equilibrium situation, with the thermal relaxation behavior of the system at positive temperature. The spectral analysis to follow on the Liouville operator of the spin-boson model employs a new deformation technique, the translation deformation, which generates isolated eigenvalues of the unperturbed system and allows standard perturbation theory. The price to pay for this simplified analysis is that the used deformation puts strong restrictions on possible coupling functions and only allows the study of the high temperature regime.

These restrictions could be lifted by Bach, Fröhlich and Sigal in [8] who not only extended the degree of freedom for the spin but, more significantly, could prove RtE for the spin-boson system uniformly in the positive temperature of the boson reservoir. They adopted the concept of spectral theory from [24], and, to tackle the uniformity in the temperature, they fell back upon the renormalization group developed in [6]. The application of the renormalization procedure to the positive temperature framework is outlined in [8] and enabled the authors to get along with dilation as spectral deformation which requires much less regularity of the coupling functions.

An alternative technical approach to study RtE for the spin-boson model was given by Merkli in [32] who transferred the concept of *Mourre estimates*, also known as *positive commutators*, to the spectral investigation of the positive temperature situation. This technique incorporates the generator of the translation deformation and therefore represents an infinitesimal version of the translation deformation technique. The strategy yields a technical improvement w.r.t. [24] in the sense that the assumptions on the regularity of the coupling functions can be relaxed (Merkli gets along with sufficient smoothness instead of analyticity of the coupling functions w.r.t. translation), however, it proves RtE in a weaker version, namely in the ergodic mean sense. The positive commutators cannot overcome the restriction to high temperatures.

The papers [12, 13] by Dereziński and Jakšić are another contribution to the spectral analysis of thermal Pauli-Fierz systems in relation to their thermal relaxation.

So far we discussed models where the reservoirs are given in a configuration of finite boson density. Such a setup prohibits the macroscopic occupation of the ground state, i.e., the description of Bose-Einstein condensation is excluded. The recent work [33] of Merkli deals with a Bose gas at thermal equilibrium which is so dense that it builds a condensate. The coupling of a particle system to the zero modes of the boson gas (which correspond to the condensate) exhibits a technical challenge. The infrared behavior of the coupling functions treated in the references mentioned above do not provide a framework which allows an effective coupling of the particle system to the condensate. Merkli managed to introduce a model of an interacting particle-condensate system for which he could prove the existence of a stable equilibrium. Hereby, stability is meant in the following sense: any initial condition close to the equilibrium state of the interacting system converges towards the equilibrium of the uncoupled system if one takes successively the limit of large time and then the limit of small coupling. The RtE property of the system is still an open problem. The analysis is based on positive commutator techniques.

Further contributions to the particle systems interacting with a thermal photon reservoir are the works of Fröhlich and Merkli, [15], and Fröhlich, Merkli and Sigal, [16], on *thermal ionization*. The investigated model describes an idealized atom, consisting of a finite number of eigenvalues lying below the ionization threshold of a continuous spectrum, which is brought into contact with a *black-body radiator* at sufficient high positive temperature. The authors show that such a system does not possess any time-translation invariant state of positive temperature and that the expectation value of any finite-dimensional projection in an arbitrary initial state of positive temperature tends to zero under the time evolution. This phenomenon is known as thermal ionization. Unlike for the spin-boson model, where the particle system only possesses bound states, the existence of continuous spectrum gives the idealized atom the opportunity to leave eventually any equilibrium configuration which is in contrast to the thermal relaxation behavior of the models discussed so far. The statements in [15, 16] are established by studying the Liouville operator using positive commutators.

RtE has a zero temperature analogue: *return into the ground state*. The relevant physical process is no longer thermal relaxation but *radiative decay*. It is well known that atomic systems (non-relativistic electrons in a Coulomb potential of a nucleus) – while possessing stable excited energy levels in the absence of an electromagnetic field – have no stable states except for the ground state when interacting with photons. It goes back to studies of Bach, Fröhlich and Sigal, [5, 7], and ourselves, [36], on a realistic model for non-relativistic quantum electrodynamics that the excited states are replaced by *metastable states* which decay quasi-exponentially as predicted by the Fermi golden rule. However, it is not clear how to prove that these states relax into the ground state which would correspond to the "thermal" equilibrium at temperature zero. The analytical difficulties are founded by the fact that the zero temperature framework does not give us a spectral theory for RtE at hand.

The zero temperature analogue of thermal ionization is the *photoelectric effect* which was studied by Bach, Klopp and Zenk in [9]. Building up on a simplified model for an atom consisting of a single bound state and continuum the authors managed to prove the ionization of the atom by photons as a non-statistical phenomenon. The observations are in agreement with Einstein's prediction that the ejection of an electron from its formation only occurs if the energy of the incoming photon cloud exceeds the ionization threshold such that the energy surplus can be transformed into kinetic energy of the travelling electron. The ionization mechanism at zero temperature therefore differs significantly from the ionization by thermal fluctuations where the photons do not have to overcome an ionization threshold.

We go over to discuss the systems appearing in the literature which are closely related to the setup in the present thesis. The impulsion to study the thermal relaxation of particles interacting with several bosonic heat reservoirs came from the contribution [28]. In that paper Jakšić and Pillet investigate a spin which interacts with finitely many fermionic reservoirs which are at different positive temperatures. Since the system is from the beginning not close to equilibrium the spectral theory for thermal relaxation behavior developed in [24] is not applicable. In other words, the standard Liouville operator does not carry a priori any information about the relaxation of a system far from equilibrium. Jakšić and Pillet found a remedy by connecting the dynamical issues with an equivalent non-self-adjoint generator of the time evolution, the C-Liouvillean. The spectrum of the C-Liouville operator allows predictions about the time development of states (close to the configuration where the Fermi reservoirs are at different temperatures) towards a limit state. Further, that relaxed state, the NESS, can be characterized in terms of the C-Liouville operator. This concept, with some modifications to improve the validity of statements uniformly in the temperature, has been adopted by ourselves within this thesis. While formally the setup in [28] looks similar to our model of a particle system interacting with bosonic reservoirs at different temperatures, our situation exhibits much more difficulties of a technical kind. This has primarily to do with the fact that creation and annihilation processes, as they enter the interaction between the particle system and the reservoirs, are bounded operations in the fermionic case unlike in the bosonic case. While the C-Liouville operator is accretive in the fermionic case, the unboundedness of bosonic creation and annihilation operators causes that the connection of the C-Liouvillean to the time evolution it generates is a priori unclear. This fact poses many technical subtleties which are to be tackled to transfer the concepts of [28] to our situation. Jakšić and Pillet acknowledge in their paper that the extension to bosonic reservoirs is an important and, until recently, an open problem. A further discrepancy between [28] and our work is that we can study the relaxation uniformly in the reservoir temperatures. This could be achieved firstly by modifying the C-Liouville operator and secondly by employing the renormalization group technique. Following the arguments of [26, 27, 28, 29] we show for our system that the entropy production in the NESS is strictly positive when the temperatures of the reservoirs differ sufficiently.

Finally we mention some references for non-equilibrium situations in quantum spin systems. A quantum spin system can roughly be characterized as a countably infinite collection of sites where spins are fixed and which interact with their neighbors. The set of sites appears as a division into finite many subsets where one of the subsets contains only finite many sites, representing the “particle” system, while the others are infinite, standing for the reservoirs or simply for the environment. These reservoirs can now be prepared at equilibrium of different temperatures. This kind of situation is studied by Ruelle in [42] on non-negativity of the entropy production. It is the work which introduces the notion of a NESS as it is used in [26] and also in this work, c.f. Remark 1.16. A special case of this setup, the *two-sided XY chain*, was analyzed on positivity of the entropy production by Aschbacher and Pillet in [3].

Parts of this thesis have been elaborated in cooperation with Merkli and Sigal. The corresponding results are to appear in [34].

Acknowledgements

In the first place I would like to thank my doctoral advisor, *Prof. Dr. VB*, for the continuation of the excellent supervision of my development as researcher beyond my graduation. I am indebted to Prof. VB for encouraging me to broaden my scientific horizon by spending two years of my doctoral studies at the University of Toronto. I am grateful to him for the various kinds of support – of professional, financial and mental type – which I received in the course of the research stay in Toronto and back at the University of Mainz. Prof. VB really understood to give me the feeling of being an important member of his group and that my work was appreciated. I am grateful for recommending me to *Prof. Dr. IMS* and for introducing me to the field of non-equilibrium statistical quantum mechanics.

I would like to thank *Prof. Dr. IMS* for integrating me as a visiting Ph.D. student into his group at the University of Toronto and for the great opportunity to collaborate with him on a project which forms the basis to this thesis. Prof. IMS gave me a new perspective of scientific working and he was a very good teacher and Ph.D. advisor during my stay in Toronto. His professional and financial support was crucial for the successful proceeding of my dissertation. His commitment for my status at the department of mathematics and for my social settlement was outstanding. I am indebted to Prof. IMS for offering me an excellent environment for fruitful research in a very interesting field of mathematical physics. I am grateful for all these many positive experiences in Toronto.

I am grateful to *Dr. MM* who joined the collaboration. His enthusiasm for the project was a great incitation for me. I am happy that Dr. MM was both a fellow researcher and a mentor for me at the same time. I am indebted to Dr. MM for uncountable discussions on the topic in more or less formal environments. Dr. MM gave me a strong mental support, he was always open for urgent calls to give immediate advice in critical phases. I appreciate his invitations to the ETH Zürich and to the McGill University in Montréal.

The remarkable situation of having a troika of advisors and mentors guaranteed the best possible supervision of my research work.

Representatively for the assistance of my coworkers I would like to thank my colleagues *WB* and *Dr. OM* for helpful discussions on mathematical details. I thank *AR* for careful proofreading of the thesis and for helpful advice on layout and style issues. *JD* and *MH* helped me with LaTeX-related problems. The research group for *Numerical Mathematics* at the University of Mainz allowed me to use their printing facilities to produce colored pictures. This support is acknowledged.

I am grateful to everybody who accompanied me on the way to my doctoral degree, in particular during my stay in Canada. I was very happy about having had the company of a *beloved person* in Toronto to lean on. I enjoyed the many visits of *my family* and *my friends*. Thanks to *my parents* who enabled everything to support my academic career. Knowing that my parents would always believe in me achieving my goals was a strong encouragement.

The *IHP network "Analysis and Quantum"*, supported by the European Union, invited me to several conferences where I was able to present my results to an international audience of experts. This support of communicating my research activities is highly appreciated.

The visit at the University of Toronto for a period of two years in relation with the research on this project was funded by the *German Academic Exchange Service, DAAD*, under a HSP-III scholarship. I am very grateful for this support.

Part I

Model, Theory, Results

1 The Positive Temperature Model of Particles Interacting with Radiation

Statistical quantum mechanics (StQM) is the discipline of quantum physics describing systems at positive temperature. The notion of temperature implies an uncertainty of the occupation of a quantum mechanical state. This lack of information is interpreted as thermal fluctuation and is mathematically handled with probabilistic methods. A quantum mechanical state, usually represented as an element of a Hilbert space, gets replaced by a density matrix over the same Hilbert space representing an ensemble of possible configurations, the eigenvectors of the matrix, and their assigned occupation probability, given by the eigenvalues. The temperature enters this framework as a parameter which characterizes the occupation probability distributions for so-called *thermal equilibrium* configurations.

Since a physical state is defined by the expectation values of observable quantities one measures in this state the mathematical notion of a state in statistical quantum mechanics is a positive linear functional on an algebra of non commuting observables. Since for large quantum systems (e.g., for those with infinitely many degrees of freedom) not every such functional can be brought into relation with a density matrix this concept of states of the quantum system goes even beyond the generalization from Hilbert space vectors to density matrices. The corresponding mathematical framework is provided through the theory of *C^*-algebras* which represents a well established field of research. The richness of this theory offers many opportunities for the study of abstract and concrete quantum statistical systems. We will introduce the reader to the field of C^*-algebras and its techniques in the first part of this chapter, based on the monographs [10, 11, 17]. The paper [14] gives a good review of problems related to *W^*-dynamical systems*.

Building up on the abstract notion we are presenting the quantum statistical model of a particle system which is interacting with several heat reservoirs at not necessarily the same temperatures. The model we are going to use is a slight extension of the model proposed in [8], where the interaction of the particle system was restricted to a single reservoir.

1.1 Operator Algebraic Approach to Statistical Quantum Mechanics

1.1.1 States and Dynamics on C^*- and W^*-Algebras

The physical measurands, the *observables*, of a quantum system on an underlying Hilbert space are usually given by operators on that Hilbert space. The collection of observables forms an algebra. Inspired by the observation that the bounded operators on a Hilbert space form a C^**-algebra* (for a definition of C^*-algebras we refer the reader to [10, Sect. 2.1]) we henceforth assume that the set of physical quantum observables of interest is given by an abstract C^*-algebra $\mathcal{A}$. A physical state (configuration) of the system is given by a (mathematical) state on the algebra of observables. The subset of the dual space $\mathcal{A}^*$ containing all positive, linear functionals of norm one,

$$\mathcal{E}(\mathcal{A}) := \{(\omega : \mathcal{A} \to \mathbb{C}) \in \mathcal{A}^* \mid \|\omega\|_{\mathcal{A}^*} = 1,\, \omega(A^*A) \geq 0,\, \forall\, A \in \mathcal{A}\},$$

is the convex set of all states of our system assigning real expectation (measured) values to the self adjoint observables among the elements of $\mathcal{A}$ and therefore characterizes the system's configurations. Refer to [10, Sect. 2.3.2] for convexity of $\mathcal{E}(\mathcal{A})$ and further properties of states on C^*-algebras.

In various situations a given C^*-algebra $\mathcal{A}$ features the existence of a *predual*, i.e., there exists a Banach space whose dual space is isomorphic to the algebra $\mathcal{A}$. Such an algebra is called a W^**-algebra*. Its predual is uniquely determined up to isomorphisms and is denoted by $\mathcal{A}_*$. Since $\mathcal{A}_*$ is isometrically imbedded into $\mathcal{A}^*$ the structure of a W^*-algebra gives rise to a new topology on $\mathcal{A}$, the *σ-weak topology*, generated by the system of semi-norms $\{A \mapsto |\omega(A)| \mid \omega \in \mathcal{A}_*\}$. Further, the predual allows a distinction of states over $\mathcal{A}$, the so-called *normal states* which are collected in the set

$$\mathcal{N}(\mathcal{A}) := \mathcal{E}(\mathcal{A}) \cap \mathcal{A}_* = \{\omega \in \mathcal{A}_* \hookrightarrow \mathcal{A}^* \mid \|\omega\|_{\mathcal{A}^*} = 1,\, \omega(A^*A) \geq 0,\, \forall\, A \in \mathcal{A}\}.$$

Apparently, the normal states are exactly the states which are continuous w.r.t. the σ-weak topology on $\mathcal{A}$. We remark that a C^*-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, realized as a subalgebra of the bounded operators on a separable Hilbert space $\mathcal{H}$, is a W^*-algebra if and only if the algebra is weakly closed within $\mathcal{B}(\mathcal{H})$. Von Neumann's

Bicommutant Theorem (see [10, Thm. 2.4.11]) allows us to express the weak closure of a subset $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ as the *bicommutant*

$$\mathcal{M}'' = (\mathcal{M}')'$$

where the *commutant* of a subset $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is defined by

$$\mathcal{M}' := \{M' \in \mathcal{B}(\mathcal{H}) \,|\, [M, M'] = 0 \,\forall\, M \in \mathcal{M}\}.$$

Thus, the criterion for a W^*-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ can be rephrased as

$$\mathcal{A}'' = \mathcal{A}.$$

The concrete realization of a W^*-algebra on a Hilbert space is referred to as *von Neumann algebra*. Its predual

$$\mathcal{A}_* = \mathcal{L}^1(\mathcal{H}) / \left\{\rho \in \mathcal{L}^1(\mathcal{H}) \,\middle|\, \mathrm{tr}(\rho A) = 0 \,\forall\, A \in \mathcal{A}\right\}$$

is the Banach space $\mathcal{L}^1(\mathcal{H})$ of trace class operators on $\mathcal{H}$ where we identify two operators $\rho \sim \rho'$ if the corresponding functionals $\mathcal{A} \ni A \mapsto \mathrm{tr}(\rho A), \mathrm{tr}(\rho' A)$ coincide. The normal states are given by

$$A \mapsto \mathrm{tr}\,(\rho A)$$

for $\rho = \rho^* \in \mathcal{L}^1(\mathcal{H})$ being a *density matrix*, i.e., $0 \leq \rho \leq 1$ and $\mathrm{tr}(\rho) = 1$. We remark that for each C^*-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ the commutant algebra $\mathcal{A}' \subseteq \mathcal{B}(\mathcal{H})$ is weakly closed and therefore a von Neumann algebra.

A *dynamics* on the C^*- (or W^*-) algebra $\mathcal{A}$ of observables is introduced by a group $\alpha = \{\alpha^t\}_{t \in \mathbb{R}}$ of automorphisms α^t on $\mathcal{A}$. The dynamics therefore evolves the observables (Heisenberg picture). This time evolution can be lifted to a state $\omega \in \mathcal{E}(\mathcal{A})$, the evolved state is then given by

$$\alpha^t * \omega := \omega \circ \alpha^t.$$

A state ω is called *stationary* or *time invariant* w.r.t. α if for all $t \in \mathbb{R}$

$$\alpha^t * \omega = \omega \circ \alpha^t = \omega$$

holds. The pair $(\mathcal{A}, \alpha)$ is called a *C^*-dynamical system*, if the group α is strongly continuous, i.e.,

$$\mathbb{R} \ni t \mapsto \alpha^t(A) \tag{1.1}$$

is continuous as a map from $\mathbb{R}$ to $\mathcal{A}$ in the C^*-norm topology for all $A \in \mathcal{A}$. In the case that $\mathcal{A}$ is a W^*-algebra we call the pair $(\mathcal{A}, \alpha)$ a *W^*-dynamical system* if the group α is pointwise σ-weak continuous, i.e., the map (1.1) is continuous for all $A \in \mathcal{A}$ while the space $\mathcal{A}$ is equipped with the σ-weak topology. Note that if $\mathcal{A} = \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and $H = H^*$ is an unbounded self-adjoint operator on $\mathcal{H}$ then $t \mapsto e^{-iHt}$ is strongly continuous on $\mathcal{H}$, and therefore $t \mapsto \alpha^t := e^{iHt}(\,\cdot\,)e^{-iHt}$ constitutes a W^*-dynamical system $(\mathcal{A}, \alpha)$ which is not a C^*-dynamical system.

Since not every time invariant state describes a thermal equilibrium configuration at a given temperature $T > 0$, we have to provide a distinguishing definition for a *thermal equilibrium state*. The notion of a *KMS (Kubo-Martin-Schwinger) state* turned out to be the right definition as an equilibrium state and generalizes the *Gibbs characterization* for a finite system as considered in Section 1.2. Given a C^*- (or W^*-) dynamical system $(\mathcal{A}, \alpha)$, then a state $\omega \in \mathcal{E}(\mathcal{A})$ (which has to be normal in the W^* case) is called an α-KMS state w.r.t. the inverse temperature $\beta = T^{-1}$, or short, an (α, β)-KMS state, if for each pair $A, B \in \mathcal{A}$ there is a function $F_{A,B}$ which is analytic on the domain $D_\beta := \{z \in \mathbb{C} \,|\, 0 < \mathrm{Im}(z) < \beta\}$ and continuous on the closure $\overline{D}_\beta$ satisfying the *KMS condition*,

$$F_{A,B}(t) = \omega(A\alpha^t(B)) \qquad \text{and} \qquad F_{A,B}(t + i\beta) = \omega(\alpha^t(B)A) \tag{1.2}$$

for all $t \in \mathbb{R}$. One can show that the α-KMS states for any inverse temperature β are stationary w.r.t α while the opposite conclusion is not true. Henceforth, we understand by a stationary state a state which is time invariant and by an equilibrium state a KMS state.

1.1.2 GNS Representation and Tomita-Takesaki Theory

To study the dynamical behavior of a state ω on a C^*-algebra $\mathcal{A}$ under the group $\alpha = \{\alpha^t\}_{t\in\mathbb{R}}$ of automorphisms, it is useful to represent the abstract C^*-algebra on a Hilbert space. The *GNS (Gelfand-Naimark-Segal) representation* provides a procedure how to construct canonically a representation $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$ into the bounded operators on a Hilbert space $\mathcal{H}_\omega$, such that

$$\omega(A) = \langle\, \Omega_\omega \,|\, \pi_\omega(A)\Omega_\omega \,\rangle_{\mathcal{H}_\omega}\,,$$

where the vector representative $\Omega_\omega \in \mathcal{H}_\omega$ is *cyclic* w.r.t. $\pi_\omega(\mathcal{A})$, i.e., the set $\{\pi_\omega(A)\Omega_\omega \,|\, A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\omega$. For an exposition of the GNS construction refer to [10, Sect. 2.3.3] and [17, Sect. III.2.2], as well as to [18]. In the case that ω is a faithful state, i.e., it holds

$$\omega(A^*A) = 0 \qquad \Longleftrightarrow \qquad A = 0,$$

the vector Ω_ω is also *separating* for $\pi_\omega(\mathcal{A})$, i.e., $\pi_\omega(A)\Omega_\omega = 0$ implies already $A = 0$. Note that all KMS states are faithful.

Since the weak closure of the image of $\mathcal{A}$ under π_ω, the algebra $\pi_\omega(\mathcal{A})''$, is a von Neumann algebra, it makes sense to extend the notion of normal states to *π_ω-normality.* A state η on $\mathcal{A}$ is called *relative normal w.r.t.* ω or *ω-normal* or *π_ω-normal*, if there is a density matrix $\rho \in \mathcal{L}^1(\mathcal{H}_\omega)$ such that

$$\eta(A) = \operatorname{tr}(\rho\, \pi_\omega(A)) .$$

The states on $\mathcal{A}$ which are ω-normal are collected in the set

$$\mathcal{N}_\omega(\mathcal{A}) := \{\eta \in \mathcal{E}(\mathcal{A}) \,|\, \exists\, \rho \in \mathcal{L}^1(\mathcal{H}_\omega) : \eta = \operatorname{tr}(\rho \pi_\omega(\,\cdot\,))\} .$$

The physical significance of relative normality is that two states normal w.r.t. each other have a finite *relative entropy* which means that one state can be prepared out of the other one by changing the entropy only by a finite amount. The mathematical concept of relative entropy of two relatively normal states is discussed further down.

We will see in a moment that ω-normal states can always be represented as vector states in the representation π_ω provided that ω is faithful. For this insight we need the standard form associated with the GNS triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$. Let us assume that Ω_ω is cyclic and separating for

$$\mathcal{M}_\omega := \pi_\omega(\mathcal{A})''$$

as it is guaranteed for a faithful state ω. The assignment

$$S_\omega : \mathcal{M}_\omega\Omega_\omega \longrightarrow \mathcal{M}_\omega\Omega_\omega, \qquad A\Omega_\omega \mapsto A^*\Omega_\omega$$

is well defined as an anti-linear operator with dense domain. It is closable and the closure shall also be denoted by S_ω. The adjoint anti-linear operator $F_\omega = S_\omega^*$ is given by the closure of

$$F_\omega : \mathcal{M}'_\omega\Omega_\omega \longrightarrow \mathcal{M}'_\omega\Omega_\omega, \qquad A'\Omega_\omega \mapsto A'^*\Omega_\omega .$$

The polar decomposition of S_ω,

$$S_\omega = J_\omega \Delta_\omega^{1/2},$$

defines the anti-unitary operator J_ω, the so-called *modular conjugation*, and the self-adjoint, positive *modular operator*

$$\Delta_\omega := S_\omega^* S_\omega .$$

The modular conjugation obeys

$$J_\omega = J_\omega^* = J_\omega^{-1} \qquad \text{and} \qquad J_\omega \Delta_\omega^{1/2} = \Delta_\omega^{-1/2} J_\omega .$$

It is due to a theorem of *Tomita-Takesaki* (see [10, Thm. 2.5.14] and [17, Sect. V.2.1, Thm. 2.1.1]) that

$$\begin{aligned} J_\omega \mathcal{M}_\omega J_\omega &= \mathcal{M}_\omega' \qquad \text{and} \\ \Delta_\omega^{it} \mathcal{M}_\omega \Delta_\omega^{-it} &= \mathcal{M}_\omega \qquad \forall\, t \in \mathbb{R}. \end{aligned} \tag{1.3}$$

The first relation of (1.3) implies that

$$\pi_\omega' : \mathcal{A} \to \mathcal{M}_\omega' \subseteq \mathcal{B}(\mathcal{H}_\omega), \qquad \pi_\omega'(A) := J_\omega \pi_\omega(A) J_\omega$$

is an anti-linear representation of $\mathcal{A}$ into the bounded operator on $\mathcal{H}_\omega$ commuting with the representation π_ω. The group $\sigma_\omega = \{\sigma_\omega^t\}_{t\in\mathbb{R}}$,

$$\sigma_\omega^t : \mathcal{M}_\omega \to \mathcal{M}_\omega, \qquad \sigma_\omega^t(A) := \Delta_\omega^{it} A \Delta_\omega^{-it},$$

is referred to as *modular automorphism group*. We denote by

$$\mathcal{L}_\omega := -\operatorname*{s-lim}_{t\to 0} \frac{\Delta_\omega^{it} - \mathbb{1}_{\mathcal{H}_\omega}}{it} = -\ln(\Delta_\omega)$$

the generator of the group $\{\Delta_\omega^{it}\}_{t\in\mathbb{R}}$.

We further introduce the *natural positive cone* associated with the pair $(\mathcal{M}_\omega, \Omega_\omega)$,

$$\mathcal{P}_\omega := \overline{\{ A J_\omega A \Omega_\omega \mid A \in \mathcal{M}_\omega \}}.$$

It is a consequence of Tomita-Takesaki's theory that

$$\begin{aligned} J_\omega \xi &= \xi \qquad \forall\, \xi \in \mathcal{P}_\omega \qquad \text{and} \\ \Delta_\omega^{it} \mathcal{P}_\omega &= \mathcal{P}_\omega \qquad \forall\, t \in \mathbb{R}. \end{aligned}$$

The crucial property of $\mathcal{P}_\omega$ is that each ω-normal state η has a unique vector representative in $\mathcal{P}_\omega$, i.e., there exists a unique $\xi \in \mathcal{P}_\omega$ such that

$$\eta(A) = \langle\, \xi \mid \pi_\omega(A) \xi \,\rangle_{\mathcal{H}_\omega}.$$

Furthermore, the modular conjugation J_η and the positive cone $\mathcal{P}_\eta$ associated with an ω-normal faithful state η obey

$$J_\eta = J_\omega \qquad \text{and} \qquad \mathcal{P}_\eta = \mathcal{P}_\omega \tag{1.4}$$

by [10, Prop. 2.5.30]

Given two faithful ω-normal states $\eta_j = \langle\, \xi_j \mid \pi(\,\cdot\,)\xi_j \,\rangle_{\mathcal{H}_\omega}$, $j = 1, 2$, with $\xi_j \in \mathcal{P}_\omega$ being the unique vector representatives from the natural cone, we define the *relative modular operator*

$$\Delta_{\eta_1,\eta_2} := S_{\eta_1,\eta_2}^* S_{\eta_1,\eta_2}$$

where the anti-linear operator S_{η_1,η_2} is the closure of the map

$$\mathcal{M}_\omega \xi_2 \to \mathcal{M}_\omega \xi_1, \qquad A\xi_2 \mapsto A^*\xi_1. \tag{1.5}$$

The fact that the states η_1 and η_2 are faithful implies that the vectors ξ_1, ξ_2 are separating for $\mathcal{M}_\omega = \pi_\omega(A)''$. Since the vectors ξ_j are chosen from $\mathcal{P}_\omega$ we know from [10, Prop. 2.5.30] that the vectors are cyclic w.r.t. $\mathcal{M}_\omega$ as well. This makes the assignment (1.5) well defined as an anti-linear operator on a dense domain. With the help of the relative modular operator we introduce the notion of *relative entropy* of the state η_2 w.r.t. η_1,

$$\mathrm{Ent}(\eta_2|\eta_1) := \langle\, \xi_2 \,|\, \log\left(\Delta_{\eta_1,\eta_2}\right) \xi_2 \,\rangle_{\mathcal{H}_\omega}\,.$$

The relative entropy of two states not normal w.r.t. each other is set to be $-\infty$. Therefore, relative normality measures how far two states are separated in a entropy sense. Fundamental properties of the relative entropy are

$$\mathrm{Ent}(\eta_2|\eta_1) \leq 0$$

for all relative ω-normal states η_1, η_2 and

$$\mathrm{Ent}(\eta_2|\eta_1) = 0 \qquad \Longleftrightarrow \qquad \eta_1 = \eta_2,$$

c.f. [11, Sect. 6.2.3].

We are going to apply the modular structure to lift the time evolution $\alpha = \{\alpha^t\}_{t\in\mathbb{R}}$ given on the C^*-algebra $\mathcal{A}$ to $\pi_\omega(\mathcal{A})''$. For $(\mathcal{A}, \alpha)$ being a C^*- or W^*-dynamical system and ω a faithful state (which is assumed to be normal in the W^*-algebra context) we find a strongly continuous group $t \mapsto U_\omega(t)$ of unitary operators $U_\omega(t)$ on $\mathcal{H}_\omega$ leaving the positive cone $\mathcal{P}_\omega$ invariant,

$$U_\omega(t)\mathcal{P}_\omega \subseteq \mathcal{P}_\omega \qquad \forall\, t \in \mathbb{R},$$

such that

$$\pi_\omega\left(\alpha^t(A)\right) = U_\omega(t)\, \pi_\omega(A)\, U_\omega(-t) \qquad \forall\, A \in \mathcal{A},$$

c.f. [10, Cor. 2.5.32] The infinitesimal generator

$$L_\omega := \operatorname*{s-lim}_{t\to 0} \frac{U_\omega(t) - \mathbb{1}_{\mathcal{H}_\omega}}{it}$$

is the self-adjoint *standard Liouville operator* or *standard Liouvillean* associated to the dynamical datum $(\mathcal{A}, \alpha, \omega)$. It is worth to note that the Liouville operator anti-commutes with the modular conjugation,

$$L_\omega J_\omega = -J_\omega L_\omega$$

and therefore the group $\{e^{iL_\omega t}\}_{t\in\mathbb{R}}$ commutes with J_ω,

$$e^{iL_\omega t}J_\omega = J_\omega e^{iL_\omega t}$$

because of the anti-linear nature of J_ω.

Because of the invariance of the positive cone $\mathcal{P}_\omega$ under $U_\omega(t) = e^{iL_\omega t}$ and the unique representation of ω-normal states by vectors in $\mathcal{P}_\omega$, the elements of the kernel of the Liouville operator are in a one-to-one correspondence to the α-stationary, ω-normal states. This is meant in the sense that

$$\xi \mapsto \langle\, \xi \,|\, \pi_\omega(\,\cdot\,)\xi \,\rangle_{\mathcal{H}_\omega}$$

is a bijection from the set $\left\{\xi \in \ker(L_\omega) \cap \mathcal{P}_\omega \,\middle|\, \|\xi\|_{\mathcal{H}_\omega} = 1\right\}$ in the set of all α-stationary, ω_0-normal states, c.f. [10, Thm. 2.5.31]. Moreover, it holds

$$\ker(L_\omega) = \{0\} \qquad \Longleftrightarrow \qquad \left\{\eta \in \mathcal{N}_\omega(\mathcal{A}) \,\middle|\, \eta \circ \alpha^t = \eta \,\forall\, t \in \mathbb{R}\right\} = \emptyset \tag{1.6}$$

and

$$\begin{aligned} &\ker(L_\omega) = \mathbb{C}\Psi, \quad \|\Psi\|_{\mathcal{H}_\omega} = 1 \\ \Longleftrightarrow \qquad &\left\{\eta \in \mathcal{N}_\omega(\mathcal{A}) \,\middle|\, \eta \circ \alpha^t = \eta \,\forall\, t \in \mathbb{R}\right\} = \{\langle\, \Psi \,|\, \pi_\omega(\,\cdot\,)\Psi \,\rangle\}. \end{aligned} \tag{1.7}$$

The article [14] provides a good summary of the above connections, we refer the reader in particular to [14, Thm. 2.12]. Therefore, the standard Liouville operator L_ω – or rather its spectrum – is the appropriate object to study the dynamics of ω-normal states. Note, that the Liouville operator for a fixed dynamics α does not change if we build it w.r.t. an ω-normal state η instead w.r.t. ω itself, i.e.,

$$L_\omega = L_\eta.$$

This goes back to (1.4) and the fact that η is represented in terms of π_ω.

So far we did not discuss to which extend information about KMS states is encoded in the Liouville operator. However, the state ω is an α-KMS state to the inverse temperature $\beta > 0$ if and only if α is lifted to the modular automorphism group σ in the sense that for all $A \in \mathcal{A}$

$$\pi_\omega\left(\alpha^{-\beta t}(A)\right) = \Delta_\omega^{it}\pi_\omega(A)\Delta_\omega^{-it}$$

holds. This is equivalent to

$$\Delta_\omega = e^{-\beta L_\omega} \qquad \text{or} \qquad \mathcal{L}_\omega = \beta L_\omega. \tag{1.8}$$

Thus, the operator $\mathcal{L}_\omega$ is the Liouville operator of the time evolution under which ω becomes a KMS state w.r.t. the inverse temperature one.

1.1.3 Local Perturbations and Structural Stability of KMS States

We perform a *local perturbation* on a C^*- or W^*-dynamical system $(\mathcal{A}, \alpha)$. For a self-adjoint element $P = P^* \in \mathcal{A}$ we define the perturbed automorphism group $\alpha_P = \{\alpha_P^t\}_{t\in\mathbb{R}}$ by a *Dyson series*

$$\alpha_P^t(A) := \alpha^t(A) + \sum_{n=1}^{\infty} i^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \left[\alpha^{t_n}(P), \left[\ldots, \left[\alpha^{t_1}(P), \alpha^t(A)\right]\right]\right],$$

which is well defined by [11, Prop. 5.4.1]. For a state ω on $\mathcal{A}$ we build the standard form $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega, \mathcal{P}_\omega, L_\omega)$, where L_ω is the Liouville operator w.r.t. the unperturbed time evolution α. The Liouville operator $L_{P,\omega}$ corresponding to ω w.r.t. the perturbed time evolution α_P is given by

$$L_{P,\omega} := L_\omega + \pi_\omega(P) - \pi'_\omega(P). \tag{1.9}$$

If we assume that ω is an (α, β)-KMS state then the *structural stability* of KMS states, c.f. [11, Thm. 5.4.4], implies that $\Omega_\omega \in \mathcal{D}(e^{-\beta(L_\omega + \pi_\omega(P))/2})$ and that $(\mathcal{H}_\omega, \pi_\omega, \Omega_{P,\omega}, \mathcal{P}_\omega, L_{P,\omega})$ with

$$\Omega_{P,\omega} := \frac{e^{-\beta(L_\omega+\pi_\omega(P))/2}\Omega_\omega}{\|e^{-\beta(L_\omega+\pi_\omega(P))/2}\Omega_\omega\|}$$

is the modular information of the state

$$\omega_P := \langle\, \Omega_{P,\omega} \,|\, \pi_\omega(\,\cdot\,)\Omega_{P,\omega}\,\rangle$$

which is an (α_P, β)-KMS state. This state is the only ω-normal (α_P, β)-KMS state if and only if $\mathcal{M}_\omega = \pi_\omega(\mathcal{A})''$ is a factor, i.e., $\mathcal{M}_\omega \cap \mathcal{M}'_\omega = \mathbb{C}\mathbb{1}_{\mathcal{H}_\omega}$.

Given a faithful ω-normal state $\eta = \langle\, \xi \,|\, \pi_\omega(\,\cdot\,)\xi\,\rangle$ with $\xi \in \mathcal{P}_\omega$ we study the relative entropy of the time evolved state $\eta \circ \alpha_P^t$ w.r.t. the reference state ω which we assume to be invariant under the unperturbed time evolution $\{\alpha^t\}_{t\in\mathbb{R}}$. It is a result from [26, 29] that

$$\begin{aligned} \mathrm{Ent}\left(\eta \circ \alpha_P^t | \omega\right) &= \mathrm{Ent}(\eta|\omega) - \int_0^t ds \,\left\langle\, \xi \,\middle|\, e^{iL_{P,\omega}s}[\pi(P), i\mathcal{L}_\omega]e^{-iL_{P,\omega}s}\xi \,\right\rangle \\ &\equiv \mathrm{Ent}(\eta|\omega) - \int_0^t ds\, \eta \circ \alpha_P^s(\delta_\omega(P)), \end{aligned} \tag{1.10}$$

a relation which we are henceforth referring to as *entropy production formula*, where we implicitly assume that there exists a derivation δ_ω on $\mathcal{A}$ such that P is in its domain and that

$$\pi\left(\delta_\omega(P)\right) = [\pi(P), i\mathcal{L}_\omega]$$

holds. The observable

$$\mathfrak{s}_{P,\omega} := \delta_\omega(P) \tag{1.11}$$

is then called *entropy production rate observable* of the perturbed system w.r.t. the state ω. This name is motivated by the following relation which one obtains by differentiating (1.10),

$$\partial_t \operatorname{Ent}(\eta \circ \alpha_P^t | \omega) = -\eta \circ \alpha_P^t(\mathfrak{s}_{P,\omega}).$$

The *entropy production rate functional*

$$\operatorname{Ep}_\omega : \mathcal{N}_\omega(\mathcal{A}) \to \mathbb{R}, \qquad \operatorname{Ep}_\omega(\eta) := \eta(\mathfrak{s}_{P,\omega})$$

assigns the expectation value of the entropy production rate to an ω_0-normal state.

Having the abstract structure of quantum statistical models at hand we transfer it to the concrete system involved in this work. We proceed with our demonstration as follows. At first, we consider the sub-systems, the particle system (Section 1.2), and the photon reservoir (Section 1.3), separately. We review the zero temperature model of each sub-system by defining a suitable Hilbert space and a Hamilton operator. In a second step we introduce the physically relevant measurands, the algebra of observables, and the set of states on this algebra. The dynamics on observables will be given in the Heisenberg picture generated by the Hamiltonian of the corresponding sub-system. We identify a distinguished state which describes the system at thermal equilibrium at a given temperature, the *Gibbs state*, or, more general, the KMS state. For the equilibrium state we choose a suitable GNS representation of the observables as bounded operators on a suitable Hilbert space, occasionally referred to as *thermal Hilbert space*, and introduce the modular structure.

Equipped with the positive temperature model for each sub-system we plug them together to get all the mathematical objects to describe simultaneously the dynamics of the (so far non-interacting) sub-systems. However, in the case that the sub-systems are at different temperatures the system is in a stationary state but not in an equilibrium state. In a last step we add interaction to the model and study how the subsystems, once at thermal equilibrium, evolve under the interacting dynamics.

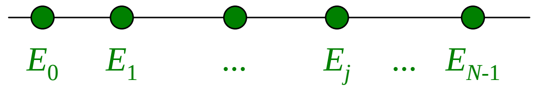

Figure 1.1: Spectrum of the particle Hamiltonian.

1.2 The Particle System

1.2.1 Particle Hilbert Space and Hamiltonian

We choose the particle system to be a simplified model for an atom or a molecule with finite many energy levels (and no continuous spectrum). This model is described on a particle Hilbert space

$$\mathcal{H}_{\mathrm{p}} = \mathbb{C}^N.$$

The particle dynamics is generated by the self-adjoint particle Hamiltonian H_{p} which is diagonalized in an orthonormal basis $\{\varphi_j\}_{j=0,1,\dots,N-1}$ in $\mathcal{H}_{\mathrm{p}}$,

$$H_{\mathrm{p}}\varphi_j = E_j\varphi_j.$$

For convenience we assume that the eigenvalues are increasingly ordered,

$$E_0 \le E_1 \le \cdots \le E_{N-1} \le 0 \tag{1.12}$$

and are represented in the sequence $\{E_j\}_{j=0,1,\dots,N-1}$ corresponding to their degree of degeneracy. In particular, the particle system has a ground state φ_0 with ground state energy E_0. The spectrum of H_{p} is illustrated in Figure 1.1.

1.2.2 Particle Observables and Time Evolution

For the particle system we choose the algebra of observables as

$$\mathcal{A}_{\mathrm{p}} := \mathcal{B}(\mathcal{H}_{\mathrm{p}}),$$

the W^*-algebra of bounded operators on $\mathcal{H}_{\mathrm{p}}$. The set of states on $\mathcal{A}_{\mathrm{p}}$ is given by

$$\mathcal{E}(\mathcal{A}_{\mathrm{p}}) = \left\{\eta \in \mathcal{A}_{\mathrm{p}}^* \,\middle|\, \exists\, \rho = \rho^* \in \mathcal{L}^1(\mathcal{H}_{\mathrm{p}}),\, 0 \le \rho \le 1,\, \mathrm{tr}(\rho) = 1 : \eta = \mathrm{tr}(\rho(\,\cdot\,))\right\},$$

thus all states on $\mathcal{A}_{\mathrm{p}}$ are normal. Therefore, the relative entropy between two different states is finite. We call such a system a *finite system*.

The Hamilton operator H_p induces a one-parameter group $\alpha_\mathrm{p} = \{\alpha_\mathrm{p}^t\}_{t\in\mathbb{R}}$ of automorphisms on $\mathcal{A}_\mathrm{p}$, the time evolution in the Heisenberg picture, via

$$\alpha_\mathrm{p}^t(A) := e^{iH_\mathrm{p}t} A e^{-iH_\mathrm{p}t}.$$

The Planck constant is set one. Note, that α_p is strongly continuous as a family of operators on $\mathcal{A}_\mathrm{p}$ and therefore $(\mathcal{A}_\mathrm{p}, \alpha_\mathrm{p})$ defines a C^*- and W^*-dynamical system.

1.2.3 Gibbs State and GNS Representation

The *entropy* of a finite system in a state $\omega = [A \mapsto \mathrm{tr}(\rho A)]$ for a density matrix ρ is given by

$$S(\omega) := -\,\mathrm{tr}\left(\rho \ln(\rho)\right). \tag{1.13}$$

A thermal equilibrium is characterized as a state which maximizes the entropy under certain constraints. For the *canonical ensemble*, which we are considering here, the mean energy $\omega(H_\mathrm{p})$ is specified (while apparently the concept of particle number fluctuation is not given for a finite system). Maximizing (1.13) under $\omega(H_\mathrm{p}) = \mathrm{const}$ we obtain the equilibrium state ω_p given by

$$\omega_\mathrm{p}(A) := Z^{-1}\,\mathrm{tr}\left(e^{-\beta_\mathrm{p}H_\mathrm{p}}A\right), \tag{1.14}$$

where the Lagrangian multiplier β_p plays the role of the inverse temperature of the particle system and the normalization factor

$$Z \equiv Z(\beta_\mathrm{p}) := \mathrm{tr}\left(e^{-\beta_\mathrm{p}H_\mathrm{p}}\right) \tag{1.15}$$

represents the *partition function*. The equilibrium state (1.14) is referred to as *Gibbs state* in which the energy levels are occupied by a *Boltzmann distribution*. The finiteness of $Z(\beta_\mathrm{p})$ is the criterion for the existence of the Gibbs state and it is the characterization of a so-called *finite system*. We denote by

$$\rho_\mathrm{p} := Z^{-1} e^{-\beta_\mathrm{p}H_\mathrm{p}}$$

the equilibrium density matrix. We note that the Gibbs state (1.14) is time invariant, i.e.,

$$\omega_\mathrm{p} \circ \alpha_\mathrm{p}^t = \omega_\mathrm{p}$$

for all $t \in \mathbb{R}$, because of the cyclicity of the trace and the commutativity of ρ_p with $e^{iH_\mathrm{p}t}$.

To be consistent with the definition of an equilibrium state in Section 1.1.1 we have to verify that the Gibbs state fulfils the KMS condition (1.2). Given two observables $A, B \in \mathcal{A}_\mathrm{p}$ we define the function

$$F_{A,B} : \mathbb{C} \to \mathbb{C}, \qquad F_{A,B}(z) := \omega_\mathrm{p}\left(A\alpha_\mathrm{p}^z(B)\right) = Z^{-1}\,\mathrm{tr}\left(e^{-\beta_\mathrm{p}H_\mathrm{p}} A e^{izH_\mathrm{p}} B e^{-izH_\mathrm{p}}\right)$$

which is obviously an entire function fulfilling $F_{A,B}(t) = \omega_{\mathrm{p}}(A\alpha_{\mathrm{p}}^t(B))$ and also

$$\begin{aligned} F_{A,B}(t+i\beta_{\mathrm{p}}) &= Z^{-1}\,\mathrm{tr}\left(e^{-\beta_{\mathrm{p}}H_{\mathrm{p}}}Ae^{-\beta_{\mathrm{p}}H_{\mathrm{p}}}e^{itH_{\mathrm{p}}}Be^{-itH_{\mathrm{p}}}e^{\beta_{\mathrm{p}}H_{\mathrm{p}}}\right) \\ &= Z^{-1}\,\mathrm{tr}\left(e^{-\beta_{\mathrm{p}}H_{\mathrm{p}}}e^{itH_{\mathrm{p}}}Be^{-itH_{\mathrm{p}}}A\right) \\ &= \omega_{\mathrm{p}}\left(\alpha_{\mathrm{p}}^t(B)A\right) \end{aligned}$$

for all $t \in \mathbb{R}$. The KMS condition therefore generalizes the Gibbs characterization of equilibrium states. For systems with infinite partition function (1.15) (in particular, if the Hamiltonian has continuous spectrum) the Gibbs state cannot be defined and the KMS characterization replaces it.

We go over to write the Gibbs state ω_{p} in its GNS representation. The GNS construction provides us with a faithful $*$-representation π_{p} of the algebra $\mathcal{A}_{\mathrm{p}}$ as bounded operators on a suitable Hilbert space $\mathcal{H}_{\mathrm{p}}^2$ such that ω_{p} can be expressed as a vector state,

$$\omega_{\mathrm{p}}(A) = \langle\, \Omega_{\mathrm{p}} \mid \pi_{\mathrm{p}}(A)\Omega_{\mathrm{p}}\,\rangle_{\mathcal{H}_{\mathrm{p}}^2}\,,$$

with a cyclic and separating (w.r.t. $\pi_{\mathrm{p}}(\mathcal{A}_{\mathrm{p}})''$) vector representative $\Omega_{\mathrm{p}} \in \mathcal{H}_{\mathrm{p}}^2$. We start the GNS construction by defining the Hilbert space $\mathcal{H}_{\mathrm{p}}^2$ on which the representation will act,

$$\mathcal{H}_{\mathrm{p}}^2 := \mathcal{L}^2(\mathcal{H}_{\mathrm{p}}),$$

the space of Hilbert Schmidt operators on $\mathcal{H}_{\mathrm{p}}$. We are aware that for the finite dimensional Hilbert space $\mathcal{H}_{\mathrm{p}}$ the operator spaces $\mathcal{B}(\mathcal{H}_{\mathrm{p}})$ and $\mathcal{L}^2(\mathcal{H}_{\mathrm{p}})$ coincide. However, for the sake of a more general exhibition of the GNS construction for Gibbs states over infinite dimensional Hilbert spaces we differentiate between these two spaces. The Hilbert Schmidt operators naturally form a Hilbert space with inner product

$$\langle\, A \mid B\,\rangle_{\mathcal{H}_{\mathrm{p}}^2} := \mathrm{tr}(A^*B).$$

Next, we introduce the representation π_{p} by

$$\pi_{\mathrm{p}} : \mathcal{A}_{\mathrm{p}} \to \mathcal{B}(\mathcal{H}_{\mathrm{p}}^2), \quad \pi_{\mathrm{p}}(A)B := AB, \tag{1.16}$$

for $A \in \mathcal{A}_{\mathrm{p}} = \mathcal{B}(\mathcal{H}_{\mathrm{p}})$, $B \in \mathcal{H}_{\mathrm{p}}^2 = \mathcal{L}^2(\mathcal{H}_{\mathrm{p}})$. Since $\mathcal{L}^2(\mathcal{H}_{\mathrm{p}})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H}_{\mathrm{p}})$ the representation π_{p} is well defined. It is obvious that π_{p} is injective and it is easy to see that $\pi_{\mathrm{p}}(A)$ is a bounded operator on $\mathcal{H}_{\mathrm{p}}^2$,

$$\begin{aligned} \|\pi_{\mathrm{p}}(A)B\|_{\mathcal{H}_{\mathrm{p}}^2}^2 &= \mathrm{tr}((AB)^*AB) = \mathrm{tr}(A^*ABB^*) \\ &\le \|A^*A\|_{\mathcal{A}_{\mathrm{p}}}\,\mathrm{tr}(B^*B) = \|A\|_{\mathcal{A}_{\mathrm{p}}}^2\,\|B\|_{\mathcal{H}_{\mathrm{p}}^2}^2\,. \end{aligned}$$

We show that π_{p} is a $*$-morphism which immediately implies that π_{p} is norm preserving (see, e.g., [10, Sect. 2.3.1]). Let $A \in \mathcal{A}_{\mathrm{p}}$ and $B, C \in \mathcal{H}_{\mathrm{p}}^2$, then

$$\begin{aligned} \langle\, C \mid \pi_{\mathrm{p}}(A)^*B\,\rangle_{\mathcal{H}_{\mathrm{p}}^2} &= \langle\, \pi_{\mathrm{p}}(A)C \mid B\,\rangle_{\mathcal{H}_{\mathrm{p}}^2} = \mathrm{tr}\left((AC)^*B\right) \\ &= \mathrm{tr}(C^*A^*B) = \langle\, C \mid \pi_{\mathrm{p}}(A^*)B\,\rangle_{\mathcal{H}_{\mathrm{p}}^2}\,. \end{aligned}$$

We distinguish a vector in $\mathcal{H}_\mathrm{p}^2$ by defining

$$\Omega_\mathrm{p} := \rho_\mathrm{p}^{1/2} = Z^{-1/2} e^{-\beta_\mathrm{p} H_\mathrm{p}/2},$$

note that Ω_p is Hilbert Schmidt because ρ_p is trace class, and observe that

$$\langle \Omega_\mathrm{p} \,|\, \pi_\mathrm{p}(A)\Omega_\mathrm{p} \rangle_{\mathcal{H}_\mathrm{p}^2} = \mathrm{tr}\left(\left[\rho_\mathrm{p}^{1/2}\right]^* A \rho_\mathrm{p}^{1/2} \right) = \mathrm{tr}(\rho_\mathrm{p} A) = \omega_\mathrm{p}(A). \tag{1.17}$$

It remains to show that Ω_p is cyclic and separating for $\pi_\mathrm{p}(\mathcal{A}_\mathrm{p})$. To this end consider $0 = \pi_\mathrm{p}(A)\Omega_\mathrm{p} = A\rho_\mathrm{p}^{1/2}$. The operator $\rho_\mathrm{p}^{1/2}$ is invertible on a dense domain, so A vanishes on a dense domain. It follows $A = 0$ by continuity and therefore Ω_p is a separating vector. To show cyclicity consider the projections $P_\nu = \sum_{j=0}^{\nu} |\varphi_j\rangle \langle \varphi_j| \in \mathcal{H}_\mathrm{p}^2$. The set $\{AP_\nu \,|\, A \in \mathcal{A}_\mathrm{p}, \nu = 0, 1, \ldots\}$ is a dense subset of $\mathcal{H}_\mathrm{p}^2$. Note that $\rho_\mathrm{p}^{1/2}$ is bounded invertible on the range of each P_ν and therefore $A\rho_\mathrm{p}^{-1/2} P_\nu$ is an element of $\mathcal{A}_\mathrm{p}$ and $\pi_\mathrm{p}(A\rho_\mathrm{p}^{-1/2} P_\nu)\Omega_\mathrm{p} = AP_\nu$. This shows that Ω_p is cyclic.

It is convenient to modify the representation using the following isometric isomorphism

$$V : \mathcal{L}^2(\mathcal{H}_\mathrm{p}) \to \mathcal{H}_\mathrm{p} \otimes \mathcal{H}_\mathrm{p}, \quad |\phi\rangle \langle \overline{\psi}| \mapsto \phi \otimes \psi,$$

from the Hilbert Schmidt operators into the tensor product of $\mathcal{H}_\mathrm{p}$ with itself. Thereby, $\overline{\psi}$ denotes the complex conjugation of ψ in the basis $\{\varphi_j\}_{j=0,1,\ldots,N-1}$, i.e.,

$$\overline{\sum_{j=0}^{N-1} a_j \varphi_j} := \sum_{j=0}^{N-1} \overline{a_j} \varphi_j.$$

The isomorphism V transforms the GNS representation,

$$V\pi_\mathrm{p}(A)V^{-1}\phi \otimes \psi = V\,|A\phi\rangle \langle \overline{\psi}| = (A\phi) \otimes \psi = (A \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}})\phi \otimes \psi,$$

and the cyclic vector,

$$V\Omega_\mathrm{p} = Z^{-1/2} V \sum_{j=0}^{N-1} e^{-\beta_\mathrm{p} E_j/2} |\varphi_j\rangle \langle \varphi_j| = Z^{-1/2} \sum_{j=0}^{N-1} e^{-\beta_\mathrm{p} E_j/2} \varphi_j \otimes \varphi_j.$$

Identifying $\mathcal{H}_\mathrm{p}^2$ with $\mathcal{H}_\mathrm{p} \otimes \mathcal{H}_\mathrm{p}$, π_p with $V\pi_\mathrm{p}(\,\cdot\,)V^{-1}$, and Ω_p with $V\Omega_\mathrm{p}$ we can express the GNS representation by

$$\pi_\mathrm{p} : \mathcal{A}_\mathrm{p} \to \mathcal{H}_\mathrm{p}^2 = \mathcal{H}_\mathrm{p} \otimes \mathcal{H}_\mathrm{p}, \quad \pi_\mathrm{p}(A) := A \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}},$$

$$\Omega_\mathrm{p} := Z^{-1/2} \sum_{j=0}^{N-1} e^{-\beta_\mathrm{p} E_j/2} \varphi_j \otimes \varphi_j.$$

1.2.4 Modular Structure of the Particle System

The cyclic and separating property of Ω_p w.r.t. $\mathcal{M}_\mathrm{p} := \pi_\mathrm{p}(\mathcal{A}_\mathrm{p})'' = \mathcal{B}(\mathcal{H}_\mathrm{p}) \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}}$ allows us to introduce the anti-linear map

$$S_\mathrm{p} : \mathcal{M}_\mathrm{p}\Omega_\mathrm{p} \to \mathcal{M}_\mathrm{p}\Omega_\mathrm{p}, \qquad (A \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}})\Omega_\mathrm{p} \mapsto (A^* \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}})\Omega_\mathrm{p}.$$

Introducing the anti-unitary operator

$$J_\mathrm{p} : \mathcal{H}_\mathrm{p}^2 \to \mathcal{H}_\mathrm{p}^2, \qquad J_\mathrm{p}(\phi \otimes \psi) := \overline{\psi} \otimes \overline{\phi},$$

and the self-adjoint, positive operator

$$\Delta_\mathrm{p} := e^{-\beta_\mathrm{p} H_\mathrm{p}} \otimes e^{\beta_\mathrm{p} H_\mathrm{p}},$$

we can decompose

$$S_\mathrm{p} = J_\mathrm{p}\Delta_\mathrm{p}^{1/2} \tag{1.18}$$

into the product of the particle modular conjugation and the particle modular operator because of

$$\begin{aligned}
J_\mathrm{p}\Delta_\mathrm{p}^{1/2}(A \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}})\Omega_\mathrm{p} &= J_\mathrm{p}(e^{-\beta_\mathrm{p} H_\mathrm{p}/2} A \otimes e^{\beta_\mathrm{p} H_\mathrm{p}/2})\Omega_\mathrm{p} \\
&= Z^{-1/2} \sum_{j=0}^{N-1} e^{-\beta_\mathrm{p} E_j/2} J_\mathrm{p}(e^{-\beta_\mathrm{p} H_\mathrm{p}/2} A \otimes e^{\beta_\mathrm{p} H_\mathrm{p}/2})(\varphi_j \otimes \varphi_j) \\
&= Z^{-1/2} \sum_{j=0}^{N-1} J_\mathrm{p}(e^{-\beta_\mathrm{p} H_\mathrm{p}/2} A\varphi_j \otimes \varphi_j) \\
&= Z^{-1/2} \sum_{j=0}^{N-1} (\varphi_j \otimes \overline{e^{-\beta_\mathrm{p} H_\mathrm{p}/2} A\varphi_j}) \\
&= Z^{-1/2} \sum_{j,k=0}^{N-1} (\varphi_j \otimes e^{-\beta_\mathrm{p} E_k/2} \overline{\langle \varphi_k \mid A\varphi_j \rangle \varphi_k}) \\
&= Z^{-1/2} \sum_{j,k=0}^{N-1} e^{-\beta_\mathrm{p} E_k/2} (\langle \varphi_j \mid A^*\varphi_k \rangle \, \varphi_j \otimes \varphi_k) \\
&= Z^{-1/2} \sum_{k=0}^{N-1} e^{-\beta_\mathrm{p} E_k/2} (A^*\varphi_k \otimes \varphi_k) \\
&= (A^* \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}})\Omega_\mathrm{p} \\
&= S_\mathrm{p}(A \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}})\Omega_\mathrm{p}.
\end{aligned}$$

Thus, the operator J_p is the modular conjugation and Δ_p the modular operator for the particle system.

We obtain the anti-linear representation

$$\pi'_{\mathrm{p}} : \mathcal{A}_{\mathrm{p}} \to \mathcal{B}(\mathcal{H}_{\mathrm{p}}^2), \qquad \pi'_{\mathrm{p}}(A) := J_{\mathrm{p}}\pi_{\mathrm{p}}(A)J_{\mathrm{p}} = \mathbb{1}_{\mathcal{H}_{\mathrm{p}}} \otimes \overline{A},$$

commuting with π_{p}, where $\overline{A}$ emerges from the matrix A by taking the complex conjugate of all entries w.r.t. the basis $\{\varphi_j\}_{j=0,1,\ldots,N-1}$, i.e.,

$$\left\langle \varphi_k \,\middle|\, \overline{A}\varphi_j \right\rangle := \overline{\langle \varphi_k \mid A\varphi_j \rangle} = \langle \varphi_j \mid A^*\varphi_k \rangle \, .$$

Further, the positive cone associated to $(\pi_{\mathrm{p}}(\mathcal{A}), \Omega_{\mathrm{p}})$ is given by

$$\mathcal{P}_{\mathrm{p}} := \overline{\{\pi_{\mathrm{p}}(A)J_{\mathrm{p}}\pi_{\mathrm{p}}(A)\Omega_{\mathrm{p}} \mid A \in \mathcal{A}_{\mathrm{p}}\}} = \left\{(A \otimes \overline{A})\Omega_{\mathrm{p}} \,\middle|\, A \in \mathcal{A}_{\mathrm{p}} \right\}.$$

1.2.5 Liouville Operator and Thermal Relaxation Properties

The Liouville operator for the particle system can easily be derived from the general relation (1.8) using the fact that ω_{p} is a β-KMS state,

$$L_{\mathrm{p}} := H_{\mathrm{p}} \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{p}}} - \mathbb{1}_{\mathcal{H}_{\mathrm{p}}} \otimes H_{\mathrm{p}},$$

and one easily verifies that

$$\pi_{\mathrm{p}}(\alpha_{\mathrm{p}}^t(A)) = e^{iL_{\mathrm{p}}t}\pi_{\mathrm{p}}(A)e^{-iL_{\mathrm{p}}t}$$

and that the group $e^{iL_{\mathrm{p}}t}$ leaves the positive cone $\mathcal{P}_{\mathrm{p}}$ invariant. The spectral representation of L_{p} reads

$$L_{\mathrm{p}} = \sum_{j,k=0}^{N-1} E_{j,k} \left|\varphi_{j,k}\right\rangle \left\langle\varphi_{j,k}\right| ,$$

where

$$\{\varphi_{j,k} := \varphi_j \otimes \varphi_k\}_{j,k=0,1,\ldots,N-1}$$

is an orthonormal basis in $\mathcal{H}_{\mathrm{p}}^2 = \mathcal{H}_{\mathrm{p}} \otimes \mathcal{H}_{\mathrm{p}}$ and

$$\mathrm{spec}(L_{\mathrm{p}}) = \{E_{j,k} := E_j - E_k \mid j,k = 0,1,\ldots,N-1\}$$

is the spectrum of L_{p}, c.f. Figure 1.2.

The abstract modular theory elaborated in Section 1.1.2 applied to the particle system implies that the stationary states are given by convex combinations of the states

$$\eta_j := \langle \varphi_{j,j} \mid \pi_{\mathrm{p}}(\,\cdot\,)\varphi_{j,j} \rangle_{\mathcal{H}_{\mathrm{p}}^2} \, ,$$

where the vectors $\varphi_{1,1}, \ldots, \varphi_{N-1,N-1}$ build an orthonormal basis of the kernel of the particle Liouville operator L_{p}. Therefore, there exist configurations of the system which are close (in the sense that the relative entropy is finite) to the equilibrium state ω_{p} not thermally relaxing into the equilibrium as time goes by. We say that the finite particle system does *not* feature the *return to equilibrium property.*

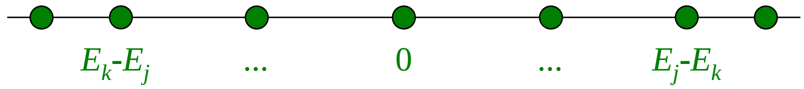

Figure 1.2: Spectrum of the particle Liouvillean.

1.3 The Photon Reservoir

1.3.1 Fock Space, CCR, and Free Field Hamiltonian

We shortly review the description of photons in *second quantization*. Standard references are [11, 38, 43]. The configurations of a single photon are given by square integrable functions over $\mathbb{R}^3$, i.e., the one-photon dynamics can be described in

$$\mathfrak{h}_1 := L^2[\mathbb{R}^3].$$

We neglect that a photon also carries a polarization degree of freedom. As it turned out, the polarization has no influence on the result about thermal relaxation and can be dropped for simplicity. The model we are describing here is rather a model for phonons or scalar bosons in general. However, we will refer to this system as photon or radiation fields – and later simply as heat reservoir. A system of ν undistinguishable photons is described in

$$\mathfrak{h}_\nu := \mathcal{S}_\nu L^2[\mathbb{R}^{3\nu}],$$

where the projection $\mathcal{S}_\nu$ onto absolutely symmetric ν-particle wave functions reflects the fact that photons are bosons and therefore obey the *Bose-Einstein statistics*. The orthogonal projection $\mathcal{S}_\nu$ is given by

$$[\mathcal{S}_\nu \psi_\nu](\vec{k}_1, \ldots, \vec{k}_\nu) := \frac{1}{\nu!} \sum_{\pi \in S_\nu} \psi_\nu(\vec{k}_{\pi 1}, \ldots, \vec{k}_{\pi \nu})$$

where $\psi_\nu \in \mathfrak{h}_\nu$ and $\vec{k}_j \in \mathbb{R}^3$ is the momentum variable of the j^{th} particle (we choose the momentum representation for convenience) and S_ν denotes the symmetric group of permutations of ν elements. The quantized photon field is realized on the *bosonic Fock space* $\mathcal{F}(\mathfrak{h}_1)$ over the one-particle space $\mathfrak{h}_1$,

$$\mathcal{H}_{\mathrm{f}} := \mathcal{F}(\mathfrak{h}_1) = \bigoplus_{\nu=0}^{\infty} \mathcal{S}_\nu \Big[\underbrace{\mathfrak{h}_1 \otimes \cdots \otimes \mathfrak{h}_1}_{n\text{-times}} \Big] = \bigoplus_{\nu=0}^{\infty} \mathfrak{h}_\nu, \tag{1.19}$$

understood as the set of all sequences $(\psi_\nu \in \mathfrak{h}_\nu)_{\nu \in \mathbb{N}_0}$ which are square summable, i.e.,

$$\sum_{\nu=0}^{\infty} \|\psi_\nu\|_{\mathfrak{h}_\nu}^2 < \infty.$$

The norm $\|\cdot\|_{\mathfrak{h}_\nu}$ is the usual L^2-norm in $\mathfrak{h}_\nu$. The zero particle (or vacuum) sector $\mathfrak{h}_0$ is set to be the complex field,

$$\mathfrak{h}_0 := \mathbb{C},$$

and is spanned by the vacuum vector

$$\Omega_{\mathrm{vac}} := (1, 0, 0, \dots) \in \mathfrak{h}_0.$$

The Fock space (1.19) equipped with the canonical inner product

$$\langle\, (\psi_\nu \in \mathfrak{h}_\nu)_\nu \,|\, (\varphi_\nu \in \mathfrak{h}_\nu)_\nu \,\rangle_{\mathcal{H}_{\mathrm{f}}} := \sum_{\nu=0}^{\infty} \langle\, \psi_\nu \,|\, \varphi_\nu \,\rangle_{\mathfrak{h}_\nu}$$

becomes a Hilbert space, where

$$\langle\, \psi_\nu \,|\, \varphi_\nu \,\rangle_{\mathfrak{h}_\nu} := \int_{\mathbb{R}^{3\nu}} \overline{\psi_\nu(k)} \varphi_\nu(k)\, d^{3\nu}k$$

is the usual L^2 inner product in $\mathfrak{h}_\nu$. The bosonic Fock space is the configuration space of a single photon field at zero temperature.

The Fock space $\mathcal{H}_{\mathrm{f}} = \mathcal{F}(\mathfrak{h}_1)$ carries a representation of the *canonical commutation relations (CCR)*. We introduce the algebra of *creation and annihilation operators*. Given a single photon state $f \in \mathfrak{h}_1$ and a ν-photon state $\psi_\nu \in \mathfrak{h}_\nu$ the creation operator $a^*(f)$ maps ψ_ν to a $(\nu+1)$-photon state by "adding" f to the ν-particle configuration via

$$a^*(f)\psi_\nu := \sqrt{\nu+1}\, \mathcal{S}_{\nu+1}\, [f \otimes \psi_\nu] \in \mathfrak{h}_{\nu+1} \tag{1.20}$$

and the annihilation operator $a(f)$ maps ψ_ν to a $(\nu-1)$-photon state by eliminating f from the ν-particle configuration via

$$a(f)\psi_\nu := \Big[(\vec{k}_1, \dots, \vec{k}_{\nu-1}) \mapsto \sqrt{\nu} \left\langle f \,\middle|\, \psi_\nu(\,\cdot\,, \vec{k}_1, \dots, \vec{k}_{\nu-1}) \right\rangle_{\mathfrak{h}_1} \Big] \in \mathfrak{h}_{\nu-1}. \tag{1.21}$$

We make the convention that creation and annihilation operators act on the vacuum state Ω_{vac} as follows,

$$\begin{aligned} a^*(f)\Omega_{\mathrm{vac}} &:= f \in \mathfrak{h}_1, \\ a(f)\Omega_{\mathrm{vac}} &:= 0. \end{aligned} \tag{1.22}$$

The definitions (1.20, 1.21) lead to the following bounds of the creation and annihilation operators,

$$\begin{aligned} \|a^*(f)\psi_\nu\|_{\mathfrak{h}_{\nu+1}} &\leq \sqrt{\nu+1}\ \|f\|_{\mathfrak{h}_1}\ \|\psi_\nu\|_{\mathfrak{h}_\nu} \\ \|a(f)\psi_\nu\|_{\mathfrak{h}_{\nu-1}} &\leq \sqrt{\nu}\ \|f\|_{\mathfrak{h}_1}\ \|\psi_\nu\|_{\mathfrak{h}_\nu} \end{aligned} \tag{1.23}$$

for $\psi_\nu \in \mathfrak{h}_\nu$. We extend the creation and annihilation operators to closed operators on the dense domain

$$\mathcal{D}(a^*(f)) = \mathcal{D}(a(f)) = \mathcal{D}(N_\mathrm{f}^{1/2}) := \left\{ (\psi_\nu \in \mathfrak{h}_\nu)_\nu \in \mathcal{F} \,\middle|\, \sum_{\nu=0}^{\infty} \nu\, \|\psi_\nu\|^2_{\mathfrak{h}_\nu} < \infty \right\}$$

by defining

$$a^\#(f)(\psi_\nu \in \mathfrak{h}_\nu)_\nu := (a^\#(f)\psi_\nu \in \mathfrak{h}_{\nu\pm1})_\nu,$$

where $a^\#(f)$ either stands for $a(f)$ or $a^*(f)$ (subsequently we will use the sharp symbol # without further remarks to make statements about creation and annihilation operators at the same time). It is easy to show that $a^*(f)$ is the adjoint operator of $a(f)$ and vice versa, c.f. [11, 35, 38]. We remark that, by (1.23), the creation and annihilation operators are relatively bounded w.r.t. the square root of the *photon number operator* N_f,

$$\begin{aligned} \left\|a^*(f)(N_\mathrm{f}+1)^{-1/2}\right\|_{\mathcal{F}(\mathfrak{h}_1)} &\leq \|f\|_{\mathfrak{h}_1}\,, \\ \left\|a(f)(N_\mathrm{f}+1)^{-1/2}\right\|_{\mathcal{F}(\mathfrak{h}_1)} &\leq \|f\|_{\mathfrak{h}_1}\,, \end{aligned} \tag{1.24}$$

where the number operator acts on a vector $\psi = (\psi_\nu \in \mathfrak{h}_\nu)_\nu$ from

$$\mathcal{D}(N_\mathrm{f}) := \left\{ (\psi_\nu \in \mathfrak{h}_\nu)_\nu \in \mathcal{F} \,\middle|\, \sum_{\nu=0}^{\infty} \|\nu\psi_\nu\|^2_{\mathfrak{h}_\nu} < \infty \right\}$$

as

$$N_\mathrm{f}\psi := (\nu\psi_\nu \in \mathfrak{h}_\nu)_\nu.$$

Further, the creation and annihilation operators fulfil the following *commutation relations*, the *CCR*,

$$\begin{aligned} [a^*(f), a^*(g)] &= [a(f), a(g)] = 0, \\ [a(f), a^*(g)] &= \langle\, f \mid g \,\rangle_{\mathfrak{h}_1}\,. \end{aligned} \tag{1.25}$$

We remark that (1.20, 1.22) imply the density of the set

$$\operatorname{span}\{a^*(f_n)\cdots a^*(f_1)\Omega_\mathrm{vac} \mid f_1,\ldots,f_n \in \mathfrak{h}_1, n \in \mathbb{N}_0\} \overset{\text{dense}}{\subseteq} \mathcal{H}_\mathrm{f} \tag{1.26}$$

in the Fock space.

Let now $\psi_\nu \in \mathfrak{h}_\nu$ be a C_0^∞-function, that is, a smooth function with compact support. For a concrete momentum $\vec{k}$ we define the "pointwise" annihilation operator $a(\vec{k})$ which eliminates a photon with the given momentum $\vec{k}$ (rather than a momentum distribution f) from the ν-photon configuration ψ_ν by

$$a(\vec{k})\psi_\nu := \sqrt{\nu}\,\psi_\nu(\vec{k},\,\cdot\,) \in \mathfrak{h}_{\nu-1}. \tag{1.27}$$

From the Equations (1.21, 1.27) one can conclude that

$$a(f) = \int_{\mathbb{R}^3} \overline{f(\vec{k})} a(\vec{k})\, d^3\vec{k}, \tag{1.28}$$

in a strong sense, initially on C_0^∞-functions in $\mathfrak{h}_\nu$, but can be extended to all states in $\mathcal{D}(N_{\mathrm{f}}^{1/2})$. The operator $a^*(\vec{k})$ which creates a photon with a concrete momentum $\vec{k}$ can be defined in the form sense as

$$\left\langle \varphi \,\middle|\, a^*(\vec{k})\psi \right\rangle_{\mathcal{F}(\mathfrak{h}_1)} := \left\langle a(\vec{k})\varphi \,\middle|\, \psi \right\rangle_{\mathcal{F}(\mathfrak{h}_1)}.$$

The operators $a^*(f)$ and $a^*(\vec{k})$ are related via

$$a^*(f) = \int_{\mathbb{R}^3} f(\vec{k}) a^*(\vec{k})\, d^3\vec{k}, \tag{1.29}$$

to be understood in the form (or weak) sense. The objects $a^*(\vec{k})$ and $a(\vec{k})$ are operator valued distributions and can be understood formally as the operators defined in (1.20) and (1.21) with a Dirac delta momentum distribution $f = \delta((\,\cdot\,) - \vec{k})$. We refer the reader to [35, 38, 44] for a more detailed discussion on this subject. We note that the CCR (1.25) translate to

$$\begin{aligned} \left[a^*(\vec{k}), a^*(\vec{k}')\right] &= \left[a(\vec{k}), a(\vec{k}')\right] = 0, \\ \left[a(\vec{k}), a^*(\vec{k}')\right] &= \delta(\vec{k} - \vec{k}'), \end{aligned} \tag{1.30}$$

the version of the CCR for the "pointwise" creation and annihilation operators.

The relativistic energy-momentum relation, the *dispersion relation*, for a free massless photon reads

$$\omega(\vec{k}) = \sqrt{\vec{k}^2} = |\vec{k}|. \tag{1.31}$$

The energy operator for a non-interacting photon field can be obtained by lifting the dispersion relation (1.31) to the Fock space $\mathcal{F}(\mathfrak{h}_1)$ by second quantization,

$$H_{\mathrm{f}} := d\Gamma(\omega) \equiv \int_{\mathbb{R}^3} a^*(\vec{k})\omega(\vec{k})a(\vec{k})\, d^3\vec{k}.$$

Figure 1.3: Spectrum of the free field Hamiltonian.

The free field Hamiltonian H_f acts on a ν-photon state $\psi_\nu \in \mathfrak{h}_\nu$ as

$$[H_\mathrm{f}\psi_\nu](\vec{k}_1, \dots, \vec{k}_\nu) = \Big(\omega(\vec{k}_1) + \dots + \omega(\vec{k}_\nu)\Big)\, \psi_\nu(\vec{k}_1, \dots, \vec{k}_\nu).$$

Zero is the only eigenvalue of H_f and its kernel is spanned by the Fock vacuum Ω_vac. The rest of the spectrum covers the positive real axis and is absolutely continuous away from zero, see Figure 1.3. The Hamiltonian is self-adjoint on its natural domain

$$\begin{aligned}\mathcal{D}(H_\mathrm{f}) \;:=\; & \Big\{(\psi_\nu \in \mathfrak{h}_\nu)_\nu \in \mathcal{H}_\mathrm{f} \,\Big| \\ & \sum_{\nu=1}^{\infty} \int_{\mathbb{R}^{3\nu}} \Big| \Big(\omega(\vec{k}_1) + \dots + \omega(\vec{k}_\nu)\Big)\, \psi_\nu(\vec{k}_1, \dots, \vec{k}_\nu)\Big|^2 \, d^{3\nu}k < \infty \Big\}.\end{aligned}$$

At the end of this section we establish a tool which will be useful for later computations. Given a measurable function $F : \mathbb{R} \to \mathbb{C}$ we can build by functional calculus a closed operator $F(H_\mathrm{f})$ defined on

$$\begin{aligned}\mathcal{D}(F(H_\mathrm{f})) \;:=\; & \Big\{(\psi_\nu \in \mathfrak{h}_\nu)_\nu \in \mathcal{F} \,\Big| \\ & \sum_{\nu=1}^{\infty} \int_{\mathbb{R}^{3\nu}} \Big| F\Big(\omega(\vec{k}_1) + \dots + \omega(\vec{k}_\nu)\Big)\, \psi_\nu(\vec{k}_1, \dots, \vec{k}_\nu)\Big|^2 \, d^{3\nu}k < \infty \Big\}\end{aligned}$$

which is acting on a ν-photon state ψ_ν as follows,

$$[F(H_\mathrm{f})\psi_\nu](\vec{k}_1, \dots, \vec{k}_\nu) = F\Big(\omega(\vec{k}_1) + \dots + \omega(\vec{k}_\nu)\Big)\, \psi_\nu(\vec{k}_1, \dots, \vec{k}_\nu).$$

Although the operator $F(H_\mathrm{f})$ is not commuting with $a(\vec{k})$ and $a^*(\vec{k})$, resp., there is a simple relation which allows us to interchange the order of applications on vectors of the Fock space. Given a ν-photon state $\psi_\nu \in \mathfrak{h}_\nu$ we have

$$\begin{aligned}
&\left[a(\vec{k})F(H_{\mathrm{f}})\psi_\nu\right](\vec{k}_1,\ldots,\vec{k}_{\nu-1})\\
&= \sqrt{\nu}\,\left[F(H_{\mathrm{f}})\psi_\nu\right](\vec{k},\vec{k}_1,\ldots,\vec{k}_{\nu-1})\\
&= \sqrt{\nu}\,F\left(\omega(\vec{k})+\omega(\vec{k}_1)+\cdots+\omega(\vec{k}_{\nu-1})\right)\psi_\nu(\vec{k},\vec{k}_1,\ldots,\vec{k}_{\nu-1})\\
&= F\left(\omega(\vec{k})+\omega(\vec{k}_1)+\cdots+\omega(\vec{k}_{\nu-1})\right)\left[a(\vec{k})\psi_\nu\right](\vec{k}_1,\ldots,\vec{k}_{\nu-1})\\
&= \left[F\left(H_{\mathrm{f}}+\omega(\vec{k})\right)a(\vec{k})\psi_\nu\right](\vec{k}_1,\ldots,\vec{k}_{\nu-1}).
\end{aligned}$$

This computation and a similarly one for $F(H_{\mathrm{f}})a^*(\vec{k})$ show that

$$\begin{aligned}
a(\vec{k})F(H_{\mathrm{f}}) &= F\left(H_{\mathrm{f}}+\omega(\vec{k})\right)a(\vec{k}),\\
a^*(\vec{k})F\left(H_{\mathrm{f}}+\omega(\vec{k})\right) &= F(H_{\mathrm{f}})a^*(\vec{k}),
\end{aligned} \tag{1.32}$$

on suitable domains, a relation which is known as *pull through formula*.

The construction of a Fock space over a general L^2-space or even over an abstract Hilbert space is explained in [11, Sect. 5.2.1-5.2.2]. Since all the concepts are literally the same as in the concrete example presented in this section we spare a further discussion on this topic and refer to the mentioned monograph for more details.

1.3.2 Dynamics on the Weyl Algebra

The measurements on the photon field correspond to the *field operators*

$$\Phi(f) := \frac{1}{\sqrt{2}}\left[a^*(f)+a(f)\right]$$

and their *canonical conjugated "momentum" operators*

$$\Pi(f) := \Phi(if) = \frac{i}{\sqrt{2}}\left[a^*(f)-a(f)\right].$$

These operators operators extend to self-adjoint operators on $\mathcal{H}_{\mathrm{f}}$. However, they are unbounded and do not establish a C^*-algebra of operators. This motivates us to go over to the *Weyl operators*

$$W(f) := e^{i\Phi(f)},$$

for $f \in \mathfrak{h}_1$, which are unitary on $\mathcal{H}_{\mathrm{f}}$ because of the self-adjointness of $\Phi(f)$. We define our algebra of photon observables as the C^*-algebra generated by $W(f)$,

$$\mathcal{A}_{\mathrm{f}} := \mathcal{W}(\mathcal{D}_{\mathrm{f}}) \equiv \overline{\operatorname{span}\left\{W(f)\mid f\in\mathcal{D}_{\mathrm{f}}\right\}}^{\|\cdot\|_{\mathcal{B}(\mathcal{H}_{\mathrm{f}})}}, \tag{1.33}$$

where $\mathcal{W}(\mathcal{D}_\mathrm{f})$ is the *Weyl algebra* over the dense set

$$\mathcal{D}_\mathrm{f} := \left\{ f \in \mathfrak{h}_1 \,\middle|\, \omega^{-1/2} f \in \mathfrak{h}_1 \right\} \tag{1.34}$$

of allowed form factors, and the closure in (1.33) is taken in the uniform norm $\|\cdot\|_{\mathcal{B}(\mathcal{H}_\mathrm{f})}$ of bounded operators on $\mathcal{H}_\mathrm{f}$. The additional assumption on the infrared behavior, $\omega^{-1/2} f \in \mathfrak{h}_1$, is necessary to define an equilibrium state on $\mathcal{A}_\mathrm{f}$ as we will see later. Note that the field operators can be approximated in a strong sense by linear combinations of Weyl operators since $\Phi(f)$ is the infinitesimal generator of the strongly continuous one parameter group $\mathbb{R} \ni t \mapsto W(tf)$.

The Weyl operators inherit the CCR from the creation and annihilation operators, they read

$$W(f)W(g) = e^{-\frac{i}{2}\mathrm{Im}\langle f \,|\, g\rangle_{\mathfrak{h}_1}} W(f+g) = e^{-i\mathrm{Im}\langle f \,|\, g\rangle_{\mathfrak{h}_1}} W(g)W(f), \tag{1.35}$$

for a proof see [11, Prop. 5.2.4]. It is worth to mention that the vacuum vector Ω_vac is a cyclic vector for the Weyl algebra $\mathcal{A}_\mathrm{f}$. This follows from (1.26) and the fact that the creation operator $a^*(f)$ can be expressed in terms of the infinitesimal generator of $t \mapsto W(tf)$ as

$$a^*(f) = \frac{1}{\sqrt{2}}[\Phi(f) - i\Phi(if)].$$

For computations to come, we also compute the vacuum expectation value of a Weyl operator. To this end we first remark that any ν-photon state $\psi_\nu \in \mathfrak{h}_\nu$ is an analytic vector for $\Phi(f)$, it holds due to (1.23)

$$\begin{aligned} \|\Phi(f)^n \psi_\nu\|_{\mathcal{H}_\mathrm{f}} &\leq \frac{2\,\|f\|_{\mathfrak{h}_1}}{\sqrt{2}} \sqrt{n+\nu+1}\, \left\|\Phi(f)^{n-1}\psi_\nu\right\|_{\mathcal{H}_\mathrm{f}} \\ &\leq \left(\sqrt{2}\,\|f\|_{\mathfrak{h}_1}\right)^n \sqrt{\frac{(n+\nu+1)!}{(\nu+1)!}}\, \|\psi_\nu\|_{\mathfrak{h}_\nu} \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{|s|^n}{n!} \|\Phi(f)^n \psi_\nu\|_{\mathcal{H}_\mathrm{f}} &\leq \|\psi_\nu\|_{\mathfrak{h}_\nu} \sum_{n=0}^{\infty} \frac{\left(\sqrt{2}\,|s|\,\|f\|_{\mathfrak{h}_1}\right)^n}{\sqrt{n!}} \sqrt{\frac{(n+\nu+1)!}{n!(\nu+1)!}} \\ &\leq 2^{(\nu+1)/2}\, \|\psi_\nu\|_{\mathfrak{h}_\nu} \sum_{n=0}^{\infty} \frac{\left(2|s|\,\|f\|_{\mathfrak{h}_1}\right)^n}{\sqrt{n!}} \\ &< \infty \end{aligned} \tag{1.36}$$

for all $s \in \mathbb{C}$. In particular Ω_{vac} is an analytic vector such that we may compute

$$\begin{aligned} \langle \Omega_{\text{vac}} \,|\, W(f)\Omega_{\text{vac}} \rangle_{\mathcal{H}_{\text{f}}} &= \sum_{j=0}^{\infty} \frac{i^j}{j!} \left\langle \Omega_{\text{vac}} \,\middle|\, \Phi(f)^j \Omega_{\text{vac}} \right\rangle_{\mathcal{H}_{\text{f}}} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \left\langle \Omega_{\text{vac}} \,\middle|\, \Phi(f)^{2j} \Omega_{\text{vac}} \right\rangle_{\mathcal{H}_{\text{f}}} = \exp\left(-\frac{1}{4}\,\|f\|^2_{\mathfrak{h}_1}\right), \end{aligned} \tag{1.37}$$

where we used $\langle \Omega_{\text{vac}} \,|\, \Phi(f)^{2j+1}\Omega_{\text{vac}} \rangle_{\mathcal{H}_{\text{f}}} = 0$ and

$$\begin{aligned} \left\langle \Omega_{\text{vac}} \,\middle|\, \Phi(f)^{2j}\Omega_{\text{vac}} \right\rangle_{\mathcal{H}_{\text{f}}} &= \frac{1}{\sqrt{2}} \left\langle \Omega_{\text{vac}} \,\middle|\, \Phi(f)^{2j-1} a^*(f)\Omega_{\text{vac}} \right\rangle_{\mathcal{H}_{\text{f}}} \\ &= \frac{1}{\sqrt{2}} \left\langle \Omega_{\text{vac}} \,\middle|\, \Phi(f)^{2j-2} a^*(f)\Phi(f)\Omega_{\text{vac}} \right\rangle_{\mathcal{H}_{\text{f}}} \\ &\quad + \frac{\|f\|^2_{\mathfrak{h}_1}}{2} \left\langle \Omega_{\text{vac}} \,\middle|\, \Phi(f)^{2j-2}\Omega_{\text{vac}} \right\rangle_{\mathcal{H}_{\text{f}}} \\ &= \ldots = \frac{2j-1}{2}\,\|f\|^2_{\mathfrak{h}_1} \left\langle \Omega_{\text{vac}} \,\middle|\, \Phi(f)^{2j-2}\Omega_{\text{vac}} \right\rangle_{\mathcal{H}_{\text{f}}} \\ &= \ldots = \frac{(2j)!}{j!4^j}\,\|f\|^{2j}_{\mathfrak{h}_1}. \end{aligned}$$

and therein the commutation relation

$$[\Phi(f), a^*(f)] = \frac{\|f\|^2_{\mathfrak{h}_1}}{\sqrt{2}}.$$

The field Hamiltonian H_{f} generates the free time evolution $\{\alpha_{\text{f}}^t\}_{t\in\mathbb{R}}$ on the Weyl algebra $\mathcal{A}_{\text{f}}$ by

$$\alpha_{\text{f}}^t(A) := e^{iH_{\text{f}}t} A e^{-iH_{\text{f}}t}. \tag{1.38}$$

For a Weyl operator $W(f)$, $f \in \mathfrak{h}_1$, we can express the time evolution explicitly using the pull through formula (1.32). Note first that

$$\begin{aligned} e^{iH_{\text{f}}t}\Phi(f)e^{-iH_{\text{f}}t} &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} e^{iH_{\text{f}}t} \left(f(\vec{k})a^*(\vec{k}) + \overline{f(\vec{k})}a(\vec{k}) \right) e^{-iH_{\text{f}}t}\, d^3\vec{k} \\ &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} \left(e^{i\omega(\vec{k})t} f(\vec{k})a^*(\vec{k}) + e^{-i\omega(\vec{k})t}\overline{f(\vec{k})}a(\vec{k}) \right) d^3\vec{k} \\ &= \Phi(e^{i\omega t} f). \end{aligned}$$

Further, we remark that the vectors $\psi = (\psi_\nu \in \mathfrak{h}_\nu)_{\nu=0,\ldots,m}$ of the Fock space which overlap with at most finitely many ν-photons sectors $\mathfrak{h}_\nu$ build a dense subset of analytic vectors for $\Phi(f)$, due to (1.36). For such a vector we may calculate

$$e^{iH_\mathrm{f}t}W(f)e^{-iH_\mathrm{f}t}\psi = \sum_{j=0}^{\infty} e^{iH_\mathrm{f}t}\frac{[i\Phi(f)]^j}{j!}e^{-iH_\mathrm{f}t}\psi = \sum_{j=0}^{\infty}\frac{[i\Phi(e^{i\omega t}f)]^j}{j!}\psi = W(e^{i\omega t}f)\psi$$

and get the time transformation law for Weyl operators as

$$\alpha_\mathrm{f}^t(W(f)) = W(e^{i\omega t}f),$$

known as *Bogoliubov transformation*. This shows that, indeed, $\{\alpha_\mathrm{f}^t\}_{t\in\mathbb{R}}$ is a group of automorphisms $\alpha_\mathrm{f}^t : \mathcal{A}_\mathrm{f} \to \mathcal{A}_\mathrm{f}$. Although the map $\mathfrak{h}_1 \ni f \mapsto W(f)$ is strongly continuous, i.e., for any $\psi \in \mathcal{H}_\mathrm{f}$ we have $W(f_n)\psi \to W(f)\psi$ whenever $f_n \to f$ in $\mathfrak{h}_1$, the group $\{\alpha_\mathrm{f}^t\}_{t\in\mathbb{R}}$ is not strongly continuous. This is due to the fact that

$$\|W(f) - \mathbb{1}_{\mathcal{H}_\mathrm{f}}\|_{\mathcal{B}(\mathcal{H}_\mathrm{f})} = 2 \quad \forall f \in \mathfrak{h}_1 \setminus \{0\},$$

see [11, Prop. 5.2.4].

1.3.3 Thermodynamic Limit and KMS State

In this section we are going to investigate the thermal equilibrium of a free photon gas of finite density at a positive temperature $T_\mathrm{f} = 1/\beta_\mathrm{f}$. The task is to introduce a representation of the CCR corresponding to a KMS state which describes a photon configuration with finite energy and particle density, in particular we study the situation away from *Bose-Einstein condensation*.

The infinite extension of the photon system is reflected by the existence of continuous spectrum of H_f which is, as we will see later, responsible for dissipative effects, i.e., energy transport to infinity. Unlike in the case of the finite particle system the operator $e^{-\beta_\mathrm{f}H_\mathrm{f}}$ is no longer trace class. Thus, it is not possible to define a Gibbs state on $\mathcal{A}_\mathrm{f}$. We therefore refer to the photon system as a *infinite system*. The construction of a KMS state for a photon gas of finite density uses the *thermodynamic limit* process which will be illustrated in the following. We use the publications [2, 18, 31] and [11, Sect. 5.2.5] as a guideline. The cyclic representation corresponding to the KMS state which results from the subsequent considerations was first derived by Araki and Woods in [2].

The occupation of configurations of photons in the infinite extended position space $\mathbb{R}^3$ is considered as an inductive limit process where we restrict the photon positions to increasing bounded regions $\Lambda \subseteq \mathbb{R}^3$. Each confined system represents a finite system which allows the Gibbs classification of a thermal equilibrium state. A controlled limit process which lets the box Λ grow to eventually include every bounded region in $\mathbb{R}^3$ yields an equilibrium state of the infinitely extended system.

For a bounded region $\Lambda \subseteq \mathbb{R}^3$ with a sufficiently regular boundary $\partial\Lambda$ we introduce the configuration Hilbert space of the confined photon field as

$$\mathcal{H}_{\mathrm{f}}^{(\Lambda)} := \mathcal{F}(\mathfrak{h}_1^{(\Lambda)})$$

which is the bosonic Fock space over the one-photon Hilbert space

$$\mathfrak{h}_1^{(\Lambda)} := \left\{ f \in \mathfrak{h}_1 \,\middle|\, \hat{f} = 0 \quad \text{a.e. on} \quad \mathbb{R}^3 \setminus \Lambda \right\}$$

of wave functions f (in the momentum representation) whose Fourier transform $\hat{f}$ is restricted to the region Λ, i.e., the photon is confined in position to Λ. A usual choice for Λ would be a box $\Lambda = [-\ell, \ell]^3$ of finite side length $2\ell > 0$.

The one-photon Hamilton operator

$$h_{\mathrm{f}}^{(\Lambda)} := H_{\mathrm{f}} \restriction_{\mathfrak{h}_1^{(\Lambda)}}$$

for the confined system is the restriction of H_{f} to $\mathfrak{h}_1^{(\Lambda)}$. We note that the Fourier transform of $h_{\mathrm{f}}^{(\Lambda)}$ is the square root of the negative Laplace operator, $\widehat{h_{\mathrm{f}}^{(\Lambda)}} = \sqrt{-\Delta}$, restricted to $L^2(\Lambda)$. Imposing classical boundary conditions the operator $h_{\mathrm{f}}^{(\Lambda)}$ extends to a self-adjoint operator with discrete spectrum. For $\Lambda = [-\ell, \ell]^3$ being a box one usually imposes periodic boundary conditions.

The extension apparently depends on the boundary conditions which describe the interaction of the photon gas with the "walls" $\partial\Lambda$ of the box where it is captured. Consequently, the dynamical behavior of the gas will depend on the boundary conditions.

Further, the operator $e^{-\beta_{\mathrm{f}} h_{\mathrm{f}}^{(\Lambda)}}$ is trace class on $\mathfrak{h}_1^{(\Lambda)}$ for any positive $\beta_{\mathrm{f}} > 0$. Choosing a *chemical potential* $\mu_{\mathrm{f}} \in \mathbb{R}$ such that the shifted one-photon Hamiltonian is stricլty positive, i.e., $h_{\mathrm{f}}^{(\Lambda)} - \mu_{\mathrm{f}} \mathbb{1}_{\mathfrak{h}_1^{(\Lambda)}} \geq C \mathbb{1}_{\mathfrak{h}_1^{(\Lambda)}} > 0$, we obtain by second quantization (w.r.t. the Fock space $\mathcal{F}(\mathfrak{h}_1^{(\Lambda)})$) an operator

$$K_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})} := d\Gamma_{\mathfrak{h}_1^{(\Lambda)}} \left(h_{\mathrm{f}}^{(\Lambda)} - \mu_{\mathrm{f}} \mathbb{1}_{\mathfrak{h}_1^{(\Lambda)}} \right) = H_{\mathrm{f}}^{(\Lambda)} - \mu_{\mathrm{f}} N_{\mathrm{f}}^{(\Lambda)}$$

which is trace class on $\mathcal{F}(\mathfrak{h}_1^{(\Lambda)})$ by [11, Prop. 5.2.27]. This operator is referred to as *generalized Hamilton operator*. Here, the operator

$$N_{\mathrm{f}}^{(\Lambda)} := d\Gamma_{\mathfrak{h}_1^{(\Lambda)}} \left(\mathbb{1}_{\mathfrak{h}_1^{(\Lambda)}} \right)$$

is the photon number operator and

$$H_{\mathrm{f}}^{(\Lambda)} := d\Gamma_{\mathfrak{h}_1^{(\Lambda)}} \left(h_{\mathrm{f}}^{(\Lambda)} \right)$$

is the Hamilton operator of the photon gas confined to Λ. The operator $h_{\mathrm{f}}^{(\Lambda)} - \mu_{\mathrm{f}}$ describes the same physics on the one-photon sector as $h_{\mathrm{f}}^{(\Lambda)}$ does. Hence, we use the generalized Hamiltonian $K_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}$ being the second quantization of $h_{\mathrm{f}}^{(\Lambda)} - \mu_{\mathrm{f}}$ to implement the dynamics of the photon gas on the Fock space $\mathcal{F}(\mathfrak{h}_1^{(\Lambda)})$.

The finiteness of the partition function,

$$Z^{(\Lambda,\mu_{\mathrm{f}})}(\beta_{\mathrm{f}}) := \operatorname{tr}\left(e^{-\beta_{\mathrm{f}} K_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}}\right) < \infty,$$

reflects that the confined gas is a finite system so that we can define the *Gibbs grand canonical ensemble state*. For the grand canonical ensemble not only the energy but also the photon number are subject to fluctuations such that it only makes sense to specify the expectation value of energy and photon number in a given state. The thermal equilibrium is distinguished as the state $\omega : A \mapsto \operatorname{tr}(\rho A)$ which maximizes the entropy functional

$$S(\omega) = -\operatorname{tr}(\rho \ln(\rho))$$

under the constraints

$$\omega\left(H_{\mathrm{f}}^{(\Lambda)}\right) = \text{const}, \qquad \omega\left(N_{\mathrm{f}}^{(\Lambda)}\right) = \text{const}. \tag{1.39}$$

It turns out that the equilibrium state $\omega_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}$ of the free photon gas in the box Λ at inverse temperature β_{f} is given by

$$\omega_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}(A) := Z^{(\Lambda,\mu_{\mathrm{f}})}(\beta_{\mathrm{f}})^{-1} \operatorname{tr}\left(e^{-\beta_{\mathrm{f}}\left(H_{\mathrm{f}}^{(\Lambda)} - \mu_{\mathrm{f}} N_{\mathrm{f}}^{(\Lambda)}\right)} A\right). \tag{1.40}$$

The inverse temperature β_{f} and the chemical potential μ_{f} play the role of the Lagrangian multipliers associated to the constraints (1.39), it holds

$$-\partial_{\beta_{\mathrm{f}}} \ln\left(Z^{(\Lambda,\mu_{\mathrm{f}})}(\beta_{\mathrm{f}})\right) = \omega_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}\left(H_{\mathrm{f}}^{(\Lambda)}\right),$$
$$\frac{1}{\beta_{\mathrm{f}}}\partial_{\mu_{\mathrm{f}}} \ln\left(Z^{(\Lambda,\mu_{\mathrm{f}})}(\beta_{\mathrm{f}})\right) = \omega_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}\left(N_{\mathrm{f}}^{(\Lambda)}\right).$$

The underlying algebra of observables is the Weyl algebra generated by form factors corresponding to positions inside the region Λ. We define the so-called *local algebra* as

$$\mathcal{A}_{\mathrm{f}}^{(\Lambda)} := \mathcal{W}\left(\mathcal{D}_{\mathrm{f}} \cap \mathfrak{h}_1^{(\Lambda)}\right) \equiv \overline{\operatorname{span}\left\{W(f) \,\middle|\, f \in \mathcal{D}_{\mathrm{f}} \cap \mathfrak{h}_1^{(\Lambda)}\right\}}^{\|\cdot\|_{\mathcal{B}(\mathcal{H}_{\mathrm{f}}^{(\Lambda)})}}. \tag{1.41}$$

The generalized Hamilton operator $H_{\mathrm{f}}^{(\Lambda)}$ then generates an automorphism group $\alpha_{\mathrm{f},(\Lambda,\mu_{\mathrm{f}})} = \{\alpha^t_{\mathrm{f},(\Lambda,\mu_{\mathrm{f}})}\}_{t\in\mathbb{R}}$ on the local algebra $\mathcal{A}_{\mathrm{f}}^{(\Lambda)}$ defined as

$$\alpha^t_{\mathrm{f},(\Lambda,\mu_{\mathrm{f}})}(A) := e^{itK_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}} A e^{-itK_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}}, \qquad A \in \mathcal{A}_{\mathrm{f}}^{(\Lambda)},$$

where the action of $\alpha^t_{\mathrm{f},(\Lambda,\mu_\mathrm{f})}$ on a Weyl operator $W(f)$, $f \in \mathfrak{h}_1^{(\Lambda)}$, is given by

$$\alpha^t_{\mathrm{f},(\Lambda,\mu_\mathrm{f})}(W(f)) = W\left(e^{it(h_\mathrm{f}^{(\Lambda)}-\mu_\mathrm{f})}f\right).$$

With the arguments of Section 1.1.2 we check that $\omega_\mathrm{f}^{(\Lambda,\mu_\mathrm{f})}$ is an $(\alpha_{\mathrm{f},(\Lambda,\mu_\mathrm{f})},\beta_\mathrm{f})$-KMS state on $\mathcal{A}_\mathrm{f}^{(\Lambda)}$, we express the KMS condition (1.2) formally as

$$\omega_\mathrm{f}^{(\Lambda,\mu_\mathrm{f})}\left(A\alpha^{t+i\beta_\mathrm{f}}_{\mathrm{f},(\Lambda,\mu_\mathrm{f})}(B)\right) = \omega_\mathrm{f}^{(\Lambda,\mu_\mathrm{f})}\left(\alpha^t_{\mathrm{f},(\Lambda,\mu_\mathrm{f})}(B)A\right)$$

for $t \in \mathbb{R}$ and $A, B \in \mathcal{A}_\mathrm{f}^{(\Lambda)}$.

It follows from the considerations made in [11, Sect. 5.2.5] that the operators

$$A_{f_1,\ldots,f_n} := a(f_1)\cdots a(f_n)e^{-\beta_\mathrm{f}K_\mathrm{f}^{(\Lambda,\mu_\mathrm{f})}/2}$$

extend to bounded operators which are Hilbert-Schmidt, that is, $A^*_{f_1,\ldots,f_n}A_{f_1,\ldots,f_n}$ are of trace class on $\mathcal{H}_\mathrm{f}^{(\Lambda)}$ for $f_1,\ldots,f_n \in \mathfrak{h}_1^{(\Lambda)}$. This allows an extension of the state (1.40) to polynomials in creation and annihilation operators. This extension is continuous in the sense that

$$\left|\omega_\mathrm{f}^{(\Lambda,\mu_\mathrm{f})}\left(a^*(f_1)\cdots a^*(f_n)a(g_1)\cdots a(g_m)\right)\right| \le C\prod_{j=1}^n \|f_j\|_{\mathfrak{h}_1^{(\Lambda)}}\prod_{k=1}^m \|g_k\|_{\mathfrak{h}_1^{(\Lambda)}} \tag{1.42}$$

for a suitable constant C. Using the relation

$$e^{-\beta_\mathrm{f}K_\mathrm{f}^{(\Lambda,\mu_\mathrm{f})}/2}a^*(f) = a^*\left(e^{-\beta_\mathrm{f}\left(h_\mathrm{f}^{(\Lambda)}-\mu_\mathrm{f}\right)/2}f\right)e^{-\beta_\mathrm{f}K_\mathrm{f}^{(\Lambda,\mu_\mathrm{f})}/2},$$

which follows from the pull through formula (1.32) where we replace $\omega(\vec{k})$ by $h_\mathrm{f}^{(\Lambda)}-\mu_\mathrm{f}$ and H_f by $K_\mathrm{f}^{(\Lambda,\mu_\mathrm{f})}$, we derive with the help of the CCR (1.25) that for the *two-point functions* holds

$$\begin{aligned}
&\omega_\mathrm{f}^{(\Lambda,\mu_\mathrm{f})}(a^*(f)a(g))\\
&= \frac{\mathrm{tr}\left(a^*\left(e^{-\beta_\mathrm{f}\left(h_\mathrm{f}^{(\Lambda)}-\mu_\mathrm{f}\right)/2}f\right)e^{-\beta_\mathrm{f}K_\mathrm{f}^{(\Lambda,\mu_\mathrm{f})}}a\left(e^{-\beta_\mathrm{f}\left(h_\mathrm{f}^{(\Lambda)}-\mu_\mathrm{f}\right)/2}g\right)\right)}{Z^{(\Lambda,\mu_\mathrm{f})}(\beta_\mathrm{f})}\\
&= \omega_\mathrm{f}^{(\Lambda,\mu_\mathrm{f})}\left(a\left(e^{-\beta_\mathrm{f}\left(h_\mathrm{f}^{(\Lambda)}-\mu_\mathrm{f}\right)/2}g\right)a^*\left(e^{-\beta_\mathrm{f}\left(h_\mathrm{f}^{(\Lambda)}-\mu_\mathrm{f}\right)/2}f\right)\right)\\
&= \omega_\mathrm{f}^{(\Lambda,\mu_\mathrm{f})}\left(a^*\left(e^{-\beta_\mathrm{f}\left(h_\mathrm{f}^{(\Lambda)}-\mu_\mathrm{f}\right)/2}f\right)a\left(e^{-\beta_\mathrm{f}\left(h_\mathrm{f}^{(\Lambda)}-\mu_\mathrm{f}\right)/2}g\right)\right)\\
&\quad + \left\langle g \,\middle|\, e^{-\beta_\mathrm{f}\left(h_\mathrm{f}^{(\Lambda)}-\mu_\mathrm{f}\right)}f\right\rangle_{\mathfrak{h}_1^{(\Lambda)}}
\end{aligned}$$

$$
\begin{aligned}
&= \quad \ldots \\
&= \quad \omega_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}\Big(a^*\Big(e^{-n\beta_{\mathrm{f}}\left(h_{\mathrm{f}}^{(\Lambda)}-\mu_{\mathrm{f}}\right)/2}f\Big)a\Big(e^{-n\beta_{\mathrm{f}}\left(h_{\mathrm{f}}^{(\Lambda)}-\mu_{\mathrm{f}}\right)/2}g\Big)\Big) \\
&\qquad + \sum_{m=1}^{n}\left\langle g \,\middle|\, e^{-m\beta_{\mathrm{f}}\left(h_{\mathrm{f}}^{(\Lambda)}-\mu_{\mathrm{f}}\right)} f \right\rangle_{\mathfrak{h}_1^{(\Lambda)}} \\
&\xrightarrow{n\to\infty} \quad \sum_{m=1}^{\infty}\left\langle g \,\middle|\, e^{-m\beta_{\mathrm{f}}\left(h_{\mathrm{f}}^{(\Lambda)}-\mu_{\mathrm{f}}\right)} f \right\rangle_{\mathfrak{h}_1^{(\Lambda)}} \\
&= \quad \left\langle g \,\middle|\, \left(e^{\beta_{\mathrm{f}}\left(h_{\mathrm{f}}^{(\Lambda)}-\mu_{\mathrm{f}}\right)}-1\right)^{-1} f \right\rangle_{\mathfrak{h}_1^{(\Lambda)}} \qquad (1.43)
\end{aligned}
$$

where we used the continuity (1.42) and

$$
\lim_{n\to\infty}\left\| e^{-n\beta_{\mathrm{f}}\left(h_{\mathrm{f}}^{(\Lambda)}-\mu_{\mathrm{f}}\right)/2} f \right\|_{\mathfrak{h}_1^{(\Lambda)}} = 0
$$

due to $\beta_{\mathrm{f}}(h_{\mathrm{f}}^{(\Lambda)} - \mu \mathbb{1}_{\mathfrak{h}_1^{(\Lambda)}}) > 0$. To compute the *one-point functions* $\omega_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}(a^*(f))$ we choose an orthonormal basis $\{\psi_{\nu,j} \,|\, \nu, j \in \mathbb{N}_0\}$ of $\mathcal{H}_{\mathrm{f}}^{(\Lambda)}$ where $\{\psi_{\nu,j}\}_{j\in\mathbb{N}_0}$ is a basis of the ν-photon sector

$$
\mathfrak{h}_{\nu}^{(\Lambda)} := \mathcal{S}_{\nu}\left[\bigotimes_{k=1}^{\nu}\mathfrak{h}_1^{(\Lambda)}\right].
$$

Since $K_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}$ leaves the ν-photon sector invariant, but the creation operator $a^*(f)$ does not, it follows

$$
\mathrm{tr}\left(e^{-\beta_{\mathrm{f}}K_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}}a^*(f)\right) = \sum_{\nu,j=0}^{\infty}\left\langle \psi_{\nu,j} \,\middle|\, e^{-\beta_{\mathrm{f}}K_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}}a^*(f)\psi_{\nu,j}\right\rangle = 0
$$

and therefore is

$$
\omega_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}(a^*(f)) = 0
$$

for all $f \in \mathfrak{h}_1^{(\Lambda)}$. Repetition of the above method shows that $\omega_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}$ applied to polynomials in creation and annihilation operators can be expressed in terms of products and sums of the two-point functions (1.43), we say that the state is *quasi-free*. We will resume the notion of quasi-freeness in Section 1.3.4. The quasi-free structure of $\omega_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}$ leads to

$$
\begin{aligned}
\omega_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}(W(f)) &= \exp\left(-\frac{\omega_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}(\Phi(f)^2)}{2}\right) \\
&= \exp\left(-\frac{1}{4}\left\langle f \,\middle|\, \left(1+2\rho_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})}\right) f\right\rangle_{\mathfrak{h}_1^{(\Lambda)}}\right)
\end{aligned}
$$

where we defined the operator

$$\rho_{\mathrm{f}}^{(\Lambda,\mu_{\mathrm{f}})} := \left(e^{\beta_{\mathrm{f}}\left(h_{\mathrm{f}}^{(\Lambda)} - \mu_{\mathrm{f}}\right)} - 1 \right)^{-1} .$$

The arguments entering here will be illustrated further below when we present an alternative way to motivate the definition of the equilibrium state. All the above statements are summarized in [11, Prop. 5.2.28].

So far, we have established the equilibrium situation of the photon gas in the box Λ. We aim to realize an equilibrium situation for the free gas in $\mathbb{R}^3$ by a controlled lifting of the confinement for a system at equilibrium. This procedure is referred to as *thermodynamic limit* and is elaborated in the sequel. We observe that the one-particle Hilbert space $\mathfrak{h}_1^{(\Lambda)}$, the Fock space $\mathcal{H}_{\mathrm{f}}^{(\Lambda)}$ and the local algebra $\mathcal{A}_{\mathrm{f}}^{(\Lambda)}$ as well are increasing with the size of the box in the sense that

$$\begin{aligned} \mathfrak{h}_1^{(\Lambda)} &\subseteq \mathfrak{h}_1^{(\Lambda')}, \\ \mathcal{H}_{\mathrm{f}}^{(\Lambda)} &\subseteq \mathcal{H}_{\mathrm{f}}^{(\Lambda')}, \\ \mathcal{A}_{\mathrm{f}}^{(\Lambda)} &\subseteq \mathcal{A}_{\mathrm{f}}^{(\Lambda')} \end{aligned}$$

for $\Lambda \subseteq \Lambda'$. Let $\Lambda_1 \subseteq \Lambda_2 \subseteq \dots$ be an increasing sequence of bounded regions with sufficient regular boundaries which converges towards the whole position space, $\Lambda_n \nearrow \mathbb{R}^3$ as $n \to \infty$, in the sense that every bounded subset of $\mathbb{R}^3$ is contained in a set Λ_n if n is large enough. We denote by

$$\mathcal{A}_{\mathrm{f}}^{(\mathrm{ql})} := \overline{\bigvee_{n\in\mathbb{N}} \mathcal{A}_{\mathrm{f}}^{(\Lambda_n)}}$$

the *quasi-local algebra* which is the norm closure of the algebra generated by the union of all $\mathcal{A}_{\mathrm{f}}^{(\Lambda_n)}$. Since we want to control intensive quantities as the energy density and the photon number density of the expanding system we require that the limit process guarantees the existence of

$$\begin{aligned} e_{\mathrm{f}} &:= \lim_{n\to\infty} \frac{\omega_{\mathrm{f}}^{(\Lambda_n,\mu_{\mathrm{f}})}\left(H_{\mathrm{f}}^{(\Lambda_n)}\right)}{|\Lambda_n|}, \\ n_{\mathrm{f}} &:= \lim_{n\to\infty} \frac{\omega_{\mathrm{f}}^{(\Lambda_n,\mu_{\mathrm{f}})}\left(N_{\mathrm{f}}^{(\Lambda_n)}\right)}{|\Lambda_n|} \end{aligned}$$

as real numbers, where $|\cdot|$ denotes the Lebesgue measure. The partition function will diverge, i.e.,

$$\lim_{n\to\infty} Z^{(\Lambda_n,\mu_{\mathrm{f}})}(\beta_{\mathrm{f}}) = \infty,$$

since the discrete eigenvalues of $H_{\mathrm{f}}^{(\Lambda)}$ accumulate to continuous spectrum reflecting that the system goes over to an infinite one. This implies that the equilibria $\omega_{\mathrm{f}}^{(\Lambda_n,\mu_{\mathrm{f}})}$ will not tend to a Gibbs state.

With the arguments of [11, Prop. 5.2.29] we construct the equilibrium state in the thermodynamic limit. Given a Weyl operator $W(f) \in \mathcal{A}_{\mathrm{f}}^{(\Lambda_m)}$ with $f \in \mathfrak{h}_1^{(\Lambda_m)}$ the local observable $W(f)$ will be contained in $\mathcal{A}_{\mathrm{f}}^{(\Lambda_n)}$ for all $n \geq m$. This allows us to consider the limit

$$\begin{aligned} \omega_{\mathrm{f}}(W(f)) &:= \lim_{\mu_{\mathrm{f}}\to 0}\lim_{n\to\infty} \omega_{\mathrm{f}}^{(\Lambda_n,\mu_{\mathrm{f}})}(W(f)) \\ &= \lim_{\mu_{\mathrm{f}}\to 0}\lim_{n\to\infty} \exp\left(-\frac{1}{4}\left\langle f \,\middle|\, \left(1+2\rho_{\mathrm{f}}^{(\Lambda_n,\mu_{\mathrm{f}})}\right) f \right\rangle_{\mathfrak{h}_1^{(\Lambda)}} \right) \\ &= \exp\left(-\frac{1}{4}\langle f \,|\, (1+2\rho_{\mathrm{f}})\, f\,\rangle_{\mathfrak{h}_1} \right) \end{aligned} \tag{1.44}$$

where the multiplication operator ρ_{f} is given by the radiation density of a *black-body radiator*

$$\rho_{\mathrm{f}}(\vec{k}) := \frac{1}{e^{\beta_{\mathrm{f}}\omega(\vec{k})}-1},$$

known as the *Planck law*. In order to describe an inert photon gas we chose the chemical potential μ_{f} to be zero in the definition (1.44) of ω_{f}. This, however, requires that the form factors f obey a more stringent infrared regularization, namely $\omega^{-1/2} f \in L^2[\mathbb{R}^3]$, to be in the domain of ρ_{f}. This was respected in the buildup of the algebras $\mathcal{A}_{\mathrm{f}}$, (1.33), and $\mathcal{A}_{\mathrm{f}}^{(\Lambda)}$, (1.41). The definition (1.44) extends to a state on the whole quasi-local algebra $\mathcal{A}_{\mathrm{f}}^{(\mathrm{ql})}$. We further have

$$\lim_{\mu_{\mathrm{f}}\to 0}\lim_{n\to\infty} \omega_{\mathrm{f}}^{(\Lambda_n,\mu_{\mathrm{f}})}\left(A\alpha^t_{\mathrm{f},(\Lambda_n,\mu_{\mathrm{f}})}(B)\right) = \omega_{\mathrm{f}}\left(A\alpha_{\mathrm{f}}^t(B)\right)$$

for $A, B \in \mathcal{A}_{\mathrm{f}}^{(\Lambda_n)}$ which implies that the formal KMS condition

$$\omega_{\mathrm{f}}\left(A\alpha_{\mathrm{f}}^{t+i\beta_{\mathrm{f}}}(B)\right) = \omega_{\mathrm{f}}\left(\alpha_{\mathrm{f}}^t(B)A\right),$$

extended to all $A, B \in \mathcal{A}_{\mathrm{f}}^{(\mathrm{ql})}$, survives the thermodynamics limit, unlike the Gibbs characterization of an equilibrium state. However, care has to be taken since the lack of continuity properties of the dynamics α_{f} does not make $(\mathcal{A}_{\mathrm{f}}^{(\mathrm{ql})}, \alpha_{\mathrm{f}})$ a C^*- nor a W^*-dynamical system such that we rephrase the KMS condition for ω_{f} once we have a suitable representation.

We shall mention that the thermodynamic limit depends on the boundary condition on the box and on the way how the thermodynamic parameters (e.g., the photon number or energy density) are controlled while the system is expanded. Different thermodynamic limit processes might yield different thermal equilibria. For instance, if we drop the requirement about the finite particle density n_{f} of the photon gas and, instead, we postulate a finite density of particles in the ground state the thermodynamic limit yields an equilibrium state of a free photon gas in the presence of a *sea of photons*, the so-called *Bose-Einstein condensate*, for details refer to [33]. The non-uniqueness of equilibrium states at a given temperature is usually referred to as a *phase transition*. The typical situation would be that, for a distinguished temperature β_{f} and a chemical potential μ_{f}, there exist two or more equilibrium states which show significantly different photon number densities. These equilibria would be considered as different coexisting phases. In our construction of a photon equilibrium state we implicitly avoided to be in a phase transition regime.

1.3.4 Empirical Definition of a KMS State and Araki-Woods Representation

While the derivation of a thermal equilibrium state for the free photon gas via thermodynamic limit is conceptional we present an alternative approach which rather bases on empirical physical arguments but reaches at the same result. We show afterwards that the obtained state actually fulfils the mathematical KMS condition. The subsequent considerations are not carried out rigorously but lead us to the definition of the equilibrium state ω_{f}.

Since our algebra of observables $\mathcal{A}_{\mathrm{f}}$ is generated by the Weyl operators it is sufficient to determine the action of ω_{f} on $W(f)$. Assuming that the map $t \mapsto \omega_{\mathrm{f}}(W(tf))$ is analytic for any $f \in \mathfrak{h}_1$ the state is defined by the functionals $(f_1, \ldots, f_n) \mapsto \omega_{\mathrm{f}}(\Phi(f_1) \cdots \Phi(f_n))$ with $f_1, \ldots, f_n \in \mathfrak{h}_1$ and $n \in \mathbb{N}$. We want to narrow down all the possible choices of ω_{f} by using that the photons are not interacting with each other. To this end, we express

$$\omega_{\mathrm{f}}(\Phi(f_1)\cdots\Phi(f_n)) = \sum_{j=1}^{n} \sum_{\substack{I\subseteq\{1,\ldots,n\},\\ I\ni 1,\, \#I=j}} \omega^{(j)}_{\mathrm{trunc}}(f_i, i\in I)\,\omega_{\mathrm{f}}\Big(\prod_{\substack{i=0,\\ i\notin I}}^{n} \Phi(f_i)\Big)$$

in terms of the so-called *truncated functionals* $\omega^{(j)}_{\mathrm{trunc}}(f_i, i \in I)$. Note that I is regarded as an ordered subset of $\{1, \ldots, n\}$. The truncated functionals can be extracted from the state ω_{f} recursively, e.g.,

$$\begin{aligned}
\omega^{(1)}_{\mathrm{trunc}}(f_1) &= \omega_{\mathrm{f}}(\Phi(f_1)),\\
\omega^{(2)}_{\mathrm{trunc}}(f_1, f_2) &= \omega_{\mathrm{f}}(\Phi(f_1)\Phi(f_2)) - \omega_{\mathrm{f}}(\Phi(f_1))\omega_{\mathrm{f}}(\Phi(f_2)),
\end{aligned}$$

and represent the correlations between the operations of creating and annihilating of photons with given momentum distributions $f_1, \ldots, f_n$. To ensure that the state ω_{f} describes non-interacting photons we require that all multiple correlation are vanishing, i.e., $\omega^{(j)}_{\mathrm{trunc}}(f_i, i \in I) = 0$ for all $j \geq 3$. Such a state is called *quasi-free*. This assumption was physically justified for our photon model in Section 1.3.3 and in [11, Sect. 5.2.5] where the equilibrium state of the free boson gas was derived as the thermodynamic limit of Gibbs states over finite systems.

So far we did not use that the equilibrium state has to be time invariant. Using that $\alpha^t_{\mathrm{f}}(a^*(\vec{k})) = e^{i\omega(\vec{k})t}a^*(\vec{k})$ and $\alpha^t_{\mathrm{f}}(a(\vec{k})) = e^{-i\omega(\vec{k})t}a(\vec{k})$ the invariance of ω_{f} under (1.38) implies that $\omega_{\mathrm{f}}(a^{\#}(\vec{k})) \propto \delta(\vec{k})$ where either $a^{\#}(\vec{k}) = a^*(\vec{k})$ or $a^{\#}(\vec{k}) = a(\vec{k})$. It follows with (1.28, 1.29) and the linearity of ω_{f} that $\omega_{\mathrm{f}}(a^*(f)) \propto f(0)$ and $\omega_{\mathrm{f}}(a(f)) \propto \overline{f}(0)$. However, the evaluation of a L^2 function at zero is not well defined such that the proportional constant must be zero. The same argument extends to the application of ω_{f} to several creation and annihilation operators and shows that $\omega_{\mathrm{f}}(a^*(f_1)\cdots a^*(f_n)) = \omega_{\mathrm{f}}(a(f_1)\cdots a(f_n)) = 0$. In particular, $\omega_{\mathrm{f}}(\Phi(f)) = 0$ and therefore the only non-zero truncated functional is

$$\omega^{(2)}_{\mathrm{trunc}}(f_1, f_2) = \omega_{\mathrm{f}}(\Phi(f_1)\Phi(f_2)).$$

We use this to express $\omega_{\mathrm{f}}(\Phi(f)^n)$ in terms of $\omega_{\mathrm{f}}(\Phi(f)^2)$. For even exponents we get

$$\begin{aligned}\omega_{\mathrm{f}}(\Phi(f)^{2n}) &= \sum_{\substack{I\subseteq\{1,\ldots,2n\},\\ I\ni 1,\, \#I=2}} \omega_{\mathrm{f}}(\Phi(f)^2)\omega_{\mathrm{f}}(\Phi(f)^{2n-2}) \\ &= (2n-1)\omega_{\mathrm{f}}(\Phi(f)^2)\omega_{\mathrm{f}}(\Phi(f)^{2n-2}) = \frac{(2n)!}{n!2^n}\omega_{\mathrm{f}}(\Phi(f)^2)^n,\end{aligned}$$

and for odd exponents we compute similarly

$$\begin{aligned}\omega_{\mathrm{f}}(\Phi(f)^{2n+1}) &= \sum_{\substack{I\subseteq\{1,\ldots,2n+1\},\\ I\ni 1,\, \#I=2}} \omega_{\mathrm{f}}(\Phi(f)^2)\omega_{\mathrm{f}}(\Phi(f)^{2n-1}) \\ &= 2n\omega_{\mathrm{f}}(\Phi(f)^2)\omega_{\mathrm{f}}(\Phi(f)^{2n-1}) = 2^n n!\omega_{\mathrm{f}}(\Phi(f)^2)^n\omega_{\mathrm{f}}(\Phi(f)) \\ &= 0,\end{aligned}$$

and therefore

$$\begin{aligned}\omega_{\mathrm{f}}(W(f)) &= \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!}\omega_{\mathrm{f}}(\Phi(f)^{2n}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\omega_{\mathrm{f}}(\Phi(f)^2)^n \\ &= \exp\left(-\frac{\omega_{\mathrm{f}}(\Phi(f)^2)}{2}\right).\end{aligned}$$

It remains to fix $\omega_{\mathrm{f}}(\Phi(f)^2)$. This requires another physical input implementing that ω_{f} describes the radiation of non-interacting photons in a thermal equilibrium at inverse temperature β_f. Given that $\omega_\mathrm{f}(a^*(\vec{k})a(\vec{k}'))$ is the measured particle density of photons with momentum $\vec{k}, \vec{k}'$, resp., we get by the *Planck law*

$$\omega_\mathrm{f}(a^*(\vec{k})a(\vec{k}')) = \delta(\vec{k} - \vec{k}')\rho_\mathrm{f}(\vec{k}), \tag{1.45}$$

where

$$\rho_\mathrm{f}(\vec{k}) = \frac{1}{e^{\beta_\mathrm{f}\omega(\vec{k})} - 1}$$

is the radiation density of a *black-body radiator*. We remark at this point that we sometimes use the radial symmetry of ρ_f to interpret

$$\rho_\mathrm{f}(E) \equiv \rho_\mathrm{f}(E\hat{k}) = \frac{1}{e^{\beta E} - 1}$$

as a function of a positive variable E, where $\hat{k}$ is a unit vector. Equation (1.45) can be extended to

$$\omega_\mathrm{f}(a^*(f)a(g)) = \langle\, g \mid \rho_\mathrm{f} f \,\rangle_{\mathfrak{h}_1}$$

for $f, g \in \mathfrak{h}_1$ and therefore

$$\omega_\mathrm{f}(\Phi(f)^2) = \frac{1}{2}\omega_\mathrm{f}(a^*(f)^2 + a(f)^2 + 2a^*(f)a(f) + \|f\|^2_{\mathfrak{h}_1}) = \frac{1}{2}\left\|\sqrt{1+2\rho_\mathrm{f}}\, f\right\|^2_{\mathfrak{h}_1}.$$

Finally we are in the position to specify ω_f on the Weyl algebra,

$$\omega_\mathrm{f}(W(f)) = \exp\left(-\frac{1}{4}\left\|\sqrt{1+2\rho_\mathrm{f}}\, f\right\|^2_{\mathfrak{h}_1}\right) \tag{1.46}$$

which coincides with the derivation (1.44) of an equilibrium state with the thermodynamic limit procedure. We note that the definition (1.46) requires that f fulfils the infrared behavior $\omega^{-1/2} f \in \mathfrak{h}_1$ what we already respected when we defined the Weyl algebra in (1.33). Because of the CCR for Weyl operators, (1.35), and linearity, the assignment (1.46) extends to polynomials in $W(f_1), \dots, W(f_n)$. It remains to show that ω_f is well defined as a state on the Weyl algebra. We will prove the state properties of ω_f with the help of a theorem of Araki and Segal.

A representation $(\mathcal{H}, \pi)$ of the Weyl algebra $\mathcal{A}_\mathrm{f} = \overline{\mathrm{span}\,\{W(f) \mid f \in \mathcal{D}_\mathrm{f}\}}$ into the bounded operators on a Hilbert space $\mathcal{H}$ is called *regular* if the functions $\mathbb{R} \ni \tau \mapsto \pi(W(\tau f))$ are strongly continuous for any $f \in \mathcal{D}_\mathrm{f}$. Regularity of the representation is, by Stone's Theorem (c.f. [40, Thm. VIII.8]), equivalent to the existence of a self-adjoint operator $\Phi_\pi(f)$ on $\mathcal{H}$, referred to as the field operator in the representation π, such that for all $\tau \in \mathbb{R}$

$$\pi(W(\tau f)) = e^{i\tau\Phi_\pi(f)}$$

holds. For a cyclic representation $(\mathcal{H}, \pi, \Psi)$ of the Weyl algebra $\mathcal{A}_\mathrm{f}$, i.e., if Ψ is cyclic w.r.t. $\pi(\mathcal{A}_\mathrm{f})''$, we can associate a *generating functional* by

$$\mathcal{Z} : \mathcal{D}_\mathrm{f} \to \mathbb{C}, \quad \mathcal{Z}(f) := \langle \Psi \,|\, \pi(W(f))\Psi \rangle_\mathcal{H}$$

where $\langle \cdot \,|\, \cdot \rangle_\mathcal{H}$ is the inner product of $\mathcal{H}$. The following theorem, due to Araki and Segal, c.f. [1, Thm. 4.2, 4.3], characterizes the generating functional of a regular and cyclic representation of the Weyl algebra.

Theorem 1.1 (Generating Functional & Representation) *Let $\mathcal{D}$ be a dense subspace of $\mathfrak{h}_1$. A map $\mathcal{Z} : \mathcal{D} \to \mathbb{C}$ is the generating functional of a regular and cyclic representation $(\mathcal{H}, \pi, \Psi)$ of the Weyl algebra $\mathcal{W}(\mathcal{D})$ over $\mathcal{D}$ if and only if the following conditions are satisfied:*

(i) $\mathcal{Z}(0) = 1$.

(ii) The map $\mathbb{R} \ni \tau \mapsto \mathcal{Z}(\tau f)$ is continuous for each $f \in \mathcal{D}$.

(iii) For any $f_1, \ldots, f_n \in \mathcal{D}$ and any $a_1, \ldots, a_n \in \mathbb{C}$ $(n \in \mathbb{N})$ we have

$$\sum_{j,k=1}^{n} \mathcal{Z}(f_j - f_k) e^{\frac{i}{2}\mathrm{Im}\langle f_j \,|\, f_k \rangle_{\mathfrak{h}_1}} a_j \overline{a}_k \geq 0.$$

Furthermore, if the conditions (i) – (iii) are fulfilled, the representation $(\mathcal{H}, \pi, \Psi)$ is uniquely determined, up to unitary equivalence, by the generating functional $\mathcal{Z}$.

We apply the above theorem to the map $\mathcal{Z}(f) := \omega_\mathrm{f}(W(f))$ as given in (1.46). It is obvious that the conditions (i) and (ii) are fulfilled. An elementary computation shows that (iii) also holds in this specific case. By Theorem 1.1, we are given a Hilbert space $\mathcal{H}_\mathrm{f}^2$, a representation $\pi_\mathrm{f} : \mathcal{A}_\mathrm{f} \to \mathcal{B}(\mathcal{H}_\mathrm{f}^2)$ and a cyclic vector $\Omega_\mathrm{f} \in \mathcal{H}_\mathrm{f}^2$ such that ω_f can be written as

$$\omega_\mathrm{f}(A) = \langle \Omega_\mathrm{f} \,|\, \pi_\mathrm{f}(A)\Omega_\mathrm{f} \rangle_{\mathcal{H}_\mathrm{f}^2} \tag{1.47}$$

for each $A \in \mathcal{A}_\mathrm{f}$.

The above theorem guarantees the existence of a regular, cyclic representation $(\mathcal{H}_\mathrm{f}^2, \pi_\mathrm{f}, \Omega_\mathrm{f})$ for the state ω_f. We now specify the representation. The representation Hilbert space is given – similar to the particle case considered in Section. 1.2.3 – as the tensor product of the zero temperature Fock space $\mathcal{H}_\mathrm{f} = \mathcal{F}(\mathfrak{h}_1)$ with itself,

$$\mathcal{H}_\mathrm{f}^2 := \mathcal{H}_\mathrm{f} \otimes \mathcal{H}_\mathrm{f}.$$

The Weyl operators in $\mathcal{A}_{\mathrm{f}}$ are mapped into the algebra of bounded operators on $\mathcal{H}_{\mathrm{f}}^2$ via the assignment

$$\pi_{\mathrm{f}}(W(f)) := W(\sqrt{1+\rho_{\mathrm{f}}}\, f) \otimes W(\sqrt{\rho_{\mathrm{f}}}\, \overline{f}) \tag{1.48}$$

which is extended linearly to the whole algebra. Note that, because of

$$\begin{aligned}
&\pi_{\mathrm{f}}(W(f))\pi_{\mathrm{f}}(W(g)) \\
&= W(\sqrt{1+\rho_{\mathrm{f}}}\, f)W(\sqrt{1+\rho_{\mathrm{f}}}\, g) \otimes W(\sqrt{\rho_{\mathrm{f}}}\, \overline{f})W(\sqrt{\rho_{\mathrm{f}}}\, \overline{g}) \\
&= e^{-\frac{i}{2}\left[\mathrm{Im}\left\langle \sqrt{1+\rho_{\mathrm{f}}}\, f \,\middle|\, \sqrt{1+\rho_{\mathrm{f}}}\, g \right\rangle_{\mathfrak{h}_1} + \mathrm{Im}\left\langle \sqrt{\rho_{\mathrm{f}}}\, \overline{f} \,\middle|\, \sqrt{\rho_{\mathrm{f}}}\, \overline{g} \right\rangle_{\mathfrak{h}_1}\right]} \\
&\quad \times W(\sqrt{1+\rho_{\mathrm{f}}}\, [f+g]) \otimes W(\sqrt{\rho_{\mathrm{f}}}\, [\overline{f}+\overline{g}]) \\
&= e^{-\frac{i}{2}\langle f \,|\, g \rangle_{\mathfrak{h}_1}} \pi_{\mathrm{f}}(W(f+g)) = \pi_{\mathrm{f}}(W(f)W(g)),
\end{aligned}$$

the map π_{f} is multiplicative and a representation of the CCR. This representation was first specified by Araki and Woods in [2] and will therefore be referred to as *Araki-Woods representation*, it is the GNS representation for the state ω_{f}. The proof, that (1.48) is indeed well defined as representation, is given in this reference. The field operator $\Phi_{\mathrm{aw}}(f)$ of the Araki-Woods representation can be derived from (1.48) as the derivative (in a strong sense) of the strongly continuous group $\mathbb{R} \ni \tau \mapsto \pi_{\mathrm{f}}(W(\tau f))$. Using the Leibniz rule we get

$$\Phi_{\mathrm{aw}}(f) := i^{-1}\partial_\tau|_{\tau=0}\pi_{\mathrm{f}}(W(\tau f)) = \Phi(\sqrt{1+\rho_{\mathrm{f}}}\, f) \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} + \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \otimes \Phi(\sqrt{\rho_{\mathrm{f}}}\, \overline{f}).$$

The corresponding creation and annihilation operators of the representation, $a^*_{\mathrm{aw}}(f)$ and $a_{\mathrm{aw}}(f)$, resp., are given by

$$\begin{aligned}
a^*_{\mathrm{aw}}(f) :=& \frac{1}{\sqrt{2}}[\Phi_{\mathrm{aw}}(f) - i\Phi_{\mathrm{aw}}(if)] \\
=& a^*(\sqrt{1+\rho_{\mathrm{f}}}\, f) \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} + \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \otimes a(\sqrt{\rho_{\mathrm{f}}}\, \overline{f}), \\
a_{\mathrm{aw}}(f) :=& \frac{1}{\sqrt{2}}[\Phi_{\mathrm{aw}}(f) + i\Phi_{\mathrm{aw}}(if)] \\
=& a(\sqrt{1+\rho_{\mathrm{f}}}\, f) \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} + \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \otimes a^*(\sqrt{\rho_{\mathrm{f}}}\, \overline{f}).
\end{aligned}$$

This allows us to extend the representation π_{f} from Weyl operators to polynomials in creation and annihilation operators by setting

$$\pi_{\mathrm{f}}(a^*(f)) := a^*_{\mathrm{aw}}(f), \qquad \pi_{\mathrm{f}}(a(f)) := a_{\mathrm{aw}}(f). \tag{1.49}$$

Note that the creation and annihilation operator in the Araki-Woods representation also obey the CCR, i.e.,

$$\begin{aligned}
&[a^*_{\mathrm{aw}}(f), a^*_{\mathrm{aw}}(g)] = [a_{\mathrm{aw}}(f), a_{\mathrm{aw}}(g)] = 0, \\
&[a_{\mathrm{aw}}(f), a^*_{\mathrm{aw}}(g)] = \langle\, f \mid g \,\rangle_{\mathfrak{h}_1}.
\end{aligned} \tag{1.50}$$

The vector

$$\Omega_{\mathrm{f}} := \Omega_{\mathrm{vac}} \otimes \Omega_{\mathrm{vac}},$$

built of the vacuum vectors of the Fock space factors, will play the role of the cyclic vector. The proof of the cyclicity of Ω_{f} can be found in [2, p. 648] and uses similar arguments as the proof that Ω_{vac} is cyclic for the Weyl algebra $\mathcal{A}_{\mathrm{f}}$, given in Sect. 1.3.2. With the help of (1.37) we show that the triple $(\mathcal{H}_{\mathrm{f}}^2, \pi_{\mathrm{f}}, \Omega_{\mathrm{f}})$ actually reproduces the generating functional in the sense of (1.47),

$$\begin{aligned}
&\langle\, \Omega_{\mathrm{f}} \mid \pi_{\mathrm{f}}(W(f))\Omega_{\mathrm{f}} \,\rangle_{\mathcal{H}_{\mathrm{f}}^2} \\
&= \Big\langle\, \Omega_{\mathrm{vac}} \,\Big|\, W(\sqrt{1+\rho_{\mathrm{f}}}\, f)\Omega_{\mathrm{vac}} \,\Big\rangle_{\mathcal{H}_{\mathrm{f}}} \langle\, \Omega_{\mathrm{vac}} \mid W(\sqrt{\rho_{\mathrm{f}}}\, \overline{f})\Omega_{\mathrm{vac}} \,\rangle_{\mathcal{H}_{\mathrm{f}}} \\
&= \exp\Big(-\frac{1}{4}\Big\|\sqrt{1+\rho_{\mathrm{f}}}\, f\Big\|_{\mathfrak{h}_1}^2\Big) \exp\Big(-\frac{1}{4}\,\|\sqrt{\rho_{\mathrm{f}}}\, f\|_{\mathfrak{h}_1}^2\Big) \\
&= \omega_{\mathrm{f}}(W(f)).
\end{aligned}$$

To complete the section we check explicitly the KMS condition (1.2) for the state ω_{f}. We remark that the dynamical system $(\mathcal{A}_{\mathrm{f}}, \alpha_{\mathrm{f}})$ is neither a C^*- (the evolution $\{\alpha_{\mathrm{f}}^t\}_{t\in\mathbb{R}}$ is not strongly continuous in the C^*-topoply) nor a W^*-dynamical system (the algebra $\mathcal{A}_{\mathrm{f}}$ is not weakly closed in $\mathcal{B}(\mathcal{H}_{\mathrm{f}})$). In Section 1.3.6 we will address this issue by transferring the problem from the system $(\mathcal{A}_{\mathrm{f}}, \alpha_{\mathrm{f}})$ to a W^*-dynamical system on the W^*-algebra $\pi_{\mathrm{f}}(\mathcal{A}_{\mathrm{f}})''$ using the representation π_{f}. For the moment we define for this particular system a KMS state to be a state which fulfills (1.2) for a function $F_{A,B}$ associated with elements $A, B \in \mathcal{A}_{\mathrm{f}}$. Because of the CCR for Weyl operators (1.35) the linear combinations

$$A = \sum_{j=1}^{\infty} c_j W(f_j), \qquad B = \sum_{k=1}^{\infty} d_k W(g_k),$$

with $c_j, d_k \in \mathbb{C}$ and $f_j, g_k \in \mathcal{D}_{\mathrm{f}}$, span the algebra $\mathcal{A}_{\mathrm{f}}$. For such a pair $A, B \in \mathcal{A}_{\mathrm{f}}$ we compute, for $t \in \mathbb{R}$,

$$\begin{aligned}
\omega_{\mathrm{f}}(A\alpha_{\mathrm{f}}^t(B)) &= \omega_{\mathrm{f}}\Big(\sum_{j,k=1}^{\infty} c_j W(f_j) d_k W(e^{i\omega t} g_k)\Big) \\
&= \sum_{j,k=1}^{\infty} c_j d_k \exp\Big(-\frac{i}{2}\,\mathrm{Im}\,\langle\, f_j \mid e^{i\omega t} g_k \,\rangle_{\mathfrak{h}_1}\Big)\omega_{\mathrm{f}}\Big(W(f_j + e^{i\omega t} g_k)\Big) \\
&= \sum_{j,k=1}^{\infty} c_j d_k \exp\Big(-\frac{i}{2}\,\mathrm{Im}\,\langle\, f_j \mid e^{i\omega t} g_k \,\rangle_{\mathfrak{h}_1}\Big) \\
&\qquad \times \exp\Big(-\frac{1}{4}\Big\|\sqrt{1+2\rho_{\mathrm{f}}}\,(f_j + e^{i\omega t} g_k)\Big\|_{\mathfrak{h}_1}^2\Big)
\end{aligned}$$

$$= \sum_{j,k=1}^{\infty} c_j d_k \omega_{\mathrm{f}}(W(f_j))\omega_{\mathrm{f}}(W(g_k)) \exp\Big(-\frac{1}{2}\langle f_j \,|\, e^{i\omega t} g_k \rangle_{\mathfrak{h}_1}\Big)$$
$$\times \exp\Big(-\frac{1}{2}\Big(\langle f_j \,|\, \rho_{\mathrm{f}} e^{i\omega t} g_k \rangle_{\mathfrak{h}_1} + \langle g_k \,|\, \rho_{\mathrm{f}} e^{-i\omega t} f_j \rangle_{\mathfrak{h}_1}\Big)\Big).$$

Because of $|e^{i\omega(\vec{k})s}| \leq 1$ and $|\rho_{\mathrm{f}}(\vec{k})e^{-i\omega(\vec{k})s}| \leq \frac{\text{const.}}{\omega(\vec{k})}$ for $s \in \mathbb{C}$ with $0 \leq \mathrm{Im}(s) \leq \beta_{\mathrm{f}}$ and since $f_j, g_k \in \mathcal{D}_{\mathrm{f}}$, the map $\mathbb{R} \ni t \mapsto \omega_{\mathrm{f}}(A\alpha_{\mathrm{f}}^t(B))$ has, by dominated convergence theorem, an analytic continuation to $D_{\beta_{\mathrm{f}}} = \{z \in \mathbb{C} \,|\, 0 < \mathrm{Im}(z) < \beta_{\mathrm{f}}\}$. We denote the continuation by $F_{A,B}$ and observe that $F_{A,B}$ is continuous on the boundaries of $D_{\beta_{\mathrm{f}}}$. We compute

$$\begin{aligned}
F_{A,B}(t+i\beta_{\mathrm{f}}) &= \sum_{j,k=1}^{\infty} c_j d_k \omega_{\mathrm{f}}(W(f_j))\omega_{\mathrm{f}}(W(g_k)) \exp\Big(-\frac{1}{2}\langle f_j \,|\, e^{i\omega t - \beta_{\mathrm{f}}\omega} g_k \rangle_{\mathfrak{h}_1}\Big) \\
&\quad \times \exp\Big(-\frac{1}{2}\Big(\langle f_j \,|\, \rho_{\mathrm{f}} e^{i\omega t - \beta_{\mathrm{f}}\omega} g_k \rangle_{\mathfrak{h}_1} + \langle g_k \,|\, \rho_{\mathrm{f}} e^{-i\omega t + \beta_{\mathrm{f}}\omega} f_j \rangle_{\mathfrak{h}_1}\Big)\Big) \\
&= \sum_{j,k=1}^{\infty} c_j d_k \omega_{\mathrm{f}}(W(f_j))\omega_{\mathrm{f}}(W(g_k)) \\
&\quad \exp\Big(-\frac{1}{2}\Big(\langle g_k \,|\, \rho_{\mathrm{f}} e^{\beta_{\mathrm{f}}\omega} e^{-i\omega t} f_j \rangle_{\mathfrak{h}_1}\Big)\Big) \\
&\quad \times \exp\Big(-\frac{1}{2}\Big(\langle f_j \,|\, [1+\rho_{\mathrm{f}}] e^{-\beta_{\mathrm{f}}\omega} e^{i\omega t} g_k \rangle_{\mathfrak{h}_1}\Big)\Big) \\
&= \sum_{j,k=1}^{\infty} c_j d_k \omega_{\mathrm{f}}(W(f_j))\omega_{\mathrm{f}}(W(g_k)) \exp\Big(-\frac{1}{2}\langle e^{i\omega t} g_k \,|\, f_j \rangle_{\mathfrak{h}_1}\Big) \\
&\quad \times \exp\Big(-\frac{1}{2}\Big(\langle e^{i\omega t} g_k \,|\, \rho_{\mathrm{f}} f_j \rangle_{\mathfrak{h}_1} + \langle f_j \,|\, \rho_{\mathrm{f}} e^{i\omega t} g_k \rangle_{\mathfrak{h}_1}\Big)\Big) \\
&= \sum_{j,k=1}^{\infty} c_j d_k \exp\Big(-\frac{i}{2}\mathrm{Im}\,\langle e^{i\omega t} g_k \,|\, f_j \rangle_{\mathfrak{h}_1}\Big) \\
&\qquad \times \exp\Big(-\frac{1}{4}\Big\|\sqrt{1+2\rho_{\mathrm{f}}}\,(e^{i\omega t} g_k + f_j)\Big\|_{\mathfrak{h}_1}^2\Big) \\
&= \sum_{j,k=1}^{\infty} c_j d_k \exp\Big(-\frac{i}{2}\mathrm{Im}\,\langle e^{i\omega t} g_k \,|\, f_j \rangle_{\mathfrak{h}_1}\Big)\omega_{\mathrm{f}}\Big(W(e^{i\omega t} g_k + f_j)\Big) \\
&= \omega_{\mathrm{f}}\Big(\sum_{j,k=1}^{\infty} d_k c_j W(e^{i\omega t} g_k) W(f_j)\Big) \\
&= \omega_{\mathrm{f}}(\alpha_{\mathrm{f}}^t(B)A),
\end{aligned} \tag{1.51}$$

where we used that

$$\rho_{\mathrm{f}} e^{\beta_{\mathrm{f}}\omega} = 1 + \rho_{\mathrm{f}}. \tag{1.52}$$

We stress that the GNS representation $(H_{\rm f}^2, \pi_{\rm f}, \Omega_{\rm f})$ of the state $\omega_{\rm f}$ depends on the inverse temperature $\beta_{\rm f}$ of the equilibrium it is describing. We remark that for $\omega_{\rm f}$ and $\omega_{\rm f}'$ being the equilibrium states as constructed above at different inverse temperatures $\beta_{\rm f} \neq \beta_{\rm f}'$, resp., the corresponding GNS representations are not unitary equivalent. This implies that thermal equilibria at different temperatures are not normal w.r.t. each other, i.e., their relative entropy is infinite.

1.3.5 Modular Structure of the Photon System

The access to the modular structure of the photon system is given by the anti-linear operator

$$S_{\rm f} : \mathcal{M}_{\rm f}\Omega_{\rm f} \to \mathcal{M}_{\rm f}\Omega_{\rm f}, \qquad A\Omega_{\rm f} \mapsto A^*\Omega_{\rm f}.$$

where $S_{\rm f}$ is initially given on $\mathcal{M}_{\rm f}\Omega_{\rm f}$ with

$$\mathcal{M}_{\rm f} := \pi_{\rm f}(\mathcal{A}_{\rm f})''$$

but extends to a closed operator also denoted by $S_{\rm f}$. Note that the vector $\Omega_{\rm f}$ is both cyclic and separating for $\mathcal{M}_{\rm f}$ and therefore $S_{\rm f}$ is well defined on a dense domain. The aim is to decompose $S_{\rm f}$ in a anti-unitary operator $J_{\rm f}$ and a positive operator $\Delta_{\rm f}^{1/2}$ as discussed in Section 1.1.2. We introduce the positive operator

$$\Delta_{\rm f} := e^{-\beta_{\rm f} H_{\rm f}} \otimes e^{\beta_{\rm f} H_{\rm f}}$$

and note that

$$\begin{aligned} & J_{\rm f}\left[(a^*(f_n)\cdots a^*(f_1)) \otimes (a^*(g_m)\cdots a^*(g_1))\Omega_{\rm f}\right] \\ := \quad & (a^*(\overline{g}_m)\cdots a^*(\overline{g}_1)) \otimes (a^*(\overline{f}_n)\cdots a^*(\overline{f}_1))\Omega_{\rm f} \end{aligned}$$

defines an anti-unitary operator $J_{\rm f}$ on $\mathcal{H}_{\rm f}^2$. Hereby, the conjugation $\overline{f}$ of a function $f \in L^2[\mathbb{R}^3]$ is the usual (pointwise) complex conjugation. We check the relation

$$S_{\rm f} = J_{\rm f}\Delta_{\rm f}^{1/2} \tag{1.53}$$

by explicit computations. Let $W(f) \in \mathcal{A}_{\rm f}$ with $e^{\beta_{\rm f}\omega/2} f \in \mathcal{D}_{\rm f}$. Expanding the Weyl operator in a power series, using that $\Omega_{\rm f}$ is an analytic vector for $\Phi_{\rm aw}(f)$, we obtain

$$\begin{aligned} \pi_{\rm f}(W(f))\Omega_{\rm f} &= \sum_{k=0}^{\infty} \frac{i^k}{k!} \Phi_{\rm aw}(f)^k \Omega_{\rm f} \\ &= \sum_{k=0}^{\infty} \frac{i^k}{2^{k/2}k!} \Big[(a^*(\sqrt{1+\rho_{\rm f}}\, f) + a(\sqrt{1+\rho_{\rm f}}\, f)) \otimes \mathbb{1}_{\mathcal{H}_{\rm f}} \\ &\qquad\qquad + \mathbb{1}_{\mathcal{H}_{\rm f}} \otimes (a^*(\sqrt{\rho_{\rm f}}\, \overline{f}) + a(\sqrt{\rho_{\rm f}}\, \overline{f})) \Big]^k \Omega_{\rm f}. \end{aligned}$$

With the help of the pull through formula (1.32) and the fact that $\Delta_{\mathrm{f}}^{1/2}\Omega_{\mathrm{f}} = \Omega_{\mathrm{f}}$ we get, using relation (1.52),

$$
\begin{aligned}
& J_{\mathrm{f}}\Delta_{\mathrm{f}}^{1/2}\pi_{\mathrm{f}}(W(f))\Omega_{\mathrm{f}} \\
= \quad & J_{\mathrm{f}} \sum_{k=0}^{\infty} \frac{i^k}{2^{k/2}k!} \Big[(a^*(\sqrt{1+\rho_{\mathrm{f}}}\, e^{-\beta_{\mathrm{f}}\omega/2} f) + a(\sqrt{1+\rho_{\mathrm{f}}}\, e^{\beta_{\mathrm{f}}\omega/2} f)) \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \\
& \qquad + \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \otimes (a^*(\sqrt{\rho_{\mathrm{f}}}\, e^{\beta_{\mathrm{f}}\omega/2}\overline{f}) + a(\sqrt{\rho_{\mathrm{f}}}\, e^{-\beta_{\mathrm{f}}\omega/2}\overline{f})) \Big]^k \Omega_{\mathrm{f}} \\
= \quad & \sum_{k=0}^{\infty} \frac{(-i)^k}{2^{k/2}k!} \\
& \times J_{\mathrm{f}} \Big[(a^*(\sqrt{\rho_{\mathrm{f}}}\, f) + a(\sqrt{\rho_{\mathrm{f}}}\, f)) \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \\
& \quad + \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \otimes (a^*(\sqrt{1+\rho_{\mathrm{f}}}\, \overline{f}) + a(\sqrt{1+\rho_{\mathrm{f}}}\, \overline{f})) \\
& \quad + a(\sqrt{\rho_{\mathrm{f}}}\, (e^{\beta_{\mathrm{f}}\omega} - 1) f) \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} + \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \otimes a(\sqrt{1+\rho_{\mathrm{f}}}\, (e^{-\beta_{\mathrm{f}}\omega} - 1)\overline{f}) \Big]^k \Omega_{\mathrm{f}} \\
= \quad & \sum_{k=0}^{\infty} \frac{(-i)^k}{2^{k/2}k!} \\
& \Big[\mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \otimes (a^*(\sqrt{\rho_{\mathrm{f}}}\, \overline{f}) + a(\sqrt{\rho_{\mathrm{f}}}\, \overline{f})) \\
& \quad + (a^*(\sqrt{1+\rho_{\mathrm{f}}}\, f) + a(\sqrt{1+\rho_{\mathrm{f}}}\, f)) \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \\
& \quad + \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \otimes a(\sqrt{\rho_{\mathrm{f}}}\, (e^{\beta_{\mathrm{f}}\omega} - 1)\overline{f}) + a(\sqrt{1+\rho_{\mathrm{f}}}\, (e^{-\beta_{\mathrm{f}}\omega} - 1) f) \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \Big]^k \Omega_{\mathrm{f}}.
\end{aligned}
$$

We abbreviate

$$
\begin{aligned}
A &:= (a^*(\sqrt{1+\rho_{\mathrm{f}}}\, f) + a(\sqrt{1+\rho_{\mathrm{f}}}\, f)) \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} + \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \otimes (a^*(\sqrt{\rho_{\mathrm{f}}}\, \overline{f}) + a(\sqrt{\rho_{\mathrm{f}}}\, \overline{f})), \\
B &:= a(\sqrt{1+\rho_{\mathrm{f}}}\, (e^{-\beta_{\mathrm{f}}\omega} - 1) f) \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} + \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \otimes a(\sqrt{\rho_{\mathrm{f}}}\, (e^{\beta_{\mathrm{f}}\omega} - 1)\overline{f})
\end{aligned}
$$

and note that $B\Omega_{\mathrm{f}} = 0$ and

$$
\begin{aligned}
[B, A] &= \Big[a(\sqrt{1+\rho_{\mathrm{f}}}\, (e^{-\beta_{\mathrm{f}}\omega} - 1) f), a^*(\sqrt{1+\rho_{\mathrm{f}}}\, f) \Big] \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \\
&\quad + \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \otimes \big[a(\sqrt{\rho_{\mathrm{f}}}\, (e^{\beta_{\mathrm{f}}\omega} - 1)\overline{f}), a^*(\sqrt{\rho_{\mathrm{f}}}\, \overline{f}) \big] \\
&= \big\langle f \,\big|\, (1+\rho_{\mathrm{f}})(e^{-\beta_{\mathrm{f}}\omega} - 1) f \big\rangle_{\mathfrak{h}_1} + \big\langle \overline{f} \,\big|\, \rho_{\mathrm{f}}(e^{\beta_{\mathrm{f}}\omega} - 1)\overline{f} \big\rangle_{\mathfrak{h}_1} \\
&= \langle f \,|\, [(\rho_{\mathrm{f}} - 1 - \rho_{\mathrm{f}}) + (1 + \rho_{\mathrm{f}} - \rho_{\mathrm{f}})]\, f \rangle_{\mathfrak{h}_1} \\
&= 0.
\end{aligned}
$$

This implies that $[A+B]^k\Omega_\mathrm{f} = A^k\Omega_\mathrm{f}$ and we may write

$$\begin{aligned} J_\mathrm{f}\Delta_\mathrm{f}^{1/2}\pi_\mathrm{f}(W(f))\Omega_\mathrm{f} &= \sum_{k=0}^\infty \frac{(-i)^k}{2^{k/2}k!}\Big[(a^*(\sqrt{1+\rho_\mathrm{f}}\,f) + a(\sqrt{1+\rho_\mathrm{f}}\,f)) \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}} \\ &\qquad + \mathbb{1}_{\mathcal{H}_\mathrm{f}} \otimes (a^*(\sqrt{\rho_\mathrm{f}}\,\overline{f}) + a(\sqrt{\rho_\mathrm{f}}\,\overline{f}))\Big]^k \Omega_\mathrm{f} \\ &= \pi_\mathrm{f}(W(-f))\Omega_\mathrm{f} \\ &= S_\mathrm{f}\pi_\mathrm{f}(W(f))\Omega_\mathrm{f}. \end{aligned}$$

This proves the polar decomposition (1.53) of S_f in the product of the modular conjugation J_f and the square root of the modular operator Δ_f.

The modular conjugation J_f enables us to introduce the anti-linear representation

$$\pi'_\mathrm{f} : \mathcal{A}_\mathrm{f} \to \mathcal{B}(\mathcal{H}_\mathrm{f}^2), \qquad \pi'_\mathrm{f}(A) := J_\mathrm{f}\pi_\mathrm{f}(A)J_\mathrm{f}$$

which acts on a Weyl operator as

$$\pi'_\mathrm{f}(W(f)) = W(\sqrt{\rho_\mathrm{f}}\,f) \otimes W(\sqrt{1+\rho_\mathrm{f}}\,\overline{f}).$$

The fact that π'_f is commuting with π_f follows from Tomita-Takesaki theory but can also be verified with the help of the CCR in the version (1.35) for Weyl operators,

$$\begin{aligned} \pi'_\mathrm{f}(W(f))\pi_\mathrm{f}(W(g)) &= W(\sqrt{\rho_\mathrm{f}}\,f)W(\sqrt{1+\rho_\mathrm{f}}\,g) \otimes W(\sqrt{1+\rho_\mathrm{f}}\,\overline{f})W(\sqrt{\rho_\mathrm{f}}\,\overline{g}) \\ &= e^{-i\,\mathrm{Im}\left[\left\langle \sqrt{\rho_\mathrm{f}}\,f \,\middle|\, \sqrt{1+\rho_\mathrm{f}}\,g\right\rangle_{\mathfrak{h}_1} + \left\langle \sqrt{1+\rho_\mathrm{f}}\,\overline{f} \,\middle|\, \sqrt{\rho_\mathrm{f}}\,\overline{g}\right\rangle_{\mathfrak{h}_1}\right]} \\ &\quad \times W(\sqrt{1+\rho_\mathrm{f}}\,g)W(\sqrt{\rho_\mathrm{f}}\,f) \otimes W(\sqrt{\rho_\mathrm{f}}\,\overline{g})W(\sqrt{1+\rho_\mathrm{f}}\,\overline{f}) \\ &= e^{-i\,\mathrm{Im}\left[2\,\mathrm{Re}\left\langle f \,\middle|\, \sqrt{\rho_\mathrm{f}(1+\rho_\mathrm{f})}\,g\right\rangle_{\mathfrak{h}_1}\right]} \\ &\quad \times W(\sqrt{1+\rho_\mathrm{f}}\,g)W(\sqrt{\rho_\mathrm{f}}\,f) \otimes W(\sqrt{\rho_\mathrm{f}}\,\overline{g})W(\sqrt{1+\rho_\mathrm{f}}\,\overline{f}) \\ &= \pi_\mathrm{f}(W(g))\pi'_\mathrm{f}(W(f)). \end{aligned}$$

As a consequence we get the relation

$$\pi'_\mathrm{f}(\mathcal{A}_\mathrm{f})' \supseteq \pi_\mathrm{f}(\mathcal{A}_\mathrm{f})'',$$

Tomita-Takesaki theory even yields

$$\pi'_\mathrm{f}(\mathcal{A}_\mathrm{f})' = \pi_\mathrm{f}(\mathcal{A}_\mathrm{f})''.$$

The corresponding positive cone is given by

$$\begin{aligned} \mathcal{P}_\mathrm{f} &:= \overline{\{\pi_\mathrm{f}(A)J_\mathrm{f}\pi_\mathrm{f}(A)\Omega_\mathrm{f} \mid A \in \mathcal{A}_\mathrm{f}\}} \\ &= \overline{\Big\{ \sum_{j,k=1}^n c_j\overline{c}_k W(\sqrt{1+\rho_\mathrm{f}}\,f_j + \sqrt{\rho_\mathrm{f}}\,f_k) \otimes W(\sqrt{\rho_\mathrm{f}}\,\overline{f}_j + \sqrt{1+\rho_\mathrm{f}}\,\overline{f}_k)\Omega_\mathrm{f} \,\Big|} \\ &\qquad \overline{\Big|\, c_1,\dots,c_n \in \mathbb{C},\ f_1,\dots,f_n \in \mathcal{D}_\mathrm{f},\ n \in \mathbb{N}\Big\}}. \end{aligned}$$

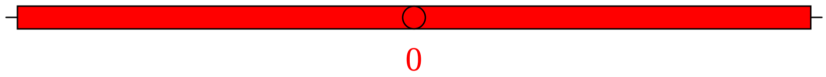

Figure 1.4: Spectrum of the free field Liouvillean associated with ω_{f}.

1.3.6 Liouville Operator and Return to Equilibrium

The free field Liouville operator associated with the state ω_{f} and the time evolution α_{f} is given by the self-adjoint operator

$$L_{\mathrm{f}} := H_{\mathrm{f}} \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} - \mathbb{1}_{\mathcal{H}_{\mathrm{f}}} \otimes H_{\mathrm{f}}.$$

To verify this statement we have to check that L_{f} implements the time evolution on the representation space in the sense

$$e^{iL_{\mathrm{f}}t}\pi_{\mathrm{f}}(A)e^{-iL_{\mathrm{f}}t} = \pi_{\mathrm{f}}(\alpha_{\mathrm{f}}^t(A)), \tag{1.54}$$

for all $A \in \mathcal{A}_{\mathrm{f}}$, and that the group $e^{iL_{\mathrm{f}}t}$ leaves the positive cone $\mathcal{P}_{\mathrm{f}}$ invariant. Given a Weyl operator $W(f) \in \mathcal{A}_{\mathrm{f}}$ we can express its evolution under α_{f} as

$$\begin{aligned}
\pi_{\mathrm{f}}(\alpha_{\mathrm{f}}^t(W(f))) &= \pi_{\mathrm{f}}(W(e^{i\omega t}f)) = W\left(e^{i\omega t}\sqrt{1+\rho_{\mathrm{f}}}\, f\right) \otimes W\left(\sqrt{\rho_{\mathrm{f}}}\,\overline{e^{i\omega t} f}\right) \\
&= \left[e^{iH_{\mathrm{f}}t}W\left(\sqrt{1+\rho_{\mathrm{f}}}\, f\right)e^{-iH_{\mathrm{f}}t}\right] \otimes \left[e^{-iH_{\mathrm{f}}t}W\left(\sqrt{\rho_{\mathrm{f}}}\,\overline{f}\right)e^{iH_{\mathrm{f}}t}\right] \\
&= e^{iL_{\mathrm{f}}t}\pi_{\mathrm{f}}(W(f))e^{-iL_{\mathrm{f}}t}.
\end{aligned}$$

An extension of this relation to all observables from the Weyl algebra establishes (1.54). It further holds for $A \in \mathcal{A}_{\mathrm{f}}$,

$$\begin{aligned}
e^{iL_{\mathrm{f}}t}\pi_{\mathrm{f}}(A)J_{\mathrm{f}}\pi_{\mathrm{f}}(A)\Omega_{\mathrm{f}} &= \pi_{\mathrm{f}}(\alpha_{\mathrm{f}}^t(A))J_{\mathrm{f}}e^{iL_{\mathrm{f}}t}\pi_{\mathrm{f}}(A)\Omega_{\mathrm{f}} \\
&= \pi_{\mathrm{f}}(\alpha_{\mathrm{f}}^t(A))J_{\mathrm{f}}\pi_{\mathrm{f}}(\alpha_{\mathrm{f}}^t(A))\Omega_{\mathrm{f}} \in \mathcal{P}_{\mathrm{f}}
\end{aligned}$$

using that apparently $L_{\mathrm{f}}\Omega_{\mathrm{f}} = 0$ and $iL_{\mathrm{f}}J_{\mathrm{f}} = J_{\mathrm{f}}iL_{\mathrm{f}}$. By continuity of $e^{iL_{\mathrm{f}}t}$ and the fact that $\mathcal{P}_{\mathrm{f}}$ is closed it follows that the group generated by L_{f} leaves the cone invariant. The above considerations justify that L_{f} is referred to as standard Liouville operator corresponding to the state ω_{f}. Its spectral properties can be summarized as

$$\ker(L_{\mathrm{f}}) = \mathbb{C}\Omega_{\mathrm{f}}, \qquad \mathrm{spec}_{\mathrm{ac}}(L_{\mathrm{f}}) = \mathbb{R} \setminus \{0\} \tag{1.55}$$

and are illustrated in Figure 1.4.

As already announced we have to reconsider the notion of KMS states for the photon system since $(\mathcal{A}_\mathrm{f}, \alpha_\mathrm{f})$ is neither a C^*- nor a W^*-dynamical system such that the concepts of Section 1.1.2 do not apply directly. We are exclusively interested in the dynamical behavior of states close to the equilibrium at inverse temperature β_f in a relative entropy sense, i.e., we study states $\eta = \langle \xi \,|\, \pi_\mathrm{f}(\,\cdot\,)\xi \,\rangle_{\mathcal{H}_\mathrm{f}^2}$, $\xi \in \mathcal{P}_\mathrm{f}$, on $\mathcal{A}_\mathrm{f}$ which are ω_f-normal. The fact that these states are expressible in terms of the representation π_f and the link (1.54) between the dynamics α_f on $\mathcal{A}_\mathrm{f}$ and the dynamics $[\pi_\mathrm{f}\alpha_\mathrm{f}] = \{[\pi_\mathrm{f}\alpha_\mathrm{f}]^t\}_{t\in\mathbb{R}}$ on $\pi_\mathrm{f}(\mathcal{A}_\mathrm{f})''$, given by

$$[\pi_\mathrm{f}\alpha_\mathrm{f}]^t(A) := e^{iL_\mathrm{f}t} A e^{-iL_\mathrm{f}t},$$

suggests that we transfer the dynamical considerations to the system $(\pi_\mathrm{f}(\mathcal{A}_\mathrm{f})'', [\pi_\mathrm{f}\alpha_\mathrm{f}])$ which turns out to be W^*-dynamical. Identifying an ω_f-normal state $\eta = \langle \xi \,|\, \pi_\mathrm{f}(\,\cdot\,)\xi \,\rangle_{\mathcal{H}_\mathrm{f}^2}$ on $\mathcal{A}_\mathrm{f}$ with the normal state $\langle \xi \,|\, (\,\cdot\,)\xi \,\rangle_{\mathcal{H}_\mathrm{f}^2}$ on $\pi_\mathrm{f}(\mathcal{A}_\mathrm{f})''$ we call η an $(\alpha_\mathrm{f}, \beta_\mathrm{f})$-KMS state if $\langle \xi \,|\, (\,\cdot\,)\xi \,\rangle$ is a $([\pi_\mathrm{f}\alpha_\mathrm{f}], \beta_\mathrm{f})$-KMS state. The relation (1.54) and the computations (1.51) imply that ω_f is a β_f-KMS state w.r.t. α_f in this sense.

The absence of linear independent (w.r.t. Ω_f) vectors in the kernel of L_f implies that there are no ω_f-normal stationary states beside the equilibrium state itself, c.f. (1.7). Further, the fact that the remaining spectrum of L_f is absolutely continuous implies that any ω_f-normal state converges under the time evolution $\{\alpha_\mathrm{f}^t\}_{t\in\mathbb{R}}$ towards the equilibrium as $t \to \infty$. This property is referred to as *return to equilibrium property* and can be understood by the following arguments. The fact that Ω_f is separating for $\pi_\mathrm{f}(\mathcal{A}_\mathrm{f})'' = \pi_\mathrm{f}'(\mathcal{A}_\mathrm{f})'$ implies that Ω_f is cyclic w.r.t. $\pi_\mathrm{f}'(\mathcal{A}_\mathrm{f})$, c.f. [10, Prop. 2.5.3]. Given an ω_f-normal state

$$\eta := \langle \xi \,|\, \pi_\mathrm{f}(\,\cdot\,)\xi \,\rangle_{\mathcal{H}_\mathrm{f}^2}$$

with $\xi \in \mathcal{P}_\mathrm{f}$ we find an approximating state

$$\eta_\varepsilon := \langle \pi_\mathrm{f}'(B)\Omega_\mathrm{f} \,|\, \pi_\mathrm{f}(\,\cdot\,)\pi_\mathrm{f}'(B)\Omega_\mathrm{f} \,\rangle_{\mathcal{H}_\mathrm{f}^2}$$

with $B \in \mathcal{A}_\mathrm{f}$, $\|\pi_\mathrm{f}'(B)\Omega_\mathrm{f}\|_{\mathcal{H}_\mathrm{f}^2} = 1$ and $\|\eta - \eta_\varepsilon\|_{\mathcal{A}_\mathrm{f}^*} < \varepsilon$ for a given $\varepsilon > 0$. This approximation behaves under the time evolution as follows,

$$\begin{aligned}
\eta_\varepsilon(\alpha_\mathrm{f}^t(A)) &= \left\langle \pi_\mathrm{f}'(B)\Omega_\mathrm{f} \,\middle|\, \pi_\mathrm{f}(\alpha_\mathrm{f}^t(A))\pi_\mathrm{f}'(B)\Omega_\mathrm{f} \right\rangle_{\mathcal{H}_\mathrm{f}^2} \\
&= \left\langle \pi_\mathrm{f}'(B^*B)\Omega_\mathrm{f} \,\middle|\, \pi_\mathrm{f}(\alpha_\mathrm{f}^t(A))\Omega_\mathrm{f} \right\rangle_{\mathcal{H}_\mathrm{f}^2} \\
&= \left\langle \pi_\mathrm{f}'(B^*B)\Omega_\mathrm{f} \,\middle|\, e^{iL_\mathrm{f}t}\pi_\mathrm{f}(A)\Omega_\mathrm{f} \right\rangle_{\mathcal{H}_\mathrm{f}^2} \\
&\xrightarrow{t\to\infty} \langle \pi_\mathrm{f}'(B^*B)\Omega_\mathrm{f} \,|\, \Omega_\mathrm{f} \,\rangle_{\mathcal{H}_\mathrm{f}^2} \langle \Omega_\mathrm{f} \,|\, \pi_\mathrm{f}(A)\Omega_\mathrm{f} \,\rangle_{\mathcal{H}_\mathrm{f}^2} \\
&= \omega_\mathrm{f}(A),
\end{aligned}$$

where we used that the group $e^{iL_\mathrm{f}t}$ converges weakly towards the orthogonal projection $|\Omega_\mathrm{f}\rangle\langle\Omega_\mathrm{f}|$ onto the kernel of L_f due to the spectral properties (1.55) of L_f.

Because $\|\eta \circ \alpha_{\mathrm{f}}^t - \eta_\varepsilon \circ \alpha_{\mathrm{f}}^t\|_{\mathcal{A}_{\mathrm{f}}^*} < \varepsilon$ uniformly in $t \in \mathbb{R}$ we obtain

$$\begin{aligned} |\eta(\alpha_{\mathrm{f}}^t(A)) - \omega_{\mathrm{f}}(\alpha_{\mathrm{f}}^t(A))| &\leq |\eta(\alpha_{\mathrm{f}}^t(A)) - \eta_\varepsilon(\alpha_{\mathrm{f}}^t(A))| + |\eta_\varepsilon(\alpha_{\mathrm{f}}^t(A)) - \omega_{\mathrm{f}}(\alpha_{\mathrm{f}}^t(A))| \\ &\leq \varepsilon + |\eta_\varepsilon(\alpha_{\mathrm{f}}^t(A)) - \omega_{\mathrm{f}}(\alpha_{\mathrm{f}}^t(A))| \\ &\xrightarrow{t\to\infty} \varepsilon, \end{aligned}$$

where $\varepsilon > 0$ can be chosen arbitrarily small. This implies the return to equilibrium property

$$\lim_{t\to\infty} \eta(\alpha_{\mathrm{f}}^t(A)) = \omega_{\mathrm{f}}(A) \qquad \text{for all} \qquad A \in \mathcal{A}_{\mathrm{f}}.$$

We add some remarks. As illustrated above the thermal relaxation behavior of states which have finite relative entropy compared with the equilibrium state is predicted by spectral properties of the Liouville operator L_{f}. This concept was elaborated by Jakšić and Pillet in [24]. The absolutely continuous spectrum of L_{f} away from a simple zero eigenvalue goes back to the corresponding spectral properties of H_{f} and it encodes the dissipative character of the system. The continuous spectrum, as it is typical for infinite systems, enables the mechanism of sending energy to infinity which is necessary to force relative normal states into equilibrium by thermal relaxation.

All states relative normal w.r.t. the equilibrium are in its region of attraction. We cannot expect that the equilibrium state ω_{f} at inverse temperature β_{f} is also attracting for states which are separated by an infinite relative entropy. For instance, any KMS state ω_{f}' corresponding to an inverse temperature $\beta_{\mathrm{f}}' \neq \beta$ is stationary under α_{f} and therefore will not converge towards ω_{f}. This, in turn, implies that KMS states of the photon field at different temperatures cannot be normal w.r.t. each other, unlike in the case of the finite particle system where all states are normal.

1.4 The Composed Particle-Photon System

The aim of this work is to study a particle system, as described in Section 1.2, interacting with several photon reservoirs, as introduced in Section 1.3. The setup of the joint system is the following. Each photon reservoir, still decoupled from the particle system, will be prepared in a thermal equilibrium state – or close to it in the relative entropy sense. Each reservoir therefore tends to thermally relax into an equilibrium configuration as long as it is not interacting with the other constituents, c.f. Section 1.3.6. However, we do not require that these reservoir equilibria are w.r.t. the same temperature. In fact, we are most interested in the situation where the reservoir temperatures do not coincide. A state describing that situation would be far from equilibrium since it cannot be normal w.r.t. several equilibria of the

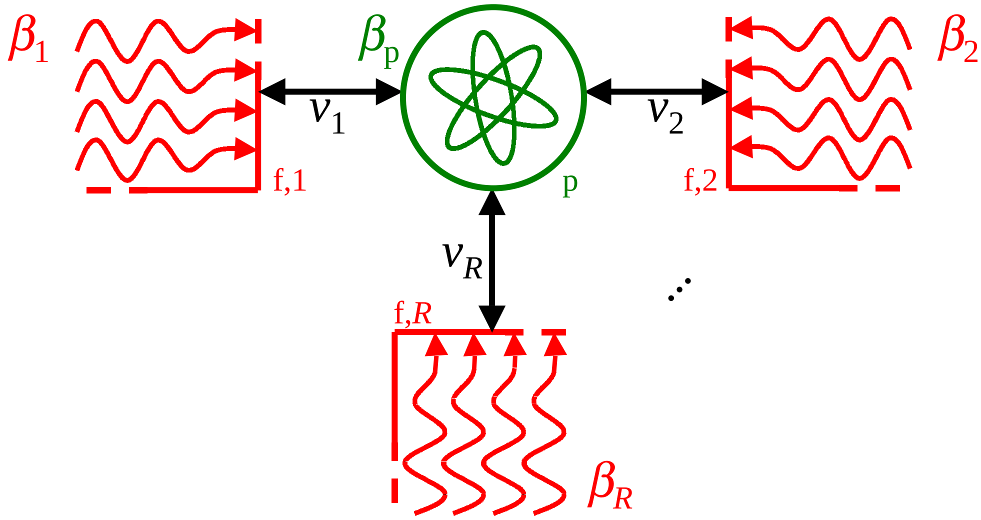

Figure 1.5: Model of the composed particle-photon system.

photon subsystems at different temperatures. We discussed that in Section 1.3.6. The setup is illustrated in Figure 1.5.

1.4.1 Zero and Positive Temperature Framework

The Hilbert space of configurations of a single particle system coexisting with $R \in \mathbb{N}$ reservoirs at zero temperature is given by the tensor product of the particle Hilbert space $\mathcal{H}_\mathrm{p}$ with R copies of the bosonic Fock space $\mathcal{H}_\mathrm{f}$,

$$\mathcal{H} := \mathcal{H}_\mathrm{p} \otimes \left[\bigotimes_{r=1}^{R} \mathcal{H}_\mathrm{f} \right].$$

The Hamilton operator H_0 of the non-interacting system is given by the sum of the Hamiltonians of the sub-systems

$$H_0 := H_\mathrm{p} + \sum_{r=1}^{R} H_{\mathrm{f},r},$$

where

$$H_{\mathrm{f},r} := \mathbb{1}_{\mathcal{H}_\mathrm{p}} \otimes [\mathbb{1}_{\mathcal{H}_\mathrm{f}} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}} \otimes \underbrace{H_\mathrm{f}}_{r} \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}}]$$

is the Hamilton operator only acting on the r^{th} reservoir and we abbreviate

$$H_{\text{p}} \equiv H_{\text{p}} \otimes \left[\bigotimes_{r=1}^{R} \mathbb{1}_{\mathcal{H}_{\text{f}}} \right].$$

The observables of the composed system are collected in the C^*-algebra

$$\begin{aligned} \mathcal{A} &:= \mathcal{A}_{\text{p}} \otimes \left[\bigotimes_{r=1}^{R} \mathcal{A}_{\text{f}} \right] \\ &= \overline{\operatorname{span}\{ A \otimes W(f_1) \otimes \cdots \otimes W(f_R) \mid A \in \mathcal{A}_{\text{p}},\ f_1, \ldots, f_R \in \mathcal{D}_{\text{f}} \}}^{\| \cdot \|_{\mathcal{B}(\mathcal{H})}} \end{aligned}$$

on which the free time evolution

$$\alpha_0^t : \mathcal{A} \to \mathcal{A}, \qquad \alpha_0^t(A) := e^{iH_0 t} A e^{-iH_0 t}, \qquad t \in \mathbb{R},$$

acts.

The initial setup of the joint particle-photon system as described in the introduction to this section is realized by a state which is normal w.r.t.

$$\omega_0 := \omega_{\text{p}} \otimes \left[\bigotimes_{r=1}^{R} \omega_{\text{f},r} \right].$$

Hereby, the state $\omega_{\text{f},r}$ is the KMS state (1.44, 1.46) of the r^{th} reservoir at inverse temperature $\beta_r > 0$. We do not display the temperature dependence of the state ω_0. The particle component ω_{p}, actually, can be chosen arbitrarily (all states of the particle subsystem are normal) but is fixed as the Gibbs state (1.14) at inverse temperature $\beta_{\text{p}} > 0$. The particle temperature therefore should not be thought of as a parameter determining the thermodynamics but as a degree of freedom used to adapt the description of the system at the analysis. For convenience we will later choose the particle temperature the same as the minimal temperature of the reservoirs. Apparently, the state ω_0 is invariant under the free time evolution α_0. Using the results (1.17, 1.47) of the previous sections we write the state in its GNS representation

$$\omega_0 = \langle \Omega_0 \mid \pi(\,\cdot\,)\Omega_0 \rangle,$$

where

$$\pi := \pi_{\text{p}} \otimes \left[\bigotimes_{r=1}^{R} \pi_{\text{f},r} \right] : \mathcal{A} \to \mathcal{B}(\mathcal{H}^2) \tag{1.56}$$

is a representation of $\mathcal{A}$ into the bounded operators on the Hilbert space

$$\mathcal{H}^2 := \mathcal{H}_{\text{p}}^2 \otimes \left[\bigotimes_{r=1}^{R} \mathcal{H}_{\text{f}}^2 \right] \cong \mathcal{H} \otimes \mathcal{H}. \tag{1.57}$$

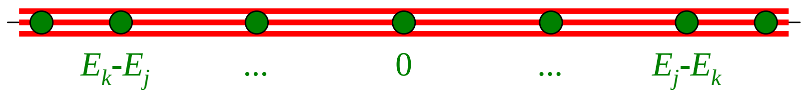

Figure 1.6: Spectrum of the free standard Liouville operator associated with ω_0.

The inner product $\langle \cdot \mid \cdot \rangle$ without index shall refer to the Hilbert space $\mathcal{H}^2$. The representations $\pi_{\mathrm{f},r}$ are copies of the Araki-Woods representation π_{f} introduced in (1.48) corresponding to the equilibrium of the photon gas at inverse temperature $\beta_{\mathrm{f}} = \beta_r$, i.e.,

$$\pi_{\mathrm{f},r}(W(f)) := W(\sqrt{1+\rho_{\mathrm{f},r}}\, f) \otimes W(\sqrt{\rho_{\mathrm{f},r}}\, \overline{f}),$$

where $\rho_{\mathrm{f},r}$ is the radiation density at inverse temperature β_r,

$$\rho_{\mathrm{f},r}(\vec{k}) := \frac{1}{e^{\beta_r \omega(\vec{k})} - 1}.$$

The vector representative

$$\Omega_0 := \Omega_{\mathrm{p}} \otimes \left[\bigotimes_{r=1}^{R} \Omega_{\mathrm{f}} \right] \in \mathcal{H}^2 \tag{1.58}$$

of ω_0 is cyclic and separating for the algebra $\pi(\mathcal{A})''$ which follows from the cyclic and separating properties of Ω_{p} and Ω_{f} w.r.t. $\pi_{\mathrm{p}}(\mathcal{A}_{\mathrm{p}})''$ and $\pi_{\mathrm{f},r}(\mathcal{A}_{\mathrm{f}})''$, resp. We remark that the vector Ω_0 carries, through the factor Ω_{p}, a dependence on the particle temperature β_{p} while the representation π, along with the factors $\pi_{\mathrm{f},r}$, is dependent on the reservoir temperatures β_r.

The modular structure associated with the state ω_0 is given by the modular conjugation

$$J := J_{\mathrm{p}} \otimes \left[\bigotimes_{r=1}^{R} J_{\mathrm{f}} \right]$$

and the modular operator

$$\Delta_0 := \Delta_{\mathrm{p}} \otimes \left[\bigotimes_{r=1}^{R} \Delta_{\mathrm{f},r} \right] = e^{-\mathcal{L}_0},$$

where $\Delta_{\mathrm{f},r} := e^{-\beta_r L_{\mathrm{f}}}$ and

$$\mathcal{L}_0 := \beta_{\mathrm{p}} L_{\mathrm{p}} + \sum_{r=1}^{R} \beta_r L_{\mathrm{f},r} \tag{1.59}$$

is a rescaled free Liouville operator, using the notation

$$L_{\mathrm{f},r} := \mathbb{1}_{\mathcal{H}_\mathrm{p}^2} \otimes [\mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \otimes \underbrace{L_\mathrm{f}}_{r} \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2}]$$

for the free field Liouville operator only acting on the r^{th} reservoir. The relation

$$S_0 \pi(A)\Omega_0 := J\Delta_0^{1/2}\pi(A)\Omega_0 = \pi(A^*)\Omega_0 \tag{1.60}$$

follows directly from previous considerations, refer to (1.18, 1.53, 1.56, 1.58). The free rescaled Hamilton operator corresponding to (1.59) is given by

$$H_{\mathrm{resc}} := \beta_\mathrm{p} H_\mathrm{p} + \sum_{r=1}^{R} \beta_r H_{\mathrm{f},r}$$

and it is used to implement the rescaled time evolution

$$\sigma_0^t : \mathcal{A} \to \mathcal{A}, \qquad \sigma_0^t(A) := e^{iH_{\mathrm{resc}}t} A e^{-iH_{\mathrm{resc}}t}, \qquad t \in \mathbb{R}, \tag{1.61}$$

under which the state ω_0 becomes a $(\sigma_0, 1)$-KMS state. The evolution $\sigma_0 = \{\sigma_0^t\}_{t\in\mathbb{R}}$ is lifted to the modular automorphism group associated with ω_0 via π, i.e.,

$$\pi \circ \sigma_0^t = e^{i\mathcal{L}_0 t} \pi(\,\cdot\,) e^{-i\mathcal{L}_0 t}.$$

The Liouville operator L_0 corresponding to ω_0 w.r.t. the free evolution $\{\alpha_0^t\}_{t\in\mathbb{R}}$ is given by

$$L_0 := L_\mathrm{p} + \sum_{r=1}^{R} L_{\mathrm{f},r},$$

its spectrum is illustrated in Figure 1.6. Note that each eigenvalue of L_p is also an eigenvalue of L_0 to which R complete real lines of continuous spectrum, resulting from the spectra of $L_{\mathrm{f},r}$, are attached. The eigenvalues are therefore covered by $N^2 \times R$ layers of continuum, hence they are embedded to a high degree. We remark that ω_0 is no α_0-KMS state to any temperature unless the reservoir temperatures are all the same. This statement has to be interpreted in the sense of Section 1.3.6, i.e., the state $\langle\, \Omega_0 \mid (\,\cdot\,)\Omega_0 \,\rangle$ on the W^*-algebra $\pi(\mathcal{A})''$ is no KMS state to any inverse temperature w.r.t. the evolution $[\pi\alpha_0] = \{[\pi\alpha_0]^t\}_{t\in\mathbb{R}}$ with

$$[\pi\alpha_0]^t(A) := e^{iL_0 t} A e^{-iL_0 t}$$

for $A \in \pi(\mathcal{A})''$. The anti-linear representation resulting from the modular structure commuting with π is defined in the usual way,

$$\pi' : \mathcal{A} \to \mathcal{B}(\mathcal{H}^2), \qquad \pi'(A) := J\pi(A)J.$$

The positive cone is denoted by

$$\mathcal{P} := \overline{\{\, AJA\Omega_0 \mid A \in \pi(\mathcal{A})'' \,\}}.$$

1.4.2 Jakšić-Pillet Glued Representation

This subsection rather provides some technical and notational tools to handle the particle-photon model. For the sake of simplicity in the presentation of the model we rewrite the Hilbert space $\mathcal{H}^2$ as given in (1.57) and the representation π mapping into the bounded operators on $\mathcal{H}^2$ using an isometric isomorphism which *glues* several Fock spaces together. This procedure goes back to the paper [23] of Jakšić and Pillet and is therefore known as *Jakšić-Pillet gluing*.

We start considering the factor $\mathcal{H}_\mathrm{f}^2 = \mathcal{H}_\mathrm{f} \otimes \mathcal{H}_\mathrm{f} = \mathcal{F}(\mathfrak{h}_1) \otimes \mathcal{F}(\mathfrak{h}_1)$ corresponding to a single photon reservoir which appears as a tensor product of two bosonic Fock space over $\mathfrak{h}_1$. The *exponential law for Fock spaces* provides us with a unitary isomorphism

$$V : \mathcal{F}(\mathfrak{h}_1) \otimes \mathcal{F}(\mathfrak{h}_1) \xrightarrow{\cong} \mathcal{F}(\mathfrak{h}_1 \oplus \mathfrak{h}_1), \tag{1.62}$$

where the target space $\mathcal{F}(\mathfrak{h}_1 \oplus \mathfrak{h}_1)$ is the bosonic Fock space over $\mathfrak{h}_1 \oplus \mathfrak{h}_1$. Its vacuum is denoted by $\Omega_{\mathfrak{h}_1 \oplus \mathfrak{h}_1}$ and the creation and annihilation operators by $a^*_{\mathfrak{h}_1 \oplus \mathfrak{h}_1}(f \oplus g)$ and $a_{\mathfrak{h}_1 \oplus \mathfrak{h}_1}(f \oplus g)$, resp. The isomorphism (1.62) is given by

$$\begin{aligned} V[\Omega_\mathrm{vac} \otimes \Omega_\mathrm{vac}] &:= \Omega_{\mathfrak{h}_1 \oplus \mathfrak{h}_1}, \\ V a^*(f) \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}} V^{-1} &:= a^*_{\mathfrak{h}_1 \oplus \mathfrak{h}_1}(f \oplus 0), \\ V \mathbb{1}_{\mathcal{H}_\mathrm{f}} \otimes a^*(g) V^{-1} &:= a^*_{\mathfrak{h}_1 \oplus \mathfrak{h}_1}(0 \oplus g). \end{aligned}$$

Next, we note that

$$\mathfrak{h}_1 \oplus \mathfrak{h}_1 = L^2[\mathbb{R}^3, d^3\vec{k}] \oplus L^2[\mathbb{R}^3, d^3\vec{k}] \cong L^2[\mathbb{R} \times S^2, d(u, \Sigma)]$$

such that we understand (for a given pair $f, g \in L^2[\mathbb{R}^3, d^3k]$) the direct sum $f \oplus g$ as a square integrable function over $\mathbb{R} \times S^2$ given by

$$[f \oplus g](u, \Sigma) := \begin{cases} u f(u\Sigma), & u \geq 0, \\ u g(-u\Sigma), & u < 0, \end{cases} \qquad (u, \Sigma) \in \mathbb{R} \times S^2.$$

Applying the exponential law for Fock spaces a second time we can represent the R-fold tensor product $\bigotimes_{r=1}^R \mathcal{H}_\mathrm{f}^2$ as a single bosonic Fock space

$$\bigotimes_{r=1}^R \mathcal{H}_\mathrm{f}^2 \cong \mathcal{F}\Big(\bigoplus_{r=1}^R L^2[\mathbb{R} \times S^2, d(u, \Sigma)] \Big) \cong \mathcal{F}\left(L^2[\mathbb{R} \times S^2 \times \mathbb{N}_1^R, d(u, \Sigma, r)] \right),$$

where $\mathbb{N}_1^R := \{1, \ldots, R\}$. Identifying the original Araki-Woods representation with the glued one we may express the creation and annihilation operators acting on the various reservoirs as

$$\begin{aligned} &\sum_{r=1}^R \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \otimes \underbrace{[a^\#(f_r) \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}} + \mathbb{1}_{\mathcal{H}_\mathrm{f}} \otimes a^\#(g_r)]}_{r} \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \\ &\equiv \; a^\#_\mathrm{gl}(f \oplus g), \end{aligned}$$

where $f = (f_1, \ldots, f_R)$, $g = (g_1, \ldots, g_R)$ and

$$[f \oplus g](u, \Sigma, r) := [f_r \oplus g_r](u, \Sigma) = \begin{cases} u f_r(u\Sigma), & u \geq 0, \\ u g_r(-u\Sigma), & u < 0 \end{cases}$$

and

$$a_{\rm gl}(h) = \sum_{r=1}^{R} \int_{\mathbb{R}^3} du \int_{S^2} d\Sigma\, \overline{h(u, \Sigma, r)} a_{\rm gl}(u, \Sigma, r)$$
$$a^*_{\rm gl}(h) = \sum_{r=1}^{R} \int_{\mathbb{R}^3} du \int_{S^2} d\Sigma\, h(u, \Sigma, r) a^*_{\rm gl}(u, \Sigma, r),$$

for $h \in L^2[\mathbb{R} \times S^2 \times \mathbb{N}_1^R]$ being the glued creation and annihilation operators, resp. We denote by

$$a^{\#}_{\rm gl}(u, \Sigma, r)$$
$$\equiv \quad u \times \begin{cases} \mathbb{1}_{\mathcal{H}^2_{\rm f}} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}^2_{\rm f}} \otimes \underbrace{[a^{\#}(u\Sigma) \otimes \mathbb{1}_{\mathcal{H}_{\rm f}}]}_{r} \otimes \mathbb{1}_{\mathcal{H}^2_{\rm f}} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}^2_{\rm f}}, & u \geq 0, \\ \mathbb{1}_{\mathcal{H}^2_{\rm f}} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}^2_{\rm f}} \otimes \underbrace{[\mathbb{1}_{\mathcal{H}_{\rm f}} \otimes a^{\#}(-u\Sigma)]}_{r} \otimes \mathbb{1}_{\mathcal{H}^2_{\rm f}} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}^2_{\rm f}}, & u < 0 \end{cases}$$

the pointwise creation and annihilation operators on the glued space $\mathcal{F}(L^2[\mathbb{R} \times S^2 \times \mathbb{N}_1^R])$. The corresponding field and Weyl operators are given by

$$\Phi_{\rm gl}(h) := \frac{1}{\sqrt{2}} \left[a^*_{\rm gl}(h) + a_{\rm gl}(h) \right],$$
$$W_{\rm gl}(h) := e^{i\Phi_{\rm gl}(h)}.$$

The vacuum is denoted, for notational simplicity, by $\Omega_{\rm vac}$ also. For notational convenience we abbreviate

$$(\Upsilon, dy) := (\mathbb{R} \times S^2 \times \mathbb{N}_1^R, d(u, \Sigma, r)) \tag{1.63}$$

such that we finally achieve the structure

$$\mathcal{H}^2 \equiv \mathcal{H}^2_{\rm p} \otimes \mathcal{F}(L^2[\Upsilon, dy]).$$

The pointwise creation and annihilation operators fulfil the CCR

$$\begin{aligned} \left[a^*_{\rm gl}(u, \Sigma, r), a^*_{\rm gl}(u', \Sigma', r') \right] &= [a_{\rm gl}(u, \Sigma, r), a_{\rm gl}(u', \Sigma', r')] = 0, \\ \left[a_{\rm gl}(u, \Sigma, r), a^*_{\rm gl}(u', \Sigma', r') \right] &= \delta_{r,r'} \delta(u - u') \delta(\Sigma - \Sigma'), \end{aligned} \tag{1.64}$$

as one verifies by applying the CCR (1.30) for a, a^*.

To benefit from these notational simplifications we rewrite the representation π in terms of objects of the glued Fock space $\mathcal{F}(\Upsilon)$. Let's abbreviate

$$\begin{aligned}
a_r^\#(\vec{k}) &:= \mathbb{1}_{\mathcal{H}_\mathrm{f}} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}} \otimes \underbrace{a^\#(\vec{k})}_{r} \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}}, \\
a_r^\#(f) &:= \mathbb{1}_{\mathcal{H}_\mathrm{f}} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}} \otimes \underbrace{a^\#(f)}_{r} \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}}, \\
\Phi_r(f) &:= a_r(f) + a_r^*(f), \\
W_r(f) &:= e^{i\Phi_r(f)}.
\end{aligned}$$

For form factors $f = (f_1, \ldots, f_R) \in \mathcal{D}_\mathrm{f}^R$ and $A \in \mathcal{A}_\mathrm{p}$ we obtain

$$\begin{aligned}
&\pi\Big(\sum_{r=1}^R A \otimes \Phi_r(f_r)\Big) \\
&= \sum_{r=1}^R \pi_\mathrm{p}(A) \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \otimes \Big[\underbrace{\Phi(\sqrt{1+\rho_{\mathrm{f},r}}\, f_r) \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}} + \mathbb{1}_{\mathcal{H}_\mathrm{f}} \otimes \Phi(\sqrt{\rho_{\mathrm{f},r}}\, \overline{f}_r)}_{r}\Big] \\
&\qquad \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \\
&= \pi_\mathrm{p}(A) \otimes \Phi_\mathrm{gl}\big((\sqrt{1+\rho_{\mathrm{f},r}}\, f_r)_{r=1,\ldots,R} \oplus (\sqrt{\rho_{\mathrm{f},r}}\, \overline{f}_r)_{r=1,\ldots,R}\big) \\
&= \pi_\mathrm{p}(A) \otimes \Phi_\mathrm{gl}(\mathfrak{g}(f)),
\end{aligned}$$

where the gluing function $\mathfrak{g}$ is defined as

$$\begin{gathered}
\mathfrak{g} : \mathcal{D}_\mathrm{f}^R \to L^2[\Upsilon], \\
f = (f_1, \ldots, f_R) \mapsto \mathfrak{g}(f) := (\sqrt{1+\rho_{\mathrm{f},r}}\, f_r)_{r=1,\ldots,R} \oplus (\sqrt{\rho_{\mathrm{f},r}}\, \overline{f}_r)_{r=1,\ldots,R},
\end{gathered} \tag{1.65}$$

i.e.,

$$\mathfrak{g}(f)(u,\Sigma,r) = \sqrt{\frac{u}{1-e^{-\beta_r u}}} \times \begin{cases} \sqrt{u}\, f_r(u\Sigma), & u \geq 0, \\ (-\sqrt{-u})\, \overline{f}_r(-u\Sigma), & u < 0. \end{cases}$$

Note that $\mathfrak{g}$ incorporates the reservoir temperatures β_r as the representation π does. This finally leads to the following form of the representation π,

$$\pi\Big(A \otimes W(f_1) \otimes \cdots \otimes W(f_R)\Big) = \pi_\mathrm{p}(A) \otimes W_\mathrm{gl}(\mathfrak{g}(f)).$$

The anti-linear representation π' can be treated in the same way and we find that

$$\pi'\Big(A \otimes W(f_1) \otimes \cdots \otimes W(f_R)\Big) = \pi'_\mathrm{p}(A) \otimes W_\mathrm{gl}(\mathfrak{g}'(f)),$$

where

$$\begin{aligned}
\mathfrak{g}'(f)(u,\Sigma,r) &:= \overline{-\mathfrak{g}(f)(-u,\Sigma,r)} \\
&= \sqrt{\frac{u}{e^{\beta_r u}-1}} \times \begin{cases} \sqrt{u}\, \overline{f}_r(u\Sigma), & u \geq 0, \\ (-\sqrt{-u})\, f_r(-u\Sigma), & u < 0. \end{cases}
\end{aligned}$$

The field Liouville operators have the following form in the glued representation,

$$L_{\mathrm{f},r} = d\Gamma_{\mathrm{gl}}\left((u,\Sigma,j)\mapsto \delta_{j,r}u\right) \equiv \int_{\mathbb{R}} du \int_{S^2} d\Sigma\, a^*_{\mathrm{gl}}(u,\Sigma,r)u a_{\mathrm{gl}}(u,\Sigma,r),$$

and the free Liouville operator can be expressed as

$$L_0 = L_{\mathrm{p}} + L_{\mathrm{res}}$$

where

$$L_{\mathrm{res}} := d\Gamma_{\mathrm{gl}}\left((u,\Sigma,r)\mapsto u\right) \equiv \int_{\Upsilon} d(u,\Sigma,r)\, a^*_{\mathrm{gl}}(u,\Sigma,r)u a_{\mathrm{gl}}(u,\Sigma,r). \tag{1.66}$$

is the contribution of all photon fields.

The commutation relations (1.64) allow the same reasoning as in Section 1.3.1 which leads to the pull through formula for the glued Fock space. We state this important formula. Let $F : \mathbb{R} \to \mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)$ be a measurable, operator valued function and let

$$d\Gamma_{\mathrm{gl}}(\lambda) \equiv \int_{\Upsilon} dy\, a^*_{\mathrm{gl}}(y)\lambda(y)a_{\mathrm{gl}}(y)$$

be the second quantization of the measurable function $\lambda : \Upsilon \to \mathbb{R}$, in particular, $d\Gamma_{\mathrm{gl}}(\lambda)$ is a self-adjoint operator. Then, the following commutation relations hold true,

$$\begin{aligned} a_{\mathrm{gl}}(y)F\left(d\Gamma_{\mathrm{gl}}(\lambda)\right) &= F\left(d\Gamma_{\mathrm{gl}}(\lambda)+\lambda(y)\right)a_{\mathrm{gl}}(y),\\ a^*_{\mathrm{gl}}(y)F\left(d\Gamma_{\mathrm{gl}}(\lambda)+\lambda(y)\right) &= F\left(d\Gamma_{\mathrm{gl}}(\lambda)\right)a^*_{\mathrm{gl}}(y), \end{aligned} \tag{1.67}$$

to be understood in a weak sense on suitable domains.

1.4.3 Particle-Photon Interaction

We consider photon creation and annihilation processes of first order as interaction between the particle system and the photon reservoirs. The interaction operator is given by

$$v_r := \sqrt{2}\,\Phi_r(G_r) := a_r(G_r) + a^*_r(G_r), \tag{1.68}$$

where $G_r \in L^2[\mathbb{R}^3; \mathcal{B}(\mathcal{H}_{\mathrm{p}})]$, $r = 1,\ldots,R$, are $\mathcal{B}(\mathcal{H}_{\mathrm{p}})$-valued, square integrable functions and

$$\begin{aligned} a_r(F) &:= \int_{\mathbb{R}^3} d^3\vec{k}\, F(\vec{k})^* \otimes a_r(\vec{k}),\\ a^*_r(F) &:= \int_{\mathbb{R}^3} d^3\vec{k}\, F(\vec{k}) \otimes a^*_r(\vec{k}) \end{aligned} \tag{1.69}$$

is the extension of creation and annihilation operators for the r^{th} reservoir to $\mathcal{B}(\mathcal{H}_{\rm p})$-valued form factors F. The definition (1.69) has to be understood in the sense of (1.28) and (1.29), i.e., in a weak sense. The operators v_r, $r = 1, \ldots, R$, are self-adjoint and appear as a perturbation of the free Hamilton operator H_0,

$$H := H_0 + gv \qquad \text{with} \qquad v := \sum_{r=1}^{R} v_r, \tag{1.70}$$

where $g \in \mathbb{R}$ is the coupling constant. For notational convenience we take $g \geq 0$. To ensure the self-adjointness of the interacting Hamilton operator H we assume that the following hypothesis is fulfilled.

Hypothesis I-1.2 (Self-Adjointness of H) *We assume that the coupling functions G_r, $r = 1, \ldots, R$, obey the following weighted L^2-norm,*

$$\int_{\mathbb{R}^3} d^3\vec{k} \left[1 + \frac{1}{\omega(\vec{k})}\right] \left\| G_r(\vec{k}) \right\|^2_{\mathcal{B}(\mathcal{H}_{\rm p})} < \infty. \tag{1.71}$$

This assumption results in

Lemma 1.3 (Essential Self-Adjointness of H) *Under the assumption of Hypothesis I-1.2, the perturbed Hamilton operator H given in (1.70) is essentially self-adjoint for g sufficiently small.*

Proof. With exactly the same arguments as in the proof of Lemma A.4 we see that the assumptions (1.71) on the coupling functions G_r imply that

$$\left\| \left[a_r(G_r) + a_r^*(G_r)\right] (H_{{\rm f},r} + 1)^{-1} \right\|_{\mathcal{B}(\mathcal{H})} \leq \int_{\mathbb{R}^3} d^3\vec{k} \left[1 + \frac{1}{\omega(\vec{k})}\right] \left\| G_r(\vec{k}) \right\|^2_{\mathcal{B}(\mathcal{H}_{\rm p})}.$$

Therefore the interaction gv is relatively bounded w.r.t. H_0 with relative bound

$$\left\| gv\,(H_0 - E_0 + 1)^{-1} \right\|_{\mathcal{B}(\mathcal{H})} \leq g \sum_{r=1}^{R} \int_{\mathbb{R}^3} d^3\vec{k} \left[1 + \frac{1}{\omega(\vec{k})}\right] \left\| G_r(\vec{k}) \right\|^2_{\mathcal{B}(\mathcal{H}_{\rm p})} < 1$$

for g sufficiently small. The essential self-adjointness of H follows by the Kato-Rellich Theorem, see [38, Sect. X.2, Thm. X.12]. ∎

The operator H therefore extends to a self-adjoint operator whose extension is denoted by the same symbol. This allows us to introduce a Heisenberg time evolution $\alpha = \{\alpha^t\}_{t\in\mathbb{R}}$ on bounded operators A on $\mathcal{H}$ by

$$\alpha^t(A) := e^{iHt} A e^{-iHt}$$

which can be expressed as a Dyson series expansion

$$\alpha^t(A) = \alpha_0^t(A) + \sum_{n=1}^{\infty}(ig)^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \left[\alpha_0^{t_n}(v), \left[\ldots, \left[\alpha_0^{t_1}(v), \alpha_0^t(A)\right]\right]\right]. \quad (1.72)$$

The extension of the unperturbed time evolution α_0 to unbounded operators v_r is understood as

$$\alpha_0^t(v_r) := e^{iH_0 t} v_r e^{-iH_0 t} = a_r\left(e^{i\omega t}\alpha_{\mathrm{p}}^t(G_r(\,\cdot\,))\right) + a_r^*\left(e^{i\omega t}\alpha_{\mathrm{p}}^t(G_r(\,\cdot\,))\right). \quad (1.73)$$

The convergence of the r.h.s. of (1.72) towards $\alpha^t(A)$ holds strongly on vectors of the form $\psi = \psi_{\mathrm{p}} \otimes [W(g_1)\Omega_{\mathrm{vac}}] \otimes \cdots \otimes [W(g_R)\Omega_{\mathrm{vac}}]$ and for observables of the form $A = A_{\mathrm{p}} \otimes W(f_1) \otimes \cdots \otimes W(f_R)$, building a dense subalgebra in $\mathcal{A}$, as one proves with similar arguments as those presented in Lemma B.1(i). Since the Dyson series expansion (1.72) only serves as a motivation for a redefinition of the perturbed time evolution and never enters the analysis we content ourselves with this remark about the well-definedness of (1.72). We note that the evolution group $\{\alpha^t\}_{t\in\mathbb{R}}$ is well defined on bounded operators, however, it does not necessarily leave the Weyl operators invariant (and therefore the algebra $\mathcal{A}$ neither). Using the *Trotter product formula* we can see that for an observable $A \in \mathcal{A}$, the following holds true,

$$\alpha^t(A) = \operatorname*{s-lim}_{n\to\infty} \left[e^{iH_0 t/n} e^{igvt/n}\right]^n A \left[e^{-igvt/n} e^{-iH_0 t/n}\right]^n.$$

Since $e^{igvt/n}$ is in the weak closure of the algebra $\mathcal{A}$ and since further $\alpha_0^t = e^{iH_0 t}(\,\cdot\,)e^{-iH_0 t}$ leaves $\mathcal{A}$ invariant, we can conclude that $\alpha^t(A)$ lies in the weak closure $\mathcal{A}''$ of $\mathcal{A}$ (this is in fact a trivial result because $\mathcal{A}'' = \mathcal{B}(\mathcal{H})$ as one easily verifies). This implies that

$$\alpha^t(\mathcal{A}) \subseteq \mathcal{A}'' = \mathcal{B}(\mathcal{H}).$$

The invariance of $\mathcal{A}$ under α poses a subtle difficulty in the respect that we aim to evolve ω_0-normal states under α. The problem is that the state ω_0 has no normal extension on $\mathcal{A}''$, i.e., it is not σ-weakly continuous extendable to $\mathcal{A}''$. Thus, the composition $\omega_0 \circ \alpha^t$ does not define a priori a state on $\mathcal{A}$. We will bypass this problem by interpreting the time evolution α not as a group of automorphism on observables but as a group of transformations acting on ω_0-normal states. Hence we have to redefine for any $\eta \in \mathcal{N}_{\omega_0}(\mathcal{A})$ what we understand by $\alpha^t * \eta$ rather then considering the ill defined composition $\eta \circ \alpha^t$.

We give some motivation for the definition of the evolution on ω_0-normal states. Let $\eta = \langle\, \xi \,|\, \pi(\,\cdot\,)\xi \,\rangle$ be an ω_0-normal state where $\xi \in \mathcal{P}$. The task is to make sense of $\pi \circ \alpha^t(A)$ for $A \in \mathcal{A}$ in order to understand $\eta \circ \alpha^t$, i.e., we define the action $\alpha^t * \pi$ of α on π. For this purpose we already extended the representation π_{f} to creation and annihilation operators. For operator-valued form factors $F \in L^2[\mathbb{R}^3; \mathcal{B}(\mathcal{H}_\mathrm{p})]$ with $\omega^{-1/2}F \in L^2[\mathbb{R}^3; \mathcal{B}(\mathcal{H}_\mathrm{p})]$ we may define in a similar way to (1.49),

$$\begin{aligned}
\pi(a_r(F)) &:= \int_{\mathbb{R}^3} d^3\vec{k}\, \pi\left(F(\vec{k})^* \otimes a_r(\vec{k})\right) \\
&= \int_{\mathbb{R}^3} d^3\vec{k} \Big\{ \left[F(\vec{k})^* \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}}\right] \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \\
&\qquad \otimes \Big[\underbrace{\sqrt{1+\rho_{\mathrm{f},r}(\vec{k})}\, a(\vec{k}) \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}}}_{r}\Big] \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \\
&\qquad + \left[F(\vec{k})^* \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}}\right] \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \\
&\qquad \otimes \Big[\underbrace{\sqrt{\rho_{\mathrm{f},r}(\vec{k})}\, \mathbb{1}_{\mathcal{H}_\mathrm{f}} \otimes a^*(\vec{k})}_{r}\Big] \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_\mathrm{f}^2} \Big\}
\end{aligned} \tag{1.74}$$

and

$$\pi(a_r^*(F)) := \pi(a_r(F))^*.$$

In the glued version the representation reads

$$\pi\Big(\sum_{r=1}^{R} \Phi_r(F_r)\Big) = \Phi_\mathrm{gl}(\mathfrak{g}(F)) \tag{1.75}$$

for $F = (F_1, \ldots, F_R)$, where the gluing function $\mathfrak{g}$ extends to

$$\begin{aligned}
&\mathfrak{g} : \mathcal{B}(\mathcal{H}_\mathrm{p}) \otimes \mathcal{D}_\mathrm{f}^R \to \mathcal{B}(\mathcal{H}_\mathrm{p}^2) \otimes \Upsilon, \\
&\mathfrak{g}(F)(u, \Sigma, r) := \sqrt{\frac{u}{1-e^{-\beta_r u}}} \times \begin{cases} \sqrt{u}\,[F_r(u\Sigma) \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}}], & u \geq 0, \\ (-\sqrt{-u})\,[F_r(-u\Sigma)^* \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}}], & u < 0. \end{cases}
\end{aligned} \tag{1.76}$$

and the glued field operator handles operator valued form factors $F \in L^2[\mathbb{R} \times S^2 \times \mathbb{N}_1^R; \mathcal{B}(\mathcal{H}_\mathrm{p}^2)]$ as

$$\Phi_\mathrm{gl}(F) := \frac{1}{\sqrt{2}} \int_{\Upsilon} dy\, \left[F(y)^* \otimes a_\mathrm{gl}(y) + F(y) \otimes a_\mathrm{gl}^*(y)\right].$$

Correspondingly, we can extend the anti-linear representation π' by setting

$$\pi'\Big(\sum_{r=1}^{R} \Phi_r(F_r)\Big) := J\pi\Big(\sum_{r=1}^{R} \Phi_r(F_r)\Big)J = \Phi_\mathrm{gl}(\mathfrak{g}'(F))$$

where $F = (F_1, \ldots, F_R)$ and

$$\mathfrak{g}'(F)(u, \Sigma, r) := -J_\mathrm{p}\mathfrak{g}(F)(-u, \Sigma, r)J_\mathrm{p}.$$

It is easy to verify that

$$[\pi(\Phi_r(F)), \pi'(\Phi_r(G))] = 0$$

for all $F, G \in L^2[\mathbb{R}^3; \mathcal{B}(\mathcal{H}_{\rm p})]$ with $\omega^{-1/2}F, \omega^{-1/2}G \in L^2[\mathbb{R}^3; \mathcal{B}(\mathcal{H}_{\rm p})]$.

Having extended π to the polynomial algebra of creation and annihilation operators with $\mathcal{B}(\mathcal{H}_{\rm p})$-valued form factors we are in the position to apply the ω_0-normal η to $a_r^{\#}(\mathcal{G})$ by making the canonical convention

$$\eta\left(a_r^{\#}(G)\right) := \left\langle \xi \,\middle|\, \pi(a_r^{\#}(G))\xi \right\rangle \tag{1.77}$$

where we assume that ξ is in the form domain of $\pi(a_r^{\#}(G))$.

The above considerations and Equation (1.73) allow us to apply the representation π to the Dyson expansion on the r.h.s. of (1.72). We first observe that

$$\begin{aligned}
&\pi\left(\left[\alpha_0^{t_n}(v), \left[\ldots, \left[\alpha_0^{t_1}(v), \alpha_0^t(A)\right]\right]\right]\right) \\
&\quad = \left[\pi(\alpha_0^{t_n}(v)), \left[\ldots, \left[\pi(\alpha_0^{t_1}(v)), \pi(\alpha_0^t(A))\right]\right]\right] \\
&\quad = \Big[e^{iL_0 t_n}[\pi(v) - \pi'(v)]e^{-iL_0 t_n}, \Big[\\
&\qquad \ldots, \left[e^{iL_0 t_1}[\pi(v) - \pi'(v)]e^{-iL_0 t_1}, e^{iL_0 t}\pi(A)e^{-iL_0 t}\right]\Big]\Big].
\end{aligned}$$

We now define $\alpha^t * \pi \equiv \pi \circ \alpha^t \equiv \pi(\alpha^t(\,\cdot\,))$ as

$$\begin{aligned}
\alpha^t * \pi(A) \;:=\;& \pi(\alpha_0^t(A)) + \sum_{n=1}^{\infty}(ig)^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \\
& \times\pi\left(\left[\alpha_0^{t_n}(v), \left[\ldots, \left[\alpha_0^{t_1}(v), \alpha_0^t(A)\right]\right]\right]\right) \\
=\;& e^{iL_0 t}\pi(A)e^{-iL_0 t} + \sum_{n=1}^{\infty}(ig)^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \\
& \times\Big[e^{iL_0 t_n}[\pi(v) - \pi'(v)]e^{-iL_0 t_n}, \Big[\\
& \quad \ldots, \left[e^{iL_0 t_1}[\pi(v) - \pi'(v)]e^{-iL_0 t_1}, e^{iL_0 t}\pi(A)e^{-iL_0 t}\right]\Big]\Big]
\end{aligned} \tag{1.78}$$

where all limit procedures have to be understood in a strong sense on vectors of the form $\psi = [B_{\rm p} \otimes W_{\rm gl}(f)]\Omega_0$ and for observables of the form $A = A_p \otimes W(f_1) \otimes \cdots \otimes W(f_R)$, refer to Lemma B.1(i) for a similar situation. To simplify the expression (1.78) we introduce the perturbed Liouville operator

$$L := L_0 + gI$$

where the perturbation part is given by

$$I := \pi(v) - \pi'(v) = a_{\rm gl}(\mathcal{G} - \mathcal{G}') + a^*_{\rm gl}(\mathcal{G} - \mathcal{G}') \tag{1.79}$$

with

$$\mathcal{G} := \mathfrak{g}(G_1, \ldots, G_R),$$

$$\mathcal{G}(u, \Sigma, r) = \sqrt{\frac{u}{1 - e^{-\beta_r u}}} \times \begin{cases} \sqrt{u}\, G_r(u\Sigma) \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}}, & u \geq 0, \\ (-\sqrt{-u}) G_r(-u\Sigma)^* \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}}, & u < 0, \end{cases} \tag{1.80}$$

and

$$\mathcal{G}' := \mathfrak{g}'(G_1, \ldots, G_R),$$

$$\mathcal{G}'(u, \Sigma, r) = \sqrt{\frac{u}{e^{\beta_r u} - 1}} \times \begin{cases} \sqrt{u}\, \mathbb{1}_{\mathcal{H}_\mathrm{p}} \otimes \overline{G_r(u\Sigma)}^*, & u \geq 0, \\ (-\sqrt{-u}) \mathbb{1}_{\mathcal{H}_\mathrm{p}} \otimes \overline{G_r(-u\Sigma)}, & u < 0, \end{cases} \tag{1.81}$$

We remark that this is the analog to the perturbed Liouville operator (1.9) for the C^*-dynamical case although in our case the perturbation v is not a bounded operator of the algebra $\mathcal{A}$. To ensure self-adjointness of L we require the following properties of G_r.

Hypothesis II-1.4 (Self-Adjointness of L) *We assume that the coupling functions G_r, $r = 1, \ldots, R$, obey the following bounds,*

$$\int_{\mathbb{R}^3} d^3\vec{k}\, \left[\omega(\vec{k}) + \omega(\vec{k})^{-2}\right] \left\| G_r(\vec{k}) \right\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p})} < \infty.$$

Lemma 1.5 *Let $F \in L^2[\Upsilon; \mathcal{B}(\mathcal{H}_\mathrm{p}^2)]$ be an operator valued function on the glued space Υ obeying the weighted L^2 norm*

$$\int_{\Upsilon} d(u, \Sigma, r)\, \left(|u| + |u|^{-1}\right) \|F(u, \Sigma, r)\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)} < \infty. \tag{1.82}$$

Then the operator

$$L_0 + a_\mathrm{gl}(F) + a^*_\mathrm{gl}(F) \tag{1.83}$$

is essentially self-adjoint on the domain of the auxiliary operator

$$L_\mathrm{aux} := d\Gamma_\mathrm{gl}\left((u, \Sigma, r) \mapsto |u|\right) \equiv \int_{\Upsilon} d(u, \Sigma, r)\, a^*_\mathrm{gl}(u, \Sigma, r) |u| a_\mathrm{gl}(u, \Sigma, r).$$

Proof. Let $\varphi, \psi \in \mathcal{D}(L_\mathrm{aux})$. We estimate

$$\begin{aligned} &\left|\langle \psi \mid \left(L_0 + a_\mathrm{gl}(F) + a^*_\mathrm{gl}(F)\right) \varphi \rangle\right| \\ &\leq \left[\left\| \frac{L_\mathrm{res} + L_\mathrm{p}}{L_\mathrm{aux} + 1} \right\| + \left\| (L_\mathrm{aux} + 1)^{-1/2} \left(a_\mathrm{gl}(F) + a^*_\mathrm{gl}(F)\right) (L_\mathrm{aux} + 1)^{-1/2} \right\| \right] \\ &\quad \times \left\| (L_\mathrm{aux} + 1)^{1/2} \psi \right\| \left\| (L_\mathrm{aux} + 1)^{1/2} \varphi \right\| \\ &\leq [1 + \|L_\mathrm{p}\| + C] \left\| (L_\mathrm{aux} + 1)^{1/2} \psi \right\| \left\| (L_\mathrm{aux} + 1)^{1/2} \varphi \right\|, \end{aligned}$$

where we used that L_{res} is relatively L_{aux} bounded with relative bound smaller than 1 and where the constant C is given by

$$\begin{aligned} C^2 &:= 2 \int_\Upsilon d(u,\Sigma,r)\, |u|^{-1} \, \|F(u,\Sigma,r)\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)} \\ &\leq 2 \int_\Upsilon d(u,\Sigma,r)\, \left(|u| + |u|^{-1}\right) \|F(u,\Sigma,r)\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)} < \infty \end{aligned}$$

due to Lemma A.4 (for $(\delta,\tau) = (i\frac{\pi}{2}, 0)$). On the domain $\mathcal{D}(L_{\mathrm{aux}}^{3/2})$ we have the commutator relation

$$\left[L_{\mathrm{aux}}, L_0 + a_{\mathrm{gl}}(F) + a^*_{\mathrm{gl}}(F)\right] = \left[L_{\mathrm{aux}}, a_{\mathrm{gl}}(F) + a^*_{\mathrm{gl}}(F)\right] = a^*_{\mathrm{gl}}(|u|F) - a_{\mathrm{gl}}(|u|F)$$

because of the pull through formula (1.67). Therefore, for $\varphi, \psi \in \mathcal{D}(L_{\mathrm{aux}}^{3/2})$, we obtain

$$\begin{aligned} &\left|\left\langle \psi \,\middle|\, \left[L_{\mathrm{aux}} + 1, L_0 + a_{\mathrm{gl}}(F) + a^*_{\mathrm{gl}}(F)\right] \varphi \right\rangle\right| \\ &\quad \leq C' \left\|(L_{\mathrm{aux}}+1)^{1/2}\psi\right\| \left\|(L_{\mathrm{aux}}+1)^{1/2}\varphi\right\|, \end{aligned}$$

where, again by Lemma A.4,

$$\begin{aligned} C'^2 &:= 2 \int_\Upsilon d(u,\Sigma,r)\, |u|^{-1} \, \||u|F(u,\Sigma,r)\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)} \\ &\leq 2 \int_\Upsilon d(u,\Sigma,r)\, \left(|u| + |u|^{-1}\right) \|F(u,\Sigma,r)\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)} < \infty. \end{aligned}$$

A variant of the *Nelson's Commutator Theorem* as presented in [38, Thm. X.36'] implies that the operator (1.83) is essentially self-adjoint on $\mathcal{D}(L_{\mathrm{aux}})$. ■

Corollary 1.6 (Self-Adjointness of L) *Under the assumptions of Hypothesis II-1.4, the operators*

$$\begin{aligned} L &= L_0 + g\left[\pi(v) - \pi'(v)\right], \\ L^{(\ell)} &:= L_0 + g\pi(v), \\ L^{(\mathrm{r})} &:= L_0 - g\pi'(v), \\ K^{(s)} &:= L_0 + g\left[\pi(v) - \pi'\left(\sigma_0^s \circ \alpha_0^{-\beta s}(v)\right)\right], \qquad s \in \mathbb{R},\ \beta > 0, \end{aligned}$$

are essentially self-adjoint on the domain of the operator L_{aux}. Recall that the group $(\sigma_0^s)_{s\in\mathbb{R}}$ entering $K^{(s)}$ was introduced in (1.61).

Proof. The assumption of Hypothesis II-1.4 implies that the form factors $\mathcal{G}$ and $\mathcal{G}'$ appearing in the definition (1.79) of the perturbation $I = \pi(v) - \pi'(v)$ obey the bound (1.82),

$$\begin{aligned}
&\int_{\Upsilon} d(u,\Sigma,r)\ \left(|u| + |u|^{-1}\right) \|\mathcal{G}(u,\Sigma,r)\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)} \\
&= \sum_{r=1}^{R} \int_0^\infty u^2 du \int_{S^2} d\Sigma\, \frac{|u| + |u|^{-1}}{1 - e^{-\beta_r u}} \|G_r(u\Sigma)\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p})} \\
&\quad - \sum_{r=1}^{R} \int_{-\infty}^{0} u^2 du \int_{S^2} d\Sigma\, \frac{|u| + |u|^{-1}}{1 - e^{-\beta_r u}} \|G_r(-u\Sigma)\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p})} \\
&= \sum_{r=1}^{R} \int_{R^3} d^3\vec{k} \left(\omega(\vec{k}) + \omega(\vec{k})^{-1}\right) \left(\frac{1}{1 - e^{-\beta_r \omega(\vec{k})}} + \frac{1}{e^{\beta_r \omega(\vec{k})} - 1}\right) \left\|G_r(\vec{k})\right\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p})} \\
&\leq \sum_{r=1}^{R} \int_{R^3} d^3\vec{k} \left(\omega(\vec{k}) + \omega(\vec{k})^{-1}\right) \left(2\beta_r^{-1}\omega(\vec{k})^{-1} + 1\right) \left\|G_r(\vec{k})\right\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p})} \\
&\leq \sum_{r=1}^{R} \left(2\beta_r^{-1} + 1\right) \int_{R^3} d^3\vec{k} \left(\omega(\vec{k}) + 1 + \omega(\vec{k})^{-1} + \omega(\vec{k})^{-2}\right) \left\|G_r(\vec{k})\right\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p})} \\
&< \infty.
\end{aligned}$$

The same estimate holds for $\mathcal{G}'$. Thus, Lemma 1.5 implies that L is essentially self-adjoint on $\mathcal{D}(L_\mathrm{aux})$. The assertion for the other operators is proved in the same way. ∎

Corollary 1.6 guarantees the existence of the strongly continuous one parameter group $\{e^{iLt}\}_{t\in\mathbb{R}}$. We find that the expression (1.78) is the Dyson series expansion of $e^{iLt}\pi(A)e^{-iLt}$, i.e.,

$$\alpha^t * \pi(A) = e^{iLt}\pi(A)e^{-iLt}. \tag{1.84}$$

This statement, again, employs the kind of arguments as given in Lemma B.1(i). The evolution α is lifted to an automorphism group $[\pi\alpha] = \{[\pi\alpha]^t\}_{t\in\mathbb{R}}$ on $\pi(\mathcal{A})''$ given by

$$[\pi\alpha]^t(A) := e^{iLt}Ae^{-iLt}, \qquad A \in \pi(\mathcal{A})''.$$

We take this relation as a definition and consider (1.73, 1.78) as its formal motivation rather than a mathematically rigorous derivation.

The excursion leading to the relation (1.84) motivates us to define the time evolution of an arbitrary ω_0-normal state $\eta = \langle\, \xi \mid \pi(\,\cdot\,)\xi \,\rangle$ by

$$\alpha^t * \eta := \left\langle\, \xi \,\middle|\, \alpha^t * \pi(\,\cdot\,)\xi \,\right\rangle = \left\langle\, \xi \,\middle|\, e^{iLt}\pi(\,\cdot\,)e^{-iLt}\xi \,\right\rangle,$$

having in mind that $\alpha^t * \eta$ does not result from an automorphism α^t on $\mathcal{A}$. Nevertheless, we will henceforth write $\eta \circ \alpha^t$ instead of $\alpha^t * \eta$ always being aware in which sense the composition has to be understood and that it only makes sense for ω_0-normal states.

The invariance of $\mathcal{A}$ under α^t has a further consequence that $(\mathcal{A}, \alpha)$ does not define a dynamical system of any kind. The modular theory derived through Section 1.1 is not directly applicable. We find a remedy in the same spirit of Section 1.3.6 where we discussed the interpretation of $(\mathcal{A}_\mathrm{f}, \alpha_\mathrm{f})$ as a dynamical system. We first remark the following

Lemma 1.7 *The lifted system* $(\pi(\mathcal{A})'', [\pi\alpha])$ *is a* W^**-dynamical system.*

Proof. We have to verify that $[\pi\alpha]$ leaves $\pi(\mathcal{A})''$ invariant. We choose $A \in \pi(\mathcal{A})''$ and denote $L^{(\ell)} = L_0 + V$ where $V := g\pi(v)$ and write $V' = JVJ = g\pi'(v)$. By Corollary 1.6 the operator $L^{(\ell)}$ has a self-adjoint extension and we obtain with the help of the Trotter product formula,

$$e^{iL^{(\ell)}t} A e^{-iL^{(\ell)}t} = \operatorname*{s-lim}_{n\to\infty} \left[e^{iL_0 t/n} e^{iVt/n}\right]^n A \left[e^{-iVt/n} e^{-iL_0 t/n}\right]^n .$$

We show that $e^{iVt/n} \in \pi(\mathcal{A})''$. To this end we choose an arbitrary $B' \in \pi(\mathcal{A})'$ and observe that

$$\left\langle \psi \mid \left[e^{iVt/n}, B'\right] \varphi \right\rangle = \sum_{k=0}^{\infty} \frac{(igt)^k}{n^k k!} \left\langle \psi \mid [\pi(v)^k, B'] \varphi \right\rangle = 0,$$

for ψ, φ being from a dense set of analytic vectors for the self-adjoint operator $\pi(v)$ using that $\pi(v)$ and B' commute. Since $[e^{iVt/n}, B']$ is a bounded operator we conclude that the commutator vanishes on the whole space and therefore $e^{iVt/n} \in \pi(\mathcal{A})''$. Since L_0, as the unperturbed Liouville operator, generates an automorphism group leaving $\pi(\mathcal{A})''$ invariant we obtain

$$\left[e^{iL_0 t/n} e^{iVt/n}\right]^n A \left[e^{-iVt/n} e^{-iL_0 t/n}\right]^n \in \pi(\mathcal{A})''.$$

The strong limit does not leave the weak closure such that

$$e^{iL^{(\ell)}t} A e^{-iL^{(\ell)}t} \in \pi(\mathcal{A})''.$$

A second application of the Trotter product formula yields

$$\begin{aligned} [\pi\alpha]^t(A) &= \operatorname*{s-lim}_{n\to\infty} \left[e^{iL^{(\ell)}t/n} e^{-iV't/n}\right]^n A \left[e^{iV't/n} e^{-iL^{(\ell)}t/n}\right]^n \\ &= e^{iL^{(\ell)}t} A e^{-iL^{(\ell)}t} \in \pi(\mathcal{A})'' \end{aligned}$$

where we used that $e^{-iV't/n} = Je^{iVt/n}J \in \pi(\mathcal{A})'$ commutes with A. It is obvious that $t \mapsto e^{iLt}Ae^{-iLt}$ is strongly continuous for every $A \in \pi(\mathcal{A})''$ which is the appropriate continuity property for W^*-dynamical systems. ∎

We further have an identification between ω_0-normal states on $\mathcal{A}$ and normal states on $\pi(\mathcal{A})''$ via

$$\langle\, \xi \mid \pi(\,\cdot\,)\xi \,\rangle \qquad \leftrightarrow \qquad \langle\, \xi \mid (\,\cdot\,)\xi \,\rangle\,.$$

The state $A \mapsto \langle\, \Omega_0 \mid A\Omega_0 \,\rangle$ on $\pi(\mathcal{A})''$ is already given in its GNS representation and its modular structure consists of the same data $(\pi(\mathcal{A})'', \mathcal{H}^2, J, \mathcal{P})$ as the modular structure of ω_0. Since we only study ω_0-normal states we effectively have a W^*-algebra framework at hand. All the concepts have to be transferred to the system $(\pi(\mathcal{A})'', [\pi\alpha])$. For instance, by an ω_0-normal (α,β)-KMS state on $\mathcal{A}$ we understand a state $\langle\, \xi \mid \pi(\,\cdot\,)\xi \,\rangle$ such that $\langle\, \xi \mid (\,\cdot\,)\xi \,\rangle$ is a $([\pi\alpha],\beta)$-KMS state on $\pi(\mathcal{A})''$.

We check that the group $\{e^{iLt}\}_{t\in\mathbb{R}}$ leaves the positive cone $\mathcal{P}$ invariant. Let $A \in \pi(\mathcal{A})''$ and decompose the operator $L = L_0 + V - V'$ where $V := g\pi(v)$ and $V' := g\pi'(v)$. We remark that $e^{iVt} \in \pi(\mathcal{A})''$ and $e^{-iV't} = Je^{iVt}J \in \pi(\mathcal{A})'$. Since L_0 is a Liouville operator associated with the cone $\mathcal{P}$ it leaves the cone invariant. Applying the Trotter product formula yields

$$e^{iLt}AJA\Omega_0 = \lim_{n\to\infty} \left[e^{iL_0t/n}e^{iVt/n}e^{-iV't/n} \right]^n AJA\Omega_0.$$

We consider

$$\begin{aligned} \left[e^{iL_0t/n}e^{iVt/n}e^{-iV't/n} \right] AJA\Omega_0 &= e^{iL_0t/n}e^{iVt/n}Ae^{-iV't/n}JA\Omega_0 \\ &= e^{iL_0t/n}\left[e^{iVt/n}A\right] J \left[e^{iVt/n}A\right] \Omega_0 \in \mathcal{P} \end{aligned}$$

which implies inductively that $\left[e^{iL_0t/n}e^{iVt/n}e^{-iV't/n}\right]^n AJA\Omega_0 \in \mathcal{P}$. Since $\mathcal{P}$ is closed we obtain

$$e^{iLt}AJA\Omega_0 \in \mathcal{P}.$$

The invariance of $\mathcal{P}$ under e^{iLt} and (1.84) justify that L is called the standard Liouville operator associated with ω_0 w.r.t. the perturbed time evolution α.

1.4.4 Heat Fluxes and Entropy Production Rate

To classify the physical system in stationary states away from thermal equilibrium we introduce the notion of *heat fluxes* and *entropy production rate*. By the heat flux of one of the subsystems we understand the net flow of energy into the corresponding subsystem, i.e., the change of energy in time. The initial heat flux through the particle system, when the coupling to the reservoirs is switched on, is given by

$$\phi_{\mathrm{p}} := \partial_t|_{t=0}\alpha^t(H_{\mathrm{p}}) = i[H, H_{\mathrm{p}}] = i[gv, H_{\mathrm{p}}]$$

and the initial heat flux of the r^{th} reservoir by

$$\phi_{\mathrm{f},r} := \partial_t|_{t=0}\alpha^t(H_{\mathrm{f},r}) = i[H, H_{\mathrm{f},r}] = i[gv_r, H_{\mathrm{f},r}] = g\left[a_r(-i\omega G_r) + a_r^*(-i\omega G_r)\right].$$

The total flux of the system is introduced as

$$\phi_{\mathrm{tot}} := \phi_{\mathrm{p}} + \sum_{r=1}^{R} \phi_{\mathrm{f},r} = \partial_t|_{t=0}\alpha^t(H_0) = i[H, H_0] = -i[H, gv].$$

We remark that ϕ_{p}, $\phi_{\mathrm{f},r}$ and ϕ_{tot} are given in terms of creation and annihilation operators and can therefore be plugged into ω_0-normal states $\eta = \langle\, \xi \mid \pi(\,\cdot\,)\xi\,\rangle$, for ξ in the form domain of $\pi(\phi_{\mathrm{p}})$, $\pi(\phi_{\mathrm{f},r})$, $\pi(\phi_{\mathrm{tot}})$, resp., using the extension of π to polynomials in creation and annihilation operators as discussed in the previous section, c.f. (1.77). We further note that while $H_{\mathrm{p}} \in \mathcal{A}_{\mathrm{p}}$ is a proper observable the reservoir Hamiltonians $H_{\mathrm{f},r}$ cannot be expressed in terms of creation and annihilation operators and therefore applying of ω_0-normal states to $H_{\mathrm{f},r}$ is not possible. Doing so formally would lead to infinite expressions which stresses the fact that the photon reservoirs accumulate an infinite amount of energy.

We further introduce the entropy production rate observable

$$\mathfrak{s} := \beta_{\mathrm{p}}\phi_{\mathrm{p}} + \sum_{r=1}^{R} \beta_r \phi_{\mathrm{f},r}$$

which describes the initial change of entropy when the reservoirs at given inverse temperatures $\beta_1, \ldots, \beta_R$ are brought into interaction with the particle system at inverse temperature β_{p}. This definition is the translation of the thermodynamic "slogan"

$$dS = \beta dQ,$$

describing the entropy change dS in relation to the heat transfer dQ, to our situation. Using the definition of the fluxes we see that

$$\mathfrak{s} = i\left[gv, \beta_{\mathrm{p}} H_{\mathrm{p}} + \sum_{r=1}^{R} \beta_r H_{\mathrm{f},r}\right] = [gv, iH_{\mathrm{resc}}],$$

and further

$$\pi(\mathfrak{s}) = [g\pi(v), i\mathcal{L}_0],$$

in consistency with the definition (1.11) of the entropy production rate in Section 1.1.3. However, in the case of bosonic interaction neither the perturbation v nor the entropy production rate $\mathfrak{s}$ are elements from the algebra $\mathcal{A}$ but we may compute their expectation value in an ω_0-normal state. The entropy production formula (1.10) is also valid in the situation of unbounded entropy production observables, it reads in this context

$$\mathrm{Ent}\left(\eta \circ \alpha^t|\omega_0\right) = \mathrm{Ent}(\eta|\omega_0) - \int_0^t ds\, \eta \circ \alpha^s(\mathfrak{s}) \tag{1.85}$$

for any ω_0-normal state $\eta = \langle\, \xi \mid \pi(\,\cdot\,)\xi\,\rangle$ with ξ from the form domain of $\pi(\mathfrak{s})$. The proof of this relation is literally the same as the proof given in [26] for C^*- and W^*-dynamical systems with bounded interactions from the algebra, except for the convergence of the employed Dyson series which is only strong in our context.

1.4.5 Technical Requirements

We provide a set of requirements on the mathematical objects of the considered system. These assumptions are essential for the analysis we perform on the system. The first hypothesis deals with the regime of parameters which might influence the observation of the studied phenomena, that are the reservoir temperatures and the coupling constant. The coupling constant g is treated as a perturbative parameter, i.e., it is chosen sufficiently small. All results shall hold uniformly for low temperatures while the high temperature regime is excluded for simplicity. Further, several results will only hold for small temperature differences which, however, need not to be small compared with the strength of the coupling.

Hypothesis III-1.8 (Parameter Range of g and β_r) *The coupling constant $g \in \mathbb{R}$ is a perturbative parameter which is assumed to be sufficiently small,*

$$0 < |g| \ll 1.$$

For notational convenience we chose $g > 0$.

Without loss of generality we assume that the inverse reservoir temperatures are ordered as

$$0 < \underline{\beta} \leq \beta_{\min} := \beta_R \leq \cdots \leq \beta_1 =: \beta_{\max} < \infty$$

where $\beta_{\min}$ is assumed to be uniformly bounded away from zero by a positive constant $\underline{\beta}$ while there is no upper bound on the inverse temperatures. Further, the temperature differences are assumed to be sufficiently small compared to a constant, i.e.,

$$|\beta_{\max} - \beta_{\min}| \ll 1.$$

The inverse temperature difference is not a perturbative parameter on the same scale as g*, i.e., we allow that* $|\beta_{\max} - \beta_{\min}| \gg g$*.*

The next assumption concerns the degeneracy of the particle energy levels.

Hypothesis IV-1.9 (Non-Degeneracy of the Particle System) *The eigenvalues of the particle Hamiltonian are assumed to be non-degenerate, i.e.,*

$$E_0 < E_1 < \cdots < E_{N-1}.$$

We need an assumption which guarantees that the photon reservoirs are effectively coupled to the particle system such that thermal relaxation can occur. In this context we understand by an effective coupling that any transition from a higher particle energy level to a lower one under the emission of photons carrying away the energy difference of the energy levels is allowed.

Hypothesis V-1.10 (Fermi Golden Rule Condition) *The coupling functions obey*

$$\gamma_{\mathrm{FGR}} := 2\pi \min_{r=1}^{R} \left[\min_{\substack{m,n=0,\\ m>n}}^{N-1} E_{m,n}^2 \int_{S^2} d\Sigma \left| \langle \varphi_n \,|\, G_r(E_{m,n}\Sigma)\varphi_m \rangle_{\mathcal{H}_{\mathrm{p}}} \right|^2 \right] > 0. \tag{1.86}$$

The number γ_{FGR} *is referred to as* Fermi golden rule level shift*.*

Note that only the transition probabilities $|\langle \varphi_n \,|\, G_r(E_{m,n}\Sigma)\varphi_m \rangle_{\mathcal{H}_{\mathrm{p}}}|^2$, $m > n$, from a higher particle energy level E_m down to lower level E_n under emission of a photon of the difference energy $E_{m,n} = E_m - E_n$ have to be accounted for the Fermi golden rule condition. None of the transitions to a lower energy level may be prohibited. It is plausible that excitation processes of the particle system, i.e., transitions from lower to higher energy levels, play no rule for the thermal relaxation properties. We remark further that the Fermi golden rule level shift γ_{FGR} is only strictly positive if the number of possible transitions is finite. Taking the infimum in (1.86) over infinite many energy levels, i.e., $N = \infty$, we typically obtain a vanishing Fermi golden rule level shift. Our considerations are therefore restricted to particle systems with finitely many degrees of freedom. It is still an open problem to study the thermal relaxation for setups where the particle system has infinitely many energy levels.

For our analysis on the interacting system we need conditions considerably stronger than those assumed in the Hypotheses I-1.2 and II-1.4. The following Hypotheses will provide the necessary conditions on the coupling functions for our work and will henceforth be assumed to be fulfilled. Some mathematical analysis on the interacting system requires a transformation of the interacting part of the perturbed Liouville operator with the help of the rescaled free evolution α_0, we need at various places that $\pi(\alpha_0^s(v))$ is well defined as analytic functions in s for $|\operatorname{Im}(s)| \leq \beta_{\max}/2$. To this end we intensify the Hypothesis I-1.2.

Hypothesis VI-1.11 (α_0-Analyticity of the Perturbation) *We assume that the coupling functions G_r, $r = 1, \ldots, R$, obey the following weighted L^2-norm,*

$$\int_{\mathbb{R}^3} d^3\vec{k} \left[1 + \frac{1}{\omega(\vec{k})}\right] \left\| e^{(\beta_{\max}/2+\varepsilon_0)\omega(\vec{k})} G_r(\vec{k}) \right\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p})} < \infty \tag{1.87}$$

for a small constant $\varepsilon_0 > 0$.

We remark that, while imposing the more stringent condition (1.87) on the coupling functions to lie in a weighted L^2-space, we do not require that the corresponding weighted L^2-norms of G_r are bounded uniformly in the temperature. In fact, the weighted L^2-norm (1.87) grows exponentially in the inverse temperature $\beta_{\max}$. However, this does no pose a problem since this norm never appears in the context of the perturbation theory. The Hypothesis VI-1.11 is not crucial for the validity of our results and it can be avoided by an approximation of the coupling functions by rapidly decreasing functions. The gained results are uniform in the approximation such that it can finally be removed. This strategy was carried out in [34]. In the present work we restrict ourselves to the less general case of coupling functions to avoid further technical inconveniences.

A main ingredient for our analysis is an analyticity property of the coupling functions G_r. Recall that

$$\mathcal{G} = \mathfrak{g}(G_1, \ldots, G_R) \in L^2[\Upsilon; \mathcal{B}(\mathcal{H}_\mathrm{p}^2)]$$

denotes the glued coupling function. On the space $L^2[\Upsilon]$ of square integrable functions over Υ, given in (1.63), we introduce a unitary two-parameter family $\{\mathfrak{D}(\theta)\}_{\theta\in\mathbb{R}^2}$ by

$$[\mathfrak{D}(\theta)f]\,(u, \Sigma, r) := e^{\delta\,\mathrm{sgn}(u)/2} f(j_\theta(u), \Sigma, r), \qquad \theta = (\delta, \tau) \in \mathbb{R}^2,$$

where

$$j_\theta(u) := e^{\delta\,\mathrm{sgn}(u)} u + \tau \tag{1.88}$$

and sgn is the signum function, $\mathrm{sgn}(u) = u/|u|$. The parameter δ parameterizes a dilation of the function f while τ is a translation. For $f \in L^2[\Upsilon]$ we abbreviate

$$f_\theta := \mathfrak{D}(\theta) f.$$

The family $\{\mathfrak{D}(\theta)\}_{\theta\in\mathbb{R}^2}$ implements a *spectral deformation* of the considered operators. We require the following properties of the coupling functions $\mathcal{G}$ to be fulfilled.

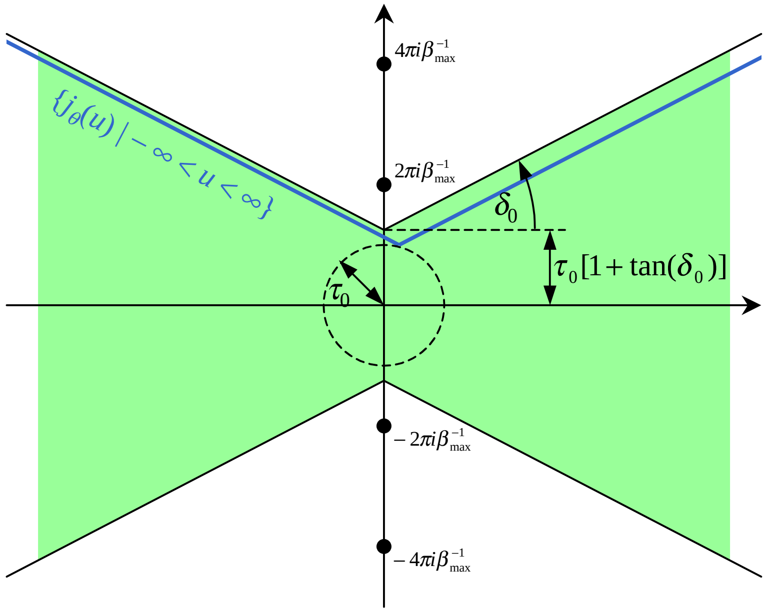

Figure 1.7: Illustration of the set $\mathcal{U}_{\delta_0,\tau_0}$.

Hypothesis VII-1.12 (Deformation Analyticity & Regularization) *The function $u \mapsto \mathcal{G}(u,\Sigma,r) \in \mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)$ has, for a.e. $(\Sigma,r) \in S^2 \times \mathbb{N}_1^R$, an analytic continuation on the complex domain*

$$\mathcal{U}_{\delta_0,\tau_0} := \{ j_\theta(u) \,|\, \theta \in D_{\delta_0,\tau_0},\ u \in \mathbb{R} \} \tag{1.89}$$

where

$$D_{\delta_0,\tau_0} := \left\{ (\delta,\tau) \in \mathbb{C}^2 \,\Big|\, |\operatorname{Im}(\delta)| < \delta_0,\ |\tau| < \tau_0 \right\}, \tag{1.90}$$

for fixed positive constants $\delta_0, \tau_0 > 0$ fulfilling

$$\frac{\pi}{8} < \delta_0 < \frac{\pi}{4} \qquad \text{and} \qquad \tau_0 \leq 2\pi\beta_{\max}^{-1}. \tag{1.91}$$

The continuation shall also be denoted by $\mathcal{U}_{\delta_0,\tau_0} \ni z \mapsto \mathcal{G}(z,\Sigma,r)$. Further, we assume that there are positive constants $C_1 > 0$ and $0 < C_2 < 2\pi\beta_{\max}^{-1}$ such that

$$\operatorname*{ess-sup}_{(\Sigma,r)\in S^2\times\mathbb{N}_1^R} \|\mathcal{G}(z,\Sigma,r)\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)}, \leq C_1|z|^\nu \qquad \text{for } |z| \leq C_2 \tag{1.92}$$

for a positive infrared (IR) regularization $\nu \geq 1$. *Moreover, we require the following* ultraviolet (UV) behavior *to be fulfilled,*

$$\operatorname*{ess-sup}_{(\Sigma,r)\in S^2\times\mathbb{N}_1^R} \|\mathcal{G}(z,\Sigma,r)\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} \leq C_3 e^{-a|z|^2} \qquad \text{for } |z| \geq C_4,\ z \in \mathcal{U}_{\delta_0,\tau_0} \tag{1.93}$$

where $a, C_3, C_4 > 0$ are positive constants with $a > 0$.

We remark that the UV regularization (1.93) covers the assumption on the decay of the coupling functions in Hypothesis VI-1.11. We stress that the assumptions on the UV behavior may be weakened by approximating $\mathcal{G}$ by rapidly decaying functions as done in [34]. Again, we spare the technical expenditure to focus on the essential problems.

Remark 1.13 *Under the assumptions of Hypothesis VII-1.12 the dominated convergence theorem implies that the map*

$$D_{\delta_0,\tau_0} \to L^2[\Upsilon; \mathcal{B}(\mathcal{H}_\mathrm{p}^2)], \qquad \theta \mapsto e^{isj\theta}\mathcal{G}_\theta$$

is analytic – pointwise a.e. and in the L^2 sense – for any $s \in \mathbb{C}$.

To illustrate the requirements of Hypothesis VII-1.12 we specify a class of coupling functions obeying the above conditions.

Proposition 1.14 (Class of Analytic Perturbations) *Let $\delta_0, \tau_0 > 0$ satisfy (1.91) and*

$$\frac{\tau_0}{\cos(\delta_0)} < \frac{\pi}{\beta_{\max}}. \tag{1.94}$$

For $r = 1, \ldots, R$, let $M_r = M_r^* \in \mathcal{B}(\mathcal{H}_\mathrm{p})$ be self-adjoint matrices and $g_r : \mathbb{R}^+ \to \mathbb{C}$ functions on the positive real axis. Assume that the functions g_r have analytic continuations (also denoted by g_r) onto the domain $\{z \in \mathcal{U}_{\delta_0,\tau_0} \,|\, \mathrm{Re}(z) > 0\}$. Further, we require that the g_r have continuous extensions onto the imaginary axis and take real values there, i.e., $g_r(ix) \in \mathbb{R}$ for real x. Moreover, we assume that g_r fulfills a certain UV regularization, i.e., there are positive constants $a, C > 0$ such that $|g_r(z)| \le Ce^{-a|z|^2}$, provided that $z \in \mathcal{U}_{\delta_0,\tau_0}$ and $\mathrm{Re}(z) > 0$. Choose a number $p \in \left\{\frac{1}{2} + n \,\middle|\, n \in \mathbb{N}_0\right\}$ as the actual infrared regularization of the coupling functions G_r. Then the functions

$$\vec{k} \mapsto G_r(\vec{k}) := i^{p+\frac{3}{2}}|\vec{k}|^p g_r(|\vec{k}|)M_r \tag{1.95}$$

fulfill the requirements of Hypothesis VII-1.12. In particular, we may choose

$$G_r(\vec{k}) = \sqrt{|\vec{k}|}\, e^{-a|\vec{k}|^2} M_r$$

with $\frac{a}{\cos(2\delta_0)} > 0$.

Proof. We first remark that, for any $r = 1, \ldots, R$, the function

$$z \mapsto \frac{z}{1 - e^{-\beta_r z}}$$

has simple poles in $\pm 2n\pi i\beta_r^{-1}$ for $n \in \mathbb{N}$, and is analytic elsewhere. In particular, it has an analytic continuation into the origin of the complex plane and is non-zero on its domain. We show that under the conditions (1.94) on the parameters τ_0, δ_0 the set $\mathcal{U}_{\delta_0,\tau_0}$ does not contain any of the poles. By definition (1.88), we have for $\theta \in D_{\delta_0,\tau_0}$, $u \in \mathbb{R}$ and $z := j_\theta(u)$,

$$\frac{\mathrm{Im}(z) - \mathrm{Im}(\tau)}{\mathrm{Re}(z) - \mathrm{Re}(\tau)} = \mathrm{sgn}(u) \tan(\mathrm{Im}(\delta)),$$

i.e., the point z lies on the line going through τ with slope $\mathrm{sgn}(u) \tan(\mathrm{Im}(\delta))$. This line intersects with the imaginary axis in

$$b := i \left[\mathrm{Im}(\tau) - \mathrm{Re}(\tau) \,\mathrm{sgn}(u) \tan(\mathrm{Im}(\delta))\right]$$

where

$$|b| \leq |\,\mathrm{Im}(\tau)| + |\,\mathrm{Re}(\tau)||\tan(\mathrm{Im}(\delta))| \leq \tau_0[1 + \tan(\delta_0)] \leq \frac{2\tau_0}{\cos(\delta_0)} < 2\pi\beta_{\mathrm{max}}^{-1}.$$

This implies that $z = j_\theta(u)$ does not hit any poles. An illustration of the above considerations is given in Figure 1.7. Hence, there exists an analytic function

$$\mathcal{U}_{\delta_0,\tau_0} \ni z \mapsto \sqrt{\frac{z}{1 - e^{-\beta_r z}}}$$

being the extension of $\mathbb{R} \ni u \mapsto \sqrt{u(1 - e^{-\beta_r u})^{-1}}$. We now extend the functions g_r across the imaginary axis on the domain $\mathcal{U}_{\delta_0,\tau_0}$ by setting

$$g_r(-x + iy) := \overline{g_r(x + iy)}$$

for $x > 0$, $y \in \mathbb{R}$ and $x + iy \in \mathcal{U}_{\delta_0,\tau_0}$. Using the continuity of g_r on the imaginary axis and the fact that the g_r take real values there, the Schwarz reflection principle (c.f. [41, Thm. 11.17]) implies that the functions g_r are analytic on $\mathcal{U}_{\delta_0,\tau_0}$. Therefore, the functions

$$f_r : \mathcal{U}_{\delta_0,\tau_0} \to \mathbb{C}, \qquad f_r(z) := (iz)^{p+\frac{1}{2}} g_r(z)$$

are analytic as well because $p \in \frac{1}{2} + \mathbb{N}$. We now construct an operator valued function

$$\mathcal{G} : \mathcal{U}_{\delta_0,\tau_0} \times S^2 \times \mathbb{N}_1^R \to \mathcal{B}(\mathcal{H}_\mathrm{p}^2),$$
$$\mathcal{G}(z, \Sigma, r) := i\sqrt{\frac{z}{1 - e^{-\beta_r z}}}\, f_r(z)[M_r \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}}]$$

which is analytic in the variable z by the above considerations and is the image of the coupling functions G_r, given in (1.95), under the gluing map $\mathfrak{g}$,

$$\begin{aligned}
&\mathfrak{g}(G_1,\ldots,G_R)(u,\Sigma,r)\\
&= \sqrt{\frac{u}{1-e^{-\beta_r u}}} \times \begin{cases} \sqrt{u}\, i^{p+\frac{3}{2}} u^p g_r(u)[M_r \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}}], & u \geq 0,\\ (-\sqrt{-u})(-i)^{p+\frac{3}{2}}(-u)^p \overline{g_r(-u)}[M_r \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}}], & u < 0. \end{cases}\\
&= i\sqrt{\frac{u}{1-e^{-\beta_r u}}}\,(iu)^{p+\frac{1}{2}} \times \begin{cases} g_r(u)[M_r \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}}], & u \geq 0,\\ \overline{g_r(-u)}[M_r \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}}], & u < 0. \end{cases}\\
&= i\sqrt{\frac{u}{1-e^{-\beta_r u}}}\, f_r(u)[M_r \otimes \mathbb{1}_{\mathcal{H}_\mathrm{p}}]\\
&= \mathcal{G}(u,\Sigma,r)
\end{aligned}$$

for $(u,\Sigma,r) \in \Upsilon$.

It remains to check the (IR) and the (UV) behavior of the constructed coupling functions. Expanding $\mathcal{G}$ around zero gives for $|z| \leq \min\{1,\tau_0\}$,

$$\|\mathcal{G}(z,\Sigma,r)\|_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)} \leq |z|^{p+\frac{1}{2}} \sup_{|\zeta|\leq\tau_0} \left| \sqrt{\frac{\zeta}{1-e^{-\beta_r\zeta}}}\, g_r(\zeta)\right| \|M_r\|_{\mathcal{B}(\mathcal{H}_\mathrm{p})} \leq C_1|z|^\nu$$

for a positive constant $C_1 < \infty$ since the supremum is taken over an analytic function. Hereby, the IR regularization ν of the glued functions and the IR regularization p of the physical coupling functions are related by

$$\nu = p + \frac{1}{2}.$$

This establishes the IR regularization. Further, for $|z| \geq 1$, we obtain

$$\begin{aligned}
\|\mathcal{G}(z,\Sigma,r)\|_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)} &\leq Ce^{-(a-\varepsilon)|z|^2} \sup_{\substack{\zeta\in\mathcal{U}_{\delta_0,\tau_0},\\ |\zeta|\geq 1}} \left| \sqrt{\frac{\zeta}{1-e^{-\beta_r\zeta}}}\right| |\zeta|^{p+\frac{1}{2}} e^{-\varepsilon|\zeta|^2}\, \|M_r\|_{\mathcal{B}(\mathcal{H}_\mathrm{p})}\\
&\leq C_3 e^{-(a-\varepsilon)|z|^2}
\end{aligned}$$

for some positive constant $C_3 < \infty$ and $\varepsilon > 0$ such that $a - \varepsilon > 0$, which is the UV regularity. ∎

1.4.6 Thermal Relaxation Behavior: A Survey of Results

Within the composed model described above two different types of systems with contradicting thermal relaxation properties compete. While the finite particle system possesses stationary states in a neighborhood of the equilibrium (in the relative entropy sense), a single infinite extended photon system would drive any state which is normal w.r.t. the equilibrium towards the latter. Since we allow that several photonic systems are close to equilibria at different temperatures the tendency of a return to equilibrium even leads to a competition among the reservoirs concerning the relaxation behavior once they are coupled through the particle system. Due to this competition of comparable reservoirs we cannot expect that the interacting system will reach a thermal equilibrium. It is more plausible to think about the infinitely extended reservoirs as staying at their own temperatures while the total system approaches a time-invariant state featuring stationary heat transfer among the photon systems through the particle system. The existence of heat fluxes suggests that this stationary state cannot be a thermal equilibrium though it describes the configuration after the initial configuration has thermally relaxed. We refer to this state as a *non-equilibrium stationary state* or simply *NESS*. We will further show, for sufficient large temperature differences in the reservoirs, that the standard Liouville operator of the interacting system has a trivial kernel. This leads to the conclusion that there are no α-stationary states which are normal w.r.t. ω_0. Hence, the NESS, as a stationary state, is separated from ω_0 by an infinite amount of relative entropy. This in turn implies that the entropy production rate in the NESS w.r.t. ω_0 must be strictly positive.

For the initial reservoir configurations being all close to the same equilibrium, i.e., the reservoir temperatures $\beta := \beta_1 = \cdots = \beta_R$ coincide, we expect that the coupled system has an attracting equilibrium state at inverse temperature β. The proof of this result can be reduced to the work of Bach, Fröhlich, Sigal in [8], but is also re-derived as a special case within this work.

In either case the initial configuration of the particle system does not play any role, the finite system is forced by the infinite systems which control the thermodynamic processes.

The above considerations are summarized in a mathematical language in the following theorem which presents the main result of this thesis.

Theorem 1.15 (Relaxation & Thermodynamic Characteristics) *Assume that the conditions of the Hypotheses I-1.2-VII-1.12 are fulfilled. There exists a state $\tilde{\omega}$ (i.e., a linear, positive, normalized functional) on a $*$-subalgebra $\mathcal{A}_1$, which is dense in $\mathcal{A}$ w.r.t. the strong topology in $\mathcal{B}(\mathcal{H}^2)$, and there are positive numbers $g_0 > 0$ and $\delta\beta_0 > 0$, both independent of $\beta_1, \ldots, \beta_R$, with the following properties: For $0 < g < g_0$ and $|\beta_{\max} - \beta_{\min}| < \delta\beta_0$ and for any ω_0-normal state $\eta \in \mathcal{N}_{\omega_0}(\mathcal{A})$ holds*

$$\lim_{t\to\infty} \eta \circ \alpha^t(A) = \tilde{\omega}(A) \tag{1.96}$$

for all $A \in \mathcal{A}_1$. Further, there exists a dense subset $\mathcal{N}^{\mathrm{ana}} \subset \mathcal{N}_{\omega_0}(\mathcal{A})$ of ω_0-normal states such that the convergence of $\eta \in \mathcal{N}^{\mathrm{ana}}$ towards $\tilde{\omega}$ under the time evolution is *exponentially fast, i.e., there is a positive constant* $d > 0$ *and a decay rate given by* $\tau_{\text{dec}} = \frac{g^2 d}{2+\beta_{\max}}$ *such that*

$$\lim_{t\to\infty} e^{\tau_{\text{dec}} t} \left|\eta \circ \alpha^t(A) - \tilde{\omega}(A)\right| = 0 \tag{1.97}$$

for all $A \in \mathcal{A}_1$*. Moreover, the state* $\tilde{\omega}$

- *extends to the unique* ω_0*-normal* (α, β)*-KMS state on* $\mathcal{A}$ *in case that* $\beta_1 = \cdots = \beta_R =: \beta$*. Then, the entropy production rate* $\mathrm{Ep}_{\omega_0}(\tilde{\omega})$ *in the state* $\tilde{\omega}$ *w.r.t.* ω_0 *vanishes.*
- *has no* ω_0*-normal extension to* $\mathcal{A}$ *and it has a strictly positive entropy production rate,* $\mathrm{Ep}_{\omega_0}(\omega) > 0$*, in the case that* $|\beta_{\max} - \beta_{\min}|$ *is sufficiently large w.r.t. the coupling constant in the sense of (3.6). Further, the state* $\tilde{\omega}$ *features non-vanishing stationary heat fluxes, there exist* $r, r' \in \{1, \ldots, R\}$ *such that*

$$\tilde{\omega}(\phi_{\mathrm{f},r}) > 0 \qquad \textit{and} \qquad \tilde{\omega}(\phi_{\mathrm{f},r'}) < 0.$$

The verification of the statements of Theorem 1.15 goes back to various results which are the fruits of the spectral theory for NESS derived in Chapter 2. The subalgebra $\mathcal{A}_1$ is realized as a subset of $\mathcal{A}^{\text{ana}}$ which in turn is defined in (2.34). The state $\tilde{\omega}$ is introduced in (2.35) and Corollary 2.14. The thermal relaxation (1.96, 1.97) of the system is content of Theorem 2.11. The thermodynamic characteristics of the state as KMS condition, entropy production rate and heat fluxes are results of Corollary 2.16 and Proposition 2.17.

Remark 1.16 (Non-Equilibrium Stationary State) *The state* $\tilde{\omega}$ *of Theorem 1.15 is the pointwise time limit on observables from the subalgebra* $\mathcal{A}_1$*. This motivates us to refer to* $\tilde{\omega}$ *as a stationary state. However, we pointed out in Section 1.4.3 that the perturbed time evolution* α *is not an automorphism group on the algebra* $\mathcal{A}$*, and therefore cannot be applied to arbitrary states on* $\mathcal{A}$*, but rather acts as a time evolution on* ω_0*-normal states. Hence, the notion of time evolution of the state* $\tilde{\omega}$ *is not explained for large temperature differences (in that case* $\tilde{\omega}$ *is not normal w.r.t.* ω_0*) while, in the equal temperature situation, the question of stationarity is well posed and can be answered positively (in that case* $\tilde{\omega}$ *is a KMS state). We remark that any concept for* ω_0*-normal states which survives the long time limit can be adapted to the state* $\tilde{\omega}$*. This vague statement shall mean the following with regard to the time evolution* $\alpha^t * \tilde{\omega}$ *of the state* $\tilde{\omega}$*,*

$$\alpha^t * \tilde{\omega}(A) := \lim_{s\to\infty} \omega_0 \circ \alpha^s(\alpha^t(A))$$

for all $A \in \mathcal{A}_1$*. Equation (1.96) then trivially implies the stationarity of* $\tilde{\omega}$ *w.r.t.* α *on* $\mathcal{A}_1$*.*

The notion of a NESS originally goes back to Ruelle, [42], who classified a non-equilibrium stationary state as a weak-$*$-limit point of the ergodic means of the initial configuration ω_0 evolving under α, i.e., the set of NESS's is the set of weak-$*$-limit points of

$$\left\{ \frac{1}{T} \int_0^T dt\, \alpha^t \circ \omega_0 \,\middle|\, T > 0 \right\}.$$

Since in our situation all ω_0-normal states converge weakly on a subalgebra $\mathcal{A}_1$ towards a limit state the state $\tilde{\omega}$ is a NESS, and in fact the only one, in the sense of Ruelle if we restrict to the observables in $\mathcal{A}_1$.

2 Dynamical and Thermal Properties of NESS and Their Spectral Theory

In Section 1.3.6 we briefly pointed out the spectral theory for equilibrium states. The thermal relaxation to the equilibrium state could be derived from the fact that the standard Liouville operator L_{f} associated with the KMS state ω_{f} in the appropriate representation π_{f} has a simple null eigenvector while the rest of its spectrum is absolutely continuous. We crucially used that the vector representative Ω_{f} of the equilibrium state is cyclic for that very representation and is left invariant by the group $e^{iL_{\mathrm{f}}t}$ generated by the Liouville operator. The combination of both properties, the modular structure of ω_{f} and the nature of the spectrum of L_{f}, allowed us to study the evolution of ω_{f}-normal states and in particular to compute the long time behavior $t \to \infty$ in terms of the weak limit of the group $e^{iL_{\mathrm{f}}t}$. This connection between the thermal relaxation properties of equilibria and the spectral properties of their Liouville operators was established by Jakšić and Pillet in [24] for the description of a single bosonic reservoir in interaction with a particle system. The same approach was used by Bach, Fröhlich and Sigal in [8] to study the return to equilibrium property in a more general temperature regime.

Subsequently, we will outline a modified strategy to connect the thermal relaxation properties of a system, not necessarily being close to an equilibrium state, with spectral properties of a suitable generator of the time evolution. The approach we are describing goes back to a work of Jakšić and Pillet, published in [28], and was modified by Merkli, Sigal, and the present author in [34].

We start our discussion by reviewing the α_0-invariant state

$$\omega_0 = \langle\, \Omega_0 \,|\, \pi(\,\cdot\,)\Omega_0 \,\rangle$$

introduced in Section 1.4.1 describing a system of R photon reservoirs and a particle system at different temperatures and which localizes the initial configurations of the system. Given an ω_0-normal state $\eta = \langle\, \xi \,|\, \pi(\,\cdot\,)\xi \,\rangle$ with $\xi \in \mathcal{P}$ we defined the evolution of η under the interacting dynamics by

$$\eta \circ \alpha^t = \langle \xi \mid e^{iLt} \pi(\,\cdot\,) e^{-iLt} \xi \rangle .$$

We remark that the choice of the generator L was pinned down by the request that it implements the perturbed time evolution on the representation space,

$$e^{iLt}\pi(A)e^{-iLt} = \alpha^t * \pi(A),$$

where $\alpha^t * \pi(A)$ was defined in (1.78), and that the group e^{iLt} leaves the cone $\mathcal{P}$ invariant. However, if we are only demanding that a generator, say K, implements the time evolution, i.e.,

$$e^{iKt}\pi(A)e^{-iKt} = \alpha^t * \pi(A) = e^{iLt}\pi(A)e^{-iLt} \tag{2.1}$$

we get a much wider selection of possible generators K. The fact that $\pi(A)$ commutes with $e^{iL_0\tau}\pi'(w)e^{-iL_0\tau} = \pi'(\alpha_0^\tau(w))$ for all $\tau \in \mathbb{R}$ and w being a polynomial in creation and annihilation operators shows that the ansatz

$$K = L_0 + \pi(v) - \pi'(w)$$

fulfills (2.1) if we understand its r.h.s. as an expansion in a Dyson series, i.e.,

$$\begin{aligned} e^{iKt}\pi(A)e^{-iKt} &= \pi(\alpha_0^t(A)) + \sum_{n=1}^{\infty}(ig)^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \\ &\quad \times \Big[e^{iL_0 t_n}[\pi(v) - \pi'(w)]e^{-iL_0 t_n}, \Big[\\ &\qquad \ldots, \Big[e^{iL_0 t_1}[\pi(v) - \pi'(w)]e^{-iL_0 t_1}, \pi(\alpha_0^t(A)) \Big] \Big]\Big] \\ &= e^{iLt}\pi(A)e^{-iLt}. \end{aligned}$$

This expression is only a formal relation.

The degree of freedom in the definition of K can now be used to require that K annihilates a given unit vector, say $\tilde{\Omega} \in \mathcal{H}^2$,

$$K\tilde{\Omega} = 0.$$

Assuming that $\tilde{\Omega}$ is both cyclic and separating for $\pi(\mathcal{A})''$ and that w can be chosen in such a way that $K\tilde{\Omega} = 0$, the operator K becomes the infinitesimal generator of a one parameter group $\{U(t)\}_{t\in\mathbb{R}}$,

$$U(t)[\pi(A)\tilde{\Omega}] := \alpha^t * \pi(A)\tilde{\Omega} = e^{iLt}\pi(A)e^{-iLt}\tilde{\Omega}, \qquad A \in \mathcal{A},$$

which is well defined due to the separating property of $\tilde{\Omega}$ on the dense domain $\pi(\mathcal{A})''\tilde{\Omega}$. We remark that this group is unitary if and only if the state

$$\hat{\omega} := \left\langle \tilde{\Omega} \,\middle|\, \pi(\,\cdot\,)\tilde{\Omega} \right\rangle$$

is α-stationary, namely, for $A, B \in \mathcal{A}$ holds

$$\begin{aligned}
&\left\langle U(t)\pi(B)\tilde{\Omega} \,\middle|\, U(t)\pi(A)\tilde{\Omega} \right\rangle \\
&\quad = \quad \left\langle \tilde{\Omega} \,\middle|\, e^{iLt}\pi(B^*A)e^{-iLt}\tilde{\Omega} \right\rangle \\
&\quad = \quad \hat{\omega} \circ \alpha^t(B^*A) \\
&\left\{ \begin{array}{c} = \\ \neq \end{array} \right\} \quad \hat{\omega}(B^*A) \left\{ \begin{array}{l} \text{for } \hat{\omega} \circ \alpha^t = \hat{\omega} \\ \text{for } \hat{\omega} \circ \alpha^t \neq \hat{\omega} \text{ and some } A, B \in \mathcal{A},\, t \in \mathbb{R} \end{array} \right. \\
&\quad = \quad \left\langle \pi(B)\tilde{\Omega} \,\middle|\, \pi(A)\tilde{\Omega} \right\rangle .
\end{aligned}$$

In the case of α-invariance of $\hat{\omega}$ the ω_0-normality of $\hat{\omega}$ and (1.6) imply that the standard Liouville L operator has a non-trivial kernel. However, as we will show later in Section 3.1, Proposition 3.3, the Liouville operator has a trivial kernel for reservoir temperature differences sufficiently large. This in turn implies that $\{U(t)\}_{t\in\mathbb{R}}$ is a non-unitary group and therefore the generator K is not self-adjoint.

Before we give the specific choice of the vector $\tilde{\Omega}$ we formally display the link between spectral properties of K and the long time behavior of ω_0-normal states under the time evolution α. The fact that $\tilde{\Omega}$ is separating for $\pi(\mathcal{A})''$ is equivalent to $\tilde{\Omega}$ being cyclic w.r.t. $\pi(\mathcal{A})'$, c.f. [10, Prop. 2.5.4]. Given an ω_0-normal state

$$\eta := \langle\, \xi \mid \pi(\,\cdot\,)\xi \,\rangle$$

with $\xi \in \mathcal{P}$ we find an approximation

$$\eta_\varepsilon := \left\langle \pi'(B)\tilde{\Omega} \,\middle|\, \pi(\,\cdot\,)\pi'(B)\tilde{\Omega} \right\rangle$$

with $B \in \mathcal{A}$, $\left\|\pi'(B)\tilde{\Omega}\right\| = 1$ and $\|\eta - \eta_\varepsilon\|_{\mathcal{A}^*} < \varepsilon$ for a given $\varepsilon > 0$. The time evolution acts on η_ε as follows,

$$\begin{aligned}
\eta_\varepsilon \circ \alpha^t(A) \quad &= \quad \left\langle \pi'(B)\tilde{\Omega} \,\middle|\, \pi(\alpha^t(A))\pi'(B)\tilde{\Omega} \right\rangle \\
&= \quad \left\langle \pi'(B^*B)\tilde{\Omega} \,\middle|\, \pi(\alpha^t(A))\tilde{\Omega} \right\rangle \\
&= \quad \left\langle \pi'(B^*B)\tilde{\Omega} \,\middle|\, e^{iKt}\pi(A)\tilde{\Omega} \right\rangle \\
&\xrightarrow{t\to\infty} \quad \left\langle \pi'(B^*B)\tilde{\Omega} \,\middle|\, \tilde{\Omega} \right\rangle \left\langle \tilde{\Omega}^* \,\middle|\, \pi(A)\tilde{\Omega} \right\rangle \\
&=: \quad \tilde{\omega}(A),
\end{aligned}$$

using that $\pi(\alpha^t(A)) = e^{iLt}\pi(A)e^{-iLt} \in \pi(\mathcal{A})''$, where the last three lines need some clarification. Since K is not self-adjoint the group e^{iKt} is a priori not defined and a first task would be to understand this group in an appropriate sense. Secondly, we have to understand the convergence of e^{iKt} as $t \to \infty$ in a weak sense. Again, since K is not self-adjoint it is not clear that the group converges towards the projection $\left|\tilde{\Omega}\right\rangle \left\langle \tilde{\Omega}^*\right|$ on the null space of K. Here, the object $\left|\tilde{\Omega}\right\rangle \left\langle \tilde{\Omega}^*\right|$ formally built of the eigenvectors of K and K^*,

$$K\tilde{\Omega} = 0, \qquad K^*\tilde{\Omega}^* = 0,$$

is not a bounded projection operator. In fact, $\tilde{\Omega}^*$ is not a vector in $\mathcal{H}^2$ but rather an element $\left\langle \tilde{\Omega}^*\right|$ from the dual space $\mathcal{D}'_{\mathfrak{D}-\mathrm{a}}$ of an appropriate dense set $\mathcal{D}_{\mathfrak{D}-\mathrm{a}}$ in $\mathcal{H}^2$ fulfilling

$$\left\langle \tilde{\Omega}^*\right| K\psi = 0$$

for all $\psi \in \mathcal{D}(K)$ such that $K\psi \in \mathcal{D}_{\mathfrak{D}-\mathrm{a}}$. All these statements can be made rigorous in the language of *resonance eigenvectors*, as we will see later. This has the consequence that the state $\tilde{\omega}$ is in general (i.e., for large reservoir temperature differences) not an ω_0-normal state.

Subsequently, we will work out the sketched strategy with mathematical rigor.

2.1 The C-Liouville Operator

We start to work out the approach discussed above. Our first observation is that in the case of equal temperatures $\beta := \beta_\mathrm{p} = \beta_1 = \cdots = \beta_R$ the perturbed system possesses an (α, β)-KMS state

$$\omega := \langle\, \Omega \,|\, \pi(\,\cdot\,)\Omega\, \rangle \tag{2.2}$$

with Ω given by

$$\Omega := \frac{e^{-\beta L^{(\ell)}/2}\Omega_0}{\left\| e^{-\beta L^{(\ell)}/2}\Omega_0 \right\|} \tag{2.3}$$

where

$$L^{(\ell)} := L_0 + g\pi(v) \tag{2.4}$$

is the *(left) Radon-Nikodym operator*. Lemma B.3 ensures that (2.3) is well defined. For notational completeness we also introduce the *right Radon-Nikodym operator*,

$$L^{(\mathrm{r})} := -JL^{(\ell)}J = L_0 - g\pi'(v). \tag{2.5}$$

The KMS-property of ω follows formally from the structural stability, Section 1.1.3, and the fact that ω_0 is an α_0-KMS state for equal temperatures. However, in Section 1.1.3 we only considered bounded perturbations from the algebra $\mathcal{A}$. Chapter B provides the necessary technicalities to transfer the modular theory to our situation acknowledging the unboundedness of the perturbation. Lemma B.5 guarantees that Ω is in the kernel of L and therefore ω is an α-stationary state and further, by Proposition B.7, we know that ω is an (α, β)-KMS state.

In the general case where the reservoir temperatures can be chosen arbitrarily the Liouville operator cannot be expected to possess a non-trivial kernel. Inspired by the structure of the null vector of L for the equal temperature case we fix a vector by

$$\tilde{\Omega} := \frac{e^{-\beta L^{(\ell)}/2}\Omega_0}{\left\|e^{-\beta L^{(\ell)}/2}\Omega_0\right\|} \tag{2.6}$$

where $\beta \in [0, \beta_{\max}]$ is a reference parameter, later chosen to match the maximal inverse temperature $\beta_{\max}$. Note that (2.3) and (2.6) do not coincide as long as the temperatures are not the same in all reservoirs since π, and therefore also $L^{(\ell)}$, depends on the inverse temperatures. Lemma B.3 ensures that (2.6) is well defined.

Since the vector $\tilde{\Omega}$ is cyclic and separating w.r.t. $\pi(\mathcal{A})''$ by Lemma B.4 we may define a one parameter group $\{U(t)\}_{t\in\mathbb{R}}$ acting on the dense domain $\pi(\mathcal{A})''\tilde{\Omega}$ by

$$U(t)[\pi(A)\tilde{\Omega}] := \pi \circ \alpha^t(A)\tilde{\Omega} = e^{iLt}\pi(A)e^{-iLt}\tilde{\Omega}, \qquad A \in \mathcal{A}. \tag{2.7}$$

Our aim is to study the infinitesimal generator of that group.

Proposition 2.1 (Infinitesimal Generator of $U(t)$) *The group $\mathbb{R} \ni t \mapsto U(t)$ is strongly differentiable on vectors in $\mathcal{D}(L^{(\ell)}) \cap \pi(\mathcal{A})''\tilde{\Omega}$. Its infinitesimal generator K reads*

$$K := \operatorname*{s-lim}_{t\to 0} \frac{U(t) - \mathbb{1}_{\mathcal{H}^2}}{it} = L_0 + g\left[\pi(v) - \pi'\left(\gamma_0^{i/2}(v)\right)\right], \tag{2.8}$$

where

$$\gamma_0^t := \sigma_0^t \circ \alpha_0^{-\beta t} = e^{i(H_{\text{resc}}-\beta H_0)t}(\,\cdot\,)e^{-i(H_{\text{resc}}-\beta H_0)t}$$

and recall the definition (1.61) of σ_0^t. It holds in particular

$$K\tilde{\Omega} = 0.$$

Proof. For $A \in \pi(\mathcal{A})''$ such that $A\tilde{\Omega} \in \mathcal{D}(L^{(\ell)})$ we may compute

$$\begin{aligned} \partial_t|_{t=0}U(t)[A\tilde{\Omega}] &= \partial_t|_{t=0}e^{iLt}Ae^{-iLt}\tilde{\Omega} = \partial_t|_{t=0}e^{iL^{(\ell)}t}Ae^{-iL^{(\ell)}t}\tilde{\Omega} \\ &= [iL^{(\ell)}, A]\tilde{\Omega} = i(L^{(\ell)}A\tilde{\Omega} - AL^{(\ell)}\tilde{\Omega}). \end{aligned}$$

Since $\tilde{\Omega} \in \mathcal{D}(L^{(\ell)})$, as one proves similar to Lemma B.3, the last line is well defined and $t \mapsto U(t)$ is strongly differentiable on $\mathcal{D}(L^{(\ell)}) \cap \pi(\mathcal{A})''\tilde{\Omega}$. For the generator K holds

$$\begin{aligned}
(K - L^{(\ell)})[A\tilde{\Omega}] &= -AL^{(\ell)}\tilde{\Omega} \\
&= -\frac{Ae^{-\beta L^{(\ell)}/2}L^{(\ell)}\Omega_0}{\left\|e^{-\beta L^{(\ell)}/2}\Omega_0\right\|} \\
&= -\frac{g}{\left\|e^{-\beta L^{(\ell)}/2}\Omega_0\right\|}Ae^{-\beta L^{(\ell)}/2}\pi(v)\Omega_0 \\
&= -\frac{g}{\left\|e^{-\beta L^{(\ell)}/2}\Omega_0\right\|}Ae^{-\beta L^{(\ell)}/2}Je^{-\mathcal{L}_0/2}S_0\pi(v)\Omega_0 \\
&= -\frac{g}{\left\|e^{-\beta L^{(\ell)}/2}\Omega_0\right\|}Ae^{-\beta L^{(\ell)}/2}\pi'\left(\sigma_0^{i/2}(v)\right)\Omega_0 \\
&= -gAe^{-\beta L^{(\ell)}/2}\pi'\left(\sigma_0^{i/2}(v)\right)e^{\beta L^{(\ell)}/2}\tilde{\Omega} \\
&= -gA\pi'\left(\alpha_0^{-i\beta/2} \circ \sigma_0^{i/2}(v)\right)\tilde{\Omega} \\
&= -g\pi'\left(\gamma_0^{i/2}(v)\right)A\tilde{\Omega}
\end{aligned}$$

where we used Lemma B.2(i) and the commutativity between A and $\pi'(\gamma_0^{i/2}(v))$ and $[L^{(\ell)}, \pi'(\gamma_0^{i/2}(v))] = [L_0, \pi'(\gamma_0^{i/2}(v))]$ which is responsible for

$$e^{-\beta L^{(\ell)}/2}\pi'\left(\sigma_0^{i/2}(v)\right)e^{\beta L^{(\ell)}/2}\tilde{\Omega} = e^{-\beta L_0/2}\pi'\left(\sigma_0^{i/2}(v)\right)e^{\beta L_0/2}\tilde{\Omega}.$$

For the well-definedness of $\pi'(\sigma_0^{i/2}(v))$ we employed Hypothesis VI-1.11. The relation (2.8) therefore holds on $\mathcal{D}(L^{(\ell)}) \cap \pi(\mathcal{A})''\tilde{\Omega}$. ■

The generator K is referred to as *C-Liouville operator* underlining that it generates the perturbed time evolution α lifted to $\pi(A)''$, at least formally, but it is not the standard Liouville operator for the representation π in the sense that its group $U(t)$ does not necessarily leave the positive cone $\mathcal{P}$ invariant. We remark that the standard Liouville operator L and the C-Liouville operator are the same if and only if $\gamma_0^t = \mathbb{1}_{\mathcal{A}}$ for all $t \in \mathbb{R}$, i.e., if $\beta = \beta_{\mathrm{p}} = \beta_1 = \cdots = \beta_R$. In the case of non-equal temperatures the operator K is not self-adjoint.

We remark that the choice of the vector $\tilde{\Omega}$ as the null vector of the C-Liouvillean is somewhat arbitrary. From a conceptional point of view the vector Ω_0 would be a canonical candidate since its cyclic and separating properties are at hand and do not require an extended technical argumentation. The C-Liouville operator built up on this vector would read

$$K_{\Omega_0} = L_0 + g\left[\pi(v) - \pi'\left(\sigma_0^{i/2}(v)\right)\right].$$

This choice was made in [28] where the concept of spectral theory for NESS was introduced. The crucial disadvantage of this approach, however, is that the perturbation part of K_{Ω_0} carries an exponential weight in the inverse temperatures,

$$\sigma_0^{i/2}(v) = e^{-\left(\beta_{\mathrm{p}} H_{\mathrm{p}} + \sum_{r=1}^{R} \beta_r H_{\mathrm{f},r}\right)/2} v e^{\left(\beta_{\mathrm{p}} H_{\mathrm{p}} + \sum_{r=1}^{R} \beta_r H_{\mathrm{f},r}\right)/2},$$

such that the perturbative analysis requires that $\beta_{\max}$ is small which restricts the considerations to a low temperature regime. Incorporating an approximation $\tilde{\Omega}$ of the (non-existent) KMS state into the definition of K leads to a perturbation which is only weighted exponentially in the temperature differences,

$$\gamma_0^{i/2}(v) = e^{-\left((\beta_{\mathrm{p}}-\beta) H_{\mathrm{p}} + \sum_{r=1}^{R} (\beta_r-\beta) H_{\mathrm{f},r}\right)/2} v e^{\left((\beta_{\mathrm{p}}-\beta) H_{\mathrm{p}} + \sum_{r=1}^{R} (\beta_r-\beta) H_{\mathrm{f},r}\right)/2}.$$

It was respected in Hypothesis III-1.8 that the differences in the inverse temperatures have to be small.

For simplicity the further considerations are done under the assumption that

$$\beta = \beta_{\mathrm{p}} = \beta_{\max}. \tag{2.9}$$

2.2 Representation of the Schrödinger Time Evolution $U(t)$

Formally, we can consider the group $U(t)$ as the exponential function of K, i.e., $U(t) = e^{iKt}$. The group acts on vectors of $\pi(\mathcal{A})''\tilde{\Omega}$ as a time evolution in the Schrödinger picture. The ansatz allows to transfer the study of a Heisenberg time evolution $t \mapsto \alpha^t$, acting on observables, to the study of a Schrödinger evolution $t \mapsto U(t)$ on vectors. This shifts the analysis into a Hilbert space framework. However, since K is in general not self-adjoint in the non-equilibrium situation the group $U(t)$ is in general not unitary. The situation is even more subtle. The imaginary part of the generator K for $\beta_{\max} > \beta_{\min}$ is neither bounded from below nor from above such that an interpretation of $U(t)$ as a C_0 semigroup does not apply. It is the aim of this section to represent the group $U(t)$ in terms of its generator K.

2.2.1 The Family $K^{(s)}$ of Generators

Since the lack of self-adjointness of K causes the troubles we go over to consider a family of operators given by

$$s \mapsto K^{(s)} := L_0 + gI^{(s)} \tag{2.10}$$

with

$$I^{(s)} := \pi(v) - \pi'\left(\gamma_0^{\overline{s}}(v)\right) = a_{\mathrm{gl}}^*\left(\mathcal{G} - \mathcal{G}'_{(s\delta\vec{\beta})}\right) + a_{\mathrm{gl}}\left(\mathcal{G} - \mathcal{G}'_{(\overline{s}\delta\vec{\beta})}\right)$$

for s in the parameter set

$$\mathbb{S}_{\varepsilon_0} := \left\{ z \in \mathbb{C} \,\middle|\, |z| < \frac{1}{2} + \varepsilon_0 \right\} \tag{2.11}$$

for a small positive number $\varepsilon_0 > 0$. The glued coupling functions are given by (1.80, 1.81) and the notation

$$\begin{aligned}
\mathcal{G}_{(\vec{\kappa})}(u, \Sigma, r) &:= e^{i\kappa_r u}[\alpha_{\mathrm{p}}^{\kappa_{\mathrm{p}}} \otimes \mathbb{1}_{\mathcal{A}_{\mathrm{p}}}](\mathcal{G}(u, \Sigma, r)) \\
&= \sqrt{\frac{u}{1 - e^{-\beta_r u}}}\, e^{i\kappa_r u} \\
&\quad \times \begin{cases} \sqrt{u}\, \alpha_{\mathrm{p}}^{\kappa_{\mathrm{p}}}\left(G_r(u\Sigma)\right) \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{p}}}, & u \geq 0, \\ (-\sqrt{-u})\alpha_{\mathrm{p}}^{\kappa_{\mathrm{p}}}\left(G_r(-u\Sigma)^*\right) \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{p}}}, & u < 0, \end{cases}
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
\mathcal{G}'_{(\vec{\kappa})}(u, \Sigma, r) &:= e^{i\kappa_r u}[\mathbb{1}_{\mathcal{A}_{\mathrm{p}}} \otimes \alpha_{\mathrm{p}}^{-\kappa_{\mathrm{p}}}](\mathcal{G}'(u, \Sigma, r)) \\
&= \sqrt{\frac{u}{e^{\beta_r u} - 1}}\, e^{i\kappa_r u} \\
&\quad \times \begin{cases} \sqrt{u}\, \mathbb{1}_{\mathcal{H}_{\mathrm{p}}} \otimes \alpha_{\mathrm{p}}^{-\kappa_{\mathrm{p}}}\left(\overline{G_r(u\Sigma)}^*\right), & u \geq 0, \\ (-\sqrt{-u})\mathbb{1}_{\mathcal{H}_{\mathrm{p}}} \otimes \alpha_{\mathrm{p}}^{-\kappa_{\mathrm{p}}}\left(\overline{G_r(-u\Sigma)}\right), & u < 0, \end{cases}
\end{aligned} \tag{2.13}$$

for $\vec{\kappa} = (\kappa_{\mathrm{p}}, \kappa_1, \ldots, \kappa_R) \in \mathbb{C}^{R+1}$ and

$$\delta\vec{\beta} := (\delta\beta_{\mathrm{p}}, \delta\beta_1, \ldots, \delta\beta_R) := (\beta_{\mathrm{p}} - \beta, \beta_1 - \beta, \ldots, \beta_R - \beta).$$

Hypothesis VII-1.12 and Remark 1.13 then imply that $\mathcal{G}'_{(s\delta\vec{\beta})} \in L^2[\Upsilon]$ for all $s \in \mathbb{S}_\varepsilon$ such that the expression (2.10) is well defined as a family of operators on a dense domain. By Corollary 1.6 we know that $K^{(s)}$ extends to a self-adjoint operator, also denoted by $K^{(s)}$, for $s \in \mathbb{R}$. Since $L - K^{(s)} = \pi'(\gamma_0^{\overline{s}}(v) - v)$ and $e^{i\pi'(\gamma_0^{\overline{s}}(v)-v)t} \in \pi(\mathcal{A})'$, we obtain with the Trotter product formula

$$e^{iK^{(s)}t}\pi(A)e^{-iK^{(s)}t} = \pi \circ \alpha^t(A) = e^{iLt}\pi(A)e^{-iLt} \qquad \text{for all } s, t \in \mathbb{R}, \tag{2.14}$$

where the exponential functions in $K^{(s)}$ are well defined as unitary operators due to the self-adjointness of $K^{(s)}$ for real s. We remark that $[0,1] \ni \varsigma \mapsto K^{(-i\varsigma/2)}$ is a path in the space of generators of the Heisenberg evolution connecting the standard Liouville operator L with the C-Liouville operator K in the sense that

$$K^{(-i/2)} = K \qquad \text{and} \qquad K^{(0)} = L.$$

The next lemma shows a useful decomposition of $e^{iK^{(s)}t}$ into known operators.

Lemma 2.2 *Let be $s,t\in\mathbb{R}$. The unitary operator $e^{iK^{(s)}t}$ has the following decomposition,*

$$e^{iK^{(s)}t}=e^{iL^{(\ell)}t}\Gamma_0^{-is}e^{-iL_0t}e^{iL^{(\mathrm{r})}t}\Gamma_0^{is},$$

where

$$\Gamma_0:=\Delta_0e^{\beta L_0}=e^{-(\mathcal{L}_0-\beta L_0)}.$$

Proof. We first remark that $K^{(s)}=L_0+V-V'^{(s)}$ and $L^{(\ell)}=L_0+V$ where $V:=g\pi(v)$ and $V'^{(s)}:=g\pi'(\gamma_0^{\bar{s}}(v))$. We further note that $V'^{(s)}=\Gamma_0^{-is}V'\Gamma_0^{is}$ where $V':=g\pi'(v)$. This enables us to rewrite, using the Trotter product formula,

$$\begin{aligned}
e^{-iL^{(\ell)}t}e^{iK^{(s)}t} &= \operatorname*{s-lim}_{n\to\infty}\left[e^{-iL_0t/n}e^{-iVt/n}\right]^n\left[e^{iVt/n}e^{i(L_0-V'^{(s)})t/n}\right]^n\\
&= \operatorname*{s-lim}_{n\to\infty}\left[e^{-iL_0t/n}e^{-iVt/n}\right]^{n-2}e^{-iL_0t/n}\\
&\qquad\times\left(e^{-iVt/n}e^{-iL_0t/n}e^{i(L_0-V'^{(s)})t/n}e^{iVt/n}\right)\\
&\qquad\times e^{i(L_0-V'^{(s)})t/n}\left[e^{iVt/n}e^{i(L_0-V'^{(s)})t/n}\right]^{n-2}\\
&= \operatorname*{s-lim}_{n\to\infty}\left[e^{-iL_0t/n}e^{-iVt/n}\right]^{n-3}e^{-iL_0t/n}\\
&\qquad\times\left(e^{-iVt/n}e^{-iL_02t/n}e^{i(L_0-V'^{(s)})2t/n}e^{iVt/n}\right)\\
&\qquad\times e^{i(L_0-V'^{(s)})t/n}\left[e^{iVt/n}e^{i(L_0-V'^{(s)})t/n}\right]^{n-3}\\
&= \dots\\
&= e^{-iL_0t}e^{it(L_0-V'^{(s)})}\\
&= e^{-iL_0t}\exp\left(it\Gamma_0^{-is}(L_0-V')\Gamma_0^{is}\right)\\
&= e^{-iL_0t}\Gamma_0^{-is}e^{i(L_0-V')t}\Gamma_0^{is}\\
&= \Gamma_0^{-is}e^{-iL_0t}e^{iL^{(\mathrm{r})}t}\Gamma_0^{is},
\end{aligned}$$

which proves the assertion. It remains to clarify the manipulations of the parentheses in the third and the sixth line. Set $\tau=t/n$ and consider

$$\begin{aligned}
&e^{-iV\tau}e^{-iL_0\tau}e^{i(L_0-V'^{(s)})\tau}e^{iV\tau}\\
&\quad= e^{-iV\tau}e^{-iL_0\tau}\operatorname*{s-lim}_{m\to\infty}\left[e^{iL_0\tau/m}e^{-iV'^{(s)}\tau/m}\right]^m e^{iV\tau}\\
&\quad= e^{-iV\tau}e^{-iL_0\tau}\operatorname*{s-lim}_{m\to\infty}\left[e^{iL_0\tau/m}e^{-iV'^{(s)}\tau/m}\right]^{m-1}e^{iL_0\tau/m}e^{iV\tau}e^{-iV'^{(s)}\tau/m}\\
&\quad= e^{-iV\tau}e^{-iL_0\tau}\operatorname*{s-lim}_{m\to\infty}\left[e^{iL_0\tau/m}e^{-iV'^{(s)}\tau/m}\right]^{m-1}\\
&\qquad\qquad\times\left(e^{iL_0\tau/m}e^{iV\tau}e^{-iL_0\tau/m}\right)\left[e^{iL_0\tau/m}e^{-iV'^{(s)}\tau/m}\right]\\
&\quad= \dots
\end{aligned}$$

$$\begin{aligned}
&= e^{-iV\tau}e^{-iL_0\tau}\left(e^{iL_0\tau}e^{iV\tau}e^{-iL_0\tau}\right)\operatorname*{s-lim}_{m\to\infty}\left[e^{iL_0\tau/m}e^{-iV'^{(s)}\tau/m}\right]^m \\
&= e^{-iV\tau}e^{-iL_0\tau}\left(e^{iL_0\tau}e^{iV\tau}e^{-iL_0\tau}\right)e^{i(L_0-V'^{(s)})\tau} \\
&= e^{-iL_0\tau}e^{i(L_0-V'^{(s)})\tau},
\end{aligned}$$

where we used that $e^{iL_0\tilde{\tau}}e^{iV\tau}e^{-iL_0\tilde{\tau}} = \exp(ig\pi(\alpha_0^{\tilde{\tau}}(v)))$ commutes with $e^{-iV'^{(s)}\tau/m}$ for all $\tilde{\tau} \in \mathbb{R}$. ■

For what follows, we assume that $s \in \mathbb{R}$ is chosen to be real. Our goal is to find a representation of $\left\langle \varphi \,\middle|\, e^{iK^{(s)}t}\psi \right\rangle$ for suitable vectors $\varphi, \psi \in \mathcal{H}^2$ which has a well defined extension to complex values of the parameter s, in particular for $s = -i/2$. The following lemma presents the appropriate representation.

Lemma 2.3 *For any pair of vectors $\varphi, \psi \in \mathcal{H}^2$ and for $s \in \mathbb{R}$, $t \geq 0$, we can write*

$$\left\langle \varphi \,\middle|\, e^{iK^{(s)}t}\psi \right\rangle = \frac{1}{2\pi i}\int\limits_{\mathbb{R}-i\varepsilon} dz \left\langle \varphi \,\middle|\, \left(z - K^{(s)}\right)^{-1}\psi \right\rangle e^{izt} \tag{2.15}$$

for any $\varepsilon > 0$, where the integral has to be understood as an improper Riemann integral.

Proof. Define for fixed $\varphi, \psi \in \mathcal{H}^2$ and $s \in \mathbb{R}$ the function

$$f(t) := \left\langle \varphi \,\middle|\, e^{iK^{(s)}t}\psi \right\rangle$$

and consider for $\mathrm{Im}(z) < 0$ its Fourier-Laplace transform

$$\hat{f}(z) := \int\limits_0^\infty dt\, f(t)e^{-izt} = -i\left\langle \varphi \,\middle|\, \left(z - K^{(s)}\right)^{-1}\psi \right\rangle.$$

For $\varepsilon > 0$ and $t \geq 0$ we obtain as inverse relation

$$\begin{aligned}
f(t) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dx\, \hat{f}(x - i\varepsilon)e^{i(x-i\varepsilon)t} \\
&= \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} dx \left\langle \varphi \,\middle|\, \left(x - i\varepsilon - K^{(s)}\right)^{-1}\psi \right\rangle e^{i(x-i\varepsilon)t}.
\end{aligned}$$

■

Next, we show that the l.h.s. of (2.15) has an analytic continuation to complex values of the parameter s.

Proposition 2.4 *For any $A \in \mathcal{A}$ and any $t \in \mathbb{R}$ the vector valued function*

$$\mathbb{R} \ni s \mapsto e^{iK^{(s)}t}\pi(A)\tilde{\Omega}$$

has an analytic continuation on the set $\mathbb{S}_{\frac{\varepsilon_0}{4}}$, denoted by $\xi : \mathbb{S}_{\frac{\varepsilon_0}{4}} \to \mathcal{H}^2$. Further, at the point $s = -i/2$ the function takes the value

$$\xi(-i/2) = \pi(\alpha^t(A))\tilde{\Omega} = U(t)[\pi(A)\tilde{\Omega}].$$

Proof. Due to (2.14) we get

$$e^{iK^{(s)}t}\pi(A)\tilde{\Omega} = \pi(\alpha^t(A))e^{iK^{(s)}t}\tilde{\Omega}$$

such that it is sufficient to consider the analyticity of $s \mapsto e^{iK^{(s)}t}\tilde{\Omega}$. Yet, by Lemma 2.2, we may focus on the function

$$\begin{aligned} s \mapsto\ & \Gamma_0^{-is}e^{-iL_0t}e^{iL^{(\mathrm{r})}t}\Gamma_0^{is}\tilde{\Omega} \qquad (2.16)\\ =\ & \sum_{n=0}^{\infty}(-ig)^n \int\limits_{0\le\tau_n\le\cdots\le\tau_1\le t} d\tau_1\cdots d\tau_n\, \Gamma_0^{-is}\pi'\left(\alpha_0^{-\tau_1}(v)\cdots\alpha_0^{-\tau_n}(v)\right)\Gamma_0^{is}\tilde{\Omega} \\ =\ & \sum_{n=0}^{\infty}(-ig)^n \int\limits_{0\le\tau_n\le\cdots\le\tau_1\le t} d\tau_1\cdots d\tau_n\, \pi'\left(\gamma_0^s\left[\alpha_0^{-\tau_1}(v)\cdots\alpha_0^{-\tau_n}(v)\right]\right) \\ & \times\sum_{m=0}^{\infty}(-g)^m \int\limits_{0\le\varsigma_1\le\cdots\le\varsigma_m\le\beta/2} d\varsigma_1\cdots d\varsigma_m\, \pi\left(\alpha_0^{i\varsigma_m}(v)\cdots\alpha_0^{i\varsigma_1}(v)\right)\frac{\Omega_0}{\left\|e^{-\beta L^{(\ell)}/2}\Omega_0\right\|}. \end{aligned}$$

Hereby we used Lemma B.1(i) to write $\tilde{\Omega} = \left\|e^{-\beta L^{(\ell)}/2}\Omega_0\right\|^{-1}e^{-\beta L^{(\ell)}/2}e^{\beta L_0/2}\Omega_0$ as Dyson series. The expansion of $e^{-iL_0t}e^{iL^{(\mathrm{r})}t}$ in a Dyson series is standard. We check the absolute convergence of this series. First we make use of the Jakšić-Pillet glued representation to write

$$\pi'\left(\gamma_0^s\circ\alpha_0^{-\tau_j}(v)\right) = a_{\mathrm{gl}}^*\left(\mathcal{G}'_{\left(s\delta\vec{\beta}-\tau_j\vec{1}\right)}\right) + a_{\mathrm{gl}}\left(\mathcal{G}'_{\left(s\delta\vec{\beta}-\tau_j\vec{1}\right)}\right)$$

and

$$\pi\left(\alpha_0^{i\varsigma_j}(v)\right) = a_{\mathrm{gl}}^*\left(\mathcal{G}_{\left(i\varsigma_j\vec{1}\right)}\right) + a_{\mathrm{gl}}\left(\mathcal{G}_{\left(-i\varsigma_j\vec{1}\right)}\right),$$

where $\vec{1} = (1,1,\ldots,1) \in \mathbb{C}^{R+1}$, recall definitions (2.12, 2.13). We recall the standard estimate of creation and annihilation operators on a vector ψ_n of the n-"photon" sector in the Fock space over $L^2[\Upsilon]$,

$$\left\|a_{\mathrm{gl}}^{\#}(F)\varphi\otimes\psi_n\right\| \le \sqrt{n+1}\,\|F\|_{L^2[\Upsilon;\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)]}\,\|\varphi\otimes\psi_n\|,$$

where $\varphi \in \mathcal{H}_{\mathrm{p}}^2$ and $F \in L^2[\Upsilon; \mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)]$, c.f. Lemma A.3. Since the coupling functions $\mathcal{G}'_{\left(s\delta\vec{\beta}-\tau_j\vec{1}\right)}$ are a.e. pointwise differentiable w.r.t. s for $|\operatorname{Im}(s)| < 1/2 + \varepsilon_0/2$ with derivative

$$\partial_s \mathcal{G}'_{\left(s\delta\vec{\beta}-\tau_j\vec{1}\right)}(u,\Sigma,r) = i\delta\beta_r u \mathcal{G}'_{\left(s\delta\vec{\beta}-\tau_j\vec{1}\right)}(u,\Sigma,r) + \left[i\delta\beta_{\mathrm{p}} L_{\mathrm{p}}, \mathcal{G}'_{\left(s\delta\vec{\beta}-\tau_j\vec{1}\right)}(u,\Sigma,r)\right]$$

and having the uniform norm bounds, guaranteed by Hypothesis VII-1.12,

$$\begin{aligned}
b_1 &:= \sup_{|\operatorname{Im}(s)|\leq 1/2+\varepsilon_0/2} \left\| \mathcal{G}'_{\left(s\delta\vec{\beta}-\tau_j\vec{1}\right)} \right\|_{L^2[\Upsilon,\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)]} \\
&= \sup_{|\operatorname{Im}(s)|\leq 1/2+\varepsilon_0/2} \left\| \mathcal{G}'_{\left(s\delta\vec{\beta}\right)} \right\|_{L^2[\Upsilon,\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)]} < \infty, \\
b_1' &:= \sup_{|\operatorname{Im}(s)|\leq 1/2+\varepsilon_0/4} \left\| \partial_s \mathcal{G}'_{\left(s\delta\vec{\beta}-\tau_j\vec{1}\right)} \right\|_{L^2[\Upsilon,\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)]} \\
&\leq \sup_{r=1,\ldots,R} \left(|\delta\beta_r| + 2|\delta\beta_{\mathrm{p}}| \, \|L_{\mathrm{p}}\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} \right) b_1 < \infty, \\
b_2 &:= \sup_{|\varsigma|\leq\beta/2} \left\| \mathcal{G}_{\left(i\varsigma\vec{1}\right)} \right\|_{L^2[\Upsilon,\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)]} < \infty,
\end{aligned}$$

we obtain by the dominated convergence theorem the complex differentiability of

$$\$_{\frac{\varepsilon_0}{4}} \ni s \mapsto \pi'\left(\gamma_0^{\bar{s}}\left[\alpha_0^{-\tau_1}(v)\cdots\alpha_0^{-\tau_n}(v)\right]\right)\pi\left(\alpha_0^{i\varsigma_m}(v)\cdots\alpha_0^{i\varsigma_1}(v)\right)\Omega_0.$$

Here, we took the anti-linear nature of π' into account which requires the complex conjugation of the parameter s inside π'. A further uniform estimate,

$$\begin{aligned}
&\sup_{\substack{|\operatorname{Im}(s)|\leq 1/2+\varepsilon_0/2,\\ 0\leq\tau_n\leq\cdots\leq\tau_1\leq t,\\ 0\leq\varsigma_1\leq\cdots\leq\varsigma_m\leq\beta/2}} \left\|\pi'\left(\gamma_0^{\bar{s}}\left[\alpha_0^{-\tau_1}(v)\cdots\alpha_0^{-\tau_n}(v)\right]\right)\pi\left(\alpha_0^{i\varsigma_m}(v)\cdots\alpha_0^{i\varsigma_1}(v)\right)\Omega_0\right\| \\
&\qquad\qquad \leq \sqrt{(n+m+1)!}\,(2b_1)^n(2b_2)^m,
\end{aligned}$$

finally ensures the analyticity of (2.16) because of the convergence of the following series,

$$\begin{aligned}
&\sum_{n,m=0}^{\infty} g^{n+m} \int\limits_{\substack{0\leq\tau_n\leq\cdots\leq\tau_1\leq t,\\ 0\leq\varsigma_1\leq\cdots\leq\varsigma_m\leq\beta/2}} d\tau_1\cdots d\tau_n\, d\varsigma_1\cdots d\varsigma_m \sqrt{(n+m+1)!}\,(2b_1)^n(2b_2)^m \\
&\quad= \sum_{n,m=0}^{\infty} g^{n+m}(2b_1)^n(2b_2)^m \frac{t^n\beta^m}{2^m n! m!}\sqrt{(n+m+1)!} \\
&\quad= \sum_{n,m=0}^{\infty} g^{n+m}(2b_1)^n b_2^m t^n \beta^m \sqrt{\frac{n+1}{n!m!}\binom{n+m+1}{m}}
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{n,m=0}^{\infty} g^{n+m}(2b_1)^n b_2^m t^n \beta^m \sqrt{\frac{n+1}{n!m!}2^{n+m+1}} \\
&= \sqrt{2}\left(\sum_{n=0}^{\infty}\frac{\sqrt{n+1}}{\sqrt{n!}}\left(\sqrt{8}gb_1t\right)^n\right)\left(\sum_{m=0}^{\infty}\frac{1}{\sqrt{m!}}\left(\sqrt{2}gb_2\beta\right)^m\right) \\
&< \infty.
\end{aligned}
$$

We now evaluate at the point $s = -i/2$, i.e., we compute

$$
\begin{aligned}
&\Gamma_0^{-1/2} e^{-iL_0 t} e^{iL^{(\mathrm{r})}t}\Gamma_0^{1/2}\tilde{\Omega} \\
&= \Gamma_0^{-1/2} e^{-iL_0 t} e^{iL^{(\mathrm{r})}t} e^{\beta L_0/2} J S_0 e^{-\beta L^{(\ell)}/2}\frac{\Omega_0}{\left\|e^{-\beta L^{(\ell)}/2}\Omega_0\right\|} \\
&= \Gamma_0^{-1/2} e^{-iL_0 t} e^{iL^{(\mathrm{r})}t} e^{\beta L_0/2} J e^{\beta L_0/2} e^{-\beta L^{(\ell)}/2}\frac{\Omega_0}{\left\|e^{-\beta L^{(\ell)}/2}\Omega_0\right\|} \\
&= \Gamma_0^{-1/2} J e^{-iL_0 t} e^{iL^{(\ell)}t} e^{-\beta L^{(\ell)}/2}\frac{\Omega_0}{\left\|e^{-\beta L^{(\ell)}/2}\Omega_0\right\|} \\
&= S_0 e^{-(it-\beta/2)L_0} e^{(it-\beta/2)L^{(\ell)}}\frac{\Omega_0}{\left\|e^{-\beta L^{(\ell)}/2}\Omega_0\right\|} \\
&= e^{(-it-\beta/2)L^{(\ell)}}\frac{\Omega_0}{\left\|e^{-\beta L^{(\ell)}/2}\Omega_0\right\|} \\
&= e^{-iL^{(\ell)}t}\tilde{\Omega},
\end{aligned}
$$

where we used the Lemma B.2(iii) twice and the relations $L^{(\mathrm{r})}J = -JL^{(\ell)}$ and $\Delta_0^{-1/2}J = J\Delta_0^{1/2} = S_0$. We finally obtain

$$
\xi(-i/2) = \pi(\alpha^t(A))e^{iL^{(\ell)}t}\Gamma_0^{-1/2}e^{-iL_0 t}e^{iL^{(\mathrm{r})}t}\Gamma_0^{1/2}\tilde{\Omega} = \pi(\alpha^t(A))\tilde{\Omega}.
$$

■

The next task is to check the analyticity of the r.h.s. of (2.15) in s. The problematic part is that the perturbation $I^{(s)}$ in $K^{(s)} = L_0 + gI^{(s)}$ is not relatively L_0-bounded which prevents standard arguments showing analyticity of $s \mapsto K^{(s)}$. Instead, we apply a spectral deformation to $K^{(s)}$ to gain analytic properties.

2.2.2 Spectral Deformation of $K^{(s)}$

We start by defining the *spectral deformation* on the space $\mathcal{F}(L^2[\Upsilon])$ which was previewed in Section 1.4.5 in order to state the requirements on the coupling functions.

We can use neither the deformation by dilation introduced in [8] nor the translation applied in [23, 28] but we combine both types. Such a combination was already mentioned in [8] and becomes essential in our work. The dilation deformation is used to make the operator $K^{(s)}$ sectorial while the translation separates the eigenvalues from the continuous spectrum, c.f. Figures 2.1 and 3.2. The first feature will be useful to integrate the resolvent of K along the real axis to obtain an integral representation of $U(t)$, c.f. Proposition 2.9, while the isolation of eigenvalues allows the computation of the projection on the null space of K, c.f. Proposition 2.10. This effort has to be made since K is not self-adjoint.

Define a unitary *translation group* $\{\mathfrak{D}_{\mathrm{t}}(\tau)\}_{\tau\in\mathbb{R}}$ on $L^2[\Upsilon]$ which acts on a given function $f \in L^2[\Upsilon]$ as

$$[\mathfrak{D}_{\mathrm{t}}(\tau)f](u,\Sigma,r) := f(u+\tau,\Sigma,r).$$

We further introduce a unitary *group of dilations* $\{\mathfrak{D}_{\mathrm{d}}(\delta)\}_{\delta\in\mathbb{R}}$ which is defined on a function $f \in L^2[\Upsilon]$ by

$$[\mathfrak{D}_{\mathrm{d}}(\delta)f](u,\Sigma,r) := e^{\delta\,\mathrm{sgn}(u)/2} f(e^{\delta\,\mathrm{sgn}(u)}u,\Sigma,r).$$

We compose both transformation to an operation which first translates and then dilates a function (note that translation and dilation do not commute such that the order of application has to be respected). The operation

$$\mathfrak{D}(\theta) := \mathfrak{D}_{\mathrm{d}}(\delta)\mathfrak{D}_{\mathrm{t}}(\tau) \qquad \text{for } \theta = (\delta,\tau) \in \mathbb{R}^2$$

defines a two parameter family $\{\mathfrak{D}(\theta)\}_{\theta\in\mathbb{R}^2}$ of unitary operators given on $f \in L^2[\Upsilon]$ by

$$[\mathfrak{D}(\theta)f](u,\Sigma,r) = e^{\delta\,\mathrm{sgn}(u)/2} f(j_\theta(u),\Sigma,r), \tag{2.17}$$

where

$$j_\theta(u) = e^{\delta\,\mathrm{sgn}(u)}u + \tau. \tag{2.18}$$

The family $\{\mathfrak{D}(\theta)\}_{\theta\in\mathbb{R}^2}$ can be lifted to the Fock space $\mathcal{F}(L^2[\Upsilon])$ by

$$\mathfrak{D}(\theta)\left[a^*_{\mathrm{gl}}(f_n)\cdots a^*_{\mathrm{gl}}(f_1)\Omega_0\right] := a^*_{\mathrm{gl}}(\mathfrak{D}(\theta)f_n)\cdots a^*_{\mathrm{gl}}(\mathfrak{D}(\theta)f_1)\Omega_0 \tag{2.19}$$

for $f_1,\ldots,f_n \in L^2[\Upsilon]$. In order to extend the family $\{\mathfrak{D}(\theta)\}_{\theta\in\mathbb{R}^2}$ to the space $\mathcal{H}^2$ we identify $\mathfrak{D}(\theta) \equiv \mathbb{1}_{\mathcal{H}^2_{\mathrm{p}}} \otimes \mathfrak{D}(\theta)$ such that particle variables remain uninfluenced by the transformation. For a vector $\psi \in \mathcal{F}(L^2[\Upsilon])$ (and in particular for $f \in L^2[\Upsilon]$) we abbreviate

$$\psi_\theta := \mathfrak{D}(\theta)\psi, \qquad f_\theta := \mathfrak{D}(\theta)f,$$

resp., and for an operator A on $\mathcal{F}(L^2[\Upsilon])$ we introduce the notation

$$A_\theta := \mathfrak{D}(\theta)A\mathfrak{D}(\theta)^{-1}.$$

The above notations also apply to vectors $\psi \in \mathcal{H}^2$, form factors $f \in L^2[\Upsilon; \mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)]$ and operators A on $\mathcal{H}^2$. We remark that the spectral properties of A remain unchanged under conjugation with the unitary operator $\mathfrak{D}(\theta)$, for $\theta \in \mathbb{R}^2$, i.e., the spectrum, pure point spectrum and continuous spectrum are invariant,

$$\mathrm{spec}(A_\theta) = \mathrm{spec}(A), \qquad \mathrm{spec}_{\mathrm{pp}}(A_\theta) = \mathrm{spec}_{\mathrm{pp}}(A), \qquad \mathrm{spec}_{\mathrm{c}}(A_\theta) = \mathrm{spec}_{\mathrm{c}}(A),$$

for $\theta \in \mathbb{R}^2$. Due to the unitarity of $\mathfrak{D}(\theta)$ we also have the invariance of matrix elements of an operator A for vectors $\varphi, \psi \in \mathcal{H}^2$ in the sense that

$$\langle \varphi_\theta \,|\, A_\theta \psi_\theta \rangle = \langle \varphi \,|\, A\psi \rangle \qquad \text{for } \theta \in \mathbb{R}^2. \tag{2.20}$$

However, by extending the family $\{\mathfrak{D}(\theta)\}_{\theta \in \mathbb{R}^2}$ to complex parameters $\theta \in \mathbb{C}$ we lose the unitarity of the transformation $\mathfrak{D}(\theta)$. Assuming that the family of operators $\theta \mapsto A_\theta = \mathfrak{D}(\theta) A \mathfrak{D}(\theta)^{-1}$ has an analytic continuation to complex values of θ in an appropriate sense (to be discussed later) we keep the invariance of the pure point spectrum while the continuous spectrum, in general, is deformed,

$$\mathrm{spec}_{\mathrm{pp}}(A_\theta) = \mathrm{spec}_{\mathrm{pp}}(A), \qquad \mathrm{spec}_{\mathrm{c}}(A_\theta) \neq \mathrm{spec}_{\mathrm{c}}(A) \qquad \text{for } \theta \in \mathbb{C}^2.$$

This observation motivates the nomenclature *spectral deformation* associated with the family $\{\mathfrak{D}(\theta)\}_\theta$. The benefit of the concept of spectral deformation is that the relation (2.20) extends by analytic continuation to

$$\langle \varphi_{\overline{\theta}} \,|\, A_\theta \psi_\theta \rangle = \langle \varphi \,|\, A\psi \rangle \qquad \text{for } \theta \in \mathbb{C}^2$$

under the assumption that all functions $\theta \mapsto \varphi_\theta, \psi_\theta, A_\theta$ are analytic, again in the appropriate sense. This relation has a very useful application where we try to rewrite

$$\left\langle \varphi \,\middle|\, \left(z - K^{(s)}\right)^{-1} \psi \right\rangle = \left\langle \varphi_{\overline{\theta}} \,\middle|\, \left(z - K_\theta^{(s)}\right)^{-1} \psi_\theta \right\rangle \tag{2.21}$$

aiming to be in the position to control the spectrum of the deformed operator $K_\theta^{(s)} = \mathfrak{D}(\theta) K^{(s)} \mathfrak{D}(\theta)^{-1}$ for complex s while the spectrum of $K^{(s)}$ is not accessible to us.

We now fill in the blanks and launch the rigorous consideration by computing the deformation of the operators which play a role in our analysis. We collect the corresponding results in a lemma.

Lemma 2.5 *Let $\theta = (\delta, \tau) \in \mathbb{R}^2$.*

(i) The spectral deformation $L_{0,\theta} = \mathfrak{D}(\theta) L_0 \mathfrak{D}(\theta)^{-1}$ of the free Liouville operator L_0 is given by

$$L_{0,\theta} = L_{\mathrm{p}} + \cosh(\delta) L_{\mathrm{res}} + \sinh(\delta) L_{\mathrm{aux}} + \tau N_{\mathrm{res}} \tag{2.22}$$

where

$$L_{\mathrm{aux}} = d\Gamma_{\mathrm{gl}}((u,\Sigma,r)\mapsto |u|) \equiv \int_{\Upsilon} d(u,\Sigma,r)\, a^*_{\mathrm{gl}}(u,\Sigma,r)|u|a_{\mathrm{gl}}(u,\Sigma,r) \qquad (2.23)$$

is an auxiliary operator and

$$N_{\mathrm{res}} := d\Gamma_{\mathrm{gl}}(1) \equiv \int_{\Upsilon} dy\, a^*_{\mathrm{gl}}(y)a_{\mathrm{gl}}(y) \qquad (2.24)$$

is the number operator on the bosonic Fock space $\mathcal{F}(L^2[\Upsilon])$.

(ii) The spectral deformation $I^{(s)}_\theta = \mathfrak{D}(\theta)I^{(s)}\mathfrak{D}(\theta)^{-1}$, $s\in \mathbb{S}_{\varepsilon_0}$, of the operator $I^{(s)}$ is given by

$$I^{(s)}_\theta = a^*_{\mathrm{gl}}\left(F^{(s)}_\theta\right) + a_{\mathrm{gl}}\left(F^{(\overline{s})}_\theta\right), \qquad (2.25)$$

where $F^{(s)}_\theta = [\mathcal{G} - \mathcal{G}'_{(s\delta\vec{\beta})}]_\theta$, i.e.,

$$F^{(s)}_\theta(u,\Sigma,r) = e^{\delta\,\mathrm{sgn}(u)/2}\left[\mathcal{G} - \mathcal{G}'_{(s\delta\vec{\beta})}\right](j_\theta(u),\Sigma,r). \qquad (2.26)$$

Proof.

(i) The definition (2.17, 2.19) of $\mathfrak{D}(\theta)$ and (1.66) of L_{res} imply that

$$\begin{aligned}
L_{0,\theta} &= L_{\mathrm{p}} + \mathfrak{D}(\theta)d\Gamma_{\mathrm{gl}}((u,\Sigma,r)\mapsto u)\mathfrak{D}(\theta)^{-1} = L_{\mathrm{p}} + d\Gamma_{\mathrm{gl}}((u,\Sigma,r)\mapsto j_\theta(u)) \\
&= L_{\mathrm{p}} + d\Gamma_{\mathrm{gl}}((u,\Sigma,r)\mapsto e^{\delta\,\mathrm{sgn}(u)}u) + d\Gamma_{\mathrm{gl}}((u,\Sigma,r)\mapsto \tau) \\
&= L_{\mathrm{p}} + \tau N_{\mathrm{res}} + \int_0^\infty du \int_{S^2\times\mathbb{N}_1^R} d(\Sigma,r)\, a^*_{\mathrm{gl}}(u,\Sigma,r)e^{\delta/2}u a_{\mathrm{gl}}(u,\Sigma,r) \\
&\quad + \int_{-\infty}^0 du \int_{S^2\times\mathbb{N}_1^R} d(\Sigma,r)\, a^*_{\mathrm{gl}}(u,\Sigma,r)e^{-\delta/2}u a_{\mathrm{gl}}(u,\Sigma,r) \\
&= L_{\mathrm{p}} + \tau N_{\mathrm{res}} \\
&\quad + \int_{\Upsilon} d(u,\Sigma,r)\, a^*_{\mathrm{gl}}(u,\Sigma,r)\left[\cosh(\delta)u + \sinh(\delta)|u|\right] a_{\mathrm{gl}}(u,\Sigma,r) \\
&= L_{\mathrm{p}} + \tau N_{\mathrm{res}} + \cosh(\delta)L_{\mathrm{res}} + \sinh(\delta)L_{\mathrm{aux}}.
\end{aligned}$$

(ii) The assertion follows immediately from the definition (2.17, 2.19).

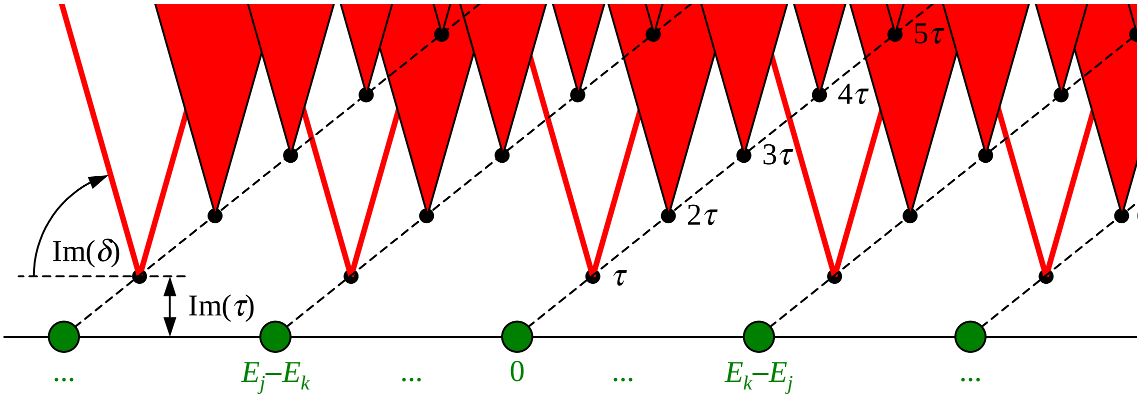

Figure 2.1: The spectrum of $L_{0,\theta}$ consists of the eigenvalues of L_{p}, paired with a V-shaped line and countably many cones of continuous spectrum arising from the spectrum of $\cos(\delta')L_{\mathrm{res}} + i\sin(\delta')L_{\mathrm{aux}}$, lined up along $E_k - E_j + \tau\mathbb{N}$. The apex angle of the cones is $\pi - 2\,\mathrm{Im}(\delta)$.

■

We describe the spectrum of the deformed unperturbed Liouville operator, $L_{0,\theta}$.

Proposition 2.6 (Spectrum of $L_{0,\theta}$) *The definition (2.22) of $L_{0,\theta}$ extends for complex deformation parameters $\theta \in \mathbb{C}^2$. The spectrum of $L_{0,\theta}$ is then given by*

$$\begin{aligned}
&\mathrm{spec}\,(L_{0,\theta}) \\
&\quad= \mathrm{spec}\,(L_{\mathrm{p}}) + \{0\} \cup (\tau + \{z \in \mathbb{C} \mid |\,\mathrm{Re}(z)| = \mathrm{sgn}(\sin(\delta'))|\cot(\delta')|\,\mathrm{Im}(z)\,\}) \\
&\qquad\qquad \cup \bigcup_{n=2}^{\infty} (n\tau + \{z \in \mathbb{C} \mid |\,\mathrm{Re}(z)| \le \mathrm{sgn}(\sin(\delta'))|\cot(\delta')|\,\mathrm{Im}(z)\,\})
\end{aligned}$$

as illustrated in Figure 2.1, where $\delta' := \mathrm{Im}(\delta)$. In particular, for $\mathrm{Im}(\tau), \sin(\delta') > 0$ and $|\,\mathrm{Re}(\tau)| \le \cot(\delta')\,\mathrm{Im}(\tau)$, the spectrum takes the form

$$\begin{aligned}
\mathrm{spec}\,(L_{0,\theta}) &= \mathrm{spec}\,(L_{\mathrm{p}}) + \{0\} \cup (\tau + \{z \in \mathbb{C} \mid |\,\mathrm{Re}(z)| = \cot(\delta')\,\mathrm{Im}(z)\,\}) \\
&\qquad\qquad \cup (2\tau + \{z \in \mathbb{C} \mid |\,\mathrm{Re}(z)| \le \cot(\delta')\,\mathrm{Im}(z)\,\}) \\
&\subseteq \mathrm{spec}\,(L_{\mathrm{p}}) + \{0\} \cup (\tau + \{z \in \mathbb{C} \mid |\,\mathrm{Re}(z)| \le \cot(\delta')\,\mathrm{Im}(z)\,\})\,,
\end{aligned}$$

and the eigenvalues of L_{p} are isolated eigenvalues of $L_{0,\theta}$ separated from the rest of the spectrum by a gap $\mathrm{Im}(\tau)$.

Proof. We first remark that

$$L_{0,\theta} = \mathfrak{D}_{\mathrm{d}}(\mathrm{Re}(\delta))L_{0,(i\delta',\tau)}\mathfrak{D}_{\mathrm{d}}(-\,\mathrm{Re}(\delta))$$

where $\mathfrak{D}_{\mathrm{d}}(\mathrm{Re}(\delta))$ is a unitary operator. Because of the invariance of the spectrum under unitary conjugation we henceforth may assume that $\mathrm{Re}(\delta) = 0$. Since L_{res}, L_{aux} and N_{res} only act on photon variables and L_{p} only on particle variables we have

$$\mathrm{spec}\,(L_{0,\theta}) = \mathrm{spec}\,(L_{\mathrm{p}}) + \mathrm{spec}\,(\cos(\delta')L_{\mathrm{res}} + i\sin(\delta')L_{\mathrm{aux}} + \tau N_{\mathrm{res}})\,.$$

We now decompose the Fock space $\mathcal{F}(L^2[\Upsilon])$ into orthogonal subspaces, $\mathrm{ran}(P_{[N_{\mathrm{res}}=n]})$, $n \in \mathbb{N}_0$. It then holds

$$\begin{aligned}
&\mathrm{spec}\,(\cosh(\delta)L_{\mathrm{res}} + \sinh(\delta)L_{\mathrm{aux}} + \tau N_{\mathrm{res}}) \\
&= \bigcup_{n=0}^{\infty} \Big[n\tau + \mathrm{spec}\Big([\cos(\delta')L_{\mathrm{res}} + i\sin(\delta')L_{\mathrm{aux}}] \restriction_{\mathrm{ran}(P_{[N_{\mathrm{res}}=n]})} \Big) \Big]\,.
\end{aligned}$$

We recall the definitions (1.66, 2.23, 2.24) which imply that

$$\begin{aligned}
&\mathrm{spec}\Big([\cos(\delta')L_{\mathrm{res}} + i\sin(\delta')L_{\mathrm{aux}}] \restriction_{\mathrm{ran}(P_{[N_{\mathrm{res}}=n]})} \Big) \\
&= \{ z \in \mathbb{C} \,|\, |\mathrm{Re}(z)| \le \mathrm{sgn}(\sin(\delta'))|\cot(\delta')|\,\mathrm{Im}(z) \}\,,
\end{aligned}$$

for $n \ge 2$ since $|L_{\mathrm{res}}| \le L_{\mathrm{aux}}$, and

$$\begin{aligned}
&\mathrm{spec}\Big([\cos(\delta')L_{\mathrm{res}} + i\sin(\delta')L_{\mathrm{aux}}] \restriction_{\mathrm{ran}(P_{[N_{\mathrm{res}}=1]})} \Big) \\
&= \{ \cos(\delta')u + i\sin(\delta')|u| \,|\, u \in \mathbb{R} \} \\
&= \{ z \in \mathbb{C} \,|\, |\mathrm{Re}(z)| = \mathrm{sgn}(\sin(\delta'))|\cot(\delta')|\,\mathrm{Im}(z) \}\,,
\end{aligned}$$

and

$$\mathrm{spec}\Big([\cos(\delta')L_{\mathrm{res}} + i\sin(\delta')L_{\mathrm{aux}}] \restriction_{\mathrm{ran}(P_{[N_{\mathrm{res}}=0]})} \Big) = \{0\}.$$

For $\mathrm{Im}(\tau), \sin(\delta') > 0$ and $|\mathrm{Re}(\tau)| \le \cot(\delta')\,\mathrm{Im}(\tau)$ we have

$$\begin{aligned}
&(n+1)\tau + \{ z \in \mathbb{C} \,|\, |\mathrm{Re}(z)| \le \cot(\delta')\,\mathrm{Im}(z) \} \\
&\subseteq\ n\tau + \{ z \in \mathbb{C} \,|\, |\mathrm{Re}(z)| \le \cot(\delta')\,\mathrm{Im}(z) \}
\end{aligned}$$

for $n \ge 1$. In particular, the eigenvalues in $\mathrm{spec}(L_{\mathrm{p}})$ are separated from the rest of the spectrum by a gap given by $\mathrm{Im}(\tau)$. ∎

In what follows, a particular class of *deformation analytic vectors* will play a crucial role. Henceforth, we fix the numbers

$$\frac{\pi}{8} < \delta_0 < \frac{\pi}{4} \qquad \text{and} \qquad 0 < \tau_0 < 2\pi\beta_{\max}^{-1}$$

and define the domains

$$\begin{aligned}
D_{\delta_0,\tau_0} &:= \big\{ (\delta,\tau) \in \mathbb{C}^2 \,\big|\, |\mathrm{Im}(\delta)| < \delta_0,\ |\mathrm{Im}(\tau)| < \tau_0 \big\}\,, \\
D^{+}_{\delta_0,\tau_0} &:= D^{+}_{\delta_0} \times S^{+}_{\tau_0} \subseteq \mathbb{C}^2,
\end{aligned}$$

where

$$D_{\delta_0}^{+} := \{\delta \in \mathbb{C} \,|\, 0 < \mathrm{Im}(\delta) < \delta_0\},$$
$$S_{\tau_0}^{+} := \left\{\tau \in \mathbb{C} \,\middle|\, 0 < \frac{|\tau|}{2} < \mathrm{Im}(\tau) < \tau_0\right\}.$$

A vector $\psi \in \mathcal{H}^2$ is called *deformation analytic* if the map $\mathbb{R}^2 \ni (\delta, \tau) = \theta \mapsto \psi_\theta$ has an analytic continuation on the domain D_{δ_0,τ_0} in each variable separately. We denote by

$$\mathcal{D}_{\mathfrak{D}-\mathrm{a}} := \left\{\psi \in \mathcal{H}^2 \,\middle|\, \psi \text{ is deformation analytic}\right\}$$

the subspace of all deformation analytic vectors. A vector $\psi \in \mathcal{D}_{\mathfrak{D}-\mathrm{a}}$ is called *deformation analytic in* $\mathcal{D}(L_{\mathrm{aux}} + N_{\mathrm{res}})$ if $\psi \in \mathcal{D}(L_{\mathrm{aux}} + N_{\mathrm{res}})$ and if further the map $\theta \mapsto (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)\psi_\theta$ has an analytic continuation on the domain D_{δ_0,τ_0} in each variable separately. The deformation analytic vectors in $\mathcal{D}(L_{\mathrm{aux}} + N_{\mathrm{res}})$ are collected in the subspace

$$\mathcal{D}_{\mathfrak{D}-\mathrm{a}}^{\mathrm{aux}} := \{\psi \in \mathcal{D}_{\mathfrak{D}-\mathrm{a}} \,|\, \psi \text{ is deformation analytic in } \mathcal{D}(L_{\mathrm{aux}} + N_{\mathrm{res}})\}.$$

Remark 2.7 (Restriction of Translation Parameter) *The explicit construction of deformation analytic coupling functions in Proposition 1.14 points out the limitation of the analyticity in the translation parameter τ. The incorporation of the factor $\sqrt{u(1-e^{-\beta_r u})^{-1}}$ into the glued coupling functions creates poles $\pm 2n\pi i\beta_r^{-1}$, $n \in \mathbb{N}$, which restrict the imaginary part of the translation parameter to be small compared to the reservoir temperature β_r^{-1}. Since our study of the system shall also cover the low temperature regime the translation parameter* $\mathrm{Im}(\tau)$ *eventually has to be very small. This complicates spectral analysis of the operator K_θ by the following reason. The translation parameter creates a spectral gap between the eigenvalues of $L_{0,\theta}$ and the rest of the unperturbed spectrum of order* $\mathrm{Im}(\tau)$*, c.f. Proposition 2.6 and Figure 2.1. To study the low temperature regime the magnitude of the perturbation becomes significantly larger than the gap and standard perturbation theory does not make predictions about the perturbation of the eigenvalues. We treat this situation with renormalization group techniques in Chapter 4 and 5.*

One could think about regularizing the coupling functions G_r such that we find analytic continuation of $\mathcal{G} = \mathfrak{g}(G_1, \ldots, G_R)$ around the poles $\pm 2n\pi i\beta_r^{-1}$, $n \in \mathbb{N}$, $r = 1, \ldots, R$. This, however, would make the particle-photon interaction dependent on the temperature under which the system is studied and which is not a convincing approach. Nevertheless, in Section C.3 we construct a class of form factors, dependent on the reservoir temperatures, which are mapped under the gluing to analytic functions not featuring the poles. These functions are used to build a strongly dense subalgebra of observables for which the thermal relaxation behavior of the system can be studied.

We present a lemma which makes the relation (2.21) rigorous.

Lemma 2.8 *Let* $\varphi, \psi \in \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$. *Then, for* $\mathrm{Im}(z) \leq -2$ *and* $s \in \mathbb{R}$, *we may rewrite*

$$\left\langle \varphi \,\middle|\, \left(z - K^{(s)}\right)^{-1} \psi \right\rangle = \left\langle \varphi_{\overline{\theta}} \,\middle|\, \left(z - K^{(s)}_\theta\right)^{-1} \psi_\theta \right\rangle$$

where $\theta = (\delta, \tau) \in D^+_{\delta_0, \tau_0}$.

Proof. Let $\delta_1 \in \mathbb{R}$ and denote $\theta' = (\delta_1, 0)$. Since $\mathfrak{D}_{\mathrm{d}}(\delta_1)$ is unitary we obtain

$$\begin{aligned}
&\left\langle \varphi_{\overline{\theta}} \,\middle|\, \left(z - K^{(s)}_\theta\right)^{-1} \psi_\theta \right\rangle \\
&= \left\langle \mathfrak{D}_{\mathrm{d}}(\delta_1)\varphi_{\overline{\theta}} \,\middle|\, \left(z - \mathfrak{D}_{\mathrm{d}}(\delta_1) K^{(s)}_\theta \mathfrak{D}_{\mathrm{d}}(\delta_1)^{-1}\right)^{-1} \mathfrak{D}_{\mathrm{d}}(\delta_1)\psi_\theta \right\rangle \\
&= \left\langle \varphi_{\overline{\theta+\theta'}} \,\middle|\, \left(z - K^{(s)}_{\theta+\theta'}\right)^{-1} \psi_{\theta+\theta'} \right\rangle .
\end{aligned}$$

Thus, the map

$$\begin{aligned}
\delta \ \mapsto\ & \left\langle \varphi_{\overline{\theta}} \,\middle|\, \left(z - K^{(s)}_\theta\right)^{-1} \psi_\theta \right\rangle \\
=\ & \Big\langle (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)\varphi_{\overline{\theta}} \,\Big|\, (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} \left(z - K^{(s)}_\theta\right)^{-1} (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} \\
& \times (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)\psi_\theta \Big\rangle
\end{aligned}$$

is constant along $\mathbb{R} + id$ with $0 < d < \delta_0$. Theorem C.5 and the assumptions on φ, ψ imply that the above map is analytic so that it is independent of δ. By continuity (we refer again to Theorem C.5) we obtain

$$\begin{aligned}
&\left\langle \varphi_{\overline{\theta}} \,\middle|\, \left(z - K^{(s)}_\theta\right)^{-1} \psi_\theta \right\rangle \\
&= \lim_{\delta \to 0} \left\langle \varphi_{\overline{\theta}} \,\middle|\, \left(z - K^{(s)}_\theta\right)^{-1} \psi_\theta \right\rangle \\
&= \left\langle \varphi_{(0,\overline{\tau})} \,\middle|\, \left(z - K^{(s)}_{(0,\tau)}\right)^{-1} \psi_{(0,\tau)} \right\rangle \\
&= \left\langle \mathfrak{D}_{\mathrm{t}}(\tau_1)\varphi_{(0,\overline{\tau})} \,\middle|\, \left(z - \mathfrak{D}_{\mathrm{t}}(\tau_1) K^{(s)}_{(0,\tau)} \mathfrak{D}_{\mathrm{t}}(\tau_1)^{-1}\right)^{-1} \mathfrak{D}_{\mathrm{t}}(\tau_1)\psi_{(0,\tau)} \right\rangle \\
&= \left\langle \varphi_{(0,\overline{\tau+\tau_1})} \,\middle|\, \left(z - K^{(s)}_{(0,\tau+\tau_1)}\right)^{-1} \psi_{(0,\tau+\tau_1)} \right\rangle
\end{aligned}$$

for $\tau_1 \in \mathbb{R}$ due to the unitarity of $\mathfrak{D}_{\mathrm{t}}(\tau_1)$. The same analyticity argument shows that the above expression is independent of τ and by continuity we finally get

$$\begin{aligned}\left\langle \varphi_{\overline{\theta}} \,\middle|\, \left(z - K_{\theta}^{(s)}\right)^{-1} \psi_{\theta} \right\rangle &= \lim_{\tau \to 0} \left\langle \varphi_{(0,\overline{\tau})} \,\middle|\, \left(z - K_{(0,\tau)}^{(s)}\right)^{-1} \psi_{(0,\tau)} \right\rangle \\ &= \left\langle \varphi \,\middle|\, \left(z - K^{(s)}\right)^{-1} \psi \right\rangle.\end{aligned}$$

■

2.2.3 Integral Representation of $U(t)$

We assemble the lemmata of the last sections to express the group $U(t)$, in a weak sense, as a Cauchy-like integral representation in terms of the generator K.

Proposition 2.9 (Integral Representation of $U(t)$) *Let $A \in \mathcal{A}$ such that $\pi(A)\tilde{\Omega} \in \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$ and let $\varphi \in \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$. Moreover, choose $\theta = (i\delta', i\tau') \in \mathcal{D}^{+}_{\delta_0,\tau_0}$ with $\delta' \in [\frac{\pi}{8}, \frac{\pi}{4}]$ and $\tau' > 0$. For $t > 0$ we have*

$$\left\langle \varphi \,\middle|\, U(t)\pi(A)\tilde{\Omega} \right\rangle = \frac{1}{2\pi i} \int_{\mathbb{R}-2i} dz \left\langle \varphi_{\overline{\theta}} \,\middle|\, (z - K_{\theta})^{-1} [\pi(A)\tilde{\Omega}]_{\theta} \right\rangle e^{izt} \qquad (2.27)$$

where the integration has to be understood as improper Riemann integration.

Proof. By the Lemmata 2.3 and 2.8 we may write for $s \in \mathbb{R}$,

$$\begin{aligned}\left\langle \varphi \,\middle|\, e^{iK^{(s)}t}\pi(A)\tilde{\Omega} \right\rangle &= \frac{1}{2\pi i} \int_{\mathbb{R}-2i} dz \left\langle \varphi \,\middle|\, \left(z - K^{(s)}\right)^{-1} \pi(A)\tilde{\Omega} \right\rangle e^{izt} \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}-2i} dz \left\langle \varphi_{\overline{\theta}} \,\middle|\, \left(z - K_{\theta}^{(s)}\right)^{-1} [\pi(A)\tilde{\Omega}]_{\theta} \right\rangle e^{izt}. \qquad (2.28)\end{aligned}$$

The l.h.s. of (2.28) has an analytic continuation in s on the set $\mathbb{S}_{\frac{\varepsilon_0}{4}}$ due to Proposition 2.4. The integral on the r.h.s. of (2.28) is well defined and analytic in $s \in \mathbb{S}_{\frac{\varepsilon_0}{4}}$ by Theorem C.6(ii) since $\varphi_{\overline{\theta}}, [\pi(A)\tilde{\Omega}]_{\theta} \in \mathcal{D}(L_{\mathrm{aux}} + N_{\mathrm{res}})$. Evaluating both sides at $s = -i/2$ gives

$$\left\langle \varphi \,\middle|\, U(t)\pi(A)\tilde{\Omega} \right\rangle = \frac{1}{2\pi i} \int_{\mathbb{R}-2i} dz \left\langle \varphi_{\overline{\theta}} \,\middle|\, (z - K_{\theta})^{-1} [\pi(A)\tilde{\Omega}]_{\theta} \right\rangle e^{izt},$$

c.f. Proposition 2.4. ■

2.3 Characterization of the NESS as a Resonance State

2.3.1 Weak Long Time Limit of the Group $U(t)$

Proposition 2.9 connects the dynamical behavior of ω_0-normal states with the resolvent, and therefore with spectral properties, of the generator of the group $U(t)$ or rather its deformation K_θ. The subsequent Chapters 3, 4, 5 are devoted to the analysis of the spectrum of K_θ. We use the spectral information provided in Theorem 3.1 to compute the weak limit of the group $U(t)$.

Proposition 2.10 (Weak Limit of $U(t)$) *We assume that $g > 0$ is sufficiently small and $|\beta_{\max} - \beta_{\min}| \ll 1$. Let $A \in \mathcal{A}$ such that $\pi(A)\tilde{\Omega} \in \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$ and let $\varphi \in \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$. Moreover, choose $\theta = (i\delta', i\tau') \in \mathcal{D}^{+}_{\delta_0,\tau_0}$ with $\delta' \in [\frac{\pi}{8}, \frac{\pi}{4}]$ and $\tau' \sim \frac{g^2}{2+\beta_{\max}}$ as given in (4.3). Then the group $\{U(t)\}_{t\in\mathbb{R}}$ has a weak long time limit as $t \to \infty$ in the sense that*

$$\lim_{t\to\infty} \left\langle \varphi \,\middle|\, U(t)\pi(A)\tilde{\Omega} \right\rangle = \left\langle \varphi_{\overline{\theta}} \,\middle|\, \tilde{\Omega}_\theta \right\rangle \left\langle \tilde{\Omega}^*_{\overline{\theta}} \,\middle|\, [\pi(A)\tilde{\Omega}]_\theta \right\rangle$$

*where $\tilde{\Omega}_\theta$ is the spectral deformation of $\tilde{\Omega}$ and an eigenvector of K_θ corresponding to the simple, isolated zero eigenvalue. The vectors $\tilde{\Omega}_\theta, \tilde{\Omega}^*_{\overline{\theta}}$ are the $\mathfrak{D}(\theta)$-*resonance eigenvectors *of K and K^*, resp., corresponding to the* zero resonance *specified by*

$$K_\theta\tilde{\Omega}_\theta = 0, \qquad (K_\theta)^*\tilde{\Omega}^*_{\overline{\theta}} = 0, \qquad \left\langle \tilde{\Omega}^*_{\overline{\theta}} \,\middle|\, \tilde{\Omega}_\theta \right\rangle = 1. \tag{2.29}$$

Moreover, the convergence is exponentially fast, i.e., there exist a decay rate $\tau_{\mathrm{dec}} = \tau' d$, where $d > 0$ is a positive constant, such that

$$\lim_{t\to\infty} e^{\tau_{\mathrm{dec}}t} \left| \left\langle \varphi \,\middle|\, U(t)\pi(A)\tilde{\Omega} \right\rangle - \left\langle \varphi_{\overline{\theta}} \,\middle|\, \tilde{\Omega}_\theta \right\rangle \left\langle \tilde{\Omega}^*_{\overline{\theta}} \,\middle|\, [\pi(A)\tilde{\Omega}]_\theta \right\rangle \right| = 0.$$

Proof. As a consequence of Theorem 3.1 we know that the spectrum of K_θ is contained in a half plane supplemented by a point, i.e.,

$$\mathrm{spec}\,(K_\theta) \subseteq \left\{ E^{(-i/2)}_{0,g} \right\} \cup \left\{ z \in \mathbb{C} \,\middle|\, \mathrm{Im}(z) \geq \mathrm{Im}\left(E^{(-i/2)}_{0,g}\right) + 2\tau_{\mathrm{dec}} \right\} \tag{2.30}$$

where $\tau_{\mathrm{dec}} = \tau' d$ with a positive constant $d > 0$. Here, $E^{(-i/2)}_{0,g}$ is the simple, isolated eigenvalue of $K_\theta = K^{(-i/2)}_\theta$ for which either $E^{(-i/2)}_{0,g} = 0$ or $\mathrm{Im}(E^{(-i/2)}_{0,g}) \leq -2\tau_{\mathrm{dec}}$ holds. Note that $\mathrm{Im}(E^{(-i/2)}_{0,g}) > -2\tau_{\mathrm{dec}}$ and $E^{(-i/2)}_{0,g} \neq 0$ together with Theorem 3.1 would contradict that zero is an eigenvalue of K_θ by Proposition C.16. We set $\psi := \pi(A)\tilde{\Omega}$ and apply Proposition 2.9. With Cauchy's integral formula we can deform the integration contour in (2.27) and pick the residue of the integrand associated with the eigenvalue $E_{0,g}^{(-i/2)}$,

$$\begin{aligned} \langle \varphi \,|\, U(t)\psi \,\rangle &= \frac{1}{2\pi i} \oint\limits_{\left|E_{0,g}^{(-i/2)} - z\right| = \tau_{\text{dec}}} dz \, \left\langle\, \varphi_{\overline{\theta}} \,\middle|\, (z - K_\theta)^{-1} \, \psi_\theta \,\right\rangle e^{izt} \\ &\quad + \frac{1}{2\pi i} \int\limits_{\mathbb{R} + E_{0,g}^{(-i/2)} + i\tau_{\text{dec}}} dz \, \left\langle\, \varphi_{\overline{\theta}} \,\middle|\, (z - K_\theta)^{-1} \, \psi_\theta \,\right\rangle e^{izt}, \end{aligned} \tag{2.31}$$

where the second integral is understood as improper Riemann integral which is convergent by the same reasoning as in the proof to Theorem C.6(ii) using the oscillatory factor e^{izt} and integration by parts as well as the numerical range estimates provided in Proposition A.9(ii). We remark that the integrals along the vertical lines $x + i[-2, \operatorname{Im}(E_{0,g}^{(-i/2)}) + \tau_{\text{dec}}]$ as $x \to \pm\infty$, connecting the contours $\mathbb{R} - 2i$ and $\mathbb{R} + E_{0,g}^{(-i/2)} + i\tau_{\text{dec}}$, also contribute to (2.31). However, since the integrand decays as

$$\left|\left\langle\, \varphi_{\overline{\theta}} \,\middle|\, (z - K_\theta)^{-1} \, \psi_\theta \,\right\rangle e^{izt}\right| \leq \frac{e^{-\left[\operatorname{Im}\left(E_{0,g}^{(-i/2)}\right) + \tau_{\text{dec}}\right]t} \, \|\varphi_{\overline{\theta}}\| \, \|\psi_\theta\|}{\operatorname{dist}\left(z; \operatorname{NumRan}\left(K_\theta\right)\right)} \leq \frac{C}{|x|},$$

for $z = x + E_{0,g}^{(-i/2)} + i\tau_{\text{dec}}$ and a positive constant C, by Proposition A.9(ii), the contributions of the vertical lines are zero in the limit $x \to \pm\infty$.

Now, we observe that

$$P_\theta := \frac{1}{2\pi i} \oint\limits_{\left|E_{0,g}^{(-i/2)} - z\right| = \tau_{\text{dec}}} dz \, (z - K_\theta)^{-1}$$

is a non-orthogonal projection operator fulfilling

$$\left(K_\theta - E_{0,g}^{(-i/2)}\right) P_\theta = \frac{1}{2\pi i} \oint\limits_{\left|E_{0,g}^{(-i/2)} - z\right| = \tau_{\text{dec}}} dz \left[\frac{z - E_{0,g}^{(-i/2)}}{z - K_\theta} - \mathbb{1}_{\mathcal{H}^2} \right] = 0,$$

since the integrand is a holomorphic function inside the integration contour ($E_{0,g}^{(-i/2)}$ is a simple, isolated eigenvalue), and

$$P_\theta \Psi = \frac{1}{2\pi i} \oint\limits_{\left|E_{0,g}^{(-i/2)} - z\right| = \tau_{\text{dec}}} dz \, \left(z - E_{0,g}^{(-i/2)}\,\right)^{-1} \Psi = \Psi$$

for all $\Psi \in \ker(K_\theta - E_{0,g}^{(-i/2)})$. Thus, the range of P_θ coincides with the kernel of $(K_\theta - E_{0,g}^{(-i/2)})$ and since $E_{0,g}^{(-i/2)}$ is a simple eigenvalue of K_θ the operator is a rank one projection. We remark that the fact that $E_{0,g}^{(-i/2)}$ is a simple, isolated eigenvalue of K_θ implies that its complex conjugate $\overline{E_{0,g}^{(-i/2)}}$ is a simple, isolated eigenvalue of the adjoint operator $(K_\theta)^*$. Therefore, we find vectors Ψ_θ and $\Psi^*_{\overline{\theta}}$ obeying

$$K_\theta \Psi_\theta = E_{0,g}^{(-i/2)} \Psi_\theta, \qquad (K_\theta)^* \Psi^*_{\overline{\theta}} = \overline{E_{0,g}^{(-i/2)}} \Psi^*_{\overline{\theta}},$$

such that

$$P_\theta = |\Psi_\theta\rangle \langle \Psi^*_{\overline{\theta}}| \, .$$

Since $P_\theta \neq 0$ is a projection,

$$|\Psi_\theta\rangle \langle \Psi^*_{\overline{\theta}}| = P_\theta = P_\theta^2 = |\Psi_\theta\rangle \left\langle \Psi^*_{\overline{\theta}} \,\middle|\, \Psi_\theta \right\rangle \langle \Psi^*_{\overline{\theta}}| \, ,$$

the vectors $\Psi_\theta, \Psi^*_{\overline{\theta}}$ are normalized by

$$\left\langle \Psi^*_{\overline{\theta}} \,\middle|\, \Psi_\theta \right\rangle = 1.$$

Thus, we can rewrite the first integral in (2.31) as

$$\begin{aligned}
&\frac{1}{2\pi i} \oint\limits_{\left|E_{0,g}^{(-i/2)} - z\right| = \tau_{\text{dec}}} dz \left\langle \varphi_{\overline{\theta}} \,\middle|\, (z - K_\theta)^{-1} \psi_\theta \right\rangle e^{izt} \\
&= \frac{1}{2\pi i} \oint\limits_{\left|E_{0,g}^{(-i/2)} - z\right| = \tau_{\text{dec}}} dz \left\langle \varphi_{\overline{\theta}} \,\middle|\, (z - K_\theta)^{-1} \psi_\theta \right\rangle e^{iE_{0,g}^{(-i/2)} t} \\
&\quad + \frac{1}{2\pi i} \oint\limits_{\left|E_{0,g}^{(-i/2)} - z\right| = \tau_{\text{dec}}} dz \left\langle \varphi_{\overline{\theta}} \,\middle|\, (z - K_\theta)^{-1} \psi_\theta \right\rangle \left[e^{izt} - e^{iE_{0,g}^{(-i/2)} t} \right] \\
&= \left\langle \varphi_{\overline{\theta}} \,\middle|\, \Psi_\theta \right\rangle \left\langle \Psi^*_{\overline{\theta}} \,\middle|\, \psi_\theta \right\rangle e^{iE_{0,g}^{(-i/2)} t}
\end{aligned}$$

using that the function $z \mapsto \left\langle \varphi_{\overline{\theta}} \,\middle|\, (z - K_\theta)^{-1} \psi_\theta \right\rangle \left[e^{izt} - e^{iE_{0,g}^{(-i/2)} t} \right]$ is holomorphic inside the integration contour. This is due to the fact that $E_{0,g}^{(-i/2)}$ is a simple, isolated eigenvalue of K_θ. Back to (2.31) we get

$$\begin{aligned}
&\left\langle \varphi \,\middle|\, U(t) \psi \right\rangle \qquad (2.32) \\
&= e^{iE_{0,g}^{(-i/2)} t} \Bigg[\left\langle \varphi_{\overline{\theta}} \,\middle|\, \Psi_\theta \right\rangle \left\langle \Psi^*_{\overline{\theta}} \,\middle|\, \psi_\theta \right\rangle \\
&\qquad + \frac{e^{-\tau_{\text{dec}} t}}{2\pi i} \int\limits_{\mathbb{R}} dx \left\langle \varphi_{\overline{\theta}} \,\middle|\, \left(x + E_{0,g}^{(-i/2)} + i\tau_{\text{dec}} - K_\theta \right)^{-1} \psi_\theta \right\rangle e^{ixt} \Bigg].
\end{aligned}$$

Again, the integral is defined in the improper Riemann sense and we get by integration by parts for $t > 0$,

$$\left| \int_{\mathbb{R}} dx \left\langle \varphi_{\overline{\theta}} \,\middle|\, \left(x + E_{0,g}^{(-i/2)} + i\tau_{\mathrm{dec}} - K_\theta\right)^{-1} \psi_\theta \right\rangle e^{ixt} \right| \tag{2.33}$$

$$\begin{aligned} &= \left| \int_{\mathbb{R}} dx \left\langle \varphi_{\overline{\theta}} \,\middle|\, \left(x + E_{0,g}^{(-i/2)} + i\tau_{\mathrm{dec}} - K_\theta\right)^{-2} \psi_\theta \right\rangle \frac{e^{ixt}}{it} \right| \\ &\leq \frac{\|\varphi_{\overline{\theta}}\| \, \|\psi_\theta\|}{t} \int_{\mathbb{R}} dx \left\| \left(x + E_{0,g}^{(-i/2)} + i\tau_{\mathrm{dec}} - K_\theta\right)^{-2} \right\| \\ &\xrightarrow{t\to\infty} 0, \end{aligned}$$

where the last integral converges due to Proposition A.9(ii).

To compute the limit $t \to \infty$ in (2.32) we first rule out the case $\mathrm{Im}(E_{0,g}^{(-i/2)}) < 0$. Under the assumption that $\mathrm{Im}(E_{0,g}^{(-i/2)}) < 0$ we obtain $\lim_{t\to\infty} |\langle \varphi \,|\, U(t)\psi \rangle| = \infty$. In the special case $A = \mathbb{1}_{\mathcal{B}(\mathcal{H}^2)}$ and $\varphi = \tilde{\Omega}$ (note that $\tilde{\Omega} \in \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$ by Theorem C.14) we have $\langle \varphi \,|\, U(t)\psi \rangle = 1$ in contradiction to the divergence. Thus, it follows $E_{0,g}^{(-i/2)} = 0$. Since $\tilde{\Omega}_\theta \in \ker(K_\theta)$ by Proposition C.16 we can choose $\Psi_\theta = \tilde{\Omega}_\theta$ and $\Psi^*_{\overline{\theta}} = \tilde{\Omega}^*_{\overline{\theta}}$ such that (2.29) holds.

With (2.32, 2.33) we finally have

$$\begin{aligned} & e^{\tau_{\mathrm{dec}} t} \left| \langle \varphi \,|\, U(t)\psi \rangle - \left\langle \varphi_{\overline{\theta}} \,\middle|\, \tilde{\Omega}_\theta \right\rangle \left\langle \tilde{\Omega}^*_{\overline{\theta}} \,\middle|\, \psi_\theta \right\rangle \right| \\ &\leq \frac{\|\varphi_{\overline{\theta}}\| \, \|\psi_\theta\|}{2\pi t} \int_{\mathbb{R}} dx \left\| \left(x + E_{0,g}^{(-i/2)} + i\tau_{\mathrm{dec}} - K_\theta\right)^{-2} \right\| \\ &\xrightarrow{t\to\infty} 0. \end{aligned}$$

∎

2.3.2 Thermal Relaxation to a NESS

The group $\{U(t)\}_{t\in\mathbb{R}}$ was originally defined in (2.7) on the dense set $\pi(\mathcal{A})''\tilde{\Omega}$. Its long time behavior, however, could only be studied on vectors $\pi(A)\tilde{\Omega} \in \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$, c.f. Proposition 2.10. This has as a consequence that the relaxation behavior of ω_0-normal states can only be quantified on observables from the set

$$\mathcal{A}^{\mathrm{ana}} := \left\{ A \in \mathcal{A} \,\middle|\, \pi(A)\tilde{\Omega} \in \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}} \right\}. \tag{2.34}$$

We will show later in Section C.3 that $\mathcal{A}^{\text{ana}}$ contains a strongly dense $*$-subalgebra $\mathcal{A}_1$ in $\mathcal{A}$ such that the vector $\tilde{\Omega}$ is cyclic for $\pi(\mathcal{A}_1)$ and $\pi'(\mathcal{A}_1)$, i.e., $\pi(\mathcal{A}_1)\tilde{\Omega}$ and $\pi'(\mathcal{A}_1)\tilde{\Omega}$ are dense sets in $\mathcal{H}^2$. On $\mathcal{A}^{\text{ana}}$ we define a linear functional

$$\tilde{\omega} : \mathcal{A}^{\text{ana}} \to \mathbb{C}, \qquad \omega(A) := \left\langle \tilde{\Omega}^*_{\overline{\theta}} \,\middle|\, [\pi(A)\tilde{\Omega}]_\theta \right\rangle \tag{2.35}$$

where $\tilde{\Omega}^*_{\overline{\theta}}, \tilde{\Omega}_\theta$ are the zero $\mathfrak{D}(\theta)$-resonance eigenvectors of K and K^* specified in (2.29). It is not clear from the definition (2.35) neither that $\tilde{\omega}$ defines a positive bounded functional nor that $\tilde{\omega}$ is independent of the spectral deformation parameter θ although its definition incorporates the zero $\mathfrak{D}(\theta)$-resonance eigenvectors which depend on the spectral deformation. We verify these properties in Corollary 2.14 and therefore refer to $\tilde{\omega}$ as a state on $\mathcal{A}_1$. Note that, even if the the functional $\tilde{\omega}$ is independent of θ, it is not possible to remove the deformation by setting $\theta = 0$ since, in general, the vector $\tilde{\Omega}^*_{\overline{\theta}}$ diverges as $\theta \to 0$, i.e., it would leave the Hilbert space . This reflects that, in general, $\tilde{\omega}$ is not an ω_0-normal state and cannot be expressed in terms of the representation π.

It is worth to note that we may extend $\tilde{\omega}$ from observables in $\mathcal{A}^{\text{ana}}$ to linear combinations in creation and annihilation operators $a_r^\#(F)$, $F \in L^2[\mathbb{R}^3; \mathcal{B}(\mathcal{H}_{\text{p}})]$, as long as the vector $\pi(a_r^\#(F))\tilde{\Omega}$ belongs to $\mathcal{D}^{\text{aux}}_{\mathfrak{D}-\text{a}}$. This simply goes back to the extension of π to creation and annihilation operators in the sense of (1.74) such that we define

$$\begin{aligned} \tilde{\omega}(a_r^\#(F)) &:= \left\langle \tilde{\Omega}^*_{\overline{\theta}} \,\middle|\, [\pi(a_r^\#(F))\tilde{\Omega}]_\theta \right\rangle \\ &= \lim_{t\to\infty} \omega_0\left(\alpha^t(a_r^\#(F))\right) = \lim_{t\to\infty} \left\langle \Omega_0 \,\middle|\, e^{iLt}\pi(a_r^\#(F))e^{-iLt}\Omega_0 \right\rangle. \end{aligned}$$

The interaction $v_r = a_r(G_r) + a_r^*(G_r)$, the heat fluxes ϕ_{p} and $\phi_{\text{f},r} = g[a_r(-i\omega G_r) + a_r^*(-i\omega G_r)]$ as well as the entropy production rate $\mathfrak{s} = \beta_{\text{p}}\phi_{\text{p}} + \sum_{r=1}^R \beta_r\phi_{\text{f},r}$ belong to this class of operators. The proof that these operators are deformation analytic in the appropriate sense is based on the proof of Theorem C.14. Hence, it makes sense to talk about the expectation value of the interaction energy $\tilde{\omega}(gv_r)$, of the net heat fluxes $\tilde{\omega}(\phi_{\text{f},r})$ of the reservoirs and, in particular, of the entropy production rate w.r.t. ω_0

$$\text{Ep}_{\omega_0}(\tilde{\omega}) = \tilde{\omega}(\mathfrak{s})$$

in the state $\tilde{\omega}$.

In the equal temperature situation, i.e., $\beta_{\max} = \beta_{\min}$, we have $K_\theta = L_\theta$ and $K = L$. Further, the vector $\Omega = \tilde{\Omega}|_{\beta_{\max}=\beta_{\min}=\beta} \in \ker(L)$ is the vector representative of the unique ω_0-normal perturbed KMS state, c.f. Proposition B.7. Applying Proposition C.16 in the equal temperature situation implies $L_{\overline{\theta}}\Omega_{\overline{\theta}} = 0$. On the other hand we have, due to $(L_\theta)^* = L_{\overline{\theta}}$, for the kernel $\ker(L_{\overline{\theta}}) = \mathbb{C}\Omega^*_{\overline{\theta}}$. This implies that $\tilde{\Omega}^*_{\overline{\theta}} = \lambda \Omega_{\overline{\theta}}$ for a suitable factor $\lambda \in \mathbb{C}$ which turns out to be one by

$$1 = \left\langle \tilde{\Omega}^*_{\overline{\theta}} \,\middle|\, \Omega_\theta \right\rangle = \lambda \langle \Omega_{\overline{\theta}} \,|\, \Omega_\theta \rangle = \lambda \|\Omega\|^2 = \lambda$$

using the fact that $\Omega \in \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$ by Theorem C.14 which allows to remove the spectral deformation. This implies that

$$\tilde{\omega}(A) = \langle \Omega_{\overline{\theta}} \,|\, [\pi(A)\Omega]_\theta \rangle = \langle \Omega \,|\, \pi(A)\Omega \rangle = \omega(A)$$

for all $A \in \mathcal{A}^{\mathrm{ana}}$, i.e., the state $\tilde{\omega}$ becomes the unique ω_0-normal (α, β)-KMS state which extends to the whole algebra $\mathcal{A}$. Facing the upcoming theorem about thermal relaxation it is consistent with [8] that the attracting state $\tilde{\omega}$ is the KMS state of the perturbed system.

We are prepared to state the main theorem about the thermal relaxation properties of our system in connection with spectral properties of the C-Liouville operator.

Theorem 2.11 (Thermal Relaxation to NESS) *We assume that $g > 0$ is sufficiently small and $|\beta_{\max} - \beta_{\min}| \ll 1$. Any ω_0-normal state $\eta \in \mathcal{N}_{\omega_0}(\mathcal{A})$ converges under the time evolution α pointwise on $\mathcal{A}^{\mathrm{ana}}$ towards the functional $\tilde{\omega}$, i.e.,*

$$\lim_{t\to\infty} \eta\left(\alpha^t(A)\right) = \tilde{\omega}(A) \qquad \text{for all } A \in \mathcal{A}^{\mathrm{ana}}.$$

Moreover, there exists a dense subset $\mathcal{N}^{\mathrm{ana}} \subseteq \mathcal{N}_{\omega_0}(\mathcal{A})$ of ω_0-normal states which converge exponentially fast, i.e., there exists a positive decay rate

$$\tau_{\mathrm{dec}} = \frac{g^2}{2 + \beta_{\max}} \frac{\gamma_{\mathrm{eq}}}{1920 \mathcal{C}^2_{\chi_1}}, \tag{2.36}$$

with γ_{eq} being a positive constant defined in (4.46) and $\mathcal{C}_{\chi_1} > 0$ introduced in (4.6), such that

$$\lim_{t\to\infty} e^{\tau_{\mathrm{dec}} t} \left|\eta\left(\alpha^t(A)\right) - \tilde{\omega}(A)\right| = 0$$

for all $\eta \in \mathcal{N}^{\mathrm{ana}}$ and $A \in \mathcal{A}^{\mathrm{ana}}$.

Proof. We specify the set

$$\mathcal{N}^{\mathrm{ana}} := \left\{ \eta \in \mathcal{N}_{\omega_0}(\mathcal{A}) \,\middle|\, \exists\, B \in \mathcal{A}_1 :\, \eta = \left\langle \pi'(B)\tilde{\Omega} \,\middle|\, \pi(\,\cdot\,)\pi'(B)\tilde{\Omega} \right\rangle \right\}$$

where the $*$-algebra $\mathcal{A}_1 \subseteq \mathcal{A}^{\mathrm{ana}} \subseteq \mathcal{A}$ is defined in (C.13). The set $\mathcal{N}^{\mathrm{ana}}$ is dense in $\mathcal{N}_{\omega_0}(\mathcal{A})$ by the following argument. Let η be an arbitrary ω_0-normal state. Then there exists a unit vector ξ from the positive cone $\mathcal{P}$ such that $\eta = \langle \xi \,|\, \pi(\,\cdot\,)\xi \rangle$. By Proposition C.10 the vector $\tilde{\Omega}$ is cyclic for $\pi'(\mathcal{A}_1)$, i.e., for a given $\varepsilon > 0$ there exists an observable $B \in \mathcal{A}_1$ such that $\left\|\xi - \pi'(B)\tilde{\Omega}\right\| < \varepsilon/2$ and $\left\|\pi'(B)\tilde{\Omega}\right\| = 1$. For $A \in \mathcal{A}$ with $\|A\|_{\mathcal{A}} = 1$ holds

$$\begin{aligned}
&\left|\eta(A) - \left\langle \pi'(B)\tilde{\Omega} \,\middle|\, \pi(A)\pi'(B)\tilde{\Omega} \right\rangle\right| \\
&\quad \leq \left|\left\langle \xi \,\middle|\, \pi(A)(\xi - \pi'(B)\tilde{\Omega}) \right\rangle\right| + \left|\left\langle \xi - \pi'(B)\tilde{\Omega} \,\middle|\, \pi(A)\pi'(B)\tilde{\Omega} \right\rangle\right| \\
&\quad \leq \left\|\xi - \pi'(B)\tilde{\Omega}\right\| \left(\|\xi\| + \left\|\pi'(B)\tilde{\Omega}\right\|\right) \|\pi(A)\| \\
&\quad < \varepsilon,
\end{aligned}$$

i.e., $\left\|\eta - \left\langle \pi'(B)\tilde{\Omega} \,\middle|\, \pi(\,\cdot\,)\pi'(B)\tilde{\Omega} \right\rangle\right\|_{\mathcal{A}_*} < \varepsilon$.

We now choose a state $\eta = \left\langle \pi'(B)\tilde{\Omega} \,\middle|\, \pi(\,\cdot\,)\pi'(B)\tilde{\Omega} \right\rangle \in \mathcal{N}^{\text{ana}}$ with $B \in \mathcal{A}_1$. Since $\mathcal{A}_1$ is a $*$-algebra by Proposition C.10 we have $B^*B \in \mathcal{A}_1$. Due to Theorem C.14 holds $\pi'(B^*B)\tilde{\Omega} \in \mathcal{D}^{\text{aux}}_{\mathfrak{D}-\text{a}}$. We choose $A \in \mathcal{A}^{\text{ana}}$ and apply Proposition 2.10,

$$\begin{aligned}
\lim_{t\to\infty} \eta(\alpha^t(A)) &= \lim_{t\to\infty} \left\langle \pi'(B^*B)\tilde{\Omega} \,\middle|\, U(t)\pi(A)\tilde{\Omega} \right\rangle \\
&= \left\langle [\pi'(B^*B)\tilde{\Omega}]_{\overline{\theta}} \,\middle|\, \tilde{\Omega}_\theta \right\rangle \left\langle \tilde{\Omega}^*_{\overline{\theta}} \,\middle|\, [\pi(A)\tilde{\Omega}]_\theta \right\rangle.
\end{aligned}$$

Hereby, we used that $\pi'(B)$ commutes with $\pi(\alpha^t(A)) = e^{iLt}\pi(A)e^{-iLt} \in \pi(\mathcal{A})''$. Since both vectors, $[\pi'(B^*B)\tilde{\Omega}]_\theta$ and $\tilde{\Omega}_\theta$, are analytic in each variable of $\theta = (\delta, \tau)$ we conclude that the function

$$\theta \mapsto p(\theta) := \left\langle [\pi'(B^*B)\tilde{\Omega}]_{\overline{\theta}} \,\middle|\, \tilde{\Omega}_\theta \right\rangle$$

is analytic. Because the deformation $\mathfrak{D}(\theta)$ is unitary for $\theta \in \mathbb{R}^2$, the function p is constant on $\mathbb{R}^2$,

$$\begin{aligned}
p(\theta) &= \left\langle [\pi'(B^*B)\tilde{\Omega}]_\theta \,\middle|\, \tilde{\Omega}_\theta \right\rangle = \left\langle \mathfrak{D}(\theta)\pi'(B^*B)\tilde{\Omega} \,\middle|\, \mathfrak{D}(\theta)\tilde{\Omega} \right\rangle \\
&= \left\langle \pi'(B^*B)\tilde{\Omega} \,\middle|\, \tilde{\Omega} \right\rangle = \left\langle \pi'(B)\tilde{\Omega} \,\middle|\, \pi'(B)\tilde{\Omega} \right\rangle = \eta(\mathbb{1}_{\mathcal{B}(\mathcal{H}^2)}) = 1.
\end{aligned}$$

Due to analyticity, the function p is constant on its whole domain. This implies that

$$\lim_{t\to\infty} \eta(\alpha^t(A)) = \tilde{\omega}(A).$$

Further, the convergence is exponentially fast with the given decay rate τ_{dec} by Proposition 2.10 since $\pi'(B^*B)\tilde{\Omega} \in \mathcal{D}^{\text{aux}}_{\mathfrak{D}-\text{a}}$.

Now, we consider the time evolution of an arbitrary ω_0-normal state η. Let $\varepsilon > 0$ and $A \in \mathcal{A}^{\text{ana}}$. There exists a state $\eta_\varepsilon \in \mathcal{N}^{\text{ana}}$ such that $\|\eta - \eta_\varepsilon\|_{\mathcal{A}_*} < \varepsilon/(\|A\|_{\mathcal{A}} + 1)$.

It holds

$$\begin{aligned}
|\eta(\alpha^t(A)) - \tilde{\omega}(A)| &\leq |\eta(\alpha^t(A)) - \eta_\varepsilon(\alpha^t(A))| + |\eta_\varepsilon(\alpha^t(A)) - \tilde{\omega}(A)| \\
&\leq \|\eta - \eta_\varepsilon\|_{\mathcal{A}_*} \left\|\alpha^t(A)\right\|_{\mathcal{A}} + |\eta_\varepsilon(\alpha^t(A)) - \tilde{\omega}(A)| \\
&\leq \varepsilon \frac{\|A\|_{\mathcal{A}}}{\|A\|_{\mathcal{A}} + 1} + |\eta_\varepsilon(\alpha^t(A)) - \tilde{\omega}(A)| \\
&\leq \varepsilon + |\eta_\varepsilon(\alpha^t(A)) - \tilde{\omega}(A)| \\
&\xrightarrow{t\to\infty} \varepsilon,
\end{aligned}$$

based on the convergence properties for states $\eta_\varepsilon \in \mathcal{N}^{\mathrm{ana}}$ studied above. This implies

$$\lim_{t\to\infty} \eta(\alpha^t(A)) = \tilde{\omega}(A),$$

however, the convergence needs not to be exponentially fast. ■

Remark 2.12 (Exponential Decay Rate) *Proposition 2.10 illustrates that the translation parameter τ' as chosen in (2.36) is proportional to the exponential rate τ_{dec} of convergence towards the NESS. The relation (2.36) implies that the rate of convergence gets larger if the reservoirs are stronger coupled to the particle system, i.e., if g increases, or if the thermal fluctuations grow, i.e., if the minimal reservoir temperature $T_{\min} = \beta_{\max}^{-1}$ increases. It is noteworthy that*

$$\tau_{\mathrm{dec}} \overset{g\to 0}{\sim} g^2 \qquad \text{and} \qquad \tau_{\mathrm{dec}} \overset{\beta_{\max}\to\infty}{\sim} \beta_{\max}^{-1} = T_{\min}.$$

Hence, for weak coupling the decay is governed by the strength of the interaction while in the low temperature regime the thermal fluctuation dominates the relaxation process.

Corollary 2.13 *We make the same assumptions as in Theorem 2.11. Let $\eta = \langle\, \xi \,|\, \pi(\,\cdot\,)\xi \,\rangle$, $\xi \in \mathcal{P}$, be an ω_0-normal state. Let Φ stand for the entropy production rate $\mathfrak{s}$, the heat fluxes $\phi_{\mathrm{f},r}$ or the interaction energy v and assume that ξ is in the form domain of Φ. Then the expectation value of Φ in the state η converges under the time evolution towards $\tilde{\omega}(\Phi)$,*

$$\lim_{t\to\infty} \eta\left(\alpha^t(\Phi)\right) = \tilde{\omega}(\Phi).$$

Moreover, if we choose η from the class $\mathcal{N}^{\mathrm{ana}}$ then the convergence is exponentially fast,

$$\lim_{t\to\infty} e^{\tau_{\mathrm{dec}} t} \left|\eta\left(\alpha^t(\Phi)\right) - \tilde{\omega}(\Phi)\right| = 0,$$

where the rate of convergence τ_{dec} is given in (2.36).

Proof. Since $\pi(\Phi)\tilde{\Omega} \in \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$ we conclude the convergence properties with the same arguments as those presented in the proof of Theorem 2.11. ∎

Corollary 2.14 *The definition (2.35) of $\tilde{\omega}$ is independent of the spectral deformation and it holds*

$$\sup_{A\in\mathcal{A}^{\mathrm{ana}}\setminus\{0\}} \frac{|\tilde{\omega}(A)|}{\|A\|_{\mathcal{A}}} = 1 \qquad \textit{and} \qquad \tilde{\omega}(A^*A) \geq 0$$

for $A \in \mathcal{A}^{\mathrm{ana}}$ with $A^*A \in \mathcal{A}^{\mathrm{ana}}$. In particular, $\tilde{\omega}$ is a state (i.e., a positive, normalized linear functional) on the $*$-subalgebra $\mathcal{A}_1 \subseteq \mathcal{A}^{\mathrm{ana}} \subseteq \mathcal{A}$ given in (C.13).

Proof. Let $A \in \mathcal{A}^{\mathrm{ana}}$. By Theorem 2.11 holds

$$\tilde{\omega}(A) = \lim_{t\to\infty} \omega_0(\alpha^t(A))$$

which is independent of the spectral deformation parameters θ. We further observe that

$$|\tilde{\omega}(A)| = \lim_{t\to\infty} \left|\left\langle e^{-iLt}\Omega_0 \,\middle|\, \pi(A)e^{-iLt}\Omega_0 \right\rangle\right| \leq \|A\|_{\mathcal{A}}$$

and, for $A^*A \in \mathcal{A}^{\mathrm{ana}}$,

$$\tilde{\omega}(A^*A) = \lim_{t\to\infty} \omega_0(\alpha^t(A^*A)) \geq 0$$

because $\omega_0 \circ \alpha^t$ is a state on $\mathcal{A}$. Since $\tilde{\omega}(\mathbb{1}_{\mathcal{B}(\mathcal{H}^2)}) = \omega_0(\mathbb{1}_{\mathcal{B}(\mathcal{H}^2)}) = 1$ the functional is a state on each $*$-subalgebra contained in $\mathcal{A}^{\mathrm{ana}}$. ∎

Theorem 2.11 describes the state $\tilde{\omega}$ as the limit point of ω_0-normal states propagating under α. This means that each state which is close (in the relative entropy sense) to the preparation of the subsystems at equilibrium at inverse temperatures $\beta_{\mathrm{p}}, \beta_1, \ldots, \beta_R$, resp., will converge under the interacting time evolution towards $\tilde{\omega}$ on a subalgebra of observables. This motivates us to refer to $\tilde{\omega}$ as the *non-equilibrium stationary state*, or simply *NESS*, attracting all configurations with finite relative entropy w.r.t. ω_0. We point out that the attribute "non-equilibrium" could be misleading since $\tilde{\omega}$ becomes a KMS state in the equal temperature situation as one expects. Nevertheless we keep this notion since, in general, the state $\tilde{\omega}$ will be far from being a thermal equilibrium. We will substantiate this in the subsequent section by computing the thermodynamic characteristics of the system in the state $\tilde{\omega}$ such as non-vanishing stationary heat fluxes and positive entropy production rate which speak for a non-equilibrium situation.

We further note that stationarity of $\tilde{\omega}$ w.r.t. α is not a defined concept either. Recall that the perturbed time evolution is not given on the algebra of observables (it does not leave the algebra invariant, we refer to the discussion about this issue in Section 1.4.3 and Remark 1.16) but rather on ω_0-normal states. Hence, for $\tilde{\omega}$ not being ω_0-normal its evolution w.r.t. α is not explained. Given an observable $A \in \mathcal{A}^{\mathrm{ana}}$ we are able to compute the expectation value in the state $\tilde{\omega}$. At given time $t \in \mathbb{R}$ the observable will have propagated under the Heisenberg evolution to $\alpha^t(A)$ and thereby may have left the set $\mathcal{A}^{\mathrm{ana}}$. The application of the state ω_0 to $\alpha^t(A)$ is possible with the interpretation $\omega_0(\alpha^t(A)) = \omega_0 \circ \alpha^t(A)$ where $\omega_0 \circ \alpha^t = \left\langle e^{-iLt}\Omega_0 \,\middle|\, \pi(\,\cdot\,)e^{-iLt}\Omega_0 \right\rangle$ is an ω_0-normal state. The time evolution of $\tilde{\omega}$ then can be understood in the sense

$$\tilde{\omega}\left(\alpha^t(A)\right) = \lim_{s\to\infty} \omega_0 \circ \alpha^s(\alpha^t(A)).$$

The state $\tilde{\omega}$ is then α-stationary by concept.

As a last observation concerning the nature of the NESS we consider the case that $\tilde{\omega}$ is ω_0-normal, i.e., there exists a vector $\xi \in \mathcal{P}$ such that $\tilde{\omega}(A) = \langle\, \xi \,|\, \pi(A)\xi \,\rangle$ for $A \in \mathcal{A}^{\mathrm{ana}}$. Then the NESS extends uniquely to a normal state $\eta := \langle\, \xi \,|\, \pi(\,\cdot\,)\xi \,\rangle$ on the whole algebra $\mathcal{A}$, using that $\pi(\mathcal{A}^{\mathrm{ana}})$ is strongly dense in $\pi(\mathcal{A})$ by the arguments of the proof to Proposition C.10. The state η is α-stationary on $\mathcal{A}^{\mathrm{ana}}$,

$$\eta(\alpha^t(A)) = \eta \circ \alpha^t(A) = \lim_{s\to\infty} \eta \circ \alpha^{s+t}(A) = \tilde{\omega}(A) = \eta(A).$$

Since $\mathcal{A}^{\mathrm{ana}}$ is strongly dense in $\mathcal{A}$ the time invariance extends to all observables by the following reason. Let $A \in \mathcal{A}$, then there exists a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}^{\mathrm{ana}}$ with $\lim_{n\to\infty} \pi(A_n)\psi = \pi(A)\psi$ for all $\psi \in \mathcal{H}^2$ and it holds

$$\begin{aligned} \eta\left(\alpha^t(A)\right) &= \left\langle e^{-iLt}\xi \,\middle|\, \pi(A)e^{-iLt}\xi \right\rangle = \lim_{n\to\infty} \left\langle e^{-iLt}\xi \,\middle|\, \pi(A_n)e^{-iLt}\xi \right\rangle \\ &= \lim_{n\to\infty} \eta\left(\alpha^t(A_n)\right) = \lim_{n\to\infty} \eta(A_n) = \lim_{n\to\infty} \langle\, \xi \,|\, \pi(A_n)\xi \,\rangle \\ &= \eta(A). \end{aligned}$$

The invariance of the normal state η implies that $\xi \in \ker(L)$, see the discussion of Section 1.1.2 and [10, Thm. 2.5.31]. Let η' be another α-stationary ω_0-normal state. Then η' coincides with η on $\mathcal{A}^{\mathrm{ana}}$ by Theorem 2.11 and

$$\eta'(A) = \lim_{t\to\infty} \eta'\left(\alpha^t(A)\right) = \tilde{\omega}(A) = \lim_{t\to\infty} \eta\left(\alpha^t(A)\right) = \eta(A), \qquad A \in \mathcal{A}^{\mathrm{ana}},$$

and due to the strong density of $\mathcal{A}^{\mathrm{ana}}$ in $\mathcal{A}$ it is $\eta' = \eta$ on $\mathcal{A}$. Hence, we conclude

$$\ker(L) = \mathbb{C}\xi$$

with the help of (1.7). Vice versa, if ξ is an eigenstate of L corresponding to an eigenvalue $E \in \mathbb{R}$ then the state $\eta := \langle\, \xi \,|\, \pi(\,\cdot\,)\xi \,\rangle$ is apparently α-stationary and it follows that $\eta(A) = \tilde{\omega}(A)$ for all $A \in \mathcal{A}^{\mathrm{ana}}$, thus

$$\ker(L) \neq \{0\} \qquad \Longleftrightarrow \qquad \dim(\ker(L)) = 1. \tag{2.37}$$

Under the assumptions of Theorem 2.11 the above considerations lead to a remarkable set of the equivalences,

$$\begin{aligned}
& \exists\, \xi \in \mathcal{H}^2 : \tilde{\omega}(A) = \langle\, \xi \,|\, \pi(A)\xi\, \rangle \quad \forall\, A \in \mathcal{A}^{\mathrm{ana}} \\
\Longleftrightarrow \quad & \exists\, \xi \in \mathcal{H}^2 \setminus \{0\} : \ker(L) = \mathbb{C}\xi \\
\Longleftrightarrow \quad & \mathrm{spec}_{\mathrm{pp}}(L) \neq \emptyset \\
\Longleftrightarrow \quad & \ker(L) \neq \{0\} \\
\Longleftrightarrow \quad & \mathrm{spec}_{\mathrm{pp}}(L) = \{0\},
\end{aligned} \tag{2.38}$$

where the last equivalence goes back to [21, Thm. 1.1] saying that under the assumption that the standard Liouville operator has a simple zero eigenvalue then $\mathrm{spec}_{\mathrm{pp}}(L)$ is an additive subgroup of $\mathbb{R}$. However, by Proposition 3.3, there is no point spectrum outside a finite box around zero, hence the group of eigenvalues is the trivial group only consisting of zero itself.

2.3.3 Thermodynamic Characterization of the NESS

After having introduced the NESS of the interacting systems and its dynamical (attracting) properties in the previous section we now focus on the thermodynamic characteristics of the state $\tilde{\omega}$. We compute the expectation value of the entropy production rate in this state and the net heat fluxes of the subsystems.

Proposition 2.15 *Let $g > 0$ be sufficiently small and $|\beta_{\max} - \beta_{\min}| \ll 1$. The entropy production rate w.r.t. ω_0 in the state $\tilde{\omega}$ is non-negative,*

$$\mathrm{Ep}_{\omega_0}(\tilde{\omega}) \geq 0,$$

and it further holds

$$\mathrm{Ep}_{\omega_0}(\tilde{\omega}) = 0 \qquad \Longleftrightarrow \qquad 0 \in \mathrm{spec}_{\mathrm{pp}}(L).$$

Proof. The entropy production formula (1.85) implies

$$\frac{1}{t}\int_0^t ds\, \omega_0 \circ \alpha^s(\mathfrak{s}) = -\frac{1}{t}\,\mathrm{Ent}\left(\omega_0 \circ \alpha^t | \omega_0\right) \geq 0$$

for all $t > 0$, using that $\mathrm{Ent}(\omega_0|\omega_0) = 0$. The l.h.s. converges towards $\mathrm{Ep}_{\omega_0}(\tilde{\omega})$ as $t \to \infty$,

$$\begin{aligned}
\left| \frac{1}{t} \int_0^t ds\, \omega_0 \circ \alpha^s(\mathfrak{s}) - \mathrm{Ep}_{\omega_0}(\tilde{\omega}) \right| &\leq \frac{1}{t} \int_0^t ds\ |\omega_0 \circ \alpha^s(\mathfrak{s}) - \tilde{\omega}(\mathfrak{s})| \\
&= \frac{1}{t} \int_0^T ds\ |\omega_0 \circ \alpha^s(\mathfrak{s}) - \tilde{\omega}(\mathfrak{s})| + \frac{1}{t} \int_T^t ds\ |\omega_0 \circ \alpha^s(\mathfrak{s}) - \tilde{\omega}(\mathfrak{s})| \\
&< \frac{1}{t} \int_0^T ds\ |\omega_0 \circ \alpha^s(\mathfrak{s}) - \tilde{\omega}(\mathfrak{s})| + \frac{t-T}{t} \varepsilon \\
&\xrightarrow{t \to \infty} \varepsilon,
\end{aligned}$$

where, for a given $\varepsilon > 0$, the number $T > 0$ is chosen such that $|\omega_0 \circ \alpha^s(\mathfrak{s}) - \mathrm{Ep}_{\omega_0}(\tilde{\omega})| < \varepsilon$ for all $t \geq T$, c.f. Corollary 2.13. This proves that the entropy production rate in $\tilde{\omega}$ is non-negative,

$$\mathrm{Ep}_{\omega_0}(\tilde{\omega}) = \lim_{t \to \infty} \frac{1}{t} \int_0^t ds\, \omega_0 \circ \alpha^s(\mathfrak{s}) \geq 0.$$

We now assume that $0 \in \mathrm{spec}_{\mathrm{pp}}(L)$. As a consequence of (2.38) we know that $\tilde{\omega}$ coincides on $\mathcal{A}^{\mathrm{ana}}$ with an ω_0-normal, α-invariant state η. An application of the entropy production formula (1.85) yields

$$\mathrm{Ep}_{\omega_0}(\tilde{\omega}) = \tilde{\omega}(\mathfrak{s}) = \lim_{t \to \infty} \eta\left(\alpha^t(\mathfrak{s})\right) = \eta(\mathfrak{s}) = -\partial_t|_{t=0}\, \mathrm{Ent}\left(\eta \circ \alpha^t|\omega_0\right) = 0.$$

Vice versa, if $\mathrm{Ep}_{\omega_0}(\tilde{\omega}) = 0$ then the state $\tilde{\eta} := \left\langle \tilde{\Omega} \,\middle|\, \pi(\,\cdot\,)\tilde{\Omega} \right\rangle \in \mathcal{N}^{\mathrm{ana}}$ obeys

$$\begin{aligned}
\mathrm{Ent}\left(\tilde{\eta} \circ \alpha^t|\omega_0\right) &= \mathrm{Ent}(\tilde{\eta}|\omega_0) - \int_0^t ds\ \left[\tilde{\eta} \circ \alpha^s(\mathfrak{s}) - \mathrm{Ep}_{\omega_0}(\tilde{\omega})\right] \\
&\geq \mathrm{Ent}(\tilde{\eta}|\omega_0) - \int_0^t ds\, e^{-\tau_{\mathrm{dec}} s} \left| e^{\tau_{\mathrm{dec}} s} \left[\tilde{\eta} \circ \alpha^s(\mathfrak{s}) - \tilde{\omega}(\mathfrak{s})\right]\right| \\
&\geq -C,
\end{aligned}$$

for a positive constant $C < \infty$, because of Corollary 2.13 where the rate τ_{dec} of exponential convergence is given in (2.36). By [37, Prop. 5.27.] the set

$$\{\gamma \in \mathcal{N}_{\omega_0}(\mathcal{A}) \,|\, \mathrm{Ent}(\gamma|\omega_0) \geq -C\,\}$$

is compact w.r.t. the weak topology on the $\mathcal{N}_{\omega_0}(\mathcal{A})$ (note that the definition of the relative entropy in [37] differs by a relative sign from ours). This means that there exists an ω_0-normal state η and a sequence $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \tilde{\eta}\left(\alpha^{t_n}(A)\right) = \eta(A)$$

for all $A \in \mathcal{A}$. This implies that $\tilde{\omega}(A) = \eta(A)$ for all $A \in \mathcal{A}^{\mathrm{ana}}$ and the relation (2.38) yields $0 \in \mathrm{spec}_{\mathrm{pp}}(L)$. ■

Corollary 2.16 *Let $g > 0$ be sufficiently small. If $\beta_{\max} = \beta_{\min}$ then the entropy production rate in the state $\omega = \tilde{\omega}|_{\beta_{\max}=\beta_{\min}}$ is zero,*

$$\mathrm{Ep}_{\omega_0}(\omega) = 0.$$

If $|\beta_{\max} - \beta_{\min}|$ is sufficiently large compared to the coupling constant g in the sense of (3.6) then the entropy production rate in the state $\tilde{\omega}$ is strictly positive,

$$\mathrm{Ep}_{\omega_0}(\tilde{\omega}) > 0.$$

Proof. This follows immediately from the Proposition 2.15 and Proposition 3.3 describing the spectrum of the standard Liouville operator L depending on the temperature difference $\beta_{\max} - \beta_{\min}$. ■

The positivity of the entropy production rate in the NESS has consequences on the heat fluxes through the reservoirs.

Proposition 2.17 (Energy Conservation and Stationary Fluxes) *Let $g > 0$ be sufficiently small and $|\beta_{\max} - \beta_{\min}| \ll 1$. The energy flux ϕ_{p} of the particle system and the total energy flux ϕ_{tot} vanish in the NESS,*

$$\tilde{\omega}\left(\phi_{\mathrm{p}}\right) = 0 \qquad \text{and} \qquad \tilde{\omega}\left(\phi_{\mathrm{tot}}\right) = 0.$$

If $0 \notin \mathrm{spec}_{\mathrm{pp}}(L)$, e.g., if the temperature difference $\beta_{\max} - \beta_{\min}$ is sufficiently large compared to the coupling constant g in the sense of (3.6), then there are non-vanishing stationary heat fluxes through the reservoirs, i.e., there exist $r, r' \in 1, \ldots, R$ such that

$$\tilde{\omega}\left(\phi_{\mathrm{f},r}\right) > 0 \qquad \text{and} \qquad \tilde{\omega}\left(\phi_{\mathrm{f},r'}\right) < 0. \tag{2.39}$$

If further only two reservoirs at inverse temperatures $\beta_{\max} = \beta_1 > \beta_2 = \beta_{\min}$ are coupled to the particle system then the stationary heat flux goes from the hotter (r=2) to the colder (r=1) reservoir, i.e.,

$$\tilde{\omega}(\phi_{\mathrm{f},1}) > 0, \qquad \tilde{\omega}(\phi_{\mathrm{f},2}) = -\tilde{\omega}(\phi_{\mathrm{f},1}) < 0.$$

Proof. We first observe that the energy flux of the particle system vanishes in the NESS. By definition is $\phi_\mathrm{p} = \partial_t|_{t=0}\alpha^t(H_\mathrm{p})$ where H_p acts trivially on the photon variables and therefore $H_\mathrm{p} \in \mathcal{A}_1$. Therefore holds

$$\tilde{\omega}\left(\phi_\mathrm{p}\right) = \lim_{s\to\infty} \omega_0\left(\partial_t|_{t=0}\alpha^{t+s}(H_\mathrm{p})\right) = \partial_t|_{t=0} \lim_{s\to\infty} \omega_0\left(\alpha^{t+s}(H_\mathrm{p})\right) = \partial_t|_{t=0}\tilde{\omega}(H_\mathrm{p}) = 0.$$

We now consider the total flux observable $\phi_\mathrm{tot} = -\partial_t|_{t=0}\alpha^t(v)$. With the same arguments as before we obtain

$$\tilde{\omega}\left(\phi_\mathrm{tot}\right) = -\lim_{s\to\infty} \omega_0\left(\partial_t|_{t=0}\alpha^{t+s}(v)\right) = -\partial_t|_{t=0} \lim_{s\to\infty} \omega_0\left(\alpha^{t+s}(v)\right) = -\partial_t|_{t=0}\tilde{\omega}(v) = 0$$

using Corollary 2.13, and therefore

$$\sum_{r=1}^{R} \tilde{\omega}(\phi_{\mathrm{f},r}) = 0. \tag{2.40}$$

We now assume that $0 \notin \mathrm{spec}_\mathrm{pp}(L)$. This implies that $\sum_{r=1}^{R} \beta_r\tilde{\omega}(\phi_{\mathrm{f},r}) = \mathrm{Ep}_{\omega_0}(\tilde{\omega}) > 0$. There exists a label $r = 1, \ldots, R$ with $\tilde{\omega}(\phi_{\mathrm{f},r}) > 0$. To fulfil the flux balance (2.40) we find another reservoir label $r' = 1, \ldots, R$ such that $\tilde{\omega}(\phi_{\mathrm{f},r'}) < 0$.

For $R = 2$ we can express one reservoir heat flux as the negative of the other one. The positivity of the entropy production rate in $\tilde{\omega}$ yields

$$(\beta_1 - \beta_2)\tilde{\omega}(\phi_{\mathrm{f},1}) = \mathrm{Ep}_{\omega_0}(\tilde{\omega}) > 0.$$

Since $\beta_1 > \beta_2$ it follows that $\tilde{\omega}(\phi_{\mathrm{f},1}) > 0$ and therefore $\tilde{\omega}(\phi_{\mathrm{f},2}) < 0$. ■

The Proposition 2.17 describes that the particle system does not accumulate or provide energy in the NESS and that the total energy is preserved while the infinitely large extended reservoirs feature stationary heat fluxes when they were prepared at different temperatures. This is in no contradiction to the time invariance of the NESS. The reservoir energy “observables”, the Hamiltonians $H_{\mathrm{f},r}$, do not belong to $\mathcal{A}$ and not even to the polynomial algebra of creation and annihilation operators. A formal application of an ω_0-normal state (or the time limit $\tilde{\omega}$) yields an infinite expectation value reflecting the fact that the reservoirs contain an infinite amount of energy. Roughly speaking, the amount of energy stays the same, namely infinite, over finite time intervals even if the reservoirs show a non-vanishing energy flux.

In the case that two reservoirs are coupled to the particle system it is possible to compute perturbatively the energy flux from the hotter into the colder reservoir. We anticipate a result from [34] saying that the net flux of the first reservoir in the NESS is given in leading order as

$$\tilde{\omega}(\phi_{\mathrm{f},1}) = g^2\phi' + o(g^2)\mathcal{O}\left(\beta_\mathrm{max} - \beta_\mathrm{min}\right) \tag{2.41}$$

where

$$\phi' := \lim_{\varepsilon \searrow 0} 2\,\mathrm{Re} \left\langle \Omega_0^* \,\middle|\, \pi(v_1) i L_{\mathrm{f},1} (L_0 + i\varepsilon)^{-1} e^{-\beta_1 L_0/2} \pi(v_1) \Omega_0 \right\rangle .$$

Hereby, the vectors Ω_0, Ω_0^* are given by

$$\Omega_0 = \Omega_\mathrm{p} \otimes \Omega_\mathrm{vac}, \qquad \Omega_0^* = \zeta \otimes \Omega_\mathrm{vac}$$

where Ω_p, ζ are the zero eigenvectors of the level shift operator $\Lambda_0^{(-i/2)}$ and its adjoint $(\Lambda_0^{(-i/2)})^*$, discussed in the Sections 3.3.2 and 3.5. While Ω_p was already introduced in Section 1.2 the vector $\zeta \in \ker(L_\mathrm{p})$ is fixed through

$$\left(\Lambda_0^{(-i/2)}\right)^* \zeta = 0, \qquad \langle \zeta \,|\, \Omega_\mathrm{p} \rangle_{\ker(L_\mathrm{p})} = 1.$$

An explicit expression for ϕ' is given by

$$\phi' = 2\pi \sum_{\substack{j,k=1,\\ j>k}}^{N-1} \int_{S^2} d\Sigma \, \frac{E_{j,k}^3}{e^{\beta_1 E_{j,k}/2} - e^{-\beta_1 E_{j,k}/2}} \left| G_1(E_{j,k}\Sigma) \right|^2 \left[\kappa_j e^{-\beta_1 E_k} - \kappa_k e^{-\beta_1 E_j} \right],$$

where $\kappa_j > 0$ are the coefficients of the expansion of ζ in the basis $\{\varphi_{j,j}\}_{j=0,\ldots,N-1}$ of $\ker(L_\mathrm{p})$,

$$\zeta = \sum_{j=0}^{N-1} \kappa_j \varphi_{j,j},$$

for the particular choice $\beta_\mathrm{p} = \beta_1$, compare with (2.9). The proof of (2.41) presented in [34] uses the perturbation theory for the resonance eigenvectors $\tilde{\Omega}_\theta$ and $\tilde{\Omega}_{\bar{\theta}}^*$. The expansion of these vectors is carried out under the assumption that the reservoir temperatures are sufficiently high. To avoid this restriction one would have to apply a renormalization process. Since such an analysis would be a complete project for itself we content ourselves with presenting the results of the perturbative computation of the heat flux.

Part II

Spectral Analysis

3 Spectral Analysis of the Operators $K_\theta^{(s)}$

This chapter is devoted to the analysis of the spectrum of the operators $K_\theta^{(s)}$. In fact, this chapter (along with the subsequent ones) is the technically most involved and it provides all necessary properties of the C-Liouvillean K to study the evolution of the group $t \mapsto U(t) \equiv e^{iKt}$.

The global picture of the spectrum of $K_\theta^{(s)}$ is given by a numerical range estimate, c.f. Proposition A.9 and Lemmata C.1 and C.2. Roughly speaking, the numerical range, and therefore also the spectrum, is confined to a truncated cone with apex angle $\pi - 2\,\mathrm{Im}(\delta)$ as illustrated in Figure 3.1. The sectorial property of $K_\theta^{(s)}$ is already sufficient to make the representation of the group $U(t)$ in terms of integration over the resolvent $(z - K_\theta)^{-1}$ a well defined expression, c.f. Lemma 2.8 and (2.27) of Proposition 2.9. The proof of the sectorial location of the spectrum goes back to rather standard relative bound estimates on the perturbation as they are provided in Appendix A.

However, the spectral property of $K_\theta^{(s)}$ along the real axis, in particular around the origin of the complex plane, is not accessible only with estimates on the perturbation. It is the structure of the null space and the absence of spectrum on the positive and negative real axis which determine the long time limit of the group $U(t)$. Therefore, we need to apply more subtle tools in order to obtain insight into spectral regions close to the real axis. The closest neighborhood of particle eigenvalues, i.e., points from the spectrum of the particle Liouvillean L_p, can be treated by the *Feshbach technique* transferring the analysis to an equivalent problem on a spectral subspace. That way we gain clarification about the spectrum around $\mathrm{spec}(L_\mathrm{p}) \setminus \{0\}$, namely we can show that all the spectrum moves from the real axis into the upper half plane, provided that $\mathrm{Im}(\delta), \mathrm{Im}(\tau) > 0$. This procedure is the main task of the present chapter. The strategy in proving the shift of the eigenvalues uses arguments as in [8], however, the non-self-adjointness of $K^{(s)}$ for $s \notin \mathbb{R}$ poses some difficulties. They have been dealt with in the project [34] and enter this work in Section 3.3.

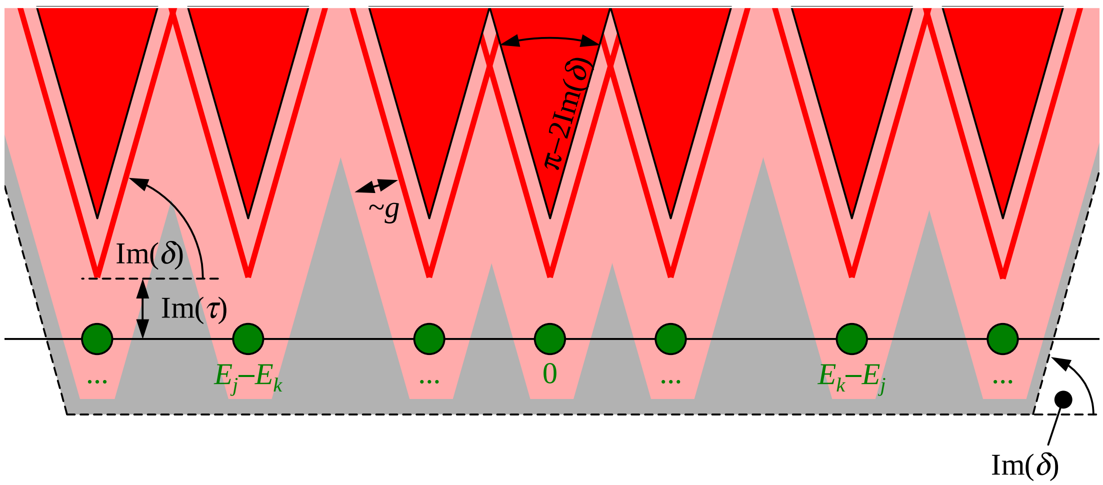

Figure 3.1: The spectrum of $L_{0,\theta}$ for the parameter choice (3.1, 3.2) consists of the eigenvalues of L_p, paired with cones of continuous spectrum arising from the spectrum of $(L_\mathrm{res})_\theta$. The apex angle of the cones is $\pi - 2\,\mathrm{Im}(\delta)$ and they are separated from the eigenvalues by a shift $\mathrm{Im}(\tau)$. The relative bound of the perturbation $gI_\theta^{(s)}$ w.r.t. $\mathrm{Im}(L_{0,\theta})$ confines the spectrum of $K_\theta^{(s)}$ to the pink region, contained in a truncated cone of apex angle $\pi - 2\,\mathrm{Im}(\delta)$.

The zero eigenvalue of L_p plays a special role. This is related to its $N = \dim \mathcal{H}_\mathrm{p}$-fold degeneration. The spectral analysis near the origin requires multiple applications of the Feshbach technique. The importance of a *Feshbach iteration procedure* is accounted by outsourcing it into Chapter 4. The following Chapter 5 provides the corresponding spectral interpretation of the Feshbach procedure and relates it to $K_\theta^{(s)}$.

The spectral information arising from the analysis of the subsequent chapters is summarized in Theorem 3.1. Before stating the main theorem we first do some preparation work.

Throughout this chapter, and also the subsequent ones, we assume the following choice of parameters. Let the deformation parameters $\theta = (i\delta', i\tau') \in (i\mathbb{R}^+)^2$ fulfill

$$\delta' \in \left[\frac{\pi}{8}, \frac{\pi}{4}\right] \qquad \text{and} \qquad \tau' \in (0, 2\pi\beta_{\max}^{-1}), \tag{3.1}$$

and let the parameter s obey

$$s \in \mathbb{S}_{\varepsilon_0} \tag{3.2}$$

as in (2.11). All spectral considerations of this chapter are done on a scale $\rho > 0$. Since that scale shall measure close environments of the particle eigenvalues we require that it is small compared to the distance of two neighboring points from the spectrum of L_{p}, i.e.,

$$\rho < \min_{\substack{e,e'\in\operatorname{spec}(L_{\mathrm{p}})\\ e\neq e'}} |e-e'| =: d_{L_{\mathrm{p}}}. \tag{3.3}$$

In fact, we will fix the parameter ρ by coupling it to g,

$$\rho := g^{2/3(1+\tilde{\varepsilon})} \tag{3.4}$$

where $0 < \tilde{\varepsilon} < 1/4$. This guarantees the order relation

$$\frac{g^2}{\sin(\delta')} \ll \rho \ll d_{L_{\mathrm{p}}}$$

for g sufficiently small. We use the scale ρ to define disjoint subsets of $\mathbb{C}$ on which different approaches are undertaken to study the contained spectrum of $K_\theta^{(s)}$. We introduce the set

$$\mathcal{S} := \left\{ z \in \mathbb{C} \,\middle|\, \operatorname{Im}(z) \leq \frac{\sin(\delta')}{2}\rho \right\},$$

on which a refined spectral analysis is performed. The complement of $\mathcal{S}$ is far enough inside the upper half plane such that the spectrum inside $\mathbb{C}\setminus\mathcal{S}$ is described sufficiently detailed for our purposes by the rough numerical range localization given in Proposition A.9. To study the spectrum in $\mathcal{S}$ we decompose

$$\mathcal{S} = \Big[\bigcup_{e\in\operatorname{spec}(L_{\mathrm{p}})} \mathcal{S}_e \Big] \cup \overline{\mathcal{S}},$$

where, for $e \in \operatorname{spec}(L_{\mathrm{p}})$,

$$\begin{aligned} \mathcal{S}_e &:= \{ z \in \mathcal{S} \,|\, |z-e| \leq 4\rho \}, \\ \overline{\mathcal{S}} &:= \mathcal{S} \setminus \bigcup_{e\in\operatorname{spec}(L_{\mathrm{p}})} \mathcal{S}_e. \end{aligned} \tag{3.5}$$

The sets $\overline{\mathcal{S}}$, $\mathcal{S}_e$ for $e \neq 0$ and $\mathcal{S}_0$ are shown in Figure 3.2. The color code refers to the different approaches to tackle the spectral analysis: on the blueish area $\overline{\mathcal{S}}$ we prove absence of the spectrum by ordinary expansion of the resolvent in a Neumann series, the greyish regions are treated with the Feshbach technique, and the greenish region is subject to Feshbach iteration.

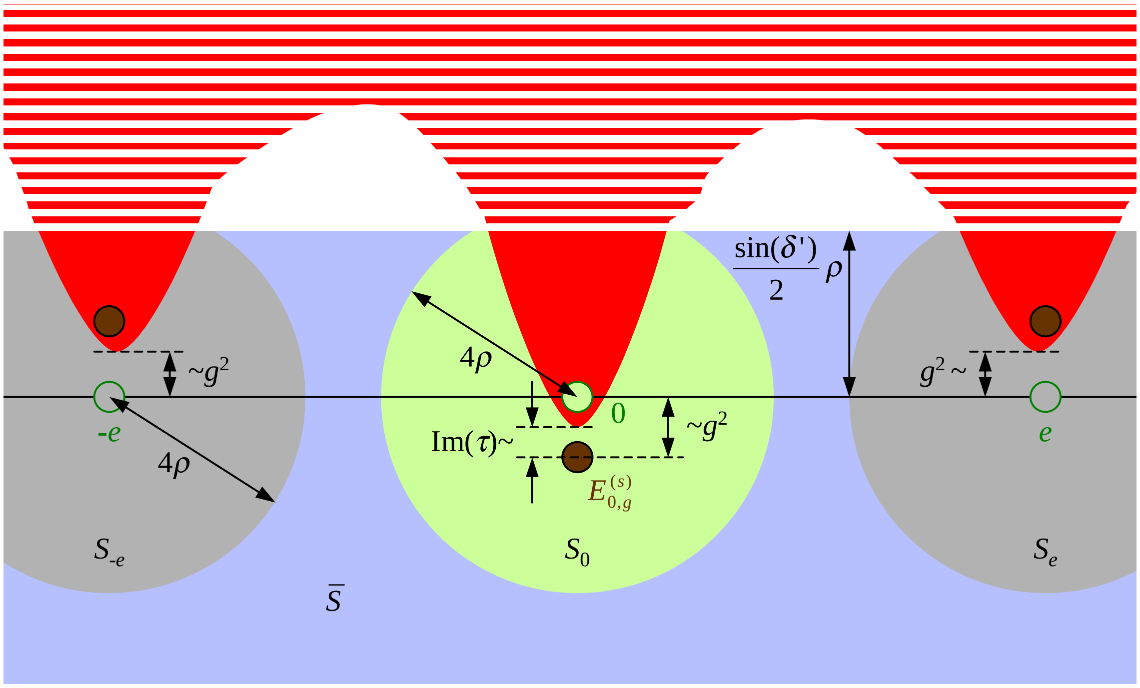

Figure 3.2: Illustration of the spectrum of $K_\theta^{(s)}$: Absence of spectrum in $\overline{\mathcal{S}}$ (away from particle eigenvalues); localization of the spectrum in the upper half plane in $\mathcal{S}_e$ near non-zero particle eigenvalues $e \in \mathrm{spec}(L_\mathrm{p})\setminus\{0\}$; isolated, simple eigenvalue in the neighborhood of zero. No prediction is made about the region outside $\mathcal{S}$.

3.1 Spectral Picture of the Standard and the C-Liouville Operator

Theorem 3.1 (Spectrum of $K_\theta^{(s)}$) *Let the parameters $\theta = (i\delta', i\tau')$ and s obey (3.1) and (3.2). Further, we fix the translation parameter by*

$$\tau' := \frac{g^2 \gamma_{\mathrm{eq}}}{2 + \beta_{\mathrm{max}}},$$

in accordance with (3.1), where the positive constant γ_{eq} is explained in (4.46). Under these conditions and the further requirement that g is small enough and $|\beta_{\mathrm{max}} - \beta_{\mathrm{min}}| \ll 1$, the spectrum of $K_\theta^{(s)}$ can be described as follows.

(i) There exists a complex number $E_{0,g}^{(s)} \in \mathcal{S}_0$ *of norm*

$$\left| E_{0,g}^{(s)} \right| \leq 4g^2 \left\| \Gamma_{\mathrm{eq}} \right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))}$$

which is a simple, isolated eigenvalue of $K_\theta^{(s)}$. *The level shift operator* Γ_{eq} *is defined in (4.44) whereby its norm is estimated uniformly in the inverse temperatures.*

(ii) The eigenvalue $E_{0,g}^{(s)}$ *has the lowest imaginary part among all other spectral points in a neighborhood of order* ρ, *it holds*

$$\left[\mathrm{spec}\left(K_\theta^{(s)} \right) \setminus \left\{ E_{0,g}^{(s)} \right\} \right] \cap \mathcal{S}_0 \subseteq \left\{ z \in \mathbb{C} \,\middle|\, \mathrm{Im}(z) \geq \mathrm{Im}\left(E_{0,g}^{(s)} \right) + 2\tau_{\mathrm{dec}} \right\}.$$

The gap τ_{dec} *is proportional to the translation parameter,*

$$\tau_{\mathrm{dec}} := \frac{\tau'}{1920 \mathcal{C}_{\chi_1}^2},$$

where the positive constant $\mathcal{C}_{\chi_1}$ *is introduced in (4.6).*

(iii) For $e \in \mathrm{spec}(L_{\mathrm{p}}) \setminus \{0\}$ *the spectrum of* $K_\theta^{(s)}$ *inside the region* $\mathcal{S}_e$ *is shifted completely into the upper half plane by a distance of order* g^2, *it holds*

$$\mathrm{spec}\left(K_\theta^{(s)} \right) \cap \mathcal{S}_e \subseteq \left\{ z \in \mathbb{C} \,\middle|\, \mathrm{Im}(z) \geq g^2 \frac{\gamma_{\mathrm{FGR}}}{4} \right\}$$

where the Fermi golden rule level shift γ_{FGR} *was introduced in (1.86).*

(iv) There is no spectrum inside $\overline{\mathcal{S}}$, *i.e.,*

$$\mathrm{spec}\left(K_\theta^{(s)} \right) \cap \overline{\mathcal{S}} = \emptyset.$$

(v) Globally, the spectral information of $K_\theta^{(s)}$ *is accessible via the numerical range,*

$$\begin{aligned} &\mathrm{NumRan}\left(K_\theta^{(s)} \right) \\ &\subseteq \left\{ z \in \mathbb{C} \,\middle|\, \mathrm{Im}(z) \geq -1 + \max\left\{ \frac{\sin(\delta')}{8} (|\,\mathrm{Re}(z) - \|L_{\mathrm{p}}\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})}\,|), 0 \right\} \right\}. \end{aligned}$$

Further, for any $z \notin \mathrm{spec}(K_\theta^{(s)})$ *there exists a positive constant* $C < \infty$ *such that*

$$\left\| \left(z - K_\theta^{(s)} \right)^{-1} \right\| \leq \frac{C}{|\,\mathrm{Re}(z)| + 1}$$

for fixed $\mathrm{Im}(z)$ *and* $|\,\mathrm{Re}(z)|$ *sufficiently large.*

Proof. Theorem 3.1 is the technically most challenging theorem of this work, its proof spreads over the subsequent chapters. At this point we give the references to the corresponding proofs.

(i) The statement about the spectrum of $K_\theta^{(s)}$ inside $\mathcal{S}_0$ is a direct consequence of Theorem 5.8(ii). It employs the Sections 3.3, 3.5 and the Chapters 4, 5.

(ii) The separation of the eigenvalue $E_{0,g}^{(s)}$ from the rest of the spectrum in $\mathcal{S}_0$ follows directly from Theorem 5.8(iii).

(iii) The assertion about the spectrum in $\mathcal{S}_e$ is a consequence of Theorem 3.19. The result is based on the analysis of the Sections 3.3 and 3.4.

(iv) The absence of spectrum inside $\overline{\mathcal{S}}$ is considered in Section 3.2 and is guaranteed by Proposition 3.4.

(v) The global localization and the resolvent estimate goes back to the numerical range estimate provided in Proposition A.9.

■

We study the localization of the eigenvalue $E_{0,g}^{(-i/2)}$ of the deformed C-Liouvillean K_θ in more detail.

Proposition 3.2 (Kernel of the deformed C-Liouvillean K_θ) *Under the same assumptions of Theorem 3.1 and for the particular choice $s = -\frac{i}{2}$ we have $E_{0,g}^{(-i/2)} = 0$ and the kernel of $K_\theta^{(-i/2)} = K_\theta$ is spanned by $\tilde{\Omega}_\theta$, i.e.,*

$$\ker(K_\theta) = \mathbb{C}\tilde{\Omega}_\theta.$$

Proof. It follows from Proposition C.16 that zero is also an eigenvalue of $K_\theta = K_\theta^{(-i/2)}$. The knowledge about the spectrum in $\mathcal{S}_0$ provided by Theorem 3.1 implies that either $E_{0,g}^{(-i/2)} = 0$ or $\mathrm{Im}(E_{0,g}^{(-i/2)}) \leq -2\tau_{\mathrm{dec}}$ since otherwise zero would not appear in the spectrum of K_θ. However, in the proof to Proposition 2.10 we showed that $\mathrm{Im}(E_{0,g}^{(-i/2)}) < 0$ contradicts that K is the generator of the group $U(t)$. Note that we did not use the present proposition within the proof of Proposition 2.10. Hence, we have $E_{0,g}^{(-i/2)} = 0$. Due to the simplicity of the eigenvalue and $K_\theta\tilde{\Omega}_\theta = 0$, we conclude that the kernel of K_θ is spanned by $\tilde{\Omega}_\theta$. ■

While it is impossible to deduct the spectrum of the C-Liouvillean K from K_θ we are able to make conclusion about the spectrum of the self-adjoint standard Liouville operator L from its deformation $L_\theta = K_\theta^{(0)}$.

Proposition 3.3 (Absolutely Continuous Spectrum of L) *For $g > 0$ sufficiently small and $|\beta_{\max} - \beta_{\min}| \ll 1$ the spectrum of the self-adjoint standard Liouville operator L outside the interval $[-4\rho, 4\rho]$ is absolutely continuous. In particular, there is no eigenvalue outside $[-4\rho, 4\rho]$.*

- *If further $\beta_{\max} = \beta_{\min}$, i.e., all reservoirs are at the same temperature, the operator L has a simple eigenvalue at zero and the rest of the spectrum is absolutely continuous, it holds*

$$\ker(L) = \mathbb{C}\tilde{\Omega}|_{\beta_{\max}=\beta_{\min}=\beta}, \qquad \mathrm{spec}_{\mathrm{ac}}(L) = \mathbb{R} \setminus \{0\}.$$

- *If further*

$$g^{\tilde{\varepsilon}} \ll (\beta_{\max} - \beta_{\min})^2 \left[1 - \frac{Z(2\overline{\beta})}{Z(\overline{\beta})^2}\right], \tag{3.6}$$

where $\overline{\beta} := (\beta_{\min} + \beta_{\max})/2$, i.e., the reservoir temperature differences are sufficiently large compared to the coupling constant, the whole spectrum of L is absolutely continuous,

$$\mathrm{spec}(L) = \mathrm{spec}_{\mathrm{ac}}(L) = \mathbb{R}.$$

Then, in particular, the standard Liouvillean possesses no eigenvalues.

Proof. Assume that θ obeys the assumptions of Theorem 3.1, then the same theorem implies that for the spectrum of L_θ holds,

$$\mathrm{spec}\,(L_\theta) \setminus \mathcal{S}_0 \subseteq \left\{ z \in \mathbb{C} \,\middle|\, \mathrm{Im}(z) \geq g^2 \frac{\gamma_{\mathrm{FGR}}}{4} \right\}.$$

If further $\beta_{\max} = \beta_{\min}$ we have $L_\theta = K_\theta$ and therefore Proposition 3.2 applies. Thus, zero is an isolated simple eigenvalue of L_θ and the rest of the spectrum is shifted into the upper half plane by $2\tau_{\mathrm{dec}}$.

On the other hand, if the restriction (3.6) holds then the spectrum of L_θ is completely in the upper half plane by Proposition 3.24, it is

$$\mathrm{spec}\,(L_\theta) \subseteq \{ z \in \mathbb{C} \,|\, \mathrm{Im}(z) \geq c \}$$

where

$$c := g^2 \frac{\gamma_{\mathrm{FGR}}}{4} \min \left\{ 1, \frac{(\beta_{\max} - \beta_{\min})^2 d_{L_{\mathrm{p}}}}{16} \left[1 - \frac{Z(2\overline{\beta})}{Z(\overline{\beta})^2} \right] \right\}.$$

Let $(a, b) \subseteq \mathbb{R}$ be a bounded interval. The interval can be chosen arbitrarily if we assume the extra condition (3.6). We choose it from $\mathbb{R} \setminus \{0\}$ for $\beta_{\max} = \beta_{\min}$ and in the general case we exclude that (a,b) overlaps with $[-4\rho, 4\rho]$. For $\varphi \in \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$ we consider the function

$$z \mapsto f(z) := \left\langle \varphi \,\middle|\, (z-L)^{-1} \varphi \right\rangle$$

which is analytic on the lower half plane. For $\mathrm{Im}(z) \leq -2$ we may rewrite

$$f(z) = \left\langle \varphi_{\overline{\theta}} \,\middle|\, (z-L_\theta)^{-1} \varphi_\theta \right\rangle$$

using Lemma 2.8. Due to the spectral properties of L_θ the function f has an analytic continuation on the domain $(a,b) + i(-\infty, c')$ for a positive constant c'. Hence, we have

$$\sup_{0<\epsilon<c'} \int_a^b dx \, \left|\mathrm{Im} \left\langle \varphi \,\middle|\, (x+i\epsilon-L)^{-1} \varphi \right\rangle\right| = \sup_{0<\epsilon<c'} \int_a^b dx \, |\mathrm{Im}\, f(x+i\epsilon)| < \infty.$$

Since the vector φ was chosen from the dense set $\mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$ we can apply [39, Thm. XIII.20] and obtain that (a,b) is contained in the absolutely continuous part of the spectrum.

It remains to show that zero is a simple eigenvalue of L in the case $\beta_{\max} = \beta_{\min}$. Since, under this extra condition, the standard Liouville operator L coincides with the C-Liouville operator K we conclude that

$$\Omega := \tilde{\Omega}|_{\beta_{\max}=\beta_{\min}=\beta} = \frac{e^{-\beta L^{(\ell)}/2}\Omega_0}{\left\| e^{-\beta L^{(\ell)}/2}\Omega_0 \right\|} \in \ker(L),$$

hence, zero is an eigenvalue of L. The simplicity follows from (2.37). ■

In this chapter and the subsequent ones the operator $K_\theta^{(s)}$ will be estimated in terms of the operator

$$\begin{aligned} M_{[\theta]} &:= d\Gamma_{\mathrm{gl}}((u,\Sigma,r) \mapsto m_\theta(u)) = \mathrm{Im}(L_{0,\theta}) = \sin(\delta')L_{\mathrm{aux}} + \tau' N_{\mathrm{res}}, \\ m_\theta(u) &:= \sin(\delta')|u| + \tau'. \end{aligned} \tag{3.7}$$

Note that the above definition coincides with the definitions (A.9) and (A.10) for $\theta = (i\delta', i\tau') \in (i\mathbb{R}^+)^2$ such that $\sin(\delta') > 0$.

3.2 Spectrum away from Particle Eigenvalues

In this section we analyze the set $\mathrm{spec}(K_\theta^{(s)}) \cap \overline{\mathcal{S}}$. We will find that the operator $K_\theta^{(s)}$ has no spectrum in $\overline{\mathcal{S}}$.

Proposition 3.4 *For $z \in \overline{\mathcal{S}}$ and $g^2 \ll \rho \sin(\delta') \ll d_{L_\mathrm{p}} = \min_{\substack{e,e' \in \mathrm{spec}(L_\mathrm{p}) \\ e \neq e'}} |e - e'|$, the operator $(K_\theta^{(s)} - z)$ is invertible and its inverse is a bounded operator. Thus, we have*

$$\mathrm{spec}\left(K_\theta^{(s)}\right) \cap \overline{\mathcal{S}} = \emptyset.$$

Proof. We show that the inverse operator of $(K_\theta - z)$ can be expressed as a norm convergent Neumann series,

$$\begin{aligned} &\left(K_\theta^{(s)} - z\right)^{-1} \qquad (3.8) \\ &= \frac{(M_{[\theta]} + \rho)^{1/2}}{L_{0,\theta} - z} \\ &\quad \times \sum_{n=0}^{\infty} \left\{ -g(M_{[\theta]} + \rho)^{-1/2} I_\theta^{(s)} (M_{[\theta]} + \rho)^{-1/2} \frac{M_{[\theta]} + \rho}{L_{0,\theta} - z} \right\}^n (M_{[\theta]} + \rho)^{-1/2}. \end{aligned}$$

To estimate the series we consider two cases. First, we assume that $|\mathrm{Im}(z)| \leq \frac{\sin(\delta')}{2}\rho$ and with (3.5), we conclude that $|\mathrm{Re}(z) - e| \geq \sqrt{(4\rho)^2 - \left(\frac{\sin(\delta')}{2}\rho\right)^2} \geq 3\rho$ for all $e \in \mathrm{spec}(L_\mathrm{p})$. Having in mind that $M_{[\theta]} = \mathrm{Im}(L_{0,\theta}) = \sin(\delta')L_\mathrm{res} + \tau' N_\mathrm{res}$ and $\mathrm{Re}(L_{0,\theta}) = L_\mathrm{p} + \cos(\delta')L_\mathrm{aux}$ for $\theta = (i\delta', i\tau')$ and that L_p, L_res, N_res and L_aux are pairwise commuting self-adjoint operators with $|L_\mathrm{res}| \leq L_\mathrm{aux}$, the application of functional calculus yields

$$\begin{aligned} \left\| \frac{M_{[\theta]} + \rho}{L_{0,\theta} - z} \right\| &\leq \sup_{\substack{0 \leq m \leq \rho, \\ |\ell| \leq m/\sin(\delta'), \\ e \in \mathrm{spec}(L_\mathrm{p})}} \frac{m + \rho}{|e - \mathrm{Re}(z) + \cos(\delta')\ell|} + \sup_{m > \rho} \frac{m + \rho}{|m - \mathrm{Im}(z)|} \\ &\leq \frac{2\rho}{3\rho - \cot(\delta')\rho} + \frac{2\rho}{\rho - \frac{\sin(\delta')}{2}\rho} \leq 8, \end{aligned}$$

because $\frac{\pi}{8} \leq \delta' \leq \frac{\pi}{4}$. The second case considers $\mathrm{Im}(z) < -\frac{\sin(\delta')}{2}\rho$,

$$\left\| \frac{M_{[\theta]} + \rho}{L_{0,\theta} - z} \right\| \leq \sup_{m \geq 0} \frac{m + \rho}{|m - \mathrm{Im}(z)|} \leq \sup_{m \geq 0} \frac{m + \rho}{m + \frac{\sin(\delta')}{2}\rho} = \frac{2}{\sin(\delta')} < 8.$$

Together with Lemma A.5, we see that

$$\left\| g(M_{[\theta]} + \rho)^{-1/2} I_\theta^{(s)} (M_{[\theta]} + \rho)^{-1/2} \frac{M_{[\theta]} + \rho}{L_{0,\theta} - z} \right\| \leq C \frac{g}{\sqrt{\rho \sin(\delta')}} \leq \frac{1}{2},$$

for a positive constant $C < \infty$. This ensures the norm convergence of the series

(3.8) and we end up with the following estimate for the resolvent,

$$\left\| \left(K_\theta^{(s)} - z \right)^{-1} \right\| \leq \frac{16}{\rho}.$$

■

3.3 Spectrum in the Neighborhood of Particle Eigenvalues

We go over to study the set $\mathrm{spec}(K_\theta) \cap \mathcal{S}_e$ for a particle eigenvalue $e \in \mathrm{spec}(L_\mathrm{p})$. Choosing $z \in \mathcal{S}_e$, close to an eigenvalue e of the unperturbed Liouville operator $L_{0,\theta}$, it is not possible to prove invertibility of $(K_\theta - z)$ with the help of a Neumann series expansion. We apply the *smooth Feshbach map* as introduced in [4], c.f. Appendix E, to study spectral properties of $(K_\theta - z)$. This results into the Theorem 3.19 and Proposition 3.24.

To apply the smooth Feshbach map we first choose a smooth cutoff function $\Theta : \mathbb{R}_0^+ \to [0,1]$ with

$$\mathrm{supp}(\Theta) = [0,1], \qquad \text{and} \qquad \Theta(x) = 1 \iff x \in \left[0, \frac{7}{8}\right]. \tag{3.9}$$

We use the function Θ to define a smooth partition of the one,

$$\chi_\rho^2(x) + \overline{\chi}_\rho^2(x) = 1 \qquad \forall\, x \in \mathbb{R}_0^+,$$

where

$$\begin{aligned} \chi_\rho(x) &:= \sin\left(\frac{\pi}{2}\Theta(\rho^{-1}x)\right), \\ \overline{\chi}_\rho(x) &:= \cos\left(\frac{\pi}{2}\Theta(\rho^{-1}x)\right), \end{aligned} \tag{3.10}$$

are smooth functions $\chi_\rho, \overline{\chi}_\rho : \mathbb{R}_0^+ \to [0,1]$ with

$$\begin{aligned} \mathrm{supp}(\chi_\rho) &= [0,\rho], \qquad \chi_\rho(x) = 1 \iff x \in \left[0, \frac{7}{8}\rho\right], \qquad \text{and} \\ \mathrm{supp}(\overline{\chi}_\rho) &= \left[\frac{7}{8}\rho, \infty\right), \qquad \overline{\chi}_\rho(x) = 1 \iff x \in [\rho, \infty). \end{aligned} \tag{3.11}$$

The parameter $\rho > 0$ plays the role of a cutoff parameter. We introduce via spectral calculus a smooth "projection" operator

$$\Xi_{e,\rho} := \chi_\rho\left(|L_\mathrm{p} - e| \otimes \mathbb{1}_{\mathcal{F}(L^2[\Upsilon])} + \mathbb{1}_{\mathcal{H}_\mathrm{p}^2} \otimes M_{[\theta]} \right) \in \mathcal{B}\left(\mathcal{H}^2\right),$$

where, recall, $M_{[\theta]} = \mathrm{Im}(L_{0,\theta}) = \sin(\delta')L_{\mathrm{aux}} + \tau' N_{\mathrm{res}}$. Because of (3.3), the operator $\Xi_{e,\rho}$ can also be expressed as

$$\Xi_{e,\rho} = P_{[L_{\mathrm{p}}=e]} \otimes \chi_\rho(M_{[\theta]}) \tag{3.12}$$

with $P_{[L_{\mathrm{p}}=e]}$ being the (sharp) orthogonal projection on the eigenspace of L_{p} corresponding to the eigenvalue e. We further define a $\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)$-valued function

$$X_{e,\rho} : u \mapsto \chi_\rho(u) P_{[L_{\mathrm{p}}=e]}$$

and write $M_{[\theta]} = \int_0^\infty u dP(u)$ in its spectral representation. This enables us to rewrite

$$\Xi_{e,\rho} = X_{e,\rho}(M_{[\theta]}) \equiv \int_0^\infty X_{e,\rho}(u) \otimes dP(u)$$

in the sense of generalized spectral calculus. We also introduce the complementary smooth "projector",

$$\begin{aligned}
\overline{\Xi}_{e,\rho} &:= \sqrt{\mathbb{1} - \Xi_{e,\rho}^2} = \overline{\chi}_\rho \left(|L_{\mathrm{p}} - e| \otimes \mathbb{1}_{\mathcal{F}(L^2[\Upsilon])} + \mathbb{1}_{\mathcal{H}_{\mathrm{p}}^2} \otimes M_{[\theta]} \right) \\
&= \sum_{j,k=0}^{N-1} \int_0^\infty \overline{\chi}_\rho(|E_{j,k} - e| + u) \left|\varphi_{j,k}\right\rangle \left\langle \varphi_{j,k}\right| \otimes dP(u) \\
&= \int_0^\infty \overline{X}_{e,\rho}(u) \otimes dP(u) \equiv \overline{X}_{e,\rho}(M_{[\theta]}),
\end{aligned}$$

where the $\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)$-valued function $\overline{X}_{e,\rho}$ is given by

$$\overline{X}_{e,\rho} : u \mapsto \sum_{j,k=0}^{N-1} \overline{\chi}_\rho(|E_{j,k} - e| + u) \left|\varphi_{j,k}\right\rangle \left\langle \varphi_{j,k}\right| .$$

The relation (3.12) allows us to write

$$\overline{\Xi}_{e,\rho} = \sqrt{P_{[L_{\mathrm{p}}=e]}^\perp \otimes \mathbb{1}_{\mathcal{F}(L^2[\Upsilon])} + P_{[L_{\mathrm{p}}=e]} \otimes \overline{\chi}_\rho^2(M_{[\theta]})}\,,$$

where $P_{[L_{\mathrm{p}}=e]}^\perp := \mathbb{1} - P_{[L_{\mathrm{p}}=e]} = P_{[L_{\mathrm{p}}\neq e]}$ is the complementary projection w.r.t. $P_{[L_{\mathrm{p}}=e]}$.

We conclude this notational part by introducing orthogonal projections on the range of the operators $\Xi_{e,\rho}$ and $\overline{\Xi}_{e,\rho}$. We define

$$\begin{aligned}
&P_{e,\rho} : \text{ orthogonal projection on ran}\,(\Xi_{e,\rho}), && P_{e,\rho}^\perp := \mathbb{1} - P_{e,\rho}, \\
&\overline{P}_{e,\rho} : \text{ orthogonal projection on ran}\left(\overline{\Xi}_{e,\rho}\right), && \overline{P}_{e,\rho}^\perp := \mathbb{1} - \overline{P}_{e,\rho},
\end{aligned}$$

and note that

$$\begin{aligned}
P_{e,\rho} &= P_{\left[|L_{\mathrm{p}}-e|\otimes \mathbb{1}_{\mathcal{F}(L^2[\Upsilon])}+\mathbb{1}_{\mathcal{H}_{\mathrm{p}}^2}\otimes M_{[\theta]}<\rho\right]} \\
&= P_{[L_{\mathrm{p}}=e]}\otimes P_{\left[M_{[\theta]}<\rho\right]}, \\
\overline{P}_{e,\rho} &= P_{\left[|L_{\mathrm{p}}-e|\otimes \mathbb{1}_{\mathcal{F}(L^2[\Upsilon])}+\mathbb{1}_{\mathcal{H}_{\mathrm{p}}^2}\otimes M_{[\theta]}>\frac{7}{8}\rho\right]}, \\
P_{e,\rho}^{\perp} &= P_{[L_{\mathrm{p}}\neq e]}\otimes P_{\left[M_{[\theta]}<\rho\right]} + \mathbb{1}_{\mathcal{H}_{\mathrm{p}}^2}\otimes P_{\left[M_{[\theta]}\geq\rho\right]}, \\
\overline{P}_{e,\rho}^{\perp} &= P_{\left[|L_{\mathrm{p}}-e|\otimes \mathbb{1}_{\mathcal{F}(L^2[\Upsilon])}+\mathbb{1}_{\mathcal{H}_{\mathrm{p}}^2}\otimes M_{[\theta]}\leq\frac{7}{8}\rho\right]} \\
&= P_{[L_{\mathrm{p}}=e]}\otimes P_{\left[M_{[\theta]}\leq\frac{7}{8}\rho\right]}.
\end{aligned}$$

Since the operator $M_{[\theta]}$ has no singular spectrum away from zero, the projections $\overline{P}_{e,\rho}^{\perp}$ and $P_{e,\frac{7}{8}\rho}$ coincide and therefore

$$\overline{P}_{e,\rho} = P_{e,\frac{7}{8}\rho}^{\perp} = P_{[L_{\mathrm{p}}\neq e]}\otimes P_{\left[M_{[\theta]}<\frac{7}{8}\rho\right]} + \mathbb{1}_{\mathcal{H}_{\mathrm{p}}^2}\otimes P_{\left[M_{[\theta]}\geq\frac{7}{8}\rho\right]}. \tag{3.13}$$

3.3.1 First Application of the Smooth Feshbach Map

We use the freshly introduced notation to define the first application of the smooth Feshbach map, discussed in Appendix E, to the operator family

$$K_\theta^{(s)} = L_{0,\theta} + gI_\theta^{(s)}, \qquad s\in\mathbb{S}_{\varepsilon_0}.$$

Considering the operator $L_{0,\theta}$ as the unperturbed part which commutes with $\Xi_{e,\rho}$ and $\overline{\Xi}_{e,\rho}$ and $gI_\theta^{(s)}$ as a perturbation we get for each $z\in\mathcal{S}_e$ a $\Xi_{e,\rho}$-Feshbach pair $(K_\theta^{(s)}-z, L_{0,\theta}-z)$. To this pair we can apply the smooth Feshbach map $\mathfrak{F}_{\Xi_{e,\rho}}$,

$$\begin{aligned}
\mathfrak{F}_{\Xi_{e,\rho}}\left(K_\theta^{(s)}-z, L_{0,\theta}-z\right) \;=\;& L_{0,\theta}-z+g\Xi_{e,\rho}I_\theta^{(s)}\Xi_{e,\rho} \\
& -g^2\Xi_{e,\rho}I_\theta^{(s)}\overline{\Xi}_{e,\rho}\left(K_\theta^{(s)}-z\right)_{\overline{\Xi}_{e,\rho}}^{-1}\overline{\Xi}_{e,\rho}I_\theta^{(s)}\Xi_{e,\rho},
\end{aligned} \tag{3.14}$$

where

$$\left(K_\theta^{(s)}-z\right)_{\overline{\Xi}_{e,\rho}} = \left[L_{0,\theta}-z+g\overline{\Xi}_{e,\rho}I_\theta^{(s)}\overline{\Xi}_{e,\rho}\right]\restriction_{\mathrm{ran}\left(\overline{P}_{e,\rho}\right)}$$

is the operator of interest with the perturbation regularized in a spectral neighborhood of the particle eigenvalue e. The image (3.14) under the Feshbach map has to be understood as an operator on $\mathrm{ran}\,(\Xi_{e,\rho}) = \mathrm{ran}\,(P_{e,\rho})$. The endeavor of what follows is to show that (3.14) is well defined, i.e., that the resolvent $\left(K_\theta^{(s)}-z\right)_{\overline{\Xi}_{e,\rho}}^{-1}$ exists, and that (3.14) defines a bounded operator on $\mathrm{ran}\,(P_{e,\rho})$.

Lemma 3.5 *If $g^2 \ll \rho \sin(\delta') \ll d_{L_\mathrm{p}}$ and $z \in \mathcal{S}_e$ than the operator $\left(K_\theta^{(s)} - z\right)_{\Xi_{e,\rho}}$ is invertible on* ran $\left(\Xi_{e,\rho}\right)$ *and its inverse is bounded.*

Proof. We expand in a Neumann series,

$$\begin{aligned}
&\overline{P}_{e,\rho}\left(K_\theta^{(s)} - z\right)_{\Xi_{e,\rho}}^{-1}\overline{P}_{e,\rho} \qquad (3.15)\\
&= \overline{P}_{e,\rho}\frac{(M_{[\theta]}+\rho)^{1/2}}{L_{0,\theta}-z}\\
&\quad\times\sum_{n=0}^{\infty}\left\{-g\Xi_{e,\rho}(M_{[\theta]}+\rho)^{-1/2}I_\theta^{(s)}(M_{[\theta]}+\rho)^{-1/2}\Xi_{e,\rho}\frac{M_{[\theta]}+\rho}{L_{0,\theta}-z}\right\}^n\\
&\quad\times(M_{[\theta]}+\rho)^{-1/2}\overline{P}_{e,\rho}\,.
\end{aligned}$$

We prove the norm convergence of this series by estimating the terms separately. First, we decompose $\overline{P}_{e,\rho} = P_1 + P_2$ corresponding to (3.13) where

$$P_1 := P_{[L_\mathrm{p}\neq e]} \otimes P_{\left[M_{[\theta]}<\frac{7}{8}\rho\right]}, \qquad P_2 := \mathbb{1}_{\mathcal{H}_\mathrm{p}^2} \otimes P_{\left[M_{[\theta]}\geq\frac{7}{8}\rho\right]}$$

and consider

$$\left\|\frac{M_{[\theta]}+\rho}{L_{0,\theta}-z}\Xi_{e,\rho}\right\| \leq \left\|\frac{M_{[\theta]}+\rho}{L_{0,\theta}-z}\overline{P}_{e,\rho}\right\| \leq \left\|\frac{M_{[\theta]}+\rho}{L_{0,\theta}-z}P_1\right\| + \left\|\frac{M_{[\theta]}+\rho}{L_{0,\theta}-z}P_2\right\|. \qquad (3.16)$$

These operator norms can be estimated via functional calculus using that L_p, L_res, N_res and L_aux are pairwise commuting self-adjoint operators. Since $|L_\mathrm{res}| \leq L_\mathrm{aux}$, we get

$$\begin{aligned}
\left\|\frac{M_{[\theta]}+\rho}{L_{0,\theta}-z}P_1\right\| &\leq \sup_{\substack{m<\frac{7}{8}\rho,\\ e'\in\mathrm{spec}(L_\mathrm{p})\setminus\{e\},\\ |\ell|\leq m/\sin(\delta')}} \frac{m+\rho}{|e'+\ell\cos(\delta')-\mathrm{Re}(z)|}\\
&\leq \sup_{e'\in\mathrm{spec}(L_\mathrm{p})\setminus\{e\}} \frac{\frac{15}{8}\rho}{|e'-e|-|e-\mathrm{Re}(z)|-\frac{7}{8}\rho\cot(\delta')}\\
&\leq \frac{\frac{15}{8}\rho}{d_{L_\mathrm{p}}-4\rho-\frac{7}{8}\rho\cot\delta'} \leq \frac{15}{4d_{L_\mathrm{p}}}\rho,
\end{aligned}$$

where we used that $\rho\cot(\delta') \ll d_{L_\mathrm{p}}$ and ρ sufficiently small such that $d_{L_\mathrm{p}} - 4\rho - \frac{7}{8}\rho\cot(\delta') > \frac{d_{L_\mathrm{p}}}{2}$. The second norm in (3.16) is treated similarly,

$$\begin{aligned}
\left\|\frac{M_{[\theta]}+\rho}{L_{0,\theta}-z}P_2\right\| &\leq \sup_{m\geq\frac{7}{8}\rho}\frac{m+\rho}{|m-\mathrm{Im}(z)|} = \sup_{m\geq\frac{7}{8}\rho}\left[1+\frac{\rho+\mathrm{Im}(z)}{m-\mathrm{Im}(z)}\right]\\
&\leq 1+\frac{\rho+\frac{\sin(\delta')}{2}\rho}{\frac{7}{8}\rho-\frac{\sin(\delta')}{2}\rho} \leq 5.
\end{aligned}$$

Altogether, we get

$$\left\| \frac{M_{[\theta]} + \rho}{L_{0,\theta} - z} \overline{\Xi}_{e,\rho} \right\| \leq 6. \tag{3.17}$$

Further, we estimate

$$\left\| (M_{[\theta]} + \rho)^{-1/2} I_\theta^{(s)} (M_{[\theta]} + \rho)^{-1/2} \right\| \leq \frac{C}{\sqrt{\rho \sin(\delta')}},$$

for a positive constant $C < \infty$, where we made use of the relative bounds on the perturbation $I_\theta^{(s)}$ provided in Lemma A.5. Finally, we have

$$\left\| g \overline{\Xi}_{e,\rho} (M_{[\theta]} + \rho)^{-1/2} I_\theta^{(s)} (M_{[\theta]} + \rho)^{-1/2} \overline{\Xi}_{e,\rho} \frac{M_{[\theta]} + \rho}{L_{0,\theta} - z} \right\| \leq 7C \frac{g}{\sqrt{\rho \sin(\delta')}} \leq \frac{1}{2} \tag{3.18}$$

for $g^2 \ll \rho \sin(\delta')$ which ensures the norm convergence of the Neumann series (3.15). Thus, $\left(K_\theta^{(s)} - z\right)_{\overline{\Xi}_{e,\rho}}$ is invertible with

$$\left\| \overline{P}_{e,\rho} \left(K_\theta^{(s)} - z\right)_{\overline{\Xi}_{e,\rho}}^{-1} \overline{P}_{e,\rho} \right\| \leq \frac{12}{\rho}.$$

■

We can conclude with the help of Lemma A.5 that $\Xi_{e,\rho} I_\theta^{(s)} \Xi_{e,\rho}$, $\Xi_{e,\rho} I_\theta^{(s)} \overline{\Xi}_{e,\rho}$ and $\overline{\Xi}_{e,\rho} I_\theta^{(s)} \Xi_{e,\rho}$ all extend to bounded operators on $\mathrm{ran}(\Xi_{e,\rho})$. Therefore, the operators $K_\theta^{(s)} - z$ and $L_{0,\theta} - z$ build a $\Xi_{e,\rho}$-Feshbach pair in the sense of Appendix E, and in particular the operator (3.14) is well defined as an element of $\mathcal{B}(\mathrm{ran}(\Xi_{e,\rho}))$. Theorem E.1 then provides a spectral link between $K_\theta^{(s)}$ and its image under the Feshbach map $\mathfrak{F}_{\Xi_{e,\rho}}$.

Proposition 3.6 (Isospectral Link) *The operator $\mathfrak{F}_{\Xi_{e,\rho}}(K_\theta^{(s)} - z, L_{0,\theta} - z)$ has the same spectral properties as the original operator $K_\theta^{(s)}$ in the sense that*

$$\begin{aligned} & z \in \mathrm{spec}\left(K_\theta^{(s)}\right) \cap \mathcal{S}_e \\ \iff\quad & 0 \in \mathrm{spec}\left(\mathfrak{F}_{\Xi_{e,\rho}}\left(K_\theta^{(s)} - z, L_{0,\theta} - z\right)\right) \quad \text{and} \quad z \in \mathcal{S}_e. \end{aligned} \tag{3.19}$$

Proposition 3.6 suggests to study the spectrum of the operator (3.14). The remaining part of this chapter is devoted to this task. A main result is the following proposition.

Proposition 3.7 *Let $z \in \mathcal{S}_e$ and $s \in \mathbb{S}_{\varepsilon_0}$. For $\theta = (i\delta', i\tau')$ as chosen in (3.1) and $g^2 \ll \rho \sin(\delta') \ll d_{L_p}$, the operator $\mathfrak{F}_{\Xi_{e,\rho}}(K_\theta^{(s)} - z, L_{0,\theta} - z)$ defined in (3.14) is of the form*

$$\begin{aligned} &\mathfrak{F}_{\Xi_{e,\rho}}(K_\theta^{(s)} - z, L_{0,\theta} - z) \qquad (3.20)\\ &= P_{e,\rho}\left[L_{0,\theta} - z + g^2 \Lambda_e^{(s)} \otimes \chi_\rho^2(M_{[\theta]})\right] P_{e,\rho} + g^2 \mathcal{O}\left(g^{-1}\rho^{\nu+1/2} + g\rho^{-1/2} + \rho^\nu\right), \end{aligned}$$

where the level shift operator $\Lambda_e^{(s)} \in \mathcal{B}(\mathrm{ran}(P_{[L_p=e]}))$ *is given by*

$$\begin{aligned} \Lambda_e^{(s)} &:= -\lim_{\varepsilon \searrow 0} P_{[L_p=e]} \int_\Upsilon dy \left[\mathcal{G}(y) - \mathcal{G}'_{(\bar{s}\delta\vec{\beta})}(y)\right]^* \qquad (3.21)\\ &\qquad \times (L_p - e + u + i\varepsilon)^{-1}\left[\mathcal{G}(y) - \mathcal{G}'_{(s\delta\vec{\beta})}(y)\right] P_{[L_p=e]} \end{aligned}$$

and the remainder term $g^2\mathcal{O}\left(g^{-1}\rho^{\nu+1/2} + g\rho^{-1/2} + \rho^\nu\right) = \mathcal{O}\left(g^{2+\tilde{\varepsilon}}\right)$ is estimated uniformly in $z \in \mathcal{S}_e$ and uniformly in the inverse temperatures. Hereby, the number $\nu \geq 1$ is the exponent of the infrared regularization of the glued coupling function $\mathcal{G}$ introduced in (1.92), Hypothesis VII-1.12.

Proof. We first observe that due to Lemma A.5

$$g\left\|\Xi_{e,\rho} I_\theta^{(s)} \Xi_{e,\rho}\right\| \leq g\left\|P_{e,\rho} I_\theta^{(s)} P_{e,\rho}\right\| = \mathcal{O}\left(g\rho^{\nu+1/2}\right)$$

the operator (3.14) can be written as

$$\begin{aligned} &\mathfrak{F}_{\Xi_{e,\rho}}(K_\theta^{(s)} - z, L_{0,\theta} - z)\\ &= P_{e,\rho}\left[L_{0,\theta} - z - g^2 \Xi_{e,\rho} I_\theta^{(s)} \overline{\Xi}_{e,\rho}\left(K_\theta^{(s)} - z\right)^{-1}_{\overline{\Xi}_{e,\rho}} \overline{\Xi}_{e,\rho} I_\theta^{(s)} \Xi_{e,\rho}\right] P_{e,\rho}\\ &\quad + g^2\mathcal{O}\left(g^{-1}\rho^{\nu+1/2}\right). \qquad (3.22) \end{aligned}$$

Hence, it remains to consider the resolvent term in (3.22). The term is – up to small errors – the level shift operator. The remaining part of the proof is tackled step by step by elaborating the following Lemmata 3.8 – 3.14. ■

In order to estimate the resolvent in (3.22) we introduce some notation,

$$\begin{aligned} \left[V_\theta\right]_{\mathfrak{c}} &:= a^*_{\mathrm{gl}}\left(\mathcal{G}_\theta\right), & \left[V_\theta\right]_{\mathfrak{a}} &= a_{\mathrm{gl}}\left(\mathcal{G}_{\overline{\theta}}\right),\\ \left[W_\theta^{(s)}\right]_{\mathfrak{c}} &:= a^*_{\mathrm{gl}}\left(\mathcal{G}'_{(s\delta\vec{\beta}),\theta}\right), & \left[W_\theta^{(s)}\right]_{\mathfrak{a}} &:= a_{\mathrm{gl}}\left(\mathcal{G}'_{(\bar{s}\delta\vec{\beta}),\theta}\right), \qquad (3.23)\\ \left[I_\theta^{(s)}\right]_{\mathfrak{c}} &:= \left[V_\theta\right]_{\mathfrak{c}} - \left[W_\theta^{(s)}\right]_{\mathfrak{c}}, & \left[I_\theta^{(s)}\right]_{\mathfrak{a}} &:= \left[V_\theta\right]_{\mathfrak{a}} - \left[W_\theta^{(s)}\right]_{\mathfrak{a}}. \end{aligned}$$

The indices $\mathfrak{c}$ and $\mathfrak{a}$ stand for creation and annihilation part, resp.

Lemma 3.8 *On* $\mathrm{ran}(P_{e,\rho})$, *we have*

$$\begin{aligned}&\Xi_{e,\rho} I_\theta^{(s)} \overline{\Xi}_{e,\rho} \left(K_\theta^{(s)} - z\right)^{-1}_{\overline{\Xi}_{e,\rho}} \overline{\Xi}_{e,\rho} I_\theta^{(s)} \Xi_{e,\rho} \\ &= \Xi_{e,\rho} \left[I_\theta^{(s)}\right]_{\mathfrak{a}} \overline{\Xi}_{e,\rho} \left(K_\theta^{(s)} - z\right)^{-1}_{\overline{\Xi}_{e,\rho}} \overline{\Xi}_{e,\rho} \left[I_\theta^{(s)}\right]_{\mathfrak{c}} \Xi_{e,\rho} + \mathcal{O}\left(\rho^\nu\right).\end{aligned}$$

Proof. We start the proof observing that

$$\begin{aligned}&\Xi_{e,\rho} I_\theta^{(s)} \overline{\Xi}_{e,\rho} \left(K_\theta^{(s)} - z\right)^{-1}_{\overline{\Xi}_{e,\rho}} \overline{\Xi}_{e,\rho} I_\theta^{(s)} \Xi_{e,\rho} \\ &\quad - \Xi_{e,\rho} \left[I_\theta^{(s)}\right]_{\mathfrak{a}} \overline{\Xi}_{e,\rho} \left(K_\theta^{(s)} - z\right)^{-1}_{\overline{\Xi}_{e,\rho}} \overline{\Xi}_{e,\rho} \left[I_\theta^{(s)}\right]_{\mathfrak{c}} \Xi_{e,\rho} \\ &= \Xi_{e,\rho} \left[I_\theta^{(s)}\right]_{\mathfrak{c}} \overline{\Xi}_{e,\rho} \left(K_\theta^{(s)} - z\right)^{-1}_{\overline{\Xi}_{e,\rho}} \overline{\Xi}_{e,\rho} \left[I_\theta^{(s)}\right]_{\mathfrak{a}} \Xi_{e,\rho} \\ &\quad + \Xi_{e,\rho} \left[I_\theta^{(s)}\right]_{\mathfrak{a}} \overline{\Xi}_{e,\rho} \left(K_\theta^{(s)} - z\right)^{-1}_{\overline{\Xi}_{e,\rho}} \overline{\Xi}_{e,\rho} \left[I_\theta^{(s)}\right]_{\mathfrak{a}} \Xi_{e,\rho} \\ &\quad + \Xi_{e,\rho} \left[I_\theta^{(s)}\right]_{\mathfrak{c}} \overline{\Xi}_{e,\rho} \left(K_\theta^{(s)} - z\right)^{-1}_{\overline{\Xi}_{e,\rho}} \overline{\Xi}_{e,\rho} \left[I_\theta^{(s)}\right]_{\mathfrak{c}} \Xi_{e,\rho}. \qquad (3.24)\end{aligned}$$

We compute representatively for all addends in (3.24),

$$\begin{aligned}&\left\| \Xi_{e,\rho} \left[I_\theta^{(s)}\right]_{\mathfrak{a}} \overline{\Xi}_{e,\rho} \left(K_\theta^{(s)} - z\right)^{-1}_{\overline{\Xi}_{e,\rho}} \overline{\Xi}_{e,\rho} \left[I_\theta^{(s)}\right]_{\mathfrak{a}} \Xi_{e,\rho} \right\| \\ &= \left\| \Xi_{e,\rho} \left[I_\theta^{(s)}\right]_{\mathfrak{a}} (M_{[\theta]} + \rho)^{-1/2} \frac{(M_{[\theta]} + \rho)\overline{\Xi}_{e,\rho}}{L_{0,\theta} - z} \right. \\ &\qquad \times \sum_{n=0}^{\infty} \left\{ -g \overline{\Xi}_{e,\rho} (M_{[\theta]} + \rho)^{-1/2} I_\theta^{(s)} (M_{[\theta]} + \rho)^{-1/2} \overline{\Xi}_{e,\rho} \frac{M_{[\theta]} + \rho}{L_{0,\theta} - z} \right\}^n \\ &\qquad \left. \times (M_{[\theta]} + \rho)^{-1/2} \overline{\Xi}_{e,\rho} \left[I_\theta^{(s)}\right]_{\mathfrak{a}} \Xi_{e,\rho} \right\| \\ &\leq \left\| \Xi_{e,\rho} (M_{[\theta]} + \rho)^{1/2} \right\| \left\| (M_{[\theta]} + \rho)^{-1/2} \left[I_\theta^{(s)}\right]_{\mathfrak{a}} (M_{[\theta]} + \rho)^{-1/2} \right\| \left\| \frac{(M_{[\theta]} + \rho)\overline{\Xi}_{e,\rho}}{L_{0,\theta} - z} \right\| \\ &\qquad \times \sum_{n=0}^{\infty} \left\| g \overline{\Xi}_{e,\rho} (M_{[\theta]} + \rho)^{-1/2} I_\theta^{(s)} (M_{[\theta]} + \rho)^{-1/2} \overline{\Xi}_{e,\rho} \frac{M_{[\theta]} + \rho}{L_{0,\theta} - z} \right\|^n \\ &\qquad \times \left\| (M_{[\theta]} + \rho)^{-1/2} \overline{\Xi}_{e,\rho} \right\| \left\| \left[I_\theta^{(s)}\right]_{\mathfrak{a}} \Xi_{e,\rho} \right\| \\ &= \mathcal{O}\left(\rho^\nu\right),\end{aligned}$$

where the estimates (3.17), (3.18), (A.14) and (A.16) enter in the last line. ∎

In the next step we replace the perturbed by the unperturbed resolvent.

Lemma 3.9 *We have*

$$\begin{aligned}&\Xi_{e,\rho}\left[I_\theta^{(s)}\right]_{\mathfrak{a}}\overline{\Xi}_{e,\rho}\left(K_\theta^{(s)}-z\right)^{-1}_{\overline{\Xi}_{e,\rho}}\overline{\Xi}_{e,\rho}\left[I_\theta^{(s)}\right]_{\mathfrak{c}}\Xi_{e,\rho}\\ &= \Xi_{e,\rho}\left[I_\theta^{(s)}\right]_{\mathfrak{a}}\overline{\Xi}_{e,\rho}\left(L_{0,\theta}\overline{P}_{e,\rho}-z\right)^{-1}\overline{\Xi}_{e,\rho}\left[I_\theta^{(s)}\right]_{\mathfrak{c}}\Xi_{e,\rho}+\mathcal{O}\left(g\rho^{-1/2}\right).\end{aligned}$$

Proof. We compute the difference

$$\begin{aligned}&\Big\|\Xi_{e,\rho}\left[I_\theta^{(s)}\right]_{\mathfrak{a}}\overline{\Xi}_{e,\rho}\left(K_\theta^{(s)}-z\right)^{-1}_{\overline{\Xi}_{e,\rho}}\overline{\Xi}_{e,\rho}\left[I_\theta^{(s)}\right]_{\mathfrak{c}}\Xi_{e,\rho}\\ &\quad-\Xi_{e,\rho}\left[I_\theta^{(s)}\right]_{\mathfrak{a}}\overline{\Xi}_{e,\rho}\left(L_{0,\theta}\overline{P}_{e,\rho}-z\right)^{-1}\overline{\Xi}_{e,\rho}\left[I_\theta^{(s)}\right]_{\mathfrak{c}}\Xi_{e,\rho}\Big\|\\ &= g\left\|\Xi_{e,\rho}\left[I_\theta^{(s)}\right]_{\mathfrak{a}}\overline{\Xi}_{e,\rho}\left(K_\theta^{(s)}-z\right)^{-1}_{\overline{\Xi}_{e,\rho}}\overline{\Xi}_{e,\rho}I_\theta^{(s)}\overline{\Xi}_{e,\rho}\left(L_{0,\theta}\overline{P}_{e,\rho}-z\right)^{-1}\overline{\Xi}_{e,\rho}\left[I_\theta^{(s)}\right]_{\mathfrak{c}}\Xi_{e,\rho}\right\|\\ &\le g\left\|\Xi_{e,\rho}(M_{[\theta]}+\rho)^{1/2}\right\|\left\|(M_{[\theta]}+\rho)^{-1/2}\left[I_\theta^{(s)}\right]_{\mathfrak{a}}(M_{[\theta]}+\rho)^{-1/2}\right\|\left\|\frac{(M_{[\theta]}+\rho)\overline{\Xi}_{e,\rho}}{L_{0,\theta}-z}\right\|\\ &\quad\times\sum_{n=0}^{\infty}\left\|g\overline{\Xi}_{e,\rho}(M_{[\theta]}+\rho)^{-1/2}I_\theta^{(s)}(M_{[\theta]}+\rho)^{-1/2}\overline{\Xi}_{e,\rho}\frac{M_{[\theta]}+\rho}{L_{0,\theta}-z}\right\|^n\\ &\quad\times\left\|\overline{\Xi}_{e,\rho}(M_{[\theta]}+\rho)^{-1/2}I_\theta^{(s)}(M_{[\theta]}+\rho)^{-1/2}\right\|\left\|\frac{(M_{[\theta]}+\rho)\overline{\Xi}_{e,\rho}^2}{L_{0,\theta}-z}\right\|\\ &\quad\times\left\|(M_{[\theta]}+\rho)^{-1/2}\left[I_\theta^{(s)}\right]_{\mathfrak{c}}(M_{[\theta]}+\rho)^{-1/2}\right\|\left\|(M_{[\theta]}+\rho)^{1/2}\Xi_{e,\rho}\right\|\\ &= \mathcal{O}\left(g\rho^{-1/2}\right).\end{aligned}$$

■

Next, we show that the leading contributions to the level shift operator are the level shift operators for the single reservoirs and that reservoir correlations can be neglected.

Lemma 3.10 *We have*

$$\begin{aligned}&\Xi_{e,\rho}\left[I_\theta^{(s)}\right]_{\mathfrak{a}}\overline{\Xi}_{e,\rho}\left(L_{0,\theta}\overline{P}_{e,\rho}-z\right)^{-1}\overline{\Xi}_{e,\rho}\left[I_\theta^{(s)}\right]_{\mathfrak{c}}\Xi_{e,\rho}\\ &= \Xi_{e,\rho}\sum_{r=1}^{R}a_{\mathrm{gl},r}\left(F_{\overline{\theta}}^{(\overline{s})}(\,\cdot\,,\,\cdot\,,r)\right)\overline{\Xi}_{e,\rho}\left(L_{0,\theta}\overline{P}_{e,\rho}-z\right)^{-1}\overline{\Xi}_{e,\rho}a^*_{\mathrm{gl},r}\left(F_\theta^{(s)}(\,\cdot\,,\,\cdot\,,r)\right)\Xi_{e,\rho}\\ &\quad+\mathcal{O}\left(\rho^\nu\right),\end{aligned}$$

where $a^*_{\mathrm{gl},r}(u,\Sigma) := a^*_{\mathrm{gl}}(u,\Sigma,r)$ *and* $a_{\mathrm{gl},r}(u,\Sigma) := a_{\mathrm{gl}}(u,\Sigma,r)$ *are the creation and annihilation operators for the* r^{th} *reservoir.*

Proof. For $j,k \in \{1,\ldots,R\}$, $j \neq k$, we will show that

$$\begin{aligned} R_{j,k} &:= \Xi_{e,\rho} a_{\mathrm{gl},j}\left(F_{\overline{\theta}}^{(\overline{s})}(\,\cdot\,,\,\cdot\,,j)\right) \overline{\Xi}_{e,\rho}\left(L_{0,\theta}\overline{P}_{e,\rho} - z\right)^{-1} \overline{\Xi}_{e,\rho} a_{\mathrm{gl},k}^{*}\left(F_{\theta}^{(s)}(\,\cdot\,,\,\cdot\,,k)\right)\Xi_{e,\rho} \\ &= \mathcal{O}\left(\rho^{\nu}\right). \end{aligned}$$

To this end we introduce the projections

$$P_\rho^{(\ell)} := P_{\left[M_{[\theta]}^{(\ell)} \leq \rho\right]}, \qquad \ell = j,k,$$

only acting on the ℓ^{th} reservoir, where

$$M_{[\theta]}^{(\ell)} = d\Gamma_{\mathrm{gl}}((u,\Sigma,r) \mapsto \delta_{r,\ell} m_\theta(u))$$

is the part of $M_{[\theta]}$ which acts on the variables of the ℓ^{th} reservoir. As a consequence, we get

$$[P_\rho^{(j)}, a_{\mathrm{gl},k}^*(F)] = 0 \qquad \text{for } j \neq k.$$

Note that

$$\Xi_{e,\rho} = \Xi_{e,\rho} P_\rho^{(k)} = P_\rho^{(j)} \Xi_{e,\rho}$$

and therefore

$$\begin{aligned} R_{j,k} &= \Xi_{e,\rho} a_{\mathrm{gl},j}\left(F_{\overline{\theta}}^{(\overline{s})}(\,\cdot\,,\,\cdot\,,j)\right) P_\rho^{(j)} \overline{\Xi}_{e,\rho} \\ &\quad \times \left(L_{0,\theta}\overline{P}_{e,\rho} - z\right)^{-1} \overline{\Xi}_{e,\rho} P_\rho^{(k)} a_{\mathrm{gl},k}^{*}\left(F_{\theta}^{(s)}(\,\cdot\,,\,\cdot\,,k)\right)\Xi_{e,\rho}. \end{aligned}$$

Taking the norm gives

$$\begin{aligned} \|R_{j,k}\| &\leq \left\| \Xi_{e,\rho} a_{\mathrm{gl},j}\left(\mathbf{1}_{[m_\theta \leq \rho]} F_{\overline{\theta}}^{(\overline{s})}(\,\cdot\,,\,\cdot\,,j)\right)(M_{[\theta]}+\rho)^{-1/2} \right\| \left\| \frac{(M_{[\theta]}+\rho)\overline{\Xi}_{e,\rho}^2}{L_{0,\theta}\overline{P}_{e,\rho} - z} \right\| \\ &\quad \times \left\| (M_{[\theta]}+\rho)^{-1/2} a_{\mathrm{gl},k}^{*}\left(\mathbf{1}_{[m_\theta \leq \rho]} F_{\theta}^{(s)}(\,\cdot\,,\,\cdot\,,k)\right)\Xi_{e,\rho} \right\| \\ &= \mathcal{O}\left(\rho^{\nu}\right), \end{aligned}$$

where we used (3.17), Lemma A.4, (A.11), and arguments elaborated in the proof of Lemma A.5 to see that

$$\begin{aligned} &\left\| \Xi_{e,\rho} a_{\mathrm{gl},j}\left(\mathbf{1}_{[m_\theta \leq \rho]} F_{\overline{\theta}}^{(\overline{s})}(\,\cdot\,,\,\cdot\,,j)\right)(M_{[\theta]}+\rho)^{-1/2} \right\| \\ &\leq \frac{1}{\sin(\delta')} \left[\int\limits_{m_\theta(u) \leq \rho} du \int\limits_{S^2} d\Sigma \frac{\left\| F_{\overline{\theta}}^{(\overline{s})}(u,\Sigma,j) \right\|^2}{|j_\theta(u)|} \right]^{1/2} \\ &\leq \frac{C}{\sin(\delta')} \left[\int\limits_{m_\theta(u) \leq \rho} du |j_\theta(u)|^{2\nu-1} \right]^{1/2} = \mathcal{O}\left(\rho^{\nu}\right), \end{aligned}$$

for a constant C. ∎

The next lemma allows us to integrate out the photon variables.

Lemma 3.11 *We have*

$$\begin{aligned}
&\Xi_{e,\rho}\sum_{r=1}^{R} a_{\mathrm{gl},r}\left(F_{\overline{\theta}}^{(\overline{s})}(\,\cdot\,,\,\cdot\,,r)\right)\overline{\Xi}_{e,\rho}\left(L_{0,\theta}\overline{P}_{e,\rho}-z\right)^{-1}\overline{\Xi}_{e,\rho}a^*_{\mathrm{gl},r}\left(F_{\theta}^{(s)}(\,\cdot\,,\,\cdot\,,r)\right)\Xi_{e,\rho}\\
&=\ \Xi_{e,\rho}\int_{\Upsilon} dy\, F_{\overline{\theta}}^{(\overline{s})}(y)^*\overline{\Xi}_{e,\rho}(u)\left(L_{0,\theta}+j_\theta(u)-z\right)^{-1}\overline{\Xi}_{e,\rho}(u)F_{\theta}^{(s)}(y)\Xi_{e,\rho}+\mathcal{O}\left(\rho^{1+2\nu}\right)
\end{aligned}$$

where

$$\overline{\Xi}_{e,\rho}(u) := \overline{X}_{e,\rho}\left(M_{[\theta]}+m_\theta(u)\right)=\overline{\chi}_\rho\left(|L_{\mathrm{p}}-e|+M_{[\theta]}+m_\theta(u)\right). \tag{3.25}$$

Proof. We are going to apply the pull through formula (1.67). Pulling creation and annihilation operators through the cutoff operator $\overline{\Xi}_{e,\rho}$ gives rise to the operators of the type (3.25) and

$$\overline{\Xi}_{e,\rho}(u,u') := \overline{X}_{e,\rho}\left(M_{[\theta]}+m_\theta(u)+m_\theta(u')\right).$$

With this notation at hand we can compute

$$\begin{aligned}
&\Xi_{e,\rho}\sum_{r=1}^{R} a_{\mathrm{gl},r}\left(F_{\overline{\theta}}^{(\overline{s})}(\,\cdot\,,\,\cdot\,,r)\right)\overline{\Xi}_{e,\rho}\left(L_{0,\theta}\overline{P}_{e,\rho}-z\right)^{-1}\overline{\Xi}_{e,\rho}a^*_{\mathrm{gl},r}\left(F_{\theta}^{(s)}(\,\cdot\,,\,\cdot\,,r)\right)\Xi_{e,\rho}\\
&=\ \Xi_{e,\rho}\int_{\Upsilon} dy\int_{\Upsilon} dy'\,\delta_{r,r'}F_{\overline{\theta}}^{(\overline{s})}(y)^*\otimes a_{\mathrm{gl}}(y)\\
&\qquad\qquad\times\overline{\Xi}_{e,\rho}\left(L_{0,\theta}\overline{P}_{e,\rho}-z\right)^{-1}\overline{\Xi}_{e,\rho}F_{\theta}^{(s)}(y')\otimes a^*_{\mathrm{gl}}(y')\Xi_{e,\rho}\\
&=\ \Xi_{e,\rho}\int_{\Upsilon} dy\, F_{\overline{\theta}}^{(\overline{s})}(y)^*\overline{\Xi}_{e,\rho}(u)\left(L_{0,\theta}+j_\theta(u)-z\right)^{-1}\overline{\Xi}_{e,\rho}(u)F_{\theta}^{(s)}(y)\Xi_{e,\rho}\\
&\quad+\Xi_{e,\rho}\int_{\Upsilon} dy\int_{\Upsilon} dy'\,\delta_{r,r'}F_{\overline{\theta}}^{(\overline{s})}(y)^*\otimes a^*_{\mathrm{gl}}(y')\overline{\Xi}_{e,\rho}(u,u')\\
&\qquad\qquad\times\left(L_{0,\theta}+j_\theta(u)+j_\theta(u')-z\right)^{-1}\overline{\Xi}_{e,\rho}(u,u')F_{\theta}^{(s)}(y')\otimes a_{\mathrm{gl}}(y)\Xi_{e,\rho}.
\end{aligned}$$

To estimate the second term we choose vectors φ, ψ of norm $\|\varphi\| = \|\psi\| = 1$ and compute in a similar way to the proofs of the Lemmata A.4 and A.5,

$$\begin{aligned}
&\left|\left\langle \varphi \,\middle|\, \Xi_{e,\rho} \int_\Upsilon dy \int_\Upsilon dy'\, \delta_{r,r'} F_{\overline{\theta}}^{(\overline{s})}(y)^* \otimes a_{\mathrm{gl}}^*(y') \overline{\Xi}_{e,\rho}(u,u') \left(L_{0,\theta} + j_\theta(u) + j_\theta(u') - z\right)^{-1} \right.\right.\\
&\qquad \left.\left. \times \overline{\Xi}_{e,\rho}(u,u') F_\theta^{(s)}(y') \otimes a_{\mathrm{gl}}(y) \Xi_{e,\rho}\, \psi \right\rangle\right| \\
&\leq \left[\int_\Upsilon dy\, \mathbf{1}_{[m_\theta(u)\leq\rho]} \frac{\left\|F_{\overline{\theta}}^{(\overline{s})}(y)^*\right\|^2}{|j_\theta(u)|} \int_\Upsilon dy'\, \mathbf{1}_{[m_\theta(u')\leq\rho]} \frac{\left\|F_\theta^{(s)}(y')\right\|^2}{|j_\theta(u')|} \right.\\
&\qquad \left. \times\, \delta_{r,r'} \left\| \overline{\Xi}_{e,\rho}(u,u') \left(L_{0,\theta} + j_\theta(u) + j_\theta(u') - z\right)^{-1} \overline{\Xi}_{e,\rho}(u,u') \right\|^2 \right]^{1/2} \\
&\qquad \times \left[\int_\Upsilon dy\, |j_\theta(u)|\, \|a_{\mathrm{gl}}(y)\Xi_{e,\rho}\varphi\|^2 \right]^{1/2} \left[\int_\Upsilon dy\, |j_\theta(u)|\, \|a_{\mathrm{gl}}(y)\Xi_{e,\rho}\psi\|^2 \right]^{1/2} \\
&\leq \sup_{m_\theta(u), m_\theta(u')\leq\rho} \left\| \overline{\Xi}_{e,\rho}^2 \left(L_{0,\theta} + j_\theta(u) + j_\theta(u') - z\right)^{-1} \right\| \\
&\qquad \times \left[\int_\Upsilon dy\, \mathbf{1}_{[m_\theta(u)\leq\rho]} \frac{\left\|F_{\overline{\theta}}^{(\overline{s})}(y)^*\right\|^2}{|j_\theta(u)|} \right]^{1/2} \left[\int_\Upsilon dy'\, \mathbf{1}_{[m_\theta(u')\leq\rho]} \frac{\left\|F_\theta^{(s)}(y')\right\|^2}{|j_\theta(u')|} \right]^{1/2} \\
&\qquad \times \frac{1}{\sin^2(\delta')} \left\|M_{[\theta]}\Xi_{e,\rho}\varphi\right\| \left\|M_{[\theta]}\Xi_{e,\rho}\psi\right\| \\
&= \mathcal{O}\left(\rho^{1+2\nu}\right),
\end{aligned}$$

where we made use of the functional calculus to obtain

$$\begin{aligned}
&\sup_{m_\theta(u), m_\theta(u')\leq\rho} \left\| \overline{\Xi}_{e,\rho}^2 \left(L_{0,\theta} + j_\theta(u) + j_\theta(u') - z\right)^{-1} \right\| \\
&\leq \sup_{m_\theta(u), m_\theta(u')\leq\rho} \left\| \left(L_{0,\theta} + j_\theta(u) + j_\theta(u') - z\right)^{-1} \overline{P}_{e,\rho} \right\| \\
&\leq \sup_{m_\theta(u), m_\theta(u')\leq\rho} \left\| \left(L_{\mathrm{p}} + \cos(\delta')[L_{\mathrm{res}} + u + u'] - \mathrm{Re}(z)\right)^{-1} P_{[L_{\mathrm{p}}\neq e]} \otimes P_{\left[M_{[\theta]}\leq\frac{7}{8}\rho\right]} \right\| \\
&\quad + \sup_{m_\theta(u), m_\theta(u')\leq\rho} \left\| \left(M_{[\theta]} + m_\theta(u) + m_\theta(u') - \mathrm{Im}(z)\right)^{-1} P_{\left[M_{[\theta]}\geq\frac{7}{8}\rho\right]} \right\| \\
&\leq \frac{1}{d_{L_{\mathrm{p}}} - (3\cot(\delta') + 4)\,\rho} + \frac{8}{(7 - 4\sin(\delta'))\rho} = \mathcal{O}\left(\rho^{-1}\right).
\end{aligned}$$

∎

The next task is to replace z by e (recall that $|z-e| \le 4\rho$) and $L_{0,\theta}$ by L_{p}.

Lemma 3.12 *We have*

$$\begin{aligned} &\Xi_{e,\rho} \int_{\Upsilon} dy\, F_{\overline{\theta}}^{(\overline{s})}(y)^* \overline{\Xi}_{e,\rho}(u) \left(L_{0,\theta} + j_\theta(u) - z\right)^{-1} \overline{\Xi}_{e,\rho}(u) F_\theta^{(s)}(y) \Xi_{e,\rho} \qquad (3.26)\\ &= \Xi_{e,\rho} \int_{\Upsilon} dy\, F_{\overline{\theta}}^{(\overline{s})}(y)^* \overline{\Xi}_{e,\rho}(u) \left(L_{\mathrm{p}} + j_\theta(u) - e\right)^{-1} \overline{\Xi}_{e,\rho}(u) F_\theta^{(s)}(y) \Xi_{e,\rho} + \mathcal{O}\left(\rho + \rho^{2\nu}\right). \end{aligned}$$

Proof. We compute the resulting difference term,

$$\begin{aligned} &\Xi_{e,\rho} \int_{\Upsilon} dy\, F_{\overline{\theta}}^{(\overline{s})}(y)^* \overline{\Xi}_{e,\rho}(u) \left(L_{0,\theta} + j_\theta(u) - z\right)^{-1} \overline{\Xi}_{e,\rho}(u) F_\theta^{(s)}(y) \Xi_{e,\rho} \\ &\quad - \Xi_{e,\rho} \int_{\Upsilon} dy\, F_{\overline{\theta}}^{(\overline{s})}(y)^* \overline{\Xi}_{e,\rho}(u) \left(L_{\mathrm{p}} + j_\theta(u) - e\right)^{-1} \overline{\Xi}_{e,\rho}(u) F_\theta^{(s)}(y) \Xi_{e,\rho} \\ &= -\Xi_{e,\rho} \int_{\Upsilon} dy\, F_{\overline{\theta}}^{(\overline{s})}(y)^* \overline{\Xi}_{e,\rho}(u) \left(L_{0,\theta} + j_\theta(u) - z\right)^{-1} \\ &\qquad \times \left[\cos(\delta') L_{\mathrm{res}} + i M_{[\theta]} + e - z\right] \left(L_{\mathrm{p}} + j_\theta(u) - e\right)^{-1} \overline{\Xi}_{e,\rho}(u) F_\theta^{(s)}(y) \Xi_{e,\rho} \\ &= -\Xi_{e,\rho} \int_{\Upsilon} dy\, F_{\overline{\theta}}^{(\overline{s})}(y)^* P_{\left[M_{[\theta]} \le \rho\right]} \overline{P}_{e,\rho}(u) \overline{\Xi}_{e,\rho}(u) \left(L_{0,\theta} + j_\theta(u) - z\right)^{-1} \\ &\qquad \times \left[\cos(\delta') L_{\mathrm{res}} + i M_{[\theta]} + e - z\right] \left(L_{\mathrm{p}} + j_\theta(u) - e\right)^{-1} \\ &\qquad \times \overline{\Xi}_{e,\rho}(u) \overline{P}_{e,\rho}(u) P_{\left[M_{[\theta]} \le \rho\right]} F_\theta^{(s)}(y) \Xi_{e,\rho}, \qquad (3.27) \end{aligned}$$

where $\overline{P}_{e,\rho}(u)$ is the orthogonal projection on $\mathrm{ran}(\overline{\Xi}_{e,\rho}(u))$ given by

$$\overline{P}_{e,\rho}(u) := P_{[L_{\mathrm{p}} \ne e]} \otimes \mathbb{1}_{\mathcal{F}(L^2[\Upsilon])} + P_{[L_{\mathrm{p}} = e]} \otimes P_{\left[M_{[\theta]} + m_\theta(u) \ge \frac{7}{8}\rho\right]}.$$

Introducing two further projections,

$$\begin{aligned} P_1 &:= P_{[L_{\mathrm{p}} \ne e]} \otimes P_{\left[M_{[\theta]} \le \rho\right]}, \\ P_2(u) &:= P_{[L_{\mathrm{p}} = e]} \otimes P_{\left[\frac{7}{8}\rho - m_\theta(u) \le M_{[\theta]} \le \rho\right]}, \end{aligned}$$

we can decompose

$$\overline{P}_{e,\rho}(u) P_{\left[M_{[\theta]} \le \rho\right]} = P_1 + P_2(u).$$

Thus, we compute the integrand in (3.27) on the range of the projections P_1 and $P_2(u)$. First, we consider

$$\begin{aligned}
&\left\| P_1 \left(L_{0,\theta} + j_\theta(u) - z\right)^{-1} \left[\cos(\delta')L_{\mathrm{res}} + iM_{[\theta]} + e - z\right] \left(L_{\mathrm{p}} + j_\theta(u) - e\right)^{-1} P_1 \right\| \\
&\quad \leq \sup_{\substack{e' \in \mathrm{spec}(L_{\mathrm{p}})\setminus\{e\}, \\ 0 \leq m \leq \rho, \\ |\ell| \leq m/\sin(\delta')}} \frac{|e - z| + \cos(\delta')|\ell| + m}{|e' + \cos(\delta')\ell + im + j_\theta(u) - z||e' + j_\theta(u) - e|} \\
&\quad \leq \frac{6\rho}{\sin(\delta')} \sup_{\substack{e' \in \mathrm{spec}(L_{\mathrm{p}})\setminus\{e\}, \\ 0 \leq m \leq \rho, \\ |\ell| \leq m/\sin(\delta')}} \frac{1}{|e' + \cos(\delta')\ell + im + j_\theta(u) - z||e' + j_\theta(u) - e|}.
\end{aligned}$$

To estimate the remaining fraction we distinguish two cases. Assume that $|u| \leq \frac{d_{L_{\mathrm{p}}}}{2}$ which implies that

$$\begin{aligned}
|e' - z + \cos(\delta')\ell + im + j_\theta(u)| &\geq |e' - e + \cos(\delta')\ell + \cos(\delta')u + \mathrm{Re}(e - z)| \\
&\geq |e' - e| - \cos(\delta')(|\ell| + |u|) - |e - z| \\
&\geq \frac{d_{L_{\mathrm{p}}}}{2} + \mathcal{O}(\rho)
\end{aligned}$$

for ℓ being of order ρ, and

$$|e' - e + j_\theta(u)| \geq |e' - e + \cos(\delta')u| \geq \frac{d_{L_{\mathrm{p}}}}{2}.$$

The complementary case, $|u| > \frac{d_{L_{\mathrm{p}}}}{2}$, yields

$$\begin{aligned}
|e' - z + \cos(\delta')\ell + im + j_\theta(u)| &\geq |m + m_\theta(u) - \mathrm{Im}(z)| \\
&\geq \sin(\delta')|u| - |m| - |e - z| \\
&\geq \frac{\sin(\delta')}{2} d_{L_{\mathrm{p}}} + \mathcal{O}(\rho)
\end{aligned}$$

and

$$|e' - e + j_\theta(u)| \geq |m_\theta(u)| \geq \frac{\sin(\delta')}{2} d_{L_{\mathrm{p}}}.$$

Either case suggests that

$$\begin{aligned}
&\sup_{u \in \mathbb{R}} \left\| P_1 \left(L_{0,\theta} + j_\theta(u) - z\right)^{-1} \left[\cos(\delta')L_{\mathrm{res}} + iM_{[\theta]} + e - z\right] \left(L_{\mathrm{p}} + j_\theta(u) - e\right)^{-1} P_1 \right\| \\
&\quad = \mathcal{O}(\rho).
\end{aligned}$$

We go over to compute

$$\begin{aligned}
&\left\| P_2(u) \left(L_{0,\theta} + j_\theta(u) - z\right)^{-1} \left[\cos(\delta')L_{\mathrm{res}} + iM_{[\theta]} + e - z\right] \left(L_{\mathrm{p}} + j_\theta(u) - e\right)^{-1} P_2(u) \right\| \\
&\quad \leq \frac{6\rho}{\sin(\delta')} \left\| \frac{P_2(u)}{j_\theta(u)\left(\cos(\delta')L_{\mathrm{res}} + iM_{[\theta]} + j_\theta(u) + e - z\right)} \right\| \\
&\quad \leq \frac{6\rho}{\sin(\delta')} \sup_{\substack{\frac{7}{8}\rho - m_\theta(u) \leq m \leq \rho, \\ |\ell| \leq \rho/\sin(\delta')}} \frac{1}{|j_\theta(u)|\,|\cos(\delta')\ell + im + j_\theta(u) + e - z|}.
\end{aligned}$$

Again, we consider two cases to estimate the fraction. First, we choose $u \in \mathbb{R}$ such that $|j_\theta(u)| > \frac{12}{\sin(\delta')}\rho$. This allows us to find an upper bound for

$$\frac{1}{|j_\theta(u)|\,|\cos(\delta')\ell + im + j_\theta(u) + e - z|} \le \frac{1}{|j_\theta(u)|\left(|j_\theta(u)| - \frac{6}{\sin(\delta')}\rho\right)} \le \frac{2}{|j_\theta(u)|^2}.$$

Let now $|j_\theta(u)| \le \frac{12}{\sin(\delta')}\rho$ which implies that the integration parameter is restricted to

$$|u| \le \frac{|\operatorname{Re}(j_\theta(u))|}{\cos(\delta')} \le \frac{|j_\theta(u)|}{\cos(\delta')} \le \frac{12}{\cos(\delta')\sin(\delta')}\rho$$

and further holds

$$\begin{aligned}\frac{1}{|j_\theta(u)|\,|\cos(\delta')\ell + im + j_\theta(u) + e - z|} &\le \frac{1}{|j_\theta(u)||m + m_\theta(u) - \operatorname{Im}(z)|}\\ &\le \frac{1}{|j_\theta(u)|\left(\frac{7}{8}\rho - \frac{\sin(\delta')}{2}\rho\right)} \le \frac{8}{3\rho|j_\theta(u)|},\end{aligned}$$

since $m + m_\theta(u) \ge \frac{7}{8}\rho$ and $\operatorname{Im}(z) \le \frac{\sin(\delta')}{2}\rho$.

Applying all this knowledge to (3.27), we end up with

$$\begin{aligned}\|(3.27)\| &\le \frac{C'}{\sin(\delta')}\Bigg[\rho\int_\Upsilon dy\,\left\|F_{\overline{\theta}}^{(\overline{s})}(y)^*\right\|\left\|F_\theta^{(s)}(y)\right\|\\ &\quad + \rho\int_\Upsilon dy\,\mathbf{1}_{\left[|j_\theta(u)|>\frac{12}{\sin(\delta')}\rho\right]}\frac{\left\|F_{\overline{\theta}}^{(\overline{s})}(y)^*\right\|\left\|F_\theta^{(s)}(y)\right\|}{|j_\theta(u)|^2}\\ &\quad + \int_\Upsilon dy\,\mathbf{1}_{\left[|j_\theta(u)|\le\frac{12}{\sin(\delta')}\rho\right]}\frac{\left\|F_{\overline{\theta}}^{(\overline{s})}(y)^*\right\|\left\|F_\theta^{(s)}(y)\right\|}{|j_\theta(u)|}\Bigg]\\ &= \mathcal{O}\left(\rho + \rho^{2\nu}\right).\end{aligned}$$

■

Now, we remove the cutoff operators $\overline{\Xi}_{e,\rho}(u)$ from (3.26)

Lemma 3.13 *We have*

$$\begin{aligned}&\Xi_{e,\rho}\int_\Upsilon dy\,F_{\overline{\theta}}^{(\overline{s})}(y)^*\overline{\Xi}_{e,\rho}(u)\left(L_\mathrm{p} + j_\theta(u) - e\right)^{-1}\overline{\Xi}_{e,\rho}(u)F_\theta^{(s)}(y)\Xi_{e,\rho} \qquad (3.28)\\ &= \Xi_{e,\rho}\int_\Upsilon dy\,F_{\overline{\theta}}^{(\overline{s})}(y)^*\left(L_\mathrm{p} + j_\theta(u) - e\right)^{-1}F_\theta^{(s)}(y)\Xi_{e,\rho} + \mathcal{O}\left(\rho^{2\nu}\right).\end{aligned}$$

Proof. We show that the contribution of $\mathbb{1} - \overline{\Xi}^2_{e,\rho}(u) = \Xi^2_{e,\rho}(u)$ to (3.28) is small, where we define

$$\Xi_{e,\rho}(u) := P_{[L_{\mathrm{p}}=e]} \otimes \chi_\rho \left(M_{[\theta]} + m_\theta(u) \right).$$

Computing the differences, we obtain

$$\begin{aligned}
&\Bigg\| \Xi_{e,\rho} \int_\Upsilon dy\, F_{\overline{\theta}}^{(\overline{s})}(y)^* \overline{\Xi}_{e,\rho}(u) \left(L_{\mathrm{p}} + j_\theta(u) - e \right)^{-1} \overline{\Xi}_{e,\rho}(u) F_\theta^{(s)}(y) \Xi_{e,\rho} \\
&\qquad - \Xi_{e,\rho} \int_\Upsilon dy\, F_{\overline{\theta}}^{(\overline{s})}(y)^* \left(L_{\mathrm{p}} + j_\theta(u) - e \right)^{-1} F_\theta^{(s)}(y) \Xi_{e,\rho} \Bigg\| \\
&= \left\| \Xi_{e,\rho} \int_\Upsilon dy\, F_{\overline{\theta}}^{(\overline{s})}(y)^* \Xi^2_{e,\rho}(u) \left(L_{\mathrm{p}} + j_\theta(u) - e \right)^{-1} F_\theta^{(s)}(y) \Xi_{e,\rho} \right\| \\
&= \left\| \Xi_{e,\rho} \int_\Upsilon dy\, \frac{F_{\overline{\theta}}^{(\overline{s})}(y)^* \Xi^2_{e,\rho}(u) F_\theta^{(s)}(y)}{j_\theta(u)} \Xi_{e,\rho} \right\| \\
&\le \int_\Upsilon dy\, \mathbf{1}_{[m_\theta(u) \le \rho]} \frac{\left\| F_{\overline{\theta}}^{(\overline{s})}(y)^* \right\| \left\| F_\theta^{(s)}(y) \right\|}{|j_\theta(u)|} = \mathcal{O}\left(\rho^{2\nu} \right).
\end{aligned}$$

■

As a last task, we remove the spectral parameters $\theta = (i\delta', i\tau')$. Moreover we will show that

Lemma 3.14 *The r.h.s. of (3.28) up to the error terms is independent of θ and can be represented as*

$$\begin{aligned}
&\Xi_{e,\rho} \int_\Upsilon dy\, F_{\overline{\theta}}^{(\overline{s})}(y)^* \left(L_{\mathrm{p}} + j_\theta(u) - e \right)^{-1} F_\theta^{(s)}(y) \Xi_{e,\rho} \\
&= \lim_{\varepsilon \searrow 0} \Xi_{e,\rho} \int_\Upsilon dy\, F^{(\overline{s})}(y)^* \left(L_{\mathrm{p}} + u - e + i\varepsilon \right)^{-1} F^{(s)}(y) \Xi_{e,\rho}.
\end{aligned} \tag{3.29}$$

Proof. First, note that

$$\left(L_{\mathrm{p}} + j_\theta(u) - e \right)^{-1} = \lim_{\varepsilon \searrow 0} \left(L_{\mathrm{p}} + j_\theta(u) - e + i\varepsilon \right)^{-1}$$

in norm topology since $\mathrm{Im}(j_\theta(u)) > 0$ for our choice of $\theta = (\delta, \tau)$ with $\mathrm{Im}(\delta), \mathrm{Im}(\tau) > 0$. For $\varepsilon > 0$, we have analyticity of $\theta \mapsto \left(L_{\mathrm{p}} + j_\theta(u) - e + i\varepsilon \right)^{-1}$ in δ and τ separately on the domain $0 < \mathrm{Im}(\delta) < \frac{\pi}{4}$ and $\mathrm{Im}(\tau) > 0$, because of

$$\partial_\delta \left(L_\mathrm{p} + j_\theta(u) - e + i\varepsilon\right)^{-1} = -\frac{e^{\delta\,\mathrm{sgn}(u)}|u|}{\left(L_\mathrm{p} + j_\theta(u) - e + i\varepsilon\right)^2}$$

and

$$\partial_\tau \left(L_\mathrm{p} + j_\theta(u) - e + i\varepsilon\right)^{-1} = -\left(L_\mathrm{p} + j_\theta(u) - e + i\varepsilon\right)^{-2},$$

using that

$$\left\|\left(L_\mathrm{p} + j_\theta(u) - e + i\varepsilon\right)^{-2}\right\| \le \varepsilon^{-2},$$

uniformly in θ. Further, we stress that $\theta = (\delta, \tau) \mapsto \left(L_\mathrm{p} + j_\theta(u) - e + i\varepsilon\right)^{-1}$ is continuous as $\mathrm{Im}(\delta), \mathrm{Im}(\tau) \searrow 0$. By dominated convergence theorem we have analyticity of

$$\theta = (\delta, \tau) \mapsto \Xi_{e,\rho} \int\limits_{\Upsilon} dy\, F_{\overline{\theta}}^{(\overline{s})}(y)^* \left(L_\mathrm{p} + j_\theta(u) - e + i\varepsilon\right)^{-1} F_\theta^{(s)}(y) \Xi_{e,\rho} \tag{3.30}$$

for $0 < \mathrm{Im}(\delta) < \frac{\pi}{4}$, $0 < \mathrm{Im}(\tau) < 2\pi\beta_{\mathrm{max}}^{-1}$, and continuity for $\mathrm{Im}(\delta), \mathrm{Im}(\tau) \searrow 0$. Let now $\tilde{\theta} = (\tilde{\delta}, \tilde{\tau}) \in \mathbb{R}^2$. Substituting the integration variables,

$$\begin{aligned} j_{\tilde{\theta}}(u) = e^{\tilde{\delta}}u + \tilde{\tau} &\mapsto u \qquad , u \ge 0, \\ j_{\tilde{\theta}}(u) = e^{-\tilde{\delta}}u + \tilde{\tau} &\mapsto u \qquad , u < 0, \end{aligned}$$

we see easily that (3.30) is invariant under a translation $\theta \mapsto \theta + \tilde{\theta}$ for $\tilde{\theta} \in \mathbb{R}^2$. Because of analyticity the function (3.30) is independent of $\theta = (\delta, \tau)$ as long as $0 < \mathrm{Im}(\delta) < \frac{\pi}{4}$ and $0 < \mathrm{Im}(\tau) < 2\pi\beta_{\mathrm{max}}^{-1}$. Continuity finally yields that

$$\begin{aligned} &\Xi_{e,\rho} \int\limits_{\Upsilon} dy F_{\overline{\theta}}^{(\overline{s})}(y)^* \left(L_\mathrm{p} + j_\theta(u) - e + i\varepsilon\right)^{-1} F_\theta^{(s)}(y) \Xi_{e,\rho} \\ &= \Xi_{e,\rho} \int\limits_{\Upsilon} dy F^{(\overline{s})}(y)^* \left(L_\mathrm{p} + u - e + i\varepsilon\right)^{-1} F^{(s)}(y) \Xi_{e,\rho}. \end{aligned}$$

The assertion follows since, for $\theta = (i\delta', i\tau')$ with $\frac{\pi}{8} < \delta' < \frac{\pi}{4}$ and $0 < \tau' < 2\pi\beta_{\mathrm{max}}^{-1}$, the limit procedure $\varepsilon \searrow 0$ and the integration commute by the dominated convergence theorem,

$$\begin{aligned} &\lim_{\varepsilon \searrow 0} \Xi_{e,\rho} \int\limits_{\Upsilon} dy F_{\overline{\theta}}^{(\overline{s})}(y)^* \left(L_\mathrm{p} + j_\theta(u) - e + i\varepsilon\right)^{-1} F_\theta^{(s)}(y) \Xi_{e,\rho} \\ &= \Xi_{e,\rho} \int\limits_{\Upsilon} dy F_{\overline{\theta}}^{(\overline{s})}(y)^* \left(L_\mathrm{p} + j_\theta(u) - e\right)^{-1} F_\theta^{(s)}(y) \Xi_{e,\rho}. \end{aligned}$$

■

3.3.2 The Level Shift Operator

After having extracted the leading orders of the Feshbach operator $\mathfrak{F}_{\Xi_{e,\rho}}(K_\theta^{(s)} - z, L_{0,\theta} - z)$ in Proposition 3.7, it becomes necessary to study the level shift operator $\Lambda_e^{(s)}$ defined in (3.21) in order to understand the spectrum of $\mathfrak{F}_{\Xi_{e,\rho}}(K_\theta^{(s)} - z, L_{0,\theta} - z)$.

The aim of this subsection is to understand the level shift operator $\Lambda_e^{(s)}$ of the R-reservoir system as the sum of contributions from each reservoir. Moreover, it is the goal of the subsequent consideration to study the qualitative deviation of the level shift operator from the equal temperature case $\delta\vec{\beta} = (\beta_{\mathrm{p}} - \beta, 0, \ldots, 0)$.

Let us introduce the notation

$$\begin{aligned}\Lambda_{e,r}^{(s)} \;:=\; & -\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}\times S^2} d(u,\Sigma)\, P_{[L_{\mathrm{p}}=e]}\left[\mathcal{G}(u,\Sigma,r) - \mathcal{G}'_{(\bar{s}\delta\vec{\beta})}(u,\Sigma,r)\right]^* \\ & \times (L_{\mathrm{p}} - e + u + i\varepsilon)^{-1}\left[\mathcal{G}(u,\Sigma,r) - \mathcal{G}'_{(s\delta\vec{\beta})}(u,\Sigma,r)\right] P_{[L_{\mathrm{p}}=e]}\end{aligned} \tag{3.31}$$

for the level shift operator of the r^{th} reservoir such that

$$\Lambda_e^{(s)} = \sum_{r=1}^{R}\Lambda_{e,r}^{(s)}.$$

Having the notation

$$\begin{aligned} A(b) &:= e^{ibH_{\mathrm{p}}}\otimes \mathbb{1}_{\mathcal{H}_{\mathrm{p}}} = \pi_{\mathrm{p}}\left(e^{ibH_{\mathrm{p}}}\right), \\ A'(b) &:= \mathbb{1}_{\mathcal{H}_{\mathrm{p}}}\otimes e^{ibH_{\mathrm{p}}} = \pi'_{\mathrm{p}}\left(e^{-i\bar{b}H_{\mathrm{p}}}\right)\end{aligned}$$

at hand we can rewrite (3.31) as

$$\begin{aligned}&\Lambda_{e,r}^{(s)} \\ &= -\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}\times S^2} d(u,\Sigma)\, P_{[L_{\mathrm{p}}=e]}\left[\mathcal{G}(y)^* - e^{-is\delta\beta_r u}A'(-s\delta\beta_{\mathrm{p}})\mathcal{G}'(y)^*A'(s\delta\beta_{\mathrm{p}})\right] \\ &\qquad \times(L_{\mathrm{p}} - e + u + i\varepsilon)^{-1}\left[\mathcal{G}(y) - e^{is\delta\beta_r u}A'(-s\delta\beta_{\mathrm{p}})\mathcal{G}'(y)A'(s\delta\beta_{\mathrm{p}})\right] P_{[L_{\mathrm{p}}=e]},\end{aligned} \tag{3.32}$$

for $y = (u, \Sigma, r)$. The level shift operator $\Lambda_{e,r}^{(s)}$ of a single reservoir emerges from the single reservoir equilibrium situation $\delta\beta_r = \delta\beta_{\mathrm{p}} = 0$ via conjugation, as the following lemma states.

Lemma 3.15 *On the range of $P_{[L_{\mathrm{p}}=e]}$, the level shift operator $\Lambda_{e,r}^{(s)}$ corresponding to the r^{th} reservoir can be expressed as*

$$\Lambda_{e,r}^{(s)} = A(s(\beta_r - \beta_{\mathrm{p}}))\Lambda_{e,r}^{(0)}A(-s(\beta_r - \beta_{\mathrm{p}})),$$

where

$$\Lambda_{e,r}^{(0)} = -\lim_{\varepsilon\searrow 0} \int_{\mathbb{R}\times S^2} d(u,\Sigma)\, P_{[L_\mathrm{p}=e]}\, [\mathcal{G}(y)-\mathcal{G}'(y)]^* \times (L_\mathrm{p}-e+u+i\varepsilon)^{-1}\, [\mathcal{G}(y)-\mathcal{G}'(y)]\, P_{[L_\mathrm{p}=e]}.$$

Proof. Let $\varepsilon > 0$ and set $\mathcal{R}_\varepsilon \equiv \mathcal{R}_\varepsilon(u,\Sigma) := (L_\mathrm{p}+u-e+i\varepsilon)^{-1}$. Also, we abbreviate $P_{[e]} := P_{[L_\mathrm{p}=e]}$. We take into account that

$$[A(b),\mathcal{G}'(y)] = 0 = [A'(b),\mathcal{G}(y)] \tag{3.33}$$

for all $b \in \mathbb{C}$ and further

$$A(b)A'(-b) = e^{ibL_\mathrm{p}}, \tag{3.34}$$

which implies that

$$P_{[e]}A(b) = e^{ibe}P_{[e]}A'(b). \tag{3.35}$$

We are going to use the above relations to expand the product representation (3.32) of $\Lambda_{e,r}^{(s)}$ into four addends,

$$\begin{aligned}
(\mathcal{G}\mathcal{G}) &:= P_{[e]}\mathcal{G}(y)^*\mathcal{R}_\varepsilon\mathcal{G}(y)P_{[e]} = P_{[e]}\mathcal{G}(y)^*A'(b)\mathcal{R}_\varepsilon A'(-b)\mathcal{G}(y)P_{[e]}\\
&= P_{[e]}A(b)\mathcal{G}(y)^*\mathcal{R}_\varepsilon\mathcal{G}(y)A(-b)P_{[e]}\\
&\quad \text{for any } b\in\mathbb{C},\\
(\mathcal{G}'\mathcal{G}') &:= P_{[e]}e^{-is\delta\beta_r u}A'(-s\delta\beta_\mathrm{p})\mathcal{G}'(y)^*A'(s\delta\beta_\mathrm{p})\\
&\quad \times\mathcal{R}_\varepsilon e^{is\delta\beta_r u}A'(-s\delta\beta_\mathrm{p})\mathcal{G}'(y)A'(s\delta\beta_\mathrm{p})P_{[e]}\\
&= P_{[e]}A'(-s\delta\beta_\mathrm{p})\mathcal{G}'(y)^*\mathcal{R}_\varepsilon\mathcal{G}'(y)A'(s\delta\beta_\mathrm{p})P_{[e]}\\
&= P_{[e]}A(b)\mathcal{G}'(y)^*\mathcal{R}_\varepsilon\mathcal{G}'(y)A(-b)P_{[e]}\\
&\quad \text{for any } b\in\mathbb{C},\\
(\mathcal{G}\mathcal{G}') &:= P_{[e]}\mathcal{G}(y)^*\mathcal{R}_\varepsilon e^{is\delta\beta_r u}A'(-s\delta\beta_\mathrm{p})\mathcal{G}'(y)A'(s\delta\beta_\mathrm{p})P_{[e]}\\
&= P_{[e]}A(-s\delta\beta_\mathrm{p})\mathcal{G}(y)^*\mathcal{R}_\varepsilon e^{is\delta\beta_r u}\mathcal{G}'(y)A(s\delta\beta_\mathrm{p})P_{[e]},\\
(\mathcal{G}'\mathcal{G}) &:= P_{[e]}e^{-is\delta\beta_r u}A'(-s\delta\beta_\mathrm{p})\mathcal{G}'(y)^*A'(s\delta\beta_\mathrm{p})\mathcal{R}_\varepsilon\mathcal{G}(y)P_{[e]}\\
&= P_{[e]}A(-s\delta\beta_\mathrm{p})\mathcal{G}'(y)^*\mathcal{R}_\varepsilon e^{-is\delta\beta_r u}\mathcal{G}(y)A(s\delta\beta_\mathrm{p})P_{[e]},
\end{aligned}$$

where we made use of (3.33, 3.34, 3.35) at several points. Choosing $b = s(\beta_r-\beta_\mathrm{p})$, the terms $(\mathcal{G}\mathcal{G})$ and $(\mathcal{G}'\mathcal{G}')$ have already the required structure. Therefore, we focus on $(\mathcal{G}\mathcal{G}')$ and $(\mathcal{G}'\mathcal{G})$. Applying (3.34, 3.35) again, we can transform

$$\begin{aligned}
(\mathcal{G}\mathcal{G}') &= P_{[e]}A(-s\delta\beta_\mathrm{p})A'(s\delta\beta_r)\mathcal{G}(y)^*\mathcal{R}_\varepsilon e^{is\delta\beta_r(L_\mathrm{p}+u)}\mathcal{G}'(y)A(-s(\delta\beta_r-\delta\beta_\mathrm{p}))P_{[e]}\\
&= P_{[e]}A(s(\delta\beta_r-\delta\beta_\mathrm{p}))\mathcal{G}(y)^*\mathcal{R}_\varepsilon e^{is\delta\beta_r(L_\mathrm{p}+u-e)}\mathcal{G}'(y)A(-s(\delta\beta_r-\delta\beta_\mathrm{p}))P_{[e]}
\end{aligned}$$

and, equivalently,

$$(\mathcal{G}'\mathcal{G}) = P_{[e]}A(s(\delta\beta_r-\delta\beta_\mathrm{p}))\mathcal{G}'(y)^*\mathcal{R}_\varepsilon e^{-is\delta\beta_r(L_\mathrm{p}+u-e)}\mathcal{G}(y)A(-s(\delta\beta_r-\delta\beta_\mathrm{p}))P_{[e]}.$$

This allows us to write

$$\Lambda_{e,r}^{(s)} = P_{[e]} A(s(\beta_r - \beta_{\rm p})) \left[\Lambda_{e,r}^{(0)} + \lim_{\varepsilon \searrow 0} \int\limits_{\mathbb{R}\times S^2} d(u,\Sigma)\, R_\varepsilon(u,\Sigma) \right] A(-s(\beta_r - \beta_{\rm p})) P_{[e]},$$

where

$$\begin{aligned} R_\varepsilon \equiv R_\varepsilon(u,\Sigma) \;&:=\; \mathcal{G}(y)^* \mathcal{R}_\varepsilon \left[e^{is\delta\beta_r(L_{\rm p}+u-e)} - \mathbb{1}_{\mathcal{H}_{\rm p}^2} \right] \mathcal{G}'(y) \\ &\quad + \mathcal{G}'(y)^* \mathcal{R}_\varepsilon \left[e^{-is\delta\beta_r(L_{\rm p}+u-e)} - \mathbb{1}_{\mathcal{H}_{\rm p}^2} \right] \mathcal{G}(y). \end{aligned}$$

Thus, the assertion is proved by showing that

$$\lim_{\varepsilon \searrow 0} \int\limits_{\mathbb{R}\times S^2} d(u,\Sigma)\, P_{[e]} R_\varepsilon(u,\Sigma) P_{[e]} = 0.$$

We are approaching this task by noting that

$$\mathcal{R}_\varepsilon = -i \int\limits_0^\infty dt\, e^{i(L_{\rm p}+u-e+i\varepsilon)t}.$$

Therefore,

$$\begin{aligned} &P_{[e]} \mathcal{G}'(y)^* \mathcal{R}_\varepsilon \left[e^{-is\delta\beta_r(L_{\rm p}+u-e)} - \mathbb{1}_{\mathcal{H}_{\rm p}^2} \right] \mathcal{G}(y) P_{[e]} \\ &= -i \int\limits_0^\infty dt\, P_{[e]} \mathcal{G}'(y)^* e^{i(u+i\varepsilon)t} \Big[e^{-is\delta\beta_r u} e^{iL_{\rm p}(t-s\delta\beta_r)} \mathcal{G}(y) e^{-iL_{\rm p}(t-s\delta\beta_r)} \\ &\qquad\qquad - e^{iL_{\rm p}t} \mathcal{G}(y) e^{-iL_{\rm p}t} \Big] P_{[e]} \\ &= -i \int\limits_0^\infty dt\, P_{[e]} e^{i(u+i\varepsilon)t} \Big[e^{-is\delta\beta_r u} e^{iL_{\rm p}(t-s\delta\beta_r)} \mathcal{G}(y) e^{-iL_{\rm p}(t-s\delta\beta_r)} \\ &\qquad\qquad - e^{iL_{\rm p}t} \mathcal{G}(y) e^{-iL_{\rm p}t} \Big] \mathcal{G}'(y)^* P_{[e]} \\ &= -i \int\limits_0^\infty dt\, P_{[e]} \mathcal{G}(y) e^{i(-L_{\rm p}+u+e+i\varepsilon)t} \left[e^{-is\delta\beta_r(-L_{\rm p}+u+e)} - \mathbb{1}_{\mathcal{H}_{\rm p}^2} \right] \mathcal{G}'(y)^* P_{[e]} \\ &= P_{[e]} \mathcal{G}(y) \left(-L_{\rm p}+u+e+i\varepsilon\right)^{-1} \left[e^{-is\delta\beta_r(-L_{\rm p}+u+e)} - \mathbb{1}_{\mathcal{H}_{\rm p}^2} \right] \mathcal{G}'(y)^* P_{[e]}, \end{aligned}$$

where we used in the second and third line that the expression in parenthesis acts trivially on the right factor of the tensor product space $\mathcal{H}_{\rm p}^2 = \mathcal{H}_{\rm p} \otimes \mathcal{H}_{\rm p}$ and therefore

$\mathcal{G}'(y)^*$ commutes through the integral. So far, we have

$$\begin{aligned} P_{[e]} R_\varepsilon(u,\Sigma) P_{[e]} &= P_{[e]} \Big\{ \mathcal{G}(y)^* \left(L_\mathrm{p} + u - e + i\varepsilon\right)^{-1} \left[e^{is\delta\beta_r(L_\mathrm{p}+u-e)} - \mathbb{1}_{\mathcal{H}_\mathrm{p}^2} \right] \mathcal{G}'(y) \\ &\quad - \mathcal{G}(y) \left(L_\mathrm{p} - u - e - i\varepsilon\right)^{-1} \left[e^{is\delta\beta_r(L_\mathrm{p}-u-e)} - \mathbb{1}_{\mathcal{H}_\mathrm{p}^2} \right] \mathcal{G}'(y)^* \Big\} P_{[e]}. \end{aligned}$$

Note, that

$$\begin{aligned} \mathcal{G}(-u,\Sigma,r) &= -e^{-\beta_r u/2} \mathcal{G}(u,\Sigma,r)^* \qquad \text{and} \\ \mathcal{G}'(-u,\Sigma,r) &= -e^{\beta_r u/2} \mathcal{G}'(u,\Sigma,r)^*. \end{aligned}$$

Integrating over $\mathbb{R} \times S^2 \ni (u,\Sigma)$ and performing a transformation of variables, $u \mapsto -u$, for the second addend, we get

$$\begin{aligned} &\int\limits_{\mathbb{R}\times S^2} d(u,\Sigma)\, P_{[e]} R_\varepsilon(u,\Sigma) P_{[e]} \\ &= \int\limits_{\mathbb{R}\times S^2} d(u,\Sigma)\, P_{[e]} \mathcal{G}(y)^* \left[\left(L_\mathrm{p} + u - e + i\varepsilon\right)^{-1} - \left(L_\mathrm{p} + u - e - i\varepsilon\right)^{-1} \right] \\ &\qquad \times \left[e^{is\delta\beta_r(L_\mathrm{p}+u-e)} - \mathbb{1}_{\mathcal{H}_\mathrm{p}^2} \right] \mathcal{G}'(y) P_{[e]} \\ &= \int\limits_{\mathbb{R}\times S^2} d(u,\Sigma)\, P_{[e]} \mathcal{G}(y)^* \frac{-2i\varepsilon}{\left(L_\mathrm{p} + u - e\right)^2 + \varepsilon^2} \left[e^{is\delta\beta_r(L_\mathrm{p}+u-e)} - \mathbb{1}_{\mathcal{H}_\mathrm{p}^2} \right] \mathcal{G}'(y) P_{[e]} \\ &\xrightarrow{\varepsilon \to 0} 2\pi \sum_{j,k=1}^{N} \int\limits_{\mathbb{R}\times S^2} d(u,\Sigma)\, P_{[e]} \mathcal{G}(y)^* \left|\varphi_{j,k}\right\rangle \left\langle \varphi_{j,k}\right| \delta\left(E_{j,k} + u - e\right) \\ &\qquad \times \left[1 - e^{is\delta\beta_r(E_{j,k}+u-e)} \right] \mathcal{G}'(y) P_{[e]} \\ &= 0, \end{aligned}$$

where $\delta(\,\cdot\,)$ denotes the delta distribution. ∎

Lemma 3.15 allows us to derive properties of the level shift operator $\Lambda_{e,r}^{(s)}$ from the case $s = 0$.

3.4 Spectrum in the Neighborhood of Non-Zero Eigenvalues

In this section we study the level shift operator $\Lambda_e^{(s)}$ associated with non-zero particle eigenvalues $e \neq 0$. The localization of the spectrum of $\Lambda_e^{(s)}$ in the upper half plane allows the conclusion about the absence of eigenvalues of $K_\theta^{(s)}$ on the real axis in $\mathcal{S}_e$.

We launch this section by considering the space $P_{[L_\mathrm{p}=e]}\mathcal{H}_\mathrm{p}^2$. For a fixed $e \in \mathrm{spec}(L_\mathrm{p})$ we introduce the sets

$$\begin{aligned}\mathcal{I}_e &:= \left\{(j,k) \in \{0,1,\dots,N-1\}^2 \,\middle|\, E_{j,k} = e\right\},\\ \mathcal{I}_e^{(1)} &:= \{j \in \{0,1,\dots,N-1\} \,|\, \exists\, k \in \{0,1,\dots,N-1\} :\, E_{j,k} = e\},\\ \mathcal{I}_e^{(2)} &:= \{k \in \{0,1,\dots,N-1\} \,|\, \exists\, j \in \{0,1,\dots,N-1\} :\, E_{j,k} = e\},\end{aligned}$$

and the corresponding projections

$$\begin{aligned} p_e^{(1)} &: \mathcal{I}_e \to \mathcal{I}_e^{(1)}, \qquad (j,k) \mapsto j,\\ p_e^{(2)} &: \mathcal{I}_e \to \mathcal{I}_e^{(2)}, \qquad (j,k) \mapsto k.\end{aligned}$$

Apparently, the projections $p_e^{(1)}$ and $p_e^{(2)}$ are surjective. The Hypothesis IV-1.9 ensures that the projections are also injective, namely, let $(j,k),(m,n) \in \mathcal{I}_e$ with $j = p_e^{(1)}(j,k) = p_e^{(1)}(m,n) = m$. It is $E_j - E_k = E_{j,k} = e = E_{m,n} = E_m - E_n$ and, since $j = m$, we have $E_k = E_n$. The non-degeneracy of the particle eigenvalues then implies that $k = n$. The same argument shows the injectivity of $p_e^{(2)}$. Therefore, we find a bijection $b_e : \mathcal{I}_e^{(1)} \to \mathcal{I}_e^{(2)}$ such that

$$\mathcal{I}_e = \left\{(j, b_e(j)) \,\middle|\, j \in \mathcal{I}_e^{(1)}\right\}.$$

This implies that

$$\left\{\varphi_{j,b_e(j)}\right\}_{j\in\mathcal{I}_e^{(1)}}$$

is an orthonormal basis of the eigenspace $P_{[L_\mathrm{p}=e]}\mathcal{H}_\mathrm{p}^2$ of L_p corresponding to the eigenvalue e. For $e \in \mathrm{spec}(L_\mathrm{p}) \setminus \{0\}$, we derive spectral properties of the level shift operator $\Lambda_{e,r}^{(0)}$ using the matrix element representation in this basis.

Lemma 3.16 *The spectrum of the level shift operator $\Lambda_{e,r}^{(0)}$ for $e \in \mathrm{spec}(L_\mathrm{p}) \setminus \{0\}$ is contained in the upper half plane, it holds*

$$\mathrm{Im}\left\langle \psi \,\middle|\, \Lambda_{e,r}^{(0)}\psi \right\rangle_{\mathcal{H}_\mathrm{p}^2} \geq \gamma_e^r \,\|\psi\|_{\mathcal{H}_\mathrm{p}^2}^2$$

for all $\psi \in P_{[L_\mathrm{p}=e]}\mathcal{H}_\mathrm{p}^2$, where

$$\gamma_e^r := \frac{1}{2}\left[\min_{m=0}^{N-2} e^{-\beta_r E_{N-1,m}/2}\eta_{N-1,m}^r + \min_{m=1}^{N-1} e^{\beta_r E_{m,0}/2}\eta_{m,0}^r\right]$$

and

$$\eta_{j,k}^r := \eta_{k,j}^r := 2\pi E_{j,k}^2 \sqrt{\rho_{\mathrm{f},r}(E_{j,k})\,(1+\rho_{\mathrm{f},r}(E_{j,k}))}\int_{S^2} d\Sigma\, \left|G_r(E_{j,k}\Sigma)_{k,j}\right|^2 \tag{3.36}$$

for $j > k$.

Proof. The matrix elements of $\Lambda^{(0)}_{e,r}$ in the basis $\{\varphi_{j,b_e(j)}\}_{j\in\mathcal{I}^{(1)}_e}$ are given by

$$\begin{aligned}
&\big(\Lambda^{(0)}_{e,r}\big)_{m,n}\\
&:= \big\langle \varphi_{m,b_e(m)} \,\big|\, \Lambda^{(0)}_{e,r}\varphi_{n,b_e(n)} \big\rangle_{\mathcal{H}^2_{\mathrm{p}}}\\
&= -\lim_{\varepsilon\searrow 0}\sum_{k,\ell=0}^{N-1}\int_{\mathbb{R}\times S^2} d(u,\Sigma)\, \big\langle \varphi_{m,b_e(m)} \,\big|\, F(y)^*\varphi_{k,\ell} \big\rangle_{\mathcal{H}^2_{\mathrm{p}}} \big\langle \varphi_{k,\ell} \,\big|\, F(y)\varphi_{n,b_e(n)} \big\rangle_{\mathcal{H}^2_{\mathrm{p}}}\\
&\qquad\qquad \times (E_{k,\ell}-e+u+i\varepsilon)^{-1},
\end{aligned}$$

where $y=(u,\Sigma,r)\in\Upsilon$ and, recall, $F=\mathcal{G}-\mathcal{G}'$. We abbreviate

$$\mathcal{G}(y)_{j,k} := \langle \varphi_{j,k} \,|\, \mathcal{G}(y)\varphi_{k,k} \rangle_{\mathcal{H}^2_{\mathrm{p}}} \qquad \text{and} \qquad \mathcal{G}'(y)_{j,k} := \langle \varphi_{j,j} \,|\, \mathcal{G}'(y)\varphi_{j,k} \rangle_{\mathcal{H}^2_{\mathrm{p}}}$$

to write

$$\begin{aligned}
&\sum_{k,\ell=0}^{N-1} \big\langle \varphi_{m,b_e(m)} \,\big|\, F(y)^*\varphi_{k,\ell} \big\rangle_{\mathcal{H}^2_{\mathrm{p}}} \big\langle \varphi_{k,\ell} \,\big|\, F(y)\varphi_{n,b_e(n)} \big\rangle_{\mathcal{H}^2_{\mathrm{p}}} (E_{k,\ell}-e+u+i\varepsilon)^{-1}\\
&= \sum_{k,\ell=0}^{N-1} \Big[\overline{\mathcal{G}(y)_{k,m}}\mathcal{G}(y)_{k,n}\delta_{\ell,b_e(m)}\delta_{\ell,b_e(n)} + \overline{\mathcal{G}'(y)_{\ell,b_e(m)}}\mathcal{G}'(y)_{\ell,b_e(n)}\delta_{k,m}\delta_{k,n}\\
&\qquad - \overline{\mathcal{G}'(y)_{\ell,b_e(m)}}\mathcal{G}(y)_{k,n}\delta_{k,m}\delta_{\ell,b_e(n)} - \overline{\mathcal{G}(y)_{k,m}}\mathcal{G}'(y)_{\ell,b_e(n)}\delta_{\ell,b_e(m)}\delta_{k,n}\Big]\\
&\qquad \times (E_{k,\ell}-e+u+i\varepsilon)^{-1}\\
&= \delta_{b_e(m),b_e(n)}\sum_{k=0}^{N-1}\overline{\mathcal{G}(y)_{k,m}}\mathcal{G}(y)_{k,n}(E_{k,b_e(m)}-e+u+i\varepsilon)^{-1}\\
&\quad +\delta_{m,n}\sum_{\ell=0}^{N-1}\overline{\mathcal{G}'(y)_{\ell,b_e(m)}}\mathcal{G}'(y)_{\ell,b_e(n)}(E_{m,\ell}-e+u+i\varepsilon)^{-1}\\
&\quad -\overline{\mathcal{G}'(y)_{b_e(n),b_e(m)}}\mathcal{G}(y)_{m,n}(E_{m,b_e(n)}-e+u+i\varepsilon)^{-1}\\
&\quad -\overline{\mathcal{G}(y)_{n,m}}\mathcal{G}'(y)_{b_e(m),b_e(n)}(E_{n,b_e(m)}-e+u+i\varepsilon)^{-1}\\
&= \delta_{m,n}\sum_{k=0}^{N-1}\Big[|\mathcal{G}(y)_{k,m}|^2(E_{k,b_e(m)}-e+u+i\varepsilon)^{-1}\\
&\qquad\qquad + |\mathcal{G}(-y)_{k,b_e(m)}|^2(E_{m,k}-e+u+i\varepsilon)^{-1}\Big]\\
&\quad +\mathcal{G}(-y)_{b_e(n),b_e(m)}\mathcal{G}(y)_{m,n}(E_{m,b_e(n)}-e+u+i\varepsilon)^{-1}\\
&\quad +\overline{\mathcal{G}(y)_{n,m}\mathcal{G}(-y)_{b_e(m),b_e(n)}}(E_{n,b_e(m)}-e+u+i\varepsilon)^{-1}
\end{aligned}$$

using that $\delta_{b_e(m),b_e(n)} = \delta_{m,n}$, due to the bijectivity of b_e, and that $\mathcal{G}'(y) = -J_{\mathrm{p}}\mathcal{G}(-y)J_{\mathrm{p}}$ and therefore

$$\mathcal{G}'(y)_{i,j} = -\overline{\mathcal{G}(-y)_{i,j}}, \tag{3.37}$$

we interpret $-y = (-u, \Sigma, r)$ for $y = (u, \Sigma, r) \in \Upsilon$. The integration over $(u, \Sigma) \in \mathbb{R} \times S^2$ and a transformation of variables $u \mapsto -u$ in the integration of the second addend gives

$$\begin{aligned}
&\big(\Lambda_{e,r}^{(0)}\big)_{m,n} \qquad (3.38)\\
&= -\delta_{m,n} \lim_{\varepsilon \searrow 0} \sum_{k=0}^{N-1} \int\limits_{\mathbb{R}\times S^2} d(u,\Sigma) \Big[|\mathcal{G}(y)_{k,m}|^2 (E_{k,b_e(m)} - e + u + i\varepsilon)^{-1} \\
&\qquad - |\mathcal{G}(y)_{k,b_e(m)}|^2 (E_{k,m} + e + u - i\varepsilon)^{-1} \Big] \\
&\quad - \lim_{\varepsilon \searrow 0} \int\limits_{\mathbb{R}\times S^2} d(u,\Sigma) \Big[\mathcal{G}(-y)_{b_e(n),b_e(m)} \mathcal{G}(y)_{m,n} (E_{m,b_e(n)} - e + u + i\varepsilon)^{-1} \\
&\qquad + \overline{\mathcal{G}(y)_{n,m} \mathcal{G}(-y)_{b_e(m),b_e(n)}} (E_{n,b_e(m)} - e + u + i\varepsilon)^{-1} \Big].
\end{aligned}$$

We now compute the matrix elements of $\mathrm{Im}(\Lambda_{e,r}^{(0)})$,

$$\begin{aligned}
&\frac{1}{2i}\Big[\big(\Lambda_{e,r}^{(0)}\big)_{m,n} - \overline{\big(\Lambda_{e,r}^{(0)}\big)_{n,m}} \Big] \qquad (3.39)\\
&= \delta_{m,n} \lim_{\varepsilon \searrow 0} \sum_{k=0}^{N-1} \int\limits_{\mathbb{R}\times S^2} d(u,\Sigma) \Big[|\mathcal{G}(y)_{k,m}|^2 \frac{\varepsilon}{(E_{k,b_e(m)} - e + u)^2 + \varepsilon^2} \\
&\qquad + |\mathcal{G}(y)_{k,b_e(m)}|^2 \frac{\varepsilon}{(E_{k,m} + e + u)^2 + \varepsilon^2} \Big] \\
&\quad + \lim_{\varepsilon \searrow 0} \int\limits_{\mathbb{R}\times S^2} d(u,\Sigma) \Big[\mathcal{G}(-y)_{b_e(n),b_e(m)} \mathcal{G}(y)_{m,n} \frac{\varepsilon}{(E_{m,b_e(n)} - e + u)^2 + \varepsilon^2} \\
&\qquad + \overline{\mathcal{G}(y)_{n,m} \mathcal{G}(-y)_{b_e(m),b_e(n)}} \frac{\varepsilon}{(E_{n,b_e(m)} - e + u)^2 + \varepsilon^2} \Big] \\
&= \pi \delta_{m,n} \sum_{k=0}^{N-1} \int\limits_{S^2} d\Sigma \Big[|\mathcal{G}(E_{m,k}, \Sigma, r)_{k,m}|^2 + |\mathcal{G}(E_{b_e(m),k}, \Sigma, r)_{k,b_e(m)}|^2 \Big] \\
&\quad + \pi \int\limits_{S^2} d\Sigma \Big[\mathcal{G}(E_{b_e(m),b_e(n)}, \Sigma, r)_{b_e(n),b_e(m)} \mathcal{G}(E_{n,m}, \Sigma, r)_{m,n} \\
&\qquad + \overline{\mathcal{G}(E_{m,n}, \Sigma, r)_{n,m} \mathcal{G}(E_{b_e(n),b_e(m)}, \Sigma, r)_{b_e(m),b_e(n)}} \Big].
\end{aligned}$$

where we used the relations

$$\begin{aligned}
E_{k,b_e(m)} - e &= E_{k,b_e(m)} - E_{m,b_e(m)} = E_{k,m}, \\
E_{m,b_e(n)} - e &= E_{m,b_e(n)} - E_{m,b_e(m)} = E_{b_e(m),b_e(n)}
\end{aligned}$$

for all $m, n \in \mathcal{I}_e^{(1)}$ and $k = 0, 1, \ldots, N-1$. We use the matrix element representation to show strict positivity $\mathrm{Im}(\Lambda_{e,r}^{(0)})$. Let $\psi \in P_{[L_\mathrm{p}=e]}\mathcal{H}_\mathrm{p}^2$ be a unit vector, i.e., there exist complex numbers κ_j, $j \in \mathcal{I}_e^{(1)}$, such that $\sum_{j\in\mathcal{I}_e^{(1)}} |\kappa_j|^2 = 1$ and $\psi = \sum_{j\in\mathcal{I}_e^{(1)}} \kappa_j \varphi_{j,b_e(j)}$. Using the abbreviation

$$B_{j,k}^r(\Sigma) := \sqrt{\pi}\,\mathcal{G}(E_{j,k}, \Sigma, r)_{k,j}$$

we compute

$$\begin{aligned}
&\mathrm{Im}\left\langle \psi \,\middle|\, \Lambda_{e,r}^{(0)}\psi \right\rangle_{\mathcal{H}_\mathrm{p}^2} \\
&\quad= \frac{1}{2i} \sum_{m,n\in\mathcal{I}_e^{(1)}} \overline{\kappa}_m \kappa_n \left[\left(\Lambda_{e,r}^{(0)}\right)_{m,n} - \overline{\left(\Lambda_{e,r}^{(0)}\right)_{n,m}} \right] \\
&\quad= \sum_{m\in\mathcal{I}_e^{(1)}} |\kappa_m|^2 \sum_{k=0}^{N-1} \int_{S^2} d\Sigma \, \left[\left|B_{m,k}^r(\Sigma)\right|^2 + \left|B_{b_e(m),k}^r(\Sigma)\right|^2 \right] \\
&\qquad+ \sum_{m,n\in\mathcal{I}_e^{(1)}} \overline{\kappa}_m \kappa_n \int_{S^2} d\Sigma \, \left[B_{b_e(m),b_e(n)}^r(\Sigma) B_{n,m}^r(\Sigma) + \overline{B_{m,n}^r(\Sigma) B_{b_e(n),b_e(m)}^r(\Sigma)} \right] \\
&\quad= \sum_{m,n\in\mathcal{I}_e^{(1)}} \int_{S^2} d\Sigma \left[|\kappa_m|^2 \left|B_{m,n}^r(\Sigma)\right|^2 + |\kappa_m|^2 \left|B_{b_e(m),b_e(n)}^r(\Sigma)\right|^2 \right. \\
&\qquad\qquad+ \overline{\kappa}_m \kappa_n B_{b_e(m),b_e(n)}^r(\Sigma) B_{n,m}^r(\Sigma) \\
&\qquad\qquad\left. + \overline{\overline{\kappa}_n \kappa_m B_{m,n}^r(\Sigma) B_{b_e(n),b_e(m)}^r(\Sigma)} \right] \\
&\qquad+ \sum_{m\in\mathcal{I}_e^{(1)}} |\kappa_m|^2 \int_{S^2} d\Sigma \left[\sum_{k\notin\mathcal{I}_e^{(1)}} \left|B_{m,k}^r(\Sigma)\right|^2 + \sum_{k\notin\mathcal{I}_e^{(2)}} \left|B_{b_e(m),k}^r(\Sigma)\right|^2 \right]. \qquad (3.40)
\end{aligned}$$

We consider the two sums in (3.40) separately. We estimate the first sum,

$$\begin{aligned}
&\sum_{m,n\in\mathcal{I}_e^{(1)}} \int_{S^2} d\Sigma \left[|\kappa_m|^2 \left|B_{m,n}^r(\Sigma)\right|^2 + |\kappa_m|^2 \left|B_{b_e(m),b_e(n)}^r(\Sigma)\right|^2 \right. \\
&\qquad\qquad+ \overline{\kappa}_m \kappa_n B_{b_e(m),b_e(n)}^r(\Sigma) B_{n,m}^r(\Sigma) \\
&\qquad\qquad\left. + \overline{\overline{\kappa}_n \kappa_m B_{m,n}^r(\Sigma) B_{b_e(n),b_e(m)}^r(\Sigma)} \right] \\
&= \frac{1}{2} \sum_{m,n\in\mathcal{I}_e^{(1)}} \int_{S^2} d\Sigma \left[\left|\kappa_m B_{m,n}^r(\Sigma)\right|^2 + \left|\overline{\kappa}_m B_{b_e(m),b_e(n)}^r(\Sigma)\right|^2 \right. \\
&\qquad\qquad+ \left|\kappa_n B_{n,m}^r(\Sigma)\right|^2 + \left|\overline{\kappa}_n B_{b_e(n),b_e(m)}^r(\Sigma)\right|^2 \\
&\qquad\qquad+ 2\,\mathrm{Re}\left(\overline{\kappa}_m \kappa_n B_{b_e(m),b_e(n)}^r(\Sigma) B_{n,m}^r(\Sigma)\right)
\end{aligned}$$

$$
\begin{aligned}
& \qquad + 2\,\mathrm{Re}\left(\overline{\kappa}_n \kappa_m B^r_{m,n}(\Sigma) B^r_{b_e(n),b_e(m)}(\Sigma)\right)\Big] \\
= \; & \frac{1}{2} \sum_{m,n \in \mathcal{I}_e^{(1)}} \int_{S^2} d\Sigma \left[\left| \overline{\kappa}_m B^r_{b_e(m),b_e(n)}(\Sigma) + \overline{\kappa_n B^r_{n,m}(\Sigma)} \right|^2 \right. \\
& \qquad \left. + \left| \overline{\kappa}_n B^r_{b_e(n),b_e(m)}(\Sigma) + \overline{\kappa_m B^r_{m,n}(\Sigma)} \right|^2 \right] \\
\geq \; & 0. \qquad (3.41)
\end{aligned}
$$

To compute the second sum in (3.40) we assume that $e < 0$, the other case $e > 0$ is treated in the same way. Since $E_{m,b_e(m)} = e < 0$ it follows that $m < b_e(m)$ for all $m \in \mathcal{I}_e^{(1)}$. This implies that $N-1 \notin \mathcal{I}_e^{(1)}$ and that $b_e(m) \neq 0$ for all $m \in \mathcal{I}_e^{(1)}$, i.e., $0 \notin \mathcal{I}_e^{(2)}$. This observation is used to estimate

$$
\begin{aligned}
& \sum_{m \in \mathcal{I}_e^{(1)}} |\kappa_m|^2 \int_{S^2} d\Sigma \left[\sum_{k \notin \mathcal{I}_e^{(1)}} \left| B^r_{m,k}(\Sigma) \right|^2 + \sum_{k \notin \mathcal{I}_e^{(2)}} \left| B^r_{b_e(m),k}(\Sigma) \right|^2 \right] \\
\geq \; & \sum_{m \in \mathcal{I}_e^{(1)}} |\kappa_m|^2 \int_{S^2} d\Sigma \left[\left| B^r_{m,N-1}(\Sigma) \right|^2 + \left| B^r_{b_e(m),0}(\Sigma) \right|^2 \right] \\
\geq \; & \min_{m \in \mathcal{I}_e^{(1)}} \int_{S^2} d\Sigma \left| B^r_{m,N-1}(\Sigma) \right|^2 + \min_{m \in \mathcal{I}_e^{(2)}} \int_{S^2} d\Sigma \left| B^r_{m,0}(\Sigma) \right|^2 \\
\geq \; & \min_{m=0}^{N-2} \int_{S^2} d\Sigma \left| B^r_{m,N-1}(\Sigma) \right|^2 + \min_{m=1}^{N-1} \int_{S^2} d\Sigma \left| B^r_{m,0}(\Sigma) \right|^2 . \qquad (3.42)
\end{aligned}
$$

We remark that the same lower bound (3.42) is derived in the case $e > 0$ such that all further considerations are done for both cases, $e > 0$ and $e < 0$. Since $E_{m,N-1} \leq 0$ and $E_{m,0} \geq 0$ we obtain

$$
\begin{aligned}
& \int_{S^2} d\Sigma \left| B^r_{m,N-1}(\Sigma) \right|^2 \\
= \; & \pi \int_{S^2} d\Sigma \left| \mathcal{G}(E_{m,N-1}, \Sigma, r)_{N-1,m} \right|^2 \\
= \; & \pi \frac{E^2_{N-1,m}}{|1 - e^{\beta_r E_{N-1,m}}|} \int_{S^2} d\Sigma \left| G_r(E_{N-1,m}\Sigma)_{m,N-1} \right|^2 \\
= \; & \pi E^2_{N-1,m} e^{-\beta_r E_{N-1,m}/2} \sqrt{\rho_{\mathrm{f},r}(E_{N-1,m}) \left(1 + \rho_{\mathrm{f},r}(E_{N-1,m})\right)} \\
& \times \int_{S^2} d\Sigma \left| G_r(E_{N-1,m}\Sigma)_{m,N-1} \right|^2 \\
= \; & \frac{1}{2} e^{-\beta_r E_{N-1,m}/2} \eta^r_{N-1,m} \qquad (3.43)
\end{aligned}
$$

and

$$\begin{aligned}
\int_{S^2} d\Sigma \left|B^r_{m,0}(\Sigma)\right|^2 &= \pi \int_{S^2} d\Sigma \left|\mathcal{G}(E_{(m,0}, \Sigma, r)_{0,m}\right|^2 \\
&= \pi \frac{E^2_{m,0}}{|1 - e^{-\beta_r E_{m,0}}|} \int_{S^2} d\Sigma \, |G_r(E_{m,0}\Sigma)_{0,m}|^2 \\
&= \pi E^2_{m,0} e^{\beta_r E_{m,0}/2} \sqrt{\rho_{\mathrm{f},r}(E_{m,0})\,(1 + \rho_{\mathrm{f},r}(E_{m,0}))} \\
&\quad \times \int_{S^2} d\Sigma \, |G_r(E_{m,0}\Sigma)_{0,m}|^2 \\
&= \frac{1}{2} e^{\beta_r E_{m,0}/2} \eta^r_{m,0} \qquad (3.44)
\end{aligned}$$

Plugging (3.41, 3.42, 3.43, 3.44) into (3.40) gives

$$\mathrm{Im} \left\langle \psi \mid \Lambda^{(0)}_{e,r} \psi \right\rangle_{\mathcal{H}^2_{\mathrm{p}}} \geq \frac{1}{2} \left[\min_{m=0}^{N-2} e^{-\beta_r E_{N-1,m}/2} \eta^r_{N-1,m} + \min_{m=1}^{N-1} e^{\beta_r E_{m,0}/2} \eta^r_{m,0} \right].$$

■

We check that the gap γ^r_e quantifying the spectral shift of $\Lambda^{(0)}_{e,r}$ into the upper half plane is positive uniformly in the β_r.

Lemma 3.17 *(i) The gap γ^r_e for $e \neq 0$ is strictly positive uniformly in $\beta_r > 0$, it holds*

$$\inf_{\beta_r > 0} \gamma^r_e \geq \min_{n=1}^{N-1} \pi E^2_{n,0} \int_{S^2} d\Sigma \, |G_r(E_{n,0}\Sigma)_{0,n}|^2 \geq \frac{\gamma_{\mathrm{FGR}}}{2} > 0$$

where the Fermi golden rule level shift γ_{FGR} was defined in (1.86) and assumed to be positive.

(ii) The matrix $\Lambda^{(0)}_{e,r}$ is bounded uniformly in $\beta_r \to \infty$,

$$\limsup_{\beta_r \to \infty} \left\| \Lambda^{(0)}_{e,r} \right\|_{\mathcal{B}(P_{[L_{\mathrm{p}}=e]} \mathcal{H}^2_{\mathrm{p}})} < \infty,$$

i.e., it is bounded uniformly for the inverse temperature in the parameter range as specified in Hypothesis III-1.8. This statement holds at first for $e \neq 0$ and is proved here under this assumption. In Lemma 3.22(ii) we show that the estimate also holds in the case $e = 0$.

Proof.

(i) For $m = 0, \ldots, N-2$ and $n = 1, \ldots, N-1$ we compute

$$\begin{aligned}
&\frac{1}{2}\left[e^{-\beta_r E_{N-1,m}/2}\eta^r_{N-1,m} + e^{\beta_r E_{m,0}/2}\eta^r_{m,0}\right] \\
&= \pi\Bigg[E^2_{N-1,m}\rho_{\mathrm{f},r}(E_{N-1,m})\int_{S^2} d\Sigma\ \left|G_r(E_{N-1,m}\Sigma)_{m,N-1}\right|^2 \\
&\qquad + E^2_{n,0}\left(1+\rho_{\mathrm{f},r}(E_{n,0})\right)\int_{S^2} d\Sigma\ \left|G_r(E_{n,0}\Sigma)_{0,n}\right|^2\Bigg] \\
&\xrightarrow{\beta_r\to\infty} \pi E^2_{n,0}\int_{S^2} d\Sigma\ \left|G_r(E_{n,0}\Sigma)_{0,n}\right|^2 .
\end{aligned}$$

Since $\rho_{\mathrm{f},r}(E) = (e^{\beta_r E}-1)^{-1}$ decreases monotonically in β_r for $E > 0$ we conclude that

$$\gamma^r_e \geq \min_{n=1}^{N-1} \pi E^2_{n,0}\int_{S^2} d\Sigma\ \left|G_r(E_{n,0}\Sigma)_{0,n}\right|^2 \geq \frac{\gamma_{\mathrm{FGR}}}{2},$$

which is strictly positive by the Fermi golden rule condition, Hypothesis V-1.10.

(ii) We first consider the imaginary part of $\Lambda^{(0)}_{e,r}$ whose matrix elements have been computed in (3.39). We remark that for any j, k, m, n with $j > k$ holds

$$\left|\mathcal{G}(E_{j,k},\Sigma,r)_{m,n}\right| = \frac{E_{j,k}}{\sqrt{1-e^{-\beta_r E_{j,k}}}}\left|G_r(E_{j,k}\Sigma)_{m,n}\right| \xrightarrow{\beta_r\to\infty} E_{j,k}\left|G_r(E_{j,k}\Sigma)_{m,n}\right|$$

and

$$\left|\mathcal{G}(E_{k,j},\Sigma,r)_{m,n}\right| = \frac{E_{j,k}}{\sqrt{e^{\beta_r E_{j,k}}-1}}\left|G_r(E_{j,k}\Sigma)_{n,m}\right| \xrightarrow{\beta_r\to\infty} 0$$

for a.e. $\Sigma \in S^2$. This implies that every matrix element of $\mathrm{Im}(\Lambda^{(0)}_{e,r})$ is uniformly bounded for $\beta_r \to \infty$ and this implies the uniform bound of

$$\limsup_{\beta_r\to\infty}\left\|\mathrm{Im}(\Lambda^{(0)}_{e,r})\right\|_{\mathcal{B}(P_{[L_\mathrm{p}=e]}\mathcal{H}^2_\mathrm{p})} < \infty.$$

We go over to consider the real part. To this end we recall the matrix element representation (3.38) of $\Lambda_{e,r}^{(0)}$ to compute

$$\begin{aligned}
&\frac{1}{2}\left[\left(\Lambda_{e,r}^{(0)}\right)_{m,n} + \overline{\left(\Lambda_{e,r}^{(0)}\right)_{n,m}}\right] \\
&= \delta_{m,n} \lim_{\varepsilon \searrow 0} \sum_{k=0}^{N-1} \int\limits_{\mathbb{R}\times S^2} d(u,\Sigma) \left[- |\mathcal{G}(y)_{k,m}|^2 \frac{E_{k,m}+u}{(E_{k,m}+u)^2+\varepsilon^2} \right. \\
&\qquad \left. + |\mathcal{G}(y)_{k,b_e(m)}|^2 \frac{E_{k,b_e(m)}+u}{(E_{k,b_e(m)}+u)^2+\varepsilon^2} \right] \\
&\quad - \lim_{\varepsilon \searrow 0} \int\limits_{\mathbb{R}\times S^2} d(u,\Sigma) \left[\mathcal{G}(-y)_{b_e(n),b_e(m)} \mathcal{G}(y)_{m,n} \frac{E_{m,n}+u}{(E_{m,n}+u)^2+\varepsilon^2} \right. \\
&\qquad \left. + \overline{\mathcal{G}(y)_{n,m} \mathcal{G}(-y)_{b_e(m),b_e(n)}} \frac{E_{n,m}+u}{(E_{n,m}+u)^2+\varepsilon^2} \right].
\end{aligned}$$

Since for $u > 0$

$$\begin{aligned}
|\mathcal{G}(u,\Sigma,r)_{m,n}| &\xrightarrow{\beta_r \to \infty} u\,|G_r(u\Sigma)_{m,n}|\,, \\
|\mathcal{G}(-u,\Sigma,r)_{m,n}| &\xrightarrow{\beta_r \to \infty} 0,
\end{aligned}$$

the above principle values stay bounded for $\beta_r \to \infty$. This implies the uniform bound

$$\limsup_{\beta_r \to \infty} \left\| \mathrm{Re}(\Lambda_{e,r}^{(0)}) \right\|_{\mathcal{B}(P_{[L_\mathrm{p}=e]}\mathcal{H}_\mathrm{p}^2)} < \infty.$$

∎

The Lemmata 3.15 and 3.16 allow to describe the spectral properties of the full level shift operator $\Lambda_e^{(s)}$.

Proposition 3.18 *(i) Assume that the particle temperature coincides with the temperature of one of the reservoirs, i.e., $\beta_\mathrm{p} = \beta_{r'}$ for some $r' = 1, \ldots, R$. For $|\beta_{\max} - \beta_{\min}| \ll 1$, the imaginary part of the level shift operator $\Lambda_e^{(s)}$ corresponding to the operator $K^{(s)}$ associated with the eigenvalue $e \in \mathrm{spec}(L_\mathrm{p}) \setminus \{0\}$ is strictly positive, it holds*

$$\mathrm{Im} \left\langle \psi \,\middle|\, \Lambda_e^{(s)} \psi \right\rangle_{\mathcal{H}_\mathrm{p}^2} \geq \gamma_e^{r'} \, \|\psi\|_{\mathcal{H}_\mathrm{p}^2}^2 \geq \frac{\gamma_{\mathrm{FGR}}}{2} \, \|\psi\|_{\mathcal{H}_\mathrm{p}^2}^2$$

for all $\psi \in P_{[L_\mathrm{p}=e]}\mathcal{H}_\mathrm{p}^2$. In particular, we have positivity of the imaginary part of the level shift operator $\Lambda_e^{(-i/2)}$ corresponding to the C-Liouville operator $K = K^{(-i/2)}$. If the eigenvalue $e \in \mathrm{spec}(L_\mathrm{p}) \setminus \{0\}$ is non-degenerate than the same estimate holds without the condition $|\beta_{\max} - \beta_{\min}| \ll 1$.

(ii) *The imaginary part of the level shift operator $\Lambda_e^{(0)}$ corresponding to the standard Liouville operator $L = K^{(0)}$ associated with the particle eigenvalue $e \neq 0$ obeys the estimate*

$$\operatorname{Im} \langle \psi \mid \Lambda_e^{(0)} \psi \rangle_{\mathcal{H}_\mathrm{p}^2} \geq \left(\sum_{r=1}^{R} \gamma_e^r \right) \|\psi\|^2_{\mathcal{H}_\mathrm{p}^2} \geq R \frac{\gamma_{\mathrm{FGR}}}{2} \|\psi\|^2_{\mathcal{H}_\mathrm{p}^2} ,$$

for all $\psi \in P_{[L_\mathrm{p}=e]} \mathcal{H}_\mathrm{p}^2$.

Proof.

(i) We set

$$\xi_r := A(-\overline{s}(\beta_r - \beta_\mathrm{p}))\psi$$

and write with the help of Lemma 3.15

$$\begin{aligned}
&\operatorname{Im} \langle \psi \mid \Lambda_e^{(s)} \psi \rangle_{\mathcal{H}_\mathrm{p}^2} \\
&= \operatorname{Im} \left\langle \psi \,\middle|\, \Lambda_{e,r'}^{(0)} \psi \right\rangle_{\mathcal{H}_\mathrm{p}^2} + \sum_{\substack{r=1,\\ r\neq r'}}^{R} \operatorname{Im} \langle \xi_r \mid \Lambda_{e,r}^{(0)} A(-2i \operatorname{Im}(s)(\beta_r - \beta_\mathrm{p}))\xi_r \rangle \\
&= \operatorname{Im} \left\langle \psi \,\middle|\, \Lambda_{e,r'}^{(0)} \psi \right\rangle_{\mathcal{H}_\mathrm{p}^2} + \sum_{\substack{r=1,\\ r\neq r'}}^{R} \operatorname{Im} \langle \xi_r \mid \Lambda_{e,r}^{(0)} \xi_r \rangle \\
&\quad + \sum_{\substack{r=1,\\ r\neq r'}}^{R} \operatorname{Im} \langle \xi_r \mid \Lambda_{e,r}^{(0)} \left[A(-2i \operatorname{Im}(s)(\beta_r - \beta_\mathrm{p})) - \mathbb{1} \right] \xi_r \rangle \\
&\geq \gamma_e^{r'} \|\psi\|^2_{\mathcal{H}_\mathrm{p}^2} + \sum_{\substack{r=1,\\ r\neq r'}}^{R} \left[\gamma_e^r - C |2 \operatorname{Im}(s)(\beta_{\max} - \beta_{\min})| \left\| \Lambda_e^{(s)} \right\|_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)} \right] \|\xi_r\|^2_{\mathcal{H}_\mathrm{p}^2} \\
&\geq \gamma_e^{r'} \|\psi\|^2_{\mathcal{H}_\mathrm{p}^2} \geq \frac{\gamma_{\mathrm{FGR}}}{2} \|\psi\|^2_{\mathcal{H}_\mathrm{p}^2}
\end{aligned}$$

for the positive constant $C := \|H_\mathrm{p}\|_{\mathcal{B}(\mathcal{H}_\mathrm{p})} \exp(2|\operatorname{Im}(s)|(\beta_{\max} - \beta_{\min}) \|H_\mathrm{p}\|_{\mathcal{B}(\mathcal{H}_\mathrm{p})}) < \infty$ and $|\beta_{\max} - \beta_{\min}| \ll 1$ sufficiently small. Hereby, we used that $\left\| \Lambda_e^{(s)} \right\|_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)}$ is bounded uniformly in β_r from compact subsets of $\mathbb{R}^+$, refer to Lemma 3.17.

If the eigenvalue $e \in \operatorname{spec}(L_\mathrm{p}) \setminus \{0\}$ is simple then $\Lambda_e^{(s)}$ is a complex number rather than a matrix and is holds by Lemma 3.15 $\Lambda_e^{(s)} = \sum_{r=1}^{R} \Lambda_{e,r}^{(0)}$. Hence, the dependence of the temperature differences drops out.

(ii) Lemma 3.15 implies that

$$\begin{aligned}\operatorname{Im}\langle \psi \mid \Lambda_e^{(0)}\psi \rangle_{\mathcal{H}_\mathrm{p}^2} &= \sum_{r=1}^{R} \operatorname{Im}\langle \psi \mid \Lambda_{e,r}^{(0)}\psi \rangle_{\mathcal{H}_\mathrm{p}^2} \geq \left(\sum_{r=1}^{R} \operatorname{Im}\gamma_e^r\right) \|\psi\|^2_{\mathcal{H}_\mathrm{p}^2} \\ &\geq R\frac{\gamma_\mathrm{FGR}}{2}\|\psi\|^2_{\mathcal{H}_\mathrm{p}^2}.\end{aligned}$$

■

The operator $\Lambda_e^{(s)}$ carries the name level shift operator since it is accountable for the shift of the eigenvalue e of the unperturbed deformed Liouvillean $L_{0,\theta}$ into the upper half plane. This is due to the fact that the operator $K_\theta^{(s)}$ has the same spectral properties as the image $\mathfrak{F}_{\Xi_{e,\rho}}(K_\theta^{(s)}-z, L_{0,\theta}-z)+z$ under the Feshbach map for $z \in \mathcal{S}_e$, c.f. Theorem E.1. Further, by Proposition 3.7, the Feshbach operator is in leading order the free Liouville operator $L_{0,\theta}$, restricted to the eigenspace $\ker(L_\mathrm{p}-e)\cap \operatorname{ran}(P_{[M_{[\theta]}<\rho]})$, and the correction term of lowest order is the operator $g^2\Lambda_e^{(s)} \otimes \chi_\rho^2(M_{[\theta]})$, having strictly positive imaginary part. Figure 3.3 illustrates how the addition of the operator $g^2\Lambda_e^{(s)}$ to the free part $L_{0,\theta}$ affects a shift of the spectrum into the upper half plane. The gap between the spectrum and the real axis is so big that higher order correction terms cannot destroy it. The full Feshbach operator has spectrum separated from the real axis by a gap of order g^2, c.f. Figure 3.4. The next theorem uses this observation to describe the spectrum of $K_\theta^{(s)}$ in $\mathcal{S}_e$.

Theorem 3.19 (Spectrum of $K_\theta^{(s)}$ in $\mathcal{S}_e$) *Under the assumptions of Theorem 3.1, the spectrum of $K_\theta^{(s)}$ inside the region $\mathcal{S}_e$ around a non-zero particle eigenvalue $e \in \operatorname{spec}(L_\mathrm{p})\setminus\{0\}$ can be located by*

$$\operatorname{spec}\left(K_\theta^{(s)}\right)\cap \mathcal{S}_e \subseteq \left\{ z\in \mathcal{S}_e \,\middle|\, \operatorname{Im}(z) \geq g^2\frac{\gamma_\mathrm{FGR}}{4}\right\}.$$

Proof. The isospectrality of the smooth Feshbach map $\mathfrak{F}_{\Xi_{e,\rho}}$ implies that $z \in \operatorname{spec}\left(K_\theta^{(s)}\right)\cap\mathcal{S}_e$ if and only if $z\in \operatorname{spec}(\mathfrak{F}_{\Xi_{e,\rho}}(K_\theta^{(s)}-z, L_{0,\theta}-z)+z)$ and $z\in\mathcal{S}_e$. By Proposition 3.7 we know that

$$\mathfrak{F}_{\Xi_{e,\rho}}(K_\theta^{(s)}-z, L_{0,\theta}-z)+z = P_{e,\rho}\left[L_{0,\theta}+g^2\Lambda_e^{(s)}\otimes\chi_\rho^2(M_{[\theta]})\right]P_{e,\rho}+\mathcal{O}\left(g^{2+\tilde{\varepsilon}}\right) \quad (3.45)$$

where the remainder term is estimated uniformly in $z \in \mathcal{S}_e$. We compute the numerical range of the imaginary part of this operator. To this end let $\psi \in \operatorname{ran}(P_{e,\rho})$ and decompose $\psi = \psi_1 + \psi_1^\perp$ where $\psi_1 \in \operatorname{ran}(P_{[M_{[\theta]}\leq \frac{7}{8}\rho]})$ and $\psi_1^\perp \in \operatorname{ran}(P_{[M_{[\theta]}> \frac{7}{8}\rho]})$

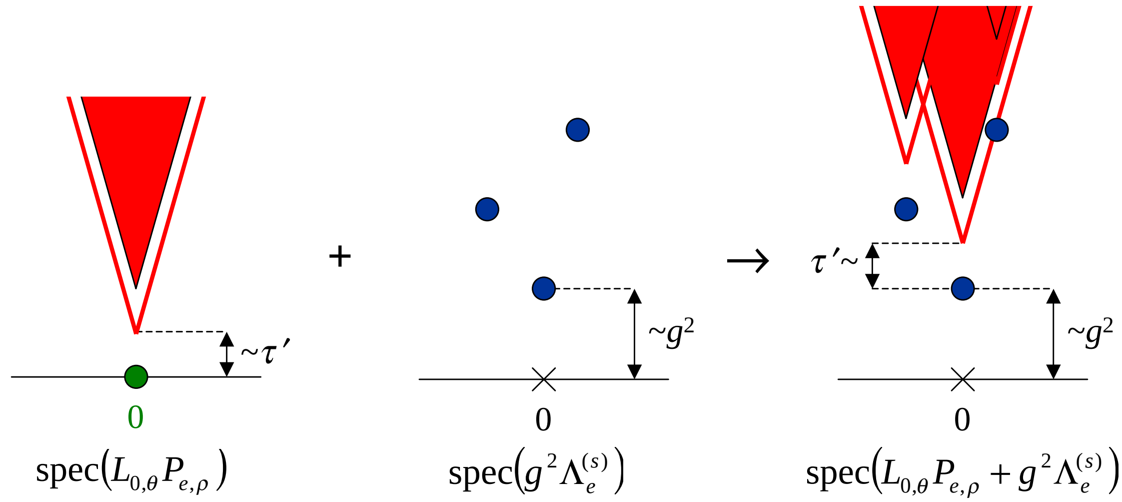

Figure 3.3: Composing the spectrum of the leading orders of $\mathfrak{F}_{\Xi_{e,\rho}}(K_\theta^{(s)} - z, L_{0,\theta} - z) + z$ out of the free operator $L_{0,\theta}$ and the level shift operator $\Lambda_e^{(s)}$.

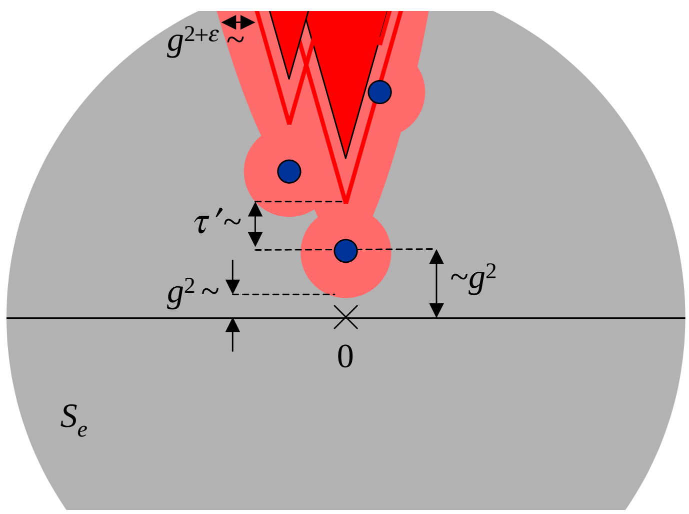

Figure 3.4: Spectrum of the operator $K_\theta^{(s)}$ inside the region $\mathcal{S}_e$ for $e \neq 0$.

and compute

$$\begin{aligned}
&\operatorname{Im} \left\langle \psi \,\middle|\, \left[L_{0,\theta} + g^2 \Lambda_e^{(s)} \otimes \chi_\rho^2(M_{[\theta]})\right] \psi \right\rangle \\
&\quad = \left\langle \psi_1 \,\middle|\, \left[M_{[\theta]} + g^2 \operatorname{Im}\left(\Lambda_e^{(s)}\right) \otimes \chi_\rho^2(M_{[\theta]})\right] \psi_1 \right\rangle \\
&\qquad + \left\langle \psi_1^\perp \,\middle|\, \left[M_{[\theta]} + g^2 \operatorname{Im}\left(\Lambda_e^{(s)}\right) \otimes \chi_\rho^2(M_{[\theta]})\right] \psi_1^\perp \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&\geq g^2 \left\langle \psi_1 \,\middle|\, \mathrm{Im}\left(\Lambda_e^{(s)}\right) \psi_1 \right\rangle + \left\langle \psi_1^\perp \,\middle|\, M_{[\theta]} \psi_1^\perp \right\rangle \\
&\geq g^2 \frac{\gamma_{\mathrm{FGR}}}{2} \left\|\psi_1\right\|^2 + \frac{7}{8}\rho \left\|\psi_1^\perp\right\|^2 \\
&\geq g^2 \frac{\gamma_{\mathrm{FGR}}}{2} \left(\left\|\psi_1\right\|^2 + \left\|\psi_1^\perp\right\|^2\right) \\
&= g^2 \frac{\gamma_{\mathrm{FGR}}}{2} \left\|\psi\right\|^2
\end{aligned}
$$

where we used that $M_{[\theta]} \geq 0$ and $\mathrm{Im}(\Lambda_e^{(s)}) \otimes \chi_\rho^2(M_{[\theta]}) \geq 0$ and $\mathrm{Im}(\Lambda_e^{(s)}) \geq \frac{\gamma_{\mathrm{FGR}}}{2}$, by Proposition 3.18, and $\rho \gg g^2$. Together with (3.45) we have

$$
\begin{aligned}
&\mathrm{Im} \left\langle \psi \,\middle|\, \left[\mathfrak{F}_{\Xi_{e,\rho}}(K_\theta^{(s)} - z, L_{0,\theta} - z) + z\right] \psi \right\rangle \\
&\quad \geq g^2 \left(\frac{\gamma_{\mathrm{FGR}}}{2} + \mathcal{O}\left(g^{\tilde{\varepsilon}}\right)\right) \left\|\psi\right\|^2 \\
&\quad \geq g^2 \frac{\gamma_{\mathrm{FGR}}}{4} \left\|\psi\right\|^2 ,
\end{aligned}
$$

for g sufficiently small. This implies that

$$
\mathrm{NumRan}\left(\mathfrak{F}_{\Xi_{e,\rho}}(K_\theta^{(s)} - z, L_{0,\theta} - z) + z\right) \subseteq \left\{\zeta \in \mathbb{C} \,\middle|\, \mathrm{Im}(\zeta) \geq g^2 \frac{\gamma_{\mathrm{FGR}}}{4} \right\}
$$

and hence

$$
\mathrm{spec}\left(\mathfrak{F}_{\Xi_{e,\rho}}(K_\theta^{(s)} - z, L_{0,\theta} - z) + z\right) \subseteq \left\{\zeta \in \mathbb{C} \,\middle|\, \mathrm{Im}(\zeta) \geq g^2 \frac{\gamma_{\mathrm{FGR}}}{4} \right\}
$$

by [30, Cor. 3.3.]. The isospectrality of the Feshbach map leads to the assertion. ■

3.5 Spectrum in the Neighborhood of the Zero Eigenvalue

In this section we focus on the analysis of the spectrum of $K_\theta^{(s)}$ in the neighborhood of zero. The main purpose is to compute explicitly the level shift operator $\Lambda_0^{(s)}$ whose properties will allow a further application of the Feshbach map in Chapter 4 – in order to study the spectrum of $K_\theta^{(s)}$ on smaller scales.

Lemma 3.20 *The level shift operator $\Lambda_{0,r}^{(0)}$ associated with the zero eigenvalue $e = 0$ of L_{p} is anti-selfadjoint, i.e.,*

$$
\Lambda_{0,r}^{(0)} = i\Gamma_{0,r} \qquad \textit{with} \qquad \Gamma_{0,r} = \Gamma_{0,r}^*.
$$

Moreover, the matrix elements of $\Gamma_{0,r}$ *in the orthonormal basis* $\{\varphi_{j,j}\}_{j=0,1,\dots,N-1}$ *of the kernel of* L_{p} *are given by*

$$\left(\Gamma_{0,r}\right)_{m,n} := \langle\, \varphi_{m,m} \,|\, \Gamma_{0,r}\varphi_{n,n}\,\rangle_{\mathcal{H}_{\mathrm{p}}^2} = \delta_{m,n} \sum_{\substack{k=0,\\ k\neq m}}^{N-1} e^{\beta_r E_{m,k}/2}\eta^r_{m,k} - (1-\delta_{m,n})\eta^r_{m,n}, \quad (3.46)$$

where the positive numbers $\eta^r_{j,k}$ *have been defined in (3.36). Further, the matrix elements fulfill*

$$\left(\Gamma_{0,r}\right)_{m,n} = \left(\Gamma_{0,r}\right)_{n,m} \in \mathbb{R}.$$

Proof. The matrix elements of $\Lambda^{(0)}_{0,r}$ are given by

$$\begin{aligned}\left(\Lambda^{(0)}_{0,r}\right)_{m,n} &:= \left\langle \varphi_{m,m} \,\middle|\, \Lambda^{(0)}_{0,r}\varphi_{n,n} \right\rangle_{\mathcal{H}_{\mathrm{p}}^2} \\ &= -\lim_{\varepsilon\searrow 0} \sum_{k,\ell=0}^{N-1} \int_{\mathbb{R}\times S^2} d(u,\Sigma)\, \langle\, \varphi_{m,m} \,|\, F(y)^*\varphi_{k,\ell}\,\rangle_{\mathcal{H}_{\mathrm{p}}^2} \langle\, \varphi_{k,\ell} \,|\, F(y)\varphi_{n,n}\,\rangle_{\mathcal{H}_{\mathrm{p}}^2} \\ &\qquad\qquad \times (E_{k,\ell}+u+i\varepsilon)^{-1},\end{aligned}$$

where $y=(u,\Sigma,r)\in\Upsilon$ and, recall $F=\mathcal{G}-\mathcal{G}'$. We abbreviate

$$\mathcal{G}(y)_{j,k} := \langle\, \varphi_{j,k} \,|\, \mathcal{G}(y)\varphi_{k,k}\,\rangle_{\mathcal{H}_{\mathrm{p}}^2} \qquad \text{and} \qquad \mathcal{G}'(y)_{j,k} := \langle\, \varphi_{j,j} \,|\, \mathcal{G}'(y)\varphi_{j,k}\,\rangle_{\mathcal{H}_{\mathrm{p}}^2}$$

to express

$$\begin{aligned}&\sum_{k,\ell=0}^{N-1} \langle\, \varphi_{m,m} \,|\, F(y)^*\varphi_{k,\ell}\,\rangle_{\mathcal{H}_{\mathrm{p}}^2} \langle\, \varphi_{k,\ell} \,|\, F(y)\varphi_{n,n}\,\rangle_{\mathcal{H}_{\mathrm{p}}^2} (E_{k,\ell}+u+i\varepsilon)^{-1} \\ &= \sum_{k,\ell=0}^{N-1} \Big[\overline{\mathcal{G}(y)_{k,m}}\mathcal{G}(y)_{k,n}\delta_{\ell,m}\delta_{\ell,n} + \overline{\mathcal{G}'(y)_{\ell,m}}\mathcal{G}'(y)_{\ell,n}\delta_{k,m}\delta_{k,n} \\ &\qquad - \overline{\mathcal{G}'(y)_{\ell,m}}\mathcal{G}(y)_{k,n}\delta_{k,m}\delta_{\ell,n} - \overline{\mathcal{G}(y)_{k,m}}\mathcal{G}'(y)_{\ell,n}\delta_{\ell,m}\delta_{k,n}\Big](E_{k,\ell}+u+i\varepsilon)^{-1} \\ &= \delta_{m,n}\sum_{k=0}^{N-1}\Big[\overline{\mathcal{G}(y)_{k,m}}\mathcal{G}(y)_{k,n}(E_{k,m}+u+i\varepsilon)^{-1} \\ &\qquad + \overline{\mathcal{G}'(y)_{k,m}}\mathcal{G}'(y)_{k,n}(E_{m,k}+u+i\varepsilon)^{-1}\Big] \\ &\quad -\overline{\mathcal{G}'(y)_{n,m}}\mathcal{G}(y)_{m,n}(E_{m,n}+u+i\varepsilon)^{-1} - \overline{\mathcal{G}(y)_{n,m}}\mathcal{G}'(y)_{m,n}(E_{n,m}+u+i\varepsilon)^{-1} \\ &= \delta_{m,n}\sum_{k=0}^{N-1}\Big[\overline{\mathcal{G}(y)_{k,m}}\mathcal{G}(y)_{k,n}(E_{k,m}+u+i\varepsilon)^{-1} \\ &\qquad + \mathcal{G}(-y)_{k,m}\overline{\mathcal{G}(-y)_{k,n}}(E_{m,k}+u+i\varepsilon)^{-1}\Big] \\ &\quad +\mathcal{G}(-y)_{n,m}\mathcal{G}(y)_{m,n}(E_{m,n}+u+i\varepsilon)^{-1} + \overline{\mathcal{G}(y)_{n,m}\mathcal{G}(-y)_{m,n}}(E_{n,m}+u+i\varepsilon)^{-1},\end{aligned}$$

where we used (3.37). The integration over $(u,\Sigma) \in \mathbb{R} \times S^2$ and a transformation of variables $u \mapsto -u$ while integrating the second and fourth addend yields

$$
\begin{aligned}
\left(\Lambda_{0,r}^{(0)}\right)_{m,n} &= -\delta_{m,n} \lim_{\varepsilon \searrow 0} \sum_{k=0}^{N-1} \int_{\mathbb{R}\times S^2} d(u,\Sigma) \left[\overline{\mathcal{G}(y)_{k,m}} \mathcal{G}(y)_{k,n} (E_{k,m} + u + i\varepsilon)^{-1} \right. \\
&\qquad \left. - \mathcal{G}(y)_{k,m} \overline{\mathcal{G}(y)_{k,n}} (E_{m,k} + u - i\varepsilon)^{-1} \right] \\
&\quad - \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}\times S^2} d(u,\Sigma) \left[\mathcal{G}(-y)_{n,m} \mathcal{G}(y)_{m,n} (E_{m,n} + u + i\varepsilon)^{-1} \right. \\
&\qquad \left. - \overline{\mathcal{G}(-y)_{n,m} \mathcal{G}(y)_{m,n}} (E_{m,n} + u - i\varepsilon)^{-1} \right] \\
&= \delta_{m,n} \lim_{\varepsilon \searrow 0} \sum_{k=0}^{N-1} \int_{\mathbb{R}\times S^2} d(u,\Sigma)\, |\mathcal{G}(y)_{k,m}|^2 \frac{2i\varepsilon}{(E_{k,m}+u)^2 + \varepsilon^2} \\
&\quad - \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}\times S^2} d(u,\Sigma)\, 2i \operatorname{Im} \left[\mathcal{G}(-y)_{n,m} \mathcal{G}(y)_{m,n} (E_{m,n} + u + i\varepsilon)^{-1} \right] \\
&= \delta_{m,n} \lim_{\varepsilon \searrow 0} \sum_{k=0}^{N-1} \int_{\mathbb{R}\times S^2} d(u,\Sigma)\, |\mathcal{G}(y)_{k,m}|^2 \frac{2i\varepsilon}{(E_{k,m}+u)^2 + \varepsilon^2} \\
&\quad + \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}\times S^2} d(u,\Sigma)\, \mathcal{G}(-y)_{n,m} \mathcal{G}(y)_{m,n} \frac{2i\varepsilon}{(E_{m,n}+u)^2 + \varepsilon^2} \\
&= 2\pi i\, \delta_{m,n} \sum_{k=0}^{N-1} \int_{S^2} d\Sigma\, |\mathcal{G}(E_{m,k},\Sigma,r)_{k,m}|^2 \\
&\quad + 2\pi i \int_{S^2} d\Sigma\, \mathcal{G}(E_{m,n},\Sigma,r)_{n,m} \mathcal{G}(E_{n,m},\Sigma,r)_{m,n} \\
&= 2\pi i \left[\delta_{m,n} \sum_{\substack{k=0,\\ k\neq m}}^{N-1} \int_{S^2} d\Sigma\, |\mathcal{G}(E_{m,k},\Sigma,r)_{k,m}|^2 \right. \\
&\qquad \left. + (1-\delta_{m,n}) \int_{S^2} d\Sigma\, \mathcal{G}(E_{m,n},\Sigma,r)_{n,m} \mathcal{G}(E_{n,m},\Sigma,r)_{m,n} \right].
\end{aligned}
$$

Here, we used that

$$
\mathcal{G}(-y)_{n,m}\mathcal{G}(y)_{m,n} = -\frac{u^2}{\sqrt{e^{\beta_r u} + e^{-\beta_r u} - 2}} \times \begin{cases} |G_r(u\Sigma)_{m,n}|^2, & u \geq 0, \\ |G_r(-u\Sigma)_{n,m}|^2, & u < 0, \end{cases}
$$

is negative, and in particular real, as one easily checks by explicit computations. We now define $\Gamma_{0,r}$ in terms of its matrix elements,

$$\begin{aligned}(\Gamma_{0,r})_{m,n} &:= 2\pi\Bigg[\delta_{m,n}\sum_{\substack{k=0,\\k\neq m}}^{N-1}\int_{S^2} d\Sigma\;|\mathcal{G}(E_{m,k},\Sigma,r)_{k,m}|^2\\&\qquad+(1-\delta_{m,n})\int_{S^2} d\Sigma\,\mathcal{G}(E_{m,n},\Sigma,r)_{n,m}\mathcal{G}(E_{n,m},\Sigma,r)_{m,n}\Bigg]\in\mathbb{R},\end{aligned}$$

such that $\Lambda_{0,r}^{(0)} = i\Gamma_{0,r}$ and apparently $(\Gamma_{0,r})_{m,n} = (\Gamma_{0,r})_{n,m}$. This implies that $\Gamma_{0,r}$ is selfadjoint. To give a more explicit representation to $(\Gamma_{0,r})_{m,n}$ we remark that, for $j > k$,

$$\begin{aligned}&2\pi\int_{S^2} d\Sigma\,\mathcal{G}(E_{k,j},\Sigma,r)_{j,k}\mathcal{G}(E_{j,k},\Sigma,r)_{k,j}\\&= -\frac{2\pi E_{j,k}^2}{\sqrt{e^{\beta_r E_{j,k}}+e^{-\beta_r E_{j,k}}-2}}\int_{S^2} d\Sigma\;|G_r(E_{j,k}\Sigma)_{k,j}|^2\\&= -2\pi E_{j,k}^2\sqrt{\rho_{\mathrm{f},r}(E_{j,k})\,(1+\rho_{\mathrm{f},r}(E_{j,k}))}\int_{S^2} d\Sigma\;|G_r(E_{j,k}\Sigma)_{k,j}|^2\\&= -\eta_{j,k}^r = -\eta_{k,j}^r.\end{aligned}$$

Further, we note that

$$|\mathcal{G}(y)_{j,k}|^2 = \frac{u^2}{|1-e^{-\beta_r u}|}\times\begin{cases}|G_r(u\Sigma)_{j,k}|^2, & u\geq 0,\\ |G_r(-u\Sigma)_{k,j}|^2, & u<0,\end{cases}$$

and therefore, for $k\neq m$,

$$\begin{aligned}&\int_{S^2} d\Sigma\;|\mathcal{G}(E_{m,k},\Sigma,r)_{k,m}|^2\\&= \int_{S^2} d\Sigma\,\frac{E_{m,k}^2}{|1-e^{-\beta_r E_{m,k}}|}\times\begin{cases}|G_r(E_{m,k}\Sigma)_{k,m}|^2, & m>k,\\ |G_r(E_{k,m}\Sigma)_{m,k}|^2, & m<k,\end{cases}\\&= \int_{S^2} d\Sigma\,\frac{E_{m,k}^2 e^{\beta_r E_{m,k}/2}}{\sqrt{(1-e^{-\beta_r E_{m,k}})(e^{\beta_r E_{m,k}}-1)}}\times\begin{cases}|G_r(E_{m,k}\Sigma)_{k,m}|^2, & m>k,\\ |G_r(E_{k,m}\Sigma)_{m,k}|^2, & m<k,\end{cases}\\&= -e^{\beta_r E_{m,k}/2}\int_{S^2} d\Sigma\,\mathcal{G}(E_{k,m},\Sigma,r)_{m,k}\mathcal{G}(E_{m,k},\Sigma,r)_{k,m}\\&= \frac{1}{2\pi}\eta_{m,k}^r.\end{aligned}$$

Thus, we get

$$\left(\Gamma_{0,r}\right)_{m,n} = \delta_{m,n} \sum_{\substack{k=0,\\ k\neq m}}^{N-1} e^{\beta_r E_{m,k}/2} \eta^r_{m,k} - (1-\delta_{m,n}) \eta^r_{m,n}.$$

■

The *detailed balance* structure (3.46) of the matrix elements of $\Gamma_{0,r}$ gives rise to the knowledge about positivity of $\Gamma_{0,r}$ and uniqueness of the zero eigenvalue.

Lemma 3.21 *The self-adjoint operator* $\Gamma_{0,r}$ *is non-negative and it has a simple zero eigenvalue. The kernel of* $\Gamma_{0,r}$ *is spanned by the vector*

$$\Omega_{\mathrm{p}}(\beta_r) := Z(\beta_r)^{-1/2} \sum_{k=0}^{N-1} e^{-\beta_r E_k/2} \varphi_{k,k}.$$

Moreover, the gap between the zero and the next positive eigenvalue is at least

$$\gamma_0^r := Z(\beta_r) \left(\min_{\substack{m,n=0,\\ m>n}}^{N-1} \mu^r_{m,n} \right),$$

where

$$\mu^r_{j,k} := e^{\beta_r (E_j+E_k)/2} \eta^r_{j,k} = \mu^r_{k,j}.$$

In other words,

$$\langle \psi \mid \Gamma_{0,r} \psi \rangle_{\mathcal{H}^2_{\mathrm{p}}} \geq \gamma_0^r \, \|\psi\|^2_{\mathcal{H}^2_{\mathrm{p}}}$$

for all $\psi \in \ker(L_{\mathrm{p}})$ *with* $\psi \perp \Omega_{\mathrm{p}}(\beta_r)$*.*

Proof. We compute

$$\begin{aligned}
Z(\beta_r)^{1/2} \Gamma_{0,r} \Omega_{\mathrm{p}}(\beta_r) &= \sum_{n=0}^{N-1} e^{-\beta_r E_n/2} \Gamma_{0,r} \varphi_{n,n} \\
&= \sum_{m,n=0}^{N-1} e^{-\beta_r E_n/2} \left(\Gamma_{0,r}\right)_{m,n} \varphi_{m,m} \\
&= \sum_{m=0}^{N-1} \left[\sum_{\substack{k=0,\\ k\neq m}}^{N-1} e^{-\beta_r E_m/2} e^{\beta_r E_{m,k}/2} \eta^r_{m,k} - \sum_{\substack{n=0,\\ n\neq m}}^{N-1} e^{-\beta_r E_n/2} \eta^r_{m,n} \right] \varphi_{m,m} \\
&= 0.
\end{aligned}$$

Therefore, the matrix $\Gamma_{0,r}$ has a zero eigenvector with strictly positive components. Since further the matrix has strictly negative off-diagonal elements $(\Gamma_{0,r})_{m,n} = -\eta^r_{m,n}$ for $m \neq n$, we obtain by a *Perron-Frobenius argument* that zero is a simple eigenvalue being the bottom of the spectrum of $\Gamma_{0,r}$. For this insight we consider the matrix

$$\left(\max_{m=0}^{N-1} (\Gamma_{0,r})_{m,m}\right) \mathbb{1}_{\ker(L_\mathrm{p})} - \Gamma_{0,r}$$

having non-negative matrix elements and we note that the positive vector $\Omega_\mathrm{p}(\beta_r)$ is an eigenvector corresponding to the eigenvalue $\max_{m=0}^{N-1} (\Gamma_{0,r})_{m,m}$. The application of [20, Hauptsatz 1.8] implies that $\max_{m=0}^{N-1} (\Gamma_{0,r})_{m,m}$ is a simple eigenvalue and further that it is the eigenvalue with the largest absolute value.

To estimate the gap between the zero eigenvalue and the rest of the spectrum we choose $\psi = \sum_{j=0}^{N-1} \kappa_j \varphi_{j,j} \perp \Omega_\mathrm{p}(\beta_r)$ and we compute

$$\begin{aligned}
\langle \psi \,|\, \Gamma_{0,r}\psi \rangle_{\mathcal{H}^2_\mathrm{p}} &= \sum_{m,n=0}^{N-1} \overline{\kappa}_m \kappa_n (\Gamma_{0,r})_{m,n} \\
&= \sum_{m=0}^{N-1} \sum_{\substack{k=0,\\ k\neq m}}^{N-1} |\kappa_m|^2 e^{\beta_r E_{m,k}/2} \eta^r_{m,k} - \sum_{m=0}^{N-1} \sum_{\substack{n=0,\\ n\neq m}}^{N-1} \overline{\kappa}_m \kappa_n \eta^r_{m,n} \\
&= \sum_{m,n=0}^{N-1} \mu^r_{m,n} \left[\left| e^{-\beta_r E_n/2} \kappa_m \right|^2 - \overline{e^{-\beta_r E_n/2} \kappa_m} e^{-\beta_r E_m/2} \kappa_n \right] \\
&= \sum_{\substack{m,n=0,\\ m>n}}^{N-1} \mu^r_{m,n} \left| e^{-\beta_r E_n/2} \kappa_m - e^{-\beta_r E_m/2} \kappa_n \right|^2 \\
&\geq \left(\min_{\substack{m,n=0,\\ m>n}}^{N-1} \mu^r_{m,n} \right) \sum_{\substack{m,n=0,\\ m>n}}^{N-1} \left| e^{-\beta_r E_n/2} \kappa_m - e^{-\beta_r E_m/2} \kappa_n \right|^2 \\
&= \left(\min_{\substack{m,n=0,\\ m>n}}^{N-1} \mu^r_{m,n} \right) \left[\sum_{n=0}^{N-1} e^{-\beta_r E_n} \sum_{m=0}^{N-1} |\kappa_m|^2 - \left| \sum_{m=0}^{N-1} e^{-\beta_r E_m/2} \kappa_m \right|^2 \right] \\
&= \left(\min_{\substack{m,n=0,\\ m>n}}^{N-1} \mu^r_{m,n} \right) \left[Z(\beta_r) \, \|\psi\|^2_{\mathcal{H}^2_\mathrm{p}} - Z(\beta_r) \left| \langle \psi \,|\, \Omega_\mathrm{p}(\beta_r) \rangle_{\mathcal{H}^2_\mathrm{p}} \right|^2 \right] \\
&= \left(\min_{\substack{m,n=0,\\ m>n}}^{N-1} \mu^r_{m,n} \right) Z(\beta_r) \, \|\psi\|^2_{\mathcal{H}^2_\mathrm{p}} .
\end{aligned}$$

∎

For our results it is important that the gap γ_0^r is strictly positive, uniformly in $\beta_r > 0$. This fact is given by the next lemma.

Lemma 3.22 *(i) The gap γ_0^r is strictly positive uniformly in $\beta_r > 0$, it holds*

$$\inf_{\beta_r>0} \gamma_0^r \geq \min_{\substack{m,n=0,\\ m>n}}^{N-1} 2\pi E_{m,n}^2 \int_{S^2} d\Sigma \, \left|G_r(E_{m,n}\Sigma)_{n,m}\right|^2 \geq \gamma_{\mathrm{FGR}} > 0$$

where the Fermi golden rule level shift γ_{FGR} was defined and assumed to be positive in (1.86).

(ii) The matrix $\Gamma_{0,r}$ is bounded uniformly in $\beta_r \to \infty$,

$$\limsup_{\beta_r\to\infty} \|\Gamma_{0,r}\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} \leq \max_{m=0}^{N-1} \sum_{\substack{k=0,\\ k\neq m}}^{N-1} 2\pi E_{m,k}^2 \int_{S^2} d\Sigma \, \left|G_r(E_{m,k}\Sigma)_{k,m}\right|^2 < \infty,$$

i.e., it is bounded uniformly for the inverse temperature in the parameter range as specified in Hypothesis III-1.8.

Proof.

(i) We consider for $m > n$

$$\begin{aligned} Z(\beta_r)\mu_{m,n}^r &= \sum_{j=0}^{N-1} e^{-\beta_r E_j} e^{\beta_r(E_m+E_n)/2}\eta_{m,n}^r \\ &\geq e^{\beta_r(E_m-E_n)/2}\eta_{m,n}^r \\ &= 2\pi E_{m,n}^2 \left(1+\rho_{\mathrm{f},r}(E_{m,n})\right) \int_{S^2} d\Sigma \, \left|G_r(E_{m,n}\Sigma)_{n,m}\right|^2 \\ &\xrightarrow{\beta_r\to\infty} 2\pi E_{m,n}^2 \int_{S^2} d\Sigma \, \left|G_r(E_{m,n}\Sigma)_{n,m}\right|^2 . \end{aligned}$$

Since $\rho_{\mathrm{f},r}(E_{m,n}) = (e^{\beta_r E_{m,n}} - 1)^{-1}$ is monotonically decreasing in β_r we obtain

$$\gamma_0^r \geq \min_{\substack{m,n=0,\\ m>n}}^{N-1} 2\pi E_{m,n}^2 \int_{S^2} d\Sigma \, \left|G_r(E_{m,n}\Sigma)_{n,m}\right|^2 \geq \gamma_{\mathrm{FGR}} > 0,$$

while the positivity of the quantity of the r.h.s. is ensured through the assumption of the Fermi golden rule condition, Hypothesis V-1.10.

(ii) For $j > k$ we have

$$e^{\pm\beta_r E_{j,k}/2}\eta^r_{j,k} \leq e^{\beta_r E_{j,k}/2}\eta^r_{j,k} = 2\pi E^2_{j,k}\left(1+\rho_{\mathrm{f},r}(E_{j,k})\right)\int_{S^2} d\Sigma\,\left|G_r(E_{j,k}\Sigma)_{k,j}\right|^2$$

and therefore

$$\begin{aligned}
&\|\Gamma_{0,r}\|_{\mathcal{B}(\ker(L_\mathrm{p}))}\\
&\leq \max_{m,n=0}^{N-1}\left|(\Gamma_{0,r})_{m,n}\right| = \max_{m=0}^{N-1}\sum_{\substack{k=0,\\ k\neq m}}^{N-1} e^{\beta_r E_{m,k}/2}\eta^r_{m,k}\\
&\leq \max_{m=0}^{N-1}\sum_{\substack{k=0,\\ k\neq m}}^{N-1} 2\pi E^2_{m,k}\left(1+\rho_{\mathrm{f},r}(E_{m,k})\right)\int_{S^2} d\Sigma\,\left|G_r(E_{m,k}\Sigma)_{k,m}\right|^2\\
&\xrightarrow{\beta_r\to\infty} \max_{m=0}^{N-1}\sum_{\substack{k=0,\\ k\neq m}}^{N-1} 2\pi E^2_{m,k}\int_{S^2} d\Sigma\,\left|G_r(E_{m,k}\Sigma)_{k,m}\right|^2\\
&< \infty.
\end{aligned}$$

■

The Lemmata 3.15, 3.20 and 3.21 allow to localize the numerical range of the level shift operator.

Proposition 3.23 *(i) The level shift operator $\Lambda_0^{(-i/2)}$ corresponding to the C-Liouville operator $K = K^{(-i/2)}$ has zero as simple eigenvalue and its kernel is spanned by the eigenvector $\Omega_\mathrm{p} \equiv \Omega_\mathrm{p}(\beta_\mathrm{p})$. If further the particle temperature coincides with the temperature of one of the reservoirs, i.e., $\beta_\mathrm{p} = \beta_{r'}$ for some $r' = 1, \ldots, R$, and $|\beta_{\max} - \beta_{\min}| \ll 1$ is sufficiently small then we have the estimate*

$$\mathrm{Im}\left\langle \psi \,\middle|\, \Lambda_0^{(-i/2)}\psi \right\rangle_{\mathcal{H}^2_\mathrm{p}} \geq \gamma_0^{r'}\,\|\psi\|^2_{\mathcal{H}^2_\mathrm{p}} \geq \gamma_{\mathrm{FGR}}\,\|\psi\|^2_{\mathcal{H}^2_\mathrm{p}}$$

for all $\psi \in \ker(L_\mathrm{p})$ with $\psi \perp \Omega_\mathrm{p}$.

(ii) In the case that the particle and the reservoir temperatures coincide, i.e., $\beta = \beta_\mathrm{p} = \beta_1 = \cdots = \beta_R$, the anti-self-adjoint level shift operator $\Lambda_0^{(0)}$ corresponding to the standard Liouville operator $L = K^{(0)}$ has zero as simple eigenvalue and its kernel is spanned by the eigenvector $\Omega_\mathrm{p}(\beta)$. For $\Omega_\mathrm{p}(\beta) \perp \psi \in \ker(L_\mathrm{p})$ we have the estimate

$$-i\left\langle \psi \,\middle|\, \Lambda_0^{(0)}\psi \right\rangle_{\mathcal{H}^2_\mathrm{p}} \geq \left(\sum_{r=1}^R \gamma_0^r\right)\|\psi\|^2_{\mathcal{H}^2_\mathrm{p}} \geq R\gamma_{\mathrm{FGR}}\,\|\psi\|^2_{\mathcal{H}^2_\mathrm{p}}\,.$$

(iii) In the case that at least two reservoirs are at different temperatures, i.e., $\beta_{\min} < \beta_{\max}$*, the anti-selfadjoint level shift operator* $\Lambda_0^{(0)}$ *corresponding to the standard Liouville operator* $L = K^{(0)}$ *obeys the following estimate,*

$$-i \left\langle \psi \,\middle|\, \Lambda_0^{(0)} \psi \right\rangle_{\mathcal{H}_{\mathrm{p}}^2} \geq \gamma_{\mathrm{FGR}} \left[1 - \langle\, \Omega_{\mathrm{p}}(\beta_{\min}) \,|\, \Omega_{\mathrm{p}}(\beta_{\max}) \,\rangle_{\mathcal{H}_{\mathrm{p}}^2} \right] \|\psi\|_{\mathcal{H}_{\mathrm{p}}^2}^2$$

for all $\psi \in \ker(L_{\mathrm{p}})$*. Hereby, the distance of the eigenvalue with the lowest imaginary part from the real axis is of order*

$$\begin{aligned} &\gamma_{\mathrm{FGR}} \left[1 - \langle\, \Omega_{\mathrm{p}}(\beta_{\min}) \,|\, \Omega_{\mathrm{p}}(\beta_{\max}) \,\rangle_{\mathcal{H}_{\mathrm{p}}^2} \right] \\ &\geq \left[(\beta_{\max} - \beta_{\min})^2 - \mathcal{O}\left(|\beta_{\max} - \beta_{\min}|^3 \right) \right] \frac{\gamma_{\mathrm{FGR}} d_{L_{\mathrm{p}}}}{16} \left[1 - \frac{Z(2\overline{\beta})}{Z(\overline{\beta})^2} \right] \end{aligned} \tag{3.47}$$

where $\overline{\beta} := (\beta_{\max} + \beta_{\min})/2$*. In particular,* $\Lambda_0^{(0)}$ *has no zero eigenvalue.*

Proof.

(i) We note that

$$\begin{aligned} A\left(\frac{i}{2}(\beta_r - \beta_{\mathrm{p}}) \right) \Omega_{\mathrm{p}}(\beta_{\mathrm{p}}) &= Z(\beta_{\mathrm{p}})^{-1/2} \sum_{j=0}^{N-1} e^{-\beta_{\mathrm{p}} E_j/2} \left[e^{-(\beta_r - \beta_{\mathrm{p}}) H_{\mathrm{p}}/2} \varphi_j \right] \otimes \varphi_j \\ &= Z(\beta_{\mathrm{p}})^{-1/2} \sum_{j=0}^{N-1} e^{-\beta_r E_j/2} \varphi_{j,j} \\ &= \sqrt{\frac{Z(\beta_r)}{Z(\beta_{\mathrm{p}})}}\, \Omega_{\mathrm{p}}(\beta_r). \end{aligned}$$

By Lemmata 3.15, 3.20 and 3.21 we get

$$\begin{aligned} \Lambda_0^{(-i/2)} \Omega_{\mathrm{p}}(\beta_{\mathrm{p}}) &= i \sum_{r=1}^{R} A\left(-\frac{i}{2}(\beta_r - \beta_{\mathrm{p}}) \right) \Gamma_{0,r} A\left(\frac{i}{2}(\beta_r - \beta_{\mathrm{p}}) \right) \Omega_{\mathrm{p}}(\beta_{\mathrm{p}}) \\ &= i \sum_{r=1}^{R} \sqrt{\frac{Z(\beta_r)}{Z(\beta_{\mathrm{p}})}}\, A\left(-\frac{i}{2}(\beta_r - \beta_{\mathrm{p}}) \right) \Gamma_{0,r} \Omega_{\mathrm{p}}(\beta_r) \\ &= 0. \end{aligned}$$

Since $\Gamma_{0,r}$ has purely negative off-diagonal elements so has the matrix $-i\Lambda_0^{(-i/2)}$

because of

$$\begin{aligned}\left(-i\Lambda_0^{(-i/2)}\right)_{m,n} &:= \left\langle \varphi_{m,m} \,\middle|\, (-i)\Lambda_0^{(-i/2)}\varphi_{n,n} \right\rangle_{\mathcal{H}_\mathrm{p}^2} \\ &= \sum_{r=1}^{R} \left\langle A\left(-\frac{i}{2}(\beta_r - \beta_\mathrm{p})\right)\varphi_{m,m} \,\middle|\, \Gamma_{0,r} A\left(\frac{i}{2}(\beta_r - \beta_\mathrm{p})\right)\varphi_{n,n} \right\rangle_{\mathcal{H}_\mathrm{p}^2} \\ &= \sum_{r=1}^{R} e^{(\beta_r - \beta_\mathrm{p})E_{m,n}/2} \left(\Gamma_{0,r}\right)_{m,n}.\end{aligned}$$

An application of [20, Hauptsatz 1.8] similar to the one in the proof of Lemma 3.21 yields that zero is a simple eigenvalue and that no other eigenvalue of $-i\Lambda_0^{(-i/2)}$ has negative real part.

Let now $\Omega_\mathrm{p}(\beta_\mathrm{p}) \perp \psi \in \ker(L_\mathrm{p})$. For $r = 1, \dots, R$, the vector

$$\xi_r := A\left(-\frac{i}{2}(\beta_r - \beta_\mathrm{p})\right)\psi$$

is orthogonal to the vector $\Omega_\mathrm{p}(\beta_r) = Z(\beta_r)^{-1/2} Z(\beta_\mathrm{p})^{1/2} A\left(\frac{i}{2}(\beta_r - \beta_\mathrm{p})\right)\Omega_\mathrm{p}(\beta_\mathrm{p})$. With this notation we obtain with Lemma 3.21

$$\begin{aligned}&\operatorname{Im}\left\langle \psi \,\middle|\, \Lambda_0^{(-i/2)}\psi \right\rangle_{\mathcal{H}_\mathrm{p}^2} \\ &= \langle \psi \,|\, \Gamma_{0,r'}\psi \rangle_{\mathcal{H}_\mathrm{p}^2} + \sum_{\substack{r=1,\\ r\neq r'}}^{R} \operatorname{Im}\langle \xi_r \,|\, \Gamma_{0,r} A\left(i(\beta_r - \beta_\mathrm{p})\right)\xi_r \rangle_{\mathcal{H}_\mathrm{p}^2} \\ &= \langle \psi \,|\, \Gamma_{0,r'}\psi \rangle_{\mathcal{H}_\mathrm{p}^2} + \sum_{\substack{r=1,\\ r\neq r'}}^{R} \langle \xi_r \,|\, \Gamma_{0,r}\xi_r \rangle_{\mathcal{H}_\mathrm{p}^2} \\ &\quad + \sum_{\substack{r=1,\\ r\neq r'}}^{R} \operatorname{Im}\langle \xi_r \,|\, \Gamma_{0,r}\left[A\left(i(\beta_r - \beta_\mathrm{p}) - \mathbb{1}\right)\right]\xi_r \rangle_{\mathcal{H}_\mathrm{p}^2} \\ &\geq \gamma_0^{r'}\, \|\psi\|_{\mathcal{H}_\mathrm{p}^2}^2 + \sum_{\substack{r=1,\\ r\neq r'}}^{R} \left[\gamma_0^r - \|\Gamma_{0,r}\left[A\left(i(\beta_r - \beta_\mathrm{p})\right) - \mathbb{1}\right]\|_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)}\right] \|\xi_r\|_{\mathcal{H}_\mathrm{p}^2}^2 \\ &\geq \gamma_0^{r'}\, \|\psi\|_{\mathcal{H}_\mathrm{p}^2}^2 + \sum_{\substack{r=1,\\ r\neq r'}}^{R} \left[\gamma_0^r - C|\beta_{\max} - \beta_{\min}|\, \|\Gamma_{0,r}\|_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)}\right] \|\xi_r\|_{\mathcal{H}_\mathrm{p}^2}^2 \\ &\geq \gamma_0^{r'}\, \|\psi\|_{\mathcal{H}_\mathrm{p}^2}^2 \geq \gamma_{\mathrm{FGR}}\, \|\psi\|_{\mathcal{H}_\mathrm{p}^2}^2\end{aligned}$$

for the positive constant $C := \|H_\mathrm{p}\|_{\mathcal{B}(\mathcal{H}_\mathrm{p})} \exp((\beta_{\max} - \beta_{\min})\, \|H_\mathrm{p}\|_{\mathcal{B}(\mathcal{H}_\mathrm{p})}) < \infty$, for $|\beta_{\max} - \beta_{\min}|$ sufficiently small. Hereby, we used Lemma 3.22 to estimate γ_0^r and $\|\Gamma_{0,r}\|_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)}$ uniformly in β_r from a compact set in $\mathbb{R}^+$.

(ii) Since $\Lambda_0^{(0)} = i\sum_{r=1}^{R}\Gamma_{0,r}$ and further $\Gamma_{0,r}\Omega_{\mathrm{p}}(\beta) = \Gamma_{0,r}\Omega_{\mathrm{p}}(\beta_r) = 0$ we see immediately that $\Lambda_0^{(0)}\Omega_{\mathrm{p}}(\beta) = 0$. The application of Lemma 3.21 implies for $\psi \perp \Omega_{\mathrm{p}}(\beta)$

$$-i\left\langle \psi \,\middle|\, \Lambda_0^{(0)}\psi \right\rangle_{\mathcal{H}_{\mathrm{p}}^2} = \sum_{r=1}^{R} \langle \psi \,|\, \Gamma_{0,r}\psi \rangle_{\mathcal{H}_{\mathrm{p}}^2} \geq \sum_{r=1}^{R} \gamma_0^r \,\|\psi\|_{\mathcal{H}_{\mathrm{p}}^2}^2 \geq R\gamma_{\mathrm{FGR}}\,\|\psi\|_{\mathcal{H}_{\mathrm{p}}^2}^2 .$$

(iii) Throughout the proof we assume that $\psi \in \ker(L_{\mathrm{p}})$ with $\|\psi\|_{\mathcal{H}_{\mathrm{p}}^2} = 1$. We first observe that holds

$$-i\left\langle \psi \,\middle|\, \Lambda_0^{(0)}\psi \right\rangle_{\mathcal{H}_{\mathrm{p}}^2} \geq \sum_{r=1,R} \langle \psi \,|\, \Gamma_{0,r}\psi \rangle_{\mathcal{H}_{\mathrm{p}}^2} .$$

By Lemmata 3.21 and 3.22 we have

$$\langle \psi \,|\, \Gamma_{0,r}\psi \rangle_{\mathcal{H}_{\mathrm{p}}^2} \geq \gamma_{\mathrm{FGR}}\left[1 - \left|\langle \Omega_{\mathrm{p}}(\beta_r) \,|\, \psi \rangle_{\mathcal{H}_{\mathrm{p}}^2}\right|^2\right].$$

for a unit vector ψ. Hence, we obtain

$$\begin{aligned} &-i\left\langle \psi \,\middle|\, \Lambda_0^{(0)}\psi \right\rangle_{\mathcal{H}_{\mathrm{p}}^2} \\ &\geq \gamma_{\mathrm{FGR}}\left[2 - \left|\langle \Omega_{\mathrm{p}}(\beta_{\mathrm{max}}) \,|\, \psi \rangle_{\mathcal{H}_{\mathrm{p}}^2}\right|^2 - \left|\langle \Omega_{\mathrm{p}}(\beta_{\mathrm{min}}) \,|\, \psi \rangle_{\mathcal{H}_{\mathrm{p}}^2}\right|^2\right]. \end{aligned}$$

The aim of the subsequent considerations is to maximize the function $\psi \mapsto |\langle \Omega_1 \,|\, \psi \rangle_{\mathcal{H}_{\mathrm{p}}^2}|^2 + |\langle \Omega_2 \,|\, \psi \rangle_{\mathcal{H}_{\mathrm{p}}^2}|^2$ under the constraint $\|\psi\|_{\mathcal{H}_{\mathrm{p}}^2} = 1$ where $\Omega_1 := \Omega_{\mathrm{p}}(\beta_{\mathrm{max}})$ and $\Omega_2 := \Omega_{\mathrm{p}}(\beta_{\mathrm{min}})$ are unit vectors. Since $\beta_{\mathrm{max}} > \beta_{\mathrm{min}}$ the vectors Ω_1 and Ω_2 span a two-dimensional space and we can express the vector ψ in this basis, i.e., $\psi = a_1\Omega_1 + a_2\Omega_2 + \eta$ where $\eta \perp \mathrm{span}(\Omega_1, \Omega_2)$. We obtain

$$\begin{aligned} &\left|\langle \Omega_1 \,|\, \psi \rangle_{\mathcal{H}_{\mathrm{p}}^2}\right|^2 + \left|\langle \Omega_2 \,|\, \psi \rangle_{\mathcal{H}_{\mathrm{p}}^2}\right|^2 \\ &= \left|a_1 + a_2\,\langle \Omega_1 \,|\, \Omega_2 \rangle_{\mathcal{H}_{\mathrm{p}}^2}\right|^2 + \left|a_2 + a_1\,\langle \Omega_1 \,|\, \Omega_2 \rangle_{\mathcal{H}_{\mathrm{p}}^2}\right|^2 \\ &= \left(1 + \langle \Omega_1 \,|\, \Omega_2 \rangle_{\mathcal{H}_{\mathrm{p}}^2}^2\right)\left[|a_1|^2 + |a_2|^2\right] + 4\,\langle \Omega_1 \,|\, \Omega_2 \rangle_{\mathcal{H}_{\mathrm{p}}^2}\,\mathrm{Re}(\overline{a}_1 a_2) \end{aligned} \tag{3.48}$$

using that

$$\begin{aligned} \langle \Omega_1 \,|\, \Omega_2 \rangle_{\mathcal{H}_{\mathrm{p}}^2} &= Z(\beta_{\mathrm{min}})^{-1/2} Z(\beta_{\mathrm{max}})^{-1/2} \sum_{j=0}^{N-1} e^{-E_j(\beta_{\mathrm{min}}+\beta_{\mathrm{max}})/2} \\ &= \frac{Z\left(\frac{\beta_{\mathrm{min}}+\beta_{\mathrm{max}}}{2}\right)}{\sqrt{Z(\beta_{\mathrm{min}})Z(\beta_{\mathrm{max}})}} \end{aligned}$$

is strictly positive. The constraint $\|\psi\|_{\mathcal{H}^2_\mathrm{p}} = 1$ translates to

$$1 = \|\psi\|^2_{\mathcal{H}^2_\mathrm{p}} = |a_1|^2 + |a_2|^2 + 2\,\langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_\mathrm{p}}\,\mathrm{Re}(\overline{a}_1 a_2) + \|\eta\|^2_{\mathcal{H}^2_\mathrm{p}}\,. \qquad (3.49)$$

Plugging (3.49) into (3.48), we get

$$\begin{aligned} &\left|\langle\,\Omega_1\,|\,\psi\,\rangle_{\mathcal{H}^2_\mathrm{p}}\right|^2 + \left|\langle\,\Omega_2\,|\,\psi\,\rangle_{\mathcal{H}^2_\mathrm{p}}\right|^2 \\ &\quad = \; 2\left(1 - \|\eta\|^2_{\mathcal{H}^2_\mathrm{p}}\right) - \left(1 - \langle\,\Omega_1\,|\,\Omega_2\,\rangle^2_{\mathcal{H}^2_\mathrm{p}}\right)\left[|a_1|^2 + |a_2|^2\right]. \end{aligned}$$

This expression increases if $|a_1|^2 + |a_2|^2$ becomes smaller. Fixing the absolute value of a_1, a_2 the sum of squares $|a_1|^2 + |a_2|^2$ is minimal under the constraint (3.49) for $a_1, a_2 \in \mathbb{R}^+$. In order to maximize (3.48) we maximize the function

$$\begin{aligned} f &: (\mathbb{R}^+_0)^3 \to \mathbb{R}^+_0, \\ f(a_1, a_2, c) &:= 2(1 - c^2) - \left(1 - \langle\,\Omega_1\,|\,\Omega_2\,\rangle^2_{\mathcal{H}^2_\mathrm{p}}\right)\left[a_1^2 + a_2^2\right]. \end{aligned}$$

on the manifold

$$\left\{(a_1, a_2, c) \in (\mathbb{R}^+_0)^3 \;\middle|\; a_1^2 + a_2^2 + 2\,\langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_\mathrm{p}}\, a_1 a_2 + c^2 = 1\right\}.$$

With the help of a Lagrange multiplier λ we find the coordinates of the critical point to fulfil

$$\begin{aligned} -\left(1 - \langle\,\Omega_1\,|\,\Omega_2\,\rangle^2_{\mathcal{H}^2_\mathrm{p}}\right) a_1 - \lambda\left[a_1 + a_2\,\langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_\mathrm{p}}\right] &= 0, \\ -\left(1 - \langle\,\Omega_1\,|\,\Omega_2\,\rangle^2_{\mathcal{H}^2_\mathrm{p}}\right) a_2 - \lambda\left[a_2 + a_1\,\langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_\mathrm{p}}\right] &= 0, \\ -2c - \lambda c &= 0, \\ a_1^2 + a_2^2 + 2\,\langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_\mathrm{p}}\, a_1 a_2 + c^2 &= 1. \end{aligned}$$

The first two equations yield

$$\begin{aligned} \left(\left(1 - \langle\,\Omega_1\,|\,\Omega_2\,\rangle^2_{\mathcal{H}^2_\mathrm{p}}\right) + \lambda\left(1 + \langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_\mathrm{p}}\right)\right)[a_1 + a_2] &= 0, \\ \left(\left(1 - \langle\,\Omega_1\,|\,\Omega_2\,\rangle^2_{\mathcal{H}^2_\mathrm{p}}\right) + \lambda\left(1 - \langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_\mathrm{p}}\right)\right)[a_1 - a_2] &= 0 \end{aligned}$$

which implies, since $a_1 + a_2 > 0$ under the constraint, that

$$\lambda = \frac{\langle\,\Omega_1\,|\,\Omega_2\,\rangle^2_{\mathcal{H}^2_\mathrm{p}} - 1}{1 + \langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_\mathrm{p}}} = \langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_\mathrm{p}} - 1 \neq -2$$

which in turn yields that $c = 0$ and $a_1 = a_2$ and therefore

$$a_1^2 + a_2^2 = \frac{1}{1 + \langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_\mathrm{p}}}.$$

This gives

$$\left|\langle\,\Omega_1\,|\,\psi\,\rangle_{\mathcal{H}^2_{\mathrm{p}}}\right|^2+\left|\langle\,\Omega_2\,|\,\psi\,\rangle_{\mathcal{H}^2_{\mathrm{p}}}\right|^2\leq 2-\frac{1-\langle\,\Omega_1\,|\,\Omega_2\,\rangle^2_{\mathcal{H}^2_{\mathrm{p}}}}{1+\langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_{\mathrm{p}}}}=1+\langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_{\mathrm{p}}}$$

and finally

$$-i\left\langle\,\psi\,\middle|\,\Lambda_0^{(0)}\psi\,\right\rangle_{\mathcal{H}^2_{\mathrm{p}}}\geq\gamma_{\mathrm{FGR}}\left[1-\langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_{\mathrm{p}}}\right].$$

We now compute the order of the gap. We start observing that

$$\begin{aligned}1-\langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_{\mathrm{p}}} &= \frac{1-\langle\,\Omega_1\,|\,\Omega_2\,\rangle^2_{\mathcal{H}^2_{\mathrm{p}}}}{1+\langle\,\Omega_1\,|\,\Omega_2\,\rangle_{\mathcal{H}^2_{\mathrm{p}}}}\geq\frac{1-\langle\,\Omega_1\,|\,\Omega_2\,\rangle^2_{\mathcal{H}^2_{\mathrm{p}}}}{2}\\ &= \frac{1}{2}-\frac{Z(\overline{\beta})^2}{2Z(\overline{\beta}-\delta\beta)Z(\overline{\beta}+\delta\beta)}\\ &=: f(\delta\beta)\end{aligned}$$

where $\delta\beta:=(\beta_{\max}-\beta_{\min})/2$. Apparently, the function $f:\mathbb{R}_0^+\to\mathbb{R}$ is real-analytic and it holds $f(0)=0$. We compute higher orders,

$$f'(x)=\frac{Z(\overline{\beta})^2\left[Z(\overline{\beta}-x)Z'(\overline{\beta}+x)-Z'(\overline{\beta}-x)Z(\overline{\beta}+x)\right]}{2\left[Z(\overline{\beta}-x)Z(\overline{\beta}+x)\right]^2}.$$

Set $h(x):=Z(\overline{\beta}-x)Z'(\overline{\beta}+x)-Z'(\overline{\beta}-x)Z(\overline{\beta}+x)$, it is $h(0)=0$ and also $f'(0)=0$. We compute the second derivative of f,

$$\begin{aligned}&f''(x)\\ &\quad=\frac{Z(\overline{\beta})^2}{2\left[Z(\overline{\beta}-x)Z(\overline{\beta}+x)\right]^3}\\ &\qquad\times\Big\{\left[Z''(\overline{\beta}-x)Z(\overline{\beta}+x)-2Z'(\overline{\beta}-x)Z'(\overline{\beta}+x)+Z(\overline{\beta}-x)Z''(\overline{\beta}+x)\right]\\ &\qquad\quad\times Z(\overline{\beta}-x)Z(\overline{\beta}+x)\\ &\qquad\quad-2h(x)\partial_x\left[Z(\overline{\beta}-x)Z(\overline{\beta}+x)\right]\Big\},\end{aligned}$$

and, evaluating at $x = 0$ gives,

$$\begin{aligned}
f''(0) &= \frac{Z''(\overline{\beta})Z(\overline{\beta}) - Z'(\overline{\beta})^2}{Z(\overline{\beta})^2} \\
&= Z(\overline{\beta})^{-2} \sum_{j,k=0}^{N-1} [E_j^2 - E_j E_k] e^{-\overline{\beta}(E_j+E_k)} \\
&= Z(\overline{\beta})^{-2} \sum_{\substack{j,k=0,\\ j<k}}^{N-1} \left([E_j^2 - E_j E_k] + [E_k^2 - E_j E_k]\right) e^{-\overline{\beta}(E_j+E_k)} \\
&= Z(\overline{\beta})^{-2} \sum_{\substack{j,k=0,\\ j<k}}^{N-1} (E_j - E_k)^2 e^{-\overline{\beta}(E_j+E_k)} \\
&\geq \frac{d_{L_\mathrm{p}}}{2Z(\overline{\beta})^2} \left[\sum_{j,k=0}^{N-1} e^{-\overline{\beta}(E_j+E_k)} - \sum_{j=0}^{N-1} e^{-2\overline{\beta}E_j}\right] \\
&= \frac{d_{L_\mathrm{p}}}{2} \left[1 - \frac{Z(2\overline{\beta})}{Z(\overline{\beta})^2}\right] \\
&> 0.
\end{aligned}$$

The expansion of f implies that

$$\gamma_{\mathrm{FGR}} \left[1 - \langle\, \Omega_1 \,|\, \Omega_2 \,\rangle_{\mathcal{H}_\mathrm{p}^2}\right] \geq \left(\delta\beta^2 - \mathcal{O}\left(\delta\beta^3\right)\right) \frac{\gamma_{\mathrm{FGR}} d_{L_\mathrm{p}}}{4} \left[1 - \frac{Z(2\overline{\beta})}{Z(\overline{\beta})^2}\right]$$

which is the assertion (3.47).

■

We use the previous propositions in order to describe the spectrum of the deformed standard Liouville operator $L_\theta = K_\theta^{(0)}$ under further restrictions on the reservoir temperatures.

Proposition 3.24 (Spectrum of L_θ in $\mathcal{S}_0$) *We make the same assumption as under Theorem 3.1. Further, we assume that $s = 0$ and*

$$g^{\tilde{\varepsilon}} \ll 1 - \langle\, \Omega_\mathrm{p}(\beta_{\min}) \,|\, \Omega_\mathrm{p}(\beta_{\max}) \,\rangle_{\mathcal{H}_\mathrm{p}^2},$$

in particular $\beta_{\max} > \beta_{\min}$. We remark that, by (3.47), this condition can be expressed as

$$g^{\tilde{\varepsilon}} \ll (\beta_{\max} - \beta_{\min})^2 \left[1 - \frac{Z(2\overline{\beta})}{Z(\overline{\beta})^2}\right],$$

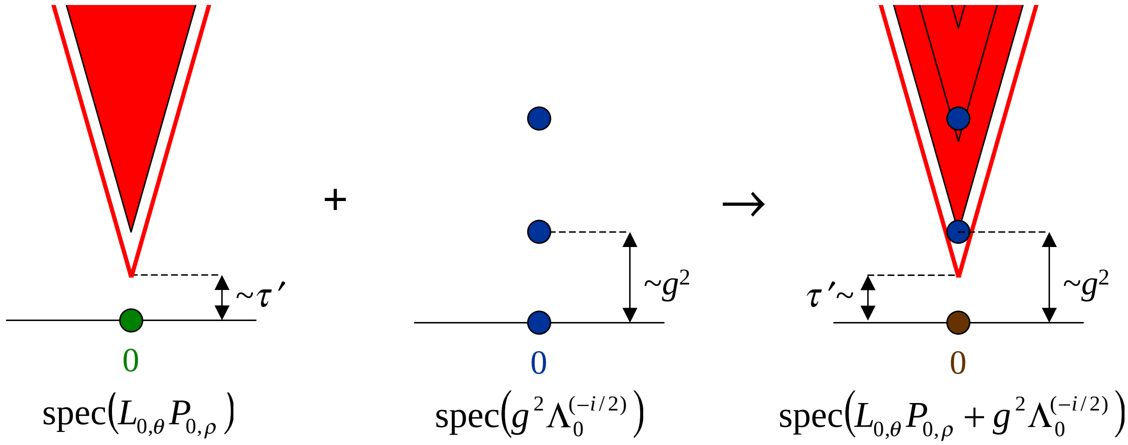

Figure 3.5: Composing the spectrum of the leading orders of $\mathfrak{F}_{\Xi_{0,\rho}}(K_\theta - z, L_{0,\theta} - z) + z$ out of the free operator $L_{0,\theta}$ and the level shift operator $\Lambda_0^{(-i/2)}$.

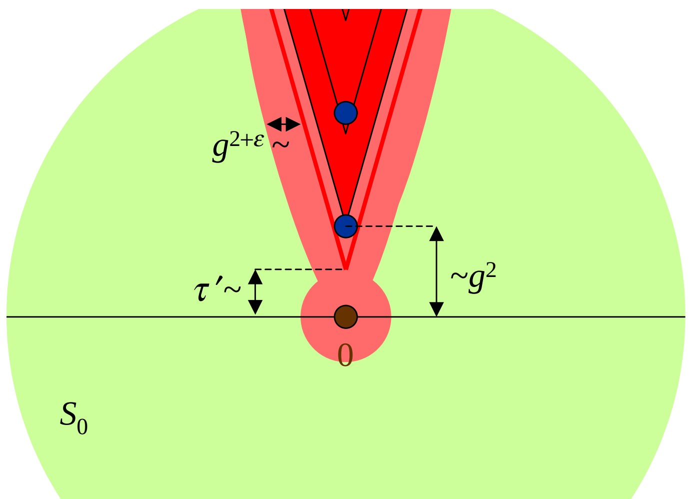

Figure 3.6: Localization of the spectrum of K_θ up to order g^2: the isolated zero eigenvalue disappears in a "cloud of possible spectrum".

where $\overline{\beta} = (\beta_{\max} + \beta_{\min})/2$. *Then, the spectrum of* $L_\theta = K_\theta^{(0)}$ *inside the region* $\mathcal{S}_0$ *can be located by*

$$\operatorname{spec}(L_\theta) \cap \mathcal{S}_0 \subseteq \left\{ z \in \mathcal{S}_0 \,\middle|\, \operatorname{Im}(z) \geq g^2 (\beta_{\max} - \beta_{\min})^2 \frac{\gamma_{\mathrm{FGR}} d_{L_{\mathrm{p}}}}{64} \left[1 - \frac{Z(2\overline{\beta})}{Z(\overline{\beta})^2} \right] \right\}.$$

Proof. The isospectrality of the smooth Feshbach map $\mathfrak{F}_{\Xi_{0,\rho}}$ implies that $z \in \operatorname{spec}(L_\theta) \cap \mathcal{S}_0$ if and only if $z \in \operatorname{spec}(\mathfrak{F}_{\Xi_{0,\rho}}(L_\theta - z, L_{0,\theta} - z) + z) \cap \mathcal{S}_0$. By Proposition 3.7 we know that

$$\mathfrak{F}_{\Xi_{0,\rho}}(L_\theta - z, L_{0,\theta} - z) + z = P_{0,\rho}\left[L_{0,\theta} + g^2\Lambda_0^{(0)} \otimes \chi_\rho^2(M_{[\theta]})\right] P_{0,\rho} + \mathcal{O}\left(g^{2+\tilde{\varepsilon}}\right) \quad (3.50)$$

where the remainder term is estimated uniformly in $z \in \mathcal{S}_0$. We compute the numerical range of the imaginary part of this operator. To this end let $\psi \in \operatorname{ran}(P_{0,\rho})$ and decompose $\psi = \psi_1 + \psi_1^\perp$ where $\psi_1 \in \operatorname{ran}(P_{[M_{[\theta]} \leq \frac{7}{8}\rho]})$ and $\psi_1^\perp \in \operatorname{ran}(P_{[M_{[\theta]} > \frac{7}{8}\rho]})$ and compute

$$\begin{aligned}
&\operatorname{Im}\left\langle \psi \,\middle|\, \left[L_{0,\theta} + g^2\Lambda_0^{(0)} \otimes \chi_\rho^2(M_{[\theta]})\right] \psi \right\rangle \\
&= \left\langle \psi_1 \,\middle|\, \left[M_{[\theta]} + g^2 \operatorname{Im}\left(\Lambda_0^{(0)}\right) \otimes \chi_\rho^2(M_{[\theta]})\right] \psi_1 \right\rangle \\
&\quad + \left\langle \psi_1^\perp \,\middle|\, \left[M_{[\theta]} + g^2 \operatorname{Im}\left(\Lambda_0^{(0)}\right) \otimes \chi_\rho^2(M_{[\theta]})\right] \psi_1^\perp \right\rangle \\
&\geq g^2 \left\langle \psi_1 \,\middle|\, \operatorname{Im}\left(\Lambda_0^{(0)}\right) \psi_1 \right\rangle + \langle \psi_1^\perp \,|\, M_{[\theta]}\psi_1^\perp \rangle \\
&\geq g^2\gamma_{\mathrm{FGR}}\left[1 - \langle \Omega_{\mathrm{p}}(\beta_{\min}) \,|\, \Omega_{\mathrm{p}}(\beta_{\max}) \rangle_{\mathcal{H}_{\mathrm{p}}^2}\right] \|\psi_1\|^2 + \frac{7}{8}\rho \left\|\psi_1^\perp\right\|^2 \\
&\geq g^2\gamma_{\mathrm{FGR}}\left[1 - \langle \Omega_{\mathrm{p}}(\beta_{\min}) \,|\, \Omega_{\mathrm{p}}(\beta_{\max}) \rangle_{\mathcal{H}_{\mathrm{p}}^2}\right] \left(\|\psi_1\|^2 + \left\|\psi_1^\perp\right\|^2\right) \\
&= g^2\gamma_{\mathrm{FGR}}\left[1 - \langle \Omega_{\mathrm{p}}(\beta_{\min}) \,|\, \Omega_{\mathrm{p}}(\beta_{\max}) \rangle_{\mathcal{H}_{\mathrm{p}}^2}\right] \|\psi\|^2
\end{aligned}$$

where we used that $M_{[\theta]} \geq 0$ and $\operatorname{Im}(\Lambda_0^{(0)}) \otimes \chi_\rho^2(M_{[\theta]}) \geq 0$ and $\operatorname{Im}(\Lambda_0^{(0)}) \geq \gamma_{\mathrm{FGR}}[1 - \langle \Omega_{\mathrm{p}}(\beta_{\min}) \,|\, \Omega_{\mathrm{p}}(\beta_{\max}) \rangle_{\mathcal{H}_{\mathrm{p}}^2}]$, by Proposition 3.23(iii), and $\rho \gg g^2$. Together with (3.50) we have

$$\begin{aligned}
&\operatorname{Im}\langle \psi \,|\, [\mathfrak{F}_{\Xi_{0,\rho}}(L_\theta - z, L_{0,\theta} - z) + z]\, \psi \rangle \\
&\geq g^2\gamma_{\mathrm{FGR}}\left(1 - \langle \Omega_{\mathrm{p}}(\beta_{\min}) \,|\, \Omega_{\mathrm{p}}(\beta_{\max}) \rangle_{\mathcal{H}_{\mathrm{p}}^2} + \mathcal{O}\left(g^{\tilde{\varepsilon}}\right)\right) \|\psi\|^2 \\
&\geq g^2\gamma_{\mathrm{FGR}} \frac{1 - \langle \Omega_{\mathrm{p}}(\beta_{\min}) \,|\, \Omega_{\mathrm{p}}(\beta_{\max}) \rangle_{\mathcal{H}_{\mathrm{p}}^2}}{2} \|\psi\|^2 \\
&\geq g^2(\beta_{\max} - \beta_{\min})^2 \frac{\gamma_{\mathrm{FGR}} d_{L_{\mathrm{p}}}}{64} \left[1 - \frac{Z(2\overline{\beta})}{Z(\overline{\beta})^2}\right] \|\psi\|^2
\end{aligned}$$

for g sufficiently small and $|\beta_{\max} - \beta_{\min}| \ll 1$. The isospectrality of the Feshbach map leads to the assertion. ∎

The previous proposition describes the spectrum of $K_\theta^{(0)} = L_\theta$ under additional temperature constraints. The second special case of interest is the particular choice

$s = -i/2$. Employing the isospectrality in the sense of Proposition 3.6 we can study the operator

$$
\begin{aligned}
&\mathfrak{F}_{\Xi_{0,\rho}}(K_\theta - z, L_{0,\theta} - z) + z \\
&= P_{0,\rho}\left[L_{0,\theta} + g^2 \Lambda_0^{(-i/2)} \otimes \chi_\rho^2(M_{[\theta]})\right] P_{0,\rho} + \mathcal{O}\left(g^{2+\tilde{\varepsilon}}\right) \\
&= P_{0,\rho}\left[L_{0,\theta} + g^2 \Lambda_0^{(-i/2)} - g^2 \Lambda_0^{(-i/2)} \otimes \overline{\chi}_\rho^2(M_{[\theta]})\right] P_{0,\rho} + \mathcal{O}\left(g^{2+\tilde{\varepsilon}}\right)
\end{aligned}
$$

instead of $K_\theta = K_\theta^{(-i/2)}$. Since $L_{0,\theta}$ and $\Lambda_0^{(-i/2)}$ act on different variables the spectrum of $L_{0,\theta} \restriction_{\mathrm{ran}(P_{0,\rho})} + g^2 \Lambda_0^{(-i/2)}$ is given by

$$
\begin{aligned}
&\mathrm{spec}\left(L_{0,\theta} \restriction_{\mathrm{ran}(P_{0,\rho})} + g^2 \Lambda_0^{(-i/2)}\right) \\
&= \mathrm{spec}\left(L_{0,\theta} \restriction_{\mathrm{ran}(P_{0,\rho})}\right) + g^2\, \mathrm{spec}\left(\Lambda_0^{(-i/2)}\right) \\
&\subseteq \{0\} \cup \left\{\zeta \in \mathbb{C} \,\middle|\, \mathrm{Im}(\zeta) \geq \tan(\tilde{\delta})|\,\mathrm{Re}(\zeta)| + \min\{\tau', \mathcal{O}\left(g^2\right)\}\right\} \\
&=: A_\theta,
\end{aligned}
$$

by Proposition 3.23(i), where

$$
\tan(\tilde{\delta}) = \min\left\{\tan(\delta'), \frac{\left\|\mathrm{Im}(\Lambda_0^{(-i/2)})\right\|_{\mathcal{B}(\ker(Lp))}}{\left\|\mathrm{Re}(\Lambda_0^{(-i/2)})\right\|_{\mathcal{B}(\ker(Lp))}}\right\}.
$$

Therefore, in leading orders, the operator $\mathfrak{F}_{\Xi_{0,\rho}}(K_\theta - z, L_{0,\theta} - z) + z$ has a simple zero eigenvalue separated from the rest of the spectrum by a gap given by $\min\{\tau', \mathcal{O}\left(g^2\right)\}$, see Figure 3.5. However, the gap is smaller than the deformation parameter τ' which in turn is proportional to the minimal temperature of the reservoirs. Thus, in the situation where the temperature is small compared to the coupling constant, i.e., $\tau' \ll \mathcal{O}\left(g^2\right)$, the higher order corrections to $\mathfrak{F}_{\Xi_{0,\rho}}(K_\theta - z, L_{0,\theta} - z) + z$ destroy the localization of an isolated eigenvalue, see Figure 3.6.

The study of the spectrum around zero for both operators, K_θ and L_θ, without any additional constraints on the parameters $\beta_{\max}, \beta_{\min}$ and g, requires a more sophisticated analytical technique. The *renormalization transformation* in Chapter 4 provides such a tool which allows the spectral analysis on smaller scales and delivers detailed results about the spectrum near the origin.

4 Smooth Feshbach Iteration and Renormalization

The analysis of the spectrum of $K_\theta^{(s)}$ in the nearest neighborhood of zero is done iteratively on decreasing scales. The iterative process requires a sequence of Hilbert spaces $(\mathcal{H}^{(n)})_{n=1,2,\ldots}$, on each Hilbert space $\mathcal{H}^{(n)}$ acts a family $\mathbb{C} \supseteq B_{1/4} \ni z \mapsto K^{(n)}[z]$ of bounded operators. The operator family $K^{(n)}$ encodes the spectral information of its predecessor family $K^{(n-1)}$ on a small scale around the origin. The transition between the operator families is done by the *renormalization transformation*. The iteration of the renormalization transformation generates a discrete flow of operator families on the sequence of Hilbert spaces and, connected to the operator flow, a flow of spectral information representing the spectrum of the initial data $K_\theta^{(s)}$ of the iterative process on smaller and smaller scales. The concept of the *renormalization group (RG)* based on the Feshbach map was invented by Bach, Fröhlich and Sigal in [6] for applications to spectral problems in quantum field theory. It entered the analysis of concrete models in quantum electrodynamics and quantum statistical mechanics in [5, 7, 6, 36]. A technical refinement was achieved by the same authors in collaboration with Chen in [4] by employing the smooth Feshbach map instead of the standard one. The present chapter is devoted to an adaption of this technique to our concrete problem. The main modifications compared to [4] are that the analysis of the positive temperature system requires an additional control parameter, this was already discussed in [8]. Further, one of the control parameters does not scale properly in our situation such that the renormalization transformation does leave the underlying Hilbert spaces invariant but works on a decreasing sequence of spaces which eventually collapse to dimension one. Hence, we get along with finitely many iteration steps and do not care for limit processes. The first challenge, however, is to fit the operator $K_\theta^{(s)}$ of interest into the framework provided in [4] as done in Section 4.2. We launch this chapter by introducing the necessary preliminaries.

4.1 Sequence of Hilbert Spaces and Banach Spaces of Operators

Throughout this chapter we make the same assumptions on the parameters s, $\theta = (\delta, \tau)$ and ρ as in the previous one, recall in particular (3.1, 3.2, 3.4). We introduce two small positive numbers, ρ_* and ρ_{**}, which measure the underlying scale of the renormalization transformation. In the first step of renormalization, the scale is given through

$$0 < \rho_* = \mathcal{O}\left(\frac{g^2}{\rho}\right),$$

for the exact definition see (4.43), where in the successive steps the scale is given by

$$0 < \rho_{**} \leq \frac{1}{20}.$$

We remark that the parameter ρ_{**} will be chosen independently of the coupling constant g and of the scale $\rho = g^{2/3(1+\tilde{\varepsilon})}$ of the previous chapter, compare with the defining relation (4.5). In fact, the number ρ_{**} is considered to be large w.r.t. g and ρ, i.e., $g, \rho \ll \rho_{**}$. Associated to the scales ρ_* and ρ_{**} we define a sequence of Hilbert spaces

$$\mathcal{H}^{(1)} \hookleftarrow \mathcal{H}^{(2)} \supseteq \cdots \supseteq \mathcal{H}^{(n)} \supseteq \mathcal{H}^{(n+1)} \supseteq \cdots \supseteq \mathcal{H}^{(\mathcal{N}-1)} \supsetneqq \mathcal{H}^{(\mathcal{N})}$$

by

$$\mathcal{H}^{(n)} := \mathcal{H}^{(n)}_{<\infty} \otimes \left[P_{\left[M_{[\theta_n]} \leq 1\right]} \mathcal{F}(L^2[\Upsilon])\right],$$

where

$$\mathcal{H}^{(1)}_{<\infty} := \ker(L_{\mathrm{p}}),$$
$$\mathcal{H}^{(n)}_{<\infty} := \mathbb{C}, \qquad n = 2, 3, \ldots, \mathcal{N},$$

is a finite dimensional Hilbert space representing the particle degrees of freedom in the n^{th} step, and

$$\theta_n := (i\delta', i\tau'_n) := \left(i\delta', i\frac{\tau'}{\rho_{[n]}}\right) \in (i\mathbb{R}^+)^2, \qquad \rho_{[n]} := \begin{cases} \rho, & n = 1 \\ \rho\rho_*\rho_{**}^{n-2}, & n \geq 2, \end{cases} \tag{4.1}$$

are sequences of effective deformation parameters and scales, resp. The estimating operator in the n^{th} iteration step therefore reads

$$M_{[\theta_n]} = d\Gamma_{\mathrm{gl}}\left[m_{\theta_n}\right] = \sin(\delta')L_{\mathrm{aux}} + \frac{\tau'}{\rho_{[n]}} N_{\mathrm{res}}$$

where, recall,

$$m_{\theta_n}(u) = \sin(\delta')|u| + \frac{\tau'}{\rho_{[n]}}.$$

The number of iteration steps $\mathcal{N}$ is chosen such that

$$\frac{\tau'}{\rho_{[\mathcal{N}-1]}} \leq 1 < \frac{\tau'}{\rho_{[\mathcal{N}]}}$$

and therefore the infinite dimensional Hilbert spaces $\mathcal{H}^{(n)}$, $n = 1, \ldots, \mathcal{N}-1$, collapse to a one dimensional space

$$\mathcal{H}^{(\mathcal{N})} = \ker(N_{\mathrm{res}}) = \mathbb{C}\Omega_0.$$

Without loss of generality we assume that $\mathcal{N} \geq 3$, otherwise we choose τ' sufficiently small. This is possible since we nowhere require that the translation deformation parameter τ' is sufficiently large, unlike for the dilation parameter δ'. In fact, the condition $\mathcal{N} \geq 3$ results in

$$\tau' \leq \rho_{[\mathcal{N}-1]} \leq \rho\rho_* = \mathcal{O}\left(g^2\right). \tag{4.2}$$

Hence, to comply with this requirement, we choose from now on

$$\tau' := \frac{g^2 \gamma_{\mathrm{eq}}}{2 + \beta_{\max}} < \min\{\rho\rho_*, 2\pi\beta_{\max}^{-1}\}, \tag{4.3}$$

for g sufficiently small, where the constant γ_{eq} is defined in (4.46) and enters the concrete definition (4.43) of ρ_*. We further define the sets

$$\begin{aligned}
\mathcal{Q}^{(n)} &:= \left\{ (q_1, q_2) \in \mathbb{R}^2 \;\middle|\; \tan(\delta')|q_1| + \frac{\tau'}{\rho_{[n]}} \leq 1,\ q_2 \in [0,1] \right\}, \\
\mathcal{M}^{(n)} &:= \{(u, \Sigma, r) \in \Upsilon \mid m_{\theta_n}(u) \leq 1\}.
\end{aligned}$$

Next, we introduce Banach spaces $\mathfrak{W}^{(n)}_{R,S}$, $R + S \in \mathbb{N}_0$, of functions

$$w^{(n)}_{R,S} : \mathcal{Q}^{(n)} \times \{\mathcal{M}^{(n)}\}^{R+S} \to \mathcal{H}^{(n)}_{<\infty}$$

which are continuously differentiable w.r.t. the variable $q \in \mathcal{Q}^{(n)}$, i.e.,

$$w^{(n)}_{R,S}[\,\cdot\,, Y^{(R,S)}] \in C^1\left(\mathcal{Q}^{(n)}; \mathcal{H}^{(n)}_{<\infty}\right)$$

for almost every $Y^{(R,S)} \in \{\mathcal{M}^{(n)}\}^{R+S}$. Further, the functions are required to be totally symmetric w.r.t. the variables $y^{(R)} \in \{\mathcal{M}^{(n)}\}^R$ and $\tilde{y}^{(S)} \in \{\mathcal{M}^{(n)}\}^S$. Finally, the functions obey the norm bound

$$\begin{aligned} \left\| w_{R,S}^{(n)} \right\|_{(n)}^{\#} &:= \sqrt{ \left\| w_{R,S}^{(n)} \right\|_{(n)}^{2} + \left\| \left| \nabla_q w_{R,S}^{(n)} \right| \right\|_{(n)}^{2} } < \infty, \\ \left\| w_{R,S}^{(n)} \right\|_{(n)}^{2} &:= \int_{\{\mathcal{M}^{(n)}\}^{R+S}} \frac{dY^{(R,S)}}{m_{\theta_n}\left(Y^{(R,S)}\right)^{3+2\mu}} \sup_{q\in\mathcal{Q}^{(n)}} \left\| w_{R,S}^{(n)}\left[q; Y^{(R,S)}\right] \right\|_{\mathcal{B}(\mathcal{H}_{<\infty}^{(n)})}^{2}, \end{aligned} \tag{4.4}$$

where the norm of the gradient has to be understood as

$$\left\| \left| \nabla_q w_{R,S}^{(n)}[\,\cdot\,] \right| \right\|_{\mathcal{B}(\mathcal{H}_{<\infty}^{(n)})}^{2} = \sum_{j=1,2} \left\| \partial_{q_j} w_{R,S}^{(n)}[\,\cdot\,] \right\|_{\mathcal{B}(\mathcal{H}_{<\infty}^{(n)})}^{2}.$$

Here, we make use of the notation introduced in Appendix D prior to Theorem D.3. The parameter μ appearing in the definition (4.4) of the norm $\|\cdot\|_{(n)}^{\#}$ will be used in Section 4.4 to establish the contracting property of the renormalization procedure and it is assumed to obey

$$\frac{1}{2} \leq \mu < \min\{1, \nu\},$$

where ν is the infrared regularization of the coupling functions, c.f. Hypothesis VII-1.12. The number μ and the parameter ρ_{**} are related through

$$\rho_{**} = (16\mathcal{C}_{\chi_1})^{-2/\mu} \leq \frac{1}{20} \tag{4.5}$$

where the incorporated constant $\mathcal{C}_{\chi_1} \geq 1$ only depends on the cutoff function χ_1 introduced in Section 3.3, Equation (3.10), and is fixed such that

$$106 + \sup_{x\in[0,1]} \left[2|\chi_1'(x)| + 14|\overline{\chi}_1'(x)|\right] \leq \mathcal{C}_{\chi_1} \tag{4.6}$$

holds. The constant $\mathcal{C}_{\chi_1}$ plays a significant role in the proof of Lemma 4.10 and of Proposition 4.4. The parameter μ represents the infrared regularization of the form factors needed for the renormalization procedure. The direct sum

$$\mathfrak{W}_{\#}^{(n)} := \bigoplus_{R+S\geq 0} \mathfrak{W}_{R,S}^{(n)} \tag{4.7}$$

is defined as the space of all sequences $\left(w_{R,S}^{(n)} \in \mathfrak{W}_{R,S}^{(n)}\right)_{R+S\geq 0}$ with finite weighted ℓ^1-norm

$$\left\| \left(w_{R,S}^{(n)}\right)_{R+S\geq 0} \right\|_{(n),\xi}^{\#} := \sum_{R+S\geq 0} \xi^{-(R+S)} \left\| w_{R,S}^{(n)} \right\|_{(n)}^{\#}$$

where the weight is given by

$$\xi := \frac{\sqrt{\rho_{**}}}{4\mathcal{C}_{\chi_1}} = \frac{1}{4\mathcal{C}_{\chi_1}(16\mathcal{C}_{\chi_1})^{1/\mu}} \leq \frac{1}{4}. \tag{4.8}$$

The space $\left(\mathfrak{W}_{\#}^{(n)}, \|\cdot\|_{(n),\xi}^{\#}\right)$ is a Banach space.

This space can canonically be embedded into the bounded operators on $\mathcal{H}^{(n)}$. This embedding is given by

$$\begin{aligned} \mathcal{W}_{(n)} : \mathfrak{W}_{\#}^{(n)} &\to \mathcal{B}\left(\mathcal{H}^{(n)}\right), \\ \mathcal{W}_{(n)}\left[\left(w_{R,S}^{(n)}\right)_{R+S\geq 0}\right] &:= \sum_{R+S\geq 0} \mathcal{W}_{(n)}\left[w_{R,S}^{(n)}\right] \end{aligned} \tag{4.9}$$

where $\mathcal{W}_{(n)}[w_{R,S}^{(n)}] := P^{(n)}\mathcal{W}_{[\theta_n]}[w_{R,S}^{(n)}]P^{(n)}$, i.e.,

$$\mathcal{W}_{(n)}\left[w_{0,0}^{(n)}\right] := P^{(n)}\mathcal{W}_{[\theta_n]}\left[w_{0,0}^{(n)}\right]P^{(n)} = P^{(n)}w_{0,0}^{(n)}\left[\Lambda_{[\theta_n]}\right]P^{(n)},$$

and, for $R + S \geq 1$,

$$\begin{aligned} \mathcal{W}_{(n)}\left[w_{R,S}^{(n)}\right] &:= P^{(n)}\mathcal{W}_{[\theta_n]}\left[w_{R,S}^{(n)}\right]P^{(n)} \\ &= P^{(n)} \int\limits_{\{\mathcal{M}^{(n)}\}^{R+S}} \frac{dY^{(R,S)}}{m_{\theta_n}\left(Y^{(R,S)}\right)^{1/2}} a_{\mathrm{gl}}^*\left(y^{(R)}\right) w_{R,S}^{(n)}\left[\Lambda_{[\theta_n]}; Y^{(R,S)}\right] a_{\mathrm{gl}}\left(\tilde{y}^{(S)}\right) P^{(n)} \end{aligned} \tag{4.10}$$

are the *Wick monomials* of order (R, S) corresponding to $w_{R,S}^{(n)}$ and

$$P^{(n)} := \begin{cases} P_{[L_{\mathrm{p}}=0]} \otimes P_{\left[M_{[\theta_1]}\leq 1\right]}, & n = 1, \\ P_{\left[M_{[\theta_n]}\leq 1\right]}, & n = 2, 3, \ldots, \mathcal{N}, \end{cases}$$

is the orthogonal projection on $\mathcal{H}^{(n)}$. Note that the definition (4.10) is consistent on $\mathrm{ran}(P^{(n)})$ with the definition of $\mathcal{W}_{[\theta_n]}[w_{R,S}^{(n)}]$ given in (D.6). Hereby, $w_{R,S}^{(n)}\left[\Lambda_{[\theta_n]}; Y^{(R,S)}\right]$ is defined via functional calculus where

$$\begin{aligned} \Lambda_{[\theta_n]} &:= \left(\cos(\delta')L_{\mathrm{res}}, M_{[\theta_n]}\right) \\ &\equiv d\Gamma_{\mathrm{gl}}(\lambda_{\theta_n}) \equiv \int\limits_{\Upsilon} d(u,\Sigma,r)\, a_{\mathrm{gl}}^*(u,\Sigma,r)\lambda_{\theta_n}(u)a_{\mathrm{gl}}(u,\Sigma,r) \end{aligned} \tag{4.11}$$

is a pair of commuting self-adjoint operators and

$$\lambda_{\theta_n}(u) := (\cos(\delta')u, m_{\theta_n}(u)) = \left(\cos(\delta')u, \sin(\delta')|u| + \frac{\tau'}{\rho_{[n]}}\right).$$

Note that

$$\mathrm{spec}\left(\cos(\delta')L_{\mathrm{res}} \restriction_{\mathrm{ran}\, P_{\left[M_{[\theta_n]}\leq 1\right]}}\right) \times \mathrm{spec}\left(M_{[\theta_n]} \restriction_{\mathrm{ran}\, P_{\left[M_{[\theta_n]}\leq 1\right]}}\right) = \mathcal{Q}^{(n)} \tag{4.12}$$

is the set of control parameters controlling the dependence of the Wick monomials on the free operators $\cos(\delta')L_{\mathrm{res}}$ and $M_{[\theta_n]}$.

We will need later the *partially integrated Wick monomials* acting on $\mathcal{H}^{(n)}$,

$$\begin{aligned} &\mathcal{W}^{(p,q)}_{(n)}\left[w^{(n)}_{R+p,S+q}\right]\left(\lambda; y^{(p)}_*, \tilde{y}^{(q)}_*\right) \qquad (4.13)\\ :=\ & P^{(n)}\mathcal{W}^{(p,q)}_{[\theta_n]}\left[w^{(n)}_{R+p,S+q}\right]\left(\lambda; y^{(p)}_*, \tilde{y}^{(q)}_*\right)P^{(n)} \\ =\ & P^{(n)}\int_{\{\mathcal{M}^{(n)}\}^{R+S}} \frac{dY^{(R,S)}}{m_{\theta_n}(Y^{(R,S)})^{1/2}} a^*_{\mathrm{gl}}\left(y^{(R)}\right) \\ & \qquad \times w^{(n)}_{R+p,S+q}\left[\Lambda_{[\theta_n]}+\lambda; y^{(R)}, y^{(p)}_*, \tilde{y}^{(S)}, \tilde{y}^{(q)}_*\right] a_{\mathrm{gl}}\left(\tilde{y}^{(S)}\right)P^{(n)}. \end{aligned}$$

The next proposition guarantees that the embedding $\mathcal{W}_{(n)}$ is well defined. Henceforth, we assume that the parameters μ, ξ are chosen as in (4.5, 4.8).

Proposition 4.1 *For $R+S \geq 1$, the assignment (4.10) is well defined as a map from $\mathfrak{W}^{(n)}_{R,S}$ into the bounded operators on $\mathcal{H}^{(n)}$. Furthermore, for $w^{(n)}_{R,S} \in \mathfrak{W}^{(n)}_{R,S}$, the following norm bound is obeyed,*

$$\left\|\mathcal{W}_{(n)}\left[w^{(n)}_{R,S}\right]\right\|_{\mathcal{B}(\mathcal{H}^{(n)})} \leq \frac{1}{\sqrt{R^R S^S}}\left\|w^{(n)}_{R,S}\right\|_{(n)}.$$

Before proving the Proposition 4.1 we first provide a lemma.

Lemma 4.2 *Let $\psi \in \mathcal{H}^{(n)}$, $R \geq 1$, $a > 0$ and $y^{(R)} = (y_1, \ldots, y_R) \in \Upsilon^R$. Then the following inequality holds true,*

$$\left\|a_{\mathrm{gl}}\left(y^{(R)}\right)\psi\right\|_{\mathcal{H}^{(n)}} \leq \frac{1}{R^{aR} m_{\theta_n}(u^{(R)})^a}\left\|a_{\mathrm{gl}}\left(y^{(R)}\right)\psi\right\|_{\mathcal{H}^{(n)}}. \qquad (4.14)$$

Proof. An application of the pull through formula (1.67) yields

$$\begin{aligned} \left\|a_{\mathrm{gl}}\left(y^{(R)}\right)\psi\right\|_{\mathcal{H}^{(n)}} &= \left\|a_{\mathrm{gl}}\left(y^{(R)}\right)P_{\left[M_{[\theta_n]}\leq 1\right]}\psi\right\|_{\mathcal{H}^{(n)}} \\ &= \left\|P_{\left[M_{[\theta_n]}+\sum_{j=1}^R m_{\theta_n}(u_j)\leq 1\right]}a_{\mathrm{gl}}\left(y^{(R)}\right)\psi\right\|_{\mathcal{H}^{(n)}} \\ &\leq \left\|P_{\left[\sum_{j=1}^R m_{\theta_n}(u_j)\leq 1\right]}a_{\mathrm{gl}}\left(y^{(R)}\right)\psi\right\|_{\mathcal{H}^{(n)}} \end{aligned}$$

where $y_j = (u_j, \Sigma_j, r_j)$. To conclude the proof we define an auxiliary function $f(x) := x\ln(x)$. Since f is convex we obtain for positive numbers $a_1, \ldots, a_R \in (0,1]$ with $\sum_{j=1}^R a_j = 1$,

$$\sum_{j=1}^R \ln(a_j^{-1}) = \sum_{j=1}^R a_j f(a_j^{-1}) \geq f\left(\sum_{j=1}^R a_j a_j^{-1}\right) = f(R) = R\ln(R).$$

This implies $a_1 \cdots a_R \leq R^{-R}$. For $\sum_{j=1}^{R} m_{\theta_n}(u_j) \leq 1$ set $a_j := \lambda m_{\theta_n}(u_j) \in (0,1]$ where $\lambda \geq 1$ is chosen such that $\sum_{j=1}^{R} a_j = 1$. It then holds

$$m_{\theta_n}(u^{(R)}) = \prod_{j=1}^{R} m_{\theta_n}(u_j) = \prod_{j=1}^{R} \lambda^{-1} a_j \leq (\lambda R)^{-R} \leq R^{-R}.$$

and therefore

$$1 \leq R^{-aR} m_{\theta_n}(u^{(R)})^{-a}.$$

This implies (4.14). ■

Proof of Proposition 4.1. Let $\psi, \varphi \in \mathcal{H}^{(n)}$ be arbitrary. The application of Lemma 4.2 with $a := 1/2 + \mu$ allows the following computation,

$$\begin{aligned}
&\left| \left\langle \psi \,\middle|\, \mathcal{W}_{(n)}\left[w_{R,S}^{(n)}\right] \varphi \right\rangle_{\mathcal{H}^{(n)}} \right| \\
&= \left| \int_{\{\mathcal{M}^{(n)}\}^{R+S}} \frac{dY^{(R,S)}}{m_{\theta_n}\left(Y^{(R,S)}\right)^{1/2}} \left\langle \psi \,\middle|\, a_{\mathrm{gl}}^*\left(y^{(R)}\right) w_{R,S}^{(n)}\left[\Lambda_{[\theta_n]}; Y^{(R,S)}\right] a_{\mathrm{gl}}\left(\tilde{y}^{(S)}\right) \varphi \right\rangle \right| \\
&\leq \int_{\{\mathcal{M}^{(n)}\}^{R+S}} \frac{dY^{(R,S)}}{m_{\theta_n}\left(Y^{(R,S)}\right)^{1/2}} \sup_{q \in \mathcal{Q}^{(n)}} \left\| w_{R,S}^{(n)}\left[q; Y^{(R,S)}\right] \right\|_{\mathcal{B}(\mathcal{H}_{<\infty}^{(n)})} \\
&\qquad \times \left\| a_{\mathrm{gl}}\left(y^{(R)}\right) \psi \right\| \left\| a_{\mathrm{gl}}\left(\tilde{y}^{(S)}\right) \varphi \right\| \\
&\leq \frac{1}{\sqrt{R^R S^S}} \int_{\{\mathcal{M}^{(n)}\}^{R+S}} dY^{(R,S)} \frac{\sup_{q \in \mathcal{Q}^{(n)}} \left\| w_{R,S}^{(n)}\left[q; Y^{(R,S)}\right] \right\|_{\mathcal{B}(\mathcal{H}_{<\infty}^{(n)})}}{m_{\theta_n}\left(Y^{(R,S)}\right)^{1+\mu}} \\
&\qquad \times \left\| a_{\mathrm{gl}}\left(y^{(R)}\right) \psi \right\| \left\| a_{\mathrm{gl}}\left(\tilde{y}^{(S)}\right) \varphi \right\| \\
&\leq \sqrt{\frac{N_R(\psi) N_S(\varphi)}{R^R S^S}} \\
&\quad \times \left[\int_{\{\mathcal{M}^{(n)}\}^{R+S}} \frac{dY^{(R,S)}}{m_{\theta_n}\left(Y^{(R,S)}\right)^{3+2\mu}} \sup_{q \in \mathcal{Q}^{(n)}} \left\| w_{R,S}^{(n)}\left[q; Y^{(R,S)}\right] \right\|_{\mathcal{B}(\mathcal{H}_{<\infty}^{(n)})}^2 \right]^{1/2} \\
&= \sqrt{\frac{N_R(\psi) N_S(\varphi)}{R^R S^S}} \left\| w_{R,S}^{(n)} \right\|_{(n)},
\end{aligned}$$

where

$$N_R(\psi) := \int_{\{\mathcal{M}^{(n)}\}^R} dy^{(R)} m_{\theta_n}(y^{(R)}) \left\| a_{\mathrm{gl}}\left(y^{(R)}\right) \psi \right\|^2 .$$

We show inductively over $R \in \mathbb{N}$ that

$$N_R(\psi) \leq \left\| (M_{[\theta_n]} + \varepsilon)^{R/2} \psi \right\|^2$$

for any $\varepsilon > 0$. This relation is obviously true for $R = 1$ since $N_1(\psi) = \langle \psi \mid M_{[\theta_n]} \psi \rangle$. Further, we have for $y^{(R)} = (y_1, \ldots, y_{R-1}, y_R) = (y^{(R-1)}, y_R)$, $y_j = (u_j, \Sigma_j, r_j) \in \Upsilon$,

$$\begin{aligned} N_R(\psi) &= \int\limits_{\{\mathcal{M}^{(n)}\}^{R-1}} dy^{(R-1)} m_{\theta_n}(y^{(R-1)}) \\ &\quad \times \int\limits_{\mathcal{M}^{(n)}} dy_R \left\langle a_{\mathrm{gl}}\left(y^{(R-1)}\right) \psi \,\middle|\, \left[a^*_{\mathrm{gl}}(y_R) m_{\theta_n}(y_R) a_{\mathrm{gl}}(y_R)\right] a_{\mathrm{gl}}\left(y^{(R-1)}\right) \psi \right\rangle \\ &= \int\limits_{\{\mathcal{M}^{(n)}\}^{R-1}} dy^{(R-1)} m_{\theta_n}(y^{(R-1)}) \left\| M^{1/2}_{[\theta_n]} a_{\mathrm{gl}}\left(y^{(R-1)}\right) \psi \right\|^2 \\ &= \int\limits_{\{\mathcal{M}^{(n)}\}^{R-1}} dy^{(R-1)} m_{\theta_n}(y^{(R-1)}) \\ &\quad \times \left\| \left(\frac{M_{[\theta_n]}}{M_{[\theta_n]} + \varepsilon + \sum_{j=1}^{R-1} m_{\theta_n}(u_j)} \right)^{1/2} a_{\mathrm{gl}}\left(y^{(R-1)}\right) (M_{[\theta_n]} + \varepsilon)^{1/2} \psi \right\|^2 \\ &\leq \int\limits_{\{\mathcal{M}^{(n)}\}^{R-1}} dy^{(R-1)} m_{\theta_n}(y^{(R-1)}) \left\| a_{\mathrm{gl}}\left(y^{(R-1)}\right) (M_{[\theta_n]} + \varepsilon)^{1/2} \psi \right\|^2 \\ &= N_{R-1}\left((M_{[\theta_n]} + \varepsilon)^{1/2} \psi \right), \end{aligned}$$

where we used the pull through formula (1.67). We finally obtain

$$\begin{aligned} \left\| \mathcal{W}_{(n)} \left[w^{(n)}_{R,S} \right] \right\|_{\mathcal{B}(\mathcal{H}^{(n)})} &= \sup_{\|\psi\| = \|\varphi\| = 1} \left| \left\langle \psi \,\middle|\, \mathcal{W}_{(n)} \left[w^{(n)}_{R,S} \right] \varphi \right\rangle_{\mathcal{H}^{(n)}} \right| \\ &\leq \sqrt{\frac{(1+\varepsilon)^{R+S}}{R^R S^S}} \left\| w^{(n)}_{R,S} \right\|_{(n)} \\ &\xrightarrow{\varepsilon \to 0} \frac{1}{\sqrt{R^R S^S}} \left\| w^{(n)}_{R,S} \right\|_{(n)}. \end{aligned}$$

■

An immediate consequence of Proposition 4.1 is

Corollary 4.3 *The map* $\mathcal{W}_{(n)} \; :\longrightarrow \; \mathcal{B}(\mathcal{H}^{(n)})$ *is well defined and for* $\underline{w}^{(n)} = (w^{(n)}_{R,S})_{R+S \geq 0} \in \mathfrak{W}^{(n)}_{\#}$ *holds*

$$\left\| \mathcal{W}_{(n)} \left[\underline{w}^{(n)} \right] \right\|_{\mathcal{B}(\mathcal{H}^{(n)})} \leq \left\| \underline{w}^{(n)} \right\|^{\#}_{(n),\xi}$$

and

$$\left\| \mathcal{W}_{(n)} \left[\underline{w}^{(n)} \right] \right\|_{\mathcal{B}(\mathcal{H}^{(n)})} \leq \xi \left\| \underline{w}^{(n)} \right\|^{\#}_{(n),\xi}$$

if $w^{(n)}_{0,0} = 0$.

The Wick monomials of order $(0,0)$ are functions in $\Lambda_{[\theta_n]}$. We decompose the corresponding space of functions as

$$\mathfrak{W}^{(n)}_{0,0} = \mathcal{H}^{(n)}_{<\infty} \oplus \mathcal{T}^{(n)} = \left\{ \left(w^{(n)}_{0,0}[0], w^{(n)}_{0,0}[\,\cdot\,] - w^{(n)}_{0,0}[0] \right) \,\middle|\, w^{(n)}_{0,0} \in \mathfrak{W}^{(n)}_{0,0} \right\}$$

into a direct sum of all possible $\mathcal{H}^{(n)}_{<\infty}$-valued offsets of the functions of $(0,0)$-order and a space

$$\mathcal{T}^{(n)} := \left\{ T \in C^1 \left(\mathcal{Q}^{(n)}; \mathcal{H}^{(n)}_{<\infty} \right) \,\middle|\, T(0) = 0,\ \|T\|_{\mathcal{T}^{(n)}} := \sup_{q \in \mathcal{Q}^{(n)}} \left\| |\nabla_q T(q)| \right\|_{\mathcal{B}(\mathcal{H}^{(n)}_{<\infty})} < \infty \right\}$$

of differentiable functions vanishing at zero. The space $\mathcal{T}^{(n)}$ is a Banach space equipped with the norm $\|\cdot\|_{\mathcal{T}^{(n)}}$. That way we can rewrite

$$\mathfrak{W}^{(n)}_{\#} = \mathcal{H}^{(n)}_{<\infty} \oplus \mathcal{T}^{(n)} \oplus \bigoplus_{R+S \geq 1} \mathfrak{W}^{(n)}_{R,S}.$$

Components in $\mathcal{T}^{(n)}$ are assigned to functions of the free operators $\Lambda_{[\theta_n]} = (\cos(\delta') L_{\mathrm{res}}, M_{[\theta_n]})$,

$$\left(w^{(n)}_{(0,0)}[\,\cdot\,] - w^{(n)}_{(0,0)}[0] \right) \mapsto \left(w^{(n)}_{(0,0)}[\Lambda_{[\theta_n]}] - w^{(n)}_{(0,0)}[0] \right) =: T^{(n)} \left[\Lambda_{[\theta_n]} \right],$$

while components belonging to $\mathcal{H}^{(n)}_{<\infty}$ are mapped under $\mathcal{W}_{(n)}$, in the case $n = 2, 3, \ldots, \mathcal{N}$, to multiples of the identity operator,

$$w^{(n)}_{(0,0)}[0] \mapsto w^{(n)}_{(0,0)}[0] \mathbb{1}_{\mathcal{H}^{(n)}} =: -E^{(n)} \mathbb{1}_{\mathcal{H}^{(n)}},$$

representing a spectral shift. In the case $n = 1$ the component $w^{(1)}_{0,0}[0]$ is mapped to

$$w^{(1)}_{(0,0)}[0] \mapsto w^{(1)}_{(0,0)}[0] \otimes \mathbb{1}_{\mathcal{F}(L^2[\Upsilon])} =: -E^{(1)} \otimes \mathbb{1}_{\mathcal{F}(L^2[\Upsilon])},$$

which represents a multidimensional level shift operator on the particle variables. The non-scalar situation of the spectral shift requires a separate consideration of the renormalization transformation acting on $\mathfrak{W}_{\#}^{(1)}$ as we will see later. The components of $\mathfrak{W}_{R,S}^{(n)}$, $R+S \geq 1$, are mapped to Wick monomials,

$$w_{R,S}^{(n)} \mapsto \mathcal{W}_{(n)}\left[w_{R,S}^{(n)}\right] =: W^{(n)},$$

which represent the space of perturbations to the free operators $T^{(n)}[\Lambda_{[\theta_n]}] - E^{(n)}$. Henceforth we will understand an element $\underline{w}^{(n)} = (w_{R,S}^{(n)})_{R+S\geq 0} \in \mathfrak{W}_{\#}^{(n)}$ as an operator

$$K^{(n)} := \mathcal{W}_{(n)}\left[\underline{w}^{(n)}\right] = T^{(n)}\left[\Lambda_{[\theta_n]}\right] - E^{(n)} + W^{(n)} \in \mathcal{B}(\mathcal{H}^{(n)}).$$

The aim of this chapter is the spectral analysis of operators from the class $\mathcal{W}_{(n)}[\mathfrak{W}_{\#}^{(n)}] \subseteq \mathcal{B}(\mathcal{H}^{(n)})$. The renormalization procedure requires that we are able to treat operators which depend analytically rather than linearly on a spectral parameter z. Therefore, we introduce the Banach space $\mathfrak{W}^{(n)}$ of analytic functions

$$B_{1/4} \ni z \mapsto \underline{w}^{(n)}[z] \in \mathfrak{W}_{\#}^{(n)}$$

where we set

$$B_r := \{\zeta \in \mathbb{C} \mid |\zeta| < r\}.$$

The space $\mathfrak{W}^{(n)}$ is equipped with the supremum-norm,

$$\left\|\underline{w}^{(n)}[\,\cdot\,]\right\|_{(n),\xi} := \sup_{z\in B_{1/4}} \left\|\underline{w}^{(n)}[z]\right\|_{(n),\xi}^{\#}.$$

By $\mathcal{W}_{(n)}\left[\mathfrak{W}^{(n)}\right]$ we understand the space of analytic functions $B_{1/4} \ni z \mapsto K^{(n)}[z] \in \mathcal{W}_{(n)}\left[\mathfrak{W}_{\#}^{(n)}\right]$ on $B_{1/4}$ with values in $\mathcal{W}_{(n)}\left[\mathfrak{W}_{\#}^{(n)}\right]$.

4.2 Initial Data for the Renormalization Procedure

It is the aim of the present chapter to iterate the application of the smooth Feshbach map in order to study the spectrum of $K_\theta^{(s)}$ on smaller scales. As we will see in the subsequent sections the smooth Feshbach map, embedded in the renormalization transformation, links operators from the class $\mathcal{W}_{(n)}[\mathfrak{W}_{\#}^{(n)}] \subseteq \mathcal{B}(\mathcal{H}^{(n)})$ to operators from $\mathcal{W}_{(n+1)}[\mathfrak{W}_{\#}^{(n+1)}] \subseteq \mathcal{B}(\mathcal{H}^{(n+1)})$, preserving certain spectral properties. To be in position to apply the renormalization procedure to the object of interest, namely the operator $K_\theta^{(s)}$ or rather its image $\mathfrak{F}_{\Xi_{0,\rho}}(K_\theta^{(s)} - z, L_{0,\theta} - z)$ under the first application of the smooth Feshbach map (given in (3.14) and discussed in Section 3.3), we have to fit it into the framework of Banach spaces of operators just described in the previous Section 4.1. To this end it is necessary to rewrite $\mathfrak{F}_{\Xi_{0,\rho}}(K_\theta^{(s)} - z, L_{0,\theta} - z)$ in terms of the free operator $L_{0,\theta}$ and the Wick monomials.

We recall the Neumann series expansion (3.15) of the restricted resolvent $(K_\theta^{(s)} - z)^{-1}_{\overline{\Xi}_{0,\rho}}$ to rewrite the defining expression (3.14) of $\mathfrak{F}_{\Xi_{0,\rho}}(K_\theta^{(s)} - z, L_{0,\theta} - z)$ as a series,

$$\begin{aligned} &\mathfrak{F}_{\Xi_{0,\rho}}\left(K_\theta^{(s)} - z, L_{0,\theta} - z\right) \\ &= L_{0,\theta} - z - \sum_{L=1}^{\infty}(-g)^L \Xi_{0,\rho} I_\theta^{(s)} \left(\frac{\overline{\Xi}_{0,\rho}^2}{L_{0,\theta} - z} I_\theta^{(s)}\right)^{L-1} \Xi_{0,\rho}, \end{aligned}$$

where the absolute convergence of the series is guaranteed by the arguments of the proof to Lemma 3.5. For a fixed $L \in \mathbb{N}$ we set

$$\begin{aligned} \tilde{F}_0(\lambda_1, \lambda_2) &:= X_{0,\rho}(\lambda_2) =: \tilde{F}_L(\lambda_1, \lambda_2), \\ \tilde{F}_\ell(\lambda_1, \lambda_2) &:= \frac{\overline{X}_{0,\rho}^2(\lambda_2)}{T^{(0)}[\lambda_1, \lambda_2; z] - E^{(0)}[z]}, \qquad \ell = 1, \ldots, L-1, \end{aligned}$$

where we abbreviate

$$\begin{aligned} T^{(0)}[\lambda_1, \lambda_2; z] &:= \lambda_1 + i\lambda_2, \\ E^{(0)}[z] &:= z, \end{aligned}$$

and

$$\begin{aligned} w_{1,0}^{(0)}[\lambda_1, \lambda_2, y; z] &:= g F_\theta^{(s)}(y) m_\theta(u)^{1/2}, \\ w_{0,1}^{(0)}[\lambda_1, \lambda_2, \tilde{y}; z] &:= g F_{\overline{\theta}}^{(\overline{s})}(\tilde{y})^* m_\theta(\tilde{u})^{1/2}, \\ w_{R,S}^{(0)} &:= 0, \qquad R + S \geq 2 \end{aligned} \tag{4.15}$$

with $F_\theta^{(s)} = [\mathcal{G} - \mathcal{G}'_{(s\delta\vec{\beta})}]_\theta$ as introduced in (2.26) and $y = (u, \Sigma, r), \tilde{y} = (\tilde{u}, \tilde{\Sigma}, \tilde{r}) \in \Upsilon$. We recall the set of notations of Chapter D. Having these notations at hand we can write

$$W^{(0)}[z] := g I_\theta^{(s)} = \mathcal{W}_{[\theta]}\left[\left(w_{R,S}^{(0)}[\,\cdot\,; z]\right)_{R+S\geq 1}\right]$$

(see definition (D.6)) and

$$\begin{aligned} &g^L \Xi_{0,\rho} I_\theta^{(s)} \left(\frac{\overline{\Xi}_{0,\rho}^2}{L_{0,\theta} - z} I_\theta^{(s)}\right)^{L-1} \Xi_{0,\rho} \\ &= \tilde{F}_0(\Lambda_{[\theta]}) W^{(0)}[z] \tilde{F}_1(\Lambda_{[\theta]}) W^{(0)}[z] \cdots W^{(0)}[z] \tilde{F}_{L-1}(\Lambda_{[\theta]}) W^{(0)}[z] \tilde{F}_L(\Lambda_{[\theta]}) \end{aligned}$$

where $\Lambda_{[\theta]} = (\cos(\delta') L_{\mathrm{res}}, M_{[\theta]})$ is introduced in Section 4.1 and Appendix D, see (4.11) and (D.1), resp. The application of Theorem D.3 yields

$$\begin{aligned} \mathfrak{F}_{\Xi_{0,\rho}}\left(K_\theta^{(s)} - z, L_{0,\theta} - z\right) &= \mathcal{W}_{[\theta]}\left[\left(\hat{w}_{R,S}^{(1)}[z]\right)_{R+S\geq 0}\right] \\ &\equiv \hat{T}^{(1)}[\Lambda_{[\theta]}; z] - \hat{E}^{(1)}[z] + \hat{W}^{(1)}[z] \end{aligned}$$

where

$$\begin{aligned}\hat{E}^{(1)}[z] &:= -\hat{w}^{(1)}_{0,0}[0;z],\\ \hat{T}^{(1)}[\lambda;z] &:= \hat{w}^{(1)}_{0,0}[\lambda;z]-\hat{w}^{(1)}_{0,0}[0;z],\\ \hat{W}^{(1)}[z] &:= \mathcal{W}_{[\theta]}\left[\left(\hat{w}^{(1)}_{R,S}[\,\cdot\,;z]\right)_{R+S\geq 1}\right]\end{aligned}$$

and the integral kernels $\hat{w}^{(1)}_{R,S}$ are the symmetrization (in the sense of (D.8)) of the functions

$$\begin{aligned}&\tilde{w}^{(1)}_{R,S}[\lambda;Y^{(R,S)};z]\\ &:= \sum_{L=1}^{\infty}(-1)^{L-1}\sum_{\substack{r_1+\cdots+r_L=R,\\ s_1+\cdots+s_L=S}}\ \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ r_\ell+p_\ell+s_\ell+q_\ell=1}}\left[\prod_{\ell=1}^{L}\binom{r_\ell+p_\ell}{r_\ell}\binom{s_\ell+q_\ell}{s_\ell}\right]\\ &\quad\times X_{0,\rho}\left(\lambda_2+\left[\eta^{(\theta)}_0\left(Y^{(R,S)}\right)\right]_2\right)\\ &\quad\times\Bigg\langle\mathcal{W}^{(r_1,s_1)}_{[\theta]}\left[w^{(0)}_{r_1+p_1,s_1+q_1}\right]\left(\lambda+\eta^{(\theta)}_1\left(Y^{(R,S)}\right);y^{(r_1)}_1,\tilde{y}^{(s_1)}_1;z\right)\\ &\qquad\times\frac{\overline{X}^2_{0,\rho}\left(M_{[\theta]}+\lambda_2+\left[\eta^{(\theta)}_1\left(Y^{(R,S)}\right)\right]_2+\sum_{j=1}^{s_1}m_\theta\left(\tilde{u}^{(s_1)}_{1,j}\right)\right)}{T^{(0)}\left[\Lambda_{[\theta]}+\lambda+\eta^{(\theta)}_1\left(Y^{(R,S)}\right)+\sum_{j=1}^{s_1}\lambda_\theta\left(\tilde{u}^{(s_1)}_{1,j}\right);z\right]-E^{(0)}[z]}\\ &\qquad\cdots\\ &\qquad\times\frac{\overline{X}^2_{0,\rho}\left(M_{[\theta]}+\lambda_2+\left[\eta^{(\theta)}_{L-1}\left(Y^{(R,S)}\right)\right]_2+\sum_{j=1}^{s_{L-1}}m_\theta\left(\tilde{u}^{(s_{L-1})}_{L-1,j}\right)\right)}{T^{(0)}\left[\Lambda_{[\theta]}+\lambda+\eta^{(\theta)}_{L-1}\left(Y^{(R,S)}\right)+\sum_{j=1}^{s_1}\lambda_\theta\left(\tilde{u}^{(s_{L-1})}_{L-1,j}\right);z\right]-E^{(0)}[z]}\\ &\qquad\times\mathcal{W}^{(r_L,s_L)}_{[\theta]}\left[w^{(0)}_{r_L+p_L,s_L+q_L}\right]\left(\lambda+\eta^{(\theta)}_L\left(Y^{(R,S)}\right);y^{(r_L)}_L,\tilde{y}^{(s_L)}_L;z\right)\Bigg\rangle_{\Omega_{\mathrm{vac}}}\\ &\quad\times X_{0,\rho}\left(\lambda_2+\left[\eta^{(\theta)}_L\left(Y^{(R,S)}\right)\right]_2+\sum_{j=1}^{s_L}m_\theta\left(\tilde{u}^{(s_L)}_{L,j}\right)\right),\end{aligned}$$

for $R+S\geq 1$ and

$$\begin{aligned}&\hat{w}^{(1)}_{0,0}[\lambda;z]\\ &:= T^{(0)}[\lambda;z]-E^{(0)}[z]\\ &\quad+\sum_{L=2}^{\infty}(-1)^{L-1}\sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ p_\ell+q_\ell=1}}X_{0,\rho}\left(\lambda_2\right)\\ &\quad\times\Bigg\langle\mathcal{W}^{(0,0)}_{[\theta]}\left[w^{(0)}_{p_1,q_1}\right](\lambda;z)\,\frac{\overline{X}^2_{0,\rho}\left(M_{[\theta]}+\lambda_2\right)}{T^{(0)}\left[\Lambda_{[\theta]}+\lambda;z\right]-E^{(0)}[z]}\end{aligned}$$

$$\times \cdots \times \frac{\overline{X}_{0,\rho}^2 \left(M_{[\theta]} + \lambda_2\right)}{T^{(0)}\left[\Lambda_{[\theta]} + \lambda; z\right] - E^{(0)}[z]} \mathcal{W}_{[\theta]}^{(0,0)} \left[w_{p_L,q_L}^{(0)}\right](\lambda; z) \Big\rangle_{\Omega_{\mathrm{vac}}} X_{0,\rho}(\lambda_2),$$

where $\lambda = (\lambda_1, \lambda_2)$, $[(\lambda_1, \lambda_2)]_j := \lambda_j$ and the functions $\eta_\ell^{(\theta)}$ are defined in (D.10). The notation for the partially integrated Wick monomials $\mathcal{W}_{[\theta]}^{(r_\ell,s_\ell)}\left[w_{r_\ell+p_\ell,s_\ell+q_\ell}^{(0)}\right]$ is explained in (D.7), the vacuum expectation value $\langle \cdot \rangle_{\Omega_{\mathrm{vac}}}$ in (D.2).

The operator $\mathfrak{F}_{\Xi_{0,\rho}}(K_\theta^{(s)} - z, L_{0,\theta} - z)$ is defined on $\mathrm{ran}(P_{0,\rho}) = \ker(L_{\mathrm{p}}) \otimes P_{\left[M_{[\theta]} \leq \rho\right]} \mathcal{F}[L^2(\Upsilon)]$. We use the unitary rescaling operator S_ρ defined in (D.12) and the rescaling map $\mathfrak{S}_\rho$ acting on an operator A like

$$\mathfrak{S}_\rho(A) = \rho^{-1} S_\rho A S_\rho^{-1},$$

given in (D.13), Appendix D.2, to blow up the domain. We refer to Appendix D.2 for a detailed discussion of $\mathfrak{S}_\rho$, in particular one shall consult (D.14) for the scaling properties of the bosonic variables. As a consequence we get with the functional calculus

$$S_\rho P_{\left[M_{[\theta]} \leq \rho\right]} \mathcal{F}(L^2[\Upsilon]) = P_{\left[S_\rho M_{[\theta]} S_\rho^{-1} \leq \rho\right]} S_\rho \mathcal{F}(L^2[\Upsilon]) = P_{\left[\rho M_{[\theta_1]} \leq \rho\right]} \mathcal{F}(L^2[\Upsilon])$$

where $\theta_1 = (i\delta', i\rho^{-1}\tau')$. We define the family of operators

$$K^{(1)}[z] := \mathfrak{S}_\rho\left(\mathfrak{F}_{\Xi_{0,\rho}}\left(K_\theta^{(s)} - Z^{(0)}[z], L_{0,\theta} - Z^{(0)}[z]\right)\right), \qquad |z| < \frac{1}{4}, \tag{4.16}$$

which lives on the space

$$\mathcal{H}^{(1)} = \ker(L_{\mathrm{p}}) \otimes \left[P_{\left[M_{[\theta_1]} \leq 1\right]} \mathcal{F}(L^2[\Upsilon])\right].$$

The spectral parameter z is adjusted by the function

$$Z^{(0)} : B_{1/4} \to B_{\rho/4}, \qquad Z^{(0)}[z] := \rho z. \tag{4.17}$$

The isospectral property of the Feshbach map, Theorem E.1, and the fact that $\mathfrak{S}_\rho$ leaves the dimension of the kernel invariant implies the following equivalence,

$$Z^{(0)}[z] \in \mathrm{spec}\left(K_\theta^{(s)}\right) \qquad \Longleftrightarrow \qquad 0 \in \mathrm{spec}\left(K^{(1)}[z]\right) \tag{4.18}$$

for all $z \in B_{1/4}$.

Consulting Proposition D.4 we see that this family of operators can be written as

$$K^{(1)}[z] = \mathcal{W}_{[\theta_1]}\left[\left(w_{R,S}^{(1)}[\,\cdot\,; z]\right)_{R+S \geq 0}\right] = T^{(1)}[\Lambda_{[\theta_1]}; z] - E^{(1)}[z] + W^{(1)}[z], \tag{4.19}$$

for

$$
\begin{aligned}
E^{(1)}[z] &:= -w_{0,0}^{(1)}[0,z] = -\rho^{-1}\hat{E}^{(1)}[\rho z],\\
T^{(1)}[\lambda;z] &:= w_{0,0}^{(1)}[\lambda,z] - w_{0,0}^{(1)}[0,z] = \rho^{-1}\hat{T}^{(1)}[\rho\lambda;\rho z],\\
W^{(1)}[z] &:= \mathcal{W}_{[\theta_1]}\left[\left(w_{R,S}^{(1)}[\,\cdot\,;z]\right)_{R+S\geq 1}\right].
\end{aligned}
\tag{4.20}
$$

The rescaling procedure transfers to the integral kernels which are modified by $\mathfrak{s}_\rho$, given in (D.15). The rescaled integral kernels read

$$
\begin{aligned}
w_{R,S}^{(1)}[\lambda;Y^{(R,S)};z] &:= \mathfrak{s}_\rho\left(\hat{w}_{R,S}^{(1)}[\,\cdot\,;\rho z]\right)[\lambda;Y^{(R,S)}] \\
&= \rho^{-1}\sum_{L=1}^{\infty}(-1)^{L-1}\sum_{\substack{r_1+\cdots+r_L=R,\\ s_1+\cdots+s_L=S}}\;\sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ r_\ell+p_\ell+s_\ell+q_\ell=1}}\left[\prod_{\ell=1}^{L}\binom{r_\ell+p_\ell}{r_\ell}\binom{s_\ell+q_\ell}{s_\ell}\right] \\
&\quad \times V^{(1)}_{\underline{r,p,s,q}_L}[\lambda;Y^{(R,S)};z]
\end{aligned}
\tag{4.21}
$$

where we abbreviate for, fixed $L\in\mathbb{N}$, the tuple $\underline{r,p,s,q}_L := (r_\ell,p_\ell,s_\ell,q_\ell)_{\ell=1}^{L}\in(\mathbb{N}_0)^{4L}$. The function $V^{(1)}_{\underline{r,p,s,q}_L}$ is the symmetrization (in the sense of (D.8)) of

$$
\begin{aligned}
&\tilde{V}^{(1)}_{\underline{r,p,s,q}_L}[\lambda;Y^{(R,S)};z] := \\
&\quad X_{0,1}\left(\lambda_2+\left[\eta_0^{(\theta_1)}\left(Y^{(R,S)}\right)\right]_2\right) \\
&\quad \times\Big\langle \mathcal{W}_{[\theta]}^{(r_1,s_1)}\left[w^{(0)}_{r_1+p_1,s_1+q_1}\right]\left(\rho\left(\lambda+\eta_1^{(\theta_1)}\left(Y^{(R,S)}\right);y_1^{(r_1)},\tilde{y}_1^{(s_1)};z\right)\right) \\
&\qquad \times\frac{\overline{X}_{0,1}^{2}\left(\rho^{-1}M_{[\theta]}+\lambda_2+\left[\eta_1^{(\theta_1)}\left(Y^{(R,S)}\right)\right]_2+\sum_{j=1}^{s_1}m_{\theta_1}\left(\tilde{u}_{1,j}^{(s_1)}\right)\right)}{T^{(0)}\left[\Lambda_{[\theta]}+\rho\left(\lambda+\eta_1^{(\theta_1)}(Y^{(R,S)})+\sum_{j=1}^{s_1}\lambda_{\theta_1}\left(\tilde{u}_{1,j}^{(s_1)}\right)\right);\rho z\right]-E^{(0)}[\rho z]} \\
&\qquad \cdots \\
&\qquad \times\frac{\overline{X}_{0,1}^{2}\left(\rho^{-1}M_{[\theta]}+\lambda_2+\left[\eta_{L-1}^{(\theta_1)}\left(Y^{(R,S)}\right)\right]_2+\sum_{j=1}^{s_{L-1}}m_{\theta_1}\left(\tilde{u}_{L-1,j}^{(s_{L-1})}\right)\right)}{T^{(0)}\left[\Lambda_{[\theta]}+\rho\left(\lambda+\eta_{L-1}^{(\theta_1)}(Y^{(R,S)})+\sum_{j=1}^{s_1}\lambda_{\theta_1}\left(\tilde{u}_{L-1,j}^{(s_{L-1})}\right)\right);\rho z\right]-E^{(0)}[\rho z]} \\
&\qquad \times\mathcal{W}_{[\theta]}^{(r_L,s_L)}\left[w^{(0)}_{r_L+p_L,s_L+q_L}\right]\left(\rho\left(\lambda+\eta_L^{(\theta_1)}\left(Y^{(R,S)}\right);y_L^{(r_L)},\tilde{y}_L^{(s_L)};z\right)\right)\Big\rangle_{\Omega_{\text{vac}}} \\
&\quad \times X_{0,1}\left(\lambda_2+\left[\eta_L^{(\theta_1)}\left(Y^{(R,S)}\right)\right]_2+\sum_{j=1}^{s_L}m_{\theta_1}\left(\tilde{u}_{L,j}^{(s_L)}\right)\right)
\end{aligned}
\tag{4.22}
$$

for $R+S \geq 1$ and

$$
\begin{aligned}
& w_{0,0}^{(1)}[\lambda; z] \qquad\qquad (4.23)\\
&:= \rho^{-1}\left(T^{(0)}[\rho\lambda;\rho z] - E^{(0)}[\rho z]\right) \\
&\quad +\rho^{-1}\sum_{L=2}^{\infty}(-1)^{L-1}\sum_{\substack{p_1,q_1,\ldots,p_L,q_L: \\ p_\ell+q_\ell=1}} X_{0,1}(\lambda_2) \\
&\quad \times\Bigg\langle \mathcal{W}_{[\theta]}^{(0,0)}\left[w_{p_1,q_1}^{(0)}\right](\rho\lambda;\rho z)\,\frac{\overline{X}_{0,1}^2\left(\rho^{-1}M_{[\theta]}+\lambda_2\right)}{T^{(0)}\left[\Lambda_{[\theta]}+\rho\lambda;\rho z\right]-E^{(0)}[\rho z]} \\
&\qquad \times\cdots\times\frac{\overline{X}_{0,1}^2\left(\rho^{-1}M_{[\theta]}+\lambda_2\right)}{T^{(0)}\left[\Lambda_{[\theta]}+\rho\lambda;\rho z\right]-E^{(0)}[\rho z]}\mathcal{W}_{[\theta]}^{(0,0)}\left[w_{p_L,q_L}^{(0)}\right](\rho\lambda;\rho z)\Bigg\rangle_{\Omega_{\mathrm{vac}}} \\
&\quad \times X_{0,1}(\lambda_2)\,.
\end{aligned}
$$

One easily checks the above relations using that

$$\eta^{(\theta)}(\rho Y^{(R,S)}) = \rho\eta^{(\theta_1)}(Y^{(R,S)}), \qquad m_\theta(\rho u) = \rho m_{\theta_1}(u)$$

and

$$X_{0,\rho}(\rho x) = X_{0,1}(x), \qquad \overline{X}_{0,\rho}(\rho x) = \overline{X}_{0,1}(x).$$

Note that the first term in the expansion of $w_{0,0}^{(1)}[\lambda; z]$ is given by

$$\rho^{-1}\left(T^{(0)}[\rho\lambda;\rho z] - E^{(0)}[\rho z]\right) = T^{(0)}[\lambda; z] - E^{(0)}[z] = (\lambda_1 + i\lambda_2) - z. \qquad (4.24)$$

We connect the representation (3.20) with (4.19) by making the following observation. Because of

$$\left\langle \mathcal{W}_{[\theta_1]}\left[\left(w_{R,S}^{(1)}[\,\cdot\,; z]\right)_{R+S\geq 1}\right]\right\rangle_{\Omega_{\mathrm{vac}}} = 0, \qquad \left\langle T^{(1)}[\Lambda_{[\theta_1]}; z]\right\rangle_{\Omega_{\mathrm{vac}}} = T^{(1)}[0; z] = 0$$

the term $E^{(1)}[z]$ in (4.19) is determined by

$$E^{(1)}[z] = -\left\langle K^{(1)}[z]\right\rangle_{\Omega_{\mathrm{vac}}}$$

and therefore, with the help of (3.20),

$$
\begin{aligned}
E^{(1)}[z] &= -\left\langle \mathfrak{S}_\rho\left(\mathfrak{F}_{\Xi_{0,\rho}}(K_\theta^{(s)} - \rho z, L_{0,\theta} - \rho z)\right)\right\rangle_{\Omega_{\mathrm{vac}}} \\
&= z - \frac{g^2}{\rho}\Lambda_0^{(s)} + \mathcal{O}\left(\frac{g^{2+\tilde{\varepsilon}}}{\rho}\right), \qquad (4.25)
\end{aligned}
$$

where the remainder term $\mathcal{O}\left(g^{2+\tilde{\varepsilon}}\rho^{-1}\right) = \mathcal{O}\left(g^{(4+\tilde{\varepsilon})/3}\right)$ is estimated uniformly in $|z| < 1/4$.

We now show that the operator family $z \mapsto K^{(1)}[z]$ qualifies as initial data for the renormalization procedure, i.e., the integral kernels $w^{(1)}_{R,S}$ generating $K^{(1)}$ and defined in (4.21) build an element in $\mathfrak{W}^{(1)}$.

Proposition 4.4 *The sequence* $\underline{w}^{(1)} := \left(-E^{(1)}, T^{(1)}, w^{(1)}_{R,S}\right)_{R+S\geq 1}$ *of integral kernels given in (4.20, 4.21, 4.23) obeys the following bounds,*

$$\begin{aligned} \left\| \left(w^{(1)}_{R,S}\right)_{R+S\geq 1} \right\|_{(1),\xi} &= \mathcal{O}\left(g\rho^{\mu}\right) = \mathcal{O}\left(g^{1+2\mu/3(1+\tilde{\varepsilon})}\right), \\ \sup_{z\in B_{1/4}} \left\|E^{(1)}[z] - z\right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} &\leq \frac{g^2}{\rho} \left\|\Lambda^{(s)}_0\right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} + \mathcal{O}\left(\frac{g^{2+\tilde{\varepsilon}}}{\rho}\right) \\ &= \mathcal{O}\left(g^{1+(1-2\tilde{\varepsilon})/3}\right), \\ \sup_{\substack{z\in B_{1/4}, \\ \lambda\in\mathcal{Q}^{(1)}}} \left\| \left| \nabla_{\lambda} T^{(1)}[\lambda; z] - \binom{1}{i} \right| \right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} &= \mathcal{O}\left(\frac{g^2}{\rho^2}\right) = \mathcal{O}\left(g^{(2-4\tilde{\varepsilon})/3}\right). \end{aligned} \tag{4.26}$$

The last estimate in (4.26) can be improved,

$$\sup_{\substack{z\in B_{1/4}, \\ \lambda\in\mathcal{Q}^{(1)}}} \left\|T^{(1)}[\lambda; z] - (\lambda_1 + i\lambda_2)\right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} = \mathcal{O}\left(g\rho^{\mu} + \frac{g^2}{\rho}\right). \tag{4.27}$$

Proof. Recall the definition (4.21) of $w^{(1)}_{R,S}$ in terms of the functions $\tilde{V}^{(1)}_{\underline{r,p,s,q}_L}$ given in (4.22). Plugging the bounds on $\tilde{V}^{(1)}_{\underline{r,p,s,q}_L}$ provided in Lemma 4.10(iii) (see below) into (4.21) we get

$$\begin{aligned} &\sup_{z\in B_{1/4}} \left\|w^{(1)}_{R,S}[\,\cdot\,;z]\right\|^{\#}_{(1)} \\ &\leq \sum_{L=1}^{\infty} \mathcal{C}_{\chi_1}(L+1)\left(\frac{\mathcal{C}_{\chi_1}}{\rho}\right)^L \left(2\rho^{1+\mu}\right)^{R+S} \sum_{\substack{r_1+\cdots+r_L=R, \\ s_1+\cdots+s_L=S}} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L: \\ r_\ell+p_\ell+s_\ell+q_\ell=1}} \left(2gM(\underline{\omega}^{(0)})\right)^L \end{aligned}$$

where $1 \leq \mathcal{C}_{\chi_1} < \infty$ is the constant introduced in (4.6), only depending on the cutoff function χ_1. We further used that $\binom{j+k}{j} \leq 2^{j+k}$ and $\prod_{\ell=1}^{L} 2^{p_\ell+q_\ell} \leq 2^L$

since $r_\ell + p_\ell + s_\ell + q_\ell = 1$. Summing over $R + S \geq 1$ yields

$$\begin{aligned}
&\left\| \left(w^{(1)}_{R,S} \right)_{R+S\geq 1} \right\|_{(1),\xi} \\
&= \sum_{R+S\geq 1} \xi^{-(R+S)} \sup_{z\in B_{1/4}} \left\| w^{(1)}_{R,S}[\,\cdot\,;z] \right\|^{\#}_{(1)} \\
&\leq 2\mathcal{C}_{\chi_1}\rho^{1+\mu} \sum_{L=1}^{\infty} (L+1) \left(\frac{\mathcal{C}_{\chi_1}}{\rho} \right)^L \sum_{R+S\geq 1} \xi^{-(R+S)} \\
&\quad \times \sum_{\substack{r_1+\cdots+r_L=R,\\ s_1+\cdots+s_L=S}} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ r_\ell+p_\ell+s_\ell+q_\ell=1}} \left(2gM(\underline{\omega}^{(0)}) \right)^L \\
&\leq 2\mathcal{C}_{\chi_1}\rho^{1+\mu} \sum_{L=1}^{\infty} (L+1) \left(\frac{\mathcal{C}_{\chi_1}}{\rho} \right)^L \left[2\xi^{-1} gM(\underline{\omega}^{(0)}) \sum_{r+s=1} \left(\sum_{p=0}^{r} \xi^p \right) \left(\sum_{q=0}^{s} \xi^q \right) \right]^L \\
&\leq 2\mathcal{C}_{\chi_1}\rho^{1+\mu} \sum_{L=1}^{\infty} (L+1) \left[\frac{6\mathcal{C}_{\chi_1} gM(\underline{\omega}^{(0)})}{\rho\xi(1-\xi)^2} \right]^L \\
&\leq \frac{96\mathcal{C}^2_{\chi_1}\rho^{\mu} gM(\underline{\omega}^{(0)})}{\xi(1-\xi)^2}
\end{aligned}$$

where we used

$$\sum_{L=1}^{\infty} (L+1)x^L = \frac{d}{dx} \sum_{L=0}^{\infty} x^L - 1 = \frac{1}{(1-x)^2} - 1 = x\frac{2-x}{(1-x)^2} \leq 8x, \qquad 0 \leq x \leq \frac{1}{2}.$$

This relation is applicable due to (3.4, 4.5, 4.8) and

$$\frac{g}{\rho} \frac{6\mathcal{C}_{\chi_1} M(\underline{\omega}^{(0)})}{\xi(1-\xi)^2} \leq g^{1/6} \frac{128M(\underline{\omega}^{(0)})}{3} \mathcal{C}^2_{\chi_1} (16\mathcal{C}_{\chi_1})^{1/\mu} \leq \frac{1}{2},$$

for g sufficiently small. Under the assumptions (3.4, 4.5, 4.8) on the involved parameters we have

$$\begin{aligned}
\left\| \left(w^{(1)}_{R,S} \right)_{R+S\geq 1} \right\|_{(1),\xi} &\leq \frac{96\mathcal{C}^2_{\chi_1}\rho^{\mu} gM(\underline{\omega}^{(0)})}{\xi(1-\xi)^2} \\
&\leq g^{1+2\mu/3(1+\tilde{\varepsilon})} 1536 M(\underline{\omega}^{(0)}) \mathcal{C}^3_{\chi_1} (16\mathcal{C}_{\chi_1})^{1/\mu}
\end{aligned}$$

The relation (4.25) allows a simple estimate of

$$\sup_{z\in B_{1/4}} \left\| E^{(1)}[z] - z \right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} \leq \frac{g^2}{\rho} \left\| \Lambda^{(s)}_0 \right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} + \mathcal{O}\left(\frac{g^{2+\tilde{\varepsilon}}}{\rho} \right).$$

It remains to estimate the deviation of $T^{(1)}$ from the function $(\lambda_1,\lambda_2)\mapsto\lambda_1+i\lambda_2$. To this end we consider the representation (4.23) and recall the relation (4.24) to compute

$$\begin{aligned}
&\sup_{\substack{z\in B_{1/4},\\ \lambda\in\mathcal{Q}^{(1)}}}\left|\left|\left|\nabla_\lambda T^{(1)}[\lambda;z]-\begin{pmatrix}1\\ i\end{pmatrix}\right|\right|\right|_{\mathcal{B}(\ker(L_{\rm p}))}\\
\le\;& \rho^{-1}\sum_{L=2}^{\infty}\sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ p_\ell+q_\ell=1}}\sup_{\substack{z\in B_{1/4},\\ \lambda\in\mathcal{Q}^{(1)}}}\left|\left|\left|\nabla_\lambda V^{(1)}_{\underline{0,p,0,q}_L}[\lambda;z]\right|\right|\right|_{\mathcal{B}(\ker(L_{\rm p}))}\\
\le\;& \sum_{L=2}^{\infty}\mathcal{C}_{\chi_1}(L+1)\left(\frac{\mathcal{C}_{\chi_1}}{\rho}\right)^L\sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ p_\ell+q_\ell=1}}\left(gM(\underline{\omega}^{(0)})\right)^L\\
\le\;& \mathcal{C}_{\chi_1}\sum_{L=2}^{\infty}(L+1)\left(\frac{2\mathcal{C}_{\chi_1}gM(\underline{\omega}^{(0)})}{\rho}\right)^L\\
\le\;& 12\mathcal{C}_{\chi_1}\left(\frac{2\mathcal{C}_{\chi_1}gM(\underline{\omega}^{(0)})}{\rho}\right)^2\\
=\;& \frac{g^2}{\rho^2}48\mathcal{C}_{\chi_1}^3M(\underline{\omega}^{(0)})^2,
\end{aligned}$$

where we used that

$$\sum_{L=2}^{\infty}(L+1)x^L=\frac{d}{dx}\left[\sum_{L=0}^{\infty}x^L-x-x^2\right]=\frac{1}{(1-x)^2}-1-2x=x^2\frac{3-2x}{(1-x)^2}\le 12x^2,$$

for $0\le x\le\frac{1}{2}$, and

$$\frac{2\mathcal{C}_{\chi_1}gM(\underline{\omega}^{(0)})}{\rho}\le\frac{1}{2}$$

for g sufficiently small.

To establish (4.27) we recall the definition (4.12) of $\mathcal{Q}^{(1)}$ and apply functional calculus to the pair $\Lambda_{[\theta_1]}=(\cos(\delta')L_{\rm res},M_{[\theta_1]})$ of normal operators,

$$\begin{aligned}
&\sup_{\lambda\in\mathcal{Q}^{(1)}}\left\|T^{(1)}[\lambda;z]-(\lambda_1+i\lambda_2)\right\|_{\mathcal{B}(\ker(L_{\rm p}))}\\
=\;& \left\|T^{(1)}[\Lambda_{[\theta_1]};z]-L_{0,\theta_1}\right\|_{\mathcal{B}(\mathcal{H}^{(1)})}\\
\le\;& \left\|K^{(1)}[z]-L_{0,\theta_1}+z-\frac{g^2}{\rho}\Lambda_0^{(s)}\otimes\chi_1^2(M_{[\theta_1]})\right\|_{\mathcal{B}(\mathcal{H}^{(1)}))}\\
&+\left\|E^{(1)}[z]-z+\frac{g^2}{\rho}\Lambda_0^{(s)}\right\|_{\mathcal{B}(\ker(L_{\rm p}))}+\frac{g^2}{\rho}\left\|\Lambda_0^{(s)}\otimes\overline{\chi}_1^2(M_{[\theta_1]})\right\|_{\mathcal{B}(\mathcal{H}^{(1)})}
\end{aligned}$$

$$
\begin{aligned}
&+\xi \left\| \left(w_{R,S}^{(1)} \right)_{R+S\geq 1} \right\|_{(1),\xi} \\
&= \mathcal{O}\left(\frac{g^2}{\rho}\right) + \mathcal{O}\left(g\rho^{\mu}\right),
\end{aligned}
$$

uniformly in $z \in B_{1/4}$, by (4.25, 4.26), Proposition 3.7 and Corollary 4.3.

The analyticity of $z \mapsto \underline{w}^{(1)}[z]$ follows from Lemma 4.10(iii) and the absolute convergence (uniformly in the parameter $z \in B_{1/4}$) of the above series of analytic functions $z \mapsto V^{(1)}_{\underline{r,p,s,q}_L}[\,\cdot\,; z]$. ∎

4.3 The Renormalization Transformation

We are now prepared to introduce the renormalization transformation $\mathcal{R}^{(1)}_{\rho_*}, \mathcal{R}^{(n)}_{\rho_{**}}$, $n = 2, 3, \ldots, \mathcal{N}$, which acts on suitable poly-discs in $\mathfrak{W}^{(n)}$ given by

$$
\begin{aligned}
\mathcal{D}^{(n)}(\varepsilon, \eta) \;:=\; \Bigg\{ & \underline{w}^{(n)} = \left(-E^{(n)}[\,\cdot\,], T^{(n)}[\,\cdot\,], \left(w^{(n)}_{R,S}[\,\cdot\,] \right)_{R+S\geq 1} \right) \in \mathfrak{W}^{(n)} \,\Bigg|\, \\
& \sup_{\substack{z \in B_{1/4}, \\ q \in \mathcal{Q}^{(n)}}} \left\| \left| \nabla_q T^{(n)}[q; z] - \begin{pmatrix} 1 \\ i \end{pmatrix} \right| \right\|_{\mathcal{B}(\mathcal{H}^{(n)}_{<\infty})} \leq \varepsilon, \\
& \sup_{z \in B_{1/4}} \left\| E^{(n)}[z] - z \right\|_{\mathcal{B}(\mathcal{H}^{(n)}_{<\infty})} \leq \eta, \; \left\| \left(w^{(n)}_{R,S}[\,\cdot\,] \right)_{R+S\geq 1} \right\|_{(n),\xi} \leq \eta \Bigg\}
\end{aligned}
$$

for suitable $\varepsilon, \eta > 0$. This poly-disc is a collection of all elements $\underline{w}^{(n)}$ in $\mathfrak{W}^{(n)}$ which are close to the element $\underline{w}^{(n)}_* := (z \mapsto -z, (q_1, q_2; z) \mapsto (q_1 + iq_2), 0)$, i.e., $\mathcal{W}_{(n)}[\underline{w}^{(n)}[z]]$ is close to the operator

$$
\mathcal{W}_{(n)}[\underline{w}^{(n)}_*[z]] = \cos(\delta') L_{\text{res}} + iM_{[\theta_n]} - z = L_{0,\theta_n} \restriction_{\mathcal{H}^{(n)}} - z.
$$

The renormalization transformation is a composition of three operations: a *decimation of degrees of freedom* via the smooth Feshbach map followed by a *rescaling procedure* and an *adjustment of spectral parameters*. These operations are explained in subsequent subsections.

4.3.1 Adjustment of Spectral Parameters

The first ingredient to the renormalization transformation, the smooth Feshbach map, is only defined for spectral parameters on a scale ρ_{**} (ρ_* for the first iteration step) rather than for all $z \in B_{1/4}$, reflecting the fact that the Feshbach map allows the spectral analysis on a smaller scale. We forestall the needed transformation of spectral parameters from $B_{1/4}$ to a proper scale before we discuss the Feshbach map in Section 4.3.2

For $\underline{w}^{(1)} = (-E^{(1)}, T^{(1)}, (w^{(1)}_{R,S})_{R+S\geq 1})$ as defined in (4.21, 4.23) and for $\underline{w}^{(n)} = (-E^{(n)}, T^{(n)}, (w^{(n)}_{R,S})_{R+S\geq 1}) \in \mathcal{D}^{(n)}(\varepsilon,\eta)$, $n = 2, 3, \ldots, \mathcal{N}$, we define the complex sets

$$D\left[\underline{w}^{(1)}\right] := \left\{ z \in B_{1/4} \,\middle|\, \left|\left\langle E^{(1)}\right\rangle_{\Omega_{\mathrm{p}}}[z]\right| < \rho_*/4 \right\} = \left\langle E^{(1)}\right\rangle_{\Omega_{\mathrm{p}}}^{-1}[B_{\rho_*/4}], \tag{4.28}$$

and

$$D\left[\underline{w}^{(n)}\right] := \left\{ z \in B_{1/4} \,\middle|\, \left|E^{(n)}[z]\right| < \rho_{**}/4 \right\} = {E^{(n)}}^{-1}[B_{\rho_{**}/4}], \tag{4.29}$$

for $n = 2, 3, \ldots, \mathcal{N}$, as the collection of all spectral parameters z which allow an application of the Feshbach map to $\mathcal{W}_{(n)}\left[\underline{w}^{(n)}[z]\right]$. Hereby, the function $\left\langle E^{(1)}\right\rangle_{\Omega_{\mathrm{p}}}$ is defined by

$$\left\langle E^{(1)}\right\rangle_{\Omega_{\mathrm{p}}} : B_{1/4} \to \mathbb{C}, \qquad \left\langle E^{(1)}\right\rangle_{\Omega_{\mathrm{p}}}[z] := \left\langle\, \Omega_{\mathrm{p}} \,\middle|\, E^{(1)}[z]\Omega_{\mathrm{p}} \,\right\rangle_{\ker(L_{\mathrm{p}})}$$

and it inherits the analytic properties from $E^{(1)}$. As a simple consequence we obtain the following lemma which locates the set $D\left[\underline{w}^{(n)}\right]$ and describes the mapping properties of $\left\langle E^{(1)}\right\rangle_{\Omega_{\mathrm{p}}}$ and $E^{(n)}$ on $D\left[\underline{w}^{(n)}\right]$.

Lemma 4.5 *Let $n = 1, \ldots, \mathcal{N}$ and set $\tilde{\rho} := \rho_*$ in the case $n = 1$ and $\tilde{\rho} := \rho_{**}$ for $n = 2, 3, \ldots, \mathcal{N}$. Assume $0 < \tilde{\rho} \leq \frac{1}{20}$ (in addition to the previous assumptions on ρ_* and ρ_{**}). Let $\underline{w}^{(1)}$ as defined in (4.21, 4.23) and choose $\underline{w}^{(n)} \in \mathcal{D}^{(n)}(\varepsilon,\eta)$ with $0 < \eta \leq \tilde{\rho}/16$ for $n = 2, 3, \ldots, \mathcal{N}$. We assume that g is sufficiently small and that $|\beta_{\max} - \beta_{\min}| \ll 1$.*

(i) Then, the following inclusion holds true

$$B_{3\tilde{\rho}/16} \subseteq B_{\tilde{\rho}/4-\eta} \subseteq D\left[\underline{w}^{(n)}\right] \subseteq B_{\tilde{\rho}/4+\eta} \subseteq B_{5\tilde{\rho}/16}. \tag{4.30}$$

(ii) We have

$$\begin{aligned} \left|\partial_z \left\langle E^{(1)}\right\rangle_{\Omega_{\mathrm{p}}}[z] - 1\right| &\leq 7\eta \leq \frac{7}{16}\tilde{\rho}, \qquad && n = 1, \\ \left|\partial_z E^{(n)}[z] - 1\right| &\leq 7\eta \leq \frac{7}{16}\tilde{\rho}, \qquad && n = 2, 3, \ldots, \mathcal{N}, \end{aligned} \tag{4.31}$$

for all $|z| \leq \frac{1}{32}$.

(iii) The function

$$R^{(n)} : D\left[\underline{w}^{(n)}\right] \to B_{1/4}, \qquad R^{(n)}[z] := \begin{cases} \rho_*^{-1}\left\langle E^{(1)}\right\rangle_{\Omega_{\mathrm{p}}}[z], & n=1, \\ \rho_{**}^{-1}E^{(n)}[z], & n=2,3,\ldots,\mathcal{N} \end{cases}$$

is biholomorphic, i.e., it is bijective and its inverse

$$Z^{(n)} := \left[R^{(n)}\right]^{-1} : B_{1/4} \to D\left[\underline{w}^{(n)}\right] \tag{4.32}$$

is a holomorphic function obeying

$$\left|\partial_z Z^{(n)}[z] - \tilde{\rho}\right| \leq 12\tilde{\rho}\eta \tag{4.33}$$

for all $z \in B_{1/4}$.

Proof.

(i) We start with the observation that, in the case $n = 1$,

$$\begin{aligned} \left|\left\langle E^{(1)}\right\rangle_{\Omega_{\mathrm{p}}}[z] - z\right| &= \frac{g^2}{\rho}\left|\left\langle \Omega_{\mathrm{p}} \,\middle|\, \Lambda_0^{(s)}\Omega_{\mathrm{p}} \right\rangle\right| + \mathcal{O}\left(\frac{g^{2+\tilde{\varepsilon}}}{\rho}\right) \\ &= \frac{g^2}{\rho}\left[\left|\left\langle \Omega_{\mathrm{p}} \,\middle|\, \left(\Lambda_0^{(s)} - \Lambda_0^{(s)}|_{\beta_{\max}=\beta_{\min}=\beta_{\mathrm{p}}}\right)\Omega_{\mathrm{p}} \right\rangle\right| + \mathcal{O}\left(g^{\tilde{\varepsilon}}\right)\right] \\ &= \rho_*\mathcal{O}\left(|\beta_{\max} - \beta_{\min}| + g^{\tilde{\varepsilon}}\right) \\ &< \eta \end{aligned}$$

for $|\beta_{\max} - \beta_{\min}| \ll 1$ and g sufficiently small, recall (4.25). We used that the level shift operator $\Lambda_0^{(s)}|_{\beta_{\max}=\beta_{\min}=\beta_{\mathrm{p}}}$ in the equal temperature situation has Ω_{p} as zero eigenvector,

$$\Lambda_0^{(s)}|_{\beta_{\max}=\beta_{\min}=\beta_{\mathrm{p}}}\Omega_{\mathrm{p}} = i\sum_{r=1}^{R}\Gamma_{0,r}|_{\beta_r=\beta_{\mathrm{p}}}\Omega_{\mathrm{p}} = 0$$

and that $\Lambda_0^{(s)} - \Lambda_0^{(s)}|_{\beta_{\max}=\beta_{\min}=\beta_{\mathrm{p}}}$ is of order $|\beta_{\max} - \beta_{\min}|$. We first prove the inclusion (4.30). Let $z \in B_{\tilde{\rho}/4-\eta}$, then, in the case $n = 1$,

$$\left|\left\langle E^{(1)}\right\rangle_{\Omega_{\mathrm{p}}}[z]\right| \leq \left|\left\langle E^{(1)}\right\rangle_{\Omega_{\mathrm{p}}}[z] - z\right| + |z| < \eta + \frac{\rho_*}{4} - \eta = \frac{\rho_*}{4},$$

and, for $n = 2, 3, \ldots, \mathcal{N}$,

$$\left|E^{(n)}[z]\right| \leq \left|E^{(n)}[z] - z\right| + |z| < \eta + \frac{\rho_{**}}{4} - \eta = \frac{\rho_{**}}{4}.$$

For $z \in D\left[\underline{w}^{(n)}\right]$ we have, for $n = 1$,

$$|z| \leq \left|\left\langle E^{(1)}\right\rangle_{\Omega_{\mathrm{p}}}[z] - z\right| + \left|\left\langle E^{(1)}\right\rangle_{\Omega_{\mathrm{p}}}[z]\right| < \eta + \frac{\rho_*}{4},$$

and, for $n = 2, 3, \ldots, \mathcal{N}$,

$$|z| \leq \left|E^{(n)}[z] - z\right| + \left|E^{(n)}[z]\right| < \eta + \frac{\rho_{**}}{4}.$$

(ii) We go over to prove (4.31). We restrict ourselves to the case $n = 2, 3, \ldots, \mathcal{N}$, the case $n = 1$ is similar. Let $|z| \leq \frac{1}{32}$. Cauchy's Integral formula applied to the holomorphic function $E^{(n)}$ and (4.30) yield, for $\rho_{**}/4 + \eta < a < 1/4$,

$$\begin{aligned}
\left|\partial_z E^{(n)}[z] - 1\right| &= \left|\frac{1}{2\pi i} \oint\limits_{|\zeta|=a} \frac{E^{(n)}[\zeta] - \zeta}{(\zeta - z)^2}\, dz\right| \\
&\leq a \sup_{|\zeta| \leq a} \left|E^{(n)}[\zeta] - \zeta\right| \sup_{|\zeta|=a} \frac{1}{|\zeta - z|^2} \\
&< \frac{a\eta}{(a - |z|)^2} < \frac{a\eta}{(a - \frac{1}{32})^2} \\
&\leq 7\eta \leq \frac{7}{16}\rho_{**} \leq \frac{7}{320}
\end{aligned}$$

where we chose $a = \frac{7}{32}$.

(iii) The relation (4.31) implies that the function $z \mapsto E^{(n)}[z]$ is injective on $B_{1/32}$ with holomorphic inverse.

Now, we show surjectivity of $E^{(n)} : D\left[\underline{w}^{(n)}\right] \to B_{\tilde{\rho}/4}$. We first observe that $E^{(n)}\left[D\left[\underline{w}^{(n)}\right]\right] \subseteq E^{(n)}\left[B_{1/40}\right]$ since $D\left[\underline{w}^{(n)}\right] \subseteq B_{5\tilde{\rho}/16} \subseteq B_{1/40}$ (for $\tilde{\rho} \leq \frac{1}{20}$) and $E^{(n)}$ is injective on the even bigger ball $B_{1/32}$. Assume that $E^{(n)}\left[D\left[\underline{w}^{(n)}\right]\right] \subsetneqq B_{\tilde{\rho}/4}$, i.e., there exists $\zeta \in E^{(n)}\left[B_{1/40} \setminus D\left[\underline{w}^{(n)}\right]\right] \cap B_{\tilde{\rho}/4}$. Thus, $z := {E^{(n)}}^{-1}[\zeta] \in B_{1/40} \setminus D\left[\underline{w}^{(n)}\right]$. This implies $\left|E^{(n)}[z]\right| = |\zeta| < \tilde{\rho}/4$ which is in contradiction to $z \notin D\left[\underline{w}^{(n)}\right]$.

The bijectivity of $R^{(n)} = \tilde{\rho}^{-1} E^{(n)} : D\left[\underline{w}^{(n)}\right] \to B_{1/4}$ guarantees the existence of the inverse function $Z^{(n)} : B_{1/4} \to D\left[\underline{w}^{(n)}\right]$ and its derivative fulfils

$$\begin{aligned}
\left|\partial_z Z^{(n)}[z] - \tilde{\rho}\right| &= \left|\frac{1}{{R^{(n)}}'\left[Z^{(n)}[z]\right]} - \tilde{\rho}\right| = \tilde{\rho}\left|\frac{1 - {E^{(n)}}'\left[Z^{(n)}[z]\right]}{{E^{(n)}}'\left[Z^{(n)}[z]\right]}\right| \\
&\leq \tilde{\rho}\frac{\left|1 - {E^{(n)}}'\left[Z^{(n)}[z]\right]\right|}{1 - \left|1 - {E^{(n)}}'\left[Z^{(n)}[z]\right]\right|} \leq 7\tilde{\rho}\eta\frac{1}{1 - 7\eta} \\
&\leq 12\tilde{\rho}\eta
\end{aligned}$$

for all $z \in B_{1/4}$, where the prime stands for the derivative w.r.t. the spectral parameter.

∎

The function $z \mapsto Z^{(n)}[z]$ is the appropriate adjustment of spectral parameters from the disc $B_{1/4}$ to $D\left[\underline{w}^{(n)}\right]$ which is comparable to a ball of radius $\tilde{\rho}/4 = \rho_*/4$, for $n = 1$, and $\tilde{\rho}/4 = \rho_{**}/4$, for $n = 2, 3, \ldots, \mathcal{N}$. The implication of the estimate (4.33) is that the map $Z^{(n)}$ first shrinks the domain $B_{1/4}$ by a factor $\tilde{\rho}$ and then performs a parallel shift given by the complex number $Z^{(n)}[0]$ – up to higher order correction terms. The following corollary rephrases the fact in a mathematical language.

Corollary 4.6 *Let $n = 1, \ldots, \mathcal{N}$ and set $\tilde{\rho} := \rho_*$ in the case $n = 1$ and $\tilde{\rho} := \rho_{**}$ for $n = 2, 3, \ldots, \mathcal{N}$. Under the assumption of Lemma 4.5 we have for the adjustment function $Z^{(n)}$ defined in (4.32) the following expansion,*

$$\left| Z^{(n)}[z] - Z^{(n)}[\zeta] - \tilde{\rho}(z - \zeta) \right| \leq 12\tilde{\rho}\eta|z - \zeta|,$$

for all $z, \zeta \in B_{1/4}$.

4.3.2 Decimation of Degrees of Freedom via Smooth Feshbach Map: Iteration Step $n = 2, 3, \ldots, \mathcal{N}$

Given an operator of type

$$K^{(n)}[z] = \mathcal{W}_{(n)}\left[\underline{w}^{(n)}[z]\right] = T^{(n)}\left[\Lambda_{[\theta_n]}; z\right] - E^{(n)}[z] + W^{(n)}[z], \tag{4.34}$$

gained by an element $\underline{w}^{(n)} \in \mathcal{D}^{(n)}(\varepsilon, \eta)$, the renormalization procedure shall provide a method to obtain detailed information about the spectrum of $K^{(n)}[z]$ around the origin on decreasing scales which is not accessible by standard perturbative arguments treating $T^{(n)}\left[\Lambda_{[\theta_n]}; z\right] - E^{(n)}[z]$ as a free operators (whose spectrum is considered to be understood) and $W^{(n)}[z]$ as an interaction term. The smooth Feshbach map provides an opportunity to encode the spectral information of $K^{(n)}[z]$ on a scale ρ_{**} as the spectrum of an operator which lives on a spectral subspace. Transferring the analysis of $K^{(n)}[z]$ to an operator on a spectral subspace can be understood as an effective decimation of degrees of freedom.

The application of the smooth Feshbach map requires a smooth cutoff function $\chi_{\rho_{**}} : \mathbb{R}_0^+ \to [0, 1]$ as introduced in Section 3.3, Equations (3.9) through (3.11). Given this function we define a smooth "projection" operator

$$\Xi_{\rho_{**}}^{(n)} := \chi_{\rho_{**}}\left(M_{[\theta_n]}\right),$$

and its complementary "projection" operator by

$$\overline{\Xi}^{(n)}_{\rho_{**}} := \sqrt{\mathbb{1}_{\mathcal{H}^{(n)}} - \Xi^{(n)\,2}_{\rho_{**}}} = \overline{\chi}_{\rho_{**}}(M_{[\theta_n]}).$$

We further introduce orthogonal projections on the range of the operators $\Xi^{(n)}_{\rho_{**}}$ and $\overline{\Xi}^{(n)}_{\rho_{**}}$. We define

$$\begin{aligned} P^{(n)}_{\rho_{**}} &: \text{orthogonal projection on ran}\left(\Xi^{(n)}_{\rho_{**}}\right), & P^{(n)\perp}_{\rho_{**}} &:= \mathbb{1}_{\mathcal{H}^{(n)}} - P^{(n)}_{\rho_{**}}, \\ \overline{P}^{(n)}_{\rho_{**}} &: \text{orthogonal projection on ran}\left(\overline{\Xi}^{(n)}_{\rho_{**}}\right), & \overline{P}^{(n)\perp}_{\rho_{**}} &:= \mathbb{1}_{\mathcal{H}^{(n)}} - \overline{P}^{(n)}_{\rho_{**}}, \end{aligned}$$

and note that

$$\begin{aligned} P^{(n)}_{\rho_{**}} &= P_{\left[M_{[\theta_n]} < \rho_{**}\right]}, \\ \overline{P}^{(n)}_{\rho_{**}} &= P_{\left[M_{[\theta_n]} > \frac{7}{8}\rho_{**}\right]}, \\ P^{(n)\perp}_{\rho_{**}} &= P_{\left[M_{[\theta_n]} \geq \rho_{**}\right]}, \\ \overline{P}^{(n)\perp}_{\rho_{**}} &= P_{\left[M_{[\theta_n]} \leq \frac{7}{8}\rho_{**}\right]}. \end{aligned}$$

Subsequently, we show that for $\underline{w}^{(n)} \in \mathcal{D}^{(n)}(\varepsilon, \eta)$ and $z \in D[\underline{w}^{(n)}]$ the operator $K^{(n)}[z] = T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z] + W^{(n)}[z]$, $n = 2, 3, \ldots, \mathcal{N}$, given in (4.34) together with the unperturbed part $T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]$ build a $\Xi^{(n)}_{\rho_{**}}$-Feshbach pair in the sense of Appendix E.

Lemma 4.7 *Let $0 < \rho_{**} \leq \frac{1}{20}$ (in addition to the previous assumptions on ρ_{**}) and choose for $n = 2, 3, \ldots \mathcal{N}$ an element $\underline{w}^{(n)} = \left(-E^{(n)}, T^{(n)}, (w^{(n)}_{R,S})_{R+S\geq 1}\right) \in \mathcal{D}^{(n)}(\varepsilon, \eta)$ with $0 < \varepsilon, \eta \leq \rho_{**}/16$.*

(i) For $q = (q_1, q_2) \in \mathcal{Q}^{(n)}$ with $q_2 \in [\frac{7}{8}\rho_{**}, 1]$ holds

$$\left|T^{(n)}[q; z] - E^{(n)}[z]\right| \geq \frac{\rho_{**}}{2}$$

for all $z \in D[\underline{w}^{(n)}]$ and therefore $T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]$ is invertible on ran $\left(\overline{\Xi}^{(n)}_{\rho_{**}}\right)$ *and its inverse is bounded by*

$$\left\| \frac{\overline{P}^{(n)}_{\rho_{**}}}{T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]} \right\|_{\mathcal{B}(\mathcal{H}^{(n)})} \leq \frac{2}{\rho_{**}}.$$

(ii) For any $z \in D[\underline{w}^{(n)}]$ the operator

$$K^{(n)}[z] := \mathcal{W}_{(n)}\left[\underline{w}^{(n)}\right] = T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z] + W^{(n)}[z]$$

together with the free part $T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]$ build a $\Xi^{(n)}_{\rho_{**}}$-Feshbach pair, i.e., the operator

$$K^{(n)}[z]_{\overline{\Xi}^{(n)}_{\rho_{**}}} = T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z] + \overline{\Xi}^{(n)}_{\rho_{**}} W^{(n)}[z] \overline{\Xi}^{(n)}_{\rho_{**}}$$

is bounded invertible on the range of $\overline{\Xi}^{(n)}_{\rho_{**}}$.

Proof.

(i) We compute for $q = (q_1, q_2) \in \mathcal{Q}^{(n)}$ with $q_2 \in [\frac{7}{8}\rho_{**}, 1]$ and $z \in D[\underline{w}^{(n)}]$,

$$\begin{aligned}
\left|T^{(n)}[q; z] - E^{(n)}[z]\right| &\geq |q_1 + iq_2| - \left|T^{(n)}[q; z] - (q_1 + iq_2)\right| - |E^{(n)}[z]| \\
&\geq |q| - \sup_{\tilde{q} \in \mathcal{Q}^{(n)}} \left|\nabla_{\tilde{q}} T^{(n)}[\tilde{q}; z] - \begin{pmatrix} 1 \\ i \end{pmatrix}\right| |q| - |E^{(n)}[z]| \\
&\geq |q|(1 - \varepsilon) - \frac{\rho_{**}}{4} \geq \frac{7}{8}\rho_{**}\left(1 - \frac{\rho_{**}}{16}\right) - \frac{\rho_{**}}{4} \\
&\geq \frac{\rho_{**}}{2}.
\end{aligned}$$

This estimate and the functional calculus imply

$$\begin{aligned}
\left\|\frac{\overline{P}^{(n)}_{\rho_{**}}}{T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]}\right\|_{\mathcal{B}(\mathcal{H}^{(n)})} &= \sup_{(q_1, q_2) \in \mathcal{Q}^{(n)}} \left|\frac{P_{[q_2 \geq \frac{7}{8}\rho_{**}]}}{T^{(n)}[q_1, q_2; z] - E^{(n)}[z]}\right| \\
&\leq \frac{2}{\rho_{**}}.
\end{aligned}$$

(ii) Apparently, the operators $\Xi^{(n)}_{\rho_{**}}, \overline{\Xi}^{(n)}_{\rho_{**}}$ and $T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]$ commute mutually. Since further $W^{(n)}[z]$ is a bounded operator, it is sufficient to prove invertibility of $K^{(n)}[z]_{\overline{\Xi}^{(n)}_{\rho_{**}}}$ on ran $\left(\overline{\Xi}^{(n)}_{\rho_{**}}\right)$ to show that $(K^{(n)}[z], T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z])$ is a $\Xi^{(n)}_{\rho_{**}}$-Feshbach pair. We invert $K^{(n)}[z]_{\overline{\Xi}^{(n)}_{\rho_{**}}}$ by expansion in a norm convergent Neumann series,

$$\overline{P}^{(n)}_{\rho_{**}} K^{(n)}[z]^{-1}_{\overline{\Xi}^{(n)}_{\rho_{**}}} \overline{P}^{(n)}_{\rho_{**}} \tag{4.35}$$

$$\begin{aligned}
&= \overline{P}^{(n)}_{\rho_{**}} \left(1 + \frac{\overline{\Xi}^{(n)}_{\rho_{**}}}{T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]} W^{(n)}[z] \overline{\Xi}^{(n)}_{\rho_{**}}\right)^{-1} \frac{\overline{P}^{(n)}_{\rho_{**}}}{T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]} \\
&= \overline{P}^{(n)}_{\rho_{**}} \sum_{L=0}^{\infty} \left(-\frac{\overline{\Xi}^{(n)}_{\rho_{**}}}{T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]} W^{(n)}[z] \overline{\Xi}^{(n)}_{\rho_{**}}\right)^{L} \frac{\overline{P}^{(n)}_{\rho_{**}}}{T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]},
\end{aligned}$$

where

$$
\begin{aligned}
&\left\| \frac{\overline{\Xi}^{(n)}_{\rho_{**}}}{T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]} W^{(n)}[z] \overline{\Xi}^{(n)}_{\rho_{**}} \right\|_{\mathcal{B}(\mathcal{H}^{(n)})} \\
&\leq \left\| \frac{\overline{P}^{(n)}_{\rho_{**}}}{T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]} \right\|_{\mathcal{B}(\mathcal{H}^{(n)})} \left\| W^{(n)}[z] \right\|_{\mathcal{B}(\mathcal{H}^{(n)})} \\
&\leq \frac{2}{\rho_{**}} \xi \left\| \left(w^{(n)}_{R,S} \right)_{R+S \geq 1} \right\|_{(n),\xi} \leq \frac{2}{\rho_{**}} \xi \eta \\
&\leq \frac{\xi}{8} < 1
\end{aligned}
$$

where we use the embedding Corollary 4.3.

■

Lemma 4.7 guarantees that the smooth Feshbach map $\mathfrak{F}_{\Xi^{(n)}_{\rho_{**}}}$ may be applied to the pair $(K^{(n)}[z], T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z])$ for $K^{(n)}[z]$ given in (4.34) with $\underline{w}^{(n)} \in \mathcal{D}^{(n)}(\rho_{**}/16, \rho_{**}/16)$ and $z \in D[\underline{w}^{(n)}]$. We use the expansion (4.35) to rewrite the image under the Feshbach map as

$$
\begin{aligned}
&\mathfrak{F}_{\Xi^{(n)}_{\rho_{**}}} \left(K^{(n)}[z], T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z] \right) \qquad (4.36) \\
&= T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z] + \Xi^{(n)}_{\rho_{**}} W^{(n)}[z] \Xi^{(n)}_{\rho_{**}} \\
&\quad - \Xi^{(n)}_{\rho_{**}} W^{(n)}[z] \overline{\Xi}^{(n)}_{\rho_{**}} K^{(n)}[z]^{-1}_{\overline{\Xi}^{(n)}_{\rho_{**}}} \overline{\Xi}^{(n)}_{\rho_{**}} W^{(n)}[z] \Xi^{(n)}_{\rho_{**}} \\
&= T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z] + \Xi^{(n)}_{\rho_{**}} W^{(n)}[z] \Xi^{(n)}_{\rho_{**}} \\
&\quad - \sum_{L=0}^{\infty} \Xi^{(n)}_{\rho_{**}} W^{(n)}[z] \overline{\Xi}^{(n)}_{\rho_{**}} \left(- \frac{\overline{\Xi}^{(n)}_{\rho_{**}}}{T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]} W^{(n)}[z] \overline{\Xi}^{(n)}_{\rho_{**}} \right)^L \\
&\qquad \times \frac{\overline{\Xi}^{(n)}_{\rho_{**}}}{T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]} W^{(n)}[z] \Xi^{(n)}_{\rho_{**}} \\
&= T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z] \\
&\quad + \sum_{L=0}^{\infty} \Xi^{(n)}_{\rho_{**}} W^{(n)}[z] \left(- \frac{{\overline{\Xi}^{(n)}_{\rho_{**}}}^2}{T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]} W^{(n)}[z] \right)^L \Xi^{(n)}_{\rho_{**}}.
\end{aligned}
$$

4.3.3 Rescaling Bosonic Variables and Renormalization Transformation: Iteration Step $n = 2, 3, \ldots, \mathcal{N}$

For a given $\underline{w}^{(n)} \in \mathcal{D}^{(n)}(\varepsilon, \eta)$, $\varepsilon, \eta \leq \rho_{**}/16$, we could build in the previous section an operator $\mathfrak{F}_{\Xi^{(n)}_{\rho_{**}}}\left(K^{(n)}[z], T^{(n)}[\Lambda_{[\theta_n]}; z] - E^{(n)}[z]\right)$ given in (4.36) which acts on the reduced Hilbert space $P^{(n)}_{\rho_{**}}\mathcal{H}^{(n)} = P_{\left[M_{[\theta_n]} \leq \rho_{**}\right]}\mathcal{F}(L^2[\Upsilon])$. The aim of this section is to rescale the bosonic variables L_{aux}, L_{res} and N_{res} within the operator (4.36) such that it lives on a spectral subspace where the variable $M_{[\theta_n]}$ is of order one rather than of order ρ_{**}. To this end we employ the unitary rescaling operator $S_{\rho_{**}}$ defined in (D.12) and the rescaling map $\mathfrak{S}_{\rho_{**}}$ acting on an operator A like

$$\mathfrak{S}_{\rho_{**}}(A) = \rho_{**}^{-1} S_{\rho_{**}} A S_{\rho_{**}}^{-1}.$$

We refer to Appendix D.2 and (D.14) for details on $\mathfrak{S}_{\rho_{**}}$. Consequently we get with the functional calculus

$$\begin{aligned} S_{\rho_{**}} P^{(n)}_{\rho_{**}} \mathcal{H}^{(n)} &= S_{\rho_{**}} P_{\left[M_{[\theta_n]} \leq \rho_{**}\right]}\mathcal{F}(L^2[\Upsilon]) = P_{\left[S_{\rho_{**}} M_{[\theta_n]} S_{\rho_{**}}^{-1} \leq \rho_{**}\right]} S_{\rho_{**}} \mathcal{F}(L^2[\Upsilon]) \\ &= P_{\left[\rho_{**} M_{[\theta_{n+1}]} \leq \rho_{**}\right]}\mathcal{F}(L^2[\Upsilon]) = \mathcal{H}^{(n+1)}. \end{aligned}$$

The fact that $M_{[\theta_n]} = \sin(\delta') L_{\text{aux}} + \rho_{[n]}^{-1}\tau' N_{\text{res}}$ does not scale properly under $\mathfrak{S}_{\rho_{**}}$ (the operator L_{aux} scales as ρ_{**}^0 while the operator N_{res} scales as ρ_{**}^{-1}) but that the translation parameter $\rho_{[n]}^{-1}\tau'$ is blown up by a factor ρ_{**}^{-1} is the reason why the rescaling operator $S_{\rho_{**}}$ does not bring back the space $P^{(n)}_{\rho_{**}}\mathcal{H}^{(n)}$ to the space $\mathcal{H}^{(n)}$ but maps it to a subspace $\mathcal{H}^{(n+1)}$.

The renormalization transformation $\mathcal{R}^{(n)}_{\rho_{**}}$ incorporates the decimation of degrees of freedom via Feshbach map $\mathfrak{F}_{\Xi^{(n)}_{\rho_{**}}}$, the rescaling $\mathfrak{S}_{\rho_{**}}$ of bosonic variables and the adjustment $Z^{(n)}$ of spectral parameters to map an element $\underline{w}^{(n)} \in \mathcal{D}^{(n)}(\varepsilon, \eta)$ for $\varepsilon, \eta \leq \rho_{**}/16$, or rather the associated family of operators $B_{1/4} \ni z \mapsto K^{(n)}[z] = \mathcal{W}_{(n)}[\underline{w}^{(n)}[\,\cdot\,; z]]$ on $\mathcal{H}^{(n)}$, to an operator family $B_{1/4} \ni z \mapsto K^{(n+1)}[z]$ on $\mathcal{H}^{(n+1)}$. The assignment is as follows,

$$\begin{aligned} \mathcal{R}^{(n)}_{\rho_{**}} &: \mathcal{D}^{(n)}(\varepsilon, \eta) \to \mathcal{W}_{(n+1)}\left[\mathfrak{W}^{(n+1)}\right], \\ \left(\mathcal{R}^{(n)}_{\rho_{**}}[\underline{w}^{(n)}]\right)[z] &:= \mathfrak{S}_{\rho_{**}}\left[\mathfrak{F}_{\Xi^{(n)}_{\rho_{**}}}\left(K^{(n)}[\zeta], T^{(n)}[\Lambda_{[\theta_n]}; \zeta] - E^{(n)}[\zeta]\right)\right], \\ \zeta &:= Z^{(n)}[z] \in D[\underline{w}^{(n)}], \end{aligned} \tag{4.37}$$

where $\underline{w}^{(n)} = (-E^{(n)}, T^{(n)}, (w^{(n)}_{R,S})_{R+S\geq 1})$ and $K^{(n)}[z] = \mathcal{W}_{(n)}[\underline{w}^{(n)}[\,\cdot\,; z]$. The assignment (4.37) is the definition of the frequently mentioned *renormalization transformation*. With the help of (4.36), Theorem D.3, Proposition D.4 and the same arguments as used in Section 4.2 we see that the image

$$\mathcal{R}^{(n)}_{\rho_{**}}[\underline{w}^{(n)}] = \mathcal{W}_{(n+1)}\left[\underline{w}^{(n+1)}\right]$$

under the renormalization transformation is generated by a sequence $\underline{w}^{(n+1)} = (w_{R,S}^{(n+1)})_{R+S\geq 0}$ of integral kernels given by

$$\begin{aligned} & w_{R,S}^{(n+1)}[\lambda; Y^{(R,S)}; z] \qquad (4.38) \\ := \;& \rho_{**}^{-1} \sum_{L=1}^{\infty} (-1)^{L-1} \sum_{\substack{r_1+\cdots+r_L=R, \\ s_1+\cdots+s_L=S}} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L: \\ r_\ell+p_\ell+s_\ell+q_\ell\geq 1}} \left[\prod_{\ell=1}^{L} \binom{r_\ell+p_\ell}{r_\ell} \binom{s_\ell+q_\ell}{s_\ell} \right] \\ & \times V_{\underline{r,p,s,q}_L}^{(n+1)}[\lambda; Y^{(R,S)}; z], \end{aligned}$$

for $R+S\geq 1$, and

$$\begin{aligned} & w_{0,0}^{(n+1)}[\lambda; z] \qquad (4.39) \\ := \;& \rho_{**}^{-1} \left(T^{(n)}[\rho_{**}\lambda; Z^{(n)}[z]] - \rho_{**} z \right) \\ & + \rho_{**}^{-1} \sum_{L=2}^{\infty} (-1)^{L-1} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L: \\ p_\ell+q_\ell\geq 1}} \chi_1^2(\lambda_2) \\ & \times \Big\langle \mathcal{W}_{(n)}^{(0,0)} \left[w_{p_1,q_1}^{(n)} \right] \left(\rho_{**}\lambda; Z^{(n)}[z] \right) \frac{\overline{\chi}_1^2 \left(\rho_{**}^{-1} M_{[\theta_n]} + \lambda_2 \right)}{T^{(n)}\left[\Lambda_{[\theta_n]} + \rho_{**}\lambda; Z^{(n)}[z] \right] - \rho_{**} z} \times \cdots \\ & \quad \times \frac{\overline{\chi}_1^2 \left(\rho_{**}^{-1} M_{[\theta_n]} + \lambda_2 \right)}{T^{(n)}\left[\Lambda_{[\theta_n]} + \rho_{**}\lambda; Z^{(n)}[z] \right] - \rho_{**} z} \mathcal{W}_{(n)}^{(0,0)} \left[w_{p_L,q_L}^{(n)} \right] \left(\rho_{**}\lambda; Z^{(n)}[z] \right) \Big\rangle_{\Omega_{\mathrm{vac}}}. \end{aligned}$$

The functions $V_{\underline{r,p,s,q}_L}^{(n+1)}$ appearing in (4.38) are the symmetrization (in the sense of (D.8)) of

$$\begin{aligned} & \tilde{V}_{\underline{r,p,s,q}_L}^{(n+1)}[\lambda; Y^{(R,S)}; z] \qquad (4.40) \\ := \;& \Big\langle \chi_1 \left(\rho_{**}^{-1} M_{[\theta_n]} + \lambda_2 + \left[\eta_0^{(\theta_{n+1})}(Y^{(R,S)}) \right]_2 \right) \\ & \times \mathcal{W}_{(n)}^{(r_1,s_1)} \left[w_{r_1+p_1,s_1+q_1}^{(n)} \right] \left(\rho_{**} \left(\lambda + \eta_1^{(\theta_{n+1})}(Y^{(R,S)}); y_1^{(r_1)}, \tilde{y}_1^{(s_1)} \right); Z^{(n)}[z] \right) \\ & \times \frac{\overline{\chi}_1^2 \left(\lambda_2 + \left[\eta_1^{(\theta_{n+1})}(Y^{(R,S)}) \right]_2 + \sum_{j=1}^{s_1} m_{\theta_{n+1}} \left(\tilde{u}_{1,j}^{(s_1)} \right) \right)}{T^{(n)} \left[\rho_{**} \left(\lambda + \eta_1^{(\theta_{n+1})}(Y^{(R,S)}) + \sum_{j=1}^{s_1} \lambda_{\theta_{n+1}} \left(\tilde{u}_{1,j}^{(s_1)} \right) \right); Z^{(n)}[z] \right] - \rho_{**} z} \\ & \cdots \\ & \times \frac{\overline{\chi}_1^2 \left(\lambda_2 + \left[\eta_{L-1}^{(\theta_{n+1})}(Y^{(R,S)}) \right]_2 + \sum_{j=1}^{s_{L-1}} m_{\theta_{n+1}} \left(\tilde{u}_{L-1,j}^{(s_{L-1})} \right) \right)}{T^{(n)} \left[\rho_{**} \left(\lambda + \eta_{L-1}^{(\theta_{n+1})}(Y^{(R,S)}) + \sum_{j=1}^{s_{L-1}} \lambda_{\theta_{n+1}} \left(\tilde{u}_{L-1,j}^{(s_{L-1})} \right) \right); Z^{(n)}[z] \right] - \rho_{**} z} \\ & \times \mathcal{W}_{(n)}^{(r_L,s_L)} \left[w_{r_L+p_L,s_L+q_L}^{(n)} \right] \left(\rho_{**} \left(\lambda + \eta_L^{(\theta_{n+1})}(Y^{(R,S)}); y_L^{(r_L)}, \tilde{y}_L^{(s_L)} \right); Z^{(n)}[z] \right) \\ & \times \chi_1 \left(\rho_{**}^{-1} M_{[\theta_n]} + \lambda_2 + \left[\eta_L^{(\theta_{n+1})}(Y^{(R,S)}) \right]_2 + \sum_{j=1}^{s_L} m_{\theta_{n+1}} \left(\tilde{u}_{L,j}^{(s_L)} \right) \right) \Big\rangle_{\Omega_{\mathrm{vac}}}. \end{aligned}$$

Hereby we used (D.15) of Proposition D.4 and

$$\eta_\ell^{(\theta_n)}(\rho_{**}Y^{(R,S)}) = \rho_{**}\eta_\ell^{(\theta_{n+1})}(Y^{(R,S)}), \qquad m_{\theta_n}(\rho_{**}u) = \rho_{**}m_{\theta_{n+1}}(u)$$

and

$$\chi_{\rho_{**}}(\rho_{**}x) = \chi_1(x), \qquad \overline{\chi}_{\rho_{**}}(\rho_{**}x) = \overline{\chi}_1(x).$$

Identifying the sequence $\underline{w}^{(n+1)}$ with the operator $\mathcal{W}_{(n+1)}[\underline{w}^{(n+1)}]$ it generates we can understand the renormalization transformation $\mathcal{R}_{\rho_{**}}^{(n)}$ as a map

$$\begin{aligned} \mathcal{R}_{\rho_{**}}^{(n)} &: \mathcal{D}^{(n)}(\varepsilon,\eta) \to \mathfrak{W}^{(n+1)}, \\ \mathcal{R}_{\rho_{**}}^{(n)}[\underline{w}^{(n)}] &:= \underline{w}^{(n+1)} \end{aligned} \tag{4.41}$$

with $\underline{w}^{(n+1)}$ given in (4.38) and (4.39). Setting

$$K^{(n+1)}[z] := \mathcal{W}_{(n+1)}[\underline{w}^{(n+1)}[\,\cdot\,; z]]$$

we observe that the definition (4.37, 4.41), the isospectral property of the smooth Feshbach map provided in Theorem E.1 and the invariance of the kernel under the rescaling $\mathfrak{S}_{\rho_{**}}$ lead to the spectral link between $K^{(n)}$ and its image $K^{(n+1)}$ under the renormalization transformation,

$$\begin{aligned} 0 \in \operatorname{spec}\left(K^{(n)}\left[Z^{(n)}[z]\right]\right) \qquad &\Longleftrightarrow \qquad 0 \in \operatorname{spec}\left(K^{(n+1)}[z]\right), \\ 0 \in \operatorname{spec}_{\mathrm{pp}}\left(K^{(n)}\left[Z^{(n)}[z]\right]\right) \qquad &\Longleftrightarrow \qquad 0 \in \operatorname{spec}_{\mathrm{pp}}\left(K^{(n+1)}[z]\right) \end{aligned} \tag{4.42}$$

for all $z \in B_{1/4}$. Thus, the spectral information of $K^{(n)}[\zeta]$ for parameters $\zeta := Z^{(n)}[z] \in Z^{(n)}[B_{1/4}] = D[\underline{w}^{(n)}] \supseteq B_{\rho_{**}/4-\eta}$, i.e., in a ρ_{**}-neighborhood of zero, is encoded as the spectral information of $K^{(n+1)}[z]$ for spectral parameters from the ball $B_{1/4}$. Therefore, studying the spectral properties of the family $z \mapsto K^{(n+1)}[z]$ on a scale ρ_{**}^0 enables us to analyze the spectrum of $K^{(n)}[z]$ on a scale ρ_{**}. In Section 4.4 we show that, under certain conditions, $\mathcal{R}_{\rho_{**}}^{(n)}$ even maps $\mathcal{D}^{(n)}(\varepsilon,\eta)$ into $\mathcal{D}^{(n+1)}(\varepsilon+\eta/2,\eta/2)$ which allows an iteration of the renormalization transformation and generates a discrete flow, c.f. Section 4.5. Linking the spectral information of each iteration step with the previous one enables us to understand the spectrum of $K^{(n)}[\zeta]$ for ζ on an arbitrary scale ρ_{**}^k as long as we incorporate k iteration steps. This recursive localization of the spectrum is worked out in Chapter 5.

4.3.4 Decimation of Degrees of Freedom via Smooth Feshbach Map: Iteration Step $n = 1$

The application of the smooth Feshbach map to $K^{(1)}[z]$ (introduced in Section 4.2) is different from the previous sections since the decimation of degrees of freedom is not only restricted to the bosonic variables but acts also on the particle sector. Given the cutoff function χ_1 and the scaling parameter

$$\rho_* := \frac{g^2}{2\rho}\gamma_{\mathrm{eq}} \tag{4.43}$$

(the constant γ_{eq} is defined in (4.46) below) we define the smooth "projection" operator

$$\Xi^{(1)}_{\rho_*} := \chi_{\rho_*}\left(\Gamma_{\mathrm{eq}} \otimes \mathbb{1}_{\mathcal{F}[\Upsilon]} + \mathbb{1}_{\ker(L_{\mathrm{p}})} \otimes M_{[\theta_1]}\right),$$

where

$$\Gamma_{\mathrm{eq}} := \operatorname{Im}\left(\Lambda_0^{(s)}|_{\beta_{\max}=\beta_{\min}=\beta_{\mathrm{p}}}\right)$$

is the imaginary part of the level shift operator $\Lambda_0^{(s)}$ associated with $K^{(s)}$ for the equilibrium case where all reservoir temperatures β_r coincide with the particle temperature β_{p} (we refer to Section 3.3.2 to recall the properties of the level shift operator). Due to Lemmata 3.15 and 3.20 the operator Γ_{eq} is independent of s,

$$\Gamma_{\mathrm{eq}} = \sum_{r=1}^{R} \Gamma_{0,r}|_{\beta_r=\beta_{\mathrm{p}}}, \tag{4.44}$$

and it holds

$$\Lambda_0^{(s)} - i\Gamma_{\mathrm{eq}} = \mathcal{O}\left(s(\beta_{\max} - \beta_{\min})\right) \tag{4.45}$$

if we let β_{p} coincide with a reservoir temperature $\beta_{r'}$ as postulated in (2.9). Because of Proposition 3.23(ii) the operator Γ_{eq} has a one dimensional kernel spanned by Ω_{p} and all other eigenvalues are separated from zero by a gap

$$\inf\left[\operatorname{spec}(\Gamma_{\mathrm{eq}}) \setminus \{0\}\right] \geq \gamma_{\mathrm{eq}} := \sum_{r=1}^{R} \gamma_0^r \geq R\gamma_{\mathrm{FGR}} \tag{4.46}$$

which is strictly positive by Hypothesis V-1.10, uniformly in the inverse temperatures. Therefore, the operator $\Xi^{(1)}_{\rho_*}$ can be written as

$$\Xi^{(1)}_{\rho_*} = |\Omega_{\mathrm{p}}\rangle\langle\Omega_{\mathrm{p}}| \otimes \chi_{\rho_*}(M_{[\theta_1]})$$

if $\rho_* < \gamma_{\mathrm{eq}}$ which is fulfilled for g sufficiently small which we henceforth assume. Thus, the operator $\Xi^{(1)}_{\rho_*}$ can be expressed via functional calculus as

$$\Xi^{(1)}_{\rho_*} = X^{(1)}_{\rho_*}(M_{[\theta_1]}),$$

where

$$X^{(1)}_{\rho_*} : u \mapsto \chi_{\rho_*}(u)\,|\Omega_{\mathrm{p}}\rangle\langle\Omega_{\mathrm{p}}|,$$

is a smooth function $\mathbb{R}_0^+ \to \mathcal{B}(\ker(L_\mathrm{p}))$. The complementary "projection" operator is defined as

$$\overline{\Xi}_{\rho_*}^{(1)} := \sqrt{\mathbb{1}_{\mathcal{H}^{(n)}} - {\Xi_{\rho_*}^{(1)}}^2} = \overline{X}_{\rho_*}^{(1)}(M_{[\theta_1]}),$$

where

$$\overline{X}_{\rho_*}^{(1)} : u \mapsto \sum_{j,k=0}^{N-1} \overline{\chi}_{\rho_*}\left((\Gamma_\mathrm{eq})_{j,k} + u\right) |\varphi_{j,j}\rangle \langle \varphi_{k,k}|$$

is a smooth function with values in $\mathcal{B}(\ker(L_\mathrm{p}))$ and $(\Gamma_\mathrm{eq})_{j,k} := \langle \varphi_{j,j} \mid \Gamma_\mathrm{eq}\varphi_{k,k} \rangle_{\ker(L_\mathrm{p})}$ are the matrix elements of Γ_eq in the orthonormal basis $\{\varphi_{j,j}\}_{j=0,\dots,N-1}$ of $\ker(L_\mathrm{p})$.

We conclude this notational part by introducing orthogonal projections on the range of the operators $\Xi_{\rho_*}^{(1)}$ and $\overline{\Xi}_{\rho_*}^{(1)}$. We define

$$P_{\rho_*}^{(1)} : \text{ orthogonal projection on ran}\left(\Xi_{\rho_*}^{(1)}\right), \qquad P_{\rho_*}^{(1)\perp} := \mathbb{1}_{\ker(L_\mathrm{p})} - P_{\rho_*}^{(1)},$$
$$\overline{P}_{\rho_*}^{(1)} : \text{ orthogonal projection on ran}\left(\overline{\Xi}_{\rho_*}^{(1)}\right), \qquad \overline{P}_{\rho_*}^{(1)\perp} := \mathbb{1}_{\ker(L_\mathrm{p})} - \overline{P}_{\rho_*}^{(1)},$$

and note that

$$\begin{aligned}
P_{\rho_*}^{(1)} &= P_{\left[\Gamma_\mathrm{eq}\otimes \mathbb{1}_{\mathcal{F}[\Upsilon]} + \mathbb{1}_{\ker(L_\mathrm{p})}\otimes M_{[\theta_1]} < \rho_*\right]} \\
&= |\Omega_\mathrm{p}\rangle \langle \Omega_\mathrm{p}| \otimes P_{\left[M_{[\theta_1]} < \rho_*\right]}, \\
\overline{P}_{\rho_*}^{(1)} &= P_{\left[\Gamma_\mathrm{eq}\otimes \mathbb{1}_{\mathcal{F}[\Upsilon]} + \mathbb{1}_{\ker(L_\mathrm{p})}\otimes M_{[\theta_1]} > \frac{7}{8}\rho_*\right]}, \\
&= \left(|\Omega_\mathrm{p}\rangle \langle \Omega_\mathrm{p}|\right)^\perp \otimes P_{\left[M_{[\theta_1]} < \frac{7}{8}\rho_*\right]} + \mathbb{1}_{\mathcal{H}_\mathrm{p}^2} \otimes P_{\left[M_{[\theta_1]} \geq \frac{7}{8}\rho_*\right]}, \\
P_{\rho_*}^{(1)\perp} &= P_{\left[\Gamma_\mathrm{eq}\otimes \mathbb{1}_{\mathcal{F}[\Upsilon]} + \mathbb{1}_{\ker(L_\mathrm{p})}\otimes M_{[\theta_1]} \geq \rho_*\right]} \\
&= \left(|\Omega_\mathrm{p}\rangle \langle \Omega_\mathrm{p}|\right)^\perp \otimes P_{\left[M_{[\theta_1]} < \rho_*\right]} + \mathbb{1}_{\mathcal{H}_\mathrm{p}^2} \otimes P_{\left[M_{[\theta_1]} \geq \rho_*\right]}, \\
\overline{P}_{\rho_*}^{(1)\perp} &= P_{\left[\Gamma_\mathrm{eq}\otimes \mathbb{1}_{\mathcal{F}[\Upsilon]} + \mathbb{1}_{\ker(L_\mathrm{p})}\otimes M_{[\theta_1]} \leq \frac{7}{8}\rho_*\right]} \\
&= |\Omega_\mathrm{p}\rangle \langle \Omega_\mathrm{p}| \otimes P_{\left[M_{[\theta_1]} \leq \frac{7}{8}\rho_*\right]},
\end{aligned}$$

where $\left(|\Omega_\mathrm{p}\rangle \langle \Omega_\mathrm{p}|\right)^\perp := \mathbb{1}_{\ker(L_\mathrm{p})} - |\Omega_\mathrm{p}\rangle \langle \Omega_\mathrm{p}|$.

In what follows we proceed similar to Section 4.3.2 by applying the smooth Feshbach map to the operator $K^{(1)}[z]$. To this end we have to extract a "free part" from $K^{(1)}[z]$ which, together with $K^{(1)}[z]$, forms a $\Xi_{\rho_*}^{(1)}$-Feshbach pair in the sense of Appendix E. The difficulty in this situation is that the canonical choice $T^{(1)}[\Lambda_{[\theta_1]}; z] - E^{(1)}[z]$ as a free part does not qualify since it is not necessarily commuting with the cutoff operator $\Xi_{\rho_*}^{(1)}$. However, the leading orders

$$\breve{T}^{(1)}[\Lambda_{[\theta_1]}; z] = L_{0,\theta_1} \restriction_{\mathcal{H}^{(1)}} - z + i\frac{g^2}{\rho}\Gamma_\mathrm{eq} \otimes \chi_1^2(M_{[\theta_1]})$$

of $T^{(1)}[\Lambda_{[\theta_1]}; z] - E^{(1)}[z]$ are a suitable choice as we will prove below. Hereby, we define

$$\breve{T}^{(1)}[\lambda; z] := (\lambda_1 + i\lambda_2) - z + i\frac{g^2}{\rho}\chi_1^2(\lambda_2)\Gamma_{\mathrm{eq}}.$$

Apparently, $\breve{T}^{(1)}[\Lambda_{[\theta_1]}; z]$ commutes with $\Xi_{\rho_*}^{(1)}$ and $\overline{\Xi}_{\rho_*}^{(1)}$. The perturbative part of $K^{(1)}[z]$ which results from distinguishing $\breve{T}^{(1)}[\Lambda_{[\theta_1]}; z]$ as free part is then given by

$$\begin{aligned}
\breve{W}^{(1)}[z] &:= K^{(1)}[z] - \breve{T}^{(1)}[\Lambda_{[\theta_1]}; z] = K^{(1)}[z] - \left(L_{0,\theta_1} - z + i\frac{g^2}{\rho}\Gamma_{\mathrm{eq}} \otimes \chi_1^2(M_{[\theta_1]})\right) \\
&= \frac{g^2}{\rho}\left(\Lambda_0^{(s)} - i\Gamma_{\mathrm{eq}}\right) \otimes \chi_1^2(M_{[\theta_1]}) + \mathcal{O}\left(\frac{g^{2+\tilde{\varepsilon}}}{\rho}\right) \\
&= \frac{g^2}{\rho}\mathcal{O}\left(g^{\tilde{\varepsilon}} + s(\beta_{\max} - \beta_{\min})\right) \\
&= \rho_*\mathcal{O}\left(g^{\tilde{\varepsilon}} + s(\beta_{\max} - \beta_{\min})\right), \qquad (4.47)
\end{aligned}$$

because of (4.25, 4.45). Alternatively, we may express $\breve{W}^{(1)}[z]$ in terms of Wick monomials

$$\breve{W}^{(1)}[z] = \mathcal{W}_{(1)}\left[\left(\breve{w}_{R,S}^{(1)}[\,\cdot\,; z]\right)_{R+S\geq 0}\right]$$

where

$$\begin{aligned}
\breve{w}_{0,0}^{(1)}[\lambda; z] &:= w_{0,0}^{(1)}[\lambda; z] - (\lambda_1 + i\lambda_2) + z - i\frac{g^2}{\rho}\chi_1^2(\lambda_2)\Gamma_{\mathrm{eq}}, \\
\breve{w}_{R,S}^{(1)} &:= w_{R,S}^{(1)}, \qquad R + S \geq 1.
\end{aligned}$$

Note that $\underline{\breve{w}}^{(1)} := (\breve{w}_{R,S}^{(1)})_{R+S\geq 0}$ also has a contribution $\breve{w}_{0,0}^{(1)}$ to the $\mathfrak{W}_{0,0}^{(1)}$-sector of order

$$\begin{aligned}
&\sup_{\substack{z\in B_{1/4},\\ \lambda\in\mathcal{Q}^{(1)}}} \left\|\breve{w}_{0,0}^{(1)}[\lambda; z]\right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} \qquad (4.48)\\
&= \sup_{z\in B_{1/4}} \left\|T^{(1)}[\Lambda_{[\theta_1]}; z] - E^{(1)}[z] - \breve{T}^{(1)}[\Lambda_{[\theta_1]}; z]\right\|_{\mathcal{B}(\mathcal{H}^{(1)})} \\
&= \sup_{z\in B_{1/4}} \left\|\breve{W}^{(1)}[z] - W^{(1)}[z]\right\|_{\mathcal{B}(\mathcal{H}^{(1)})} \\
&= \mathcal{O}\left(g\rho^{\mu} + \frac{g^2}{\rho}(g^{\tilde{\varepsilon}} + |\beta_{\max} - \beta_{\min}|)\right),
\end{aligned}$$

using functional calculus, recall the definition (4.12) of $\mathcal{Q}^{(1)}$, and using (4.26, 4.47) together with Corollary 4.3.

Lemma 4.8 *(i) Let $z \in D[\underline{w}^{(1)}]$. Then the operator $\breve{T}^{(1)}[\Lambda_{[\theta_1]}; z]$ is bounded invertible on* $\mathrm{ran}(\overline{P}^{(1)}_{\rho_*})$ *and the resolvent obeys the following norm bound,*

$$\left\| \breve{T}^{(1)}[\Lambda_{[\theta_1]}; z]^{-1} \overline{P}^{(1)}_{\rho_*} \right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} \leq \frac{3}{\rho_*}.$$

(ii) For $z \in D[\underline{w}^{(1)}]$ and $|\beta_{\max} - \beta_{\min}| \ll 1$ and g sufficiently small the operator

$$K^{(1)}[z]_{\Xi^{(1)}_{\rho_*}} := \breve{T}^{(1)}[\Lambda_{[\theta_1]}; z] + \Xi^{(1)}_{\rho_*} \breve{W}^{(1)}[z] \Xi^{(1)}_{\rho_*}$$

is bounded invertible on the $\mathrm{ran}(\overline{P}^{(1)}_{\rho_*})$. *Therefore, the pair $(K^{(1)}[z], \breve{T}^{(1)}[\Lambda_{[\theta_1]}; z])$ is a $\Xi^{(1)}_{\rho_*}$-Feshbach pair.*

Proof.

(i) We start by decomposing the projection $\overline{P}^{(1)}_{\rho_*} = P_1 + P_2$ where

$$P_1 := (|\Omega_{\mathrm{p}}\rangle \langle \Omega_{\mathrm{p}}|)^{\perp} \otimes P_{\left[M_{[\theta_1]} < \frac{7}{8}\rho_*\right]}, \qquad P_2 := \mathbb{1}_{\ker(L_{\mathrm{p}})} \otimes P_{\left[M_{[\theta_1]} \geq \frac{7}{8}\rho_*\right]}$$

and compute the norm of the resolvent $\breve{T}^{(1)}[\Lambda_{[\theta_1]}; z]^{-1} = (L_{0,\theta_1} - z + i\frac{g^2}{\rho}\Gamma_{\mathrm{eq}})^{-1}$ separately on each of the sub-ranges. We start considering

$$\begin{aligned}
&\left\| \left(L_{0,\theta_1} - z + i\frac{g^2}{\rho}\Gamma_{\mathrm{eq}} \otimes \chi_1^2(M_{[\theta_1]}) \right)^{-1} P_1 \right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} \\
&\leq \sup_{\substack{0 \leq m < \frac{7}{8}\rho_*, \\ e \in \mathrm{spec}(\Gamma_{\mathrm{eq}}) \setminus \{0\}}} \frac{1}{\left| m - \mathrm{Im}(z) + \frac{g^2}{\rho} e \right|} \leq \frac{1}{\frac{g^2}{\rho}\gamma_{\mathrm{eq}} - |\,\mathrm{Im}(z)|} \\
&\leq \frac{1}{\frac{g^2}{\rho}\gamma_{\mathrm{eq}} - \frac{5}{16}\rho_*} \leq \frac{1}{\rho_*},
\end{aligned}$$

using Lemma 4.5(i), and go over to estimate

$$\begin{aligned}
&\left\| \left(L_{0,\theta_1} - z + i\frac{g^2}{\rho}\Gamma_{\mathrm{eq}} \otimes \chi_1^2(M_{[\theta_1]}) \right)^{-1} P_2 \right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} \\
&\leq \sup_{\substack{m \geq \frac{7}{8}\rho_*, \\ e \in \mathrm{spec}(\Gamma_{\mathrm{eq}})}} \frac{1}{\left| m - \mathrm{Im}(z) + \frac{g^2}{\rho} e \chi_1^2(m) \right|} \leq \frac{1}{\frac{7}{8}\rho_* - |\,\mathrm{Im}(z)|} \\
&\leq \frac{1}{\frac{7}{8}\rho_* - \frac{5}{16}\rho_*} \leq \frac{2}{\rho_*}.
\end{aligned}$$

Altogether, we obtain

$$\left\| \left(L_{0,\theta_1} - z + i\frac{g^2}{\rho}\Gamma_{\mathrm{eq}} \otimes \chi_1^2(M_{[\theta_1]}) \right)^{-1} \overline{P}_{\rho_*}^{(1)} \right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} \leq \frac{3}{\rho_*}.$$

(ii) We recall the estimate (4.47) of the perturbative part $\breve{W}^{(1)}[z]$ to estimate

$$\begin{aligned} & \left\| \Xi_{\rho_*}^{(1)} \breve{W}^{(1)}[z] \Xi_{\rho_*}^{(1)} \breve{T}^{(1)}[\Lambda_{[\theta_1]}; z]^{-1} \overline{P}_{\rho_*}^{(1)} \right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} \\ \leq\ & \frac{3}{\rho_*} \left\| \breve{W}^{(1)}[z] \right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} \\ =\ & \mathcal{O}\left(g^{\tilde{\varepsilon}} + |\beta_{\max} - \beta_{\min}| \right) < 1 \end{aligned} \tag{4.49}$$

for $|\beta_{\max} - \beta_{\min}| \ll 1$ and g sufficiently small. This relative bound and (i) allow the expansion of $K^{(1)}[z]^{-1}_{\overline{\Xi}^{(1)}_{\rho_*}}$ into a norm convergent Neumann series,

$$\begin{aligned} & \overline{P}_{\rho_*}^{(1)} K^{(1)}[z]^{-1}_{\overline{\Xi}^{(1)}_{\rho_*}} \overline{P}_{\rho_*}^{(1)} \\ =\ & \breve{T}^{(1)}[\Lambda_{[\theta_1]}; z]^{-1} \sum_{L=0}^{\infty} \left(-\overline{\Xi}_{\rho_*}^{(1)} \breve{W}^{(1)}[z] \overline{\Xi}_{\rho_*}^{(1)} \breve{T}^{(1)}[\Lambda_{[\theta_1]}; z]^{-1} \right)^L . \end{aligned} \tag{4.50}$$

■

Due to Lemma 4.8 the smooth Feshbach map is applicable to the $\Xi_{\rho_*}^{(1)}$-Feshbach pair $(K^{(1)}[z], \breve{T}^{(1)}[\Lambda_{[\theta_1]}; z])$ for $z \in D[\underline{w}^{(1)}]$, we refer to Appendix E for details on the Feshbach map. With the help of the expansion (4.50) we can write

$$\begin{aligned} & \mathfrak{F}_{\Xi_{\rho_*}^{(1)}} \left(K^{(1)}[z], \breve{T}^{(1)}[\Lambda_{[\theta_1]}; z] \right) \\ =\ & \breve{T}^{(1)}[\Lambda_{[\theta_1]}; z] + \Xi_{\rho_*}^{(1)} \breve{W}^{(1)}[z] \Xi_{\rho_*}^{(1)} \\ & - \Xi_{\rho_*}^{(1)} \breve{W}^{(1)}[z] \overline{\Xi}_{\rho_*}^{(1)} K^{(1)}[z]^{-1}_{\overline{\Xi}^{(1)}_{\rho_*}} \overline{\Xi}_{\rho_*}^{(1)} \breve{W}^{(1)}[z] \Xi_{\rho_*}^{(1)} \\ =\ & \breve{T}^{(1)}[\Lambda_{[\theta_1]}; z] + \Xi_{\rho_*}^{(1)} \breve{W}^{(1)}[z] \Xi_{\rho_*}^{(1)} \\ & - \sum_{L=0}^{\infty} \Xi_{\rho_*}^{(1)} \breve{W}^{(1)}[z] \overline{\Xi}_{\rho_*}^{(1)} \left(- \frac{\overline{\Xi}_{\rho_*}^{(1)}}{\breve{T}^{(1)}[\Lambda_{[\theta_1]}; z]} \breve{W}^{(1)}[z] \overline{\Xi}_{\rho_*}^{(1)} \right)^L \\ & \qquad \times \frac{\overline{\Xi}_{\rho_*}^{(1)}}{\breve{T}^{(1)}[\Lambda_{[\theta_1]}; z]} \breve{W}^{(1)}[z] \Xi_{\rho_*}^{(1)} \\ =\ & \breve{T}^{(1)}[\Lambda_{[\theta_1]}; z] \\ & + \sum_{L=0}^{\infty} \Xi_{\rho_*}^{(1)} \breve{W}^{(1)}[z] \left(- \frac{\overline{\Xi}_{\rho_*}^{(1)\,2}}{\breve{T}^{(1)}[\Lambda_{[\theta_1]}; z]} \breve{W}^{(1)}[z] \right)^L \Xi_{\rho_*}^{(1)} \end{aligned} \tag{4.51}$$

and understand it as an operator on $\mathrm{ran}(P^{(1)}_{\rho_*})$.

4.3.5 Rescaling Bosonic Variables and Renormalization Transformation: Iteration Step $n = 1$

The rescaling of the operator $\mathfrak{F}_{\Xi^{(1)}_{\rho_*}}(K^{(1)}[z], \breve{T}^{(1)}[\Lambda_{[\theta_1]}; z])$ and the underlying Hilbert space $P^{(1)}_{\rho_*}\mathcal{H}^{(1)}$ on which the operator acts is performed in exactly the same way as in Section 4.2 and 4.3.3. We directly go over to define the first application of the renormalization transformation $\mathcal{R}^{(1)}_{\rho_*}$ to the initial data $\underline{w}^{(1)} = (w^{(1)}_{R,S})_{R+S\geq 0}$,

$$\mathcal{R}^{(1)}_{\rho_*}\left[\underline{w}^{(1)}\right] := \underline{w}^{(2)},$$

where the sequence $\underline{w}^{(2)} := (w^{(2)}_{R,S})_{R+S\geq 0}$ is given by

$$\begin{aligned} & w^{(2)}_{R,S}[\lambda; Y^{(R,S)}; z] \\ & := \rho_*^{-1}\sum_{L=1}^{\infty}(-1)^{L-1}\sum_{\substack{r_1+\cdots+r_L=R,\\ s_1+\cdots+s_L=S}}\ \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ r_\ell+p_\ell+s_\ell+q_\ell\geq 0}}\left[\prod_{\ell=1}^{L}\binom{r_\ell+p_\ell}{r_\ell}\binom{s_\ell+q_\ell}{s_\ell}\right] \\ & \quad \times V^{(2)}_{\underline{r,p,s,q}_L}[\lambda; Y^{(R,S)}; z], \end{aligned} \tag{4.52}$$

for $R + S \geq 1$, and

$$\begin{aligned} & w^{(2)}_{0,0}[\lambda; z] \\ & := \rho_*^{-1}\left\langle \Omega_{\mathrm{p}}\,\middle|\,\left(\breve{T}^{(1)}\left[\rho_*\lambda; Z^{(1)}[z]\right] - \chi_1^2(\lambda_2)\breve{w}^{(1)}_{0,0}\left[\rho_*\lambda; Z^{(1)}[z]\right]\right)\Omega_{\mathrm{p}}\right\rangle_{\ker(L_{\mathrm{p}})} \\ & \quad +\rho_*^{-1}\sum_{L=2}^{\infty}(-1)^{L-1}\sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ p_\ell+q_\ell\geq 0}}\chi_1^2(\lambda_2) \\ & \quad \times\left\langle \Omega_0\,\middle|\,\mathcal{W}^{(0,0)}_{(1)}\left[\breve{w}^{(1)}_{p_1,q_1}\right]\left(\rho_*\lambda; Z^{(1)}[z]\right)\frac{\overline{X}_1^{(1)^2}\left(\rho_*^{-1}M_{[\theta_1]} + \lambda_2\right)}{\breve{T}^{(1)}\left[\Lambda_{[\theta_1]} + \rho_*\lambda; Z^{(1)}[z]\right]}\right. \\ & \qquad \left.\times\cdots\times\frac{\overline{X}_1^{(1)^2}\left(\rho_*^{-1}M_{[\theta_1]} + \lambda_2\right)}{\breve{T}^{(1)}\left[\Lambda_{[\theta_1]} + \rho_*\lambda; Z^{(1)}[z]\right]}\mathcal{W}^{(0,0)}_{(1)}\left[\breve{w}^{(1)}_{p_L,q_L}\right]\left(\rho_*\lambda; Z^{(1)}[z]\right)\Omega_0\right\rangle. \end{aligned} \tag{4.53}$$

The functions $V^{(2)}_{\underline{r,p,s,q}_L}$ appearing in (4.52) are the symmetrization (in the sense of (D.8)) of

$$\begin{aligned}
&\tilde{V}^{(2)}_{\underline{r,p,s,q}_L}[\lambda; Y^{(R,S)}; z] \qquad (4.54)\\
&:= \left\langle \Omega_0 \left| \chi_1\left(\rho_*^{-1} M_{[\theta_1]} + \lambda_2 + \left[\eta_0^{(\theta_2)}(Y^{(R,S)})\right]_2\right)\right.\right.\\
&\quad \times \mathcal{W}^{(r_1,s_1)}_{(1)}\left[\breve{w}^{(1)}_{r_1+p_1,s_1+q_1}\right]\left(\rho_*\left(\lambda + \eta_1^{(\theta_2)}(Y^{(R,S)}); y_1^{(r_1)}, \tilde{y}_1^{(s_1)}\right); Z^{(1)}[z]\right)\\
&\quad \times \frac{\overline{X}_1^{(1)^2}\left(\lambda_2 + \left[\eta_1^{(\theta_2)}(Y^{(R,S)})\right]_2 + \sum_{j=1}^{s_1} m_{\theta_2}\left(\tilde{u}^{(s_1)}_{1,j}\right)\right)}{\breve{T}^{(1)}\left[\rho_*\left(\lambda + \eta_1^{(\theta_2)}(Y^{(R,S)}) + \sum_{j=1}^{s_1}\lambda_{\theta_2}\left(\tilde{u}^{(s_1)}_{1,j}\right)\right); Z^{(1)}[z]\right]}\\
&\quad \cdots\\
&\quad \times \frac{\overline{X}_1^{(1)^2}\left(\lambda_2 + \left[\eta_{L-1}^{(\theta_2)}(Y^{(R,S)})\right]_2 + \sum_{j=1}^{s_{L-1}} m_{\theta_2}\left(\tilde{u}^{(s_{L-1})}_{L-1,j}\right)\right)}{\breve{T}^{(1)}\left[\rho_*\left(\lambda + \eta_{L-1}^{(\theta_2)}(Y^{(R,S)}) + \sum_{j=1}^{s_{L-1}}\lambda_{\theta_2}\left(\tilde{u}^{(s_{L-1})}_{L-1,j}\right)\right); Z^{(1)}[z]\right]}\\
&\quad \times \mathcal{W}^{(r_L,s_L)}_{(1)}\left[\breve{w}^{(1)}_{r_L+p_L,s_L+q_L}\right]\left(\rho_*\left(\lambda + \eta_L^{(\theta_2)}(Y^{(R,S)}); y_L^{(r_L)}, \tilde{y}_L^{(s_L)}\right); Z^{(1)}[z]\right)\\
&\quad \left.\times \chi_1\left(\lambda_2 + \left[\eta_L^{(\theta_2)}(Y^{(R,S)})\right]_2 + \sum_{j=1}^{s_L} m_{\theta_2}\left(\tilde{u}^{(s_L)}_{L,j}\right)\right)\Omega_0\right\rangle.
\end{aligned}$$

Note that the additional term $\chi_1^2(\lambda_2)\breve{w}^{(1)}_{0,0}\left[\rho_*\lambda; Z^{(1)}[z]\right]$ appearing in (4.53) and the fact that the summation indices $\underline{r,p,s,q}_L$ and $\underline{0,p,0,q}_L$ in (4.52, 4.53), resp., range over $r_\ell + p_\ell + s_\ell + q_\ell \geq 0$ instead of $r_\ell + p_\ell + s_\ell + q_\ell \geq 1$ are due to the existence of the $\breve{w}^{(1)}_{0,0}$-contribution in $\underline{\breve{w}}^{(1)}$.

One convinces oneself by consulting (4.51), Theorem D.3, Proposition D.4 and by comparing with the elaborations in the Sections 4.2 and 4.3.3 that $\underline{w}^{(2)}$ is chosen such that

$$\mathcal{W}_{(2)}\left[\left(\mathcal{R}^{(1)}_{\rho_*}[\underline{w}^{(1)}]\right)[z]\right] = \mathfrak{S}_{\rho_*}\left[\mathfrak{F}_{\Xi^{(1)}_{\rho_*}}\left(K^{(1)}[Z^{(1)}[z]], \breve{T}^{(1)}[\Lambda_{[\theta_1]}; Z^{(1)}[z]]\right)\right] \qquad (4.55)$$

holds. We set

$$K^{(2)}[z] := \mathcal{W}_{(2)}[\underline{w}^{(2)}[\,\cdot\,; z]]$$

and observe that due to (4.55) and Theorem E.1 the families $z \mapsto K^{(1)}[z]$ and $z \mapsto K^{(2)}[z]$ are spectrally linked in the sense that

$$\begin{aligned}
0 \in \mathrm{spec}\left(K^{(1)}\left[Z^{(1)}[z]\right]\right) \quad &\Longleftrightarrow \quad 0 \in \mathrm{spec}\left(K^{(2)}[z]\right)\\
0 \in \mathrm{spec}_{\mathrm{pp}}\left(K^{(1)}\left[Z^{(1)}[z]\right]\right) \quad &\Longleftrightarrow \quad 0 \in \mathrm{spec}_{\mathrm{pp}}\left(K^{(2)}[z]\right)
\end{aligned} \qquad (4.56)$$

for all $z \in B_{1/4}$.

We close this section with a remark about the leading term in (4.53). We expand (setting $\zeta := Z^{(1)}[z]$)

$$\begin{aligned}
&\rho_*^{-1} \left\langle \Omega_\mathrm{p} \,\middle|\, \left(\breve{T}^{(1)} \left[\rho_*\lambda; Z^{(1)}[z]\right] - \chi_1^2(\lambda_2) \breve{w}_{0,0}^{(1)} \left[\rho_*\lambda; Z^{(1)}[z]\right] \right) \Omega_\mathrm{p} \right\rangle_{\ker(L_\mathrm{p})} \qquad (4.57)\\
&= \left\langle \Omega_\mathrm{p} \,\middle|\, \left\{ \left[(\lambda_1 + i\lambda_2) - \frac{\zeta}{\rho_*} + i \frac{g^2}{\rho\rho_*} \chi_1^2(\lambda_2) \Gamma_\mathrm{eq} \right] \right.\right.\\
&\qquad \left.\left. + \chi_1^2(\lambda_2) \left[\frac{1}{\rho_*} w_{0,0}^{(1)}[\rho_*\lambda; \zeta] - (\lambda_1 + i\lambda_2) + \frac{\zeta}{\rho_*} - i \frac{g^2}{\rho\rho_*} \chi_1^2(\lambda_2) \Gamma_\mathrm{eq} \right] \right\} \Omega_\mathrm{p} \right\rangle_{\ker(L_\mathrm{p})}\\
&= (\lambda_1 + i\lambda_2) - \overline{\chi}_1^2(\lambda_2) \frac{\zeta}{\rho_*}\\
&\quad + \chi_1^2(\lambda_2) \frac{1}{\rho_*} \left\langle \Omega_\mathrm{p} \,\middle|\, \left[T^{(1)}[\rho_*\lambda; \zeta] - \rho_*(\lambda_1 + i\lambda_2) - E^{(1)}[\zeta] \right] \Omega_\mathrm{p} \right\rangle_{\ker(L_\mathrm{p})}\\
&= -z + (\lambda_1 + i\lambda_2) + \overline{\chi}_1^2(\lambda_2) \frac{\left\langle E^{(1)} \right\rangle_{\Omega_\mathrm{p}} [\zeta] - \zeta}{\rho_*}\\
&\quad + \chi_1^2(\lambda_2) \frac{1}{\rho_*} \left\langle \Omega_\mathrm{p} \,\middle|\, \left[T^{(1)}[\rho_*\lambda; \zeta] - \rho_*(\lambda_1 + i\lambda_2) \right] \Omega_\mathrm{p} \right\rangle_{\ker(L_\mathrm{p})} .
\end{aligned}$$

This decomposition will be useful to show in Theorem 4.9(ii) that $\underline{w}^{(2)}$ belongs to a poly-disc $\mathcal{D}^{(2)}(\varepsilon_2, \eta_2)$.

4.4 Contracting Property of the Renormalization Transformation

The iteration of the renormalization transformation requires a control of the mapping properties of $\mathcal{R}_{\rho_{**}}^{(n)}$. We show that $\mathcal{R}_{\rho_{**}}^{(n)}$ maps a poly-disc $\mathcal{D}^{(n)}(\varepsilon, \eta)$ into a poly-disc $\mathcal{D}^{(n+1)}(\varepsilon', \eta')$ for suitable ε', η', i.e., $\mathcal{R}_{\rho_{**}}^{(n)}$ maps its domain into the domain of $\mathcal{R}_{\rho_{**}}^{(n+1)}$.

Theorem 4.9 *(i) For $\underline{w}^{(n)} \in \mathcal{D}^{(n)}(\varepsilon, \eta)$, $n \geq 2$, with $\varepsilon, \eta \leq \frac{\rho_{**}}{8\mathcal{C}_{\chi_1}}$ we have*

$$\mathcal{R}_{\rho_{**}}^{(n)} \left[\underline{w}^{(n)}\right] \in \mathcal{D}^{(n+1)} \left(\varepsilon + \frac{\eta}{2}, \frac{\eta}{2} \right).$$

(ii) For $|\beta_{\max} - \beta_{\min}| \ll 1$ and g sufficiently small we have

$$\underline{w}^{(2)} = \mathcal{R}_{\rho_*}^{(1)} \left[\underline{w}^{(1)}\right] \in \mathcal{D}^{(2)}(\varepsilon_2, \eta_2)$$

where

$$\varepsilon_2 := \frac{\rho_{**}}{32\mathcal{C}_{\chi_1}}, \qquad \eta_2 := \frac{\rho_{**}}{32\mathcal{C}_{\chi_1}}$$

and $\underline{w}^{(2)} = (w_{R,S}^{(2)})_{R+S\geq 0}$ is given in (4.52, 4.53).

The contracting property of the renormalization transformation $\mathcal{R}^{(n)}_{\rho_{**}}$ uses the following lemma to a great extend.

Lemma 4.10 *Fix* $L \in \mathbb{N}$ *and* $\underline{r,p,s,q}_L = (r_\ell, p_\ell, s_\ell, q_\ell)_{\ell=1}^L \in (\mathbb{N}_0)^{4L}$ *and set* $R := \sum_{\ell=1}^L r_\ell$ *and* $S := \sum_{\ell=1}^L s_\ell$.

(i) Let $n = 2, 3, \ldots, \mathcal{N}$ *and* $\underline{w}^{(n)} = \left(w^{(n)}_{R,S}\right)_{R+S\geq 0} \in \mathcal{D}^{(n)}(\varepsilon, \eta)$ *for* $\varepsilon, \eta \leq \rho_{**}/16$. *For* $r_\ell + p_\ell + s_\ell + q_\ell \geq 1$ *we consider the function*

$$\begin{aligned}
&\tilde{V}^{(n+1)}_{\underline{r,p,s,q}_L}[\lambda; Y^{(R,S)}; z] \\
&= \Bigg\langle F_0\left(\rho_{**}^{-1}\Lambda_{[\theta_n]} + \lambda\right) \\
&\qquad \times \prod_{\ell=1}^{L} \Bigg\{ \mathcal{W}^{(r_\ell, s_\ell)}_{(n)}\left[w^{(n)}_{r_\ell+p_\ell, s_\ell+q_\ell}\right] \\
&\qquad\qquad \left(\rho_{**}\left(\lambda + \eta_\ell^{(\theta_{n+1})}(Y^{(R,S)}); y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}\right); Z^{(n)}[z]\right) \\
&\qquad\qquad \times F_\ell\left(\rho_{**}^{-1}\Lambda_{[\theta_n]} + \lambda\right)\Bigg\}\Bigg\rangle_{\Omega_{\mathrm{vac}}}
\end{aligned}$$

with

$$\begin{aligned}
F_0(\lambda) &:= \chi_1\left(\lambda_2 + \left[\eta_0^{(\theta_{n+1})}(Y^{(R,S)})\right]_2\right), \\
F_L(\lambda) &:= \chi_1\left(\lambda_2 + \left[\eta_L^{(\theta_{n+1})}(Y^{(R,S)})\right]_2 + \sum_{j=1}^{s_L} m_{\theta_{n+1}}\left(\tilde{u}^{(s_L)}_{L,j}\right)\right), \\
F_\ell(\lambda) &:= \frac{\overline{\chi}_1^2\left(\lambda_2 + \left[\eta_\ell^{(\theta_{n+1})}(Y^{(R,S)})\right]_2 + \sum_{j=1}^{s_\ell} m_{\theta_{n+1}}\left(\tilde{u}^{(s_\ell)}_{\ell,j}\right)\right)}{T^{(n)}\left[\rho_{**}\left(\lambda + \eta_\ell^{(\theta_{n+1})}(Y^{(R,S)}) + \sum_{j=1}^{s_\ell} \lambda_{\theta_{n+1}}\left(\tilde{u}^{(s_\ell)}_{\ell,j}\right)\right); Z^{(n)}[z]\right] - \rho_{**} z}
\end{aligned}$$

for $\ell = 1, \ldots, L-1$, *as defined in (4.40). Then, the function* $B_{1/4} \ni z \mapsto \tilde{V}^{(n+1)}_{\underline{r,p,s,q}_L}[\,\cdot\,; z]$ *is analytic with values in* $\mathfrak{W}^{(n+1)}_{R,S}$ *and obeys the following bound,*

$$\begin{aligned}
&\rho_{**}^{-1}\left\|\tilde{V}^{(n+1)}_{\underline{r,p,s,q}_L}[\,\cdot\,; z]\right\|^{\#}_{(n+1)} \\
&\quad \leq \; 2(L+1)\mathcal{C}_{\chi_1}^{L+1}\rho_{**}^{(1+\mu)(R+S)-L}\prod_{\ell=1}^{L}\frac{\left\|w^{(n)}_{r_\ell+p_\ell, s_\ell+q_\ell}[\,\cdot\,; Z^{(n)}[z]]\right\|^{\#}_{(n)}}{\sqrt{p_\ell^{p_\ell} q_\ell^{q_\ell}}}.
\end{aligned}$$

using the convention that $0^0 := 1$.

(ii) *For* $r_\ell + p_\ell + s_\ell + q_\ell \geq 0$ *we consider the function* $\tilde{V}^{(2)}_{\underline{r,p,s,q}_L}$ *given in (4.54). Then the function* $B_{1/4} \ni z \mapsto \tilde{V}^{(2)}_{\underline{r,p,s,q}_L}[\,\cdot\,; z]$ *is analytic with values in* $\mathfrak{W}^{(2)}_{R,S}$ *and obeys the following bound,*

$$\rho_*^{-1} \left\| \tilde{V}^{(2)}_{\underline{r,p,s,q}_L}[\,\cdot\,; z] \right\|^{\#}_{(2)} \leq 2(L+1)\mathcal{C}^{L+1}_{\chi_1} \rho_*^{(1+\mu)(R+S)-L} \prod_{\ell=1}^{L} \frac{\left\| \breve{w}^{(1)}_{r_\ell+p_\ell, s_\ell+q_\ell}[\,\cdot\,; Z^{(1)}[z]] \right\|^{\#}_{(1)}}{\sqrt{p_\ell^{p_\ell} q_\ell^{q_\ell}}}.$$

(iii) *For* $r_\ell + p_\ell + s_\ell + q_\ell = 1$ *we consider the function* $\tilde{V}^{(1)}_{\underline{r,p,s,q}_L}$ *given in (4.22). Then the function* $B_{1/4} \ni z \mapsto \tilde{V}^{(1)}_{\underline{r,p,s,q}_L}[\,\cdot\,; z]$ *is analytic with values in* $\mathfrak{W}^{(1)}_{R,S}$ *and obeys the following bound,*

$$\rho^{-1} \left\| \tilde{V}^{(1)}_{\underline{r,p,s,q}_L}[\,\cdot\,; z] \right\|^{\#}_{(1)} \leq (L+1)\mathcal{C}^{L+1}_{\chi_1} \rho^{(1+\mu)(R+S)-L} \left(g M(\underline{\omega}^{(0)}) \right)^L ,$$

where

$$M(\underline{\omega}^{(0)}) := \left[\max_{\tilde{F} \in \left\{ F^{(s)}_\theta, F^{(\overline{s})*}_{\overline{\theta}} \right\}} \int_\Upsilon dy \, \left[m_\theta(u)^{-1} + m_\theta(u)^{-1-2\mu} \right] \left\| \tilde{F}(y) \right\|^2_{\mathcal{B}(\mathcal{H}^2_{\mathrm{p}})} \right]^{1/2}$$

and $F^{(s)}_\theta = [\mathcal{G} - \mathcal{G}'_{(s\delta\vec{\beta})}]_\theta$ *is explained in (2.26). The number* $M(\underline{\omega}^{(0)})$ *is finite for* $\frac{1}{2} \leq \mu < \nu$ *where* ν *is the infrared regularization of Hypothesis VII-1.12.*

Proof.

(i) Since the cutoff function χ_1 is smooth with compact support we have

$$|F_\ell(\lambda)| + \sum_{j=1,2} |\partial_{\lambda_j} F_\ell(\lambda)| \leq \mathcal{C}_{\chi_1}, \qquad \ell = 0, L$$

and (using the abbreviation $\tilde{\lambda}^{(\ell)} := \eta_\ell^{(\theta_{n+1})}(Y^{(R,S)}) + \sum_{j=1}^{s_\ell} \lambda_{\theta_{n+1}}(\tilde{u}_{\ell,j}^{(s_\ell)})$)

$$\begin{aligned}
&|F_\ell(\lambda)| + \sum_{j=1,2} |\partial_{\lambda_j} F_\ell(\lambda)| \\
&\leq \left| \frac{\overline{\chi}_1^2 \left(\lambda_2 + \tilde{\lambda}_2^{(\ell)}\right)}{T^{(n)}\left[\rho_{**}\left(\lambda + \tilde{\lambda}^{(\ell)}\right); Z^{(n)}[z]\right] - \rho_{**} z} \right| \\
&\quad + \sum_{j=1,2} \left| \frac{\overline{\chi}_1^2 \left(\lambda_2 + \tilde{\lambda}_2^{(\ell)}\right) \rho_{**}(\partial_{\lambda_j} T^{(n)}) \left[\rho_{**}\left(\lambda + \tilde{\lambda}^{(\ell)}\right); Z^{(n)}[z]\right]}{\left(T^{(n)}\left[\rho_{**}\left(\lambda + \tilde{\lambda}^{(\ell)}\right); Z^{(n)}[z]\right] - \rho_{**} z\right)^2} \right| \\
&\quad + \left| \frac{2\overline{\chi}_1 \left(\lambda_2 + \tilde{\lambda}_2^{(\ell)}\right) \partial_{\lambda_2} \overline{\chi}_1 \left(\lambda_2 + \tilde{\lambda}_2^{(\ell)}\right)}{T^{(n)}\left[\rho_{**}\left(\lambda + \tilde{\lambda}^{(\ell)}\right); Z^{(n)}[z]\right] - \rho_{**} z} \right| \\
&\leq \mathcal{C}_{\chi_1} \rho_{**}^{-1}
\end{aligned}$$

for the positive constant $\mathcal{C}_{\chi_1} \geq 1$ given in (4.6), only depending on the cutoff function χ_1. We made use of Lemma 4.7(i) to estimate $\left| T^{(n)}\left[\rho_{**}\left(\lambda + \tilde{\lambda}^{(\ell)}\right); Z^{(n)}[z]\right] - \rho_{**} z \right| \geq \frac{\rho_{**}}{2}$. Next, we observe that (writing η_ℓ instead of $\eta_\ell^{(\theta_{n+1})}(Y^{(R,S)})$)

$$\begin{aligned}
&\left| \tilde{V}^{(n+1)}_{\underline{r,p,s,q}_L}[\lambda; Y^{(R,S)}; z] \right| \qquad (4.58)\\
&\leq \prod_{\ell=0}^{L} \left\| F_\ell \left(\rho_{**}^{-1} \Lambda_{[\theta_n]} + \lambda\right) \right\|_{\mathcal{B}(\mathcal{H}^{(n)})} \\
&\quad \times \prod_{\ell=1}^{L} \left\| \mathcal{W}_{(n)}^{(r_\ell, s_\ell)} \left[w^{(n)}_{r_\ell+p_\ell, s_\ell+q_\ell} \right] \left(\rho_{**} \left(\lambda + \eta_\ell; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)} \right); Z^{(n)}[z] \right) \right\|_{\mathcal{B}(\mathcal{H}^{(n)})} \\
&\leq \frac{\mathcal{C}_{\chi_1}^{L+1}}{\rho_{**}^{L-1}} \\
&\quad \times \prod_{\ell=1}^{L} \left\| \mathcal{W}_{(n)}^{(r_\ell, s_\ell)} \left[w^{(n)}_{r_\ell+p_\ell, s_\ell+q_\ell} \right] \left(\rho_{**} \left(\lambda + \eta_\ell; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)} \right); Z^{(n)}[z] \right) \right\|_{\mathcal{B}(\mathcal{H}^{(n)})} .
\end{aligned}$$

By product rule we obtain

$$\begin{aligned}
&\partial_{\lambda_j} \tilde{V}^{(n+1)}_{\underline{r,p,s,q}_L}[\lambda; Y^{(R,S)}; z] \qquad (4.59)\\
&= \left[\tilde{V}^{(n+1)}_{\underline{r,p,s,q}_L} \right]^{(1)}_{\lambda_j} [\lambda; Y^{(R,S)}; z] + \left[\tilde{V}^{(n+1)}_{\underline{r,p,s,q}_L} \right]^{(2)}_{\lambda_j} [\lambda; Y^{(R,S)}; z],
\end{aligned}$$

where

$$\left[\tilde{V}^{(n+1)}_{\underline{r,p,s,q}_L}\right]^{(1)}_{\lambda_j}[\lambda; Y^{(R,S)}; z]$$

$$:= \sum_{k=1}^{L} \Big\langle \prod_{\ell=1}^{k-1} \Big\{ F_{\ell-1}(\rho_{**}^{-1}\Lambda_{[\theta_n]} + \lambda) \times \mathcal{W}^{(r_\ell,s_\ell)}_{(n)}\left[w^{(n)}_{r_\ell+p_\ell,s_\ell+q_\ell}\right]\left(\rho_{**}\left(\lambda+\eta_\ell; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}\right); Z^{(n)}[z]\right)\Big\}$$

$$\times \partial_{\lambda_j} F_k(\rho_{**}^{-1}\Lambda_{[\theta_n]} + \lambda)$$

$$\times \prod_{\ell=k+1}^{L} \Big\{ \mathcal{W}^{(r_\ell,s_\ell)}_{(n)}\left[w^{(n)}_{r_\ell+p_\ell,s_\ell+q_\ell}\right]\left(\rho_{**}\left(\lambda+\eta_\ell; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}\right); Z^{(n)}[z]\right) \times F_\ell\left(\rho_{**}^{-1}\Lambda_{[\theta_n]} + \lambda\right)\Big\}\Big\rangle_{\Omega_{\mathrm{vac}}}$$

and

$$\left[\tilde{V}^{(n+1)}_{\underline{r,p,s,q}_L}\right]^{(2)}_{\lambda_j}[\lambda; Y^{(R,S)}; z]$$

$$:= \sum_{k=1}^{L} \Big\langle F_0(\rho_{**}^{-1}\Lambda_{[\theta_n]} + \lambda)$$

$$\times \prod_{\ell=1}^{k-1} \Big\{ \mathcal{W}^{(r_\ell,s_\ell)}_{(n)}\left[w^{(n)}_{r_\ell+p_\ell,s_\ell+q_\ell}\right]\left(\rho_{**}\left(\lambda+\eta_\ell; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}\right); Z^{(n)}[z]\right) \times F_\ell(\rho_{**}^{-1}\Lambda_{[\theta_n]} + \lambda)\Big\}$$

$$\times \rho_{**}\mathcal{W}^{(r_k,s_k)}_{(n)}\left[\partial_{\lambda_j} w^{(n)}_{r_k+p_k,s_k+q_k}\right]\left(\rho_{**}\left(\lambda+\eta_k; y_k^{(r_k)}, \tilde{y}_k^{(s_k)}\right); Z^{(n)}[z]\right)$$

$$\times \prod_{\ell=k+1}^{L} \Big\{ F_\ell\left(\rho_{**}^{-1}\Lambda_{[\theta_n]} + \lambda\right) \times \mathcal{W}^{(r_\ell,s_\ell)}_{(n)}\left[w^{(n)}_{r_\ell+p_\ell,s_\ell+q_\ell}\right]\left(\rho_{**}\left(\lambda+\eta_\ell; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}\right); Z^{(n)}[z]\right)\Big\}$$

$$\times F_L\left(\rho_{**}^{-1}\Lambda_{[\theta_n]} + \lambda\right)\Big\rangle_{\Omega_{\mathrm{vac}}}.$$

The two terms $[\tilde{V}^{(n+1)}_{\underline{r,p,s,q}_L}]^{(1,2)}_{\lambda_j}[\lambda; Y^{(R,S)}; z]$ are estimated as follows,

$$\left|\left[\tilde{V}^{(n+1)}_{\underline{r,p,s,q}_L}\right]^{(1)}_{\lambda_j}[\lambda; Y^{(R,S)}; z]\right| \tag{4.60}$$

$$\leq \sum_{k=0}^{L} \left\|\partial_{\lambda_j} F_k(\rho_{**}^{-1}\Lambda_{[\theta_n]} + \lambda)\right\|_{\mathcal{B}(\mathcal{H}^{(n)})} \prod_{\substack{\ell=0,\\ \ell\neq k}}^{L} \left\|F_\ell(\rho_{**}^{-1}\Lambda_{[\theta_n]} + \lambda)\right\|_{\mathcal{B}(\mathcal{H}^{(n)})}$$

$$\times \prod_{\ell=1}^{L} \left\|\mathcal{W}_{(n)}^{(r_\ell,s_\ell)}\left[w_{r_\ell+p_\ell,s_\ell+q_\ell}^{(n)}\right]\left(\rho_{**}\left(\lambda+\eta_\ell; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}\right); Z^{(n)}[z]\right)\right\|_{\mathcal{B}(\mathcal{H}^{(n)})}$$

$$\leq (L+1)\frac{\mathcal{C}_{\chi_1}^{L+1}}{\rho_{**}^{L-1}}$$

$$\times \prod_{\ell=1}^{L} \left\|\mathcal{W}_{(n)}^{(r_\ell,s_\ell)}\left[w_{r_\ell+p_\ell,s_\ell+q_\ell}^{(n)}\right]\left(\rho_{**}\left(\lambda+\eta_\ell; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}\right); Z^{(n)}[z]\right)\right\|_{\mathcal{B}(\mathcal{H}^{(n)})}$$

and

$$\left|\left[\tilde{V}_{\underline{r,p,s,q}_L}^{(n+1)}\right]_{\lambda_j}^{(2)}[\lambda; Y^{(R,S)}; z]\right| \tag{4.61}$$

$$\leq \prod_{\ell=0}^{L} \left\|F_\ell(\rho_{**}^{-1}\Lambda_{[\theta_n]} + \lambda)\right\|_{\mathcal{B}(\mathcal{H}^{(n)})}$$

$$\times \sum_{k=1}^{L} \left\{\rho_{**}\left\|\mathcal{W}_{(n)}^{(r_k,s_k)}\left[\partial_{\lambda_j} w_{r_k+p_k,s_k+q_k}^{(n)}\right]\right.\right.$$

$$\left.\left(\rho_{**}\left(\lambda+\eta_k; y_k^{(r_k)}, \tilde{y}_k^{(s_k)}\right); Z^{(n)}[z]\right)\right\|_{\mathcal{B}(\mathcal{H}^{(n)})}$$

$$\left.\times \prod_{\substack{\ell=1,\\ \ell\neq k}}^{L} \left\|\mathcal{W}_{(n)}^{(r_\ell,s_\ell)}\left[w_{r_\ell+p_\ell,s_\ell+q_\ell}^{(n)}\right]\left(\rho_{**}\left(\lambda+\eta_\ell; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}\right); Z^{(n)}[z]\right)\right\|_{\mathcal{B}(\mathcal{H}^{(n)})}\right\}$$

$$\leq \frac{\mathcal{C}_{\chi_1}^{L+1}}{\rho_{**}^{L-2}}$$

$$\times \sum_{k=1}^{L} \left\{\left\|\mathcal{W}_{(n)}^{(r_k,s_k)}\left[\partial_{\lambda_j} w_{r_k+p_k,s_k+q_k}^{(n)}\right]\right.\right.$$

$$\left.\left(\rho_{**}\left(\lambda+\eta_k; y_k^{(r_k)}, \tilde{y}_k^{(s_k)}\right); Z^{(n)}[z]\right)\right\|_{\mathcal{B}(\mathcal{H}^{(n)})}$$

$$\left.\times \prod_{\substack{\ell=1,\\ \ell\neq k}}^{L} \left\|\mathcal{W}_{(n)}^{(r_\ell,s_\ell)}\left[w_{r_\ell+p_\ell,s_\ell+q_\ell}^{(n)}\right]\left(\rho_{**}\left(\lambda+\eta_\ell; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}\right); Z^{(n)}[z]\right)\right\|_{\mathcal{B}(\mathcal{H}^{(n)})}\right\}.$$

Inserting the estimates (4.60, 4.61) into (4.59) and using (4.58) we obtain

$$\sup_{\lambda\in\mathcal{Q}^{(n+1)}} \left|\tilde{V}_{\underline{r,p,s,q}_L}^{(n+1)}[\lambda; Y^{(R,S)}; z]\right|^2 + \sup_{\lambda\in\mathcal{Q}^{(n+1)}} \sum_{j=1,2} \left|\partial_{\lambda_j}\tilde{V}_{\underline{r,p,s,q}_L}^{(n+1)}[\lambda; Y^{(R,S)}; z]\right|^2$$

$$\leq 3(L+1)^2 \frac{\mathcal{C}_{\chi_1}^{2(L+1)}}{\rho_{**}^{2L-2}} \times \prod_{\ell=1}^{L} \Bigg\{ \sup_{\lambda \in \mathcal{Q}^{(n+1)}} \Big\| \mathcal{W}_{(n)}^{(r_\ell,s_\ell)} \left[w_{r_\ell+p_\ell,s_\ell+q_\ell}^{(n)} \right] \left(\rho_{**}\left(\lambda+\eta_\ell; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}\right); Z^{(n)}[z] \right) \Big\|_{\mathcal{B}(\mathcal{H}^{(n)})}^2 + \sup_{\lambda \in \mathcal{Q}^{(n+1)}} \sum_{j=1,2} \Big\| \mathcal{W}_{(n)}^{(r_\ell,s_\ell)} \left[\partial_{\lambda_j} w_{r_\ell+p_\ell,s_\ell+q_\ell}^{(n)} \right] \left(\rho_{**}\left(\lambda+\eta_\ell; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}\right); Z^{(n)}[z] \right) \Big\|_{\mathcal{B}(\mathcal{H}^{(n)})}^2 \Bigg\}.$$

Integration against the measure $m_{\theta_{n+1}}(Y^{(R,S)})^{-3-2\mu}\, dY^{(R,S)}$ yields

$$\begin{aligned}
&\left(\Big\| \tilde{V}_{\underline{r,p,s,q}_L}^{(n+1)}[\,\cdot\,; z] \Big\|_{(n+1)}^{\#} \right)^2 \qquad (4.62) \\
&= \int_{\{\mathcal{M}^{(n+1)}\}^{R+S}} \frac{dY^{(R,S)}}{m_{\theta_{n+1}}(Y^{(R,S)})^{3+2\mu}} \Bigg\{ \sup_{\lambda \in \mathcal{Q}^{(n+1)}} \Big| \tilde{V}_{\underline{r,p,s,q}_L}^{(n+1)}[\lambda; Y^{(R,S)}; z] \Big|^2 + \sup_{\lambda \in \mathcal{Q}^{(n+1)}} \sum_{j=1,2} \Big| \partial_{\lambda_j} \tilde{V}_{\underline{r,p,s,q}_L}^{(n+1)}[\lambda; Y^{(R,S)}; z] \Big|^2 \Bigg\} \\
&\leq 3(L+1)^2 \frac{\mathcal{C}_{\chi_1}^{2(L+1)}}{\rho_{**}^{2L-2}} \prod_{\ell=1}^{L} \Bigg\{ \int_{\{\mathcal{M}^{(n+1)}\}^{r_\ell+s_\ell}} \frac{dY^{(r_\ell,s_\ell)}}{m_{\theta_{n+1}}(Y^{(r_\ell,s_\ell)})^{3+2\mu}} \\
&\quad \times \Bigg\{ \sup_{\lambda \in \mathcal{Q}^{(n+1)}} \Big\| \mathcal{W}_{(n)}^{(r_\ell,s_\ell)} \left[w_{r_\ell+p_\ell,s_\ell+q_\ell}^{(n)} \right] \left(\rho_{**}\left(\lambda+\eta_\ell; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}\right); Z^{(n)}[z] \right) \Big\|_{\mathcal{B}(\mathcal{H}^{(n)})}^2 \\
&\quad + \sup_{\lambda \in \mathcal{Q}^{(n+1)}} \sum_{j=1,2} \Big\| \mathcal{W}_{(n)}^{(r_\ell,s_\ell)} \left[\partial_{\lambda_j} w_{r_\ell+p_\ell,s_\ell+q_\ell}^{(n)} \right] \left(\rho_{**}\left(\lambda+\eta_\ell; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}\right); Z^{(n)}[z] \right) \Big\|_{\mathcal{B}(\mathcal{H}^{(n)})}^2 \Bigg\} \Bigg\} \\
&\leq 3(L+1)^2 \mathcal{C}_{\chi_1}^{2(L+1)} \rho_{**}^{(2+2\mu)(R+S)-2L+2} \prod_{\ell=1}^{L} \Bigg\{ \int_{\{\mathcal{M}^{(n)}\}^{r_\ell+s_\ell}} \frac{dY^{(r_\ell,s_\ell)}}{m_{\theta_n}(Y^{(r_\ell,s_\ell)})^{3+2\mu}} \\
&\quad \times \Bigg\{ \sup_{\lambda \in \mathcal{Q}^{(n)}} \Big\| \mathcal{W}_{(n)}^{(r_\ell,s_\ell)} \left[w_{r_\ell+p_\ell,s_\ell+q_\ell}^{(n)} \right] \left(\lambda; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}; Z^{(n)}[z] \right) \Big\|_{\mathcal{B}(\mathcal{H}^{(n)})}^2
\end{aligned}$$

$$+ \sup_{\lambda \in \mathcal{Q}^{(n)}} \sum_{j=1,2} \Big\| \mathcal{W}_{(n)}^{(r_\ell, s_\ell)} \left[\partial_{\lambda_j} w^{(n)}_{r_\ell+p_\ell, s_\ell+q_\ell} \right]$$

$$\left(\lambda; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}; Z^{(n)}[z] \right) \Big\|^2_{\mathcal{B}(\mathcal{H}^{(n)})} \Bigg\} \Bigg\}.$$

where we performed the transformation $Y^{(r_\ell,s_\ell)} \mapsto \rho_{**}^{-1} Y^{(r_\ell,s_\ell)}$ of integration variables and used that

$$\begin{aligned} m_{\theta_{n+1}}(\rho_{**}^{-1} Y^{(r_\ell,s_\ell)}) &= \rho_{**}^{-r_\ell - s_\ell} m_{\theta_n}(Y^{(r_\ell,s_\ell)}), \\ \rho_{**} \mathcal{M}^{(n+1)} &\subseteq \mathcal{M}^{(n)}, \\ \rho_{**}(\lambda + \eta_\ell) &\in \mathcal{Q}^{(n)} \qquad \forall\, \lambda \in \mathcal{Q}^{(n+1)}. \end{aligned}$$

We apply Proposition 4.1 to see that

$$\begin{aligned} &\int_{\{\mathcal{M}^{(n)}\}^{r_\ell+s_\ell}} \frac{dY^{(r_\ell,s_\ell)}}{m_{\theta_n}(Y^{(r_\ell,s_\ell)})^{3+2\mu}} \\ &\qquad \times \sup_{\lambda \in \mathcal{Q}^{(n)}} \Big\| \mathcal{W}_{(n)}^{(r_\ell,s_\ell)} \left[w^{(n)}_{r_\ell+p_\ell, s_\ell+q_\ell} \right] \left(\lambda; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}; Z^{(n)}[z] \right) \Big\|^2_{\mathcal{B}(\mathcal{H}^{(n)})} \\ &\quad \leq \frac{1}{p_\ell^{p_\ell} q_\ell^{q_\ell}} \\ &\qquad \times \int_{\{\mathcal{M}^{(n)}\}^{r_\ell+s_\ell}} \frac{dY^{(r_\ell,s_\ell)}}{m_{\theta_n}(Y^{(r_\ell,s_\ell)})^{3+2\mu}} \Big\| w^{(n)}_{r_\ell+p_\ell, s_\ell+q_\ell}[\,\cdot\,, y_\ell^{(r_\ell)}, \,\cdot\,, \tilde{y}_\ell^{(s_\ell)}, Z^{(n)}[z]] \Big\|^2_{(n)} \\ &\quad = \frac{1}{p_\ell^{p_\ell} q_\ell^{q_\ell}} \Big\| w^{(n)}_{r_\ell+p_\ell, s_\ell+q_\ell}[Z^{(n)}[z]] \Big\|^2_{(n)} \end{aligned}$$

Inserting this estimate in (4.62) finally gives

$$\begin{aligned} &\rho_{**}^{-1} \Big\| \tilde{V}^{(n+1)}_{\underline{r,p,s,q}_L}[\,\cdot\,; z] \Big\|^{\#}_{(n+1)} \\ &\quad \leq \; 2(L+1) \mathcal{C}_{\chi_1}^{L+1} \rho_{**}^{(1+\mu)(R+S)-L} \prod_{\ell=1}^{L} \frac{\Big\| w^{(n)}_{r_\ell+p_\ell, s_\ell+q_\ell}[Z^{(n)}[z]] \Big\|^{\#}_{(n)}}{\sqrt{p_\ell^{p_\ell} q_\ell^{q_\ell}}}. \end{aligned}$$

The analyticity of $z \mapsto \tilde{V}^{(n+1)}_{\underline{r,p,s,q}_L}[\,\cdot\,; z]$ follows from the analytic properties of the functions $z \mapsto T^{(n)}[\,\cdot\,; z], E^{(n)}[z], w^{(n)}_{R,S}[\,\cdot\,; z]$ and the above estimates.

(ii) The proof is the same as under (i).

(iii) The strategy in proving the second assertion only differs from the proof of (i) due to the fact that the Wick monomials

$$\mathcal{W}_{[\theta]}^{(r_\ell,s_\ell)} \left[w^{(0)}_{r_\ell+p_\ell, s_\ell+q_\ell} \right] \left(\rho \left(\lambda + \eta_\ell^{(\theta_1)} \left(Y^{(R,S)} \right); y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}; z \right) \right)$$

in (4.22) are not bounded operators on $\mathcal{H}^2 = \mathcal{H}_\mathrm{p}^2 \otimes \mathcal{F}(L^2[\Upsilon])$. However, by sandwiching the Wick monomials with the operator $(M_{[\theta]}+1)^{-1/2}$ we obtain a bounded operator

$$\begin{aligned}\widetilde{\mathcal{W}}_\ell \equiv\\ &(M_{[\theta]}+1)^{-1/2}\mathcal{W}_{[\theta]}^{(r_\ell,s_\ell)}\left[w^{(0)}_{r_\ell+p_\ell,s_\ell+q_\ell}\right]\left(\rho\left(\lambda+\eta_\ell^{(\theta_1)}\left(\Upsilon^{(R,S)}\right);y_\ell^{(r_\ell)},\tilde{y}_\ell^{(s_\ell)};z\right)\right)\\ &\times(M_{[\theta]}+1)^{-1/2},\end{aligned}$$

c.f. Lemma A.5 and recall that, by definition (4.15), $w^{(0)}_{R,S} = 0$ holds for $R+S \geq 2$. Introducing the operator-valued functions (we use the abbreviation $\tilde{\lambda}^{(\ell)} := \eta_\ell^{(\theta_1)}(Y^{(R,S)}) + \sum_{j=1}^{s_\ell}\lambda_{\theta_1}(\tilde{u}^{(s_\ell)}_{\ell,j})$ again)

$$\begin{aligned}\hat{F}_0(\lambda) &:= X_{0,1}\left(\lambda_2+\left[\eta_0^{(\theta_1)}\left(\Upsilon^{(R,S)}\right)\right]_2\right),\\ \hat{F}_L(\lambda) &:= X_{0,1}\left(\lambda_2+\tilde{\lambda}_2^{(L)}\right),\\ \hat{F}_\ell(\lambda) &:= \frac{(M_{[\theta]}+1)\overline{X}^2_{0,1}\left(\lambda_2+\tilde{\lambda}_2^{(\ell)}\right)}{T^{(0)}\left[\rho\left(\lambda+\tilde{\lambda}^{(\ell)}\right);\rho z\right]-E^{(0)}[\rho z]}\end{aligned}$$

for $\ell = 1,\ldots,L-1$, we may write

$$\tilde{V}^{(1)}_{\underline{r,p,s,q}_L}[\lambda;Y^{(R,S)};z] = \left\langle \hat{F}_0\left(\rho^{-1}\Lambda_{[\theta]}+\lambda\right)\prod_{\ell=1}^{L}\left\{\widetilde{\mathcal{W}}_\ell\hat{F}_\ell\left(\rho^{-1}\Lambda_{[\theta]}+\lambda\right)\right\}\right\rangle_{\Omega_\mathrm{vac}}.$$

We observe that, for $\ell = 1,\ldots,L-1$,

$$\left\|\hat{F}_\ell(\rho^{-1}\Lambda_{[\theta]}+\lambda)\right\| = \left\|\frac{M_{[\theta]}+1}{\rho\left(L_{0,\theta}+\tilde{\lambda}^{(\ell)}-z\right)}\overline{X}^2_{0,1}\left(\lambda_2+\tilde{\lambda}_2^{(\ell)}\right)\right\| \leq \frac{\mathcal{C}_{\chi_1}}{\rho}$$

with $\mathcal{C}_{\chi_1} \geq 1$ given in (4.6), as one shows with the same arguments as those used in the proof of Lemma 3.5, Equations (3.16) - (3.17). The same kind of estimate holds for the derivatives $\partial_{\lambda_j}F_\ell(\rho^{-1}\Lambda_{[\theta]}+\lambda)$. Proceeding as under (i) and making use of $\partial_{\lambda_j}w^{(0)}_{R,S} = 0$ we arrive at

$$\begin{aligned}&\left(\left\|\tilde{V}^{(1)}_{\underline{r,p,s,q}_L}[\,\cdot\,;z]\right\|^{\#}_{(1)}\right)^2\\ &\leq\ (L+1)^2\mathcal{C}_{\chi_1}^{2(L+1)}\rho^{(2+2\mu)(R+S)-2L+2}\\ &\quad\times\prod_{\ell=1}^{L}\left\{\int_{\Upsilon^{r_\ell+s_\ell}}\frac{dY^{(r_\ell,s_\ell)}}{m_\theta(Y^{(r_\ell,s_\ell)})^{3+2\mu}}\right.\end{aligned}$$

$$\times \sup_{\lambda\in\mathbb{R}^2} \left\| (M_{[\theta]}+1)^{-1/2} \mathcal{W}_{[\theta]}^{(r_\ell,s_\ell)} \left[w^{(0)}_{r_\ell+p_\ell,s_\ell+q_\ell} \right] \left(\lambda; y_\ell^{(r_\ell)}, \tilde{y}_\ell^{(s_\ell)}; \rho z \right) \right.$$

$$\left. \left. \times (M_{[\theta]}+1)^{-1/2} \right\|^2 \right\}$$

$$\leq (L+1)^2 \mathcal{C}_{\chi_1}^{2(L+1)} \rho^{(2+2\mu)(R+S)-2L+2}$$

$$\times \left[g^2 \max_{F\in\left\{F_\theta^{(s)}, F_{\overline{\theta}}^{(\overline{s})*}\right\}} \int_\Upsilon dy \, \left[m_\theta(u)^{-1} + m_\theta(u)^{-1-2\mu} \right] \|F(y)\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)} \right]^L$$

where we used that (for $r_\ell = s_\ell = 0$)

$$\left\| (M_{[\theta]}+1)^{-1/2} \mathcal{W}_{[\theta]}^{(0,0)} \left[w^{(0)}_{p_\ell,q_\ell} \right] (\lambda; \rho z) \, (M_{[\theta]}+1)^{-1/2} \right\|^2$$

$$\leq \max_{F\in\left\{F_\theta^{(s)}, F_{\overline{\theta}}^{(\overline{s})*}\right\}} g^2 \left\| (M_{[\theta]}+1)^{-1/2} a_{\mathrm{gl}}^{\#}(F) \, (M_{[\theta]}+1)^{-1/2} \right\|^2$$

$$\leq g^2 \max_{F\in\left\{F_\theta^{(s)}, F_{\overline{\theta}}^{(\overline{s})*}\right\}} \int_\Upsilon dy \, \frac{\|F(y)\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)}}{m_\theta(u)},$$

refer to Lemma A.4, and (for $p_\ell = q_\ell = 0$)

$$\int_\Upsilon \frac{dy}{m_\theta(u)^{3+2\mu}} \left\| (M_{[\theta]}+1)^{-1/2} \mathcal{W}_{[\theta]}^{(r_\ell,s_\ell)} \left[w^{(0)}_{r_\ell,s_\ell} \right] (\lambda; y; \rho z) \, (M_{[\theta]}+1)^{-1/2} \right\|^2$$

$$\leq \max_{F\in\left\{F_\theta^{(s)}, F_{\overline{\theta}}^{(\overline{s})*}\right\}} \int_\Upsilon \frac{dy}{m_\theta(u)^{3+2\mu}} g^2 m_\theta(u)^2 \|F(y)\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)}$$

$$\leq g^2 \max_{F\in\left\{F_\theta^{(s)}, F_{\overline{\theta}}^{(\overline{s})*}\right\}} \int_\Upsilon dy \, \frac{\|F(y)\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)}}{m_\theta(u)^{1+2\mu}}.$$

The analyticity of $z \mapsto \tilde{V}^{(1)}_{\underline{r,p,s,q}_L}[\,\cdot\,; z]$ is obvious.

∎

Proof of Theorem 4.9.

(i) Recall the definition (4.38) of $w^{(n+1)}_{R,S}$ in terms of the functions $\tilde{V}^{(n+1)}_{\underline{r,p,s,q}_L}$.

Lemma 4.10(i) implies that

$$\begin{aligned}
&\left\| w^{(n+1)}_{R,S}[\,\cdot\,;z] \right\|^{\#}_{(n+1)} \\
&\leq \sum_{L=1}^{\infty} 2\mathcal{C}_{\chi_1}(L+1)\left(\frac{\mathcal{C}_{\chi_1}}{\rho_{**}}\right)^L \left(2\rho_{**}^{1+\mu}\right)^{R+S} \sum_{\substack{r_1+\cdots+r_L=R,\\ s_1+\cdots+s_L=S}} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ r_\ell+p_\ell+s_\ell+q_\ell\geq 1}} \\
&\quad \times \prod_{\ell=1}^{L}\left[\left(\frac{2}{\sqrt{p_\ell}}\right)^{p_\ell}\left(\frac{2}{\sqrt{q_\ell}}\right)^{q_\ell} \left\| w^{(n)}_{r_\ell+p_\ell,s_\ell+q_\ell}[\,\cdot\,;Z^{(n)}[z]] \right\|^{\#}_{(n)}\right]
\end{aligned}$$

where we used that $\binom{j+k}{j} \leq 2^{j+k}$. Summation over $R+S\geq 1$ yields,

$$\begin{aligned}
&\left\| \left(w^{(n+1)}_{R,S}\right)_{R+S\geq 1} \right\|_{(n+1),\xi} \\
&= \sum_{R+S\geq 1} \xi^{-(R+S)} \sup_{z\in B_{1/4}} \left\| w^{(n+1)}_{R,S}[\,\cdot\,;z] \right\|^{\#}_{(n+1)} \\
&\leq 4\mathcal{C}_{\chi_1}\rho_{**}^{1+\mu} \sum_{L=1}^{\infty}(L+1)\left(\frac{\mathcal{C}_{\chi_1}}{\rho_{**}}\right)^L \sum_{R+S\geq 1} \sum_{\substack{r_1+\cdots+r_L=R,\\ s_1+\cdots+s_L=S}} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ r_\ell+p_\ell+s_\ell+q_\ell\geq 1}} \\
&\quad \times \prod_{\ell=1}^{L}\Bigg[\left(\frac{2\xi}{\sqrt{p_\ell}}\right)^{p_\ell}\left(\frac{2\xi}{\sqrt{q_\ell}}\right)^{q_\ell} \xi^{-(r_\ell+p_\ell+s_\ell+q_\ell)} \\
&\qquad \times \sup_{z\in B_{1/4}} \left\| w^{(n)}_{r_\ell+p_\ell,s_\ell+q_\ell}[\,\cdot\,;Z^{(n)}[z]] \right\|^{\#}_{(n)}\Bigg] \\
&\leq 4\mathcal{C}_{\chi_1}\rho_{**}^{1+\mu} \sum_{L=1}^{\infty}(L+1)\left(\frac{\mathcal{C}_{\chi_1}}{\rho_{**}}\right)^L \\
&\quad \times \left[\sum_{r+s\geq 1}\sum_{p=0}^{r}\left(\frac{2\xi}{\sqrt{p}}\right)^p \sum_{q=0}^{s}\left(\frac{2\xi}{\sqrt{q}}\right)^q \xi^{-(r+s)} \sup_{z\in B_{1/4}} \left\| w^{n}_{r,s}[\,\cdot\,;Z^{(n)}[z]] \right\|^{\#}_{(n)}\right]^L \\
&\leq 4\mathcal{C}_{\chi_1}\rho_{**}^{1+\mu} \sum_{L=1}^{\infty}(L+1)\left[\frac{\mathcal{C}_{\chi_1}}{\rho_{**}(1-2\xi)^2}\left\| \left(w^{(n)}_{R,S}\right)_{R+S\geq 1} \right\|_{(n),\xi}\right]^L \\
&\leq 128\mathcal{C}_{\chi_1}^2\rho_{**}^{\mu} \left\| \left(w^{(n)}_{R,S}\right)_{R+S\geq 1} \right\|_{(n),\xi} \\
&\leq \frac{\eta}{2},
\end{aligned}$$

where we used

$$\sum_{L=1}^{\infty}(L+1)x^L = \frac{d}{dx}\sum_{L=0}^{\infty}x^L - 1 = \frac{1}{(1-x)^2} - 1 = x\frac{2-x}{(1-x)^2} \leq 8x,$$

$0 \le x \le \frac{1}{2}$, and

$$\frac{\mathcal{C}_{\chi_1}}{\rho_{**}(1-2\xi)^2} \left\| \left(w_{R,S}^{(n)} \right)_{R+S \ge 1} \right\|_{(n),\xi} \le \frac{4\mathcal{C}_{\chi_1}\eta}{\rho_{**}} \le \frac{1}{2}$$

since $\xi \le 1/4$ and $\eta \le \frac{\rho_{**}}{8\mathcal{C}_{\chi_1}}$ and $128\mathcal{C}_{\chi_1}^2 \rho_{**}^{\mu} = 128\mathcal{C}_{\chi_1}^2 (16\mathcal{C}_{\chi_1})^{-2} \le 1/2$.

Next, we estimate for $T^{(n+1)}[\lambda; z] := w_{0,0}^{(n+1)}[\lambda; z] - w_{0,0}^{(n+1)}[0; z]$,

$$\begin{aligned}
&\sup_{\substack{z \in B_{1/4}, \\ \lambda \in \mathcal{Q}^{(n+1)}}} \left| \nabla_\lambda T^{(n+1)}[\lambda; z] - \begin{pmatrix} 1 \\ i \end{pmatrix} \right| \\
&\le \sup_{\substack{z \in B_{1/4}, \\ \lambda \in \mathcal{Q}^{(n+1)}}} \left| \left(\nabla_\lambda T^{(n)} \right) [\rho_{**}\lambda; Z^{(n)}[z]] - \begin{pmatrix} 1 \\ i \end{pmatrix} \right| \\
&\quad + \sum_{L=2}^{\infty} \sum_{\substack{p_1, q_1, \dots, p_L, q_L: \\ p_\ell + q_\ell \ge 1}} \rho_{**}^{-1} \sup_{\substack{z \in B_{1/4}, \\ \lambda \in \mathcal{Q}^{(n+1)}}} \left| \nabla_\lambda V_{\underline{0,p,0,q}_L}^{(n+1)}[\lambda; z] \right| \\
&\le \sup_{\substack{z \in B_{1/4}, \\ \lambda \in \mathcal{Q}^{(n)}}} \left| \nabla_\lambda T^{(n)}[\lambda; z] - \begin{pmatrix} 1 \\ i \end{pmatrix} \right| \\
&\quad + 2\mathcal{C}_{\chi_1} \sum_{L=2}^{\infty} (L+1) \left[\frac{\mathcal{C}_{\chi_1}}{\rho_{**}} \sum_{p+q \ge 1} \sup_{z \in B_{1/4}} \left\| w_{p,q}^{(n)}[\,\cdot\,; z] \right\|_{(n)}^{\#} \right]^L \\
&\le \sup_{\substack{z \in B_{1/4}, \\ \lambda \in \mathcal{Q}^{(n)}}} \left| \nabla_\lambda T^{(n)}[\lambda; z] - \begin{pmatrix} 1 \\ i \end{pmatrix} \right| \\
&\quad + 2\mathcal{C}_{\chi_1} \sum_{L=2}^{\infty} (L+1) \left[\frac{\mathcal{C}_{\chi_1}\xi}{\rho_{**}} \left\| \left(w_{R,S}^{(n)} \right)_{R+S \ge 1} \right\|_{(n),\xi} \right]^L \\
&\le \sup_{\substack{z \in B_{1/4}, \\ \lambda \in \mathcal{Q}^{(n)}}} \left| \nabla_\lambda T^{(n)}[\lambda; z] - \begin{pmatrix} 1 \\ i \end{pmatrix} \right| \\
&\quad + 24\mathcal{C}_{\chi_1} \left[\frac{\mathcal{C}_{\chi_1}\xi}{\rho_{**}} \left\| \left(w_{R,S}^{(n)} \right)_{R+S \ge 1} \right\|_{(n),\xi} \right]^2 \\
&\le \varepsilon + \frac{\eta}{2},
\end{aligned}$$

where we used

$$\sum_{L=2}^{\infty} (L+1) x^L = \frac{d}{dx} \left[\sum_{L=0}^{\infty} x^L - x - x^2 \right] = x^2 \frac{3-2x}{(1-x)^2} \le 12x^2,$$

for $0 \leq x \leq \frac{1}{2}$, and

$$\frac{\mathcal{C}_{\chi_1}\xi}{\rho_{**}} \left\| \left(w_{R,S}^{(n)} \right)_{R+S\geq 1} \right\|_{(n),\xi} \leq \frac{\mathcal{C}_{\chi_1}\xi}{\rho_{**}} \frac{\rho_{**}}{8\mathcal{C}_{\chi_1}} = \frac{\xi}{8} \leq 1/2$$

and

$$24\frac{\mathcal{C}_{\chi_1}^3\xi^2}{\rho_{**}^2} \left\| \left(w_{R,S}^{(n)} \right)_{R+S\geq 1} \right\|_{(n),\xi}^2 \leq 24\frac{\mathcal{C}_{\chi_1}^3\xi^2}{\rho_{**}^2} \frac{\rho_{**}}{8\mathcal{C}_{\chi_1}}\eta = 3\frac{\mathcal{C}_{\chi_1}^2\xi^2}{\rho_{**}}\eta = \frac{3\eta}{16} \leq \frac{\eta}{2}.$$

Finally, we see that $E^{(n+1)}[z] := -w_{0,0}^{(n+1)}[0; z]$ deviates from z by

$$\begin{aligned} \sup_{z\in B_{1/4}} \left| E^{(n+1)}[z] - z \right| &\leq \sum_{L=2}^{\infty} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L: \\ p_\ell+q_\ell\geq 1}} \rho_{**}^{-1} \sup_{z\in B_{1/4}} \left| V_{\underline{0,p,0,q}_L}^{(n+1)}[0; z] \right| \\ &\leq 24\mathcal{C}_{\chi_1} \left[\frac{\mathcal{C}_{\chi_1}\xi}{\rho_{**}} \left\| \left(w_{R,S}^{(n)} \right)_{R+S\geq 1} \right\|_{(n),\xi} \right]^2 \\ &\leq \frac{\eta}{2}. \end{aligned}$$

(ii) With the same arguments as under (i) and with the help of Lemma 4.10(ii) we obtain

$$\begin{aligned} &\left\| \left(w_{R,S}^{(2)} \right)_{R+S\geq 1} \right\|_{(2),\xi} \\ &\leq 4\mathcal{C}_{\chi_1}\rho_*^{1+\mu} \sum_{L=1}^{\infty} (L+1) \left[\frac{\mathcal{C}_{\chi_1}}{\rho_*(1-2\xi)^2} \left\| \left(\breve{w}_{R,S}^{(1)} \right)_{R+S\geq 0} \right\|_{(1),\xi} \right]^L \\ &\leq 128\mathcal{C}_{\chi_1}^2\rho_*^{\mu} \left\| \left(\breve{w}_{R,S}^{(1)} \right)_{R+S\geq 0} \right\|_{(1),\xi} \\ &< \eta_2, \end{aligned}$$

for $|\beta_{\max} - \beta_{\min}| \ll 1$ and g sufficiently small where we used that

$$\begin{aligned} &\rho_*^{-1} \left\| \left(\breve{w}_{R,S}^{(1)} \right)_{R+S\geq 0} \right\|_{(1),\xi} \\ &= \rho_*^{-1} \left[\sup_{\substack{z\in B_{1/4}, \\ \lambda\in\mathcal{Q}^{(1)}}} \left\| \breve{w}_{0,0}^{(1)}[\lambda; z] \right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} + \left\| \left(w_{R,S}^{(1)} \right)_{R+S\geq 1} \right\|_{(1),\xi} \right] \\ &= \rho_*^{-1}\mathcal{O}\left(g\rho^{\mu} + \frac{g^2}{\rho}(g^{\tilde{\varepsilon}} + |\beta_{\max} - \beta_{\min}|) \right) \end{aligned} \tag{4.63}$$

$$\begin{aligned}
&= \mathcal{O}\left(\frac{\rho^{1+\mu}}{g} + g^{\tilde{\varepsilon}} + |\beta_{\max} - \beta_{\min}|\right) \\
&= \mathcal{O}\left(g^{2(1+\tilde{\varepsilon})(1+\mu)/3-1} + g^{\tilde{\varepsilon}} + |\beta_{\max} - \beta_{\min}|\right) \\
&\leq \frac{1}{2}
\end{aligned}$$

due to Proposition 4.4, Equation (4.48) and since $\mu \geq 1/2$ and $\tilde{\varepsilon} > 0$.

Next, for $T^{(2)}[\lambda; z] := w_{0,0}^{(2)}[\lambda; z] - w_{0,0}^{(2)}[0; z]$, we have with the help of (4.57)

$$\begin{aligned}
&\sup_{\substack{z\in B_{1/4},\\ \lambda\in\mathcal{Q}^{(2)}}} \left|\nabla_\lambda T^{(2)}[\lambda; z] - \begin{pmatrix}1\\ i\end{pmatrix}\right| \\
&\leq \sup_{\substack{z\in B_{1/4},\\ \lambda\in\mathcal{Q}^{(2)}}} \chi_1^2(\lambda_2) \left| \left\langle \Omega_{\mathrm{p}} \,\middle|\, \left| \left(\nabla_\lambda T^{(1)}\right)[\rho_*\lambda; Z^{(1)}[z]] - \begin{pmatrix}1\\ i\end{pmatrix}\right| \Omega_{\mathrm{p}} \right\rangle_{\ker(L_{\mathrm{p}})} \right| \\
&\quad + \rho_*^{-1} \sup_{\substack{z\in B_{1/4},\\ \lambda\in\mathcal{Q}^{(2)}}} \left|\partial_{\lambda_2}\chi_1^2(\lambda_2)\right| \Bigg\{ \\
&\qquad\qquad \left| \left\langle \Omega_{\mathrm{p}} \,\middle|\, \left[T^{(1)}[\rho_*\lambda; Z^{(1)}[z]] - \rho_*(\lambda_1 + i\lambda_2)\right] \Omega_{\mathrm{p}} \right\rangle_{\ker(L_{\mathrm{p}})} \right| \\
&\qquad\qquad + \left| \left\langle E^{(1)}\right\rangle_{\Omega_{\mathrm{p}}}[Z^{(1)}[z]] - Z^{(1)}[z] \right| \Bigg\} \\
&\quad + \sum_{L=2}^{\infty} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ p_\ell+q_\ell\geq 0}} \rho_*^{-1} \sup_{\substack{z\in B_{1/4},\\ \lambda\in\mathcal{Q}^{(2)}}} \left|\nabla_\lambda V^{(2)}_{\underline{0,p,0,q}_L}[\lambda; z]\right| \\
&\leq \sup_{\substack{z\in B_{1/4},\\ \lambda\in\mathcal{Q}^{(1)}}} \left\|\left|\nabla_\lambda T^{(1)}[\lambda; z] - \begin{pmatrix}1\\ i\end{pmatrix}\right|\right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} \\
&\quad + \frac{\mathcal{C}_{\chi_1}}{\rho_*} \sup_{\substack{z\in B_{1/4},\\ \lambda\in\mathcal{Q}^{(1)}}} \Bigg\{ \left\|T^{(1)}[\lambda; z] - (\lambda_1 + i\lambda_2)\right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} \\
&\qquad\qquad + \left\|E^{(1)}[z] - z + \frac{g^2}{\rho}\Lambda_0^{(s)}\right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} \Bigg\} \\
&\quad + 24\mathcal{C}_{\chi_1}\left[\frac{\mathcal{C}_{\chi_1}\xi}{\rho_*}\left\|\left(\breve{w}^{(1)}_{R,S}\right)_{R+S\geq 0}\right\|_{(1),\xi}\right]^2 \\
&= \mathcal{O}\left(\frac{g^2}{\rho^2} + \frac{\rho^{1+\mu}}{g} + g^{\tilde{\varepsilon}} + |\beta_{\max} - \beta_{\min}|\right) \\
&< \varepsilon_2
\end{aligned}$$

for $|\beta_{\max} - \beta_{\min}| \ll 1$ and g sufficiently small, using Proposition 4.4 and (4.25, 4.63).

Finally, we have for $E^{(2)}[z] := -w^{(2)}_{0,0}[0; z]$,

$$\sup_{z \in B_{1/4}} \left| E^{(2)}[z] - z \right| \leq 24 \mathcal{C}_{\chi_1} \left[\frac{\mathcal{C}_{\chi_1} \xi}{\rho_*} \left\| \left(\breve{w}^{(1)}_{R,S} \right)_{R+S \geq 0} \right\|_{(1),\xi} \right]^2 < \eta_2 .$$

■

4.5 Flow under the Renormalization Transformation

The repeated application of the renormalization transformation is possible under the assumptions of Theorem 4.9. The iteration generates a discrete *renormalization group flow* of integral kernels,

$$\underline{w}^{(n+1)} := \begin{cases} \mathcal{R}^{(1)}_{\rho_*}[\underline{w}^{(1)}], & n = 1, \\ \mathcal{R}^{(n)}_{\rho_{**}}[\underline{w}^{(n)}], & n = 2, 3, \dots, \mathcal{N} - 1, \end{cases}$$

associated with the initial value $\underline{w}^{(1)}$ as given in (4.21, 4.23). To the flow

$$\left(\underline{w}^{(n)} \right)_{n=1,\dots,\mathcal{N}} = \left(-E^{(n)}, T^{(n)}, \left(w^{(n)}_{R,S} \right)_{R+S \geq 1} \right)_{n=1,\dots,\mathcal{N}}$$

we assign a flow $(K^{(n)})_{n=1,\dots,\mathcal{N}}$ of families of bounded operators given by

$$B_{1/4} \ni z \mapsto K^{(n)}[z] := \mathcal{W}_{(n)} \left[\underline{w}^{(n)}[\,\cdot\,; z] \right] = T^{(n)}[\Lambda_{[\theta]}; z] - E^{(n)}[z] + W^{(n)}[z] \qquad (4.64)$$

where

$$W^{(n)}[z] := \mathcal{W}_{(n)} \left[\left(w^{(n)}_{R,S}[\,\cdot\,; z] \right)_{R+S \geq 1} \right] .$$

The first addend, $T^{(n)}[\Lambda_{[\theta_n]}; z]$, of the decomposition (4.64) is the dominating part. The gradient of the function $T^{(n)}$ fulfills

$$\sup_{\substack{z \in B_{1/4}, \\ \lambda \in \mathcal{Q}^{(n)}}} \left|\left|\left| \nabla_\lambda T^{(n)}[\lambda; z] - \begin{pmatrix} 1 \\ i \end{pmatrix} \right|\right|\right|_{\mathcal{B}(\mathcal{H}^{(n)}_{<\infty})} \leq \varepsilon_n$$

which in turn implies that

$$\left\| T^{(n)}[\lambda; z] - (\lambda_1 + i\lambda_2) \right\|_{\mathcal{B}(\mathcal{H}^{(n)}_{<\infty})} \leq |\lambda| \varepsilon_n \leq [1 + \cot(\delta')] \varepsilon_n \qquad (4.65)$$

for all $\lambda = (\lambda_1, \lambda_2) \in \mathcal{Q}^{(n)}$ and $z \in B_{1/4}$. Thus, the operator $T^{(n)}[\Lambda_{[\theta_n]}; z]$ is given, up to an error of order $(1+\cot(\delta'))\varepsilon_n$, by the free Liouville operator $\cos(\delta')L_{\mathrm{res}}+iM_{[\theta_n]} = L_{0,\theta_n} \upharpoonright_{\mathcal{H}^{(n)}}$.

The second part in the decomposition (4.64) is simply a complex number, in the case $n = 2, 3, \ldots, \mathcal{N}$, or a matrix, in the case $n = 1$, where the assignment $B_{1/4} \ni z \mapsto E^{(n)}[z]$ is analytic. This part describes a shift of the spectrum of $K^{(n)}[z]$ in the complex plane w.r.t. the spectrum of the free Liouville operator L_{0,θ_n}. For $E^{(n)}[z]$ we have the bound

$$\sup_{z\in B_{1/4}} \left\|E^{(n)}[z] - z\right\|_{\mathcal{B}(\mathcal{H}^{(n)}_{<\infty})} \leq \eta_n. \tag{4.66}$$

The last part of the decomposition (4.64) is a small perturbation of the spectrum whose concrete form is not of particular interest in this section and in what follows. We only need that

$$\sup_{z\in B_{1/4}} \left\|W^{(n)}[z]\right\|_{\mathcal{B}(\mathcal{H}^{(n)})} \leq \eta_n. \tag{4.67}$$

The bounds ε_n and η_n are due to Proposition 4.4 and Theorem 4.9 and are explicitly given by

$$\begin{aligned}
\eta_1 &:= 4\rho_* \frac{\|\Gamma_{\mathrm{eq}}\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))}}{\gamma_{\mathrm{eq}}}, & \varepsilon_1 &:= \mathcal{O}\left(\frac{g^2}{\rho^2}\right), \\
\eta_2 &:= \frac{\rho_{**}}{32\mathcal{C}_{\chi_1}}, & \varepsilon_2 &:= \frac{\rho_{**}}{32\mathcal{C}_{\chi_1}}, \\
\eta_n &:= \frac{\eta_2}{2^{n-2}}, \qquad n \geq 3, & \varepsilon_n &:= \varepsilon_2 + \eta_2 \sum_{k=1}^{n-2} 2^{-k}, \qquad n \geq 3.
\end{aligned} \tag{4.68}$$

We remark that $\eta_1 \geq 4\rho_* \gg g\rho^\mu$ (since γ_{eq} is smaller than the gap between the lowest and second smallest eigenvalue of Γ_{eq}) and

$$\begin{aligned}
&\frac{g^2}{\rho}\left\|\Lambda_0^{(s)}\right\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} + \mathcal{O}\left(\frac{g^{2+\tilde\varepsilon}}{\rho} + g\rho^\mu\right) \\
&= 2\rho_* \frac{\|\Gamma_{\mathrm{eq}}\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} + \mathcal{O}\left(|\beta_{\max} - \beta_{\min}|\right)}{\gamma_{\mathrm{eq}}} + \mathcal{O}\left(\frac{g^{2+\tilde\varepsilon}}{\rho}\right) \\
&\leq \eta_1
\end{aligned}$$

for $|\beta_{\max} - \beta_{\min}| \ll 1$ and g sufficiently small.

We stress that the operator family $K^{(\mathcal{N})}$ for $n = \mathcal{N}$ takes the form

$$K^{(\mathcal{N})}[z] = -E^{(\mathcal{N})}[z]$$

since $\mathcal{H}^{(\mathcal{N})} = \ker(N_{\mathrm{res}}) = \mathbb{C}\Omega_0$ and $T^{(n)}[\Lambda_{[\theta_\mathcal{N}]}; z] \restriction_{\ker(N_{\mathrm{res}})} = T^{(n)}[0; z] \restriction_{\ker(N_{\mathrm{res}})} = 0$ and $\langle \Omega_0 \,|\, W^{(n)}[z]\Omega_0 \rangle = 0$.

The spectra of the families $z \mapsto K^{(n)}[z]$ are linked through

$$\begin{aligned} 0 \in \mathrm{spec}\left(K^{(n+k)}[z]\right) &\iff 0 \in \mathrm{spec}\left(K^{(n)}\left[Z^{(n)} \circ \cdots \circ Z^{(n+k-1)}[z]\right]\right), \\ 0 \in \mathrm{spec}_{\mathrm{pp}}\left(K^{(n+k)}[z]\right) &\iff 0 \in \mathrm{spec}_{\mathrm{pp}}\left(K^{(n)}\left[Z^{(n)} \circ \cdots \circ Z^{(n+k-1)}[z]\right]\right) \end{aligned} \tag{4.69}$$

for all $z \in B_{1/4}$, see (4.42, 4.56) where the functions $Z^{(n)}$ are connected to $E^{(n)}$ by (4.32). For completion we recall the relation (4.18) and the definition $Z^{(0)}[z] = \rho z$ to see that

$$\begin{aligned} Z^{(0)} \circ \cdots \circ Z^{(n-1)}[z] \in \mathrm{spec}\left(K_\theta^{(s)}\right) &\iff 0 \in \mathrm{spec}\left(K^{(n)}[z]\right), \\ Z^{(0)} \circ \cdots \circ Z^{(n-1)}[z] \in \mathrm{spec}_{\mathrm{pp}}\left(K_\theta^{(s)}\right) &\iff 0 \in \mathrm{spec}_{\mathrm{pp}}\left(K^{(n)}[z]\right). \end{aligned} \tag{4.70}$$

Since $Z^{(0)} \circ \cdots \circ Z^{(n-1)}\left[B_{1/4}\right]$ is comparable with a ball of radius $\rho_{[n]}/4$ the incorporation of more iteration steps in the renormalization procedure allows the study of the spectrum of $K_\theta^{(s)}$ on smaller and smaller scales. The process of extracting spectral information from higher iteration steps is worked out in the next chapter.

5 Recursive Localization of the Spectrum of $K_\theta^{(s)}$ on Decreasing Scales

The renormalization group flow of operator families $(z \mapsto K^{(n)}[z])_{n=1,\dots,\mathcal{N}}$ introduced in (4.64), Section 4.5, and the associated flow of spectral information given by the relations (4.69, 4.70) provide a tool to zoom into any arbitrary small spectral neighborhood of zero of the operator $K_\theta^{(s)}$. The zooming procedure works as follows. Each renormalization iteration step n the operator $K^{(n)}[z]$ decomposes due to (4.64) in the same way into a free part $T^{(n)}[\Lambda_{[\theta_n]}; z]$ (which is in leading order the spectrally deformed, free Liouville operator L_{0,θ_n} whose spectrum is confined to a cone), a spectral shift $E^{(n)}[z]$ (only deviating slightly from the spectral parameter z itself), and a small perturbation $W^{(n)}[z]$. Further, in each step all these terms are controlled by bounds (4.65, 4.66, 4.67) of the same order. Therefore, we can locate the spectrum of $K^{(n)}[z]$ in a shifted, smeared out cone. However, the error terms do not allow predictions about the spectrum within the band of "smearing". In particular, the closest neighborhood of the tip of the cone is not accessible with this rough analysis of the spectrum. The isospectral link (4.69) between the iteration steps implies that the spectrum of $K^{(n)}[z]$ around the tip of the cone on a scale $\rho_{**}^k/4$ can be regained from the spectrum of $K^{(n+k)}[z]$ by blowing up the $\rho_{**}^k/4$-neighborhood to the full circle of radius $1/4$ using the function $(Z^{(n)} \circ \cdots \circ Z^{(n+k-1)})^{-1}$. Thus, after having magnified the spectrum on the scale $\rho_{**}^k/4$ it looks the same as the spectrum on the scale $1/4$. Figure 5.1 illustrates the magnifying process.

The magnifying procedure can now be used to assemble a finer picture of the spectrum of $K^{(n)}[z]$ by piling up the spectrum on scales $\rho_{**}^k/4$ obtained by shrinking the smeared out cone including the spectrum of $K^{(n+k)}[z]$. This process is done recursively. As a result we obtain that the error bands to the smeared out cone in a $\rho_{**}^k/4$-neighborhood around the tip can be reduced by a factor ρ_{**}^k which significantly improves the spectral picture close to the tip, we refer to the Figure 5.2.

In the very last step of renormalization the family of operators $z \mapsto K^{(\mathcal{N})}[z] =$

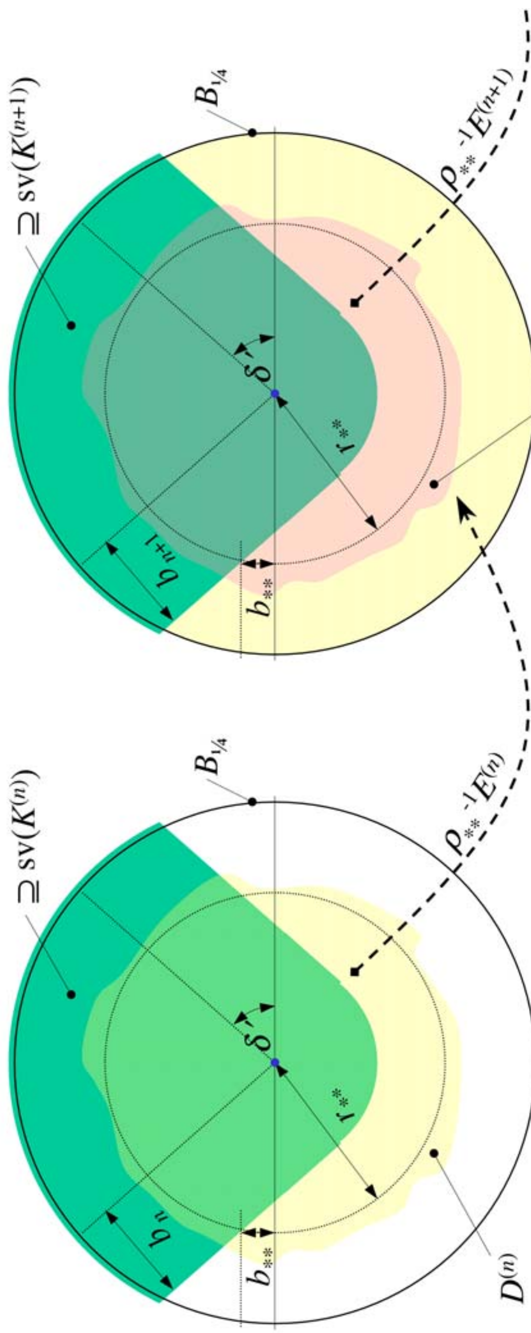

Figure 5.1: The spectral information of $K^{(n)}$ inside the yellowish shaded area $D^{(n)}$ is encoded as the spectral information of $K^{(n+1)}$ in the yellowish ball $B_{1/4}$. The area $D^{(n)}$ is blown up through the function ${Z^{(n)}}^{-1}$. The reddish shaded area $D^{(n+1)}$ is subject to further analysis employing the spectral link with the spectrum of $K^{(n+2)}$.

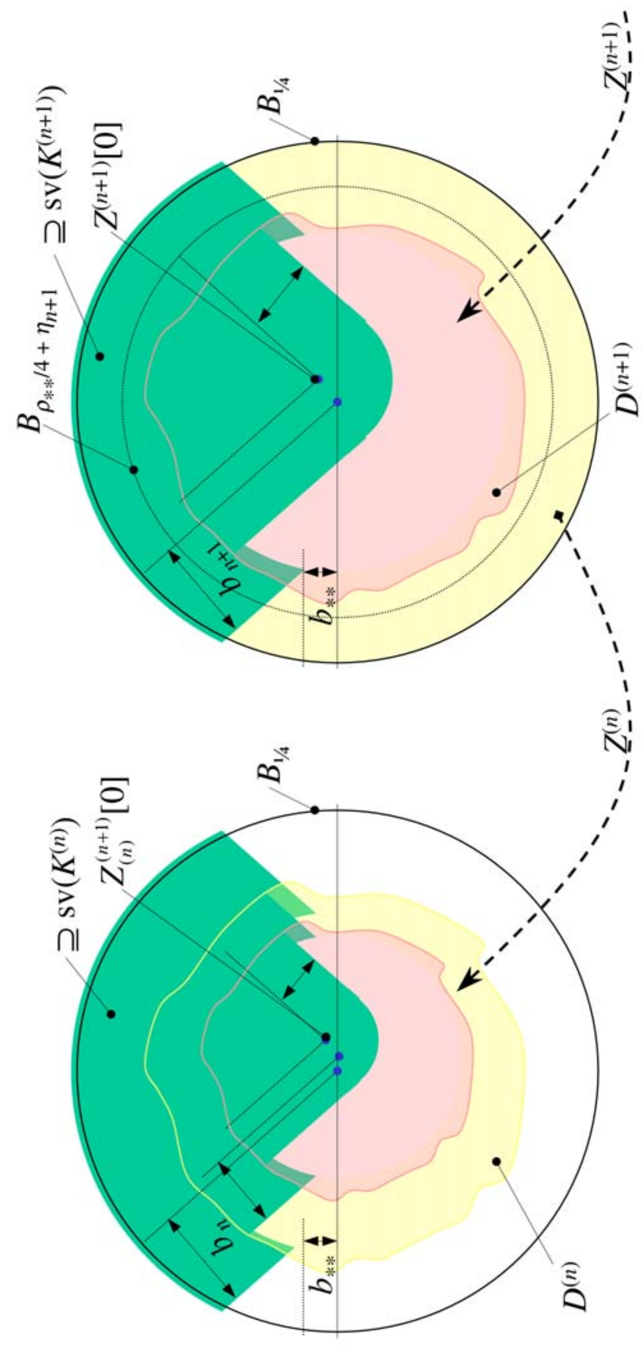

Figure 5.2: The refined information about the spectrum of $K^{(n+1)}$ inside the yellowish ball $B_{1/4}$ is transferred via the function $Z^{(n)}$ to the yellow area $D^{(n)}$ which allows a refinement of the localization of the spectrum of $K^{(n)}$.

$-E^{(\mathcal{N})}[z]$ reduces to a scalar function and therefore the spectrum consists of a single point inside the ball $B_{1/4}$. This isolation of the spectral point survives the reassembling of the spectrum and finally leads to an isolated eigenvalue of the operator $K_\theta^{(s)}$.

In this chapter we make the same assumptions on the parameters s, $\theta = (i\delta', i\tau')$, ρ, ρ_*, ρ_{**} as in the previous ones, recall (3.1, 3.2, 3.4, 4.3, 4.5, 4.43).

5.1 Spectrum of the Operator Families $(K^{(n)})_n$ and Isospectral Link

The information about the operator family $K^{(n)}$ collected in Section 4.5 allows us to state a first – though not very detailed – result about the spectral properties of $K^{(n)}$.

Lemma 5.1 *Introduce the notation*

$$b_n := 2\eta_n + \frac{2}{\sin(\delta')}\varepsilon_n, \qquad n = 1, \ldots, \mathcal{N} - 1,$$

and define the cone

$$A_{\delta'} := \{\zeta \in \mathbb{C} \mid \mathrm{Im}(\zeta) \geq \tan(\delta')|\,\mathrm{Re}(\zeta)|\,\}. \tag{5.1}$$

For $n = 1, \ldots, \mathcal{N} - 1$, we define the sets

$$C^{(n)} := \left\{ z \in B_{1/4} \,\middle|\, \mathrm{dist}(z; A_{\delta'}) \leq b_n \right\}.$$

The spectrum of $K^{(n)}[z]$, $n = 1, \ldots, \mathcal{N} - 1$, inside the ball $B_{1/4}$ can be located as follows,

$$\mathrm{spec}(K^{(n)}[z] + z) \cap B_{1/4} \subseteq C^{(n)} \tag{5.2}$$

and

$$\mathrm{spec}(K^{(\mathcal{N})}[z]) = \left\{-E^{(\mathcal{N})}[z]\right\}. \tag{5.3}$$

Proof. The assertion is obvious for the case $n = \mathcal{N}$. Let $n = 1, \ldots, \mathcal{N} - 1$ and write

$$K^{(n)}[z] + z = L_{0,\theta_n} \restriction_{\mathcal{H}^{(n)}} + R^{(n)}[z],$$

where the remainder term $R^{(n)}[z] = T^{(n)}[\Lambda_{[\theta_n]}; z] - \big(\cos(\delta') L_{\mathrm{res}} + i M_{[\theta_n]}\big) + z - E^{(n)}[z] + W^{(n)}[z]$ can be estimated as

$$\sup_{z \in B_{1/4}} \left\| R^{(n)}[z] \right\| \leq 2\eta_n + [1 + \cot(\delta')]\varepsilon_n \leq b_n$$

because of (4.65, 4.66, 4.67). Since we know that $L_{0,\theta_n} \restriction_{\mathcal{H}^{(n)}}$ is a normal bounded operator on $\mathcal{H}^{(n)}$ and further

$$\operatorname{spec}(L_{0,\theta_n} \restriction_{\mathcal{H}^{(n)}}) \subseteq A_{\delta'}$$

we conclude through application of the following Lemma 5.2 that (5.2) holds. ■

Lemma 5.2 *Let H be a normal bounded operator on a Hilbert space and let I be a bounded operator on the same Hilbert space. The spectrum of the sum $(H+I)$ can be located as*

$$\operatorname{spec}(H+I) \subseteq \{z \in \mathbb{C} \mid \operatorname{dist}(z; \operatorname{spec}(H)) \leq \|I\|\}.$$

Proof. Let $z \in \mathbb{C}$ with $\operatorname{dist}(z; \operatorname{spec}(H)) > \|I\|$. In particular $z \notin \operatorname{spec}(H)$ and therefore $(H-z)$ has a bounded inverse $(H-z)^{-1}$. Since H is normal so is $(H-z)^{-1}$ and we have by functional calculus a norm estimate $\|(H-z)^{-1}\| \leq [\operatorname{dist}(z; \operatorname{spec}(H))]^{-1}$. Thus the operator $I(H-z)^{-1}$ has by assumption a norm strictly smaller than one which implies the convergence of the Neumann series

$$(H-z)^{-1} \sum_{n=0}^{\infty} \left[-I(H-z)^{-1}\right]^n = (H-z)^{-1} \left[\mathbb{1} + I(H-z)^{-1}\right]^{-1} = (H+I-z)^{-1}$$

and therefore $z \notin \operatorname{spec}(H+I)$. ■

The spectrum of each operator $K^{(n)}[z]$ is not of primary interest for our analysis. The notion of *singular values* of an operator family $D \ni z \mapsto F[z]$ on a domain $D \subseteq \mathbb{C}$ is more suitable for our purposes. We define the set $\operatorname{sv}(F)$ of singular values in the following way,

$$\operatorname{sv}(F) := \{z \in D \mid \operatorname{spec}(F[z]) \ni 0\}. \tag{5.4}$$

For notational convenience we will henceforth refer to the set of singular values as the spectrum of the family of operators. Correspondingly, the point spectrum $\operatorname{sv}_{\mathrm{pp}}$ of the operator family F is defined as

$$\operatorname{sv}_{\mathrm{pp}}(F) := \{z \in D \mid \operatorname{spec}_{\mathrm{pp}}(F[z]) \ni 0\}. \tag{5.5}$$

Note that the set of singular values of a family of the type $\mathbb{C} \ni z \mapsto F[z] = F - z$ coincides with the spectrum of F, i.e., $\operatorname{sv}(F[\,\cdot\,]) = \operatorname{spec}(F)$ and $\operatorname{sv}_{\mathrm{pp}}(F[\,\cdot\,]) = \operatorname{spec}_{\mathrm{pp}}(F)$. Lemma 5.1 has its analogue in the following corollary.

Corollary 5.3 *For $n = 1, \ldots, \mathcal{N}-1$ we have the relation*

$$\operatorname{sv}(K^{(n)}) \subseteq C^{(n)} \tag{5.6}$$

and

$$\operatorname{sv}(K^{(\mathcal{N})}) = \operatorname{sv}_{\mathrm{pp}}(K^{(\mathcal{N})}) = \left\{z \in B_{1/4} \,\middle|\, E^{(\mathcal{N})}[z] = 0\right\}. \tag{5.7}$$

The aim of the subsequent consideration is to connect the spectral properties of a family $K^{(k)}$ with the properties of its successors $K^{(n)}$, $n \geq k$, in order to pull back the information to the spectrum of $K^{(k)}$. That way, we successively refine the spectral information in a neighborhood of zero on smaller and smaller scales.

We recall that each family $K^{(n)}$ of the form (4.64) comes with a biholomorphic function

$$Z^{(n)} : B_{1/4} \to D^{(n)},$$

defined in (4.32), associated with the analytic function $z \mapsto E^{(n)}[z]$ which maps the ball $B_{1/4}$ of radius 1/4 onto the set

$$D^{(n)} := D[\underline{w}^{(n)}]$$

introduced in (4.28, 4.29). Since

$$\sup_{z\in B_{1/4}} \left|\partial_z Z^{(n)}[z] - \tilde{\rho}\right| \leq 12\tilde{\rho}\eta_n, \tag{5.8}$$

by (4.33), Lemma 4.5(iii), the function $Z^{(n)}$ is – up to small corrections – a rescaling function which shrinks the ball $B_{1/4}$ by a factor $\tilde{\rho} := \rho_*$, for $n = 1$, and $\tilde{\rho} := \rho_{**}$, for $n = 2, 3, \ldots, \mathcal{N}$, down to the domain $D^{(n)}$ which is almost a ball of radius $\tilde{\rho}/4$, namely

$$B_{\tilde{\rho}/4-\eta_n} \subseteq D^{(n)} \subseteq B_{\tilde{\rho}/4+\eta_n},$$

by Lemma 4.5(i). The function $Z^{(n)}$ relates the spectra of $K^{(n)}$ and $K^{(n+1)}$ in the following sense,

$$\begin{aligned} \mathrm{sv}\left(K^{(n+k)}\right) &= \mathrm{sv}\left(K^{(n)} \circ Z^{(n)} \circ \cdots \circ Z^{(n+k-1)}\right), \\ \mathrm{sv}_{\mathrm{pp}}\left(K^{(n+k)}\right) &= \mathrm{sv}_{\mathrm{pp}}\left(K^{(n)} \circ Z^{(n)} \circ \cdots \circ Z^{(n+k-1)}\right), \end{aligned} \tag{5.9}$$

due to (4.69). Therefore, the analysis of the spectrum of $K^{(n)}$ in a $\tilde{\rho}/4$-neighborhood of zero can be replaced by the analysis of the spectrum of $K^{(n+1)}$ in a ball of radius 1/4. Since we can locate the spectrum of each family $K^{(n)}$ on a scale 1/4 we can recursively locate the spectrum of the initial family $K_\theta^{(s)}$ on arbitrary scales $\rho_{[n]}$ by considering the subsequent n families.

5.2 Reassembling the Spectrum

We perform the recursive localization. First we observe that the last application of the renormalization procedure produces an operator family on a one dimensional space and that therefore the spectrum is completely understood,

$$\mathrm{sv}(K^{(\mathcal{N})}) = \mathrm{sv}_{\mathrm{pp}}(K^{(\mathcal{N})}) = \left\{Z^{(\mathcal{N})}(0)\right\}. \tag{5.10}$$

To describe the pullback of information from a family to its predecessor family we introduce some notation. Set

$$\begin{aligned} b_* &:= 2\eta_1 + \frac{2}{\sin(\delta')}(\varepsilon_1 + \eta_1), & b_{**} &:= 2\eta_2 + \frac{2}{\sin(\delta')}(\varepsilon_2 + \eta_2), \\ r_* &:= \frac{2b_*}{\sin(\delta')}, & r_{**} &:= \frac{2b_{**}}{\sin(\delta')}. \end{aligned}$$

Note that $b_1 \leq b_*$ and $b_n \leq b_{**}$ for $n \geq 2$. A simple geometric argument illustrated in Figure 5.3 shows that

$$\begin{aligned} &\mathrm{Im}(z) \geq b_* \qquad \text{for all } z \in C^{(1)} \setminus B_{r_*}, \\ &\mathrm{Im}(z) \geq b_{**} \qquad \text{for all } z \in C^{(n)} \setminus B_{r_{**}}, \qquad n = 2, 3, \ldots, \mathcal{N} - 1 \end{aligned} \tag{5.11}$$

Since $\delta' \in [\frac{\pi}{8}, \frac{\pi}{4}]$ and

$$144\varepsilon_2 + 196\eta_2 < \rho_{**}$$

because of (4.6, 4.68) we have

$$\begin{aligned} B_{r_*} &\subseteq B_{\rho_*/4-\eta_1} \subseteq D^{(1)}, \\ B_{r_{**}} &\subseteq B_{\rho_{**}/4-\eta_n} \subseteq D^{(n)}, \end{aligned} \tag{5.12}$$

thus, B_{r_*}, $B_{r_{**}}$ are included in the image of $B_{1/4}$ under $Z^{(1)}$, $Z^{(n)}$, resp. We are prepared to define recursively subsets $\Sigma^{(n)}$ of $C^{(n)}$ by

$$\begin{aligned} \Sigma^{(\mathcal{N})} &:= \left\{ Z^{(\mathcal{N})}(0) \right\}, \\ \Sigma^{(n)} &:= \left(C^{(n)} \setminus B_{r_{**}} \right) \cup Z^{(n)} \left[\Sigma^{(n+1)} \right], \quad n = 2, \ldots, \mathcal{N} - 1, \\ \Sigma^{(1)} &:= \left(C^{(1)} \setminus B_{r_*} \right) \cup Z^{(1)} \left[\Sigma^{(2)} \right]. \end{aligned}$$

The isospectral link (5.9) between the families $K^{(n)}$ and the localization (5.6, 5.7, 5.10, 5.12) imply that

$$\mathrm{sv}(K^{(n)}) \subseteq \Sigma^{(n)}. \tag{5.13}$$

For convenience we introduce an abbreviation for the composition of the functions $Z^{(k)}, \ldots, Z^{(n)}$ for $k \leq n$,

$$Z^{(n)}_{(k)} := Z^{(k)} \circ \cdots \circ Z^{(n)} : B_{1/4} \to D^{(k)},$$

and use this notation to introduce the set

$$\dot{\Sigma}^{(n)} := \Sigma^{(n)} \setminus \left\{ Z^{(\mathcal{N})}_{(n)}(0) \right\},$$

which includes all spectral points of $K^{(n)}$ except the one originating from the eigenvalue of $K^{(\mathcal{N})}$.

The goal of the further considerations is to show that $Z^{(\mathcal{N})}_{(n)}(0)$ is the point with lowest imaginary part among all points in $\Sigma^{(n)}$ and that it is uniformly (on the scale $\rho_{**}^{\mathcal{N}-k}$) separated from $\dot{\Sigma}^{(n)}$. Before we state the corresponding proposition we first provide some preparatory lemmata.

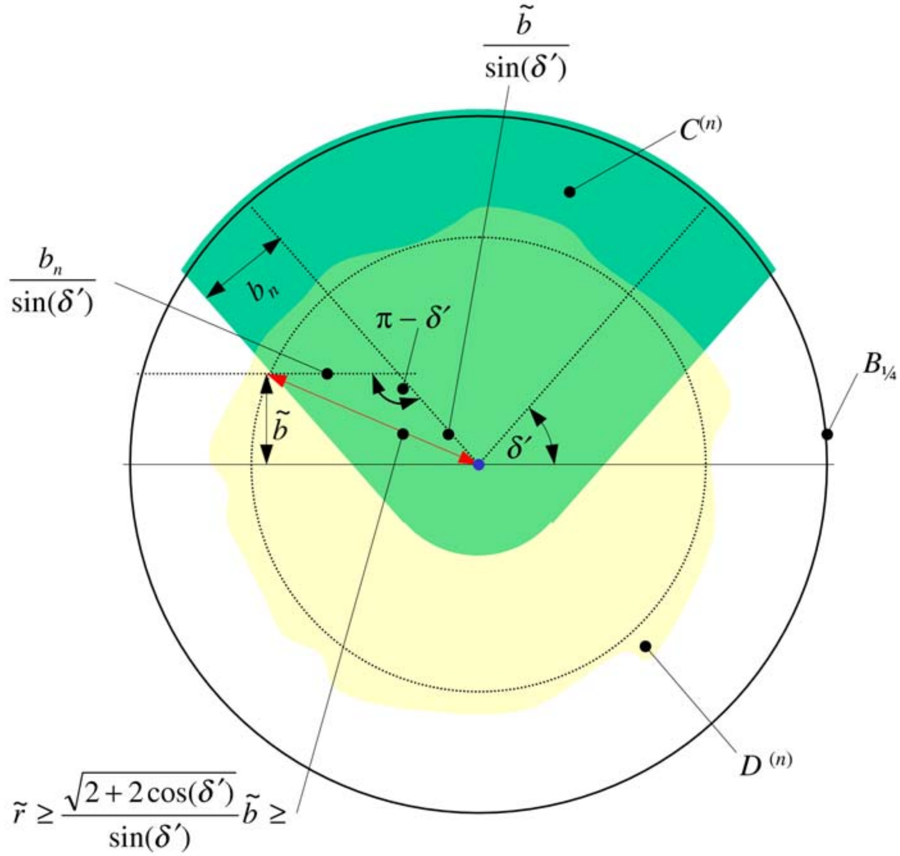

Figure 5.3: Relations between the parameters $\tilde r \in \{r_*, r_{**}\}$, $\tilde b \in \{b_*, b_{**}\}$, b_n, and δ'.

Lemma 5.4 *For $n = 2, \ldots, \mathcal{N}$ holds*

$$\begin{aligned} \left|Z_{(n)}^{(\mathcal{N})}(0)\right| &\leq 2\eta_n, \\ \left|Z_{(1)}^{(\mathcal{N})}(0)\right| &\leq \eta_1 + 4\rho_*\eta_2 \leq 2\eta_1. \end{aligned} \tag{5.14}$$

Proof. We first remark that for $\zeta = Z^{(k)}(0)$ the following estimate holds,

$$|\zeta| = \left|\zeta - E^{(k)}[\zeta]\right| \leq \eta_k,$$

and that (5.8) implies that

$$\left|Z^{(k)}(z) - Z^{(k)}(\zeta) - \tilde\rho(z-\zeta)\right| \leq 12\tilde\rho\eta_k|z-\zeta|, \tag{5.15}$$

which was already stated in Corollary 4.6, for $\tilde{\rho} := \rho_*$, if $n = 1$, and $\tilde{\rho} := \rho_*$ for $n \geq 2$. Define $\zeta_k := Z^{(\mathcal{N})}_{(\mathcal{N}-k)}(0)$ and note that

$$\begin{aligned}
|\zeta_0| &= \left|Z^{(\mathcal{N})}(0)\right| \leq \eta_{\mathcal{N}}, \\
|\zeta_{k+1}| &= \left|Z^{(\mathcal{N}-k-1)}(\zeta_k)\right| \\
&\leq \left|Z^{(\mathcal{N}-k-1)}(\zeta_k) - Z^{(\mathcal{N}-k-1)}(0) - \rho_{**}\zeta_k\right| + \left|Z^{(\mathcal{N}-k-1)}(0)\right| + \rho_{**}|\zeta_k| \\
&\leq \rho_{**}(12\eta_{\mathcal{N}-k-1} + 1)|\zeta_k| + \eta_{\mathcal{N}-k-1} \\
&\leq 2\rho_{**}|\zeta_k| + \eta_{\mathcal{N}-k-1}, \qquad\qquad k = 0, \dots, \mathcal{N} - 3, \\
|\zeta_{\mathcal{N}-1}| &= \left|Z^{(1)}(\zeta_{\mathcal{N}-2})\right| \\
&\leq \left|Z^{(1)}(\zeta_{\mathcal{N}-2}) - Z^{(1)}(0) - \rho_*\zeta_{\mathcal{N}-2}\right| + \left|Z^{(1)}(0)\right| + \rho_*|\zeta_{\mathcal{N}-2}| \\
&\leq \rho_*(12\eta_1 + 1)|\zeta_{\mathcal{N}-2}| + \eta_1 \\
&\leq 2\rho_*|\zeta_{\mathcal{N}-2}| + \eta_1,
\end{aligned}$$

where we used that $12\eta_k < 1$. This implies the estimate

$$|\zeta_k| \leq \sum_{j=0}^{k} (2\rho_{**})^j \eta_{\mathcal{N}-k+j} = \eta_{\mathcal{N}-k} \sum_{j=0}^{k} \rho_{**}^j < \frac{\eta_{\mathcal{N}-k}}{1 - \rho_{**}} < 2\eta_{\mathcal{N}-k},$$

for $k \leq \mathcal{N} - 2$, since $\rho_{**} < 1/2$, and

$$|\zeta_{\mathcal{N}-1}| \leq 2\rho_*|\zeta_{\mathcal{N}-2}| + \eta_1 \leq \eta_1 + 4\rho_*\eta_2 \leq 2\eta_1,$$

due to the definition (4.68) of η_1 and $\eta_2 < 1$. The assertion follows by choosing $k = \mathcal{N} - n$. ∎

A direct consequence of Lemma 5.4 and (5.11) is the following

Corollary 5.5 *For $n = 2, \dots, \mathcal{N} - 1$ holds*

$$\operatorname{Im}\left(z - Z^{(\mathcal{N})}_{(n)}(0)\right) \geq b_{**} - 2\eta_n \geq \frac{2}{\sin(\delta')}(\varepsilon_2 + \eta_2) \qquad \text{for all } z \in C^{(n)} \setminus B_{r_{**}}$$

and

$$\operatorname{Im}\left(z - Z^{(\mathcal{N})}_{(1)}(0)\right) \geq b_* - 2\eta_1 \geq \frac{2}{\sin(\delta')}(\varepsilon_1 + \eta_1) \qquad \text{for all } z \in C^{(1)} \setminus B_{r_*}.$$

The next Lemma describes how two points move w.r.t. each other under the iterative application of functions $Z^{(n)}$.

Lemma 5.6 *Let $z, \zeta \in B_{1/4}$ and $1 \leq k \leq n \leq \mathcal{N}$. For $k \geq 2$ holds*

$$\left|Z^{(n)}_{(k)}(z) - Z^{(n)}_{(k)}(\zeta) - \rho_{**}^{n-k+1}(z - \zeta)\right| \leq 36\rho_{**}^{n-k+1}\eta_k|z - \zeta| \tag{5.16}$$

and
$$\left| Z_{(1)}^{(n)}(z) - Z_{(1)}^{(n)}(\zeta) - \rho_* \rho_{**}^{n-1}(z-\zeta) \right| \le 36 \rho_* \rho_{**}^{n-1} \eta_1 |z-\zeta| \tag{5.17}$$
and
$$\left| Z_{(0)}^{(n)}(z) - Z_{(0)}^{(n)}(\zeta) - \rho_{[n+1]}(z-\zeta) \right| \le 36 \rho_{[n+1]} \eta_1 |z-\zeta| \tag{5.18}$$
where $Z^{(0)}[z] = \rho z$ is the function defined in (4.17) and the scale $\rho_{[n]}$ was introduced in (4.1).

Proof. The case $n = 1$ in (5.17) follows directly from Corollary 4.6. Henceforth, we assume $n \ge 2$. Introduce $j := n - k$ and denote

$$\begin{aligned} r_j &:= Z_{(n-j)}^{(n)}(z) - Z_{(n-j)}^{(n)}(\zeta) - \rho_{**}^{j+1}(z-\zeta), \qquad j \le n-2, \\ r_{n-1} &:= Z_{(1)}^{(n)}(z) - Z_{(1)}^{(n)}(\zeta) - \rho_* \rho_{**}^{j}(z-\zeta). \end{aligned}$$

Corollary 4.6 implies that

$$|r_0| = \left| Z^{(n)}(z) - Z^{(n)}(\zeta) - \rho_{**}(z-\zeta) \right| \le 12 \rho_{**} \eta_n |z-\zeta|.$$

We prove inductively that (5.16, 5.17) hold. Set $\tilde\rho := \rho_*$ if $j = n-1$ and $\tilde\rho := \rho_{**}$ if $j \le n-2$. Applying Corollary 4.6 again we find

$$\begin{aligned} |r_j| &= \left| Z^{(n-j)}\left(Z_{(n-j+1)}^{(n)}(z) \right) - Z^{(n-j)}\left(Z_{(n-j+1)}^{(n)}(\zeta) \right) - \tilde\rho \rho_{**}^{j}(z-\zeta) \right| \\ &\le \Big| Z^{(n-j)}\left(Z_{(n-j+1)}^{(n)}(z) \right) - Z^{(n-j)}\left(Z_{(n-j+1)}^{(n)}(\zeta) \right) \\ &\qquad\qquad - \tilde\rho \left(Z_{(n-j+1)}^{(n)}(z) - Z_{(n-j+1)}^{(n)}(\zeta) \right) \Big| \\ &\quad + \tilde\rho \left| Z_{(n-j+1)}^{(n)}(z) - Z_{(n-j+1)}^{(n)}(\zeta) - \rho_{**}^{j}(z-\zeta) \right| \\ &\le 12 \tilde\rho \eta_{n-j} \left| Z_{(n-j+1)}^{(n)}(z) - Z_{(n-j+1)}^{(n)}(\zeta) \right| + \tilde\rho |r_{j-1}| \\ &\le 12 \tilde\rho \eta_{n-j} \left(|r_{j-1}| + \rho_{**}^{j} |z-\zeta| \right) + \tilde\rho |r_{j-1}| \\ &= \tilde\rho (12 \eta_{n-j} + 1) |r_{j-1}| + 12 \tilde\rho \rho_{**}^{j} \eta_{n-j} |z-\zeta| \\ &\le \frac{5\tilde\rho}{4} |r_{j-1}| + 12 \tilde\rho \rho_{**}^{j} \eta_{n-j} |z-\zeta|, \end{aligned}$$

where we use that $12\eta_{n-j} < 1/4$. This recursive estimate allows us to find a bound on $|r_j|$,

$$\begin{aligned} |r_j| &\le 12 \tilde\rho \rho_{**}^{j} |z-\zeta| \sum_{m=0}^{j} \left(\frac{5}{4}\right)^m \eta_{n-j+m} < 12 \tilde\rho \rho_{**}^{j} |z-\zeta| \eta_{n-j} \sum_{m=0}^{\infty} \left(\frac{5}{8}\right)^m \\ &= \frac{12}{1-\frac{5}{8}} \tilde\rho \rho_{**}^{j} \eta_{n-j} |z-\zeta| < 36 \tilde\rho \rho_{**}^{j} \eta_{n-j} |z-\zeta| \\ &= 36 \tilde\rho \rho_{**}^{n-k} \eta_k |z-\zeta|. \end{aligned}$$

The assertion (5.18) follows by multiplying the inequality (5.17) with ρ. ∎

We state the main Proposition of this chapter.

Proposition 5.7 (Spectral Gap) *Let $n = 1, \ldots, \mathcal{N} - 1$ and $z \in \dot{\Sigma}^{(n)}$. Then, the point $Z_{(n)}^{(\mathcal{N})}(0)$ is separated from z in the following way,*

$$\mathrm{Im}\left(z - Z_{(n)}^{(\mathcal{N})}(0)\right) \geq \frac{\rho_{**}^{\mathcal{N}-n}}{60\mathcal{C}_{\chi_1}},$$

for $n \geq 2$. For $n = 1$ holds

$$\mathrm{Im}\left(z - Z_{(1)}^{(\mathcal{N})}(0)\right) \geq \frac{\rho_* \rho_{**}^{\mathcal{N}-2}}{60\mathcal{C}_{\chi_1}}.$$

The Proposition 5.7 is illustrated in Figure 5.4.

Proof. If $n = 1$ and $z \in C^{(1)} \setminus B_{r_*}$ then Corollary 5.5 implies

$$\mathrm{Im}\left(z - Z_{(1)}^{(\mathcal{N})}(0)\right) \geq \frac{2}{\sin(\delta')}(\varepsilon_1 + \eta_1) \geq 8\rho_* \geq \frac{\rho_* \rho_{**}^{\mathcal{N}-2}}{60\mathcal{C}_{\chi_1}},$$

if $n \geq 2$ and $z \in C^{(n)} \setminus B_{r_{**}}$ then

$$\mathrm{Im}\left(z - Z_{(n)}^{(\mathcal{N})}(0)\right) \geq \frac{2}{\sin(\delta')}(\varepsilon_2 + \eta_2) \geq \frac{\rho_{**}}{8\mathcal{C}_{\chi_1}} \geq \frac{\rho_{**}^{\mathcal{N}-n}}{60\mathcal{C}_{\chi_1}}$$

holds by application of Corollary 5.5. Otherwise, since $z \neq Z_{(n)}^{(\mathcal{N})}(0)$, there exists a $k = 1, \ldots, \mathcal{N} - 1 - n$ and an element $\zeta \in C^{(n+k)} \setminus B_{r_{**}}$, such that $z = Z_{(n)}^{(n+k-1)}(\zeta)$. Set $\xi := Z_{(n+k)}^{(\mathcal{N})}(0)$ and compare the vectors ξ and ζ. Due to Lemma 5.4 we know that $|\xi| \leq 2\eta_{n+k} \leq 2\eta_2$ and together with (5.11) we get

$$\mathrm{Im}(\zeta - \xi) \geq b_{**} - 2\eta_2.$$

We rewrite the difference $\zeta - \xi$ in polar coordinates, i.e.,

$$\mathrm{Im}(\zeta - \xi) = \sin(\alpha)|\zeta - \xi|.$$

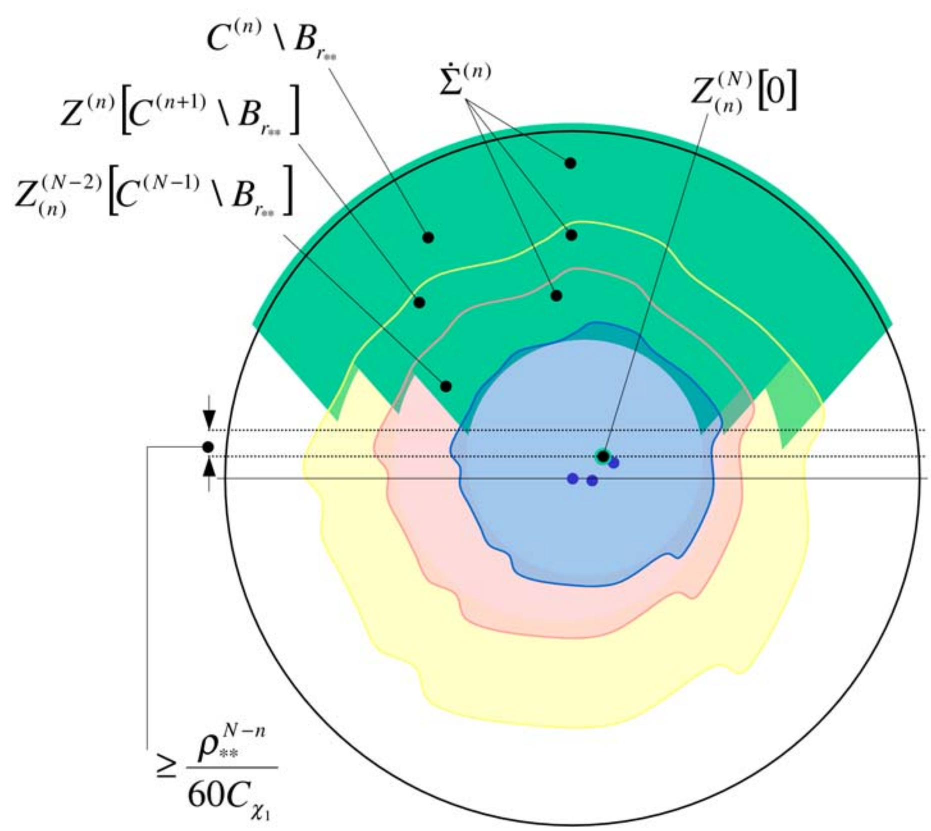

Figure 5.4: Spectral gap between $Z_{(n)}^{(\mathcal{N})}(0)$ and $\dot{\Sigma}^{(n)}$.

A simple geometric argument (we refer the reader to Figure 5.3 for details) shows

$$\begin{aligned}
|\tan(\alpha)| &= \frac{|\operatorname{Im}(\zeta-\xi)|}{|\operatorname{Re}(\zeta-\xi)|} \geq \frac{b_{**}-2\eta_2}{r_{**}+2\eta_2} = \frac{b_{**}-2\eta_2}{\frac{2}{\sin(\delta')}b_{**}+2\eta_2} \\
&= \sin(\delta')\frac{b_{**}-2\eta_2}{2b_{**}+2\sin(\delta')\eta_2} = \sin(\delta')\frac{\frac{2}{\sin(\delta')}(\eta_2+\varepsilon_2)}{\frac{4}{\sin(\delta')}(\eta_2+\varepsilon_2)+4\eta_2+2\sin(\delta')\eta_2} \\
&= \sin(\delta')\frac{\eta_2+\varepsilon_2}{2(\eta_2+\varepsilon_2)+2\eta_2\sin(\delta')+\eta_2\sin^2(\delta')} \geq \sin(\delta')\frac{\eta_2+\varepsilon_2}{5\eta_2+2\varepsilon_2} \\
&> \frac{\sin(\delta')}{5}.
\end{aligned}$$

Since $\mathrm{Im}(\zeta-\xi)$ is positive we conclude

$$\sin(\alpha)=\frac{|\tan\alpha|}{\sqrt{1+\tan^2(\alpha)}}\geq\frac{\sin(\delta')}{\sqrt{25+\sin^2(\delta')}}>\frac{1}{6}\sin(\delta')$$

because $\delta'\in[\pi/8,\pi/4]$. Now we get with Lemma 5.6

$$\begin{aligned}
\mathrm{Im}\left(z-Z_{(n)}^{(\mathcal{N})}(0)\right) &= \mathrm{Im}\left(Z_{(n)}^{(n+k-1)}(\zeta)-Z_{(n)}^{(n+k-1)}(\xi)\right)\\
&\geq \tilde{\rho}\rho_{**}^{k-1}\,\mathrm{Im}(\zeta-\xi)-36\tilde{\rho}\rho_{**}^{k-1}\eta_n|\zeta-\xi|\\
&= \tilde{\rho}\rho_{**}^{k-1}|\zeta-\xi|(\sin(\alpha)-36\eta_n)\\
&\geq \tilde{\rho}\rho_{**}^{k-1}(r_{**}-2\eta_2)\left(\frac{\sin(\delta')}{6}-36\eta_n\right)\\
&= \tilde{\rho}\rho_{**}^{k-1}\left(\frac{4(\eta_2+\varepsilon_2)}{\sin^2(\delta')}+\frac{4}{\sin(\delta')}\eta_2-2\eta_2\right)\left(\frac{\sin(\delta')}{6}-36\eta_n\right)\\
&\geq \tilde{\rho}\rho_{**}^{k-1}(6\eta_2+4\varepsilon_2)\left(\frac{\sin(\delta')}{6}-36\eta_n\right)\\
&\geq \tilde{\rho}\rho_{**}^{k}\mathcal{C}_{\chi_1}^{-1}\left(\frac{\sin(\delta')}{20}-\frac{45}{4}\eta_n\right)\\
&\geq \frac{\tilde{\rho}\rho_{**}^{\mathcal{N}-1-n}}{60\mathcal{C}_{\chi_1}},
\end{aligned}$$

where $\tilde{\rho}:=\rho_*$ for $n=1$ and $\tilde{\rho}:=\rho_{**}$ otherwise, and where we used that $\delta'\in[\pi/8,\pi/4]$ and $45\eta_n/4\leq 1/600$. ■

A direct consequence of the Proposition 5.7 is the following statement which describes the spectrum of $K_\theta^{(s)}$ in a ball $B_{4\rho}$.

Theorem 5.8 (Spectrum of $K_\theta^{(s)}$ in $\mathcal{S}_0$) *Under the assumptions of this chapter on the parameters $\theta=(i\delta',i\tau')$ and s we have the following spectral picture of $K_\theta^{(s)}$, for $|\beta_{\max}-\beta_{\min}|\ll 1$ and g sufficiently small.*

(i) The spectrum of the operator $K_\theta^{(s)}$ inside the ball $\mathcal{S}_0\subseteq B_{4\rho}$ is contained in the set

$$\Sigma^{(0)}:=\left(C^{(0)}\setminus B_{\rho/4}\right)\cup Z^{(0)}\left[\Sigma^{(1)}\right]\subseteq C^{(0)}$$

where the set

$$C^{(0)}:=\{z\in\mathcal{S}_0\,|\,\mathrm{dist}\,(z;\{0\}\cup[i\tau'+A_{\delta'}])\leq b_0\,\}$$

is illustrated in Figure 3.6. The cone $A_{\delta'}$ is given in (5.1) and $b_0=\mathcal{O}\left(g^2\right)$.

(ii) The operator $K_\theta^{(s)}$ has a simple, isolated eigenvalue $E_{0,g}^{(s)} := Z_{(0)}^{(\mathcal{N})}(0)$ with

$$\left|E_{0,g}^{(s)}\right| \leq 4g^2 \|\Gamma_{\text{eq}}\|_{\mathcal{B}(\ker(L_\text{p}))}.$$

(iii) The eigenvalue $E_{0,g}^{(s)}$ is separated from the rest of the spectrum in the sense that

$$\operatorname{Im}\left(z - E_{0,g}^{(s)}\right) \geq \frac{\rho_{[\mathcal{N}]}}{60\mathcal{C}_{\chi 1}}$$

for all $z \in \Sigma^{(0)} \setminus \{E_{0,g}^{(s)}\}$. The gap can be estimated by

$$\frac{\rho_{[\mathcal{N}]}}{60\mathcal{C}_{\chi 1}} \geq 2\tau_{\text{dec}} := \frac{\tau'}{960\mathcal{C}_{\chi 1}^2} = \frac{g^2\gamma_{\text{eq}}}{(960\mathcal{C}_{\chi 1}^2)(2+\beta_{\max})} \sim \frac{g^2\gamma_{\text{eq}}}{\beta_{\max}}$$

as $\beta_{\max} \to \infty$.

Proof.

(i) For $z \in \mathcal{S}_0 = B_{4\rho}$ the isospectral relation (3.19) holds, i.e., $z \in \operatorname{spec}(K_\theta^{(s)}) \cap \mathcal{S}_0$ if and only if

$$z \in \operatorname{spec}\left(\mathfrak{F}_{\Xi_{0,\rho}}\left(K_\theta^{(s)} - z, L_{0,\theta} - z\right) + z\right) \cap \mathcal{S}_0.$$

Recall that the Feshbach operator can be expanded as

$$\begin{aligned}
&\mathfrak{F}_{\Xi_{0,\rho}}\left(K_\theta^{(s)} - z, L_{0,\theta} - z\right) \\
&= P_{0,\rho}\left[L_{0,\theta} + g^2\Lambda_0^{(s)} \otimes \chi_\rho^2(M_{[\theta]})\right] P_{0,\rho} + \mathcal{O}\left(g^{2+\tilde{\varepsilon}}\right) \\
&= \cos(\delta')L_{\text{res}} + iM_{[\theta]} + ig^2\Gamma_{\text{eq}} + g^2\mathcal{O}\left(g^{\tilde{\varepsilon}} + |\beta_{\max} - \beta_{\min}| + 1\right).
\end{aligned}$$

by Proposition 3.7. The operator $\cos(\delta')L_{\text{res}} + iM_{[\theta]} + ig^2\Gamma_{\text{eq}}$ is a normal, bounded operator on $\operatorname{ran}(P_{0,\rho})$ and its spectrum is given by

$$\begin{aligned}
&\operatorname{spec}\left(\cos(\delta')L_{\text{res}} + iM_{[\theta]} + ig^2\Gamma_{\text{eq}}\right) \\
&= \bigcup_{e\in\operatorname{spec}(\Gamma_{\text{eq}})} \left[ig^2e + \operatorname{spec}\left(\cos(\delta')L_{\text{res}} + iM_{[\theta]}\right)\right] \\
&\subseteq \bigcup_{e\in\operatorname{spec}(\Gamma_{\text{eq}})} \left[ig^2e + \{0\} \cup (i\tau' + A_{\delta'})\right] \\
&= \{0\} \cup \left(i\frac{g^2\gamma_{\text{eq}}}{2+\beta_{\max}} + A_{\delta'}\right)
\end{aligned}$$

since $\min(\mathrm{spec}(\Gamma_{\mathrm{eq}}) \setminus \{0\}) \geq g^2\gamma_{\mathrm{eq}} > \tau'$. The Lemma 5.2 implies that $\mathrm{spec}(K_\theta^{(s)}) \cap \mathcal{S}_0 \subseteq C^{(0)}$. By (4.70) we have

$$\mathrm{spec}\left(K_\theta^{(s)}\right) \cap B_{\rho/4} = \mathrm{sv}\left(K^{(1)} \circ Z^{(0)^{-1}}\right).$$

Since $\mathrm{sv}(K^{(1)}) \subseteq \Sigma^{(1)}$ we arrive at the assertion.

(ii) Relation (5.10) suggests that $Z^{(\mathcal{N})}(0)$ is in the pure point spectrum of the family $K^{(\mathcal{N})}$ and with the help of (5.9) we obtain

$$E_{0,g}^{(s)} = Z_{(0)}^{(\mathcal{N})}(0) = \rho Z_{(1)}^{(\mathcal{N})}(0) \in \mathrm{spec}_{\mathrm{pp}}\left(K_\theta^{(s)}\right).$$

Further the dimension of the kernel of $K^{(\mathcal{N})}[Z^{(\mathcal{N})}[0]]$ is trivially one and since the Feshbach map and therefore also the renormalization transformation is multiplicity preserving (refer to Theorem E.1) the eigenvalue $E_{0,g}^{(s)}$ of $K_\theta^{(s)}$ is simple. Lemma 5.4 localizes the position of the eigenvalue,

$$\left|E_{0,g}^{(s)}\right| = \rho\left|Z_{(1)}^{(\mathcal{N})}\right| \leq 2\rho\eta_1 = 8\rho\rho_* \frac{\|\Gamma_{\mathrm{eq}}\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))}}{\gamma_{\mathrm{eq}}} = 4g^2\,\|\Gamma_{\mathrm{eq}}\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))}.$$

(iii) Replacing in Figure 5.3 the labels b_n by b_0 and r_* by $\rho/4$ we see easily that

$$\begin{aligned}\mathrm{Im}(z) &\geq -b_0\cos(\delta') + \sin(\delta')\sqrt{\left(\frac{\rho}{4}\right)^2 - b_0^2} \\ &\geq -\sin(\delta')\frac{\rho}{16} + \sin(\delta')\sqrt{\left(\frac{\rho}{4}\right)^2 - \left(\frac{\rho}{8}\right)^2} \geq \sin(\delta')\frac{\rho}{16}\end{aligned}$$

for all $z \in C^{(0)} \setminus B_{\rho/4}$, since $b_0 = \mathcal{O}(g^2) \ll \rho$. This results in

$$\mathrm{Im}\left(z - E_{0,g}^{(s)}\right) \geq \sin(\delta')\frac{\rho}{16} - 4g^2\,\|\Gamma_{\mathrm{eq}}\|_{\mathcal{B}(\ker(L_{\mathrm{p}}))} \geq \sin(\delta')\frac{\rho}{32} \geq \frac{\rho_{[\mathcal{N}]}}{60\mathcal{C}_{\chi_1}},$$

for $z \in C^{(0)} \setminus B_{\rho/4}$, because $g^2 \ll \rho$ and $\rho \gg \rho_{[\mathcal{N}]}$ for $\mathcal{N} \geq 3$.

Now choose $z := \rho\zeta \in \mathrm{spec}(K_\theta^{(s)}) \cap B_{\rho/4}$ with $z \neq E_{0,g}^{(s)}$. The first part of this theorem implies that $\zeta \in \dot{\Sigma}^{(1)}$ and Proposition 5.7 yields

$$\begin{aligned}\mathrm{Im}\left(z - E_{0,g}^{(s)}\right) &= \rho\,\mathrm{Im}\left(\zeta - Z_{(1)}^{(\mathcal{N})}(0)\right) \geq \frac{\rho_{[\mathcal{N}]}}{60\mathcal{C}_{\chi_1}} = \frac{\rho_{[\mathcal{N}-1]}\rho_{**}}{60\mathcal{C}_{\chi_1}} \\ &= \frac{\tau'}{(16\mathcal{C}_{\chi_1})^{2/\mu}60\mathcal{C}_{\chi_1}} \geq \frac{\tau'}{960\mathcal{C}_{\chi_1}^2} = 2\tau_{\mathrm{dec}} \\ &= \frac{g^2\gamma_{\mathrm{eq}}}{(960\mathcal{C}_{\chi_1}^2)(2+\beta_{\max})} = \frac{g^2\gamma_{\mathrm{eq}}}{\beta_{\max}}\left[\frac{1}{(960\mathcal{C}_{\chi_1}^2)(1+2\beta_{\max}^{-1})}\right],\end{aligned}$$

for $\mu \geq 1/2$. Recall the definitions (4.3) of τ' and (4.5) of ρ_{**}.

■

Part III

Appendices

A Relative Bounds on the Perturbation

In this appendix we provide the relative bounds of the interaction part of the Liouville operators $K_\theta^{(s)}$ which enter the analysis in the main text. The relative bounds allow a first rough spectral localization of $K_\theta^{(s)}$, i.e., as a result, we obtain a localization of the numerical range, c.f. Proposition A.9. We carefully display the dependence of the relative norms on the reservoir temperatures to avoid interferences with the coupling constant. The employed estimates are of standard type and have already been applied partially in [8].

The operators we are dealing with in the main text are defined on the Hilbert space

$$\mathcal{H} = \mathcal{H}_{\mathrm{p}}^2 \otimes \mathcal{F}(L^2[\Upsilon]),$$

where

$$\mathcal{H}_{\mathrm{p}}^2 = \mathcal{H}_{\mathrm{p}} \otimes \mathcal{H}_{\mathrm{p}} \equiv \mathbb{C}^{N\times N} \otimes \mathbb{C}^{N\times N}$$

is the positive temperature particle Hilbert space and $\mathcal{F}(L^2[\Upsilon])$ is the Fock space over

$$(\Upsilon, dy) \equiv (\mathbb{R} \times S^2 \times \mathbb{N}_1^R, d(u, \Sigma, r)).$$

describing the photon configurations at positive temperature. The family of operators studied in Chapter 3 is of the form

$$K_\theta^{(s)} = L_{0,\theta} + g I_\theta^{(s)}, \qquad s \in \mathbb{S}_{\varepsilon_0}, \tag{A.1}$$

where $\theta = (\delta, \tau)$ are from a suitable subset in $\mathbb{C}^2$ while $\mathbb{S}_{\varepsilon_0}$ is given in (2.11). The deformed free Liouville operator was introduced as

$$\begin{aligned} L_{0,\theta} &= L_{\mathrm{p}} + \mathbb{1}_{\mathcal{H}_{\mathrm{p}}^2} \otimes [\cosh(\delta) L_{\mathrm{res}} + \sinh(\delta) L_{\mathrm{aux}} + \tau N_{\mathrm{res}}] \\ &\equiv L_{\mathrm{p}} + \cosh(\delta) L_{\mathrm{res}} + \sinh(\delta) L_{\mathrm{aux}} + \tau N_{\mathrm{res}} \end{aligned}$$

(we henceforth omit trivial tensor products with the identity operator) with

$$\begin{aligned} L_{\mathrm{res}} &= d\Gamma_{\mathrm{gl}}((u, \Sigma, r) \mapsto u), \\ L_{\mathrm{aux}} &= d\Gamma_{\mathrm{gl}}((u, \Sigma, r) \mapsto |u|), \\ N_{\mathrm{res}} &= d\Gamma_{\mathrm{gl}}((u, \Sigma, r) \mapsto 1). \end{aligned}$$

The spectrally deformed perturbation operator $I_\theta^{(s)}$ is given by

$$I_\theta^{(s)} = a_{\rm gl}^* \left(\mathcal{G}_\theta - \mathcal{G}'_{(s\delta\vec{\beta}),\theta} \right) + a_{\rm gl} \left(\mathcal{G}_{\overline{\theta}} - \mathcal{G}'_{(\overline{s}\delta\vec{\beta}),\overline{\theta}} \right),$$

where the incorporated coupling functions are of the form

$$\begin{aligned}
\mathcal{G}(u,\Sigma,r) &= \sqrt{\frac{u}{1-e^{-\beta_r u}}} \times \begin{cases} \sqrt{u}\, G_r(u\Sigma) \otimes \mathbb{1}_{\mathcal{H}_{\rm p}}, & u \geq 0, \\ (-\sqrt{-u}) G_r(-u\Sigma)^* \otimes \mathbb{1}_{\mathcal{H}_{\rm p}}, & u < 0, \end{cases} \\
\mathcal{G}'_{(\vec{\kappa})}(u,\Sigma,r) &= \sqrt{\frac{u}{e^{\beta_r u}-1}}\, e^{i\kappa_r u} \\
&\quad \times \begin{cases} \sqrt{u}\, \mathbb{1}_{\mathcal{H}_{\rm p}} \otimes \alpha_{\rm p}^{-\kappa_{\rm p}} \left(\overline{G_r(u\Sigma)}^* \right), & u \geq 0, \\ (-\sqrt{-u}) \mathbb{1}_{\mathcal{H}_{\rm p}} \otimes \alpha_{\rm p}^{-\kappa_{\rm p}} \left(\overline{G_r(-u\Sigma)} \right), & u < 0, \end{cases}
\end{aligned}$$

for $\vec{\kappa} = (\kappa_{\rm p}, \kappa_1, \ldots, \kappa_R) \in \mathbb{C}^{R+1}$ and

$$\delta\vec{\beta} = (\delta\beta_{\rm p}, \delta\beta_1, \ldots, \delta\beta_R) = (\beta_{\rm p} - \beta, \beta_1 - \beta, \ldots, \beta_R - \beta).$$

The spectrally deformed functions are defined through composition with the function

$$u \mapsto j_\theta(u) = e^{\delta \,\mathrm{sgn}(u)} u + \tau,$$

namely

$$\begin{aligned}
\mathcal{G}_\theta(u,\Sigma,r) &= e^{\delta\,\mathrm{sgn}(u)/2} \mathcal{G}(j_\theta(u),\Sigma,r), \\
\mathcal{G}'_{(\vec{\kappa}),\theta}(u,\Sigma,r) &= e^{\delta\,\mathrm{sgn}(u)/2} \mathcal{G}'_{(\vec{\kappa})}(j_\theta(u),\Sigma,r).
\end{aligned}$$

The well-definedness of $\mathcal{G}_\theta$ and $\mathcal{G}'_{(s\delta\vec{\beta}),\theta}$ as $L^2[\Upsilon]$-functions is guaranteed by Hypothesis VII-1.12 for $s \in \mathbb{S}_{\varepsilon_0}$ and $\theta \in D_{\delta_0,\tau_0}$, defined in (1.90). One even has the following uniform bounds provided in Lemma A.1. We introduce the abbreviation

$$F^{(s)} := \mathcal{G} - \mathcal{G}'_{(s\delta\vec{\beta})}$$

which allows us to simplify

$$I_\theta^{(s)} = a_{\rm gl}^* \left(F_\theta^{(s)} \right) + a_{\rm gl} \left(F_{\overline{\theta}}^{(\overline{s})} \right). \tag{A.2}$$

Lemma A.1 *Under the assumptions of Hypothesis VII-1.12 we have*

$$\int_\Upsilon dy \left[1 + \frac{1}{|j_\theta(u)|^\varrho} \right] \left\| F_\theta^{(s)}(y) \right\|^2_{\mathcal{B}(\mathcal{H}_{\rm p}^2)} \leq C_\varrho e^{E|s(\beta_{\max}-\beta_{\min})| + |s(\beta_{\max}-\beta_{\min})|^2/(2a)} \tag{A.3}$$

for $\varrho < 2\nu + 1$, uniformly in $\theta \in D_{\delta_0,\tau_0}$, $s \in \mathbb{S}_{\varepsilon_0}$ and uniformly in the inverse temperatures β_r, i.e., the constant $C_\varrho < \infty$ can be chosen independently of s, θ, β_r. The positive constant E is given by $E := 4 \|H_{\rm p}\|_{\mathcal{B}(\mathcal{H}_{\rm p})} = -4E_0$ and $a > 0$ is the UV regularization postulated in Hypothesis VII-1.12, Equation (1.93).

Proof. We observe that

$$F^{(s)}(z,\Sigma,r) = \mathcal{G}(z,\Sigma,r) - e^{is(\beta_r-\beta)z}\left[\mathbb{1}_{\mathcal{A}_{\mathrm{p}}} \otimes \alpha_{\mathrm{p}}^{-s(\beta_{\mathrm{p}}-\beta)}\right](\mathcal{G}'(z,\Sigma,r))$$

is pointwise analytic in $z \in \mathcal{U}_{\delta_0,\tau_0}$, defined in (1.89), for almost every $(\Sigma,r) \in S^2 \times \mathbb{N}_1^R$ and obeys the bound

$$\begin{aligned}\left\|F^{(s)}(z,\Sigma,r)\right\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} &\leq \|\mathcal{G}(z,\Sigma,r)\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)}\left[1+e^{|s|(|\beta_r-\beta||z|+2|\beta_{\mathrm{p}}-\beta|)}\right] \\ &\leq C|z|^{\nu}e^{-a|z|^2+|s(\beta_r-\beta)||z|+2\|H_{\mathrm{p}}\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})}|s(\beta_{\mathrm{p}}-\beta)|}\end{aligned}$$

with $C<\infty$ being a positive constant, $a>0$ and $\nu \geq 1$, due to Hypothesis VII-1.12. We now perform spectral deformation on $F^{(s)}$. A first remark is that the integral (A.3) does not depend on $\mathrm{Re}(\delta)$ as one sees by transforming the variables of integration, $u \mapsto e^{-\mathrm{Re}(\delta)\,\mathrm{sgn}(u)}u$ (the independence of $\mathrm{Re}(\delta)$ is connected to the unitarity of the dilation group $\mathfrak{D}_{\mathrm{d}}(\delta)$ for a real parameter δ). We henceforth assume $\mathrm{Re}(\delta)=0$ and obtain the bound

$$\begin{aligned}&\left[1+|j_\theta(u)|^{-\varrho}\right]\left\|F_\theta^{(s)}(u,\Sigma,r)\right\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} \\ &= \left[1+|j_\theta(u)|^{-\varrho}\right]\left\|F^{(s)}(j_\theta(u),\Sigma,r)\right\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} \\ &\leq C^2\left[|j_\theta(u)|^{2\nu}+|j_\theta(u)|^{2\nu-\varrho}\right]e^{-2a|j_\theta(u)|^2+2|s(\beta_r-\beta)||j_\theta(u)|+4\|H_{\mathrm{p}}\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})}|s(\beta_{\mathrm{p}}-\beta)|} \\ &\leq C^2\left[|j_\theta(u)|^{2\nu}+|j_\theta(u)|^{2\nu-\varrho}\right]e^{4\|H_{\mathrm{p}}\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})}|s(\beta_{\mathrm{p}}-\beta)|} \\ &\quad\times\exp\left(-2a(|u|-|\tau|)^2+2|s(\beta_r-\beta)|(|u|+|\tau|)\right) \\ &= C^2\left[|j_\theta(u)|^{2\nu}+|j_\theta(u)|^{2\nu-\varrho}\right]e^{4\|H_{\mathrm{p}}\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})}|s(\beta_{\mathrm{p}}-\beta)|-2a|\tau|^2+2|s(\beta_r-\beta)\tau|} \\ &\quad\times\exp\left(-2a\left[|u|^2-|u|\left(2|\tau|+\frac{|s(\beta_r-\beta)|}{a}\right)\right]\right) \\ &= C^2\left[|j_\theta(u)|^{2\nu}+|j_\theta(u)|^{2\nu-\varrho}\right]e^{4\|H_{\mathrm{p}}\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})}|s(\beta_{\mathrm{p}}-\beta)|+2|s(\beta_r-\beta)\tau|} \\ &\quad\times\exp\left(2|s(\beta_r-\beta)\tau|+\frac{|s(\beta_r-\beta)|^2}{2a}\right) \\ &\quad\times\exp\left(-2a\left[|u|-\left(|\tau|+\frac{|s(\beta_r-\beta)|}{2a}\right)\right]^2\right).\end{aligned} \tag{A.4}$$

We consider for $c:=|\tau|+|s(\beta_r-\beta)|/(2a)$ and any power $\zeta\in\mathbb{R}$ the integral

$$\begin{aligned}&\sup_{c\geq 0}\int_{|u|\geq 1} du \int_{S^2\times\mathbb{N}_1^R} d(\Sigma,r)\,|u|^{\zeta}\exp\left(-2a[|u|-c]^2\right) \\ &\leq 2C'\sup_{c\geq 0}\int_{\mathbb{R}^+\times S^2\times\mathbb{N}_1^R} d(u,\Sigma,r)\,\exp\left(-2a'[u-c]^2\right) \\ &\leq 2C'\int_{\Upsilon} d(u,\Sigma,r)\,\exp\left(-2a'u^2\right) \leq \frac{8\pi^{3/2}RC'}{\sqrt{2a'}}\end{aligned} \tag{A.5}$$

where $C' < \infty$ is a positive constant independent of c and $0 < a' < a$. Further, for $\zeta \geq 0$, we compute the integral

$$\sup_{c\geq 0} \int_{|u|\leq 1} du \int_{S^2\times\mathbb{N}_1^R} d(\Sigma, r)\, |j_\theta(u)|^\zeta \exp\left(-2a[|u|-c]^2\right)$$
$$\leq \int_{|u|\leq 1} du \int_{S^2\times\mathbb{N}_1^R} d(\Sigma, r)\, (|u| + |\tau|)^\zeta \leq 4\pi R[1+\tau_0]^\zeta \tag{A.6}$$

and for $0 \leq \zeta < 1$, we consider the integral

$$\sup_{c\geq 0} \int_{|u|\leq 1} du \int_{S^2\times\mathbb{N}_1^R} d(\Sigma, r)\, |j_\theta(u)|^{-\zeta} \exp\left(-2a[|u|-c]^2\right)$$
$$\leq\ 2\int_0^1 du \int_{S^2\times\mathbb{N}_1^R} d(\Sigma, r)\, |u - |\tau||^{-\zeta} \leq \frac{8\pi R}{1-\zeta}\left[|1-|\tau||^{1-\zeta} + |\tau|^{1-\zeta}\right]. \tag{A.7}$$

The estimate (A.4) together with the integral bounds (A.5, A.6, A.7) yields

$$\int_\Upsilon dy \left[1 + \frac{1}{|j_\theta(u)|^\varrho}\right] \left\|F_\theta^{(s)}(y)\right\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)}$$
$$\leq\ C''(1+\tau_0)^{2\nu} e^{|s(\beta_\mathrm{max}-\beta_\mathrm{min})|(4\|H_\mathrm{p}\|_{\mathcal{B}(\mathcal{H}_\mathrm{p})}+4\tau_0)} e^{|s(\beta_\mathrm{max}-\beta_\mathrm{min})|^2/(2a)}$$
$$\leq\ C''(1+\tau_0)^{2\nu} e^{8\pi|s|} e^{4\|H_\mathrm{p}\|_{\mathcal{B}(\mathcal{H}_\mathrm{p})}|s(\beta_\mathrm{max}-\beta_\mathrm{min})|} e^{|s(\beta_\mathrm{max}-\beta_\mathrm{min})|^2/(2a)}$$

using that $\beta_\mathrm{p}, \beta \in [\beta_\mathrm{min}, \beta_\mathrm{max}]$ and $\tau_0(\beta_\mathrm{max} - \beta_\mathrm{min}) \leq 2\pi(\beta_\mathrm{max} - \beta_\mathrm{min})/\beta_\mathrm{max} \leq 2\pi$, where $C'' < \infty$ is a positive constant independent of s and θ and the inverse temperatures. We finally observe that $|s| < 1/2 + \varepsilon_0$. ■

Remark A.2 *The Lemma A.1 suggest that the effective strength of the perturbation* $I_\theta^{(s)}$, *(A.2), of the operator* $K_\theta^{(s)}$, *(A.1), is given by the effective coupling constant*

$$g_{[s]} := g e^{E|s(\beta_\mathrm{max}-\beta_\mathrm{min})| + |s(\beta_\mathrm{max}-\beta_\mathrm{min})|^2/(2a)}.$$

It is crucial that $g_{[s]}$ *grows as the temperature difference increases but is independent of the order of the inverse temperatures* $\beta_\mathrm{min}, \beta_\mathrm{max}$. *This allows the treatment of the low temperature regime, i.e., large magnitudes of* β_min, *with the means of perturbation theory, as long as the reservoir temperatures do not differ to much. This is in contrast to the analysis in [28] where the coupling functions of the C-Liouville operator are weighted exponentially in the inverse temperatures. The analysis of the operator* $K_\theta^{(s)}$, *however, requires at various places that* $|\beta_\mathrm{max} - \beta_\mathrm{min}| \ll 1$ *such that effectively the smallness of* $g_{[s]}$ *is equivalent to the smallness of* g.

We focus on the relative bound of the perturbation $I_\theta^{(s)}$. We first recall the relative bounds of creation and annihilation operators w.r.t. the number operator.

Lemma A.3 *Let $F \in L^2[\Upsilon; \mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)]$ be an over Υ square integrable, $\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)$-valued function. The creation and annihilation operators obey the following relative bound,*

$$\left\| a_{\mathrm{gl}}^{\#}(F)(N_{\mathrm{res}}+1)^{-1/2} \right\| \leq \|F\|_{L^2[\Upsilon;\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)]} \,. \tag{A.8}$$

Proof. The assertion follows directly from the definition, compare with (1.24). ■

Subsequently, we derive relative bounds on the perturbation $I_\theta^{(s)}$ w.r.t. the positive operator

$$M_{[\theta]} := e^{|\operatorname{Re}(\delta)|}|\sin(\operatorname{Im}(\delta))| L_{\mathrm{aux}} + |\tau| N_{\mathrm{res}} = d\Gamma_{\mathrm{gl}}((u,\Sigma,r) \mapsto m_\theta(u)), \tag{A.9}$$

where

$$m_\theta(u) := |u| e^{|\operatorname{Re}(\delta)|}|\sin(\operatorname{Im}(\delta))| + |\tau|. \tag{A.10}$$

Lemma A.4 *Let $\theta = (\delta,\tau)$ with $0 < |\operatorname{Im}(\delta)| \leq \frac{\pi}{2}$ and $|\operatorname{Im}(\tau)| < 2\pi\beta_{\max}^{-1}$. Let $F \in L^2[\Upsilon, dy; \mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)]$ be an over Υ square integrable, $\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)$-valued function fulfilling $\int_\Upsilon dy \left(1 + \frac{1}{|j_\theta(u)|}\right) \|F(y)\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} < \infty$.*

(i) The creation and annihilation operators obey the following relative bounds,

$$\left.\begin{array}{r} \sup_{d>0} \left\| a_{\mathrm{gl}}(F)(M_{[\theta]}+d)^{-1/2} \right\|, \\ \sup_{d>0} \left\| (M_{[\theta]}+d)^{-1/2} a_{\mathrm{gl}}^*(F) \right\| \end{array}\right\} \tag{A.11}$$
$$\leq \left[\int_\Upsilon dy \frac{\|F(y)\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)}}{m_\theta(u)} \right]^{1/2} \leq \frac{1}{|\sin(\operatorname{Im}(\delta))|^{1/2}} \left[\int_\Upsilon dy \frac{\|F(y)\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)}}{|j_\theta(u)|} \right]^{1/2}$$

and

$$\left.\begin{array}{r} \left\| a_{\mathrm{gl}}^*(F)(M_{[\theta]}+1)^{-1/2} \right\|, \\ \left\| (M_{[\theta]}+1)^{-1/2} a_{\mathrm{gl}}(F) \right\| \end{array}\right\} \tag{A.12}$$
$$\leq \left[\int_\Upsilon dy \left(1 + \frac{1}{m_\theta(u)}\right) \|F(y)\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} \right]^{1/2}$$
$$\leq \left[\int_\Upsilon dy \left(1 + \frac{1}{|\sin(\operatorname{Im}(\delta))||j_\theta(u)|}\right) \|F(y)\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} \right]^{1/2} .$$

(ii) Let $d > 0$ and $P_{[M_{[\theta]} \leq d]}$ be the orthogonal projection on the spectral subspace of $M_{[\theta]}$ corresponding to spectral subset $[0, d]$. The creation and annihilation operators fulfill the following bounds on the range of the projection $P_{[M_{[\theta]} \leq d]}$,

$$\left.\begin{array}{l} \left\| a_{\rm gl}(F) P_{[M_{[\theta]} \leq d]} \right\|, \\ \left\| P_{[M_{[\theta]} \leq d]} a^*_{\rm gl}(F) \right\| \end{array}\right\} \leq \sqrt{2d} \left[\int_{\Upsilon} dy \mathbf{1}_{[m_\theta(u) \leq d]} \frac{\|F(y)\|^2_{\mathcal{B}(\mathcal{H}^2_{\rm p})}}{m_\theta(u)} \right]^{1/2} \tag{A.13}$$

$$\leq \sqrt{\frac{2d}{|\sin(\mathrm{Im}(\delta))|}} \left[\int_{\Upsilon} dy \mathbf{1}_{[m_\theta(u) \leq d]} \frac{\|F(y)\|^2_{\mathcal{B}(\mathcal{H}^2_{\rm p})}}{|j_\theta(u)|} \right]^{1/2} .$$

Proof.

(i) We start proving (A.11). Let $\psi \in \mathcal{H}$ be a unit vector, $\|\psi\| = 1$, and $d > 0$. We have

$$\begin{aligned} & \left\| a_{\rm gl}(F)(M_{[\theta]} + d)^{-1/2} \psi \right\| \\ \leq & \int_{\Upsilon} dy \, \|F(y)\|_{\mathcal{B}(\mathcal{H}^2_{\rm p})} \left\| a_{\rm gl}(y)(M_{[\theta]} + d)^{-1/2} \psi \right\| \\ \leq & \left[\int_{\Upsilon} dy \frac{\|F(y)\|^2_{\mathcal{B}(\mathcal{H}^2_{\rm p})}}{m_\theta(u)} \right]^{1/2} \\ & \times \left[\int_{\Upsilon} dy \left\langle (M_{[\theta]} + d)^{-1/2} \psi \,\middle|\, a^*_{\rm gl}(y) m_\theta(u) a_{\rm gl}(y) (M_{[\theta]} + d)^{-1/2} \psi \right\rangle \right]^{1/2} \\ = & \left[\int_{\Upsilon} dy \frac{\|F(y)\|^2_{\mathcal{B}(\mathcal{H}^2_{\rm p})}}{m_\theta(u)} \right]^{1/2} \left\langle \psi \,\middle|\, \frac{M_{[\theta]}}{M_{[\theta]} + d} \psi \right\rangle^{1/2} \\ \leq & \left[\int_{\Upsilon} dy \frac{\|F(y)\|^2_{\mathcal{B}(\mathcal{H}^2_{\rm p})}}{m_\theta(u)} \right]^{1/2} \leq \frac{1}{|\sin(\mathrm{Im}(\delta))|^{1/2}} \left[\int_{\Upsilon} dy \frac{\|F(y)\|^2_{\mathcal{B}(\mathcal{H}^2_{\rm p})}}{|j_\theta(u)|} \right]^{1/2} , \end{aligned}$$

where we used that

$$m_\theta(u) \geq |\sin(\mathrm{Im}(\delta))| \left(|u| e^{\mathrm{Re}(\delta)} + |\tau| \right) \geq |\sin(\mathrm{Im}(\delta))| |j_\theta(u)|.$$

To obtain (A.12) we compute for $\|\psi\| = 1$,

$$
\begin{aligned}
&\left\|a^*_{\mathrm{gl}}(F)(M_{[\theta]}+1)^{-1/2}\psi\right\|^2 \\
&= \int_\Upsilon dy' \int_\Upsilon dy \left\langle F(y')(M_{[\theta]}+1)^{-1/2}\psi \,\middle|\, F(y)\otimes a_{\mathrm{gl}}(y')a^*_{\mathrm{gl}}(y)(M_{[\theta]}+1)^{-1/2}\psi \right\rangle \\
&\le \int_\Upsilon dy\, \|F(y)\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} \left\|(M_{[\theta]}+1)^{-1/2}\psi\right\|^2 \\
&\quad + \int_\Upsilon dy' \int_\Upsilon dy \Big\langle F(y')\otimes a_{\mathrm{gl}}(y)(M_{[\theta]}+1)^{-1/2}\psi \,\Big| \\
&\qquad\qquad F(y)\otimes a_{\mathrm{gl}}(y')(M_{[\theta]}+1)^{-1/2}\psi \Big\rangle \\
&\le \int_\Upsilon dy\, \|F(y)\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} + \left[\int_\Upsilon dy\, \|F(y)\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} \left\|a_{\mathrm{gl}}(y)(M_{[\theta]}+1)^{-1/2}\psi\right\|\right]^2 \\
&\le \int_\Upsilon dy \left(1+\frac{1}{m_\theta(u)}\right) \|F(y)\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} \\
&\le \int_\Upsilon dy \left(1+\frac{1}{|\sin(\mathrm{Im}(\delta))||j_\theta(u)|}\right) \|F(y)\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)},
\end{aligned}
$$

where we used the pull through formula (1.67) and the CCR (1.64).

To prove the second parts of (A.11, A.12), we observe that these relations concern the corresponding adjoint operators of the estimated ones and therefore the assertion follows from the above estimates.

(ii) Since $P_{[M_{[\theta]}\le d]}$ is a projection we can write, using the pull through formula,

$$
\begin{aligned}
a_{\mathrm{gl}}(F)P_{[M_{[\theta]}\le d]} &= \int_\Upsilon dy P_{[M_{[\theta]}+m_\theta(u)\le d]} F(y)\otimes a_{\mathrm{gl}}(y) P_{[M_{[\theta]}\le d]} \\
&= \int_\Upsilon dy \mathbf{1}_{[m_\theta(u)\le d]} F(y)\otimes a_{\mathrm{gl}}(y) P_{[M_{[\theta]}\le d]} \\
&= a_{\mathrm{gl}}\left(\mathbf{1}_{[m_\theta\le d]}F\right) P_{[M_{[\theta]}\le d]}.
\end{aligned}
$$

The estimate (A.13) is implied by (A.11),

$$
\begin{aligned}
\left\|a_{\mathrm{gl}}(F)P_{[M_{[\theta]}\le d]}\right\| &\le \left\|a_{\mathrm{gl}}(F)P_{[M_{[\theta]}\le d]}(M_{[\theta]}+d)^{-1/2}\right\| \left\|(M_{[\theta]}+d)^{1/2}P_{[M_{[\theta]}\le d]}\right\| \\
&\le \sqrt{2d}\, \left\|a_{\mathrm{gl}}\left(\mathbf{1}_{[m_\theta\le d]}F\right)(M_{[\theta]}+d)^{-1/2}\right\|
\end{aligned}
$$

$$\leq \sqrt{2d} \left[\int_{\Upsilon} dy \mathbf{1}_{[m_\theta(u) \leq d]} \frac{\|F(y)\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)}}{m_\theta(u)} \right]^{1/2}$$

$$\leq \sqrt{\frac{2d}{|\sin(\mathrm{Im}(\delta))|}} \left[\int_{\Upsilon} dy \mathbf{1}_{[m_\theta(u) \leq d]} \frac{\|F(y)\|^2_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)}}{|j_\theta(u)|} \right]^{1/2} .$$

The second assertion is proved in the same manner.

■

A consequence of Lemma A.4 is the relative (form) bound of the perturbation $I_\theta^{(s)}$ w.r.t. $M_{[\theta]}$.

Lemma A.5 (Relative Form Bound of $I_\theta^{(s)}$) *Under the assumptions of Hypothesis VII-1.12, and for $\theta \in D_{\delta_0,\tau_0}$ with $|\mathrm{Im}(\delta)| > 0$, the perturbation $I_\theta^{(s)}$ obeys the following relative bounds.*

(i) For any $d > 0$ holds

$$g \left\| (M_{[\theta]} + d)^{-1/2} I_\theta^{(s)} (M_{[\theta]} + d)^{-1/2} \right\| \leq \frac{C g_{[s]}}{\sqrt{d |\sin(\mathrm{Im}(\delta))|}} \tag{A.14}$$

where $C < \infty$ is a positive constant independent of θ, s, d and the inverse temperatures. Further, we have

$$\left. \begin{array}{l} g \left\| I_\theta^{(s)} (M_{[\theta]} + 1)^{-1/2} \right\| , \\ g \left\| (M_{[\theta]} + 1)^{-1/2} I_\theta^{(s)} \right\| \end{array} \right\} \leq \frac{C' g_{[s]}}{\sqrt{|\sin(\mathrm{Im}(\delta))|}} \tag{A.15}$$

for a positive constant $C' < \infty$, uniformly in θ, s and the inverse temperatures.

(ii) If we further assume that $\frac{\pi}{8} < |\mathrm{Im}(\delta)| < \frac{\pi}{4}$, then there is a positive constant $C < \infty$ such that

$$\left. \begin{array}{l} g \left\| \left[I_\theta^{(s)} \right]_\mathfrak{a} P_{[M_{[\theta]} \leq d]} \right\| , \\ g \left\| P_{[M_{[\theta]} \leq d]} \left[I_\theta^{(s)} \right]_\mathfrak{c} \right\| \end{array} \right\} \leq C g_{[s]} \left(\frac{d}{|\mathrm{Im}(\sin(\delta))|} \right)^{\nu + 1/2} , \tag{A.16}$$

recall the notation (3.23),

$$\left[I_\theta^{(s)} \right]_\mathfrak{a} = a_{\mathrm{gl}} \left(F_{\overline{\theta}}^{(\overline{s})} \right) , \qquad \left[I_\theta^{(s)} \right]_\mathfrak{c} = a^*_{\mathrm{gl}} \left(F_\theta^{(s)} \right) ,$$

and therefore also

$$g\left\|P_{\left[M_{[\theta]}\le d\right]}I_\theta^{(s)}P_{\left[M_{[\theta]}\le d\right]}\right\|\le 2Cg_{[s]}\left(\frac{d}{|\operatorname{Im}(\sin(\delta))|}\right)^{\nu+1/2}, \tag{A.17}$$

where $\nu\ge 1$ is the infrared regularization of the coupling functions as defined in (1.92).

Proof.

(i) Recall that $I_\theta^{(s)}=a_{\mathrm{gl}}^*\left(F_\theta^{(s)}\right)+a_{\mathrm{gl}}\left(F_{\overline{\theta}}^{(\overline{s})}\right)$. Lemma A.4, Equation (A.11), implies that

$$\begin{aligned}
&g\left\|(M_{[\theta]}+d)^{-1/2}I_\theta^{(s)}(M_{[\theta]}+d)^{-1/2}\right\|\\
&\le\ g\sqrt{\frac{2}{|\sin(\operatorname{Im}(\delta))|}}\left[\int_\Upsilon dy\frac{\left\|F_\theta^{(s)}(y)\right\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)}+\left\|F_{\overline{\theta}}^{(\overline{s})}(y)\right\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)}}{|j_\theta(u)|}\right]^{1/2}\\
&\quad\times\left\|(M_{[\theta]}+d)^{-1/2}\right\|\\
&\le\ \frac{Cg_{[s]}}{\sqrt{d|\sin(\operatorname{Im}(\delta))|}},
\end{aligned}$$

which is due to Lemma A.1. The estimate (A.15) follows in the same way using Lemmata A.1 and A.4 and (A.12).

(ii) By applying Lemma A.4, Equation (A.13), we obtain

$$g\left\|P_{\left[M_{[\theta]}\le d\right]}\left[I_\theta^{(s)}\right]_{\mathfrak{c}}\right\|\le g\sqrt{\frac{2d}{|\sin(\operatorname{Im}(\delta))|}}\left[\int_\Upsilon dy\mathbf{1}_{[m_\theta(u)\le d]}\frac{\left\|F_\theta^{(s)}(y)\right\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)}}{|j_\theta(u)|}\right]^{1/2}.$$

Note that $m_\theta(u)\le d$ implies $|u|\le\frac{d}{|\sin(\operatorname{Im}(\delta))|}$ and therefore the integration parameter u is restricted to a compact region independent of $\frac{\pi}{8}<|\operatorname{Im}(\delta)|<\frac{\pi}{4}$. Using that due to Hypothesis VII-1.12

$$\left[\int_{S^2\times\mathbb{N}_1^R}d(\Sigma,r)\left\|F^{(s)}(z,\Sigma,r)\right\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)}\right]^{1/2}\sim|z|^\nu\qquad\text{as } z\to 0,$$

uniformly in s, and

$$|j_\theta(u)|\le e^{|\operatorname{Re}(\delta)|}|u|+|\tau|\le\frac{m_\theta(u)}{|\sin(\operatorname{Im}(\delta))|},$$

we can estimate the integrand

$$
\begin{aligned}
& g \int\limits_{S^2\times\mathbb{N}_1^R} d(\Sigma,r)\frac{\left\|F_\theta^{(s)}(y)\right\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)}}{|j_\theta(u)|} \\
\leq \;& g \int\limits_{S^2\times\mathbb{N}_1^R} d(\Sigma,r)\frac{\left\|F^{(s)}(j_\theta(u),\Sigma,r)\right\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)}}{|j_\theta(u)|} \\
\leq \;& C g_{[s]} |j_\theta(u)|^{2\nu-1} \leq C g_{[s]} \left(\frac{m_\theta(u)}{|\sin(\mathrm{Im}(\delta))|}\right)^{2\nu-1} .
\end{aligned}
$$

The last integration over u finally yields

$$
g\left\|P_{[M_{[\theta]}\leq d]}\left[I_\theta^{(s)}\right]_{\mathfrak{c}}\right\| \leq C' g_{[s]}\left(\frac{d}{|\sin(\mathrm{Im}(\delta))|}\right)^{\nu+1/2} .
$$

The other relations are derived in the same way.

∎

Lemma A.6 *Under the assumptions of Hypothesis VII-1.12 and for* $\theta \in D_{\delta_0,\tau_0}$ *with* $|\mathrm{Im}(\delta)| > 0$*, the commutator* $\left[M_{[\theta]}, I_\theta^{(s)}\right]$ *obeys for* $d > 0$ *the following relative bound,*

$$
g\left\|(M_{[\theta]}+d)^{-1/2}\left[M_{[\theta]}, I_\theta^{(s)}\right](M_{[\theta]}+d)^{-1/2}\right\| \leq \frac{C g_{[s]}}{\sqrt{d}}, \tag{A.18}
$$

where $C < \infty$ *is a positive constant independent of* θ*,* s*,* d *and the inverse reservoir temperatures.*

Proof. Since

$$
\left[M_{[\theta]}, I_\theta^{(s)}\right] = \left[M_{[\theta]}, a_{\mathrm{gl}}^*\left(F_\theta^{(s)}\right) + a_{\mathrm{gl}}\left(F_{\overline{\theta}}^{(\overline{s})}\right)\right] = a_{\mathrm{gl}}^*\left(m_\theta F_\theta^{(s)}\right) - a_{\mathrm{gl}}\left(m_\theta F_{\overline{\theta}}^{(\overline{s})}\right)
$$

we obtain by Equation (A.11) of Lemma A.4

$$
\begin{aligned}
& g\left\|(M_{[\theta]}+d)^{-1/2}\left[M_{[\theta]}, I_\theta^{(s)}\right](M_{[\theta]}+d)^{-1/2}\right\| \\
&\quad \leq \; g\left\|(M_{[\theta]}+d)^{-1/2} a_{\mathrm{gl}}^*\left(m_\theta F_\theta^{(s)}\right)\right\| \left\|(M_{[\theta]}+d)^{-1/2}\right\| \\
&\qquad + g\left\|(M_{[\theta]}+d)^{-1/2}\right\| \left\|a_{\mathrm{gl}}\left(m_\theta F_{\overline{\theta}}^{(\overline{s})}\right)(M_{[\theta]}+d)^{-1/2}\right\| \\
&\quad \leq \; g d^{-1/2}\left[\int_\Upsilon dy\, m_\theta(u)\left(\left\|F_\theta^{(s)}(y)\right\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} + \left\|F_{\overline{\theta}}^{(\overline{s})}(y)\right\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)}\right)\right]^{1/2} \\
&\quad \leq \; \frac{C g_{[s]}}{\sqrt{d}},
\end{aligned}
$$

by the same arguments as used in the proof of Lemma A.1. ■

Lemma A.7 *Under the assumptions of Hypothesis VII-1.12 and for* $\theta \in D_{\delta_0,\tau_0}$ *with* $|\operatorname{Im}(\delta)| > 0$, *we obtain the following relative bounds,*

$$
\left.
\begin{aligned}
& g\left\|\left(I_\theta^{(s)} - I^{(s)}\right)(M_{[\theta]}+1)^{-1/2}\right\|, \\
& g\left\|(M_{[\theta]}+1)^{-1/2}\left(I_\theta^{(s)} - I^{(s)}\right)\right\|
\end{aligned}
\right\}
\leq C g_{[s]}\left(\frac{|\delta|}{\sqrt{\sin(|\operatorname{Im}(\delta)|)}} + 1\right)
\tag{A.19}
$$

for a positive constant $0 < C < \infty$, *uniformly in* θ, s *and the inverse reservoir temperatures.*

Proof. Consider the derivative of the coupling functions $F_\theta^{(s)} = \mathcal{G}_\theta - \mathcal{G}'_{(s\delta\vec{\beta}),\theta}$ w.r.t. the spectral parameters,

$$
\nabla_\theta F_\theta^{(s)}(u,\Sigma,r) := \left(\partial_\delta F_\theta^{(s)}(u,\Sigma,r), \partial_\tau F_\theta^{(s)}(u,\Sigma,r)\right),
$$

and denote

$$
\begin{aligned}
\nabla_\theta I_\theta^{(s)} &:= \left(\partial_\delta I_\theta^{(s)}, \partial_\tau I_\theta^{(s)}\right) \\
&= \left(a_{\mathrm{gl}}^*\left(\partial_\delta F_\theta^{(s)}\right) + a_{\mathrm{gl}}\left(\partial_{\overline{\delta}} F_{\overline{\theta}}^{(\overline{s})}\right), a_{\mathrm{gl}}^*\left(\partial_\tau F_\theta^{(s)}\right) + a_{\mathrm{gl}}\left(\partial_{\overline{\tau}} F_{\overline{\theta}}^{(\overline{s})}\right)\right) \\
&\equiv a_{\mathrm{gl}}^*\left(\nabla_\theta F_\theta^{(s)}\right) + a_{\mathrm{gl}}\left(\nabla_{\overline{\theta}} F_{\overline{\theta}}^{(\overline{s})}\right).
\end{aligned}
$$

We compute the derivatives explicitly,

$$\begin{aligned}\partial_\delta F_\theta^{(s)}(u,\Sigma,r) &= \partial_\delta \left[e^{\delta\,\mathrm{sgn}(u)/2} F^{(s)}(j_\theta(u),\Sigma,r)\right]\\ &= \frac{1}{2}\,\mathrm{sgn}(u) F_\theta^{(s)}(u,\Sigma,r) + \left[\partial_u F^{(s)}\right]_\theta (u,\Sigma,r)\partial_\delta j_\theta(u)\\ &= \mathrm{sgn}(u)\left(\frac{1}{2}F_\theta^{(s)}(u,\Sigma,r) + e^{\delta\,\mathrm{sgn}(u)} u \left[\partial_u F^{(s)}\right]_\theta (u,\Sigma,r)\right)\end{aligned}$$

and

$$\begin{aligned}\partial_\tau F_\theta^{(s)}(u,\Sigma,r) &= \partial_\tau \left[e^{\delta\,\mathrm{sgn}(u)/2} F^{(s)}(j_\theta(u),\Sigma,r)\right]\\ &= e^{\delta\,\mathrm{sgn}(u)/2}\left[\partial_u F^{(s)}\right](j_\theta(u),\Sigma,r)\\ &= \left[\partial_u F^{(s)}\right]_\theta (u,\Sigma,r).\end{aligned}$$

By Lemma A.4 and under the assumptions of Hypothesis VII-1.12 we obtain for the relative bound

$$\begin{aligned}& g\left\|\left[I_\theta^{(s)} - I_{(0,\tau)}^{(s)} - \partial_\delta I_\theta^{(s)}\delta\right](M_{[\theta]}+1)^{-1/2}\right\|\\ &= g\Big\|\Big[a_{\mathrm{gl}}^*\left(F_\theta^{(s)} - F_{(0,\tau)}^{(s)} - \partial_\delta F_\theta^{(s)}\delta\right)\\ &\qquad + a_{\mathrm{gl}}\left(F_{\overline{\theta}}^{(\overline{s})} - F_{(0,\overline{\tau})}^{(\overline{s})} - \partial_{\overline{\delta}} F_{\overline{\theta}}^{(\overline{s})}\overline{\delta}\right)\Big](M_{[\theta]}+1)^{-1/2}\Big\|\\ &\le gC\Bigg[\int_\Upsilon dy\left(1+\frac{1}{\sin(|\operatorname{Im}(\delta)|)|j_\theta(u)|}\right)\\ &\qquad\times\Bigg(\left\|F_\theta^{(s)}(y) - F_{(0,\tau)}^{(s)}(y) - \partial_\delta F_\theta^{(s)}(y)\delta\right\|_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)}^2\\ &\qquad + \left\|F_{\overline{\theta}}^{(\overline{s})}(y) - F_{(0,\overline{\tau})}^{(\overline{s})}(y) - \partial_{\overline{\delta}} F_{\overline{\theta}}^{(\overline{s})}(y)\overline{\delta}\right\|_{\mathcal{B}(\mathcal{H}_\mathrm{p}^2)}^2\Bigg)\Bigg]^{1/2}\\ &= o\left(\frac{|\delta| g_{[s]}}{\sqrt{\sin(|\operatorname{Im}(\delta)|)}}\right),\end{aligned}$$

for a positive constant C, by dominated convergence theorem. Hereby we used the analyticity of $\theta \mapsto F_\theta^{(s)}$ in the L^2-sense as guaranteed by Hypothesis VII-1.12 and Remark 1.13. It further holds

$$\begin{aligned}& g\left\|\left[I_{(0,\tau)}^{(s)} - I^{(s)} - \partial_\tau I_{(0,\tau)}^{(s)}\tau\right](M_{[\theta]}+1)^{-1/2}\right\|\\ &\le g\Big\|\Big[a_{\mathrm{gl}}^*\left(F_{(0,\tau)}^{(s)} - F^{(s)} - \partial_\tau F_{(0,\tau)}^{(s)}\tau\right)\\ &\qquad + a_{\mathrm{gl}}\left(F_{(0,\overline{\tau})}^{(\overline{s})} - F^{(\overline{s})} - \partial_{\overline{\tau}} F_{(0,\overline{\tau})}^{(\overline{s})}\overline{\tau}\right)\Big](|\tau| N_{\mathrm{res}}+1)^{-1/2}\Big\|\end{aligned}$$

$$
\begin{aligned}
& \times \left\| \left(\frac{|\tau| N_{\mathrm{res}} + 1}{M_{[\theta]} + 1} \right)^{-1/2} \right\| \\
\leq \; & g \left\| \frac{F^{(s)}_{(0,\tau)} - F^{(s)} - \partial_\tau F^{(s)}_{(0,\tau)} \tau}{|\tau|} \right\|_{L^2[\Upsilon; \mathcal{B}(\mathcal{H}^2_{\mathrm{p}})]} \\
& + g \left\| \frac{F^{(\overline{s})}_{(0,\overline{\tau})} - F^{(\overline{s})} - \partial_{\overline{\tau}} F^{(\overline{s})}_{(0,\overline{\tau})} \overline{\tau}}{|\tau|} \right\|_{L^2[\Upsilon; \mathcal{B}(\mathcal{H}^2_{\mathrm{p}})]} \\
= \; & g_{[s]} o(1),
\end{aligned}
$$

as $\tau \longrightarrow 0$, due to the analyticity of $\theta \mapsto F^{(s)}_\theta$ and a similar argument as in the proof of Lemma A.1. We further get the estimates

$$
\begin{aligned}
g \left\| \partial_\delta I^{(s)}_\theta \delta (M_{[\theta]} + 1)^{-1/2} \right\| &= \mathcal{O} \left(\frac{|\delta| g_{[s]}}{\sqrt{\sin(|\operatorname{Im}(\delta)|)}} \right), \\
g \left\| \partial_\tau I^{(s)}_{(0,\tau)} \tau (M_{[\theta]} + 1)^{-1/2} \right\| &= \mathcal{O} \left(g_{[s]} \right), \qquad \text{as } \tau \longrightarrow 0.
\end{aligned}
$$

All the above computations result in

$$
\begin{aligned}
& g \left\| \left(I^{(s)}_\theta - I^{(s)} \right) (M_{[\theta]} + 1)^{-1/2} \right\| \\
\leq \; & g \left\| \left(I^{(s)}_\theta - I^{(s)}_{(0,\tau)} \right) (M_{[\theta]} + 1)^{-1/2} \right\| + g \left\| \left(I^{(s)}_{(0,\tau)} - I^{(s)} \right) (M_{[\theta]} + 1)^{-1/2} \right\| \\
\leq \; & C' g_{[s]} \left(\frac{|\delta|}{\sqrt{\sin(|\operatorname{Im}(\delta)|)}} + 1 \right)
\end{aligned}
$$

for a positive constant C'. The estimate on $g \left\| (M_{[\theta]} + 1)^{-1/2} \left(I^{(s)}_\theta - I^{(s)} \right) \right\|$ is proven in the same way. ■

Corollary A.8 *Under the assumptions of Hypothesis VII-1.12, for $\theta \in D_{\delta_0, \tau_0}$ with $|\operatorname{Im}(\delta)| > 0$, and for $s \in \mathbb{R}$, we obtain the following relative bounds,*

$$
\left.
\begin{aligned}
& g \left\| \operatorname{Im} \left(I^{(s)}_\theta \right) (M_{[\theta]} + 1)^{-1/2} \right\|, \\
& g \left\| (M_{[\theta]} + 1)^{-1/2} \operatorname{Im} \left(I^{(s)}_\theta \right) \right\|
\end{aligned}
\right\} \leq C g_{[s]} \left(\frac{|\delta|}{\sqrt{\sin(|\operatorname{Im}(\delta)|)}} + 1 \right) \tag{A.20}
$$

for a positive constant $0 < C < \infty$, uniformly in θ and in s on compact subsets of $\mathbb{R}$.

Proof. We remark that ${I^{(s)}}^* = I^{(s)}$ for $s \in \mathbb{R}$ and therefore

$$
\operatorname{Im} \left(I^{(s)}_\theta \right) = \operatorname{Im} \left(I^{(s)}_\theta - I^{(s)} \right) = \frac{\left(I^{(s)}_\theta - I^{(s)} \right) - \left(I^{(s)}_{\overline{\theta}} - I^{(s)} \right)}{2i}.
$$

The assertion follows by Lemma A.7. ∎

The relative bounds serve us to locate the numerical range, $\mathrm{NumRan}(K_\theta^{(s)})$, of the operator $K_\theta^{(s)}$.

Proposition A.9 (Numerical Range of $K_\theta^{(s)}$) *Let* $\theta = (\delta, \tau) \in \mathbb{C}^2$ *obey* $|\mathrm{Im}(\delta)| < \frac{\pi}{4}$ *and* $\frac{|\tau|}{2} \leq \mathrm{Im}(\tau) < 2\pi\beta_{\max}^{-1}$. *Moreover, we assume that* $s \in \mathbb{S}_{\frac{\varepsilon_0}{2}}$ *and either of the further conditions,*

(i) Let $s \in \mathbb{R}$.

(ii) Let $|\mathrm{Im}(\delta)| \in [\frac{\pi}{8}, \frac{\pi}{4}]$.

Then, for sufficient small $g_{[s]}$, *the operator* $K_\theta^{(s)}$ *is sectorial, i.e., its numerical range lies in the following sector,*

$$\mathrm{NumRan}\left(K_\theta^{(s)}\right) \subseteq \left\{ z \in \mathbb{C} \,\middle|\, \mathrm{sgn}(\delta')\,\mathrm{Im}(z) \geq -1 + \max\left\{ \frac{|\sin(\delta')|}{8} \left(|\mathrm{Re}(z)| - \|L_\mathrm{p}\|\right), 0 \right\} \right\},$$

where $\delta' := \mathrm{Im}(\delta)$.

Proof. We first exclude $\delta' = 0$ and we restrict ourselves to study the case $\delta' > 0$. The complementary case is treated by considering $K_\theta^{(s)} = K_{\overline{\theta}}^{(\overline{s})*}$. Further, we may assume that $\mathrm{Re}(\delta) = 0$. This assumption is no restriction of generality since $K_\theta^{(s)} = \mathfrak{D}_\mathrm{d}(\mathrm{Re}\,(\delta)) K_{\theta'}^{(s)} \mathfrak{D}_\mathrm{d}(\mathrm{Re}\,(\delta))^{-1}$ for $\theta' := (i\,\mathrm{Im}(\delta), \tau)$ and the numerical range is invariant under conjugation with the unitary operator $\mathfrak{D}_\mathrm{d}(\mathrm{Re}(\delta))$. Subsequently, we assume $\delta = i\delta' \in i\mathbb{R}^+$.

Let $\psi \in \mathcal{D}(K_\theta^{(s)})$ be a unit vector, $\|\psi\| = 1$. We compute

$$\begin{aligned}
&\left|\mathrm{Re}\left\langle \psi \,\middle|\, K_\theta^{(s)}\psi \right\rangle\right| \\
&= \left|\left\langle \psi \,\middle|\, \left[L_\mathrm{p} + \cos(\delta')L_\mathrm{res} + \mathrm{Re}(\tau)N_\mathrm{res} + g\,\mathrm{Re}\left(I_\theta^{(s)}\right)\right]\psi \right\rangle\right| \\
&\leq \|L_\mathrm{p}\| + \left[\left\|\frac{\cos(\delta')L_\mathrm{res} + \mathrm{Re}(\tau)N_\mathrm{res}}{\sin(\delta')L_\mathrm{aux} + |\tau|N_\mathrm{res} + 1}\right\| \right. \\
&\qquad \left. + g\left\|(M_{[\theta]}+1)^{-1/2}\,\mathrm{Re}\left(I_\theta^{(s)}\right)(M_{[\theta]}+1)^{-1/2}\right\|\right] \left\langle \psi \,\middle|\, (M_{[\theta]}+1)\psi \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \|L_{\mathrm{p}}\| + \left[\frac{1}{\sin(\delta')} + \frac{g_{[s]}C}{\sqrt{\sin(\delta')}}\right] \left\langle \psi \,\middle|\, (M_{[\theta]}+1)\psi \right\rangle \\
&\leq \|L_{\mathrm{p}}\| + \frac{2}{\sin(\delta')} \left\langle \psi \,\middle|\, (M_{[\theta]}+1)\psi \right\rangle, \qquad \text{(A.21)}
\end{aligned}
$$

for a positive constant C because of Lemma A.5 and for $g_{[s]}$ sufficiently small. Further, we get

$$
\begin{aligned}
&\mathrm{Im} \left\langle \psi \,\middle|\, K_\theta^{(s)} \psi \right\rangle \\
&= \left\langle \psi \,\middle|\, \left[\sin(\delta') L_{\mathrm{aux}} + \mathrm{Im}(\tau) N_{\mathrm{res}} + g\, \mathrm{Im}\left(I_\theta^{(s)}\right)\right] \psi \right\rangle \\
&\geq \left[1 - \left\| \frac{(\mathrm{Im}(\tau) - |\tau|) N_{\mathrm{res}}}{\sin(\delta') L_{\mathrm{aux}} + |\tau| N_{\mathrm{res}} + 1} \right\| \right. \\
&\qquad \left. - g \left\| (M_{[\theta]}+1)^{-1/2}\, \mathrm{Im}\left(I_\theta^{(s)}\right) (M_{[\theta]}+1)^{-1/2} \right\| \right] \left\langle \psi \,\middle|\, (M_{[\theta]}+1)\psi \right\rangle - 1 \\
&\geq \left[\frac{\mathrm{Im}(\tau)}{|\tau|} - g \left\| (M_{[\theta]}+1)^{-1/2}\, \mathrm{Im}\left(I_\theta^{(s)}\right) (M_{[\theta]}+1)^{-1/2} \right\| \right] \left\langle \psi \,\middle|\, (M_{[\theta]}+1)\psi \right\rangle - 1
\end{aligned}
$$

We continue this estimation for the different assumptions made in the statement of the corollary.

(i) Under the assumptions of (i) and with the help of Corollary A.8 we obtain

$$
\begin{aligned}
\mathrm{Im} \left\langle \psi \,\middle|\, K_\theta^{(s)} \psi \right\rangle &\geq \left[\frac{1}{2} - g_{[s]}C'\right] \left\langle \psi \,\middle|\, (M_{[\theta]}+1)\psi \right\rangle - 1 \\
&\geq \frac{1}{4} \left\langle \psi \,\middle|\, (M_{[\theta]}+1)\psi \right\rangle - 1 \geq -1, \qquad \text{(A.22)}
\end{aligned}
$$

for a positive constant C' and for $g_{[s]}$ sufficiently small.

(ii) Under the assumption of (ii) we cannot apply Corollary A.8, instead we estimate using Lemma A.5

$$
\begin{aligned}
\mathrm{Im} \left\langle \psi \,\middle|\, K_\theta^{(s)} \psi \right\rangle &\geq \left[\frac{1}{2} - \frac{g_{[s]}C'}{\sqrt{\sin(\delta')}}\right] \left\langle \psi \,\middle|\, (M_{[\theta]}+1)\psi \right\rangle - 1 \\
&\geq \frac{1}{4} \left\langle \psi \,\middle|\, (M_{[\theta]}+1)\psi \right\rangle - 1 \geq -1, \qquad \text{(A.23)}
\end{aligned}
$$

for a positive constant C' and for $g_{[s]}$ sufficiently small because $\sin(\delta') \geq \frac{1}{3}$.

Plugging the inequality (A.21) into (A.22, A.23) we finally get

$$\operatorname{Im}\left\langle \psi \,\middle|\, K_\theta^{(s)}\psi \right\rangle \geq -1 + \max\left\{ \frac{\sin(\delta')}{8}\left(\left| \operatorname{Re}\left\langle \psi \,\middle|\, K_\theta^{(s)}\psi \right\rangle \right| - \|L_{\mathrm{p}}\| \right), 0 \right\}.$$

We take up the case $\delta' = 0$ which is paired with the additional assumption (i), i.e., $s \in \mathbb{R}$. We note that $\delta \mapsto K_\theta^{(s)}$ is strongly continuous, thus

$$\left\langle \psi \,\middle|\, K_{(0,\tau)}^{(s)}\psi \right\rangle = \lim_{\delta' \searrow 0} \left\langle \psi \,\middle|\, K_{(i\delta',\tau)}^{(s)}\psi \right\rangle.$$

From here follows

$$\begin{aligned}
\operatorname{Im}\left\langle \psi \,\middle|\, K_{(0,\tau)}^{(s)}\psi \right\rangle &= \lim_{\delta' \searrow 0} \operatorname{Im}\left\langle \psi \,\middle|\, K_{(i\delta',\tau)}^{(s)}\psi \right\rangle \\
&\geq \lim_{\delta' \searrow 0} \frac{\sin(\delta')}{8}\left(\left| \operatorname{Re}\left\langle \psi \,\middle|\, K_{(i\delta',\tau)}^{(s)}\psi \right\rangle \right| - \|L_{\mathrm{p}}\| \right) - 1 \\
&= -1.
\end{aligned}$$

∎

B Technicalities of the Modular Structure of the Interacting System

The perturbation theory for KMS states and their structural stability under local perturbation discussed in [11] and outlined in Section 1.1.3 does not directly apply to the situation where the perturbation does not belong to the underlying C^*-algebra but is given in terms of unbounded field operators. Subsequently we prove that all results extend not only formally but also in a rigorous way to our situation. Further, in the non-equilibrium situation there is no concept like structural stability, however, some techniques can be borrowed from the equilibrium situation but need an adaption to our setup. We provide these technical lemmata which allow a carefree application to advanced computations in the main text.

B.1 Dyson Series Expansions and the Domain of the Operator S_0

We recall the definitions (2.4, 2.5, 1.59) of the operators $L^{(\ell)}$, $L^{(\mathrm{r})}$ and $\mathcal{L}_0$.

Lemma B.1 (Dyson Series Expansion) *Assume that* $|\operatorname{Im}(s)| \leq \beta_{\max}/2$ *and* $|\operatorname{Im}(z)| \leq 1/2$ *with* $|\operatorname{Im}(s) + \beta_{\max}\operatorname{Im}(z)| \leq \beta_{\max}/2$.

(i) The operators $e^{iz\mathcal{L}_0}e^{isL^{(\ell)}}e^{-isL_0}e^{-iz\mathcal{L}_0}$ *and* $e^{-iz\mathcal{L}_0}e^{-isL_0}e^{isL^{(\ell)}}e^{iz\mathcal{L}_0}$ *are densely defined and their domains include linear combinations of vectors* $A\Omega_0$ *with* $A = A_{\mathrm{p}} \otimes W_{\mathrm{gl}}(F)$ *where* $A_{\mathrm{p}} \in \mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)$ *and* $F \in L^2[\Upsilon]$ *has compact support. Moreover we may express the application of these operators on* $A\Omega_0$ *as a Dyson*

series,

$$\begin{aligned}
&e^{iz\mathcal{L}_0}e^{isL^{(\ell)}}e^{-isL_0}e^{-iz\mathcal{L}_0}A\Omega_0 \\
&= \sum_{n=0}^{\infty}(isg)^n \int\limits_{0\le\varsigma_n\le\cdots\le\varsigma_1\le 1} d\varsigma_1\cdots d\varsigma_n\, \pi\left(\sigma_0^z\left(\alpha_0^{s\varsigma_n}(v)\cdots\alpha_0^{s\varsigma_1}(v)\right)\right)A\Omega_0, \\
&e^{-iz\mathcal{L}_0}e^{-isL_0}e^{isL^{(\ell)}}e^{iz\mathcal{L}_0}A\Omega_0 \\
&= \sum_{n=0}^{\infty}(isg)^n \int\limits_{0\le\varsigma_n\le\cdots\le\varsigma_1\le 1} d\varsigma_1\cdots d\varsigma_n\, \pi\left(\sigma_0^{-z}\left(\alpha_0^{-s\varsigma_1}(v)\cdots\alpha_0^{-s\varsigma_n}(v)\right)\right)A\Omega_0.
\end{aligned}$$

(ii) The operators $e^{iz\mathcal{L}_0}e^{isL^{(\ell)}}e^{-isL_0}e^{-iz\mathcal{L}_0}$ *and* $e^{-iz\mathcal{L}_0}e^{-isL_0}e^{isL^{(\ell)}}e^{iz\mathcal{L}_0}$ *are closable and their closures*

$$\begin{aligned}
D_{s,z}^{(1)} &:= \left[e^{iz\mathcal{L}_0}e^{isL^{(\ell)}}e^{-isL_0}e^{-iz\mathcal{L}_0}\right]^{**}, \\
D_{s,z}^{(2)} &:= \left[e^{-iz\mathcal{L}_0}e^{-isL_0}e^{isL^{(\ell)}}e^{iz\mathcal{L}_0}\right]^{**}
\end{aligned}$$

have trivial kernels.

(iii) Let $A' \in \pi(\mathcal{A})'$*. Then, the vector* $A'\Omega_0$ *is in the domain of the operators* $D_{s,z}^{(1)}$ *and* $D_{s,z}^{(2)}$ *and the following relations hold true,*

$$\begin{aligned}
D_{s,z}^{(1)}A'\Omega_0 &= A'e^{iz\mathcal{L}_0}e^{isL^{(\ell)}}e^{-isL_0}e^{-iz\mathcal{L}_0}\Omega_0 \\
&= A'e^{iz\mathcal{L}_0}e^{isL^{(\ell)}}\Omega_0, \\
D_{s,z}^{(2)}A'\Omega_0 &= A'e^{-iz\mathcal{L}_0}e^{-isL_0}e^{isL^{(\ell)}}e^{iz\mathcal{L}_0}\Omega_0 \\
&= A'e^{-iz\mathcal{L}_0}e^{-isL_0}e^{isL^{(\ell)}}\Omega_0.
\end{aligned}$$

Proof.

(i) We only consider the operator $e^{iz\mathcal{L}_0}e^{isL^{(\ell)}}e^{-isL_0}e^{-iz\mathcal{L}_0}$, the second assertion is proved in the same way. We first check the analyticity of $A\Omega_0$ for the operator L_0 and $\mathcal{L}_0$. Since Ω_0 is an analytic vector for $\Phi_{\mathrm{gl}}(F)$ we may write

$$A\Omega_0 = A_{\mathrm{p}} \otimes \sum_{m=0}^{\infty}\frac{i^m}{m!}\Phi_{\mathrm{gl}}(F)^m\Omega_0.$$

For F supported on $[-\rho,\rho]\times S^2\times \mathbb{N}_1^R$ we have

$$\left\|L_0^k\Phi_{\mathrm{gl}}(F)^m\Omega_0\right\| \le \left(\|L_{\mathrm{p}}\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} + m\rho\right)^k \|\Phi_{\mathrm{gl}}(F)^m\Omega_0\|$$

and therefore, by Lemma A.3,

$$\begin{aligned}
&\left\| L_0^k A\Omega_0 \right\| \\
&\leq \; \|A_{\rm p}\|_{\mathcal{B}(\mathcal{H}_{\rm p}^2)} \sum_{m=0}^{\infty} \frac{\left(\|L_{\rm p}\|_{\mathcal{B}(\mathcal{H}_{\rm p}^2)} + m\rho \right)^k}{m!} \|\Phi_{\rm gl}(F)^m \Omega_0\| \\
&\leq \; \|A_{\rm p}\|_{\mathcal{B}(\mathcal{H}_{\rm p}^2)} \sum_{m=0}^{\infty} \frac{\left(\|L_{\rm p}\|_{\mathcal{B}(\mathcal{H}_{\rm p}^2)} + m\rho \right)^k}{m!} \sqrt{(m+1)!} \left(\sqrt{2}\, \|F\|_{L^2[\Upsilon]} \right)^m .
\end{aligned}$$

For $\varsigma \in \mathbb{C}$ we obtain the following estimate,

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{|\varsigma|^k}{k!} \left\| L_0^k A\Omega_0 \right\| \\
&\leq \; \|A_{\rm p}\|_{\mathcal{B}(\mathcal{H}_{\rm p}^2)} \sum_{m=0}^{\infty} \sqrt{m+1}\, \frac{\left(\sqrt{2}\, \|F\|_{L^2[\Upsilon]} \right)^m}{\sqrt{m!}} \sum_{k=0}^{\infty} \frac{|\varsigma|^k \left(\|L_{\rm p}\|_{\mathcal{B}(\mathcal{H}_{\rm p}^2)} + m\rho \right)^k}{k!} \\
&= \; \|A_{\rm p}\|_{\mathcal{B}(\mathcal{H}_{\rm p}^2)} \sum_{m=0}^{\infty} \sqrt{m+1}\, \frac{\left(\sqrt{2}\, \|F\|_{L^2[\Upsilon]} \right)^m}{\sqrt{m!}} \exp\left(|\varsigma| \left(\|L_{\rm p}\|_{\mathcal{B}(\mathcal{H}_{\rm p}^2)} + m\rho \right) \right) \\
&\leq \; \exp\left(|\varsigma|\, \|L_{\rm p}\|_{\mathcal{B}(\mathcal{H}_{\rm p}^2)} \right) \|A_{\rm p}\|_{\mathcal{B}(\mathcal{H}_{\rm p}^2)} \sum_{m=0}^{\infty} \sqrt{m+1}\, \frac{\left(\sqrt{2}\, \exp\left(|\varsigma|\rho\right) \|F\|_{L^2[\Upsilon]} \right)^m}{\sqrt{m!}} \\
&< \; \infty
\end{aligned}$$

which shows that $A\Omega_0$ is an analytic vector for L_0. The same way, we can verify that $A\Omega_0$ is an analytic vector for any linear combination of the free Liouville operators L_0 and $\mathcal{L}_0$. In particular, $A\Omega_0$ is in the domain of $e^{-isL_0}e^{-iz\mathcal{L}_0} = e^{-i(sL_0+z\mathcal{L}_0)}$ and the map $\varsigma \mapsto e^{-i\varsigma L_0}e^{-iz\mathcal{L}_0}A\Omega_0$ is differentiable.

Let $P_{\left[|L^{(\ell)}|\leq\lambda\right]}$ denote the projection on the spectral subspace of $L^{(\ell)}$ corresponding to the interval $[-\lambda,\lambda] \subseteq \mathbb{R}$. Obviously, the vector $P_{\left[|L^{(\ell)}|\leq\lambda\right]} e^{-isL_0}e^{-iz\mathcal{L}_0}A\Omega_0$ is in the domain of $e^{isL^{(\ell)}}$ and we have analyticity of the map

$$\mathbb{C} \ni \varsigma \mapsto e^{i\varsigma L^{(\ell)}} P_{\left[|L^{(\ell)}|\leq\lambda\right]} e^{-i\varsigma L_0} e^{-iz\mathcal{L}_0} A\Omega_0$$

and we may write

$$\begin{aligned}
&e^{isL^{(\ell)}} P_{\left[|L^{(\ell)}|\leq\lambda\right]} e^{-isL_0} e^{-iz\mathcal{L}_0} A\Omega_0 \\
&= \; P_{\left[|L^{(\ell)}|\leq\lambda\right]} e^{-iz\mathcal{L}_0} A\Omega_0 + \int_0^1 d\varsigma\, \partial_\varsigma \left[P_{\left[|L^{(\ell)}|\leq\lambda\right]} e^{is\varsigma L^{(\ell)}} e^{-is\varsigma L_0} e^{-iz\mathcal{L}_0} A\Omega_0 \right]
\end{aligned}$$

$$= P_{\left[|L^{(\ell)}|\leq\lambda\right]}\Big(e^{-iz\mathcal{L}_0}A\Omega_0 + \int_0^1 d\varsigma\, e^{is\varsigma L^{(\ell)}}\,[isg\pi(v)]\,e^{-is\varsigma L_0}e^{-iz\mathcal{L}_0}A\Omega_0\Big)$$

$$= P_{\left[|L^{(\ell)}|\leq\lambda\right]}\Big(e^{-iz\mathcal{L}_0}A\Omega_0 + isg\int_0^1 d\varsigma\, e^{is\varsigma L^{(\ell)}}e^{-is\varsigma L_0}\pi\left(\alpha_0^{s\varsigma}(v)\right)e^{-iz\mathcal{L}_0}A\Omega_0\Big).$$

We aim to apply the above relation iteratively. To this end we remark that the map

$$\varsigma_2 \mapsto e^{-is\varsigma_2 L_0}\pi\left(\alpha_0^{s\varsigma}(v)\right)e^{-iz\mathcal{L}_0}A\Omega_0.$$

is differentiable for $\varsigma_2 \in [0,\varsigma]$ using Hypothesis VI-1.11. With the same arguments as above we get the expansion

$$\begin{aligned} & e^{isL^{(\ell)}}P_{\left[|L^{(\ell)}|\leq\lambda\right]}e^{-isL_0}e^{-iz\mathcal{L}_0}A\Omega_0 \\ &= P_{\left[|L^{(\ell)}|\leq\lambda\right]}\sum_{n=0}^{\infty}(isg)^n \\ &\quad\times\int\limits_{0\leq\varsigma_n\leq\cdots\leq\varsigma_1\leq 1} d\varsigma_1\cdots d\varsigma_n\,\pi\left(\alpha_0^{s\varsigma_n}(v)\cdots\alpha_0^{s\varsigma_1}(v)\right)e^{-iz\mathcal{L}_0}A\Omega_0. \end{aligned} \tag{B.1}$$

We now show that the series on the r.h.s. of (B.1) converges even after dropping the projection $P_{\left[|L^{(\ell)}|\leq\lambda\right]}$. We first observe that

$$e^{-iz\mathcal{L}_0}A\Omega_0 = \alpha_{\mathrm{p}}^{-z\beta_{\mathrm{p}}}(A_{\mathrm{p}})\otimes\sum_{m=0}^{\infty}\frac{i^m}{m!}\left[\frac{1}{\sqrt{2}}\left(a_{\mathrm{gl}}^*(F^+)+a_{\mathrm{gl}}(F^-)\right)\right]^m\Omega_0$$

where

$$\begin{aligned} F^+(u,\Sigma,r) &:= e^{-iz\beta_r u}F(u,\Sigma,r), \\ F^-(u,\Sigma,r) &:= e^{-i\bar{z}\beta_r u}F(u,\Sigma,r) \end{aligned}$$

obeying

$$\left\|F^{\pm}\right\|_{L^2[\Upsilon]} \leq e^{\beta_{\max}\rho/2}\left\|F\right\|_{L^2[\Upsilon]} =: b_1.$$

We further use that

$$\pi\left(\alpha_0^{s\varsigma_j}(v)\right) = a_{\mathrm{gl}}^*\left(\mathcal{G}_{\left(s\varsigma_j\vec{1}\right)}\right) + a_{\mathrm{gl}}\left(\mathcal{G}_{\left(\bar{s}\varsigma_j\vec{1}\right)}\right),$$

where $\vec{1} = (1,1,\ldots,1) \in \mathbb{C}^{R+1}$, recall (2.12), and that the uniform bound

$$b_2 := \sup_{|\operatorname{Im}(\varsigma)|\leq\beta_{\max}/2,}\left\|\mathcal{G}_{\left(\varsigma\vec{1}\right)}\right\|_{L^2[\Upsilon;\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)]} < \infty$$

is guaranteed by the Hypotheses VI-1.11 and VII-1.12. Moreover, the map $\varsigma \mapsto \mathcal{G}_{(\varsigma\vec{1})} \in L^2[\Upsilon; \mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)]$ is continuous by the same hypotheses and the dominated convergence theorem. Together with the relative bound of creation and annihilation operators w.r.t. the number operator, see Lemma A.3, we obtain continuity of the map

$$(\varsigma_1, \ldots, \varsigma_n) \mapsto \pi\left(\sigma_0^z\left(\alpha_0^{s\varsigma_n}(v)\cdots\alpha_0^{s\varsigma_1}(v)\right)\right)\left[A_{\mathrm{p}} \otimes \left(a_{\mathrm{gl}}^*(F^+) + a_{\mathrm{gl}}(F^-)\right)^m\right]\Omega_0$$

which makes the integrals in (B.1) well defined. Also by Lemma A.3 we get the estimate

$$\sup_{|\operatorname{Im}(\varsigma_j)| \le \beta_{\max},\, j=1,\ldots,n} \left\|\pi\left(\alpha_0^{s\varsigma_n}(v)\cdots\alpha_0^{s\varsigma_1}(v)\right)\left[A_{\mathrm{p}} \otimes \left(a_{\mathrm{gl}}^*(F^+) + a_{\mathrm{gl}}(F^-)\right)^m\right]\Omega_0\right\|$$
$$\le \|A_{\mathrm{p}}\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}}^2)} \sqrt{(n+m+1)!}\,(2b_1)^n(2b_2)^m .$$

The absolute convergence of the series in (B.1) follows by the next estimate,

$$\begin{aligned}
&\sum_{n,m=0}^{\infty} |sg|^n \int\limits_{0 \le \varsigma_n \le \cdots \le \varsigma_1 \le 1} d\varsigma_1 \cdots d\varsigma_n \frac{\sqrt{(n+m+1)!}\,(2b_1)^n(\sqrt{2}\,b_2)^m}{m!} \\
&= \sum_{n,m=0}^{\infty} |2b_1 sg|^n (\sqrt{2}\,b_2)^m \frac{\sqrt{(n+m+1)!}}{n!m!} \\
&= \sum_{n,m=0}^{\infty} \frac{\sqrt{n+1}\,|2b_1 sg|^n}{\sqrt{n!}} \frac{(\sqrt{2}\,b_2)^m}{\sqrt{m!}} \sqrt{\binom{m+n+1}{m}} \\
&\le 2\sum_{n=0}^{\infty} \frac{\sqrt{n+1}\,|4b_1 sg|^n}{\sqrt{n!}} \sum_{m=0}^{\infty} \frac{(\sqrt{8}\,b_2)^m}{\sqrt{m!}} \\
&< \infty,
\end{aligned}$$

where we used that $\binom{m+n+1}{m} \le 2^{n+m+1}$.

Now we remove in (B.1) the cutoff λ. We note that $\operatorname{s-lim}_{\lambda\to\infty} P_{[|L^{(\ell)}| \le \lambda]} = \mathbb{1}_{\mathcal{H}^2}$. Reconsidering (B.1) and using that $e^{isL^{(\ell)}}$ is a closed operator we obtain that

$$e^{-isL_0} e^{-iz\mathcal{L}_0} A\Omega_0 = \lim_{\lambda\to\infty} P_{[|L^{(\ell)}| \le \lambda]} e^{-isL_0} e^{-iz\mathcal{L}_0} A\Omega_0$$

is in the domain of $e^{isL^{(\ell)}}$ and

$$\begin{aligned}
&e^{isL^{(\ell)}} e^{-isL_0} e^{-iz\mathcal{L}_0} A\Omega_0 \\
&= \sum_{n=0}^{\infty} (isg)^n \int\limits_{0 \le \varsigma_n \le \cdots \le \varsigma_1 \le 1} d\varsigma_1 \cdots d\varsigma_n\, \pi\left(\alpha_0^{s\varsigma_n}(v)\cdots\alpha_0^{s\varsigma_1}(v)\right) e^{-iz\mathcal{L}_0} A\Omega_0 .
\end{aligned} \tag{B.2}$$

Finally, to meet the assertion, we show that the vector given in (B.2) is in the domain of the operator $e^{iz\mathcal{L}_0}$. This follows by the closedness of $e^{iz\mathcal{L}_0}$ and

$$\begin{aligned} & e^{iz\mathcal{L}_0} e^{isL^{(\ell)}} e^{-isL_0} e^{-iz\mathcal{L}_0} A\Omega_0 \qquad \text{(B.3)} \\ &= \sum_{n=0}^{\infty} (isg)^n \int_{0\le\varsigma_n\le\cdots\le\varsigma_1\le 1} d\varsigma_1 \cdots d\varsigma_n \, e^{iz\mathcal{L}_0} \pi\left(\alpha_0^{s\varsigma_n}(v)\cdots\alpha_0^{s\varsigma_1}(v)\right) e^{-iz\mathcal{L}_0} A\Omega_0 \\ &= \sum_{n=0}^{\infty} (isg)^n \int_{0\le\varsigma_n\le\cdots\le\varsigma_1\le 1} d\varsigma_1 \cdots d\varsigma_n \, \pi\left(\sigma_0^z\left(\alpha_0^{s\varsigma_n}(v)\cdots\alpha_0^{s\varsigma_1}(v)\right)\right) A\Omega_0 \end{aligned}$$

using that the r.h.s. of (B.3) converges because

$$\pi\left(\sigma_0^z \circ \alpha_0^{s\varsigma_j}(v)\right) = a^*_{\mathrm{gl}}\left(\mathcal{G}_{(z\vec{\beta}+s\varsigma_j\vec{1})}\right) + a_{\mathrm{gl}}\left(\mathcal{G}_{(\bar{z}\vec{\beta}+\bar{s}\varsigma_j\vec{1})}\right),$$

$\vec{\beta} = (\beta_{\mathrm{p}}, \beta_1, \ldots, \beta_R) \in \mathbb{C}^{R+1}$, obeys for $|\operatorname{Im}(s) + \beta_{\max}\operatorname{Im}(z)| \le \beta_{\max}/2$ a similar uniform L^2-bound,

$$\sup_{\substack{|\operatorname{Im}(\varsigma)|\le\beta_{\max}/2,\\ |\operatorname{Im}(\zeta)|\le 1/2:\\ |\operatorname{Im}(\varsigma)+\beta_{\max}\operatorname{Im}(\zeta)|\le\beta_{\max}/2}} \left\|\mathcal{G}_{(\zeta\vec{\beta}+\varsigma\vec{1})}\right\|_{L^2[\Upsilon;\mathcal{B}(\mathcal{H}^2_{\mathrm{p}})]} < \infty,$$

as in the above elaboration.

(ii) We only focus on the operator $e^{iz\mathcal{L}_0} e^{isL^{(\ell)}} e^{-isL_0} e^{-iz\mathcal{L}_0}$. Its closability follows from the fact that the adjoint operator is densely defined. To this end we observe that

$$\left[e^{iz\mathcal{L}_0} e^{isL^{(\ell)}} e^{-isL_0} e^{-iz\mathcal{L}_0}\right]^* \supseteq e^{i\bar{z}\mathcal{L}_0} e^{i\bar{s}L_0} e^{-i\bar{s}L^{(\ell)}} e^{-i\bar{z}\mathcal{L}_0}$$

where the r.h.s. is defined on a dense set by (i).

To show that the closure $D^{(1)}_{s,z}$ has a trivial kernel it suffices to show that the range of the operator $\left[e^{iz\mathcal{L}_0} e^{isL^{(\ell)}} e^{-isL_0} e^{-iz\mathcal{L}_0}\right]^*$ is dense. This, in turn, follows from

$$\begin{aligned} \operatorname{ran}\left(\left[e^{iz\mathcal{L}_0} e^{isL^{(\ell)}} e^{-isL_0} e^{-iz\mathcal{L}_0}\right]^*\right) &\supseteq \operatorname{ran}\left(e^{i\bar{z}\mathcal{L}_0} e^{i\bar{s}L_0} e^{-i\bar{s}L^{(\ell)}} e^{-i\bar{z}\mathcal{L}_0}\right) \\ &= \mathcal{D}\left(e^{i\bar{z}\mathcal{L}_0} e^{i\bar{s}L^{(\ell)}} e^{-i\bar{s}L_0} e^{-i\bar{z}\mathcal{L}_0}\right) \end{aligned}$$

which, again by (i), is dense.

(iii) Again, we only consider the operator $e^{iz\mathcal{L}_0} e^{isL^{(\ell)}} e^{-isL_0} e^{-iz\mathcal{L}_0}$. Note that $\mathcal{M}' := \pi(\mathcal{A})'$ is the strong closure of linear combinations of operators of the type

$\pi'(B') = B'_{\mathrm{p}} \otimes W_{\mathrm{gl}}(F')$, $F \in L^2[\Upsilon]$ having compact support, considered under (i) (we refer to the von Neumann density theorem, see [10, Cor. 2.4.15]). Therefore, there are elements $B^{j,k}_{\mathrm{p}} \in \mathcal{B}(\mathcal{H}^2_{\mathrm{p}})$ and $F_{j,k} \in L^2[\Upsilon]$ compactly supported such that $\text{s-lim}_{j\to\infty} B'_j = A'$ where $B'_j := \sum_{k=1}^{K_j} B^{j,k}_{\mathrm{p}} \otimes W_{\mathrm{gl}}(F_{j,k}) \in \mathcal{M}'$. The first part of this lemma implies that the vectors $B'_j\Omega_0$ are in the domain of $e^{iz\mathcal{L}_0}e^{isL^{(\ell)}}e^{-isL_0}e^{-iz\mathcal{L}_0}$ and allows the following expansion in a Dyson series,

$$
\begin{aligned}
&e^{iz\mathcal{L}_0}e^{isL^{(\ell)}}e^{-isL_0}e^{-iz\mathcal{L}_0}B'_j\Omega_0 \\
&= \sum_{n=0}^{\infty}(isg)^n \int_{0\le\varsigma_n\le\cdots\le\varsigma_1\le 1} d\varsigma_1\cdots d\varsigma_n\,\pi\left(\sigma_0^z\left(\alpha_0^{s\varsigma_n}(v)\cdots\alpha_0^{s\varsigma_1}(v)\right)\right)B'_j\Omega_0 \\
&= B'_j\sum_{n=0}^{\infty}(isg)^n \int_{0\le\varsigma_n\le\cdots\le\varsigma_1\le 1} d\varsigma_1\cdots d\varsigma_n\,\pi\left(\sigma_0^z\left(\alpha_0^{s\varsigma_n}(v)\cdots\alpha_0^{s\varsigma_1}(v)\right)\right)\Omega_0 \\
&= B'_j e^{iz\mathcal{L}_0}e^{isL^{(\ell)}}e^{-isL_0}e^{-iz\mathcal{L}_0}\Omega_0 \\
&\xrightarrow{j\to\infty} A' e^{iz\mathcal{L}_0}e^{isL^{(\ell)}}e^{-isL_0}e^{-iz\mathcal{L}_0}\Omega_0,
\end{aligned}
$$

using that B'_j commutes with $\pi\left(\sigma_0^z\left(\alpha_0^{s\varsigma_n}(v)\cdots\alpha_0^{s\varsigma_1}(v)\right)\right)$ and that the Dyson series converges on Ω_0. The assertion follows because $D^{(1)}_{s,z}$ is the closure of the operator $e^{iz\mathcal{L}_0}e^{isL^{(\ell)}}e^{-isL_0}e^{-iz\mathcal{L}_0}$.

■

For computational purposes it is necessary to understand how the anti-linear operator S_0 extends from the dense set $\pi(\mathcal{A})''\Omega_0$.

Lemma B.2 (Domain of $\mathcal{S}_0$) *(i) Let $F_r \in L^2[\mathbb{R}^3;\mathcal{B}(\mathcal{H}_{\mathrm{p}})]$, $r = 1,\ldots,R$, with $\omega^{-1/2}F_r \in L^2[\mathbb{R}^3;\mathcal{B}(\mathcal{H}_{\mathrm{p}})]$. Further, choose $A \in \mathcal{A}$ such that $\pi(A)\Omega_0 \in \mathcal{D}(N^{1/2}_{\mathrm{res}})$, where N_{res} is the "photon" number operator on the bosonic Fock space $\mathcal{F}(L^2[\Upsilon])$ over $L^2[\Upsilon]$ (which ensures that $\pi(A)\Omega_0$ is in the domain of the operator $\pi\left(a^{\#}_r(F_r)\right)$ which can be expressed in terms of creation and annihilation operators over $\mathcal{F}(L^2[\Upsilon])$, see definition (1.75) and Lemma A.3). Then the vector $\pi\left(a^{\#}_r(F_r)\right)\pi(A)\Omega_0$ is in the domain of S_0, introduced in (1.60), and it holds the following identity,*

$$S_0\left[\pi\left(a^{\#}_r(F_r)\right)\pi(A)\Omega_0\right] = \pi(A^*)\pi\left(a^{\#}_r(F_r)^*\right)\Omega_0.$$

(ii) Let $F^{(1)}_{r_1},\ldots,F^{(n)}_{r_n} \in L^2[\mathbb{R}^3;\mathcal{B}(\mathcal{H}_{\mathrm{p}})]$, $r_j = 1,\ldots,R$, with $\omega^{-1/2}F^{(j)}_{r_j} \in L^2[\mathbb{R}^3;\mathcal{B}(\mathcal{H}_{\mathrm{p}})]$. Then the vector $\pi\left(a^{\#_1}_{r_1}(F^{(1)}_{r_1})\right)\cdots\pi\left(a^{\#_n}_{r_n}(F^{(n)}_{r_n})\right)\Omega_0$ is in the domain of S_0 and it holds

$$S_0\left[\pi\left(a^{\#_1}_{r_1}(F^{(1)}_{r_1})\right)\cdots\pi\left(a^{\#_n}_{r_n}(F^{(n)}_{r_n})\right)\Omega_0\right] = \pi\left(a^{\#_n}_{r_n}(F^{(n)}_{r_n})^*\right)\cdots\pi\left(a^{\#_1}_{r_1}(F^{(1)}_{r_1})^*\right)\Omega_0.$$

(iii) Let $|\operatorname{Im}(s)| \leq \beta_{\max}/2$*. Then, the vectors* $e^{sL^{(\ell)}}\Omega_0$ *and* $e^{-sL_0}e^{sL^{(\ell)}}\Omega_0$ *are in the domain of* S_0 *and we have*

$$S_0\left[e^{sL^{(\ell)}}\Omega_0\right] = e^{-\bar{s}L_0}e^{\bar{s}L^{(\ell)}}\Omega_0 \qquad \textit{and}$$
$$S_0\left[e^{-sL_0}e^{sL^{(\ell)}}\Omega_0\right] = e^{\bar{s}L^{(\ell)}}\Omega_0.$$

Proof.

(i) We only consider the case $a_r^{\#}(F_r) = a_r(F_r)$, the other case is treated in the same way. Since $F_r \in L^2[\mathbb{R}^3;\mathcal{B}(\mathcal{H}_\mathrm{p})] \cong L^2[\mathbb{R}^3] \otimes \mathcal{B}(\mathcal{H}_\mathrm{p})$ we express

$$F_r = \lim_{n\to\infty}\sum_{m=1}^{N_n} f_m^n M_m^n$$

with $f_m^n, \omega^{-1/2}f_m^n \in L^2[\mathbb{R}^3]$ and $M_m^n \in \mathcal{B}(\mathcal{H}_\mathrm{p})$. Using the notation $F_m^n := (0,\dots,0,\underbrace{f_m^n}_{r},0,\dots,0)$, we may write

$$\begin{aligned}
\psi &:= \pi\left(a_r(F_r)\right)\pi(A)\Omega_0\\
&= \lim_{n\to\infty}\sum_{m=1}^{N_n}\pi\left(a_r(f_m^n M_m^n)\right)\pi(A)\Omega_0\\
&= \lim_{n\to\infty}\sum_{m=1}^{N_n}\left[\pi_\mathrm{p}(M_m^n)\otimes a_\mathrm{gl}\left(\mathfrak{g}(F_m^n)\right)\right]\pi(A)\Omega_0.
\end{aligned}$$

Note that the annihilation operator $a_\mathrm{gl}(\mathfrak{g}(F_m^n))$ can be expressed as a strong limit of Weyl operators by

$$\begin{aligned}
a_\mathrm{gl}(\mathfrak{g}(F_m^n)) &= \frac{\Phi_\mathrm{gl}(\mathfrak{g}(F_m^n)) + i\Phi_\mathrm{gl}(i\mathfrak{g}(F_m^n))}{\sqrt{2}}\\
&= \operatorname*{s-lim}_{t\to 0}\frac{W_\mathrm{gl}(t\mathfrak{g}(F_m^n)) + iW_\mathrm{gl}(it\mathfrak{g}(F_m^n)) - (1+i)\mathbb{1}}{\sqrt{2}\,t}
\end{aligned}$$

such that ψ can be written as the limit

$$\psi = \lim_{\substack{n\to\infty,\\ t\to 0}}\psi_{n,t}$$

of vectors

$$\begin{aligned}
\psi_{n,t} &:= \sum_{m=1}^{N_n}\pi_\mathrm{p}(M_m^n)\otimes\frac{W_\mathrm{gl}(t\mathfrak{g}(F_m^n)) + iW_\mathrm{gl}(it\mathfrak{g}(F_m^n)) - (1+i)\mathbb{1}}{\sqrt{2}\,t}\\
&\qquad \times\pi(A)\Omega_0\\
&= \frac{1}{\sqrt{2}\,t}\sum_{m=1}^{N_n}\pi\left(\left[M_m^n\otimes\left(W_r(tf_m^n) + iW_r(itf_m^n) - (1+i)\mathbb{1}\right)\right]A\right)\Omega_0
\end{aligned}$$

with $[M_m^n \otimes (W_r(tf_m^n) + iW_r(itf_m^n) - (1+i)\mathbb{1})]A \in \mathcal{A}$, thus $\psi_{n,t} \in \mathcal{D}(S_0)$. Applying S_0 to $\psi_{n,t}$ gives

$$\begin{aligned} S_0\psi_{n,t} & \\ &= \frac{1}{\sqrt{2}\,t}\sum_{m=1}^{N_n} \pi\left(A^*\left[M_m^{n\,*} \otimes (W_r(-tf_m^n) - iW_r(-itf_m^n) - (1-i)\mathbb{1})\right]\right)\Omega_0 \\ &\xrightarrow[t\to 0]{n\to\infty} \pi(A^*)\pi\left(a_r^*(F_r)\right)\Omega_0 \end{aligned}$$

where we just did the above steps backwards using that

$$a_{\mathrm{gl}}^*(\mathfrak{g}(F_m^n)) = \operatorname*{s-lim}_{t\to 0} \frac{W_{\mathrm{gl}}(t\mathfrak{g}(F_m^n)) - iW_{\mathrm{gl}}(it\mathfrak{g}(F_m^n)) - (1-i)\mathbb{1}}{\sqrt{2}\,t}.$$

Since S_0 is a closed operator we conclude that $\psi \in \mathcal{D}(S_0)$ and

$$S\psi = \pi(A^*)\pi\left(a_r^*(F_r)\right)\Omega_0,$$

as claimed.

(ii) The assertion is proved in the same way as under (i), we omit the proof.

(iii) We remark that the vector $e^{sL^{(\ell)}}\Omega_0$ can be expanded in a series as

$$\begin{aligned} e^{sL^{(\ell)}}\Omega_0 &= e^{sL^{(\ell)}}e^{-s\beta L_0}\Omega_0 \\ &= \sum_{n=0}^{\infty}(sg)^n \int_{0\le\varsigma_n\le\cdots\le\varsigma_1\le 1} d\varsigma_1\cdots d\varsigma_n\,\pi\left(\alpha_0^{-is\varsigma_n}(v)\cdots\alpha_0^{-is\varsigma_1}(v)\right)\Omega_0, \end{aligned}$$

c.f. Lemma B.1(i). By (ii) we know that $\pi\left(\alpha_0^{-is\varsigma_n}(v)\cdots\alpha_0^{-is\varsigma_1}(v)\right)\Omega_0 \in \mathcal{D}(S_0)$ and

$$S_0\left[\pi\left(\alpha_0^{-is\varsigma_n}(v)\cdots\alpha_0^{-is\varsigma_1}(v)\right)\Omega_0\right] = \pi\left(\alpha_0^{i\overline{s}\varsigma_1}(v)\cdots\alpha_0^{i\overline{s}\varsigma_n}(v)\right)\Omega_0.$$

The closedness of S_0 implies that $e^{sL^{(\ell)}}\Omega_0 \in \mathcal{D}(S_0)$ with

$$\begin{aligned} S_0\left[e^{sL^{(\ell)}}\Omega_0\right] &= \sum_{n=0}^{\infty}(\overline{s}g)^n \int_{0\le\varsigma_n\le\cdots\le\varsigma_1\le 1} d\varsigma_1\cdots d\varsigma_n\,\pi\left(\alpha_0^{i\overline{s}\varsigma_1}(v)\cdots\alpha_0^{i\overline{s}\varsigma_n}(v)\right)\Omega_0 \\ &= e^{-\overline{s}L_0}e^{\overline{s}L^{(\ell)}}\Omega_0. \end{aligned}$$

The second assertion follows by $S_0^{-1} = S_0$.

∎

B.2 Existence of the Perturbed KMS State

We ensure that the vector representative of the perturbed KMS state can be defined in our framework. We present a more general lemma which considers the non-equilibrium case where the reservoir temperatures $\beta_1, \ldots, \beta_R$ do not necessarily coincide. It is a generalization of [8, Thm. IV.3] to different reservoir temperatures.

Lemma B.3 *The vector* Ω_0 *is in the domain of the operator* $e^{-\beta L^{(\ell)}/2}$ *for all* $\beta \in [0, \beta_{\max}]$ *and the image obeys the norm bound*

$$\left\| e^{-\beta L^{(\ell)}/2}\Omega_0 \right\| \leq \max\left\{1, \frac{Z(\beta)}{Z(\beta_{\mathrm{p}})}\right\} \tag{B.4}$$
$$\times \exp\left(\frac{g^2\beta^2}{2} \sum_{r=1}^{R} \int_{\mathbb{R}^3} d^3\vec{k} \left[1 + \frac{2}{\beta_r \omega(\vec{k})}\right] \left\| e^{(\beta_{\max}-\beta_r)\omega(\vec{k}_r)/2} G_r(\vec{k}) \right\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})} \right) < \infty.$$

Hence, the norm of $e^{-\beta L^{(\ell)}/2}\Omega_0$ *is bounded uniformly in the inverse temperatures as long as* $|\beta_{\max} - \beta_{\min}|$ *is bounded.*

Proof. It follows from Lemma B.1(i) that $\Omega_0 \in \mathcal{D}(e^{-\beta L^{(\ell)}/2})$. To prove the norm bound we show that $\left\langle \Omega_0 \,\middle|\, e^{-\beta L^{(\ell)}} \Omega_0 \right\rangle_{\mathcal{H}^2}$ is bounded by the square of the r.h.s. of (B.4). By expansion in a Dyson-series, see Lemma B.1(i), we get

$$\begin{aligned}
&\left\langle \Omega_0 \,\middle|\, e^{-\beta L^{(\ell)}} \Omega_0 \right\rangle_{\mathcal{H}^2} \\
&= \sum_{n=0}^{\infty} (-g)^n \int_{0\leq s_n \leq \cdots \leq s_1 \leq \beta} ds_1 \cdots ds_n \left\langle \Omega_0 \,\middle|\, \pi(\alpha_0^{is_n}(v) \cdots \alpha_0^{is_1}(v))\Omega_0 \right\rangle_{\mathcal{H}^2} \\
&= \sum_{n=0}^{\infty} g^{2n} \int_{0\leq s_{2n} \leq \cdots \leq s_1 \leq \beta} ds_1 \cdots ds_{2n} \left\langle \Omega_0 \,\middle|\, \pi(\alpha_0^{is_{2n}}(v) \cdots \alpha_0^{is_1}(v))\Omega_0 \right\rangle_{\mathcal{H}^2} .
\end{aligned} \tag{B.5}$$

Note that

$$\begin{aligned}
\pi(\alpha_0^{is}(v)) &= \sum_{r=1}^{R} \int_{\mathbb{R}^3} d^3\vec{k}\, \pi\Big(e^{-sH_{\mathrm{p}}} G_r(\vec{k}) e^{sH_{\mathrm{p}}} \otimes e^{-s\omega(\vec{k})} a_r^*(\vec{k}) \\
&\qquad + e^{-sH_{\mathrm{p}}} G_r(\vec{k})^* e^{sH_{\mathrm{p}}} \otimes e^{s\omega(\vec{k})} a_r(\vec{k}) \Big) \\
&= \sum_{r=1}^{R} \sum_{\sigma=\pm} \int d^3\vec{k}\, \pi\left(e^{-sH_{\mathrm{p}}} G^{\sigma}(\vec{k}, r) e^{sH_{\mathrm{p}}} \otimes a^{\sigma}(\vec{k}, r, s) \right),
\end{aligned} \tag{B.6}$$

where

$$\begin{aligned} a^+(\vec{k}, r, s) &:= e^{-s\omega(\vec{k})} a_r^*(\vec{k}), \\ a^-(\vec{k}, r, s) &:= e^{s\omega(\vec{k})} a_r(\vec{k}), \\ G^+(\vec{k}, r) &:= G_r(\vec{k}), \\ G^-(\vec{k}, r) &:= G_r(\vec{k})^*. \end{aligned}$$

Plugging (B.6) into (B.5) we get

$$\begin{aligned} &\left\langle \Omega_0 \,\middle|\, e^{-\beta L^{(\ell)}} \Omega_0 \right\rangle \\ &= \sum_{n=0}^{\infty} g^{2n} \sum_{\sigma_1,\dots,\sigma_{2n}=\pm} \int_{0\le s_{2n}\le\cdots\le s_1\le\beta} ds_1 \cdots ds_{2n} \sum_{r_1,\dots,r_{2n}=1}^{R} \int_{\mathbb{R}^3} d^3\vec{k}_1 \cdots \int_{\mathbb{R}^3} d^3\vec{k}_{2n} \\ &\quad \Big\langle \Omega_0 \Big| \pi\Big[e^{-s_{2n}H_{\mathrm{p}}} G^{\sigma_{2n}}(\vec{k}_{2n}, r_{2n}) e^{(s_{2n}-s_{2n-1})H_{\mathrm{p}}} \cdots e^{(s_2-s_1)H_{\mathrm{p}}} G^{\sigma_1}(\vec{k}_1, r_1) e^{s_1 H_{\mathrm{p}}} \\ &\qquad \otimes a^{\sigma_{2n}}(\vec{k}_{2n}, r_{2n}, s_{2n}) \cdots a^{\sigma_1}(\vec{k}_1, r_1, s_1) \Big] \Omega_0 \Big\rangle \\ &= \sum_{n=0}^{\infty} g^{2n} \sum_{\sigma_1,\dots,\sigma_{2n}=\pm} \int_{0\le s_{2n}\le\cdots\le s_1\le\beta} ds_1 \cdots ds_{2n} \sum_{r_1,\dots,r_{2n}=1}^{R} \int_{\mathbb{R}^3} d^3\vec{k}_1 \cdots \int_{\mathbb{R}^3} d^3\vec{k}_{2n} \\ &\quad Z(\beta_{\mathrm{p}})^{-1} \operatorname{tr}\Big(e^{-(\beta_{\mathrm{p}}-s_1+s_{2n})H_{\mathrm{p}}} G^{\sigma_{2n}}(\vec{k}_{2n}, r_{2n}) e^{(s_{2n}-s_{2n-1})H_{\mathrm{p}}} \times \cdots \\ &\qquad \times e^{(s_2-s_1)H_{\mathrm{p}}} G^{\sigma_1}(\vec{k}_1, r_1) \Big) \\ &\quad \times [\omega_{\mathrm{f},1} \otimes \cdots \otimes \omega_{\mathrm{f},R}] \Big(a^{\sigma_{2n}}(\vec{k}_{2n}, r_{2n}, s_{2n}) \cdots a^{\sigma_1}(\vec{k}_1, r_1, s_1) \Big). \end{aligned} \tag{B.7}$$

We are going to apply the Hölder's inequality for the trace,

$$\operatorname{tr}(A_1 B_1 \cdots A_m B_m) \le \prod_{j=1}^{m} \|B_j\| \prod_{j=1}^{m} \operatorname{tr}(|A_j|^{p_j})^{1/p_j},$$

where $p_j > 0$ and $p_1^{-1} + \cdots + p_m^{-1} = 1$, to control the contribution of the trace in (B.7). To this end, we distinguish two cases. First we assume that $\beta_{\mathrm{p}} \ge \beta$. We define $p_1 = -\beta_{\mathrm{p}}/(-\beta_{\mathrm{p}} + s_1 - s_{2n})$, $p_j = -\beta_{\mathrm{p}}/(s_{2n+2-j} - s_{2n+1-j})$, $j = 2, \dots, 2n$, which are all positive since $0 \le s_j - s_{j+1} \le \beta \le \beta_{\mathrm{p}}$ and their inverses sum up to one. With Hölder's inequality we obtain

$$\begin{aligned} &Z(\beta_{\mathrm{p}})^{-1} \operatorname{tr}\Big(e^{-(\beta_{\mathrm{p}}-s_1+s_{2n})H_{\mathrm{p}}} G^{\sigma_{2n}}(\vec{k}_{2n}, r_{2n}) e^{(s_{2n}-s_{2n-1})H_{\mathrm{p}}} \cdots e^{(s_2-s_1)H_{\mathrm{p}}} G^{\sigma_1}(\vec{k}_1, r_1) \Big) \\ &\qquad \le Z(\beta_{\mathrm{p}})^{-1} \operatorname{tr}\left(e^{-\beta_{\mathrm{p}} H_{\mathrm{p}}}\right) \prod_{j=1}^{2n} \left\| G^{\sigma_j}(\vec{k}_j, r_j) \right\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})} \\ &\qquad = \prod_{j=1}^{2n} \left\| G^{\sigma_j}(\vec{k}_j, r_j) \right\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})}. \end{aligned} \tag{B.8}$$

In the complementary case, $\beta_{\mathrm{p}} < \beta$, we define $p_1 = -\beta/(-\beta + s_1 - s_{2n})$, $p_j = -\beta/(s_{2n+2-j} - s_{2n+1-j})$, $j = 2, \ldots, 2n$, which are again positive. Note, that

$$\mathrm{tr}\left(e^{-(\beta_{\mathrm{p}}-s_1+s_{2n})p_1 H_{\mathrm{p}}}\right) = \mathrm{tr}\left(e^{-\beta H_{\mathrm{p}}} e^{\beta \frac{\beta-\beta_{\mathrm{p}}}{\beta-s_1+s_{2n}} H_{\mathrm{p}}}\right) \leq \mathrm{tr}\left(e^{-\beta H_{\mathrm{p}}}\right)$$

since $H_{\mathrm{p}} \leq 0$ by assumption, see (1.12). Applying the Hölder inequality therefore gives

$$\begin{aligned} Z(\beta_{\mathrm{p}})^{-1}\, \mathrm{tr}\left(e^{-(\beta_{\mathrm{p}}-s_1+s_{2n})H_{\mathrm{p}}} G^{\sigma_{2n}}(\vec{k}_{2n}, r_{2n}) e^{(s_{2n}-s_{2n-1})H_{\mathrm{p}}} \cdots e^{(s_2-s_1)H_{\mathrm{p}}} G^{\sigma_1}(\vec{k}_1, r_1)\right) \\ \leq \; Z(\beta_{\mathrm{p}})^{-1}\, \mathrm{tr}\left(e^{-\beta H_{\mathrm{p}}}\right) \prod_{j=1}^{2n} \left\| G^{\sigma_j}(\vec{k}_j, r_j)\right\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})} \\ = \; \frac{Z(\beta)}{Z(\beta_{\mathrm{p}})} \prod_{j=1}^{2n} \left\| G^{\sigma_j}(\vec{k}_j, r_j)\right\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})}. \end{aligned} \tag{B.9}$$

Since the reservoir state $\omega_{\mathrm{res}} := \omega_{\mathrm{f},1} \otimes \cdots \otimes \omega_{\mathrm{f},R}$ is quasi-free we get with Wick's theorem, Lemma D.1,

$$\begin{aligned} &\omega_{\mathrm{res}}\left(a^{\sigma_{2n}}(\vec{k}_{2n}, r_{2n}, s_{2n}) \cdots a^{\sigma_1}(\vec{k}_1, r_1, s_1)\right) \\ &= \sum_{\tau \in P_{2n}} \prod_{j=1}^{n} \omega_{\mathrm{res}}\Big(a^{\sigma_{\tau(2j)}}(\vec{k}_{\tau(2j)}, r_{\tau(2j)}, s_{\tau(2j)}) \\ &\qquad \times a^{\sigma_{\tau(2j-1)}}(\vec{k}_{\tau(2j-1)}, r_{\tau(2j-1)}, s_{\tau(2j-1)})\Big), \end{aligned} \tag{B.10}$$

where P_{2n} is the set of all permutations $\tau \in S_{2n}$ which fulfil $\tau(1) < \tau(3) < \cdots < \tau(2n-1)$ and $\tau(2j-1) < \tau(2j)$ for $j = 1, \ldots, n$. We refer to the elaborations in Section 1.3.4, in particular to the definition (1.45) of ω_{f}. The only non-vanishing contributions in (B.10) are

$$\begin{aligned} \omega_{\mathrm{res}}(a^-(\vec{k}, r, s) a^+(\vec{k}', r', s')) &= \delta_{r,r'} \delta(\vec{k} - \vec{k}') \frac{e^{(\beta_r + s - s')\omega(\vec{k})}}{e^{\beta_r \omega(\vec{k})} - 1}, \\ \omega_{\mathrm{res}}(a^+(\vec{k}, r, s) a^-(\vec{k}', r', s')) &= \delta_{r,r'} \delta(\vec{k} - \vec{k}') \frac{e^{(-s + s')\omega(\vec{k})}}{e^{\beta_r \omega(\vec{k})} - 1}. \end{aligned}$$

For application in (B.7) we consider the case $0 \leq s \leq s' \leq \beta$. We introduce the abbreviation $\delta s := s' - s \in [0, \beta]$ to calculate

$$\begin{aligned} &\sum_{\sigma, \sigma' = \pm} \omega_{\mathrm{res}}(a^{\sigma}(\vec{k}, r, s) a^{\sigma'}(\vec{k}', r', s')) \\ &\quad = \delta_{r,r'} \delta(\vec{k} - \vec{k}') \frac{e^{(\beta_r - \delta s)\omega(\vec{k})} + e^{\delta s \omega(\vec{k})}}{e^{\beta_r \omega(\vec{k})} - 1} \end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{x\in[1,e^{\beta\omega(\vec{k})}]} \delta_{r,r'}\delta(\vec{k}-\vec{k}')\frac{e^{\beta_r\omega(\vec{k})}x^{-1}+x}{e^{\beta_r\omega(\vec{k})}-1} \\
&\leq \delta_{r,r'}\delta(\vec{k}-\vec{k}')\frac{e^{\beta_{\max}\omega(\vec{k})}+1}{e^{\beta_r\omega(\vec{k})}-1} \\
&\leq \delta_{r,r'}\delta(\vec{k}-\vec{k}')e^{(\beta_{\max}-\beta_r)\omega(\vec{k})}\coth(\beta_r\omega(\vec{k})/2). \qquad \text{(B.11)}
\end{aligned}
$$

Using Equations (B.8, B.9, B.10, B.11) and

$$
\#P_{2n} = (2n-1)(2n-3)\cdots 1 = \frac{(2n)!}{2^n n!}
$$

we can perform the sum over σ_j in (B.7) to estimate

$$
\begin{aligned}
&\left\langle \Omega_0 \,\middle|\, e^{-\beta L^{(\ell)}}\Omega_0 \right\rangle_{\mathcal{H}^2} \\
&\leq \max\left\{1, \frac{Z(\beta)}{Z(\beta_{\mathrm{p}})}\right\} \sum_{n=0}^{\infty} g^{2n} \int\limits_{0\leq s_{2n}\leq\cdots\leq s_1\leq\beta} ds_1\cdots ds_{2n} \\
&\quad\times \sum_{r_1,\ldots,r_{2n}=1}^{R} \int_{\mathbb{R}^3} d^3\vec{k}_1 \cdots \int_{\mathbb{R}^3} d^3\vec{k}_{2n} \prod_{j=1}^{2n} \left\| G^{\pm}(\vec{k}_j, r_j) \right\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})} \\
&\quad\times \frac{(2n)!}{2^n n!} \prod_{j=1}^{n} \delta_{r_{2j-1},r_{2j}}\delta(\vec{k}_{2j-1}-\vec{k}_{2j}) e^{(\beta_{\max}-\beta_{r_{2j}})\omega(\vec{k}_{r_{2j}})} \coth(\beta_{r_{2j}}\omega(\vec{k}_{2j})/2) \\
&= \max\left\{1, \frac{Z(\beta)}{Z(\beta_{\mathrm{p}})}\right\} \\
&\quad\times \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{g^2\beta^2}{2} \sum_{r=1}^{R} \int_{\mathbb{R}^3} d^3\vec{k}\, \coth(\beta_r\omega(\vec{k})/2) \left\| e^{(\beta_{\max}-\beta_r)\omega(\vec{k}_r)/2} G_r(\vec{k}) \right\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})} \right]^n \\
&= \max\left\{1, \frac{Z(\beta)}{Z(\beta_{\mathrm{p}})}\right\} \\
&\quad\times \exp\left(\frac{g^2\beta^2}{2} \sum_{r=1}^{R} \int_{\mathbb{R}^3} d^3\vec{k}\, \coth(\beta_r\omega(\vec{k})/2) \left\| e^{(\beta_{\max}-\beta_r)\omega(\vec{k}_r)/2} G_r(\vec{k}) \right\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})} \right) \\
&\leq \max\left\{1, \frac{Z(\beta)}{Z(\beta_{\mathrm{p}})}\right\} \\
&\quad\times \exp\left(g^2\beta^2 \sum_{r=1}^{R} \int_{\mathbb{R}^3} d^3\vec{k} \left[1 + \frac{2}{\beta_r\omega(\vec{k})}\right] \left\| e^{(\beta_{\max}-\beta_r)\omega(\vec{k}_r)/2} G_r(\vec{k}) \right\|^2_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})} \right),
\end{aligned}
$$

where we used that $\coth(x)/2 \leq 1 + 1/x$. This expression is finite due to the assumptions of Hypothesis VI-1.11. ∎

Lemma B.4 (Cyclic & Separating Property of $\tilde{\Omega}$) *The vector*

$$\tilde{\Omega} := \frac{e^{-\beta L^{(\ell)}/2}\Omega_0}{\left\| e^{-\beta L^{(\ell)}/2}\Omega_0 \right\|}$$

is cyclic and separating for $\mathcal{M} := \pi(\mathcal{A})''$ and $\mathcal{M}' := \pi(\mathcal{A})'$ for all $\beta \in [0, \beta_{\max}]$.

Proof. We first prove the cyclicity of $\tilde{\Omega}$ w.r.t. $\mathcal{M}$. We remark that it is equivalent to prove the separating property of $\tilde{\Omega}$ for $\mathcal{M}' = \pi(\mathcal{A})'$, see [10, Prop. 2.5.3]. To this end, choose $A' \in \mathcal{M}'$ such that $A'\tilde{\Omega} = 0$. An application of Lemma B.1(iii) implies that $A'\Omega_0$ is in the domain of $D^{(1)}_{\frac{i\beta}{2},0}$ and it holds

$$D^{(1)}_{\frac{i\beta}{2},0} A'\Omega_0 = A' e^{-\beta L^{(\ell)}/2}\Omega_0 = 0$$

which implies that $A'\Omega_0 = 0$ because $D^{(1)}_{\frac{i\beta}{2},0}$ has trivial kernel, c.f. Lemma B.1(ii). Due to the cyclic property of Ω_0 w.r.t. $\mathcal{M}$ we conclude that $A' = 0$.

It remains to prove that $\tilde{\Omega}$ is also separating for $\mathcal{M}$. Let $A \in \mathcal{M}$ be chosen such that $A\tilde{\Omega} = 0$. Since $JAJ \in \mathcal{M}'$ and because of Lemma B.1(iii) the vector $JAJ\Omega_0$ is in the domain of the operator $D^{(2)}_{\frac{i\beta}{2},-\frac{i}{2}}$ and

$$D^{(2)}_{\frac{i\beta}{2},-\frac{i}{2}} JAJ\Omega_0 = JAJ e^{-\mathcal{L}_0/2} e^{\beta L_0/2} e^{-\beta L^{(\ell)}/2} e^{\mathcal{L}_0/2}\Omega_0.$$

Using the modular data (J, Δ_0, S_0) associated with the state $\omega_0 = \langle\, \Omega_0 \,|\, \pi(\,\cdot\,)\Omega_0 \,\rangle$ (see (1.60)) and an application of Lemma B.2(iii) gives

$$\begin{aligned} JD^{(2)}_{\frac{i\beta}{2},-\frac{i}{2}} JA\Omega_0 &= AJe^{-\mathcal{L}_0/2} e^{\beta L_0/2} e^{-\beta L^{(\ell)}/2}\Omega_0 \\ &= AS_0 e^{\beta L_0/2} e^{-\beta L^{(\ell)}/2}\Omega_0 \\ &= Ae^{-\beta L^{(\ell)}/2}\Omega_0 \\ &= 0. \end{aligned}$$

This implies $A\Omega_0 = 0$ since J is invertible and $D^{(2)}_{\frac{i\beta}{2},-\frac{i}{2}}$ has a trivial kernel, see Lemma B.1(ii). This, in turn, implies $A = 0$ due to the separating property of Ω_0. ∎

The subsequent consideration are done under the assumption that all reservoir temperatures coincide, i.e., $\beta := \beta_{\mathrm{p}} = \beta_1 = \cdots = \beta_R$. We aim to establish $\omega = \langle\, \Omega \,|\, \pi(\,\cdot\,)\Omega \,\rangle$ with $\Omega = \tilde{\Omega}|_{\beta_{\max}=\beta_{\min}=\beta}$, given in (2.2), as the perturbed KMS state.

Lemma B.5 (Zero Eigenvector of L) *For $\beta := \beta_{\mathrm{p}} = \beta_1 = \cdots = \beta_R$, the vector $\Omega = \tilde{\Omega}|_{\beta_{\max}=\beta_{\min}=\beta}$ given in (2.3) is in the kernel of the perturbed Liouville operator L.*

Proof. Note that $L = L^{(\ell)} - g\pi'(v)$ and therefore

$$\begin{aligned} Le^{-\beta L^{(\ell)}/2}\Omega_0 &= [L^{(\ell)} - g\pi'(v)]e^{-\beta L^{(\ell)}/2}\Omega_0 \\ &= e^{-\beta L^{(\ell)}/2}[L^{(\ell)} - ge^{\beta L^{(\ell)}/2}\pi'(v)e^{-\beta L^{(\ell)}/2}]\Omega_0 \\ &= ge^{-\beta L^{(\ell)}/2}[\pi(v) - Je^{-\beta L^{(\mathrm{r})}/2}\pi(v)e^{\beta L^{(\mathrm{r})}/2}J]\Omega_0 \\ &= ge^{-\beta L^{(\ell)}/2}[\pi(v) - Je^{-\beta L_0/2}\pi(v)e^{\beta L_0/2}]\Omega_0 \\ &= ge^{-\beta L^{(\ell)}/2}[\pi(v) - Je^{-\beta L_0/2}\pi(v)]\Omega_0, \end{aligned}$$

where we used that $e^{-\beta L^{(\mathrm{r})}/2}\pi(v)e^{\beta L^{(\mathrm{r})}/2} = e^{-\beta L_0/2}\pi(v)e^{\beta L_0/2}$ as one checks by an explicit expansion in a Dyson series. Since $\omega_0 = \langle\, \Omega_0 \,|\, \pi(\,\cdot\,)\Omega_0\,\rangle$ is an (α_0, β)-KMS state we obtain

$$Je^{-\beta L_0/2}\pi(v)\Omega_0 = J\Delta_0^{1/2}\pi(v)\Omega_0 = S_0\pi(v)\Omega_0 = \pi(v^*)\Omega_0 = \pi(v)\Omega_0$$

using that the anti-linear operator S_0 can be extended to $\pi(v)\Omega_0$, see Lemma B.2(i). We finally get $Le^{-\beta L^{(\ell)}/2}\Omega_0 = 0$. ■

Next, we prove the invariance of Ω under the modular conjugation J.

Lemma B.6 *The vector Ω is a fix point of the modular conjugation J, i.e., $J\Omega = \Omega$, for equal temperatures $\beta := \beta_{\mathrm{p}} = \beta_1 = \cdots = \beta_R$.*

Proof. We use that $J = S_0 e^{\beta L_0/2}$ (since $\omega_0 = \langle\, \Omega_0 \,|\, \pi(\,\cdot\,)\Omega_0\,\rangle$ is an (α_0, β)-KMS state) to obtain

$$Je^{-\beta L^{(\ell)}/2}\Omega_0 = S_0 e^{\beta L_0/2}e^{-\beta L^{(\ell)}/2}\Omega_0 = e^{-\beta L^{(\ell)}/2}\Omega_0,$$

where we used Lemma B.2(iii). ■

It follows the KMS property of ω.

Proposition B.7 (KMS Property of ω) *In the equal temperature case $\beta := \beta_{\mathrm{p}} = \beta_1 = \cdots = \beta_R$ the operator*

$$\Delta := e^{-\beta L}$$

is the modular operator associated with the state $\omega = \langle\, \Omega \,|\, \pi(\,\cdot\,)\Omega\,\rangle$. This implies that ω is an (α, β)-KMS state, i.e., the state $A \mapsto \langle\, \Omega \,|\, A\Omega\,\rangle$ is an (α, β)-KMS state for the W^*-dynamical system $(\pi(\mathcal{A})'', t \mapsto e^{iLt}(\,\cdot\,)e^{-iLt})$ in the sense of Section 1.1.1. Following the arguments of Section 1.1.3 it is the only ω_0-normal KMS-state since $\pi(\mathcal{A})''$ is a factor.

Proof. We have to verify that the operator

$$S := J\Delta^{1/2}$$

fulfills the relation

$$S\, A\Omega = A^*\Omega$$

for all $A \in \pi(\mathcal{A})''$. For a given $A \in \pi(\mathcal{A})''$ with $A\Omega \in \mathcal{D}(e^{-\beta L/2})$ holds

$$e^{-\beta L/2} A\Omega = e^{-\beta L/2} A e^{\beta L/2}\Omega = e^{-\beta L^{(\ell)}/2} A e^{\beta L^{(\ell)}/2}\Omega = C^{-1} e^{-\beta L^{(\ell)}/2} A\Omega_0,$$

where $C := \left\| e^{-\beta L^{(\ell)}/2}\Omega_0 \right\|$. The above relation can be checked by an explicit expansion of $e^{-\beta L/2} A e^{\beta L/2}$ and $e^{-\beta L^{(\ell)}/2} A e^{\beta L^{(\ell)}/2}$ in a Dyson series using that A commutes with $\pi'(v)$. Using the modular structure for $\omega_0 = \langle\, \Omega_0 \,|\, \pi(\,\cdot\,)\Omega_0 \,\rangle$ and the relation $JL^{(\ell)} = -L^{(\mathrm{r})}J$ we obtain

$$\begin{aligned} Je^{-\beta L/2} A\Omega &= C^{-1} J e^{-\beta L^{(\ell)}/2} A\Omega_0 = C^{-1} e^{\beta L^{(\mathrm{r})}/2} J A\Omega_0 \\ &= C^{-1} e^{\beta L^{(\mathrm{r})}/2} e^{-\beta L_0/2} S_0 A\Omega_0 = C^{-1} e^{\beta L^{(\mathrm{r})}/2} e^{-\beta L_0/2} A^*\Omega_0 \\ &= C^{-1} A^* e^{\beta L^{(\mathrm{r})}/2} e^{-\beta L_0/2}\Omega_0 = C^{-1} A^* J e^{-\beta L^{(\ell)}/2}\Omega_0 \\ &= A^* J\Omega = A^*\Omega. \end{aligned}$$

Here we used Lemma B.6 and that

$$\begin{aligned} e^{\beta L^{(\mathrm{r})}/2} e^{-\beta L_0/2} A^*\Omega_0 &= J e^{-\beta L^{(\ell)}/2} e^{\beta L_0/2} (JA^*J)\Omega_0 \\ &= J(JA^*J) e^{-\beta L^{(\ell)}/2} e^{\beta L_0/2}\Omega_0 \\ &= A^* e^{\beta L^{(\mathrm{r})}/2} e^{-\beta L_0/2}\Omega_0, \end{aligned}$$

see Lemma B.1(iii). Thus, the operator $e^{-\beta L}$ is the modular operator associated with ω and since L is the perturbed Liouville operator w.r.t. ω we conclude that the (α, β)-KMS condition for ω is fulfilled. ■

C Analytic Continuations of Operators and Vectors

In Lemma 2.3 of Section 2.2 we managed to find a resolvent representation for the unitary group $e^{iK^{(s)}t}$ for real s. The task, however, is to extend the relation (2.15) to complex parameters s, in particular we want to find a representation of the group $U(t) = e^{iKt}$ for $K = K^{(-i/2)}$. While we could show the analyticity in s of the l.h.s. of (2.15) already in Section 2.2 it is not until now that we address the issue of analyticity of the r.h.s. of (2.15). Although the spectrum of $K^{(s)}$ is located on the real axis for $s \in \mathbb{R}$ we lose control over it as soon as we complexify the parameter. The problem here is the lack of coercivity – the perturbation $I^{(s)}$ is not relatively L_0-bounded. We bypass that difficulty by performing a spectral deformation on $K^{(s)}$ as introduced in Section 2.2.2. The advantage of this particular deformation is that the deformed perturbation $I_\theta^{(s)}$ is now relatively bounded w.r.t. the deformed free Liouville operator $L_{0,\theta}$ which in turn becomes sectorial. This requires that the dilation parameter δ, which is responsible for the rotation of the continuous spectrum into the upper half plane, has sufficiently large imaginary part.

Throughout the whole chapter we fix

$$0 < \delta_0 < \frac{\pi}{4}, \qquad 0 < \tau_0 < 2\pi\beta_{\max}^{-1}$$

and we introduce the following notation

$$D^+_{\delta_0,\tau_0} := D^+_{\delta_0} \times S^+_{\tau_0} \subseteq \mathbb{C}^2$$

for a subset of

$$D_{\delta_0,\tau_0} = \left\{ (\delta,\tau) \in \mathbb{C}^2 \;\middle|\; |\operatorname{Im}(\delta)| < \delta_0,\, |\tau| < \tau_0 \right\}$$

where

$$\begin{aligned} D^+_{\delta_0} &:= \{ \delta \in \mathbb{C} \,|\, 0 < \operatorname{Im}(\delta) < \delta_0 \}, \\ S^+_{\tau_0} &:= \{ \tau \in \mathbb{C} \,|\, 0 < |\tau| < 2|\operatorname{Im}(\tau)| < \tau_0 \}. \end{aligned}$$

C.1 Spectral Deformation Analyticity of $K_\theta^{(s)}$

In this section we provide analyticity results of the resolvent $\left(z - K^{(s)}\right)^{-1}$ under spectral deformation. In this section we follow closely the arguments of [8, App. A]. The operator $M_{[\theta]}$ defined in (A.9) will play a crucial role in the estimations. First we consider the invertibility of $\left(z - K_\theta^{(s)}\right)^{-1}$ for spectral parameters far away from the real axis.

Lemma C.1 *Let* $\mathrm{Im}(z) \leq -2$ *and choose the deformation parameters as* $\theta = (\delta, \tau) \in \overline{D^+_{\delta_0,\tau_0}}$. *Choose* $s \in \mathbb{S}_{\frac{\varepsilon_0}{2}}$.

(i) Assume further that $s \in \mathbb{R}$.

(ii) Assume further that $\mathrm{Im}(\delta) \in [\frac{\pi}{8}, \frac{\pi}{4}]$.

Then the operator $\left(z - K_\theta^{(s)}\right)$ *is invertible on a dense set and its inverse extends to a bounded operator with norm*

$$\left\| \left(z - K_\theta^{(s)}\right)^{-1} \right\| \leq \frac{1}{\mathrm{dist}\left\{z, \mathrm{NumRan}\left(K_\theta^{(s)}\right)\right\}}.$$

Proof. Our first observations is that we may assume without loss of generality that $\delta = i\delta'$ is purely imaginary. Note that $K_\theta^{(s)} = \mathfrak{D}_{\mathrm{d}}(\mathrm{Re}(\delta)) K_{\theta'}^{(s)} \mathfrak{D}_{\mathrm{d}}(\mathrm{Re}(\delta))^{-1}$, where $\theta' := (i\,\mathrm{Im}(\delta), \tau)$, and $\mathfrak{D}_{\mathrm{d}}(\mathrm{Re}(\delta))$ is a unitary operator. Since the numerical range and the norm of an operator remain invariant under unitary conjugation we may assume $\mathrm{Re}(\delta) = 0$. In this case we have $\mathrm{Im}(L_0) = M_{[\theta]} + (\mathrm{Im}(\tau) - |\tau|)N_{\mathrm{res}}$, note that $\sin(\mathrm{Im}(\delta)) \geq 0$.

Now, we consider

$$\begin{aligned} K_\theta^{(s)*} &= \left[L_{\mathrm{p}} + \cos(\delta) L_{\mathrm{res}} + \mathrm{Re}(\tau) N_{\mathrm{res}} + g\,\mathrm{Re}\left(I_\theta^{(s)}\right)\right] \\ &\quad -i\left[\sin(\delta') L_{\mathrm{aux}} + \mathrm{Im}(\tau) N_{\mathrm{res}} + g\,\mathrm{Im}\left(I_\theta^{(s)}\right)\right]. \end{aligned}$$

For $\psi \in \mathcal{D}(L_{\text{aux}} + N_{\text{res}}) \subseteq \mathcal{D}(K_\theta^{(s)*})$ we have

$$\begin{aligned}
&\operatorname{Im} \left\langle (M_{[\theta]}+1)^{-1}\psi \,\middle|\, K_\theta^{(s)*}\psi \right\rangle \\
&= \frac{1}{2i}\left[\left\langle (M_{[\theta]}+1)^{-1}\psi \,\middle|\, gI_\theta^{(s)*}\psi \right\rangle - \left\langle gI_\theta^{(s)*}\psi \,\middle|\, (M_{[\theta]}+1)^{-1}\psi \right\rangle\right] \\
&\quad - \operatorname{Im}\left\langle (M_{[\theta]}+1)^{-1}\psi \,\middle|\, i[\sin(\delta')L_{\text{aux}} + \operatorname{Im}(\tau)N_{\text{res}}]\psi \right\rangle \\
&= \frac{1}{2i}\left\langle \psi \,\middle|\, \left[(M_{[\theta]}+1)^{-1}, g\operatorname{Re}\left(I_\theta^{(s)}\right)\right]\psi \right\rangle - \left\langle \psi \,\middle|\, \frac{\sin(\delta')L_{\text{aux}} + \operatorname{Im}(\tau)N_{\text{res}}}{M_{[\theta]}+1}\psi \right\rangle \\
&\quad + \frac{1}{2i}\left[\left\langle ig\operatorname{Im}\left(I_\theta^{(s)}\right)\psi \,\middle|\, (M_{[\theta]}+1)^{-1}\psi \right\rangle - \left\langle (M_{[\theta]}+1)^{-1}\psi \,\middle|\, ig\operatorname{Im}\left(I_\theta^{(s)}\right)\psi \right\rangle\right] \\
&= \frac{1}{2i}\left\langle \psi \,\middle|\, \left[(M_{[\theta]}+1)^{-1}, g\operatorname{Re}\left(I_\theta^{(s)}\right)\right]\psi \right\rangle - \left\langle \psi \,\middle|\, \frac{\sin(\delta')L_{\text{aux}} + \operatorname{Im}(\tau)N_{\text{res}}}{M_{[\theta]}+1}\psi \right\rangle \\
&\quad - \operatorname{Re}\left\langle g\operatorname{Im}\left(I_\theta^{(s)}\right)\psi \,\middle|\, (M_{[\theta]}+1)^{-1}\psi \right\rangle . \qquad \text{(C.1)}
\end{aligned}$$

We estimate the terms separately. First, we consider

$$\begin{aligned}
&\left|\left\langle \psi \,\middle|\, \left[(M_{[\theta]}+1)^{-1}, g\operatorname{Re}\left(I_\theta^{(s)}\right)\right]\psi \right\rangle\right| \\
&= \left|\left\langle (M_{[\theta]}+1)^{-1/2}\psi \,\middle|\, (M_{[\theta]}+1)^{-1/2}\left[M_{[\theta]}, g\operatorname{Re}\left(I_\theta^{(s)}\right)\right](M_{[\theta]}+1)^{-1}\psi \right\rangle\right| \\
&\le Cg\left\|(M_{[\theta]}+1)^{-1/2}\psi\right\|^2 \\
&= Cg\left\langle \psi \,\middle|\, (M_{[\theta]}+1)^{-1}\psi \right\rangle , \qquad \text{(C.2)}
\end{aligned}$$

for some positive constant C, using Lemma A.6. To estimate the last term of (C.1) we consider the two cases of additional assumptions.

(i) Since $s \in \mathbb{R}$ Corollary A.8 implies that

$$\left|\left\langle g\operatorname{Im}\left(I_\theta^{(s)}\right)\psi \,\middle|\, (M_{[\theta]}+1)^{-1}\psi \right\rangle\right| \le gC'\left\|\psi\right\|^2 , \qquad \text{(C.3)}$$

for a positive constant C'.

(ii) Since $\operatorname{Im}(\delta) \in [\frac{\pi}{8}, \frac{\pi}{4}]$ and therefore $\sin(\operatorname{Im}(\delta)) \ge \frac{1}{3}$ we obtain

$$\left|\left\langle g\operatorname{Im}\left(I_\theta^{(s)}\right)\psi \,\middle|\, (M_{[\theta]}+1)^{-1}\psi \right\rangle\right| \le \frac{gC'}{|\sin(\operatorname{Im}(\delta))|}\left\|\psi\right\|^2 \le 3gC'\left\|\psi\right\|^2 , \qquad \text{(C.4)}$$

for a positive constant C', using Lemma A.5.

Plugging the estimates (C.2, C.3, C.4) into (C.1) gives

$$\begin{aligned}
&\operatorname{Im}\left\langle (M_{[\theta]}+1)^{-1}\psi \,\middle|\, K_\theta^{(s)^*}\psi \right\rangle \\
&\leq\; -\left\langle \psi \,\middle|\, \frac{\sin(\delta')L_{\text{aux}}+\operatorname{Im}(\tau)N_{\text{res}}}{M_{[\theta]}+1}\psi \right\rangle + Cg\,\langle\, \psi \,|\, (M_{[\theta]}+1)^{-1}\psi \,\rangle \\
&\quad +C'g\,\|\psi\|^2 \\
&=\; (1+Cg)\,\langle\, \psi \,|\, (M_{[\theta]}+1)^{-1}\psi \,\rangle + (C'g-1)\,\|\psi\|^2 \\
&\quad +\left\langle \psi \,\middle|\, \frac{(|\tau|-\operatorname{Im}(\tau))\,N_{\text{res}}}{M_{[\theta]}+1}\psi \right\rangle \\
&\leq\; (1+Cg)\,\langle\, \psi \,|\, (M_{[\theta]}+1)^{-1}\psi \,\rangle + \left(C'g-\frac{\operatorname{Im}(\tau)}{|\tau|}\right)\|\psi\|^2 \\
&\leq\; (1+Cg)\,\langle\, \psi \,|\, (M_{[\theta]}+1)^{-1}\psi \,\rangle + \left(C'g-\frac{1}{2}\right)\|\psi\|^2 \\
&\leq\; (1+Cg)\,\langle\, \psi \,|\, (M_{[\theta]}+1)^{-1}\psi \,\rangle, \qquad\qquad \text{(C.5)}
\end{aligned}$$

for g sufficiently small. The above inequality extends to all $\psi \in \mathcal{D}(K_\theta^{(s)^*})$. We now aim to show that the kernel of $(z-K_\theta^{(s)})^*$ is trivial which in turn implies that the range of $(z-K_\theta^{(s)})$ is dense. To this end we choose $\psi \in \ker\left[(z-K_\theta^{(s)})^*\right]$. With the help of (C.5) we end up with

$$\begin{aligned}
\operatorname{Im}(z)\,\langle\, (M_{[\theta]}+1)^{-1}\psi \,|\, \psi \,\rangle \;&=\; -\operatorname{Im}\langle\, (M_{[\theta]}+1)^{-1}\psi \,|\, \overline{z}\psi \,\rangle \\
&=\; -\operatorname{Im}\left\langle (M_{[\theta]}+1)^{-1}\psi \,\middle|\, K_\theta^{(s)^*}\psi \right\rangle \\
&\geq\; -(1+Cg)\,\langle\, (M_{[\theta]}+1)^{-1}\psi \,|\, \psi \,\rangle \\
&\geq\; -\frac{3}{2}\,\langle\, (M_{[\theta]}+1)^{-1}\psi \,|\, \psi \,\rangle
\end{aligned}$$

for $g \ll 1$. Since $\operatorname{Im}(z) \leq -2$ we conclude that $\psi = 0$.

The localization of the numerical range of $K_\theta^{(s)}$, Proposition A.9, along with [19, Prop. 19.7] imply that $z \in \operatorname{spec}(K_\theta^{(s)})$ and

$$\left\| \left(z-K_\theta^{(s)}\right)^{-1} \right\| \leq \frac{1}{\operatorname{dist}\left\{z, \operatorname{NumRan}\left(K_\theta^{(s)}\right)\right\}}.$$

■

The previous lemma has an important implication on the decay of the resolvent $\left(z-K_\theta^{(s)}\right)^{-1}$ as $\operatorname{Re}(z) \to \infty$.

Lemma C.2 *Under the assumptions of Lemma C.1 on the parameters* $\theta = (\delta, \tau)$ *and* s *we have*

$$\operatorname{spec}\left(K_\theta^{(s)}\right) \subseteq \{ z \in \mathbb{C} \,|\, \operatorname{Im}(z) > -2 \}$$

and for $x \in \mathbb{R}$ *with* $|x| \geq 2\,\|L_{\mathrm{p}}\|$ *holds*

$$\left\| \left((x-2i) - K_\theta^{(s)} \right)^{-n} \right\| \leq \left[\frac{1}{|x|+1} \max \left\{ \sqrt{8}\,, \frac{16\sqrt{2}}{\sin(\operatorname{Im}(\delta))} \right\} \right]^n$$

for each $n \in \mathbb{N}$.

Proof. Since $\operatorname{NumRan}\left(K_\theta^{(s)}\right) \subseteq \{ z \in \mathbb{C} \,|\, \operatorname{Im}(z) \geq -1 \}$ by Proposition A.9 we obtain

$$\operatorname{spec}\left(K_\theta^{(s)}\right) \subseteq \{ z \in \mathbb{C} \,|\, \operatorname{Im}(z) > -2 \}$$

using Lemma C.1. To prove the norm bound it is sufficient to consider the case $n = 1$. We use Proposition A.9 to estimate for $\zeta \in \operatorname{NumRan}\left(K_\theta^{(s)}\right)$ and $\delta' := \operatorname{Im}(\delta)$,

$$\begin{aligned}
|(x-2i) - \zeta| &\geq \frac{1}{\sqrt{2}} \left[|x - \operatorname{Re}(\zeta)| + |2 + \operatorname{Im}(\zeta)| \right] \\
&\geq \frac{1}{\sqrt{2}} \left[|x - \operatorname{Re}(\zeta)| + 1 + \max \left\{ 0, \frac{\sin(\delta')}{8} (|\operatorname{Re}(\zeta)| - \|L_{\mathrm{p}}\|) \right\} \right] \\
&\geq \frac{1}{\sqrt{2}} \left[|x - \operatorname{Re}(\zeta)| + 1 + \max \left\{ 0, \frac{\sin(\delta')}{8} \left(\frac{|x|}{2} - |x - \operatorname{Re}(\zeta)| \right) \right\} \right] \\
&\geq \begin{cases} \frac{1}{\sqrt{2}} \left[\frac{|x|}{2} + 1 \right], & \frac{|x|}{2} \leq |x - \operatorname{Re}(\zeta)|, \\ \frac{1}{\sqrt{2}} \left[\frac{\sin(\delta')|x|}{16} + 1 \right], & \frac{|x|}{2} > |x - \operatorname{Re}(\zeta)| \end{cases} \\
&\geq [|x| + 1] \min \left\{ \frac{1}{\sqrt{8}}, \frac{\sin(\delta')}{16\sqrt{2}} \right\}.
\end{aligned}$$

Lemma C.1 finally yields

$$\begin{aligned}
\left\| \left((x-2i) - K_\theta^{(s)} \right)^{-1} \right\| &\leq \frac{1}{\operatorname{dist}\left\{ (x-2i), \operatorname{NumRan}\left(K_\theta^{(s)}\right) \right\}} \\
&\leq \frac{1}{|x|+1} \max \left\{ \sqrt{8}\,, \frac{16\sqrt{2}}{\sin(\delta')} \right\}.
\end{aligned}$$

∎

We need another preparatory lemma before we are in position to state the theorem about spectral deformation analyticity.

Lemma C.3 *Choose the parameters* $\theta = (\delta, \tau) \in \mathbb{C}^2$ *and* $s \in \mathbb{C}$ *as in Lemma C.1 and assume that* $\mathrm{Im}(z) \leq -2$. *Then, for* g *sufficiently small, the operators*

$$B^{\pm}_{\theta,s} := (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{\mp 1} \left(z - K^{(s)}_{\theta}\right)^{-1} (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{\pm 1},$$

defined on $\mathcal{D}(L_{\mathrm{aux}} + N_{\mathrm{res}})$, *extend to bounded operators of norm*

$$\begin{aligned} \left\|B^{\pm}_{\theta,s}\right\| &\leq e^{2|\mathrm{Re}(\delta)|} \left[1 + \mathcal{O}(g)\right] \left\| \left(z - K^{(s)}_{\theta}\right)^{-1} \right\| \\ &\leq \frac{e^{2|\mathrm{Re}(\delta)|} \left[1 + \mathcal{O}(g)\right]}{\mathrm{dist}\left\{z, \mathrm{NumRan}\left(K^{(s)}_{\theta}\right)\right\}}. \end{aligned} \tag{C.6}$$

Proof. We may restrict our considerations on the case that $\delta = i\delta$ is purely imaginary, by the following argument. Let $\theta' := (i\,\mathrm{Im}(\delta), \tau)$, then $\theta = (\mathrm{Re}(\delta), 0) + \theta'$ holds and therefore

$$\begin{aligned} B^{\pm}_{\theta,s} &= (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{\mp 1} \mathfrak{D}_{\mathrm{d}}(\mathrm{Re}(\delta)) \left(z - K^{(s)}_{\theta'}\right)^{-1} \mathfrak{D}_{\mathrm{d}}(\mathrm{Re}(\delta))^{-1} \\ &\quad \times (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{\pm 1} \\ &= \mathfrak{D}_{\mathrm{d}}(\mathrm{Re}(\delta)) \left(d\Gamma_{\mathrm{gl}}\left(e^{-\mathrm{Re}(\delta)\,\mathrm{sgn}(u)}|u|\right) + N_{\mathrm{res}} + 1\right)^{\mp 1} \left(z - K^{(s)}_{\theta'}\right)^{-1} \\ &\quad \times \left(d\Gamma_{\mathrm{gl}}\left(e^{-\mathrm{Re}(\delta)\,\mathrm{sgn}(u)}|u|\right) + N_{\mathrm{res}} + 1\right)^{\pm 1} \mathfrak{D}_{\mathrm{d}}(\mathrm{Re}(\delta))^{-1}. \end{aligned}$$

Since $\mathfrak{D}_{\mathrm{d}}(\mathrm{Re}(\delta))$ is unitary we get

$$\begin{aligned} \left\|B^{\pm}_{\theta,s}\right\| &= \Big\| \left(d\Gamma_{\mathrm{gl}}\left(e^{-\mathrm{Re}(\delta)\,\mathrm{sgn}(u)}|u|\right) + N_{\mathrm{res}} + 1\right)^{\mp 1} \left(z - K^{(s)}_{\theta'}\right)^{-1} \\ &\qquad \times \left(d\Gamma_{\mathrm{gl}}\left(e^{-\mathrm{Re}(\delta)\,\mathrm{sgn}(u)}|u|\right) + N_{\mathrm{res}} + 1\right)^{\pm 1} \Big\| \\ &\leq \left\| \left(\frac{L_{\mathrm{aux}} + N_{\mathrm{res}} + 1}{\left(d\Gamma_{\mathrm{gl}}\left(e^{\mp|\mathrm{Re}(\delta)|\,\mathrm{sgn}(u)}|u|\right) + N_{\mathrm{res}} + 1\right)} \right)^{\pm 1} \right\| \left\|B^{\pm}_{\theta',s}\right\| \\ &\quad \times \left\| \left(\frac{L_{\mathrm{aux}} + N_{\mathrm{res}} + 1}{\left(d\Gamma_{\mathrm{gl}}\left(e^{\pm|\mathrm{Re}(\delta)|\,\mathrm{sgn}(u)}|u|\right) + N_{\mathrm{res}} + 1\right)} \right)^{\mp 1} \right\| \\ &\leq e^{2|\mathrm{Re}(\delta)|} \left\|B^{\pm}_{\theta',s}\right\|. \end{aligned}$$

From now on we assume that $\delta = i\delta' \in i\mathbb{R}$.

Consider

$$\begin{aligned} &\left(z - K^{(s)}_{\theta}\right)^{-1} (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} - (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} \left(z - K^{(s)}_{\theta}\right)^{-1} \\ &= \left(z - K^{(s)}_{\theta}\right)^{-1} (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} \left[K^{(s)}_{\theta}, L_{\mathrm{aux}} + N_{\mathrm{res}}\right] \\ &\quad \times (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} \left(z - K^{(s)}_{\theta}\right)^{-1}. \end{aligned}$$

Both sides are bounded operators, because the commutator is relatively $(L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)$-bounded. This, in turn, can be seen by rewriting

$$\left[K_\theta^{(s)}, L_{\mathrm{aux}} + N_{\mathrm{res}}\right] = g\left[I_\theta^{(s)}, L_{\mathrm{aux}} + N_{\mathrm{res}}\right]$$

and using that the commutator on the r.h.s. can be expressed as linear combination of creation and annihilation operators. Those are, due to Lemma A.3, relatively bounded w.r.t. N_{res} with relative bound given by the norms of the coupling functions. Therefore

$$\left\|(L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1}\left[K_\theta^{(s)}, L_{\mathrm{aux}} + N_{\mathrm{res}}\right]\right\| = \mathcal{O}(g)\,.$$

Thus, the operator

$$A := \left(z - K_\theta^{(s)}\right)^{-1}(L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1}\left[K_\theta^{(s)}, L_{\mathrm{aux}} + N_{\mathrm{res}}\right]$$

is bounded with norm $\|A\| = \mathcal{O}(g)$ where we used Lemma C.1 and that

$$\operatorname{dist}\left\{z, \operatorname{NumRan}\left(K_\theta^{(s)}\right)\right\} \geq 1 \qquad \text{for } \operatorname{Im}(z) \leq -2,$$

see Proposition A.9. Thus,

$$(1 + A)(L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1}\left(z - K_\theta^{(s)}\right)^{-1}(L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{+1} = \left(z - K_\theta^{(s)}\right)^{-1}$$

and

$$\begin{aligned} B_{\theta,s}^{\pm} &= (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1}\left(z - K_\theta^{(s)}\right)^{-1}(L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{+1} \\ &= (1 + A)^{-1}\left(z - K_\theta^{(s)}\right)^{-1} \end{aligned}$$

which is bounded with norm

$$\left\|B_{\theta,s}^{\pm}\right\| \leq [1 + \mathcal{O}(g)]\left\|\left(z - K_\theta^{(s)}\right)^{-1}\right\| \leq \frac{1 + \mathcal{O}(g)}{\operatorname{dist}\left\{z, \operatorname{NumRan}\left(K_\theta^{(s)}\right)\right\}}.$$

The proof for $B_{\theta,s}^{-}$ is similar. ∎

Proposition C.4 (Strong Analyticity of $K_\theta^{(s)}$) *Let $s \in \mathbb{S}_{\varepsilon_0}$. Then the map*

$$D_{\delta_0,\tau_0} \ni \theta \mapsto K_\theta^{(s)}$$

is strongly analytic on $\mathcal{D}(L_{\mathrm{aux}} + N_{\mathrm{res}})$, in each variable separately.

Proof. We set

$$\begin{aligned} \partial_\delta L_{0,\theta} &= \sinh(\delta) L_{\rm res} + \cosh(\delta) L_{\rm aux}, \\ \partial_\tau L_{0,\theta} &= N_{\rm res}, \end{aligned} \tag{C.7}$$

and

$$\begin{aligned} \partial_{\theta_j} K_\theta^{(s)} &= \partial_{\theta_j} L_{0,\theta} + g \partial_{\theta_j} I_\theta^{(s)} \\ &= \partial_{\theta_j} L_{0,\theta} + g \left[a_{\rm gl}^* \left(\partial_{\theta_j} F_\theta^{(s)} \right) + a_{\rm gl} \left(\partial_{\overline{\theta}_j} F_{\overline{\theta}}^{(\overline{s})} \right) \right] \end{aligned} \tag{C.8}$$

with θ_j standing for δ or τ, resp. All above operators are relative bounded w.r.t. $(L_{\rm aux} + N_{\rm res} + 1)$. We consider, representatively, the differentiability w.r.t. δ only. We choose $\theta = (\delta, \tau) \in D_{\delta_0, \tau_0}$ and $\theta' = (\delta', 0)$ and compute

$$\begin{aligned} &\left\| \left[\frac{L_{0,\theta+\theta'} - L_{0,\theta}}{\delta'} - \partial_\delta L_{0,\theta} \right] (L_{\rm aux} + N_{\rm res} + 1)^{-1} \right\| \\ &\leq \quad \left| \frac{\cosh(\delta+\delta') - \cosh(\delta)}{\delta'} - \sinh(\delta) \right| \left\| \frac{L_{\rm res}}{L_{\rm aux} + N_{\rm res} + 1} \right\| \\ &\qquad + \left| \frac{\sinh(\delta+\delta') - \sinh(\delta)}{\delta'} - \cosh(\delta) \right| \left\| \frac{L_{\rm aux}}{L_{\rm aux} + N_{\rm res} + 1} \right\| \\ &\xrightarrow{\delta' \to 0} \quad 0, \end{aligned}$$

because of $|L_{\rm res}| \leq |L_{\rm aux}|$, and

$$\begin{aligned} &\left\| \left[\frac{I_{\theta+\theta'}^{(s)} - I_\theta^{(s)}}{\delta'} - \partial_\delta I_\theta^{(s)} \right] (L_{\rm aux} + N_{\rm res} + 1)^{-1} \right\| \\ &\leq \quad \left\| a_{\rm gl}^* \left(\frac{F_{\theta+\theta'}^{(s)} - F_\theta^{(s)}}{\delta'} - \partial_\delta F_\theta^{(s)} \right) (L_{\rm aux} + N_{\rm res} + 1)^{-1} \right\| \\ &\qquad + \left\| a_{\rm gl} \left(\frac{F_{\overline{\theta}+\overline{\theta'}}^{(\overline{s})} - F_{\overline{\theta}}^{(\overline{s})}}{\overline{\delta'}} - \partial_{\overline{\delta}} F_{\overline{\theta}}^{(\overline{s})} \right) (L_{\rm aux} + N_{\rm res} + 1)^{-1} \right\| \\ &\leq \quad \left\| \frac{F_{\theta+\theta'}^{(s)} - F_\theta^{(s)}}{\delta'} - \partial_\delta F_\theta^{(s)} \right\|_{L^2[\Upsilon; \mathcal{B}(\mathcal{H}_{\rm p}^2)]} + \left\| \frac{F_{\overline{\theta}+\overline{\theta'}}^{(\overline{s})} - F_{\overline{\theta}}^{(\overline{s})}}{\overline{\delta'}} - \partial_{\overline{\delta}} F_{\overline{\theta}}^{(\overline{s})} \right\|_{L^2[\Upsilon; \mathcal{B}(\mathcal{H}_{\rm p}^2)]} \\ &\xrightarrow{\delta' \to 0} \quad 0, \end{aligned}$$

which holds true since $\theta \mapsto F_\theta^{(s)}$ is analytic because of Hypothesis VII-1.12 and Remark 1.13. Hence, the operator $K_\theta^{(s)}$ is strongly analytic on $\mathcal{D}(L_{\rm aux} + N_{\rm res})$ in δ. The analyticity w.r.t. τ is proved in the same way. ∎

From the Lemmata C.1, C.3 and Proposition C.4 we obtain the analyticity of the regularized resolvent $(L_{\rm aux} + N_{\rm res} + 1)^{-1}(z - K_\theta^{(s)})^{-1}(L_{\rm aux} + N_{\rm res} + 1)^{-1}$ in the deformation parameters. The awareness of the need to regularize the resolvent in order to show deformation analyticity originates from [8, App. A].

Theorem C.5 (Deformation Analyticity) *Let* $s \in \mathbb{R}$ *and* $\mathrm{Im}(z) \leq -2$. *The regularized resolvent*

$$D^+_{\delta_0,\tau_0} \ni \theta \mapsto R^{(s)}_\theta := (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} \left(z - K^{(s)}_\theta\right)^{-1} (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} \quad \text{(C.9)}$$

as a function of $\theta = (\delta, \tau)$ *is analytic, in each variable separately, on the domain* $D^+_{\delta_0,\tau_0}$. *More precisely, for fixed* $\delta \in \overline{D^+_{\delta_0}}$ *the map* $\tau \mapsto R^{(s)}_{(\delta,\tau)}$ *is analytic on the domain* $S^+_{\tau_0}$ *and for fixed* $\tau \in S^+_{\tau_0}$ *the map* $\delta \mapsto R^{(s)}_{(\delta,\tau)}$ *is analytic on the domain* $D^+_{\delta_0}$. *Moreover, the map* $\theta \mapsto R^{(s)}_\theta$ *is continuously extendable to* $\partial D^+_{\delta_0,\tau_0}$.

Proof. Note that, due to Proposition C.4, the operators $L_{0,\theta}$ and $K^{(s)}_\theta$ are strongly differentiable w.r.t. $\theta = (\delta, \tau)$ with the partial derivatives given in (C.7, C.8). We consider, representatively, the differentiability w.r.t. δ only. Let $\theta = (\delta, \tau) \in \mathcal{D}^+_{\delta_0,\tau_0}$, denote $\theta' = (\delta', 0)$ and consider

$$\begin{aligned}
&(L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} \Bigg[\frac{\left(z - K^{(s)}_{\theta+\theta'}\right)^{-1} - \left(z - K^{(s)}_\theta\right)^{-1}}{\delta'} \\
&\qquad - \left(z - K^{(s)}_\theta\right)^{-1} \partial_\delta K^{(s)}_\theta \left(z - K^{(s)}_\theta\right)^{-1} \Bigg] (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} \\
= \;& (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} \Bigg[\left(z - K^{(s)}_{\theta+\theta'}\right)^{-1} \frac{K^{(s)}_{\theta+\theta'} - K^{(s)}_\theta}{\delta'} \\
&\qquad - \left(z - K^{(s)}_\theta\right)^{-1} \partial_\delta K^{(s)}_\theta \Bigg] \left(z - K^{(s)}_\theta\right)^{-1} (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} \\
= \;& B^+_{\theta+\theta',s} (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} \left[\frac{K^{(s)}_{\theta+\theta'} - K^{(s)}_\theta}{\delta'} - \partial_\delta K^{(s)}_\theta \right] (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} B^-_{\theta,s} \\
& + B^+_{\theta+\theta',s} (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} \left[K^{(s)}_{\theta+\theta'} - K^{(s)}_\theta \right] \left(z - K^{(s)}_\theta\right)^{-1} \\
& \quad \times \partial_\delta K^{(s)}_\theta (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1} B^-_{\theta,s} \\
=: \;& Q_{\delta'}.
\end{aligned}$$

Because of Lemma C.3 the operators $B^-_{\theta,s}$ and $B^+_{\theta+\theta',s}$ are uniformly bounded as $\delta' \to 0$ such that there is a positive constant C with

$$\begin{aligned}
&\|Q_{\delta'}\| \\
&\leq \quad C\Bigg[\left\|(L_{\text{aux}}+N_{\text{res}}+1)^{-1}\left[\frac{K^{(s)}_{\theta+\theta'}-K^{(s)}_{\theta}}{\delta'}-\partial_\delta K^{(s)}_\theta\right](L_{\text{aux}}+N_{\text{res}}+1)^{-1}\right\| \\
&\qquad +\left\|(L_{\text{aux}}+N_{\text{res}}+1)^{-1}\left[K^{(s)}_{\theta+\theta'}-K^{(s)}_{\theta}\right]\right\| \\
&\qquad \times\left\|\partial_\delta K^{(s)}_\theta (L_{\text{aux}}+N_{\text{res}}+1)^{-1}\right\|\Bigg] \\
&\xrightarrow{\delta'\to 0} \quad 0.
\end{aligned}$$

Here, we used the strong analyticity of $\theta \mapsto K^{(s)}_\theta$ on the domain $\mathcal{D}(L_{\text{aux}}+N_{\text{res}})$ as provided in Proposition C.4. This concludes the proof of analyticity of (C.9) in δ. To prove continuity in $\theta \in D^+_{\delta_0,\tau_0}$ we consider

$$\begin{aligned}
&(L_{\text{aux}}+N_{\text{res}}+1)^{-1}\left[\left(z-K^{(s)}_{\theta+\theta'}\right)^{-1}-\left(z-K^{(s)}_{\theta}\right)^{-1}\right](L_{\text{aux}}+N_{\text{res}}+1)^{-1} \\
&= \quad B^+_{\theta+\theta',s}(L_{\text{aux}}+N_{\text{res}}+1)^{-1}\left[K^{(s)}_{\theta+\theta'}-K^{(s)}_{\theta}\right](L_{\text{aux}}+N_{\text{res}}+1)^{-1}B^-_{\theta,s} \\
&\xrightarrow{\delta'\to 0} \quad 0,
\end{aligned}$$

by the same arguments as above. The analyticity and continuity w.r.t. τ is proved in the same way. ∎

C.2 Analytic Continuation of $K^{(s)}_\theta$ in s

Theorem C.6 *Let* $\mathrm{Im}(z) \leq -2$ *and choose* $\theta = (\delta,\tau) \in \overline{D^+_{\delta_0,\tau_0}}$ *such that* $\delta = i\delta'$ *with* $\delta' \in [\frac{\pi}{8}, \frac{\pi}{4}]$ *and* $\mathrm{Re}(\tau) = 0$.

(i) Then the regularized resolvent

$$\begin{aligned}
\mathbb{S}_{\frac{\varepsilon_0}{2}} \ni s \mapsto R^{(s)}_\theta &\equiv R^{(s)}_\theta(z) \\
&= (L_{\text{aux}}+N_{\text{res}}+1)^{-1}\left(z-K^{(s)}_\theta\right)^{-1}(L_{\text{aux}}+N_{\text{res}}+1)^{-1}
\end{aligned}$$

is analytic in s.

(ii) Then, for $\varphi, \psi \in \mathcal{H}^2$, the improper Riemann integral

$$\mathbb{S}_{\frac{\varepsilon_0}{2}} \ni s \mapsto p(s) := \int_{\mathbb{R}-2i} dz \left\langle \varphi \,\middle|\, R_\theta^{(s)}(z)\psi \right\rangle e^{izt}$$

is analytic in s.

Proof.

(i) Note that $K_\theta^{(s)} = L_{0,\theta} + g\left[\pi(v) - \pi'\left(\gamma_0^{\bar{s}}(v)\right)\right]_\theta$ such that

$$\begin{aligned} \partial_s K_\theta^{(s)} &= -g\partial_s \pi'\left(\gamma_0^{\bar{s}}(v)\right)_\theta \\ &= -g\left[a_{\mathrm{gl}}^*\left(\partial_s \mathcal{G}'_{(s\delta\vec{\beta}),\theta}\right) + a_{\mathrm{gl}}\left(\partial_{\bar{s}} \mathcal{G}'_{(\bar{s}\delta\vec{\beta}),\bar{\theta}}\right)\right], \end{aligned}$$

in a strong sense on $\mathcal{D}(L_{\mathrm{aux}} + N_{\mathrm{res}})$. The derivatives of the coupling functions are explicitly given by

$$\begin{aligned} \partial_s \mathcal{G}'_{(s\delta\vec{\beta}),\theta}(u,\Sigma,r) &= i\delta\beta_r j_\theta(u)\mathcal{G}'_{(s\delta\vec{\beta}),\theta}(u,\Sigma,r) + i\delta\beta_{\mathrm{p}}\left[L_{\mathrm{p}}, \mathcal{G}'_{(s\delta\vec{\beta}),\theta}(u,\Sigma,r)\right] \\ &= i\left(\delta\beta_r u\mathcal{G}'_{(s\delta\vec{\beta})} + \delta\beta_{\mathrm{p}}\left[L_{\mathrm{p}}, \mathcal{G}'_{(s\delta\vec{\beta})}(\,\cdot\,)\right]\right)_\theta (u,\Sigma,r), \end{aligned}$$

recall the definition (2.13). We abbreviate

$$\begin{aligned} \partial_s R_\theta^{(s)} := \; & (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1}\left(z - K_\theta^{(s)}\right)^{-1}\left[\partial_s K_\theta^{(s)}\right]\left(z - K_\theta^{(s)}\right)^{-1} \\ & \times (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)^{-1}. \end{aligned}$$

Let $s \in \mathbb{S}_{\frac{\varepsilon_0}{2}}$ and $s' \in \mathbb{C}$ such that $s + s' \in \mathbb{S}_{\frac{\varepsilon_0}{2}}$. With the same arguments as in the proof of Theorem C.5 we obtain that

$$\lim_{s' \to 0} \frac{R_\theta^{(s+s')} - R_\theta^{(s)}}{s'} = \partial_s R_\theta^{(s)} \tag{C.10}$$

using the analyticity of $s \mapsto \mathcal{G}'_{(s\delta\vec{\beta}),\theta}$ in the L^2 sense due to the Hypotheses VI-1.11 and VII-1.12 and the dominated convergence theorem.

(ii) We first prove the convergence of the improper Riemann integral. For $z = x - 2i \in \mathbb{R} - 2i$ with $|x| \geq 2\|L_{\mathrm{p}}\|$ we have by Lemma C.2 the following estimate on the matrix element,

$$\left|\left\langle \varphi \,\middle|\, \left(z - K_\theta^{(s)}\right)^{-n} \psi \right\rangle\right| \leq \frac{68^n \|\varphi\| \|\psi\|}{(|x|+1)^n},$$

for $n \in \mathbb{N}$, which in turn shows that the improper Riemann integral

$$\begin{aligned}
&\lim_{a,b\to\infty} \int_{-a}^{b} dx \left\langle \varphi \,\middle|\, \left((x-2i) - K_\theta^{(s)}\right)^{-1} \psi \right\rangle e^{i(x-2i)t} \\
&= \lim_{a,b\to\infty} \left\langle \varphi \,\middle|\, \left((x-2i) - K_\theta^{(s)}\right)^{-1} \psi \right\rangle \frac{e^{i(x-2i)t}}{it} \Bigg|_{x=-a}^{x=b} \\
&\quad + \lim_{a,b\to\infty} \int_{-a}^{b} dx \left\langle \varphi \,\middle|\, \left((x-2i) - K_\theta^{(s)}\right)^{-2} \psi \right\rangle \frac{e^{i(x-2i)t}}{it} \\
&= \int_{-\infty}^{\infty} dx \left\langle \varphi \,\middle|\, \left((x-2i) - K_\theta^{(s)}\right)^{-2} \psi \right\rangle \frac{e^{i(x-2i)t}}{it}
\end{aligned}$$

converges uniformly in s. We now consider for $s, s+s' \in \mathbb{S}_{\frac{\varepsilon_0}{2}}$

$$\begin{aligned}
&\left| \frac{p(s+s') - p(s)}{s'} - \int_{\mathbb{R}-2i} dz \left\langle \varphi \,\middle|\, \partial R_\theta^{(s)}(z) \psi \right\rangle e^{izt} \right| \\
&= \left| \int_{\mathbb{R}-2i} dz \left\langle \varphi \,\middle|\, \frac{R_\theta^{(s+s')}(z) - R_\theta^{(s)}(z)}{s'} - \partial_s R_\theta^{(s)}(z) \psi \right\rangle e^{izt} \right| \\
&\leq \left\| (L_{\text{aux}} + N_{\text{res}} + 1)^{-1} \left[\frac{K_\theta^{(s+s')} - K_\theta^{(s)}}{s'} - \partial_s K_\theta^{(s)} \right] (L_{\text{aux}} + N_{\text{res}} + 1)^{-1} \right\| \\
&\quad \times \int_{\mathbb{R}-2i} dz \left\| B^+_{\theta,s+s'}(z) \right\| \left\| B^-_{\theta,s}(z) \right\| \|\varphi\| \, \|\psi\| \\
&\quad + \left\| (L_{\text{aux}} + N_{\text{res}} + 1)^{-1} \left[K_\theta^{(s+s')} - K_\theta^{(s)} \right] \right\| \\
&\quad \times \left\| \partial_\delta K_\theta^{(s)} \, (L_{\text{aux}} + N_{\text{res}} + 1)^{-1} \right\| \\
&\quad \times \int_{\mathbb{R}-2i} dz \left\| B^+_{\theta,s+s'}(z) \right\| \left\| B^-_{\theta,s}(z) \right\| \left\| \left(z - K_\theta^{(s)}\right)^{-1} \right\| \|\varphi\| \, \|\psi\| \\
&\xrightarrow{s'\to 0} 0,
\end{aligned}$$

where we used the relation (C.10) and the strong differentiability of $s \mapsto K_\theta^{(s)}$. Further we used the following uniform norm bound for $B^\pm_{\theta,s} \equiv B^\pm_{\theta,s}(z)$,

$$\left\| B^\pm_{\theta,s}(z) \right\| \leq [1 + \mathcal{O}(g)] \left\| \left(z - K_\theta^{(s)}\right)^{-1} \right\| \leq \frac{C}{|x| + 1}$$

for $z = x - 2i \in \mathbb{R} - 2i$ and for a positive constant $C < \infty$, see Lemmata C.3, and the bound on the resolvent provided by Lemma C.1. This implies the absolute convergence of

$$\int_{\mathbb{R}-2i} dz \, \left\|B^{+}_{\theta,s+s'}(z)\right\| \left\|B^{-}_{\theta,s}(z)\right\| \leq \int_{\mathbb{R}} dx \, \frac{C^2}{(|x|+1)^2} < \infty$$

and

$$\int_{\mathbb{R}-2i} dz \, \left\|B^{+}_{\theta,s+s'}(z)\right\| \left\|B^{-}_{\theta,s}(z)\right\| \left\|\left(z - K^{(s)}_{\theta}\right)^{-1}\right\| \leq \int_{\mathbb{R}} dx \, \frac{C^3}{(|x|+1)^3} < \infty,$$

uniformly in s'.

∎

C.3 Deformation Analytic Observables

In Section 2.3.1 we introduced a functional ω which was defined on a subset $\mathcal{A}^{\mathrm{ana}}$ of observables, defined in (2.34). The observables collected in $\mathcal{A}^{\mathrm{ana}}$ are referred to as deformation analytic observables and are characterized as those elements $A \in \mathcal{A}$ for which $\pi(A)\tilde{\Omega} \in \mathcal{D}(L_{\mathrm{aux}} + N_{\mathrm{res}})$ and the deformation $\theta \mapsto (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)[\pi(A)\tilde{\Omega}]_\theta$ is analytic in each variable separately, i.e., $\pi(A)\tilde{\Omega} \in \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$. It is the aim of this section to prove that $\mathcal{A}^{\mathrm{ana}}$ contains a strongly dense $*$-subalgebra $\mathcal{A}_1$ in $\mathcal{A}$. To this end we will proceed as follows. We construct a set D^{ana} which is dense in $L^2[\mathbb{R}^3]^R$ and is mapped under the gluing function $\mathfrak{g}$, defined in (1.65), to a dense set R^{ana} in $\mathrm{ran}(\mathfrak{g}) \subseteq L^2[\Upsilon]$. The span of observables of the type $A := A_{\mathrm{p}} \otimes W(f_1) \otimes \cdots \otimes W(f_R)$ for $A_{\mathrm{p}} \in \mathcal{A}_{\mathrm{p}}$ and $(f_1, \ldots f_R) \in D^{\mathrm{ana}}$ is then strongly dense in $\mathcal{A}$. In a further step we show that $\pi(A)\tilde{\Omega} = [A_{\mathrm{p}} \otimes \mathbb{1}_{\mathcal{H}_{\mathrm{p}}}] \otimes W_{\mathrm{gl}}(\mathfrak{g}(f_1, \ldots, f_R))$ has the required analytic properties.

We equip the R-fold cartesian product $\mathcal{X} := \mathcal{D}_{\mathrm{f}}^R$ of the space $\mathcal{D}_{\mathrm{f}}$ defined in (1.34) with the norm

$$\|(f_1, \ldots, f_R)\|_{\mathcal{X}} := \left(\sum_{r=1}^{R} \int_{\mathbb{R}} d^3\vec{k} \left[1 + \frac{1}{|\vec{k}|}\right] |f_r(\vec{k})|^2 \right)^{1/2}$$

which makes it to a Banach space.

Lemma C.7 *The gluing function* $\mathfrak{g} : \mathcal{X} \to L^2[\Upsilon]$ *is a bounded (continuous) real-linear map and has a bounded inverse* $\mathfrak{g}^{-1} : \mathrm{ran}(\mathfrak{g}) \to \mathcal{X}$ *with norm bounds*

$$\begin{aligned} \|\mathfrak{g}(f)\|_{L^2[\Upsilon]} &\leq 2\max\{1, \beta_{\min}^{-1/2}\}\, \|f\|_{\mathcal{X}}\,, \\ \left\|\mathfrak{g}^{-1}(F)\right\|_{\mathcal{X}} &\leq \sqrt{2}\max\{1, \beta_{\max}^{1/2}\}\, \|F\|_{L^2[\Upsilon]}\,. \end{aligned} \tag{C.11}$$

Proof. For $f = (f_1, \ldots, f_R) \in \mathcal{X}$ we have

$$\begin{aligned} \|\mathfrak{g}(f)\|^2_{L^2[\Upsilon]} &= \sum_{r=1}^{R} \int_0^\infty du \int_{S^2} d\Sigma\, \frac{u^2}{1 - e^{-\beta_r u}} |f_r(u\Sigma)|^2 \\ &\quad - \sum_{r=1}^{R} \int_{-\infty}^0 du \int_{S^2} d\Sigma\, \frac{u^2}{1 - e^{-\beta_r u}} |f_r(-u\Sigma)|^2 \\ &= \sum_{r=1}^{R} \int_0^\infty du \int_{S^2} d\Sigma \left[\frac{u^2}{1 - e^{-\beta_r u}} + \frac{u^2}{e^{\beta_r u} - 1}\right] |f_r(u\Sigma)|^2 \\ &= \sum_{r=1}^{R} \int_{\mathbb{R}^3} d^3\vec{k}\, \frac{e^{\beta_r|\vec{k}|} - e^{-\beta_r|\vec{k}|}}{(1 - e^{-\beta_r|\vec{k}|})(e^{\beta_r|\vec{k}|} - 1)} |f_r(\vec{k})|^2 \\ &= \sum_{r=1}^{R} \int_{\mathbb{R}^3} d^3\vec{k}\, \frac{1 + e^{-\beta_r|\vec{k}|}}{1 - e^{-\beta_r|\vec{k}|}} |f_r(\vec{k})|^2. \end{aligned}$$

For $x > 0$ we consider the function

$$x \mapsto t(x) := \frac{1 + e^{-x}}{1 - e^{-x}} = \frac{\cosh(\frac{x}{2})}{\sinh(\frac{x}{2})}, \qquad t'(x) = -\frac{1}{2\sinh^2(\frac{x}{2})} < 0,$$

which is strictly monotonously decreasing such that

$$\frac{1 + e^{-x}}{1 - e^{-x}} \geq \lim_{\tilde{x} \to \infty} t(\tilde{x}) = 1.$$

We also have

$$\frac{1 + e^{-x}}{1 - e^{-x}} \geq \frac{1}{x} \frac{x}{1 - e^{-x}} = \frac{1}{x}[e^{-\xi}]^{-1} \geq \frac{1}{x}$$

for $\xi \in [0, x]$, thus

$$\frac{1 + e^{-x}}{1 - e^{-x}} \geq \frac{1}{2}\left[1 + \frac{1}{x}\right].$$

Further, for $x \leq \ln(2)$,

$$\frac{1 + e^{-x}}{1 - e^{-x}} \leq \frac{2}{1 - e^{-x}} = \frac{2}{x} \frac{x}{1 - e^{-x}} = \frac{2e^{\xi}}{x} \leq \frac{2e^{\ln(2)}}{x} \leq 4\left[1 + \frac{1}{x}\right],$$

and, for $x \geq \ln(2)$,

$$\frac{1+e^{-x}}{1-e^{-x}} \leq t(\ln(2)) = \frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 3 \leq 4\left[1+\frac{1}{x}\right].$$

Altogether, we have for $x > 0$,

$$\frac{1}{2}\left[1+\frac{1}{x}\right] \leq \frac{1+e^{-x}}{1-e^{-x}} \leq 4\left[1+\frac{1}{x}\right].$$

This gives the following bound,

$$\begin{aligned} &\frac{1}{2}\min\{1,\beta_{\max}^{-1}\}\sum_{r=1}^{R}\int_{\mathbb{R}^3} d^3\vec{k}\left[1+\frac{1}{|\vec{k}|}\right]|f_r(\vec{k})|^2 \\ \leq\ & \sum_{r=1}^{R}\int_{\mathbb{R}^3} d^3\vec{k}\,\frac{1+e^{-\beta_r|\vec{k}|}}{1-e^{-\beta_r|\vec{k}|}}|f_r(\vec{k})|^2 \\ \leq\ & 4\max\{1,\beta_{\min}^{-1}\}\sum_{r=1}^{R}\int_{\mathbb{R}^3} d^3\vec{k}\left[1+\frac{1}{|\vec{k}|}\right]|f_r(\vec{k})|^2. \end{aligned}$$

Therefore, the gluing function $\mathfrak{g} : \mathcal{X} \to L^2[\Upsilon]$ is a bounded (continuous) real-linear map and has a bounded inverse $\mathfrak{g}^{-1} : \operatorname{ran}(\mathfrak{g}) \to \mathcal{X}$ with the norm bounds given in (C.11). ∎

We are going over to describe the range of $\mathfrak{g}$.

Lemma C.8 *The range of the gluing function $\mathfrak{g}$ is given by*

$$\operatorname{ran}(\mathfrak{g}) = \left\{F \in L^2[\Upsilon] \,\middle|\, F(u,\Sigma,r) = -e^{\beta_r u/2}\overline{F}(-u,\Sigma,r)\ a.e.\right\}.$$

Proof. While the inclusion "$\subseteq$" comes from the definition of $\mathfrak{g}$, we check the inclusion "$\supseteq$". For $F \in L^2[\Upsilon]$ we define $f_r : \mathbb{R}^3 \to \mathbb{C}$, $r = 1,\ldots,R$, by

$$f_r(\vec{k}) := \frac{\sqrt{1-e^{-\beta_r|\vec{k}|}}}{|\vec{k}|}F\left(|\vec{k}|,\frac{\vec{k}}{|\vec{k}|},r\right).$$

We show that $f := (f_1, \ldots, f_R) \in \mathcal{X}$,

$$\begin{aligned}
&\sum_{r=1}^{R} \int_{\mathbb{R}^3} d^3\vec{k} \left[1 + \frac{1}{|\vec{k}|}\right] |f_r(\vec{k})|^2 \\
&\quad \leq \sum_{r=1}^{R} \int_{\mathbb{R}^+} u^2 du \int_{S^2} d\Sigma \left[1 + \frac{1}{u}\right] \frac{1 - e^{-\beta_r u}}{u^2} |F(u, \Sigma, r)|^2 \\
&\quad \leq \sum_{r=1}^{R} \int_{\mathbb{R}^+} du \int_{S^2} d\Sigma \, \frac{u+1}{u} (1 - e^{-\beta_r u}) |F(u, \Sigma, r)|^2 \\
&\quad \leq \sup_{\substack{u \in \mathbb{R}^+, \\ r \in \mathbb{N}_1^R}} \left[\frac{u+1}{u}(1 - e^{-\beta_r u})\right] \int_{\Upsilon} dy \, |F(y)|^2 \\
&\quad = C \, \|F\|^2_{L^2[\Upsilon]}
\end{aligned}$$

using that

$$C := \sup_{\substack{u \in \mathbb{R}^+, \\ r \in \mathbb{N}_1^R}} \left[\frac{u+1}{u}(1 - e^{-\beta_r u})\right] < \infty$$

since

$$\lim_{u \to 0} \frac{u+1}{u}(1 - e^{-\beta_r u}) = \beta_r, \qquad \lim_{u \to \infty} \frac{u+1}{u}(1 - e^{-\beta_r u}) = 1.$$

If further $F(u, \Sigma, r) = -e^{\beta_r u/2} \overline{F}(-u, \Sigma, r)$ a.e. then f is the pre-image of F under $\mathfrak{g}$. ■

We now consider the subset $R^0 \subseteq \operatorname{ran}(\mathfrak{g})$ given by

$$\begin{aligned}
R^0 \; := \; \Big\{ (u, \Sigma, r) \mapsto e^{\beta_r u/4} h(u, \Sigma, r) \,\Big|\, & h \in L^\infty[\Upsilon], \, \exists \, M > 0 : \, \mathbf{1}_{[|u| \geq M]} h = 0 \text{ a.e.}, \\
& h(u, \Sigma, r) = -\overline{h}(-u, \Sigma, r) \text{ a.e.} \Big\}.
\end{aligned}$$

Note that R^0 is dense in $\operatorname{ran}(\mathfrak{g})$. For $F(u, \Sigma, r) = e^{\beta_r u/4} h(u, \Sigma, r)$, $F \in R^0$, we define

$$h_\varepsilon(u, \Sigma, r) := G_\varepsilon * h(u, \Sigma, r) := \int_{\mathbb{R}} du \, G_\varepsilon(x - u) h(x, \Sigma, r),$$

for $\varepsilon > 0$, the convolution of h with the Gaussian

$$G_\varepsilon(x) := \frac{1}{\varepsilon\sqrt{\pi}} e^{-\frac{x^2}{\varepsilon^2}}$$

w.r.t. the variable u. Note that

$$h_\varepsilon(u,\Sigma,r) = -\overline{h}_\varepsilon(-u,\Sigma,r)$$

as one sees as follows,

$$\begin{aligned} h_\varepsilon(u,\Sigma,r) &= \int\limits_{\mathbb{R}} dx\, G_\varepsilon(x-u)h(x,\Sigma,r) = \int\limits_{\mathbb{R}} dx\, G_\varepsilon(x+u)h(-x,\Sigma,r) \\ &= -\int\limits_{\mathbb{R}} dx\, G_\varepsilon(x-(-u))\overline{h}(x,\Sigma,r) = -\overline{h}_\varepsilon(-u,\Sigma,r). \end{aligned}$$

Further, we consider the decay of $G_\varepsilon * h(u,\Sigma,r)$ as $|u|\to\infty$. Let $M>0$ such that $\mathbf{1}_{[|u|\geq M]}h = 0$ a.e., therefore,

$$|G_\varepsilon * h(u,\Sigma,r)| \leq \int\limits_{-M}^{M} dx\, G_\varepsilon(x-u)|h(x,\Sigma,r)| \leq 2M\, \|h\|_{L^\infty[\Upsilon]}\, G_\varepsilon(\zeta-u)$$

for a.e. $(\Sigma,r)\in S^2\times\mathbb{N}_1^R$ and a suitable $\zeta\in[-M,M]$. For $u\geq M$ we have

$$|G_\varepsilon * h(u,\Sigma,r)| \leq 2M\, \|h\|_{L^\infty[\Upsilon]}\, \frac{\exp(-\frac{1}{\varepsilon^2}(u-M)^2)}{\varepsilon\sqrt{\pi}},$$

and for $u\leq -M$,

$$|G_\varepsilon * h(u,\Sigma,r)| \leq 2M\, \|h\|_{L^\infty[\Upsilon]}\, \frac{\exp(-\frac{1}{\varepsilon^2}(u+M)^2)}{\varepsilon\sqrt{\pi}},$$

which implies, for $|u|\geq M$,

$$|G_\varepsilon * h(u,\Sigma,r)| \leq 2M\, \|h\|_{L^\infty[\Upsilon]}\, \frac{\exp(-\frac{1}{\varepsilon^2}(u^2-2M|u|+M^2))}{\varepsilon\sqrt{\pi}}, \tag{C.12}$$

for a.e. $(\Sigma,r)\in S^2\times\mathbb{N}_1^R$.

Lemma C.9 *The set*

$$R^{\mathrm{ana}} := \Big\{(u,\Sigma,r)\mapsto e^{\beta_r u/4}G_\varepsilon * h(u,\Sigma,r)\;\Big|\;(u,\Sigma,r)\mapsto e^{\beta_r u/4}h(u,\Sigma,r)\in R^0,\ \varepsilon>0\Big\}$$

is a dense subset of $\mathrm{ran}(\mathfrak{g})$.

Proof. It follows from the decay properties (C.12) of $G_\varepsilon * h$ that $(u,\Sigma,r) \mapsto e^{\beta_r u/4} G_\varepsilon * h(u,\Sigma,r) \in L^2[\Upsilon]$ for $(u,\Sigma,r) \mapsto e^{\beta_r u/4} h(u,\Sigma,r) \in R^0$, thus $R^{\mathrm{ana}} \subseteq \mathrm{ran}(\mathfrak{g})$. In order to proof the density we first note that, for a function $h \in L^2[\Upsilon]$, holds,

$$\begin{aligned}
\|G_\varepsilon * h\|^2_{L^2[\Upsilon]} &= \int_\Upsilon d(u,\Sigma,r) \left| \int_\mathbb{R} dx\, G_\varepsilon(x-u) h(x,\Sigma,r) \right|^2 \\
&\leq \int_\Upsilon d(u,\Sigma,r) \int_\mathbb{R} dx\, G_\varepsilon(x-u) \int_\mathbb{R} dx\, G_\varepsilon(x-u) |h(x,\Sigma,r)|^2 \\
&= \|G_\varepsilon\|_{L^1[\mathbb{R}]} \int_\Upsilon d(x,\Sigma,r)\, |h(x,\Sigma,r)|^2 \int_\Upsilon du\, G_\varepsilon(x-u) \\
&= \|G_\varepsilon\|^2_{L^1[\mathbb{R}]} \|h\|^2_{L^2[\Upsilon]} = \|h\|^2_{L^2[\Upsilon]} \,.
\end{aligned}$$

We now show that $G_\varepsilon * h \xrightarrow{\varepsilon \to 0} h$ in the L^2-sense. For a given h and $\eta > 0$ we find a continuous, compactly supported function $f \in L^2[\Upsilon]$ with $\|h - f\|_{L^2[\Upsilon]} < \eta/2$. Since

$$\begin{aligned}
\|G_\varepsilon * h - h\|_{L^2[\Upsilon]} &\leq \|f - h\|_{L^2[\Upsilon]} + \|G_\varepsilon * f - f\|_{L^2[\Upsilon]} + \|G_\varepsilon * (f-h)\|_{L^2[\Upsilon]} \\
&\leq 2\,\|f - h\|_{L^2[\Upsilon]} + \|G_\varepsilon * f - f\|_{L^2[\Upsilon]} \\
&< \eta + \|G_\varepsilon * f - f\|_{L^2[\Upsilon]} \,,
\end{aligned}$$

the proof of the convergence of $G_\varepsilon * h$ towards h may be reduced to continuous, compactly supported functions h,

$$\begin{aligned}
\|G_\varepsilon * h - h\|^2_{L^2[\Upsilon]} &= \int_\Upsilon d(u,\Sigma,r) \left| \int_\mathbb{R} dx\, G_\varepsilon(x)[h(u+x,\Sigma,r) - h(u,\Sigma,r)] \right|^2 \\
&\leq \int_\Upsilon d(u,\Sigma,r) \int_\mathbb{R} dx\, G_1(x)\, |h(u+\varepsilon x,\Sigma,r) - h(u,\Sigma,r)|^2 \\
&\xrightarrow{\varepsilon \to 0} 0.
\end{aligned}$$

This concludes the proof. ∎

Using that $\mathfrak{g}^{-1} : \mathrm{ran}(\mathfrak{g}) \to \mathcal{X}$ is continuous, the set

$$D^{\mathrm{ana}} := \mathfrak{g}^{-1}(R^{\mathrm{ana}})$$

is dense in $\mathcal{X}$. Since further $\mathcal{X}$ is dense in $L^2[\mathbb{R}^3]^R$ the set D^{ana} is dense in $L^2[\mathbb{R}^3]^R$, as well. We build the collection $\mathcal{A}_1$ of observables defined by

$$\mathcal{A}_1 := \mathrm{span}\,\{ A_\mathrm{p} \otimes W(f_1) \cdots W(f_R) \mid A_\mathrm{p} \in \mathcal{A}_\mathrm{p},\, (f_1,\ldots,f_R) \in D^{\mathrm{ana}} \} \,. \qquad \text{(C.13)}$$

We remark that the selection D^{ana} of coupling functions and the collection $\mathcal{A}_1$ of observables, along with the function $\mathfrak{g}$, depend on the reservoir temperatures.

Proposition C.10 (Density of Analytic Observables) *The set $\mathcal{A}_1$ is a strongly dense $*$-algebra in $\mathcal{A}$. Moreover, the vector $\tilde{\Omega}$ is cyclic w.r.t. $\pi(\mathcal{A}_1)$ and $\pi'(\mathcal{A}_1)$, i.e., the spaces $\pi(\mathcal{A}_1)\tilde{\Omega}$ and $\pi'(\mathcal{A}_1)\tilde{\Omega}$ are dense in $\mathcal{H}^2$.*

Proof. By definition, the collection $\mathcal{A}_1$ of observables is a linear space. Further, due to the CCR (1.35) for Weyl operators, the product $W(f)W(g)$ of two Weyl operators $W(f), W(g)$ can be expressed as a multiple of a single Weyl operator $W(f+g)$ obtained by adding the form factors f and g. Since the space D^{ana} is linear, the set $\mathcal{A}_1$ is closed under multiplication. Moreover, the space $\mathcal{A}_1$ is closed under conjugation because $W(f)^* = W(-f)$. Therefore, the collection $\mathcal{A}_1$ forms a $*$-algebra. Since

$$\mathcal{A} = \overline{\operatorname{span}\{A_{\text{p}} \otimes W(f_1)\cdots W(f_R) \mid A_{\text{p}} \in \mathcal{A}_{\text{p}},\, (f_1,\ldots,f_R) \in \mathcal{X}\}}^{\|\cdot\|_{\mathcal{B}(\mathcal{H})}}$$

the algebra $\mathcal{A}_1$ is strongly dense in $\mathcal{A}$. This follows from the density of D^{ana} in $\mathcal{X}$ and the fact that $W(g_n) \to W(g)$ strongly if $g_n \to g$, see [11, Prop. 5.2.4].

We go over to proof the cyclicity of $\tilde{\Omega}$ w.r.t. $\pi(\mathcal{A}_1)$. By construction we have

$$\pi(\mathcal{A}_1) = \operatorname{span}\left\{[A_{\text{p}} \otimes \mathbb{1}_{\mathcal{H}_{\text{p}}}] \otimes W_{\text{gl}}(g) \;\middle|\; A_{\text{p}} \in \mathcal{A}_{\text{p}},\, g \in R^{\text{ana}}\right\}$$

and

$$\pi(\mathcal{A}) = \overline{\operatorname{span}\left\{[A_{\text{p}} \otimes \mathbb{1}_{\mathcal{H}_{\text{p}}}] \otimes W_{\text{gl}}(g) \;\middle|\; A_{\text{p}} \in \mathcal{A}_{\text{p}},\, g \in \operatorname{ran}(\mathfrak{g})\right\}}^{\|\cdot\|_{\mathcal{B}(\mathcal{H})}}$$

and due to the density of R^{ana} in $\operatorname{ran}(\mathfrak{g})$ we conclude with the same arguments as above that $\pi(\mathcal{A}_1)$ is strongly dense in $\pi(\mathcal{A})$. This, in turn, implies that $\pi(\mathcal{A}_1)'' = \pi(\mathcal{A})''$, i.e., the weak closures coincide. It follows by von Neumann's density theorem, [10, Cor. 2.4.15.], that the $*$-algebra $\pi(\mathcal{A}_1)$ is strongly dense in its weak closure $\pi(\mathcal{A})''$.

Now, let $\psi \in \mathcal{H}^2$ and $\varepsilon > 0$ arbitrary. Since $\tilde{\Omega}$ is cyclic w.r.t. $\pi(\mathcal{A})''$ by Lemma B.4 we find an element $B \in \pi(\mathcal{A})''$ such that $\left\|\psi - B\tilde{\Omega}\right\| < \varepsilon/2$. Since $\pi(\mathcal{A}_1)$ is strongly dense in $\pi(\mathcal{A})''$ there is an observable $A \in \mathcal{A}_1$ with $\left\|\pi(A)\tilde{\Omega} - B\tilde{\Omega}\right\| < \varepsilon/2$. It follows that

$$\left\|\psi - \pi(A)\tilde{\Omega}\right\| \le \left\|\psi - B\tilde{\Omega}\right\| + \left\|B\tilde{\Omega} - \pi(A)\tilde{\Omega}\right\| < \varepsilon.$$

The cyclicity of $\tilde{\Omega}$ w.r.t. $\pi'(\mathcal{A}_1)$ is proved in the same way using that

$$\pi'(\mathcal{A}_1) = \operatorname{span}\left\{[\mathbb{1}_{\mathcal{H}_{\text{p}}} \otimes \overline{A}_{\text{p}}] \otimes W_{\text{gl}}(g) \;\middle|\; A_{\text{p}} \in \mathcal{A}_{\text{p}},\, g \in \mathfrak{g}'(D^{\text{ana}})\right\}$$

and

$$\pi'(\mathcal{A}) = \overline{\operatorname{span}\left\{[\mathbb{1}_{\mathcal{H}_{\text{p}}} \otimes \overline{A}_{\text{p}}] \otimes W_{\text{gl}}(g) \;\middle|\; A_{\text{p}} \in \mathcal{A}_{\text{p}},\, g \in \operatorname{ran}(\mathfrak{g}')\right\}}^{\|\cdot\|_{\mathcal{B}(\mathcal{H})}}$$

where $\mathfrak{g}'(D^{\mathrm{ana}})$ is dense in $\operatorname{ran}(\mathfrak{g}')$, and that $\tilde{\Omega}$ is is cyclic w.r.t. $\pi(\mathcal{A})'$ by Lemma B.4. ■

The next task is to study the analytic deformation properties of observables $A \in \mathcal{A}_1$. Let $f \in D^{\mathrm{ana}}$ and write the function $F := \mathfrak{g}(f) \in R^{\mathrm{ana}}$ as $F(u,\Sigma,r) = e^{\beta_r u/4} G_\varepsilon * h(u,\Sigma,r)$. Due to regularization of h by convolution with the entire function G_ε the function $u \mapsto F(u,\Sigma,r)$ extends to an entire function on $\mathbb{C}$ for a.e. $(\Sigma,r) \in S^2 \times \mathbb{N}_1^R$, also denoted by $z \mapsto F(z,\Sigma,r)$. Therefore, we can build the deformation

$$[\mathfrak{g}(f)]_\theta(u,\Sigma,r) = F_\theta(u,\Sigma,r) = e^{\delta\,\mathrm{sgn}(u)/2} F(j_\theta(u),\Sigma,r)$$

where, recall, $j_\theta(u) = e^{\delta\,\mathrm{sgn}(u)} u + \tau$, for $\theta = (\delta,\tau) \in \mathbb{C}$. We show that the function $F \in R^{\mathrm{ana}}$ remains an L^2-function even after spectral deformation.

Lemma C.11 *Let $f \in D^{\mathrm{ana}}$ and set $F := \mathfrak{g}(f) \in R^{\mathrm{ana}}$. Let $\theta = (\delta,\tau) \in \mathbb{C}^2$ with $|\operatorname{Im}(\delta)| < \frac{\pi}{4}$. Then, the deformed function F_θ stays in L^2, i.e.,*

$$F_\theta = [\mathfrak{g}(f)]_\theta \in L^2[\Upsilon].$$

Proof. We write $F(u,\Sigma,r) = e^{\beta_r u/4} G_\varepsilon * h(u,\Sigma,r)$ with $\varepsilon > 0$ and $(u,\Sigma,r) \mapsto e^{\beta_r u/4} h(u,\Sigma,r) \in R^0$. Since

$$\|F_\theta\|^2_{L^2[\Upsilon]} = \int_\Upsilon d(u,\Sigma,r)\ \left|e^{\delta\,\mathrm{sgn}(u)} e^{\beta_r j_\theta(u)/2}\right| \left|\int_{\mathbb{R}} dx\, G_\varepsilon(x - j_\theta(u)) h(x,\Sigma,r)\right|^2$$

we may assume $\operatorname{Re}(\delta) = 0$ as one sees by performing a transformation of integration variables $u \mapsto e^{-\operatorname{Re}(\delta)\,\mathrm{sgn}(u)} u$. We estimate

$$\|F_\theta\|^2_{L^2[\Upsilon]} \le \int_\Upsilon d(u,\Sigma,r)\, e^{\beta_r \operatorname{Re}(j_\theta(u))/2} \left[\int_{\mathbb{R}} dx\, |G_\varepsilon(x - j_\theta(u))||h(x,\Sigma,r)|\right]^2.$$

Using that $|e^z| = e^{\operatorname{Re}(z)}$ and $\operatorname{Re}(z^2) = \operatorname{Re}(z)^2 - \operatorname{Im}(z)^2$ and therefore $|e^{z^2}| = e^{\operatorname{Re}(z)^2 - \operatorname{Im}(z)^2}$ we get

$$|G_\varepsilon(x - j_\theta(u))| = \exp\left(\frac{\operatorname{Im}(j_\theta(u))^2}{\varepsilon^2}\right) G_\varepsilon(x - \operatorname{Re}(j_\theta(u)))$$

and, with this relation and (C.12),

$$
\begin{aligned}
\|F_\theta\|^2_{L^2[\Upsilon]} &\le \int_\Upsilon d(u,\Sigma,r)\,\exp\left(\frac{2\operatorname{Im}(j_\theta(u))^2}{\varepsilon^2} + \frac{\beta_r}{2}\operatorname{Re}(j_\theta(u))\right) \\
&\qquad\qquad \times G_\varepsilon * |h|(\operatorname{Re}(j_\theta(u)),\Sigma,r)^2 \\
&\le \frac{4M^2\,\|h\|^2_{L^\infty[\Upsilon]}}{\varepsilon^2\pi} e^{-2\frac{M^2}{\varepsilon^2}} \\
&\quad \times \int_\Upsilon d(u,\Sigma,r)\,\exp\left(-\frac{2}{\varepsilon^2}\left[\operatorname{Re}(j_\theta(u))^2 - \operatorname{Im}(j_\theta(u))^2\right]\right. \\
&\qquad\qquad \left. + \frac{4M}{\varepsilon^2}|\operatorname{Re}(j_\theta(u))| + \frac{\beta_r}{2}\operatorname{Re}(j_\theta(u))\right) \\
&< \infty
\end{aligned}
\tag{C.14}
$$

where $M > 0$ is chosen such that $\mathbf{1}_{[|u|\ge M]}h = 0$ a.e. Further, we used that

$$
\begin{aligned}
\operatorname{Re}(j_\theta(u)) &= \cos(\delta')u + \operatorname{Re}(\tau), \\
\operatorname{Im}(j_\theta(u)) &= \sin(\delta')|u| + \operatorname{Im}(\tau),
\end{aligned}
$$

for $\delta' := \operatorname{Im}(\delta)$, and therefore

$$
\begin{aligned}
&\operatorname{Re}(j_\theta(u))^2 - \operatorname{Im}(j_\theta(u))^2 \\
&\quad = \cos(2\delta')u^2 + 2\cos(\delta')\operatorname{Re}(\tau)u - 2\sin(\delta')\operatorname{Im}(\tau)|u| + \operatorname{Re}(\tau)^2 - \operatorname{Im}(\tau)^2.
\end{aligned}
$$

The fact that the coefficient $\cos(2\delta')$ of the leading order u^2 is strictly positive for $|\delta'| < \frac{\pi}{4}$ guarantees the finiteness of the integration over u in (C.14). ∎

The next statement characterizes the deformation analytic properties of functions in R^{ana}.

Lemma C.12 *Let $f \in D^{\mathrm{ana}}$ and set $F := \mathfrak{g}(f) \in R^{\mathrm{ana}}$. The function*

$$\theta = (\delta,\tau) \mapsto F_\theta = [\mathfrak{g}(f)]_\theta$$

is analytic in the L^2-sense (in each variable separately) on the domain $|\operatorname{Im}(\delta)| < \frac{\pi}{4}$ and $\tau \in \mathbb{C}$.

Proof. We write, as usual, $F(u,\Sigma,r) = e^{\beta_r u/4} G_\varepsilon * h(u,\Sigma,r)$ with $\varepsilon > 0$ and $(u,\Sigma,r) \mapsto e^{\beta_r u/4} h(u,\Sigma,r) \in R^0$. First, we note that it is obvious that $\theta \mapsto F_\theta(y)$ is pointwise analytic, for a.e. $y \in \Upsilon$, and that (with $\theta_1 = \delta$, $\theta_2 = \tau$)

$$\begin{aligned}\partial_{\theta_j} F_\theta(u,\Sigma,r) &= \partial_{\theta_j}\left[e^{\delta\,\mathrm{sgn}(u)/2} e^{\beta_r j_\theta(u)/4}\int_{\mathbb{R}} dx\,\frac{1}{\varepsilon\sqrt{\pi}} e^{-\frac{(x-j_\theta(u))^2}{\varepsilon^2}} h(x,\Sigma,r)\right] \\ &= e^{\delta\,\mathrm{sgn}(u)/2+\beta_r j_\theta(u)/4}\int_{\mathbb{R}} dx\,\frac{1}{\varepsilon\sqrt{\pi}} e^{-\frac{(x-j_\theta(u))^2}{\varepsilon^2}} h(x,\Sigma,r) \\ &\quad\times\left\{\left[\frac{\beta_r}{4}+\frac{2(x-j_\theta(u))}{\varepsilon^2}\right]\partial_{\theta_j} j_\theta(u) + \delta_{1,j}\frac{\mathrm{sgn}(u)}{2}\right\}.\end{aligned}$$

Further, the derivatives of $j_\theta(u)$ are given as

$$\begin{aligned}&\partial_\delta j_\theta(u) = \mathrm{sgn}(u) e^{\delta\,\mathrm{sgn}(u)} u = e^{\delta\,\mathrm{sgn}(u)}|u|, \\ &\partial_\tau j_\theta(u) = 1,\end{aligned}$$

such that

$$\begin{aligned}&\partial_\delta F_\theta(u,\Sigma,r) = e^{\delta\,\mathrm{sgn}(u)}|u|\left[\partial_u F\right]_\theta(u,\Sigma,r) + \frac{\mathrm{sgn}(u)}{2} F_\theta(u,\Sigma,r), \\ &\partial_\tau F_\theta(u,\Sigma,r) = \left[\partial_u F\right]_\theta(u,\Sigma,r).\end{aligned}$$

The same arguments as in the proof of Lemma C.11 show that both, $\partial_\delta F_\theta$ and $\partial_\tau F_\theta$, are functions in $L^2[\Upsilon]$ for $|\operatorname{Im}(\delta)| < \frac{\pi}{4}$ and $|\operatorname{Im}(\tau)| < 2\pi\beta_{\max}^{-1}$. By dominated convergence theorem we can conclude that

$$\frac{1}{|\theta'|}\left\| F_{\theta+\theta'} - F_\theta - (\nabla_\theta F_\theta)\cdot\theta'\right\|_{L^2[\Upsilon]} \to 0$$

as $|\theta'| \to 0$. ■

Remark C.13 *The coupling functions $f \in D^{\mathrm{ana}}$ are tailor-made such that the glued image $\theta \mapsto [\mathfrak{g}(f)]_\theta$ is smoothed out around the poles $\pm 2n\pi i\beta_r^{-1}$, $n \in \mathbb{N}$, $r = 1,\dots,R$, which usually appear under the gluing. Therefore, the function $(\delta,\tau) \mapsto [\mathfrak{g}(f)]_\theta$ is entire in τ. The smoothing and therefore also the construction of the set D^{ana} is dependent on the inverse reservoir temperatures. We refer to the discussion of Remark 2.7.*

We now justify that the elements of $\mathcal{A}_1$ are deformation analytic observables.

Theorem C.14 (Deformation Analytic Observables) *For any observable $A \in \mathcal{A}_1$ the vectors $\pi(A)\tilde{\Omega}$ and $\pi'(A)\tilde{\Omega}$ are in the domain of the operator $(L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)$ and the functions*

$$\mathbb{R}^2 \ni \theta = (\delta,\tau) \mapsto (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)\left[\pi(A)\tilde{\Omega}\right]_\theta$$

and

$$\mathbb{R}^2 \ni \theta = (\delta, \tau) \mapsto (L_{\rm aux} + N_{\rm res} + 1) \left[\pi'(A)\tilde{\Omega}\right]_\theta$$

have analytic continuations (in each variable separately) to the domain

$$D_{\delta_0,\tau_0} = \left\{(\delta,\tau) \in \mathbb{C}^2 \;\middle|\; |\,{\rm Im}(\delta)| < \delta_0,\ |\tau| < \tau_0 \right\}$$

for $\frac{\pi}{8} < \delta_0 < \frac{\pi}{4}$ *and* $\tau_0 \le 2\pi\beta_{\rm max}^{-1}$. *In other words,*

$$\pi(\mathcal{A}_1)\tilde{\Omega} \subseteq \mathcal{D}^{\rm aux}_{\mathfrak{D}-{\rm a}} \qquad \textit{and} \qquad \pi'(\mathcal{A}_1)\tilde{\Omega} \subseteq \mathcal{D}^{\rm aux}_{\mathfrak{D}-{\rm a}}.$$

Proof. With the help of Lemma B.1(i) we write the vector $\tilde{\Omega}$ as a Dyson series,

$$\tilde{\Omega} = C \sum_{n=0}^{\infty} (-g)^n \int\limits_{0\le s_n \le \cdots \le s_1 \le \beta/2} ds_1 \cdots ds_n\, \pi\left(\alpha_0^{is_n}(v) \cdots \alpha_0^{is_1}(v)\right)\Omega_0,$$

where $C := \left\| e^{-\beta L^{(\ell)}/2}\Omega_0 \right\|^{-1}$ and

$$\pi\left(\alpha_0^{is_j}(v)\right) = a^*_{\rm gl}\left(\mathcal{G}_{(is_j\vec{1})}\right) + a_{\rm gl}\left(\mathcal{G}_{(-is_j\vec{1})}\right),$$

where $\vec{1} = (1,1,\ldots,1) \in \mathbb{C}^{R+1}$, recall the notation (2.12) of $\mathcal{G}_{(\vec{\kappa})}$. Now let $A_{\rm p} \in \mathcal{A}_{\rm p}$ and $f = (f_1,\ldots,f_R) \in D^{\rm ana}$. Set $A := A_{\rm p} \otimes W(f_1) \otimes \cdots \otimes W(f_R) \in \mathcal{A}_1$ and $F := \mathfrak{g}(f)$ and expand $\pi(A) = \pi_{\rm p}(A_{\rm p}) \otimes W_{\rm gl}(\mathfrak{g}(f))$ in a series,

$$\pi(A) = \pi_{\rm p}(A_{\rm p}) \otimes \sum_{m=0}^{\infty} \frac{i^m}{2^{m/2} m!} \left[a^*_{\rm gl}(F) + a_{\rm gl}(F)\right]^m,$$

where the convergence is meant in a strong sense. Therefore we may write

$$\begin{aligned} \pi(A)\tilde{\Omega} \;=\; & C\pi_{\rm p}(A_{\rm p}) \otimes \sum_{m,n=0}^{\infty} \frac{i^m(-g)^n}{2^{m/2} m!} \int\limits_{0\le s_n \le \cdots \le s_1 \le \beta/2} ds_1 \cdots ds_n \\ & \times \left[a^*_{\rm gl}(F) + a_{\rm gl}(F)\right]^m \prod_{j=0}^{n-1} \left[a^*_{\rm gl}\left(\mathcal{G}_{(is_{n-j}\vec{1})}\right) + a_{\rm gl}\left(\mathcal{G}_{(-is_{n-j}\vec{1})}\right)\right]\Omega_0. \end{aligned}$$

We apply spectral deformation to every single addend in the above series. We remark that $\theta \mapsto F_\theta = [\mathfrak{g}(f)]_\theta$ is analytic in the L^2 sense, by Lemma C.12, and so is $\theta \mapsto [\mathcal{G}_{(\pm is_j\vec{1})}]_\theta$ by Hypothesis VII-1.12 and Remark 1.13. The relative bound of creation and annihilation operators w.r.t. $N_{\rm res}^{1/2}$ as provided in Lemma A.3 guarantees that

$$\theta \mapsto \left[a^*_{\rm gl}(F_\theta) + a_{\rm gl}(F_{\overline{\theta}})\right]^m \prod_{j=0}^{n-1} \left[a^*_{\rm gl}\left([\mathcal{G}_{(is_{n-j}\vec{1})}]_\theta\right) + a_{\rm gl}\left([\mathcal{G}_{(-is_{n-j}\vec{1})}]_{\overline{\theta}}\right)\right]\Omega_0$$

inherits the analytic properties from $\theta \mapsto F_\theta$ and $\theta \mapsto [\mathcal{G}_{(\pm i s_j \vec{1})}]_\theta$. With the arguments as given in the proof of Lemma B.1(i) we see that the series

$$\begin{aligned} [\pi(A)\tilde{\Omega}]_\theta &= C\pi_{\rm p}(A_{\rm p}) \otimes \sum_{m,n=0}^{\infty} \frac{i^m(-g)^n}{2^{m/2}m!} \int\limits_{0\le s_n\le\cdots\le s_1\le\beta/2} ds_1\cdots ds_n \\ &\quad \times \left[a_{\rm gl}^*(F_\theta) + a_{\rm gl}(F_{\overline{\theta}})\right]^m \prod_{j=0}^{n-1} \left[a_{\rm gl}^*\left([\mathcal{G}_{(i s_{n-j}\vec{1})}]_\theta\right) + a_{\rm gl}\left([\mathcal{G}_{(-i s_{n-j}\vec{1})}]_{\overline{\theta}}\right)\right] \Omega_0 \end{aligned}$$

converges uniformly on each compact subset of D_{δ_0,τ_0} which results in the analyticity of $\theta \mapsto [\pi(A)\tilde{\Omega}]_\theta$. To estimate

$$\left\| [\mathcal{G}_{(i s_j \vec{1})}]_\theta \right\|_{L^2[\Upsilon;\mathcal{B}(\mathcal{H}_{\rm p}^2)]} \le C' e^{5\beta_{\max}/2}$$

uniformly in $0 \le s_j \le \beta_{\max}/2$ and $\theta \in D_{\delta_0,\tau}$ we employ the same arguing as in the proof to Lemma A.1.

So far we have proved that $\pi(A)\tilde{\Omega} \in \mathcal{D}_{\mathfrak{D}-{\rm a}}$. We now consider

$$\begin{aligned} &(L_{\rm aux} + N_{\rm res})[\pi(A)\tilde{\Omega}]_\theta \\ &= C\pi_{\rm p}(A_{\rm p}) \otimes \sum_{m,n=0}^{\infty} \frac{i^m(-g)^n}{2^{m/2}m!} \int\limits_{0\le s_n\le\cdots\le s_1\le\beta/2} ds_1\cdots ds_n \\ &\quad \times \Bigg\{ \sum_{\ell=0}^{m-1} \left[a_{\rm gl}^*(F_\theta) + a_{\rm gl}(F_{\overline{\theta}})\right]^\ell \left[a_{\rm gl}^*((|u|+1)F_\theta) - a_{\rm gl}((|u|+1)F_{\overline{\theta}})\right] \\ &\qquad\qquad \times \left[a_{\rm gl}^*(F_\theta) + a_{\rm gl}(F_{\overline{\theta}})\right]^{m-1-\ell} \\ &\qquad\qquad \times \prod_{j=0}^{n-1} \left[a_{\rm gl}^*\left([\mathcal{G}_{(i s_{n-j}\vec{1})}]_\theta\right) + a_{\rm gl}\left([\mathcal{G}_{(-i s_{n-j}\vec{1})}]_{\overline{\theta}}\right)\right] \\ &\qquad + \left[a_{\rm gl}^*(F_\theta) + a_{\rm gl}(F_{\overline{\theta}})\right]^m \\ &\qquad \times \sum_{\ell=0}^{n-1} \left\{ \prod_{j=0}^{\ell-1} \left[a_{\rm gl}^*\left([\mathcal{G}_{(i s_{n-j}\vec{1})}]_\theta\right) + a_{\rm gl}\left([\mathcal{G}_{(-i s_{n-j}\vec{1})}]_{\overline{\theta}}\right)\right] \right\} \\ &\qquad\qquad \times \left[a_{\rm gl}^*\left((|u|+1)[\mathcal{G}_{(i s_{n-\ell}\vec{1})}]_\theta\right) - a_{\rm gl}\left((|u|+1)[\mathcal{G}_{(-i s_{n-\ell}\vec{1})}]_{\overline{\theta}}\right)\right] \\ &\qquad\qquad \times \left\{ \prod_{j=\ell+1}^{n-1} \left[a_{\rm gl}^*\left([\mathcal{G}_{(i s_{n-j}\vec{1})}]_\theta\right) + a_{\rm gl}\left([\mathcal{G}_{(-i s_{n-j}\vec{1})}]_{\overline{\theta}}\right)\right] \right\} \Bigg\} \Omega_0 \end{aligned}$$

where we used the pull through formula to commute $(L_{\rm aux} + N_{\rm res})$ with the creation and annihilation operators,

$$\begin{aligned}
[L_{\mathrm{aux}} + N_{\mathrm{res}}, a^*_{\mathrm{gl}}(F_\theta)] &= \int_\Upsilon dy\, F_\theta(y) \otimes [L_{\mathrm{aux}} + N_{\mathrm{res}}, a^*_{\mathrm{gl}}(y)] \\
&= \int_\Upsilon d(u,\Sigma,r)\, F_\theta(u,\Sigma,r) \otimes a^*_{\mathrm{gl}}(y)(|u|+1) \\
&= a^*_{\mathrm{gl}}((|u+1|)F_\theta).
\end{aligned}$$

As before, we find uniform bounds on

$$\begin{aligned}
\|(|u|+1)F_\theta\|_{L^2[\Upsilon]} + \|F_\theta\|_{L^2[\Upsilon]} &\le C_1, \\
\left\|(|u|+1)[\mathcal{G}_{(\pm is_j\vec{1})}]_\theta\right\|_{L^2[\Upsilon]} + \left\|[\mathcal{G}_{(\pm is_j\vec{1})}]_\theta\right\|_{L^2[\Upsilon]} &\le C_2,
\end{aligned}$$

which allow an estimate

$$\begin{aligned}
&\left\|(L_{\mathrm{aux}} + N_{\mathrm{res}})[\pi(A)\tilde{\Omega}]_\theta\right\| \\
&\le C\,\|A_{\mathrm{p}}\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})} \sum_{m,n=0}^\infty \frac{(m+n)\sqrt{(m+n+1)!}}{m!n!} \left(\frac{C_1}{\sqrt{2}}\right)^m \left(\frac{g\beta C_2}{2}\right)^n \\
&= C\,\|A_{\mathrm{p}}\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})} \sum_{m,n=0}^\infty \frac{m+n}{\sqrt{n!}} \sqrt{\binom{m+n+1}{m+1}} \sqrt{\frac{m+1}{m!}} \left(\frac{C_1}{\sqrt{2}}\right)^m \left(\frac{g\beta C_2}{2}\right)^n \\
&\le C\,\|A_{\mathrm{p}}\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})} \sum_{m,n=0}^\infty \frac{(m+1)(n+1)}{\sqrt{n!}} 2^{m+n+1} \sqrt{\frac{m+1}{m!}} \left(\frac{C_1}{\sqrt{2}}\right)^m \left(\frac{g\beta C_2}{2}\right)^n \\
&= 2C\,\|A_{\mathrm{p}}\|_{\mathcal{B}(\mathcal{H}_{\mathrm{p}})} \sum_{m=0}^\infty \frac{(m+1)^{3/2}}{\sqrt{m!}} (\sqrt{2}\,C_1)^m \sum_{n=0}^\infty \frac{n+1}{\sqrt{n!}} (g\beta C_2)^n \\
&< \infty,
\end{aligned}$$

uniformly in θ on compact subsets of D_{δ_0,τ_0}. We conclude that $\theta \mapsto (L_{\mathrm{aux}} + N_{\mathrm{res}} + 1)[\pi(A)\tilde{\Omega}]_\theta$ is analytic, again using that $\theta \mapsto (|u|+1)F_\theta$ and $\theta \mapsto (|u|+1)[\mathcal{G}_{(\pm is_j\vec{1})}]_\theta$ are analytic functions in the L^2-sense.

Since an arbitrary element from $\mathcal{A}_1$ is a (finite) linear combination of elements of the type $A_{\mathrm{p}} \otimes W(f_1) \otimes \cdots \otimes W(f_R)$ with $(f_1, \ldots, f_R) \in D^{\mathrm{ana}}$ we conclude that $\pi(\mathcal{A}_1)\tilde{\Omega} \subseteq \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$. The proof of $\pi'(\mathcal{A}_1)\tilde{\Omega} \subseteq \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$ uses the same arguments. ∎

Corollary C.15 *The set $\mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$ is dense in $\mathcal{H}^2$.*

Proof. By the Theorem C.14 we know that $\pi(\mathcal{A}_1)\tilde{\Omega} \subseteq \mathcal{D}^{\mathrm{aux}}_{\mathfrak{D}-\mathrm{a}}$ while Proposition C.10 guarantees that $\pi(\mathcal{A}_1)\tilde{\Omega}$ is dense in $\mathcal{H}^2$. ∎

Proposition C.16 (Zero Eigenvector of K_θ) *For any $\theta \in D_{\delta_0,\tau_0}$, the vector $\tilde{\Omega}_\theta$ is in the kernel of K_θ, i.e.,*

$$K_\theta \tilde{\Omega}_\theta = 0.$$

Proof. The previous Theorem C.14 implies that $\tilde{\Omega} \in \mathcal{D}(L_{\text{aux}} + N_{\text{res}})$ and therefore $\tilde{\Omega} \in \mathcal{D}(K_\theta)$. Further, we know that the map

$$\theta \mapsto (L_{\text{aux}} + N_{\text{res}} + 1)\tilde{\Omega}_\theta$$

is analytic, separately in the parameters δ and τ. So is the map

$$\theta \mapsto K_\theta (L_{\text{aux}} + N_{\text{res}} + 1)^{-1}$$

by Proposition C.4. Therefore the function

$$f : D_{\delta_0,\tau_0} \to \mathcal{H}^2, \qquad f(\theta) := K_\theta \tilde{\Omega}_\theta$$

is analytic. For arbitrary $\tau \in \mathbb{R}$ holds

$$f(0,\tau) = \mathfrak{D}_{\text{t}}(\tau) K \mathfrak{D}_{\text{t}}(-\tau) \mathfrak{D}_{\text{t}}(\tau) \tilde{\Omega} = \mathfrak{D}_{\text{t}}(\tau) K \tilde{\Omega} = 0,$$

since $\tilde{\Omega} \in \ker(K)$ by construction. Hence, by analyticity, $f(0,\tau) = 0$ for all $(0,\tau) \in D_{\delta_0,\tau_0}$. Let $\delta \in \mathbb{R}$ with $(\delta,\tau) \in D_{\delta_0,\tau_0}$, then

$$f(\delta,\tau) = \mathfrak{D}_{\text{d}}(\delta) K_{(0,\tau)} \mathfrak{D}_{\text{d}}(-\delta) \mathfrak{D}_{\text{d}}(\delta) \tilde{\Omega}_{(0,\tau)} = \mathfrak{D}_{\text{d}}(\delta) f(0,\tau) = 0.$$

Finally, by analyticity, we conclude that $K_\theta \tilde{\Omega}_\theta = f(\theta) = 0$ for all $\theta \in D_{\delta_0,\tau_0}$. ■

D Manipulations on Wick Monomials

D.1 Wick Ordering

The aim of the present chapter is to provide a technical tool – known as the *Wick ordering procedure* – which allows to rewrite arbitrary products of creation and annihilation operators and free operators in a standard form by commuting creation operators to the very left and annihilation operators to the right side of a product. The results presented here are taken from [6, App. A] and are adapted to our situation. In what follows we consider a Hilbert space

$$\tilde{\mathcal{H}} := \mathcal{H}_{<\infty} \otimes \mathcal{F}(L^2[\Upsilon]),$$

where $\mathcal{H}_{<\infty}$ is an arbitrary finite dimensional Hilbert space. In the applications in the main text we choose either $\mathcal{H}_{<\infty} = \mathcal{H}_{\mathrm{p}}^2 = \mathbb{C}^{N^2}$ or $\mathcal{H}_{<\infty} = \ker(L_{\mathrm{p}}) \cong \mathbb{C}^N$ or $\mathcal{H}_{<\infty} = \ker(\Gamma_{\mathrm{eq}}) = \mathbb{C}\Omega_{\mathrm{p}}$. The space $\mathcal{F}(L^2[\Upsilon])$ is the bosonic Fock space over $L^2[\Upsilon]$ where

$$(\Upsilon, dy) = (\mathbb{R} \times S^2 \times \mathbb{N}_1^R, d(u, \Sigma, r))$$

as introduced in (1.63) on which the creation and annihilation operators $a^*_{\mathrm{gl}}(y)$ and $a_{\mathrm{gl}}(y)$, resp., act. We recall the notation

$$\begin{aligned} \Lambda_{[\theta]} &= \big(\cos(\delta')L_{\mathrm{res}}, M_{[\theta]}\big) \\ &\equiv d\Gamma_{\mathrm{gl}}(\lambda_\theta) \equiv \int_\Upsilon d(u,\Sigma,r)\, a^*_{\mathrm{gl}}(u,\Sigma,r)\lambda_\theta(u)a_{\mathrm{gl}}(u,\Sigma,r) \end{aligned} \tag{D.1}$$

where

$$\lambda_\theta(u) = (\cos(\delta')u, m_\theta(u)) = (\cos(\delta')u, \sin(\delta')|u| + \tau')$$

for

$$\theta = (i\delta', i\tau') \in (i\mathbb{R}^+)^2,$$

the definition of m_θ was given in (3.7, A.10).

We first introduce some notation. Let $n \in \mathbb{N}$ and fix a multi index $\varsigma = (\varsigma_1, \dots, \varsigma_n) \in \{\pm\}^n$. We denote $a^+_{\mathrm{gl}}(y_j) := a^*_{\mathrm{gl}}(y_j)$ and $a^-_{\mathrm{gl}}(y_j) := a_{\mathrm{gl}}(y_j)$ for a set of points $y_1, \dots, y_n \in \Upsilon$. Further we introduce the abbreviation $\mathbb{N}_1^n := \{1, \dots, n\}$ and for a subset $\mathbb{D} \subseteq \mathbb{N}_1^n$ we define $\mathbb{D}^\pm := \{j \in \mathbb{D} \,|\, \varsigma_j = \pm\}$. The *Wick ordered product* $:\cdot:$ of creation and annihilation operators $a_{\mathrm{gl}}^{\varsigma_j}(y_j)$, $j \in \mathbb{D}$, is defined as

$$:\prod_{\substack{j=1,\\ j\in\mathbb{D}}}^{n} a_{\mathrm{gl}}^{\varsigma_j}(y_j): \;:=\; \prod_{j\in\mathbb{D}^+} a_{\mathrm{gl}}^{+}(y_j) \prod_{j\in\mathbb{D}^-} a_{\mathrm{gl}}^{-}(y_j).$$

Here and henceforth, we make the convention $\prod_{j=1}^n A_j := A_1 \cdots A_n$ about the order of products of operators A_j. The expectation value $\langle A \rangle_{\Omega_{\mathrm{vac}}} \in \mathcal{B}(\mathcal{H}_{<\infty})$ of an operator on $\tilde{\mathcal{H}}$ in the vacuum state Ω_{vac} is defined as

$$\langle A \rangle_{\Omega_{\mathrm{vac}}} := \sum_{j,k=1}^{\dim(\mathcal{H}_{<\infty})} |\psi_j\rangle \, \langle\, \psi_j \otimes \Omega_{\mathrm{vac}} \,|\, A\, \psi_k \otimes \Omega_{\mathrm{vac}} \,\rangle_{\tilde{\mathcal{H}}} \, \langle \psi_k| \,, \tag{D.2}$$

where $\{\psi_j\}_{j=1,\dots,\dim(\mathcal{H}_{<\infty})}$ is an orthonormal basis of $\mathcal{H}_{<\infty}$.

The *Wick's theorem* allows to convert arbitrary products of creation and annihilation operators into sums of Wick ordered products, it reads as follows,

Lemma D.1 (Wick's Theorem) *For $n \in \mathbb{N}$ choose $y_1, \dots, y_n \in \Upsilon$ and $(\varsigma_1, \dots, \varsigma_n) \in \{\pm\}^n$. It holds*

$$\prod_{j=1}^{n} a_{\mathrm{gl}}^{\varsigma_j}(y_j) = \sum_{\mathbb{D}\subseteq\mathbb{N}_1^n} \left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}}}^{n} a_{\mathrm{gl}}^{\varsigma_j}(y_j) \right\rangle_{\Omega_{\mathrm{vac}}} :\prod_{j\in\mathbb{D}} a_{\mathrm{gl}}^{\varsigma_j}(y_j): \tag{D.3}$$

in the sense of operator valued distributions.

Proof. We prove the assertion inductively over n. The statement is obviously true for $n = 1$. Now assume that (D.3) holds for all products with $n \geq 1$ factors. We consider the l.h.s. of (D.3) with $n + 1$ factors. We first assume that $\varsigma_{n+1} = -$, then

$$\left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}}}^{n+1} a_{\mathrm{gl}}^{\varsigma_j}(y_j) \right\rangle_{\Omega_{\mathrm{vac}}} = 0$$

for any subset $\mathbb{D} \subseteq \mathbb{N}_1^{n+1}$ with $n+1 \notin \mathbb{D}$. The induction hypothesis yields

$$\begin{aligned}
\prod_{j=1}^{n+1} a_{\rm gl}^{\varsigma_j}(y_j) &= \sum_{\mathbb{D}\subseteq\mathbb{N}_1^n} \left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}}}^{n} a_{\rm gl}^{\varsigma_j}(y_j) \right\rangle_{\Omega_{\rm vac}} \boldsymbol{:}\prod_{j\in\mathbb{D}} a_{\rm gl}^{\varsigma_j}(y_j)\boldsymbol{:} a_{\rm gl}^{-}(y_{n+1}) \\
&= \sum_{\mathbb{D}\subseteq\mathbb{N}_1^n} \left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}}}^{n} a_{\rm gl}^{\varsigma_j}(y_j) \right\rangle_{\Omega_{\rm vac}} \boldsymbol{:}\prod_{j\in\mathbb{D}} a_{\rm gl}^{\varsigma_j}(y_j) a_{\rm gl}^{-}(y_{n+1})\boldsymbol{:} \\
&= \sum_{\mathbb{D}\subseteq\mathbb{N}_1^{n+1}} \left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}}}^{n+1} a_{\rm gl}^{\varsigma_j}(y_j) \right\rangle_{\Omega_{\rm vac}} \boldsymbol{:}\prod_{j\in\mathbb{D}} a_{\rm gl}^{\varsigma_j}(y_j)\boldsymbol{:}.
\end{aligned}$$

We go over to consider the case $\varsigma_{n+1} = +$. We remark that the CCR can be represented as

$$[a_{\rm gl}^{\varsigma_k}(y_k), a_{\rm gl}^{+}(y_{n+1})] = \left\langle a_{\rm gl}^{\varsigma_k}(y_k) a_{\rm gl}^{+}(y_{n+1}) \right\rangle_{\Omega_{\rm vac}}$$

and therefore, for $\mathbb{D} \subseteq \mathbb{N}_1^n$,

$$\left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}}}^{n} a_{\rm gl}^{\varsigma_j}(y_j) a_{\rm gl}^{+}(y_{n+1}) \right\rangle_{\Omega_{\rm vac}} = \sum_{k\notin\mathbb{D}} \left\langle a_{\rm gl}^{\varsigma_k}(y_k) a_{\rm gl}^{+}(y_{n+1}) \right\rangle_{\Omega_{\rm vac}} \left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}\cup\{k\}}}^{n} a_{\rm gl}^{\varsigma_j}(y_j) \right\rangle_{\Omega_{\rm vac}}.$$

Using the induction hypothesis we get

$$\begin{aligned}
\prod_{j=1}^{n+1} a_{\rm gl}^{\varsigma_j}(y_j) &= a_{\rm gl}^{+}(y_{n+1}) \prod_{j=1}^{n} a_{\rm gl}^{\varsigma_j}(y_j) + \sum_{k=1}^{n} \left\langle a_{\rm gl}^{\varsigma_k}(y_k) a_{\rm gl}^{+}(y_{n+1}) \right\rangle_{\Omega_{\rm vac}} \prod_{\substack{j=1,\\ j\neq k}}^{n} a_{\rm gl}^{\varsigma_j}(y_j) \\
&= \sum_{\mathbb{D}\subseteq\mathbb{N}_1^n} \left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}}}^{n} a_{\rm gl}^{\varsigma_j}(y_j) \right\rangle_{\Omega_{\rm vac}} \boldsymbol{:} a_{\rm gl}^{+}(y_{n+1}) \prod_{j\in\mathbb{D}} a_{\rm gl}^{\varsigma_j}(y_j)\boldsymbol{:} \\
&\quad + \sum_{k=1}^{n} \sum_{\mathbb{D}\subseteq\mathbb{N}_1^n\setminus\{k\}} \left\langle a_{\rm gl}^{\varsigma_k}(y_k) a_{\rm gl}^{+}(y_{n+1}) \right\rangle_{\Omega_{\rm vac}} \left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}\cup\{k\}}}^{n} a_{\rm gl}^{\varsigma_j}(y_j) \right\rangle_{\Omega_{\rm vac}} \\
&\quad\quad \times \boldsymbol{:}\prod_{j\in\mathbb{D}} a_{\rm gl}^{\varsigma_j}(y_j)\boldsymbol{:} \\
&= \sum_{\mathbb{D}\subseteq\mathbb{N}_1^n} \left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}}}^{n} a_{\rm gl}^{\varsigma_j}(y_j) \right\rangle_{\Omega_{\rm vac}} \boldsymbol{:} a_{\rm gl}^{+}(y_{n+1}) \prod_{j\in\mathbb{D}} a_{\rm gl}^{\varsigma_j}(y_j)\boldsymbol{:} \\
&\quad + \sum_{\mathbb{D}\subseteq\mathbb{N}_1^n} \left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}}}^{n} a_{\rm gl}^{\varsigma_j}(y_j) a_{\rm gl}^{+}(y_{n+1}) \right\rangle_{\Omega_{\rm vac}} \boldsymbol{:}\prod_{j\in\mathbb{D}} a_{\rm gl}^{\varsigma_j}(y_j)\boldsymbol{:}
\end{aligned}$$

$$= \sum_{\mathbb{D}\subseteq\mathbb{N}_1^{n+1}} \left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}}}^{n+1} a_{\rm gl}^{\varsigma_j}(y_j) \right\rangle_{\Omega_{\rm vac}} :\prod_{j\in\mathbb{D}} a_{\rm gl}^{\varsigma_j}(y_j): .$$

■

Lemma D.2 *For $n \in \mathbb{N}$ we choose points $y_j = (u_j, \Sigma_j, r_j) \in \Upsilon$ and measurable functions $F_j : \mathbb{R} \times [0,\infty) \to \mathcal{B}(\mathcal{H}_{<\infty})$, $j = 1,\ldots,n$. Fix a multi index $\varsigma = (\varsigma_1,\ldots,\varsigma_n) \in \{\pm\}^n$. Then the following equality holds,*

$$\prod_{j=1}^{n} \left[a_{\rm gl}^{\varsigma_j}(y_j) F_j(\Lambda_{[\theta]})\right]$$
$$= \sum_{\mathbb{D}\subseteq\mathbb{N}_1^n} \left[\prod_{j\in\mathbb{D}^+} a_{\rm gl}^{+}(y_j)\right]$$
$$\times \left\langle \prod_{j=1}^{n} \left[a_{\rm gl}^{\varsigma_j}(y_j)\right]^{\mathbf{1}_{\mathbb{N}_1^n\setminus\mathbb{D}}(j)} F_j\left(\Lambda_{[\theta]} + \lambda + \sum_{\substack{i=1,\\ i\in\mathbb{D}^-}}^{j} \lambda_\theta(u_i) + \sum_{\substack{i=j+1,\\ i\in\mathbb{D}^+}}^{n} \lambda_\theta(u_i)\right)\right\rangle_{\Omega_{\rm vac}}\Bigg|_{\lambda=\Lambda_{[\theta]}}$$
$$\times \left[\prod_{j\in\mathbb{D}^-} a_{\rm gl}^{-}(y_j)\right] \tag{D.4}$$

in the sense of operator valued distribution. Hereby, $F_j(\Lambda_{[\theta]})$ is defined via functional calculus using that the components of $\Lambda_{[\theta]}$ are self-adjoint operators commuting with each other. The indicator function $\mathbf{1}_{\mathbb{D}}(j)$ for a subset $\mathbb{D} \subseteq \mathbb{N}_1^n$ is defined to be one if $j \in \mathbb{D}$ and to be zero otherwise such that for an operator A holds $[A]^{\mathbf{1}_{\mathbb{D}}(j)} = A$ for $j \in \mathbb{D}$ and $[A]^{\mathbf{1}_{\mathbb{D}}(j)} = \mathbb{1}$ for $j \notin \mathbb{D}$.

Proof. With a twofold application of the pull through formula (1.67) and of Lemma D.1 we obtain

$$\prod_{j=1}^{n} \left[a_{\rm gl}^{\varsigma_j}(y_j) F_j(\Lambda_{[\theta]})\right]$$
$$= \prod_{j=1}^{n} a_{\rm gl}^{\varsigma_j}(y_j) \prod_{j=1}^{n} F_j\left(\Lambda_{[\theta]} + \sum_{i=j+1}^{n} \varsigma_i\lambda_\theta(u_i)\right)$$
$$= \sum_{\mathbb{D}\subseteq\mathbb{N}_1^n} \left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}}}^{n} a_{\rm gl}^{\varsigma_j}(y_j) \right\rangle_{\Omega_{\rm vac}} :\prod_{\substack{j=1,\\ j\in\mathbb{D}}}^{n} a_{\rm gl}^{\varsigma_j}(y_j): \prod_{j=1}^{n} F_j\left(\Lambda_{[\theta]} + \sum_{i=j+1}^{n} \varsigma_i\lambda_\theta(u_i)\right)$$
$$= \sum_{\mathbb{D}\subseteq\mathbb{N}_1^n} \prod_{j\in\mathbb{D}^+} a_{\rm gl}^{+}(y_j) \left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}}}^{n} a_{\rm gl}^{\varsigma_j}(y_j) \right\rangle_{\Omega_{\rm vac}}$$

$$\times \prod_{j=1}^{n} F_j\left(\Lambda_{[\theta]} + \sum_{i=j+1}^{n} \varsigma_i \lambda_\theta(u_i) + \sum_{i\in\mathbb{D}^-} \lambda_\theta(u_i)\right) \prod_{j\in\mathbb{D}^-} a^-_{\rm gl}(y_j).$$

We now use that for a measurable function $F : \mathbb{R}\times[0,\infty) \to \mathcal{B}(\mathcal{H}_{<\infty})$ and any operator A on $\tilde{\mathcal{H}}$ the relation

$$\langle A\rangle_{\Omega_{\rm vac}} F(0,0) = \left\langle A\, F(\Lambda_{[\theta]})\right\rangle_{\Omega_{\rm vac}}$$

and therefore

$$\langle A\rangle_{\Omega_{\rm vac}} F(\Lambda_{[\theta]}) = \left\langle A\, F(\Lambda_{[\theta]}+\lambda)\right\rangle_{\Omega_{\rm vac}}\Big|_{\lambda=\Lambda_{[\theta]}}$$

holds. This and another application of the pull through formula leads to

$$\begin{aligned}
&\prod_{j=1}^{n}\left[a^{\varsigma_j}_{\rm gl}(y_j)F_j(\Lambda_{[\theta]})\right]\\
&\quad= \sum_{\mathbb{D}\subseteq\mathbb{N}_1^n}\prod_{j\in\mathbb{D}^+} a^+_{\rm gl}(y_j)\\
&\qquad\times\left\langle \prod_{\substack{j=1,\\ j\notin\mathbb{D}}}^{n} a^{\varsigma_j}_{\rm gl}(y_j)\prod_{j=1}^{n}F_j\left(\Lambda_{[\theta]}+\lambda+\sum_{i=j+1}^{n}\varsigma_i\lambda_\theta(u_i)+\sum_{i\in\mathbb{D}^-}\lambda_\theta(u_i)\right)\right\rangle_{\Omega_{\rm vac}}\Bigg|_{\lambda=\Lambda_{[\theta]}}\\
&\qquad\times\prod_{j\in\mathbb{D}^-}a^-_{\rm gl}(y_j)\\
&\quad= \sum_{\mathbb{D}\subseteq\mathbb{N}_1^n}\prod_{j\in\mathbb{D}^+} a^+_{\rm gl}(y_j)\\
&\qquad\times\left\langle \prod_{j=1}^{n}\left[a^{\varsigma_j}_{\rm gl}(y_j)\right]^{\mathbf{1}_{\mathbb{N}_1^n\setminus\mathbb{D}}(j)} F_j\left(\Lambda_{[\theta]}+\lambda+\sum_{\substack{i=j+1,\\ i\in\mathbb{D}}}^{n}\varsigma_i\lambda_\theta(u_i)+\sum_{i\in\mathbb{D}^-}\lambda_\theta(u_i)\right)\right\rangle_{\Omega_{\rm vac}}\Bigg|_{\lambda=\Lambda_{[\theta]}}\\
&\qquad\times\prod_{j\in\mathbb{D}^-}a^-_{\rm gl}(y_j).
\end{aligned}$$

The assertion follows by the next consideration

$$\begin{aligned}
\sum_{\substack{i=j+1,\\ i\in\mathbb{D}}}^{n}\varsigma_i\lambda_\theta(u_i)+\sum_{i\in\mathbb{D}^-}\lambda_\theta(u_i) &= \sum_{\substack{i=j+1,\\ i\in\mathbb{D}^+}}^{n}\lambda_\theta(u_i)-\sum_{\substack{i=j+1,\\ i\in\mathbb{D}^-}}^{n}\lambda_\theta(u_i)+\sum_{i\in\mathbb{D}^-}\lambda_\theta(u_i)\\
&= \sum_{\substack{i=j+1,\\ i\in\mathbb{D}^+}}^{n}\lambda_\theta(u_i)+\sum_{\substack{i=1,\\ i\in\mathbb{D}^-}}^{j}\lambda_\theta(u_i).
\end{aligned}$$

■

The previous result allows us to "normal order" products of Wick monomials. Before we state the corresponding theorem we first provide some notational tools. Let $L, S, R \in \mathbb{N}$ and let $(r_1, \ldots, r_L), (s_1, \ldots, s_L) \in \mathbb{N}^L$ be vectors of natural numbers whose components fulfil $r_1 + \cdots + r_L = R$ and $s_1 + \cdots + s_L = S$. For vectors $y^{(R)} \in \Upsilon^R$ and $\tilde{y}^{(S)} \in \Upsilon^S$ we write

$$\begin{aligned} Y^{(R,S)} &= \left(y^{(R)}, \tilde{y}^{(S)}\right) \in \Upsilon^{R+S}, \\ y^{(R)} &= \left(y_1^{(r_1)}, \ldots, y_L^{(r_L)}\right) \in \Upsilon^R, & y_\ell^{(r_\ell)} &= \left(y_{\ell,1}^{(r_\ell)}, \ldots, y_{\ell,r_\ell}^{(r_\ell)}\right) \in \Upsilon^{r_\ell}, \\ \tilde{y}^{(S)} &= \left(\tilde{y}_1^{(s_1)}, \ldots, \tilde{y}_L^{(s_L)}\right) \in \Upsilon^S, & \tilde{y}_\ell^{(s_\ell)} &= \left(\tilde{y}_{\ell,1}^{(s_\ell)}, \ldots, \tilde{y}_{\ell,s_\ell}^{(s_\ell)}\right) \in \Upsilon^{s_\ell} \end{aligned}$$

and

$$y_{\ell,j}^{(r_\ell)} = \left(u_{\ell,j}^{(r_\ell)}, \Sigma_{\ell,j}^{(r_\ell)}, r_{\ell,j}^{(r_\ell)}\right), \qquad \tilde{y}_{\ell,j}^{(r_\ell)} = \left(\tilde{u}_{\ell,j}^{(s_\ell)}, \tilde{\Sigma}_{\ell,j}^{(s_\ell)}, \tilde{r}_{\ell,j}^{(s_\ell)}\right).$$

We further abbreviate

$$\begin{aligned} dY^{(R,S)} &= dy^{(R)} d\tilde{y}^{(S)}, \\ dy^{(R)} &= \prod_{\ell=1}^{L} dy_\ell^{(r_\ell)}, & dy_\ell^{(r_\ell)} &= \prod_{j=1}^{r_\ell} dy_{\ell,j}^{(r_\ell)}, \\ d\tilde{y}^{(S)} &= \prod_{\ell=1}^{L} d\tilde{y}_\ell^{(s_\ell)}, & d\tilde{y}_\ell^{(s_\ell)} &= \prod_{j=1}^{s_\ell} d\tilde{y}_{\ell,j}^{(s_\ell)} \end{aligned}$$

and

$$\begin{aligned} a_{\mathrm{gl}}^*\left(y^{(R)}\right) &= \prod_{\ell=1}^{L} a_{\mathrm{gl}}^*\left(y_\ell^{(r_\ell)}\right), & a_{\mathrm{gl}}^*\left(y_\ell^{(r_\ell)}\right) &= \prod_{j=1}^{r_\ell} a_{\mathrm{gl}}^*\left(y_{\ell,j}^{(r_\ell)}\right), \\ a_{\mathrm{gl}}\left(\tilde{y}^{(R)}\right) &= \prod_{\ell=1}^{L} a_{\mathrm{gl}}\left(\tilde{y}_\ell^{(s_\ell)}\right), & a_{\mathrm{gl}}\left(\tilde{y}_\ell^{(s_\ell)}\right) &= \prod_{j=1}^{s_\ell} a_{\mathrm{gl}}\left(\tilde{y}_{\ell,j}^{(s_\ell)}\right) \end{aligned}$$

and

$$\begin{aligned} m_\theta\left(Y^{(R,S)}\right) &= m_\theta\left(y^{(R)}\right) m_\theta\left(\tilde{y}^{(S)}\right), \\ m_\theta\left(y^{(R)}\right) &= \prod_{\ell=1}^{L} m_\theta\left(y_\ell^{(r_\ell)}\right), & m_\theta\left(y_\ell^{(r_\ell)}\right) &= \prod_{j=1}^{r_\ell} m_\theta\left(u_{\ell,j}^{(r_\ell)}\right), \\ m_\theta\left(\tilde{y}^{(S)}\right) &= \prod_{\ell=1}^{L} m_\theta\left(\tilde{y}_\ell^{(s_\ell)}\right), & m_\theta\left(\tilde{y}_\ell^{(s_\ell)}\right) &= \prod_{j=1}^{s_\ell} m_\theta\left(\tilde{u}_{\ell,j}^{(s_\ell)}\right) \end{aligned}$$

With these notations at hand we define the Wick monomials. To this end we choose a sequence of measurable form factors

$$w_{R,S} : \mathbb{R} \times [0, \infty) \times \Upsilon^{R+S} \to \mathcal{B}(\mathcal{H}_{<\infty}), \qquad R + S \geq 0,$$

such that for almost every $\lambda \in \mathbb{R} \times [0,\infty)$ the functions $(y_1, \ldots, y_{R+S}) \mapsto w_{R,S}[\lambda; y_1, \ldots, y_{R+S}]$ are square integrable over Υ^{R+S}. The map $\mathcal{W}_{[\theta]}$ assigns an operator on $\tilde{\mathcal{H}}$, the series of *Wick monomials*, to the sequence of form factors by

$$\mathcal{W}_{[\theta]}[(w_{R,S})_{R+S\geq 0}] := \sum_{R+S\geq 0} \mathcal{W}_{[\theta]}[w_{R,S}]$$

with

$$\mathcal{W}_{[\theta]}[w_{0,0}] := w_{0,0}\left[\Lambda_{[\theta]}\right] \tag{D.5}$$

and

$$\mathcal{W}_{[\theta]}[w_{R,S}] := \int_{\Upsilon^{R+S}} \frac{dY^{(R,S)}}{m_\theta\left(Y^{(R,S)}\right)^{1/2}} a^*_{\mathrm{gl}}\left(y^{(R)}\right) w_{R,S}\left[\Lambda_{[\theta]}; Y^{(R,S)}\right] a_{\mathrm{gl}}\left(\tilde{y}^{(S)}\right) \tag{D.6}$$

for $R+S \geq 1$. Finally, we introduce *partially integrated Wick monomials*

$$\mathcal{W}^{(p,q)}_{[\theta]}\left[w_{R+p,S+q}\right] : \mathbb{R} \times [0,\infty) \times \Upsilon^p \times \Upsilon^q \to \{\text{operators on } \tilde{\mathcal{H}}\} \tag{D.7}$$

defined by

$$\begin{aligned}
&\mathcal{W}^{(p,q)}_{[\theta]}\left[w_{R+p,S+q}\right]\left(\lambda; y^{(p)}_*, \tilde{y}^{(q)}_*\right) \\
&:= \int_{\Upsilon^{R+S}} \frac{dY^{(R,S)}}{m_\theta(Y^{(R,S)})^{1/2}} \\
&\qquad \times a^*_{\mathrm{gl}}\left(y^{(R)}\right) w_{R+p,S+q}\left[\Lambda_{[\theta]} + \lambda; y^{(R)}, y^{(p)}_*, \tilde{y}^{(S)}, \tilde{y}^{(q)}_*\right] a_{\mathrm{gl}}\left(\tilde{y}^{(S)}\right)
\end{aligned}$$

for $y^{(p)}_* \in \Upsilon^p$ and $\tilde{y}^{(q)}_* \in \Upsilon^q$.

The above operators are well defined and even bounded on suitable subspaces $\mathcal{H}_{<\infty} \otimes P_{\left[M_{[\theta]}\leq 1\right]}\mathcal{F}(L^2[\Upsilon])$ of $\tilde{\mathcal{H}}$, the corresponding statement is made in Section 4.1, Proposition 4.1. In this appendix we are rather interested in algebraic properties and manipulations of these operators. Thus, the next theorem has to be understood as an algebraic statement – having in mind that it may be read as a relation for bounded operators when it finds application in Chapter 4.

Theorem D.3 (Ordering of Wick Monomials) *For $L \in \mathbb{N}$ we choose measurable functions $F_0, F_1, \ldots, F_L : \mathbb{R} \times [0,\infty) \to \mathcal{B}(\mathcal{H}_{<\infty})$. Further, let $w_{R,S} : \mathbb{R} \times [0,\infty) \times \Upsilon^R \times \Upsilon^S \to \mathcal{B}(\mathcal{H}_{<\infty})$, $R+S \geq 0$, be a sequence of measurable form factors such that for almost every $\lambda \in \mathbb{R} \times [0,\infty)$ the functions $w_{R,S}[\lambda; \cdot]$ are square integrable over Υ^{R+S}. Further, we assume that the functions $y^{(R)} \mapsto w_{R,S}[\lambda; y^{(R)}, \tilde{y}^{(S)}]$ and $\tilde{y}^{(S)} \mapsto w_{R,S}[\lambda; y^{(R)}, \tilde{y}^{(S)}]$ are totally symmetric under permutation of the variables $y^{(R)} = \left(y^{(R_1)}_1, \ldots, y^{(R_L)}_L\right)$ and $\tilde{y}^{(S)} = \left(\tilde{y}^{(S_1)}_1, \ldots, \tilde{y}^{(S_L)}_L\right)$, resp.*

Set $W := \mathcal{W}_{[\theta]}[(w_{R,S})_{R+S\geq m}]$ *for* $m \in \mathbb{N}_0$ *and identify* $F_\ell \equiv F_\ell(\Lambda_{[\theta]})$. *Then, the alternating product* $F_0 W F_1 W \cdots W F_{L-1} W F_L$ *can be expressed as a series of ordered Wick monomials,*

$$F_0 W F_1 W \cdots W F_{L-1} W F_L = \sum_{R+S\geq 0} \mathcal{W}_{[\theta]}[\hat{w}_{R,S}],$$

where the integral kernels $\hat{w}_{R,S}$ *are the symmetrization*

$$\begin{aligned} &\hat{w}_{R,S}[\lambda; y_1, \ldots y_R, \tilde{y}_1, \ldots, \tilde{y}_S] \\ &\quad := \sum_{\substack{\pi\in S_R,\\ \eta\in S_S}} \frac{1}{R!S!} \tilde{w}_{R,S}[\lambda; y_{\pi 1}, \ldots, y_{\pi R}, \tilde{y}_{\eta 1}, \ldots, \tilde{y}_{\eta S}] \end{aligned} \tag{D.8}$$

of the functions $\tilde{w}_{R,S}$ *given by*

$$\begin{aligned} &\tilde{w}_{R,S}[\lambda; Y^{(R,S)}] \\ &:= \sum_{\substack{r_1+\cdots+r_L=R,\\ s_1+\cdots+s_L=S}} \sum_{\substack{p_1,q_1,\ldots,p_L,q_L:\\ r_\ell+p_\ell+s_\ell+q_\ell\geq m}} \left[\prod_{\ell=1}^{L} \binom{r_\ell+p_\ell}{r_\ell}\binom{s_\ell+q_\ell}{s_\ell}\right] \\ &\qquad \times F_0\left(\lambda + \eta_0^{(\theta)}\left(Y^{(R,S)}\right)\right) \\ &\qquad \times \Big\langle \mathcal{W}_{[\theta]}^{(r_1,s_1)}\left[w_{r_1+p_1,s_1+q_1}\right]\left(\lambda + \eta_1^{(\theta)}\left(Y^{(R,S)}\right); y_1^{(r_1)}, \tilde{y}_1^{(s_1)}\right) \\ &\qquad\quad \times F_1\left(\Lambda_{[\theta]} + \lambda + \eta_1^{(\theta)}\left(Y^{(R,S)}\right) + \sum_{j=1}^{s_1} \lambda_\theta\left(\tilde{u}_{1,j}^{(s_1)}\right)\right) \\ &\qquad\quad \cdots \\ &\qquad\quad \times F_{L-1}\left(\Lambda_{[\theta]} + \lambda + \eta_{L-1}^{(\theta)}\left(Y^{(R,S)}\right) + \sum_{j=1}^{s_{L-1}} \lambda_\theta\left(\tilde{u}_{L-1,j}^{(s_{L-1})}\right)\right) \\ &\qquad\quad \times \mathcal{W}_{[\theta]}^{(r_L,s_L)}\left[w_{r_L+p_L,s_L+q_L}\right]\left(\lambda + \eta_L^{(\theta)}\left(Y^{(R,S)}\right); y_L^{(r_L)}, \tilde{y}_L^{(s_L)}\right) \Big\rangle_{\Omega_{\mathrm{vac}}} \\ &\qquad \times F_L\left(\lambda + \eta_L^{(\theta)}\left(Y^{(R,S)}\right) + \sum_{j=1}^{s_L} \lambda_\theta(\tilde{u}_{L,j})\right), \end{aligned} \tag{D.9}$$

where

$$\eta_\ell^{(\theta)}\left(Y^{(R,S)}\right) := \sum_{j=1}^{\ell-1}\sum_{i=1}^{s_j} \lambda_\theta\left(\tilde{u}_{j,i}^{(s_j)}\right) + \sum_{j=\ell+1}^{L}\sum_{i=1}^{r_j} \lambda_\theta\left(u_{j,i}^{(r_j)}\right). \tag{D.10}$$

Proof. We introduce the abbreviation $W_{R,S} := \mathcal{W}_{[\theta]}[w_{R,S}]$ to write

$$F_0 W F_1 W \cdots W F_{L-1} W F_L = \sum_{\substack{R_1+S_1 \geq m, \\ R_L+\ddot{S}_L \geq m}} F_0 W_{R_1,S_1} F_1 W_{R_2,S_2} \cdots W_{R_L,S_L} F_L.$$

We compute each addend of this series separately. Let $R = \sum_{\ell=1}^{L} R_\ell$ and $S = \sum_{\ell=1}^{L} S_\ell$. Then

$$\begin{aligned}
& F_0 W_{R_1,S_1} F_1 W_{R_2,S_2} \cdots W_{R_L,S_L} F_L \\
&= \int_{\Upsilon^{R+S}} \frac{dY^{(R,S)}}{m_\theta(Y^{(R,S)})^{1/2}} \\
&\qquad \times F_0[\Lambda_{[\theta]}] a^*_{\mathrm{gl}}\left(y_1^{(R_1)}\right) w_{R_1,S_1}\left[\Lambda_{[\theta]}; y_1^{(R_1)}, \tilde{y}_1^{(S_1)}\right] a_{\mathrm{gl}}\left(\tilde{y}_1^{(S_1)}\right) F_1[\Lambda_{[\theta]}] \\
&\qquad \cdots \\
&\qquad \times F_{L-1}[\Lambda_{[\theta]}] a^*_{\mathrm{gl}}\left(y_L^{(R_L)}\right) w_{R_L,S_L}\left[\Lambda_{[\theta]}; y_L^{(R_L)}, \tilde{y}_L^{(S_L)}\right] a_{\mathrm{gl}}\left(\tilde{y}_L^{(S_L)}\right) F_L[\Lambda_{[\theta]}].
\end{aligned}$$

We now apply Lemma D.2 which allows to write the above expression as a sum, indexed by subsets $\mathbb{D}$ of $\mathbb{N}_1^{R+S} = \{1, \ldots, R+S\}$, over Wick ordered products. To handle combinatorial difficulties we go over to represent the sets $\mathbb{D}$ and $\mathbb{N}_1^{R+S}$ as

$$\begin{aligned}
\mathbb{N}_1^{R+S} &\equiv \left[\bigcup_{\ell=1}^{L} \mathcal{N}_{R,\ell}\right] \cup \left[\bigcup_{\ell=1}^{L} \tilde{\mathcal{N}}_{S,\ell}\right], \qquad && \mathcal{N}_{R,\ell} = \{(R,\ell,j) \mid j = 1, \ldots, R_L\}, \\
& && \tilde{\mathcal{N}}_{S,\ell} = \left\{(S,\ell,j)\tilde{} \;\middle|\; j = 1, \ldots, S_L\right\}, \\
\mathbb{D} &\equiv \left[\bigcup_{\ell=1}^{L} \mathbb{D}_{R,\ell}\right] \cup \left[\bigcup_{\ell=1}^{L} \tilde{\mathbb{D}}_{S,\ell}\right], && \mathbb{D}_{R,\ell} = \mathbb{D} \cap \mathcal{N}_{R,\ell}, \\
& && \tilde{\mathbb{D}}_{S,\ell} = \mathbb{D} \cap \tilde{\mathcal{N}}_{S,\ell}.
\end{aligned}$$

Using this representation the summation over subsets of $\mathbb{N}_1^{R+S}$ is replaced by

$$\sum_{\mathbb{D} \subseteq \mathbb{N}_1^{R+S}} \equiv \sum_{\mathbb{D}_{R,1} \subseteq \mathcal{N}_{R,1}} \sum_{\tilde{\mathbb{D}}_{S,1} \subseteq \tilde{\mathcal{N}}_{S,1}} \cdots \sum_{\mathbb{D}_{R,L} \subseteq \mathcal{N}_{R,L}} \sum_{\tilde{\mathbb{D}}_{S,L} \subseteq \tilde{\mathcal{N}}_{S,L}}. \tag{D.11}$$

Each subset $\mathbb{D}_{R,\ell}$ specifies those $r_\ell := \#D_{R,\ell} \leq R_\ell$ variables $\{y_{\ell,j} \mid (R,\ell,j) \in \mathbb{D}_{R,\ell}\}$ that are Wick ordered outside the vacuum expectation value (appearing in the creation operators to the left), and those $R_\ell - r_\ell = \#(\mathcal{N}_{R,\ell} \setminus \mathbb{D}_{R,\ell})$ variables $\{y_{\ell,j} \mid (R,\ell,j) \notin \mathbb{D}_{R,\ell}\}$ that appear in the vacuum expectation value in (D.4). The subset $\tilde{\mathbb{D}}_{S,\ell}$ correspondingly specifies the variables of the annihilation operators that are Wick ordered outside the vacuum expectation value. We consider a special term contributing to the sum,

$$\begin{aligned}
\mathbb{D}_{R,\ell} &= \{(R,\ell,j) \mid j = 1, \ldots, r_\ell\}, \\
\mathcal{N}_{R,\ell} \setminus \mathbb{D}_{R,\ell} &= \{(R,\ell,j) \mid j = r_\ell + 1, \ldots, R_\ell\}, \\
\tilde{\mathbb{D}}_{S,\ell} &= \left\{(S,\ell,j)\tilde{} \;\middle|\; j = 1, \ldots, s_\ell\right\}, \\
\tilde{\mathcal{N}}_{S,\ell} \setminus \tilde{\mathbb{D}}_{S,\ell} &= \left\{(S,\ell,j)\tilde{} \;\middle|\; j = s_\ell + 1, \ldots, S_\ell\right\}.
\end{aligned}$$

According to Lemma D.2 the contribution generated by this term is given by

$$
\begin{aligned}
&\prod_{\ell=1}^{L}\left[\prod_{j=1}^{r_\ell}\int_\Upsilon \frac{dy_{\ell,j}}{m_\theta(u_{\ell,j})^{1/2}}\prod_{j=1}^{s_\ell}\int_\Upsilon \frac{d\tilde{y}_{\ell,j}}{m_\theta(\tilde{u}_{\ell,j})^{1/2}}\right]\left[\prod_{\ell=1}^{L}\prod_{j=1}^{r_\ell} a^*_{\mathrm{gl}}(y_{\ell,j})\right]\\
&\quad\times\Bigg\{F_0\left(\lambda+\sum_{\ell=1}^{L}\sum_{j=1}^{r_\ell}\lambda_\theta(u_{\ell,j})\right)\\
&\qquad\times\Bigg\langle\left[\prod_{j=r_1+1}^{R_1}\int_\Upsilon \frac{dy_{1,j}}{m_\theta(u_{1,j})^{1/2}}\prod_{j=s_1+1}^{S_1}\int_\Upsilon \frac{d\tilde{y}_{1,j}}{m_\theta(\tilde{u}_{1,j})^{1/2}}\right]\left[\prod_{j=r_1+1}^{R_1} a^*_{\mathrm{gl}}(y_{1,j})\right]\\
&\qquad\qquad\times w_{R_1,S_1}\left(\Lambda_{[\theta]}+\lambda+\eta_1^{(\theta)};y_1^{(R_1)},\tilde{y}_1^{(S_1)}\right)\left[\prod_{j=s_1+1}^{S_1} a_{\mathrm{gl}}(\tilde{y}_{1,j})\right]\\
&\qquad\quad\times F_1\left(\Lambda_{[\theta]}+\lambda+\eta_1^{(\theta)}+\sum_{j=1}^{S_1}\lambda_\theta(\tilde{u}_{1,j})\right)\\
&\qquad\quad\cdots\\
&\qquad\quad\times F_{L-1}\left(\Lambda_{[\theta]}+\lambda+\eta_{L-1}^{(\theta)}+\sum_{j=1}^{S_{L-1}}\lambda_\theta(\tilde{u}_{L-1,j})\right)\\
&\qquad\quad\times\left[\prod_{j=r_L+1}^{R_L}\int_\Upsilon \frac{dy_{L,j}}{m_\theta(u_{L,j})^{1/2}}\prod_{j=s_L+1}^{S_L}\int_\Upsilon \frac{d\tilde{y}_{L,j}}{m_\theta(\tilde{u}_{L,j})^{1/2}}\right]\left[\prod_{j=r_L+1}^{R_L} a^*_{\mathrm{gl}}(y_{L,j})\right]\\
&\qquad\qquad\times w_{R_L,S_L}\left(\Lambda_{[\theta]}+\lambda+\eta_L^{(\theta)};y_L^{(R_L)},\tilde{y}_L^{(S_L)}\right)\left[\prod_{j=s_L+1}^{S_L} a_{\mathrm{gl}}(\tilde{y}_{1,j})\right]\Bigg\rangle_{\Omega_{\mathrm{vac}}}\\
&\quad\Bigg\}\Bigg|_{\lambda=\Lambda_{[\theta]}}\left[\prod_{\ell=1}^{L}\prod_{j=1}^{s_\ell} a_{\mathrm{gl}}(\tilde{y}_{\ell,j})\right]\\
&=\prod_{\ell=1}^{L}\left[\prod_{j=1}^{r_\ell}\int_\Upsilon \frac{dy_{\ell,j}}{m_\theta(u_{\ell,j})^{1/2}}\prod_{j=1}^{s_\ell}\int_\Upsilon \frac{d\tilde{y}_{\ell,j}}{m_\theta(\tilde{u}_{\ell,j})^{1/2}}\right]\left[\prod_{\ell=1}^{L}\prod_{j=1}^{r_\ell} a^*_{\mathrm{gl}}(y_{\ell,j})\right]\\
&\quad\times\Bigg\{F_0\left(\lambda+\eta_0^{(\theta)}\right)\\
&\qquad\times\Bigg\langle\mathcal{W}_{[\theta]}^{(r_1,s_1)}\left[w_{R_1,S_1}\right]\left(\lambda+\eta_1^{(\theta)};y_{1,*}^{(r_1)},\tilde{y}_{1,*}^{(s_1)}\right)\\
&\qquad\quad\times F_1\left(\Lambda_{[\theta]}+\lambda+\eta_1^{(\theta)}+\sum_{j=1}^{S_1}\lambda_\theta(\tilde{u}_{1,j})\right)\\
&\qquad\quad\cdots
\end{aligned}
$$

$$\times F_{L-1}\left(\Lambda_{[\theta]}+\lambda+\eta_{L-1}^{(\theta)}+\sum_{j=1}^{S_{L-1}}\lambda_\theta(\tilde{u}_{L-1,j})\right)$$
$$\times \mathcal{W}_{[\theta]}^{(r_L,s_L)}\left[w_{R_L,S_L}\right]\left(\lambda+\eta_1^{(\theta)};y_{L,*}^{(r_L)},\tilde{y}_{L,*}^{(s_L)}\right)\Big\rangle_{\Omega_{\rm vac}}\Bigg\}\Bigg|_{\lambda=\Lambda_{[\theta]}}$$
$$\times\left[\prod_{\ell=1}^{L}\prod_{j=1}^{s_\ell}a_{\rm gl}(\tilde{y}_{\ell,j})\right]$$
$$=:\quad A\left[(R_\ell,r_\ell;S_\ell,s_\ell)_{\ell=1}^{L}\right],$$

where

$$\eta_\ell^{(\theta)}:=\sum_{j=1}^{\ell-1}\sum_{i=1}^{s_j}\lambda_\theta\left(\tilde{u}_{j,i}\right)+\sum_{j=\ell+1}^{L}\sum_{i=1}^{r_j}\lambda_\theta\left(u_{j,i}\right)$$

and

$$y_{\ell,*}^{r_\ell}:=(y_{\ell,1},\ldots,y_{\ell,r_\ell}),\qquad \tilde{y}_{\ell,*}^{s_\ell}:=(\tilde{y}_{\ell,1},\ldots,\tilde{y}_{\ell,s_\ell}).$$

Since the integral kernels $w_{R_\ell,S_\ell}\left(\lambda;y_\ell^{(R_\ell)},\tilde{y}_\ell^{(S_\ell)}\right)$ are totally symmetric in the variables $y_\ell^{(R_\ell)}$ and $\tilde{y}_\ell^{(S_\ell)}$, resp., the contribution of subsets $\mathbb{D}_{R,\ell}$, $\tilde{\mathbb{D}}_{S,\ell}$, $\ell=1,\ldots,L$, to the sum (D.11) only depends on $r_\ell=\#\mathbb{D}_{R,\ell}$ and $s_\ell=\#\tilde{\mathbb{D}}_{S,\ell}$. Counting the subsets leads to the following expression,

$$F_0W_{R_1,S_1}F_1W_{R_2,S_2}\cdots W_{R_L,S_L}F_L$$
$$=\sum_{r_1=0}^{R_1}\cdots\sum_{r_L=0}^{R_L}\sum_{s_1=0}^{S_1}\cdots\sum_{s_L=0}^{S_L}\left[\prod_{\ell=1}^{L}\binom{R_\ell}{r_\ell}\binom{S_\ell}{s_\ell}\right]A\left[(R_\ell,r_\ell;S_\ell,s_\ell)_{\ell=1}^{L}\right]$$

and therefore

$$F_0WF_1W\cdots WF_{L-1}WF_L$$
$$=\sum_{\substack{R_1+S_1\geq m,\\ \vdots\\ R_L+S_L\geq m}}\sum_{r_1=0}^{R_1}\cdots\sum_{r_L=0}^{R_L}\sum_{s_1=0}^{S_1}\cdots\sum_{s_L=0}^{S_L}\left[\prod_{\ell=1}^{L}\binom{R_\ell}{r_\ell}\binom{S_\ell}{s_\ell}\right]A\left[(R_\ell,r_\ell;S_\ell,s_\ell)_{\ell=1}^{L}\right].$$

Embracing all terms $A\left[(R_\ell,r_\ell;S_\ell,s_\ell)_{\ell=1}^{L}\right]$ in the above sum which feature the same numbers of $r_1+\cdots+r_L$ free creation and $s_1+\cdots+s_L$ free annihilation operators which are not contracted in the vacuum expectation value and sorting the sum by those numbers we obtain

$$F_0WF_1W\cdots WF_{L-1}WF_L=\sum_{R+S\geq 0}\mathcal{W}_{[\theta]}[\tilde{w}_{R,S}],$$

where the functions $\tilde{w}_{R,S}$ are given in (D.9). Permuting the integration variables in the definition (D.6) of $\mathcal{W}_{[\theta]}$ we see that $\mathcal{W}_{[\theta]}[\tilde{w}_{R,S}]=\mathcal{W}_{[\theta]}[\hat{w}_{R,S}]$ which concludes the proof. ∎

D.2 Rescaling

In this section we introduce a map which allows rescaling of the operators $\Lambda_{[\theta]}$. For a fixed $\rho > 0$ we define the unitary rescaling operator S_ρ on $\mathcal{F}(L^2[\Upsilon])$ by

$$S_\rho \left[a^*_{\mathrm{gl}}(f_n) \cdots a^*_{\mathrm{gl}}(f_1)\Omega_0\right] := a^*_{\mathrm{gl}}(S_\rho f_n) \cdots a^*_{\mathrm{gl}}(S_\rho f_1)\Omega_0 \tag{D.12}$$

where

$$[S_\rho f](u, \Sigma, r) := \rho^{1/2} f(\rho u, \Sigma, r)$$

for $f \in L^2[\Upsilon]$. It is a simple consequence that

$$S_\rho a^{\#}_{\mathrm{gl}}(u, \Sigma, r) S_\rho^{-1} = \rho^{-1/2} a^{\#}_{\mathrm{gl}}(\rho^{-1} u, \Sigma, r)$$

and therefore

$$S_\rho d\Gamma_{\mathrm{gl}}[f] S_\rho^{-1} = \rho^{-1/2} d\Gamma_{\mathrm{gl}}[S_\rho f]$$

for any $f \in \mathcal{F}(L^2[\Upsilon])$. We define a rescaling map $\mathfrak{S}_\rho$ acting on operators, given by

$$\mathfrak{S}_\rho(A) := \rho^{-1} S_\rho A S_\rho^{-1}. \tag{D.13}$$

The free operators transform under $\mathfrak{S}_\rho$ like

$$\begin{aligned} \mathfrak{S}_\rho(N_{\mathrm{res}}) &= \rho^{-1} N_{\mathrm{res}}, \\ \mathfrak{S}_\rho(L_{\mathrm{res}}) &= L_{\mathrm{res}}, \\ \mathfrak{S}_\rho(L_{\mathrm{aux}}) &= L_{\mathrm{aux}}, \\ \mathfrak{S}_\rho(M_{[(i\delta', i\tau')]}) &= \left(\cos(\delta') L_{\mathrm{aux}} + \rho^{-1}\tau' N_{\mathrm{res}}\right) = M_{[(i\delta', i\rho^{-1}\tau')]}. \end{aligned} \tag{D.14}$$

We aim to apply the rescaling to Wick monomials gained by a sequence $(w_{R,S})_{R+S\geq 0}$ of form factors.

Proposition D.4 (Rescaling of Form Factors) *Let $(w_{R,S})_{R+S\geq 0}$ be a sequence of form factors obeying the conditions of Theorem D.3 and $\theta = (i\delta', i\tau') \in (i\mathbb{R}^+)^2$. The corresponding series of Wick monomials $\mathcal{W}_{[\theta]}\left[(w_{R,S})_{R+S\geq 0}\right]$ transforms under rescaling as*

$$\mathfrak{S}_\rho\left(\mathcal{W}_{[\theta]}\left[(w_{R,S})_{R+S\geq 0}\right]\right) = \mathcal{W}_{[\theta_1]}\left[\left(\mathfrak{s}_\rho(w_{R,S})\right)_{R+S\geq 0}\right],$$

where the map $\mathfrak{s}_\rho$ acts on a form factor $w_{R,S}$ as

$$[\mathfrak{s}_\rho(w_{R,S})](\lambda; Y^{(R,S)}) := \rho^{-1} w_{R,S}(\rho\lambda; \rho Y^{(R,S)}) \tag{D.15}$$

and $\theta_1 := (i\delta', i\rho^{-1}\tau')$.

Proof. We obtain via functional calculus

$$
\begin{aligned}
&\mathfrak{S}_\rho\left(\mathcal{W}_{[\theta]}[w_{R,S}]\right)\\
&= \rho^{-1-\frac{R+S}{2}} \int\limits_{\Upsilon^{R+S}} \frac{dY^{(R,S)}}{m_\theta\left(Y^{(R,S)}\right)^{1/2}} a^*_{\mathrm{gl}}\left(\rho^{-1}y^{(R)}\right)\\
&\qquad \times S_\rho w_{R,S}\left[\cos(\delta')L_{\mathrm{res}}, M_{[\theta]}; Y^{(R,S)}\right] S_\rho^{-1} a_{\mathrm{gl}}\left(\rho^{-1}\tilde{y}^{(S)}\right)\\
&= \rho^{-1-\frac{R+S}{2}} \int\limits_{\Upsilon^{R+S}} \frac{dY^{(R,S)}}{m_\theta\left(Y^{(R,S)}\right)^{1/2}} a^*_{\mathrm{gl}}\left(\rho^{-1}y^{(R)}\right)\\
&\qquad \times w_{R,S}\left[\rho\cos(\delta')L_{\mathrm{res}}, \rho M_{[\theta_1]}; Y^{(R,S)}\right] a_{\mathrm{gl}}\left(\rho^{-1}\tilde{y}^{(S)}\right)\\
&= \rho^{-1+\frac{R+S}{2}} \int\limits_{\Upsilon^{R+S}} \frac{dY^{(R,S)}}{m_\theta\left(\rho Y^{(R,S)}\right)^{1/2}} a^*_{\mathrm{gl}}\left(y^{(R)}\right) w_{R,S}\left[\rho\Lambda_{[\theta_1]}; \rho Y^{(R,S)}\right] a_{\mathrm{gl}}\left(\tilde{y}^{(S)}\right)\\
&= \rho^{-1} \int\limits_{\Upsilon^{R+S}} \frac{dY^{(R,S)}}{m_{\theta_1}\left(Y^{(R,S)}\right)^{1/2}} a^*_{\mathrm{gl}}\left(y^{(R)}\right) w_{R,S}\left[\rho\Lambda_{[\theta_1]}; \rho Y^{(R,S)}\right] a_{\mathrm{gl}}\left(\tilde{y}^{(S)}\right)\\
&= \mathcal{W}_{[\theta_1]}\left[\mathfrak{s}_\rho(w_{R,S})\right]
\end{aligned}
$$

using that $m_\theta\left(\rho Y^{(R,S)}\right) = \rho^{R+S} m_{\theta_1}\left(Y^{(R,S)}\right)$.

∎

We remark that the rescaling of the bosonic variables effects the deformation parameter θ in the assignment $\mathcal{W}_{[\theta]}[w_{R,S}]$. The translation parameter τ appearing in $M_{[\theta]}$, describing the separation of the eigenvalues from the continuous spectrum, effectively increases by a factor ρ^{-1}, it is $\tau' \mapsto \rho^{-1}\tau'$.

E The Smooth Feshbach Map

In this appendix we review the technique of the *Smooth Feshbach Map* as introduced in [4]. Given a separable Hilbert space $\mathcal{H}$ and a closed operator H on $\mathcal{H}$, the smooth Feshbach map allows to transfer the analysis of the nature of the spectrum of H near zero to the study of the spectrum of an operator which lives on a proper subspace.

Let $\Xi = \Xi^* \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator on $\mathcal{H}$ which is bounded as $0 \leq \Xi \leq 1$. The operator Ξ may be realized as an orthogonal projection operator or as a smooth cutoff function of a self-adjoint operator. The latter is chosen as a realization in the applications in the Chapters 3 and 4 which explains the notion of *smooth* Feshbach map. We define via functional calculus the complementary operator

$$\overline{\Xi} := \sqrt{\mathbb{1}_{\mathcal{H}} - \Xi^2}\,.$$

For two closed operators H, T defined on the same domain in $\mathcal{H}$ representing a perturbed operator $H = T + W$ and its unperturbed version T we define

$$H_{\Xi} := T + \Xi W \Xi, \qquad H_{\overline{\Xi}} := T + \overline{\Xi} W \overline{\Xi}$$

where we further assume that Ξ and $\overline{\Xi}$ leave the domain of T invariant and commute with T. For our purposes we require that $T\restriction_{\mathrm{ran}(\Xi)}$ is a bounded operator on the range of Ξ. We call such a pair (H, T) a *Feshbach pair associated with* Ξ, or simply a Ξ-*Feshbach pair*, if the operators $H_{\overline{\Xi}}$ and $H_{\overline{\Xi}}^*$ are bounded invertible on the range of $\overline{\Xi}$ and further the operators

$$|H_{\overline{\Xi}}|^{-1/2}\,\overline{\Xi} W \Xi, \qquad \Xi W \overline{\Xi}\,|H_{\overline{\Xi}}|^{-1/2}$$

extend to bounded operators on $\mathcal{H}$ where $|H_{\overline{\Xi}}| := \sqrt{H_{\overline{\Xi}}^* H_{\overline{\Xi}}}$. For a Ξ-Feshbach pair (T, W) we can define the *Feshbach operator* associated to Ξ, H, T by

$$\mathfrak{F}_{\Xi}(H, T) := T + \Xi(H - T)\Xi - \Xi(H - T)\overline{\Xi} H_{\overline{\Xi}}^{-1} \overline{\Xi}(H - T)\Xi.$$

The smooth Feshbach map

$$\mathfrak{F}_{\Xi} : \{(H, T) \mid (H, T) \text{ is a } \Xi\text{-Feshbach pair}\} \to \mathcal{B}(\mathrm{ran}(\Xi))$$

assigns a bounded operator on $\mathrm{ran}(\Xi)$ to a Ξ-Feshbach pair (H, T).

The benefit of the smooth Feshbach map is its isospectral property.

Theorem E.1 (Isospectrality of the Smooth Feshbach Map) *Let Ξ be a positive operator on the separable Hilbert space $\mathcal{H}$, bounded from below and above by $0 \le \Xi \le 1$ and let $\overline{\Xi} = \sqrt{\mathbb{1}_{\mathcal{H}} - \Xi^2}$ the complementary operator. Let (H, T) be a Feshbach pair associated with Ξ. Then we have the following spectral relation between the operator H and its image $\mathfrak{F}_\Xi(H,T)$ under the smooth Feshbach map,*

(i) H is bounded invertible if and only if $\mathfrak{F}_\Xi(H,T)$ is bounded invertible on $\mathrm{ran}(\Xi)$.

(ii) If $\psi \in \ker(H) \setminus \{0\}$ then $\Xi\psi \in \ker(\mathfrak{F}_\Xi(H,T)) \setminus \{0\}$.

(iii) If $\varphi \in \ker(\mathfrak{F}_\Xi(H,T)) \setminus \{0\}$ then $(\Xi - \overline{\Xi} H_{\overline{\Xi}}^{-1} \overline{\Xi}(H-T)\Xi)\varphi \in \ker(H) \setminus \{0\}$.

(iv) The multiplicity of the zero eigenvalue is conserved, i.e., $\dim(\ker(H)) = \dim(\ker(\mathfrak{F}_\Xi(H,T)))$.

Proof. The proof to this assertion is purely algebraical and rather lengthy. We therefore omit it and refer the reader to [4, Thm. 2.1] where the original proof can be found. ■

Corollary E.2 (Reconstruction of Eigenvectors) *Let Ξ be an operator as in Theorem E.1 and let (H,T) be a Ξ-Feshbach pair. Assume that zero is a simple eigenvalue of H and let $\psi \neq 0$ be the corresponding eigenvector. Then the eigenvector can be reconstructed as*

$$\psi = \lambda \left(\Xi - \overline{\Xi} H_{\overline{\Xi}}^{-1} \overline{\Xi}(H-T)\Xi \right) \Xi\psi$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ is a suitable scalar. If further the eigenvector obeys $\psi \notin \mathrm{ran}(\overline{\Xi})$ then holds $\lambda = 1$.

Proof. By Theorem E.1(ii) follows that $\Xi\psi \in \ker(\mathfrak{F}_\Xi(H,T)) \setminus \{0\}$. A further application of Theorem E.1(iii) implies that $(\Xi - \overline{\Xi} H_{\overline{\Xi}}^{-1} \overline{\Xi}(H-T)\Xi)\Xi\psi \in \ker(H) \setminus \{0\}$, hence it is a multiple of ψ,

$$\psi = \lambda \left(\Xi - \overline{\Xi} H_{\overline{\Xi}}^{-1} \overline{\Xi}(H-T)\Xi \right) \Xi\psi,$$

$\lambda \in \mathbb{C} \setminus \{0\}$, due to the simplicity of the zero eigenvalue. Since $\psi = \Xi^2\psi + \overline{\Xi}^2\psi$ we obtain via projection $P = P^*$ on the orthogonal complement of $\mathrm{ran}(\overline{\Xi})$,

$$P\Xi^2\psi = \lambda P \Xi^2 \psi.$$

We consider the case $P\Xi^2\psi = 0$ which implies $P\psi = 0$, hence $\psi \in \mathrm{ran}\,\overline{\Xi}$. Vice versa, if $\psi \notin \mathrm{ran}\,\overline{\Xi}$ then we conclude $P\Xi^2\psi \neq 0$ and therefore $\lambda = 1$. ■

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List of Figures

Index

Curriculum Vitae

Name	Matthias Mück
Date of Birth	05. July 1974
Place of Birth	Wiesbaden

School Education

Elementary School	**1981 - 1985:** Attendance at the *Hermann Löns-Schule* in Wiesbaden-Bierstadt
High School ("Gymnasium")	**1985 - 1994:** Attendance at the *Gutenberg Schule* in Wiesbaden
	June 1994: Graduation from High School with the "Abitur" Degree

University Education

Basic Studies	**October 1994:** Enrollment at the *Johannes Gutenberg-Universität (JoGU)* in Mainz for the Diploma Program at the Department of Physics
	February 1995: Enrollment at the *JoGU* in Mainz for the Diploma Program at the Department of Mathematics
	October 1996: Intermediate Diploma in Mathematics
	April 1997: Intermediate Diploma in Physics
Advanced Studies	**October 1999 - April 2000:** Diploma Thesis in Mathematical Physics
	June 2000: Graduation as "Diplom- Mathematiker"

Doctoral Studies

June 2000: Enrollment at the *JoGU* in Mainz for the Ph.D. Program at the Department of Mathematics

January 2001 - December 2002: Scholar of the *German Academic Exchange Service*, *DAAD*, under a HSP-III Scholarship

January 2001 - December 2002: Research Stay at the *University of Toronto (UofT)*

Other Activities in Academia and Industry

Student Trainee

February - March, September - October 1995, March - April 1996: Trainee at the *Hoechst AG*, Kelsterbach

Academic Positions

April 1997 - February 2000: Teaching Assistant at the Department of Mathematics

February - April 1998: Research Assistant at the *Mainz Neutrino Mass Experiment*

October - December 2000, January 2003 - now: Scientific Employee at the Department of Mathematics

January 2001 - December 2002: Teaching Assistant at the *UofT*, Department of Mathematics

Summer Schools

August - October 1998: Attendance at the International Student Program of the *Association for Heavy Ion Research*, *GSI*, in Darmstadt

Industrial Workshops

June 2001: Attendance at the following workshops of the *Pacific Institute for the Mathematical Sciences (PIMS)*: *Industrial Problem Solving Workshop (IPSW)* in Seattle and at the *Graduate Math Modelling Camp (GMMC)* in Victoria

May - June 2002: Attendance at the *IPSW* in Vancouver and at the *GMMC* in Burnaby