Logarithmically smooth deformations of strict normal crossing logarithmically symplectic varieties

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Abstract

In this thesis we give a definition of the term *logarithmically symplectic variety*; to be precise, we distinguish even two types of such varieties. The general type is a triple (f, ∇, ω) comprising a log smooth morphism $f\colon X \to \operatorname{Spec} \kappa$ of log schemes together with a flat log connection $\nabla\colon L \to \Omega^1_f \otimes L$ and a (∇-closed) log symplectic form $\omega \in \Gamma(X, \Omega^2_f \otimes L)$. We define the functor of log Artin rings of log smooth deformations of such varieties (f, ∇, ω) and calculate its obstruction theory, which turns out to be given by the vector spaces $H^i(X, B^\bullet_{(f,\nabla)}(\omega))$, $i = 0, 1, 2$. Here $B^\bullet_{(f,\nabla)}(\omega)$ is the class of a certain complex of $\mathcal{O}_X$-modules in the derived category $\mathrm{D}(X/\kappa)$ associated to the log symplectic form ω. The main results state that under certain conditions a log symplectic variety can, by a flat deformation, be smoothed to a symplectic variety in the usual sense. This may provide a new approach to the construction of new examples of irreducible symplectic manifolds.

In dieser Arbeit geben wir eine Definition des Terms *logarithmisch-symplektische Varietät*; genau genommen unterscheiden wir sogar zwei Typen solcher Varietäten. Der allgemeinere Typ ist dabei ein Tripel (f, ∇, ω) bestehend aus einem log glatten Morphismus $f\colon X \to \operatorname{Spec} \kappa$ von log Schemata, zusammen mit einem flachen log Zusammenhang $\nabla\colon L \to \Omega^1_f \otimes L$ und einer (bezüglich ∇ geschlossenen) log symplektischen Form $\omega \in \Gamma(X, \Omega^2_f \otimes L)$. Wir definieren den Funktor von log Artinringen der Deformationen solcher Varietäten (f, ∇, ω) und berechnen dessen Hindernistheorie, die sich als durch die Vektorräume $H^i(X, B^\bullet_{(f,\nabla)}(\omega))$, $i = 0, 1, 2$, gegeben herausstellt. Dabei ist $B^\bullet_{(f,\nabla)}(\omega)$ die Klasse eines gewissen Komplexes von $\mathcal{O}_X$-Moduln in der derivierten Kategorie $\mathrm{D}(X/\kappa)$, der zur log symplektischen Form ω gehört. Die Hauptresultate sagen aus, dass sich unter gewissen Voraussetzungen eine log symplektische Varietät mittels einer flachen Deformation zu einer symplektischen Varietät im üblichen Sinne glätten lässt. Dies liefert möglicherweise einen neuen Ansatz für die Konstruktion neuer Beispiele irreduzibler symplektischer Mannigfaltigkeiten.

Dans cette thèse nous donnons la definition du terme *variété logarithmiquement symplectique*; par souci d'exactitude, nous distinguons même deux types de telles variétés. Le type général est un triplet (f, ∇, ω) qui se compose d'un morphisme log lisse $f\colon X \to \operatorname{Spec} \kappa$ de schémas log avec une connexion log plate $\nabla\colon L \to \Omega^1_f$ et une forme log symplectique (∇-fermée) $\omega \in \Gamma(X, \Omega^2_f \otimes L)$. Nous définissons le foncteur des anneaux artiniens des déformations log lisses de telles varietés (f, ∇, ω) et calculons sa théorie d'obstruction, qui se trouve d'être donnée par les espaces vectoriels $H^i(X, B^\bullet_{(f,\nabla)}(\omega))$, $i = 0, 1, 2$. Sachant que $B^\bullet_{(f,\nabla)}(\omega)$ est la classe d'un certain complexe des $\mathcal{O}_X$-modules dans la catégorie derivée $\mathrm{D}(X/\kappa)$ associée à la forme log symplectique ω. Les résultats principaux établissent que sous certain conditions une variété log symplectique admet une déformation plate dont la

fibre générale est une variété symplectique lisse au sens usuel. Ceci apporte potentiellement une autre approche pour la construction de nouveaux exemples de variétés symplectiques irréductibles.

Introduction

Compact hyperkähler manifolds – these are compact Kähler manifolds X the holonomy group of which is the symplectic group $\mathrm{Sp}(\dim X)$ – stand in the focus of investigations by complex algebraic geometry not only, but just since Bogomolov's decomposition theorem:

0.0.1 Theorem (De Rham, Berger, Bogomolov, Beauville, cf. [1, Thm. 1(2)])
Let X be a compact Kähler manifold with zero Ricci curvature. Then there exists a finite étale cover $f\colon \tilde{X} \to X$ such that

$$\tilde{X} \cong T \times \prod V_i \times \prod X_j,$$

where T is a complex torus, where the V_i are compact Calabi-Yau manifolds (these are compact simply connected Kähler manifolds of dimension $m_i \geq 3$ with holonomy group $\mathrm{SU}(m_i)$) and where the X_j are compact simply connected hyperkähler manifolds.

In the language of algebraic geometry, compact hyperkähler manifolds correspond to irreducible symplectic varieties which are proper over $\operatorname{Spec} \mathbb{C}$; these are defined as follows:

A *symplectic variety* is a smooth variety X over $\operatorname{Spec} \mathbb{C}$ which possesses a global 2-form $\omega \in \Gamma(X, \Omega^2_{X/\mathbb{C}})$, the so-called symplectic form, such that the associated $\mathcal{O}_X$-linear map $T_{X/\mathbb{C}} \to \Omega^1_{X/\mathbb{C}}$ is an isomorphism; equivalently, such that its associated skew-symmetric pairing $T_{X/\mathbb{C}} \otimes_{\mathcal{O}_X} T_{X/\mathbb{C}} \to \mathcal{O}_X$ is non-degenerate. This implies that the dimension of X is even. An irreducible symplectic variety is a symplectic variety X the symplectic form ω of which generates the ring $H^0(X, \Omega^\bullet_{X/\mathbb{C}})$ as an $H^0(X, \mathcal{O}_X)$-algebra. In particular, such a variety is simply-connected.

The decomposition theorem then takes the following form:

0.0.2 Theorem (Beauville, cf. [1, Thm. 2(2)])
Let X be a Kähler variety which is proper and smooth over $\operatorname{Spec} \mathbb{C}$ and the first Chern class of which is zero. Then there exists a finite étale cover $f\colon \tilde{X} \to X$ such that

$$\tilde{X} \cong T \times \prod V_i \times \prod X_j,$$

where T is a complex torus, where the V_i are projective Calabi-Yau varieties (these are simply connected varieties of dimension $m_i \geq 3$, smooth over $\operatorname{Spec} \mathbb{C}$, with trivial canonical line bundle and such that $H^0(V_i, \Omega^p_{V_i/\mathbb{C}}) = 0$ for $0 < p < m_i$) and where X_j is a proper irreducible symplectic Kähler variety.

Both proper symplectic varieties and proper Calabi-Yau varieties are subjects of a rich and prospering research. In contrast to numerous existent examples of proper Calabi-Yau varieties, there are only few examples of proper symplectic varieties known. All examples known to this day are subsequently listed up to deformation equivalence:

a) Hilbert schemes of points on a K3 surface, $\mathrm{Hilb}^n(S)$ (Beauville, [1, Thm. 3]), of dimension $2n$, with $b_2 = 23$, $n \in \mathbb{N}$;

b) Kummer varieties associated to an Abelian surface, $K_n(A)$ (Beauville, [1, Thm. 4]), of dimension $2n$, with $b_2 = 7$, $n \in \mathbb{N}$;

c) $\tilde{M}_4$ (O'Grady, [27, 2.0.2]), of dimension 10, with $b_2 = 24$;

d) $\tilde{M}$ (O'Grady, [28, 1.4]), of dimension 6, with $b_2 = 8$;

where the last two exceptional examples (exceptional, because they do not belong to series like the other examples) were constructed by K. O'Grady as resolutions of singular moduli spaces of sheaves on a projective K3 surface. Indeed, all of the above examples may be realised as moduli spaces (or resolutions of those) of sheaves on a K3 surface. However, the method of O'Grady used to construct his exceptional examples as resolutions of particular singular moduli spaces fails in all other cases of singular moduli spaces, as shown by D. Kaledin, M. Lehn and C. Sorger in [15].

One method to construct compact Calabi-Yau manifolds was introduced by Y. Kawamata and Y. Namikawa in their paper “Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi-Yau varieties” ([21]). For that purpose one regards (complex analytic) strict normal crossing varieties equipped with a particular logarithmic structure.

0.0.3 Theorem (cf. [21, 4.2])

Let X be a compact Kähler strict normal crossing variety equipped with the log structure of semi-stable type of dimension $d \geq 3$ and $X^\nu \to X$ the normalisation of X. Assume the following conditions:

a) $\omega_X \cong \mathcal{O}_X$;

b) $H^{d-1}(X, \mathcal{O}_X) = 0$;

c) $H^{d-2}(X^\nu, \mathcal{O}_{X^\nu}) = 0$.

Then X is smoothable by a flat deformation.

The idea of this method is thus to pass from the category of Calabi-Yau manifolds to the larger category of “strict normal crossing Calabi-Yau varieties” to regard deformations of such varieties the general fibre of which is a Calabi-Yau manifold in the original sense.

Based on a proposal by our advisor M. Lehn, an adaptation of this approach is the core of this thesis:

It is known that the fibres of a deformation of a compact hyperkähler manifold over a small analytic disc are again compact hyperkähler manifolds (cf. [1, 9]). Yet it seems possible, as in the case of Calabi-Yau manifolds, to start with a singular variety and to smoothen it to receive a proper symplectic variety, in principle.

As a first step, we pass from algebraic geometry to logarithmic algebraic geometry, i. e. from the category of schemes to the category of logarithmic schemes $(\underline{X}, \alpha_X)$; these are schemes $\underline{X}$ carrying a logarithmic structure α_X. Particularly, the notion of smoothness of a morphism of schemes is replaced with the wider notion of logarithmic smoothness of a morphism of log schemes. This yields that any semi-stable strict normal crossing variety is log smooth over the point $\operatorname{Spec}\mathbb{C}$ as soon as both involved schemes are regarded as log schemes equipped with particular log structures.

In this thesis we introduce the notion of a (logarithmically smooth) logarithmically symplectic scheme and particularly of a logarithmically symplectic variety which applies to the case of strict normal crossing varieties over $\operatorname{Spec}\mathbb{C}$. In doing so, we distinguish between two types of logarithmically symplectic schemes: Such of non-twisted type and such of generally twisted type. In our main results, in a way similar to that of Y. Kawamata and Y. Namikawa, we give conditions under which a logarithmically symplectic variety of the respective type deforms flatly into a smooth symplectic variety in the usual sense, that is, under which it is smoothable.

Content and Structure

In the first chapter we introduce the basic definitions of logarithmic geometry (especially for a reader not too familiar with that topic) such as the notion of log schemes and their morphisms, charts of log structures and morphisms, and log smoothness as well as basic results about these topics. All these notions and facts are taken from original articles on the topic by K. Kato and F. Kato, as well as from the excellent lecture notes on logarithmic geometry by A. Ogus, which have come to be known as the "Log book", but which are up to now available only directly from the author.

The second chapter recalls the notion of a functor of Artin rings in the sense of M. Schlessinger before introducing the analogously defined functors of log Artin rings. It then collects results from the theory of functors of log Artin rings as established by F. Kato, including the log Schlessinger conditions LH_1-LH_4 for the existence of a hull or even a universal element, and introduces the notion of log smooth deformations. Be aware, that the content of these first two chapters is not original, but that it is included in this thesis from the afore mentioned sources for the convenience of the reader.

Having chapters one and two as a basis, chapter three defines step by step the log schemes with additional data this thesis is working with, namely log schemes with line bundles, log schemes with flat log connections (of rank one) and, eventually, log symplectic schemes. Here it seems reasonable to distinguish between two kinds of such schemes, namely the log symplectic schemes of non-twisted type and of general type, as defined in this chapter. This chapter also introduces coherent sheaves and complexes of coherent sheaves associated to the additional data on the respective log scheme. These are the log Atiyah module of a line bundle, the log Atiyah complex of a flat log connection, and the T-complex and B-complex associated to a log symplectic scheme of non-twisted type and of general type, respectively. These sheaves and complexes of sheaves will deliver the obstruction theory of log smooth deformations of the respective kind of object in chapter four.

The middle of the third chapter is taken up by a discussion of the here defined notion of logarithmic Cartier divisors, which was inspired by a section in the lecture notes by A. Ogus, as far as necessary for this thesis. The chapter ends by recalling special log structures associated to schemes with certain additional data, such as the canonical log structure of strict normal crossing schemes of semi-stable type over $\operatorname{Spec} k$ from A. Ogus' lecture notes and F. Kato's work. Again, in this last section of chapter three, nothing is original.

In the fourth chapter we define a collection of deformation functors for the various objects defined in chapter three. After recalling and partly reenacting the log smooth deformation theory of log smooth schemes of F. Kato, we calculate (using Čech-(hyper-)cohomology) the obstruction theory of log smooth deformations of log schemes with line bundles, log schemes with flat log connections and log symplectic schemes. These obstruction theories are given by the (hyper-)cohomology groups $\mathbb{H}^0$, $\mathbb{H}^1$ and $\mathbb{H}^2$ of the sheaves and complexes of sheaves constructed in chapter three.

At the end of this chapter we show that any of the beforehand defined deformation functors possesses a hull in the sense of Schlessinger and that some of them are even pro-representable (but, as expected, not the functor of log smooth deformations of log symplectic schemes).

Chapter five contains the main results of this thesis. It begins with technical calculations and results proved for later use in this chapter. By introducing a logarithmic version of the T1 lifting principle as introduced by Z. Ran and Y. Kawamata and expanded by B. Fantechi and M. Manetti and by following the techniques of Y. Namikawa and Y. Kawamata including an adaptation of a result of J. Steenbrink, we are able to prove that under certain not to rigid conditions the obstructions given in chapter four vanish and that a log symplectic scheme may be deformed flatly into a smooth symplectic scheme in the usual sense.

Chapter six collects known examples of log symplectic schemes in our sense, appearing naturally; two of them having been investigated by Nagai. It does however not provide an application of our main theorems which produces new examples of symplectic varieties.

The last Chapter deals with questions that could not be satisfactorily answered within the scope of this thesis.

Contents

1 Logarithmic Geometry

In this first chapter we collect the basic definitions and results of logarithmic geometry, mostly for the convenience of the reader who has not yet had much contact with this topic. The references for all results are "Logarithmic structures of Fontaine-Illusie" by K. Kato ([19]), "Log smooth deformation theory" by F. Kato ([17]) and notably the yet unfinished "Lectures on Logarithmic Algebraic Geometry" by A. Ogus ([29]), which is an excellent introduction to logarithmic algebraic geometry, available on the authors web page. For the basic definition and results concerning monoids, see Appendix A. All monoids regarded are commutative. As it is custom in the context of logarithmic geometry we will usually abbreviate the words "logarithmic" and "logarithmically" to "log".

A scheme X may be regarded both as X_{zar} equipped with its classical Zariski-topology and $X_{\text{ét}}$ with its étale topology, which is finer due to the fact that open immersions are étale. We will draw a distinction between these two possible topologies only if necessary. When writing about an open/étale neighbourhood, we mean an open neighbourhood or an étale neighbourhood, depending on the chosen topology. By a point of a scheme, we mean a geometric point.

Since the topology on $X_{\text{ét}}$ is finer than on X_{zar}, there is a canonical continuous map $\upsilon_X\colon X_{\text{ét}} \to X_{\mathrm{zar}}$ which is the identity as a map on sets, when interpreting Zariski-open subschemes $U \subset X$ as open immersions $j_U\colon U \to X$, which are always étale. If $\mathcal{F}$ is a sheaf on $X_{\text{ét}}$, then its restriction to open immersions is given as the sheaf $\upsilon_{X*}\mathcal{F}\colon U \mapsto \Gamma(j_U, \mathcal{F})$. On the other hand, if $\mathcal{F}$ is a sheaf on X_{zar}, then $\upsilon_X^{-1}\mathcal{F}\colon (e\colon U \to X) \mapsto \Gamma(U, e^{-1}\mathcal{F})$ is a sheaf on $X_{\text{ét}}$. It is clear that $\upsilon_{X*}\upsilon_X^{-1}\mathcal{F} = \mathcal{F}$.

1.1 Logarithmic Structures

The category of sheaves of commutative monoids on a scheme X is denoted by $\underline{\mathrm{Mon}}_X$. We regard the structure sheaf $\mathcal{O}_X$ of X as an element of $\underline{\mathrm{Mon}}_X$, always with respect to its multiplication. For a monoid P we will denote by P_X its constant sheaf on X, which is the sheaf associated to the presheaf taking every open subset U (respectively, every étale morphism $U \to X$) to P. For a sheaf of monoids $\mathcal{M}$, specifying a monoid homomorphism $P \to \Gamma(X, \mathcal{M})$ is equivalent to giving a morphism of sheaves of monoids $P_X \to \mathcal{M}$.

1.1.1 Logarithmic Structures

1.1.1 Definition ([19, 1.1,1.2],[29, III.1.1.1])

Let X be a scheme. A *prelogarithmic structure* on X is a morphism of sheaves of monoids $\alpha\colon \mathcal{M} \to \mathcal{O}_X$.

It is called a *logarithmic structure* if it is a logarithmic morphism of sheaves of monoids, i. e. if the restricted morphism $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism. Due to the fact, that for any log structure we have $\alpha^{-1}(\mathcal{O}_X^\times) = \mathcal{M}^\times$, we identify $\mathcal{O}_X^\times$ with the subsheaf of units $\mathcal{M}^\times$ of $\mathcal{M}$ via this isomorphism.

If α is a (pre)log structure, we will refer to the sheaf of monoids $\mathcal{M}$ as the *(pre)logarithmic structure sheaf* of α (or of (X, α)) and denote it by $\mathcal{M}_\alpha$.

A *morphism* $\varphi\colon \beta \to \alpha$ *of prelog structures* is a morphism of sheaves of monoids $\varphi\colon \mathcal{M}_\beta \to \mathcal{M}_\alpha$ such that $\alpha \circ \varphi = \beta$. A *morphism* $\varphi\colon \beta \to \alpha$ *of* corresponding *log structures* is a morphism of prelog structures.

We write $\underline{\mathrm{preLog}}_X$ and $\underline{\mathrm{Log}}_X$ for the category of prelog structures and log structures on X, respectively. The initial object in $\underline{\mathrm{Log}}_X$ is the inclusion $\iota\colon \mathcal{O}_X^\times \to \mathcal{O}_X$, called the *trivial log structure* on X. The initial object of $\underline{\mathrm{preLog}}_X$ is the inclusion $1\colon 1 \to \mathcal{O}_X$. The final object in both categories is the identity $id\colon \mathcal{O}_X \to \mathcal{O}_X$, called the *hollow log structure* on X (cf. [29, III.1.1.3]).

In a log structure α, the sheaf of monoids $\mathcal{M}_\alpha$ may be written additively or multiplicatively. We will use the multiplicative notation mostly.

However, when written additively, a log structure α should be thought of as a sheaf $\mathcal{M}_\alpha$ of logarithms of certain regular functions together with an exponential map $\mathcal{M}_\alpha \to \mathcal{O}_X$ given by α. The set of logarithms of a function f is then $\alpha^{-1}(f)$, which might also be empty, and the logarithm of a unit is unique (cp. [29, III.1.1.2]).

For example, the trivial log structure ι on $\operatorname{Spec}\mathbb{C}$ can be written either multiplicatively as the inclusion $\mathbb{C}^\times \to \mathbb{C}$ or additively as the well-defined exponential map $\mathbb{C}/(2\pi i\mathbb{Z}) \to \mathbb{C}$, $m \mapsto \exp(m)$, since indeed $\exp\colon (\mathbb{C}/(2\pi i\mathbb{Z}), +) \to (\mathbb{C}^\times, \cdot)$ is an isomorphism with (well-defined) inverse map $\log$.

For the definition of the quotient of a monoid by a subgroup see Appendix A.

1.1.2 Definition

Let α be a log structure on X. The quotient sheaf $\overline{\mathcal{M}}_\alpha = \mathcal{M}_\alpha/\alpha^{-1}(\mathcal{O}_X^\times) = \mathcal{M}_\alpha/\mathcal{M}_\alpha^\times$ is called the *characteristic monoid sheaf* of α.

Analogously, if $\varphi\colon \beta \to \alpha$ is a morphism of log structures $\beta\colon \mathcal{M}_\beta \to \mathcal{O}_X$ and $\alpha\colon \mathcal{M}_\alpha \to \mathcal{O}_X$, we associate a sheaf of monoids $\overline{\mathcal{M}}_\varphi := \overline{\mathcal{M}}_{\alpha/\beta} := \mathcal{M}_\alpha/\varphi(\mathcal{M}_\beta)$ which we call the *relative characteristic sheaf* of φ or of α over β.

A morphism $\varphi\colon \beta \to \alpha$ of log structures induces a homomorphism of monoid sheaves $\overline{\varphi}\colon \overline{\mathcal{M}}_\beta \to \overline{\mathcal{M}}_\alpha$. The morphism φ an isomorphism if and only if $\overline{\varphi}$ is. We have $\overline{\mathcal{M}}_\varphi \cong \overline{\mathcal{M}}_\alpha/\overline{\varphi}(\overline{\mathcal{M}}_\beta)$.

Given a diagram of monoid sheaves (or prelog structures) $\mathcal{N} \leftarrow \mathcal{M} \to \mathcal{N}'$ on X we write $\mathcal{N} \otimes_{\mathcal{M}} \mathcal{N}'$ for its pushout, which is the sheaf associated to the presheaf of monoids $U \mapsto \mathcal{N}(U) \otimes_{\mathcal{M}(U)} \mathcal{N}'(U)$. If $\mathcal{M}$ is the trivial sheaf of monoids, we write $\mathcal{N} \oplus \mathcal{N}'$ instead of $\mathcal{N} \otimes_1 \mathcal{N}'$ (compare to the conventions for monoids in Appendix A).

Analogously we write $\mathcal{N} \times_{\mathcal{M}} \mathcal{N}'$ for the pullback of the corresponding diagram with reversed arrows, which is the sheaf of monoids $U \mapsto \mathcal{N}(U) \times_{\mathcal{M}(U)} \mathcal{N}'(U)$.

Let α be a prelog structure on the scheme X. The monoid sheaf $\mathcal{M}_\alpha^> := \mathcal{M}_\alpha \circledcirc \mathcal{O}_X^\times := \mathcal{M}_\alpha \otimes_{\alpha^{-1}(\mathcal{O}_X^\times)} \mathcal{O}_X^\times$ fits naturally into the diagram

$$\begin{tikzcd}
\alpha^{-1}(\mathcal{O}_X^\times) \arrow[r, hook] \arrow[d, "\alpha"] & \mathcal{M}_\alpha \arrow[d] \arrow[ddr, bend left, "\alpha"] & \\
\mathcal{O}_X^\times \arrow[r, hook] \arrow[drr, hook, bend right, "\iota"'] & \mathcal{M}_\alpha^> \arrow[dr, dotted, "\exists!"] \arrow[ul, phantom, "\lrcorner", very near start] & \\
& & \mathcal{O}_X,
\end{tikzcd}$$

coming with a unique homomorphism of monoid sheaves $\mathcal{M}_\alpha^> \to \mathcal{O}_X$, denoted $\alpha^>$ and given by the local rule $m \otimes u \mapsto u \cdot \alpha(m)$. In fact, $\alpha^> \colon \mathcal{M}_\alpha^> \to \mathcal{O}_X$ is a log structure on X. If α is already a log structure, then $\alpha \cong \alpha^>$. For the image $m \otimes 1$ of a local section m of $\mathcal{M}_\alpha$ under $\mathcal{M}_\alpha \to \mathcal{M}_{\alpha^>}$ we will usually just write m.

1.1.3 Definition

We call $\alpha^> \colon \mathcal{M}_\alpha^> \to \mathcal{O}_X$ the *log structure associated* to the prelog structure $\alpha\colon \mathcal{M}_\alpha \to \mathcal{O}_X$.

Any morphism of prelog structures $\varphi\colon \beta \to \alpha$ with α a log structure factors through $\beta^>$ uniquely. Associated to each morphism $\varphi\colon \beta \to \alpha$ of prelog structures there is a unique morphism of log structures $\varphi^> \colon \beta^> \to \alpha^>$ called the *morphism of log structures associated* to φ.

This defines a functor $(\,\cdot\,)^> \colon \underline{\mathrm{preLog}}_X \to \underline{\mathrm{Log}}_X$ which is left adjoint to the forgetful functor in the opposite direction.

1.1.4 Remark

The notation $\mathcal{M}_\alpha \circledcirc \mathcal{O}_X^\times$ is non-standard; it is used to symbolise that the sheaf of units $\mathcal{O}_X^\times$ of the structure sheaf is a subgroup sheaf in the “enlarged” sheaf $\mathcal{M}_\alpha^>$.

Let $f\colon X \to Y$ be a morphism of schemes. As usual, we denote the inverse image of a sheaf $\mathcal{G}$ on Y under f by $f^{-1}\mathcal{G}$ and the direct image of a sheaf $\mathcal{F}$ on X under f by $f_*\mathcal{F}$.

1.1.5 Definition

a) For any (pre)log structure $\beta\colon \mathcal{M}_\beta \to \mathcal{O}_Y$ on Y the inverse image morphism

$$f^{-1}\beta\colon f^{-1}\mathcal{M}_\beta \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$$

is a prelog structure on X. We denote its associated log structure

$$(f^{-1}\beta)^>\colon (f^{-1}\mathcal{M}_\beta) \circledcirc \mathcal{O}_X^\times \to \mathcal{O}_X$$

by $f^\times\beta$ and write $f^\times\mathcal{M}_\beta := (f^{-1}\mathcal{M}_\beta)^>$ for its log structure sheaf.

$f^\times\beta$ and $f^\times\mathcal{M}_\beta$ are called the *(logarithmic) pullback* of β and $\mathcal{M}_\beta$ under f, respectively.

b) For any log structure $\alpha\colon \mathcal{M}_\alpha \to \mathcal{O}_X$ on X we have the two morphisms $\mathcal{O}_Y \to f_*\mathcal{O}_X$ and $f_*\alpha\colon f_*\mathcal{M}_\alpha \to f_*\mathcal{O}_X$. We define $f_\times\mathcal{M}_\alpha$ and the morphism $f_\times\alpha$ to be the fibred product $f_*\mathcal{M}_\alpha \times_{f_*\mathcal{O}_X} \mathcal{O}_Y$ and the canonical morphism in the diagram

$$\begin{tikzcd} f_\times\mathcal{M}_\alpha \arrow[r] \arrow[d, "f_\times\alpha"] \arrow[dr, phantom, "\ulcorner"] & f_*\mathcal{M}_\alpha \arrow[d, "f_*\alpha"] \\ \mathcal{O}_Y \arrow[r] & f_*\mathcal{O}_X, \end{tikzcd}$$

respectively. Since the preimage under $f^\sharp$ of $f_*\mathcal{O}_X^\times$ is $\mathcal{O}_Y^\times$, the morphism $f_\times\alpha$ is a log structure.

$f_\times\alpha$ and $f_\times\mathcal{M}_\alpha$ are called the *(logarithmic) direct image* of α and $\mathcal{M}_\alpha$ under f, respectively.

With regard to characteristic monoid sheaves, for f and β as in the definition, there is a canonical isomorphism $f^{-1}\overline{\mathcal{M}_\beta} \cong \overline{f^\times\mathcal{M}_\beta}$ (cf. [19, 1.4.1]).

1.1.6 Proposition ([29, III.1.1.5])

Let $f\colon X \to Y$ be a morphism of schemes. The two functors $f^\times\colon \underline{\mathrm{Log}}_Y \to \underline{\mathrm{Log}}_X$ and $f_\times\colon \underline{\mathrm{Log}}_X \to \underline{\mathrm{Log}}_Y$ are adjoint functors. To be precise, there is a natural isomorphism

$$\mathrm{Hom}_{\underline{\mathrm{Log}}_X}(f^\times\beta, \alpha) \cong \mathrm{Hom}_{\underline{\mathrm{Log}}_Y}(\beta, f_\times\alpha).$$

1.1.7 Remark

Since the tensor product of sheaves of monoids involves a sheafification process, the log structure $\alpha^>$ associated to a prelog structure α depends on the chosen topology (Zariski or étale) on X, unless the structure sheaf $\mathcal{M}_\alpha$ of α is a sheaf of unit-integral monoids. This is meant in the following sense:

Let $\upsilon_X\colon X_{\text{ét}} \to X_{\text{zar}}$ be the canonical continuous map and let $\alpha\colon \mathcal{M}_\alpha \to \mathcal{O}_X$ be a prelog structure on X_{zar}. Let then $\alpha^>_{\text{zar}} := \alpha^>$ be the associated log structure to α on X_{zar} and $\alpha^>_{\text{ét}} := (\upsilon_X^{-1}\alpha)^>$ the associated log structure to $\upsilon_X^{-1}\alpha$ on $X_{\text{ét}}$.
Then we have a natural morphism of prelog structures $\upsilon_X^{-1}(\alpha^>_{\text{zar}}) \to \alpha^>_{\text{ét}}$ (where the second one is a log structure). If the structure sheaf $\mathcal{M}_\alpha$ of α is a sheaf of unit-integral monoids, then this morphism is an isomorphism, i. e. if $e\colon X' \to X$ is an étale morphism, then the restriction $\alpha^>_{\text{ét}}|_{X'_{\text{zar}}} = \upsilon_{X'*}(e^\times\alpha^>_{\text{ét}})$ is equal to the log pullback $e^\times\alpha^>_{\text{zar}} = e^\times\alpha$ as sheaves on X'_{zar}. In particular $\Gamma(e\colon X' \to X, \mathcal{M}^>_{\alpha,\text{ét}}) = \Gamma(X', e^\times\mathcal{M}_\alpha)$ (cp. [29, III.1.1.4]).

1.1.2 Charts and Coherence

Let X be a scheme and $\alpha\colon \mathcal{M}_\alpha \to \mathcal{O}_X$ a log structure on X. Let P be a monoid and $a\colon P \to \Gamma(X, \mathcal{M}_\alpha)$ a monoid homomorphism. Then a induces trivially a morphism of prelog structures $a\colon \alpha\circ a \to \alpha$, given by $a\colon P_X \to \mathcal{M}_\alpha$, where $\alpha\circ a$ is the prelog structure $P_X \xrightarrow{a} \mathcal{M}_\alpha \xrightarrow{\alpha} \mathcal{O}_X$ obtained by composition.

1.1.8 Definition
A *(global) chart* for a log structure α on a scheme X is a monoid homomorphism $a\colon P \to \Gamma(X, \mathcal{M}_\alpha)$ such that $a^>\colon (\alpha \circ a)^> \to \alpha$ is an isomorphism of log structures (then, in particular, $P \circledcirc \mathcal{O}_X^\times \cong \mathcal{M}_\alpha$).
A *chart* for α *at a point* x in X is an open/étale neighbourhood U of x together with a chart $a\colon P \to \Gamma(U, \mathcal{M})$.

Let $\mathscr{P}$ be one of the following properties of a monoid: *coherent*, *domainic*, *sharp*, *(unit-/ quasi-)integral*, *fine*, *saturated*, *toric*, *normal*, *free*.

1.1.9 Definition
Let α be a log structure on a scheme X. We say that a chart $a\colon P \to \Gamma(X, \mathcal{M}_\alpha)$ of α has the property $\mathscr{P}$ if the monoid P has the property $\mathscr{P}$.
A log structure $\alpha\colon \mathcal{M}_\alpha \to \mathcal{O}_X$ on a scheme X is called *quasi-coherent* if for any point $x \in X$ there exists a chart $a\colon P \to \Gamma(U, \mathcal{M}_\alpha)$ at x.
We say that a quasi-coherent log structure α has the property $\mathscr{P}$ if for any point $x \in X$ there exists a chart $a\colon P \to \Gamma(U, \mathcal{M}_\alpha)$ at x with that property (an exception to this rule is the case that if for any point $x \in X$ there exists a free chart, we call α *locally free*).

The restriction to an open subscheme of a quasi-coherent log structure is again quasi-coherent. Such a restriction preserves any of the above properties $\mathscr{P}$.

1.1.10 Remark
All properties $\mathscr{P}$ defined here for log structures, are defined for any sheaf of monoids [cf. Appendix A]. Indeed, a quasi-coherent log structure α is integral (respectively, saturated) if and only if $\mathcal{M}_{\alpha,x}$ is integral (saturated) for all points $x \in X$. However, a coherent log structure α does almost never have a coherent log structure sheaf $\mathcal{M}_\alpha$ in the sense that all of its stalks are coherent. This is due to the fact, that the sheaf of units $\mathcal{O}_X^\times$ in the ring of regular functions, which is not affected by the charts, has generally incoherent stalks $\mathcal{O}_{X,x}^\times$ at any point $x \in X$ (compare the remark following definition II.2.1.5 in [29]).

1.2 Logarithmic schemes

A logarithmic scheme X will be defined as a scheme with additional structure. Whenever we speak of a sheaf on X, we mean a sheaf on the underlying scheme.

1.2.1 Logarithmic schemes

1.2.1 Definition
A *logarithmic scheme* is a pair $X = (\underline{X}, \alpha_X)$, where $\underline{X}$ is a scheme and α_X is a log structure on $\underline{X}$. Given a log scheme X, we will denote its underlying scheme by $\underline{X}$ and its log structure by $\alpha_X\colon \mathcal{M}_X \to \mathcal{O}_X$, where we write $\mathcal{M}_X := \mathcal{M}_{\alpha_X}$ and $\mathcal{O}_X := \mathcal{O}_{\underline{X}}$, referring to $\mathcal{M}_X$ as the *logarithmic structure sheaf* of X.

A log subscheme of a log scheme X is a subscheme $\underline{Y} \subset \underline{X}$ with the log structure $\underline{j}^{\times}\alpha_X$ induced by its inclusion $\underline{j}\colon \underline{Y} \to \underline{X}$.

A *morphism of log schemes* $f = (\underline{f}, f^{\flat})\colon X \to Y$ is a morphism of schemes $\underline{f}\colon \underline{X} \to \underline{Y}$ together with a morphism $f^{\flat}\colon \underline{f}^{\times}\alpha_Y \to \alpha_X$ of log structures on X.

A morphism of log schemes $f\colon X \to Y$ is called *(logarithmically) strict* if $f^{\flat}\colon \underline{f}^{\times}\alpha_Y \to \alpha_X$ is an isomorphism. It is called *logarithmically dominant* if $f^{\flat}$ is injective, and *(logarithmically) semistrict* if $f^{\flat}$ is surjective.

We say that $\underline{f}$ is the underlying morphism of schemes of f. From now on, we will usually write f_*, f^*, $f_\times$, $f^\times$ etc. for the functors $\underline{f}_*$, $\underline{f}^*$, $\underline{f}_\times$, $\underline{f}^\times$ etc.

1.2.2 Remark

One may also define $f^{\flat}$ to be a morphism of sheaves of monoids such that the diagram

$$\begin{tikzcd}
\mathcal{M}_Y \arrow[r, "f^{\flat}"] \arrow[d, "\beta"] & f_*\mathcal{M}_X \arrow[d, "f_*\alpha"] \\
\mathcal{O}_Y \arrow[r, "f^{\sharp}"] & f_*\mathcal{O}_X
\end{tikzcd}$$

commutes. By Definition 1.1.5 and the adjunction property 1.1.6 this is equivalent to the definition given in 1.2.1. We will use either form of $f^{\flat}$ according to its usefulness.

We denote the category of log schemes by $\underline{\mathrm{LSch}}$. On every scheme S there exists the trivial log structure $\iota\colon \mathcal{O}_S^{\times} \to \mathcal{O}_S$ and the functor $(\,\cdot\,)^{\iota}\colon \underline{\mathrm{Sch}} \to \underline{\mathrm{LSch}}$, $S \mapsto S^{\iota} := (S, \iota)$, is fully faithful. This turns $\underline{\mathrm{Sch}}$ into a full subcategory of $\underline{\mathrm{LSch}}$. The inclusion functor $(\,\cdot\,)^{\iota}$ is right adjoint to the functor $\underline{\cdot}$ forgetting the log structure: $\mathrm{Hom}_{\underline{\mathrm{LSch}}}(X, S^{\iota}) \cong \mathrm{Hom}_{\underline{\mathrm{Sch}}}(\underline{X}, S)$, for $X \in \underline{\mathrm{LSch}}$ and $S \in \underline{\mathrm{Sch}}$ (cf. [29, III.1.2.1]).

If Z is a log scheme, then the category $\underline{\mathrm{LSch}}_Z$ is defined as the category which has as objects morphisms of log schemes $X \to Z$ and as morphisms commutative diagrams of morphisms of log schemes

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\
& Z &
\end{tikzcd},$$

which, by abuse of notation, we will denote by $X \to Y$.

In the category of log schemes the *fibred product* of a diagram of log schemes

$$X \to Y \leftarrow X'$$

may be constructed as the fibred product $\underline{X} \times_{\underline{Y}} \underline{X'}$ of the underlying schemes together with the log structure $\alpha_X \boxtimes_{\alpha_Y} \alpha_{X'}\colon \mathcal{M}_X \boxtimes_{\mathcal{M}_Y} \mathcal{M}_{X'} \to \mathcal{O}_{X \times_Y X'}$ defined by $\alpha_X \boxtimes_{\alpha_Y} \alpha_{X'} := pr_X^{\times}\alpha_X \otimes_{pr_Y^{\times}\alpha_Y} pr_{X'}^{\times}\alpha_{X'}$, where we write $pr_X\colon \underline{X} \times_{\underline{Y}} \underline{X'} \to \underline{X}$, $pr_{X'}\colon \underline{X} \times_{\underline{Y}} \underline{X'} \to \underline{X'}$ and $pr_Y\colon \underline{X} \times_{\underline{Y}} \underline{X'} \to \underline{Y}$ for the canonical projections.

1.2.3 Definition

A *prelogarithmic ring* is a monoid homomorphism $a\colon P \to A$ from a monoid P to the multiplicative monoid of a ring A. A *prelog ring homomorphism* $(\pi, \theta)\colon (a\colon P \to A) \to (b\colon Q \to B)$ is a commutative diagram

$$\begin{tikzcd} P \arrow[r, "\theta"] \arrow[d, "a"] & Q \arrow[d, "b"] \\ A \arrow[r, "\pi"] & B, \end{tikzcd}$$

where θ is a monoid homomorphism and π a ring homomorphism.

A *logarithmic ring* is a prelog ring $a\colon P \to A$ such that a is logarithmic, i. e. $a^{-1}(A^\times) \cong A^\times$; if A is a field, we speak of a (pre)log field. A *log ring homomorphism* is a prelog ring homomorphism between log rings.

Given a prelog ring $a\colon P \to A$ we denote its associated log ring $a^>\colon P^> \to A$, setting $P^> := P \circledcirc A^\times := P \otimes_{a^{-1}(A^\times)} A^\times$. If A is a ring, then we will write A^ι for the prelog ring $A^\iota\colon 1 \to A$.

1.2.4 Remark

Our definition of a log ring is that of F. Kato (cf. [18]). However, A. Ogus uses the term *log ring* for what we call *prelog ring* (cf. [29, III.1.2.3]). With regard to 1.2.5 this does not produce much of a conflict.

1.2.5 Definition

If $a\colon P \to A$ is a (pre)log ring, then $\operatorname{Spec} a$ is defined to be the log scheme X which has $\underline{X} := \operatorname{Spec} A$ as its underlying scheme and the log structure $\alpha_X\colon P_X \circledcirc \mathcal{O}_X^\times \to \mathcal{O}_X$ associated to the prelog structure $a\colon P_X \to \mathcal{O}_X$. Note that $\operatorname{Spec} a$ and $\operatorname{Spec} a^>$ are equal as log schemes. Hence the at first sight ambiguous notation $\operatorname{Spec} A^\iota$, which may be read as $\operatorname{Spec}(A^\iota)$ or $(\operatorname{Spec} A)^\iota$, denotes a unique log scheme.

More generally, we call a log scheme *affine* if its underlying scheme is an affine scheme.

A *logarithmic point* is a log scheme X such that $\underline{X} = \operatorname{Spec} k$ for a field k. The *standard prelogarithmic field* of the field k is the prelog field $\kappa\colon \mathbb{N}_0 \to k$ mapping $0 \mapsto 1$ and $n \mapsto 0$ if $n \geq 1$. The *standard logarithmic point* of the field k is $\operatorname{Spec} \kappa$.

Note that not every affine log scheme is isomorphic to the spectrum of a (pre)log ring. However, any log scheme with quasi-coherent log structure may be covered by spectra of (pre)log rings.

1.2.6 Lemma ([29, III.1.5.3])

Let $S = \operatorname{Spec} k$, with k an algebraically closed field. Since S consists only of one point, any log structure κ on S is given by a monoid homomorphism $\kappa\colon M \to k$. If M is unit-integral, then the log scheme (S, κ) is (non-canonically) isomorphic to $\operatorname{Spec} k_P$ for the sharp monoid $P := M/k^\times$.

1.2.7 Definition

Let A be a commutative ring and P a monoid. The *P-affine log scheme* over A is the log scheme $\mathbb{A}_A[P] := \operatorname{Spec}(P \to A[P])$ associated to the canonical prelog ring $P \to A[P]$. Its log structure will be denoted $\alpha_A[P]$ and called the *canonical log structure* on the scheme $\operatorname{Spec} A[P]$.

If $\theta\colon Q \to P$ is a monoid homomorphism, we write $\mathbb{A}_A[\theta]\colon \mathbb{A}_A[P] \to \mathbb{A}_A[Q]$ or simply θ for the morphism of $\operatorname{Spec} A^\iota$-log schemes given by the ring homomorphism $A[\theta]\colon A[Q] \to A[P]$. This makes $\mathbb{A}_A[\,\cdot\,]$ a functor $\underline{\mathrm{Mon}}^{\mathrm{op}} \to \underline{\mathrm{LSch}}_{\operatorname{Spec} A^\iota}$.

In the case $A = \mathbb{Z}$, we simply write $\mathbb{A}[P] := \mathbb{A}_{\mathbb{Z}}[P]$ and $\mathbb{A}[\theta] := \mathbb{A}_{\mathbb{Z}}[\theta]$. For the trivial monoid $P = 1$ we get $\mathbb{A}_A[1] = \operatorname{Spec} A^\iota$ and we have $\mathbb{A}_A[P] = \mathbb{A}_A[1] \times_{\mathbb{A}[1]} \mathbb{A}[P]$.

1.2.8 Definition

Let $h\colon Y \to \mathbb{A}[Q]$ be an $\mathbb{A}[Q]$-scheme and let $\theta\colon Q \to P$ be a monoid homomorphism. The *P-affine log scheme over h* is the pullback $Y \times_{\mathbb{A}[Q]} \mathbb{A}[P]$, denoted $\mathbb{A}_h[P]$ or $\mathbb{A}_{Y/Q}[P]$:

$$\begin{tikzcd}
\mathbb{A}_h[P] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{A}[P] \arrow[d, "\theta"] \\
Y \arrow[r, "h"] & \mathbb{A}[Q].
\end{tikzcd}$$

We will denote its log structure by $\alpha_h[P]$ and the canonical projections to $\mathbb{A}[P]$ by $h[\theta]$ and to Y by $\mathbb{A}_h[\theta]$, or θ_h for short.

If $Q = 1$ and if $Y = \operatorname{Spec} A$ is affine, then $\mathbb{A}_{Y/1}[P] = \mathbb{A}_A[P]$.

Given a log scheme X and a monoid P, the map

$$\operatorname{Hom}_{\underline{\mathrm{LSch}}}(X, \mathbb{A}[P]) \to \operatorname{Hom}_{\underline{\mathrm{Mon}}}(P, \mathit{\Gamma}(X, \mathcal{M}_X)),$$

defined by $g \mapsto a_g$, where a_g is the composition $P \to \mathit{\Gamma}(X, g^\times P) \xrightarrow{g^\flat} \mathit{\Gamma}(X, \mathcal{M}_X)$, is a group isomorphism. We denote its inverse by $a \mapsto g_a$.

This means, that specifying a global chart $a\colon P \to \mathit{\Gamma}(X, \mathcal{M}_X)$ for α_X (with $P^> \cong \mathcal{M}_X$) is equivalent to giving a strict morphism of log schemes $g\colon X \to \mathbb{A}[P]$. In fact, with the labelling of definition 1.1.8, $a^> = g^\flat$ and $\alpha_a^> = g^\times \alpha_{\mathbb{A}[P]}$.

1.2.9 Definition

We say that a log scheme X *is logarithmically $\mathscr{P}$* (where $\mathscr{P}$ is one of the properties as in section 1.1.2) if its log structure α_X has the property $\mathscr{P}$.

If $\mathscr{Q}$ is a property of schemes (respectively, of morphisms of schemes), then we say that a log scheme X (a morphism of log schemes $f\colon X \to Y$) *has the property $\mathscr{Q}$* if the underlying scheme $\underline{X}$ (the underlying morphism of schemes $\underline{f}$) has that property.

In particular, a log scheme X is *logarithmically fine* if it can be covered by fine charts. These are local charts $a\colon P \to \Gamma(U, \mathcal{M}_X)$ of α_X such that P is finitely generated and integral (i. e. maps injectively into P^{grp}). It is called *logarithmically saturated* if analogously there is a covering by charts with P integral and $P = P^{\mathrm{sat}} = \{x \in P^{\mathrm{grp}} \mid x^n \in P \text{ for some } n \in \mathbb{N}_0\}$. A *logarithmically fs* (or *logarithmically normal*) log scheme is a log fine and log saturated log scheme.

Likewise, a morphism of log schemes is called *proper*, if its underlying morphism is proper.

1.2.10 Remark

All authors mentioned in the introduction to this chapter use the convention, that a log scheme has property $\mathscr{P}$ if its log structure has $\mathscr{P}$. However, this produces conflicts, e. g. when talking about an integral log scheme, where "integral" can either be read as a property of its underlying scheme or of its log structure. Our convention helps to avoid these conflicts.

We denote the full subcategory of $\underline{\mathrm{LSch}}$, having as objects the log (quasi-)coherent log schemes, by $\underline{\mathrm{LSch}}^{(\mathrm{q})\mathrm{coh}}$. The full subcategory of $\underline{\mathrm{LSch}}^{\mathrm{coh}}$ of log fine (respectively, log fs, log locally free) log schemes is denoted $\underline{\mathrm{LSch}}^{\mathrm{f}}$ ($\underline{\mathrm{LSch}}^{\mathrm{fs}}$, $\underline{\mathrm{LSch}}^{\mathrm{lf}}$).

1.2.11 Definition

For any morphism $f\colon X \to Y$ of log schemes we define the sheaf of monoids

$$\overline{\mathcal{M}}_f := \overline{\mathcal{M}}_{\alpha_X / f^\times \alpha_Y} = \mathcal{M}_X / f^\flat(f^\times \mathcal{M}_Y)$$

on X, which we call the *relative characteristic sheaf* or the *lenience sheaf* of f (or of X over Y).

1.2.12 Lemma ([29, III.1.2.8])

If $f\colon X \to Y$ is a strict morphism of log schemes, then the map $\overline{f^\flat}\colon f^{-1}\overline{\mathcal{M}}_Y \to \overline{\mathcal{M}}_X$ induced by $f^\flat$ is an isomorphism. If it is semi-strict then $\overline{\mathcal{M}}_f = 0$. The converses of both statements are true if X is log integral.

1.2.13 Definition

Let X be a log coherent log scheme. We denote by $\mathrm{LLoc}\, X$ the reduced closed log subscheme $\operatorname{supp} \overline{\mathcal{M}}_X \subset X$ and call it the *logarithmic locus* of X or the *support* of its log structure α_X. Its open complement is denoted by $X^\times$ and called the *logarithmically trivial locus* of X.

A point $x \in X$ lies in $X^\times$ if and only if there is an open/étale neighbourhood U of x such that $\alpha_X|_U$ is trivial. The functor $(\,\cdot\,)^\times\colon X \mapsto X^\times$ from the category of log schemes with coherent log structure to schemes is right adjoint to the inclusion functor $(\,\cdot\,)^\iota$:

$$\mathrm{Hom}_{\underline{\mathrm{LSch}}^{\mathrm{coh}}}(S^\iota, X) \cong \mathrm{Hom}_{\underline{\mathrm{Sch}}}(S, X^\times)$$

1.2.14 Definition

Let X be a log coherent log scheme. The map $\ell_X\colon X \to \mathbb{N}_0$, given by $\ell_X(x) := \operatorname{rk}(\overline{\mathcal{M}}^{\mathrm{grp}}_{X,x})$, is called the *logarithmic rank* of X and the number $\ell_X(x)$ the *logarithmic rank* of X at the point x.

The set $X^{(\leq n)} := \ell_X^{-1}(\{0,1,\dots n\})$ is an open subscheme of $\underline{X}$ for each $n \in \mathbb{N}_0$. The underlying set of its reduced (closed) complement $X^{(\geq n+1)}$ is $\ell_X^{-1}(\mathbb{N}_{\geq n+1})$. We call the locally closed intersection $X^{(n)} = X^{(\leq n)} \cap X^{(\geq n)}$ with underlying set $\ell_X^{-1}(n)$ the *logarithmic locus* of X *of* n^{th} *order.* The scheme structure of each of these subschemes $X^{(n)}$ is, locally at a point x given by the ideal $I_{\alpha_{X,x}} \subset \mathcal{O}_{X,x}$, defined to be the image under α_X of the maximal ideal $\mathfrak{m}_{\mathcal{M}_{X,x}}$ of $\mathcal{M}_{X,x}$.

This defines a stratification $\underline{X} = X^{(\geq 0)} \supset X^{(\geq 1)} \supset \dots \supset X^{(\geq N)} = \emptyset$, $N \gg 0$, for $\underline{X}$. Moreover, $\operatorname{codim}_X X^{(\geq n)} \leq n$ for all n (cp. [29, II.2.1.6] and [16, 2.3]). In particular, $\operatorname{LLoc} X = X^{(\geq 1)}$ and $X^\times = X^{(0)}$.

1.2.2 Morphisms of log Schemes and Charts

As mentioned before, specifying a global chart $a\colon P \to \Gamma(X, \mathcal{M}_X)$ for a log scheme X is equivalent to giving a strict morphism of log schemes $g\colon X \to \mathbb{A}[P]$. A similar statement is true, when talking about charts of morphisms of log structures:

1.2.15 Definition

Let $f\colon X \to Y$ be a morphism of log schemes. A *(global) chart* of f subordinate to a monoid homomorphism $\theta\colon Q \to P$ is a commutative diagram

$$\begin{tikzcd} Q \arrow[r, "b"] \arrow[d, "\theta"] & \Gamma(Y, \mathcal{M}_Y) \arrow[d, "f^\flat"] \\ P \arrow[r, "a"] & \Gamma(X, \mathcal{M}_X) \end{tikzcd}$$

of monoid homomorphisms, where a and b are global charts for α_X and α_Y respectively.

Specifying a chart for f subordinate to $\theta\colon Q \to P$ is equivalent to giving a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g_a"] \arrow[d, "f"] & \mathbb{A}[P] \arrow[d, "\theta"] \\ Y \arrow[r, "g_b"] & \mathbb{A}[Q] \end{tikzcd}$$

of morphisms of log schemes, with g_a and g_b strict. We will call this the *chart diagram* of the chart.

This diagram canonically enlarges to

$$\begin{tikzcd} X \arrow[r, "g_{a/b}"] \arrow[dr, "f"'] & \mathbb{A}_b[P] \arrow[r, "g_b[\theta]"] \arrow[d, "\theta_b"] & \mathbb{A}[P] \arrow[d, "\theta"] \\ & Y \arrow[r, "g_b"] & \mathbb{A}[Q] \end{tikzcd}$$

with the right square Cartesian, $g_a = g_b[\theta] \circ g_{a/b}$ and all horizontal maps strict; here we write $\mathbb{A}_b[P]$ for $\mathbb{A}_{g_b}[P]$ and θ_b for θ_{g_b}. We will refer to it as the *extended chart diagram* of the chart.

Given a morphism of log schemes $f\colon X \to Y$, a chart $b\colon Q \to \Gamma(Y, \mathcal{M}_Y)$ for α_Y and a monoid homomorphism $\theta\colon Q \to P$, we have a group isomorphism

$$\operatorname{Hom}_{\underline{\mathrm{Mon}}_Q}(P, \Gamma(X, \mathcal{M}_X)) \cong \operatorname{Hom}_{\underline{\mathrm{LSch}}_Y}(X, \mathbb{A}_b[P]),$$

given by $a \mapsto g_{a/b}$.

1.2.16 Proposition ([19, 2.10], [17, 2.14])

Let $f\colon X \to Y$ be a morphism of log coherent log schemes and let $b\colon Q \to \Gamma(Y, \mathcal{M}_Y)$ be a coherent chart for α_Y. Then locally on X there exists a chart

$$\begin{tikzcd} Q \arrow[r, "b"] \arrow[d, "\theta"] & \Gamma(Y, \mathcal{M}_\beta) \arrow[d, "f^\flat"] \\ P \arrow[r, "a"] & \Gamma(X, \mathcal{M}_\alpha) \end{tikzcd}$$

comprising b and with P finitely generated.

1.2.17 Definition

Let $f\colon X \to Y$ be a morphism of log coherent log schemes. We denote by $\operatorname{LLoc} f$ the reduced closed log subscheme $\operatorname{supp} \overline{\mathcal{M}}_f \subset X$ and call it the *lenient locus* of f. Its open complement is denoted by $X^\times(f)$ and called the *semi-strict locus* of f (in X).

A point $x \in X$ lies in $X^\times(f)$ if and only if there is an open/étale neighbourhood $U$of x such that the restriction $f|_U : U \to Y$ is a semi-strict morphism. If f is log dominant, then one may replace semi-strict by strict and speak of the strict locus of f.

1.2.18 Definition

Let $f\colon X \to Y$ be a morphism of log coherent log schemes. We call the map $\ell_f\colon X \to \mathbb{N}_0$, given by $\ell_f(x) := \operatorname{rk}(\overline{\mathcal{M}}_{f,x}^{\mathrm{grp}})$, the *leniency* of f and the number $\ell_f(x)$ the *leniency* of f at x.

The set $X^{(\leq n)}(f) := \ell_f^{-1}(\{0, 1, \dots n\})$ is an open subscheme of X for each $n \in \mathbb{N}_0$. The underlying set of its reduced (closed) complement $X^{(\geq n+1)}(f)$ is $\ell_f^{-1}(\mathbb{N}_{\geq n+1})$. We call the locally closed intersection $X^{(n)}(f)$ of $X^{(\leq n)}(f)$ and $X^{(\geq n)}(f)$ with underlying set $\ell_X^{-1}(n)$ the *lenient locus* of f *of* n^{th} *order*. The scheme structure of each of these subschemes is locally at a point x given by the ideal $I_{\mathcal{M}_{X,x}} \subset \mathcal{O}_{X,x}$ (cf. [29, II.2.1.6], [16, 2.3]). It is $\operatorname{LLoc} f = X^{(\geq 1)}(f)$ and $X^\times(f) = X^{(0)}(f)$.

1.2.3 Log derivations and log differentials

1.2.19 Definition

Let $f\colon X \to Y$ be a morphism of log schemes and $\mathcal{E}$ an $\mathcal{O}_X$-module. A *(logarithmic) derivation* of f (or of X over Y) to $\mathcal{E}$ is a pair $\vartheta = (\underline{\vartheta}, \Theta)$, where $\underline{\vartheta} \in \mathrm{Der}_{\underline{Y}}(\underline{X}, \mathcal{E})$ and $\Theta\colon \mathcal{M}_X \to (\mathcal{E}, +)$ is a morphism of sheaves of monoids with the following properties:

a) $\alpha_X(m)\Theta(m) = \underline{\vartheta}(\alpha_X(m))$ for any local section m of $\mathcal{M}_X$ and

b) $\Theta(f^\flat(n)) = 0$ for any local section n of $f^{-1}\mathcal{M}_Y$.

The set of such derivations is denoted by $\mathrm{Der}_f(\mathcal{E})$. Then the sheaf $\mathcal{D}er_f(\mathcal{E})$ of (log) derivations of X over Y to $\mathcal{E}$ is the sheaf $U \mapsto \mathrm{Der}_{f|_U}(\mathcal{E}|_U)$, which is in fact an $\mathcal{O}_X$-module. If $\mathcal{E} = \mathcal{O}_X$, we write T_f for this sheaf and call it the *(logarithmic) tangent sheaf* of f or of X over Y.

1.2.20 Remark

Since $(\mathcal{E}, +)$ is a sheaf of groups, the map Θ naturally factors via the morphism of sheaves of groups $\Theta^{\mathrm{grp}}\colon \mathcal{M}_X^{\mathrm{grp}} \to (\mathcal{E}, +)$.

1.2.21 Definition and Proposition (cp. [29, IV.1.2.3])

Let $f\colon X \to Y$ be a morphism of log schemes. There exists an $\mathcal{O}_X$-module Ω^1_f and a universal derivation $(d, d\log) \in \mathrm{Der}_f(\Omega^1_f)$ such that for any $\mathcal{O}_X$-module $\mathcal{E}$ the canonical morphism of $\mathcal{O}_X$-modules

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_f, \mathcal{E}) \to \mathcal{D}er_f(\mathcal{E}),\ \lambda \mapsto (\lambda \circ d, \lambda \circ d\log)$$

is an isomorphism.

We write $i_{\,\cdot}(\,\cdot\,)\colon T_f \otimes_{\mathcal{O}_X} \Omega^1_f \to \mathcal{O}_X,\ \vartheta \otimes \sigma \mapsto i_\vartheta(\sigma)$, for the natural pairing induced by this isomorphism and we call Ω^1_f the *sheaf of (logarithmic) differentials* of X over Y.

1.2.22 Definition

Let Λ_X denote the $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{\mathrm{grp}}$, which we call the *sheaf of purely logarithmic differentials.*

The sheaf of log differentials may be constructed as

$$\Omega^1_f = \left[\Omega^1_{\underline{f}} \oplus \Lambda_X\right] / (\mathcal{K}_X + \mathcal{K}_f),$$

where $\Omega^1_{\underline{f}}$ is the usual sheaf of Kähler differentials and where $\mathcal{K}_X$ is the $\mathcal{O}_X$-submodule of $\Omega^1_{\underline{f}} \oplus \Lambda_X$ generated by local sections of the form

$$(d(\alpha(m)), -\alpha(m) \otimes m) \text{ for local sections } m \text{ of } \mathcal{M}_X$$

and $\mathcal{K}_f$ the image of the map

$$\mathcal{O}_X \otimes_{\mathbb{Z}} f^{-1}\mathcal{M}_Y^{\mathrm{grp}} \to 0 \oplus \Lambda_X,$$

which is the $\mathcal{O}_X$-module generated by local sections of the form $(0, 1 \otimes f^\flat(n))$ for local sections n of $f^{-1}\mathcal{M}_Y$.

The universal derivation is then given by

$$d\colon \mathcal{O}_X \xrightarrow{d} \Omega^1_{\underline{f}} \to \Omega^1_f \quad \text{ and } \quad d\log\colon \mathcal{M}_X \to \mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{\mathrm{grp}} \to \Omega_f, m \mapsto (0, 1 \otimes m).$$

Accordingly we will simply write ds for the class of $(ds, 0)$ and $d\log m$ for the class of $(0, 1 \otimes m)$. Occasionally, if $s = \alpha(m)$, we write $d\log s := d\log m$.

Given a local section $\vartheta = (\underline{\vartheta}, \Theta)$ of $\mathcal{D}er_f(\mathcal{E})$ for f and $\mathcal{E}$ as above, then ϑ is completely determined by Θ. This is due to the fact, that $\mathcal{O}_X^\times$ generates $(\mathcal{O}_X, +)$ as a sheaf of Abelian groups, because any local section of $\mathcal{O}_X$ can locally be written as the sum of at most two sections of $\mathcal{O}_X^\times$. Hence, the image of $\mathcal{M}_X$ under α_X generates $\mathcal{O}_X$ as a sheaf of Abelian groups. Any local section s of $\mathcal{O}_X$ can locally be written as $s = \alpha(m)$ or as $s = \alpha_X(m_1) + \alpha_X(m_2)$. Then $\underline{\vartheta}(s) = s\Theta(m)$ or $\underline{\vartheta}(s) = \alpha(m_1)\Theta(m_1) + \alpha(m_2)\Theta(m_2)$ (cp. [29, IV.1.2.4]). This leads to an alternative construction of Ω^1_f as a quotient of Λ_X:

1.2.23 Proposition ([29, IV.1.2.10 & 11])

Let $\mathcal{R}_X \subset \Lambda_X$ be the subsheaf of sections, which are locally of the form

$$\sum_i \alpha_X(m_i) \otimes m_i - \sum_j \alpha_X(m'_j) \otimes m'_j,$$

where $m_1, \ldots, m_k$ and $m'_1, \ldots, m'_{k'}$ are local sections of $\mathcal{M}_X$ such that $\sum_i \alpha_X(m_i) = \sum_j \alpha_X(m'_j)$ in $\mathcal{O}_X$, and let $\mathcal{R}_f \subset \Lambda_X$ be the image of the map

$$\mathcal{O}_X \otimes_{\mathbb{Z}} f^{-1}\mathcal{M}_Y^{\mathrm{grp}} \to \Lambda_X.$$

Then $\mathcal{R}_X$ and $\mathcal{R}_f$ are $\mathcal{O}_X$-submodules of Λ_X and there is a unique isomorphism

$$\Omega^1_f \cong \Lambda_X/(\mathcal{R}_X + \mathcal{R}_f).$$

We will denote the image of a local section $f \otimes m$ of Λ_X in Ω^1_f by $f d\log m$. This notation is consistent in the sense that if m is the image of a local section m' of $\mathcal{M}_X$ under $\mathcal{M}_X \to \mathcal{M}_X^{\mathrm{grp}}$, then $d\log m$, as defined here, is equal to $d\log m'$, as defined above as the image of m under the universal derivation $d\log$.

If $f\colon S^\iota \to S'^\iota$ is a morphism of schemes with trivial log structures, then the $\mathcal{O}_X$-module of usual Kähler differentials $\Omega^1_{\underline{f}}$ of $\underline{f}$ is canonically isomorphic to Ω^1_f by the fact that for any local section u of $\mathcal{M}_{S^\iota} = \mathcal{O}_S^\times$ we have $d\log u = u^{-1}du$. Hence, we will identify both modules in this case, as well as their dual modules $T_{\underline{f}}$ and T_f.

If $f\colon X \to Y$ is a morphism of log coherent log schemes, then Ω^1_f is a quasi-coherent $\mathcal{O}_X$-module. If moreover Y is Noetherian and the underlying morphism of schemes $\underline{f}\colon \underline{X} \to \underline{Y}$ is of finite type, then Ω^1_f is coherent (cf. [17, 5.1], [29, IV.1.2.9]).

1.2.24 Proposition ([29, IV.1.3.1])
For a commutative diagram of morphisms of log schemes

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

there is a natural morphism $g^*\Omega^1_f \to \Omega^1_{f'}$ (sending ds to $g^*ds = d(g^\sharp s)$ and $d\log m$ to $g^*d\log m := d\log(g^\flat m)$ for local sections s of $g^{-1}\mathcal{O}_X$ and m of $g^{-1}\mathcal{M}_X$), which is an isomorphism if the diagram is Cartesian.

If this diagram is Cartesian, then the induced homomorphism

$$f'^*\Omega^1_h \oplus g^*\Omega^1_f \to \Omega^1_{h\circ f'} = \Omega^1_{f\circ g}$$

is an isomorphism.

1.2.25 Proposition ([29, IV.2.3.1 & 2])
Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms of log schemes. Then there is an exact sequence

$$f^*\Omega^1_g \to \Omega^1_{g\circ f} \to \Omega^1_f \to 0.$$

If we replace the morphism f by a strict closed immersion $i\colon X \to Y$ defined by a sheaf of ideals I and if X, Y and Z are log quasi-integral, then there is an exact sequence

$$I/I^2 \to i^*\Omega^1_g \to \Omega^1_{i\circ g} \to 0$$

of $\mathcal{O}_X$-modules.

We call the $\mathcal{O}_X$-module I/I^2 the *conormal sheaf* of i.

Let $f\colon X \to Y$ be a morphism of log coherent log schemes. The $f^{-1}\mathcal{O}_Y$-linear map $d\colon \mathcal{O}_X \to \Omega^1_f$ fits into a complex $\Omega^\bullet_f$ with $f^{-1}\mathcal{O}_Y$-linear differentials, called the *logarithmic de Rham complex* of f:

1.2.26 Proposition ([29, V.2.1.1])
Let $f\colon X \to Y$ be a morphism of log coherent log schemes. There exists a complex of $\mathcal{O}_X$-modules $\Omega^\bullet_f$ with $f^{-1}\mathcal{O}_Y$-linear differentials $d^i\colon \Omega^i_f \to \Omega^{i+1}_f$ such that

a) $\Omega^i_f = \bigwedge^i \Omega^1_f$;

b) $d^0 = d\colon \mathcal{O}_X \to \Omega^1_f$;

c) $d^1(d\log m) = 0$ for each local section m of $\mathcal{M}_X$;

d) $d^{i+j}(\sigma \wedge \sigma') = (d^i\sigma)\wedge\sigma' + (-1)^i\sigma\wedge(d^j\sigma')$ for local sections σ of Ω^i_f and σ' of Ω^j_f.

1.2.4 Logarithmic infinitesimal thickenings

1.2.27 Definition

A *logarithmic (infinitesimal) thickening* is a strict closed immersion $i\colon T \to T'$ of log schemes such that the ideal sheaf I of T in T' is nilpotent and such that the subgroup $1 + I \subset \mathcal{O}_{T'}^{\times}$ operates freely on $\mathcal{M}_{T'}$. We say that the log thickening *has (at most) order* n if $I^{n+1} = 0$.

If T is log quasi-integral, then the action of $1 + I$ on $\mathcal{M}_{T'}$ is automatically free. For the Zariski topology, in a log thickening the underlying topological spaces of T and T' are homeomorphic and are thus identified. For the étale topology the same is true for log thickenings of finite order (cf. [29, IV.2.1.4]).

1.2.28 Proposition ([29, IV.2.1.2 & 3])

Let $i\colon T \to T'$ be a log thickening with ideal I. Then

a) The diagram
$$\begin{tikzcd} \mathcal{O}_{T'}^{\times} \arrow[r] \arrow[d, "i^{\sharp}"] & \mathcal{M}_{T'} \arrow[d, "i^{\flat}"] \\ i_*\mathcal{O}_T^{\times} \arrow[r] & i_*\mathcal{M}_T \end{tikzcd}$$
is Cartesian and cocartesian;

b) $\mathrm{Ker}(\mathcal{O}_{T'}^{\times} \to i_*\mathcal{O}_T^{\times}) = \mathrm{Ker}(\mathcal{M}_{T'}^{\mathrm{grp}} \to i_{\times}\mathcal{M}_T^{\mathrm{grp}}) = 1 + I$ (observe, that $i_*\mathcal{O}_T^{\times} = i_{\times}\mathcal{O}_T^{\times}$ when regarding $\mathcal{O}_T^{\times}$ as the structure sheaf of the trivial log structure on T);

c) The diagram
$$\begin{tikzcd} \mathcal{M}_{T'} \arrow[r] \arrow[d] & \mathcal{M}_{T'}^{\mathrm{grp}} \arrow[d] \\ \mathcal{M}_T \arrow[r] & \mathcal{M}_T^{\mathrm{grp}} \end{tikzcd}$$
is Cartesian;

d) T' is log coherent, log integral, log fine, log saturated or log free if and only if T is. If $a\colon P \to \Gamma(T', \mathcal{M}_{T'})$ is a chart for T', then $i^{\flat} \circ a\colon P \to \Gamma(T, \mathcal{M}_T)$ is a chart for T and the converse is true if T is log quasi-integral.

1.2.5 Logarithmic smoothness

Formal smoothness for schemes was defined by A. Grothendieck in [12, III.1] via the infinitesimal lifting property. A morphism of schemes is smooth if and only if it has the infinitesimal lifting property and is locally of finite presentation. In analogy one defines (formal) log smoothness.

1.2.29 Definition ([19, 3.3], [29, IV.3.1.1])

A morphism $f\colon X \to Y$ of log schemes is *formally smooth* (respectively, *formally unramified*, *formally étale*) if for every n and every n-th order log thickening $i\colon T \to T'$ with a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & X \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle f} \\ T' & \xrightarrow{\varphi'} & Y \end{array}$$

there exists at least one (at most one, exactly one) morphism $\tilde{\varphi}\colon T' \to X$ lifting φ' in this diagram.

A formally smooth (respectively, formally étale) morphism $f\colon X \to Y$ is called *logarithmically smooth* (*logarithmically étale*) if X and Y are log coherent and its underlying morphism $\underline{f}\colon \underline{X} \to \underline{Y}$ is locally of finite presentation.

The usual statements about composition and base change of smooth and étale morphisms hold for log smooth and log étale morphisms.

If f is log smooth, then Ω^1_f is locally free (cf. [29, IV.3.2.1]). A morphism f of log coherent log schemes is formally unramified if and only if $\Omega^1_f = 0$ (cf. [29, IV.3.1.3]).

1.2.30 Proposition ([29, IV.3.2.3 & 4])

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms of of log coherent log schemes.

a) If f is log smooth, then the sequence $0 \to f^*\Omega^1_g \to \Omega^1_{g\circ f} \to \Omega^1_f \to 0$ is exact and splits.

b) If $g \circ f$ is log smooth and the sequence $0 \to f^*\Omega^1_g \to \Omega^1_{g\circ f} \to \Omega^1_f \to 0$ is split exact, then f is log smooth.

c) If f is log étale, then $f^*\Omega^1_g \to \Omega^1_{g\circ f}$ is an isomorphism.

1.2.31 Proposition ([29, IV.3.1.6])

Let $f\colon X \to Y$ be a strict morphism of log coherent log schemes. If the underlying morphism of schemes $\underline{f}$ is formally smooth (respectively, formally étale, formally unramified) then the same is true of f. The converse holds if Y is log unit-integral.

This leads to the following definition:

1.2.32 Definition

Let $f\colon X \to Y$ be a morphism of log coherent log schemes. We say that f is *smooth* (respectively, *étale*, *unramified*) if f is strict and log smooth (log étale, formally unramified).

1.2.33 Proposition ([29, IV.3.1.8])

Let $\theta\colon Q \to P$ be a homomorphism of finitely generated monoids and let $f\colon \mathbb{A}_A[P] \to \mathbb{A}_A[Q]$ be the corresponding affine morphism over a commutative ring A. Then f is log smooth (respectively, log étale) if and only if $\operatorname{Ker}(\theta^{\mathrm{grp}})$ and the torsion part of $\operatorname{Cok}(\theta^{\mathrm{grp}})$ ($\operatorname{Ker}(\theta^{\mathrm{grp}})$ and $\operatorname{Cok}(\theta^{\mathrm{grp}})$) are finite groups the order of which is invertible in A.

1.2.34 Proposition ([19, 3.5 & 6], [29, IV.3.3.1])
Let $f\colon X \to Y$ be a log smooth (respectively, log étale) morphism of log coherent log schemes and let $b\colon Q \to \Gamma(Y, \mathcal{M}_Y)$ be a coherent chart for Y. Then étale locally on X there exists a chart

$$\begin{array}{ccc} X & \xrightarrow{g_a} & \mathbb{A}[P] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \theta} \\ Y & \xrightarrow{g_b} & \mathbb{A}[Q] \end{array}$$

comprising b and with P finitely generated (as in 1.2.16) and with the properties:

a) The homomorphism $\theta\colon Q \to P$ is injective and the order of the torsion part of its cokernel is invertible in $\mathcal{O}_X$ (respectively, and its cokernel is finite of order invertible in $\mathcal{O}_X$).

b) The canonical strict morphism $g_{a/b}\colon X \to \mathbb{A}_b[P]$ is étale.

Conversely, we have the following *Criterion for log smoothness (respectively, log étaleness)*:
1.2.35 Proposition ([19, 3.5], [29, IV.3.1.13])
Let $f\colon X \to Y$ be a morphism of log schemes admitting a coherent chart subordinate to $\theta\colon Q \to P$ and let x be a point in X. Assume that P and Q are finitely generated and that

a) $k(x) \otimes_{\mathbb{Z}} \mathrm{Ker}(\theta^{\mathrm{grp}}) = 0$ and $k(x) \otimes_{\mathbb{Z}} (\mathrm{Cok}(\theta^{\mathrm{grp}}))_{\mathrm{tors}} = 0$, where $(\,\cdot\,)_{\mathrm{tors}}$ denotes the torsion part (respectively, and $k(x) \otimes_{\mathbb{Z}} \mathrm{Cok}(\theta^{\mathrm{grp}}) = 0$);

b) the canonical strict morphism $g_{a/b}\colon X \to \mathbb{A}_b[P]$ is smooth (respectively, étale) in some neighbourhood of x.

Then f is log smooth (respectively, log étale) in some open/étale neighbourhood of x.

Example 1.2.1 To give an impression of what log smooth morphisms may look like, we regard the following example. Let $\underline{X} = \operatorname{Spec} \mathbb{C}[x, y]/(xy)$ and let $\underline{f}\colon \underline{X} \to \underline{Y} := \operatorname{Spec} \mathbb{C}$ be its structure morphism as scheme over $\operatorname{Spec} \mathbb{C}$. Then f is of finite type, but not smooth: Its $\mathcal{O}_X$-module of Kähler differentials $\Omega^1_{\underline{f}} = \mathcal{O}_X\langle dx, dy\rangle/\langle ydx + xdy\rangle$ is not locally free.
However, if we equip $\underline{X}$ and $\underline{Y}$ with suitable log structures, then there exists a log smooth morphism $f\colon X \to Y$ the underlying morphism of schemes of which is $\underline{f}$: We give $\underline{Y} = \operatorname{Spec} \mathbb{C}$ the log structure of the standard log point, which is the log structure associated to the prelog structure $b\colon \mathbb{N}_0 \to \mathbb{C}$, $n \mapsto 1$ if $n = 0$ and $n \mapsto 0$ if $n \geq 1$ and we give $\underline{X}$ the log structure associated to the prelog structure $a\colon \mathbb{N}_0^2 \to \mathcal{O}_X$, $(n_1, n_2) \mapsto x^{n_1}y^{n_2}$. Then the following diagram, which we will refer to as $(*)$, is commutative:

$$\begin{array}{ccc} \mathbb{N}_0^2 & \xrightarrow{a} & \mathcal{O}_X \\ {\scriptstyle \mathrm{diag}}\uparrow & & \uparrow{\scriptstyle f^\sharp} \\ \mathbb{N}_0 & \xrightarrow{b} & \mathbb{C}, \end{array}$$

where $\mathrm{diag}\colon \mathbb{N}_0 \to \mathbb{N}_0^2$ is the diagonal monoid homomorphism $n \mapsto (n, n)$.

The log structures of $Y = (\underline{Y}, \alpha_Y)$ and $X = (\underline{X}, \alpha_X)$ are given by

$$\alpha_Y : \mathbb{N}_0 \oplus \mathbb{C}^\times \to \mathbb{C}, n \oplus u \mapsto \begin{cases} u \text{ if } n = 0, \\ 0 \text{ if } n \geq 1, \end{cases}$$

and by

$$\alpha_X : \mathbb{N}_0 \circledcirc \mathcal{O}_X^\times = \mathbb{N}_0 \otimes_{a^{-1}(\mathcal{O}_X^\times)} \mathcal{O}_X^\times \to \mathcal{O}_X, (n_1, n_2) \otimes u \mapsto ux^{n_1}y^{n_2},$$

respectively. In what follows, we will write e_1 and e_2, respectively, for the generators $(1, 0)$ and $(0, 1) \in \mathbb{N}_0^2$ and also for their images in $\mathbb{N}_0^2 \circledcirc \mathcal{O}_X^\times$.

To turn $\underline{f}$ into a morphism of log schemes, we have to define the morphism of sheaves of monoids $f^\flat : \mathcal{M}_Y \to f_*\mathcal{M}_X$, which we do by $n \oplus u \mapsto (n, n) \otimes f^\sharp(u)$.

Then $f = (\underline{f}, f^\flat) \colon X \to Y$ is log smooth by the criterion for log smoothness 1.2.35: The diagram $(*)$ is by definition a chart for f subordinate to $\mathrm{diag} \colon \mathbb{N}_0 \to \mathbb{N}_0^2$. We have $\mathrm{Ker}(\mathrm{diag}^{\mathrm{grp}}) = 0$ and $(\mathrm{Cok}(\mathrm{diag}^{\mathrm{grp}}))_{\mathrm{tors}} = \mathbb{Z}_{\mathrm{tors}} = 0$, hence a) is fulfilled. The strict morphism $g_{a/b}$ in b) is given by the morphism

$$\mathbb{C} \otimes_{\mathbb{C}[\mathbb{N}_0]} \mathbb{C}[\mathbb{N}_0^2] = \mathbb{C}[e_1, e_2]/(e_1e_2) \to \mathbb{C}[x, y]/(xy), e_1 \mapsto x, e_2 \mapsto y,$$

which is clearly an isomorphism. Hence $g_{a/b}$ is smooth.

We calculate Ω_f^1: By the first construction given in 1.2.3, we have

$$\begin{aligned}
\Omega_f^1 &= \left[\Omega_{\underline{f}}^1 \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{\mathrm{grp}})\right] / \left\langle (d(\alpha(m)), -\alpha(m) \otimes m), (0, 1 \otimes f^\flat(n)) \right\rangle \\
&= \mathcal{O}_X \left\langle dx, dy, 1 \otimes e_1, 1 \otimes e_2 \right\rangle / \\
&\qquad \left\langle ydx + xdy, dx - x \cdot (1 \otimes e_1), dy - y \cdot (1 \otimes e_2), 1 \otimes e_1 + 1 \otimes e_2 \right\rangle \\
&= \mathcal{O}_X \left\langle dx, dy, d\log x, d\log y \right\rangle / \\
&\qquad \left\langle ydx + xdy, dx - xd\log x, dy - yd\log y, d\log x + d\log y \right\rangle \\
&= \mathcal{O}_X \left\langle d\log x, d\log y \right\rangle / \left\langle d\log x + d\log y \right\rangle,
\end{aligned}$$

where we write $d\log x := 1 \otimes e_1$ and $d\log y := 1 \otimes e_2$, because $x = a(e_1)$ and $y = a(e_2)$ (observe that $1 \otimes e_1 + 1 \otimes e_2 = 1 \otimes (1, 1) = 0$ because $a(1, 1) = a(\mathrm{diag}(1))$). Hence, Ω_f^1 is indeed a locally free $\mathcal{O}_X$-module of finite type. The log tangent sheaf T_f of f is $\mathcal{O}_X \left\langle x\partial_x, y\partial_y \right\rangle / \left\langle x\partial_x + y\partial_y \right\rangle$. □

1.2.6 Log flatness and log integrality

A strict morphism $f \colon X \to Y$ of log schemes is called *flat* if its underlying morphism of schemes is flat. For the definition of log flatness, we need the fppf-topology on $\underline{X}$. We say that a log scheme X has a property *fppf-locally* if there exists a strict faithfully flat morphism of log schemes $\varphi \colon V \to X$ of finite presentation such that V has that property (where V has the log structure $\alpha_V \cong \varphi^\times \alpha_X$).

1.2.36 Definition ([29, IV.3.5.1])

A morphism of log fine log schemes $f\colon X \to Y$ is *logarithmically flat* if fppf-locally on X there exists a chart $(a\colon P \to \Gamma(X, \mathcal{M}_X), a\colon Q \to \Gamma(Y, \mathcal{M}_Y))$ subordinate to an injective monoid homomorphism $\theta\colon Q \to P$ such that the strict morphism $g_{a/b}\colon X \to \mathbb{A}_b[P]$ in the extended chart diagram

$$\begin{tikzcd} X \arrow[r, "g_{a/b}"] \arrow[dr, "f"'] & \mathbb{A}_b[P] \arrow[r, "g_b[\theta]"] \arrow[d, "\theta_b"] & \mathbb{A}[P] \arrow[d, "\theta"] \\ & Y \arrow[r, "g_b"] & \mathbb{A}[Q] \end{tikzcd}$$

is flat.

The family of log flat morphisms of log fine log schemes is stable under composition and base change (cf. [29, IV.3.5.2]). A strict morphism of log schemes is log flat if and only if it is flat. Log smooth morphisms and log étale morphisms are log flat. In general, however, the underlying morphism of schemes of a log flat morphisms (and, in particular, of a log smooth morphisms) is *not* a flat morphism of schemes (cf. [17, 8.5]).

As Ogus remarks in his lecture notes, the definition of log flatness is not adapted to proving a morphism not to be log flat, because in general for a flat morphism there do exist charts, such that $g_{\alpha,\theta}$ is not flat (cf. [29, IV.3.5.2]). To prove that a morphism of log schemes is not flat, one may use the following theorem by Ogus, which collects nicely criteria for log flatness, log smoothness and log étaleness in a statement about a diagram:

1.2.37 Theorem ([29, IV.3.5.3 & 6])

Let $f\colon X \to Y$ be a morphism of Noetherian log fine log schemes locally of finite presentation. Then f is log flat (respectively, log smooth, log étale) if and only if, for every diagram

$$\begin{tikzcd} X'' \arrow[r, "\eta"] \arrow[dr, "f''"'] & X' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f"] \\ & Y' \arrow[r] & Y, \end{tikzcd}$$

in which the right square is Cartesian, in which η is log étale and in which f'' is strict, the morphism f'' is flat (respectively, smooth, étale).

1.2.38 Corollary ([29, IV.3.5.7])

A morphism of Noetherian log fine log schemes is log étale if and only if it is log flat and formally unramified.

1.2.39 Corollary ([29, IV.3.5.8])

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of Noetherian log fine log schemes. Assume that f is locally of finite presentation and that $g \circ f$ is log flat. If for every standard log point $\operatorname{Spec} k_{\mathbb{N}_0} \to Z$ (for any field k) the base changed morphism f_S is log flat (respectively, log smooth, log étale), then f is log flat / log smooth / log étale.

The connection between log flatness of a morphism and flatness of the underlying morphism is established by the notion of log integrality of a morphism.

1.2.40 Definition ([19, 4.3])

A morphism $f\colon X \to Y$ of log coherent log schemes is called *logarithmically integral* if locally on X there exists a chart $\theta\colon (b\colon Q \to \mathcal{M}_Y) \to (a\colon P \to \mathcal{M}_X)$ with integral monoids P and Q and with $\theta\colon Q \to P$ integral, i. e. such that $\mathbb{Z}[Q] \to \mathbb{Z}[P]$ is flat.

A morphism $f\colon X \to Y$ of log integral (respectively, log fine) log schemes is log integral if and only if for any log integral (log fine) log scheme Y' over Y the fibre product $X \times_Y Y'$ in <u>LSch</u> is a log integral (log fine) log scheme. ([19, 4.3.1])

1.2.41 Proposition ([19, 4.5])

Let $f\colon X \to Y$ be a morphism of log fine log schemes. If f is log flat and log integral, then its underlying morphism $\underline{f}$ of schemes is flat.

Proof: By assumption and 1.2.34, we locally have the extended chart diagram

$$\begin{tikzcd} X \arrow[r, "g_{a/b}"] \arrow[dr, "f"'] & \mathbb{A}_b[P] \arrow[r, "g_b[\theta]"] \arrow[d, "\theta_b"] & \mathbb{A}[P] \arrow[d, "\theta"] \\ & Y \arrow[r, "g_b"] & \mathbb{A}[Q] \end{tikzcd}$$

with $\underline{\theta}$ flat, hence $\underline{\theta_b}$ flat and with $g_{a/b}$ étale, hence $\underline{g_{a/b}}$ flat. In conclusion, $\underline{f}$ is flat. □

1.2.42 Proposition ([19, 4.4])

Let $f\colon X \to Y$ be a morphism of log fine log schemes. If

a) f is strict, or

b) for every point $y \in Y$ the monoid $\overline{\mathcal{M}}_{Y,\overline{y}}$ is zero or generated by one (non-zero) element, where $\overline{y}$ is the separable closure of y,

then f is log integral.

2 Log smooth deformation Theory

This chapter collects definitions and results from the theory of log smooth deformations. Its first section is combined of a brief summary of the first two sections of M. Schlessinger's "Functors of Artin Rings" ([31]) and excerpts from "Obstruction Calculus for Functors of Artin Rings" by B. Fantechi and M. Manetti ([8]). Its second section is a summary of "Functors of log Artin rings" by F. Kato ([18]) and its third section collects facts from the theory of log smooth deformations, which in large part are cited from "Logarithmic structures of Fontaine-Illusie" by K. Kato ([19]) and "Log smooth deformation theory" by F. Kato ([17]).

$\underline{\mathrm{Set}}$ denotes the category of sets. Any set with one element is denoted by $\{*\}$. We regard only covariant functors $F\colon C \to D$. A contravariant functor is hence denoted as a covariant functor $F\colon C^{\mathrm{op}} \to D$, where C^{op} is the opposite category of C.

2.1 Functors of Artin rings

2.1.1 Artin rings

Let T be a complete Noetherian local ring with residue field $k = T/\mathfrak{m}_T$, where $\mathfrak{m}_T$ denotes the maximal ideal of T. Let $\underline{\mathrm{Art}}_T$ denote the category of local Artin algebras over T with residue field k and $\widehat{\underline{\mathrm{Art}}}_T$ the category of complete Noetherian local algebras over T with residue field k (cp. [31, 1]). For every element R of $\underline{\mathrm{Art}}_T$ or $\widehat{\underline{\mathrm{Art}}}_T$, we denote its maximal ideal by $\mathfrak{m}_R$ and the natural projection $R \to k$ by π_R.
Every object in $\widehat{\underline{\mathrm{Art}}}_T$ is naturally a limit of objects in $\underline{\mathrm{Art}}_T$, namely $R = \varprojlim_{n\in\mathbb{N}_0} R/\mathfrak{m}_R^n$. The same is true for homomorphisms. Notice that $\underline{\mathrm{Art}}_T$ is a full subcategory of $\widehat{\underline{\mathrm{Art}}}_T$.

2.1.1 Definition
An *extension* in $\underline{\mathrm{Art}}_T$ (respectively, in $\widehat{\underline{\mathrm{Art}}}_T$) is a short exact sequence

$$e\colon 0 \to J \to B \to A \to 0$$

of T-modules, where $B \to A$ is a surjective homomorphism in $\underline{\mathrm{Art}}_T$ (respectively, in $\widehat{\underline{\mathrm{Art}}}_T$) with kernel J such that $J^2 = 0$, making J an A-module. We will denote the surjective homomorphism in e by $\overline{e}\colon B \to A$ and, by abuse of language, we will call it an *extension*, as well.
A morphism of extensions $\psi\colon e' \to e$ is a commutative diagram

$$\begin{tikzcd}
e'\colon 0 \arrow[r] & J' \arrow[r] \arrow[d, "\psi"] & B' \arrow[r, "\overline{e}'"] \arrow[d, "\psi"] & A' \arrow[r] \arrow[d, "\overline{\psi}"] & 0 \\
e\colon 0 \arrow[r] & J \arrow[r] & B \arrow[r, "\overline{e}"] & A \arrow[r] & 0.
\end{tikzcd}$$

The category of extensions in $\underline{\mathrm{Art}}_T$ is denoted by $\underline{\mathrm{Ext}}_T$.

An extension e is called a *small extension* if $\mathfrak{m}_B J = 0$ and a *principal small extension* if moreover J is a principal ideal, i. e. $\dim_k J \leq 1$. The full subcategories of small and principal small extensions in $\underline{\mathrm{Art}}_T$ are denoted by $\underline{\mathrm{sExt}}_T$ and $\underline{\mathrm{psExt}}_T$, respectively.

Every surjective homomorphism $B \to A$ in $\underline{\mathrm{Art}}_T$ allows a finite factorisation into principal small extensions.

2.1.2 Proposition ([31, 2.5.i])
A homomorphism $R \to S$ in $\widehat{\underline{\mathrm{Art}}}_T$ is smooth if and only if S is a power series ring over R.

2.1.2 Functors of Artin rings

2.1.3 Definition
A *functor of Artin rings* over T is a functor $F\colon \underline{\mathrm{Art}}_T \to \underline{\mathrm{Set}}$ with $F(k) = \{*\}$. The unique extension of F to a functor $\hat{F}\colon \widehat{\underline{\mathrm{Art}}}_T \to \underline{\mathrm{Set}}$ with $\hat{F} = F$ on $\underline{\mathrm{Art}}_T$ is given by $\hat{F}(R) = \varprojlim_{n\in\mathbb{N}_0} F(R/\mathfrak{m}_R^n)$ and is called the *completion* of F. A *morphism of functors of Artin rings* is a morphism of functors. We denote the category of functors of Artin rings over T and their morphisms by $\underline{\mathrm{Fun}}_T$.

For any $R \in \widehat{\underline{\mathrm{Art}}}_T$, the functor $\mathrm{Hom}_{\widehat{\underline{\mathrm{Art}}}_T}(R, \cdot)\colon \underline{\mathrm{Art}}_T \to \underline{\mathrm{Set}}$ is denoted by h_R. The constant functor of Artin rings $*\colon \underline{\mathrm{Art}}_T \to \underline{\mathrm{Set}}$, $*(A) = \{*\}$, which is equal to h_T, is the final object in $\underline{\mathrm{Fun}}_T$.

2.1.4 Remark
In the case that $T = k$, any functor of Artin rings $F\colon \underline{\mathrm{Art}}_T \to \underline{\mathrm{Set}}$ canonically lifts along the forgetful functor $\underline{\mathrm{Set}}_* \to \underline{\mathrm{Set}}$ to a functor $F\colon \underline{\mathrm{Art}}_T \to \underline{\mathrm{Set}}_*$ with the pointed sets as codomain . This is due to the fact that for each $R \in \widehat{\underline{\mathrm{Art}}}_k$ the canonical homomorphism $k \to R$ defines the image of $* \in F(k)$ in $F(R)$, called the trivial element of $F(R)$ and usually also denoted by $* \in F(R)$.
In general however, there is no canonical homomorphism $k \to T$ (such a homomorphism might not exist at all), so one has to be aware that $F(R)$ might be the empty set and, even if not, does not contain a canonical element.

Let V be a finite-dimensional k-vector-space and $A \in \underline{\mathrm{Art}}_T$. We write $A[V]^0$ for the T-algebra $T \to A \to A[V] = A \oplus V$, with $a \cdot v = \pi_A(a)v$ and $v \cdot w = 0$ for $a \in A$, $v, w \in V$. If V is the one-dimensional k-vector-space with generator ε, we write $A[\varepsilon]^0$ for $A[V]^0$.

2.1.5 Definition
The set $t_F = F(k[\varepsilon]^0)$ is called the *(pointed) tangent set* of F (at its unique point $* \in F(k)$).

We will denote the principal small extension $0 \to (\varepsilon) \to k[\varepsilon]^0 \to k \to 0$ in $\underline{\mathrm{Art}}_T$ by ε^0.

2.1.6 Definition
An F*-couple* is a pair (A, ξ) with $A \in \underline{\mathrm{Art}}_T$ and $\xi \in F(A)$. An F*-pro-couple* is an $\hat{F}$-couple, i. e. a pair (R, ξ) with $R \in \widehat{\underline{\mathrm{Art}}}_T$ and $\xi \in \hat{F}(R)$. A *morphism* $u\colon (A, \xi) \to (A', \xi')$ *of couples*

(respectively, of pro-couples) is a homomorphism $u\colon A \to B$ in $\underline{\mathrm{Art}}_T$ (respectively, in $\widehat{\underline{\mathrm{Art}}}_T$) such that $F(u)(\xi) = \xi'$ (respectively, such that $\hat{F}(u)(\xi) = \xi'$).

By the Yoneda lemma, specifying an element $\xi \in \hat{F}(R)$ is equivalent to giving a morphism of functors, $\xi\colon h_R \to F$. We will identify the element ξ and this morphism.

2.1.7 Definition

For any morphism $f\colon F \to G$ of functors of Artin rings a functor $\breve{F}_f\colon \underline{\mathrm{sExt}}_T \to \underline{\mathrm{Set}}$ is defined by setting $\breve{F}_f(e) := F(A) \times_{G(A)} G(B)$ for any small extension $e\colon B \to A$.

As remarked in [8, 2.14], for the principal small extension ε^0 we have $\breve{F}_f(\varepsilon^0) = t_G$, independently of f and F. If $G = *$, then $\breve{F}_f(e) = F(A)$ and we will write $\breve{F} := \breve{F}_f$ in this case.

2.1.8 Definition

A morphism $f\colon F \to G$ of functors of Artin rings is called *smooth* if for any small extension e the map $F(B) \to \breve{F}_f(e)$, $b \mapsto (F(\overline{e})(b), f(b))$, is surjective.

A morphism $f\colon F \to G$ of functors of Artin rings is called *étale* if it is smooth and if its differential $t_f = f(k[\varepsilon]^0)\colon t_F \to t_G$ is a bijection.

A functor of Artin rings F is called *smooth* if the canonical morphism $F \to *$ is smooth. This is equivalent to $F(\overline{e})\colon F(B) \to F(A)$ being surjective for all small extensions $\overline{e}\colon B \to A$ in $\underline{\mathrm{Art}}_T$.

2.1.9 Proposition ([31, 2.5], [32, 2.2.5])

Let $R \to S$ be a homomorphism in $\widehat{\underline{\mathrm{Art}}}_T$.

a) $h_S \to h_R$ is smooth if and only if $R \to S$ is smooth.

Let $f\colon F \to G$ and $g\colon G \to H$ be two morphisms of functors of Artin rings.

b) If f is smooth, then f and $\hat{f}\colon \hat{F} \to \hat{G}$ are both surjective.

c) If f and g are smooth, then $g \circ f$ is smooth.

d) If f is smooth, then $g \circ f$ is smooth if and only if g is smooth.

2.1.10 Definition

Let F be a functor of Artin rings and (R, ξ) an F-pro-couple. We call (R, ξ)

a) a *versal element* for F if $\xi\colon h_R \to F$ is smooth;

b) a *hull* of F or a *semi-universal element* for F if $\xi\colon h_R \to F$ is étale;

c) a *pro-representative* for F or a *universal element* for F if $\xi\colon h_R \to F$ is an isomorphism. In this case we say that h_R *pro-represents* F.

Observe that if F is a functor of Artin rings possessing a versal element, a hull or a pro-representative (R, ξ), then by 2.1.9 d) F is smooth if and only if h_R is.

Let F be a functor of Artin rings, let $A' \to A$ and $A'' \to A$ be homomorphisms in $\underline{\mathrm{Art}}_T$ and consider the canonical map

$$\Phi\colon F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'').$$

2.1.11 Definition

A functor of Artin rings F is called a *functor with good deformation theory* or *gdt functor* if it has the properties

H$_1$) Φ is a surjection, whenever $A'' \to A$ is a small extension;

H$_2$) Φ is a bijection, when $A = k$, $A'' = k[\varepsilon]^0$.

A functor of Artin rings possessing a hull (or a pro-representative) is a gdt functor (cf. [31, 2.11]). For a general treatment of fibre products of Artin rings and Noetherian rings, cf. [30].

A gdt functor F has the property that for any two finite dimensional vector spaces V and W the canonical map $F(k[V]^0 \times_k k[W]^0) \to F(k[V]^0) \times F(k[W]^0)$ is a bijection. Hence, $t_F = F(k[\varepsilon]^0)$ carries a natural vector space structure with $*$ as its zero element and is called the *tangent space* of F (cp. [31, 2.10]); moreover, $F(k[V]^0) \cong t_F \otimes_k V$ as k-vector-spaces for any finite dimensional vector space V.

2.1.12 Theorem ([31, 2.11])

A gdt functor F possesses a hull if and only if

H$_3$) $\dim_k(t_F) < \infty$.

Moreover, F is pro-representable if and only if F has the additional property

H$_4$) Φ is a bijection, when $A'' = A'$ and whenever $A' \to A$ is a small extension.

Given a T-algebra structure on $k[\![x]\!]$, let A_N denote the ring $k[x]/x^{N+1}$, $N \in \mathbb{N}_0$, together with its T-algebra structure as a quotient of $k[\![x]\!]$.

2.1.13 Definition

A morphism of functors of Artin rings $f\colon F \to G$ is called *curvilinearly smooth of order* N_0 if for all T-algebra structures on $k[\![x]\!]$ and all $N \in \mathbb{N}_0$ with $N \geq N_0$ the map $F(A_{N+1}) \to \breve{F}_f(e_N)$ is surjective, where e_N is the principal small extension

$$e_N\colon 0 \to k \xrightarrow{x^{N+1}} A_{N+1} \to A_N \to 0.$$

If f is curvilinearly smooth of order 0, we say that it is *curvilinearly smooth.*

A functor F is called *curvilinearly smooth* (*of order* N_0) if the canonical morphism $F \to *$ is curvilinearly smooth (of order N_0).

2.1.14 Remark
If $T = k$ is a field, then there is a unique k-algebra structure on $k[\![x]\!]$.

In B. Fantechis and M. Manettis paper [8] the following proposition is proven, the equivalence a) $\Leftrightarrow$ b) of which has been proven earlier by Y. Kawamata in [20].

2.1.15 Proposition ([8, 5.6 & 6.4])
Let F be a gdt functor over a field k. Then the following are equivalent:

a) F is smooth;

b) F is curvilinearly smooth;

c) F is curvilinearly smooth of order N_0 for an $N_0 \in \mathbb{N}_0$.

We are going to prove such a statement in the case that the base ring T is the ring $k[\![t]\!]$.

2.1.16 Lemma (cp. [8, 5.6])
Let $R \in \widehat{\underline{\mathrm{Art}}}_{k[\![t]\!]}$. Then the following are equivalent:

a) h_R is smooth, i. e. R is a power series algebra over $k[\![t]\!]$;

b) h_R is curvilinearly smooth;

c) h_R is curvilinearly smooth of order N_0 for some $N_0 \in \mathbb{N}_0$.

Proof: The implications a) $\Rightarrow$ b) and b) $\Rightarrow$ c) are clear. So assume that h_R is not smooth. Then $R = P/I$ for some power series algebra $P = k[\![t]\!][\![x_1, \ldots, x_n]\!]$ over $k[\![t]\!]$ and an ideal $0 \neq I \subset tP + \mathfrak{x}^2$, where $\mathfrak{x} = (x_1, \ldots, x_n) \subset P$.
Let I be generated by elements $f_1, \ldots, f_r$ of P. Then consider the set M of all monomials in the variables $t, x_1, \ldots, x_n$ appearing in the f_j with non-zero k-coefficient.
First, assume that $I = tP$. This is the case if and only if the monomial $t \in M$ and if all other monomials are divisible by t. Without loss of generality, we may assume that $f_1 = t$ is the only generator (hence, $r = 1$). Then $R = k[\![t]\!]/(t)[\![x_1, \ldots, x_n]\!]$ with the obvious $k[\![t]\!]$-algebra structure. Give to the ring $k[\![x]\!]$ the algebra structure $k[\![t]\!] \to k[\![x]\!]$, $t \mapsto x^{N_0}$. Then the homomorphism $\psi\colon k[\![t]\!][\![x_1, \ldots, x_n]\!] \to k[\![x]\!]$, $t \mapsto x^{N_0}$ induces an element $\phi \in h_R(k[x]/x^{N_0})$, because $\psi(tP) \subset (x^{N_0})$, which does not lift to an element of $h_R(k[x]/x^{N_0+1})$, because this lift would, by the definition of the algebra structure, have to map t to the non-zero element $x^{N_0} \in k[x]/x^{N_0+1}$, which is not possible.
Then, assume that $t \notin M$, but all monomials are divisible by t. Let $C \subset \mathbb{N}_0^{n+1}$ be the set of all multi-indices $J = (j_0, \ldots, j_n)$ such that the monomial $t^{j_0} x_1^{j_1} \cdot \ldots \cdot x_n^{j_n}$ appears in M. The set C is contained in $\{J \mid \sum_i j_i \geq 2\}$, because, by assumption, each monomial in M has at least degree 2. Now we may follow the proof of Fantechi and Manetti given in [8, 5.6]: There exist rational positive numbers $a_0, \ldots, a_n, b$ with $a_i < b$ such that $C \subset \left\{J \in \mathbb{N}_0^{n+1} \mid \sum_i a_i j_i \geq b\right\}$ and such that $C \cap \{J \mid \sum_i a_i j_i = b\} = \{J'\}$ for one particular

$J' = (j'_0, \dots, j'_n)$. Choose $N \in \mathbb{N}_0$ large enough such that $A_i := a_i N$ and $B := bN$ are all integers and $B \geq N_0$. Then $C \subset \{J \mid \sum_i A_i j_i \geq B\}$ and $C \cap \{J \mid \sum_i A_i j_i = B\} = \{J'\}$.

We give $k[\![x]\!]$ the algebra structure $k[\![t]\!] \to k[\![x]\!]$, $t \mapsto x^{A_0}$. The algebra homomorphism $\psi\colon k[\![t]\!][\![x_1, \dots, x_n]\!] \to k[\![x]\!]$, $t \mapsto x^{A_0}$, $x_i \mapsto x^{A_i}$, maps $t^{j'_0} x_1^{j'_1} \cdot \ldots \cdot x_n^{j'_n} \mapsto x^B$ and $\psi(t^{j_0} x_1^{j_1} \cdot \ldots \cdot x_n^{j_n}) \in (x^{B+1})$ for all $J \in C$, $J \neq J'$. Hence, $\psi(I) \subset (t^B)$, because the f_k generate I. Therefore, ψ induces an element $\phi \in h_R(k[x]/x^B)$. This element does not lift to an element of $h_R(k[x]/x^{B+1})$, because $\psi(I)$ is not contained in (x^{B+1}).

Finally, assume that $t \notin M$ and there exist monomials not divisible by t. Let $M_0 \subset M$ be the set of all monomials with non-zero k-coefficient and not divisible by t. Let $C' \subset \mathbb{N}_0^n$ be the set of all multi-indices $J = (j_1, \dots, j_n)$ such that the monomial $x_1^{j_1} \cdot \ldots \cdot x_n^{j_n}$ appears in M_0. The set C_0 is contained in $\{J \in \mathbb{N}_0^n \mid \sum_i j_i \geq 2\}$ and we may again find natural numbers A_i and $B$with $A_i < B$ and $B \geq N_0$ such that $C' \subset \{J \mid \sum_i A_i j_i \geq B\}$ and $C \cap \{J \mid \sum_i A_i j_i = B\} = \{J'\}$.

We give $k[\![x]\!]$ the algebra structure $k[\![t]\!] \to k[\![x]\!]$, $t \mapsto 0$. The algebra homomorphism $\psi\colon k[\![t]\!][\![x_1, \dots, x_n]\!] \to k[\![x]\!]$, $t \mapsto 0$, $x_i \mapsto x^{A_i}$, maps $x_1^{j'_1} \cdot \ldots \cdot x_n^{j'_n} \mapsto x^B$ and $\psi(t^{j_0} x_1^{j_1} \cdot \ldots \cdot x_n^{j_n}) \in (x^{B+1})$ for all $J \in C'$, $J \neq J'$. Hence, $\psi(I) \subset (t^B)$. Therefore, ψ induces an element $\phi \in h_R(k[x]/x^B)$. This element does not lift to an element of $h_R(k[x]/x^{B+1})$, because $\psi(I)$ is not contained in (x^{B+1}). $\square$

2.1.17 Remark

Observe, that in the last two cases of the proof it is essential that all monomial have at least degree 2. It is only therefore possible to construct the respective ψ in such a way that these monomials are mapped to zero, while their factors of degree one are mapped to non-zero elements.

2.1.18 Proposition

Let F be a functor of Artin rings over $k[\![t]\!]$ possessing a hull (R, ξ). Then the following are equivalent:

a) F is smooth;

b) F is curvilinearly smooth;

c) F is curvilinearly smooth of order N_0 for an $N_0 \in \mathbb{N}_0$.

Proof: The implications a) $\Rightarrow$ b) and b) $\Rightarrow$ c) are clear. For any T-algebra structure on $k[\![x]\!]$ the canonical map $h_R(A_{N+1}) \to h_R(A_N)$ factors as $h_R(A_{N+1}) \to h_{R\xi}(e_N) = h_R(A_N) \times_{F(A_N)} F(A_{N+1})$, which is surjective by assumption, followed by the projection to $h_R(A_N)$. The curvilinear smoothness of order N_0 of F therefore implies the curvilinear smoothness of order N_0 of h_R. By 2.1.16, h_R is smooth, which is the case if and only if F is smooth. $\square$

2.1.3 Obstruction theory of functors of Artin rings

Fantechi and Manetti give a definition of a linear obstruction theory for the case $T = k$ in [8, 3.1, 4.1 & 4.7]. The following definition is modelled accordingly.

Consider the following functors:

- $\breve{A}\colon \underline{\mathrm{Ext}}_T \to \underline{\mathrm{Art}}_T$, $e \mapsto A$;
- $\breve{B}\colon \underline{\mathrm{Ext}}_T \to \underline{\mathrm{Art}}_T$, $e \mapsto B$;
- $\breve{J}\colon \underline{\mathrm{Ext}}_T \to \underline{\mathrm{Mod}}_k$, $e \mapsto J$;
- $\underline{\mathrm{Mod}}_{\breve{A}}\colon \underline{\mathrm{Ext}}_T \to \underline{\mathrm{Cat}}$, $e \mapsto \underline{\mathrm{Mod}}_{\breve{A}(e)} = \underline{\mathrm{Mod}}_A$,

where in the first three cases $e\colon 0 \to J \to B \to A \to 0$ and where $\underline{\mathrm{Cat}}$ denotes the category of small categories.

Recall that for a morphism $f\colon F \to G$ of functors of Artin rings we have defined the functor $\breve{F}_f\colon \underline{\mathrm{Ext}}_T \to \underline{\mathrm{Set}}$ by $\breve{F}_f(e) = F(\breve{A}(e)) \times_{G(\breve{A}(e))} G(\breve{B}(e))$ and that if $G = *$, we have $\breve{F} := \breve{F}_f = F \circ \breve{A}$.

2.1.19 Definition (cp. [8, 3.1])
Let $f\colon F \to G$ be a morphism of functors of Artin rings, with G a gdt functor. A *linear small obstruction theory* (H_0, o) for f consists of a k-vector-space H_0 called the *(linear) small obstruction space* and a morphism of functors

$$o\colon \breve{F}_f \to H_0 \otimes_k \breve{J}(\,\cdot\,)$$

such that $o_{\varepsilon^0}(*) = 0$, where ε^0 denotes the extension $0 \to (\varepsilon) \to k[\varepsilon]^0 \to k \to 0$ and where $*$ denotes the unique element in $F(k)$.

A *linear small obstruction theory* for a functor of Artin rings F is a linear small obstruction theory for the morphism $F \to *$.

Given a linear small obstruction theory (H_0, o) for $f\colon F \to G$ and a small extension $e\colon 0 \to J \to B \to A \to 0$, for any element $x \in \breve{F}_f(e) = F(A) \times_{G(A)} G(B)$ contained in the image of $F(B)$ we have $o_e(x) = 0 \in H_0 \otimes_k J$ (cf. [Fantechi/Manetti, Prop 3.3]).

2.1.20 Definition (cp. [8, 4.1])
A linear small obstruction theory (H_0, o) for $f\colon F \to G$ is called *complete* if the converse holds, i. e. if for any small extension $e\colon 0 \to J \to B \to A \to 0$ an element $x \in \breve{F}_f(e) = F(A) \times_{G(A)} G(B)$ is contained in the image of $F(B)$ if and only if $o_e(x) = 0 \in H_0 \otimes_k J$.

It it enough to check this condition for principal small extensions (cf. [8, 4.2]).

We extend this definition to arbitrary extensions as follows:

2.1.21 Definition

An *obstruction theory* (H, o) *for* f consist of

a) a morphism of functors $H\colon \breve{F}_f \to \underline{\mathrm{Mod}}_{\breve{A}}$, i. e. for each $e \in \underline{\mathrm{Ext}}_T$ a map $H_e\colon \breve{F}_f(e) \to \underline{\mathrm{Mod}}_{\breve{A}(e)}$, $x \mapsto H_e(x) \in \underline{\mathrm{Mod}}_{\breve{A}(e)}$, compatible with morphisms $\psi\colon e' \to e$ of extensions;

b) the induced functor $O\colon \underline{\mathrm{Ext}}_T \to \underline{\mathrm{Set}}$ defined by $O_e := \coprod_{x\in \breve{F}_f(e)} H_e(x)$ together with the natural projection morphism (of functors) $\pi\colon O \to \breve{F}_f$;

c) a section (morphism of functors) $o\colon \breve{F}_f \to O$ of π, i. e. such that $\pi \circ o = id_{\breve{F}_f}$.

These have to fulfil the additional condition that $o_{\varepsilon^0}(*) = 0 \in H_{\varepsilon^0}(*)$. We call $\pi\colon O \to \breve{F}_f$ the *obstruction bundle for* f and the section o the *obstruction morphism.* In particular, for an extension e and an element $x \in \breve{F}_f(e)$ we call $o_e(x) \in H_e(x)$ the *obstruction of lifting* x *along* e and $H_e(x) \in \underline{\mathrm{Mod}}_{\breve{A}(e)}$ the *obstruction space of lifting* x *along* e.

2.1.22 Definition

An obstruction theory (H, o) for f is called *linear* if for the k-vector-space $H_0 := H_{\varepsilon^0}(*)$ and for every small extension e and every $x \in \breve{F}_f(e)$ there is a natural isomorphism $H_e(x) \cong H_0 \otimes_k \breve{J}(e)$, where the right side does not depend on x. H_0 is then called the *(linear) small obstruction space of* f.

2.1.23 Definition

An obstruction theory (H, o) for $f\colon F \to G$ is called *complete* if for any extension $e\colon 0 \to J \to B \to A \to 0$ an element $x \in \breve{F}_f(e) = F(A) \times_{G(A)} G(B)$ is contained in the image of $F(B)$ if and only if $o_e(x) = 0 \in H_e(x)$. In this case we call $o_e(x)$ the *complete obstruction of lifting* x *along* e.

To have a general description of such objects we make the following definition.

2.1.24 Definition

Let $F\colon C \to D$ be a functor to a category D where inverse images under morphisms are defined (e. g. $\underline{\mathrm{Set}}$, $\underline{\mathrm{Grp}}$, etc.). A *bundle of functors over* F is a functor $O\colon C \to D$ and an epimorphism of functors $\pi\colon O \to F$ such that there exists a section $o\colon F \to O$ (with $\pi \circ o = id_F$). It is denoted by (O, π, o).

Let $G\colon C \to \underline{\mathrm{Cat}}$ be a functor, which assigns to each $e \in C$ a subcategory of D. If setting $H(e)(x) := \pi(e)^{-1}(x)$ defines a morphism of functors $H\colon F \to G$, then we call π an H*-trivial* G*-bundle.* In this case, for each $e \in C$ we have $O(e) = \coprod_{x\in F(e)} H(e)(x)$ and we write $O = \coprod_F H$ for this property. If G is the trivial functor $\underline{D}\colon C \to \underline{\mathrm{Cat}}$, $e \mapsto D$ for all e, then we simply call π an H*-trivial-bundle.* It is clear that every H-trivial G-bundle is an H-trivial bundle.

In this sense an obstruction theory (H, o) for f is the same thing as an H-trivial $\underline{\mathrm{Mod}}_{\breve{A}}$-bundle $\pi\colon O = \coprod_{\breve{F}_f} H \to \breve{F}_f$ over $\breve{F}_f$ together with a section o.

2.2 Functors of log Artin rings

Recall from the first chapter (cf. 1.2.3) that a prelog ring $a\colon Q \to A$ is a monoid homomorphisms with A a ring. It is called a log ring, if moreover a induces a group isomorphism $a^{-1}(A^\times) \cong A^\times$.

2.2.1 Definition
We call a prelog ring $a\colon Q \to A$ with A a local ring *precise* if Q is a sharp monoid and if the image of its maximal ideal $a(\mathfrak{m}_Q)$ is contained in the maximal ideal of A.

2.2.1 Log Artin rings

We fix the following data to define the category of log Artin rings in the sense of F. Kato ([18]). Let

$$(\pi_T, \varrho)\colon \begin{array}{ccc} Q' & \xrightarrow{\varrho} & Q \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t_0} \\ T & \xrightarrow{\pi_T} & k \end{array}$$

be a prelog ring homomorphism, which satisfies the following conditions:

a) $t\colon Q' \to T$ is a precise complete Noetherian prelog ring and $t_0\colon Q \to k$ is a precise prelog field with Q and Q' toric monoids;

b) $\pi_T\colon T \to k$ is the natural projection from T to its residue field $k = T/\mathfrak{m}_T$;

c) $\varrho\colon Q' \to Q$ is an injective monoid homomorphism such that $Q \setminus \varrho(Q')$ is an ideal in Q and such that the cokernel of $\varrho^{\mathrm{grp}}\colon Q'^{\mathrm{grp}} \to Q^{\mathrm{grp}}$ is torsion-free.

Denote by $\mathcal{T}\colon M_\mathcal{T} \to T$ and $\kappa\colon M_\kappa \to k$ the associated log rings of t and t_0, respectively, and set $\varrho_\mathcal{T} := \varrho^{>}\colon M_\mathcal{T} \to M_\kappa$.

2.2.2 Definition
We denote by $\underline{\mathrm{LArt}}_\mathcal{T}$ (respectively, $\widehat{\underline{\mathrm{LArt}}}_\mathcal{T}$) the category described in what follows:

Its objects are log ring homomorphisms

$$(\varphi_A, \sigma_\mathcal{A})\colon \begin{array}{ccc} M_\mathcal{T} & \xrightarrow{\sigma_\mathcal{A}} & M_\mathcal{A} \\ \downarrow{\scriptstyle \mathcal{T}} & & \downarrow{\scriptstyle \mathcal{A}} \\ T & \xrightarrow{\varphi_A} & A \end{array}$$

together with a monoid homomorphism $\varrho_\mathcal{A}\colon M_\mathcal{A} \to M_\kappa$ such that

a) $\varphi_A \colon T \to A$ is an object in $\underline{\mathrm{Art}}_T$ (respectively, in $\widehat{\underline{\mathrm{Art}}}_T$);

b) the diagram

$$\begin{tikzcd}
M_{\mathcal{T}} \arrow[r, "\sigma_{\mathcal{A}}"'] \arrow[d, "\mathcal{T}"] \arrow[rr, bend left, "\rho_{\mathcal{T}}"] & M_{\mathcal{A}} \arrow[r, "\varrho_{\mathcal{A}}"'] \arrow[d, "\mathcal{A}"] & M_\kappa \arrow[d, "\kappa"] \\
T \arrow[r, "\varphi_A"] \arrow[rr, bend right, "\pi_T"'] & A \arrow[r, "\pi_A"] & k
\end{tikzcd}$$

is commutative;

c) $M_{\mathcal{A}} \otimes_{A^\times} k^\times \cong M_\kappa = Q \oplus k^\times$.

A homomorphism $\psi = (\varphi, \psi) \colon \mathcal{A} \to \mathcal{B}$ in $\underline{\mathrm{LArt}}_{\mathcal{T}}$ (respectively, $\widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}}$) is a homomorphism of log rings which is compatible with the above requirements, hence a commutative diagram

$$\begin{tikzcd}
M_{\mathcal{T}} \arrow[rr, "\sigma_{\mathcal{B}}"'] \arrow[dr, "\sigma_{\mathcal{A}}"'] \arrow[dd, "\mathcal{T}"] \arrow[rrr, bend left, "\rho_{\mathcal{T}}", to=2-4] & & M_{\mathcal{B}} \arrow[dr, "\varrho_{\mathcal{B}}"] \arrow[dd, "\mathcal{B}"] & \\
& M_{\mathcal{A}} \arrow[ur, "\psi"] \arrow[rr, "\varrho_{\mathcal{A}}"'] \arrow[dd, "\mathcal{A}"] & & M_\kappa \arrow[dd, "\kappa"] \\
T \arrow[rr, "\varphi_B"] \arrow[dr, "\varphi_A"'] \arrow[rrr, bend right=60, "\pi_T"', to=4-4] & & B \arrow[dr, "\pi_B"] & \\
& A \arrow[ur, "\varphi"] \arrow[rr, "\pi_A"] & & k
\end{tikzcd}$$

An extension (respectively, a small extension, a principal small extension) in $\underline{\mathrm{LArt}}_{\mathcal{T}}$ is a homomorphism $\psi \colon \mathcal{A}' \to \mathcal{A}$ the underlying homomorphism of which is an extension (respectively, a small extension, a principal small extension) in $\underline{\mathrm{Art}}_T$.

Just as $\underline{\mathrm{Art}}_T$ is a full subcategory of $\widehat{\underline{\mathrm{Art}}}_T$, the category $\underline{\mathrm{LArt}}_{\mathcal{T}}$ is a full subcategory of $\widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}}$. Every object (respectively, homomorphism) of $\widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}}$ is canonically the limit of objects (respectively, homomorphisms) in $\underline{\mathrm{LArt}}_{\mathcal{T}}$.

2.2.3 Definition

We define a functor $\overline{\cdot} \colon \widehat{\underline{\mathrm{Art}}}_T \to \widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}}$ by associating to each $R \in \widehat{\underline{\mathrm{Art}}}_T$ a specific object $\overline{R} \in \widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}}$ defined to be the log ring associated to the prelog ring $\overline{r} \colon Q \to R$, mapping $q \in \varrho(Q')$ to $\varphi_R(t(q))$ and $q \in Q \setminus \varrho(Q')$ to zero.

2.2.4 Remark

Note that if $\varrho \colon Q' \to Q$ is not an isomorphism, then $\mathcal{T}$ is itself *not* an object of $\widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}}$, because $M_{\mathcal{T}} \otimes_{T^\times} k^\times \ncong M_\kappa$. If ϱ is an isomorphism, then $\mathcal{T} \cong \overline{T} \in \widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}}$.

Fibred products exist in both categories $\underline{\mathrm{LArt}}_{\mathcal{T}}$ and $\widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}}$ and may be constructed as follows: For $\mathcal{A}\colon M_{\mathcal{A}} \to A$, $\mathcal{A}'\colon M_{\mathcal{A}'} \to A'$ and $\mathcal{A}''\colon M_{\mathcal{A}''} \to A''$ sitting in a diagram

$$\mathcal{A}' \to \mathcal{A} \leftarrow \mathcal{A}''$$

the fibred product is the fibred product of monoid homomorphisms

$$\mathcal{A}' \times_{\mathcal{A}} \mathcal{A}''\colon M_{\mathcal{A}'} \times_{M_{\mathcal{A}}} M_{\mathcal{A}''} \to A' \times_A A''.$$

2.2.5 Definition

We denote by $v\colon \widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}} \to \widehat{\underline{\mathrm{Art}}}_T$ the functor which forgets the log ring structure and is therefore left-adjoint to $\overline{\cdot}$.

By means of this functor, $\widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}}$ is co-fibred over $\widehat{\underline{\mathrm{Art}}}_T$. We denote the *fibre* of v over $R \in \widehat{\underline{\mathrm{Art}}}_T$ by

$$v^{-1}(R) = \{\mathcal{R}\colon M_{\mathcal{R}} \to R\}.$$

Let $A \in \underline{\mathrm{Art}}_T$. For $\mathcal{A}, \mathcal{A}' \in v^{-1}(A)$ we write

$$\mathrm{Isom}_A(\mathcal{A}, \mathcal{A}') := \{\psi\colon \mathcal{A} \to \mathcal{A}' \mid v(\psi) = id_A\} \quad \text{and} \quad \mathrm{Aut}_A(\mathcal{A}) := \mathrm{Isom}_A(\mathcal{A}, \mathcal{A})$$

and call these the *group of isomorphisms from $\mathcal{A}$ to $\mathcal{A}'$ over A* and the *group of automorphisms of $\mathcal{A}$ over A*, respectively.

The following proposition collects the results of section 2 in “Functors of log Artin rings” by F. Kato:

2.2.6 Proposition ([18, 2.2-2.10])

a) For any object R in $\widehat{\underline{\mathrm{Art}}}_T$ there exists, up to isomorphism, exactly one object $\mathcal{R}$ in $\widehat{\underline{\mathrm{LArt}}}_T$ with $v(\mathcal{R}) = R$, namely $\mathcal{R} \cong \overline{R}$.

b) For any homomorphism in $\widehat{\underline{\mathrm{LArt}}}_T$, the corresponding morphism of log schemes is strict.

c) For any $A \in \underline{\mathrm{Art}}_T$ and $\mathcal{A}, \mathcal{A}' \in v^{-1}(A)$ the set $\mathrm{Isom}_A(\mathcal{A}, \mathcal{A}')$ is non-empty. We have

$$\mathrm{Aut}_A(\mathcal{A}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{coker}(\varrho^{\mathrm{grp}}\colon Q'^{\mathrm{grp}} \to Q^{\mathrm{grp}}), \ker(\pi_A^{\times}\colon A^{\times} \to k^{\times})).$$

d) For any $\kappa, \kappa' \in v^{-1}(k)$ the set $\mathrm{Isom}_k(\kappa, \kappa')$ consists of one element.

In particular, for any $\mathcal{A} \in \underline{\mathrm{LArt}}_{\mathcal{T}}$, we have a strict closed embedding $\mathrm{Spec}\,\kappa \to \mathrm{Spec}\,\mathcal{A}$ over $\mathrm{Spec}\,\mathcal{T}$.

2.2.2 Functors of log Artin rings

For any log ring $\mathcal{R}$ and any ideal $I \subset R$ we write $\mathcal{R}/I$ for the log ring $M_{\mathcal{R}} \otimes_{R^{\times}} (R/I)^{\times} \to R/I$. This makes $\mathrm{Spec}(\mathcal{R}/I) \to \mathrm{Spec}\,\mathcal{R}$ a strict closed immersion of affine log schemes.

2.2.7 Definition

A *functor of log Artin rings* is a functor $F\colon \underline{\mathrm{LArt}}_{\mathcal{T}} \to \underline{\mathrm{Set}}$ with $F(\kappa) = \{*\}$.

The unique extension of F to a functor $\hat{F}\colon \widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}} \to \underline{\mathrm{Set}}$ with $\hat{F} = F$ on $\underline{\mathrm{LArt}}_{\mathcal{T}}$ is given by $\hat{F}(\mathcal{R}) = \varprojlim_{n\in\mathbb{N}_0} F(\mathcal{R}/\mathfrak{m}_R^n)$ and is called the *completion* of F.

2.2.8 Definition

Let V be a finite-dimensional k-vector-space, $\mathcal{A} \in \underline{\mathrm{LArt}}_{\mathcal{T}}$ and $A = v(\mathcal{A})$. We define $\mathcal{A}[V]^0$ to be the log Artin ring $M_{\mathcal{A}} \otimes_{A^\times} A[V]^\times \to A[V]$ induced by the inclusion $i\colon A \to A[V]^0$ in $\underline{\mathrm{Art}}_T$ (cp. 2.1.5). We write $\mathcal{A}[\varepsilon]^0$ when V is the one-dimensional k-vector-space with generator ε. The set $t_F = F(\kappa[\varepsilon]^0)$ is called the *(pointed) tangent set* of F (at its unique geometric point $* \in F(\kappa)$).

2.2.9 Remark

We may write the log ring $\mathcal{A}[V]^0$ as $M_{\mathcal{A}} \oplus (V, +) \to A[V]$, mapping $(m, v) \mapsto \mathcal{A}(m) \oplus v$.

For $\mathcal{R} \in \widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}}$ we define the functor $h_{\mathcal{R}} := \mathrm{Hom}_{\widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}}}(\mathcal{R}, \,\cdot\,)$. By means of the Yoneda Lemma, we identify $\hat{F}(\mathcal{R})$ with the set $\mathrm{Hom}(h_{\mathcal{R}}, F)$.

We transfer the definitions 2.1.6, 2.1.8 and 2.1.10 (of the terms *pro-couple*, *smoothness* and shades of *versality*) literally to functors of log Artin rings by replacing the categories $\underline{\mathrm{Art}}_T$ and $\widehat{\underline{\mathrm{Art}}}_T$ by $\underline{\mathrm{LArt}}_{\mathcal{T}}$ and $\widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}}$, respectively.

2.2.10 Definition

To a functor of log Artin rings F we associate a functor of Artin rings $v_*F\colon \underline{\mathrm{Art}}_T \to \underline{\mathrm{Set}}$ by defining

$$v_*F(A) := \big\{(\mathcal{A}, \xi) \,\big|\, \mathcal{A} \in v^{-1}(A),\ \xi \in F(\mathcal{A})\big\} / \sim,$$

where $(\mathcal{A}, \xi) \sim (\mathcal{A}', \xi')$ if and only if there exists a $\varphi \in \mathrm{Isom}_A(\mathcal{A}, \mathcal{A}')$ such that $F(\varphi)(\xi) = \xi'$ (indeed $v_*F(k) = \{*\}$ by 2.2.6).

We call v_*F the *pushforward* of F along v.

2.2.11 Proposition ([18, 3.1])

If F is pro-represented by $(\mathcal{R}, \xi)$, then v_*F is pro-represented by $(v(\mathcal{R}), [\mathcal{R}, \xi])$, where $[\mathcal{R}, \xi]$ denotes the class of $(\mathcal{R}, \xi)$ in $v_*F(R)$.

2.2.12 Definition

An F-pro-couple $(\mathcal{R}, \xi)$ is called a *pseudo-versal element* for F if the v_*F-pro-couple $(v(\mathcal{R}), [\mathcal{R}, \xi])$ is a versal element of the functor v_*F.

An F-pro-couple $(\mathcal{R}, \xi)$ is called a *pseudo-hull* of F or a *pseudo-semi-universal element* for F if the v_*F-pro-couple $(v(\mathcal{R}), [\mathcal{R}, \xi])$ is a hull of the functor v_*F.

An F-pro-couple $(\mathcal{R}, \xi)$ is called a a *pseudo-pro-representative* of F or *pseudo-universal element* for F if the v_*F-pro-couple $(v(\mathcal{R}), [\mathcal{R}, \xi])$ pro-represents the functor v_*F.

For any functor of log Artin rings $F\colon \underline{\mathrm{LArt}}_{\mathcal{T}} \to \underline{\mathrm{Set}}$ we define the functor $\overline{F}\colon \underline{\mathrm{LArt}}_{\mathcal{T}} \to \underline{\mathrm{Set}}$ by $\overline{F}(\mathcal{A}) := F(\mathcal{A})/\,\mathrm{Aut}_A(\mathcal{A})$ for every $\mathcal{A} \in \underline{\mathrm{LArt}}_{\mathcal{T}}$. By 2.2.6 this is a functor of log Artin rings and we have a bijection $\overline{F}(\mathcal{A}) \to v_*F(v(\mathcal{A}))$, $[\xi] \mapsto [\mathcal{A}, \xi]$.

2.2.13 Definition

A functor F of log Artin rings is called *rigid* if for all $\mathcal{A} \in \underline{\mathrm{LArt}}_{\mathcal{T}}$ and every $\varphi \in \mathrm{Aut}_A(\mathcal{A})$ the bijection $F(\varphi)\colon F(\mathcal{A}) \to F(\mathcal{A})$ has no fixed points unless $\varphi = id$.

It is called *rigid in the first order* if for any $\varphi \in \mathrm{Aut}_{k[\varepsilon]^0}(\kappa[\varepsilon]^0)$ the bijection $F(\varphi)\colon t_F \to t_F$ has no fixed points unless $\varphi = id$.

2.2.14 Remark

If $\varrho\colon Q' \to Q$ is an isomorphism, then $\mathrm{Aut}_A(\mathcal{A})$ is the one-element set containing only the identity $id_{\mathcal{A}}$. Hence in this case any functor F is rigid, $F = \overline{F}$ and the word “pseudo” can be erased everywhere (cp. [18, 3.3]).

Let F be a functor of log Artin rings, let $\mathcal{A}' \to \mathcal{A}$ and $\mathcal{A}'' \to \mathcal{A}$ be homomorphisms in $\underline{\mathrm{LArt}}_{\mathcal{T}}$ and consider the canonical map

$$\varPhi\colon F(\mathcal{A}' \times_{\mathcal{A}} \mathcal{A}'') \to F(\mathcal{A}') \times_{F(\mathcal{A})} F(\mathcal{A}'').$$

The following definition imitates the corresponding one in the first section.

2.2.15 Definition

A functor of log Artin rings F is called a *functor with good logarithmic deformation theory* or *log gdt functor* if it has the properties

LH_1: $\varPhi$ is surjective, whenever $\mathcal{A}'' \to \mathcal{A}$ is a small extension;

LH_2: $\varPhi$ is a bijection, when $\mathcal{A} = \kappa$, $\mathcal{A}'' = \kappa[\varepsilon]^0$.

As in the first section, a log gdt functor F has the property that, for any two finite dimensional vector spaces V and W, the canonical map $F(\kappa[V]^0 \times_\kappa \kappa[W]^0) \to F(\kappa[V]^0) \times F(\kappa[W]^0)$ is a bijection. Hence, $t_F = F(\kappa[\varepsilon]^0)$ carries a natural vector space structure with $*$ as its zero element and is called the *tangent space* of F (cp. [18, p. 105]); moreover, $F(\kappa[V]^0) \cong t_F \otimes_k V$ as k-vector-spaces for any finite dimensional vector space V.

2.2.16 Lemma ([18, 3.5])

Let F be a log gdt functor. Then for every small extension $e\colon \mathcal{A}' \to \mathcal{A}$ with kernel J the k-vector-space $F(\kappa[I]^0)$ acts transitively on the set

$$F(e)^{-1}(\xi) = \{\xi' \in F(\mathcal{A}') \,|\, F(e)(\xi') = \xi\}$$

for every fixed ξ.

Proof: We have an isomorphism $\mathcal{A}' \times_{\mathcal{A}} \mathcal{A}' \to \mathcal{A}' \times_\kappa \kappa[I]^0$ given by $(x, y) \mapsto (x, \overline{x} \oplus (y - x))$ (on the levels both of rings and monoids, where $\overline{x}$ denotes $\pi_{A'}(x)$ and $\varrho_{A'}(x)$, respectively). Combining this isomorphism with LH_2 yields

$$F(\mathcal{A}') \times (t_F \otimes_k I) \xrightarrow{\cong} F(\mathcal{A}') \times_{F(\mathcal{A})} F(\mathcal{A}'),$$

defining the action of an element $\eta \in t_F \otimes_k I$ on a $\xi' \in F(e)^{-1}(\xi)$ by $(\xi', \eta) \mapsto (\xi', \eta \cdot \xi')$. By LH_1 this action is transitive (cp. [Schlessinger Remark 2.13]). □

2.2.17 Proposition ([18, 3.4 & 3.12])
Let $F\colon \underline{\mathrm{LArt}}_{\mathcal{T}} \to \underline{\mathrm{Set}}$ be a functor of log Artin rings.

a) If F is pro-representable, then F is rigid.

b) If F possesses a hull, then F is rigid in the first order.

c) If F is rigid, then any pseudo-pro-representative of F pro-represents F.

d) If F is a log gdt functor which is rigid in the first order, then any pseudo-hull of F is a hull.

2.2.18 Lemma ([18, 3.7 & 3.8])

a) The natural morphism $F \to \overline{F}$ is smooth.

b) The implications i) $\Rightarrow$ ii) $\Leftrightarrow$ iii) hold between the following conditions for a morphism $\varphi\colon F \to G$ of functors of log Artin rings,

   i) $\varphi\colon F \to G$ is smooth;

   ii) $\overline{\varphi}\colon \overline{F} \to \overline{G}$ is smooth;

   iii) $v_*\varphi\colon v_*F \to v_*G$ is smooth (as a morphism of functors of Artin rings).

   If F, $\overline{F}$ and $\overline{G}$ are log gdt functors and if $\ker(t_F \to t_{\overline{F}}) \to \ker(t_G \to t_{\overline{G}})$ is surjective, then also ii) $\Rightarrow$ i).

c) In particular, F is smooth (over $*$) $\Leftrightarrow$ $\overline{F}$ is smooth $\Leftrightarrow$ v_*F is smooth (as a functor of Artin rings).

Let F be a log gdt functor. Consider the following two conditions:

LH_3: $\dim_k t_F < \infty$;

LH_4: Φ is a bijection, when $\mathcal{A}'' = \mathcal{A}'$ and whenever $\mathcal{A}' \to \mathcal{A}$ is a small extension.

2.2.19 Theorem ([18, 3.12])
Let $F\colon \underline{\mathrm{LArt}}_{\mathcal{T}} \to \underline{\mathrm{Set}}$ be a log gdt functor.

a) F possesses a pseudo-hull if and only if $\overline{F}$ satisfies LH_3.

   It possesses a hull if and only if F satisfies LH_3 and is rigid in the first order.

b) F is pseudo-pro-representable if and only if $\overline{F}$ satisfies LH_3 and LH_4.

   It is pro-representable if and only if F satisfies LH_3 and LH_4 and is rigid.

2.2.20 Remark
We may transfer the definitions 2.1.19, 2.1.20, 2.1.21, 2.1.22 and 2.1.23 of obstruction theories and their properties of linearity and completeness literally to this chapter when replacing the category $\underline{\mathrm{Art}}_T$ by $\underline{\mathrm{LArt}}_{\mathcal{T}}$. An $\mathcal{A}$-module is then a $v(\mathcal{A})$-module.

2.3 Log smooth deformations

Let $f_0 \colon X \to Y$ be a log smooth morphism of log schemes and let $i \colon Y \to \mathcal{Y}$ be a strict closed immersion. A *log smooth deformation of* f_0 *over* i is a Cartesian diagram

$$\begin{tikzcd} X \arrow[r, "\hat{i}"] \arrow[d, "f_0"] \arrow[dr, phantom, very near start, "\ulcorner"] & \mathcal{X} \arrow[d, "f"] \\ Y \arrow[r, "i"] & \mathcal{Y} \end{tikzcd}$$

of morphisms of log schemes, with f log smooth (and $\hat{i}$ automatically a strict closed immersion). The morphism f is then called a *(log smooth) lifting* of f_0 over i (or over $\mathcal{Y}$ if i is known from the context).

A morphism $\varphi \colon f' \to f$ of log smooth deformations $f \colon \mathcal{X} \to \mathcal{Y}$ and $f' \colon \mathcal{X}' \to \mathcal{Y}$ of f_0 over i is a morphism of log schemes $\varphi \colon \mathcal{X}' \to \mathcal{X}$ over $\mathcal{Y}$ such that $\varphi|_X = id_X$.

2.3.1 Lemma

Let $f_0 \colon X \to Y$ be a log integral log smooth morphism of log fine log schemes. Then every log smooth deformation $f \colon \mathcal{X} \to \mathcal{Y}$ is log integral, and thus a flat deformation of the underlying schemes.

Proof: Assume that f is not log integral. Then there exists a log integral log scheme V and a morphism of log schemes $V \to \mathcal{Y}$ such that $X_V := V \times_{\mathcal{Y}} \mathcal{X}$ is not log integral. Let $U := V \times_{\mathcal{Y}} Y$ and $X_U := U \times_Y X = U \times_V X_V$. Then U is log integral, because $U \to V$ is a strict closed immersion. Due to the log integrality of the morphism f_0, the log scheme X_U is log integral. But $X_U \to X_V$ is a strict closed immersion, so in particular $\mathcal{M}_{X_U,x} \cong \mathcal{M}_{X_V,x}$, thus either both sides are log integral or not. This contradicts the assumption. $\square$

Let $\mathcal{T}$ and κ be as in the last section and let $f_0 \colon X \to \operatorname{Spec} \kappa$ be a log integral morphism of log fine log schemes. For $\mathcal{A} \in \underline{\mathrm{LArt}}_{\mathcal{T}}$ let $i \colon \operatorname{Spec} \kappa \to \operatorname{Spec} \mathcal{A}$ be the associated strict closed immersion over $\operatorname{Spec} \mathcal{T}$.

We define the functor $\mathrm{Def}_{f_0} \colon \underline{\mathrm{LArt}}_{\mathcal{T}} \to \underline{\mathrm{Set}}$ by

$$\mathrm{Def}_{f_0}(\mathcal{A}) = \{\text{Isomorphism classes of log smooth lifting of } f_0 \text{ to } \operatorname{Spec} \mathcal{A}\}$$

and evidently for morphisms.

2.3.2 Lemma ([17, 8.3])

Let $f \colon \mathcal{X} \to \operatorname{Spec} \mathcal{A}$ be a log smooth deformation of $f_0 \colon X \to \operatorname{Spec} \kappa$ along $i \colon \operatorname{Spec} \kappa \to \operatorname{Spec} \mathcal{A}$. Then any local chart $(a \colon P_X \to \mathcal{M}_X, b \colon Q \to Q \oplus k^\times, \theta \colon Q \to P)$ of f_0 lifts to a local chart $(a \colon P_{\mathcal{X}} \to \mathcal{M}_{\mathcal{X}}, b \colon Q \to Q \oplus A^\times, \theta \colon Q \to P)$ of f.

2.3.3 Proposition ([19, 3.14])

Let $f_0\colon X \to Y$ be a log integral log smooth morphism of log fine log schemes and let $i\colon Y \to Y'$ be an infinitesimal thickening of log fine log schemes. If X is affine, then a log smooth deformation of f_0 over i exists and is unique up to isomorphism.

2.3.4 Proposition ([19, 3.14])

Let $f_0\colon X \to \operatorname{Spec}\kappa$ be a log integral log smooth morphism of log fine log schemes.

a) The tangent space of the functor of log Artin rings Def_{f_0} is $H^1(X, T_{f_0})$.

b) The vector space $H^2(X, T_{f_0})$ the small obstruction space of a complete linear obstruction theory for the functor Def_{f_0}.

Let $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension of log Artin rings ($J^2 = 0$) and let f be a lifting of f_0 over $\mathcal{A}$.

c) The group of automorphisms of a lifting $\tilde{f}$ over $\tilde{\mathcal{A}}$ inducing the identity on f is $H^0(\mathcal{X}, T_f \otimes_A J)$.

d) The set of isomorphism classes of liftings of f over $\tilde{\mathcal{A}}$ is a pseudo-torsor under $H^1(\mathcal{X}, T_f \otimes_A J)$.

e) The obstruction $o_e([f])$ to lifting f over $\tilde{\mathcal{A}}$ is an element of the obstruction space $H^2(\mathcal{X}, T_f \otimes_A J)$.

2.3.5 Theorem ([18, 4.4i)])

Suppose that f_0 is a log integral log smooth morphism of log schemes, with proper underlying morphism and such that $f_{0*}\mathcal{O}_X = \mathcal{O}_{\operatorname{Spec}\kappa}$. Then the functor Def_{f_0} satisfies LH_1, LH_2 and LH_3 and hence possesses a pseudo-hull.

2.3.6 Corollary

In the case that $\varrho\colon Q' \to Q$ is a isomorphism, suppose that f_0 is a log integral log smooth morphism of log schemes with proper underlying morphism and such that $f_{0*}\mathcal{O}_X = \mathcal{O}_{\operatorname{Spec}\kappa}$. Then the functor Def_{f_0} possesses a hull.

Proof: Since $\varrho\colon Q' \to Q$ is a isomorphism, remark 2.2.14 implies that Def_{f_0} is rigid, so any pseudo-hull is a hull. □

3 Additional Data

The purpose of this chapter is to step-by-step introduce additional data, namely line bundles, flat logarithmic connections and logarithmically symplectic forms, to the definition of a log smooth morphism of log schemes until finally reaching our definitions of a "log symplectic scheme of non-twisted type" and of a "log symplectic scheme of general type". Moreover, we introduce certain $\mathcal{O}_X$-modules and complexes of $\mathcal{O}_X$-modules, the (hyper-)cohomology of which will later turn out to yield the obstruction theory of the "functor of log symplectic deformations of a log symplectic scheme", which will be defined in chapter 4.

Following remark 1.1.7 in A. Ogus' lecture notes ([29]), we define the term "logarithmic Cartier divisor" in the second section, which generalises the well-known term "Cartier divisor". Subsequently we link these logarithmic Cartier divisor to line bundles and flat logarithmic connections.

In the last section we collect facts from works of R. Friedman ([11]), F. Kato ([17]) and A. Ogus ([29]) concerning the canonical log structure of strict normal crossing varieties.

Let X be a log scheme. An (affine) open covering $\mathcal{U} = \{X_i\}_{i\in I}$ of X consists of (affine) open subschemes $j_i\colon \underline{X}_i \to \underline{X}$ forming an (affine) open covering of $\underline{X}$ and each equipped with the induced log structure $\alpha_{X_i} = j_i^\times \alpha_X$.

A sheaf on X means a sheaf on $\underline{X}$ and we write $H^i(X,\mathcal{F})$ for $H^i(\underline{X},\mathcal{F})$ etc.

For a morphism $f\colon X \to Y$ of log schemes, we denote by $\underline{\mathrm{Comp}}_f$ the category of co-complexes of $\mathcal{O}_X$-modules on X with $f^{-1}\mathcal{O}_Y$-linear differential; by abuse of language, we will speak of complexes. For $K^\bullet \in \underline{\mathrm{Comp}}_f$, we denote by $K^\bullet[p]$ and $K^{\geq p,\bullet}$ the p-shifted complex and the p-truncated complex, respectively, of $K^\bullet$ for $p \in \mathbb{Z}$.

3.1 Line bundles

For an Abelian group G, a *pseudo-torsor* is a G-set $(S, \varrho\colon G\times S \to S)$ such that for every $s \in S$ the map $s \mapsto gs$ is bijective. A G-pseudo-torsor is called a *G-torsor* if S is non-empty.

Let X be a log scheme. *A line bundle on X* is a line bundle L on $\underline{X}$, i. e. a locally free $\mathcal{O}_X$-module of rank one.

We will use the following convention extensively: Let L be a line bundle. Then there exists an affine open covering $\mathcal{U} = \{X_i\}$ of X and isomorphisms $\psi_i\colon\ L|_{X_i} \to \mathcal{O}_{X_i}$ of $\mathcal{O}_{X_i}$-modules. We will say that this covering *trivialises* L. By abuse of notation and language, we will suppress the isomorphisms ψ_i, meaning that we will identify the line bundle L with the collection of all $\mathcal{O}_{X_i}$'s glued together by the transition functions $f_{ij} := \psi_i|_{X_{ij}} \circ \psi_j^{-1}|_{X_{ij}}$ defined on the overlaps $X_{ij} = X_i \cap X_j$. We will hence speak of a (local) section s_i of $L|_{X_i}$ meaning the (local) section $\psi_i(s_i)$ of $\mathcal{O}_{X_i}$. So saying that the local sections s_i and s_j of $L|_{X_i}$ and $L|_{X_j}$, respectively, agree on X_{ij}, we write $s_i = f_{ij}s_j$ on X_{ij}, meaning that $\psi_i(s_i)|_{X_{ij}} = f_{ij}\ \psi_j(s_j)|_{X_{ij}}$.

Although this is well-known, we will, for later use of certain notations, prove:

3.1.1 Proposition

Let X a log scheme.

a) The set of endomorphisms of a line bundle L is $H^0(X, \mathcal{O}_X)$.

b) The set of isomorphisms between two fixed line bundles on X is a pseudo-torsor under the group $H^0(X, \mathcal{O}_X^\times)$. The group of automorphisms of a line bundle is canonically isomorphic to $H^0(X, \mathcal{O}_X^\times)$.

c) The Picard group of isomorphism classes of line bundles on X, $\mathrm{Pic}(X) = \mathrm{Pic}(\underline{X})$, is canonically isomorphic to $H^1(X, \mathcal{O}_X^\times)$.

Proof: Let $\mathcal{U} = \{X_i\}_{i\in I}$ be an affine open covering of X such that $L_i := L|_{X_i}$ is isomorphic to $\mathcal{O}_{X_i}$ for each i. On X_i we identify $L_i := L|_{X_i}$ with $\mathcal{O}_{X_i}$. For each $X_{ij} = X_i \cap X_j$ there exists a unique element $f_{ij} \in \Gamma(X_{ij}, \mathcal{O}_X^\times)$, the transition function on X_{ij}, such that two local sections s_i and s_j of L over X_i and X_j, respectively, are equal on X_{ij} if and only if $s_i = f_{ij}s_j$. The Čech-1-cochain (f_{ij}) in $\mathcal{O}_X^\times$ satisfies the condition $f_{jk}f_{ij}f_{ik}^{-1} = 1$ and is therefore a cocycle. Hence it defines a class $[L] \in H^1(X, \mathcal{O}_X^\times)$.

An isomorphism of line bundles $\psi\colon L' \to L$ is given on each X_i by an automorphism $\psi_i := \psi|_{L'_i} : \mathcal{O}_{X_i} \to \mathcal{O}_{X_i}$, which is just the multiplication by a unit f_i and defines a 0-cochain (f_i) in $\mathcal{O}_X^\times$. Due to the lack of coboundaries the group of isomorphisms between two line bundles is a pseudo-torsor under $H^0(X, \mathcal{O}_X^\times)$. If $L' = L$, then this pseudo-torsor is naturally identified with the group. For an endomorphism of L, we just have to drop the condition that the f_i are units. This shows a) and b).

If f'_{ij} denotes the transition function of L' on X_{ij}, then we must have $f_{ij}f_j = f_if'_{ij}$, hence $f'_{ij} = f_{ij}f_jf_i^{-1}$. Therefore, the cocycles (f_{ij}) and (f'_{ij}) differ by the coboundary $\check{d}(f_i)$. This shows that the isomorphism class of L uniquely determines the class $[L] \in H^1(X, \mathcal{O}_X^\times)$ and vice versa. Tensoring line bundles corresponds to component-wise multiplication of the representing cocycles. This shows c). □

An $\mathcal{O}_X^\times$*-torsor* on X is a sheaf $\mathcal{T}$ of non-empty $\mathcal{O}_X^\times$-sets, such that $\mathcal{O}_X^\times$ acts regularly on $\mathcal{T}$. Necessarily, locally $\mathcal{T}$ is isomorphic to $\mathcal{O}_X^\times$ as a sheaf of $\mathcal{O}_X^\times$-sets.

Given an $\mathcal{O}_X^\times$-torsor, there exists an affine covering $\mathcal{U} = \{X_i\}_{i\in I}$ of X trivialising $\mathcal{T}$ and a cocycle of transition functions $f_{ij} \in \Gamma(X_{ij}, \mathcal{O}_X^\times)$ between the restrictions $\mathcal{T}|_{X_i} \cong \mathcal{O}_{X_i}^\times$. This gives a one-to-one correspondence between line bundles L and $\mathcal{O}_X^\times$-torsors $\mathcal{T}$ on X such that $\mathcal{T} \subset L$ as a sheaf of $\mathcal{O}_X^\times$-sets. We will denote the $\mathcal{O}_X^\times$*-subtorsor associated to a line bundle* L by $L^\times$. Then $L = L^\times \otimes_{\mathcal{O}_X^\times} \mathcal{O}_X$.

If L is a line bundle, then $L^\times$ is the sheaf of local generators of L as an $\mathcal{O}_X$-module. So every $\mathcal{O}_X^\times$-torsor is of the form $L^\times$ for the line bundle L it generates. In this regard, the Picard group is also canonically isomorphic to the group of isomorphism classes of $\mathcal{O}_X^\times$-torsors on X.

3.1.2 Definition

Let Y be a log scheme. A *scheme with line bundle over* Y is a pair (f, L), also written $f\colon (X, L) \to Y$, consisting of the following data:

a) A log smooth morphism of log fs log schemes $f\colon X \to Y$;

b) a line bundle L on X.

The pair (f, L) is called *proper and log integral* if the morphism of log schemes $f\colon X \to Y$ is proper and log integral.

A *morphism* $(f', L') \to (f, L)$ *of log schemes with line bundle over* Y is a pair (h, ψ), where

a) $h\colon X' \to X$ is a morphism of Y-log schemes;

b) $\psi\colon h^*L \to L'$ is a morphism of line bundles.

We denote the category of log schemes with line bundle over Y by $\underline{\mathrm{LLSch}}_Y$.

3.1.3 Definition

Let $f_0\colon (X, L) \to Y$ be a log scheme with line bundle over Y and let $i\colon Y \to \mathcal{Y}$ be a strict closed immersion. *A log smooth deformation of* (f_0, L) *over* i is a log scheme with line bundle $f\colon (\mathcal{X}, \mathcal{L}) \to \mathcal{Y}$ together with

a) a strict closed immersion $\hat{\imath}\colon X \to \mathcal{X}$ such that

$$\begin{tikzcd} X \arrow[r, hook, "\hat{\imath}"] \arrow[d, "f_0"] & \mathcal{X} \arrow[d, "f"] \\ Y \arrow[r, hook, "i"] & \mathcal{Y} \end{tikzcd}$$

is a log smooth deformation of the log scheme X;

b) an isomorphism $\hat{\imath}^*\mathcal{L} \cong L$.

In this case we will write $\hat{\imath}^*(f, \mathcal{L}) = (f_0, L)$ and say that we have a Cartesian diagram

$$\begin{tikzcd} (X, L) \arrow[r, hook, "\hat{\imath}"] \arrow[d, "f_0"] & (\mathcal{X}, \mathcal{L}) \arrow[d, "f"] \\ Y \arrow[r, hook, "i"] & \mathcal{Y}. \end{tikzcd}$$

3.1.1 Action of log derivations

Let $f\colon X \to Y$ be a morphism of log schemes. For every $p \in \mathbb{N}_0$ we have a canonical $f^{-1}\mathcal{O}_Y$-bilinear map

$$T_f \times \Omega^p_f \to \Omega^p_f,$$

given by the Lie derivative $(\vartheta, \sigma) \mapsto \vartheta(\sigma) := d(i_\vartheta\sigma) + i_\vartheta(d\sigma)$, defining an action of log derivations on log p-forms.

Observe that $\vartheta(\sigma_p \wedge \sigma_q) = \vartheta(\sigma_p) \wedge \sigma_q + \sigma_p \wedge \vartheta(\sigma_q)$ for all p-forms σ_p, all q-forms σ_q and all log derivations ϑ.

A compatible action of log derivations on log derivations

$$T_f \times T_f \to T_f$$

is given by $(\vartheta, \delta) \mapsto \vartheta(\delta) := [\vartheta, \delta]$. This means that for any local section $\delta \otimes \sigma$ of $T_f \otimes_{\mathcal{O}_X} \Omega_f^p$, ϑ acts according to Leibniz's rule, i. e. $\vartheta(\delta \otimes \sigma) = \vartheta(\delta) \otimes \sigma + \delta \otimes \vartheta(\sigma)$ and is compatible with the pairing $i_{\,\cdot\,}(\,\cdot\,)$ such that $\vartheta(i_\delta(\sigma)) = i_{\vartheta(\delta)}(\sigma) + i_\delta(\vartheta(\sigma))$.

3.1.2 The log Atiyah module of a line bundle

Let $f\colon X \to Y$ be a log smooth morphism of log schemes. The $f^{-1}\mathcal{O}_Y^\times$-invariant map $d\log\colon \mathcal{O}_X^\times \to \Omega_f^1$ induces the *log Chern map*

$$d\log\colon H^1(X, \mathcal{O}_X^\times) \to H^1(\Omega_f^1) = \operatorname{Ext}^1_{\mathcal{O}_X}(T_f, \mathcal{O}_X),$$

assigning to each isomorphism class $[L]$ of a line bundle L its *log Chern class* $d\log(L) \in H^1(\Omega_f^1)$ and hence an isomorphism class of short exact sequences of $\mathcal{O}_X$-modules

$$d\log(L)\colon 0 \to \mathcal{O}_X \to A_f(L) \to T_f \to 0,$$

called *the log Atiyah extension of* L. The $\mathcal{O}_X$ module which is up to isomorphism uniquely determined by this extension is called the *log Atiyah module of* L, written $A_f(L)$. It is locally free of rank $n+1$ and may be constructed as follows:

Let $\mathcal{U} = \{X_i\}$ an open affine covering as in the proof of 3.1.1 trivialising L. Writing $f|i := f|_{X_i} : X_i \to Y$, on each X_i, we identify $A_i(L) := A_f(L)|_{X_i}$ with $\mathcal{O}_{X_i} \oplus T_{f|i}$, where two sections (g_i, ϑ_i) and (g_j, ϑ_j) over X_i and X_j, respectively, are identified over X_{ij} if and only if $\vartheta_i = \vartheta_j$ and $g_i - g_j = \frac{\vartheta_i(f_{ij})}{f_{ij}} = i_{\vartheta_i}(d\log f_{ij}) = \Theta_i(f_{ij})$. This is due to the fact, that if $[L]$ is given by the cocycle (f_{ij}), then $d\log(L)$ is given by the cocycle $(\frac{df_{ij}}{f_{ij}})$ in $\check{C}^1(X, \Omega_f^1)$.

3.1.3 L-Derivations

Let $f\colon X \to Y$ be a log smooth morphism of log schemes and L a line bundle on X. For every $p \in \mathbb{N}_0$ we define an $f^{-1}\mathcal{O}_Y$-bilinear map

$$A_f(L) \times (\Omega_f^p \otimes_{\mathcal{O}_X} L) \to \Omega_f^p \otimes_{\mathcal{O}_X} L,\ (a, \sigma) \mapsto a(\sigma)$$

as follows: Let $\mathcal{U}$ be an affine open covering trivialising L. On each X_i of $\mathcal{U}$ we define $A_i(L) \times \Omega_{f|i}^p \to \Omega_{f|i}^p$ by $((g_i, \vartheta_i), \sigma_i) \mapsto (g_i, \vartheta_i)(\sigma_i) := g_i\sigma_i + \vartheta_i(\sigma_i)$, where $\vartheta_i(\sigma_i)$ is the action of log derivations on log forms defined above. This is well-defined due to $f_{ij}(g_i\sigma_i + \vartheta_i(\sigma_i)) = g_i\sigma_j + \frac{\vartheta_i(f_{ij})}{f_{ij}}\sigma_j + \vartheta_i(\sigma_j) = g_j\sigma_j + \vartheta_j(\sigma_j)$.

In relation to its action on forms, we refer to $A_f(L)$ as the *sheaf of derivations extended by* L or L*-derivations* and we call $a(\,\cdot\,)$ the Lie derivative of a. Observe that if $L = \mathcal{O}_X$, then $A_f(L) = \mathcal{O}_X \oplus \mathrm{Der}_f(\mathcal{O}_X)$. Observe also that $a(\sigma_p \wedge \sigma_q) = \pi(a)(\sigma_p) \wedge \sigma_q + \sigma_p \wedge a(\sigma_q)$ for all p-forms σ_p and all q-forms with values in L, σ_q, where $\pi\colon A_f(L) \to T_f$.

3.2 Log Schemes with flat log connection

3.2.1 Definition

Let $f\colon X \to Y$ be a morphism of log schemes. A *(rank one) logarithmic connection* ∇, also written (L, ∇), on f consists of a line bundle L on X and a map

$$\nabla\colon L \to \Omega^1_f \otimes_{\mathcal{O}_X} L$$

which satisfies Leibniz's rule $\nabla(f \cdot s) = df \otimes s + f\nabla(s)$ for local sections f of $\mathcal{O}_X$ and s of L, and therefore is $f^{-1}\mathcal{O}_Y$-linear.

To each log connection ∇ we associate a sequence of $f^{-1}\mathcal{O}_Y$-linear maps

$$L \xrightarrow{\nabla^{(0)}} \Omega^1_f \otimes_{\mathcal{O}_X} L \xrightarrow{\nabla^{(1)}} \Omega^2_f \otimes_{\mathcal{O}_X} L \xrightarrow{\nabla^{(2)}} \Omega^3_f \otimes_{\mathcal{O}_X} L \to \dots,$$

where we define $\nabla^{(0)} := \nabla$ and $\nabla^{(p)}(\sigma \otimes s) := d\sigma \otimes s + (-1)^p \sigma \wedge \nabla^{(0)}(s)$.

A log connection ∇ is called *flat* if $\nabla^{(1)}\nabla^{(0)} = 0$ (which implies $\nabla^{(p+1)}\nabla^{(p)} = 0$ for all p). In this case, the sequence above is a complex of $\mathcal{O}_X$-modules with $f^{-1}\mathcal{O}_Y$-linear differentials $\nabla := \nabla^\bullet$, which we call the *logarithmic de Rham complex associated to* ∇ and which we denote by $(\Omega^\bullet_f \otimes L, \nabla) \in \underline{\mathrm{Comp}}_f$.

A morphism $\psi\colon (L, \nabla) \to (L', \nabla')$ of flat log connections is a morphism of line bundles $\psi\colon L \to L'$ such that the diagram

$$\begin{tikzcd} L \arrow[r, "\nabla"] \arrow[d, "\psi"] & \Omega^1_f \otimes L \arrow[d, "id\otimes\psi"] \\ L' \arrow[r, "\nabla'"] & \Omega^1_f \otimes L' \end{tikzcd}$$

commutes; equivalently, such that $\psi^\bullet := id_{\Omega^\bullet_f} \otimes \psi$ is a morphism of complexes.

3.2.2 Remark

Analogously, one can define a rank n log connection by replacing the line bundle L by a vector bundle of rank n. We will regard only rank one log connections in this thesis.

3.2.3 Definition and Proposition

The isomorphism classes of flat log connections ∇ on f form an Abelian group which we denote by

$$\mathrm{LConn}(f)$$

and which we call the *Picard group of* f.

Proof: This is indeed a group with respect to the tensor product: If (L, ∇) and (L', ∇') are two flat log connections, then $(L \otimes L', \nabla \otimes \nabla')$ is a flat log connection, where $\nabla \otimes \nabla' \colon L \otimes L' \to \Omega^1_f \otimes (L \otimes L')$ is defined by the rule $(\nabla \otimes \nabla')(s \otimes s') = \nabla(s) \otimes s' + s \otimes \nabla'(s')$. The neutral element of this group is the trivial flat log connection $(\mathcal{O}_X, d)$ and the inverse (L^{-1}, ∇^{-1}) of (L, ∇) is given by the dual line bundle L^{-1} of L together with $\nabla^{-1} \colon L^{-1} \to \Omega^1_f \otimes L^{-1}$ defined by $\nabla^{-1}(\varphi)(s) = -d(\varphi(s)) - (id \otimes \varphi)(\nabla(s))$ for local sections φ of L^{-1} and s of L. The flatness of $\nabla \otimes \nabla'$ and ∇^{-1} is easy to check. □

3.2.4 Remark

We do not write $\mathrm{Pic}(f)$ for the Picard group of f to avoid confusion with the usual Picard-functor from above evaluated on a morphism of (log) schemes.

3.2.5 Definition

We denote by $(\Omega_f^{\times,\bullet}, d\log^\bullet) \in \underline{\mathrm{Comp}}_f$ the complex

$$\Omega_f^{\times,\bullet} \colon \mathcal{O}_X^\times \to \Omega_f^1 \to \Omega_f^2 \to \Omega_f^3 \to \dots,$$

where the differential is $d\log \colon \mathcal{O}_X^\times \to \Omega^1_f$ in the first place and d elsewhere. We will refer to it as the *unit complex of* f.

Observe that this complex has $f^{-1}\mathcal{O}_Y$-linear differentials at all places ≥ 1 and an $f^{-1}\mathcal{O}_Y^\times$-linear zeroth differential.

3.2.6 Proposition

a) The set of endomorphisms of a flat log connection ∇ is $\mathrm{End}_{\underline{\mathrm{Comp}}_f}(\nabla) = \mathbb{H}^0(\Omega_f^\bullet)$.

b) The set of isomorphisms between two fixed flat log connections is a pseudo-torsor under the group $\mathbb{H}^0(\Omega_f^{\times,\bullet})$. The group of automorphisms of a flat log connection is canonically isomorphic to $\mathbb{H}^0(\Omega_f^{\times,\bullet})$.

c) The monoid $\mathrm{LConn}(f)$ is an Abelian group and canonically isomorphic to $\mathbb{H}^1(\Omega_f^{\times,\bullet})$.

Proof: An endomorphism of a flat log connection (L, ∇) is an endomorphism of L, hence an element $u \in H^0(X, \mathcal{O}_X)$. Moreover, it must satisfy $u\nabla s = \nabla(us) = du \otimes s + u\nabla s$ for all local sections s of L, thus $du = 0$. This shows a). To be an automorphism, u needs to be a unit. Then $du = 0$ is equivalent to $d\log u = 0$. This shows b), because given two isomorphisms $\psi_k \colon (L, \nabla) \to (L', \nabla')$, $k = 1, 2$, of flat log connections, their difference $\psi_2^{-1}\psi_1$ is an automorphism of ∇.

Let (L, ∇) be a flat log connection on f and let $\mathcal{U}$ be an affine open covering as in the proof of 3.1.1 trivialising L. The line bundle L is given by the 1-cocycle (f_{ij}) in $\mathcal{O}_X^\times$. As $L_i := L|_{X_i} = \mathcal{O}_{X_i}$ is trivial, we have beside the connection $\nabla_i := \nabla|_{X_i} \colon \mathcal{O}_{X_i} \to \Omega^1_{f|i}$ the trivial flat connection $d \colon \mathcal{O}_{X_i} \to \Omega^1_{f|i}$. Both differ by a 1-form $\mathfrak{v}_i \in \Gamma(X_i, \Omega^1_{f|i})$ such that $\nabla_i = d + \mathfrak{v}_i \wedge \cdot$ on each X_i.

The ∇_i have the property that $\nabla_i\nabla_i = 0$ and $\nabla_i(f_{ij}\,\cdot\,) = f_{ij}\nabla_j(\,\cdot\,)$. For the v_i this implies that

a) $dv_i = 0$;

b) $v_j - v_i - d\log(f_{ij}) = 0$.

This means precisely that the 1-cochain (v_i, f_{ij}) in $\Omega_f^{\times,\bullet}$ is a cocycle in this complex and hence defines a class $[\nabla] = [L, \nabla] \in \mathbb{H}^1(\Omega_f^{\times,\bullet})$.
If (v_i', f_{ij}') denotes the cocycle of another flat log connection (L', ∇'), then we have $f_{ij}' = f_{ij}f_jf_i^{-1}$ with the notation as in the proof of 3.1.1. Moreover, $f_i\nabla_i' = \nabla_i(f_i\,\cdot\,) = df_i \wedge \cdot + f_i\nabla_i$. Hence, $v_i' - v_i = d\log f_i$. Therefore, the cocycles (v_i, f_{ij}) and (v_i', f_{ij}') differ by the coboundary $(d\log \pm \check{d})(f_i)$, which shows that the isomorphism class of (L, ∇) uniquely determines the class $[\nabla] \in H^1(X, \mathcal{O}_X^\times)$ and vice versa. Tensoring flat log connections corresponds to componentwise adding or respectively multiplying the representing cocycles. This shows c). □

3.2.7 Remark

a) $\mathrm{Pic}(X) = \mathrm{LConn}(id_X)$, because $\Omega^1_{id_X} = 0$, so

$$\Omega^{\times,\bullet}_{id_X} = \left[\mathcal{O}_X^\times \to 0 \to 0 \to \ldots\right].$$

This is the reason to call $\mathrm{LConn}(f)$ the Picard group of the morphism f.

b) We will refer to the v_i associated to ∇, as in the proof, as the *discrepancy forms of* ∇ (with respect to the open affine cover $\mathcal{U}$ trivialising L). The cocycle (v_i, f_{ij}) representing the class of ∇ will be referred to as its *discrepancy cocycle.*

3.2.8 Definition

Let Y be a log scheme. A *log scheme with flat log connection over* Y is a pair (f, ∇), also written $f\colon (X, \nabla) \to Y$, consisting of the following data:

a) A log smooth morphism of log fs log schemes $f\colon X \to Y$;

b) a flat log connection ∇ on f.

The pair (f, ∇) is called *proper and log integral*, if the morphism of log schemes $f\colon X \to Y$ is proper and log integral.
A *morphism* $(f', \nabla') \to (f, \nabla)$ *of log schemes with flat log connection over* Y is a pair (h, ψ), where

a) $h\colon f' \to f$ is a morphism of Y-log-schemes;

b) $\psi\colon h^*\nabla \to \nabla'$ is a morphism of flat log connections on f, where $h^*\nabla\colon h^*L \to h^*\Omega^1_f \otimes h^*L$.

We denote the category of log schemes with flat log connection over Y by $\underline{\mathrm{LCSch_Y}}$.

3.2.9 Definition

Let $f_0\colon (X,\nabla) \to Y$ be a log scheme with flat connection and let $i\colon Y \to \mathcal{Y}$ be a strict closed immersion. A *log smooth deformation of* (f,∇) *over* i is a log scheme with flat connection $f\colon (\mathcal{X},\Delta) \to \mathcal{Y}$ together with

a) a strict closed immersion $\hat{\imath}\colon X \to \mathcal{X}$ such that

$$\begin{tikzcd} X \arrow[r, hook, "\hat{\imath}"] \arrow[d, "f_0"] & \mathcal{X} \arrow[d, "f"] \\ Y \arrow[r, hook, "i"] & \mathcal{Y} \end{tikzcd}$$

is a log smooth deformation of the log scheme X;

b) an isomorphism of flat log connections $\hat{\imath}^*\Delta \cong \nabla$.

In this case we will simply write $\hat{\imath}^*(f,\Delta) = (f_0,\nabla)$ and say that we have a Cartesian diagram

$$\begin{tikzcd} (X,\nabla) \arrow[r, hook, "\hat{\imath}"] \arrow[d, "f_0"] & (\mathcal{X},\Delta) \arrow[d, "f"] \\ Y \arrow[r, hook, "i"] & \mathcal{Y}. \end{tikzcd}$$

3.2.1 The log Atiyah complex of a flat log connection

Let $f\colon X \to Y$ be a log smooth morphism of log schemes. The log Chern class $d\log(L)$ of the line bundle L of any flat log connection ∇ on f is trivial, for if $[\nabla]$ has a discrepancy cocycle (∇_i, f_{ij}), then the class $d\log(L)$ is represented by $(d\log f_{ij}) = \check{d}(\nabla_i)$, which is a Čech-coboundary in Ω^1_f.

This is equivalent to the Atiyah sequence splitting: In the Atiyah sequence

$$0 \to \mathcal{O}_X \xrightarrow{i} A_f(L) \xrightarrow{\pi} T_f \to 0$$

we denote the left splitting by $s = s_\nabla\colon A_f(L) \to \mathcal{O}_X$ and the right one by $t = t_\nabla\colon T_f \to A_f(L)$. On each X_i of an covering $\mathcal{U}$ trivialising L these homomorphisms are given by the mappings

$$\begin{aligned} i&\colon g_i \mapsto (g_i, 0),\\ \pi&\colon (g_i,\vartheta_i) \mapsto \vartheta_i,\\ t&\colon \vartheta_i \mapsto (i_{\vartheta_i}(\nabla_i), \vartheta_i) \text{ and}\\ s&\colon (g_i,\vartheta_i) \mapsto g_i - i_{\vartheta_i}(\nabla_i) = pr_1(g_i,\vartheta_i) - t(pr_2(g_i,\vartheta_i)), \end{aligned}$$

where pr_k is the local projection to the k-th component.

We define a morphism of complexes

$$d\log^\bullet\colon \Omega_f^{\times,\bullet} \to \Omega_f^{\geq 1,\bullet}[1]$$

as follows: In degree 0 it is the map $d\log\colon \mathcal{O}_X^\times \to \Omega_f^1$ and in degrees $p > 0$ the map $(-1)^p d\colon \Omega_f^p \to \Omega_f^{p+1}$. (Observe that the complex $\Omega_f^{\geq 1,\bullet}[1]$ has the differential $-d$.)
Recall that $\mathrm{LConn}(f) = \mathbb{H}^1(\Omega_f^{\times,\bullet})$. The image $d\log^\bullet(\nabla) \in \mathbb{H}^1(\Omega_f^{\geq 1,\bullet}[1])$ of the class $[\nabla]$ of a flat log connection will be called its *log Chern class*. We identify

$$\mathbb{H}^1(\Omega_f^{\geq 1,\bullet}[1]) = \mathrm{Ext}^1_{\underline{\mathrm{Comp}}_f}(\mathcal{O}_X[0], \Omega_f^{\geq 1,\bullet}[1]).$$

Composing $d\log^\bullet$ with the maps

$$\Psi\colon\ \mathrm{Ext}^1_{\underline{\mathrm{Comp}}_f}(\mathcal{O}_X[0], \Omega_f^{\geq 1,\bullet}[1]) \to \mathrm{Ext}^1_{\underline{\mathrm{Comp}}_f}(T_f[0], T_f[0] \otimes_{f^{-1}\mathcal{O}_Y} \Omega_f^{\geq 1,\bullet}[1]),$$

which is induced by $e \mapsto id_{T_f[0]} \otimes_{f^{-1}\mathcal{O}_Y} e$, and

$$\Phi\colon\ \mathrm{Ext}^1_{\underline{\mathrm{Comp}}_f}(T_f[0], T_f[0] \otimes_{f^{-1}\mathcal{O}_Y} \Omega_f^{\geq 1,\bullet}[1]) \to \mathrm{Ext}^1_{\underline{\mathrm{Comp}}_f}(T_f[0], \Omega_f^\bullet),$$

which is induced by the evaluation $i\,.\,(\,\cdot\,)\colon T_f \otimes_{f^{-1}\mathcal{O}_Y} \Omega_f^{p+1} \to \Omega_f^p$, finally defines a map

$$\Phi \circ \Psi \circ d\log^\bullet\colon \mathbb{H}^1(\Omega_f^{\times,\bullet}) \to \mathrm{Ext}^1_{\underline{\mathrm{Comp}}_f}(T_f[0], \Omega_f^\bullet).$$

The codomain of this map is the set of isomorphism classes of short exact sequences of complexes of $\mathcal{O}_X$-modules with $f^{-1}\mathcal{O}_Y$-linear differential of the form

$$0 \to \Omega_f^\bullet \to A^\bullet \to T_f[0] \to 0.$$

For simplicity, we will write $d\log^\bullet := \Phi \circ \Psi \circ d\log^\bullet$ and we will call the image $d\log^\bullet(\nabla)$ of $[\nabla]$ in $\mathrm{Ext}^1_{f^{-1}\mathcal{O}_Y}(T_f[0], \Omega_f^\bullet)$ the *log Atiyah extension of* ∇. The complex of $\mathcal{O}_X$-modules with $f^{-1}\mathcal{O}_Y$-linear differential which is up to isomorphism uniquely determined by this extension is called the *log Atiyah complex of* ∇ and denoted by $A_f^\bullet(\nabla)$. It may be constructed as follows:
For $p > 0$, $A_f^p(\nabla) = \Omega_f^p$, because the part of degree p of $T_f[0]$ is zero. In degree 0 we identify $A^0_{f|i} = \mathcal{O}_{X_i} \oplus T_{f|i}$, and two sections (g_i, ϑ_i) and (g_j, ϑ_j) over X_i and X_j, respectively, are equal on X_{ij} if and only if $\vartheta_i = \vartheta_j$ and $g_j - g_i + i_{\vartheta_i}(d\log^0(f_{ij}))$, because $d\log^\bullet(\nabla)$ is given by the cocycle $d\log^\bullet(v_i, f_{ij}) = (-dv_i, d\log f_{ij}) = (0, d\log f_{ij})$.
So $(A_f^\bullet(\nabla), d_A)$ is (up to isomorphism) the complex

$$A_f^\bullet(L)\colon A_f(L) \xrightarrow{d_A} \Omega_f^1 \xrightarrow{d} \Omega_f^2 \xrightarrow{d} \ldots,$$

where the first differential is $d_A := d \circ s_\nabla$. This is due to the commutativity of the first squares of both morphisms in the exact sequence $0 \to \Omega_f^\bullet \to A^\bullet \to T_f[0] \to 0$.
Since the log Atiyah extension splits, we have short exact sequences

$$0 \to \mathbb{H}^i(\Omega_f^\bullet) \to \mathbb{H}^i(A_f^\bullet(\nabla)) \to H^i(X, T_f) \to 0$$

for all i.

3.2.2 The Lie derivative of a flat log connection

Let $f\colon X \to Y$ be a log smooth morphism of log schemes and (L, ∇) a flat log connection on f. For every p we define an $f^{-1}\mathcal{O}_Y$-linear map

$$T_f \times (\Omega_f^p \otimes_{\mathcal{O}_X} L) \to \Omega_f^p \otimes_{\mathcal{O}_X} L$$

by the Lie derivative of ∇, $(\vartheta, \sigma) \mapsto \vartheta^\nabla(\sigma) := \nabla(i_\vartheta \sigma) + i_\vartheta(\nabla\sigma)$ giving an action of log derivations on log forms with values in L.

This action is linked to the action of L-derivations by

$$a(\sigma) = s(a) \cdot \sigma + \pi(a)^\nabla(\sigma)$$

for any local section a of $A_f(L)$, where s is the splitting map $A_f(L) \to \mathcal{O}_X$ and π the projection $A_f(L) \to T_f$.

3.2.3 The logarithmic Lie derivative

Let $f\colon X \to Y$ be a log smooth morphism of log schemes. We define an $f^{-1}\mathcal{O}_Y$-bilinear map

$$\cdot \log \cdot : T_f \times \Omega_X^{\times,\bullet} \to \Omega_X^{\times,\bullet}$$

by the local mapping $(\vartheta, u^\bullet) \mapsto \vartheta \log(u^\bullet) := d\log^\bullet(i_\vartheta(u^\bullet)) + i_\vartheta(d\log^\bullet u^\bullet)$. This means, that for a local section $\vartheta = (\underline{\vartheta}, \Theta)$ of T_f in degree 0 we have $\vartheta \log u = i_\vartheta(d\log u) = \Theta(u)$, in the first degree $\vartheta \log u = d\log(i_\vartheta(u)) + i_\vartheta(du)$ and in higher degrees $\vartheta \log u = \vartheta(u)$ is just the Lie derivation of that degree. We call this pairing the *logarithmic Lie derivative*.

3.2.4 ∇-Derivations

Let $f\colon X \to Y$ be a log smooth log scheme and ∇ a flat log connection f. For every $p, r \in \mathbb{N}_0$ we define an $f^{-1}\mathcal{O}_Y$-bilinear map

$$A_f^r(\nabla) \times \Omega_f^p \otimes_{\mathcal{O}_X} L \to \Omega_f^{p+r} \otimes_{\mathcal{O}_X} L,\ (a, \sigma) \mapsto a(\sigma)$$

as follows:

For $r = 0$, let $a(\sigma)$ be the action of the L-derivation $a \in A_f(L)$ on σ as defined in 3.1.3. For $r > 1$, we have $A^r(\nabla) = \Omega_f^r$ and we define $a(\sigma)$ to be $a \wedge \sigma$.

In relation to its action on forms, we refer to $A_f^\bullet(\nabla)$ as the *complex of derivations extended by* ∇ or ∇*-derivations*. Observe that if $\nabla = d$, then $A_f^\bullet(L) = \Omega_X^\bullet \oplus \mathrm{Der}_f(\mathcal{O}_X)[0]$. Observe also that $a(\sigma_p \wedge \sigma_q) = \pi(a)(\sigma_p) \wedge \sigma_q + \sigma_p \wedge a(\sigma_q)$ for all p-forms σ_p and all q-forms with values in L, σ_q (where $\pi(a) = 0$ for $a \in A_f^r(\nabla)$, $r > 0$).

If σ is a fixed ∇-closed p-form with values in L (i. e. $\nabla\sigma = 0$), then the map $b^\sigma\colon A_f^\bullet(\nabla) \to \Omega_f^\bullet \otimes_{\mathcal{O}_X} L[p]$, $a \mapsto a(\sigma)$ is a morphism of complexes, as seen in the following calculation.

In degree 0 we have locally

$$\begin{aligned}\nabla(a(\sigma)) &= \nabla(g\sigma + \vartheta(\sigma)) = dg \wedge \sigma + d(\vartheta(\sigma)) + v \wedge \vartheta(\sigma)\\ &= dg \wedge \sigma + \vartheta(d\sigma) + v \wedge \vartheta(\sigma)\\ &= dg \wedge \sigma + \vartheta(-v \wedge \sigma) + v \wedge \vartheta(\sigma) = d(g - \vartheta(v)) \wedge \sigma\\ &= d_A(a) \wedge \sigma,\end{aligned}$$

where $a = (g, \vartheta)$ in the local splitting $A_f(L) \cong \mathcal{O}_X \oplus T_f$, and in higher degrees

$$\nabla(a \wedge \sigma) = da \wedge \sigma + a \wedge \nabla\sigma = da \wedge \sigma.$$

A similar calculation shows that if a is a fixed d_A-closed ∇-derivation of degree r, then the map $a(\,\cdot\,)\colon \Omega_f^\bullet \otimes_{\mathcal{O}_X} L \to \Omega_f^\bullet \otimes_{\mathcal{O}_X} L[r]$, $\sigma \mapsto a(\sigma)$ is a morphism of complexes. For general (non-closed) fixed elements this is not the case.

3.3 Log symplectic schemes

3.3.1 Definition

Let $f\colon X \to Y$ be a log smooth morphism of log schemes and $\nabla = (L, \nabla)$ a flat log connection on f. A *logarithmically symplectic form* ω *on* f *of type* ∇ is an element $\omega \in \Gamma(X, \Omega_f^2 \otimes_{\mathcal{O}_X} L)$ such that

a) $\nabla(\omega) = 0$;

b) ω induces an isomorphism $i_{\cdot}(\omega)\colon T_f \to \Omega_f^1 \otimes_{\mathcal{O}_X} L$.

In other words, ω is an element of $\mathbb{H}^0(\Omega_f^{\geq 2,\bullet} \otimes_{\mathcal{O}_X} L)$ the associated skew-symmetric pairing $T_f \wedge_{\mathcal{O}_X} T_f \to L$ of which is non-degenerate at every point x of X. We say that a logarithmic form ω of type $\nabla = (\mathcal{O}_X, d)$ is *of non-twisted type*; if (L, ∇) is not specified, it is *of generally twisted type* or *of general type* for short.

3.3.2 Remark

As in the case of usual smooth symplectic schemes, the non-degeneracy of the pairing $T_f \wedge_{\mathcal{O}_X} T_f \to L$ implies that f is necessarily equidimensional of even dimension $\dim(f) = \dim(X) - \dim(Y) = 2n$.

Moreover, this property is equivalent to the line bundle $\Omega_f^{2n} \otimes L^{\otimes n}$ being generated by a global section, namely by $\omega^{\wedge n}$.

Let $\mathcal{U} = \{X_i\}$ an open affine covering trivialising L. On the level of Čech-cochains, a log symplectic form ω is given by a collection (ω_i), with $\omega_i \in \Gamma(X_i, \Omega_f^1)$ satisfying the following conditions:

a) $\omega_i = f_{ij}\omega_j$;

b) $d\omega_i + \varpi_i \wedge \omega_i = 0$;

c) ω_i induces an isomorphism $i \,.\, (\omega_i) \colon T_{f|i} \to \Omega^1_{f|i}$.

3.3.1 Non-twisted type

3.3.3 Definition

Let Y be a log scheme. A *logarithmically symplectic scheme of non-twisted type* over Y is a pair (f, ω), also written $f \colon (X, \omega) \to Y$, consisting of the following data:

a) A log smooth morphism of log fs log schemes $f \colon X \to Y$;

b) a log symplectic form ω on f of non-twisted type.

The pair (f, ω) is called *proper and log integral* if the morphism of log schemes $f \colon X \to Y$ is proper and log integral. It is called *simple* if, moreover, ω generates the ring $\Gamma(X, \Omega^\bullet_f)$ as an $\Gamma(Y, \mathcal{O}_Y)$-algebra.

A morphism $(f', \omega') \to (f, \omega)$ of log symplectic schemes of non-twisted type over Y is a morphism of Y-log-schemes $h \colon X' \to X$ such that $h^*\omega = \omega'$.

We denote the category of log symplectic schemes of non-twisted type over Y by $\underline{\mathrm{LSnSch}}_Y$.

3.3.4 Definition

Let $f_0 \colon (X, \omega) \to Y$ be a log symplectic scheme of non-twisted type and let $i \colon Y \to \mathcal{Y}$ be a strict closed immersion. A *logarithmically symplectic deformation of non-twisted type* of (f_0, ω) is a log symplectic scheme of non-twisted type $f \colon (\mathcal{X}, \varpi) \to \mathcal{Y}$ together with a strict closed immersion $\hat{\imath} \colon X \to \mathcal{X}$ such that

a) $$\begin{tikzcd} X \arrow[r, hook, "\hat{\imath}"] \arrow[d, "f_0"] & \mathcal{X} \arrow[d, "f"] \\ Y \arrow[r, hook, "i"] & \mathcal{Y} \end{tikzcd}$$

is a log smooth deformation of f_0;

b) $\hat{\imath}^*\varpi = \omega$.

In this case we will simply write $\hat{\imath}^*(f, \varpi) = (f_0, \omega)$ and say that we have a Cartesian diagram

$$\begin{tikzcd} (X, \omega) \arrow[r, hook, "\hat{\imath}"] \arrow[d, "f_0"] & (\mathcal{X}, \varpi) \arrow[d, "f"] \\ Y \arrow[r, hook, "i"] & \mathcal{Y}. \end{tikzcd}$$

Let k be a field. Recall from 1.2.5 that the standard log point on $\operatorname{Spec} k$ is the spectrum of the prelog ring $\kappa \colon \mathbb{N}_0 \to k$, defined by $0 \mapsto 1$ and $n \mapsto 0$ if $n \geq 1$.

3.3.5 Definition

A *logarithmically symplectic variety of non-twisted type* over a field k is a log symplectic scheme $f\colon (X,\omega) \to \operatorname{Spec}\kappa$ of non-twisted type, where $\operatorname{Spec}\kappa$ is the standard log point on $\operatorname{Spec} k$, such that $\underline{f}\colon \underline{X} \to k$ is a variety in the usual sense (i. e. $\underline{X}$ is a reduced separated Noetherian scheme and $\underline{f}$ is of finite type).

3.3.2 General type

3.3.6 Definition

Let Y be a log scheme. A *logarithmically symplectic scheme (of general type)* over Y is a triple (f,∇,ω), also written $f\colon (X,\nabla,\omega) \to Y$, consisting of the following data:

a) A log smooth morphism of log fs log schemes $f\colon X \to Y$;

b) a flat log connection ∇ on f;

c) a log symplectic form ω on f of type ∇.

The triple (f,∇,ω) is called *proper and log integral* if the morphism of log schemes $f\colon X \to Y$ is proper and log integral. It is called *simple* if, moreover, there exists a line bundle $L^{\frac{1}{2}}$ with $L = (L^{\frac{1}{2}})^{\otimes 2}$ and such that ω generates the ring $\Gamma(X, \bigwedge^{\bullet}(\Omega^1_f \otimes L^{\frac{1}{2}}))$ as an $\Gamma(Y,\mathcal{O}_Y)$-algebra.

A morphism $(f',\nabla',\omega') \to (f,\nabla,\omega)$ of log symplectic schemes over Y is a pair (h,ψ), where

a) $h\colon X' \to X$ is a morphism of Y-log schemes;

b) $\psi\colon h^*\nabla \to \nabla'$ is a morphism of flat log connections on f such that $\psi(h^*\omega) = \omega'$.

We denote the category of log symplectic schemes (of general type) over Y by $\underline{\mathrm{LSSch}}_Y$.

3.3.7 Remark

We have a natural functor $\underline{\mathrm{LSnSch}}_Y \to \underline{\mathrm{LSSch}}_Y$ sending a log symplectic scheme of non-twisted type (f,ω) to the log symplectic scheme (f,d,ω) and a morphism of log symplectic schemes of non-twisted type $h\colon (f',\omega') \to (f,\omega)$ to the morphism of log symplectic schemes of general type $(h, id_d)\colon (f',d,\omega') \to (f,d,\omega)$. Observe, that this functor is injective (on objects) and faithful (i. e. injective on morphisms), but in general not full:

In general there exist automorphisms of the trivial flat log connection $\psi\colon d \to d$ on f besides the identity. So given a morphism $(h,\psi)\colon (f',d) \to (f,d)$ of log schemes with trivial flat connection over Y with $\psi \neq id_d$ and defining $\omega' := \psi(h^*\omega)$, we have a morphism of log symplectic schemes of twisted type $(h,\psi)\colon (f',d,\omega') \to (f,d,\omega)$ not coming from a morphism of log symplectic schemes of non-twisted type, although its source and target come from log symplectic schemes of non-twisted type.

Hence $\underline{\mathrm{LSnSch}}_Y$ is a subcategory of $\underline{\mathrm{LSSch}}_Y$, but in general *not a full* subcategory.

3.3.8 Definition

Let $f_0 \colon (X, \nabla, \omega) \to Y$ be a log symplectic scheme and $i \colon Y \to \mathcal{Y}$ a strict closed immersion of log schemes. *A log symplectic deformation of* (f_0, ∇, ω) *over* i *(of general type)* is a log symplectic scheme $f \colon (\mathcal{X}, \Delta, \varpi) \to \mathcal{Y}$ together with a strict closed immersion $\hat{\imath} \colon X \to \mathcal{X}$ such that

a) $$\begin{tikzcd} X \arrow[r, hook, "\hat{\imath}"] \arrow[d, "f_0"] & \mathcal{X} \arrow[d, "f"] \\ Y \arrow[r, hook, "i"] & \mathcal{Y} \end{tikzcd}$$
is a log smooth deformation of f_0;

b) $\hat{\imath}^* \Delta = \nabla$;

c) $\hat{\imath}^* \varpi = \omega$.

In this case we will simply write $i^*(f, \Delta, \varpi) = (f_0, \nabla, \omega)$ and say that we have a Cartesian diagram

$$\begin{tikzcd} (X, \nabla, \omega) \arrow[r, hook, "\hat{\imath}"] \arrow[d, "f_0"] & (\mathcal{X}, \Delta, \varpi) \arrow[d, "f"] \\ Y \arrow[r, hook, "i"] & \mathcal{Y}. \end{tikzcd}$$

3.3.9 Definition

A log symplectic variety (of general type) over a field k is a log symplectic scheme $f \colon (X, \nabla, \omega) \to \operatorname{Spec} \kappa$ (of general type), where $\operatorname{Spec} \kappa$ is the standard log point on $\operatorname{Spec} k$, such that $\underline{f} \colon \underline{X} \to k$ is a variety in the usual sense (i. e. $\underline{X}$ is a reduced Noetherian scheme and $\underline{f}$ is of finite type).

3.3.3 The T-complex

Let $f \colon (X, \omega) \to Y$ be a log symplectic scheme of non-twisted type. To the log symplectic form $\omega \in \mathbb{H}^0(\Omega_f^{\geq 2, \bullet})$ we associate the morphism

$$t^\omega \colon T_f[0] \to \Omega_f^{\geq 2, \bullet}[2]$$

which we define by the local mapping $\vartheta \mapsto \vartheta(\omega)$ in degree 0 and the zero map elsewhere. This morphism in turn defines an element

$$t^\omega \in \operatorname{Hom}_{\mathrm{D^b}(\underline{\mathrm{Comp}}_f)}(T_f[0], \Omega_f^{\geq 2, \bullet}[2]) = \operatorname{Ext}^1_{\mathrm{D^b}(\underline{\mathrm{Comp}}_f)}(T_f[0], \Omega_f^{\geq 2, \bullet}[1])$$

which is the isomorphism class (a distinct triangle in $\mathrm{D^b}(\underline{\mathrm{Comp}}_f)$) of short exact sequences

$$t^\omega \colon 0 \to \Omega_f^{\geq 2, \bullet}[1] \to T^\bullet \to T_f[0] \to 0.$$

of complexes of $\mathcal{O}_X$-modules with $f^{-1}\mathcal{O}_Y$-linear differential, which we call the *T-sequence of ω*.

Here the complex in the middle, which we will refer to as the *T-complex of ω*, is, up to isomorphism, the complex

$$T_f^\bullet(\omega)\colon T_f \xrightarrow{t^\omega} \Omega_f^2 \xrightarrow{-d} \Omega_f^3 \xrightarrow{-d} \dots,$$

with T_f sitting in degree 0 and Ω_f^{p+1} each in degree p. The maps of the sequence t^ω are the obvious inclusion and projection, respectively. The differential d_T of the complex is given by the map t^ω in degree 0 and $-d$ elsewhere (with the sign due to shifting).

Identifying T_f with Ω_f^1 by using the isomorphism $I_\omega = i\,.\,(\omega)\colon T_f \to \Omega_f^1$ leads to a simpler description of the T-complex: Under composition with I_ω^{-1} the map $t^\omega\colon T_f[0] \to \Omega_f^{\geq 2,\bullet}[2]$ becomes simply $-d\colon \Omega_f^1[0] \to \Omega_f^{\geq 2,\bullet}[2]$ due to the fact that $t^\omega(I_\omega^{-1}(\tau)) = I_\omega^{-1}(\tau)(\omega) = di_{I_\omega^{-1}(\tau)}(\omega) = dI_\omega I_\omega^{-1}(\tau) = d\tau$. The T-complex becomes $T_f^\bullet(\omega) \cong \Omega_f^{\geq 1,\bullet}[1]$ and the T-sequence becomes

$$-d\colon 0 \to \Omega_f^{\geq 2,\bullet}[1] \to \Omega_f^{\geq 1,\bullet}[1] \to \Omega_f^1[0] \to 0.$$

In particular the T-sequence and the T-complex (up to isomorphism) are independent on the symplectic form ω as soon as the latter exists.

3.3.4 The B-complex

Let $f\colon (X,\nabla,\omega) \to Y$ be a log symplectic scheme of general type. Recall the definition of the log Atiyah complex $A_f^\bullet(\nabla)$. Let L be the line bundle of ∇. We associate to ω the morphism of complexes

$$b^\omega\colon A_f^\bullet(\nabla) \to \Omega_f^{\geq 2,\bullet} \otimes_{\mathcal{O}_X} L[2]$$

given by the action of ∇-derivations as defined in section 3.2.4.

We associate to this morphism via the identification

$$\mathrm{Hom}_{\mathrm{D^b}(\underline{\mathrm{Comp}}_f)}(A_f^\bullet(\nabla), \Omega_f^{\geq 2,\bullet} \otimes_{\mathcal{O}_X} L[2]) = \mathrm{Ext}^1_{\mathrm{D^b}(\underline{\mathrm{Comp}}_f)}(A_f^\bullet(\nabla), \Omega_f^{\geq 2,\bullet} \otimes_{\mathcal{O}_X} L[1])$$

an isomorphism class (a distinct triangle in $\mathrm{D^b}(\underline{\mathrm{Comp}}_f)$) of short exact sequences

$$b^\omega\colon 0 \to \Omega_f^{\geq 2,\bullet} \otimes_{\mathcal{O}_X} L[1] \to B^\bullet \to A_f^\bullet(\nabla) \to 0$$

of $\mathcal{O}_X$-modules which we call the *B-extension of ω*.

The complex in the middle (or rather any representative of its isomorphism class) will be called the *B-complex of ω* and denoted $B^\bullet_{(f,\nabla)}(\omega)$ or $B_f^\bullet(\omega)$ for short. It is a representative of the cone of the morphism b^ω in the derived category $\mathrm{D^b}(\underline{\mathrm{Comp}}_f)$ and may be constructed as the direct sum $B_f^0 = 0 \oplus A_f$ in degree zero and $B^p(\omega) = (\Omega_f^{p+1} \otimes_{\mathcal{O}_X} L) \oplus A_f^p(\nabla)$ for $p \geq 1$ together with the differential d_B (which is *not* the direct sum of differentials) given by $d_B(\tau,\sigma) = (-\nabla\tau - b^\omega(\sigma), d_A(\sigma))$.

A local description with respect to an open affine covering $\mathcal{U}$ trivialising L is therefore given by

$$B^p_{f|i}(\omega) := B^p_f(\omega)\Big|_{X_i} = \begin{cases} 0 \oplus A_{f|i}(L) & \text{if } p = 0 \\ \Omega^{p+1}_{f|i} \oplus \Omega^p_{f|i} & \text{if } p \geq 1 \end{cases}$$

where two sections (τ_i, σ_i) and (τ_j, σ_j) of $B^p_f(\omega)$ over X_i and X_j, respectively, are equal on X_{ij} if and only if $\sigma_i = \sigma_j$ and $f_{ij}\tau_j - \tau_i = 0$.

We may again identify T_f with $\Omega^1_f \otimes_{\mathcal{O}_X} L$ by using the isomorphism $I_\omega = i\,.\,(\omega)\colon T_f \to \Omega^1_f \otimes_{\mathcal{O}_X} L$, which leads to a simpler description of the B-complex:

Recall that the log Atiyah extension of L splits. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_f \arrow[r, "t"] \arrow[d, "I_\omega", "\cong"'] & A_f(L) \arrow[r, "s"] \arrow[l, bend left, "\pi"] \arrow[d, "\Phi_{\nabla,\omega}", "\cong"'] & \mathcal{O}_X \arrow[r] \arrow[l, bend left, "i"] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1_f \otimes L \arrow[r, shift left] & (\Omega^1_f \otimes L) \oplus \mathcal{O}_X \arrow[r, shift left] \arrow[l, shift left] & \mathcal{O}_X \arrow[r] \arrow[l, shift left] & 0,
\end{tikzcd}$$

the first row of which is the Atiyah extension read backwards through its splitting, the maps in the second row of which are the natural inclusions and projections and the isomorphism $\Phi_{\nabla,\omega}\colon A_f(L) \to (\Omega^1_f \otimes L) \oplus \mathcal{O}_X$ is defined as $\Phi_{\nabla,\omega} := (I \circ p, s)$ with inverse $\Phi^{-1}_{\nabla,\omega} = i \circ pr_2 + t \circ I^{-1} \circ pr_1$.

This gives us a simpler description of the log Atiyah complex of ∇: Define $A'^{\bullet}_f(\nabla)$ to be the complex

$$A'^{\bullet}_f(\nabla)\colon (\Omega^1_f \otimes L) \oplus \mathcal{O}_X \to \Omega^1_f \to \Omega^2_f \to \ldots,$$

where the zeroth differential is $d \circ pr_2$ and all other differentials d. Then $\Phi_{\nabla,\omega}$ induces an isomorphism $A'^{\bullet}_f(\nabla) \cong A^{\bullet}_f(\nabla)$; indeed $d_A = d \circ s = (d \circ pr_2) \circ \Phi_{\nabla,\omega}$.

So we may regard the B-extension as an extension of $A'^{\bullet}_f(\nabla)$ by $\Omega^{\geq 2,\bullet}_f \otimes L$,

$$b^\omega\colon 0 \to \Omega^{\geq 2,\bullet}_f \otimes L \xrightarrow{j} B^{\bullet}_f(\omega) \xrightarrow{\mu} A'^{\bullet}_f(\nabla) \to 0$$

The isomorphism class in the triangulated category $\mathrm{D^b}(\underline{\mathrm{Comp}}_f)$ of this sequence is a distinguished triangle, because $B^{\bullet}_f(\omega)$ represents the cone of the morphism b^ω. The next diagram shows an excerpt of this triangle, into which we fit the lower exact splitting sequence from above; for better legibility, we write $B := B_f(\omega)$, $\Omega := \Omega_f$ etc.

$$
\begin{tikzcd}[column sep=small]
 & & & & & & & & \Omega^1\otimes L \arrow[dl, shift left] \arrow[dr] & \\
 & & & & & B^0 \arrow[rr, "\Phi_{\nabla,\omega}", "\cong"'] \arrow[dr, "s"] \arrow[dd, "d_B"] & & A'^0 \arrow[ur, shift left] \arrow[rr, "b^\omega[1]"] \arrow[dl, shift left] \arrow[dd, "d\circ pr_2"] & & \dots \\
 & & & \Omega^1\otimes L \arrow[dl, shift left] \arrow[dr, "-\nabla"] \arrow[urr, bend left, "t\circ I^{-1}"] & & & \mathcal{O} \arrow[ur, shift left] \arrow[dr, "d"] & & & \\
\dots \arrow[rr, "\Phi_{\nabla,\omega}", "\cong"'] & & A'^0 \arrow[ur, shift left] \arrow[rr, "b^\omega[1]"] \arrow[dl, shift left] \arrow[dd, "-d\circ pr_2"] & & \Omega^2\otimes L \arrow[r, "j"] \arrow[dd, "-\nabla"] & B^1 \arrow[rr, "\mu"] \arrow[dd, "d_B"] & & \Omega^1 \arrow[rr, "\wedge\,\omega[1]"] \arrow[dd, "d"] & & \dots \\
 & \mathcal{O} \arrow[ur, shift left] \arrow[dr, "d"] & & & & & & & & \\
\dots \arrow[rr, "\mu"] & & \Omega^1 \arrow[rr, "\wedge\,\omega[1]"] \arrow[d, "-d"] & & \Omega^3\otimes L \arrow[r, "j"] \arrow[d, "-\nabla"] & B^2 \arrow[rr, "\mu"] \arrow[d, "d_B"] & & \Omega^2 \arrow[rr, "\wedge\,\omega[1]"] \arrow[d, "d"] & & \dots \\
 & & \vdots & & \vdots & \vdots & & \vdots & &
\end{tikzcd}
$$

By tilting the upper part of this diagram to an "upright position" and erasing the A'^0's we get the distinguished triangle

$$
\begin{tikzcd}
 & & \Omega^1\otimes L \arrow[r, "t\circ I^{-1}"] \arrow[d, "-\nabla"] & B^0 \arrow[r, "s"] \arrow[d, "d_B"] & \mathcal{O} \arrow[r, "\cdot\,\omega[1]"] \arrow[d, "d"] & \dots \\
\dots \arrow[r, "s"] & \mathcal{O} \arrow[r, "\cdot\,\omega[1]"] \arrow[d, "d"] & \Omega^2\otimes L \arrow[r, "j"] \arrow[d, "-\nabla"] & B^1 \arrow[r, "\mu"] \arrow[d, "d_B"] & \Omega^1 \arrow[r, "\wedge\,\omega[1]"] \arrow[d, "d"] & \dots \\
\dots \arrow[r, "\mu"] & \Omega^1 \arrow[r, "\wedge\,\omega[1]"] \arrow[d, "d"] & \Omega^3\otimes L \arrow[r, "j"] \arrow[d, "-\nabla"] & B^2 \arrow[r, "\mu"] \arrow[d, "d_B"] & \Omega^2 \arrow[r, "\wedge\,\omega[1]"] \arrow[d, "d"] & \dots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

which replaces b^ω by the morphism of complexes

$$\hat{\omega}\colon\, \Omega_f^\bullet \to \Omega_f^{\geq 1,\bullet}\otimes L[2],\ \sigma\mapsto \sigma\wedge\omega$$

in $\mathrm{Hom}_{\underline{\mathrm{Comp}}_f}(\Omega_f^\bullet, \Omega_f^{\geq 1,\bullet}\otimes L[2])$.

This leads to a simpler description of the isomorphism class of the complex $B_f^\bullet(\omega)$: Clearly, the isomorphism class of $B_f^\bullet(\omega)$ is the cone of the (class of the) morphism $\hat{\omega}$ in $\mathrm{D^b}(\underline{\mathrm{Comp}}_f)$. Hence, we can replace our previous representative of $B_f^\bullet(\omega)$ by the canonical representative $B_f'^\bullet(\omega)$ of this cone, which is given in each degree $p\geq 0$ by $B_f^p = (\Omega_f^{p+1}\otimes_{\mathcal{O}_X} L)\oplus\Omega_f^p$ and which has the differential d_B defined by $d_B(\tau,\sigma) = (-\nabla\tau - \sigma\wedge\omega, d\sigma)$.

Then $B_f^p = B_f'^p$ for $p\geq 1$ and the "new" degree-zero module $B_f'^0$ fits into the above diagrams in place of B_f^0, when replacing s by the natural projection $q = s\circ\Phi_{\nabla,\omega}\colon B_f'^0\to\mathcal{O}_X$ and $t\circ I^{-1}$ by the natural inclusion $j = \Phi_{\nabla,\omega}\circ t\circ I^{-1}\colon \Omega_f^1\otimes L\to B_f'^0$.

Therefore, we have the short exact sequence

$$\hat{\omega}\colon\, 0\to\Omega_f^{\geq 1,\bullet}\otimes L[1]\to B_f^\bullet(\omega)\to\Omega_f^\bullet\to 0$$

which corresponds to the element

$$\hat{\omega} \in \operatorname{Ext}^1_{\underline{\mathrm{Comp}}_f}(\Omega_f^\bullet, \Omega_f^{\geq 1,\bullet} \otimes L[1]) = \operatorname{Hom}_{\underline{\mathrm{Comp}}_f}(\Omega_f^\bullet, \Omega_f^{\geq 1,\bullet} \otimes L[2]).$$

3.4 Log Cartier divisors

Let X be a log scheme with log structure $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$. For an open subscheme $U \subset X$ we set

$$\mathcal{M}_X^{\mathrm{dom}}(U) := \left\{ m \in \Gamma(U, \mathcal{M}_X) \,\middle|\, m_x \in \mathcal{M}_{X,x}^{\mathrm{dom}} \; \forall x \in U \right\},$$

where $\mathcal{M}_{X,x}^{\mathrm{dom}}$ is the submonoid of $\mathcal{M}_{X,x}$ of non-absorbent-divisors. The presheaf $U \mapsto \mathcal{M}_X^{\mathrm{dom}}(U)$ is a subsheaf of $\mathcal{M}_X$, denoted $\mathcal{M}_X^{\mathrm{dom}}$, which consists of all sections which are not zero divisors in $\mathcal{M}_X$. Then the localisation

$$U \mapsto \Gamma(U, \mathcal{M}_X^{\mathrm{dom}})^{-1}\Gamma(U, \mathcal{M}_X)$$

is a presheaf on X. We denote by $\mathcal{M}_X^{\mathrm{rat}}$ the associated sheaf and call it the *rational monoid sheaf* of X. Its subsheaf of invertible sections, denoted $\mathcal{M}_X^{\mathrm{rat}\times}$, equals $(\mathcal{M}_X^{\mathrm{dom}})^{\mathrm{rat}\times}$.

If X carries the hollow log structure $id\colon \mathcal{O}_X \to \mathcal{O}_X$, then $\mathcal{O}_X^{\mathrm{dom}}$ is the monoid sheaf of non-zero-divisors in $\mathcal{O}_X$. In this case $\mathcal{O}_X^{\mathrm{rat}}$ equals the sheaf of rational functions $\mathcal{K}_X$ (considered as a multiplicative sheaf of monoids) and $\mathcal{O}_X^{\mathrm{rat}\times}$ its subsheaf of invertible elements $\mathcal{K}_X^\times$ (cf. [22]); for the trivial log structure $\iota\colon \mathcal{O}_X^\times \to \mathcal{O}_X$ on X we have $\mathcal{O}_X^{\times\mathrm{dom}} = \mathcal{O}_X^\times = \mathcal{O}_X^{\times\mathrm{rat}} = (\mathcal{O}^\times)_X^{\mathrm{rat}\times}$.

The localisation $\mathcal{M}_X \to \mathcal{M}_X^{\mathrm{grp}}$ factors canonically via $\mathcal{M}_X^{\mathrm{rat}}$. If X is a (unit-/quasi-)integral log scheme, then $\mathcal{M}_X^{\mathrm{rat}} = \mathcal{M}_X^{\mathrm{rat}\times}$ is the associated group sheaf $\mathcal{M}_X^{\mathrm{grp}}$ of the log structure sheaf (cf. Appendix A).

3.4.1 Log Cartier divisors and line bundles

Let X be a log scheme. Then the sheaf of units $\mathcal{O}_X^\times$ acts regularly on $\mathcal{M}_X^{\mathrm{rat}\times}$ as a sheaf of subgroups. Defining $\overline{\mathcal{M}_X^{\mathrm{rat}\times}}$ as the sheaf of $\mathcal{O}_X^\times$-orbits in $\mathcal{M}_X^{\mathrm{rat}\times}$ we thus get a short exact sequence

$$0 \to \mathcal{O}_X^\times \to \mathcal{M}_X^{\mathrm{rat}\times} \xrightarrow{\pi} \overline{\mathcal{M}_X^{\mathrm{rat}\times}} \to 0$$

of sheaves of groups.

3.4.1 Definition (cp. [29, III.1.1.7])

A *logarithmic Cartier divisor* on X is an element of the group $\mathrm{LCar}(X) := H^0(X, \overline{\mathcal{M}_X^{\mathrm{rat}\times}})$. We assign to each log Cartier divisor D the fibre $\pi^{-1}(-D)$ under $\pi\colon \mathcal{M}_X^{\mathrm{rat}\times} \to \overline{\mathcal{M}_X^{\mathrm{rat}\times}}$ of its inverse $-D$, which is a subtorsor of $\mathcal{M}_X^{\mathrm{rat}\times}$ denoted $\mathcal{M}_X(D)^\times$. Its associated line bundle $\mathcal{M}_X(D)^\times \otimes_{\mathcal{O}_X^\times} \mathcal{O}_X$, which is the $\mathcal{O}_X$-subbundle of $\mathcal{M}_X^{\mathrm{rat}\times} \otimes_{\mathcal{O}_X^\times} \mathcal{O}_X$ generated by $\mathcal{M}_X(D)^\times$, is denoted $\mathcal{M}_X(D)$.

3.4.2 Remark

Let X be a log scheme and let D be a log Cartier divisor on X. Let (m_i) be a Čech-0-cocycle in $\mathcal{M}_X^{\mathrm{rat}\times}$ with respect to an open covering $\mathcal{U} = \{X_i\}$ of X representing D. Then the $\mathcal{O}_X^\times$-torsor $\mathcal{M}_X(D)^\times$ is given by

$$\begin{aligned}&\Gamma(U, \mathcal{M}_X(D)^\times) = \\ &\qquad \left\{ m \in \Gamma(U, \mathcal{M}_X^{\mathrm{rat}\times}) \,\middle|\, m_x \cdot m_{i,x} \in \mathcal{O}_{X,x}^\times \text{ for all } x \in U \cap X_i \text{ and all } i \right\}\end{aligned}$$

and we have isomorphisms $\mathcal{M}_X(D)^\times|_{X_i} = \frac{1}{m_i}\mathcal{O}_{X_i}^\times \to \mathcal{O}_{X_i}^\times$ given by multiplication with m_i, for each i.

The line bundle $\mathcal{M}_X(D)$ is given by

$$\begin{aligned}&\Gamma(U, \mathcal{M}_X(D)) = \\ &\qquad \left\{ m \in \Gamma(U, \mathcal{M}_X^{\mathrm{rat}\times} \otimes_{\mathcal{O}_X^\times} \mathcal{O}_X) \,\middle|\, m_x \cdot m_{i,x} \in \mathcal{O}_{X,x} \text{ for all } x \in U \cap X_i \text{ and all } i \right\}\end{aligned}$$

and we have isomorphisms $\mathcal{M}_X(D)|_{X_i} = \frac{1}{m_i}\mathcal{O}_X \to \mathcal{O}_{X_i}$ given by multiplication with m_i, for each i.

3.4.3 Proposition

The natural group homomorphism $\delta_X : \mathrm{LCar}(X) \to \mathrm{Pic}(X)$ in the long exact cohomology sequence associated to the above short exact sequence assigns to each log Cartier divisor D the isomorphism class of the line bundle $\mathcal{M}_X(D)$ (respectively, the isomorphism class of the $\mathcal{O}_X^\times$-torsor $\mathcal{M}_X(D)^\times$).

Proof: In terms of Čech-cocyles we can describe the map δ_X explicitly as follows: Let $(\overline{m}_i)$ be a 0-cocycle in $\overline{\mathcal{M}_X^{\mathrm{rat}\times}}$ representing D and let m_i be a representative in $\mathcal{M}_X^{\mathrm{rat}\times}$ of $\overline{m}_i$ for each i. Then the cochain (m_i) is in general not a cocycle but on each X_{ij}, $f_{ij} := m_i m_j^{-1}$ is a unit. The 1-cochain (f_{ij}) in $\mathcal{O}_X^\times$ clearly satisfies $f_{ij} f_{jk} f_{ik}^{-1} = 1$ and hence is a cocycle, defining an isomorphism class of line bundles. If (m_i') is another cochain of representatives, then $m_i' = f_i m_i$ with units f_i. But then $f_{ij}' = f_{ij}\frac{f_j}{f_i}$, so (f_{ij}') and (f_{ij}) differ by a coboundary and define the same class in $\mathrm{Pic}(X)$.

Looking at the description of $\mathcal{M}_X(D)$ (respectively, of $\mathcal{M}_X(D)^\times$) in 3.4.2, we see that $\mathcal{M}_X(D)$ (respectively, $\mathcal{M}_X(D)^\times$) belongs to that particular isomorphism class of line bundles (respectively, of $\mathcal{O}_X^\times$-torsors).

Observe, that $f_{ij} = m_i m_j^{-1}$ does not make (f_{ij}) a coboundary in $\mathcal{O}_X^\times$ but only in $\mathcal{M}_X^{\mathrm{rat}\times}$. This, of course, corresponds to the fact that $\mathrm{Im}(\delta_X) = \mathrm{Ker}(\mathrm{Pic}(X) \to H^1(\mathcal{M}_X^{\mathrm{rat}\times}))$. □

3.4.4 Definition

We denote by $\mathrm{LCar}^0(X) \subset \mathrm{LCar}(X)$ the kernel of the map δ_X, which equals the image of $H^0(\mathcal{M}_X^{\mathrm{rat}\times}) \to \mathrm{LCar}(X)$, and we call it the group of *principal log Cartier divisors.* Naturally, $\mathrm{LCar}^0(X)$ is isomorphic to the cokernel $H^0(\mathcal{M}_X^{\mathrm{rat}\times})/H^0(\mathcal{O}_X^\times)$ of the inclusion $H^0(\mathcal{O}_X^\times) \to H^0(\mathcal{M}_X^{\mathrm{rat}\times})$.

We say that two log Cartier divisors on X are linearly equivalent if their difference is principal. The group of these equivalence classes is denoted

$$\overline{\mathrm{LCar}}(X) := \mathrm{LCar}(X)/\mathrm{LCar}^0(X).$$

The morphism δ_X induces an injection $\overline{\delta}_X : \overline{\mathrm{LCar}}(X) \to \mathrm{Pic}(X)$.

3.4.5 Definition

Let X be a log scheme and let X^{id} denote the log scheme with the same underlying scheme $\underline{X}$, but equipped with the hollow log structure. We write $\mathrm{Car}(X) := \mathrm{LCar}(X^{id})$ and $\overline{\mathrm{Car}}(X) := \overline{\mathrm{LCar}}(X^{id})$. These are the classical groups of (ordinary) Cartier divisors (and of their classes modulo linear equivalence, respectively) on $\underline{X}$.

3.4.6 Definition

Let X be a log scheme. We call a log Cartier divisor *effective* if it lies in $\mathrm{LCar}^+(X) := H^0(\overline{\mathcal{M}^{\mathrm{dom}}}_X) \subset \mathrm{LCar}(X)$.

For any log scheme X, the log structure α_X descends to a well-defined map $\overline{\alpha}_X : \overline{\mathcal{M}}_X \to \overline{\mathcal{O}}_X$ between the characteristic sheaves of α_X and of the hollow log structure id_X. Hence, we have an induced map $\overline{\alpha}_X : \mathrm{LCar}^+(X) \to H^0(\overline{\mathcal{O}}_X)$. If $\underline{X}$ is an integral scheme, then this is a map between the effective log Cartier divisors on X and its (usual) effective Cartier divisors $\mathrm{Car}^+(X) = H^0(\overline{\mathcal{O}^{\mathrm{dom}}}_X) = H^0(\overline{\mathcal{O}}_X)$. In general however, a map $\mathrm{LCar}^+(X) \to \mathrm{Car}^+(X)$ does not exist, because α might map domainic sections of $\mathcal{M}_X$ to zero-dividing sections of $\mathcal{O}_X$ (but compare 3.5.7).

3.4.7 Definition

We will call those line bundles the isomorphism classes in $\mathrm{Pic}(X)$ of which lie in the subgroup

$$\mathrm{Pic}^{\times}(X) := \mathrm{Im}(\overline{\delta}_X) = \mathrm{Im}(\delta_X) = \mathrm{Ker}(\mathrm{Pic}(X) \to H^1(X, \mathcal{M}_X^{\mathrm{rat}\times}))$$

log Cartier.

3.4.8 Lemma

For every log Cartier line bundle $L \in \mathrm{Pic}^{\times}(X)$ there exists an isomorphism $\psi\colon\ L|_{X^\times} \to \mathcal{O}_{X^\times}$ over the log trivial locus $X^\times$ of X (cf. 1.2.13).

Proof: Let L be log Cartier. Then it is equal to $\mathcal{M}_X(D)$ for some log divisor D. Since the restriction of D to the log trivial locus is trivial (because $\overline{\mathcal{M}_X^{\mathrm{rat}\times}}|_{X^\times} = 0$), we have $L|_{X^\times} = \mathcal{M}_X(D)|_{X^\times} = \mathcal{O}_{X^\times}^\times(D|_{X^\times}) = \mathcal{O}_{X^\times}^\times(0) = \mathcal{O}_{X^\times}$, so $L|_{X^\times}$ and $\mathcal{O}_{X^\times}$ are isomorphic. □

The converse is false in general. For example, take $\underline{X} = \mathbb{P}^1_k$ (with projective coordinates z_0 and z_1) with the log structure being trivial on the affine chart $X_1 = \mathbb{P}^1_{k,z_1^{-1}}$ and being induced on the chart $X_0 = \mathbb{P}^1_{k,z_0^{-1}}$ by $\mathbb{N}_{0X_0} \to \mathcal{O}_{X_0}$, $n \mapsto (\frac{z_1}{z_0})^{2n}$. Then $X^\times = X_1^\iota$. Let P

be the point given by $z_1 = 0$ in $\mathbb{P}^1_k$. Then $\mathcal{M}_X(n) = \mathcal{O}_X(2nP)$ for $n \in \mathbb{Z} = H^0(\overline{\mathcal{M}_X^{\mathrm{rat}\times}})$, but also each of the non-log-Cartier line bundles $\mathcal{O}_X((2m+1)P)$, $m \in \mathbb{Z}$, allows an isomorphism $\mathcal{O}_X((2m+1)P)|_{X^\times} \cong \mathcal{O}_{X^\times}$, because $X^\times$ is affine.

3.4.9 Lemma

Let $X' \to X \leftarrow X''$ be a diagram of log schemes. There is a canonical group isomorphism $\mathrm{LCar}(X' \times_X X'') \cong \mathrm{LCar}(X') \times_{\mathrm{LCar}(X)} \mathrm{LCar}(X'')$.

Proof: Since the fibred product (as presheaves) of sheaves is already a sheaf, we have $\mathrm{LCar}(X' \times_X X'') = H^0(X' \times_X X'', \overline{\mathcal{M}^{\mathrm{rat}\times}_{X' \times_X X''}}) = H^0(\overline{\mathcal{M}^{\mathrm{rat}\times}_{X'}}) \times_{H^0(\overline{\mathcal{M}^{\mathrm{rat}\times}_{X}})} H^0(\overline{\mathcal{M}^{\mathrm{rat}\times}_{X''}}) = \mathrm{LCar}(X') \times_{\mathrm{LCar}(X)} \mathrm{LCar}(X'')$. □

3.4.10 Lemma

Let $i\colon X \to \mathcal{X}$ be an infinitesimal thickening of log integral log schemes. Then there is a canonical isomorphism

$$\mathrm{LCar}(X) \cong \mathrm{LCar}(\mathcal{X}).$$

Proof: By lemma 1.2.12, we have $\overline{\mathcal{M}}_X^{\mathrm{grp}} \cong i^{-1}\overline{\mathcal{M}}_{\mathcal{X}}^{\mathrm{grp}}$. Since the underlying topological space of X and $\mathcal{X}$ is the same, $i^{-1} = id$, thus $\mathrm{LCar}(X) = H^0(X, \overline{\mathcal{M}}_X^{\mathrm{grp}}) \cong H^0(X, i^{-1}\overline{\mathcal{M}}_{\mathcal{X}}^{\mathrm{grp}}) = H^0(\mathcal{X}, \overline{\mathcal{M}}_{\mathcal{X}}^{\mathrm{grp}}) = \mathrm{LCar}(\mathcal{X})$. □

3.4.2 Log Cartier divisors and flat log connections

Let $f\colon X \to Y$ be a morphism of log schemes. The log derivation $d\log\colon \mathcal{M}_X \to \Omega^1_f$ extends canonically to $d\log\colon \mathcal{M}_X^{\mathrm{rat}} \to \Omega^1_f$, due to the fact that, by definition, $d\log$ factors via $\mathcal{M}_X^{\mathrm{grp}}$, hence via $\mathcal{M}_X^{\mathrm{rat}}$.

Regard the short exact sequence

$$0 \to \Omega_f^{\times,\bullet} \to M_f^\bullet \to \overline{\mathcal{M}_X^{\mathrm{rat}\times}}[0] \to 0$$

of complexes, where we define the *rational unit complex* $M_f^\bullet$ as

$$M_f^\bullet\colon \mathcal{M}_X^{\mathrm{rat}\times} \to \Omega^1_f \to \Omega^2_f \to \Omega^3_f \to \dots$$

with $d\log\colon \mathcal{M}_X^{\mathrm{rat}\times} \to \Omega^1_f$ as its first differential and d elsewhere.

This short exact sequence gives rise to a long exact sequence

$$0 \to \mathbb{H}^0(\Omega_f^{\times,\bullet}) \to \mathbb{H}^0(M_f^\bullet) \to \mathrm{LCar}(X) \xrightarrow{\delta_f} \mathrm{LConn}(f) \to \mathbb{H}^1(M_f^\bullet) \to \dots .$$

3.4.11 Definition

Let D be a log Cartier divisor. Let $\mathcal{U} = \{X_i\}$ be an open covering trivialising its associated line bundle $\mathcal{M}_X(D)$. By remark 3.4.2 we have $\mathcal{M}_X(D)|_{X_i} = \frac{1}{m_i}\mathcal{O}_X$ and we define $\nabla(\frac{1}{m_i}s) := \frac{1}{m_i}(ds - s d\log m_i)$ on X_i for each i. Since on X_{ij} we have $\nabla(\frac{1}{m_j}f_{ij}s) = f_{ij}\frac{1}{m_j}(ds + s d\log f_{ij} - s d\log m_j) = \frac{1}{m_i}(ds - s d\log m_i) = \nabla(\frac{1}{m_i}s)$, this defines a flat log connection $\nabla\colon \mathcal{M}_X(D) \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}_X(D)$ on f, which we denote by $M_f(D)$.

3.4.12 Proposition

The natural map δ_f in this long exact sequence assigns to each log Cartier divisor D the isomorphism class of the flat log connection $M_f(D)$.

Proof: We may describe δ_f explicitly in terms of Čech-cocyles as follows: Let $(\overline{m}_i)$ be a 0-cocycle in $\overline{\mathcal{M}_X^{\mathrm{rat}\times}}$ representing D and let m_i be a representative in $\mathcal{M}_X^{\mathrm{rat}\times}$ of $\overline{m}_i$ for each i. Then the cochain (m_i) induces units $f_{ij} := m_i m_j^{-1}$ on X_{ij} and 1-forms $v_i := -d\log m_i$ on X_i. The 1-cochain (f_{ij}, v_i) in $\Omega_X^{\times,\bullet}$ clearly satisfies $f_{ij}f_{jk}f_{ik}^{-1} = 1, d\log f_{ij} - \check{d}(v_i) = 0$ and $dv_i = 0$ and hence is a cocycle defining the isomorphism class of the flat log connection $M_f(D)$. If (m_i') is another cochain of representatives, then $m_i' = f_i m_i$ with units f_i. But then $f_{ij}' = f_{ij}\frac{f_j}{f_i}$ and $v_i' = v_i + d\log f_i$, so (f_{ij}', v_i') and (f_{ij}, v_i) differ by a coboundary and define the same class in $\mathrm{LConn}(X)$. From definition 3.4.11 we see that $M_f(D)$ belongs to that particular isomorphism class flat log connections.

Observe, that though $f_{ij} = m_i m_j^{-1}$ and $v_i = -d\log m_i$ do not make (f_{ij}, v_i) a coboundary in $\Omega_X^{\times,\bullet}$, they do in $M_f^\bullet$. This, of course, corresponds to the fact that $\mathrm{Im}(\delta_f) = \mathrm{Ker}(\mathrm{LConn}(f) \to H^1(M_f^\bullet))$. □

3.4.13 Definition

We denote by $\mathrm{LCar}^0(f) \subset \mathrm{LCar}(X)$ the kernel of δ_f, which equals the image of $\mathbb{H}^0(M_f^\bullet) \to \mathrm{LCar}(X)$, and call it the group of *f-principal log Cartier divisors.* Naturally, $\mathrm{LCar}^0(f)$ is isomorphic to the cokernel $\mathbb{H}^0(M_f^\bullet)/\mathbb{H}^0(\Omega_f^{\times,\bullet})$ of the inclusion $\mathbb{H}^0(\Omega_f^{\times,\bullet}) \to \mathbb{H}^0(M_f^\bullet)$.

We say that two log Cartier divisors are f-linearly equivalent if their difference is f-principal. We define

$$\overline{\mathrm{LCar}}(f) := \mathrm{LCar}(X)/\mathrm{LCar}^0(f).$$

The map δ_f induces an injection $\delta_f\colon \overline{\mathrm{LCar}}(f) \to \mathrm{LConn}(f)$.

Observe that we have to distinguish between principal and f-principal log Cartier divisors: Since $\mathbb{H}^0(M_f^\bullet) \subset H^0(\mathcal{M}_X^{\mathrm{rat}\times})$, every f-principal log Cartier divisor is principal, i. e. $\mathrm{LCar}^0(f) \subset \mathrm{LCar}^0(X)$, but in general not the other way round. Hence, we have a canonical surjection $\overline{\mathrm{LCar}}(f) \to \overline{\mathrm{LCar}}(X)$.

3.4.14 Definition

We will say that flat log connections ∇ on f the isomorphism classes in $\mathrm{LConn}(f)$ of which lie in

$$\begin{aligned}\mathrm{LConn}^\times(f) &:= \mathrm{Im}(H^0(\overline{\mathcal{M}_X^{\mathrm{rat}\times}}) \to \mathbb{H}^1(X, \Omega_f^{\times,\bullet})) = \mathrm{Ker}(\mathbb{H}^1(X, \Omega_f^{\times,\bullet}) \to \mathbb{H}^1(M_f^\bullet))\\ &= \mathrm{Im}(\delta_f\colon \overline{\mathrm{LCar}}(f) \to \mathrm{LConn}(f))\end{aligned}$$

are *log Cartier.*

3.4.15 Lemma

For any flat log Cartier connection ∇ on f there exists an isomorphism $\psi\colon\ \nabla|_{X^\times(f)} \to d|_{X^\times(f)}$ over the is the semi-strict locus $X^\times(f)$ of f (cf. 1.2.17).

Proof: Let ∇ be a flat log Cartier connection. There is a log Cartier divisor $D \in \mathrm{LCar}(X)$ such that $\nabla = M_f^\bullet(D)$. Then $\nabla|_{X^\times(f)} = M_f^\bullet(D)\Big|_{X^\times(f)} = M_f^\bullet\Big|_{X^\times(f)}(D|_{X^\times(f)}) = \Omega^{\times,\bullet}_{f|_{X^\times(f)}}(0) = d|_{X^\times(f)}$, thus there exists an isomorphism $\psi\colon\ \nabla|_{X^\times(f)} \to d|_{X^\times(f)}$. □

It is clear from the definition and the surjectivity of $\overline{\mathrm{LCar}}(f) \to \overline{\mathrm{LCar}}(X)$, that the natural map $\mathrm{LConn}^\times(f) \to \mathrm{Pic}^\times(X)$ is surjective. This means, that to every log Cartier line bundle $\mathcal{M}_X(D)$ there exists a flat log Cartier connection with line bundle $\mathcal{M}_X(D)$, namely $M_f(D)$.

3.4.16 Lemma
Given three morphisms of log schemes $X \xrightarrow{f} Y \xrightarrow{g} Z$ and a flat log Cartier connection ∇ on f, there exists a flat log Cartier connection $\tilde{\nabla}$ on $g \circ f$ which up to isomorphism restricts to ∇ on f.

Proof: We have the given flat log Cartier connection $\nabla\colon L \to \Omega^1_f \otimes L$ which we want to lift to a flat log connection $\tilde{\nabla}\colon \tilde{L} \to \Omega^1_{g\circ f} \otimes \tilde{L}$. By assumption, $\nabla = M_f(D) \in \mathrm{LConn}(f)$ for a log divisor $D \in \mathrm{LCar}(X)$. But then we simply set $\tilde{\nabla} := M_{g\circ f}(D)$. By construction, there is an isomorphism of line bundles $\psi\colon \tilde{L} \to L$ and $\tilde{\nabla}$ is such that the diagram

$$\begin{tikzcd}
\tilde{L} \arrow[r, "\tilde{\nabla}"] \arrow[d, "id_{\tilde{L}}"] & \Omega^1_{g\circ f} \otimes \tilde{L} \arrow[d, "\cdot|_f \otimes id_{\tilde{L}}"] \\
\tilde{L} \arrow[r, "\tilde{\nabla}|_f"] \arrow[d, "\psi"] & \Omega^1_f \otimes \tilde{L} \arrow[d, "id \otimes \psi"] \\
L \arrow[r, "\nabla"] & \Omega^1_f \otimes L
\end{tikzcd}$$

commutes. □

3.5 Log structures associated to other structures

In this section we introduce natural log structures on (strict) normal crossing schemes. To do this systematically, we first collect results about the log structures associated to schemes with open immersion and to schemes with Deligne-Faltings structure. Here, as in the lecture notes of Ogus ([29]), the introduction of schemes with Deligne-Faltings structure serves as a passage from log structures on schemes with open immersion to log structures on normal crossing schemes. The chapter closes with a collection of results on (strict) normal crossing schemes.

We assume all schemes to be Noetherian.

3.5.1 The compactifying log structure on schemes with open immersion

3.5.1 Definition

A *scheme with open immersion* $X = (\underline{X}, j_X)$ is a pair consisting of a scheme $\underline{X}$ (without log structure) together with an open immersion $j_X : U_X \to \underline{X}$ of a non-empty subscheme $U_X \subset \underline{X}$. We will always denote the reduced closed complement of $j_X(U_X)$ in $\underline{X}$ by Z_X and its closed immersion by $i_X : Z_X \to \underline{X}$.

A *morphism of schemes with open immersion* $f : X \to Y$ is a morphism of schemes $\underline{f} : \underline{X} \to \underline{Y}$ such that $\underline{f} \circ j_X : U_X \to \underline{Y}$ factors over j_Y as $\underline{f} : U_X \to U_Y$. We denote the category of schemes with open immersion by $\underline{\mathrm{OSch}}$.

If S is a scheme, then (S, id_S) is a scheme with open immersion. The inclusion functor $(\,\cdot\,, id_{\cdot}) : \underline{\mathrm{Sch}} \to \underline{\mathrm{OSch}}$ makes $\underline{\mathrm{Sch}}$ a full subcategory of $\underline{\mathrm{OSch}}$ and is right adjoint to the forgetful functor $X \mapsto \underline{X}$.

3.5.2 Definition

Let $X = (\underline{X}, j_X)$ be a scheme with open immersion and let ι_{U_X} denote the trivial log structure on the subscheme U_X. The direct image log structure

$$(j_X)_\times \iota_{U_X} : (j_X)_\times \mathcal{O}_{U_X}^\times \to \mathcal{O}_X$$

is called the *compactifying log structure* of X. It will be denoted $\alpha_{X^\jmath}$ and its log structure sheaf $\mathcal{M}_{X^\jmath}$. The log scheme $(\underline{X}, \jmath_X)$ defined that way will be denoted $X^\jmath$. If no confusion is likely to arise, we will even denote this log scheme by X and its log structure by α_{j_X} or simply α_X.

Let $f : X \to Y$ be a morphism of schemes with open immersion. Then there is a canonical morphism $f^\jmath : X^\jmath \to Y^\jmath$ of log schemes with the same underlying morphism of schemes defined as $f^\jmath = (\underline{f}, f^{\jmath\flat})$, with $f^{\jmath\flat} : \mathcal{M}_{Y^\jmath} = j_{Y\times}\mathcal{O}_{U_Y}^\times \to \underline{f}_\times \mathcal{M}_{X^\jmath} = (\underline{f} \circ j_X)_\times \mathcal{O}_{U_X}^\times$ induced by $\underline{f}^\sharp : \mathcal{O}_{U_Y}^\times \to \underline{f}_* \mathcal{O}_{U_X}^\times$.

3.5.3 Remark

Recall, that for an open immersion $j : U \to \underline{X}$ we have injective maps $j_*\mathcal{O}_U^\times \to j_*\mathcal{O}_U$ and $\mathcal{O}_X \to j_*\mathcal{O}_U$, which implies, in particular, that $\alpha_{X^\jmath}$ is injective. Hence, in the literature one finds the denotation $j_*\mathcal{O}_U^\times \cap \mathcal{O}_X$ for $j_\times\mathcal{O}_U^\times = j_*\mathcal{O}_U^\times \times_{j_*\mathcal{O}_U} \mathcal{O}_X$ (e. g. in [17, 4.9]).

Let $X = (\underline{X}, \alpha_X)$ be a log scheme. Then the pair $X^o = (\underline{X}, X^\times \to X)$ is a scheme with open immersion. Every morphism $f : X \to Y$ of log schemes induces a morphism of schemes with open immersion $f^o : X^o \to Y^o$, because $\underline{f}(X^\times) \subset Y^\times$.

So we have two functors

$$(\,\cdot\,)^\jmath : \underline{\mathrm{OSch}} \to \underline{\mathrm{LSch}},\ X \mapsto X^\jmath,$$

and

$$(\,\cdot\,)^o\colon \underline{\mathrm{LSch}} \to \underline{\mathrm{OSch}},\ Y \mapsto Y^o.$$

These are adjoint due to the following proposition.

3.5.4 Proposition ([29, III.1.6.2])

The natural map

$$\mathrm{Hom}_{\underline{\mathrm{LSch}}}(X^{\jmath}, Y) \to \mathrm{Hom}_{\underline{\mathrm{OSch}}}(X, Y^o),\ f \mapsto f^o,$$

is an isomorphism, where $X \in \underline{\mathrm{OSch}}$ and $Y \in \underline{\mathrm{LSch}}$.

3.5.5 Definition

Let S be a scheme. A geometric point $x \in S$ is called an *associated point* of S if $\mathfrak{m}_x$ consists entirely of zero-divisors.

If S is reduced, then its only associated points are the generic points of its irreducible components.

3.5.6 Proposition ([29, III.1.6.3])

Let $X = (\underline{X}, j_X)$ be a Noetherian scheme with open immersion such that all associated points of $\underline{X}$ are contained in U_X. Then the compactifying log structure $\alpha_{X^{\jmath}}$ is integral and there is an isomorphism $\overline{\mathcal{M}}_{X^{\jmath}} \cong \Gamma_{Z_X}(\mathrm{Car}^+_{\underline{X}})$ (induced by $\alpha_{X^{\jmath}}\colon \mathcal{M}_{X^{\jmath}} \to \mathcal{O}_X$), where $\Gamma_{Z_X}(\mathrm{Car}^+_{\underline{X}})$ denotes the sheaf of effective Cartier divisors on $\underline{X}$ with support in Z_X.

3.5.7 Corollary

Let $X = (\underline{X}, j_X)$ be a Noetherian scheme with open immersion such that all associated points of $\underline{X}$ are contained in U_X. Then $\jmath_X$ induces an isomorphism

$$\mathrm{LCar}^+(X^{\jmath}) \to H^0(\Gamma_{Z_X}(\mathrm{Car}^+_{\underline{X}})) = \left\{ D \in \mathrm{Car}^+(\underline{X}) \,\middle|\, \mathrm{supp}\, D \subset Z_X \right\}.$$

Hence, $\mathrm{LCar}(X^{\jmath}) \cong \{D \in \mathrm{Car}(\underline{X}) \,|\, \mathrm{supp}\, D \subset Z_X\}$.

Again, defining the compactifying log structure, the result depends on the choice of the topology (Zariski- or étale) (cf. Ogus Remark III.1.6.4). However, we have the following proposition.

3.5.8 Proposition ([29, III.1.6.5])

Let $X = (\underline{X}, j_X)$ be a scheme with open immersion such that all associated points of X are contained in U. Assume that the reduced complement Y of U is of pure codimension 1 and that all irreducible components of Y are regular. Then $\eta\colon X_{\text{ét}} \to X_{\text{zar}}$ induces an isomorphism

$$\eta^{\times}\alpha_{(X_{\mathrm{zar}})^{\jmath}} \to \alpha_{(X_{\text{ét}})^{\jmath}},$$

where the log structures denote the compactifying log structures constructed with respect to the Zariski and étale topology, respectively.

3.5.9 Proposition ([29, III.1.9.1])

Let $X = (\underline{X}, j_X)$ be a normal scheme with open immersion such that Z_X is purely of codimension 1 in X.

a) The stalks of $\overline{\mathcal{M}}_{X^\jmath}$ are finitely generated monoids, and they are free if X is locally factorial.

b) Working with the Zariski topology: If X is locally factorial, then $\alpha_{X^\jmath}$ is a fine log structure.

c) Working with the étale topology: If X is locally factorial and if X admits an étale covering $e\colon X' \to X$ such that each irreducible component of $e^{-1}(Z_X)$ is unibranch, then $\alpha_{X^\jmath}$ is a fine log structure.

3.5.2 The Log structure associated to Deligne-Faltings schemes

3.5.10 Definition

A *scheme with Deligne-Faltings structure* or *DF scheme* $X = (\underline{X}, l_X)$ (*of size* r) is a pair consisting of a scheme $\underline{X}$ and an r-tuple $l_X = (l_{X,1}, \ldots, l_{X,r})$ of homomorphisms of line bundles $l_{X,i}\colon L_{X,i} \to \mathcal{O}_X$ on $\underline{X}$. We write $L_X := (L_{X,1}, \ldots, L_{X,r})$.

A *morphism* $f\colon X \to Y$ *of Deligne-Faltings schemes (of size r)* is a pair $(\underline{f}, f^l)$ consisting of a morphism of schemes $\underline{f}\colon \underline{X} \to \underline{Y}$ and a family $f^l = (f^l_1, \ldots, f^l_r)$ of morphisms of line bundles $f^l_i\colon \underline{f}^* L_{Y,i} \to L_{X,i}$ such that $l_{X,i} \circ f^l_i = \underline{f}^* l_{Y,i}$ for all $i = 1, \ldots, r$. We denote the category of DF schemes of size r by $\underline{\mathrm{DFSch}}^r$.

If S is a scheme, then $(S, id_{\mathcal{O}_S})$, where $id_{\mathcal{O}_S}$ stands for the r-tuple $(id_{\mathcal{O}_S}, \ldots, id_{\mathcal{O}_S})$, is a DF scheme of size r. The inclusion functor $(\,\cdot\,, id_{\mathcal{O}_\cdot})\colon \underline{\mathrm{Sch}} \to \underline{\mathrm{DFSch}}^r$ makes $\underline{\mathrm{Sch}}$ a full subcategory of $\underline{\mathrm{DFSch}}^r$ for each r and is right adjoint to the forgetful functor $X \mapsto \underline{X}$.

3.5.11 Definition

Let $X = (\underline{X}, l_X)$ be a DF scheme. Define the sheaf $L_X^{\otimes}$ by

$$\Gamma(U, L_X^{\otimes}) := \left\{(n, s) \,\middle|\, n \in \mathbb{N}_0^r,\ s \in \Gamma(U, (L_X^{\otimes n})^\times)\right\},$$

where $L_X^{\otimes n} := L_{X,1}^{\otimes n_1} \otimes \ldots \otimes L_{X,r}^{\otimes n_r}$, and define the prelog structure

$$l\colon L_X^{\otimes} \to \mathcal{O}_X$$

by $l_X(n, s) := l_X^{\otimes n}(s)$ for a local section (n, s) of $L_X^{\otimes}$, where

$$l_X^{\otimes n} = l_{X,1}^{\otimes n_1} \otimes \ldots \otimes l_{X,r}^{\otimes n_r}\colon L_X^{\otimes n} \to \mathcal{O}_X$$

is the tensor product of the homomorphisms $l_{X,i}$. We denote the log structure associated to l_X by λ_X and call it the *DF log structure* on X. Its structure sheaf $L_X^{\otimes} \oplus \mathcal{O}_X^\times$ will be denoted $\mathcal{M}_{X^\lambda}$ and the log scheme $(\underline{X}, \lambda_X)$ defined that way X^λ.

Let $f\colon X \to Y$ be a morphism of DF schemes. Then there is a natural morphism $f^\lambda\colon X^\lambda \to Y^\lambda$ of log schemes with the same underlying morphism of schemes defined as $f^\lambda = (\underline{f}, f^{\lambda\flat})$, with $f^{\lambda\flat}\colon f^\times \mathcal{M}_{Y^\lambda} = f^{-1}L_Y^\otimes \circledcirc \mathcal{O}_X^\times \to \mathcal{M}_{X^\lambda} = L_X^\otimes \circledcirc \mathcal{O}_X^\times$ induced by the map $f^l\colon f^{-1}L_Y \to f^*L_Y \to L_X$.

Let X be a DF scheme. For each $i = 1, \dots, r$ there is a natural map $l_{X,i}^\times\colon L_{X,i}^\times \to \mathcal{M}_{X^\lambda} = L_X^\otimes \circledcirc \mathcal{O}_X^\times$, given locally by $s \mapsto (e_i, s) \otimes 1$, where e_i is the i-th generator of $\mathbb{N}_0^r$. Let $I_{X,i}$ denote the ideal sheaf in $\mathcal{O}_X$ which is the image of $l_i\colon L_i \to \mathcal{O}_X$. Then the diagram

$$\begin{tikzcd}
L_{X,i}^\times \arrow[rr, "l_{X,i}^\times"] \arrow[d, hook] & & \mathcal{M}_{X^\lambda} = L_X^\otimes \circledcirc \mathcal{O}_X^\times \arrow[d, "\lambda_X"] \\
L_{X,i} \arrow[r, two heads, "l_{X,i}"] & I_{X,i} \arrow[r, hook] & \mathcal{O}_X
\end{tikzcd}$$

is commutative.

The image $\overline{l_{X,i}^\times(s)}$ in $\overline{\mathcal{M}_{X^\lambda}}$ of a local section s of $L_{X,i}^\times$ is actually independent of the choice of s, because $L_{X,i}^\times$ is an $\mathcal{O}_X^\times$-torsor. Thus, for each i, all local images $\overline{l_{X,i}^\times(s)}$ patch together to form a global section of $\Gamma(X, \overline{\mathcal{M}}_{X^\lambda})$, which we denote by $E_{X,i}$. By the following proposition λ_X is integral. Hence, $E_{X,i} \in \Gamma(X, \overline{\mathcal{M}}_{X^\lambda}) = \mathrm{LCar}^+(X_i)$. Moreover, the image $l_{X,i}(s)$ of any local section s of $L_{X,i}^\times$ in $I_{X,i}$ is a generator for $I_{X,i}$.

We write $E_X := (E_{X,1}, \dots, E_{X,r})$ and $I_X := (I_{X,1}, \dots, I_{X,r})$. The following proposition is a reformulation of [29, III.1.7.3].

3.5.12 Proposition ([29, III.1.7.3])

Let $X = (\underline{X}, l_X)$ be a DF scheme and let $X^\lambda = (\underline{X}, \lambda_X)$ be its associated DF log scheme. Let E_X and I_X be as defined above. Let $Z_{[i]}$ be the closed subscheme of $\underline{X}$ defined by $I_{X,i}$ for each i.

a) The log structure λ_X is integral and, for each i, there are natural isomorphisms

   i) of $\mathcal{O}_X^\times$-torsors $L_{X,i}^\times \to \mathcal{M}_{X^\lambda}(-E_{X,i})^\times$;

   ii) of $\mathcal{O}_X$-modules $L_{X,i} \to \mathcal{M}_{X^\lambda}(-E_{X,i})$;

   iii) of $\mathcal{O}_X$-modules $I_{X,i} \cong (E_{X,i})$, where $(E_{X,i})$ denotes the principal ideal sheaf in $\mathcal{O}_X$ defined by $\overline{\lambda_X}(E_{X,i})$ in $\overline{\mathcal{O}}_X$ (compare remark 3.4.2).

b) Let $\overline{a}\colon \mathbb{N}_{0X}^r \to \overline{\mathcal{M}}_{X^l}$ be the monoid homomorphism defined by mapping $e_i \mapsto E_{X,i}$. Then locally on $\underline{X}$, $\overline{a}$ lifts to a chart $a\colon \mathbb{N}_{0X}^r \to \mathcal{M}_{X^l}$. For each $x \in \underline{X}$, the stalk $\overline{\mathcal{M}}_{X^l,x}$ is freely generated by the images of those $E_{X,i}$ with $\overline{\lambda_X}(E_{X,i})_x = 0$ in $\overline{\mathcal{O}}_{X,x}$.

c) Let $Z = \bigcup_i Z_{[i]}$ and denote $\nu\colon Z^\nu \to Z$ with $Z^\nu = \bigsqcup_i Z_{[i]}$ the normalisation of Z. Let U be the open complement of Z in $\underline{X}$ and $(\underline{X}, j\colon U \to X)$ the corresponding scheme with open embedding. Then there is a natural morphism of log structures

$$\lambda_X \to \alpha_X,$$

where α_X is the compactifying log structure of the scheme with open embedding $(\underline{X}, j\colon U \to X)$.

d) Suppose that $\underline{X}$ is normal, that all $l_{X,i}$ are injective and that each $Z_{[i]}$ is integral (respectively, geometrically unibranch). Then in the Zariski (in the étale) topology the morphism $\lambda_X \to \alpha_X$ is an isomorphism. In particular, the map

$$\overline{\mathcal{M}}_{X^\lambda} \to \nu_* \mathbb{N}_{0Z^\nu},\ E_{X,i} \mapsto 1_{Z_{[i]}}$$

is an isomorphism.

3.5.13 Remark

The concept of DF schemes as defined above was independently introduced by G. Faltings and P. Deligne. In a letter to L. Illusie (as reported by Illusie himself in [14, Letter of 1988]) Deligne defines a *generalised divisor* on a scheme $\underline{X}$ to be a line bundle L on X with an $\mathcal{O}_X$-linear map $u\colon L \to \mathcal{O}_X$. Such generalised divisors then lead to log structures on the scheme. Faltings introduces the same concept in [7], calling it *logarithmic structure.* We have referred to these objects as *Deligne-Faltings structures* following A. Ogus.

3.5.3 Strict normal crossing schemes and logarithmic structures of semi-stable type

(Strict) normal crossing schemes over a field k arise as limits of smooth varieties in deformation processes. As such, they naturally carry specific log structures, namely log structures of embedding type or even of semi-stable type. These arise from restricting the log structure of the total space of a deformation as a scheme with open immersion, where the open subset consists of the unity of the smooth fibres, to the singular normal crossing fibre.

3.5.14 Definition

Let V be a regular scheme. A *strict normal crossing divisor* in V is a closed reduced subscheme $\underline{X} \subset V$ such that at every geometric point $x \in \underline{V}$ there exists a regular sequence $(t_1, \ldots, t_m)$ generating the maximal ideal of $\mathcal{O}_{V,x}$ and a number $r(x) \in \mathbb{N}_0$ such that the element $t_1 \cdot \ldots \cdot t_{r(x)}$ generates the ideal of $\underline{X}$ in $\mathcal{O}_{V,x}$.

3.5.15 Definition

A scheme $\underline{X}$ is called a *strict normal crossing scheme* if for every geometric point $x \in \underline{X}$ there exists an open/étale neighbourhood $e_x\colon U_x \to \underline{X}$ of x together with an embedding $i_x\colon U_x \to V_x$ of U_x into a regular scheme V_x, such that U_x is a strict normal crossing divisor in V_x. Such a pair (e_x, i_x) is called an *SNC coordinate system for $\underline{X}$ at x.* We will write $W_x := V_x \backslash U_x$ for the complement of U_x in V_x and we will denote its open embedding by $j_x\colon W_x \to V_x$.

In the Zariski topology, more generally, $\underline{X}$ is called a *normal crossing scheme* if there exists an étale covering $e\colon \underline{X}' \to \underline{X}$ such that $\underline{X}'$ is a strict normal crossing scheme.

Let V be a regular scheme with strict normal crossing divisor $\underline{X} = \bigcup_i \underline{X}_{[i]}$ and denote the inclusion $i\colon \underline{X} \to \underline{V}$. Let $j\colon \underline{W} \to \underline{V}$ denote the open immersion of the complement $\underline{W}$ of $\underline{X}$ in $\underline{V}$ and let $\alpha_V\colon \mathcal{M}_V \to \mathcal{O}_{\underline{V}}$ be the compactifying log structure associated to the scheme with open immersion $(\underline{V}, j)$. Since $\underline{V}$ is normal, α_V is isomorphic to the DF log structure λ_V associated to the injections $l_{\underline{X},i}\colon I_{\underline{X}_{[i]}} \to \mathcal{O}_{\underline{V}}$ of the ideal sheaves of the $\underline{X}_{[i]} \subset \underline{V}$. Therefore by 3.5.12, locally at each point x of $\underline{V}$ in which $r(x)$ components of $\underline{X}$ meet we have a chart $a\colon \mathbb{N}_{0V}^{r(x)} \to \mathcal{M}$ and globally an isomorphism $\overline{\mathcal{M}_V} \cong i_*\nu_*\mathbb{N}_{0\underline{X}^\nu}$, where $\nu\colon \underline{X}^\nu \to \underline{X}$ is the normalisation of $\underline{X}$.

Since on W the log structure α_V is trivial, both sides of the isomorphism vanish outside of $\underline{X}$. Hence, the same isomorphism $\overline{\mathcal{M}} \cong \nu_*\mathbb{N}_{0X^\nu}$ holds on $\underline{X}$, replacing α_V by its restriction to $\underline{X}$, $\alpha_X := i^\times\alpha_V = i^\times j_\times\iota_W$. This log structure α_X is isomorphic to the DF log structure λ_X of the restricted maps $i^*I_i \to \mathcal{O}_X$ (compare to the introduction of section 1.8 in [29]).

Let $\underline{X}$ be a strict normal crossing scheme. By $\nu\colon X^\nu \to \underline{X}$ we denote its normalisation, where X^ν is the disjoint union of the irreducible components $X_{[1]}, \ldots, X_{[m]}$ of $\underline{X}$, each of which is regular.

For each $j = 1, \ldots, m$ and $k \neq j$ we let

$$X^{[j]} := \bigcup_{k\neq j} X_{[k]}, \qquad D_{[j]} := X_{[j]} \cap X^{[j]},$$
$$X_{[jk]} := X_{[j]} \cap X_{[k]}, \qquad X^{[jk]} := X^{[j]} \cap X^{[k]}, \qquad D_{[jk]} := X_{[jk]} \cap X^{[jk]}, \text{ etc.}$$

We put $D := \bigcup_j D_{[j]} = \bigcap_j X^{[j]} = \bigcup X_{[jk]} \subset \underline{X}$ and (by abuse of notation) $D^\nu := \nu^{-1}(D) = \bigsqcup_j D_{[j]} \subset X^\nu$. Classically, $D \subset \underline{X}$ is called the *double locus* of $\underline{X}$ (cf. [11, 1.7]).

3.5.16 Lemma ([11, 1.8, 1.10 & 1.11]; in this form: [29, III.1.8.2])

Let $\underline{X}$ be a strict normal crossing scheme with normalisation $X^\nu = X_{[1]} \sqcup \ldots \sqcup X_{[m]}$. Then (for each index j)

a) $D_{[j]}$ is a Cartier divisor in both $X_{[j]}$ and $X^{[j]}$.

b) The ideal sheaf $I_{[j]} := I_{X_{[j]}\subset X} \subset \mathcal{O}_X$ is annihilated by the ideal sheaf $I^{[j]} := I_{X^{[j]}\subset X} \subset \mathcal{O}_X$ and can be identified with the ideal sheaf $I_{D_{[j]}\subset X^{[j]}} \subset \mathcal{O}_{X^{[j]}}$.

c) The ideal sheaf $I^{[j]}$ is annihilated by the ideal $I_{[j]}$ and can be identified with the ideal sheaf $I_{D_{[j]}\subset X_{[j]}} \subset \mathcal{O}_{X_{[j]}}$.

d) Each $I_{[j]}\big|_D$ is an invertible sheaf of $\mathcal{O}_D$-modules, as is $\mathcal{O}_D(-\underline{X}) := \bigotimes_j I_{[j]}\big|_D$.

e) If $\underline{X} \to V$ is a (global) SNC coordinate system on $\underline{X}$, then $\mathcal{O}_V(-\underline{X})|_D \cong \mathcal{O}_D(-\underline{X})$ (hence, the symbol $\mathcal{O}_D(-\underline{X})$).

3.5.17 Definition ([11, 1.9])

The sheaf $\mathcal{O}_D(-\underline{X}) := \bigotimes_j I_{[j]}\big|_D$ is called the *infinitesimal conormal bundle* of D in $\underline{X}$. The *infinitesimal normal bundle* of D in $\underline{X}$ is its dual on D, denoted $\mathcal{O}_D(\underline{X})$.

3.5.18 Remark

Observe, that $\mathcal{O}_D(-\underline{X})$ is not the usual conormal sheaf $\mathcal{N}_{D\subset\underline{X}}$: Take for example $X = \operatorname{Spec}\mathbb{C}[x,y]/(xy)$. Then its components $X_{[1]}$ and $X_{[2]}$ have the ideals generated by x and by y, respectively, and D has the ideal (x,y). Hence the usual conormal sheaf is given as the quotient $(x,y)/(x,y)^2 = \mathcal{O}_D\langle x,y\rangle$ and hence not a line bundle, while the line bundle $\mathcal{O}_D(-\underline{X}) = I_{[1]}\big|_D \otimes I_{[2]}\big|_D$ is given by $(x)/(x^2)\otimes(y)/(y^2) = \mathcal{O}_X\langle xy\rangle$.

3.5.19 Theorem ([29, III.1.8.3], cp. [17, 11.7], [21, 1.1])

Let $\underline{X}$ be a strict normal crossing scheme with normalisation $X^\nu = X_{[1]}\sqcup\ldots\sqcup X_{[m]}$. Then the following categories are naturally equivalent:

a) The category of log structures α on $\underline{X}$ such that $\overline{\alpha}\colon \overline{\mathcal{M}}_\alpha \to \overline{\mathcal{O}}_X$ induces an isomorphism $\overline{\mathcal{M}}_\alpha \cong \nu_*\mathbb{N}_{0X^\nu}$ (cf. proposition 3.5.12 d);

b) the category of tupels of pairs $((\mathcal{L}_1,l_1),\ldots,(\mathcal{L}_m,l_m))$ of line bundles on $\underline{X}$ with isomorphisms $l_j\colon\ \mathcal{L}_j|_{X[j]} \to I_{[j]}$;

c) the category of tupels of pairs $((\mathcal{L}'_1,l'_1),\ldots,(\mathcal{L}'_m,l'_m))$ of line bundles L'_j on $X_{[j]}$ with isomorphisms $l'_j\colon\ \mathcal{L}_j|_{D_{[j]}} \to I_{[j]}\big|_{D_{[j]}}$;

d) the category of pairs $(\mathcal{L},l)$ of a line bundle $\mathcal{L}$ on $\underline{X}$ with an isomorphism $l\colon\ \mathcal{L}|_D \to \mathcal{O}_D(-\underline{X})$.

3.5.20 Definition

Let $\underline{X}$ be a strict normal crossing scheme. Then a log structure α on $\underline{X}$ is called a log structure *of embedding type* if for every point x in $\underline{X}$ there exists an SNC coordinate system $e_x\colon U_x \to X$ such that we have an isomorphism $e_x^\times\alpha \to i_x^\times j_{x\times}\iota_{W_x}$.

In this case, the log scheme $X = (\underline{X},\alpha)$ is called a *strict normal crossing scheme of embedding type.*

3.5.21 Corollary

Let $\underline{X}$ be a strict normal crossing scheme. Then a log structure α is of embedding type if and only if étale-locally $\overline{\mathcal{M}}_\alpha \cong \nu_*\mathbb{N}_{0X^\nu}$.

Proof: This follows from the proof of 3.5.19 in [29, III.1.8.3]: Given an isomorphism $\overline{\mathcal{M}}_\alpha \cong \nu_*\mathbb{N}_{0X^\nu}$, α itself is isomorphic to the log structure associated to the DF structure $(\underline{X}, l_X)$ given by the natural maps $l_{X,i}\colon \mathcal{M}_\alpha(-E_i) \to (E_i) \subset \mathcal{O}_X$, where $E_i \in H^0(\underline{X},\nu_*\mathbb{N}_{0X^\nu})$ is the i-th generator. On the other hand, any such log structure of embedding type induces an isomorphism $\overline{\mathcal{M}}_\alpha \cong \nu_*\mathbb{N}_{0X^\nu}$. □

3.5.22 Definition ([17, 11.6], cp. [29, III.1.8.7])
Let X be a strict normal crossing scheme. Then a log structure α on X is called a log structure *of semi-stable type* if α is of embedding type and if there exists a homomorphism of sheaves of monoids $\widehat{\mathrm{diag}} \colon \mathbb{N}_{0X} \to \mathcal{M}_\alpha$ such that

$$\begin{tikzcd} \mathbb{N}_{0X} \arrow[d, "\widehat{\mathrm{diag}}"'] \arrow[dr, "\mathrm{diag}"] & \\ \mathcal{M}_\alpha \arrow[r] & \overline{\mathcal{M}}_\alpha \cong \nu_* \mathbb{N}_{0X^\nu} \end{tikzcd}$$

commutes.

If $\underline{X}$ is a k-scheme, then the homomorphism $\mathbb{N}_{0X} \to \mathcal{M}_\alpha$ induces a log smooth morphism $f \colon (X, \alpha) \to \operatorname{Spec} \kappa$ the underlying morphism $\underline{f}$ of which is the k-scheme structure of X and such that $f^\flat = \widehat{\mathrm{diag}} \cdot id_{k^\times} : \mathbb{N}_0 \oplus k^\times \to \mathcal{M}_\alpha$, where $\operatorname{Spec} \kappa$ is the standard log point on $\operatorname{Spec} k$.

This morphism $f \colon (X, \alpha) \to \operatorname{Spec} \kappa$ of log schemes is then called a *strict normal crossing log scheme of semi-stable type over* k or simply an *SNC log scheme*.

Recall that the logarithmic rank $\ell_X(x)$ of X at a point $x \in X$ is defined to by the rank of $\overline{\mathcal{M}}^{\mathrm{grp}}_{X,x}$ and that the leniency of f at a point $x \in X$ is defined to be the rank of $\overline{\mathcal{M}}^{\mathrm{grp}}_{f,x}$. The logarithmic locus $\mathrm{LLoc}(X)$ of X is the reduced closed complement of the log trivial locus $X^\times$ and the lenient locus $\mathrm{LLoc}(f)$ of f is the reduced closed complement of $X^\times(f)$ (cf. definitions 1.2.14 and 1.2.18).

3.5.23 Corollary
Let $f \colon (X, \alpha) \to \operatorname{Spec} \kappa$ be an SNC log scheme. Then, if ℓ_X denotes the logarithmic rank of X and ℓ_f the leniency of f, for every geometric point x of X, we have $\ell_X(x) = \ell_f(x) + 1$ and this number is equal to the number of irreducible components $r(x)$ of X containing x. It follows, that the double locus D is equal to the lenient locus $\mathrm{LLoc}(f)$ of f, to the logarithmic locus of X of order ≥ 2, to the non-normal locus $\mathrm{NonN}(X)$ of the scheme $\underline{X}$, and to the singular locus $\operatorname{Sing} X$ of $\underline{X}$.

3.5.24 Proposition ([17, 11.7], [29, III.1.8.8])
If X is a strict normal crossing scheme over k, then giving X a log structure of semi-stable type is equivalent to giving an isomorphism $\varphi \colon \mathcal{O}_D \to \mathcal{O}_D(-X)$.

3.5.25 Corollary
Let $f \colon X \to \operatorname{Spec} \kappa$ be an SNC log scheme. Then f is log smooth.

Proof: This is essentially example 1.2.1 from chapter 1:

Étale locally at a point $x \in X$ where r components of X meet we have by definition the diagonal morphism $\mathrm{diag} \colon \mathbb{N}_{X0} \to \mathbb{N}_{0X}^{r(x)}$ as a chart for f with $\mathrm{Ker}(\mathrm{diag}^{\mathrm{grp}}) = 0$ and $\mathrm{Cok}(\mathrm{diag}^{\mathrm{grp}}) = \mathbb{Z}^{r(x)}/\mathrm{diag}(\mathbb{Z})$. This implies that $k(x) \otimes_{\mathbb{Z}} \mathrm{Ker}(\mathrm{diag}^{\mathrm{grp}}) = 0$ and

$k(x) \otimes_{\mathbb{Z}} (\mathrm{Cok}(\mathrm{diag}^{\mathrm{grp}}))_{\mathrm{tors}} = 0$. The strict morphism of log schemes $X \to \mathbb{A}_\kappa[\mathbb{N}_0^{r(x)}]$ is smooth (even étale), because its underlying morphism is the isomorphism of schemes

$$\operatorname{Spec} k[z_1, \dots, z_{r(x)}]/(z_1 \cdot \ldots \cdot z_{r(x)}) \to \operatorname{Spec} k \times_{\operatorname{Spec} k[\mathbb{N}_0]} \operatorname{Spec} k[\mathbb{N}_0^{r(x)}]$$

étale locally. Hence, by the criterion for log smoothness 1.2.35, f is log smooth. □

3.5.26 Definition

Let k be a field and κ the standard log point on k. An *SNC log symplectic scheme* is a log symplectic scheme $f\colon (X, \nabla, \omega) \to \operatorname{Spec} \kappa$ such that $f\colon X \to \operatorname{Spec} \kappa$ is an SNC log scheme. Analogously we define the terms *SNC log scheme with line bundle* and *SNC log scheme with flat log connection*.

An *SNC log variety* is an *SNC log scheme*, such that $\underline{f}\colon \underline{X} \to k$ is a variety in the usual sense. Analogously we define the terms *SNC log variety with line bundle*, *SNC log variety with flat log connection* and *SNC log symplectic variety*.

3.6 Log Cartier divisors on SNC log varieties

3.6.1 Regular varieties with SNC divisors

Before looking at SNC log schemes, let first $(\underline{X}, j_X)$ be a regular variety with open immersion such that Z_X is a strict normal crossing divisor and let $X = (\underline{X}, \alpha_X)$ denote its associated log scheme with compactifying log structure. We write $Z_{[1]}, \dots, Z_{[m]}$ for the irreducible components of Z_X.

The log structure $\alpha_X\colon \mathcal{M}_X \to \mathcal{O}_X$ is an injective morphism of sheaves of monoids (cf. remark 3.5.3) and has the characteristic sheaf $\overline{\mathcal{M}}_X = i_{X*}\nu_*\mathbb{N}_{0Z^\nu}$, the group of global sections of which is $H^0(\overline{\mathcal{M}}_X) = \bigoplus_{i=1}^m \mathbb{N}_0 \cdot Z_{[i]} \cong \mathbb{N}_0^m$, where we write $Z_{[i]}$ for the generator of the i-th component.

Étale-locally, this log structure admits a chart $\bigoplus_{i=1}^m \mathbb{N}_0 \cdot Z_{[i]} \to \mathcal{O}_X$, mapping the generator $Z_{[i]}$ to some local equation z_i for $Z_{[i]}$ as a subscheme of X. Since each of the $Z_{[i]}$ is Cartier in the usual sense, each of the z_i is a local element of the sheaf $\mathcal{O}_X^{\mathrm{dom}}$ of non-zero-divisors in $\mathcal{O}_X$. Hence, the log structure α_X factorises into injective morphisms $\mathcal{M}_X \to \mathcal{O}_X^{\mathrm{dom}} \to \mathcal{O}_X$, inducing an injective morphism of sheaves of groups

$$\alpha^{\mathrm{grp}}\colon \mathcal{M}_X^{\mathrm{grp}} \to (\mathcal{O}_X^{\mathrm{dom}})^{\mathrm{grp}}.$$

Since α_X is an integral log structure (cf. proposition 3.5.9), we have $\mathcal{M}_X = \mathcal{M}_X^{\mathrm{dom}}$, thus $\mathcal{M}_X^{\mathrm{grp}} = \mathcal{M}_X^{\mathrm{rat}\times}$, and since X is a regular scheme, $\mathcal{O}_X^{\mathrm{dom}} = \mathcal{O}_X^{\mathrm{rat}\times} = \mathcal{K}_X^\times$. So there is a commutative diagram of sheaves of groups with exact rows

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{O}_X^\times \arrow[r] \arrow[d, equal] & \mathcal{M}_X^{\mathrm{grp}} \arrow[r] \arrow[d, hook, "\alpha^{\mathrm{grp}}"] & \overline{\mathcal{M}}_X^{\mathrm{grp}} \arrow[r] \arrow[d, hook, "\overline{\alpha^{\mathrm{grp}}}"] & 1 \\ 1 \arrow[r] & \mathcal{O}_X^\times \arrow[r] & \mathcal{K}_X^\times \arrow[r] & \overline{\mathcal{K}_X^\times} \arrow[r] & 1, \end{tikzcd}$$

implying that the homomorphism $\delta_X : \mathrm{LCar}(X) \to \mathrm{Pic}(X)$ assigning to each log Cartier divisor D the line bundle $\mathcal{M}_X(D)$ factorises as $\mathrm{LCar}(X) \xrightarrow{\overline{\alpha^{\mathrm{grp}}}} \mathrm{Car}(\underline{X}) \xrightarrow{\delta_{\underline{X}}} \mathrm{Pic}$ with the first map being injective, where $\delta_{\underline{X}}$ assigns to each Cartier divisor (in the usual sense) D the line bundle $\mathcal{O}_X(D)$.

Moreover, we have a commutative diagram

$$\begin{tikzcd} \mathcal{M}_X \arrow[r, two heads] \arrow[d, hook, "\alpha"] & \overline{\mathcal{M}}_X \arrow[d, hook, "\overline{\alpha}"] \\ \mathcal{O}_X^{\mathrm{dom}} \arrow[r, two heads] & \overline{\mathcal{O}_X^{\mathrm{dom}}}, \end{tikzcd}$$

where the objects on the right are the quotients of the objects on the left by the free action of $\mathcal{O}_X^\times$. This implies that, analogously to just above, the restriction $H^0(\overline{\mathcal{M}_X}) \to H^1(\mathcal{O}_X^\times)$ of δ_X to effective log Cartier divisors factorises injectively via the effective Cartier divisors on $\underline{X}$ (in the usual sense) $H^0(\overline{\mathcal{O}_X^{\mathrm{dom}}})$.

Having denoted the monoid generators of $\mathrm{LCar}^+(X) = H^0(\overline{\mathcal{M}}_X)$ by $Z_{[i]}$, the group $\mathrm{LCar}(X)$ is generated by the $Z_{[i]}$ as an Abelian group, hence, $\mathrm{LCar}(X) = \bigoplus_{i=1}^m \mathbb{Z} \cdot Z_{[i]} \cong \mathbb{Z}^m$. It is clear from the description of the injective morphisms α and α^{grp} that $Z_{[i]}$ is mapped to the effective Cartier divisor $Z_{[i]}$ by the first map in the factorisation of δ_X. In particular, we may conclude the following, the first part of which is just a special case of corollary 3.5.7:

3.6.1 Proposition

Let $(\underline{X}, j_X)$ be a regular variety with open immersion such that Z_X is a strict normal crossing divisor and let $X = (\underline{X}, \alpha_X)$ denote its associated log scheme with compactifying log structure. Then

a) $\mathrm{LCar}^+(X) \to \big\{D \in \mathrm{Car}^+(\underline{X}) \,\big|\, \operatorname{supp} D \subset Z_X\big\}$, induced by $Z_{[i]} \mapsto Z_{[i]}$, is a monoid-isomorphism.

b) $\mathrm{LCar}(X) \to \{D \in \mathrm{Car}(\underline{X}) \,|\, \operatorname{supp} D \subset Z_X\}$, induced by $Z_{[i]} \mapsto Z_{[i]}$, is a group-isomorphism.

In particular, for any $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ the line bundles $\mathcal{M}_X(\sum_{i=1}^m a_i Z_{[i]})$ and $\mathcal{O}_X(\sum_{i=1}^m a_i Z_{[i]})$ are equal.

Regarding such X as a log smooth variety $f\colon X \to \operatorname{Spec} k^\iota$ over the point with trivial log structure, we may ask for the log Cartier connections on f.

We regard again the étale-local chart $\bigoplus_{i=1}^{m} \mathbb{N}_0 \cdot Z_{[i]} \to \mathcal{O}_X$, $Z_{[i]} \mapsto z_i$, from above. Writing $v := -d\log z_i := -d\log(Z_{[i]})$, the log Cartier connection $M_f(Z_{[i]})$ is in this chart given as $d + v$ for the local trivialisation $\mathcal{M}_X(Z_{[i]}) = \frac{1}{z_i}\mathcal{O}_X$. Considering this line bundle as a subsheaf $\frac{1}{z_i}\mathcal{O}_X \subset \mathcal{K}_X$, the log connection ∇ is nothing but the differential d rationally augmented by the "quotient rule" and restricted to this subsheaf: $\nabla(\frac{1}{z_i}s) = \frac{1}{z_i}(ds - sd\log z_i)$ for any local section s of $\mathcal{O}_X$ (cp. definition 3.4.11). The same applies accordingly to a general log Cartier connection $M_f(\sum_{i=1}^{m} a_i Z_{[i]}) = \bigotimes_{i=1}^{m} M_f(Z_{[i]})^{\otimes a_i}$.

Now, let $\underline{f}\colon \underline{X} \to \underline{C}$ be a flat morphism of schemes with C a regular open subscheme of a curve and X regular. Let $P \subset C$ be a closed point and assume that $Z := \underline{f}^{-1}(P)$ is a strict normal crossing divisor in $\underline{X}$. Giving both schemes $\underline{C}$ and $\underline{X}$ the compactifying log structure α_C and α_X, respectively, associated to the open immersion $\underline{C} \setminus P \to U$ and $\underline{X} \setminus Z \to X$, respectively, and defining $X := (\underline{X}, \alpha)$ and $C := (\underline{C}, \alpha_C)$, turns $\underline{f}$ into a morphism of log schemes $f\colon X \to C$.

Since Z is a fibre in the flat family $\underline{f}\colon \underline{X} \to \underline{C}$, we have $\mathcal{O}_X(Z) = \mathcal{O}_X$. Due to the fact that Z is a strict normal crossing divisor, $\mathcal{M}_X(Z) = \mathcal{O}_X$, too, in this situation. Let $\nabla = M_X(Z)$ denote the flat log connection on f associated to the log Cartier divisor Z. Since it has $\mathcal{M}_X(Z) = \mathcal{O}_X$ as line bundle, we may write $\nabla = d + v$ for some (global) 1-form $v \in \Gamma(X, \Omega^1_f)$.

A global equation $z \in \Gamma(X, \mathcal{O}_X)$ for the divisor Z is given by the image $f^\sharp(p)$ of an equation $p \in \Gamma(C, \mathcal{O}_C)$ for the point P in C under $f^\sharp\colon f^{-1}\mathcal{O}_C \to \mathcal{O}_X$. Hence, $v = -d\log z = -d\log(f^\sharp p) = 0$ as a log differential relative to f and hence, $M_f(Z)$ is the trivial flat log connection $(\mathcal{O}_X, d)$.

3.6.2 SNC log varieties

From now on, let $f\colon X \to \operatorname{Spec}\kappa$ be an SNC log variety and denote by $X_{[1]}, \ldots, X_{[m]}$ its irreducible components and by $\nu\colon X^\nu \to X$ its normalisation. Then étale-locally, α_X is the compactifying log structure α_V of an SNC coordinate system V (with Cartier divisor X) restricted to X. Therefore, $\mathrm{LCar}(X) = \bigoplus_{i=1}^{m} \mathbb{Z} \cdot X_{[i]}$.

We regard the i-th component $X_{[i]}$ of X, which is a regular scheme of finite type over k containing the Cartier divisor $D_{[i]} = \bigcup_{j \neq i} X_{[ij]}$. Denoting the closed immersion by $i'\colon D_{[i]} \to X_{[i]}$ and the open immersion of its complement by $j'\colon X_{[i]} \setminus D_{[i]} \to X_{[i]}$, one log structure on $X_{[i]}$ is the compactifying one associated to j', which we denote by $\alpha'_{X_{[i]}}$ and which has the characteristic sheaf $i'^*\nu'_*\mathbb{N}_{0D^\nu_{[i]}}$, where we write $\nu'\colon D^\nu_{[i]} \to D_{[i]}$ for the normalisation of $D_{[i]}$.

This log structure is, however, not equal to the log structure $\alpha_{X_{[i]}}$ of $X_{[i]}$ as a log subscheme of X. Nevertheless, these two log structures are related as follows:

The diagram

$$\begin{tikzcd} \nu^{-1}(X_{[i]}) = \bigsqcup_{j\neq i} X_{[ij]} \sqcup X_{[i]} \arrow[r, "i_i^\nu"] \arrow[d, "\nu|_{\dots}"] & X^\nu \arrow[d, "\nu"] \\ X_{[i]} \arrow[r, "i_i"] & X \end{tikzcd}$$

is Cartesian with ν proper. Therefore, by proper base change, the characteristic sheaf of $\alpha_{X_{[i]}}$ which is the restriction of $\overline{\mathcal{M}}_X = \nu_*\mathbb{N}_{0X^\nu}$ to its i-th component $X_{[i]}$, is $i_i^{-1}\nu_*\mathbb{N}_{0X^\nu} = \nu|_{\dots *}\, i_i^{\nu\,-1}\mathbb{N}_{0X^\nu} = \mathbb{N}_0 \oplus i'^{-1}\nu'_*\mathbb{N}_{0D^\nu_{[i]}}$. Hence, its log Cartier divisors are $\mathrm{LCar}(X_{[i]}) = \mathbb{Z}\cdot X_{[i]} \oplus \bigoplus_{j\neq i}\mathbb{Z}\cdot X_{[ij]} = \mathbb{Z}\oplus \left\{ D \in \mathrm{Car}(\underline{X}_{[i]}) \,\middle|\, \mathrm{supp}\, D \subset D_{[i]} \right\}$, where we write $X_{[i]}$ for the monoid generator of the constant $\mathbb{N}_0$-summand of $\overline{\mathcal{M}}_{X_{[i]}}$.

We know from the first part of this section that the line bundle $\mathcal{M}_{X_{[i]}}(X_{[ij]})$ is equal to $\mathcal{O}_{X_{[i]}}(X_{[ij]})$. But we also know, which line bundle is the image of the generator $X_{[i]}$ of the constant $\mathbb{Z}$-part: Due to the fact that $\mathcal{M}_X(X) = \mathcal{M}_X(\sum_{i=1}^m X_{[i]}) = \mathcal{O}_X$, we have $\mathcal{M}_X(X_{[i]}) = \mathcal{M}_X(-\sum_{j\neq i} X_{[j]})$ which restricts on $X_{[i]}$ to

$$\mathcal{M}_{X_{[i]}}(X_{[i]}) = \mathcal{M}_{X_{[i]}}(-\sum_{j\neq i} X_{[ij]}) = \mathcal{O}_X(-\sum_{j\neq i} X_{[ij]}).$$

In an étale-local SNC coordinate system $X \subset V$, the line bundle $\mathcal{M}_{X_{[i]}}(X_{[i]})$ is equal to $\mathcal{M}_V(X_{[i]})\big|_{X_{[i]}} = \mathcal{O}_V(X_{[i]})\big|_{X_{[i]}} = \mathcal{N}_{X_{[i]}\subset V}$, the normal bundle of $X_{[i]}$ in the ambient scheme V.

So we may write down the map $\delta_{[i]} : \mathrm{LCar}(X_{[i]}) \to \mathrm{Pic}(X)$ explicitly as

$$a_i X_{[i]} + \sum_{j\neq i}^m a_j X_{[ij]} \mapsto \mathcal{M}_X\Big(a_i X_{[i]} + \sum_{j\neq i}^m a_j X_{[ij]}\Big) = \mathcal{O}_{X_{[i]}}\Big(\sum_{j\neq i}(a_j - a_i) X_{[ij]}\Big).$$

Analogously, $\delta_{f|_{X_{[i]}}} : \mathrm{LCar}(X_{[i]}) \to \mathrm{LConn}(X)$ is given by

$$a_i X_{[i]} + \sum_{j\neq i}^m a_j X_{[ij]} \mapsto M_{f|X_{[i]}}\Big(a_i X_{[i]} + \sum_{j\neq i}^m a_j X_{[ij]}\Big).$$

We may explicitly describe the log Cartier connections $\mathcal{M}_{f|_{X_{[i]}}}(X_{[ij]})$ just as in the first part of this section as being rationally augmented versions of d. The log Cartier connection $\nabla_{[ii]} := \mathcal{M}_{f|_{X_{[i]}}}(X_{[i]})$ is simply the inverse flat log connection of the tensor product $\bigotimes_{j\neq i}\mathcal{M}_{f|_{X_{[i]}}}(X_{[ij]})$. It therefore may be described as follows:

In the étale-local SNC coordinate system as above, we may write the line bundle of $\nabla_{[ii]}$ in the form $\mathcal{N}_{X_{[i]}\subset V} = (\prod_j x_j)\mathcal{O}_{X_{[i]}} \subset \mathcal{K}_{X_{[i]}}$. Then, in this chart, $v = -d\log X_{[i]}$, hence, $\nabla_{[ii]}((\prod_j x_j)s) = (\prod_j x_j)(ds + \sum_j s d\log x_j)$. So again, ∇ is nothing but the differential d rationally augmented and restricted to the subsheaf.

4 Log symplectic deformation theory

Recall the definition of the category $\underline{\mathrm{LArt}}_{\mathcal{T}}$ given in chapter 2. From now on, we work in a more specific setting. Let T be a complete Noetherian k-algebra with quotient field k of characteristic 0. Let $\mathcal{T}: Q \to T$ be a precise prelog ring over T and give k the prelog ring structure $\kappa: Q \to k$ defined by composing $\mathcal{T}$ with the canonical projection $T \to k$. The monoid homomorphism ϱ of chapter 2, section 2.2.1 is then the identity map $Q \to Q$ (with $Q' = Q$). This means in particular, that for every $A \in \underline{\mathrm{Art}}_T$ the fibre $v^{-1}(A)$ consists of elements $\mathcal{A}$ which are all isomorphic (as elements in $\underline{\mathrm{LArt}}_{\mathcal{T}}$) and up to unique isomorphism given by the composition of $\mathcal{T}$ with the structure homomorphism $\varphi_A: T \to A$.

4.0.2 Corollary

In the setting just described, we may remove the term "pseudo-" from all statements about functors of log Artin rings of chapter 2, section 2.2. Compare remark 2.2.14.

4.1 Deformation functors

4.1.1 The log symplectic deformation functor and related functors

Let $f_0: X \to \operatorname{Spec} \kappa$ be a log smooth morphism of log fs log schemes. Recall from chapter 2, section 2.3 the definition of its functor of log smooth deformations, $\operatorname{Def}_{f_0}: \underline{\mathrm{LArt}}_{\mathcal{T}} \to \underline{\mathrm{Set}}$. Let L (respectively, ∇) denote a line bundle (respectively, a flat log connection) on X (respectively, on f_0). Analogously to the definition of Def_{f_0}, we denote by $\operatorname{Def}_{(f_0,L)}$ (respectively, by $\operatorname{Def}_{(f_0,\nabla)}$) the functor of isomorphism classes of infinitesimal log smooth deformations of the log scheme with line bundle (f_0, L) (respectively, of the log scheme with flat log connection (f_0, ∇)).

Moreover, we denote by $\operatorname{Def}_{(f_0,\omega)}$ (respectively, by $\operatorname{Def}_{(f_0,\nabla,\omega)}$) the functor of isomorphism classes of infinitesimal log smooth deformation of non-twisted type of the log symplectic scheme (f_0, ω) (of general type of the log symplectic scheme (f_0, ∇, ω)).

We will refer to each of these functors simply as the deformation functor of the respective kind of object.

Given a log symplectic scheme (f_0, ∇, ω) over $\operatorname{Spec} \kappa$, with $\nabla = (L, \nabla)$, there is an obvious chain of forgetful morphisms of functors, each neglecting one part of the data,

$$\operatorname{Def}_{(f_0,\nabla,\omega)} \to \operatorname{Def}_{(f_0,\nabla)} \to \operatorname{Def}_{(f_0,L)} \to \operatorname{Def}_{f_0}$$

and given a log symplectic scheme of non-twisted type (f_0, ω) over $\operatorname{Spec} \kappa$, we have a forgetful morphism $\operatorname{Def}_{(f_0,\omega)} \to \operatorname{Def}_{f_0}$ plus an obvious morphism $\operatorname{Def}_{(f_0,\omega)} \to \operatorname{Def}_{(f,d,\omega)}$, given as $[f, \varpi] \mapsto [f, d, \varpi]$. This last morphism makes $\operatorname{Def}_{(f_0,\omega)}$ a subfunctor of $\operatorname{Def}_{(f,d,\omega)}$: It is clearly injective on objects and since for any $\varphi: \mathcal{A}' \to \mathcal{A}$ we have $\varphi^* d = d$, the restriction of the map $\operatorname{Def}_{(f,d,\omega)}(\varphi)$ to $\operatorname{Def}_{(f,\omega)}(\mathcal{A}')$ is $\operatorname{Def}_{(f,\omega)}(\varphi)$.

4.1.2 The local Picard functor and related functors

Let $f\colon \mathcal{X} \to \operatorname{Spec} \mathcal{T}$ be a log smooth log scheme over $\mathcal{T}$ with $\mathcal{X}$ a log fs log scheme. For each log Artin ring $\mathcal{A} \in \underline{\mathrm{LArt}}_{\mathcal{T}}$ we denote by $\mathcal{X}_{\mathcal{A}}$ the scheme $\mathcal{X} \times_{\operatorname{Spec} \mathcal{T}} \operatorname{Spec} \mathcal{A}$ and by $f_{\mathcal{A}}$ the canonical morphism $\mathcal{X}_{\mathcal{A}} \to \operatorname{Spec} \mathcal{A}$, which is log smooth by base change. We write X for $\mathcal{X}_\kappa$ and f_0 for f_κ and we will refer to $\mathcal{X}_{\mathcal{A}}$ and $f_{\mathcal{A}}$ as the *trivial deformation with respect to f over $\mathcal{A}$ of X and f_0*, respectively.

Given a line bundle L on f_0, we denote by $\operatorname{Def}_{L|f}$ the functor of log Artin rings which assigns to each $\mathcal{A}$ the set $\operatorname{Def}_{L|f}(\mathcal{A})$ of isomorphism classes of line bundles $\mathcal{L}$ on the trivial deformation $f_{\mathcal{A}}$ over $\mathcal{A}$ together with an isomorphism $\mathcal{L} \otimes_A k \cong L$. Due to the fact that the log smooth log scheme $f_{\mathcal{A}}$ has the underlying scheme $\underline{\mathcal{X}}_A = \underline{\mathcal{X}} \times_{\operatorname{Spec} T} \operatorname{Spec} A$, we have an equality $\operatorname{Def}_{L|f}(\mathcal{A}) = \operatorname{Pic}_L(A)$, where Pic_L denotes the local Picard functor of L on $\underline{\mathcal{X}}/\operatorname{Spec} T$ (cf. [31, 3.1], [32, 3.3.1]). For this reason we call $\operatorname{Def}_{L|f}$ the *local (logarithmic) Picard functor of L on f*.

Given a flat log connection ∇ on f_0, we denote by $\operatorname{Def}_{\nabla|f}$ the functor of log Artin rings which assigns to each $\mathcal{A}$ the set $\operatorname{Def}_{\nabla|f}(\mathcal{A})$ of flat log connections Δ on the trivial deformation $f_{\mathcal{A}}$ over $\mathcal{A}$ together with an isomorphism $\Delta \otimes_A k \cong \nabla$. We call this functor the *local flat logarithmic connection functor of ∇ on f*.

Assume, that there is a line bundle $\mathcal{L}$ on $\mathcal{X}$ given. We denote by $\mathcal{L}_{\mathcal{A}}$ the element $\mathcal{L} \otimes_T A$ and we write L for $\mathcal{L}_\kappa$. Then for every $\mathcal{A} \in \underline{\mathrm{LArt}}_{\mathcal{T}}$, we have $\mathcal{L}_{\mathcal{A}} \in \operatorname{Def}_{L|f}(\mathcal{A})$ and we call $\mathcal{L}_{\mathcal{A}}$ the *trivial deformation with respect to f of L over $\mathcal{A}$*.

Analogously, assume that there is a flat log connection Δ on f given. We denote by $\Delta_{\mathcal{A}} = (\mathcal{L}_A, \Delta_{\mathcal{A}})$ the element given by the restriction $\Delta_{\mathcal{A}}\colon \mathcal{L}_{\mathcal{A}} \to \Omega^1_{f_A} \otimes_{\mathcal{O}_{\mathcal{X}_{\mathcal{A}}}} \mathcal{L}_{\mathcal{A}}$, $\Delta_{\mathcal{A}}([s]) = [\Delta(s)]$. Observe that $\Omega^1_{f_{\mathcal{A}}} \cong \Omega^1_f \otimes_T A$ by base change. We write $\nabla = (L, \nabla)$ for Δ_κ and we call $\Delta_{\mathcal{A}} \in \operatorname{Def}_{\nabla|f}(\mathcal{A})$ the *trivial deformation with respect to f of ∇ over $\mathcal{A}$*.

Given a log symplectic form ω (of general type) on $f_0\colon (X, \nabla) \to \operatorname{Spec} \kappa$, we denote by $\operatorname{Def}_{\omega|(f,\Delta)}$ the functor which assigns to each $\mathcal{A}$ the set $\operatorname{Def}_{\omega|(f,\Delta)}(\mathcal{A})$ of log symplectic forms ϖ on the trivial deformation $(f_A, \Delta_{\mathcal{A}})$ over $\mathcal{A}$ such that $\varpi \otimes_A k = \omega$. We call this functor the *local logarithmic symplectic form functor of ω on (f, Δ)*.

Given a log symplectic form ω of non-twisted type on f_0, we denote by $\operatorname{Def}_{\omega|f}$ the functor which assigns to each $\mathcal{A}$ the set $\operatorname{Def}_{\omega|f}(\mathcal{A})$ of log symplectic forms ϖ of non-twisted type on the trivial deformation $f_{\mathcal{A}}$ over $\mathcal{A}$ such that $\varpi \otimes_A k = \omega$. This functor is called the *local non-twisted logarithmic symplectic form functor of ω on f*.

Each of these functors of type $\operatorname{Def}_{F|\mathcal{G}}$ is, if defined, naturally a subfunctor of the corresponding functor of type $\operatorname{Def}_{(G,F)}(\,\cdot\,)$, where G is the restriction of $\mathcal{G}$ to κ.

4.1.1 Remark

Observe, that by replacing $\mathcal{T}$ with an element $\tilde{\mathcal{A}} \in \underline{\mathrm{LArt}}_{\mathcal{T}}$ each of the functors introduced here may be defined on the subcategory $\underline{\mathrm{LArt}}_{\tilde{\mathcal{A}}} \subset \underline{\mathrm{LArt}}_{\mathcal{T}}$ as a functor of log Artin rings $\underline{\mathrm{LArt}}_{\tilde{\mathcal{A}}} \to \underline{\mathrm{Set}}$.

4.2 Obstruction theory of log symplectic deformations

In this section we will discuss infinitesimal deformations of log schemes with line bundles, log schemes with flat log connection and log symplectic schemes (of both types) as defined in chapter 3. We will calculate the tangent spaces to the functors above and obstructions to the existence of infinitesimal deformations. In doing so we are geared to the calculations done in SGA ([12, III.6]) for flat deformations of schemes.

Let $\mathcal{A} \in \underline{\mathrm{LArt}}_{\mathcal{T}}$ be a log Artin ring and let A be its underlying Artin ring. Recall that in chapter 2 we have defined an extensions in $\underline{\mathrm{Art}}_T$ to be a short exact sequence of T-modules $e\colon 0 \to J \to B \to A \to 0$, where $B \to A$ is a surjective homomorphism in $\underline{\mathrm{Art}}_T$ with kernel J such that $J^2 = 0$. We have called such an extension small if $\mathfrak{m}_B J = 0$ and principal small if moreover J is a principal ideal. An extension (respectively, a small extension, a principal small extension) in $\underline{\mathrm{LArt}}_{\mathcal{T}}$ has been defined to be a homomorphism $\mathcal{B} \to \mathcal{A}$ the underlying homomorphism $B \to A$ of which sits in an extension (respectively, a small extension, a principal small extension) in $\underline{\mathrm{Art}}_T$ (cf. sections 2.1.1 and 2.2.1).

Let for now $f_0\colon X \to \operatorname{Spec}\kappa$ be a log smooth log fs log scheme over $\operatorname{Spec}\kappa$ and let $\mathcal{U} = \{X_i\}_{i\in I}$ be an open affine covering of X. We will write X_I for $\bigcap_{i\in I} X_i$ and $f_0|I$ for the restriction of f_0 to X_I. We denote the Čech-cochain-complex with respect to $\mathcal{U}$ of a sheaf $\mathcal{F}$ by $\check{C}^\bullet(\mathcal{U},\mathcal{F})$ (respectively, of a complex of sheaves $\mathcal{F}^\bullet$ by $\check{C}^\bullet(\mathcal{U},\mathcal{F}^\bullet)$). By a cochain in the double complex $\check{C}^\bullet(\mathcal{U},\mathcal{F}^\bullet)$, we mean a cochain in its associated total complex (analogously for a cocycle and a coboundary, respectively).

We will gradually build up the data that turns $f_0\colon X \to \operatorname{Spec}\kappa$ into a log symplectic scheme: First we regard the log smooth morphism f_0 and its log smooth deformation functor Def_{f_0}. Then we add to f_0 a log symplectic form of non-twisted type ω on f_0 and regard the deformation functor of log symplectic schemes of non-twisted type $\mathrm{Def}_{(f_0,\omega)}$. Erasing ω and adding to f_0 first a line bundle L and then later a flat log connection ∇, we regard the deformation functor of log smooth schemes with line bundle $\mathrm{Def}_{(f_0,L)}$ and the deformation functor of log smooth schemes with flat log connection $\mathrm{Def}_{(f_0,\nabla)}$, respectively. Finally, adding to f_0 and ∇ a log symplectic form ω of type ∇ on f_0, we examine the deformation functor of log symplectic schemes of general type $\mathrm{Def}_{(f_0,\nabla,\omega)}$. At the end of each stage, we draw corollaries concerning the deformation theory of the corresponding local deformations functor $\mathrm{Def}_{f|\omega}$, $\mathrm{Def}_{f|L}$, $\mathrm{Def}_{f|\nabla}$ and $\mathrm{Def}_{(f,\Delta)|\omega}$, respectively.

In each section, given a lifting η (standing for f, (f,ϖ), $(f,\mathcal{L})$ etc.) over a log Artin ring $\mathcal{A}$ of the base object η_0 (standing for f_0, (f_0,ω), (f_0,L) etc.), we calculate a first subsection entitled "Group of automorphism" the group of automorphisms of a lifting $\tilde{\eta}$ along an extension $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ over the given lifting η. In a second subsection entitled "Pseudo-torsor of liftings" we give a group G acting freely on the set of isomorphism classes of liftings $\tilde{\eta}$ of η along e, making this set a G-pseudo-torsor. This, in conclusion, yields the tangent space of the respective deformation functor. The obstruction theory (as defined in 2.1.21) of the respective functor is calculated in the respective third subsection entitled “Obstruction Theory”. In particular we identify the obstruction $o_e(\eta)$ to the existence of a lifting $\tilde{\eta}$ of η along the extension e. At the end of a section, under the heading “Conclusion”, we collect the results of the preceding three subsections.

The statements of the first following section about log smooth deformations of f_0 are well-known (cf. [19, 3.14]), but we execute the constructions also in this case to introduce notation to be used later on. Similarly, the deformation theory of smooth schemes with line bundle is well-known (cf. [32, 3.3.3]), differing from our calculations only by the absence of log structures and the exchange of smoothness and log smoothness. All other stages are developed by ourselves.

4.2.1 Log smooth deformations of log schemes

Let $f_0\colon X \to \operatorname{Spec}\kappa$ be a log fs log smooth log scheme and $f\colon \mathcal{X} \to \operatorname{Spec}\mathcal{A}$ a log smooth deformation of f_0. Let $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension of $\mathcal{A}$ and $\tilde{f}\colon \tilde{\mathcal{X}} \to \operatorname{Spec}\tilde{\mathcal{A}}$ a lifting of f over $\tilde{\mathcal{A}}$.

The kernel of $\mathcal{O}_{\tilde{\mathcal{X}}} \to \mathcal{O}_{\mathcal{X}}$ is $\mathcal{O}_{\tilde{\mathcal{X}}} \otimes_{\tilde{A}} J = \mathcal{O}_{\mathcal{X}} \otimes_A J$. Hence, we have a short exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}} \otimes_A J \to \mathcal{O}_{\tilde{\mathcal{X}}} \to \mathcal{O}_{\mathcal{X}} \to 0.$$

Since the inclusion $\operatorname{Spec}\mathcal{A} \to \operatorname{Spec}\tilde{\mathcal{A}}$ is a (first order) infinitesimal thickening, the short sequence of monoids

$$1 \to 1 + \mathcal{O}_{\mathcal{X}} \otimes_A J \to \mathcal{M}_{\tilde{\mathcal{X}}} \to \mathcal{M}_{\mathcal{X}} \to 1$$

is exact, with $1 + \mathcal{O}_{\mathcal{X}} \otimes_A J$ acting freely on $\mathcal{M}_{\tilde{\mathcal{X}}}$.

Let $\mathcal{X}_I$ and $\tilde{\mathcal{X}}_I$ denote the open affine subscheme of $\mathcal{X}$ and $\tilde{\mathcal{X}}$, respectively, lying over X_I. We write $f|I$ and $\tilde{f}|I$ for the restriction of f and $\tilde{f}$ to $\mathcal{X}_I$ and $\tilde{\mathcal{X}}_I$, respectively.

Group of automorphisms

An automorphism $\tilde{\varphi}$ of $\tilde{f}$ which induces the identity on f is given on each $\tilde{\mathcal{X}}_i$ by a ring automorphism $\tilde{\varphi}_i^*\colon \mathcal{O}_{\tilde{\mathcal{X}}_i} \to \mathcal{O}_{\tilde{\mathcal{X}}_i}$ with $\tilde{\varphi}_i^* = 1 + \underline{\vartheta}_i$, where $\underline{\vartheta}_i\colon \mathcal{O}_{\tilde{\mathcal{X}}_i} \to \mathcal{O}_{\mathcal{X}} \otimes_A J$. Due to the A-linearity of $\tilde{\varphi}^*$ and the fact that

$$ab + \underline{\vartheta}_i(ab) = \tilde{\varphi}_i^*(ab) = \tilde{\varphi}_i^*(a)\tilde{\varphi}_i^*(b) = (a + \underline{\vartheta}_i(a))(b + \underline{\vartheta}_i(b)) = ab + (b\underline{\vartheta}_i(a) + a\underline{\vartheta}_i(b))$$

for local sections a, b of $\mathcal{O}_{\tilde{\mathcal{X}}}$, $\underline{\vartheta}_i$ is an element of $\Gamma(\mathcal{X}_i, T_{\underline{f}} \otimes_A J)$, i. e. an (ordinary) A-linear derivation with values in $\mathcal{O}_{\mathcal{X}_i} \otimes_A J$.

Moreover, $\tilde{\varphi}_i$ must respect the log structure $\alpha_{\mathcal{X}_i}$ and be compatible with $\alpha_{\tilde{\mathcal{X}}_i}$. To this end, the equalities $\tilde{\varphi}^* \circ \alpha_{\tilde{\mathcal{X}}} = \alpha_{\tilde{\mathcal{X}}} \circ \tilde{\varphi}^\flat$, $\tilde{\varphi}^\flat|_{\mathcal{M}_{\mathcal{X}}} = id$ and $\alpha_{\tilde{\mathcal{X}}}|_{\mathcal{M}_{\mathcal{X}}} = \alpha_{\mathcal{X}}$ must hold.

Because of the second equality, we can write $\tilde{\varphi}_i^\flat = (1 + \Theta_i) \cdot id$, where $\cdot$ is the free action of $1 + \mathcal{O}_{\mathcal{X}} \otimes_A J$ on $\mathcal{M}_{\tilde{\mathcal{X}}}$ and Θ_i is a map $\mathcal{M}_{\tilde{\mathcal{X}}_i} \to \mathcal{O}_{\mathcal{X}_i} \otimes_A J$. We conclude from the fact that

$$
\begin{aligned}
(1+\Theta_i(ab))\cdot ab = \tilde{\varphi}_i^\flat(ab) &= \tilde{\varphi}_i^\flat(a)\tilde{\varphi}_i^\flat(b) \\
&= (1+\Theta_i(a))\cdot((1+\Theta_i(b))\cdot ab) = (1+\Theta_i(a)+\Theta_i(b))\cdot ab
\end{aligned}
$$

(observe, that the target of Θ_i is a sheaf of groups) and from the $\tilde{A}^\times$-linearity of $\tilde{\varphi}^\flat$ that

$$\Theta_i \in \operatorname{Hom}_{\underline{\mathrm{Mon}}}(\mathcal{M}_{\mathcal{X}_i}, \mathcal{O}_{\mathcal{X}_i} \otimes_A J).$$

Let a be a local section of $\mathcal{M}_{\tilde{\mathcal{X}}_i}$. Then

$$
\begin{aligned}
\alpha_{\tilde{\mathcal{X}}_i}(a) + \alpha_{\mathcal{X}_i}(a)\Theta(a) &= \alpha_{\tilde{\mathcal{X}}_i}(a)(1+\Theta_i(a)) = \alpha_{\tilde{\mathcal{X}}_i}(a)\alpha_{\tilde{\mathcal{X}}_i}(1+\Theta_i(a)) \\
&= \alpha_{\tilde{\mathcal{X}}_i}((1+\Theta_i(a))\cdot a) = \alpha_{\tilde{\mathcal{X}}_i}(\tilde{\varphi}_i^\flat(a)) = \tilde{\varphi}_i^*(\alpha_{\tilde{\mathcal{X}}_i}(a)) \\
&= \alpha_{\tilde{\mathcal{X}}_i}(a) + \vartheta_i(\alpha_{\mathcal{X}_i}(a)),
\end{aligned}
$$

where in the second equality we regard $\mathcal{O}_{\tilde{\mathcal{X}}}^\times$ as a submonoid sheaf of $\mathcal{M}_{\tilde{\mathcal{X}}}$.
Hence, $\alpha_{\mathcal{X}_i}(a)\Theta_i(a) = \vartheta_i(\alpha_{\mathcal{X}_i}(a))$ and $\Theta_i(c) = 0$ for $c \in A^\times$, so $\vartheta_i := (\underline{\vartheta}_i, \Theta_i)$ is an A-linear log derivation with values in J, thus an element of $\Gamma(\mathcal{X}_i, T_f \otimes_A J)$ and the collection $(\vartheta_i)_{i\in I}$ is a cochain in $\check{C}^0(\mathcal{U}, T_f \otimes_A J)$.
In order that the $\tilde{\varphi}_i$ define a global automorphism $\tilde{\varphi}$ they have to agree on the overlaps $\tilde{\mathcal{X}}_{ij}$, thus fulfil $\tilde{\varphi}_i = \tilde{\varphi}_j$, which implies $\vartheta_i = \vartheta_j$. This shows that (ϑ_i) is a cocycle in $\check{C}^0(X_i, T_f \otimes_A J)$ and defines, due to the lack of coboundaries, a unique element $\vartheta \in H^0(\mathcal{X}, T_f \otimes_A J)$.
Hence, the group of automorphisms of $\tilde{\mathcal{X}}$ over $\mathcal{X}$ is canonically isomorphic to the group $H^0(\mathcal{X}, T_f \otimes_A J)$.

Pseudo-torsor of liftings

In what follows, we will in the beginning strictly distinguish between $\tilde{\mathcal{X}}_{ij}$ and $\tilde{\mathcal{X}}_{ji}$ as subschemes of $\tilde{\mathcal{X}}_i$ and $\tilde{\mathcal{X}}_j$, respectively. For elements interpretable as maps the indexing ij corresponds in essence to their direction "from j to i" (i. e. from X_j to X_i, from $\mathcal{O}_{X_j}$ to $\mathcal{O}_{X_i}$ etc.).
Assume that there exists a lifting $\tilde{f}_0\colon \tilde{\mathcal{X}}_0 \to \operatorname{Spec}\tilde{A}$ of f. By the uniqueness of log fs log smooth liftings of affine log schemes (cf. proposition 2.3.3), any other lifting $\tilde{f}$ is given by the same open schemes $\tilde{\mathcal{X}}_i = \tilde{\mathcal{X}}_{0i}$ as $\tilde{\mathcal{X}}_0$, but glued differently by automorphisms $\tilde{\varphi}_{ij}$ on their overlaps $\tilde{\mathcal{X}}_{ij}$. These satisfy $\tilde{\varphi}_{ij}^* = 1 + \vartheta_{ji}$, where 1 denotes the identity on $\mathcal{O}_{\tilde{\mathcal{X}}_{ij}}$ (which is the corresponding glueing morphism of $\tilde{\mathcal{X}}_0$) and where $\vartheta_{ji} \in \Gamma(\mathcal{X}_{ij}, T_f \otimes_A J)$. (Observe the index swap when passing from $\tilde{\varphi}_{ij}$ to ϑ_{ji}.)
The relation $\tilde{\varphi}_{kj}\tilde{\varphi}_{ji}\tilde{\varphi}_{ki}^{-1} = id_{\tilde{\mathcal{X}}_{ijk}}$ has to be satisfied on $\tilde{\mathcal{X}}_{ijk}$, which means that

$$0 = \vartheta_{jk} + \vartheta_{ij} - \vartheta_{ik}.$$

Therefore, the cochain (ϑ_{ij}) is a cocycle in $\check{C}^1(\mathcal{U}, T_f \otimes_A J)$ and thus represents an element in the group $H^1(\mathcal{X}, T_f \otimes_A J)$ denoted $[\tilde{f} - \tilde{f}_0]$.

If we alter this cocycle by a coboundary, it still describes the same isomorphism class of deformations: To see this, let $\vartheta'_{ij} := \vartheta_{ij} + (\vartheta'_j - \vartheta'_i)$, with $(\vartheta'_i) \in \check{C}^0(\mathcal{U}, T_f \otimes_A J)$ and let $\tilde{f}'$ be the corresponding deformation. Then

$$(1 + \vartheta_{ij})(1 + \vartheta'_j) = 1 + (\vartheta_{ij} + \vartheta'_j) = 1 + (\vartheta'_{ij} + \vartheta'_i) = (1 + \vartheta'_i)(1 + \vartheta'_{ij}),$$

i. e. there are locally defined automorphisms $\tilde{\varphi}'_i \colon \tilde{\mathcal{X}}_i \to \tilde{\mathcal{X}}_i$ with $\tilde{\varphi}'_i|_{\mathcal{X}_i} = id_{\mathcal{X}_i}$, given by $\tilde{\varphi}'^*_i = 1 + \vartheta'_i$, and with the property that

$$\tilde{\varphi}'_j \circ \tilde{\varphi}_{ji} = \tilde{\varphi}'_{ji} \circ \tilde{\varphi}'_i.$$

Such a cochain of locally defined automorphisms $(\tilde{\varphi}'_i)$ describes just a global automorphism $\tilde{\varphi}' \colon \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$ with $\tilde{\varphi}'|_{\mathcal{X}} = id_{\mathcal{X}}$. Hence, $\tilde{f}$ and $\tilde{f}'$ are isomorphic.

We conclude that the group $G := H^1(\mathcal{X}, T_f \otimes_A J)$ acts freely on the set of isomorphism classes of liftings along e if this set is non-empty, making this set a G-pseudo-torsor. For the trivial extension $\varepsilon^0 \colon 0 \to (\varepsilon) \to \mathcal{A}[\varepsilon]^0 \to \mathcal{A} \to 0$ the set of isomorphism classes of liftings over $\mathcal{A}[\varepsilon]^0$ is thus given by $H^1(\mathcal{X}, T_f)$, with the trivial deformation $\tilde{f}_0 := f \times_{\operatorname{Spec} \mathcal{A}} \operatorname{Spec} \mathcal{A}[\varepsilon]^0$ corresponding to zero. In particular, $t_{\mathrm{Def}_{f_0}} = \mathrm{Def}_{f_0}(\kappa[\varepsilon]^0) = H^1(X, T_{f_0})$.

Obstruction theory

Now we are going to calculate the obstruction for lifting f to $\tilde{f}$ over $\tilde{\mathcal{A}}$. To this end, we start with a collection of deformations $\tilde{\mathcal{X}}_i$ of the affine open subschemes $\mathcal{X}_i \subset \mathcal{X}$ (which exists uniquely up to isomorphism by 2.3.3) and an arbitrary cochain $(\tilde{\varphi}_{ji})$, where $\tilde{\varphi}_{ji} \colon \tilde{\mathcal{X}}_{ij} \to \tilde{\mathcal{X}}_{ij}$ is an automorphism which restricts to $id_{\mathcal{X}_{ij}}$ over $\mathcal{A}$.

Then $\tilde{\varphi}^*_{ji}\tilde{\varphi}^*_{kj}(\tilde{\varphi}^*_{ki})^{-1} = 1 + \vartheta_{ijk}$, with a cochain $(\vartheta_{ijk}) \in \check{C}^2(\mathcal{U}, T_f \otimes_A J)$. Calculating

$$\begin{aligned} 1 &= \tilde{\varphi}^*_{ji}\big(\tilde{\varphi}^*_{kj}\tilde{\varphi}^*_{lk}(\tilde{\varphi}^*_{lj})^{-1}\big)(\tilde{\varphi}^*_{ji})^{-1}\left(\tilde{\varphi}^*_{ji}\tilde{\varphi}^*_{lj}(\tilde{\varphi}^*_{li})^{-1}\right)\left(\tilde{\varphi}^*_{ki}\tilde{\varphi}^*_{lk}(\tilde{\varphi}^*_{li})^{-1}\right)^{-1}\left(\tilde{\varphi}^*_{ji}\tilde{\varphi}^*_{kj}(\tilde{\varphi}^*_{ki})^{-1}\right)^{-1} \\ &= 1 + \tilde{\varphi}^*_{ji}\vartheta_{jkl}((\tilde{\varphi}^*_{ji})^{-1}\cdot) + \vartheta_{ijl} - \vartheta_{ikl} - \vartheta_{ijk} = 1 + \vartheta_{jkl} + \vartheta_{ijl} - \vartheta_{ikl} - \vartheta_{ijk}, \end{aligned}$$

we show that this cochain is in fact a 2-cocycle. If $(\tilde{\varphi}'_{ij})$ is any other collection, then we have $\tilde{\varphi}'^*_{ji} - \tilde{\varphi}^*_{ji} = \vartheta'_{ij}$ for a cochain (ϑ'_{ij}) in $\check{C}^1(\mathcal{U}, T_f \otimes_A J)$; hence, $\tilde{\varphi}'^*_{ji}\tilde{\varphi}'^*_{kj}(\tilde{\varphi}'^*_{ki})^{-1} = 1 + \vartheta_{ijk} + \vartheta'_{jk} + \vartheta'_{ij} - \vartheta'_{ik}$.

This means that the class $o_e([f]) \in H^2(\mathcal{X}, T_f \otimes_A J)$ defined by the cocycle (ϑ_{ijk}) is independent of the collection $(\tilde{\varphi}_{ij})$ chosen, because two such cocycles always differ by a coboundary. Moreover, $o_e([f])$ vanishes exactly when there exists a collection with $\vartheta_{ijk} = 0$, and this is true if and only if a lifting $\tilde{f}$ of f exists.

Since both assignments $f \mapsto H^2(\mathcal{X}, T_f \otimes_A J)$ and $e \mapsto H^2(\mathcal{X}, T_f \otimes_A \hat{J}(e))$ are functorial, if we define $H_e([f]) := H^2(\mathcal{X}, T_f \otimes_A J)$ and $o \colon V_{\mathrm{Def}_{f_0}} \to O := \coprod_{V_{\mathrm{Def}_{f_0}}} H$ by the assignment $x \in V_{\mathrm{Def}_{f_0}}(e) \mapsto o_e(x) \in H^2(\mathcal{X}, T_f \otimes_A J)$, then (H, o) is a complete obstruction theory for the deformation functor Def_{f_0} in the sense of definition 2.1.21 of chapter 2. Due to the fact, that by the projection formula for any small extension $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ we have $H_e([f]) = H^2(\mathcal{X}, T_f \otimes_A J) \cong H^2(\mathcal{X}, T_{f_0} \otimes_{\mathbb{C}} J) \cong H^2(\mathcal{X}, T_{f_0}) \otimes_{\mathbb{C}} J$, this obstruction theory (H, o) is linear; in particular, $H_0 := H_{\varepsilon^0}(f_0) = H^2(X, T_{f_0})$ is a small obstruction space for the functor Def_{f_0} in the sense of definition 2.1.19 of chapter 2.

Conclusion

4.2.1 Proposition ([19, 3.14]; cf. 2.3.4)
Let $f_0\colon X \to \operatorname{Spec}\kappa$ be a log smooth log fs log scheme.

a) The tangent space of the deformation functor Def_{f_0} is $H^1(X, T_{f_0})$.

b) The vector space $H^2(X, T_{f_0})$ is the small obstruction space of a complete linear obstruction theory for the functor Def_{f_0}.

Let $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension of log Artin rings and let f be a lifting of f_0 over $\mathcal{A}$.

c) The group of automorphisms of a lifting $\tilde{f}$ over $\tilde{\mathcal{A}}$ inducing the identity on f is $H^0(\mathcal{X}, T_f \otimes_A J)$.

d) The set of isomorphism classes of liftings of f over $\tilde{\mathcal{A}}$ is a pseudo-torsor under $H^1(\mathcal{X}, T_f \otimes_A J)$.

e) The complete obstruction $o_e([f])$ to lifting f over $\tilde{\mathcal{A}}$ is an element of the obstruction space $H^2(\mathcal{X}, T_f \otimes_A J)$.

4.2.2 Remark
This is proposition 3.14 in [19] (which we already cited in 2.3.4). It is incorrectly cited in [17], where the author claims that for all extensions e as above the torsor of liftings and obstruction space are $H^p(\mathcal{X}, T_f) \otimes_A J$, $p = 1, 2$, respectively, with the ideal J standing *outside* of the argument of the functor $H^p(\mathcal{X}, \cdot\,)$. This is true for small extensions because in this case J is a free A-module and the projection formula allows to "factor out" J. For arbitrary extensions this is generally false.

Since $f_0\colon X \to \operatorname{Spec}\kappa$ is a log smooth and log integral morphism of log schemes by proposition 1.2.42, every log smooth lifting f of f_0 over $\mathcal{A}$ has the flat lifting $\underline{f}$ of $\underline{f_0}$ over $A = v(\mathcal{A})$ as underlying morphism of schemes. Hence, there is an obvious forgetful morphism $\mathrm{Def}_{f_0} \to \mathrm{Def}_{\underline{f_0}}$, corresponding to the inclusion $T_{f_0} \subset T_{\underline{f_0}}$ (for a lifting f of f_0, to the inclusion $T_f \subset T_{\underline{f}}$), where $\mathrm{Def}_{\underline{f_0}}$ denotes the functor of log Artin rings of flat deformations of $\underline{f_0}$ over the underlying Artin ring, and we may conclude the following:

4.2.3 Corollary
Under the A-linear map $H^p(\mathcal{X}, T_f \otimes_A J) \to H^p(\mathcal{X}, T_{\underline{f}} \otimes_A J)$

a) an automorphism of a lifting $\tilde{f}$ over f is mapped to its underlying automorphism of $\underline{\tilde{f}}$ over $\underline{f}$, for $p = 0$.

b) the class $[\tilde{f} - \tilde{f}_0]$ of a lifting $\tilde{f}$ of f is mapped to the class $[\underline{\tilde{f}} - \underline{\tilde{f}_0}]$ of its underlying lifting $\underline{\tilde{f}}$ of $\underline{f}$, for $p = 1$, whenever a lifting $\tilde{f}_0$ of f is given.

c) the obstruction $o_e([f])$ of Def_{f_0} is mapped to the obstruction $o_e([\underline{f}])$ of $\operatorname{Def}_{\underline{f_0}}$, for $p = 2$.

4.2.2 Deformations of log symplectic schemes of non-twisted type

Before approaching the general case of a log symplectic scheme, we discuss the special case of log symplectic schemes of non-twisted type and their deformations. Let $f_0 \colon (X, \omega) \to \operatorname{Spec} \kappa$ be a log symplectic scheme of non-twisted type and recall the definition of its associated T-complex $T^\bullet_{f_0}(\omega)$ and T-sequence t^ω from section 3.3.3.
Let $f \colon (\mathcal{X}, \varpi) \to \operatorname{Spec} \mathcal{A}$ be a log symplectic deformation of non-twisted type of $(f_0, \omega)/\kappa$, let $e \colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension of $\mathcal{A}$ and $\tilde{f} \colon (\tilde{\mathcal{X}}, \tilde{\varpi}) \to \operatorname{Spec} \tilde{\mathcal{A}}$ a lifting of $(f, \varpi)/\mathcal{A}$ over $\tilde{\mathcal{A}}$.

Group of automorphisms

An automorphism $\tilde{\varphi}$ of $(\tilde{f}, \tilde{\varpi})/\tilde{\mathcal{A}}$ which induces the identity on $(f, \varpi)/\mathcal{A}$ is an automorphism of $\tilde{f}$, given by a cochain (ϑ_i) as in section 4.2.1 above, with the additional property that $\tilde{\varphi}^*(\tilde{\varpi}_i) = \tilde{\varpi}_i$; hence,

$$0 = \tilde{\varpi}_i + \vartheta_i(\varpi_i) - \tilde{\varpi}_i = \vartheta_i(\varpi_i),$$

i. e. $0 = (-t^{\varpi_i})(\vartheta_i)$. Therefore, (ϑ_i) is a 0-cocycle in the double complex $\check{C}^\bullet(\mathcal{U}, T^\bullet_f(\varpi))$ and, due to the lack of 0-coboundaries, uniquely defines a class in $\mathbb{H}^0(T^\bullet_f(\varpi) \otimes_A J)$.
Using the identification $T^\bullet_f(\varpi) \cong \Omega^{\geq 1, \bullet}_f[1]$ replaces the log derivations ϑ_i by 1-forms $\tau_i := i_{\vartheta_i}(\varpi_i)$ with $d\tau_i = 0$, defining a unique class in $\mathbb{H}^0(\Omega^{\geq 1, \bullet}_f[1] \otimes_A J)$.
Hence, the group of automorphisms of $(\tilde{\mathcal{X}}, \tilde{\varpi})$ over $(\mathcal{X}, \varpi)$ is canonically isomorphic to both $\mathbb{H}^0(T^\bullet_f(\varpi) \otimes_A J)$ and $\mathbb{H}^0(\Omega^{\geq 1, \bullet}_f[1] \otimes_A J)$.

Pseudo-torsor of liftings

Let $(\tilde{f}_0, \tilde{\varpi}_0)$ be a lifting of (f, ϖ) over $\tilde{\mathcal{A}}$. Any other lifting $(\tilde{f}, \tilde{\varpi})$ of (f, ϖ) is given (in relation to $(\tilde{f}_0, \tilde{\varpi}_0)$) by a cocycle (ϑ_{ij}) as before and a collection of elements $u_i \in \Gamma(\mathcal{X}_i, \Omega^2_f)$ such that $\tilde{\varpi}_i = \tilde{\varpi}_{0,i} + u_i$, which implies

$$0 = d\tilde{\varpi}_i = d(\tilde{\varpi}_{0,i} + u_i) = d\tilde{\varpi}_{0,i} + du_i.$$

Since $d\tilde{\varpi}_{0,i} = 0$, this means that $(-d)u_i = 0$.

Moreover, we must have $\tilde{\varpi}_i = \varphi_{ji}^*(\tilde{\varpi}_j)$, so

$$\begin{aligned}\tilde{\varpi}_{0,i} + u_i &= (1+\vartheta_{ij})(\tilde{\varpi}_{0,j} + u_j)\\ &= \tilde{\varpi}_{0,j} + \vartheta_{ij}(\varpi_j) + u_j,\end{aligned}$$

which, since $\tilde{\varpi}_{0,i} = \tilde{\varpi}_{0,j}$, means that $-u_j + u_i - t^{\varpi_j}(\vartheta_{ij}) = 0$.

We can rewrite these conditions for the u_i and the (ϑ_{ij}) as

a) $0 = (-d)u_i$,

b) $0 = (u_j - u_i) - (-t^{\varpi})(\vartheta_{ij})$ and

c) $0 = \check{d}(\vartheta_{ij})$.

Regarding the complex $(T_f^\bullet(\varpi), d_{T^\bullet})$, these three conditions combine to the single condition $0 = (\check{d} \pm d_{T^\bullet})(u_i, \vartheta_{ij})$ which means precisely that the cochain (u_i, ϑ_{ij}) is a 1-cocycle in $T_f^\bullet(\varpi)$ and thus defines a class $[(\tilde{f}, \tilde{\varpi}) - (\tilde{f}_0, \tilde{\varpi}_0)] \in \mathbb{H}^1(T_f^\bullet(\varpi) \otimes_A J)$.

An alteration of the cocycle by a coboundary changes only the (ϑ_{ij}) and therefore, as we already know, not the deformation class. To conclude that the set of isomorphism classes of deformations is a (pseudo) torsor under $\mathbb{H}^1(T_f^\bullet(\varpi) \otimes_A J)$, we need to verify that every lifting $\tilde{\varpi}$ of ϖ is a symplectic form.

Without further conditions $\tilde{\varpi}$ induces the map

$$i\,.\,\tilde{\varpi}\colon T_{\tilde{f}} \to \Omega^1_{\tilde{f}}$$

of which we show that it is an isomorphism by looking at the stalks. To this end, let $x \in \mathcal{X}_i$ and $\tilde{\varpi}_x = \tilde{\varpi}_{0,x} + u_x$ the germ of $\tilde{\varpi}$ at x. Let σ_x be a germ in $\Omega^1_{\tilde{f},x}$.

By assumption $i\,.\,(\tilde{\varpi}_{0,x})$ is an isomorphism. So there exists exactly one germ $\delta_x \in T_{\tilde{f}_0,x}$ with $\sigma_x = i_{\delta_x}(\tilde{\varpi}_{0,x})$. Also, for the 1-form $i_{\delta_x}(u_x) \in (\Omega^1_{\tilde{f}_0} \otimes J)_x = \Omega^1_{\tilde{f}_0,x} \otimes J$, there exists exactly one germ $\beta_x \in (T_{\tilde{f}_0} \otimes J)_x = T_{\tilde{f}_0,x} \otimes J$ with $-i_{\delta_x}(u_x) = i_{\beta_x}(\tilde{\varpi}_{0,x})$. Then $i_{\delta_x+\beta_x}(\tilde{\varpi}) = i_{\delta_x}(\tilde{\varpi}_{0,x}) + i_{\delta_x}(u_x) + i_{\beta_x}(\tilde{\varpi}_{0,x}) + i_{\beta_x}(u_x) = i_{\delta_x}(\tilde{\varpi}_{0,x}) = \sigma_x$.

This means that any lifting $\tilde{\varpi}$ of ϖ induces an isomorphism. This corresponds to the fact that units in the ring of regular functions always lift to units.

Using the identification $T_f^\bullet(\varpi) \cong \Omega_f^{\geq 1,\bullet}[1]$ replaces the log derivations ϑ_{ij} by 1-forms $\tau_{ij} := i_{\vartheta_{ij}}(\omega_j)$ with $0 = (\check{d} \pm (-d))(u_i, \tau_{ij})$, defining a unique class in $\mathbb{H}^1(\Omega_f^{\geq 1,\bullet}[1] \otimes_A J)$.

We conclude that the group $G := \mathbb{H}^1(T_f^\bullet(\varpi) \otimes_A J) = \mathbb{H}^1(\Omega_f^{\geq 1,\bullet}[1] \otimes_A J)$ acts freely on the set of isomorphism classes of liftings along e if this set is non-empty, making this set a G-pseudo-torsor. For the the trivial extension $\varepsilon^0\colon 0 \to (\varepsilon) \to \mathcal{A}[\varepsilon]^0 \to \mathcal{A} \to 0$ the set of isomorphism classes of liftings over $\mathcal{A}[\varepsilon]^0$ is thus given by $\mathbb{H}^1(T_f^\bullet) = \mathbb{H}^1(\Omega_f^{\geq 1,\bullet}[1])$. In particular $t_{\mathrm{Def}_{(f_0,\omega)}} = \mathrm{Def}_{(f_0,\omega)}(\kappa[\varepsilon]^0) = \mathbb{H}^1(T_{f_0}^\bullet) = \mathbb{H}^1(\Omega_{f_0}^{\geq 1,\bullet}[1])$.

Obstruction theory

We are now going to calculate the obstruction to lifting $(f, \varpi)/\mathcal{A}$ to $(\tilde{f}, \tilde{\varpi})/\tilde{\mathcal{A}}$ along the extension $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$. We start from a cochain $\tilde{\varphi}_{ij}$ as before and additionally a cochain of 2-forms $\tilde{\varpi}_i \in \Gamma(\tilde{\mathcal{X}}_i, \Omega^2_{\tilde{f}|i})$ with the property that $\tilde{\varpi}_i|_{\mathcal{X}_i} = \varpi_i$. Then, as before, we have elements ϑ_{ijk} satisfying the corresponding equalities as above. Additionally,

a) $\tilde{\varphi}^*_{ji}(\tilde{\varpi}_j) - \tilde{\varpi}_i = u_{ij}$ and

b) $(-d)\tilde{\varpi}_i = \varrho_i$

apply, with $u_{ij} \in \Gamma(\mathcal{X}_{ij}, \Omega^2_f \otimes_A J)$ and $\varrho_i \in \Gamma(\mathcal{X}_i, \Omega^3_f \otimes_A J)$.
We can directly see that

$$0 = (-d)(-d)\tilde{\varpi}_i = (-d)\varrho_i.$$

From the second equality we get

$$\begin{aligned}(\varrho_j - \varrho_i) &= \tilde{\varphi}^*_{ji}((-d)\tilde{\varpi}_j) - (-d)\tilde{\varpi}_i \\ &= (-d)(\tilde{\varphi}^*_{ji}\tilde{\varpi}_j - \tilde{\varpi}_i) \\ &= (-d)u_{ij};\end{aligned}$$

hence, $0 = (\varrho_j - \varrho_i) - (-d)u_{ij}$.
We calculate in two ways

$$\begin{aligned}&(\tilde{\varphi}^*_{ji}\tilde{\varpi}_j - \tilde{\varpi}_i) + \tilde{\varphi}^*_{ji}((\tilde{\varphi}^*_{kj}\tilde{\varpi}_k - \tilde{\varpi}_j)) - (\tilde{\varphi}^*_{ki}\tilde{\varpi}_k - \tilde{\varpi}_i) \\ &= u_{ij} + \tilde{\varphi}^*_{ij}(u_{jk}) - u_{ik} \\ &= u_{jk} - u_{ik} + u_{ij}\end{aligned}$$

and

$$\begin{aligned}&(\tilde{\varphi}^*_{ji}\tilde{\varpi}_j - \tilde{\varpi}_i) + \tilde{\varphi}^*_{ji}((\tilde{\varphi}^*_{kj}\tilde{\varpi}_k - \tilde{\varpi}_j)) - (\tilde{\varphi}^*_{ki}\tilde{\varpi}_k - \tilde{\varpi}_i) \\ &= \tilde{\varphi}^*_{ji}(\tilde{\varphi}^*_{kj}\tilde{\varpi}_k) - \tilde{\varphi}^*_{ki}\tilde{\varpi}_k \\ &= \left[\tilde{\varphi}^*_{ji}\tilde{\varphi}^*_{kj}\tilde{\varphi}^*_{ik}(\tilde{\varphi}^*_{ki}\tilde{\varpi}_k) - \tilde{\varphi}^*_{ki}\tilde{\varpi}_k\right] \\ &= [(1 + \vartheta_{ijk})(\tilde{\varphi}^*_{ki}\tilde{\varpi}_k) - \tilde{\varphi}^*_{ki}\tilde{\varpi}_k] \\ &= \vartheta_{ijk}(\varpi_k),\end{aligned}$$

from which we conclude that $0 = (u_{jk} - u_{ik} + u_{ij}) + (-t^{\varpi_k})(\vartheta_{ijk})$.
Considering $(\varrho_i, u_{ij}, \vartheta_{ijk})$ as a 2-cochain in the double complex $\check{C}^\bullet(\mathcal{U}, T^\bullet_f(\varpi) \otimes_A J)$, these three conditions are equivalent to

$$(\check{d} \pm d_{T^\bullet})(\varrho_i, u_{ij}\vartheta_{ijk}) = 0,$$

which makes $(\varrho_i, u_{ij}, \vartheta_{ijk})$ a 2-cocycle in $\check{C}^\bullet(\mathcal{U}, T^\bullet_f(\varpi) \otimes_A J)$ and thus defines a class $o_e([f, \varpi]) \in \mathbb{H}^2(T^\bullet_f(\varpi) \otimes_A J)$.

If $(\tilde{\varphi}'_{ij}, \tilde{\varpi}'_i)$ is any other collection, then we have, as before, $\tilde{\varphi}'^*_{ji} - \tilde{\varphi}^*_{ji} = \vartheta'_{ij}$ for a cochain (ϑ'_{ij}) in $\check{C}^1(\mathcal{U}, T_f \otimes_A J)$; hence, $\tilde{\varphi}'^*_{ji}\tilde{\varphi}'^*_{kj}(\tilde{\varphi}'^*_{ki})^{-1} = 1 + \vartheta_{ijk} + \vartheta'_{jk} + \vartheta'_{ij} - \vartheta'_{ik}$. Moreover, $\tilde{\varpi}'_i - \tilde{\varpi}_i = u'_i$ for a cochain $(u'_i) \in \check{C}^0(\mathcal{U}, \Omega^2_f \otimes_A J)$, hence

$$\tilde{\varphi}'^*_{ji}(\tilde{\varpi}'_j) - \tilde{\varpi}'_i = u_{ij} + u'_j - u'_i + \vartheta'_{ij}(\varpi_j) = u_{ij} + (u'_j - u'_i) - (-t^{\varpi})(\vartheta'_{ij})$$

and $(-d)\tilde{\varpi}'_i = \varrho_i + (-d)u'_i$.

This means that the class $o_e([f, \varpi]) \in \mathbb{H}^2(\mathcal{X}, T^\bullet_f(\varpi) \otimes_A J)$ defined by the cocycle $(\varrho_i, u_{ij}, \vartheta_{ijk})$ is independent of the collection $(\tilde{\varphi}_{ij}, \tilde{\varpi}_i)$ chosen, because two such cocycles always differ by a coboundary $(\check{d} \pm d_{T^\bullet})(u'_i, \vartheta'_{ij})$. Moreover, $o_e([f, \varpi])$ vanishes exactly when there exists a collection with $\varrho_i = 0$, $u_{ij} = 0$ and $\vartheta_{ijk} = 0$, and this is true if and only if a lifting $(\tilde{f}, \tilde{\varpi})$ of (f, ϖ) exists.

Setting $H_e([f, \varpi]) := H^2(\mathcal{X}, T^\bullet_f(\varpi) \otimes_A J)$ and $o\colon V_{\mathrm{Def}_{(f_0,\omega)}} \to O := \coprod_{V_{\mathrm{Def}_{(f_0,\omega)}}} H$, $(e, x) \mapsto o_e(x)$, defines a complete linear obstruction theory (H, o) for $\mathrm{Def}_{(f_0,\omega)}$, where again by the projection formula for any small extension $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ we have $H_e([f]) = H^2(\mathcal{X}, T^\bullet_f(\varpi) \otimes_A J) \cong H^2(\mathcal{X}, T^\bullet_{f_0}(\omega)) \otimes_{\mathbb{C}} J$; in particular, $H_0 := H_{\varepsilon^0}([f_0, \omega]) = \mathbb{H}^2(X, T^\bullet_{f_0}(\omega))$ is a small obstruction space for the functor $\mathrm{Def}_{(f_0,\omega)}$.

Using the identification $T^\bullet_f(\varpi) \cong \Omega^{\geq 1,\bullet}_f[1]$ replaces the log derivations ϑ_{ijk} by 1-forms $\tau_{ijk} := i_{\vartheta_{ijk}}(\omega_i)$ with $0 = (\check{d} \pm (-d))(\varrho_i, u_{ij}, \tau_{ijk})$, defining a unique class $o_e([f, \varpi])$ in $\mathbb{H}^2(\Omega^{\geq 1,\bullet}_f[1] \otimes_A J)$.

Conclusion

In the above section we have proven the following:

4.2.4 Proposition

Let $f_0\colon (X, \varpi) \to \operatorname{Spec} \kappa$ be a log symplectic scheme of non-twisted type.

a) The tangent space of the functor $\mathrm{Def}_{(f_0,\omega)}$ is $\mathbb{H}^1(T^\bullet_{f_0}(\omega))$.

b) The vector space $\mathbb{H}^2(T^\bullet_{f_0}(\omega))$ is the small obstruction space of a complete linear obstruction theory for $\mathrm{Def}_{(f_0,\omega)}$.

Let $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension of log rings and let $f\colon (\mathcal{X}, \varpi) \to \operatorname{Spec} \mathcal{A}$ be a lifting of $(f_0, \omega)/\kappa$ over $\mathcal{A}$.

c) The group of automorphisms of a lifting $\tilde{f}\colon (\tilde{\mathcal{X}}, \tilde{\varpi}) \to \operatorname{Spec} \tilde{\mathcal{A}}$ inducing the identity on $(f, \varpi)/\mathcal{A}$ is $\mathbb{H}^0(T^\bullet_f(\omega) \otimes_A J)$.

d) The set of isomorphism classes of liftings $(\tilde{f}, \tilde{\varpi})/\tilde{\mathcal{A}}$ is a pseudo-torsor under the additive group $\mathbb{H}^1(T^\bullet_f(\omega) \otimes_A J)$.

e) The complete obstruction $o_e([f, \varpi])$ to lifting is an element of the obstruction space $\mathbb{H}^2(T^\bullet_f(\omega) \otimes_A J)$.

Given a log smooth deformation $f\colon \mathcal{X} \to \operatorname{Spec}\mathcal{T}$ of f_0, we conclude from the above calculations, by putting all $\vartheta_I = 0$:

4.2.5 Proposition

a) The tangent space of the functor $\mathrm{Def}_{\omega|f}$ is $\mathbb{H}^0(\Omega^{\geq 2,\bullet}_{f_0}[2])$.

b) The vector space $\mathbb{H}^1(\Omega^{\geq 2,\bullet}_{f_0}[2])$ is the small obstruction space of a complete linear obstruction theory for $\mathrm{Def}_{\omega|f}$.

Let $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension of log rings and let $\varpi/\mathcal{A}$ be a lifting of ω over $\mathcal{A}$.

c) The group of automorphisms of a lifting $\tilde{\varpi}/\tilde{\mathcal{A}}$ inducing the identity on $\varpi/\mathcal{A}$ is trivial by definition (and can be identified with $\mathbb{H}^{-1}(\Omega^{\geq 2,\bullet}_{f_\mathcal{A}}[2] \otimes_A J)$).

d) The set of isomorphism classes of liftings $\tilde{\varpi}/\tilde{\mathcal{A}}$ is a pseudo-torsor under the additive group $\mathbb{H}^0(\Omega^{\geq 2,\bullet}_{f_\mathcal{A}}[2] \otimes_A J)$.

e) The complete obstruction $o_e(\varpi)$ to lifting is an element of the obstruction space $\mathbb{H}^1(\Omega^{\geq 2,\bullet}_{f_\mathcal{A}}[2] \otimes_A J)$.

Moreover, we may link the maps in the long exact cohomology sequence associated to the short exact sequence of complexes

$$t^\varpi\colon 0 \to \Omega^{\geq 2,\bullet}_f[1] \to T^\bullet_f(\varpi) \to T_f[0] \to 0$$

to the two morphisms of functors $\mathrm{Def}_{(f_0,\omega)} \to \mathrm{Def}_{f_0}$ and $\mathrm{Def}_{\omega|\tilde{f}} \to \mathrm{Def}_{(f_0,\omega)}$ (for a lifting $\tilde{f}\colon \tilde{\mathcal{X}} \to \operatorname{Spec}\tilde{\mathcal{A}}$ of f) by the following corollaries:

4.2.6 Corollary

Under the A-linear map $\mathbb{H}^p(T^\bullet_f(\omega) \otimes_A J) \to H^p(\mathcal{X}, T_f \otimes_A J)$

a) an automorphism of a lifting $(\tilde{f}, \tilde{\varpi})$ over (f, ϖ) is mapped to its underlying automorphism of $\tilde{f}$ over f, for $p = 0$.

b) the class $[(\tilde{f}, \tilde{\varpi}) - (\tilde{f}_0, \tilde{\varpi}_0)]$ of a lifting $(\tilde{f}, \tilde{\varpi})$ of (f, ϖ) is mapped to the class of its underlying lifting $\tilde{f}$ of f, for $p = 1$, whenever a lifting $(\tilde{f}_0, \tilde{\varpi}_0)$ of (f, ϖ) is given.

c) the obstruction $o_e([f, \varpi])$ of $\mathrm{Def}_{(f_0,\omega)}$ is mapped to the obstruction $o_e([f])$ of Def_{f_0}, for $p = 2$.

4.2.7 Corollary

Given a lifting $(\tilde{f}_0, \tilde{\varpi}_0)$ of (f, ϖ),

a) any automorphism of $\tilde{\varpi}_0$ over ϖ is the identity.

Under the A-linear map $\mathbb{H}^p(\Omega^{\geq 2,\bullet}_f[2] \otimes_A J) \to H^p(T^\bullet_f(\varpi) \otimes_A J)$

b) the class $[\tilde{\varpi} - \tilde{\varpi}_0]$ of a lifting $\tilde{\varpi}$ of ϖ is mapped to the class of the lifting $(\tilde{f}_0, \tilde{\varpi})$ of (f, ϖ), for $p = 1$.

c) the obstruction $o_e(\varpi)$ of $\operatorname{Def}_{\omega|\tilde{f}_0}$ is mapped to the obstruction $o_e([f,\varpi])$ of $\operatorname{Def}_{(f_0,\omega)}$, for $p=2$.

Regarding the A-linear map $t^{\varpi}\colon H^1(\mathcal{X}, T_f \otimes_A J) \to \mathbb{H}^1(\Omega_f^{\geq 2,\bullet}[2] \otimes_A J)$ induced by t^{ϖ},

d) a lifting $\tilde{\varpi}$ of ϖ exists on a lifting $\tilde{f}$ of f if and only if the class $[\tilde{f} - \tilde{f}_0]$ lies in the kernel of t^{ϖ}.

Equivalently, under the pairing $H^1(\mathcal{X}, T_f \otimes_A J) \times \mathbb{H}^0(\Omega_f^{\geq 2,\bullet}[2]) \to \mathbb{H}^1(\Omega_f^{\geq 2,\bullet}[2] \otimes_A J)$ induced by $T_f \times \Omega_f^{\geq 2,\bullet}[2] \to \Omega_f^{\geq 2,\bullet}[2]$, $(\vartheta, u^\bullet) \mapsto \vartheta(u^\bullet)$, the classes $[\tilde{f} - \tilde{f}_0]$ and $[\varpi]$ pair to zero.

Proof: For d), let $[\tilde{f} - \tilde{f}_0]$ be given by the 1-cocycle $(\tilde{\vartheta}_{ij})$ in $T_f \otimes_A J$. We have the following chain of equivalences:

$$
\begin{aligned}
&[\tilde{f} - \tilde{f}_0] \in \operatorname{Ker}(t^{\varpi}) \\
&\qquad \Leftrightarrow (t^{\varpi_i}(\tilde{\vartheta}_{ij})) = (\tilde{\vartheta}_{ij}(\varpi_i)) \text{ is a coboundary in } \check{C}^1(\mathcal{U}, \Omega_f^{\geq 2,\bullet}[2]) \\
&\qquad \Leftrightarrow (-du_i, u_j - u_i) = (\check{d} \pm (-d))(u_i) = (0, \tilde{\vartheta}_{ij}(\varpi_i)) \\
&\qquad\qquad \text{for a cochain } (u_i) \in \check{C}^0(\mathcal{U}, \Omega_f^{\geq 2,\bullet}[2]) \\
&\qquad \Leftrightarrow (u_i, \vartheta_{ij}) \text{ is a 1-cocycle in } \check{C}^\bullet(\mathcal{U}, T_f^\bullet) \\
&\qquad \Leftrightarrow (u_i, \vartheta_{ij}) \text{ defines the class } [(\tilde{f}, \tilde{\varpi}) - (\tilde{f}_0, \tilde{\varpi}_0)] \text{ of a lifting } (\tilde{f}, \tilde{\varpi}).
\end{aligned}
$$

□

4.2.8 Remark

If $f_0\colon (X,\varpi) \to \operatorname{Spec}\kappa$ is a log symplectic scheme of non-twisted type and if $f\colon (\mathcal{X},\varpi) \to \operatorname{Spec}\mathcal{A}$ is a lifting of $(f_0,\omega)/\kappa$ over $\mathcal{A}$, then by means of the isomorphism $i\,.\,(\omega)$ (respectively, $i\,.\,(\varpi)$) we may replace

a) the sheaf T_{f_0} with the sheaf $\Omega^1_{f_0}$ (respectively, the sheaf T_f with the sheaf Ω^1_f) in proposition 4.2.1 as well as in the corollaries 4.2.6 and 4.2.7.

b) the complex $T^\bullet_{f_0}(\omega)$ with the complex $\Omega_{f_0}^{\geq 1,\bullet}$ (respectively, the complex $T_f^\bullet(\varpi)$ with the complex $\Omega_f^{\geq 1,\bullet}$) and the map t^{ω} with the map d (respectively, the map t^{ϖ} with the map d) in proposition 4.2.4 and the following corollaries 4.2.6 and 4.2.7.

4.2.3 Log smooth deformations of log schemes with line bundle

Let $f_0\colon (X,L) \to \operatorname{Spec}\kappa$ be a log smooth log scheme with line bundle and recall from section 3.1.2 the definition of the log Atiyah module $A_{f_0}(L)$ and the log Atiyah extension $d\log(L)$ associated to $[L]$. Refine the open affine covering $\mathcal{U} = \{X_i\}_{i\in I}$ of X in such a way that it trivialises L.

Let $f\colon (\mathcal{X},\mathcal{L}) \to \operatorname{Spec}\mathcal{A}$ be a log smooth deformation of $(f_0,L)/\kappa$ over $\mathcal{A}$. We choose a Čech-1-cocycle in $\mathcal{O}_{\mathcal{X}}^{\times}$ corresponding to the class $[\mathcal{L}] \in \operatorname{Pic}(\mathcal{X})$ and denote it by (F_{ij}).

Let $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension and let $\tilde{f}\colon (\tilde{\mathcal{X}}, \tilde{\mathcal{L}}) \to \operatorname{Spec} \tilde{\mathcal{A}}$ be a lifting of $(f, \mathcal{L})/\mathcal{A}$ over $\tilde{\mathcal{A}}$. Then $\tilde{\mathcal{L}}$ is given by its restrictions $\tilde{\mathcal{L}}_i := \tilde{\mathcal{L}}|_{\tilde{\mathcal{X}}_i} = \mathcal{O}_{\tilde{\mathcal{X}}_i}$ together with transition maps $\tilde{\mathcal{L}}_i|_{\tilde{\mathcal{X}}_{ij}} \to \tilde{\mathcal{L}}_j|_{\tilde{\mathcal{X}}_{ji}}$.

These transition maps are given as follows: Before one passes from $\tilde{\mathcal{L}}_j|_{\tilde{\mathcal{X}}_{ji}}$ to $\tilde{\mathcal{L}}_i|_{\tilde{\mathcal{X}}_{ji}}$ by means of a multiplication with a unit $\tilde{F}_{ij} \in \Gamma(\tilde{\mathcal{X}}_{ij}, \mathcal{O}^{\times}_{\tilde{\mathcal{X}}})$, one has to pass from $\tilde{\mathcal{L}}_i|_{\tilde{\mathcal{X}}_{ji}}$ to $\tilde{\mathcal{L}}_i|_{\tilde{\mathcal{X}}_{ij}}$ by means of $\tilde{\varphi}^*_{ji}$, glueing the $\tilde{\mathcal{X}}_{ij}$ together. Hence, altogether, we have a transition map

$$\tilde{F}_{ij} \cdot \tilde{\varphi}^*_{ji}(\,\cdot\,)\colon\ \tilde{\mathcal{L}}_j|_{\tilde{\mathcal{X}}_{ji}} = \mathcal{O}_{\tilde{\mathcal{X}}_{ij}} \xrightarrow{\tilde{\varphi}^*_{ji}} \tilde{\mathcal{L}}_j|_{\tilde{\mathcal{X}}_{ij}} = \mathcal{O}_{\tilde{\mathcal{X}}_{ij}} \xrightarrow{\tilde{F}_{ij}} \tilde{\mathcal{L}}_i|_{\tilde{\mathcal{X}}_{ij}} = \mathcal{O}_{\tilde{\mathcal{X}}_{ij}}.$$

Restricted to $\mathcal{X}_{ij}$, this transition map must become the corresponding transition map of $\mathcal{L}$, i. e. we must have $F_{ij} = \tilde{F}_{ij}|_{\mathcal{X}_{ij}}$ and they must satisfy the cocycle condition

$$1 = \tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{F}_{jk})\tilde{F}^{-1}_{ik}.$$

Observe that we have a short exact sequence

$$1 \to 1 + \mathcal{O}_{\mathcal{X}} \otimes_A J \to \mathcal{O}^{\times}_{\tilde{\mathcal{X}}} \to \mathcal{O}^{\times}_{\mathcal{X}} \to 1$$

with $1 + \mathcal{O}_{\mathcal{X}} \otimes_A J$ acting freely on $\mathcal{O}^{\times}_{\tilde{\mathcal{X}}}$. The calculations made in what follows are analogue to those in section 3.3.3 in [32].

Group of automorphisms

An automorphism of $(\tilde{f}, \tilde{\mathcal{L}})/\tilde{\mathcal{A}}$ which induces the identity on $(f, \mathcal{L})/\mathcal{A}$ is given by a corresponding automorphism $\tilde{\varphi}$ of $\tilde{f}$ and a compatible automorphism $\tilde{\psi}$ of $\mathcal{L}$. The former is given on each $\tilde{\mathcal{X}}_i$ by $\tilde{\varphi}^*_i = 1 + \vartheta_i$ with $\vartheta_i = \vartheta_j$ on $\mathcal{X}_{ij}$.

If the automorphism $\tilde{\varphi}$ happens to be the identity on $\tilde{\mathcal{X}}$, then the latter automorphism ψ of $\tilde{\mathcal{L}}$, which induces the identity on $\mathcal{L}$, is given on $\tilde{\mathcal{X}}_i$ by multiplication with a unit of the form $\tilde{F}_i = 1 + g_i$, where $g_i \in \Gamma(\mathcal{X}_i, \mathcal{O}_{\mathcal{X}} \otimes_A J)$, such that it is compatible with the transition maps, i. e. $\tilde{F}_{ij}\tilde{F}_j = \tilde{F}_i\tilde{F}_{ij}$. In the general case the non-trivial automorphism $\tilde{\varphi}$ has to be joined in, so that $\tilde{F}_{ij}\tilde{F}_j = \tilde{F}_i\tilde{\varphi}^*_i(\tilde{F}_{ij})$. We calculate

$$\begin{aligned} 0 &= \tilde{F}_{ij}\tilde{F}_j - \tilde{F}_i\tilde{\varphi}^*_i(\tilde{F}_{ij}) = \tilde{F}_{ij} + F_{ij}g_j + F_{ij}g_{ij} - \tilde{F}_{ij} - \vartheta_i(F_{ij}) - F_{ij}g_i - F_{ij}g_{ij} \\ &= F_{ij}g_j - F_{ij}g_i - \vartheta_i(F_{ij}), \end{aligned}$$

thus $0 = g_j - g_i - \frac{\vartheta_i(F_{ij})}{F_{ij}} = g_j - g_i - \vartheta_i \log F_{ij}$ on $\mathcal{X}_{ij}$.

This calculation shows that the cochain (g_i, ϑ_i) is a cocycle in $\check{C}^0(\mathcal{U}, A_f(\mathcal{L}) \otimes_A J)$ and thus defines a class in $H^0(A_f(\mathcal{L}))$ which in turn uniquely determines the cocycle due to the lack of 0-coboundaries.

Hence, the group of automorphisms of $(\tilde{\mathcal{X}}, \tilde{\mathcal{L}})$ over $(\mathcal{X}, \mathcal{L})$ is canonically isomorphic to $H^0(A_f(\mathcal{L}))$.

Pseudo-torsor of liftings

Let $(\tilde{f}_0, \tilde{\mathcal{L}}_0)/\tilde{\mathcal{A}}$ be a lifting of $(f, \mathcal{L})/\mathcal{A}$ and let $(\tilde{f}, \tilde{\mathcal{L}})$ be another lifting. As the transition maps $\tilde{F}_{0,ij}$ and $\tilde{F}_{ij}$ of both deformations restrict to F_{ij} on $\mathcal{X}$, we have an equality $\tilde{F}_{ij} = \tilde{F}_{0,ij}(1 + g_{ij})$ with $g_{ij} \in \Gamma(\mathcal{X}_{ij}, \mathcal{O}_{\mathcal{X}} \otimes_A J)$. Calculating

$$\begin{aligned}1 &= \tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{F}_{jk})\tilde{F}^{-1}_{ik}\\ &= \tilde{F}_{0,ij}(1+g_{ij})(1+\vartheta_{ij})(\tilde{F}_{0,jk}(1+g_{jk}))(\tilde{F}_{0,ik})^{-1}(1-g_{ik})\\ &= 1 + (g_{ij} + \vartheta_{ij}(F_{jk}) + g_{jk} - g_{ik})\end{aligned}$$

shows that

$$0 = g_{jk} - g_{ik} + g_{ij} + \vartheta_{ij}\log F_{jk},$$

which means precisely that the cochain (g_{ij}, ϑ_{ij}) is a 1-cocycle in $\check{C}^1(\mathcal{U}, A_f(\mathcal{L}) \otimes_A J)$ and defines a class $[(\tilde{f}, \tilde{\mathcal{L}}) - (\tilde{f}_0, \tilde{\mathcal{L}}_0)] \in H^1(A_f(\mathcal{L}) \otimes_A J)$.

If we alter this cocycle by a coboundary, it still describes the same isomorphism class of deformations. To see this, let $(g'_{ij}, \vartheta'_{ij}) := (g_{ij} + (g'_j - g'_i), \vartheta_{ij} + (\vartheta'_j - \vartheta'_i))$, with $g'_j - g'_i - \vartheta'_i \log F_{ij} = 0$. We know already from the earlier calculations that this does not affect the isomorphism class of $\tilde{f}$. Let $\tilde{\mathcal{L}}'$ be the line bundle defined by the $\tilde{F}'_{ij} = \tilde{F}^0_{ij}(1 + g'_{ij})$ and put $\tilde{F}'_i := (1 + g'_i) \in \Gamma(\tilde{\mathcal{X}}_i, \mathcal{O}^\times_{\tilde{\mathcal{X}}})$. Since we have

$$\begin{aligned}\tilde{F}'_i\tilde{F}'_{ij} &= (1+g'_i)\tilde{F}^0_{ij}(1 + (g_{ij} + g'_j - g'_i))\\ &= \tilde{F}^0_{ij} + F_{ij}g'_i + F_{ij}g_{ij} + F_{ij}g'_j - F_{ij}g'_i\\ &= \tilde{F}^0_{ij} + F_{ij}g_{ij} + F_{ij}g'_j\\ &= \tilde{F}^0_{ij} + F_{ij}g_{ij} + F_{ij}\vartheta_{ij}(1) + F_{ij}g'_j\\ &= \tilde{F}_{ij}\varphi^*_{ji}(\tilde{F}'_j),\end{aligned}$$

the local multiplications with the $\tilde{F}'_i$ form an isomorphism $\tilde{\mathcal{L}}' \to \tilde{\mathcal{L}}$. Hence, the class $[(\tilde{f}, \tilde{\mathcal{L}}) - (\tilde{f}_0, \tilde{\mathcal{L}}_0)]$ defines the isomorphism class of $(\tilde{f}, \tilde{\mathcal{L}})$ uniquely.

We conclude that the group $G := H^1(\mathcal{X}, A_f(\mathcal{L}))$ acts freely on the set of isomorphism classes of liftings along e if this set is non-empty, making this set a G-pseudo-torsor. For the trivial extension $\varepsilon^0 \colon 0 \to (\varepsilon) \to \mathcal{A}[\varepsilon]^0 \to \mathcal{A} \to 0$ the set of isomorphism classes of liftings over $\mathcal{A}[\varepsilon]^0$ is thus given by $H^1(\mathcal{X}, A_f(\mathcal{L}))$. In particular $t_{\mathrm{Def}_{(f_0,L)}} = \mathrm{Def}_{(f_0,L)}(\kappa[\varepsilon]^0) = H^1(X, A_{f_0}(L))$.

Obstruction theory

We are now going to calculate the obstruction to lifting $(f, \mathcal{L})/\mathcal{A}$ to $(\tilde{f}, \tilde{\mathcal{L}})/\tilde{\mathcal{A}}$ along the extension $e \colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$. To this end, we look at an arbitrary collection $(\tilde{\varphi}_{ij}, \tilde{F}_{ij})$, where $\tilde{F}_{ij} \in \Gamma(\tilde{\mathcal{X}}_{ij}, \mathcal{O}^\times_{\tilde{\mathcal{X}}_{ij}})$ is an invertible regular function on the lifting $\tilde{\mathcal{X}}_{ij}$ of $\mathcal{X}_{ij}$, which restricts over $\mathcal{A}$ to F_{ij}; the $\tilde{\varphi}_{ij}$ are as above.

Let the cocycle (ϑ_{ijk}) in $\check{C}^2(T_f \otimes_A J)$ represent the obstruction to lifting f over $\tilde{A}$. Additionally, there exist elements $g_{ijk} \in \Gamma(\mathcal{X}_{ijk}, \mathcal{O}_{\mathcal{X}} \otimes_A J)$ such that

$$\tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{F}_{jk})\tilde{F}^{-1}_{ik} = 1 + g_{ijk}.$$

We calculate the following expression in two ways. Firstly,

$$\begin{aligned}
&\tilde{\varphi}^*_{ji}(\tilde{F}_{jk}\tilde{\varphi}^*_{kj}(\tilde{F}_{kl})\tilde{F}^{-1}_{jl}) \cdot (\tilde{F}_{ik}\tilde{\varphi}^*_{ki}(\tilde{F}_{kl})\tilde{F}^{-1}_{il})^{-1} \cdot (\tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{F}_{jl})\tilde{F}^{-1}_{il}) \cdot (\tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{F}_{jk})\tilde{F}^{-1}_{ik})^{-1}\\
&\quad = \tilde{\varphi}^*_{ji}\tilde{\varphi}^*_{kj}(\tilde{F}_{kl}) \cdot \tilde{\varphi}^*_{ki}(\tilde{F}_{kl})^{-1} = \tilde{\varphi}^*_{ji}\tilde{\varphi}^*_{kj}\tilde{\varphi}^{*-1}_{ki}\tilde{\varphi}^*_{ki}(\tilde{F}_{kl}) \cdot \tilde{\varphi}^*_{ki}(\tilde{F}_{kl})^{-1}\\
&\quad = (1 + t\vartheta_{ijk})\tilde{\varphi}^*_{ki}(\tilde{F}_{kl}) \cdot \tilde{\varphi}^*_{ki}(\tilde{F}_{kl})^{-1}\\
&\quad = 1 + \frac{\vartheta_{ijk}(\tilde{\varphi}^*_{ki}(\tilde{F}_{kl}))}{\tilde{\varphi}^*_{ki}(\tilde{F}_{kl})} = 1 + \vartheta_{ijk}\log F_{kl}.
\end{aligned}$$

Secondly,

$$\begin{aligned}
&\tilde{\varphi}^*_{ji}(\tilde{F}_{jk}\tilde{\varphi}^*_{kj}(\tilde{F}_{kl})\tilde{F}^{-1}_{jl}) \cdot (\tilde{F}_{ik}\tilde{\varphi}^*_{ki}(\tilde{F}_{kl})\tilde{F}^{-1}_{il})^{-1} \cdot (\tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{F}_{jl})\tilde{F}^{-1}_{il}) \cdot (\tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{F}_{jk})\tilde{F}^{-1}_{ik})^{-1}\\
&\quad = \tilde{\varphi}^*_{ji}(1 + g_{jkl}) \cdot (1 + g_{ikl})^{-1} \cdot (1 + g_{ijl}) \cdot (1 + g_{ijk})^{-1}\\
&\quad = (1 + g_{jkl}) \cdot (1 - g_{ikl}) \cdot (1 + g_{ijl}) \cdot (1 - g_{ijk})\\
&\quad = 1 + (g_{jkl} - g_{ikl} + g_{ijl} - g_{ijk}).
\end{aligned}$$

In conclusion,

$$0 = g_{jkl} - g_{ikl} + g_{ijl} - g_{ijk} - \vartheta_{ijk}\log F_{kl},$$

thus the cochain $(g_{ijk}, \vartheta_{ijk})$ defines a cocycle in $\check{C}^2(\mathcal{U}, A_f(L) \otimes_A J)$; hence, a class $o_e([f, \mathcal{L}]) \in H^2(\mathcal{X}, A_f(\mathcal{L}) \otimes_A J)$.

If $(\tilde{\varphi}'_{ij}, \tilde{F}'_{ij})$ is any other collection, then we have, as before, $\tilde{\varphi}'^*_{ji} - \tilde{\varphi}^*_{ji} = \vartheta'_{ij}$ for a cochain (ϑ'_{ij}) in $\check{C}^1(\mathcal{U}, T_f \otimes_A J)$; hence, $\tilde{\varphi}'^*_{ji}\tilde{\varphi}'^*_{kj}(\tilde{\varphi}'^*_{ki})^{-1} = 1 + \vartheta_{ijk} + \vartheta'_{jk} + \vartheta'_{ij} - \vartheta'_{ik}$. Moreover, $\tilde{F}'_{ij} = \tilde{F}_{ij}(1 + g'_{ij})$ for a cochain $(g'_{ij}) \in \check{C}^1(\mathcal{U}, \mathcal{O}_{\mathcal{X}} \otimes_A J)$; hence, $\tilde{F}'_{ij}\tilde{\varphi}'^*_{ji}(\tilde{F}'_{jk})\tilde{F}'^{-1}_{ik} = 1 + g_{ijk} + g'_{ij} + g'_{jk} - g'_{ik} + \vartheta_{ij}\log(F_{jk})$.

This shows that the class $o_e([f, \mathcal{L}]) \in H^2(\mathcal{X}, A_f(\mathcal{L}) \otimes_A J)$ defined by the cocycle $(g_{ijk}, \vartheta_{ijk})$ is independent of the collection $(\tilde{\varphi}_{ij}, \tilde{F}_{ij})$ chosen, because two such cocycles always differ by a coboundary $\check{d}((g'_{ij}, \vartheta'_{ij}))$. Moreover, $o_e([f, \mathcal{L}])$ vanishes exactly when there is a collection with $g_{ijk} = 0$ and $\vartheta_{ijk} = 0$, and this is true if and only if a lifting $(\tilde{f}, \tilde{\mathcal{L}})$ of $(f, \mathcal{L})$ exists.

Setting $H_e([f, \mathcal{L}]) := H^2(\mathcal{X}, A_f(\mathcal{L}) \otimes_A J)$ and $o\colon V_{\mathrm{Def}_{(f_0,L)}} \to O := \coprod_{V_{\mathrm{Def}_{(f_0,L)}}} H$, $(e, x) \mapsto o_e(x)$, defines a complete linear obstruction theory (H, o) for $\mathrm{Def}_{(f_0,L)}$, where for any small extension $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ we have $H_e([f, \mathcal{L}]) = H^2(\mathcal{X}, A_f(\mathcal{L}) \otimes_A J) \cong H^2(\mathcal{X}, A_{f_0}(L)) \otimes_{\mathbb{C}} J$; in particular, $H_0 := H_{\varepsilon^0}(f_0, L) = H^2(X, A_{f_0}(L))$ is a small obstruction space for the functor $\mathrm{Def}_{(f_0,L)}$.

Conclusion

In the above section we have proven the following:

4.2.9 Proposition
Let $f_0\colon (X,L) \to \operatorname{Spec}\kappa$ be a log smooth log scheme with line bundle.

a) The tangent space of the deformation functor $\operatorname{Def}_{(f_0,L)}$ is $H^1(X, A_{f_0}(L))$.

b) The vector space $H^2(X, A_{f_0}(L))$ is the small obstruction space of a complete linear obstruction theory for the functor $\operatorname{Def}_{(f_0,L)}$.

Let $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension of log Artin rings and $f\colon (X,\mathcal{L}) \to \operatorname{Spec}\mathcal{A}$ a lifting of $(f_0,L)/\kappa$ over $\mathcal{A}$.

c) The group of automorphisms of a lifting $\tilde{f}\colon (\tilde{\mathcal{X}},\tilde{\mathcal{L}}) \to \operatorname{Spec}\tilde{\mathcal{A}}$ inducing the identity on $(f,\mathcal{L})/\mathcal{A}$ is $H^0(\mathcal{X}, A_f(\mathcal{L}) \otimes_A J)$.

d) The set of isomorphism classes of liftings of $(f,L)/\mathcal{A}$ over $\tilde{\mathcal{A}}$ is a pseudo-torsor under $H^1(\mathcal{X}, A_f(\mathcal{L}) \otimes_A J)$.

e) The complete obstruction $o_e([f,L])$ to lifting $(f,L)/\mathcal{A}$ over $\tilde{\mathcal{A}}$ is an element of the obstruction space $H^2(\mathcal{X}, A_f(\mathcal{L}) \otimes_A J)$.

Given a deformation $f\colon \mathcal{X} \to \operatorname{Spec}\mathcal{T}$ of f_0, we may conclude from the calculations above, by putting all $\vartheta_I = 0$:

4.2.10 Proposition

a) The tangent space of the functor $\operatorname{Def}_{L|f}$ is $H^1(X,\mathcal{O}_X)$.

b) The vector space $H^2(X,\mathcal{O}_X)$ is a the small obstruction space of a complete linear obstruction theory for $\operatorname{Def}_{L|f}$.

Let $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension of log rings and let $\mathcal{L}/\mathcal{A}$ be a lifting of L over $\mathcal{A}$.

c) The group of automorphisms of a lifting $\tilde{\mathcal{L}}/\tilde{\mathcal{A}}$ inducing the identity on $\mathcal{L}/\mathcal{A}$ is $H^0(\mathcal{X}_\mathcal{A}, \mathcal{O}_{\mathcal{X}_\mathcal{A}} \otimes_T J)$.

d) The set of isomorphism classes of liftings $\tilde{\mathcal{L}}/\tilde{\mathcal{A}}$ is a pseudo-torsor under the additive group $H^1(\mathcal{X}_\mathcal{A}, \mathcal{O}_{\mathcal{X}_\mathcal{A}} \otimes_T J)$.

e) The complete obstruction $o_e([\mathcal{L}])$ to lifting is an element of the obstruction space $H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}_\mathcal{A}} \otimes_T J)$.

Moreover, we may link the maps in the long exact cohomology sequence associated to the short exact sequence of $\mathcal{O}_X$-modules

$$d\log(\mathcal{L})\colon 0 \to \mathcal{O}_X \to A_f(\mathcal{L}) \to T_f \to 0$$

to the two morphisms of functors $\operatorname{Def}_{(f_0,L)} \to \operatorname{Def}_{f_0}$ and $\operatorname{Def}_{L|\tilde{f}} \to \operatorname{Def}_{(f_0,L)}$ (for a lifting $\tilde{f}\colon \tilde{\mathcal{X}} \to \operatorname{Spec}\tilde{\mathcal{A}}$ of f) by the following corollaries:

4.2.11 Corollary
Under the A-linear map $H^p(\mathcal{X}, A_f(\mathcal{L}) \otimes_A J) \to H^p(\mathcal{X}, T_f \otimes_A J)$

a) an automorphism of a lifting $(\tilde{f}, \tilde{\mathcal{L}})$ over $(f, \mathcal{L})$ is mapped to its underlying automorphism of $\tilde{f}$ over f, for $p = 0$.

b) the class $[(\tilde{f}, \tilde{\mathcal{L}}) - (\tilde{f}_0, \tilde{\mathcal{L}}_0)]$ of a lifting $(\tilde{f}, \tilde{\mathcal{L}})$ of $(f, \mathcal{L})$ is mapped to the class of its underlying lifting $\tilde{f}$ of f, for $p = 1$, whenever a lifting $(\tilde{f}_0, \tilde{\mathcal{L}}_0)$ of $(f, \mathcal{L})$ is given.

c) the obstruction $o_e([f, \mathcal{L}])$ of $\mathrm{Def}_{(f_0, L)}$ is mapped to the obstruction $o_e([f])$ of Def_{f_0}, for $p = 2$.

4.2.12 Corollary
Given a lifting $(\tilde{f}_0, \tilde{\mathcal{L}}_0)$ of $(f, \mathcal{L})$, under the A-linear map

$$H^p(\mathcal{X}, \mathcal{O}_\mathcal{X} \otimes_A J) \to H^p(\mathcal{X}, A_f(\mathcal{L}) \otimes_A J)$$

a) any automorphism ψ of a lifting $\tilde{\mathcal{L}}$ over $\mathcal{L}$ is mapped to the automorphism (id, ψ) of $(\tilde{f}_0, \tilde{\mathcal{L}})$ over $(f, \mathcal{L})$, for $p = 0$.

b) the class $[\tilde{\mathcal{L}} - \tilde{\mathcal{L}}_0]$ of a lifting $\tilde{\mathcal{L}}$ of $\mathcal{L}$ is mapped to the class of the lifting $(\tilde{f}_0, \tilde{\mathcal{L}})$ of $(f, \mathcal{L})$, for $p = 1$.

c) the obstruction $o_e([\mathcal{L}])$ of $\mathrm{Def}_{L|\tilde{f}_0}$ is mapped to the obstruction $o_e([f, \mathcal{L}])$ of $\mathrm{Def}_{(f_0, L)}$, for $p = 2$.

Regarding the A-linear map $\cdot \log(\mathcal{L})\colon H^1(\mathcal{X}, T_f \otimes_A J) \to H^2(\mathcal{X}, \mathcal{O}_\mathcal{X} \otimes_A J)$ induced by $\cdot \log(\mathcal{L})$,

d) a lifting $\tilde{\mathcal{L}}$ of $\mathcal{L}$ exists on a lifting $\tilde{f}$ of f if and only if the class $[\tilde{f} - \tilde{f}_0]$ lies in the kernel of $\cdot \log(\mathcal{L})$.

Equivalently, under the pairing $H^1(\mathcal{X}, T_f \otimes_A J) \times H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}^\times) \to H^2(\mathcal{X}, \mathcal{O}_\mathcal{X} \otimes_A J)$ induced by $T_f \times \mathcal{O}_\mathcal{X}^\times \to \mathcal{O}_\mathcal{X}$, $(\vartheta, u) \mapsto \vartheta \log(u)$, the classes $[\tilde{f} - \tilde{f}_0]$ and $[\mathcal{L}]$ pair to zero.

Proof: For d), let $[\tilde{f} - \tilde{f}_0]$ be given by the 1-cocycle $(\tilde{\vartheta}_{ij})$ in $T_f \otimes_A J$. The element $[\tilde{f} - \tilde{f}_0] \log([\mathcal{L}])$ is the class of the 2-cocycle $(\vartheta_{ij} \log F_{jk}) = (i_{\vartheta_{ij}}(d\log F_{jk}))$ in $\mathcal{O}_\mathcal{X} \otimes_A J$, thus we have the following chain of equivalences:

$$\begin{aligned}
&[\tilde{f} - \tilde{f}_0] \in \mathrm{Ker}(\cdot \log(\mathcal{L})) \\
&\qquad \Leftrightarrow (i_{\vartheta_{ij}}(d\log F_{jk})) = (g_{jk} - g_{ik} + g_{ij}) \text{ for a cochain } (g_{ij}) \in \check{C}^1(\mathcal{U}, \mathcal{O}_\mathcal{X}) \\
&\qquad \Leftrightarrow (g_{ij}, \vartheta_{ij}) \text{ is a 1-cocycle in } A_f(\mathcal{L}) \\
&\qquad \Leftrightarrow (g_{ij}, \vartheta_{ij}) \text{ defines the class } [(\tilde{f}, \tilde{\mathcal{L}}) - (\tilde{f}_0, \tilde{\mathcal{L}}_0)] \text{ of a lifting } (\tilde{f}, \tilde{\mathcal{L}}).
\end{aligned}$$

□

4.2.4 Log smooth deformations of log schemes with flat log connection

Let $f_0\colon (X,\nabla) \to \operatorname{Spec}\kappa$ be a log smooth log scheme with flat log connection $\nabla = (\nabla, L)$. Then the log Chern classes $d\log(L)$ of $[L]$ and $d\log(\mathcal{L})$ of $\mathcal{L}$ are trivial, which means that the log Atiyah sequence $d\log(L)\colon 0 \to \mathcal{O}_X \to A_{f_0}(L) \to T_{f_0} \to 0$ splits. In particular $A_{f_0}(L) \cong T_{f_0} \oplus \mathcal{O}_X$. Accordingly, for a lifting $f\colon (\mathcal{X},\Delta) \to \operatorname{Spec}\mathcal{A}$ of f_0 over $\mathcal{A}$ with $\Delta = (\Delta,\mathcal{L})$ we have $A_f(\mathcal{L}) \cong T_f \oplus \mathcal{O}_X$.

4.2.13 Remark

Due to these splittings, we may replace $H^p(A_{f_0}(L))$ with $H^p(X,T_{f_0}) \oplus H^p(X,\mathcal{O}_X)$ (respectively, $H^p(A_f(\mathcal{L}) \otimes_A J)$ with $H^p(\mathcal{X},T_f \otimes_A J) \oplus H^p(\mathcal{X},\mathcal{O}_\mathcal{X} \otimes_A J)$) in proposition 4.2.9 and in the corollaries 4.2.11 and 4.2.12.

4.2.14 Corollary

Let $f_0\colon (X,\nabla) \to \operatorname{Spec}\kappa$ be a log smooth log scheme with flat log connection. Let $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension of log Artin rings and let $f\colon (\mathcal{X},\Delta) \to \operatorname{Spec}\mathcal{A}$ be a lifting of (f_0,∇) over $\mathcal{A}$.

a) For any lifting $\tilde{f}$ of f over $\tilde{\mathcal{A}}$ a lifting $\tilde{\mathcal{L}}$ of $\mathcal{L}$ over $\tilde{\mathcal{A}}$ exists on $\tilde{f}$, i. e. the obstruction $o_e([\mathcal{L}])$ of $\operatorname{Def}_{L|\tilde{f}}$ vanishes.

b) The obstruction $o_e([f,\mathcal{L}])$ of $\operatorname{Def}_{(f_0,L)}$ vanishes if and only if the obstruction $o_e([f])$ of Def_{f_0} does.

Proof: If a lifting $\tilde{f}_0$ of f over $\tilde{\mathcal{A}}$ exists and if $\tilde{f}$ is any such lifting, then by 4.2.12, the obstruction $o_e([\mathcal{L}])$ of $\operatorname{Def}_{L|\tilde{f}}$ vanishes if and only if $[\tilde{f} - \tilde{f}_0]$ lies in the kernel of the A-linear map $\cdot \log(\mathcal{L})\colon H^1(\mathcal{X},T_f \otimes_A J) \to H^2(\mathcal{X},\mathcal{O}_\mathcal{X} \otimes_A J)$ which, due to the splitting of the log Atiyah sequence, is the zero map. This shows a). By 4.2.11, the vanishing of $o_e([f,\mathcal{L}])$ implies that of $o_e([f])$. If on the other hand $o_e([f]) = 0$, then a lifting $\tilde{f}$ of f exists. By a), there exists also a lifting $\mathcal{L}$ of L on $\tilde{\mathcal{X}}$, thus $o_e([f,\mathcal{L}]) = 0$. $\square$

Let $f_0\colon (X,\nabla) \to \operatorname{Spec}\kappa$ be as above and recall from section 3.2.1 the definition of the log Atiyah complex $A^\bullet_{f_0}(\nabla)$ and the log Atiyah extension $d\log^\bullet(\nabla)$ associated to ∇. Let $f\colon (\mathcal{X},\Delta) \to \operatorname{Spec}\mathcal{A}$ be a lifting of $(f_0,\nabla)/\kappa$ over $\mathcal{A}$, choose a discrepancy cocycle in $\Omega^{\times,\bullet}_f$ corresponding to the class $[\Delta] \in \operatorname{LConn}(f)$ and denote it by (d_i, F_{ij}).

Let $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension and let $\tilde{f}\colon (\tilde{\mathcal{X}},\tilde{\Delta}) \to \operatorname{Spec}\tilde{\mathcal{A}}$ be a lifting of $(f,\Delta)/\mathcal{A}$ over $\tilde{\mathcal{A}}$. Then $\tilde{\Delta}$ is given by transition functions $\tilde{F}_{ij}$ as above and discrepancy forms $\tilde{d}_i$ which restrict on $\mathcal{X}$ to F_{ij} and d_i, respectively, and such that

a) $1 = \tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{F}_{jk})\tilde{F}^{-1}_{ik}$,

b) $0 = \tilde{d}_j - \tilde{d}_i - d\log\tilde{F}_{ij}$ and

c) $0 = d\tilde{d}_i$.

Observe that we have a short exact sequence

$$0 \to \Omega_f^\bullet \otimes_A J \to \Omega_{\tilde f}^{\times,\bullet} \to \Omega_f^{\times,\bullet} \to 0,$$

where the left morphism injects $\mathcal{O}_X \otimes_A J$ as $1 + \mathcal{O}_X \otimes_A J$ into $\mathcal{O}_{\tilde{\mathcal{X}}}^\times$ in degree 0.

Group of automorphisms

An automorphism of $(\tilde f, \tilde\Delta)/\tilde{\mathcal{A}}$, where $\tilde\Delta = (\tilde{\mathcal{L}}, \tilde\Delta)$ which induces the identity on $(f, \Delta)/\mathcal{A}$ is given by a corresponding automorphism $\tilde\varphi$ of $(\tilde f, \tilde{\mathcal{L}})$ which is compatible with $\tilde\Delta$. It is given by a global section $\sigma \in H^0(A_f(\mathcal{L}) \otimes_A J)$, represented by a cocycle $\sigma = (g_i, \vartheta_i)$.
Then $\tilde\varphi_i^* = 1 + \vartheta_i$, with $\vartheta_i = \vartheta_j$ on $\tilde{\mathcal{X}}_{ij}$, defines an automorphism $\tilde\varphi_i$ of $f|_{\mathcal{X}_i}$ and the multiplication with $\tilde F_i = 1 + g_i$ an automorphism of $\tilde{\mathcal{L}}_i$ with $0 = g_j - g_i - \vartheta_i \log F_{ij}$. In addition, to be compatible with $\tilde\Delta$, the data must satisfy the condition $\tilde F_i \tilde\varphi_i^*(\tilde\Delta_i(s)) = \tilde\Delta_i(\tilde F_i \tilde\varphi_i^*(s))$ for any local section s of $\tilde{\mathcal{L}}_i$. We calculate

$$\begin{aligned}
&\tilde F_i \tilde\varphi_i^*(\tilde\Delta_i(s)) - \tilde\Delta_i(\tilde F_i \tilde\varphi_i^*(s)) \\
&\quad = (1+g_i)(1+\vartheta_i)((d+\tilde d_i)(s)) - (d+\tilde d_i)((1+g_i)(1+\vartheta_i)(s)) \\
&\quad = ds + [\vartheta_i(d(s)) + \vartheta_i(d_i \wedge s) + g_i ds + \tilde d_i \wedge s] \\
&\qquad - ds - [\tilde d_i \wedge s + d\vartheta_i(s) + d_i \wedge \vartheta_i(s) + d(g_i s)] \\
&\quad = [\vartheta_i(d_i \wedge s) - d_i \wedge \vartheta_i(s) - dg_i \wedge s] \\
&\quad = [\vartheta_i(d_i) \wedge s - dg_i \wedge s] \\
&\quad = -d(g_i - i_{\vartheta_i}(d_i)) \wedge s = -d_A(g_i, \vartheta_i) \wedge s,
\end{aligned}$$

so the additional condition implies that the cochain (g_i, ϑ_i) is a 0-cocycle in the double complex $\check{C}^\bullet(\mathcal{U}, A_f^\bullet(\Delta) \otimes_A J)$. Due to the lack of coboundaries in degree 0, this cocycle uniquely determines a class in $\mathbb{H}^0(A_f^\bullet(\Delta) \otimes_A J)$.
Hence, the group of automorphisms of $(\tilde{\mathcal{X}}, \tilde\Delta)$ over $(\mathcal{X}, \Delta)$ is canonically isomorphic to $\mathbb{H}^0(A_f^\bullet(\Delta) \otimes_A J)$.

Pseudo-torsor of liftings

Let $(\tilde f_0, \tilde\Delta_0)/\tilde{\mathcal{A}}$ be a lifting of $(f, \Delta)/\mathcal{A}$ and let $(\tilde f, \tilde\Delta)$ be another lifting, with $\tilde\Delta_0 = (\tilde{\mathcal{L}}_0, \tilde\Delta_0)$ and $\tilde\Delta = (\tilde{\mathcal{L}}, \tilde\Delta)$. As the discrepancy cycles $(\tilde F_{0,ij}, \tilde d_{0,i})$ and $(\tilde F_{ij}, \tilde d_i)$ of both deformations restrict to (F_{ij}, d_i) on $\mathcal{X}$, we have equalities $\tilde F_{ij} = \tilde F_{0,ij}(1+g_{ij})$ and $\tilde d_i - \tilde d_{0,i} = \nu_i$ with $g_{ij} \in \Gamma(\mathcal{X}_{ij}, \mathcal{O}_{\mathcal{X}} \otimes_A J)$ and $\nu_i \in \Gamma(\mathcal{X}_i, \Omega_f^1 \otimes_A J)$.
Since $\tilde\Delta$ is flat by assumption, we have, for any local section s of $\tilde{\mathcal{L}}_i$,

$$\begin{aligned}
0 &= \tilde\Delta_i(\tilde\Delta_i(s)) = (\tilde\Delta_{0,i} + \nu_i)((\tilde\Delta_{0,i} + \nu_i)(s)) \\
&= \tilde\Delta_{0,i}(\tilde\Delta_{0,i}(s)) + \nu_i \wedge \tilde\Delta_{0,i}(s) + \tilde\Delta_{0,i}(\nu_i \wedge s) \\
&= 0 + \Big[\nu_i \wedge \tilde\Delta_{0,i}(s) + d\nu_i \wedge s - \nu_i \wedge \tilde\Delta_{0,i}(s)\Big] = d\nu_i \wedge s,
\end{aligned}$$

which implies $0 = d\nu_i$.

On $\mathcal{X}_{ij}$ we must also have $\tilde{\Delta}_i(s) = \tilde{\varphi}_{ij}^*(\tilde{F}_{ji}\tilde{\Delta}_j(\tilde{\varphi}_{ji}^*(\tilde{F}_{ij}s)))$ for any section s of $\tilde{\mathcal{L}}_{ij}$, so

$$\begin{aligned}
ds + \tilde{d}_{0,i} \wedge s + \nu_i \wedge s = \tilde{\Delta}_{0,i}s + \nu_i \wedge s &= \tilde{\Delta}_i(s)\\
&= \tilde{F}_{ij}\tilde{\varphi}_{ji}^*(\tilde{\Delta}_j(\tilde{F}_{ji}\tilde{\varphi}_{ij}^*(s)))\\
&= \tilde{F}_{ij}\tilde{\varphi}_{ji}^*(\tilde{F}_{ji})\tilde{\varphi}_{ji}^*(\tilde{\Delta}_j(\tilde{\varphi}_{ij}^*(s))) + \tilde{F}_{ij}d(\tilde{\varphi}_{ji}^*\tilde{F}_{ji}) \wedge s\\
&= \tilde{\varphi}_{ji}^*(\tilde{\Delta}_j(\tilde{\varphi}_{ij}^*(s))) - d\log(\tilde{F}_{ij}) \wedge s\\
&= \tilde{\varphi}_{ji}^*(\tilde{\Delta}_{0,j}(\tilde{\varphi}_{ij}^*(s)))ds + \nu_j \wedge s - d\log(\tilde{F}_{0,ij}) \wedge s - dg_{ij} \wedge s\\
&= ds + \tilde{d}_{0,j} \wedge s + \vartheta_{ij}(\tilde{d}_{0,j}) \wedge s + \nu_j \wedge s - d\log(\tilde{F}_{0,ij}) \wedge s - dg_{ij} \wedge s\\
&= ds + \tilde{d}_{0,i} \wedge s - d(g_{ij} - i_{\vartheta_{ij}}(d_j))s + \nu_j s,
\end{aligned}$$

and we conclude $0 = (\nu_j - \nu_i) - d_A(g_{ij}, \vartheta_{ij})$.

By assumption, $(\tilde{f}_0, \tilde{\mathcal{L}}_0)/\tilde{\mathcal{A}}$ and $(\tilde{f}, \tilde{\mathcal{L}})/\tilde{\mathcal{A}}$ are log smooth liftings of $(f, \mathcal{L})/\mathcal{A}$, so (g_{ij}, ϑ_{ij}) is a 1-cocycle in $\check{C}^1(\mathcal{U}, A_f(\mathcal{L}))$.

Therefore, the chain $(\nu_i, (g_{ij}, \vartheta_{ij}))$ is a cocycle in $\check{C}^\bullet(\mathcal{U}, A_f^\bullet(\Delta) \otimes_A J)$ and defines a class $[(\tilde{f}, \tilde{\Delta}) - (\tilde{f}_0, \tilde{\Delta}_0)] \in \mathbb{H}^1(\mathcal{X}, A_f^\bullet(\Delta) \otimes_A J)$.

If we alter this cocycle by a coboundary, it still describes the same isomorphism class of deformations: To see this, let $(\nu_i', (g_{ij}', \vartheta_{ij}')) := (\nu_i + dg_i', (g_{ij} + (g_i' - g_j'), \vartheta_{ij} + (\vartheta_i' - \vartheta_j')))$ with (g_i', ϑ_i') a 0-cochain in $\check{C}^\bullet(A_f^\bullet(L, \Delta))$, i. e. with $0 = g_j' - g_i' - \vartheta_i \log f_{ij}$. We already know that this does not affect the isomorphism class of the log smooth log scheme with line bundle $(\tilde{f}, \tilde{L})/\tilde{\mathcal{A}}$. Let $\tilde{\Delta}'$ be the flat log connection defined locally as $\tilde{\Delta}_i' = \tilde{\Delta}_{0,i} + \nu_i' = \tilde{\Delta}_i + dg_i$ and put $\tilde{F}_i' := 1 + g_i' \in \Gamma(\tilde{\mathcal{X}}, \mathcal{O}_{\tilde{\mathcal{X}}}^\times)$. Then

$$\begin{aligned}
\tilde{\Delta}_i(\tilde{F}_i'(s)) &= \tilde{\Delta}_i'(\tilde{F}_i'(s)) - dg_i \wedge \tilde{F}_i'(s)\\
&= \tilde{\Delta}_i'(s) + \tilde{\Delta}_i'(g_i s) - dg_i \wedge s\\
&= \tilde{\Delta}_i'(s) + g_i\tilde{\Delta}_i'(s) = \tilde{F}_i'(\tilde{\Delta}_i'(s)),
\end{aligned}$$

showing that the $\tilde{F}_i'$ define an isomorphism $\tilde{\Delta}' \to \tilde{\Delta}$. Hence, the class $[(\tilde{f}, \tilde{\Delta}) - (\tilde{f}_0, \tilde{\Delta}_0)]$ defines the isomorphism class of $(\tilde{f}, \tilde{\Delta})$ uniquely.

We conclude that the group $G := \mathbb{H}^1(\mathcal{X}, A_f^\bullet(\Delta) \otimes_A J)$ acts freely on the set of isomorphism classes of liftings along e if this set is non-empty, making this set a G-pseudo-torsor. For the trivial extension $\varepsilon^0\colon 0 \to (\varepsilon) \to \mathcal{A}[\varepsilon]^0 \to \mathcal{A} \to 0$ the set of isomorphism classes of liftings over $\mathcal{A}[\varepsilon]^0$ is thus given by $\mathbb{H}^1(A_f^\bullet(\Delta))$. In particular, $t_{\mathrm{Def}_{(f_0, \nabla)}} = \mathrm{Def}_{(f_0, \nabla)}(\kappa[\varepsilon]^0) = \mathbb{H}^1(A_{f_0}^\bullet(\nabla))$.

Obstruction theory

We are now going to calculate the obstruction to lifting $(f, \Delta)/\mathcal{A}$ to $(\tilde{f}, \tilde{\Delta})/\tilde{\mathcal{A}}$ along the extension $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$. To this end, we look at an arbitrary collection $(\tilde{\varphi}_{ij}, \tilde{F}_{ij}, \tilde{d}_i)$, where $\tilde{d}_i \in \Gamma(\tilde{\mathcal{X}}_i, \Omega^1_{\tilde{f}|i})$ are forms which restrict over $\mathcal{A}$ to d_i; all other notations are as above.

We have

a) $\tilde{\varphi}^*_{ji}\tilde{\varphi}^*_{kj}\tilde{\varphi}^{*-1}_{ki} = 1 + \vartheta_{ijk}$,

b) $\tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{F}_{jk})\tilde{F}^{-1}_{ik} = 1 + g_{ijk}$,

c) $\tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{\Delta}_j(\tilde{F}_{ji}\tilde{\varphi}^*_{ij}\,\cdot\,)) - \tilde{\Delta}_i = \nu_{ij}$ and

d) $\tilde{\Delta}_i(\tilde{\Delta}_i(\,\cdot\,)) = \kappa_i$,

where the ϑ_{ijk} and g_{ijk} are as before, $\nu_{ij} \in \Gamma(\mathcal{X}_{ij}, \Omega^1_f \otimes_A J)$ and $\kappa_i \in \Gamma(\mathcal{X}_i, \Omega^2_f \otimes_A J)$. In particular, the conditions

$$\vartheta_{jkl} - \vartheta_{ikl} + \vartheta_{ijl} - \vartheta_{ijk} = 0 \text{ and } g_{jkl} - g_{ikl} + g_{ijl} - g_{ijk} = \vartheta_{ijk}\log F_{kl}$$

are satisfied.

Reformulating the last two conditions c) and d), we get

c') $\tilde{\varphi}^*_{ji}(\tilde{d}_j) - d\log\tilde{F}_{ij} - \tilde{d}_i = \nu_{ij}$ and

d') $d\tilde{d}_i = \kappa_i$.

Applying d to both sides gives $0 = d\kappa_i$ and $0 = \kappa_j - \kappa_i - d\nu_{ij}$ for all i, j.

Abbreviating $\tilde{\varphi}^* := \tilde{\varphi}^*_{ji}\tilde{\varphi}^*_{kj}\tilde{\varphi}^{*-1}_{ki}$, we calculate on the one hand

$$\tilde{\varphi}^*(\tilde{d}_i) = (1 + \vartheta_{ijk})\tilde{d}_i = \tilde{d}_i + \vartheta_{ijk}(d_i)$$

and on the other hand

$$\begin{aligned}
\tilde{\varphi}^*(\tilde{d}_i) &= \tilde{\varphi}^*(\tilde{\varphi}^*_{ki}(\tilde{d}_k) - d\log\tilde{F}_{ik} - \nu_{ik}) \\
&= \tilde{\varphi}^*_{ji}\tilde{\varphi}^*_{kj}(\tilde{d}_k) - \tilde{\varphi}^*(d\log\tilde{F}_{ik}) - \nu_{ik} \\
&= \tilde{\varphi}^*_{ji}(\tilde{d}_j + d\log\tilde{F}_{jk} + \nu_{jk}) - (1 + \vartheta_{ijk})(d\log\tilde{F}_{ik}) - \nu_{ik} \\
&= \tilde{\varphi}^*_{ji}(\tilde{d}_j) + d\log(\tilde{\varphi}^*_{ji}(\tilde{F}_{jk})) - d\log\tilde{F}_{ik} - \vartheta_{ijk}(d\log F_{ik}) + \nu_{jk} - \nu_{ik}) \\
&= d_i + d\log\tilde{F}_{ij} + \nu_{ij} + d\log(\tilde{\varphi}^*_{ji}(\tilde{F}_{jk})) - d\log\tilde{F}_{ik} - \vartheta_{ijk}(d\log F_{ik}) + \nu_{jk} - \nu_{ik} \\
&= d_i + d\log(\tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{F}_{jk})\tilde{F}^{-1}_{ik}) - \vartheta_{ijk}(d_i) + \vartheta_{ijk}(d_k)) + \nu_{jk} - \nu_{ik} + \nu_{ij} \\
&= d_i + \vartheta_{ijk}(d_i) + dg_{ijk} - \vartheta_{ijk}(d_k) + \nu_{jk} - \nu_{ik} + \nu_{ij} \\
&= d_i + \vartheta_{ijk}(d_i) + d(g_{ijk} - i_{\vartheta_{ijk}}(d_k)) + (\nu_{jk} - \nu_{ik} + \nu_{ij}).
\end{aligned}$$

It follows that

$$0 = (\nu_{jk} - \nu_{ik} + \nu_{ij}) + d_A(g_{ijk}, \vartheta_{ijk}).$$

This, together with $d\kappa_i = 0$, $\check{d}(\kappa_i) - d(\nu_{ij}) = 0$ and $\check{d}(g_{ijk}, \vartheta_{ijk}) = (0, 0)$, shows that the collection $(\kappa_i, \nu_{ij}, (g_{ijk}, \vartheta_{ijk}))$ is a 2-cocycle in the double complex $\check{C}^\bullet(\mathcal{U}, A^\bullet_f(\Delta) \otimes_A J)$ and thus defines a class $o_e([f, \Delta]) \in \mathbb{H}^2(A^\bullet_f(\Delta) \otimes_A J)$.

If $(\tilde{\varphi}'_{ij}, \tilde{F}'_{ij}, \tilde{d}'_i)$ is any other collection, then we have, as before, $\tilde{\varphi}'^*_{ji} - \tilde{\varphi}^*_{ji} = \vartheta'_{ij}$ for a cochain $(\vartheta'_{ij}) \in \check{C}^1(\mathcal{U}, T_f \otimes_A J)$, thus $\tilde{\varphi}'^*_{ji}\tilde{\varphi}'^*_{kj}(\tilde{\varphi}'^*_{ki})^{-1} = 1 + \vartheta_{ijk} + \vartheta'_{jk} + \vartheta'_{ij} - \vartheta'_{ik}$. Moreover,

$\tilde{F}'_{ij} = \tilde{F}_{ij}(1+g'_{ij})$ for a cochain $(g'_{ij}) \in \check{C}^1(\mathcal{U}, \mathcal{O}_{\mathcal{X}} \otimes_A J)$, thus $\tilde{F}'_{ij}\tilde{\varphi}'^*_{ji}(\tilde{F}'_{jk})\tilde{F}'^{-1}_{ik} = 1 + g_{ijk} + g'_{ij} + g'_{jk} - g'_{ik} + \vartheta'_{ij}\log F_{jk}$. Finally, we have $\tilde{\Delta}'_i = \tilde{\Delta}_i + \nu'_i$ for a cochain $(\nu'_i) \in \check{C}^0(\mathcal{U}, \Omega_f \otimes_A J)$ so that

$$
\begin{aligned}
&\tilde{F}'_{ij}\tilde{\varphi}'^*_{ji}(\tilde{\Delta}'_j(\tilde{F}'_{ji}\tilde{\varphi}'^*_{ij}\,\cdot\,)) - \tilde{\Delta}'_i \\
&\quad = \tilde{F}_{ij}(1+g_{ij})(\tilde{\varphi}^*_{ji} + \vartheta'_{ij})(\tilde{\Delta}_j + \nu'_j)(\tilde{F}_{ji}(1+g_{ji})(\tilde{\varphi}^*_{ij} + \vartheta'_{ji})(\,\cdot\,)) - \tilde{\Delta}_i - \nu'_i \\
&\quad = \tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{\Delta}_j(\tilde{F}_{ji}\tilde{\varphi}^*_{ij}(\,\cdot\,))) - \tilde{\Delta}_i - \nu'_i + F_{ij}g_{ij}\Delta_j(F_{ji}(\,\cdot\,)) + F_{ij}\vartheta'_{ij}\Delta_j(F_{ji}(\,\cdot\,)) \\
&\qquad + F_{ij}\nu'_j \wedge (F_{ji}\,\cdot\,) + F_{ij}\Delta_j(F_{ji}g_{ji}\,\cdot\,) + F_{ij}\Delta_j(F_{ji}(\vartheta'_{ji})(\,\cdot\,)) \\
&\quad = \nu_{ij} + (\nu'_j - \nu'_i) - d(g_{ij} - i_{\vartheta_{ij}}(d_j))
\end{aligned}
$$

and $\tilde{\Delta}'_i(\tilde{\Delta}'_i(\,\cdot\,)) = \kappa_i + d\nu_i$.

This means that the class $o_e([f,\Delta]) \in H^2(\mathcal{X}, A_f(L) \otimes_A J)$ which is defined by the cocycle $(\kappa_i, \nu_{ij}, (g_{ijk}, \vartheta_{ijk}))$ is independent of the collection $(\tilde{\varphi}_{ij}, \tilde{F}_{ij}, \tilde{d}_i)$ chosen, because two such cocycles always differ by a coboundary $(\check{d} \pm d_A)(\nu'_i, (g'_{ij}, \vartheta'_{ij}))$. Moreover, $o_e([f,\Delta])$ vanishes exactly when there exists a collection with $\kappa_i = 0$, $\nu_{ij} = 0$, $g_{ijk} = 0$ and $\vartheta_{ijk} = 0$, and this is true if and only if a lifting $(\tilde{f}, \tilde{\Delta})$ of (f, Δ) exists.

Setting $H_e([f,\Delta]) := H^2(\mathcal{X}, A^\bullet_f(\Delta) \otimes_A J)$ and $o\colon V_{\mathrm{Def}_{(f_0,\nabla)}} \to O := \coprod_{V_{\mathrm{Def}_{(f_0,\nabla)}}} H$, $(e,x) \mapsto o_e(x)$, defines a complete linear obstruction theory (H, o) for $\mathrm{Def}_{(f_0,\nabla)}$. In particular, $H_0 := H_{\varepsilon^0}(f_0, \nabla) = H^2(X, A^\bullet_{f_0}(\nabla))$ is a small obstruction space for the functor $\mathrm{Def}_{(f_0,\nabla)}$.

Conclusion

In the above section we have proven the following:

4.2.15 Proposition

Let $f_0\colon (X, \nabla) \to \operatorname{Spec}\kappa$ be a log smooth log scheme with flat log connection.

a) The tangent space of the functor $\mathrm{Def}_{(f_0,\nabla)}$ is $\mathbb{H}^1(A^\bullet_{f_0}(\nabla))$.

b) The vector space $\mathbb{H}^2(A^\bullet_{f_0}(\nabla))$ is the small obstruction space of a complete linear obstruction theory for the functor $\mathrm{Def}_{(f_0,\nabla)}$.

Let $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension of log Artin rings and let $f\colon (\mathcal{X}, \Delta) \to \operatorname{Spec}\mathcal{A}$ be a lifting of (f_0, ∇) over $\mathcal{A}$.

c) The group of automorphisms of a lifting $\tilde{f}\colon (\tilde{\mathcal{X}}, \tilde{\Delta}) \to \operatorname{Spec}\tilde{\mathcal{A}}$ inducing the identity on $(f,\Delta)/\mathcal{A}$ is $\mathbb{H}^0(A^\bullet_f(\Delta) \otimes_A J)$.

d) The set of isomorphism classes of liftings of $(f,\Delta)/\mathcal{A}$ over $\tilde{\mathcal{A}}$ is a pseudo-torsor under the additive group $\mathbb{H}^1(A^\bullet_f(\Delta) \otimes_A J)$.

e) The complete obstruction $o_e([f,\Delta])$ to lifting $(f,\Delta)/\mathcal{A}$ over $\tilde{\mathcal{A}}$ is an element of the obstruction space $\mathbb{H}^2(A^\bullet_f(\Delta) \otimes_A J)$.

Here, the direct sums are induced by the natural splitting of the log Atiyah sequence coming from the flat log connection ∇, respectively, Δ.

Given a deformation $f\colon \mathcal{X} \to \operatorname{Spec} \mathcal{T}$ of f_0, we may conclude from the calculations above, by putting all $\vartheta_I = 0$:

4.2.16 Proposition

a) The tangent space of the functor $\operatorname{Def}_{\nabla|f}$ is $\mathbb{H}^1(\Omega^\bullet_{f_0})$.

b) The vector space $\mathbb{H}^2(\Omega^\bullet_{f_0})$ is a the small obstruction space of a complete linear obstruction theory for the functor $\operatorname{Def}_{\nabla|f}$.

Let $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension of log rings and let $\Delta/\mathcal{A}$ be a lifting of ∇ over $\mathcal{A}$.

c) The group of automorphisms of a lifting $\tilde{\Delta}/\tilde{\mathcal{A}}$ inducing the identity on $\Delta/\mathcal{A}$ is $\mathbb{H}^0(\Omega^\bullet_{f_\mathcal{A}} \otimes_T J)$.

d) The set of isomorphism classes of liftings $\tilde{\Delta}/\tilde{\mathcal{A}}$ is a pseudo-torsor under the additive group $\mathbb{H}^1(\Omega^\bullet_{f_\mathcal{A}} \otimes_T J)$.

e) The complete obstruction $o_e([\Delta])$ to lifting is an element of the obstruction space $\mathbb{H}^2(\Omega^\bullet_{f_\mathcal{A}} \otimes_T J)$.

Moreover, we may link the maps in the long exact cohomology sequence associated to the short exact sequence of complexes

$$d\log^\bullet(\Delta)\colon 0 \to \Omega^\bullet_f \to A^\bullet_f(\Delta) \to T_f \to 0$$

to the two morphisms of functors $\operatorname{Def}_{(f_0,\nabla)} \to \operatorname{Def}_{f_0}$ and $\operatorname{Def}_{\nabla|\tilde{f}} \to \operatorname{Def}_{(f_0,\nabla)}$ (for a lifting $\tilde{f}\colon \tilde{\mathcal{X}} \to \operatorname{Spec} \tilde{\mathcal{A}}$ of f) by the following corollaries:

4.2.17 Corollary

Under the A-linear map $\mathbb{H}^p(A^\bullet_f(\Delta) \otimes_A J) \to H^p(\mathcal{X}, T_f \otimes_A J)$

a) an automorphism of a lifting $(\tilde{f}, \tilde{\Delta})$ over (f, Δ) is mapped to its underlying automorphism of $\tilde{f}$ over f, for $p = 0$.

b) the class $[(\tilde{f}, \tilde{\Delta}) - (\tilde{f}_0, \tilde{\Delta}_0)]$ of a lifting $(\tilde{f}, \tilde{\Delta})$ of (f, Δ) is mapped to the class of its underlying lifting $\tilde{f}$ of f, for $p = 1$, whenever a lifting $(\tilde{f}_0, \tilde{\Delta}_0)$ of (f, Δ) is given.

c) the obstruction $o_e([f, \Delta])$ of $\operatorname{Def}_{(f_0,\nabla)}$ is mapped to the obstruction $o_e([f])$ of Def_{f_0}, for $p = 2$.

4.2.18 Corollary

Given a lifting $(\tilde{f}_0, \tilde{\Delta}_0)$ of (f, Δ), under the A-linear map

$$\mathbb{H}^p(\Omega^\bullet_f \otimes_A J) \to \mathbb{H}^p(A^\bullet_f(\Delta) \otimes_A J)$$

a) any automorphism $\psi^\bullet$ of a lifting $\tilde{\varDelta}$ over $\varDelta$ is mapped to the automorphism $(id, \psi^\bullet)$ of $(\tilde{f}_0, \tilde{\varDelta})$ over $(f, \varDelta)$, for $p = 0$.

b) the class $[\tilde{\varDelta} - \tilde{\varDelta}_0]$ of a lifting $\tilde{\varDelta}$ of $\varDelta$ is mapped to the class of the lifting $(\tilde{f}_0, \tilde{\varDelta})$ of $(f, \varDelta)$, for $p = 1$.

c) the obstruction $o_e([\varDelta])$ of $\mathrm{Def}_{\nabla|\tilde{f}}$ is mapped to the obstruction $o_e([f, \varDelta])$ of $\mathrm{Def}_{(f_0, \nabla)}$, for $p = 2$.

Regarding the A-linear map $\cdot \log(\varDelta)\colon H^1(\mathcal{X}, T_f \otimes_A J) \to \mathbb{H}^2(\varOmega_f^\bullet \otimes_A J)$ induced by $\cdot \log(\varDelta)$,

d) a lifting $\tilde{\varDelta}$ of $\varDelta$ always exists on any lifting $\tilde{f}$ of f.

Proof: For d), let $[\tilde{f} - \tilde{f}_0]$ be given by the 1-cocycle $(\tilde{\vartheta}_{ij})$ in $T_f \otimes_A J$. The element $[\tilde{f} - \tilde{f}_0]\log([\varDelta])$ is the class of the 2-cocycle $(\vartheta_{ij} \log F_{jk}, d\, i_{\vartheta_{ij}}(d_j), 0)$ in $\varOmega_f^\bullet \otimes_A J$, thus we have the following chain of equivalences:

$$
\begin{aligned}
&[\tilde{f} - \tilde{f}_0] \in \mathrm{Ker}(\,\cdot \log([\varDelta]))\\
&\qquad \Leftrightarrow (\vartheta_{ij} \log F_{jk}, d\, i_{\vartheta_{ij}}(d_j), 0) = (\check{d}g_{ij}, \check{d}\nu_i - dg_{ij}, d\nu_i)\\
&\qquad\qquad\qquad\qquad \text{for a 1-cochain } (\nu_i, g_{ij}) \text{ in } \varOmega_f^\bullet\\
&\qquad \Leftrightarrow (\nu_i, (g_{ij}, \vartheta_{ij})) \text{ is a 1-cocycle in } A_f^\bullet(\varDelta)\\
&\qquad \Leftrightarrow (\nu_i, (g_{ij}, \vartheta_{ij})) \text{ defines the class } [\tilde{f} - \tilde{f}_0, \tilde{\varDelta} - \tilde{\varDelta}_0] \text{ of a lifting } (\tilde{f}, \tilde{\varDelta}).
\end{aligned}
$$

Hence, a lifting $\tilde{\varDelta}$ of $\varDelta$ exists on a lifting $\tilde{f}$ if and only if the class $[\tilde{f} - \tilde{f}_0]$ lies in the kernel of $\cdot \log(\varDelta)$. Since, however, the short exact sequence $d\log^\bullet(\varDelta)\colon 0 \to \varOmega_f^\bullet \to A_f^\bullet(\varDelta) \to T_f \to 0$ splits, the map $\cdot \log(\varDelta)\colon H^1(\mathcal{X}, T_f \otimes_A J) \to \mathbb{H}^2(\varOmega_f^\bullet \otimes_A J)$ is the zero map, thus any $[\tilde{f} - \tilde{f}_0]$ lies in its kernel. Put differently, any $[\tilde{f} - \tilde{f}_0]$ pairs with $[\varDelta]$ to zero under the pairing $H^1(\mathcal{X}, T_f \otimes_A J) \times \mathbb{H}^1(\varOmega_f^{\times,\bullet}) \to \mathbb{H}^2(\varOmega_f^\bullet \otimes_A J)$ induced by the log Lie derivative $T_f \times \varOmega_f^{\times,\bullet} \to \varOmega_f^\bullet$, $(\vartheta, u^\bullet) \mapsto \vartheta \log(u^\bullet)$.
Explicitly, the cocycle $(\nu_i, (g_{ij}, \vartheta_{ij}))$ in the above chain of equivalences is given by

$$(t_\varDelta(0), (t_\varDelta(\vartheta_{ij}), \vartheta_{ij})) = (0, (i_{\vartheta_{ij}}(d_i), \vartheta_{ij})),$$

where (d_i, f_{ij}) is a discrepancy cocycle for $\varDelta$. □

4.2.19 Remark

Due to the natural splitting of the log Atiyah extension, we may replace $\mathbb{H}^p(A_{f_0}^\bullet(\nabla))$ with $\mathbb{H}^p(\varOmega_{f_0}^\bullet) \oplus H^p(X, T_{f_0})$ (respectively, we may replace $\mathbb{H}^p(A_f^\bullet(\varDelta) \otimes_A J)$ with $\mathbb{H}^p(\varOmega_f^\bullet \otimes_A J)$ $\oplus\, H^p(\mathcal{X}, T_f \otimes_A J))$ in proposition 4.2.15 and in the corollaries 4.2.17 and 4.2.18.

4.2.5 Deformations of log symplectic schemes of general type

Let $f_0\colon (X, \nabla, \omega) \to \operatorname{Spec}\kappa$ be a log symplectic scheme of general type. Recall from section 3.3.4 the definition of the B-complex $B^\bullet_{f_0}(\omega) = B^\bullet_{(f_0,\nabla)}(\omega)$ and the B-extension b^ω associated to ω.

4.2.20 Corollary

In all propositions and corollaries above, we may replace T_{f_0} with $\Omega^1_{f_0}$ and, if there exists a corresponding log symplectic lifting ϖ of ω, also $T_f \otimes_A J$ with $\Omega^1_f \otimes_A J$.

Let $f\colon (\mathcal{X}, \Delta, \varpi) \to \operatorname{Spec}\mathcal{A}$ be a log symplectic deformation of $(f_0, \nabla, \omega)/\kappa$ over $\mathcal{A}$ and denote $\varpi_i := \varpi|_{\mathcal{X}_i} \in \Gamma(\mathcal{X}_i, \Omega^2_f)$. Let $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension and let $\tilde{f}\colon (\tilde{\mathcal{X}}, \tilde{\Delta}, \tilde{\varpi}) \to \operatorname{Spec}\tilde{\mathcal{A}}$ be a lifting of $(f, \Delta, \varpi)/\mathcal{A}$ over $\tilde{\mathcal{A}}$. Then $\tilde{\varpi}$ is given by a collection of 2-forms $\tilde{\varpi}_i \in \Gamma(\tilde{\mathcal{X}}_i, \Omega^2_{\tilde{f}})$ which restrict on $\mathcal{X}_i$ to ϖ_i and such that

a) $\tilde{\varpi}_i = \tilde{F}_{ij}\tilde{\varpi}_j$,

b) $\tilde{\Delta}_i\tilde{\varpi}_i = 0$ and

c) $i\,.\,(\tilde{\varpi}_i)\colon T_{\tilde{f}|i} \to \Omega^1_{\tilde{f}|i}$ is an isomorphism,

with all notations as before.

Observe that we have a short exact sequence

$$0 \to (\Omega^{\geq 2,\bullet}_{f} \otimes_{\mathcal{O}_X} \mathcal{L}) \otimes_A J \to \Omega^{\geq 2,\bullet}_{\tilde{f}} \otimes_{\mathcal{O}_{\tilde{\mathcal{L}}}} \tilde{\mathcal{L}} \to \Omega^{\geq 2,\bullet}_{f} \otimes_{\mathcal{O}_X} \mathcal{L} \to 0.$$

All calculations in this chapter are based on the first description of $B^\bullet_f(\varpi)$ (with $B^0_f(\varpi) = A^0(L)$; cf. section 3.3.4). Of course all calculations go through, when using the alternate description of $B^\bullet_f(\varpi)$ (with $B^0_f(\varpi) = (\Omega^1_{f_0} \otimes_{\mathcal{O}_X} L) \oplus \mathcal{O}_X$).

Group of automorphisms

An automorphism φ of $(\tilde{f}, \tilde{\Delta}, \tilde{\varpi})/\tilde{\mathcal{A}}$ which induces the identity on $(f, \Delta, \varpi)/\mathcal{A}$ is a corresponding automorphism of $(\tilde{f}, \tilde{\Delta})/\tilde{\mathcal{A}}$, given by (g_i, ϑ_i) as above, with the additional property that $\tilde{F}_i\tilde{\varphi}^*_i(\tilde{\varpi}_i) = \tilde{\varpi}_i$. We calculate

$$0 = \tilde{\varpi}_i + g_i\varpi_i + \vartheta_i(\varpi_i) - \tilde{\varpi}_i = (g_i, \vartheta_i)(\varpi_i),$$

i. e. $0 = (-b^{\varpi_i})(g_i, \vartheta_i) = d_B(g_i, \vartheta_i)$. Therefore, (g_i, ϑ_i) is a 0-cocycle in the double complex $\check{C}^\bullet(\mathcal{U}, B^\bullet_f(\varpi))$ and, due to the lack of coboundaries, uniquely describes a class in $\mathbb{H}^0(B^\bullet_f(\varpi))$.

Hence, the group of automorphisms of $(\tilde{\mathcal{X}}, \tilde{\Delta}, \tilde{\varpi})$ over $(\mathcal{X}, \Delta, \varpi)$ is canonically isomorphic to $\mathbb{H}^0(B^\bullet_f(\varpi))$.

Pseudo-torsor of liftings

Let $(\tilde{f}_0, \tilde{\Delta}_0, \tilde{\varpi}_0)/\tilde{\mathcal{A}}$ be a lifting of $(f, \Delta, \varpi)/\mathcal{A}$ and let $(\tilde{f}, \tilde{\Delta}, \tilde{\varpi})$ be another lifting. As $\tilde{\varpi}^0$ and $\tilde{\varpi}$ both restrict to ϖ on $\mathcal{X}$, we have $\tilde{\varpi}_i = \tilde{\varpi}_i^0 + u_i$ with $u_i \in \Gamma(\mathcal{X}_i, \Omega^2_f \otimes_A J)$. The rest of the data differs by ν_i, g_{ij} and ϑ_{ij} as above.

Recall the conditions fulfilled by the ν_i, g_{ij} and ϑ_{ij}, due to the fact that by assumption $(\tilde{f}_0, \tilde{\Delta}_0)/\tilde{\mathcal{A}}$ and $(\tilde{f}, \tilde{\Delta})/\tilde{\mathcal{A}}$ are liftings of $(f, \Delta)/\mathcal{A}$.

As both $\tilde{\varpi}$ and $\tilde{\varpi}_0$ are closed by assumption, we have

$$\begin{aligned}
0 = \tilde{\Delta}_i \tilde{\varpi}_i &= (\tilde{\Delta}_{0,i} + \nu_i)(\varpi_{0,i} + u_i) \\
&= \tilde{\Delta}_{0,i}\tilde{\varpi}_{0,i} + \nu_i \wedge \tilde{\varpi}_{0,i} + \tilde{\Delta}_{0,i} u_i + \nu_i \wedge u_i \\
&= 0 + \nu_i \wedge \varpi_i + \Delta_i u_i + 0,
\end{aligned}$$

which means that $0 = -\Delta_i u_i - b^{\varpi_i}(\nu_i)$. Together with $0 = d\nu_i$ this means that $0 = d_B(u_i, \nu_i)$. Moreover, on $\tilde{\mathcal{X}}_{ij}$ we must have $\tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\varpi_j) - \varpi_i = 0$; hence,

$$\begin{aligned}
0 &= \tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{\varpi}_j) - \tilde{\varpi}_i \\
&= \tilde{F}_{0,ij}(1 + g_{ij})(1 + \vartheta_{ij})(\tilde{\varpi}_{0,j} + u_j) - \tilde{\varpi}_{0,i} - u_i \\
&= \tilde{F}_{0,ij}\tilde{\varpi}_{0,j} - \tilde{\varpi}_{0,i} + \tilde{F}_{0,ij}\vartheta_{ij}(\tilde{\varpi}_{0,j}) + \tilde{F}_{0,ij} g_{ij}\tilde{\varpi}_{0,j} + \tilde{F}_{0,ij} u_j - u_i \\
&= 0 + F_{ij}\vartheta_{ij}(\varpi_j) + F_{ij} g_{ij}\varpi_j + F_{ij} u_j - u_i \\
&= F_{ij}(g_{ij} + \vartheta_{ij})(\varpi_j) + F_{ij} u_j - u_i,
\end{aligned}$$

and we conclude $0 = (F_{ij}u_j - u_i) - F_{ij}(-b^{\varpi_j})(g_{ij}, \vartheta_{ij})$. Together with $0 = (\nu_j - \nu_i) - d_A(g_{ij}, \vartheta_{ij})$ this means that $0 = \check{d}(u_i, \nu_i) - d_B(g_{ij}, \vartheta_{ij})$. Finally, $0 = \check{d}(g_{ij}, \vartheta_{ij})$ is still valid.

Those conditions may be combined to the single one that $0 = (\check{d} \pm d_B)((u_i, \nu_i), (g_{ij}, \vartheta_{ij}))$ which means precisely that the cochain $((u_i, \nu_i), (g_{ij}, \vartheta_{ij}))$ is a 1-cocycle of the double complex $\check{C}^\bullet(\mathcal{U}, B_f^\bullet(\varpi))$ and thus defines a class $[\tilde{f}, \tilde{\Delta}, \tilde{\varpi}] \in \mathbb{H}^1(B_f^\bullet(\varpi))$.

If we alter this cocycle by a coboundary, then, due to the definition of $B_f^\bullet(\varpi)$, only the data ν_i, g_{ij} and ϑ_{ij} which are related to $(\tilde{\mathcal{X}}, \Delta)$ change, but not the u_i. Since we already know that this does not affect the isomorphism class of the log smooth log scheme with flat log connection $(\tilde{f}, \Delta)/\tilde{\mathcal{A}}$, this does not affect the isomorphism class of the log symplectic scheme $(\tilde{f}, \tilde{\Delta}, \tilde{\varpi})/\tilde{\mathcal{A}}$, either. Hence, the class $[(\tilde{f}, \tilde{\Delta}, \tilde{\varpi}) - (\tilde{f}_0, \tilde{\Delta}_0, \tilde{\varpi}_0)]$ defines the isomorphism class of $(\tilde{f}, \tilde{\Delta}, \tilde{\varpi})$ uniquely.

We conclude that the group $G := \mathbb{H}^1(B_f^\bullet(\varpi))$ acts freely on the set of isomorphism classes of liftings along e if this set is non-empty, making this set a G-pseudo-torsor. For the trivial extension $\varepsilon^0\colon 0 \to (\varepsilon) \to \mathcal{A}[\varepsilon]^0 \to \mathcal{A} \to 0$ the set of isomorphism classes of liftings over $\mathcal{A}[\varepsilon]^0$ is thus given by $\mathbb{H}^1(B_f^\bullet(\varpi))$. In particular, $t_{\mathrm{Def}_{(f_0, \nabla, \omega)}} = \mathrm{Def}_{(f_0, \nabla, \omega)}(\kappa[\varepsilon]^0) = \mathbb{H}^1(B_{f_0}^\bullet(\omega))$.

Obstruction theory

We are now going to calculate the obstruction to lifting $(f, \varDelta, \varpi)/\mathcal{A}$ to $(\tilde{f}, \tilde{\varDelta}, \tilde{\varpi})/\tilde{\mathcal{A}}$ along the extension $e\colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$. To this end, we look at an arbitrary collection $(\tilde{\varphi}_{ij}, \tilde{F}_{ij}, \tilde{d}_i, \tilde{\varpi}_i)$, where $\tilde{\varpi}_i \in \varGamma(\tilde{\mathcal{X}}_i, \varOmega^2_{\tilde{f}|i})$ are 2-forms which restrict over $\mathcal{A}$ to ϖ_i; all other notations as above.

We have

a) $\tilde{\varphi}^*_{ji}\tilde{\varphi}^*_{kj}\tilde{\varphi}^{*-1}_{ki} = 1 + \vartheta_{ijk}$,

b) $\tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{F}_{jk})\tilde{F}^{-1}_{ik} = 1 + g_{ijk}$,

c) $\tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{\varDelta}_j(\tilde{F}_{ji}\tilde{\varphi}^*_{ij}\,\cdot\,)) - \tilde{\varDelta}_i = \nu_{ij}$,

d) $\tilde{\varDelta}_i(\tilde{\varDelta}_i(\,\cdot\,)) = \kappa_i$,

e) $\tilde{F}_{ij}\tilde{\varphi}^*_{ij}\tilde{\varpi}_j - \tilde{\varpi}_i = u_{ij}$ and

f) $(-\tilde{\varDelta}_i)(\tilde{\varpi}_i) = \varrho_i$,

where the ϑ_{ijk}, g_{ijk}, ν_{ij} and κ_i have the same properties as before, $u_{ij} \in \varGamma(\mathcal{X}_{ij}, \varOmega^2_f \otimes_A J)$ and $\varrho_i \in \varGamma(\mathcal{X}_i, \varOmega^3_f \otimes_A J)$.

From the last condition we deduce that

$$\kappa_i \wedge \varpi_i = \tilde{\varDelta}_i(\tilde{\varDelta}_i(\tilde{\varpi}_i)) = -\varDelta_i(\varrho_i),$$

so $0 = (-b^{\varpi_i})(\kappa_i) + (-\varDelta_i)(\varrho_i)$. Together with $d\kappa_i = 0$ this means that $0 = d_B(\varrho_i, \kappa_i)$.

From the one but last condition we get

$$\begin{aligned}
&(F_{ij}\varrho_j - \varrho_i)\\
&= \tilde{F}_{ij}\tilde{\varphi}^*_{ji}(-\tilde{\varDelta}_j(\tilde{\varpi}_j)) + \tilde{\varDelta}_i(\tilde{\varpi}_i)\\
&= -\tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{F}_{ji}\tilde{\varphi}^*_{ij}(\tilde{\varDelta}_i(\tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{\varpi}_j)))) + \tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\nu_{ji} \wedge \tilde{\varpi}_j) + \tilde{\varDelta}_i(\tilde{\varpi}_i)\\
&= -\tilde{\varDelta}_i(\tilde{\varpi}_i) - \tilde{\varDelta}_i(u_{ij}) - F_{ij}(\nu_{ij} \wedge \varpi_j) + \tilde{\varDelta}_i(\tilde{\varpi}_i)\\
&= -(\varDelta_i(u_{ij}) + \nu_{ij} \wedge \varpi_i);
\end{aligned}$$

hence, $0 = (F_{ij}\varrho_j - \varrho_i) - (F_{ij}(-b^{\varpi_j})(\nu_{ij}) + (-\varDelta_i)(u_{ij}))$. Together with $0 = (\kappa_j - \kappa_i) - d\nu_{ij}$ this means that $0 = \check{d}(\varrho_i, \kappa_i) - d_B(u_{ij}, \nu_{ij})$.

Now we calculate in two ways:

$$\begin{aligned}
(\tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{\varpi}_j) - \tilde{\varpi}_i) + \tilde{F}_{ij}\tilde{\varphi}^*_{ji}(\tilde{F}_{jk}(\tilde{\varphi}^*_{kj}(\tilde{\varpi}_k) - \tilde{\varpi}_j)) - \tilde{F}_{ik}(\tilde{\varphi}^*_{ki}(\tilde{\varpi}_k) - \tilde{\varpi}_i)&\\
= u_{ij} + \tilde{\varphi}^*_{ij}(\tilde{F}_{ji}(u_{jk})) - u_{ik}&\\
= F_{ji}u_{jk} - u_{ik} + u_{ij}&
\end{aligned}$$

and

$$\begin{aligned}(\tilde F_{ij}\tilde\varphi_{ji}^*(\tilde\varpi_j)-\tilde\varpi_i)+\tilde F_{ij}\tilde\varphi_{ji}^*(\tilde F_{jk}(\tilde\varphi_{kj}^*(\tilde\varpi_k)-\tilde\varpi_j))-\tilde F_{ik}(\tilde\varphi_{ki}^*(\tilde\varpi_k)-\tilde\varpi_i)\\ =\tilde F_{ij}\tilde\varphi_{ji}^*(\tilde F_{jk}\tilde\varphi_{kj}^*(\tilde\varpi_k))-\tilde F_{ik}\tilde\varphi_{ki}^*(\tilde\varpi_k)\\ =\tilde F_{ij}\tilde\varphi_{ji}^*(\tilde F_{jk})\tilde\varphi_{ji}^*\tilde\varphi_{kj}^*(\tilde\varpi_k)-\tilde F_{ik}\tilde\varphi_{ki}^*(\tilde\varpi_k)\\ =\tilde F_{ik}\left[\tilde F_{ij}\tilde\varphi_{ji}^*(\tilde F_{jk})\tilde F_{ik}^{-1}\tilde\varphi_{ji}^*\tilde\varphi_{kj}^*\tilde\varphi_{ik}^*(\tilde\varphi_{ki}^*\tilde\varpi_k)-\tilde\varphi_{ki}^*(\tilde\varpi_k)\right]\\ =\tilde F_{ik}\left[(1+g_{ijk})(1+\vartheta_{ijk})(\tilde\varphi_{ki}^*\tilde\varpi_k)-\tilde\varphi_{ki}^*(\tilde\varpi_k)\right]\\ =\tilde F_{ik}\left[(g_{ijk}+\vartheta_{ijk})(\tilde\varphi_{ki}^*\tilde\varpi_k)\right]\\ =F_{ik}(g_{ijk}+\vartheta_{ijk})(\varpi_k),\end{aligned}$$

which implies $0=(F_{ij}u_{jk}+u_{ik}-u_{ij})+F_{ik}(-b^{\varpi_k})(g_{ijk},\vartheta_{ijk})$. Together with the condition $0=(\nu_{jk}-\nu_{ik}+\nu_{ij})+d_A(g_{ijk},\vartheta_{ijk})$ this means that $0=\check d(u_{ij},\nu_{ij})+d_B(g_{ijk},\vartheta_{ijk})$. Finally, we still have $0=\check d(g_{ijk},\vartheta_{ijk})$.

For the 2-cochain $((\varrho_i,\kappa_i),(u_{ij},\nu_{ij}),(g_{ijk},\vartheta_{ijk}))$ in the double complex $\check C^\bullet(\mathcal U,B_f^\bullet(\varpi)\otimes_A J)$ these conditions are equivalent to the single condition

$$(\check d\pm d_B)((\varrho_i,\kappa_i),(u_{ij},\nu_{ij}),(g_{ijk},\vartheta_{ijk}))=0$$

which makes $((\varrho_i,\kappa_i),(u_{ij},\nu_{ij}),(g_{ijk},\vartheta_{ijk}))$ a 2-cocycle in $\check C^\bullet(\mathcal U,B_f^\bullet(\varpi)\otimes_A J)$ defining a class $o_e([f,\Delta,\varpi])\in\mathbb H^2(B_f^\bullet(\varpi)\otimes_A J)$.

If $(\tilde\varphi'_{ij},\tilde F'_{ij},\tilde d'_i,\tilde\varpi'_i)$ is any other collection, then we have, with all notations as before,

$$\begin{aligned}\tilde\varphi'^*_{ji}\tilde\varphi'^*_{kj}(\tilde\varphi'^*_{ki})^{-1}&=1+\vartheta_{ijk}+\vartheta'_{jk}+\vartheta'_{ij}-\vartheta'_{ik},\\ \tilde F'_{ij}\tilde\varphi'^*_{ji}(\tilde F'_{jk})\tilde F'^{-1}_{ik}&=1+g_{ijk}+g'_{ij}+g'_{jk}-g'_{ik}+\vartheta_{ij}\log(F_{jk}),\\ \tilde F_{ij}\tilde\varphi_{ji}^*(\tilde d'_j(\tilde F_{ji}\tilde\varphi_{ij}^*\,\cdot\,))-\tilde d'_i&=\nu_{ij}+\nu'_j-\nu'_i-d(g'_{ij}-i_{\vartheta'_{ij}}(d_j))\text{ and}\\ \tilde\Delta'_i(\tilde\Delta'_i(\,\cdot\,))&=\kappa_i+d\nu'_i,\end{aligned}$$

and, moreover, $\tilde\varpi'_i-\tilde\varpi_i=u'_i$ for a cochain (u'_i) in $\check C^0(\mathcal U,(\Omega_f^2\otimes_{\mathcal O_X}\mathcal L)\otimes_A J)$. Hence,

$$\begin{aligned}\tilde F'_{ij}\tilde\varphi'^*_{ji}\tilde\varpi'_j-\tilde\varpi'_i&=u_{ij}+u'_j-u'_i+F_{ij}(g'_{ij},\vartheta'_{ij})(\varpi_j)\text{ and}\\ (-\tilde\Delta'_i)(\tilde\varpi'_i)&=\varrho_i-\nu_i\wedge\varpi_i-\Delta_i u'_i.\end{aligned}$$

This means that the class $o_e([f,\Delta,\varpi])\in\mathbb H^2(B_f^\bullet(\varpi)\otimes_A J)$ which is defined by the cocycle $((\varrho_i,\kappa_i),(u_{ij},\nu_{ij}),(g_{ijk},\vartheta_{ijk}))$ is independent of the collection $(\tilde\varphi_{ij},\tilde F_{ij},\tilde d_i,\tilde\varpi_i)$ chosen, because two such cocycles always differ by a coboundary $(\check d\pm d_B)((u'_i,\nu'_i),(g'_{ij},\vartheta'_{ij}))$. Moreover, $o_e([f,\Delta,\varpi])$ vanishes exactly when there exists a collection with $\varrho_i=0$, $u_{ij}=0$, $\kappa_i=0$, $\nu_{ij}=0$, $g_{ijk}=0$ and $\vartheta_{ijk}=0$, and this is true if and only if a lifting $(\tilde f,\tilde\Delta,\tilde\varpi)$ of (f,Δ,ϖ) exists.

Setting $H_e([f,\Delta,\varpi]):=H^2(\mathcal X,B_f^\bullet(\varpi)\otimes_A J)$ and $o\colon V_{\mathrm{Def}_{(f_0,\nabla,\omega)}}\to O:=\coprod_{V_{\mathrm{Def}_{(f_0,\nabla,\omega)}}}H$, $(e,x)\mapsto o_e(x)$, defines a complete linear obstruction theory (H,o) for $\mathrm{Def}_{(f_0,\nabla,\omega)}$. In particular, $H_0:=H_{\varepsilon^0}(f_0,\nabla,\omega)=H^2(X,B_{f_0}^\bullet(\omega))$ is a small obstruction space for the functor $\mathrm{Def}_{(f_0,\nabla,\omega)}$.

Conclusion

In the above section we have proven the following:

4.2.21 Proposition

Let $f_0 \colon (X, \nabla, \omega) \to \operatorname{Spec} \kappa$ be a log symplectic scheme of general type.

a) The tangent space of the functor $\operatorname{Def}_{(f_0, \nabla, \omega)}$ is $\mathbb{H}^1(B^\bullet_{f_0}(\omega))$.

b) The vector space $\mathbb{H}^2(B^\bullet_{f_0}(\omega))$ is a the small obstruction space of a complete linear obstruction theory for the functor $\operatorname{Def}_{(f_0, \nabla, \omega)}$.

Let $e \colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension of log Artin rings and $f \colon (\mathcal{X}, \Delta, \varpi) \to \operatorname{Spec} \mathcal{A}$ a lifting of (f_0, ∇, ω) over $\mathcal{A}$.

c) The set of automorphisms of a lifting $\tilde{f} \colon (\tilde{\mathcal{X}}, \tilde{\Delta}, \tilde{\varpi}) \to \operatorname{Spec} \tilde{\mathcal{A}}$ inducing the identity on $(f, \Delta, \varpi)/\mathcal{A}$ is $\mathbb{H}^0(B^\bullet_f(\varpi) \otimes_A J)$.

d) The set of isomorphism classes of liftings $(\tilde{f}, \tilde{\Delta}, \tilde{\varpi})/\tilde{\mathcal{A}}$ is a pseudo-torsor under the additive group $\mathbb{H}^1(B^\bullet_f(\varpi) \otimes_A J)$.

e) The complete obstruction $o_e([f, \Delta, \varpi])$ to lifting is an element in the obstruction space $\mathbb{H}^2(B^\bullet_f(\varpi) \otimes_A J)$.

Given a deformation $f \colon (\mathcal{X}, \Delta) \to \operatorname{Spec} \mathcal{T}$ of $f_0 \colon (X, \nabla) \to \operatorname{Spec} \kappa$, we may conclude from the calculations above, by putting all ϑ_I, g_I and d_I equal to zero:

4.2.22 Proposition

a) The tangent space of the functor $\operatorname{Def}_{\omega|(f,\Delta)}$ is $\mathbb{H}^0(\Omega^{\geq 2, \bullet}_{f_0} \otimes_{\mathcal{O}_X} L[2])$.

b) The vector space $\mathbb{H}^1(\Omega^{\geq 2, \bullet}_{f_0} \otimes_{\mathcal{O}_X} L[2])$ is a the small obstruction space of a complete linear obstruction theory for the functor $\operatorname{Def}_{\omega|(f,\Delta)}$.

Let $e \colon 0 \to J \to \tilde{\mathcal{A}} \to \mathcal{A} \to 0$ be an extension of log Artin rings and let $\varpi/\mathcal{A}$ be a lifting of ω over $\mathcal{A}$.

c) The group of automorphisms of a lifting $\tilde{\varpi}/\tilde{\mathcal{A}}$ inducing the identity on $\varpi/\mathcal{A}$ consists only of the identity morphism (and it may be identified with the trivial group $\mathbb{H}^{-1}(\Omega^{\geq 2, \bullet}_f \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{L}[2] \otimes_T J) = 0$).

d) The set of isomorphism classes of liftings $\tilde{\varpi}/\tilde{\mathcal{A}}$ is a pseudo-torsor under the additive group $\mathbb{H}^0(\Omega^{\geq 2, \bullet}_f \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{L}[2] \otimes_T J)$.

e) The complete obstruction $o_e(\varpi)$ to lifting is an element of the obstruction space $\mathbb{H}^1(\Omega^{\geq 2, \bullet}_f \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{L}[2] \otimes_T J)$.

Moreover, we may link the maps in the long exact cohomology sequence associated to the short exact sequence of complexes

$$b^{\varpi}\colon 0 \to \mathit{\Omega}_f^{\geq 2,\bullet} \otimes \mathcal{L}[1] \to B_f^{\bullet}(\varpi) \to A_f^{\bullet}(\mathit{\Delta}) \to 0$$

to the two morphisms of functors $\mathrm{Def}_{(f_0, \nabla, \omega)} \to \mathrm{Def}_{f_0, \nabla}$ and $\mathrm{Def}_{\omega|(\tilde{f}, \tilde{\mathit{\Delta}})} \to \mathrm{Def}_{(f_0, \nabla, \omega)}$ (for a lifting $(\tilde{f}, \tilde{\mathit{\Delta}})\colon \tilde{\mathcal{X}} \to \operatorname{Spec} \tilde{\mathcal{A}}$ of $(f, \mathit{\Delta})$) by the following corollaries:

4.2.23 Corollary
Under the A-linear map $\mathbb{H}^p(B_f^{\bullet}(\varpi) \otimes_A J) \to \mathbb{H}^p(A_f^{\bullet}(\mathit{\Delta}) \otimes_A J)$

a) an automorphism of a lifting $(\tilde{f}, \tilde{\mathit{\Delta}}, \tilde{\varpi})$ over $(f, \mathit{\Delta}, \varpi)$ is mapped to its underlying automorphism of $(\tilde{f}, \tilde{\mathit{\Delta}})$ over $(f, \mathit{\Delta})$, for $p = 0$.

b) the class $[(\tilde{f}, \tilde{\mathit{\Delta}}, \tilde{\varpi}) - (\tilde{f}_0, \tilde{\mathit{\Delta}}_0, \tilde{\varpi}_0)]$ of a lifting $(\tilde{f}, \tilde{\mathit{\Delta}}, \tilde{\varpi})$ of $(f, \mathit{\Delta}, \varpi)$ is mapped to the class of its underlying lifting $(\tilde{f}, \tilde{\mathit{\Delta}})$ of $(f, \mathit{\Delta})$, for $p = 1$, whenever a lifting $(\tilde{f}_0, \tilde{\mathit{\Delta}}_0, \tilde{\varpi}_0)$ of $(f, \mathit{\Delta}, \varpi)$ is given.

c) the obstruction $o_e([f, \mathit{\Delta}, \varpi])$ of $\mathrm{Def}_{(f_0, \nabla, \omega)}$ is mapped to the obstruction $o_e([f, \mathit{\Delta}])$ of $\mathrm{Def}_{f_0, \nabla}$, for $p = 2$.

4.2.24 Corollary
Given a lifting $(\tilde{f}_0, \tilde{\mathit{\Delta}}_0, \tilde{\varpi}_0)$ of $(f, \mathit{\Delta}, \varpi)$,

a) any automorphism of $\tilde{\varpi}_0$ over ϖ is the identity.

Under the A-linear map $\mathbb{H}^{p-1}(\mathit{\Omega}_f^{\geq 2,\bullet} \otimes \mathcal{L}[2] \otimes_A J) \to \mathbb{H}^p(B_f^{\bullet}(\varpi) \otimes_A J)$

b) the class $[\tilde{\varpi} - \tilde{\varpi}_0]$ of a lifting $\tilde{\varpi}$ of ϖ is mapped to the class of the lifting $(\tilde{f}_0, \tilde{\mathit{\Delta}}_0, \tilde{\varpi})$ of $(f, \mathit{\Delta}, \varpi)$, for $p = 1$.

c) the obstruction $o_e(\varpi)$ of $\mathrm{Def}_{\omega|(\tilde{f}, \tilde{\mathit{\Delta}})}$ is mapped to the obstruction $o_e([f, \mathit{\Delta}, \varpi])$ of $\mathrm{Def}_{(f_0, \nabla, \omega)}$, for $p = 2$.

Regarding the A-linear map $b^{\varpi}\colon \mathbb{H}^1(A_f^{\bullet}(\mathit{\Delta}) \otimes_A J) \to \mathbb{H}^1(\mathit{\Omega}_f^{\geq 2,\bullet} \otimes \mathcal{L}[2] \otimes_A J)$ induced by b^{ϖ},

d) a lifting $\tilde{\varpi}$ of ϖ exists on a lifting $(\tilde{f}, \tilde{\mathit{\Delta}})$ of $(f, \mathit{\Delta})$ if and only if the class $[(\tilde{f}, \tilde{\mathit{\Delta}}) - (\tilde{f}_0, \tilde{\mathit{\Delta}}_0)]$ lies in the kernel of b^{ϖ}.

Equivalently, under the pairing $\mathbb{H}^1(A_f^{\bullet}(\mathit{\Delta}) \otimes_A J) \times \mathbb{H}^0(\mathit{\Omega}_f^{\times \geq 2,\bullet} \otimes L[2]) \to \mathbb{H}^1(\mathit{\Omega}_f^{\geq 2,\bullet} \otimes \mathcal{L}[2] \otimes_A J)$ induced by the action of ∇-derivations $A_f^{\bullet}(\mathit{\Delta}) \times \mathit{\Omega}_f^{\times \geq 2,\bullet} \otimes L[2] \to \mathit{\Omega}_f^{\geq 2,\bullet} \otimes L[2]$, $(a^{\bullet}, u^{\bullet}) \mapsto a^{\bullet}(u^{\bullet})$, the classes $[(\tilde{f}, \tilde{\mathit{\Delta}}) - (\tilde{f}_0, \tilde{\mathit{\Delta}}_0)]$ and $[\varpi]$ pair to zero.

Proof: For d), let $[(\tilde{f}, \tilde{\mathit{\Delta}}) - (\tilde{f}_0, \tilde{\mathit{\Delta}}_0)]$ be given by the 1-cocycle $((\tilde{g}_{ij}, \tilde{\vartheta}_{ij}), \tilde{\nu}_i)$ in the complex $A_f^{\bullet}(\nabla) \otimes_A J$. The element $b^{\varpi}([\tilde{f}, \tilde{\mathit{\Delta}}])$ is the class of the 1-cocycle $(g_{ij}\varpi_j + i_{\vartheta_{ij}}(\varpi_j), \nu_i \wedge \varpi_i)$

in $\Omega_f^{\geq 2,\bullet} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{L}[2]$, thus we have the following chain of equivalences:

$$\begin{aligned}
&[(\tilde{f}, \tilde{\Delta}) - (\tilde{f}_0, \tilde{\Delta}_0)] \in \operatorname{Ker}(b^{\varpi}) \\
&\quad \Leftrightarrow (g_{ij}\varpi_j + i_{\vartheta_{ij}}(\varpi_j), \nu_i \wedge \varpi_i) = (\Delta \pm \check{d})(u_i) = (F_{ij}u_j - u_i, \Delta u_i) \\
&\qquad\qquad \text{for a 0-cochain } (u_i) \text{ in } \Omega_f^{\geq 2,\bullet} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{L}[2] \\
&\quad \Leftrightarrow ((u_i, \nu_i), (g_{ij}, \vartheta_{ij})) \text{ is a 1-cocycle in } B_f^{\bullet}(\varpi) \\
&\quad \Leftrightarrow ((u_i, \nu_i), (g_{ij}, \vartheta_{ij})) \text{ defines the class} \\
&\qquad\qquad [(\tilde{f}, \tilde{\Delta}, \tilde{\varpi}) - (\tilde{f}_0, \tilde{\Delta}_0, \tilde{\varpi}_0)] \text{ of a lifting } (\tilde{f}, \tilde{\Delta}, \tilde{\varpi}). \qquad \square
\end{aligned}$$

4.2.6 Overview over the tangent and small obstruction spaces

The tangent and small obstruction spaces calculated in this section are

$$\begin{aligned}
t_{\mathrm{Def}_{f_0}} &= H^1(X, T_{f_0}), & H_{0,\mathrm{Def}_{f_0}} &= H^2(X, T_{f_0}), \\
t_{\mathrm{Def}_{(f_0,\omega)}} &= \mathbb{H}^1(\Omega_{f_0}^{\geq 1,\bullet}[1]), & H_{0,\mathrm{Def}_{(f_0,\omega)}} &= \mathbb{H}^2(\Omega_{f_0}^{\geq 1,\bullet}[1]), \\
t_{\mathrm{Def}_{\omega|f}} &= \mathbb{H}^0(\Omega_{f_0}^{\geq 2,\bullet}[2]), & H_{0,\mathrm{Def}_{\omega|f}} &= \mathbb{H}^1(\Omega_{f_0}^{\geq 2,\bullet}[2]), \\
t_{\mathrm{Def}_{(f_0,L)}} &= H^1(X, A_{f_0}(L)), & H_{0,\mathrm{Def}_{(f_0,L)}} &= H^2(X, A_{f_0}(L)), \\
t_{\mathrm{Def}_{L|f}} &= H^1(X, \mathcal{O}_X), & H_{0,\mathrm{Def}_{L|f}} &= H^2(X, \mathcal{O}_X), \\
t_{\mathrm{Def}_{(f_0,\nabla)}} &= \mathbb{H}^1(A_{f_0}^{\bullet}(\nabla)), & H_{0,\mathrm{Def}_{(f_0,\nabla)}} &= \mathbb{H}^2(A_{f_0}^{\bullet}(\nabla)), \\
t_{\mathrm{Def}_{\nabla|f}} &= \mathbb{H}^1(\Omega_{f_0}^{\bullet}), & H_{0,\mathrm{Def}_{\nabla|f}} &= \mathbb{H}^2(\Omega_{f_0}^{\bullet}), \\
t_{\mathrm{Def}_{(f_0,\nabla,\omega)}} &= \mathbb{H}^1(B_{(f_0,\nabla)}^{\bullet}(\omega)), & H_{0,\mathrm{Def}_{(f_0,\nabla,\omega)}} &= \mathbb{H}^2(B_{(f_0,\nabla)}^{\bullet}(\omega)) \text{ and} \\
t_{\mathrm{Def}_{\omega|(f,\Delta)}} &= \mathbb{H}^0(\Omega_{f_0}^{\geq 2,\bullet} \otimes L[2]), & H_{0,\mathrm{Def}_{\omega|(f,\Delta)}} &= \mathbb{H}^1(\Omega_{f_0}^{\geq 2,\bullet} \otimes L[2]).
\end{aligned}$$

4.3 Log symplectic deformations over the standard log point

From now on we limit our considerations to the following setting. Let k be a field of characteristic zero and let T denote the power series ring $k[\![t]\!]$ in one variable. We let $\mathcal{T}\colon \mathbb{N}_0 \to T$ be the prelog ring defined by $n \mapsto t^n$. The residue field of T is k and we let $\kappa\colon \mathbb{N}_0 \to k$ denote the prelog ring given by mapping n to 1 if $n = 0$ and to 0 if not, which defines the standard log point $\operatorname{Spec}\kappa$. In particular, $\varrho\colon Q \to P$ is the identity map $\mathbb{N}_0 \to \mathbb{N}_0$ in the notation of chapter 2. Consequently, the induced morphism of log schemes $\operatorname{Spec}\kappa \to \operatorname{Spec}\mathcal{T}$ is a strict closed embedding.

For this section, let $f_0\colon (X, \nabla, \omega) \to \operatorname{Spec}\kappa$ be a proper log fs log symplectic scheme with $f_*\mathcal{O}_X = \mathcal{O}_{\operatorname{Spec}\kappa}$. The morphism $f_0\colon X \to \operatorname{Spec}\kappa$ is then integral due to proposition 1.2.42. Hence, its underlying morphism $\underline{f_0}$ is flat and, moreover, any log smooth lifting $f\colon \mathcal{X} \to \operatorname{Spec}\mathcal{A}$ of f_0 is integral by 2.3.1, thus its underlying morphism of schemes is flat. Hence $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\operatorname{Spec}\mathcal{A}}$ (cf. [31, p. 216]).

4.3.1 Existence of hulls and pro-representability

We regard the functors of log Artin rings

Def_{f_0}, $\mathrm{Def}_{(f_0,\omega)}$, $\mathrm{Def}_{\omega|f}$, $\mathrm{Def}_{(f_0,L)}$, $\mathrm{Def}_{L|f}$, $\mathrm{Def}_{(f_0,\nabla)}$, $\mathrm{Def}_{\nabla|f}$, $\mathrm{Def}_{(f_0,\nabla,\omega)}$ and $\mathrm{Def}_{\omega|(f,\Delta)}$

defined in the first two sections 4.1.2 and 4.1.1 of the current chapter. We will show in this section that every such functor it possesses a hull and that some of them are even pro-representable. This is done by verifying the log Schlessinger conditions LH_1 to LH_3 (respectively, to LH_4) (cf. chapter 2 section 2.2).

The procedure is similar to that of the preceding sections: Each of the upcoming subsections entitled "Verification of LH_n", $n = 1, \dots, 4$, comprises four "steps", in each of which we add one more datum to the log fs log smooth scheme $f_0\colon X \to \operatorname{Spec}\kappa$. Step one deals with $f_0\colon X \to \operatorname{Spec}\kappa$, step two adds a line bundle L on X, step three a flat log connection ∇ with line bundle L on f_0 and step four a log symplectic form of type ∇ on f_0.

The existence of a hull of the functor Def_{f_0} for a log smooth morphism $f_0\colon X \to \operatorname{Spec}\kappa$ with X a log fs log scheme is stated and proven by F. Kato in [17, 8.7 & § 9]. In the verification of the LH_n each first step is a repetition the basic arguments of Kato's proof. Each second step is basically an adaptation of Sernesi's proof of [32, 3.3.11] which itself is based on Schlessinger's calculations in [31, §3]. The third and fourth step are due to ourselves.

We abbreviate the functor $\mathrm{Def}_{(f_0,\nabla,\omega)}$ to Def.

Verification of LH_1

First, we show that $\mathrm{Def} = \mathrm{Def}_{(f_0,\nabla,\omega)}$ satisfies LH_1, i. e. that for every morphism $\mathcal{A}' \to \mathcal{A}$ and all small extensions $\mathcal{A}'' \to \mathcal{A}$ of log Artin rings the map

$$\Phi\colon \mathrm{Def}(\mathcal{A}' \times_{\mathcal{A}} \mathcal{A}'') \to \mathrm{Def}(\mathcal{A}') \times_{\mathrm{Def}(\mathcal{A})} \mathrm{Def}(\mathcal{A}'')$$

is surjective.

Let $\mathcal{A}' \to \mathcal{A} \leftarrow \mathcal{A}''$ be a diagram in $\underline{\mathrm{LArt}}_{\mathcal{T}}$ with $\mathcal{A}'' \to \mathcal{A}$ surjective. We take some element

$$([f', \Delta', \varpi'], [f'', \Delta'', \varpi'']) \in \mathrm{Def}(\mathcal{A}') \times_{\mathrm{Def}(\mathcal{A})} \mathrm{Def}(\mathcal{A}'')$$

which is mapped to $[f, \Delta, \varpi] \in \mathrm{Def}(\mathcal{A})$. Let $\mathcal{L}^{('/'')}$ denote the line bundle of $\Delta^{('/'')}$.

We then have a diagram of log smooth deformations

$$\begin{tikzcd}
\mathcal{X}' \arrow[d, "f'"] & & \mathcal{X}'' \arrow[d, "f''"] \\
\operatorname{Spec}\mathcal{A}' & \mathcal{X} \arrow[ul, "u'"'] \arrow[ur, hook, "u''"] \arrow[d, "f"] & \operatorname{Spec}\mathcal{A}'' \\
& \operatorname{Spec}\mathcal{A} \arrow[ul] \arrow[ur, hook] &
\end{tikzcd},$$

where u' and u'' induce isomorphisms $\mathcal{X}' \times_{\operatorname{Spec}\mathcal{A}'} \operatorname{Spec}\mathcal{A} \cong \mathcal{X} \cong \mathcal{X}'' \times_{\operatorname{Spec}\mathcal{A}''} \operatorname{Spec}\mathcal{A}$. Moreover, we get isomorphisms $u'^*\Delta' \cong \Delta \cong u''^*\Delta''$ of flat log connections (and in particular isomorphisms $u'^*\mathcal{L}' \cong \mathcal{L} \cong u''^*\mathcal{L}''$ of line bundles) and an equality $u'^*\varpi' = \varpi = u''^*\varpi''$ of 2-forms on $\mathcal{X}$.

Since each of the natural morphisms $i^{('/'')}\colon X \to \mathcal{X}^{('/'')}$ is a log infinitesimal thickening, we have $\overline{\mathcal{M}}_X \cong {i^{('/'')}}^{-1}\overline{\mathcal{M}}_{\mathcal{X}^{('/'')}}$ by lemma 1.2.12. In particular, by 3.4.10, $\operatorname{LCar}(X) \cong \operatorname{LCar}(\mathcal{X}^{('/'')})$.

By lemma 2.3.2, every local chart of X lifts to a chart on $\mathcal{X}^{('/'')}$.

Step One: Log schemes We define $\tilde{\mathcal{A}}$ to be the fibred product of log rings $\mathcal{A}' \times_{\mathcal{A}} \mathcal{A}''$. Following Kato's proof of the existence of a hull for the functor Def_{f_0} in [17, Cap. 9], there is a log smooth deformation $\tilde{f}\colon \tilde{\mathcal{X}} \to \operatorname{Spec}\tilde{\mathcal{A}}$ of f_0 over $\tilde{\mathcal{A}}$, where $\tilde{\mathcal{X}}$ is the log scheme consisting of the scheme $(|\underline{X}|, \mathcal{O}_{\mathcal{X}'} \times_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}''})$ and the log structure

$$\alpha_{\tilde{\mathcal{X}}} = \alpha_{\mathcal{X}'} \times_{\alpha_{\mathcal{X}}} \alpha_{\mathcal{X}''}\colon \mathcal{M}_{\mathcal{X}'} \times_{\mathcal{M}_{\mathcal{X}}} \mathcal{M}_{\mathcal{X}''} \to \mathcal{O}_{\mathcal{X}'} \times_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}''}$$

and with $\tilde{f} = f' \sqcup_f f''$. This is the amalgamated sum of the log smooth log schemes f' and f'' over f. By the surjectivity of $A'' \to A$ and the definition of $\alpha_{\tilde{\mathcal{X}}}$, the canonical morphism $v'\colon \mathcal{X}' \to \tilde{\mathcal{X}}$ is a strict closed immersion, making $\tilde{f}$ a log smooth deformation of f and thus of f_0.

Moreover, $\tilde{\alpha}$ is a fine log structure given locally by a chart $\tilde{a} = a' \times a''\colon P \cong P \times_P P \to \mathcal{M}_{\tilde{X}}$, where $a^{('/'')}\colon P \to \mathcal{M}_{X^{('/'')}}$ are local charts for $\mathcal{X}^{('/'')}$.

The following diagram, which we will refer to as $(*)$, shows our momentary situation:

$$\begin{tikzcd}
 & & \tilde{\mathcal{X}} \arrow[d, "\tilde{f}"] & & \\
\mathcal{X}' \arrow[urr, hook, "v'"] \arrow[d, "f'"] & & \operatorname{Spec}\tilde{\mathcal{A}} & & \mathcal{X}'' \arrow[ull, "v''"'] \arrow[d, "f''"] \\
\operatorname{Spec}\mathcal{A}' \arrow[urr, hook] & & \mathcal{X} \arrow[ull, "u'"] \arrow[urr, hook, "u''"'] \arrow[d, "f"] & & \operatorname{Spec}\mathcal{A}'' \arrow[ull] \\
 & & \operatorname{Spec}\mathcal{A} \arrow[ull] \arrow[urr, hook] & &
\end{tikzcd},$$

where arrows of the form $\hookrightarrow$ indicate strict closed immersions.

Step Two: Line bundles Following the proof of [32, 3.3.11], we define $\tilde{\mathcal{L}} := \mathcal{L}' \times_{\mathcal{L}} \mathcal{L}''$ which is a a line bundle on $\tilde{\mathcal{X}}$ with the correct restrictions to $\mathcal{X}'$ and $\mathcal{X}''$, respectively.

Hence, $(\tilde{f}, \tilde{\mathcal{L}})$ defines an element in $\operatorname{Def}_{(f_0,L)}(\tilde{\mathcal{A}})$ with

$$[\tilde{f}, \tilde{\mathcal{L}}] \mapsto ([f', \mathcal{L}'], [f'', \mathcal{L}''])$$

under the map $\operatorname{Def}_{(f_0,L)}(\tilde{\mathcal{A}}) \to \operatorname{Def}_{(f_0,L)}(\mathcal{A}') \times_{\operatorname{Def}_{(f_0,L)}(\mathcal{A})} \operatorname{Def}_{(f_0,L)}(\mathcal{A}'')$.

If the line bundles $\mathcal{L}^{(\prime/\prime\prime)}$ are log Cartier, say $\mathcal{L} = \mathcal{M}_{\mathcal{X}}(D)$, $\mathcal{L}' = \mathcal{M}_{\mathcal{X}'}(D')$ and $\mathcal{L}'' = \mathcal{M}_{\mathcal{X}''}(D'')$ for log Cartier divisors D, D' and D'' on $\mathcal{X}$, $\mathcal{X}'$ and $\mathcal{X}''$, respectively, then we must have $D' \mapsto D \mapsfrom D''$ under the isomorphisms $\mathrm{LCar}(\mathcal{X}') \cong \mathrm{LCar}(\mathcal{X}) \cong \mathrm{LCar}(\mathcal{X}'')$. Hence, $\tilde{\mathcal{L}} = \mathcal{L}' \times_{\mathcal{L}} \mathcal{L}'' \in \mathrm{Pic}(\tilde{\mathcal{X}}) = \mathrm{Pic}(\mathcal{X}') \times_{\mathrm{Pic}(\mathcal{X})} \mathrm{Pic}(\mathcal{X}'')$ is the image $\mathcal{M}_X(\tilde{D})$ under $\delta_{\tilde{\mathcal{X}}} = \delta_{\mathcal{X}'} \times_{\delta_{\mathcal{X}}} \delta_{\mathcal{X}''}$ of $\tilde{D} = (D', D'') \in \mathrm{LCar}(\mathcal{X}') \times_{\mathrm{LCar}(\mathcal{X})} \mathrm{LCar}(\mathcal{X}'') = \mathrm{LCar}(\tilde{\mathcal{X}})$. In particular, $\tilde{\mathcal{L}}$ is log Cartier.

Step Three: Flat log connection We claim that the natural map

$$\Omega^1_{\tilde{f}} \to \Omega^1_{f'} \times_{\Omega^1_f} \Omega^1_{f''},$$

given by $\sigma \mapsto (v'^*\sigma, v''^*\sigma)$, is an isomorphism.

To see this, we fix a chart of $\tilde{f}\colon \tilde{\mathcal{X}} \to \operatorname{Spec} \tilde{\mathcal{A}}$ at a point $x \in X$ subordinate to a homomorphism of monoids $\theta\colon \mathbb{N}_0 \to P$. Then, because this chart lifts to charts for $\tilde{f}$ and each $f^{(\prime/\prime\prime)}$ and by [19, 1.8], we have $\Omega^1_{\tilde{f},x} \cong \mathcal{O}_{\tilde{\mathcal{X}},x} \otimes_{\mathbb{Z}} (P^{\mathrm{grp}}/\theta(\mathbb{N}_0)^{\mathrm{grp}})$ and $\Omega^1_{f^{(\prime/\prime\prime)},x} \cong \mathcal{O}_{\mathcal{X}^{(\prime/\prime\prime)},x} \otimes_{\mathbb{Z}} (P^{\mathrm{grp}}/\theta(\mathbb{N}_0)^{\mathrm{grp}})$.

So clearly $\Omega^1_{\tilde{f},x} \to \Omega^1_{f',x} \times_{\Omega^1_{f,x}} \Omega^1_{f'',x} = (\mathcal{O}_{\mathcal{X}',x} \times_{\mathcal{O}_{\mathcal{X},x}} \mathcal{O}_{\mathcal{X}'',x}) \otimes_{\mathbb{Z}} (P^{\mathrm{grp}}/\theta(Q)^{\mathrm{grp}})$ is an isomorphism at each x and we will identify $\Omega^1_{\tilde{f}} = \Omega^1_{f'} \times_{\Omega^1_f} \Omega^1_{f''}$ via this isomorphism.

Moreover, for the line bundles $\tilde{\mathcal{L}}$ and $\mathcal{L}^{(\prime/\prime\prime)}$ as above, the natural map $\Omega^1_{\tilde{f}} \otimes_{\mathcal{O}_{\tilde{\mathcal{X}}}} \tilde{\mathcal{L}} \to (\Omega^1_{f'} \otimes_{\mathcal{O}_{\mathcal{X}'}} \mathcal{L}') \times_{(\Omega^1_f \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{L})} \Omega^1_{f'' \otimes_{\mathcal{O}_{\mathcal{X}''}} \mathcal{L}''}$ is an isomorphism which can be easily checked on its stalks as well. We will identify both sides of this isomorphism.

We construct a flat log connection $\tilde{\Delta} = (\tilde{\mathcal{L}}, \tilde{\Delta})$ on $(\tilde{\mathcal{X}}, \tilde{\mathcal{L}})$ by defining

$$\begin{aligned}\tilde{\Delta} := \Delta' \times \Delta''\colon \tilde{\mathcal{L}} = \mathcal{L}' \times_{\mathcal{L}} \mathcal{L}'' \to \Omega^2_{\tilde{f}} \otimes_{\mathcal{O}_{\tilde{\mathcal{X}}}} \tilde{\mathcal{L}} \\ = (\Omega^2_{f'} \otimes_{\mathcal{O}_{\mathcal{X}'}} \mathcal{L}') \times_{(\Omega^2_f \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{L})} (\Omega^2_{f''} \otimes_{\mathcal{O}_{\mathcal{X}''}} \mathcal{L}'').\end{aligned}$$

If $\tilde{g} = (g', g'')$ is a local section of $\mathcal{O}_{\tilde{\mathcal{X}}}$ and $\tilde{s} = (s', s'')$ a local section of $\tilde{\mathcal{L}}$, then

$$\begin{aligned}\tilde{\Delta}(\tilde{g}\tilde{s}) &= (\Delta'(g's'), \Delta''(g''s'')) = (dg' \otimes s' + g'\Delta'(s'), dg'' \otimes s'' + g''\Delta''(s'')) \\ &= (dg' \otimes s', dg'' \otimes s'') + (g'\Delta'(s'), g''\Delta''(s'')) = d\tilde{g} \otimes \tilde{s} + \tilde{g}\tilde{\Delta}(\tilde{s}),\end{aligned}$$

so indeed $\tilde{\Delta}$ is a log connection. It is flat, because

$$\tilde{\Delta}\tilde{\Delta} = (\Delta' \times \Delta'')(\Delta' \times \Delta'') = (\Delta'\Delta') \times (\Delta''\Delta'') = 0.$$

If the three flat log connections $\Delta^{(\prime/\prime\prime)}$ are log Cartier, then, just as with line bundles, $\tilde{\Delta}$ is log Cartier: If $\Delta = M_f(D)$, $\Delta' = M_{f'}(D')$ and $\Delta'' = M_{f''}(D'')$ for log Cartier divisors $D^{(\prime/\prime\prime)} \in \mathrm{LCar}(\mathcal{X}^{(\prime/\prime\prime)})$, then $\tilde{\Delta} = M_{\tilde{f}}(\tilde{D})$ for $\tilde{D} = (D', D'') \in \mathrm{LCar}(\tilde{\mathcal{X}})$.

Step Four: Log symplectic form We define $\tilde{\varpi} := (\varpi', \varpi'')$ which, via the identifications made before, is an element of $\Gamma(\tilde{\mathcal{X}}, \Omega^2_{\tilde{f}} \otimes_{\mathcal{O}_{\tilde{\mathcal{X}}}} \tilde{\mathcal{L}})$ with the properties that

a) $\tilde{\Delta}\tilde{\varpi} = (\Delta'\varpi', \Delta''\varpi'') = 0$ and that

b) $\tilde{\varpi}$ induces in the obvious manner an isomorphism

$$T_{\tilde{f}} = T_{f'} \times_{T_f} T_{f''} \xrightarrow{\cong} \Omega^1_{\tilde{f}} \otimes_{\mathcal{O}_{\tilde{\mathcal{X}}}} \tilde{\mathcal{L}} = (\Omega^1_{f'} \otimes_{\mathcal{O}_{\mathcal{X}'}} \mathcal{L}') \times_{(\Omega^1_f \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{L})} (\Omega^1_{f''} \otimes_{\mathcal{O}_{\mathcal{X}''}} \mathcal{L}'').$$

Summing everything up, we have constructed an element $[\tilde{f}, \tilde{\Delta}, \tilde{\varpi}] \in \mathrm{Def}(\tilde{\mathcal{A}})$ with the property that $\Phi([\tilde{f}, \tilde{\Delta}, \tilde{\varpi}]) = ([f', \Delta', \varpi'], [f'', \Delta'', \varpi''])$. This shows LH_1 for the functor $\mathrm{Def} = \mathrm{Def}_{(f_0, \nabla, \omega)}$.

4.3.1 Remark

For any of the other functors of log Artin rings Def defined in the sections 4.1.1 and 4.1.2, LH_1 is shown in the same way, starting with an object $(\eta', \eta'') \in \mathrm{Def}(\mathcal{A}') \times_{\mathrm{Def}(\mathcal{A})} \mathrm{Def}(\mathcal{A}'')$, with $\eta' \mapsto \eta \leftarrow\!\shortmid \eta''$ and using only those steps necessary for the data involved.

Hence, each of the functors Def_{f_0}, $\mathrm{Def}_{(f_0,\omega)}$, $\mathrm{Def}_{\omega|f}$, $\mathrm{Def}_{(f_0,L)}$, $\mathrm{Def}_{L|f}$, $\mathrm{Def}_{(f_0,\nabla)}$, $\mathrm{Def}_{\nabla|f}$, $\mathrm{Def}_{(f_0,\nabla,\omega)}$ and $\mathrm{Def}_{\omega|(f,\Delta)}$ satisfies LH_1.

For example, showing LH_1 for $\mathrm{Def}_{\nabla|f}$ involves putting $f^{('/'')} := f_{\mathcal{A}^{('/'')}}$ in the starting objects $\eta^{('/'')} = (f^{('/'')}, \Delta^{('/'')})$, then defining $\tilde{f} := f_{\mathcal{A}' \times_{\mathcal{A}} \mathcal{A}''}$ and proceeding with steps two and three as above, leaving out step four.

The same applies to the upcoming verifications of LH_2, LH_3 and, for some of the functors, of LH_4.

Verification of LH_2

Condition LH_2 is that the map $\Phi\colon \mathrm{Def}(\mathcal{A}' \times_\kappa \kappa[\varepsilon]^0) \to \mathrm{Def}(\mathcal{A}') \times \mathrm{Def}(\kappa[\varepsilon]^0)$ is bijective. Hence, we have to show that the element $[\tilde{f}, \tilde{\Delta}, \tilde{\varpi}]$ constructed in the last section is unique when $\mathcal{A} = \kappa$ and $\mathcal{A}'' = \kappa[\varepsilon]^0$.

Step One: Log Schemes This paragraph is taken from [17, § 9]. There we find the following lemma which is based on [31, 3.3 & 3.6]:

4.3.2 Lemma ([17, 9.2])

Given a diagram $(*)$ as above. If $\tilde{\tilde{\mathcal{X}}}/\tilde{\mathcal{A}}$ is another smooth lifting fitting into the diagram such that $v^{('/'')*}\tilde{\tilde{\mathcal{X}}} \cong \mathcal{X}^{('/'')}$ over $A^{('/'')}$, then the natural morphism $\tilde{\mathcal{X}} \to \tilde{\tilde{\mathcal{X}}}$ is an isomorphism of log schemes.

Now if $\tilde{\tilde{f}}\colon \tilde{\tilde{\mathcal{X}}} \to \operatorname{Spec} \tilde{\mathcal{A}}$ is any lifting of f_0 over $\tilde{\mathcal{A}}$ then we have a commutative diagram

$$\begin{tikzcd}
 & & \tilde{\tilde{\mathcal{X}}} & & \\
\mathcal{X}' \arrow[urr, hook, "w'"] & & & & \mathcal{X}'' \arrow[ull, "w''"'] \\
 & \mathcal{X} \arrow[ul, "u'"] \arrow[rr, "\varphi"] & & \mathcal{X} \arrow[ur, hook, "u''"] &
\end{tikzcd},$$

where φ is the automorphism of $\mathcal{X}/\mathcal{A}$ defined by $\mathcal{O}_{\mathcal{X}} \cong (v'u')^*\mathcal{O}_{\tilde{\tilde{\mathcal{X}}}} = (v''u'')^*\mathcal{O}_{\tilde{\tilde{\mathcal{X}}}} \cong \mathcal{O}_{\mathcal{X}}$.

If this automorphism lifts to an automorphism φ' of $\mathcal{X}'$ such that $u' \circ \varphi = \varphi' \circ u'$, then replacing w' by $w' \circ \varphi'$, we get a commutative diagram of the form $(*)$ and, by the lemma, $\tilde{\tilde{f}}$ is equal up to isomorphism to $\tilde{f}$ as described in the proof of LH_1. Now if, as LH_2 assumes, $\mathcal{A} = \kappa$, then $\varphi = id$ and $\varphi' = id$ is a lifting (compare [31, p. 220]). Hence, Def_{f_0} satisfies LH_2.

Step Two: Line bundles The uniqueness of $[\tilde{f}, \tilde{\mathcal{L}}]$ under the assumptions $\mathcal{A} = \kappa$ and $\mathcal{A}'' = \kappa[\varepsilon]^0$ of LH_2 is shown as follows: After step one, it remains to show the uniqueness of $\tilde{\mathcal{L}}$, which we do analogously to the proof of [31, 3.2, p. 218]:

If $\tilde{\tilde{\mathcal{L}}}$ is any lifting of L on $\tilde{\mathcal{X}}$, then the composition $\mathcal{L} \cong (v'u')^*\tilde{\tilde{\mathcal{L}}} = (v''u'')^*\tilde{\tilde{\mathcal{L}}} \cong \mathcal{L}$ is an automorphism of $\mathcal{L}$, which is nothing but a global section of $\mathcal{A}^\times$. Since units always lift to units under surjective ring homomorphisms between local rings, using only that $\mathcal{A}'' \to \mathcal{A}$ is surjective, this automorphism of $\mathcal{L}$ lifts to an automorphism of $\mathcal{L}'$. Using [31, 3.6], it follows that $\tilde{\tilde{\mathcal{L}}} \cong \tilde{\mathcal{L}}$.

Hence, the functors $\mathrm{Def}_{(f_0,L)}$ and $\mathrm{Def}_{L|f}$ satisfy LH_2. Observe that, since we have in this step only used the surjectivity of $\mathcal{A}' \to \mathcal{A}$, it follows that $\mathrm{Def}_{L|f}$ actually satisfies LH_4.

Step Three: Log connections Due to the universal property of the fibred product, we have

$$\mathrm{Hom}_{\mathbb{C}}(\tilde{\mathcal{L}}, \Omega_{\tilde{f}} \otimes \tilde{\mathcal{L}}) = \mathrm{Hom}_{\mathbb{C}}(\tilde{\mathcal{L}}, \Omega_{f'} \otimes \mathcal{L}') \times_{\mathrm{Hom}_{\mathbb{C}}(\tilde{\mathcal{L}}, \Omega_f \otimes \mathcal{L})} \mathrm{Hom}_{\mathbb{C}}(\tilde{\mathcal{L}}, \Omega_{f''} \otimes \mathcal{L}'').$$

Now, if $\tilde{\tilde{\Delta}}$ is any log connection on $(\tilde{f}, \tilde{\mathcal{L}})$ constructed as before, the restriction to $\mathcal{X}^{('/'')}$of which is $\Delta^{('/'')}$, then $\tilde{\tilde{\Delta}} = \tilde{\Delta}$. The uniqueness (up to isomorphism) of $(\tilde{f}, \tilde{\mathcal{L}})$ has been shown in the first two steps.

Hence, the functors $\mathrm{Def}_{(f_0,\nabla)}$ and $\mathrm{Def}_{\nabla|f}$ both satisfy LH_2. Observe that $\mathrm{Def}_{\nabla|f}$ actually satisfies LH_4.

Step Four: Log symplectic forms Let $(f^{('/'')}, \Delta^{('/'')}, \varpi^{('/'')})$ and $(\tilde{f}, \tilde{\Delta})$ be as before and let $\tilde{\varpi}$ be the log symplectic form on $(\tilde{f}, \tilde{\Delta})/\tilde{\mathcal{A}}$ defined in the proof of LH_1.

If $\tilde{\tilde{\varpi}}$ is any log symplectic form on $(\tilde{f}, \tilde{\Delta})/\tilde{\mathcal{A}}$ such that $v'^*\tilde{\tilde{\varpi}} = \varpi'$ and $v''^*\tilde{\tilde{\varpi}} = \varpi''$, then we have equalities $\varpi = (v'u')^*\tilde{\tilde{\varpi}} = (v''u'')^*\tilde{\tilde{\varpi}} = \varpi$. Thus $\tilde{\tilde{\varpi}} = (\varpi', \varpi'') = \tilde{\varpi} \in \Gamma(\Omega^2_{\tilde{f}} \otimes \tilde{\mathcal{L}}) = \Gamma(\Omega^2_{f'} \otimes \mathcal{L}') \times_{\Gamma(\Omega^2_f \otimes \mathcal{L})} \Gamma(\Omega^2_{f''} \otimes \mathcal{L}'')$, which shows the uniqueness of $\tilde{\varpi}$.

Hence, each of the functors $\mathrm{Def}_{(f_0,\omega)}$, $\mathrm{Def}_{\omega|f}$, $\mathrm{Def}_{(f_0,\nabla,\omega)}$, $\mathrm{Def}_{(\nabla,\omega)|f}$ and $\mathrm{Def}_{\omega|(f,\Delta)}$ satisfies LH_2. Observe that the functors $\mathrm{Def}_{\omega|f}$, $\mathrm{Def}_{(\nabla,\omega)|f}$ and $\mathrm{Def}_{\omega|(f,\Delta)}$ actually satisfy LH_4.

Verification of LH_3

Condition LH_3 demands that the tangent space of the respective functor of log Artin rings is finite-dimensional. The tangent spaces for the various functors have been calculated in section 4.2 and listed in 4.2.6. By assumption, $f_0\colon X \to \operatorname{Spec}\kappa$ is log smooth and the underlying morphism of schemes $\underline{f_0}$ is proper. Hence, all appearing sheaves are coherent (and in fact locally free), moreover, all appearing complexes are bounded and consequently, all of the tangent spaces are finite-dimensional vector spaces.

Verification of LH_4 for certain functors

As we have remarked in the calculations regarding LH_2, whenever we deformed only L, ∇ or ω along a fixed log scheme $f\colon \mathcal{X} \to \operatorname{Spec}\mathcal{T}$, we had no need of the assumptions made in LH_2, except for the surjectivity of $\mathcal{A}' \to \mathcal{A}$, and have therefore proven that the four functors $\mathrm{Def}_{L|f}$, $\mathrm{Def}_{\nabla|f}$, $\mathrm{Def}_{\omega|(f,\Delta)}$ and $\mathrm{Def}_{\omega|f}$ actually satisfy the condition that for every morphism $\mathcal{A}' \to \mathcal{A}$ and all small extensions $\mathcal{A}'' \to \mathcal{A}$ of log Artin rings the map

$$\Phi\colon \mathrm{Def}(\mathcal{A}' \times_{\mathcal{A}} \mathcal{A}'') \to \mathrm{Def}(\mathcal{A}') \times_{\mathrm{Def}(\mathcal{A})} \mathrm{Def}(\mathcal{A}'')$$

is bijective, which implies LH_4.

Conclusion

4.3.3 Theorem ([18, 4.4])
Let $f_0\colon X \to \operatorname{Spec}\kappa$ be a log smooth, log integral and proper morphism of log fs log schemes with $f_*\mathcal{O}_X = \mathcal{O}_{\operatorname{Spec}\kappa}$. Then the functor Def_{f_0} possesses a hull.

4.3.4 Theorem (cp. [32, 3.3.11])
Let $f_0\colon (X, L) \to \operatorname{Spec}\kappa$ be a log integral and proper scheme with line bundle with $f_*\mathcal{O}_X = \mathcal{O}_{\operatorname{Spec}\kappa}$. Then the functor $\mathrm{Def}_{(f_0,L)}$ possesses a hull.

4.3.5 Theorem
Let $f_0\colon (X, \nabla) \to \operatorname{Spec}\kappa$ be a log integral and proper scheme with flat log connection with $f_*\mathcal{O}_X = \mathcal{O}_{\operatorname{Spec}\kappa}$. Then the functor $\mathrm{Def}_{(f_0,\nabla)}$ possesses a hull.

4.3.6 Theorem
Let $f_0\colon (X, \nabla, \omega) \to \operatorname{Spec}\kappa$ be a log integral and proper log symplectic scheme with $f_*\mathcal{O}_X = \mathcal{O}_{\operatorname{Spec}\kappa}$. Then the functor $\mathrm{Def}_{(f_0,\nabla,\omega)}$ possesses a hull.

If $\nabla = d$, then $\mathrm{Def}_{(f_0,\omega)}$ possesses a hull.

4.3.7 Theorem

Let $f\colon \mathcal{X} \to \operatorname{Spec} \mathcal{T}$ be a log smooth morphism of log fs log schemes with $f_*\mathcal{O}_X = \mathcal{O}_{\operatorname{Spec} \kappa}$ and

a) let L be a line bundle on f_0. Then the functor $\operatorname{Def}_{L|f}$ is pro-representable.

b) let ∇ be a flat log connection on f_0. Then the functor $\operatorname{Def}_{\nabla|f}$ is pro-representable.

c) let ω be a log symplectic form of non-twisted type on f_0. Then the functor $\operatorname{Def}_{\omega|f}$ is pro-representable.

4.3.8 Theorem

Let $f\colon (\mathcal{X}, \Delta) \to \operatorname{Spec} \mathcal{T}$ be a log scheme with flat log connection with $f_*\mathcal{O}_X = \mathcal{O}_{\operatorname{Spec} \kappa}$ and let ω be a log symplectic form of type ∇. Then the functor $\operatorname{Def}_{\omega|(f,\Delta)}$ is pro-representable.

5 Smoothing of SNC log symplectic varieties

In this chapter we relate the sheaf of Poincaré residues to the sheaf of normalisation residues and calculate certain Ext-sheaves of these sheaves, which are later used in the main results. We prove that the tangent and obstruction spaces of the various functors of log Artin rings, which have been calculated in chapter 4, are free modules over certain of those rings. Using the logarithmic version of the T1-lifting principle, we are then able to prove our main results, stating that under certain conditions the hulls of the deformation functors $\mathrm{Def}_{(f_0,\omega)}$ and $\mathrm{Def}_{(f_0,\nabla,\omega)}$ are smooth (cf. theorems 5.4.7 and 5.4.13) and that under these conditions there exists a flat smoothing deformation of (f_0,ω) and (f_0,∇,ω), respectively (cf. theorems 5.4.9 and 5.4.14).

In this chapter, we consider the prelog ring $\mathcal{T}\colon \mathbb{N}_0 \to \mathbb{C}[\![t]\!]$, $n \mapsto t^n$. Its residue field $\mathbb{C}$ has the induced prelog ring structure $\mathcal{C}\colon \mathbb{N}_0 \to \mathbb{C}[\![t]\!] \to \mathbb{C}$ mapping $0 \mapsto 1$ and $n \mapsto 0$ for $n \geq 1$. Hence $\operatorname{Spec} \mathcal{C}$ is the standard log point on the field $\mathbb{C}$.

5.1 The Poincaré residue map for SNC log varieties

5.1.1 The Poincaré residue map

5.1.1 Definition

Let $f\colon X \to Y$ be a morphism of log schemes. We denote by τ_f the kernel and by Υ_f the cokernel of the natural map $\Omega^1_{\underline{f}} \to \Omega^1_f$. Hence, by definition, the sequence

$$0 \to \tau_f \to \Omega^1_{\underline{f}} \to \Omega^1_f \to \Upsilon_f \to 0$$

is exact. We call τ_f the *sheaf of torsion differentials* of f (cf. [11, 1.2]). We call Υ_f the *sheaf of Poincaré residues* of f and the projection $\varrho\colon \Omega^1_f \to \Upsilon_f$ the *Poincaré residue map* (cf. [29, IV.1.2.12]).

Recall that $\underline{X}^\iota$ denotes the scheme $\underline{X}$ endowed with the trivial log structure $\iota\colon \mathcal{O}^\times_{\underline{X}} \to \mathcal{O}_{\underline{X}}$. Consider the natural diagram

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[dr, "f"'] & X^Y \arrow[r] \arrow[d, "f^Y"] & \underline{X}^\iota \arrow[d, "\underline{f}^\iota"] \\
& Y \arrow[r] & \underline{Y}^\iota,
\end{tikzcd}$$

where $X^Y := \underline{X}^\iota \times_{\underline{Y}^\iota} Y$ is the scheme $\underline{X}$ equipped with the log structure $f^\times \alpha_Y$ and h the unique natural morphism. By 1.2.24, the sheaves $\Omega^1_{f^Y}$ and $\Omega^1_{\underline{f}}$ are naturally isomorphic and by 1.2.25, there is an exact sequence $\Omega^1_{\underline{f}} \to \Omega^1_f \to \Omega^1_h \to 0$ which we may complete to

$$0 \to \tau_f \to \Omega^1_{\underline{f}} \to \Omega^1_f \to \Omega^1_h \to 0.$$

In conclusion, we have the following proposition:

5.1.2 Proposition ([29, IV.2.3.4])

There is a natural isomorphism $\Upsilon_f \cong \Omega^1_h$.

If we tensor the exact sequence

$$0 \to f^\times \mathcal{M}_Y^{\mathrm{grp}} \to \mathcal{M}_X^{\mathrm{grp}} \to \overline{\mathcal{M}}_f^{\mathrm{grp}} \to 0$$

with $\mathcal{O}_X$ over $\mathbb{Z}$, we get the exact sequence

$$\mathcal{O}_X \otimes_{\mathbb{Z}} f^\times \mathcal{M}_Y^{\mathrm{grp}} \to \mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{\mathrm{grp}} \to \mathcal{O}_X \otimes_{\mathbb{Z}} \overline{\mathcal{M}}_f^{\mathrm{grp}} \to 0$$

which is also exact on the left if $\overline{\mathcal{M}}_f^{\mathrm{grp}}$ is torsion-free. This is the case, in particular, if X is torsion-free.

5.1.3 Definition

We set $\overline{\Lambda}_f := \mathcal{O}_X \otimes_{\mathbb{Z}} \overline{\mathcal{M}}_f^{\mathrm{grp}} \cong (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{\mathrm{grp}}) / \operatorname{Im}(\mathcal{O}_X \otimes_{\mathbb{Z}} f^\times \mathcal{M}_Y^{\mathrm{grp}})$.

By the first construction of log Kähler differentials, we have

$$\begin{aligned} \Upsilon_f \cong \Omega^1_h &= (\underbrace{\Omega^1_{\underline{h}}}_{=0} \oplus \Lambda_X)/(\mathcal{K}_X + \mathcal{K}_h) \\ &= \Lambda_X / (\langle \alpha(m) \otimes m \rangle + \mathcal{O}_X \otimes_{\mathbb{Z}} f^\times \mathcal{M}_Y) \\ &= \overline{\Lambda}_f / \langle [\alpha(m) \otimes m] \mid m \text{ a local section of } \mathcal{M}_X \rangle , \end{aligned}$$

so Υ_f is a natural quotient of $\overline{\Lambda}_f$.

5.1.2 The normalisation residue map for SNC varieties

Let $f \colon X \to \operatorname{Spec} \mathbb{C}$ be an SNC log variety. We denote by $\underline{\nu} \colon \underline{X}^\nu \to \underline{X}$ the normalisation of the underlying variety. Then, denoting by D^ν the preimage of the double locus, $\underline{X}^\nu$ carries naturally the compactifying log structure $\jmath^\nu$ associated to the open immersion $\jmath^\nu \colon \underline{X}^\nu \setminus D^\nu \to \underline{X}^\nu$, which makes it a log smooth scheme over the trivial log point $\operatorname{Spec} \mathbb{C}^\iota$. This log structure, unfortunately, neither makes $\underline{\nu}$ a morphism of log schemes nor $(\underline{X}^\nu, \jmath^\nu)$ a log smooth scheme over the standard log point $\operatorname{Spec} \mathbb{C}$.

The solution of this problem is to endow $\underline{X}^\nu$ with the pullback log structure $\alpha_{X^\nu} := \underline{\nu}^\times \alpha_X$. This turns the normalisation $\underline{\nu}$ into a strict morphism of log schemes $\nu \colon X^\nu \to X$.

5.1.4 Remark

This is basically the phenomenon described in section 3.6.2. In fact, by a local calculation, one shows that X^ν then is isomorphic to the pullback $(\underline{X}^\nu)^{\mathbb{C}} := (\underline{X}^\nu, \jmath^\nu) \times_{\operatorname{Spec} \mathbb{C}^\iota} \operatorname{Spec} \mathbb{C}$ of the scheme with compactifying log structure $(\underline{X}^\nu, \jmath^\nu)$ to the standard log point $\operatorname{Spec} \mathbb{C}$: The log structure sheaf of this pullback is $\mathcal{M}_{(\underline{X}^\nu)^{\mathbb{C}}} = (\mathbb{N}_0 \oplus \mathbb{C}^\times) \otimes_{\mathbb{C}^\times} \mathcal{M}_{\jmath^\nu} = \mathbb{N}_0 \oplus \mathcal{M}_{\jmath^\nu}$ with the log structure $\beta \colon \mathbb{N}_0 \oplus \mathcal{M}_{\jmath^\nu} \to \mathcal{O}_{X^\nu}$, mapping $(n, m) \mapsto \jmath^\nu(m)$ if $n = 0$ and to 0 if $n \geq 1$.

Locally at a point x in the k-th component $X_{[k]}$ of $\underline{X}^\nu$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{X^\nu} = \nu^\times \mathcal{M}_X = \mathbb{N}_0^r \circledS \mathcal{O}_{X^\nu}^\times & \longrightarrow & \mathbb{N}_0 \oplus \mathbb{N}_0^{r-1} \circledS \mathcal{O}_{X^\nu}^\times = \mathcal{M}_{(\underline{X}^\nu)^{\mathbb{C}}} \\ \downarrow{\scriptstyle \alpha_{X^\nu}} & & \downarrow{\scriptstyle \beta} \\ \mathcal{O}_{X^\nu} & = & \mathcal{O}_{X^\nu}, \end{array}$$

where the upper horizontal map is induced by the monoid isomorphism $\mathbb{N}_0^r \to \mathbb{N}_0 \oplus \mathbb{N}_0^{r-1}$, $e_j \mapsto (0, e_j)$ for $j \neq k$ and $e_k \mapsto (1, 0)$. This defines the isomorphism $X^\nu \to (\underline{X}^\nu)^{\mathbb{C}}$.

The cokernel Υ_X of $\mathcal{O}_X \to \nu_* \mathcal{O}_{X^\nu}$ will be called the *sheaf of normalisation residues* and the map π in the short exact sequence

$$0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_{X^\nu} \xrightarrow{\pi} \Upsilon_X \to 0$$

the *normalisation residue map*. Since the map $\mathcal{O}_X \to \nu_* \mathcal{O}_{X^\nu}$ is an isomorphism precisely at every normal point of X, the support of Υ_X is the double locus D.

Applying the functor $\mathcal{H}om_{\mathcal{O}_X}(\,\cdot\,, \mathcal{O}_X)$ to this sequence yields the long exact sequence

$$\begin{aligned} 0 &\to \mathcal{H}om_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \\ &\to \mathcal{E}xt^1_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) \to \mathcal{E}xt^1_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \to \dots . \end{aligned}$$

The following lemma investigates the terms in this sequence. Its calculations proceed alongside those made in [11, 2.8–2.10].

5.1.5 Lemma

We have

a) $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_{X^\nu}, \mathcal{O}_X) = 0$ for all $p \geq 1$ and

b) $\mathcal{E}xt^p_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) = 0$ for all $p \neq 1$.

Proof: Let x be a closed point of X in which $r(x)$ components meet. Denoting $R := \mathcal{O}_{X,x}$ (observe that this local ring depends on the chosen topology), we have

$$\widehat{R} := \widehat{\mathcal{O}}_{X,x} \cong \mathbb{C}[\![z_1, \dots, z_{n+1}]\!]/(z_1 \cdot \ldots \cdot z_r),$$

where $r = r(x) = \ell_X(x) = \ell_{f_0}(x) + 1$, as well as

$$\begin{aligned} \widehat{R}^\nu &:= (\nu_*\mathcal{O}_{X^\nu})^{\widehat{}}_x \cong \prod_{j=1}^r \widehat{R}/(z_j) \cdot 1_j = \frac{\bigoplus_{j=1}^r \widehat{R} \cdot 1_j}{\langle z_j \cdot 1_j \rangle} \text{ and} \\ \widehat{Y} &:= \widehat{\Upsilon}_{X,x} \cong \widehat{R}^\nu / \widehat{R} \cdot (1_1 + \ldots + 1_r) = \frac{\bigoplus_{j=1}^r \widehat{R} \cdot 1_j}{\langle z_j \cdot 1_j, 1_1 + \ldots + 1_r \rangle} \end{aligned}$$

as $\widehat{R}$-modules. The two maps in the exact sequence $0 \to \widehat{R} \to \widehat{R}^\nu \to \widehat{Y} \to 0$ are given by the diagonal $s \mapsto \sum_{j=1}^r s \cdot 1_j$ (with $z_j \cdot 1_j = 0$) and by the natural projection, respectively.

A free resolution of $\widehat{R}^\nu = (\nu_*\mathcal{O}_{X^\nu})\widehat{_x}$ is

$$P_1^\bullet \twoheadrightarrow \widehat{R}^\nu\colon \ \ldots \to \widehat{R}^{\oplus r} \xrightarrow[e_j \mapsto z_j e_j]{} \widehat{R}^{\oplus r} \xrightarrow[e_j \mapsto z_1\cdot\ldots\cdot\hat{z_j}\cdot\ldots\cdot z_r e_j]{} \widehat{R}^{\oplus r} \xrightarrow[e_j \mapsto z_j e_j]{} \widehat{R}^{\oplus r} \xrightarrow[e_j \mapsto 1_j]{} \widehat{R}^\nu.$$

Applying the functor $\mathrm{Hom}_{\widehat{R}}(\,\cdot\,, \widehat{R})$ to this sequence yields a sequence which is exact everywhere except at its zeroth entry. So $\mathrm{Ext}^i_{\widehat{R}}(\widehat{R}^\nu, \widehat{R}) = H^i(\mathrm{Hom}_{\widehat{R}}(P^\bullet, \widehat{R})) = 0$ for $i \geq 1$. Therefore, $\mathcal{E}xt^i_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) = 0$ for $i \geq 1$.

A free resolution of $\widehat{Y} = \widehat{\Upsilon}_{X,x}$ is

$$P_2^\bullet \twoheadrightarrow \widehat{Y}\colon \ \ldots \to \widehat{R}^{\oplus r} \xrightarrow[e_j \mapsto z_j e_j]{} \widehat{R}^{\oplus r} \xrightarrow[e_j \mapsto z_1\cdot\ldots\cdot\hat{z_j}\cdot\ldots\cdot z_r e_j]{} \widehat{R}^{\oplus r+1} \xrightarrow[\substack{e_j \mapsto z_j e_j \\ e_{r+1} \mapsto e_1+\ldots+e_r}]{} \widehat{R}^{\oplus r} \xrightarrow[e_j \mapsto 1_j]{} \widehat{Y}.$$

The functor $\mathrm{Hom}_{\widehat{R}}(\,\cdot\,, \widehat{R})$ applied to this resolution yields again a sequence which is exact at every entry of order higher than one. It is also exact at its zeroth entry: Here we have the zeroth map $\mathrm{Hom}_{\widehat{R}}(\widehat{R}^{\oplus r}, \widehat{R}) \to \mathrm{Hom}_{\widehat{R}}(\widehat{R}^{\oplus r+1}, \widehat{R}),\ e_j^\vee \mapsto z_j e_j^\vee + e_{r+1}^\vee$, which is injective. So $\mathrm{Ext}^i(\widehat{Y}, \widehat{R}) = H^i(\mathrm{Hom}_{\widehat{R}}(P^\bullet, \widehat{R})) = 0$ for $i \neq 1$ and, hence, $\mathcal{E}xt^i_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) = 0$ for $i \neq 1$. □

5.1.6 Remark

One may deduce 5.1.5 b) also partly from a) and the long exact sequence

$$\ldots \to \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{E}xt^p_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) \to \mathcal{E}xt^p_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) \to \ldots,$$

taking into account that $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_{X^\nu}, \mathcal{O}_X) = 0$ and $\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = 0$ for all $p \geq 1$. This then implies that $\mathcal{E}xt^p_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) = 0$ for $p \geq 2$. In this approach one needs to shows that $\mathcal{H}om_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) = 0$, separately.

5.1.7 Remark

In fact, we may calculate for $P_1^\bullet$:

$$\begin{aligned} H^0(\mathrm{Hom}_{\widehat{R}}(P_1^\bullet, \widehat{R})) &= \mathrm{Ker}(\mathrm{Hom}_{\widehat{R}}(\widehat{R}^{\oplus r}, \widehat{R}) \to \mathrm{Hom}_{\widehat{R}}(\widehat{R}^{\oplus r}, \widehat{R}),\ e_j^\vee \mapsto z_j e_j^\vee) \\ &= \langle z_1 \cdot \ldots \cdot \hat{z_j} \cdot \ldots \cdot z_r e_j^\vee \rangle \cong \widehat{I}_{D\subset X,x}. \end{aligned}$$

In $P_2^\bullet$, the first map $\mathrm{Hom}_{\widehat{R}}(\widehat{R}^{\oplus r+1}, \widehat{R}) \to \mathrm{Hom}_{\widehat{R}}(\widehat{R}^{\oplus r}, \widehat{R})$, mapping the generators $e_j^\vee \mapsto z_1 \cdot \ldots \cdot \hat{z_j} \cdot \ldots \cdot z_r e_j^\vee$ and $e_{r+1}^\vee \mapsto 0$, has the kernel $\langle z_j e_j^\vee, e_{r+1}^\vee \rangle$. The zeroth map has the image $\langle z_j e_j^\vee + e_{r+1}^\vee \rangle$, hence,

$$H^1(\mathrm{Hom}_{\widehat{R}}(P_2^\bullet, \widehat{R})) = [e_{r+1}^\vee] \cdot \widehat{R} \cong \widehat{R}/\widehat{I}_{D\subset X,x} = \widehat{\mathcal{O}}_{D,x}$$

(observe that $z_1 \cdot \ldots \cdot \hat{z_j} \cdot \ldots \cdot z_r [e_{r+1}^\vee] = 0$ for all j).

This is, of course, *not* sufficient to conclude that $\mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) \cong I_{D\subset X}$ and that $\mathcal{E}xt^1_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) \cong \mathcal{O}_D$. We will, nevertheless, see soon that this is in fact true.

5.1.8 Corollary

We have

a) $\mathrm{Ext}^p_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) = H^p(\mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X))$ for all p,

b) $\mathrm{Hom}_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) = 0$ and

c) $\mathrm{Ext}^p_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) = H^{p-1}(\mathcal{E}xt^1_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X))$ for all $p \geq 1$.

Proof: Both claims are implied by lemma 5.1.5 and the local-to-global spectral sequence $H^q(\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. □

We return to the long exact sequence

$$\begin{aligned} 0 &\to \mathcal{H}om_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \\ &\to \mathcal{E}xt^1_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) \to \mathcal{E}xt^1_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \to \dots . \end{aligned}$$

Having shown that $\mathcal{H}om_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) = 0$ and $\mathcal{E}xt^1_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) = 0$ and observing that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X$ and $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = 0$, we get a short exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) \to \mathcal{O}_X \to \mathcal{E}xt^1_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) \to 0.$$

The ideal sheaf $I_\nu := \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X)$ is called the *conductor* (or *conductor ideal sheaf*) of the normalisation $\nu\colon X^\nu \to X$ (cf. [4, 1.1 & 1.8], [23, 4.1–4.5]). It is a radical ideal which is the largest ideal sheaf on X that is also an ideal sheaf on X^ν (i. e. such that $\nu^{-1}I_\nu$ is an ideal sheaf on X^ν with $\nu_*(\nu^{-1}I_\nu) = I_\nu$). The so-called conductor subschemes $\underline{\mathrm{Spec}}_{\mathcal{O}_X}\mathcal{O}_X/I_\nu$ of X and $\underline{\mathrm{Spec}}_{\mathcal{O}_{X^\nu}}\mathcal{O}_{X^\nu}/\nu^{-1}I_\nu$ of X^ν are equal to the double locus D and its preimage D^ν, respectively.

Therefore, $I_\nu = I_D$ and consequently $\mathcal{E}xt^1_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) = i_*\mathcal{O}_D$, where $i\colon D \to X$ is the inclusion, which allows us to conclude the following:

5.1.9 Proposition

$$\mathrm{Ext}^p_{\mathcal{O}_X}(\Upsilon_X, \mathcal{O}_X) = \begin{cases} 0 & \text{if } p = 0, \\ H^{p-1}(\mathcal{O}_D) & \text{if } p \geq 1. \end{cases}$$

The sheaves of Poincaré residues and normalisation residues for SNC log varieties

Let $f\colon X \to \mathrm{Spec}\,\mathbb{C}$ still denote an SNC log variety. The connection between the sheaf of Poincaré residues Υ_f and the sheaf of normalisation residues Υ_X is given by the following statement:

5.1.10 Proposition

There is a canonical isomorphism $\Upsilon_f \cong \Upsilon_X$.

Proof: We define a morphism $\lambda\colon \Lambda_X \to \nu_*\mathcal{O}_{X^\nu}$ by the composition $\Lambda_X = \mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{\mathrm{grp}} \to \Lambda_f = \mathcal{O}_X \otimes_{\mathbb{Z}} \overline{\mathcal{M}}_X^{\mathrm{grp}} \cong \mathcal{O}_X \otimes_{\mathbb{Z}} \nu_*\mathbb{Z}_{X^\nu} \to \nu_*\mathcal{O}_{X^\nu}$, where the last arrow sends $s \otimes m$ to sm. First, we claim that λ descends to a well-defined morphism $\gamma\colon \Omega^1_f \to \Upsilon_X$ such that

$$\begin{tikzcd}
\Lambda_X \arrow[r, "\lambda"] \arrow[d, two heads] & \nu_*\mathcal{O}_{X^\nu} \arrow[d, two heads, "\pi"] \\
\Omega^1_f \arrow[r, "\gamma"] & \Upsilon_X
\end{tikzcd}$$

commutes. To this end, we have to show that any local section of the two $\mathcal{O}_X$-submodules $\mathcal{R}_f$ and $\mathcal{R}_X$ (cf. chapter 1, proposition 1.2.23) of Λ_X is mapped to zero under $\pi \circ \lambda\colon \Lambda_X \to \Upsilon_X$.

Firstly, let $s \otimes (m, c)$ be a local section of $\mathcal{O}_X \otimes_{\mathbb{Z}} f^{-1}\mathcal{M}_{\mathbb{C}}^{\mathrm{grp}} = \mathcal{O}_X \otimes_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{C}^\times)$. This section is mapped to $\pi(sm)$ which is zero, because sm lies in the image of $\mathcal{O}_X = \mathcal{O}_X \otimes_{\mathbb{Z}} \mathbb{Z}$ in $\nu_*\mathcal{O}_{X^\nu}$.

Secondly, let m be a local section of $\mathcal{M}_X$ and regard the local section $\alpha_X(m) \otimes m$ of Λ_X. Let x be a point in X and let $r = r(x)$ be the number of components meeting in x. The completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ at x (with respect to the given topology, i. e. Zariski or étale topology) is isomorphic to $\widehat{R} := \mathbb{C}[\![z_1, \dots, z_{n+1}]\!]/(z_1 \cdot \ldots \cdot z_r)$ and the log structure given by the chart $a_x\colon \mathbb{N}_0^r \to \widehat{R}$, $\underline{n} \mapsto \underline{z}^{\underline{n}}$. The image of m at x is an element $m_x = (n_1, \dots, n_r; u) \in \mathbb{N}_0^r \oplus \widehat{R}^\times$. So $(\alpha_X(m) \otimes m)_x = \alpha_{X,x}(m_x) \otimes m_x$ is mapped by λ to the tuple $(n_k\, \underline{z}^{\underline{n}})_{k=1}^r \in \bigoplus_{k=1}^r \widehat{R}/(z_k)$ which is zero, because for each k either $n_k = 0$ or $z_k \,|\, \underline{z}^{\underline{n}}$. Hence, $\pi(\lambda(\alpha_X(m) \otimes m)) = 0$ in Υ_X and, in particular, $\mathcal{R}_X$ is mapped to zero.

Our second claim is that $\gamma\colon \Omega^1_f \to \Upsilon_X$ descends further to an isomorphism $\Upsilon_f \to \Upsilon_X$. To verify this, we first have to show that the image of $\Omega^1_{\underline{f}}$ in Ω^1_f is mapped to zero by λ. As one sees in the commutative diagram

$$\begin{tikzcd}
\Lambda_{\underline{X}^\iota} \arrow[r] \arrow[dd, two heads] & \Lambda_X \arrow[r, two heads] \arrow[dd, two heads] \arrow[dr, "\lambda"'] & \overline{\Lambda}_f \arrow[dd, two heads] \arrow[d] & \\
 & & \nu_*\mathcal{O}_{X^\nu} \arrow[dd, two heads, "\pi"] & \\
\Omega^1_{\underline{f}} \arrow[r] & \Omega^1_f \arrow[r, two heads] \arrow[dr, "\gamma"'] & \Upsilon_f \arrow[d, dotted] & \\
 & & \Upsilon_X, &
\end{tikzcd}$$

the submodule $\operatorname{Im} \Omega^1_{\underline{f}} \subset \Omega^1_f$ is equal to the image of $\Lambda_{\underline{X}^\iota} \to \Lambda_X \to \Omega^1_f$. But $\Lambda_{\underline{X}^\iota}$ is mapped to zero already in $\overline{\Lambda}_f = \mathcal{O}_X \otimes_{\mathbb{Z}} \overline{\mathcal{M}}_X^{\mathrm{grp}}$ along the first row. The verification that $\Upsilon_f \to \Upsilon_X$ is an isomorphism may be done on the completion of the stalks at a point x in X. There, the morphism is given by

$$\begin{aligned}
&\widehat{R}\,\langle d\log z_1, \dots, d\log z_r \rangle \,/\, \langle z_1 d\log z_1, \dots, z_r d\log z_r, d\log z_1 + \ldots + d\log z_r \rangle \\
&\to \widehat{R}\,\langle 1_1, \dots, 1_r \rangle \,/\, \langle z_1 1_1, \dots, z_r 1_r, 1_1 + \ldots + 1_r \rangle\,, \\
&d\log z_j \mapsto 1_j
\end{aligned}$$

which is clearly an isomorphism. □

5.1.11 Corollary

$$\operatorname{Ext}^p(\Upsilon_f, \mathcal{O}_X) = \begin{cases} 0 & \text{if } p = 0, \\ H^{p-1}(\mathcal{O}_D) & \text{if } p \geq 1. \end{cases}$$

5.2 The log T1 lifting principle

In what follows, we fix notations for certain objects and morphisms in $\underline{\mathrm{LArt}}_{\mathcal{T}}$ which play a role in the so-called T1-lifting principle. The names of the rings and homomorphisms correspond by and large to those used by Fantechi and Manetti in [9]. Recall that $T = \mathbb{C}[\![t]\!]$ and that $\mathcal{T}$ is the log ring $\mathcal{T}\colon M_{\mathcal{T}} \to T$, where $M_{\mathcal{T}} = \mathbb{N}_0 \oplus T^\times$, associated to the prelog ring $t\colon \mathbb{N}_0 \to T$, $n \mapsto t^n$.

Let $A := \mathbb{C}[\![x]\!]$ and let $\varphi_A\colon T \to A$ be a T-algebra structure on A (the letter A is used differently in [9]) such that it induces the identity on the residue fields $T/(t) \to A/(x)$. We let $\mathcal{A}\colon M_{\mathcal{A}} \to A$ denote the log ring associated to the prelog ring $a\colon M_{\mathcal{T}} \to A$ defined by $a = \varphi_A \circ \mathcal{T}$ which is given by $\mathcal{A}\colon \mathbb{N}_0 \oplus A^\times \to A$, $(n, u) \mapsto u\varphi_A(t)^n$.

Let $B := \mathbb{C}[\![x, y]\!]$ and $f\colon A \to B$, $x \mapsto x + y$. We give B the log ring structure $\mathcal{B} \in \widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}}$ associated to the prelog ring $b\colon M_{\mathcal{T}} \to B$ defined by $b = f \circ \varphi_A \circ \mathcal{T}$. We denote by $g\colon \mathcal{B} \to \mathcal{A}$ the natural projection.

For every $n \in \mathbb{N}_0$, we let $\mathcal{A}_n := \mathcal{A}/(x^{n+1})$ and $\mathcal{B}_n := \mathcal{B}/(x^{n+1}, y^2)$ with the (quotient ring) log structure induced by the natural quotient homomorphisms $i\colon A \to A_n$ and $j\colon B \to B_n$, respectively. That way, i and j become homomorphisms in $\widehat{\underline{\mathrm{LArt}}}_{\mathcal{T}}$. For any $m \in \mathbb{N}_0$, we denote by the same letters i and j any natural homomorphism $i\colon \mathcal{A}_m \to \mathcal{A}_n$ and $j\colon \mathcal{B}_m \to \mathcal{B}_n$ induced by i and j, respectively (which might also be the zero homomorphism, e. g. if $m < n$). Moreover, we denote by $f\colon \mathcal{A}_m \to \mathcal{B}_n$ and $g\colon \mathcal{B}_m \to \mathcal{A}_n$ any natural homomorphisms induced by f and g, respectively.

Then the diagram

$$\begin{tikzcd} \mathcal{B}_n \arrow[r, "g"] \arrow[d, "j"] & \mathcal{A}_n \arrow[d, "i"] \\ \mathcal{B}_{n-1} \arrow[r, "g"] & \mathcal{A}_{n-1} \end{tikzcd}$$

is commutative.

For every $n \in \mathbb{N}_0$, we define $\mathcal{C}_n := \mathcal{B}_{n-1} \times_{\mathcal{A}_{n-1}} \mathcal{A}_n$. The underlying ring of this log ring is $C_n = \mathbb{C}[x, y]/(x^{n+1}, x^n y, y^2)$. It comes with two natural homomorphisms: $j'\colon \mathcal{B}_n \to \mathcal{C}_n$, which is equal to the natural projection $\mathcal{B}_n \to \mathcal{B}_n/(x^n y) = \mathcal{C}_n$, and $f'\colon \mathcal{A}_n \to \mathcal{C}_n$, which is induced by $\mathcal{A} \xrightarrow{f} \mathcal{B}_n \xrightarrow{g} \mathcal{C}_n$ and given by $x \mapsto x + y$.

We denote the natural homomorphism $\mathcal{C}_n \to \mathcal{C}_{n-1}$ by $\overline{j}$ as it is induced by j. Moreover, by definition, we have two natural projections $g' \colon \mathcal{C}_n \to \mathcal{A}_n$, with $g = g' \circ j'$, and $j'' \colon \mathcal{C}_n \to \mathcal{B}_{n-1}$, with $j = j'' \circ j'$ and $\overline{j} = j' \circ j''$.

From this definition of $\mathcal{C}_n$ it is clear that the diagram

$$\begin{tikzcd} \mathcal{A}_{n+1} \arrow[r, "i"] \arrow[d, "f"] & \mathcal{A}_n \arrow[d, "f'"] \\ \mathcal{B}_n \arrow[r, "j'"] & \mathcal{C}_n \end{tikzcd}$$

is Cartesian.

Observe that all objects and homomorphisms are fixed as soon as the T-algebra structure φ_A on $A = \mathbb{C}[\![x]\!]$ is chosen.

Recall that a log gdt functor was defined to be a functor of log Artin rings $F \colon \underline{\mathrm{LArt}_{\mathcal{T}}} \to \underline{\mathrm{Set}}$ satisfying the "logarithmic Schlessinger properties" LH_1 and LH_2 in 2.2.15.

5.2.1 Definition

We say that a log gdt functor $F \colon \underline{\mathrm{LArt}_{\mathcal{T}}} \to \underline{\mathrm{Set}}$ has the T1-lifting property if for all T-algebra structures $\varphi_A \colon T \to A$ on $A = \mathbb{C}[\![x]\!]$ the natural maps

$$\Phi_0 \colon F(\mathcal{B}_0) \to F(\mathcal{A}_0)$$

and

$$\Phi_n \colon F(\mathcal{B}_n) \to F(\mathcal{B}_{n-1}) \times_{F(\mathcal{A}_{n-1})} F(\mathcal{A}_n)$$

are surjective for all $n \in \mathbb{N}_{\geq 1}$.

5.2.2 Theorem (Ran, Kawamata, Fantechi, Manetti)

Let F be a functor of log Artin rings over $\mathcal{T}$ possessing a hull. If F has the T1-lifting property, then $F \to \operatorname{Spec} \mathbb{C}^{\iota}$ is smooth. In particular, the hull of F is smooth over $\operatorname{Spec} \mathbb{C}^{\iota}$.

Proof: Observe that $\operatorname{char} \mathbb{C} = 0$. Now that we have fixed log ring structures on A_n, B_n and C_n depending only on φ_A, the proof is literally the same as in [9] as soon as we give suitable log ring structures to the rings V_n and A'_n, which appear in that proof.

We do this as follows: We let $V := \mathbb{C}[\![x, s]\!]$ (this ring is denoted by A in Fantechi's and Manetti's proof) and $\mathcal{V}$ be the log ring associated to the prelog ring $v \colon M_{\mathcal{T}} \to V$ with $v = q \circ \varphi_A \circ \mathcal{T}$, where $q \colon A \hookrightarrow V$. Then for all $n \in \mathbb{N}_0$, the log rings $\mathcal{V}_n := \mathcal{V}/(x^{n+1}, x^2 s, s^2)$ and $\mathcal{A}'_n := \mathcal{V}/(x^{n+1}, xs, s^2)$ are defined. Note that for each $n \in \mathbb{N}_0$, there is a log ring homomorphism $\widehat{f} \colon \mathcal{V} \to \mathcal{B}$, defined by $x \mapsto x + y$ and $s \mapsto x^n$, and a log ring homomorphism $\widehat{q} \colon \mathcal{V} \to \mathcal{V}$, defined by $x \mapsto x$ and $s \mapsto x^n$, inducing correspondent morphisms between quotients of $\mathcal{V}$ as described in [9].

Fantechi's and Manetti's proof shows that $F(\mathcal{A}_{n+1}) \to F(\mathcal{A}_n)$ is surjective for all $n \geq 2$. Since F is rigid, we have $F(\mathcal{A}_n) = \overline{F}(\mathcal{A}_n) = v_* F(A_n)$, so $v_* F(A_{n+1}) \to v_* F(A_n)$ is surjective for all $n \geq 2$. By proposition 2.1.18, $v_* F$ is smooth over $* = \operatorname{Spec} \mathbb{C}$ and, by lemma 2.2.18, F is smooth over $\operatorname{Spec} \mathbb{C}^{\iota}$. □

5.3 The freeness of the obstruction spaces over the $\mathcal{A}_n$

Let φ_A be any T-algebra structure on $A = \mathbb{C}[\![x]\!]$.

Then φ_A is given either by $t \mapsto 0$ or by $t \mapsto ux^N$ for some unit $u \in A = \mathbb{C}[\![x]\!]$. In modifying A by replacing x with $u^{-1}x$, we may assume that φ_A is given by $t \mapsto 0$ or by $t \mapsto x^N$ without loss of generality. We will refer to these two cases as the 0-case and the x^N-case in what follows.

Let $(f_0, \nabla, \omega)\colon X \to \operatorname{Spec} \mathcal{C}$ be a log symplectic SNC log variety. We have calculated the tangent spaces and the obstruction spaces of the functors Def_{f_0}, $\mathrm{Def}_{(f_0,\omega)}$, $\mathrm{Def}_{(f_0,L)}$, $\mathrm{Def}_{(f_0,\nabla)}$ and $\mathrm{Def}_{(f_0,\nabla,\omega)}$ in chapter 4. In this section, we will show that for liftings $(f_k, \Delta_k, \varpi_k)$ over the $\mathcal{T}$-algebra $\mathcal{A}_k$ these spaces are free $\mathcal{A}_k$-modules. This fact will be a key stone in the proofs of our main results in the next section.

As a first step, we look at the spaces of the form $\mathbb{H}^p(\Omega_f^{\geq q,\bullet} \otimes \mathcal{L}[q])$ and $H^p(\Omega_f^q \otimes \mathcal{L})$:

5.3.1 Proposition

Let $f_0\colon (X, \nabla) \to \operatorname{Spec} \mathcal{C}$ be a SNC log variety with log Cartier connection and, for any $k \in \mathbb{N}_0$, let $f_k\colon (\mathcal{X}_k, \Delta_k) \to \operatorname{Spec} \mathcal{A}_k$ be a log smooth lifting of (f_0, ∇) over the log ring $\mathcal{A}_k$ with underlying ring A_k.

Then $\mathbb{H}^p(\Omega_{f_k}^{\geq q,\bullet} \otimes \mathcal{L}_k[q])$ is isomorphic to $\mathbb{H}^p(\Omega_{f_0}^{\geq q,\bullet} \otimes L[q]) \otimes_{\mathbb{C}} A_k$ as an A_k-module. In particular, $\mathbb{H}^p(\Omega_{f_k}^{\geq q,\bullet} \otimes \mathcal{L}_k[q])$ is a free A_k-module for all p, q.

Here, L and $\mathcal{L}_k$ denote the line bundles of ∇ and Δ_k, respectively.

By choosing $\nabla = d$ and $\Delta_k = d$, we get the following corollary as a special case:

5.3.2 Corollary

Let $f_0\colon X \to \operatorname{Spec} \mathcal{C}$ be a SNC log variety and, for any $k \in \mathbb{N}_0$, let $f_k\colon \mathcal{X}_k \to \operatorname{Spec} \mathcal{A}_k$ be a log smooth lifting of f_0 over the log ring $\mathcal{A}_k$ with underlying ring A_k.

Then $\mathbb{H}^p(\Omega_{f_k}^{\geq q,\bullet}[q])$ is isomorphic to $\mathbb{H}^p(\Omega_{f_0}^{\geq q,\bullet}[q]) \otimes_{\mathbb{C}} A_k$ as an A_k-module, for all $q, k \in \mathbb{N}_0$. In particular, $\mathbb{H}^p(\Omega_{f_k}^{\geq q,\bullet}[q])$ is a free A_k-module for all p, q.

Proof of the proposition

For a start, we regard the x^N-case. In the situation of the proposition we have a commuting diagram

$$\begin{tikzcd}
(X, \omega) \arrow[r, hook, "i"] \arrow[d, "f_0"] & (\mathcal{X}_k, \varpi_k) \arrow[d, "f_k"] \\
\operatorname{Spec} \mathcal{C} \arrow[r, hook] \arrow[dr] & \operatorname{Spec} \mathcal{A}_k \arrow[d] \\
& \operatorname{Spec} \mathbb{C}^\iota,
\end{tikzcd}$$

where $\mathbb{C}^\iota$ denotes the log ring with trivial log structure on $\mathbb{C}$.

Following [33, §2], we regard the complex $\Omega^\bullet_{f_k} \otimes \mathcal{L}_k$. By assumption, Δ_k is log Cartier, i. e. it is of the form $M_{f_k}(D_k)$ for some $D_k \in \mathrm{LCar}(\mathcal{X}_k)$. Hence, by lemma 3.4.16, there exists an "absolute version" of it, namely $\tilde{\Delta}_k := M_{\mathcal{X}_k/\mathbb{C}^\iota}(D_k)$. Associated to $\tilde{\Delta}_k$, we have its log de Rham complex

$$0 \to \mathcal{L}_k \xrightarrow{\tilde{\Delta}_k} \Omega^1_{\mathcal{X}_k/\mathbb{C}^\iota} \otimes \mathcal{L}_k \xrightarrow{\tilde{\Delta}_k} \Omega^2_{\mathcal{X}_k/\mathbb{C}^\iota} \otimes \mathcal{L}_k \xrightarrow{\tilde{\Delta}_k} \Omega^3_{\mathcal{X}_k/\mathbb{C}^\iota} \otimes \mathcal{L}_k \xrightarrow{\tilde{\Delta}_k} \dots .$$

We define the complex

$$\mathbb{L}^\bullet = i^{-1}\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota} \otimes \mathcal{L}_k[q][\xi] = i^{-1}\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota} \otimes \mathcal{L}_k[q] \otimes_{\mathbb{C}} \mathbb{C}[\xi],$$

where ξ is a formal (polynomial) variable (which should be thought of as $\xi = \log x$), with differential $d\colon \mathbb{L}^p \to \mathbb{L}^{p+1}$ given by

$$\tilde{\Delta}\colon \mathbb{L}^p \to \mathbb{L}^{p+1},\ \sum_{s=0}^{d} \sigma_s \cdot \xi^s \mapsto \tilde{\Delta}(\sigma) = \sum_{s=0}^{d} \tilde{\Delta}_k \sigma_s \cdot \xi^s + \sum_{s=1}^{d} s\sigma_s \wedge d\log x \cdot \xi^{s-1}.$$

We also define the morphism of complexes

$$\psi\colon \mathbb{L}^\bullet \to \Omega^{\geq q,\bullet}_{f_0} \otimes L[q]$$

by the composition of morphisms of complexes $\mathbb{L}^\bullet \to i^{-1}\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q] \to i^*(\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k)[q] = \Omega^{\geq q,\bullet}_{f_0} \otimes L[q]$, where the first map is the well-defined projection to the class of the zeroth coefficient. (Observe that, by identifying the topological spaces of X and $\mathcal{X}_k$, we have $i^{-1} = id$.)

5.3.3 Remark

In the special case $\nabla = d$ and $\Delta_k = d$, we have $\tilde{\Delta}_k = d$. If, moreover, $q = 0$, i. e. if we regard the ordinary log de-Rham-complexes of the f_k and $\mathcal{X}_k/\mathbb{C}^\iota$, then the complex $\mathbb{L}^\bullet$ is equal to the complex $L^\bullet_\alpha$ for $\alpha = 0$ as defined by Steenbrink in [33, 2.6].

We claim that ψ is almost a quasi-isomorphism, meaning that ψ induces isomorphisms on cohomology in all strictly positive degrees and a surjection in degree 0. We prove this on the completion of its stalks.

Let x be a closed point of X. Regard the complex $(i^{-1}\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota} \otimes \mathcal{L}_k[q])\widehat{_x} = (\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota} \otimes \mathcal{L}_k[q])\widehat{_x}$ and write $Z^{(q),\bullet} := Z^\bullet((\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota} \otimes \mathcal{L}_k[q])\widehat{_x})$ for its subcomplex of $\tilde{\Delta}^\bullet_k$-cocycles (with trivial differential $\tilde{\Delta}^\bullet_k|_{Z^{(q),\bullet}} = 0$). Obviously,

$$Z^{(q),p} = \begin{cases} Z^{(0),p+q} = Z^{p+q}((\Omega^\bullet_{\mathcal{X}_k/\mathbb{C}^\iota} \otimes \mathcal{L}_k)\widehat{_x}) & \text{if } p \geq 0, \\ 0 & \text{if } p < 0 \end{cases}$$

for all q.

For each $r \geq 0$ choose a $\mathbb{C}$-linear section

$$s^{(r)}\colon H^1((\Omega^{\geq r,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota} \otimes \mathcal{L}_k[r])\widehat{_x}) \to Z^{(r),1}.$$

The entirety of all these sections canonically defines a section of complexes with trivial differentials

$$s^{(q),\bullet}\colon H^\bullet((\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota}\otimes\mathcal{L}_k[q])\widehat{_x})\to Z^{(q),\bullet}$$

for each q, where

$$\begin{aligned}&s^{(q),0}:=id\colon H^0((\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota}\otimes\mathcal{L}_k[q])\widehat{_x})\to Z^{(q),0}\text{ and}\\&s^{(q),p}:=s^{(q+p-1)}\text{ for }p\geq 1,\end{aligned}$$

which makes sense, because

$$\begin{aligned}&H^p((\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota}\otimes\mathcal{L}_k[q])\widehat{_x})\\&\quad=\begin{cases}H^1((\Omega^{\geq q+p-1,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota}\otimes\mathcal{L}_k[q+p-1])\widehat{_x})=H^{p+q}((\Omega^{\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota}\otimes\mathcal{L}_k)\widehat{_x}) & \text{if } p\geq 1,\\ Z^{(q),0}=Z^{p+q}((\Omega^{\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota}\otimes\mathcal{L}_k)\widehat{_x}) & \text{if } p=0\end{cases}.\end{aligned}$$

Finally, $s^{(q),\bullet}$ defines a subcomplex with trivial differential

$$H^{(q),\bullet}:=\operatorname{Im}(s^{(q),\bullet})\subset Z^{(q),\bullet}\subset(\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota}\otimes\mathcal{L}_k[q])\widehat{_x}$$

of representatives of $H^\bullet((\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota}\otimes\mathcal{L}_k[q])\widehat{_x})$ for each q. By definition, the inclusion $H^{(q),\bullet}\subset(\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota}\otimes\mathcal{L}_k[q])\widehat{_x}$ is then a quasi-isomorphism.

The way we defined the sections $s^{(q),p}$ implies the following:

5.3.4 Lemma

For $p,q\geq 0$ the natural map

$$H^{(q+1),p}\to H^{(q),p+1}$$

is surjective for $p=0$ and is the identity for $p\geq 1$.

Moreover, define the subcomplex

$$H^{(q),\bullet}[\xi]=H^{(q),\bullet}\otimes_{\mathbb{C}}\mathbb{C}[\xi]\subset(\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota}\otimes\mathcal{L}_k[q])\widehat{_x}[\xi]=\widehat{L}^\bullet_x.$$

By [33, 2.12 & 2.14], this inclusion $H^{(q),\bullet}[\xi]\subset\widehat{L}^\bullet_x$ is a quasi-isomorphism, too.

The natural inclusion $H^{(q),\bullet}\subset H^{(q),\bullet}[\xi]$ is not a quasi-isomorphism. However, we have the following lemma.

5.3.5 Lemma (cp. [33, 2.15])

For $p\geq 1$ the natural inclusion $H^{(q),\bullet}\subset H^{(q),\bullet}[\xi]$ induces surjections

$$H^{(q),p}\to H^p(H^{(q),\bullet}[\xi])$$

each with a kernel isomorphic to $H^{(q-1),p}$.

Proof: Let $p\geq 1$. Following the argument of Steenbrink, any $[\sigma]\in H^p(H^{(q),\bullet}[\xi])$ has a representative $\sigma=\sum_{s=0}^d\sigma_s\cdot\xi^s$, with $\sigma_s\in H^{(q),p}=H^{(0),p+q}$. Then $0=\tilde{\Delta}\sigma=$

$\sum_{s=1}^{d} s\sigma_s \wedge d\log x \cdot \xi^{s-1}$ implies that $\sigma_s \wedge d\log x = 0$ and this is, by a lemma of de Rham (cf. [3, p. 8]), equivalent to $\sigma_s = \eta_s \wedge d\log x$ for some $\eta_s \in H^{(q-1),p} = H^{(0),p+q-1}$, $s = 1, \ldots, d$.

By the preceding lemma 5.3.4, there exist $\widehat{\eta}_s \in H^{(q),p-1}$, such that $\widehat{\eta}_s$ is mapped to η_s. If we set $\eta := \sum_{s=1}^{d} \frac{1}{s+1}\widehat{\eta}_s \cdot \xi^{s+1}$, then $\sigma = \sigma_0 + \tilde{\Delta}\eta$, with $\sigma_0 \in H^{(q),p}$, so that $[\sigma] = [\sigma_0]$. Hence, the map $H^{(q),p} \to H^p(H^{(q),\bullet}[\xi])$ is indeed surjective. Its kernel is given by those $\sigma_0 \in H^{(q),p}$ such that $\sigma_0 = \tilde{\Delta}(\zeta_0\xi) = \zeta_0 \wedge d\log x$ for some $\zeta_0 \in H^{(q-1),p}$. $\square$

Since this kernel $H^{(q-1),p}$ is a quotient of $H^{(q),p-1}$ by lemma 5.3.4, we have a commutative diagram

$$\begin{tikzcd}
H^{(q),p-1} \wedge d\log x \arrow[r, "\cong"] \arrow[d] & H^{p-1}((\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota} \otimes \mathcal{L}_k[q])^{\widehat{}}_x) \wedge d\log x \arrow[d] \\
H^{(q),p} \arrow[r, "\cong"] \arrow[d] & H^p((\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota} \otimes \mathcal{L}_k[q])^{\widehat{}}_x) \arrow[d] \\
H^p(H^{(q),\bullet}[\xi]) \cong H^p(\widehat{L}^\bullet_x) \arrow[r, "H^p(\widehat{\psi}_x)"] \arrow[d] & H^p((\Omega^{\geq q,\bullet}_{f_0} \otimes L[q])^{\widehat{}}_x) \arrow[d] \\
0 & 0.
\end{tikzcd}$$

For $p = 0$ the situation is slightly poorer. Define the map $i\colon Z^{(q),0} \to Z^0(H^{(q),\bullet}[\xi])$ by $\sigma \mapsto \sigma + 0\xi + 0\xi^2 + \ldots$. This is well-defined, because $\tilde{\Delta}(\sigma + 0\xi + 0\xi^2 + \ldots) = \tilde{\Delta}_k\sigma = 0$ (and even injective). Hence, we have a commutative diagram

$$\begin{tikzcd}
Z^{(q),0} \arrow[r, "id"] \arrow[d, "i"] & H^0((\Omega^{\geq q,\bullet}_{\mathcal{X}_k/\mathbb{C}^\iota} \otimes \mathcal{L}_k[q])^{\widehat{}}_x) \arrow[d, two heads] \\
Z^0(H^{(q),\bullet}[\xi]) \cong H^0(\widehat{L}^\bullet_x) \arrow[r, "H^0_x(\psi)"] & H^0((\widehat{\Omega}^{\geq q,\bullet}_{f_0} \otimes L[q])^{\widehat{}}_x)
\end{tikzcd}$$

showing that $H^0(\psi)\colon \mathbb{H}^0(\mathbb{L}^\bullet) \to \mathbb{H}^0(\Omega^{\geq q,\bullet}_{f_0} \otimes L[q])$ is at least surjective.

In the 0-case the same proof goes through when replacing $d\log x$ with dx.

From here onward, we follow the arguments in [21, 4.1]:

Since ψ factors, by definition, via $i^{-1}\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q] \to \Omega^{\geq q,\bullet}_{f_0} \otimes L[q]$, we have surjective maps $\mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q]) \to \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_0} \otimes L[q])$ in the long exact sequence associated to the short exact sequence

$$0 \to \Omega^{\geq q,\bullet}_{f_{k-1}} \otimes \mathcal{L}_{k-1}[q] \xrightarrow{\cdot x} \Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q] \xrightarrow{\mathrm{mod}\ x} \Omega^{\geq q,\bullet}_{f_0} \otimes L[q] \to 0,$$

which therefore splits into short exact sequences

$$0 \to \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_{k-1}} \otimes \mathcal{L}_{k-1}[q]) \xrightarrow{\cdot x} \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q]) \xrightarrow{\mathrm{mod}\ x} \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_0} \otimes L[q]) \to 0$$

of $\mathbb{C}$-vector-spaces. Hence,

$$\dim_{\mathbb{C}}(\mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q])) = \dim_{\mathbb{C}}(\mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_{k-1}} \otimes \mathcal{L}_{k-1}[q])) + \dim_{\mathbb{C}}(\mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_0} \otimes L[q])).$$

What we really want to show is that the long exact sequence associated to the short exact sequence

$$0 \to \Omega^{\geq q,\bullet}_{f_0} \otimes L[q] \xrightarrow{\cdot x^k} \Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q] \xrightarrow{\bmod x^k} \Omega^{\geq q,\bullet}_{f_{k-1}} \otimes \mathcal{L}_{k-1}[q] \to 0$$

splits into short exact sequences. To this end, we regard a piece of it:

$$\begin{tikzcd}
\mathbb{H}^{p-1}(\Omega^{\geq q,\bullet}_{f_{k-1}} \otimes \mathcal{L}_{k-1}[q]) \arrow[d, two heads] \arrow[dr] & & & \\
K' \arrow[r, hook] & \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_0} \otimes L[q]) \arrow[d, two heads] \arrow[dr] & & \\
& K \arrow[r, hook] & \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q]) \arrow[d, two heads] \arrow[dr] & \\
& & K'' \arrow[r, hook] & \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_{k-1}} \otimes \mathcal{L}_{k-1}[q]),
\end{tikzcd}$$

where the $K^{('/'')}$ are the kernels and images of the differentials in the sequence.

As $\mathbb{C}$-vector-spaces they satisfy $\dim(\mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q])) = \dim K + \dim K''$ as well as $\dim K \leq \dim \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_0} \otimes L[q])$ and $\dim K'' \leq \dim \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_{k-1}} \otimes \mathcal{L}_{k-1}[q])$. But since we already know that $\dim(\mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q])) = \dim(\mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_{k-1}} \otimes \mathcal{L}_{k-1}[q])) + \dim(\mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_0} \otimes L[q]))$, we must have equalities, so $K \cong \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_0} \otimes L[q])$ and $K'' \cong \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_{k-1}} \otimes \mathcal{L}_{k-1}[q])$ as $\mathbb{C}$-vector-spaces. Hence, K' must be zero and this long exact sequence splits into short exact sequences

$$0 \to \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_0} \otimes L[q]) \xrightarrow{\cdot x^k} \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q]) \xrightarrow{\bmod x^k} \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_{k-1}} \otimes \mathcal{L}_{k-1}[q]) \to 0$$

of $\mathbb{C}$-vector-spaces.

But this means that $\mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q]) \cong x^k \cdot \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_0} \otimes L[q]) + F$, with $F \cong \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_{k-1}} \otimes \mathcal{L}_{k-1}[q])$. By induction on k, $F \cong \bigoplus_{j=0}^{k} x^j \cdot \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_0} \otimes L[q])$, so $\mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q]) \cong A_k \otimes_{\mathbb{C}} \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_0} \otimes L[q])$. This completes the proof of proposition 5.3.1. □

5.3.6 Corollary

Let $f_k\colon (\mathcal{X}_k, \Delta_k) \to \operatorname{Spec} \mathcal{A}_k$ be a log smooth lifting of (f_0, ∇) over the log ring $\mathcal{A}_k$ and let $f_{k-1}\colon (\mathcal{X}_{k-1}, \Delta_{k-1}) \to \operatorname{Spec} \mathcal{A}_{k-1}$ be its restriction to $\mathcal{A}_{k-1}$. Then the canonical maps

$$\mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q]) \to \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_{k-1}} \otimes \mathcal{L}_{k-1}[q])$$

are surjective.

5.3.7 Corollary

Let $f_k\colon (\mathcal{X}_k, \Delta_k) \to \operatorname{Spec} \mathcal{A}_k$ be a log smooth lifting of (f_0, ∇) over the log ring $\mathcal{A}_k$. Then $H^p(\mathcal{X}_k, \Omega^q_{f_k} \otimes \mathcal{L}_k)$ is isomorphic to $H^p(X, \Omega^q_{f_0} \otimes L_k) \otimes_{\mathbb{C}} A_k$ as A_k-module for all k, p and q. In particular, $H^p(\mathcal{X}_k, \Omega^q_{f_k} \otimes \mathcal{L}_k)$ is a free A_k-module for all k, p and q.

Proof: We regard for any q and k the short exact sequence of complexes

$$0 \to \Omega^{\geq q+1,\bullet}_{f_k} \otimes \mathcal{L}_k[q] \to \Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q] \to \Omega^q_{f_k} \otimes \mathcal{L}_k[0] \to 0$$

which induces a long exact sequence

$$\ldots \to \mathbb{H}^p(\Omega^{\geq q+1,\bullet}_{f_k} \otimes \mathcal{L}_k[q]) \to \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q]) \to H^p(\mathcal{X}_k, \Omega^q_{f_k} \otimes \mathcal{L}_k[0]) \to \ldots.$$

We compare this sequence with that which we get in the case $k = 0$, tensored with A_k over $\mathbb{C}$:

$$\begin{tikzcd}
\ldots \arrow[d] & \ldots \arrow[d] \\
\mathbb{H}^{p-1}(\Omega^{\geq q+1,\bullet}_{f_k} \otimes \mathcal{L}_k[q+1]) \arrow[r, "\cong"] \arrow[d] & \mathbb{H}^{p-1}(\Omega^{\geq q+1,\bullet}_{f_0} \otimes L[q+1]) \otimes_{\mathbb{C}} A_k \arrow[d] \\
\mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_k} \otimes \mathcal{L}_k[q]) \arrow[r, "\cong"] \arrow[d] & \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_0} \otimes L[q]) \otimes_{\mathbb{C}} A_k \arrow[d] \\
H^p(\mathcal{X}_k, \Omega^q_{f_k} \otimes \mathcal{L}_k) \arrow[r] \arrow[d] & H^p(X, \Omega^q_{f_0} \otimes L) \otimes_{\mathbb{C}} A_k \arrow[d] \\
\ldots & \ldots.
\end{tikzcd}$$

By the 5-lemma, $H^p(\mathcal{X}_k, \Omega^q_{f_k} \otimes \mathcal{L}_k) \cong H^p(X, \Omega^q_{f_0} \otimes L) \otimes_{\mathbb{C}} A_k$. □

Putting $\nabla = d$ and $\Delta_k = d$, we may conclude the following:

5.3.8 Corollary

Let $f_k \colon \mathcal{X}_k \to \operatorname{Spec} \mathcal{A}_k$ be a log smooth lifting of f_0 over the log ring $\mathcal{A}_k$ and let $f_{k-1} \colon \mathcal{X}_{k-1} \to \operatorname{Spec} \mathcal{A}_{k-1}$ be its restriction to $\mathcal{A}_{k-1}$. Then the canonical map

$$\mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_k}[q]) \to \mathbb{H}^p(\Omega^{\geq q,\bullet}_{f_{k-1}}[q])$$

is surjective for all p and q.

5.3.9 Corollary

Let $f_k \colon \mathcal{X}_k \to \operatorname{Spec} \mathcal{A}_k$ be a log smooth lifting of f_0 over $\mathcal{A}_k$. Then $H^p(\mathcal{X}_k, \Omega^q_{f_k})$ is isomorphic to $H^p(X, \Omega^q_{f_0}) \otimes_{\mathbb{C}} A_k$ as A_k-module for all k, p and q. In particular, $H^p(\mathcal{X}_k, \Omega^q_{f_k})$ is a free A_k-module for all k, p and q.

If we are in the log symplectic situation (general or non-twisted), then, due to the isomorphism $T \cong \Omega^1 \otimes \mathcal{L}$ induced by the log symplectic form, we have

5.3.10 Corollary

Let $f_0 \colon (X, \nabla, \omega) \to \operatorname{Spec} \mathbb{C}$ be a log symplectic SNC log variety (of general type) and, for any $k \in \mathbb{N}_0$, let $f_k \colon (\mathcal{X}_k, \Delta_k, \varpi_k) \to \operatorname{Spec} \mathcal{A}_k$ be a log symplectic lifting (of general type) over the log ring $\mathcal{A}_k$ with underlying ring A_k.

Then $H^p(\mathcal{X}_k, T_{f_k})$ is isomorphic to $H^p(X, T_{f_0}) \otimes_{\mathbb{C}} A_k$ as A_k-module for all k, p and q. In particular, $H^p(\mathcal{X}_k, T_{f_k})$ is a free A_k-module for all k and p.

The next proposition is an analogue of proposition 5.3.1 for the B-complex:

5.3.11 Proposition

Let $f_0\colon (X, \nabla, \omega) \to \operatorname{Spec} \mathbb{C}$ be a log symplectic normal crossing variety the flat log connection of which is log Cartier and, for any $k \in \mathbb{N}_0$, let $f_k\colon (\mathcal{X}_k, \Delta_k, \varpi_k) \to \operatorname{Spec} \mathcal{A}_k$ be a log symplectic lifting of (f_0, ∇, ω) over the log ring $\mathcal{A}_k$ with underlying ring A_k.

Then $\mathbb{H}^p(B^\bullet_{f_k}(\varpi_k))$ is isomorphic to $\mathbb{H}^p(B^\bullet_{f_0}(\omega)) \otimes_{\mathbb{C}} A_k$ as an A_k-module. In particular, $\mathbb{H}^p(B^\bullet_{f_k}(\varpi_k))$ is a free A_k-module.

Proof: Consider this diagram with exact columns and rows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_0 \arrow[r] \arrow[d] & K_k \arrow[r] \arrow[d] & K_{k-1} \arrow[d] & \\
0 \arrow[r] & \mathbb{H}^p(\Omega^{\geq 1,\bullet}_{f_0} \otimes L[1]) \arrow[r] \arrow[d] & \mathbb{H}^p(\Omega^{\geq 1,\bullet}_{f_k} \otimes \mathcal{L}_k[1]) \arrow[r] \arrow[d] & \mathbb{H}^p(\Omega^{\geq 1,\bullet}_{f_{k-1}} \otimes \mathcal{L}_{k-1}[1]) \arrow[r] \arrow[d] & 0 \\
 & \mathbb{H}^p(B^\bullet_0(\omega)) \arrow[r] \arrow[d] & \mathbb{H}^p(B^\bullet_{f_k}(\varpi_k)) \arrow[r] \arrow[d] & \mathbb{H}^p(B^\bullet_{f_{k-1}}(\varpi_{k-1})) \arrow[d] & \\
0 \arrow[r] & \mathbb{H}^p(\Omega^\bullet_{f_0}) \arrow[r] \arrow[d] & \mathbb{H}^p(\Omega^\bullet_{f_k}) \arrow[r] \arrow[d] & \mathbb{H}^p(\Omega^\bullet_{f_{k-1}}) \arrow[r] \arrow[d] & 0 \\
 & K'_0 \arrow[r] \arrow[d] & K'_k \arrow[r] \arrow[d] & K'_{k-1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd}$$

where the Ks and K's denote the respective kernels and cokernels. We compare the $\mathbb{C}$-vector-space dimensions (denoted by h^p): On the one hand,

$$\begin{aligned}
h^p(B^\bullet_{f_k}(\varpi_k)) &= h^p(\Omega^{\geq 1,\bullet}_{f_k} \otimes \mathcal{L}_k[1]) + h^p(\Omega^\bullet_{f_k}) - h^p(K_k) - h^p(K'_k) \\
&\geq h^p(\Omega^{\geq 1,\bullet}_{f_0} \otimes \mathcal{L}_0[1]) + h^p(\Omega^{\geq 1,\bullet}_{f_{k-1}} \otimes \mathcal{L}_{k-1}[1]) + h^p(\Omega^\bullet_{f_0}) + h^p(\Omega^\bullet_{f_{k-1}}) \\
&\quad - h^p(K_0) - h^p(K'_0) - h^p(K_{k-1}) - h^p(K'_{k-1}) \\
&= h^p(B^\bullet_{f_0}(\varpi_0)) + h^p(B^\bullet_{f_{k-1}}(\varpi_{k-1}))
\end{aligned}$$

and on the other hand, $h^p(B^\bullet_{f_k}(\varpi_k)) \leq h^p(B^\bullet_{f_0}(\varpi_0)) + h^p(B^\bullet_{f_{k-1}}(\varpi_{k-1}))$. Therefore, the middle row must be exact, too, and we may prove $\mathbb{H}^p(B^\bullet_{f_k}(\varpi_k)) \cong \mathbb{H}^p(B^\bullet_{f_0}(\omega)) \otimes_{\mathbb{C}} A_k$ inductively again. $\square$

5.3.12 Corollary

Let $f_k\colon (\mathcal{X}_k, \Delta_k, \varpi_k) \to \operatorname{Spec} \mathcal{A}_k$ be a log smooth lifting of (f_0, ∇) over $\mathcal{A}_k$ and let $f_{k-1}\colon (\mathcal{X}_{k-1}, \Delta_{k-1}, \varpi_k) \to \operatorname{Spec} \mathcal{A}_{k-1}$ be its restriction to $\mathcal{A}_{k-1}$. Then the canonical maps

$$\mathbb{H}^p(B^\bullet_{f_k}(\varpi_k)) \to \mathbb{H}^p(B^\bullet_{f_{k-1}}(\varpi_{k-1}))$$

are surjective.

5.4 Vanishing of the obstructions and main results

5.4.1 The obstruction of log smooth liftings

Let $f_0\colon X \to \operatorname{Spec} \mathbb{C}$ be an SNC log variety with double locus D. Let $f_k\colon \mathcal{X}_k \to \operatorname{Spec} \mathcal{A}_k$ be a log smooth lifting of f_0 over $\mathcal{A}_k$ and denote $\varepsilon_k\colon 0 \to (y) \to \mathcal{B}_k \to \mathcal{A}_k \to 0$. Moreover, denote the closed immersion $i_k\colon X \to \mathcal{X}_k$.

Recall from chapter 4 that the obstruction to log smoothly lifting f_k along ε_k is an element $o_{\varepsilon_k}([f_k])$ in $H^2(T_{f_k} \otimes_{A_k} y \cdot A_k) = H^2(T_{f_k})$. Together with corollary 5.3.10 this leads to the following trivial observation:

5.4.1 Lemma

If $H^2(T_{f_0}) = 0$, then $o_{\varepsilon_k}([f_k]) = 0$ for all $k \in \mathbb{N}_0$.

However, the vanishing of the whole small obstruction space $H^2(T_{f_0})$ is a rather strong condition on f_0. In this section we will give a weaker condition that implies the vanishing of the obstruction $o_{\varepsilon_k}([f_k])$.

First, we prove the following proposition which is a generalisation of corollary 5.1.11.

5.4.2 Proposition

Let $f_0\colon X \to \operatorname{Spec} \mathbb{C}$ be an SNC log variety with double locus $D \subset \underline{X}$ and, for any $k \in \mathbb{N}_0$, let $f_k\colon \mathcal{X}_k \to \operatorname{Spec} \mathcal{A}_k$ be a log smooth lifting of f_0 over $\mathcal{A}_k$. Then

$$\operatorname{Ext}^p(\Upsilon_{f_k}, \mathcal{O}_X) = \begin{cases} 0 & \text{if } p = 0, \\ H^{p-1}(\mathcal{O}_D) & \text{if } p \geq 1. \end{cases}$$

Proof of the proposition

Since the diagram

$$\begin{array}{ccc} X & \xrightarrow{i_k} & \mathcal{X}_k \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_k} \\ X^{\mathbb{C}} & \longrightarrow & \mathcal{X}_k^{\mathcal{A}_k} \end{array}$$

is Cartesian, we get $i_k^* \Upsilon_{f_k} = i_k^* \Omega^1_h \cong \Omega^1_{h_0} = \Upsilon_{f_0}$, by the base change property of log differentials. Here, $X^{\mathbb{C}}$ and $\mathcal{X}_k^{\mathcal{A}_k}$ denote the schemes $\underline{X}$ and $\underline{\mathcal{X}_k}$ together with the log structure pulled back from $\operatorname{Spec} \mathbb{C}$ and $\operatorname{Spec} \mathcal{A}_k$, respectively.

5.4.3 Lemma

The natural morphism of $\mathcal{O}_{\mathcal{X}_k}$-modules $\Upsilon_{f_k} \to i_{k*}i_k^*\Upsilon_{f_k} \cong i_{k*}\Upsilon_{f_0}$ (induced by adjunction) is an isomorphism.

Proof: Again, we identify the topological spaces of X and $\mathcal{X}$. Let x be a closed point in X and denote

$$\widehat{R} := \widehat{\mathcal{O}}_{\mathcal{X},x} = \mathbb{C}[\![z_1, \ldots, z_n, x]\!]/(z_1 \cdot \ldots \cdot z_r - cx^N, x^k).$$

Then $(i_{k*}\mathcal{O}_X)\widehat{_x} = \widehat{\mathcal{O}}_{X,x} = \widehat{R}/(z_1 \cdot \ldots \cdot z_r)$. At the level of completion of stalks, the morphism in question, $\widehat{\Upsilon}_{f,x} \to (i_{k*}\Upsilon_{f_0})\widehat{_x} = \widehat{\Upsilon}_{f_0,x}$, is given by

$$\begin{aligned}
&\frac{\widehat{R}\,\langle d\log z_1, \ldots, d\log z_r, d\log x\rangle}{\langle z_1 d\log z_1, \ldots, z_r d\log z_r, d\log z_1 + \ldots + d\log z_r, d\log x\rangle} \\
&\qquad \cong \Big(\bigoplus_{j=1}^{r} \widehat{R}/(z_j) \cdot d\log z_j\Big) / \langle d\log z_1 + \ldots + d\log z_r\rangle \\
&\to \frac{\widehat{R}/(z_1 \cdot \ldots \cdot z_r)\,\langle d\log z_1, \ldots, d\log z_r\rangle}{\langle z_1 d\log z_1, \ldots, z_r d\log z_r, d\log z_1 + \ldots + d\log z_r\rangle} \\
&\qquad \cong \Big(\bigoplus_{j=1}^{r} \widehat{R}/(z_j) \cdot d\log z_j\Big) / \langle d\log z_1 + \ldots + d\log z_r\rangle \\
&d\log z_j \mapsto d\log z_j,\ d\log x \mapsto 0,
\end{aligned}$$

which is an isomorphism. □

Now i_k is a closed immersion, which implies that the exceptional inverse image functor $i_k^!$ is defined already on the level of $\mathcal{O}_{\mathcal{X}_k}$-modules, namely by $i_k^!\mathcal{F} = i_k^*\Gamma_X\mathcal{F}$, where $\Gamma_X\mathcal{F}$ is the sheaf of local sections of $\mathcal{F}$ with support in $X \subset \mathcal{X}_k$ (for a general morphism f of schemes $f^!$ is only defined as a functor between the associated derived categories). The functor $i_k^!$ is right-adjoint to $i_{k!}$ which, as i_k is a closed immersion, coincides with i_{k*}. For all these statements, cf. [6, p. 40–41 & p. 62–65, i. p. 3.2.11].
Hence, we may calculate

$$\begin{aligned}
\mathrm{Ext}^p_{\mathcal{O}_{\mathcal{X}_k}}(\Upsilon_{f_k}, \mathcal{O}_{\mathcal{X}_k}) &= \mathrm{Ext}^p_{\mathcal{O}_{\mathcal{X}_k}}(i_{k*}\Upsilon_{f_0}, \mathcal{O}_{\mathcal{X}_k}) \cong \mathrm{Ext}^p_{\mathcal{O}_X}(\Upsilon_{f_0}, i_k^*\Gamma_X\mathcal{O}_{\mathcal{X}_k}) \\
&= \mathrm{Ext}^p_{\mathcal{O}_X}(\Upsilon_{f_0}, i_k^*\mathcal{O}_{\mathcal{X}_k}) = \mathrm{Ext}^p_{\mathcal{O}_X}(\Upsilon_{f_0}, \mathcal{O}_X) \cong H^{p-1}(\mathcal{O}_D)
\end{aligned}$$

for $p \geq 1$ and $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}_k}}(i_{k*}\Upsilon_{f_0}, \mathcal{O}_{\mathcal{X}_k}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\Upsilon_{f_0}, \mathcal{O}_X) = 0$, which proves proposition 5.4.2. In these calculations we used that $\Gamma_X\mathcal{O}_{\mathcal{X}_k} = \mathcal{O}_{\mathcal{X}_k}$ which is true, because $X \subset \mathcal{X}_k$ is defined by a nilpotent ideal. □

Applying the functor $\mathrm{Hom}(\,\cdot\,, \mathcal{O}_{\mathcal{X}_k})$ to the short exact sequence

$$0 \to \Omega^1_{\underline{f}}/\tau^1_{\mathcal{X}_k} \to \Omega^1_{f_k} \to \Upsilon_{f_k} \to 0$$

and recalling from [11, 2.10] that $\mathrm{Ext}^p(\Omega^1_{f_k}, \mathcal{O}_{\mathcal{X}_k}) = H^p(T_{f_k})$, $\mathrm{Ext}^p(\Omega^1_{\underline{f_k}}/\tau^1_{\mathcal{X}_k}, \mathcal{O}_{\mathcal{X}_k}) = H^p(T_{\underline{f_k}})$ and $\mathrm{Ext}^p(\Upsilon_{f_k}, \mathcal{O}_X) = H^{p-1}(\mathcal{O}_D)$ by proposition 5.4.2 for all $p \in \mathbb{N}_0$, yields the long exact sequence

$$\ldots \to H^1(D, \mathcal{O}_D) \to H^2(T_{f_k}) \to H^2(T_{\underline{f_k}}) \to H^2(D, \mathcal{O}_D) \to \ldots.$$

This implies the following proposition immediately:

5.4.4 Proposition

If $H^1(\mathcal{O}_D) = 0$, then $o_{\varepsilon_k}([f_k]) = 0$.

Proof: By 4.2.3, the map $H^2(T_f) \to H^2(T_{\underline{f}})$ sends the obstruction $o_{\varepsilon_k}([f_k])$ of log smoothly lifting f_k along ε_k to the obstruction of flatly lifting the underlying flat deformation $\underline{f_k}$ of $\underline{f_0}$ along the small extension $\underline{\varepsilon_k}\colon 0 \to yA_k \to A_k[y]/y^2 \to A_k \to 0$ underlying ε_k. Since $\underline{\varepsilon_k}$ has the natural splitting $A \to A[y]/y^2$, we can always lift $\underline{f_k}$ to the trivial flat deformation $\underline{f_k}[\varepsilon]\colon \underline{\mathcal{X}_k}[\varepsilon] := \underline{\mathcal{X}_k} \times_{\operatorname{Spec} A} \operatorname{Spec} A[y]/y^2 \to \operatorname{Spec} A[y]^2$ associated to that splitting. Hence, $o_{\varepsilon_k}([f_k])$ is mapped to zero, which, since the condition $H^1(D, \mathcal{O}_D) = 0$ makes the map $H^2(T_f) \to H^2(T_{\underline{f}})$ injective, implies $o_{\varepsilon_k}([f_k]) = 0$. □

5.4.5 Remark

In [21] the authors Y. Kawamata and Y. Namikawa examine deformations of what we would call *SNC log Calabi-Yau varieties.* The two conditions of [21, 4.2] used to enforce the vanishing of the obstruction of log smooth lifting are $H^{\dim X-1}(X, \mathcal{O}_X) = 0$ and $H^{\dim X-2}(X^\nu, \mathcal{O}_{X^\nu}) = 0$. By Serre duality, these force the two maps $\operatorname{Ext}^1(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) \to \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)$ and $\operatorname{Ext}^2(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) \to \operatorname{Ext}^2(\mathcal{O}_X, \mathcal{O}_X)$ to be surjective and injective, respectively, which implies $\operatorname{Ext}^2(\Upsilon_X, \mathcal{O}_X) = H^1(D, \mathcal{O}_D) = 0$, the weaker condition directly imposed by us in proposition 5.4.4.

In our situation, regarding SNC log symplectic varieties, we may not assume the conditions of Kawamata and Namikawa to be fulfilled: Let $\underline{f_0}\colon X \to \operatorname{Spec} \mathbb{C}$ be a complex smooth symplectic variety (in the usual sense). Giving $\mathbb{C}$ the log structure of the standard log point $\operatorname{Spec} \mathcal{C}$ and X the log structure $f_0^\times \mathcal{C}$ makes f_0 a (strict) log smooth morphism and lets us view the symplectic form on $\underline{f_0}$ as a log symplectic form on the log smooth morphism f_0.

Under these assumptions, since X is normal, we have $\nu_*\mathcal{O}_{X^\nu} = \mathcal{O}_X$, so we calculate $H^{\dim X-2}(X^\nu, \mathcal{O}_{X^\nu}) = \operatorname{Ext}^2(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) = H^2(\mathcal{O}_X) \cong H^0(\Omega^2_{f_0})$, which is non-zero, because it contains the log symplectic form. For this reason, we may not impose the same conditions $H^{\dim X-1}(X, \mathcal{O}_X) = 0$ and $H^{\dim X-2}(X^\nu, \mathcal{O}_{X^\nu}) = 0$ as Kawamata and Namikawa , because the second condition would contradict our assumptions.

5.4.2 Smoothing of non-twisted log symplectic varieties

Let $f_0\colon (X, \omega) \to \operatorname{Spec} \mathcal{C}$ be an SNC log symplectic variety of non-twisted type. Let $f_k\colon (\mathcal{X}_k, \varpi_k) \to \operatorname{Spec} \mathcal{A}_k$ be a log smooth deformation of (f_0, ω) over $\mathcal{A}_k$ and denote $\varepsilon_k\colon 0 \to (y) \to \mathcal{B}_k \to \mathcal{A}_k \to 0$.

Recall from proposition 4.2.4 that the obstruction to lifting $f_k\colon (\mathcal{X}_k, \varpi_k) \to \operatorname{Spec} \mathcal{A}_k$ along ε_k is an element $o_{\varepsilon_k}([f_k, \varpi_k])$ in $\mathbb{H}^2(\Omega_f^{\geq 1, \bullet}[1] \otimes_{A_k} \varepsilon A_k) = \mathbb{H}^2(\Omega_f^{\geq 1, \bullet}[1])$.

5.4.6 Lemma
If $\mathbb{H}^1(\Omega^{\geq 2,\bullet}_{f_0}[2]) = 0$, then

$$o_{\varepsilon_k}([f_k, \varpi_k]) = 0 \quad \Leftrightarrow \quad o_{\varepsilon_k}([f_k]) = 0.$$

Proof: Under the map $\pi_k \colon \mathbb{H}^2(\Omega^{\geq 1,\bullet}_{f_k}[1]) \to H^2(T_{f_k})$, induced by the short exact sequence

$$t^{\varpi_k} \colon 0 \to \Omega^{\geq 2,\bullet}_{f_k}[1] \to \Omega^{\geq 1,\bullet}_{f_k}[1] \xrightarrow{\pi_k} T_{f_k} \to 0,$$

$o_{\varepsilon_k}([f_k, \varpi_k])$ is mapped to the obstruction $o_{\varepsilon_k}([f_k])$ of log smoothly lifting f_k over ε_k by corollary 4.2.3. Due to corollary 5.3.2 the vanishing of $\mathbb{H}^1(\Omega^{\geq 2,\bullet}_{f_0}[2]) = \mathbb{H}^2(\Omega^{\geq 2,\bullet}_{f_0}[1])$ implies that of $\mathbb{H}^2(\Omega^{\geq 2,\bullet}_{f_k}[1])$, which makes π_k injective, implying the claim. □

5.4.7 Theorem
Let $f_0 \colon (X, \omega) \to \operatorname{Spec} \mathbb{C}$ be an SNC log symplectic variety of non-twisted type with double locus D and with $f_* \mathcal{O}_{X_0} = \mathcal{O}_{\operatorname{Spec} \mathbb{C}}$. If

a) $H^1(X, \mathcal{O}_X) = 0$,

b) $\mathbb{H}^1(\Omega^{\geq 2,\bullet}_{f_0}[2]) = 0$ and

c) $H^1(D, \mathcal{O}_D) = 0$ or $H^2(X, T_{f_0}) = 0$,

then the hull of $\mathrm{Def}_{(f_0,\omega)}$ is smooth.

Proof: We have to show that $\mathrm{Def} := \mathrm{Def}_{(f_0,\omega)}$ satisfies the T1-lifting property, i. e. that $\mathrm{Def}(\mathcal{B}_k) \to \mathrm{Def}(\mathcal{A}_k) \times_{\mathrm{Def}(\mathcal{A}_{k-1})} \mathrm{Def}(\mathcal{B}_{k-1})$ is surjective for all k. If $k = 0$, then this reduces to showing the surjectivity of $\mathrm{Def}(\mathcal{B}_0) \to \mathrm{Def}(\mathcal{A}_0)$. If $H^1(D, \mathcal{O}_D) = 0$, respectively, if $H^2(X, T_f) = 0$, then we have, by proposition 5.4.4 and lemma 5.4.6, respectively by lemma 5.4.1, that $o_{\varepsilon_0}([f_0, \omega]) = 0$ for the small extension $\varepsilon_0 \colon 0 \to (y) \to \mathcal{B}_0 \to \mathcal{A}_0 \to 0$, which is equivalent to $\mathrm{Def}(\mathcal{B}_0) \to \mathrm{Def}(\mathcal{A}_0)$ being surjective.
Now let $k \geq 1$. Then we look at the diagram

$$\begin{tikzcd}
\mathrm{Def}(\mathcal{B}_k) \arrow[r] \arrow[d] & \mathrm{Def}(\mathcal{B}_{k-1}) \arrow[d] \\
\mathrm{Def}(\mathcal{A}_k) \arrow[r] & \mathrm{Def}(\mathcal{A}_{k-1}).
\end{tikzcd}$$

We regard an element $(\xi_k, \eta_{k-1}) \in \mathrm{Def}(\mathcal{A}_k) \times_{\mathrm{Def}(\mathcal{A}_{k-1})} \mathrm{Def}(\mathcal{B}_{k-1})$ with ξ_k and η_{k-1} projecting to ξ_{k-1}. Due to the vanishing of the obstruction along the small extension $\varepsilon_k \colon 0 \to (y) \to \mathcal{B}_k \to \mathcal{A}_k \to 0$ by proposition 5.4.4 and lemma 5.4.6 we can lift ξ_k to an element $\eta_k \in \mathrm{Def}(\mathcal{B}_k)$. Denoting the projection of η_k in $\mathrm{Def}(\mathcal{B}_{k-1})$ by η'_{k-1}, we need not have $\eta'_{k-1} = \eta_{k-1}$. But as the set of all $\eta \in \mathrm{Def}(\mathcal{B}_{k-1})$ mapping to $\xi_{k-1} \in \mathrm{Def}(\mathcal{A}_k)$ is a torsor under $\mathbb{H}^1(\Omega^{\geq 1,\bullet}_{f_{k-1}}[1])$ by 4.2.4, there is an element $v_{k-1} \in \mathbb{H}^1(\Omega^{\geq 1,\bullet}_{f_{k-1}}[1])$ with $v_{k-1}\eta'_{k-1} = \eta_{k-1}$. Since $\mathbb{H}^1(\Omega^{\geq 1,\bullet}_{f_k}[1]) \to \mathbb{H}^1(\Omega^{\geq 1,\bullet}_{f_{k-1}}[1])$ is surjective, there is an element $v_k \in \mathbb{H}^1(\Omega^{\geq 1,\bullet}_{f_k}[1])$ mapping to v_{k-1}. Then $v_k\eta_k$ is an element of $\mathrm{Def}(\mathcal{B}_k)$ mapping to ξ_k and $v_{k-1}\eta'_{k-1} = \eta_{k-1}$, respectively, which shows the surjectivity of $\mathrm{Def}(\mathcal{B}_k) \to \mathrm{Def}(\mathcal{A}_k) \times_{\mathrm{Def}(\mathcal{A}_{k-1})} \mathrm{Def}(\mathcal{B}_{k-1})$ for all $k \geq 1$.

By the T1-lifting principle 5.2.2, we conclude that Def is smooth. In particular, its hull is smooth. □

Let $\underline{\eta_T}\colon \operatorname{Spec}\mathbb{C}(\!(t)\!) \to \operatorname{Spec} T$ be the generic point of $\operatorname{Spec} T$. The induced strict morphism of log schemes $\eta_\mathcal{T}\colon (\operatorname{Spec}\mathbb{C}(\!(t)\!), \eta_T^\times\mathcal{T}) \to \operatorname{Spec}\mathcal{T}$ is called the *generic point* of $\operatorname{Spec}\mathcal{T}$ and we write $\operatorname{Spec}\mathcal{T}^\eta$ for $(\operatorname{Spec}\mathbb{C}(\!(t)\!), \underline{\eta_T}^\times\mathcal{T})$. By abuse of language, we will also call $\operatorname{Spec}\mathcal{T}^\eta$ the generic point of $\operatorname{Spec}\mathcal{T}$.

5.4.8 Lemma

The generic point $\operatorname{Spec}\mathcal{T}^\eta$ of $\operatorname{Spec}\mathcal{T}$ carries the trivial log structure.

Proof: This log structure has the chart $t\colon \mathbb{N}_0 \to \mathbb{C}(\!(t)\!)$ induced from $\operatorname{Spec}\mathcal{T}$, which maps the generator 1 of $\mathbb{N}_0$ to the unit $t \in \mathbb{C}(\!(t)\!)^\times$. Hence, its log structure sheaf $\mathcal{M}_{\eta_T} = \mathbb{N}_0 \circledS \mathbb{C}(\!(t)\!)^\times = \mathbb{C}(\!(t)\!)^\times$ is trivial. □

5.4.9 Theorem

Let $f_0\colon (X,\omega) \to \operatorname{Spec}\mathcal{C}$ be a log symplectic variety of non-twisted type satisfying the conditions of theorem 5.4.7. Then (f_0,ω) is formally smoothable to a log symplectic deformation $f_\infty\colon (\mathfrak{X},\mathfrak{w}) \to \operatorname{Spec}\mathcal{T}$, the generic fibre of which is a strict smooth log symplectic scheme $f'\colon (X',\omega') \to \eta_\mathcal{T}$ of non-twisted type.

In particular, $\underline{f_\infty}$ is a flat deformation of $\underline{f}_0$ and the generic fibre $\underline{f'}\colon (\underline{X'},\omega') \to \operatorname{Spec} T^\eta$ of $(\underline{f}_\infty, \mathfrak{w})$ is a (relative) smooth symplectic scheme in the usual sense.

Proof: Let $\varphi_A\colon T \to A$ be the T-algebra structure $t \mapsto x$. Then the smoothness of the hull $\mathcal{R}$ implies the existence of a morphism

$$\operatorname{Spec}\mathcal{T} = \operatorname{Spec}\varprojlim_n \mathcal{A}_n \to \operatorname{Spec}\mathcal{R} \to \mathrm{Def}_{(f_0,\omega)},$$

thus the existence of a lifting $(f_\infty,\mathfrak{w})$ in $\mathrm{Def}_{(f_0,\omega)}(\mathcal{T})$:

$$\begin{tikzcd} (X,\omega) \arrow[r, hook] \arrow[d, "f_0"] & (\mathfrak{X},\mathfrak{w}) \arrow[d, "f_\infty"] \\ \operatorname{Spec}\mathcal{C} \arrow[r, hook] & \operatorname{Spec}\mathcal{T} \end{tikzcd}$$

The generic fibre f' (the fibre over $\eta_\mathcal{T}$) of f_∞ is the morphism in the commutative diagram of morphisms of log schemes with strict open immersions as horizontal morphisms

$$\begin{tikzcd} X' := \mathfrak{X}_{\eta_\mathcal{T}} = \mathfrak{X} \times_{\operatorname{Spec}\mathcal{T}} \operatorname{Spec}\mathcal{T}^\eta \arrow[r, hook] \arrow[d, "f'"] & \mathfrak{X} \arrow[d, "f_\infty"] \\ \operatorname{Spec}\mathcal{T}^\eta \arrow[r, hook] & \operatorname{Spec}\mathcal{T} \end{tikzcd}$$

and we let ω' be the restriction of $\mathfrak{w}$ to X'.

Then f' locally admits the chart $\operatorname{diag}: \mathbb{N}_0 \to \mathbb{N}_0^r$, with $1 \in \mathbb{N}_0$ being mapped to the unit $t \in \mathbb{C}(\!(t)\!)^\times$ and its image under the chart $(1, \ldots, 1)$ mapped to the unit $t \in \mathcal{O}_{X'}^\times$. But since a product which is a unit can only have units as factors, $\mathbb{N}_0^r$ is completely mapped to $\mathcal{O}_{X'}^\times$. Hence, $\mathcal{M}_{X'} = \mathcal{M}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}^\times} \mathcal{O}_{X'}^\times = \mathbb{N}_0^r \oslash \mathcal{O}_{X'}^\times = \mathcal{O}_{X'}^\times$, which shows that $\alpha_{X'}$ is trivial. In reverse, the log structure $\alpha_{\mathfrak{X}}$ is supported only in the fibre X over the closed point $\operatorname{Spec} \mathbb{C}$ of $\operatorname{Spec} \mathcal{T}$ corresponding to the prime ideal (t).

A morphism of log schemes with trivial log structures is always strict and the (log) smoothness of such a morphism is equivalent to the smoothness of the underlying morphism. Therefore, $\underline{f'}: \underline{\mathfrak{X}_\eta} \to \operatorname{Spec} \mathbb{C}(\!(t)\!)$ is a smooth morphism of schemes in the usual sense. Due to the strictness of f', the form ω' is just a symplectic form (relative to $\underline{f'}$) in the usual sense. Hence, $\underline{f_0}$ is indeed formally smoothable to a flat deformation the generic fibre of which is symplectic. □

5.4.10 Corollary (cp. [21, 2.5])

Let $f_0: (X, \omega) \to \operatorname{Spec} \mathbb{C}$ be a SNC log symplectic variety of non-twisted type of dimension 2 with $H^1(X, \mathcal{O}_X) = 0$. Then the hull of $\operatorname{Def}_{(f_0, \omega)}$ is smooth and $\underline{f_0}$ is smoothable by a flat deformation.

Proof: By assumption, $\Omega_{f_0}^2 \cong \mathcal{O}_X$, thus $\mathbb{H}^1(\Omega_{f_0}^{\geq 2, \bullet}[2]) = H^1(X, \Omega_{f_0}^2) = H^1(X, \mathcal{O}_X) = 0$. By [11, 5.8], we have $H^2(T_{f_0}) = 0$. Using 5.4.7 and 5.4.9, the result follows. □

5.4.11 Remark

This corollary is essentially [21, 2.5], because in dimension 2 a log symplectic SNC variety is the same thing as the logarithmic degenerate Calabi-Yau variety described by Y. Kawamata and Y. Namikawa in [21], which we would call *SNC log Calabi-Yau variety* of dimension 2.

5.4.3 Smoothing of twisted log symplectic varieties

Let $f_0: (X, \nabla, \omega) \to \operatorname{Spec} \mathbb{C}$ be an SNC log symplectic variety. Let $f_k: (\mathcal{X}_k, \Delta_k, \varpi_k) \to \operatorname{Spec} \mathcal{A}_k$ be a log smooth deformation of (f_0, ∇, ω) over $\mathcal{A}_k$ and denote $\varepsilon_k: 0 \to (y) \to \mathcal{B}_k \to \mathcal{A}_k \to 0$.

Recall from chapter 4 that the obstruction to lifting $f_k: (\mathcal{X}_k, \Delta_k, \varpi_k) \to \operatorname{Spec} \mathcal{A}_k$ along ε_k is an element $o_{\varepsilon_k}([f_k, \Delta_k, \varpi_k])$ in $\mathbb{H}^2(B_{f_k}^\bullet(\varpi_k) \otimes_{A_k} \varepsilon A_k) = \mathbb{H}^2(B_{f_k}^\bullet(\varpi_k))$.

5.4.12 Lemma

If $\mathbb{H}^1(\Omega_{f_0}^{\geq 2, \bullet} \otimes L[2]) = 0$ and if ∇ is log Cartier, then

$$o_{\varepsilon_k}([f_k, \Delta_k, \varpi_k]) = 0 \quad \Leftrightarrow \quad o_{\varepsilon_k}([f_k]) = 0.$$

Proof: Under the map $\mu_k: \mathbb{H}^2(B_{f_k}^\bullet(\varpi_k)) \to \mathbb{H}^2(A_{f_k}^\bullet(\Delta_k))$ induced by the short exact sequence

$$0 \to \Omega_{f_k}^{\geq 2, \bullet} \otimes \mathcal{L}_k[1] \to B_{f_k}^\bullet(\varpi_k) \xrightarrow{\mu_k} A_{f_k}^\bullet(\Delta_k) \to 0,$$

$o_{\varepsilon_k}([f_k, \varDelta_k, \varpi_k])$ is mapped to the obstruction $o_{\varepsilon_k}([f_k, \varDelta_k])$ of lifting the log scheme with flat log connection $(f_k, \varDelta_k)$ along ε_k.

Under the surjective map $\pi_k \colon \mathbb{H}^2(A^\bullet_{f_k}(\varDelta_k)) \to H^2(T_{f_k})$ induced by the short split exact sequence

$$0 \to \varOmega^\bullet_{f_k} \to A^\bullet_{f_k}(\varDelta_k) \xrightarrow{\pi_k} T_{f_k}[0] \to 0,$$

$o_{\varepsilon_k}([f_k, \varDelta_k])$ is mapped to the obstruction $o_{\varepsilon_k}([f_k])$ of log smoothly lifting f_k along ε_k.

If ∇ is log Cartier, then it automatically lifts to any infinitesimal log smooth lifting of f_0 (cf. Lemma 3.4.10), so $o_{\varepsilon_k}([f_k, \varDelta_k])$ vanishes if and only if $o_{\varepsilon_k}([f_k])$ vanishes. By proposition 5.3.1, the vanishing of $\mathbb{H}^1(\varOmega^{\geq 2,\bullet}_{f_0} \otimes L[2]) = \mathbb{H}^2(\varOmega^{\geq 2,\bullet}_{f_0} \otimes L[1])$ implies that of $\mathbb{H}^2(\varOmega^{\geq 2,\bullet}_{f_k} \otimes \mathcal{L}_k[1])$, which makes q_k injective, thus implying the claim. □

We are now able to proof our second main result:

5.4.13 Theorem

Let $f_0 \colon (X, \nabla, \omega) \to \operatorname{Spec} \mathbb{C}$ be an SNC log symplectic variety (of general type) with double locus D and with $f_* \mathcal{O}_{X_0} = \mathcal{O}_{\operatorname{Spec} \mathbb{C}}$. If

a) ∇ is log Cartier,

b) $H^1(X, \mathcal{O}_X) = 0$,

c) $\mathbb{H}^1(\varOmega^{\geq 2,\bullet}_{f_0} \otimes L[2]) = 0$ and

d) $H^1(D, \mathcal{O}_D) = 0$ or $H^2(X, T_{f_0}) = 0$,

then the hull of $\operatorname{Def}_{(f_0, \nabla, \omega)}$ is smooth.

Proof: Let $\operatorname{Def} := \operatorname{Def}_{(f_0, \nabla, \omega)}$. By lemma 5.4.12, the vanishing of $o_{\varepsilon_k}([f_k, \varDelta_k, \varpi_k])$ is equivalent to that of $o_{\varepsilon_k}([f_k])$. By proposition 5.4.4, respectively by lemma 5.4.1, the second obstruction vanishes.

Literally as in the proof of 5.4.7, this vanishing implies the surjectivity of $\operatorname{Def}(\mathcal{B}_0) \to \operatorname{Def}(\mathcal{A}_0)$ for $k = 0$ and the surjectivity of $\operatorname{Def}(\mathcal{B}_k) \to \operatorname{Def}(\mathcal{B}_{k-1}) \times_{\operatorname{Def}(\mathcal{A}_k)} \operatorname{Def}(\mathcal{A}_{k-1})$ for $k \geq 1$, when replacing the torsor of liftings $\mathbb{H}^1(\varOmega^{\geq 1,\bullet}_{f_{k-1}}[1])$ by $\mathbb{H}^1(B^\bullet_{f_{k-1}}(\varpi_k))$.

Again, by the T1-lifting principle, we conclude that Def is smooth; hence, its hull is, too. □

5.4.14 Theorem

Let $f_0 \colon (X, \nabla, \omega) \to \operatorname{Spec} \mathbb{C}$ be a log symplectic variety (of general type) satisfying the conditions of theorem 5.4.13. Then (f_0, ∇, ω) is formally smoothable to a log symplectic deformation $f_\infty \colon (\mathfrak{X}, \mathfrak{D}, \mathfrak{w}) \to \operatorname{Spec} \mathcal{T}$ (of general type) the generic fibre of which is a strict smooth log symplectic scheme $f' \colon (X', d, \omega') \to \operatorname{Spec} \mathcal{T}^\eta$ of non-twisted type.

In particular, $\underline{f_\infty}$ is a flat deformation of $\underline{f_0}$ and the generic fibre $\underline{f'} \colon (\underline{X'}, d, \omega') \to \operatorname{Spec} T^\eta$ of $(\underline{f}_\infty, \mathfrak{D}, \mathfrak{w})$ is a (relative) smooth symplectic scheme in the usual sense.

Proof: As in the proof of 5.4.9, the smoothness of the hull $\mathcal{R}$ implies the existence of a morphism $\operatorname{Spec}\mathcal{T} \to \operatorname{Spec}\mathcal{R} \to \operatorname{Def}_{(f_0,\nabla,\omega)}$, thus the existence of a log smooth deformation $f_\infty\colon (\mathfrak{X},\mathfrak{D},\mathfrak{w}) \to \operatorname{Spec}\mathcal{T} \in \operatorname{Def}_{(f_0,\nabla,\omega)}(\mathcal{T})$.
Again, $f'\colon X' \to \operatorname{Spec}\mathcal{T}^\eta$ is a strict morphism of log schemes with trivial log structures. Since $\mathfrak{D}$ is log Cartier, its restriction must be equal to the only flat log Cartier connection $(d,\mathcal{O}_{X'})$ on X', the form $\mathfrak{w}$ thus restricts on X' to a log symplectic form ω' of non-twisted type. Due to the strictness of f', the form ω' is again just a symplectic form (relative to $\underline{f'}$) in the usual sense and $\underline{f_0}$ is formally smoothable to a flat deformation the generic fibre of which is symplectic. □

5.4.15 Corollary
Let $f_0\colon (X,\nabla,\omega) \to \operatorname{Spec}\mathbb{C}$ be a SNC log symplectic variety of general type of dimension 2 with $H^1(X,\mathcal{O}_X) = 0$ and such that ∇ is log Cartier. Then the hull of $\operatorname{Def}_{(f_0,\omega)}$ is smooth and $\underline{f_0}$ is smoothable by a flat deformation.

Proof: By assumption, $\Omega^2_{f_0} \otimes L \cong \mathcal{O}_X$, thus $\mathbb{H}^1(\Omega^{\geq 2,\bullet}_{f_0} \otimes L[2]) = H^1(X,\Omega^2_{f_0} \otimes L) = H^1(X,\mathcal{O}_X) = 0$. Again, by [11, 5.8], we have $H^2(T_{f_0}) = 0$, so using again 5.4.7 and 5.4.9, the result follows. □

6 Examples

In the preceding chapters we have developed a theory of log symplectic schemes and their log smooth deformations. In what follows, we will show that log symplectic schemes and their deformations arise naturally by looking at some examples. Before constructing these examples, we look at the blow-up of a log scheme. We assume all schemes to be Noetherian. Although one finds the section headline “Log blow-ups”, numbered II.2.5, in the lecture notes of A. Ogus, this section bears no content up to today (cf. [29, II.2.5]. Hence, all the following definitions and statements are due to ourselves.

6.1 Blow-up of a log scheme

Let X be a log scheme, let $P \subset \underline{X}$ be a closed subscheme and denote the blow-up of the underlying scheme $\underline{X}$ of X along P by $\underline{\beta}\colon \underline{X^\beta} \to \underline{X}$. Then $E := \beta^{-1}(P)$ is a divisor in $\underline{X^\beta}$ with open complement $W := \underline{X^\beta} \setminus E \cong X \setminus P$. Accordingly, we have on $\underline{X^\beta}$ on the one hand the compactifying log structure $\alpha_{W \subset \underline{X^\beta}}$ associated to the open immersion $j\colon W \to \underline{X^\beta}$ and on the other hand the pullback log structure $\underline{\beta}^\times \alpha_X$.

By the universal property of the tensor product of sheaves of monoids, the dashed morphism of sheaves of monoids in the commutative diagram

$$\begin{tikzcd}
\mathcal{O}^\times_{\underline{X^\beta}} \arrow[r, hook] \arrow[d, hook] & j_\times \mathcal{O}^\times_W \arrow[d, hook] \arrow[ddr, bend left, "\alpha_{W\subset \underline{X^\beta}}"] & \\
\underline{\beta}^\times \mathcal{M}_X \arrow[r, hook] \arrow[drr, bend right, "\underline{\beta}^\times \alpha_X"'] & \underline{\beta}^\times \mathcal{M}_X \otimes_{\mathcal{O}^\times_{\underline{X^\beta}}} j_\times \mathcal{O}^\times_W \arrow[dr, dashed] & \\
 & & \mathcal{O}_{\underline{X^\beta}}
\end{tikzcd}$$

exists uniquely and is a log structure on $\underline{X^\beta}$.

We denote this log structure on $\underline{X^\beta}$ by $\alpha_{X^\beta} := \underline{\beta}^\times \alpha_X \otimes_{\iota_{\underline{X^\beta}}} \alpha_{W \subset \underline{X^\beta}}$ and we write $X^\beta := (\underline{X^\beta}, \alpha_{X^\beta})$ for the log scheme with that log structure and underlying scheme $\underline{X^\beta}$. Since there is an obvious morphism of log structures $\underline{\beta}^\flat \colon \underline{\beta}^\times \alpha_X \to \alpha_{X^\beta}$ (namely the inclusion as a subsheaf as seen in the above diagram), $\beta := (\underline{\beta}, \underline{\beta}^\flat)\colon Y \to X$ is a morphism of log schemes which we call the *blow-up of the log scheme X along the closed subscheme $P \subset \underline{X}$*. By abuse of language, we will also call X^β the *blow-up of X along P*.

6.1.1 Lemma

Let $\underline{Y}$ be a scheme and let V and W be open subschemes of $\underline{Y}$. Then there is a natural morphism of log structures

$$\alpha_{V \subset \underline{Y}} \otimes_{\iota_{\underline{Y}}} \alpha_{W \subset \underline{Y}} \to \alpha_{V \cap W \subset \underline{Y}}.$$

If the complements $Z = \underline{Y} \setminus V$ and $E = \underline{Y} \setminus W$ are Cartier divisors in $\underline{Y}$ (i. e. non-empty closed subschemes the ideal sheaf of which is locally principal, generated by a non-zero-divisor) which have no common components, then this morphism is an isomorphism.

Proof: We denote the open immersions by the names indicated in the following Cartesian diagram:

$$\begin{tikzcd} V \cap W \arrow[r, hook, "j'_V"] \arrow[d, hook, "j'_W"'] \arrow[dr, "j_{V\cap W}"] & V \arrow[d, hook, "j_V"] \\ W \arrow[r, hook, "j_W"] & \underline{Y} \end{tikzcd}$$

We have the two canonical injective morphisms of log structures $\iota_V \to \alpha_{j'_V} = j'_{V\times}\iota_{V\cap W}$ and $\iota_W \to \alpha_{j'_W} = j'_{W\times}\iota_{V\cap W}$ on V and W, respectively, because ι_V and ι_W are the initial objects in the category of log structures on V and W, respectively. Applying the left-exact functors $j_{V\times}$ and $j_{W\times}$ to these morphisms, respectively, yields injective morphisms of log structures

$$\alpha_{j_V} = j_{V\times}\iota_V \to j_{V\times}j'_{V\times}\iota_{V\cap W} = j_{V\cap W\times}\iota_{V\cap W} = \alpha_{j_{V\cap W}} \text{ and}$$
$$\alpha_{j_W} = j_{W\times}\iota_W \to j_{W\times}j'_{W\times}\iota_{V\cap W} = j_{V\cap W\times}\iota_{V\cap W} = \alpha_{j_{V\cap W}}.$$

By the universal property of the tensor product of (pre)log structures, the morphism of log structures

$$\alpha_{j_V} \otimes_{\iota_{\underline{Y}}} \alpha_{j_W} \to \alpha_{j_{V\cap W}}$$

exists uniquely and it is given by the morphism of sheaves of monoids

$$j_{V\times}\mathcal{O}_V^\times \otimes_{\mathcal{O}_{\underline{Y}}^\times} j_{W\times}\mathcal{O}_W^\times \to j_{V\cap W\times}\mathcal{O}_{V\cap W}^\times,\ s \otimes t \mapsto s \cdot t,$$

which makes sense, because all sheaves are sheaves of submonoids of $(\mathcal{O}_{\underline{Y}}, \cdot)$.

If the complements Z and E of V and W, respectively, are Cartier divisors which have no common components, then they are locally given as the vanishing locus of non-zero-divisors $f, g \in \mathcal{O}_{\underline{Y}}$, respectively, such that if h is a local section of $\mathcal{O}_X$ with $h \mid f$ and $h \mid g$, then h lies in $\mathcal{O}_X^\times$. Let $f = f_1 \cdot \ldots \cdot f_p$ and $g = g_1 \cdot \ldots \cdot g_q$ be factorisations of f and g, respectively, into irreducible elements. For a p-tuple of natural numbers $m = (m_1, \ldots, m_p) \in \mathbb{N}_0^p$ and a q-tuple of natural numbers $n = (n_1, \ldots, n_q) \in \mathbb{N}_0^q$ we set $f^m := \prod_i f_i^{m_i}$ and $g^n := \prod_j g_j^{n_j}$.

If r is a local section of $j_{V\cap W\times}\mathcal{O}_{V\cap W}^\times$, then r is a local section of $\mathcal{O}_X$ which is invertible on $V \cap W$. Hence we may write $r = f^m g^n u$ for some local section u of $\mathcal{O}_X^\times$ and tuples $m \in \mathbb{N}_0^p$ and $n \in \mathbb{N}_0^q$. The local section $(f^m u) \otimes g^n$ of $j_{V\times}\mathcal{O}_V^\times \otimes_{\mathcal{O}_{\underline{Y}}^\times} j_{W\times}\mathcal{O}_W^\times$ is then a preimage of r. Hence this morphism is surjective.

Regard two local sections $f^m u \otimes g^n v$ and $f^{m'} u' \otimes g^{n'} v'$ of $j_{V\times}\mathcal{O}_V^\times \otimes_{\mathcal{O}_{\underline{Y}}^\times} j_{W\times}\mathcal{O}_W^\times$ both mapping to the same element $f^m g^n uv = f^{m'} g^{n'} u'v'$. By the fact that f and g are non-zero-divisors and that Z and E have no common components, this implies that $m = m'$, $n = n'$

and $uv = u'v'$. But then the two local sections $f^n u \otimes g^m v$ and $f^{m'} u' \otimes g^{n'} v'$ are equal, which shows the injectivity of the morphism. $\square$

6.1.2 Lemma

Let $\underline{f}\colon \underline{Y} \to \underline{X}$ be a morphism of schemes and let $U \subset \underline{X}$ be an open subscheme. Then there exists a natural morphism of log structures

$$\underline{f}^{\times} \alpha_{U \subset \underline{X}} \to \alpha_{f^{-1}(U) \subset \underline{Y}}.$$

If the reduced closed complement $Z = \underline{X} \setminus U$ of U is a Cartier divisor, then this morphism is an isomorphism.

Proof: We denote the morphisms by the names indicated in the following Cartesian diagram:

$$\begin{tikzcd} \underline{f}^{-1}(U) \arrow[r, hook, "\hat{j}"] \arrow[d, "\underline{f}' = \underline{f}|_U"] & \underline{Y} \arrow[d, "\underline{f}"] \\ U \arrow[r, hook, "j"] & \underline{X} \end{tikzcd}$$

Applying the functor $j_\times$ to the natural injective morphism of log structures $\iota_U \to f'_\times f'^\times \iota_U$ yields the injective morphism of log structures $\alpha_j = j_\times \iota_U \to j_\times f'_\times f'^\times \iota_U = f_\times \hat{j}_\times f'^\times \iota_U = f_\times \hat{j}_\times \iota_{f^{-1}(U)} = f_\times \alpha_{\hat{j}}$. Its adjoint morphism $f^\times \alpha_j \to \alpha_{\hat{j}}$ is the morphism we were looking for. It is given by the morphism of sheaves of monoids

$$f^\times j_{U\times} \mathcal{O}_U^\times = f^{-1} j_{U\times} \mathcal{O}_U^\times \otimes_{f^{-1}\mathcal{O}_X^\times} \mathcal{O}_Y^\times \to \hat{j}_\times \mathcal{O}_{f^{-1}(U)}^\times,\ s \otimes t \mapsto f^\sharp(s) \cdot t,$$

To show that it is an isomorphism if Z is a Cartier divisor, it is enough to consider the affine case $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then $U = \operatorname{Spec} A[\frac{1}{g}]$ and $f^{-1}(U) = \operatorname{Spec}(A[\frac{1}{g}] \otimes_A B) = \operatorname{Spec} B[\frac{1}{g}]$, where the non-zero-divisor $g \in A$ is an equation for Z. Hence, we have a cocartesian diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f^\sharp"] & A[\frac{1}{g}] \arrow[d, "f'^\sharp"] \\ B \arrow[r] & B[\frac{1}{f^\sharp(g)}]. \end{tikzcd}$$

The morphism in question is

$$(A[\tfrac{1}{g}])^\times \otimes_{A^\times} B^\times = A^\times \langle g \rangle \otimes_{A^\times} B^\times \to (B[\tfrac{1}{f^\sharp(g)}])^\times = B^\times \left\langle f^\sharp(g) \right\rangle,$$

$ag^m \otimes b \mapsto f^\sharp(a) b (f^\sharp(g))^m$, which is clearly a group isomorphism. Here, we write $A^\times \langle g \rangle$ for the subgroup of the quotient field $K(A)$ generated by $A^\times$ and g, which, since g is not an element of $A^\times$, is isomorphic to $A^\times \times \mathbb{Z}$. $\square$

6.1.3 Proposition

Let X be a log scheme the log structure of which is the compactifying log structure $\alpha_{U\subset X}$ associated to the open immersion of the complement U of a Cartier divisor Z and let X^β be the log blow-up of X along a closed subscheme P and let $E := \beta^{-1}(P)$ be its exceptional divisor and $W = X^\beta \setminus E$ its open complement. Then the log structure of X^β is isomorphic to the compactifying log structure associated to the open immersion of the subscheme $\beta^{-1}(U) \cap W = \beta^{-1}(X \setminus (P \cup Z))$.

Proof: By definition, $\alpha_{X^\beta} = \beta^\times \alpha_{U\subset X} \otimes_{\iota_{X^\beta}} \alpha_{W\subset X^\beta}$. By lemma 6.1.2, this is isomorphic to $\alpha_{\beta^{-1}(U)\subset X^\beta} \otimes_{\iota_{X^\beta}} \alpha_{W\subset X^\beta}$, which, by lemma 6.1.1 is isomorphic to $\alpha_{f^{-1}(U)\cap W\subset X^\beta}$, as claimed. □

6.1.4 Definition

Let $\underline{\beta}\colon \underline{X^\beta} \to \underline{X}$ be the blow-up of a scheme $\underline{X}$ along a closed subscheme $P \subset \underline{X}$. Let $Y \subset X$ be a closed subscheme. The *total transform* of Y is the closed subscheme $\beta^{-1}(Y) \subset X^\beta$. Its closed subscheme $\hat{Y} \subset \beta^{-1}(Y)$ defined by the ideal generated by local sections of $\mathcal{O}_{\beta^{-1}(Y)}$ which are supported in $E \cap \beta^{-1}(Y)$ is called its *strict transform.*

Giving these schemes the log structure as closed subschemes of X^β, we may speak of the total transform and the strict transform of $Y \subset X$ under β, respectively.

6.1.5 Lemma ([13, II.7.15])

Let $\underline{\beta}\colon \underline{X^\beta} \to \underline{X}$ be the blow-up of a scheme $\underline{X}$ along a closed subscheme $P \subset \underline{X}$. Let $Y \subset X$ be a closed subscheme. Then the strict transform $\hat{Y} \subset \underline{X^\beta}$ of Y is the blow-up of Y along $P \cap Y = P \times_{\underline{X}} Y \subset \underline{X} \times_{\underline{X}} \underline{X} = \underline{X}$.

6.1.6 Proposition

Let X be a log scheme the log structure of which is the compactifying log structure $\alpha_{U\subset X}$ associated to the open immersion of the complement U of a Cartier divisor Z and let $\beta\colon X^\beta \to X$ be the blow-up of X along a closed subscheme P. Let $Y \subset X$ be a closed subscheme and $\hat{Y}$ its strict transform under β in X^β. Then $\hat{Y}$ is the blow-up of Y along $P \cap Y$.

Proof: By the previous lemma 6.1.5, $\underline{\hat{Y}}$ is the blow-up $\underline{\beta_0}\colon \underline{Y^{\beta_0}} \to \underline{Y}$ of $\underline{Y}$ along $P \cap \underline{Y}$. It remains to show that the log structure of $\hat{Y}$ as a closed subscheme of X^β is $\alpha_{Y^{\beta_0}}$.

The log structure of $\hat{Y}$ is the restriction of that of X^β, so, using lemma 6.1.2 and proposition 6.1.3,

$$\begin{aligned}\alpha_{\hat{Y}} &= i^\times \alpha_{X^\beta} = i^\times \alpha_{\beta^{-1}(X\setminus(P\cup Z))\subset X^\beta}\\ &= \alpha_{i^{-1}(\beta^{-1}(X\setminus(P\cup Z)))\subset X_0^{\beta_0}} = \alpha_{\beta_0^{-1}(X_0\setminus(X_0\cap(P\cup Z)))\subset X_0^{\beta_0}}\\ &= \alpha_{X_0^{\beta_0}}.\end{aligned}$$

□

6.2 Examples constructed by blowing up

Now that we know what the blow-up of a log scheme is, we may construct our first two examples of log symplectic schemes (of general type). Let $\underline{g_0}\colon \underline{S}_0 \to \operatorname{Spec}\mathbb{C}$ be a proper smooth symplectic scheme of dimension $d = 2n$, $n \geq 1$, with symplectic form $\pi \in \Gamma(\underline{S}_0, \Omega^2_{\underline{g}_0})$.

Let $\underline{\mathcal{S}} := \underline{S}_0 \times_{\operatorname{Spec}\mathbb{C}} \operatorname{Spec}\mathbb{C}[t]$ and denote the corresponding natural second projection by $\underline{g}\colon \underline{\mathcal{S}} \to \operatorname{Spec}\mathbb{C}[t]$. Then $\underline{g}$ is trivially a (relative) smooth symplectic scheme. We identify $\underline{S}_0$ with the central fibre $\underline{\mathcal{S}} \times_{\operatorname{Spec}\mathbb{C}[t]} \operatorname{Spec} k(0)$ of the family $\underline{g}$.

We denote by U the open subscheme $\underline{\mathcal{S}} \setminus \underline{S}_0$ of $\underline{\mathcal{S}}$ and by $j\colon U \to \underline{\mathcal{S}}$ its open immersion. Then $(\underline{\mathcal{S}}, j)$ is a scheme with open immersion. As such it carries the compactifying log structure α_j. We write $\mathcal{S} := (\underline{\mathcal{S}}, \alpha_{\mathcal{S}})$ for this log scheme. Analogously, $\operatorname{Spec}\mathbb{C}[t]$ carries the log structure associated to the open immersion of the complement of its point 0, which turns it into the log affine scheme $\mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$.

Naturally, $\underline{g}$ underlies a strict log smooth morphism $g\colon \mathcal{S} \to \mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$ of log schemes since $g^{-1}(0) = S_0$ consists of a single component. Its restriction to the central fibre $g_0\colon S_0 \to \operatorname{Spec}\mathbb{C}$ is a SNC log scheme (consisting of one single component) with underlying morphism of schemes $\underline{g_0}$.

Now g is already a (rather trivial) example of a log symplectic scheme (of non-twisted type): Pulling back π to $\mathcal{S}$ via its first projection $pr_1\colon \mathcal{S} \to S_0$ defines a log symplectic form $\tilde{\pi} := pr_1^*\pi \in \Gamma(\mathcal{S}, \Omega^2_g)$ on $\mathcal{S}$ over $\mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$. We will construct two examples of SNC symplectic schemes consisting of two irreducible components based on this trivial example.

6.2.1 First example: Blowing up a point

Let s be a closed point in S_0 and let $\beta\colon \mathcal{X} \to \mathcal{S}$ be the blow-up of $\mathcal{S}$ in the point $(s, 0)$ as defined in section 6.1 above, with $\mathcal{X} := \mathcal{S}^\beta$. By proposition 6.1.3, $\mathcal{X}$ carries the compactifying log structure associated to the open immersion of the complement of the closed subscheme $X := \beta^{-1}(S_0) \subset \mathcal{X}$.

This preimage X of S_0 under β is a strict normal crossing divisor in $\mathcal{X}$, consisting of two components: $X_{[1]}$, which is the exceptional divisor of the blow-up, isomorphic to $\mathbb{P}^{2n}_{\mathbb{C}}$, and $X_{[2]}$, which is the strict transform of S_0 under β. Here, all closed subschemes carry the log structure induced from their ambient log scheme. Denoting the blow-up of S_0 in the point s by $\beta_0\colon S_0^{\beta_0} \to S_0$, we have $X_{[2]} \cong S_0^{\beta_0}$ by proposition 6.1.6; in particular, $\alpha_{X_{[2]}} = \alpha_{S_0^{\beta_0}}$.

Regard the composition $f := g \circ \beta\colon \mathcal{X} \to \mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$. As shown in chapter 3, section 3.6, f is a log smooth morphism of log schemes and its restriction $f_0 := f|_X\colon X \to \operatorname{Spec}\mathbb{C}$ is an SNC log variety.

Since $\mathcal{X}$ carries the log structure of a regular scheme with SNC divisor $X = X_{[1]} \cup X_{[2]}$, the group of its log Cartier divisors is $\operatorname{LCar}(\mathcal{X}) = \mathbb{Z} \cdot X_{[1]} \oplus Z \cdot X_{[2]} \cong \mathbb{Z}^2$. We let $\mathcal{L}$ be the line bundle $\mathcal{M}_{\mathcal{X}}(-2X_{[1]}) = \mathcal{O}_{\mathcal{X}}(-2X_{[1]})$ on $\mathcal{X}$. This line bundle comes with the flat log connection $\varDelta = (\mathcal{L}, \varDelta) = M_f(-2X_{[1]})$ which is, as remarked in section 3.6, the usual differential d restricted to the subsheaf $\mathcal{L} \subset \mathcal{O}_{\mathcal{X}}$.

We claim that $\varpi := \beta^*\tilde{\pi}$, the pullback of $\tilde{\pi}$ via β, is an element of $\varGamma(\varOmega_f^2 \otimes \mathcal{L})$ which is a log symplectic form of type $\varDelta$. Since β is an isomorphism outside of the point $(s, 0)$, it is enough to do a calculation in étale-local coordinates at $(s, 0)$:

Étale-locally at s, S_0 is isomorphic to $\mathbb{A}^{2n}_{\mathbb{C}}$, with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ and the point s corresponding to the origin, with trivial log structure and with the (log) symplectic form $\pi = \sum_{i=1}^n dx_i \wedge dy_i$. Then, étale-locally at $(s, 0)$, $\mathcal{S}$ is isomorphic to $(\mathbb{A}^{2n+1}_{\mathbb{C}}, \alpha)$, with the additional coordinate t, with log structure α associated to the prelog structure $\mathbb{N}_0 \to \mathcal{O}_{\mathbb{A}^{2n+1}_{\mathbb{C}}}, n \mapsto t^n$.

Although to our knowledge there exists no algebraic or holomorphic Darboux theorem, we assume for our calculations that this étale local situation resembles a Darboux chart in the sense, that we may write $\tilde{\pi} = \sum_{i=1}^n dx_i \wedge dy_i$ considered as a log symplectic form on $g\colon (\mathbb{A}^{2n+1}_{\mathbb{C}}, \alpha) \to \mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$ in this situation (cf. [25, 3.15 & 16] for the real analytic case). So étale-locally,

$$\underline{\mathcal{X}} = \underline{\mathcal{S}}^{\beta} = \operatorname{Proj} \frac{\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n, t][\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n, \tau]}{(x_i\tau - t\xi_i, y_i\tau - t\eta_i, x_i\xi_j - x_j\xi_i, y_i\eta_j - y_j\eta_i, x_i\eta_j - y_j\xi_i)}$$

(where the x_i, y_i and t have degree 0) with the central fibre given by

$$\underline{X} = \operatorname{Proj} \frac{\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n][\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n, \tau]}{(x_i\tau, y_i\tau, x_i\xi_j - x_j\xi_i, y_i\eta_j - y_j\eta_i, x_i\eta_j - y_j\xi_i)}$$

which consists of the two components

$$\underline{X}_{[1]} = \operatorname{Proj} \mathbb{C}[\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n, \tau] \cong \mathbb{P}^{2n}_{\mathbb{C}},$$

given by the ideal $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, and

$$\underline{X}_{[2]} = \operatorname{Proj} \frac{\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n][\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n]}{(x_i\xi_j - x_j\xi_i, y_i\eta_j - y_j\eta_i, x_i\eta_j - y_j\xi_i)} \cong \underline{S}_0^{\beta_0},$$

given by the equation $\tau = 0$. Their intersection

$$\underline{X}_{[12]} = \operatorname{Proj} \mathbb{C}[\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n] \cong \mathbb{P}^{2n-1}_{\mathbb{C}},$$

which is the double locus of $\underline{X}$, is the exceptional divisor in $X_{[2]}$.

The affine subscheme of $\mathcal{X}$ on which ξ_1 is invertible and to which we will refer as $\mathcal{X}_{\xi_1^{-1}}$ is isomorphic to

$$\operatorname{Spec} \mathbb{C}[x_1, \overline{\xi}_2, \ldots, \overline{\xi}_n, \overline{\eta}_1, \ldots, \overline{\eta}_n, \overline{\tau}]/(x_1\overline{\tau}),$$

where $\overline{\xi}_i = \frac{\xi_i}{\xi_1}$, $\overline{\eta}_i = \frac{\eta_i}{\xi_1}$ and $\overline{\tau} = \frac{\tau}{\xi_1}$, and its log structure $\alpha_{\mathcal{X}_{\xi_1^{-1}}}$ is the one associated to the prelog structure $a_{\mathcal{X}_{\xi_1^{-1}}} : \mathbb{N}_0^2 \to \mathcal{O}_{\mathcal{X}_{\xi_1^{-1}}}, (n_1, n_2) \mapsto x_1^{n_1}\overline{\tau}^{n_2}$.

In this chart we calculate

$$\begin{aligned}
\varpi = \sigma^*\tilde{\pi} &= dx_1 \wedge d(x_1\overline{\eta}_1) + \sum_{i=2}^{n} d(x_1\overline{\xi}_i) \wedge d(x_1\overline{\eta}_i) \\
&= x_1 dx_1 \wedge d\overline{\eta}_1 + \sum_{i=2}^{n} x_1\overline{\xi}_i dx_1 \wedge d\overline{\eta}_i + x_1\overline{\eta}_i d\overline{\xi}_i \wedge dx_1 + x_1^2 d\overline{\xi}_i \wedge d\overline{\eta}_i \\
&= x_1^2 \left(d\log x_1 \wedge d\overline{\eta}_1 + \sum_{i=2}^{n} \overline{\xi}_i d\log x_1 \wedge d\overline{\eta}_i + \overline{\eta}_i d\overline{\xi}_i \wedge d\log x_1 + d\overline{\xi}_i \wedge d\overline{\eta}_i \right) \\
&\in \Gamma(\mathcal{X}_{\xi_1^{-1}}, \Omega_f^2 \otimes \mathcal{O}_{\mathcal{X}}(-2X_{[1]})),
\end{aligned}$$

writing $d\log x_1$ for $d\log(1,0)$ and analogously in the other charts $\mathcal{X}_{\xi_i^{-1}}$.

The affine subscheme $\mathcal{X}_{\eta_1^{-1}}$ is isomorphic to

$$\operatorname{Spec} \mathbb{C}[y_1, \overline{\xi}_1, \ldots, \overline{\xi}_n, \overline{\eta}_2, \ldots, \overline{\eta}_n, \overline{\tau}]/(y_1\overline{\tau}),$$

where $\overline{\xi}_i = \frac{\xi_i}{\eta_1}$, $\overline{\eta}_i = \frac{\eta_i}{\eta_1}$ and $\overline{\tau} = \frac{\tau}{\eta_1}$, and its log structure $\alpha_{\mathcal{X}_{\eta_1^{-1}}}$ is the one associated to the prelog structure $a_{\mathcal{X}_{\eta_1^{-1}}} : \mathbb{N}_0^2 \to \mathcal{O}_{\mathcal{X}_{\eta_1^{-1}}}$, $(n_1, n_2) \mapsto y_1^{n_1}\overline{\tau}^{n_2}$.

In this chart we calculate

$$\begin{aligned}
\varpi = \sigma^*\tilde{\pi} &= d(\overline{\xi}_1 y_1) \wedge dy_1 + \sum_{i=2}^{n} d(y_1\overline{\xi}_i) \wedge d(y_1\overline{\eta}_i) \\
&= y_1^2 \left(d\overline{\xi_1} \wedge d\log y_1 + \sum_{i=2}^{n} \overline{\xi}_i d\log y_1 \wedge d\overline{\eta}_i + \overline{\eta}_i d\overline{\xi}_i \wedge d\log y_1 + d\overline{\xi}_i \wedge d\overline{\eta}_i \right) \\
&\in \Gamma(\mathcal{X}_{\eta_1^{-1}}, \Omega_f^2 \otimes \mathcal{O}_{\mathcal{X}}(-2X_{[1]})),
\end{aligned}$$

writing $d\log y_1$ for $d\log(1,0)$, and analogously in the other charts η_i^{-1}.

Finally, the affine subscheme τ^{-1} is isomorphic to

$$\operatorname{Spec} \mathbb{C}[t, \overline{\xi}_1, \ldots, \overline{\xi}_n, \overline{\eta}_1, \ldots, \overline{\eta}_n],$$

where $\overline{\xi}_i = \frac{\xi_i}{\tau}$ and $\overline{\eta}_i = \frac{\eta_i}{\tau}$, and its log structure $\alpha_{\tau^{-1}}$ is the one associated to the prelog structure $a_{\tau^{-1}} : \mathbb{N}_0^2 \to \mathcal{O}_{\tau_1^{-1}}$, $(n_1, n_2) \mapsto t^{n_2}$ (or the prelog structure $a_{\tau^{-1}} : \mathbb{N}_0 \to \mathcal{O}_{\tau^{-1}}$, $n \mapsto t^n$, if preferred).

In this chart we calculate

$$\varpi = \sigma^*\tilde{\pi} = t^2 \sum_{i=1}^{n} d\overline{\xi}_i \wedge d\overline{\eta}_i \in \Gamma(\mathcal{X}_{\tau^{-1}}, \Omega_f^2 \otimes \mathcal{O}_{\mathcal{X}}(-2X_{[1]}))$$

(observe that $\tau^{-1} \cap X_{[2]} = \emptyset$). This shows that ϖ is indeed an element of $\Gamma(\Omega_f^2 \otimes \mathcal{L})$.

Next, we show that it is closed under Δ: In the affine chart $\mathcal{X}_{\xi_1^{-1}}$ the log connection $\Delta = M_f(-2X_{[1]})$ is given by the discrepancy pair $(\frac{x_1^2}{x_1^2}, 2d\log x_1) = (1, 2d\log x_1)$ and thus it acts in this chart on forms σ with values in $\mathcal{L}$ by the rule $\Delta(x_1^2 \cdot \sigma) = x_1^2 \cdot (d\sigma + 2d\log x_1 \wedge \sigma)$

(cp. section 3.6). We calculate that

$$\begin{aligned}&\Delta\left(x_1^2\Big(d\log x_1\wedge d\overline{\eta}_1+\sum_{i=2}^n\overline{\xi}_i d\log x_1\wedge d\overline{\eta}_i+\overline{\eta}_i d\overline{\xi}_i\wedge d\log x_1+d\overline{\xi}_i\wedge d\overline{\eta}_i\Big)\right)\\&=x_1^2\Big(\sum_{i=2}^n d\overline{\xi}_i\wedge d\log x_1\wedge d\overline{\eta}_i+d\overline{\eta}_i\wedge d\overline{\xi}_i\wedge d\log x_1+2d\log x_1\wedge d\overline{\xi}_i\wedge d\overline{\eta}_i\Big)=0,\end{aligned}$$

and the same may be checked to be true in all other charts (here the discrepancy pair of Δ is $(1, 2d\log x_j)$, $(1, 2d\log y_j)$ and $(1, 2d\log t)$, respectively).

Finally, we have to verify that ϖ induces an isomorphism $T_f \to \Omega^1_f \otimes \mathcal{L}$. In the chart $\mathcal{X}_{\xi_1^{-1}}$ the map in question sends

$$\begin{aligned}x_1\partial_{x_1}&\mapsto x_1^2\Big(d\overline{\eta}_1+\sum_{i=2}^n\overline{\xi}_i d\overline{\eta}_i-\overline{\eta}_i d\overline{\xi}_i\Big),\\ \partial_{\overline{\xi}_i}&\mapsto x_1^2\left(\overline{\eta}_i d\log x_1+d\overline{\eta}_i\right),\\ \partial_{\overline{\eta}_1}&\mapsto x_1^2\left(-d\log x_1\right)\text{ and}\\ \partial_{\overline{\eta}_i}&\mapsto x_1^2\left(-\overline{\xi}_i d\log x_1-d\overline{\xi}_i\right).\end{aligned}$$

Therefore, its inverse is given by

$$\begin{aligned}-\partial_{\overline{\eta}_1}&\mapsto x_1^2 d\log x_1,\\ -\partial_{\overline{\eta}_i}+\overline{\xi}_i\partial_{\overline{\eta}_1}&\mapsto x_1^2 d\overline{\xi}_i,\\ \partial_{\overline{\xi}_i}+\overline{\eta}_i\partial_{\overline{\eta}_1}&\mapsto x_1^2 d\overline{\eta}_i\text{ and}\\ x_1\partial_{x_1}-\sum_{i\geq 2}\overline{\xi}_i\partial_{\overline{\xi}_i}+\overline{\eta}_i\partial_{\overline{\eta}_i}+2\overline{\xi}_i\overline{\eta}_i\partial_{\overline{\eta}_1}&\mapsto x_1^2 d\overline{\eta}_1,\end{aligned}$$

showing the bijectivity of the induced map $T_f \to \Omega^1_f \otimes \mathcal{L}$ in this chart. Proceeding analogously in all other charts, one shows that ϖ indeed induces an isomorphism.

Eventually, we have shown that $f\colon (X, \Delta, \varpi) \to \mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$ is a log symplectic scheme (of general type).

Writing $f_0 := f|_X : X \to \operatorname{Spec}\mathbb{C}$, $L := \mathcal{L}|_X$, $\nabla := \Delta|_X$ and $\omega := \varpi|_X$, the $\operatorname{Spec}\mathbb{C}$-log-scheme $f\colon (X, \nabla, \omega) \to \operatorname{Spec}\mathbb{C}$ is an example of an SNC log symplectic variety (over $\operatorname{Spec}\mathbb{C}$):

The line bundle $L = \mathcal{M}_X(-2X_{[1]}) = \mathcal{M}_X(2X_{[2]})$ obtained by restricting $\mathcal{L}$ to X is given on the two components of X as follows. On $X_{[1]}$ we have

$$L_{[1]} = \mathcal{L}|_{X_{[1]}} = \mathcal{O}_{\mathcal{X}}(2X_{[2]})\big|_{X_{[1]}} = \mathcal{O}_{X_{[1]}}(2X_{[12]}),$$

whereas on $X_{[2]}$ we have

$$L_{[2]} = \mathcal{L}|_{X_{[2]}} = \mathcal{O}_{\mathcal{X}}(-2X_{[1]})\big|_{X_{[2]}} = \mathcal{O}_{X_{[2]}}(-2X_{[12]}).$$

The restriction to $X_{[12]}$ of $L_{[1]}$ and $L_{[2]}$ is $\mathcal{N}^{\otimes 2}_{X_{[12]}\subset X_{[1]}}$ and $\mathcal{N}^{\otimes -2}_{X_{[12]}\subset X_{[2]}}$, respectively, which are isomorphic, because $\mathcal{N}_{X_{[12]}\subset X_{[1]}} \otimes_{\mathcal{O}_{X_{[12]}}} \mathcal{N}_{X_{[12]}\subset X_{[2]}} \cong \mathcal{O}_{X_{[12]}}$ by the semi-stability of X (cf. 3.5.24 and [11, 1.9]).

The flat log connection $\nabla = \Delta|_X$ is given on the components by $\nabla_{[1]} := \nabla|_{X_{[1]}}$ and $\nabla_{[2]} := \nabla|_{X_{[2]}}$ respectively, given by the augmentation of d as described in section 3.6.

The form $\omega := \varpi|_X$ is given on $X_{[1]}$ (in the chart $\mathcal{X}_{\xi_1^{-1}}$) as

$$\omega_{[1]} = -\frac{1}{\overline{\tau}^2}\Big(d\log\tau \wedge d\overline{\beta}_1 + \sum_{i=2}^{n}(\overline{\xi}_i d\log\tau \wedge d\overline{\eta}_i + \overline{\eta}_i d\overline{\xi}_i \wedge d\log\tau - d\overline{\xi}_i \wedge d\overline{\eta}_i)\Big)$$

in $\Gamma(\mathcal{X}_{[1]\,\xi_1^{-1}}, \Omega^2_{f_0} \otimes \mathcal{O}_{X_{[1]}}(2X_{[12]}))$ (observe that on $X_{[12]}$ we have $d\log x_1 = -d\log\overline{\tau}$ and $x_1^2 = \frac{1}{\overline{\tau}^2}$ for the generators of the restrictions of the line bundle) and on $X_{[2]}$ (in that chart) as

$$\omega_{[2]} = x_1^2\Big(d\log x_1 \wedge d\overline{\beta}_1 + \sum_{i=2}^{n}(\overline{\xi}_i d\log x_1 \wedge d\overline{\eta}_i + \overline{\eta}_i d\overline{\xi}_i \wedge d\log x_1 + d\overline{\xi}_i \wedge d\overline{\eta}_i)\Big)$$

in $\Gamma(\mathcal{X}_{[2]\,\xi_1^{-1}}, \Omega^2_{f_0} \otimes \mathcal{O}_{X_{[2]}}(-2X_{[12]}))$.

It inherits from ϖ the properties that it is closed under ∇ and that it induces an isomorphism $T_{f_0} \to \Omega^1_{f_0} \otimes L$.

Example 6.2.1 The above constructed $(f_0, \nabla, \omega)\colon X \to \operatorname{Spec}\mathbb{C}$ is an SNC log symplectic variety (of general type). The log symplectic scheme $(f, \Delta, \varpi)\colon \mathcal{X} \to \mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$ is a log symplectic deformation of f_0 (of general type) along the strict closed immersion $\operatorname{Spec}\mathbb{C} \to \mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$. All of its fibres other than X are smooth symplectic varieties in the sense that for each $p \neq 0 \in \mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$ the fibre $f_p\colon (\mathcal{X}_p, d, \omega_p) \to \mathbb{A}_{k(p)}[0]$ is a (strict) smooth morphism of log schemes with trivial log structures (hence, with smooth underlying morphism of schemes $\underline{f_p}$), with $\Delta_p = d$ and such that $\omega_p \in \Gamma(\mathcal{X}_p, \Omega^2_{f_p}) = \Gamma(\mathcal{X}_p, \Omega^2_{\underline{f_p}})$ is a symplectic form in the usual sense.

6.2.2 Second example: Blowing up a Lagrangian

In this example we take (S_0, π) and $g\colon (\mathcal{S}, \sigma) \to \mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$ as above, but, instead of blowing up a point, we choose a (relative) Lagrangian subscheme of $\mathcal{S}$.

So let $\Lambda \subset S_0$ be a regular Lagrangian subscheme of S_0, i. e. a regular closed subscheme of dimension $n = \frac{1}{2}\dim S_0$, with $\pi|_\Lambda = 0$. Let $\beta\colon \mathcal{X} \to \mathcal{S}$ denote the blow-up of $\mathcal{S}$ along its Lagrangian subscheme $\Lambda \times \{0\}$. The preimage X of the fibre S_0 is an SNC divisor in $\mathcal{X}$, consisting of two components: $X_{[1]}$ which is the exceptional divisor isomorphic to the projective space bundle $\mathbb{P}(\mathcal{N}_{\Lambda\times\{0\}\subset\mathcal{S}})$ (of rank n over $\Lambda \times \{0\}$), and $X_{[2]}$ which is the strict transform of S_0 under β. Denoting the blow-up of S_0 in Λ by $\beta_0\colon S_0^{\beta_0} \to S$, we have $X_{[2]} \cong S_0^{\beta_0}$.

The composition $f := g \circ \beta$ is a log smooth morphism and its restriction $f_0 := f|_X : X \to \operatorname{Spec}\mathbb{C}$ is an SNC log variety.

Let $\mathcal{L}$ be the line bundle $\mathcal{M}_{\mathcal{X}}(-X_{[1]}) = \mathcal{O}_{\mathcal{X}}(-X_{[1]})$. It comes with the flat log connection $\Delta = (\mathcal{L}, \Delta) = M_f(-X_{[1]})$ which is the usual differential d restricted to the subsheaf $\mathcal{L} \subset \mathcal{O}_{\mathcal{X}}$.

We claim that $\varpi := \beta^*\tilde{\pi}$ is an element of $\Gamma(\Omega^2_f \otimes L)$ which is a log symplectic form of type Δ. Again, this may be shown doing a local calculation in étale-local coordinates at a point $(s,0) \in \Lambda \times \{0\}$:

Assuming for our calculations a Darboux situation again, étale-locally at $s \in \Lambda$, S_0 is isomorphic to $\mathbb{A}^{2n}_{\mathbb{C}}$, with coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$, the point s corresponding to the origin and Λ given by the equations $x_1 = 0, \ldots, x_n = 0$, with trivial log structure and with the (log) symplectic form $\pi = \sum_{i=1}^n dx_i \wedge dy_i$. Then $\tilde{\pi} = \sum_{i=1}^n dx_i \wedge dy_i$ considered as a log symplectic form on g. Then, étale-locally at $(s,0)$, $\mathcal{S}$ is isomorphic to $(\mathbb{A}^{2n+1}_{\mathbb{C}}, \alpha)$, with the additional coordinate t, with log structure α associated to the prelog structure $\mathbb{N}_0 \to \mathcal{O}_{\mathbb{A}^{2n+1}_{\mathbb{C}}}$, $n \mapsto t^n$ and with $\tilde{\pi} = \sum_{i=1}^n dx_i \wedge dy_i$ considered as a log symplectic form on $g\colon (\mathbb{A}^{2n+1}_{\mathbb{C}}, \alpha) \to \mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$. So étale-locally,

$$\underline{\mathcal{X}} = \underline{\mathcal{S}^{\beta}} = \operatorname{Proj} \frac{\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n, t][\xi_1, \ldots, \xi_n, \tau]}{(x_i\tau - t\xi_i, x_i\xi_j - x_j\xi_i)},$$

with the central fibre given by

$$X = \operatorname{Proj} \frac{\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n][\xi_1, \ldots, \xi_n, \tau]}{(x_i\tau, x_i\xi_j - x_j\xi_i)}$$

which consists of the two components

$$\underline{X}_{[1]} = \operatorname{Proj} \mathbb{C}[y_1, \ldots, y_n][\xi_1, \ldots, \xi_n, \tau] \cong \Lambda \times \mathbb{P}^n_{\mathbb{C}} \cong \mathbb{P}(\mathcal{N}_{\Lambda \times \{0\} \subset \mathcal{S}}),$$

given by the ideal $(x_1, \ldots, x_n)$, and

$$\underline{X}_{[2]} = \operatorname{Proj} \frac{\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n][\xi_1, \ldots, \xi_n]}{(x_i\xi_j - x_j\xi_i)} \cong \underline{S^{\beta_0}},$$

given by the equation $\tau = 0$. Their intersection

$$\underline{X}_{[12]} = \operatorname{Proj} \mathbb{C}[y_1, \ldots, y_n][\xi_1, \ldots, \xi_n] \cong \Lambda \times \mathbb{P}^{n-1} \cong \mathbb{P}(\mathcal{N}_{\Lambda \subset S_0}) = \mathbb{P}(\mathcal{N}_{\Lambda \times \{0\} \subset \mathcal{S}}\big|_{\Lambda}),$$

which is the double locus of $\underline{X}$, is the exceptional divisor in $X_{[2]}$.

The affine subscheme of $\mathcal{X}$, on which ξ_1 is invertible (and to which we will simply refer as $\mathcal{X}_{\xi_1^{-1}}$), is isomorphic to

$$\operatorname{Spec} \mathbb{C}[x_1, \overline{\xi}_2, \ldots, \overline{\xi}_n, y_1, \ldots, y_n, \overline{\tau}]/(x_1\overline{\tau}),$$

where $\overline{\xi}_i = \frac{\xi_i}{\xi_1}$ and $\overline{\tau} = \frac{\tau}{\xi_1}$, and its log structure $\alpha_{\mathcal{X}_{\xi_1^{-1}}}$ is the one associated to the prelog structure $a_{\mathcal{X}_{\xi_1^{-1}}} : \mathbb{N}_0^2 \to \mathcal{O}_{\mathcal{X}_{\xi_1^{-1}}}$, $(n_1, n_2) \mapsto x_1^{n_1}\overline{\tau}^{n_2}$.

In this chart, writing $d\log x_1$ for $d\log e_{[1]}$ again, we calculate

$$\begin{aligned}
\varpi = \sigma^*\tilde{\pi} &= dx_1 \wedge dy_1 + \sum_{i=2}^n d(x_1\overline{\xi}_i) \wedge dy_i \\
&= x_1\Big(d\log x_1 \wedge dy_1 + \sum_{i=2}^n \overline{\xi}_i d\log x_1 \wedge dy_i + d\overline{\xi}_i \wedge dy_i\Big) \\
&\in \Gamma(\mathcal{X}_{\mathcal{X}_{\xi_1^{-1}}}, \Omega^2_f \otimes \mathcal{O}_{\mathcal{X}}(-X_{[1]}))
\end{aligned}$$

and analogously in the other charts (ξ_i^{-1}).

The affine subscheme τ^{-1} is isomorphic to

$$\operatorname{Spec}\mathbb{C}[\overline{\xi}_1,\ldots,\overline{\xi}_n,y_1,\ldots,y_n,t],$$

where $\overline{\xi}_i = \frac{\xi_i}{\tau}$, and its log structure $\alpha_{\tau^{-1}}$ and its log structure $\alpha_{\tau^{-1}}$ is the one associated to the prelog structure $a_{\tau^{-1}}\colon \mathbb{N}_0^2 \to \mathcal{O}_{\tau_1^{-1}}$, $(n_1,n_2)\mapsto t^{n_2}$ (or the prelog structure $a_{\tau^{-1}}\colon \mathbb{N}_0 \to \mathcal{O}_{\tau^{-1}}$, $n\mapsto t^n$ if preferred).

In this chart we calculate

$$\varpi = \sigma^*\tilde{\pi} = t\sum_{i=1}^n d\overline{\xi}_i \wedge dy_i \in \varGamma(\mathcal{X}_{\tau^{-1}}, \varOmega_f^2 \otimes \mathcal{O}_{\mathcal{X}}(-X_{[1]}))$$

(observe that $\tau^{-1}\cap X_{[2]} = \emptyset$). This shows that ϖ is indeed an element of $\varGamma(\varOmega_f^2\otimes\mathcal{L})$. We show that it is closed under $\varDelta$:

Indeed, in the chart $\mathcal{X}_{\xi_1^{-1}}$ the log connection $\varDelta = M_f(-X_{[1]})$ is given by the discrepancy pair $(1, d\log x_1)$, so we have

$$\begin{aligned}
&\varDelta\left(x_1\left(d\log x_1\wedge dy_1 + \sum_{i=2}^n \overline{\xi}_i d\log x_1\wedge dy_i + d\overline{\xi}_i\wedge dy_i\right)\right)\\
&= x_1\left(\sum_{i=2}^n d\overline{\xi}_i\wedge d\log x_1\wedge dy_i + d\log x_1\wedge d\overline{\xi}_i\wedge dy_i\right) = 0,
\end{aligned}$$

and the same may be checked to be true in all other charts (here the discrepancy pair of $\varDelta$ is $(1, 2d\log x_j)$ and $(1, 2d\log t)$, respectively).

Finally, we have to verify that ϖ induces an isomorphism $T_f \to \varOmega_f^1\otimes\mathcal{L}$. In the chart $\mathcal{X}_{\xi_1^{-1}}$ the map in question sends

$$\begin{aligned}
x_1\partial_{x_1} &\mapsto x_1\left(dy_1 + \sum_{i=2}^n \overline{\xi}_i dy_i\right),\\
\partial_{\overline{\xi}_i} &\mapsto x_1 dy_i,\\
\partial_{y_1} &\mapsto x_1\left(-d\log x_1\right) \text{ and}\\
\partial_{y_i} &\mapsto x_1\left(-\overline{\xi}_i d\log x_1 - d\overline{\xi}_i\right).
\end{aligned}$$

Therefore, its inverse is given by

$$\begin{aligned}
-\partial_{y_1} &\mapsto x_1 d\log x_1,\\
-\partial_{y_i} + \overline{\xi}_i\partial_{y_1} &\mapsto x_1 d\overline{\xi}_i,\\
\partial_{\overline{\xi}_i} &\mapsto x_1 dy_i \text{ and}\\
x_1\partial_{x_1} - \sum_{i\geq 2}\overline{\xi}_i\partial_{\overline{\xi}_i} &\mapsto x_1 dy_1,
\end{aligned}$$

showing the bijectivity of the induced map $T_f \to \varOmega_f^1\otimes\mathcal{L}$ in this chart. Proceeding analogously in all other charts, one shows that ϖ indeed induces an isomorphism.

Eventually, we have shown that $f\colon (X, \Delta, \varpi) \to \mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$ is a log symplectic scheme (of general type). Writing $f_0 := f|_X : X \to \operatorname{Spec} \mathbb{C}$, $L := \mathcal{L}|_X$, $\nabla := \Delta|_X$ and $\omega := \varpi|_X$, the $\operatorname{Spec}\mathbb{C}$-log-scheme $f\colon (X, \nabla, \omega) \to \operatorname{Spec}\mathbb{C}$ is an example of an SNC log symplectic variety (over $\operatorname{Spec}\mathbb{C}$):

The line bundle $L = \mathcal{M}_X(-X_{[1]}) = \mathcal{M}_X(X_{[2]})$ obtained by restricting $\mathcal{L}$ to X is given on the two components of X as follows: On $X_{[1]}$ we have

$$L_{[1]} = \mathcal{L}|_{X_{[1]}} = \mathcal{O}_{\mathcal{X}}(X_{[2]})\big|_{X_{[1]}} = \mathcal{O}_{X_{[1]}}(X_{[12]}),$$

whereas on $X_{[2]}$ we have

$$L_{[2]} = \mathcal{L}|_{X_{[2]}} = \mathcal{O}_{\mathcal{X}}(-X_{[1]})\big|_{X_{[2]}} = \mathcal{O}_{X_{[2]}}(-X_{[12]}).$$

The restriction to $X_{[12]}$ of $L_{[1]}$ and $L_{[2]}$ is $\mathcal{N}_{X_{[12]} \subset X_{[1]}}$ and $\mathcal{N}_{X^{-1}_{[12]} \subset X_{[2]}}$, respectively, which are isomorphic by the semi-stability of X.

The flat log connection $\nabla := \Delta|_X$ is given on the components by $\nabla_{[1]} := \nabla|_{X_{[1]}}$ and by $\nabla_{[2]} := \nabla|_{X_{[2]}}$, respectively, both defined by the augmentation of the differential d.

The form $\omega := \varpi|_X$ is given on $X_{[1]}$ (in the chart $\mathcal{X}_{\xi_1^{-1}}$) as

$$\omega_{[1]} = -\frac{1}{\overline{\tau}} \left(d\log \overline{\tau} \wedge dy_1 + \sum_{i=2}^{n} \left(\overline{\xi}_i d\log \overline{\tau} \wedge dy_i - d\overline{\xi}_i \wedge dy_i \right) \right)$$

in $\Gamma(\mathcal{X}_{[1]\,\xi_1^{-1}}, \Omega^2_{f_0} \otimes \mathcal{O}_{X_{[1]}}(X_{[12]}))$ (observe that on X we have $d\log x_1 = -d\log \overline{\tau}$ and $x_1 = \frac{1}{\overline{\tau}}$ for the generator of the line bundle) and on $X_{[2]}$ (in that chart) as

$$\omega_{[2]} = x_1 \left(d\log x_1 \wedge dy_1 + \sum_{i=2}^{n} \left(\overline{\xi}_i d\log x_1 \wedge dy_i + d\overline{\xi}_i \wedge dy_i \right) \right)$$

in $\Gamma(\mathcal{X}_{[2]\,\xi_1^{-1}}, \Omega^2_{f_0} \otimes \mathcal{O}_{X_{[2]}}(-X_{[12]}))$.

It inherits from ϖ the properties that it is closed under ∇ and induces an isomorphism $T_{f_0} \to \Omega^1_{f_0} \otimes L$.

Example 6.2.2 The above constructed $(f_0, \nabla, \omega)\colon X \to \operatorname{Spec}\mathbb{C}$ is an SNC log symplectic variety (of general type). The log symplectic scheme $(f, \Delta, \varpi)\colon \mathcal{X} \to \mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$ is a log symplectic deformation of f_0 (of general type) along the strict closed immersion $\operatorname{Spec}\mathbb{C} \to \mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$. All of its fibres other than X are smooth symplectic schemes in the sense, that for each $p \neq 0 \in \mathbb{A}_{\mathbb{C}}[\mathbb{N}_0]$, the fibre $f_p\colon (\mathcal{X}_p, d, \omega_p) \to \mathbb{A}_{k(p)}[0]$ is a (strict) smooth morphism of log schemes with trivial log structures (hence, with smooth underlying morphism of schemes $\underline{f_p}$), with $\Delta_p = d$ and such that $\omega_p \in \Gamma(\mathcal{X}_p, \Omega^2_{f_p}) = \Gamma(\mathcal{X}_p, \Omega^2_{\underline{f_p}})$ is a symplectic form in the usual sense.

6.3 Examples of Nagai

In [26, 4.3] Y. Nagai gives an example of a "good degeneration of compact symplectic manifold", which we will translate into the algebraic setting. In this section, all schemes are really schemes, not log schemes, unless otherwise stated. For the scheme X^β of the blow-up $\beta\colon X^\beta \to X$ along the closed subscheme $P \subset X$, we will use the classical notation $\mathrm{Bl}_P X$.
We will repeat and correct the calculations of Nagai's first example and then carry out the calculations for the second example he proposes in [26, 4.4].

6.3.1 Preparation

In this subsection we are going to carry out local calculations which we will use later to construct Nagai's examples. We write $\mathbb{A}$ instead of $\mathbb{A}_\mathbb{C}$. First, we calculate the Hilbert scheme $\mathrm{Hilb}^2(Z')$ for $Z' = \operatorname{Spec}\mathbb{C}[x,y,z]/(x,y)$ as an étale-local model for the Hilbert scheme of a normal crossing scheme near a point where two components meet. Afterwards, we calculate the Hilbert scheme $\mathrm{Hilb}^2(Z'')$ for $Z'' = \operatorname{Spec}\mathbb{C}[x,y,z]/(x,y,z)$ as a model for the Hilbert scheme of a normal crossing scheme near a point where three components meet. By a *chart* of a scheme, we mean a member of an open covering of that scheme.

Calculation of $\mathrm{Hilb}^2(\mathbb{A}^3)$ and its universal family

We begin by calculating the Hilbert scheme $H := \mathrm{Hilb}^2(\mathbb{A}^3)$ of the affine 3-space $\mathbb{A}^3$. It parametrises objects of the form $(G, \{x,y\})$ consisting of a line $G \subset \mathbb{A}^3$ together with two (not necessarily distinct) points $x, y \in G$. Hence, it is equal to the scheme of the blow-up $\mathrm{Bl}_{\overline{D}}(\mathrm{Sym}^2(\mathbb{A}^3)) = \mathrm{Bl}_D(\mathbb{A}^3 \times \mathbb{A}^3)/\mathfrak{S}_2$, where $D \subset \mathbb{A}^3 \times \mathbb{A}^3$ is the diagonal and $\overline{D} \subset \mathrm{Sym}^2(\mathbb{A}^3)$ its image. It is known that this is a regular scheme (cf. [10]).
We write $\mathbb{A}^3 \times \mathbb{A}^3 = \operatorname{Spec}\mathbb{C}[x_1+y_1, x_2+y_2, x_3+y_3, x_1-y_1, x_2-y_2, x_3-y_3]$, where the coordinates x_i belong to the first and the y_i to the second copy of $\mathbb{A}^3$. The blow-up of this scheme along its diagonal D given by the equations $x_i = y_i$ is

$$B := \mathrm{Bl}_D(\mathbb{A}^3 \times \mathbb{A}^3) = \operatorname{Proj}\mathbb{C}[x_i+y_i, x_i-y_i][\eta_i]/(\eta_i(x_j-y_j) - \eta_j(x_i-y_i)).$$

The symmetric group $\mathfrak{S}_2$ acts with its non-trivial element by

$$x_i + y_i \mapsto x_i + y_i \qquad x_i - y_i \mapsto -(x_i - y_i) \qquad \eta_i \mapsto \eta_i.$$

In the affine chart $B_{\eta_1^{-1}}$ (where η_1 is invertible) of B we set $u_2 := \frac{\eta_2}{\eta_1}$ and $u_3 := \frac{\eta_3}{\eta_1}$. Eliminating the variables $x_2 - y_2$ and $x_3 - y_3$ by means of the equations $x_2 - y_2 - u_2(x_1 - y_1) = 0$ and $x_3 - y_3 - u_3(x_1 - y_1) = 0$, this chart is the regular affine scheme

$$B_{\eta_1^{-1}} = \operatorname{Spec}\mathbb{C}[x_1+y_1, x_2+y_2, x_3+y_3, x_1-y_1, u_2, u_3]$$

and the only coordinate not invariant under $\mathfrak{S}_2$ is $x_1 - y_1$. Hence dividing out the $\mathfrak{S}_2$-action, we get

$$H_{\eta_1^{-1}} = \eta_1^{-1}/\mathfrak{S}_2 = \operatorname{Spec}\mathbb{C}[x_1+y_1, x_2+y_2, x_3+y_3, x_1y_1, u_2, u_3]$$

as the chart of H, where η_1 is invertible. Setting $a_i := x_i + y_i$, $i = 1,2,3$, and $b_1 := x_1y_1$, we get

$$H_{\eta_1^{-1}} = \operatorname{Spec}\mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3].$$

Analogously, the two other charts $H_{\eta_2^{-1}}$ and $H_{\eta_3^{-1}}$ of H are given as

$$\operatorname{Spec}\mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3] \quad \text{and} \quad \operatorname{Spec}\mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2],$$

respectively, where $b_2 := x_2y_2$, $v_1 := \frac{\eta_1}{\eta_2}$ and $v_3 := \frac{\eta_3}{\eta_2}$, and $b_3 := x_3y_3$, $w_1 := \frac{\eta_1}{\eta_3}$ and $w_2 := \frac{\eta_2}{\eta_3}$, respectively.

The glueing morphisms between these three charts are given on the charts' overlaps by the following relations of the coordinates:

$$u_2v_1 = 1,\ u_3w_1 = 1,\ v_3w_2 = 1,$$

$$u_3w_2 = u_3,\ v_1w_2 = u_2,\ u_2w_1 = w_2,\ u_3w_1 = v_3,\ u_2v_3 = u_3,\ v_2w_1 = v_1,$$

$$(a_1^2 - 4b_1) = (a_2^2 - 4b_2)v_1^2,\ (a_2^2 - 4b_2) = (a_3^2 - 4b_3)w_2^2 \text{ and } (a_3^2 - 4b_3) = (a_1^2 - 4b_1)u_3^2.$$

Therefore, H is locally isomorphic to $\mathbb{A}^6$, as expected.

Next we calculate the universal family $\Xi \subset \operatorname{Hilb}^2(\mathbb{A}^3) \times \mathbb{A}^3$ of this Hilbert scheme. This is, by definition, a scheme Ξ together with the flat projection $\pi\colon \Xi \to H$ such that the fibre $\pi^{-1}(\xi)$ over a point $\xi \in H$ projects under $pr\colon H \times \mathbb{A}^3 \to \mathbb{A}^3$ to precisely the subscheme $\xi \subset \mathbb{A}^3$. We will identify the fibre $\pi^{-1}(\xi)$ and its image $pr(\pi^{-1}(\xi)) = \xi \subset \mathbb{A}^3$.

As the Hilbert scheme H parametrises objects of the form $(G, \{x, y\})$ consisting of a line $G \subset \mathbb{A}^3$ together with two (not necessarily distinct) points $x, y \in G$, the fibre $\pi^{-1}(G, \{x, y\})$ over such a point is given either by the reduced scheme consisting of two distinct points x and y or by the point $x = y$ together with the tangent direction of the line G (i. e. a "fat point").

Now we are going to calculate the closed subscheme

$$\Xi_{\eta_1^{-1}} \subset \operatorname{Spec}\mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3, z_1, z_2, z_3] = H_{\eta_1^{-1}} \times \mathbb{A}^3$$

above the chart $H_{\eta_1^{-1}}$:

If $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are two points (which, for the moment, we assume to be distinct) on a line G, then a third point $z = (z_1, z_2, z_3)$ lies on that line if and only if the matrix

$$\begin{pmatrix} z_1 & x_1 & y_1 \\ z_2 & x_2 & y_2 \\ z_3 & x_3 & y_3 \\ 1 & 1 & 1 \end{pmatrix}$$

has a rank ≤ 2, i. e. if all of its 3×3-minors equal 0.

This gives the equations

$$\begin{aligned} z_1(x_2-y_2)-z_2(x_1-y_1)+(x_1y_2-x_2y_1)&=0\\ z_2(x_3-y_3)-z_3(x_2-y_2)+(x_2y_3-x_3y_2)&=0\\ z_3(x_1-y_1)-z_1(x_3-y_3)+(x_3y_1-x_1y_3)&=0 \end{aligned}$$

(where the fourth is contained in these three and hence omitted here). As we look at the chart $H_{\eta_1^{-1}}$, we replace (x_2-y_2) with $u_2(x_1-y_1)$ and (x_3-y_3) with $u_3(x_1-y_1)$ to get

$$\begin{aligned} z_1u_2(x_1-y_1)-z_2(x_1-y_1)+(x_1y_2-x_2y_1)&=0\\ z_2u_3(x_1-y_1)-z_3u_2(x_1-y_1)+(x_2y_3-x_3y_2)&=0\\ z_3(x_1-y_1)-z_1u_3(x_1-y_1)+(x_3y_1-x_1y_3)&=0. \end{aligned}$$

Still assuming that x and y are distinct, we may divide by (x_1-y_1) to get

$$\begin{aligned} &z_1u_2-z_2+x_2-u_2x_1=0 \text{ and}\\ &z_3-z_1u_3+x_1u_3-x_3=0 \end{aligned}$$

(the middle equation is again contained in these two). Rewriting these equations in the coordinates $a_i=x_i+y_i$, u_2, u_3, b_1 and z_i, we get

$$\begin{aligned} &z_1u_2-z_2-\tfrac{1}{2}(a_1u_2-a_2)=0 \text{ and}\\ &z_1u_3-z_3-\tfrac{1}{2}(a_1u_3-a_3)=0. \end{aligned}$$

So given a point $\xi\in\mathrm{Hilb}^2(\mathbb{A}^3)$ consisting of two distinct points $\xi=\{x,y\}$, the scheme $\operatorname{Spec}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,z_1,z_2,z_3]/(z_1u_2-z_2-\frac{1}{2}(a_1u_2-a_2),z_1u_3-z_3-\frac{1}{2}(a_1u_3-a_3))$ parametrises the line in $\mathbb{A}^3\times\{\xi\}$ through these points. This is even true when ξ consist only of one point with tangent direction (as this is the limit of two distinct points). To cut the two points x and y (respectively the point $x=y$ with tangent direction) out of the line, we need one more equation in z_1 which is given by

$$0=(z_1-x_1)(z_1-y_1)=z_1^2-a_1z_1+b_1.$$

In conclusion, $\Xi_{\eta_1^{-1}}\subset H_{\eta_1^{-1}}\times\mathbb{A}^3$ is the affine scheme

$$\Xi_{\eta_1^{-1}}=\operatorname{Spec}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,z_1,z_2,z_3]/I_\Xi$$

with the ideal I_Ξ generated by

$$z_1u_2-z_2-\tfrac{1}{2}(a_1u_2-a_2),\quad z_1u_3-z_3-\tfrac{1}{2}(a_1u_3-a_3)\quad\text{and}\quad z_1^2-a_1z_1+b_1.$$

Since the other charts lead to analogue results (with only the indices of the u's, b's and z's cyclically permuted), the universal family Ξ of H is itself a regular scheme.

Altogether, we have three charts for the universal family $\varXi \subset H \times \mathbb{A}^3$ of H (which we write down including redundant coordinates):

$$\begin{aligned}
\varXi_{\eta_1^{-1}} &= \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3, z_1, z_2, z_3]/I_{\varXi_1} \text{ with} \\
&\quad I_{\varXi_1} = \left(z_1u_2 - z_2 - \tfrac{1}{2}(a_1u_2 - a_2), z_1u_3 - z_3 - \tfrac{1}{2}(a_1u_3 - a_3), z_1^2 - a_1z_1 + b_1\right), \\
\varXi_{\eta_2^{-1}} &= \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3, z_1, z_2, z_3]/I_{\varXi_2} \text{ with} \\
&\quad I_{\varXi_2} = \left(z_2v_1 - z_1 - \tfrac{1}{2}(a_2v_1 - a_1), z_2v_3 - z_3 - \tfrac{1}{2}(a_2v_3 - a_3), z_2^2 - a_2z_2 + b_2\right) \text{ and} \\
\varXi_{\eta_3^{-1}} &= \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2, z_1, z_2, z_3]/I_{\varXi_3} \text{ with} \\
&\quad I_{\varXi_3} = \left(z_3w_1 - z_1 - \tfrac{1}{2}(a_3w_1 - a_1), z_3w_2 - z_2 - \tfrac{1}{2}(a_3w_2 - a_2), z_3^2 - a_3z_3 + b_3\right),
\end{aligned}$$

where $u_2 = \frac{\eta_2}{\eta_1}$, $u_3 = \frac{\eta_3}{\eta_1}$, $v_1 = \frac{\eta_1}{\eta_2}$, $v_3 = \frac{\eta_3}{\eta_2}$, $w_1 = \frac{\eta_1}{\eta_3}$ and $w_2 = \frac{\eta_2}{\eta_3}$ satisfy the usual relations on the open intersections.

6.3.1 Remark

The same calculation done in three variants, once for each affine chart of $\mathbb{P}^3$, yields the universal family of $\operatorname{Hilb}^2(\mathbb{P}^3)$.

Calculation of $\operatorname{Hilb}^2(\operatorname{Spec} \mathbb{C}[z_1, z_2, z_3]/(z_1z_2))$ and a semi-stable family

Let $Z' \subset \mathbb{A}^3$ denote the closed subvariety given by $z_1z_2 = 0$ and denote its two components by Z'_1 and Z'_2 given by $z_1 = 0$ and by $z_2 = 0$, respectively. We are going to calculate the scheme $H' := \operatorname{Hilb}^2(Z')$ which is a closed subscheme of $H = \operatorname{Hilb}^2(\mathbb{A}^3)$.

Instead of calculating H' directly in an analogue fashion to the above calculations for H, we use the universal family $\varXi$. Since H' is a closed subscheme of H, its universal family $\varXi'$ is a closed subscheme of $\varXi$ which consists precisely of those points of $\varXi$ belonging to a fibre that is a subscheme of $Z' \subset \mathbb{A}^3$. Hence, the equation $z_1z_2 = 0$ has to be added to those of $\varXi$.

Above $H_{\eta_1^{-1}}$ we have

$$\begin{aligned}
z_1z_2 &= z_1^2u_2 - \tfrac{1}{2}z_1(a_1u_2 - a_2) = (a_1z_1 - b_1)u_2 - \tfrac{1}{2}z_1(a_1u_2 - a_2) \\
&= \tfrac{1}{2}z_1(a_1u_2 + a_2) - b_1u_2 \text{ modulo } I_{\varXi_1},
\end{aligned}$$

and analogously in the other charts.

So the universal family $\varXi'$ of H' is given by

$$\begin{aligned}
\varXi'_{\eta_1^{-1}} &= \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3, z_1, z_2, z_3]/I_{\varXi'_1} \text{ with} \\
I_{\varXi'_1} &= \big(z_1u_2 - z_2 - \tfrac{1}{2}(a_1u_2 - a_2), z_1u_3 - z_3 - \tfrac{1}{2}(a_1u_3 - a_3), z_1^2 - a_1z_1 + b_1, \\
&\qquad \tfrac{1}{2}z_1(a_1u_2 + a_2) - b_1u_2\big), \\
\varXi'_{\eta_2^{-1}} &= \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3, z_1, z_2, z_3]/I_{\varXi'_2} \text{ with} \\
I_{\varXi'_2} &= \big(z_2v_1 - z_1 - \tfrac{1}{2}(a_2v_1 - a_1), z_2v_3 - z_3 - \tfrac{1}{2}(a_2v_3 - a_3), z_2^2 - a_2z_2 + b_2, \\
&\qquad \tfrac{1}{2}z_2(a_2u_1 + a_1) - b_2u_1\big) \text{ and}
\end{aligned}$$

$$\begin{aligned}
\Xi'_{\eta_3^{-1}} &= \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2, z_1, z_2, z_3]/I_{\Xi'_3} \text{ with}\\
I_{\Xi'_3} &= \big(z_3 w_1 - z_1 - \tfrac{1}{2}(a_3 w_1 - a_1), z_3 w_2 - z_2 - \tfrac{1}{2}(a_3 w_2 - a_2), z_3^2 - a_3 z_3 + b_3,\\
&\qquad \tfrac{1}{2} z_3(a_1 w_2 + a_2 w_1) + \tfrac{1}{4}((a_3^2 - 4b_3) w_1 w_2 - (a_1 w_2 + a_2 w_1) a_3)\big).
\end{aligned}$$

Let $A := \mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3]$ be the coordinate ring of the chart $H_{\eta_1^{-1}}$ of H and let $B := \mathbb{C}[z_1, z_2, z_3]$ the coordinate ring of $\mathbb{A}^3$. The coordinate ring of the chart $\Xi_{\eta_1^{-1}}$ of the universal family Ξ of H is then

$$A \otimes_{\mathbb{C}} B/I_{\Xi_1} \cong A \oplus A \cdot z_1$$

as an A-module. Hence, we may write $A \otimes_{\mathbb{C}} B = I_{\Xi_1} \oplus A \oplus A \cdot z_1$ as an A-module. If we write some element $f \in A \otimes_{\mathbb{C}} B$ in the corresponding form $f_0 + g + h \cdot z_1$, with $f_0 \in I_{\Xi_1}$ and $g, h \in A$, then $f \in I_{\Xi_1}$ if and only if both g and h are zero.
Since the ideal of $\Xi'_{\eta_1^{-1}}$ in $\Xi_{\eta_1^{-1}}$ is generated by the element $\frac{1}{2} z_1(a_1 u_2 + a_2) - b_1 u_2$, the ideal of $H'_{\eta_1^{-1}}$ in $H_{\eta_1^{-1}}$ is generated by the two polynomials $\frac{1}{2}(a_1 u_2 + a_2)$ and $b_1 u_2$.
Therefore, the Hilbert scheme $H' = \operatorname{Hilb}^2(Z')$ is given by the three charts

$$\begin{aligned}
H'_{\eta_1^{-1}} &= \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3]/(a_1 u_2 + a_2, b_1 u_2),\\
H'_{\eta_2^{-1}} &= \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3]/(a_2 v_1 + a_1, b_2 v_1) \text{ and}\\
H'_{\eta_3^{-1}} &= \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2]/(a_1 w_2 + a_2 w_1, (a_3^2 - 4b_3) w_1 w_2 + a_1 a_2).
\end{aligned}$$

Each of these charts may be decomposed into its irreducible components:

$$\begin{aligned}
H'_{\eta_1^{-1}} &= H'_{22,1} \cup H'_{12,1}, \text{ where}\\
&\quad H'_{22,1} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3]/(u_2, a_2) \text{ and}\\
&\quad H'_{12,1} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, u_2, u_3, b_1]/(a_1 u_2 + a_2, b_1),\\
H'_{\eta_2^{-1}} &= H'_{12,2} \cup H'_{11,2}, \text{ where}\\
&\quad H'_{11,2} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3]/(v_1, a_1) \text{ and}\\
&\quad H'_{12,2} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3]/(a_2 v_1 + a_1, b_2), \text{ and}\\
H'_{\eta_1^{-1}} &= H'_{11,3} \cup H'_{22,3} \cup H'_{12,3}, \text{ where}\\
&\quad H'_{11,3} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2]/(w_1, a_1),\\
&\quad H'_{22,3} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2]/(w_2, a_2) \text{ and}\\
&\quad H'_{12,3} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2]/J_{12,3} \text{ with}\\
J_{12,3} &= (a_1 w_2 + a_2 w_1, (a_3^2 - 4b_3) w_1 w_2 + a_1 a_2, (a_3^2 - 4b_3) w_1^2 - a_1^2, (a_3^2 - 4b_3) w_2^2 - a_2^2).
\end{aligned}$$

As we can see, H' consists in total of three irreducible components H'_{11}, H'_{12} and H'_{22} which are covered by the irreducible components of the charts as follows:

$$H'_{22} = H'_{22,1} \cup H'_{22,3}, \quad H'_{12} = H'_{12,1} \cup H'_{12,2} \cup H'_{12,3} \quad \text{and} \quad H'_{11} = H'_{11,2} \cup H'_{11,3}.$$

We examine the geometric meaning of these components: For the first component H'_{11}, we have $0 = a_1 = x_1 + y_1$ and $0 = v_1(x_2 - y_2) = x_1 - y_1$, hence, $x_1 = y_1 = 0$ in its chart $H'_{11,2}$ above $H_{\eta_2^{-1}}$. This means precisely that the support of a point $\xi \in H'_{11,2}$ is contained in Z'_1. The same argument holds for $H_{11,3}$, so $H_{11} = \mathrm{Hilb}^2(Z'_1)$. Analogously, for the second component, $H'_{22} = \mathrm{Hilb}^2(Z'_2)$.

In the chart $H'_{12,1}$ of the component H'_{12} above $H_{\eta_1^{-1}}$, we have $0 = b_1 = x_1y_1$ and $0 = a_1u_2 + a_2 = (x_1 + y_1)u_2 + (x_2 + y_2)$. This implies that $x_1 = 0$ and $y_2 = 0$, or (*not* either or) that $y_1 = 0$ and $x_2 = 0$. Hence, H'_{12} consists of those points $\xi \in H'$ the support of which consists either of two distinct points of which one is supported in Z'_1 and one in Z'_2, or is contained in $Z'_1 \cap Z'_2$.

These components intersect (in the affine charts) as listed below:

$$\begin{aligned}
H'_{\eta_1^{-1}}: \quad & H'_{11} \cap H'_{22} = \emptyset, \\
& H'_{11} \cap H'_{12} = \emptyset \text{ and} \\
& H'_{12} \cap H'_{22} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3]/(u_2, a_2, b_1), \text{ of } \dim = 3. \\
H'_{\eta_2^{-1}}: \quad & H'_{11} \cap H'_{22} = \emptyset, \\
& H'_{11} \cap H'_{12} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3]/(v_1, a_1, b_2), \text{ of } \dim = 3, \text{ and} \\
& H'_{12} \cap H'_{22} = \emptyset. \\
H'_{\eta_3^{-1}}: \quad & H'_{11} \cap H'_{22} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2]/(w_1, w_2, a_1, a_2), \text{ of } \dim = 2, \\
& H'_{11} \cap H'_{12} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2]/(w_1, a_1, (a_3^2 - 4b_3)w_2^2 - a_2^2) \text{ and} \\
& H'_{12} \cap H'_{22} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2]/(w_2, a_2, (a_3^2 - 4b_3)w_1^2 - a_1^2),
\end{aligned}$$

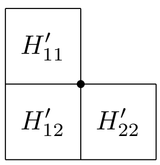
where the last two are of dimension 3. Since these intersections are all reduced regular schemes, any two components of H' meet transversally.

Finally, the intersection $H'_{11} \cap H'_{12} \cap H'_{22}$ of all three components of H' is empty above the charts $H_{\eta_1^{-1}}$ and $H_{\eta_2^{-1}}$ and is given above the chart $H_{\eta_3^{-1}}$ by

$$H'_{11,3} \cap H'_{12,3} \cap H'_{22,3} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2]/(a_1, a_2, w_1, w_2)$$

which equals the intersection of $H'_{11,3}$ and $H'_{22,3}$.

These calculations show that the situation of the components of H' in relation to each other may be visualised as follows (cp. [26, p. 419]):



Here each of the irreducible components H_{11}, H_{12} and H_{22} is 4-dimensional, each line represents a 3-dimensional subscheme (the intersections of the two adjacent components) and the point in the centre represents the 2-dimensional subscheme $H'_{11} \cap H'_{22}$.

The two components $H'_{11} = \operatorname{Hilb}^2(Z'_1)$ and $H'_{22} = \operatorname{Hilb}^2(Z'_2)$ are regular schemes. The singular locus of H'_{12} is supported, as one sees in the given charts, only above $H_{\eta_3^{-1}}$. Hence its support must be contained in the intersection of the subscheme $T_3 \subset H_{\eta_3^{-1}}$ defined by the equations $w_1 = 0$ and $w_2 = 0$ with the subscheme $H'_{12,3}$, which is

$$T_3 \cap H_{12,3} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2]/(w_1, w_2, a_1a_2, a_1^2, a_2^2).$$

Its reduction is the scheme

$$S'_3 := (T_3 \cap H'_{12,3})_{\mathrm{red}} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2]/(a_1, a_2, w_1, w_2),$$

which is precisely the intersection $H'_{11,3} \cap H'_{22,3}$. Indeed, a calculation using the computer algebra system Singular ([5], using libraries [5.a], [5.b] and [5.c]) gives the singular locus of $H_{12,3}$ as the scheme

$$\begin{aligned} &\operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2]/J_{12,3;\mathrm{sing}}, \text{ where} \\ &J_{12,3;\mathrm{sing}} = \big(a_1^2, a_1a_2, a_2^2, w_1^3, w_1^2w_2, w_1w_2^2, w_2^3, a_1w_1, a_1w_2, a_2w_1, a_2w_2, \\ &\qquad (a_3^2 - 4b_3)w_1^2, (a_3^2 - 4b_3)w_1w_2, (a_3^2 - 4b_3)w_2^2\big), \end{aligned}$$

the reduction of which is S'_3.

A smoothing family of the scheme Z' is $p\colon \mathbb{A}^3 \to \mathbb{A}^1$, given by the ring homomorphism $\mathbb{C}[t] \to \mathbb{C}[z_1, z_2, z_3]$, $t \mapsto z_1z_2$. This family induces the flat family

$$\varrho\colon \mathcal{Y} := \operatorname{Hilb}^2(\mathbb{A}^3/\mathbb{A}^1) \to \mathbb{A}^1$$

with $\mathcal{Y}_0 = H'$, which is over the charts $H_{\eta_1^{-1}}$, $H_{\eta_2^{-1}}$ and $H_{\eta_3^{-1}}$, respectively, given by the ring homomorphisms

$$\begin{aligned} &\mathbb{C}[t] \to \mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3]/(a_1u_2 + a_2), && t \mapsto -b_1u_2, \\ &\mathbb{C}[t] \to \mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3]/(a_2v_1 + a_1), && t \mapsto -b_2v_1, \text{ and} \\ &\mathbb{C}[t] \to \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2]/(a_1w_2 + a_2w_1), && t \mapsto \tfrac{1}{4}((a_3^2 - 4b_3)w_1w_2 + a_1a_2)), \end{aligned}$$

respectively, where the rings on the right hand side are the coordinate rings of the charts $\mathcal{Y}_{\eta_1^{-1}}$, $\mathcal{Y}_{\eta_2^{-1}}$ and $\mathcal{Y}_{\eta_3^{-1}}$ of $\mathcal{Y}$, respectively.
It is clear that $\mathcal{Y}$ is a closed subscheme of $H = \operatorname{Hilb}^2(\mathbb{A}^3)$, given by the equations $a_1u_2 + a_2 = 0$, $a_2v_1 + a_1 = 0$ or $a_1w_2 + a_2w_1 = 0$, respectively.
We would like to modify this family ϱ in a way such that the central fibre is a strict normal crossing divisor. This is almost the case here, except for the fact that the component H'_{12} of $\mathcal{Y}_0 = H'$ is singular, with the support of its singular locus being $H'_{11} \cap H'_{22}$. As proposed by Nagai in [26], we may resolve these singularities by blowing up the regular ambient space H along the component H'_{11} of $H' = \mathcal{Y}_0$, which contains the singular locus of H'_{12}, and then take the family $\hat{\mathcal{Y}} \to \mathbb{A}^1$ given by the strict transform of $\mathcal{Y}$ under this blow-up.

So let $\sigma\colon \hat{H} \to H$ be the blow-up of $H = \mathrm{Hilb}^2(\mathbb{A}^3)$ along the closed subscheme $H_{11} \subset H' = \mathcal{Y}_0 \subset H$ and let $\hat{\varrho} := \varrho \circ \sigma\colon \hat{\mathcal{Y}} \to \mathbb{A}^1$ be the flat family defined by composing $\sigma|_{\hat{\mathcal{Y}}}$ with ϱ.

Above the chart $H_{\eta_1^{-1}}$ the component H'_{11} is not visible in H, so the blow-up is an isomorphism here, which implies $\hat{\mathcal{Y}}_{\eta_1^{-1}} \cong \mathcal{Y}_{\eta_1^{-1}}$ and $\hat{\varrho} = \varrho$ under this isomorphism.

Above $H_{\eta_2^{-1}}$, the blow-up $\hat{H}_{\eta_2^{-1}}$ is given as

$$\operatorname{Proj} \mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3][\alpha_2, \beta_2]/(\alpha_2 v_1 - \beta_2 a_1),$$

so the preimage of $\mathcal{Y}_{\eta_2^{-1}}$ is given as

$$\operatorname{Proj} \mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3][\alpha_2, \beta_2]/(a_2 v_1 + a_1, \alpha_2 v_1 - \beta_2 a_1)$$

which is covered by the two affine schemes

$$\operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3, \overline{\beta}_2, t]/(a_1(a_2\overline{\beta}_2 + 1), v_1 - \overline{\beta}_2 a_1),$$

where $\overline{\beta}_2 = \frac{\beta_2}{\alpha_2}$, and

$$\operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3, \overline{\alpha}_2, t]/(v_1(a_2 + \overline{\beta}_2), \overline{\alpha}_2 v_1 - a_1),$$

where $\overline{\alpha}_2 = \frac{\alpha_2}{\beta_2}$.

Since the exceptional divisor of the blow-up is given by the equation $a_1 = 0$ or $v_1 = 0$, depending on the chart, the strict transform $\hat{\mathcal{Y}}_{\eta_2^{-1}}$ is given by the two affine schemes

$$\operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3, \overline{\beta}_2]/(a_2\overline{\beta}_2 + 1, v_1 - \overline{\beta}_2 a_1) \text{ and}$$
$$\operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3, \overline{\alpha}_2]/(a_2 + \overline{\beta}_2, \overline{\alpha}_2 v_1 - a_1)$$

which are both regular schemes.

The flat family $\hat{\varrho}\colon \hat{\mathcal{Y}} \to \mathbb{A}^1$ is given in both cases by the mapping $t \mapsto -b_2 v_1$.

Above the chart $H_{\eta_3^{-1}}$, the blow-up $\hat{H}_{\eta_3^{-1}}$ is

$$\operatorname{Proj} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2][\alpha_3, \beta_3]/(\alpha_3 w_1 - \beta_3 a_1),$$

so the preimage of $\mathcal{Y}_{\eta_3^{-1}}$ is

$$\operatorname{Proj} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2][\alpha_3, \beta_3]/(a_1 w_2 + a_2 w_1, \alpha_3 w_1 - \beta_3 a_1)$$

which is covered by the two affine schemes

$$\operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2, \overline{\beta}_3]/(a_1(w_2 + a_2\overline{\beta}_3), w_1 - \overline{\beta}_3 a_1),$$

where $\overline{\beta}_3 = \frac{\beta_3}{\alpha_3}$, and

$$\operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2, \overline{\alpha}_3]/(w_1(\overline{\alpha}_3 w_2 + a_2), \overline{\alpha}_3 w_1 - a_1),$$

where $\overline{\alpha}_3 = \frac{\alpha_3}{\beta_3}$.

Since the exceptional divisor of the blow-up is given by the equation $a_1 = 0$ or $w_1 = 0$, depending on the chart, the strict transform $\hat{\mathcal{Y}}_{\eta_3^{-1}}$ is covered by the two regular affine schemes

$$\operatorname{Spec}\mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2, \overline{\beta}_3]/(w_2 + a_2\overline{\beta}_3, w_1 - \overline{\beta}_3 a_1) \text{ and}$$
$$\operatorname{Spec}\mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2, \overline{\alpha}_3]/(\overline{\alpha}_3 w_2 + a_2, \overline{\alpha}_3 w_1 - a_1).$$

The flat family $\hat{\varrho}\colon \hat{\mathcal{Y}} \to \mathbb{A}^1$ is given in both cases by the mapping

$$t \mapsto \tfrac{1}{4}((a_3^2 - 4b_3)w_1 w_2 + a_1 a_2).$$

In conclusion, $\hat{\mathcal{Y}}$ is a regular scheme. The central fibre $\hat{\mathcal{Y}}_0$ of $\hat{\varrho}$ is given above $H_{\eta_1^{-1}}$ by

$$\hat{\mathcal{Y}}_{0\,\eta_1^{-1}} = \operatorname{Spec}\mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3]/(a_1 u_2 - a_2, b_1 u_2)$$

which is a strict normal crossing scheme with two (regular) components. Above $H_{\eta_2^{-1}}$ it is given by the two affine schemes

$$\operatorname{Spec}\mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3, \overline{\beta}_2]/(a_2\overline{\beta}_2 + 1, b_2 v_1, v_1 - \overline{\beta}_2 a_1) \text{ and}$$
$$\operatorname{Spec}\mathbb{C}[a_1, a_2, a_3, b_2, v_1, v_3, \overline{\alpha}_2, t]/(a_2 + \overline{\beta}_2, b_2 v_1, \overline{\alpha}_2 v_1 - a_1)$$

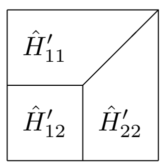
which are strict normal crossing schemes each with two (regular) components.

Above the chart $H_{\eta_3^{-1}}$, $\hat{\mathcal{Y}}_0$ is given by the two affine schemes

$$\operatorname{Spec}\mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2, \overline{\beta}_3]/(w_2 + a_2\overline{\beta}_3, (a_3^2 - 4b_3)w_1 w_2 + a_1 a_2, w_1 - \overline{\beta}_3 a_1) \text{ and}$$
$$\operatorname{Spec}\mathbb{C}[a_1, a_2, a_3, b_3, w_1, w_2, \overline{\alpha}_3]/(\overline{\alpha}_3 w_2 + a_2, (a_3^2 - 4b_3)w_1 w_2 + a_1 a_2, \overline{\alpha}_3 w_1 - a_1)$$

which are strict normal crossing schemes each with three (regular) components.

This shows that the flat family $\hat{\varrho}\colon \hat{\mathcal{Y}} \to \mathbb{A}^1$ is indeed a semi-stable family. For the situation of the components of the central fibre $\hat{\mathcal{Y}}_0$ in relation to each other we may draw the following sketch in which all components are regular of dimension 4 and where lines indicate normal intersections of dimension 3 (cp. [26, p. 421]):



,

where $\hat{H}'_{11} := \sigma^{-1}(H'_{11}) \cap \hat{\mathcal{Y}}$ and $\hat{H}'_{12}$ and $\hat{H}'_{12}$ are the strict transforms under σ of H'_{12} and H'_{22}, respectively.

The following diagram gives an overview over all objects and morphisms:

$$\begin{tikzcd}
\hat{H} \arrow[r, "\sigma"] & H = \mathrm{Hilb}^2(\mathbb{A}^3) & \\
\hat{\mathcal{Y}} \arrow[u, hook] \arrow[r, "\sigma|_{\hat{\mathcal{Y}}}"] \arrow[rr, bend left, "\hat{\varrho}"] & \mathcal{Y} = \mathrm{Hilb}^2(\mathbb{A}^3/_p\mathbb{A}^1) \arrow[u, hook] \arrow[r, "\varrho"] & \mathbb{A}^1 \\
\hat{\mathcal{Y}}_0 \arrow[u, hook] \arrow[r] & \mathcal{Y}_0 = \mathrm{Hilb}^2(Z'/_p0) = H' = \mathrm{Hilb}^2(Z') \arrow[u, hook] \arrow[r] & 0. \arrow[u, hook]
\end{tikzcd}$$

Calculation of $\mathrm{Hilb}^2(\mathrm{Spec}\,\mathbb{C}[z_1, z_2, z_3]/(z_1z_2z_3))$ **and a semi-stable family**

Let $Z'' \subset \mathbb{A}^3$ be the closed subvariety given by $z_1z_2z_3 = 0$ and denote its three components by Z''_i given by $z_i = 0$ for $i = 1, 2, 3$. Proceeding as in the case of Z', we are going to calculate the scheme $H'' := \mathrm{Hilb}^2(Z'') \subset H = \mathrm{Hilb}^2(\mathbb{A}^3)$.

Observe that due to the symmetry of Z'', i. e. its invariance under the group S_3 of permutations of the indices 1, 2 and 3, it is enough to consider one chart $H_{\eta_1^{-1}}$. The description of objects above the other charts are then given by the same equations with accordingly permuted indices. We write $R_1 := \mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3]$.

The universal family Ξ'' of H'' is given (above $H_{\eta_1^{-1}}$) by

$$\begin{aligned}
&\Xi''_{\eta_1^{-1}} = \mathrm{Spec}\, R_1[z_1, z_2, z_3]/I_{\Xi''_1}, \text{ where}\\
&\qquad I_{\Xi''_1} = \big(z_1u_2 - z_2 - \tfrac{1}{2}(a_1u_2 - a_2), z_1u_3 - z_3 - \tfrac{1}{2}(a_1u_3 - a_3), z_1^2 - a_1z_1 + b_1,\\
&\qquad\qquad \tfrac{1}{4}z_1((a_1^2 - 4b_1)u_2u_3 + a_1(a_2u_3 + a_3u_2) + a_2a_3) - \tfrac{1}{2}b_1(a_2u_3 + a_3u_2)\big)\,.
\end{aligned}$$

The Hilbert scheme $H'' = \mathrm{Hilb}^2(Z'')$ of Z'' is then given by

$$\begin{aligned}
&H''_{\eta_1^{-1}} = \mathrm{Spec}\, R_1/I_{H''_1}, \text{ where}\\
&\qquad I_{H''_1} = ((a_1^2 - 4b_1)u_2u_3 + a_1(a_2u_3 + a_3u_2) + a_2a_3, b_1(a_2u_3 + a_3u_2)).
\end{aligned}$$

It consists of 6 irreducible components:

$$H'' = H''_{11} \cup H''_{22} \cup H''_{33} \cup H''_{12} \cup H''_{13} \cup H''_{23}$$

which are each 4-dimensional. For each H''_{ij}, we describe its open affine part $H''_{ij,1}$ lying above the chart $H_{\eta_1^{-1}}$, namely

$$\begin{aligned}
H''_{11,1} &= \emptyset, & H''_{12,1} &= \mathrm{Spec}\, R_1/(b_1, a_1u_2 + a_2),\\
H''_{22,1} &= \mathrm{Spec}\, R_1/(a_2, u_2), & H''_{13,1} &= \mathrm{Spec}\, R_1/(b_1, a_1u_3 + a_3) \text{ and}\\
H''_{33,1} &= \mathrm{Spec}\, R_1/(a_3, u_3), & H''_{23,1} &= \mathrm{Spec}\, R_1/J_{23,1} \text{ with}
\end{aligned}$$

$$J_{23,1} = \big(a_2u_3 + a_3u_2, (a_1^2 - 4b_1)u_2u_3 + a_2a_3, (a_1^2 - 4b_1)u_2^2 - a_2^2, (a_1^2 - 4b_1)u_3^2 - a_3^2\big)\,.$$

Observe that if we denote the generators of $J_{23,1}$ in this order by g, h_+, h_2 and h_3, then we have $h_+^2 = h_2 h_3$ modulo g.

These components can be described analogously to the case of Z' as follows: We have $H''_{ii} = \mathrm{Hilb}^2(Z''_i)$, $i = 1, 2, 3$. The component H''_{ij} with $i < j$ consists of those subschemes ξ of length 2 of Z'' the support of which consist either of two distinct points, one lying in Z''_i and the other in Z''_j, or is contained in $Z''_i \cap Z''_j$.

The non-empty intersection of these components are given in the chart $H''_{\eta_1^{-1}}$ as follows:

$$
\begin{aligned}
&H''_{22,1} \cap H''_{33,1} = \operatorname{Spec} R_1/(a_2, a_3, u_2, u_3), && \text{regular of } \dim = 2,\\
&H''_{22,1} \cap H''_{12,1} = \operatorname{Spec} R_1/(a_2, b_1, u_2), && \text{regular of } \dim = 3,\\
&H''_{22,1} \cap H''_{13,1} = \operatorname{Spec} R_1/(a_2, b_1, u_2, a_1u_3 + a_3), && \text{regular of } \dim = 2,\\
&H''_{22,1} \cap H''_{23,1} = \operatorname{Spec} R_1/(a_2, u_2, (a_1^2 - 4b_1)u_3^2 - a_3^2), && \text{regular of } \dim = 3,\\
&H''_{33,1} \cap H''_{12,1} = \operatorname{Spec} R_1/(a_3, b_1, u_3, a_1u_2 + a_2), && \text{regular of } \dim = 2,\\
&H''_{33,1} \cap H''_{13,1} = \operatorname{Spec} R_1/(a_3, b_1, u_3), && \text{regular of } \dim = 3,\\
&H''_{33,1} \cap H''_{23,1} = \operatorname{Spec} R_1/(a_3, u_3, (a_1^2 - 4b_1)u_2^2 - a_2^2), && \text{regular of } \dim = 3,\\
&H''_{12,1} \cap H''_{13,1} = \operatorname{Spec} R_1/(b_1, a_1u_2 + a_2, a_1u_3 + a_3), && \text{regular of } \dim = 3,\\
&H''_{12,1} \cap H''_{23,1} = \operatorname{Spec} R_1/(a_1u_2 + a_2, b_1, u_2(a_1u_3 - a_3), (a_1u_3 - a_3)(a_1u_3 + a_3)) \text{ and}\\
&H''_{13,1} \cap H''_{23,1} = \operatorname{Spec} R_1/(a_1u_3 + a_3, b_1, u_3(a_1u_2 - a_2), (a_1u_2 - a_2)(a_1u_2 + a_2)),
\end{aligned}
$$

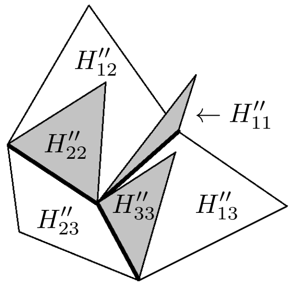
where the two last ones are not equidimensional, but consist each of two regular components; one 3-dimensional, given by the ideal $(b_1, a_1u_3 + a_3, a_1u_2 - a_2)$, respectively, by $(b_1, a_1u_2 + a_2, a_1u_3 - a_3))$, and the other one 2-dimensional, given by $(a_2, b_1, u_2, a_1u_3 + a_3)$, respectively, by $(a_2, b_1, u_2, a_1u_3 + a_3)$.

The non-empty triple intersections are

$$
\begin{aligned}
&H''_{22,1} \cap H''_{33,1} \cap H''_{12,1} = \operatorname{Spec} R_1/(a_2, a_3, b_1, u_2, u_3), && \text{regular of } \dim = 1,\\
&H''_{22,1} \cap H''_{33,1} \cap H''_{13,1} = \operatorname{Spec} R_1/(a_2, a_3, b_1, u_2, u_3), && \text{regular of } \dim = 1,\\
&H''_{22,1} \cap H''_{33,1} \cap H''_{23,1} = \operatorname{Spec} R_1/(a_2, a_3, u_2, u_3), && \text{regular of } \dim = 2,\\
&H''_{22,1} \cap H''_{12,1} \cap H''_{13,1} = \operatorname{Spec} R_1/(a_2, b_1, u_2, a_1u_3 + a_3), && \text{regular of } \dim = 2,\\
&H''_{22,1} \cap H''_{13,1} \cap H''_{23,1} = \operatorname{Spec} R_1/(a_2, b_1, u_2, a_1u_3 + a_3), && \text{regular of } \dim = 2,\\
&H''_{33,1} \cap H''_{12,1} \cap H''_{13,1} = \operatorname{Spec} R_1/(a_3, b_1, u_3, a_1u_2 + a_2), && \text{regular of } \dim = 2,\\
&H''_{33,1} \cap H''_{12,1} \cap H''_{23,1} = \operatorname{Spec} R_1/(a_3, b_1, u_3, a_1u_2 + a_2), && \text{regular of } \dim = 2,\\
&H''_{22,1} \cap H''_{12,1} \cap H''_{23,1} = \operatorname{Spec} R_1/(a_2, b_1, u_2, (a_1u_3 - a_3)(a_1u_3 + a_3)),\\
&H''_{33,1} \cap H''_{13,1} \cap H''_{23,1} = \operatorname{Spec} R_1/(a_2, b_1, u_2, (a_1u_3 - a_3)(a_1u_3 + a_3)) \text{ and}\\
&H''_{12,1} \cap H''_{13,1} \cap H''_{23,1} = \operatorname{Spec} R_1/(b_1, a_1u_2 + a_2, a_1u_3 + a_3, a_1u_2u_3),
\end{aligned}
$$

where the last three consist each of two (in the last case, three) regular components of dimension 2, which equal certain of the other triple intersections above.

The following sketch is an attempt to give an impression of the situation of all components (even H_{11} which is not visible in the chart $H''_{\eta_1^{-1}}$) in relation to each other:



Here the surfaces represent the 4-dimensional components and the fat lines represent all the 3-dimensional intersections (wherever they exist), but be aware that each of the fat lines actually stands for three different intersections (one for each pair of surfaces meeting in the picture). The point in the centre, where in the picture all components meet, stands symbolically for all intersections of dimension 2 or lower. This does, of course, not mean that all those low-dimensional intersections are equal in reality: For example, $H''_{22,1} \cap H''_{33,1} \neq H''_{22,1} \cap H''_{13,1}$. Moreover, observe that $H''_{11} \cap H''_{22} \cap H''_{33} = \emptyset$, although this is not true in the picture etc.

In the chart above $H_{\eta_1^{-1}}$, the only possibly singular component is $H''_{23,1}$. But, of course, all the three components H''_{12}, H''_{13} and H''_{23} may be singular, due to symmetry, with the singularities of H''_{12} and H''_{13} lying outside that chart. In conclusion, the (possibly non-empty) singular locus of $H''_{23,1}$ has its support in the subscheme $T_1 \subset H''_{\eta_1^{-1}}$ defined by the two equations $u_2 = 0$ and $u_3 = 0$. The intersection of T_1 with $H''_{23,1}$ is

$$T_1 \cap H''_{23,1} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3]/(u_2, u_3, a_2 a_3, a_2^2, a_3^2)$$

the reduction of which is

$$S''_1 := (T_1 \cap H''_{23,1})_{\mathrm{red}} = \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3]/(a_2, a_3, u_2, u_3),$$

which is precisely the intersection $H''_{22,1} \cap H''_{33,1}$. Indeed, another calculation using Singular ([5], using libraries [5.a], [5.b] and [5.c]) gives the singular locus of $H_{23,1}$ as the scheme

$$\begin{aligned} &\operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3]/J_{23,1;\mathrm{sing}}, \text{ where} \\ J_{23,1;\mathrm{sing}} = \big(&a_2^2, a_2 a_3, a_3^2, u_2^3, u_2^2 u_3, u_2 u_3^2, u_3^3, a_2 u_2, a_2 u_3, a_3 u_2, a_3 u_3, \\ &(a_1^2 - 4b_1)u_2^2, (a_1^2 - 4b_1)u_2 u_3, (a_1^2 - 4b_1)u_3^2\big), \end{aligned}$$

the reduction of which is equal to $S''_1 = H'_{22,1} \cap H'_{33,1}$.

In conclusion, the components H_{ii}, $i = 1, 2, 3$, are regular and the support of the singular locus of the component H_{ij} is contained in $H_{ii} \cap H_{jj}$, $1 \leq i < j \leq 3$.

A smoothing family of Z'' is $p\colon \mathbb{A}^3 \to \mathbb{A}^1$, given by the ring homomorphism $\mathbb{C}[t] \to \mathbb{C}[z_1, z_2, z_3]$, $t \mapsto z_1z_2z_3$. This family induces the flat family

$$\varrho\colon \mathcal{Y} := \operatorname{Hilb}^2(\mathbb{A}^3/\mathbb{A}^1) \to \mathbb{A}^1$$

with $\mathcal{Y}_0 = H''$, which is given above $H_{\eta_1^{-1}}$ by the ring homomorphism

$$\begin{aligned}\mathbb{C}[t] &\to \operatorname{Spec} R_1/((a_1^2 - 4b_1)u_2u_3 + a_1(a_2u_3 + a_3u_2) + a_2a_3),\\ t &\mapsto -\tfrac{1}{2}b_1(a_2u_3 + a_3u_2),\end{aligned}$$

and accordingly in the other charts.

Again, we modify the family ϱ in a way such that the central fibre is a strict normal crossing divisor. This is achieved by blowing up the ambient scheme $H = \operatorname{Hilb}^2(\mathbb{A}^3)$ of $\mathcal{Y}$ three times, once along each of the component H''_{ii}, $i = 1, 2, 3$, of $H'' = \mathcal{Y}_0$ and their strict transforms, respectively, and then taking the family $\hat{\mathcal{Y}} \to \mathbb{A}^1$ given by the strict transform of $\mathcal{Y}$ under the composition of these blow-ups.

Firstly, let $\sigma_3\colon \hat{H}_3 \to H$ be the blow-up along $H''_{33} \subset H'' \subset H$. Let then secondly $\sigma_2\colon \hat{H}_2 \to \hat{H}_3$ be the blow-up of $\hat{H}_3$ along the strict transform of H''_{22} under σ_3. Finally, let $\sigma_1\colon \hat{H}_1 \to \hat{H}_2$ be the blow-up of $\hat{H}_2$ along the strict transform of H''_{11} under $\sigma_2 \circ \sigma_3$ and denote the strict transform of $\mathcal{Y}$ under $\sigma := \sigma_1 \circ \sigma_2 \circ \sigma_3$ by $\hat{\mathcal{Y}} \subset \hat{H}_1$.

By symmetry, instead of regarding all three charts and σ as described, we may regard only the chart above $H_{\eta_1^{-1}}$ and all the compositions of three blow-ups along each of the components and their strict transforms, respectively, in all possible orders. Since neither H_{11} nor its strict transform under any blow-up is visible in this chart, we have to regard only the blow-ups $\sigma_2 \circ \sigma_3$ and $\tilde{\sigma}_3 \circ \tilde{\sigma}_2$, where $\tilde{\sigma}_2\colon \hat{\hat{H}}_2 \to H$ denotes the blow-up of H along H''_{22} and $\tilde{\sigma}_3\colon \hat{\hat{H}}_3 \to \hat{\hat{H}}_2$ the blow-up of $\hat{\hat{H}}_2$ along the strict transform of H''_{33} under $\tilde{\sigma}_2$. The difference between these two variants is just an exchange of the indices 2 and 3 for the variables a_i and u_i.

It is therefore sufficient to carry out only the two blow-ups σ_3 and σ_2 along H_{33} and the strict transform of H_{22} in the chart above $H_{\eta_1^{-1}}$:

Since the equations for $H_{33,1}$ are $a_3 = 0$ and $u_3 = 0$, the first blow-up $\hat{H}_{3,\eta_1^{-1}}$ is the scheme

$$\begin{aligned}&\operatorname{Proj} \mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3][\alpha_3, \beta_3]/\\ &\qquad((a_1^2 - 4b_1)u_2u_3 + a_1(a_2u_3 + a_3u_2) + a_2a_3, a_3\beta_3 - u_3\alpha_3),\end{aligned}$$

which is covered by the two affine schemes

$$\begin{aligned}\hat{H}_{3,\eta_1^{-1},\alpha_3^{-1}} &:= \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3, \overline{\beta}_3]/\\ &\qquad(a_3((a_1^2 - 4b_1)u_2\overline{\beta}_3 + a_1(a_2\overline{\beta}_3 + u_2) + a_2), a_3\overline{\beta}_3 - u_3),\end{aligned}$$

where $\overline{\beta}_3 = \frac{\beta_3}{\alpha_3}$, and

$$\begin{aligned}\hat{H}_{3,\eta_1^{-1},\beta_3^{-1}} &:= \operatorname{Spec} \mathbb{C}[a_1, a_2, a_3, b_1, u_2, u_3, \overline{\alpha}_3]/\\ &\qquad(u_3((a_1^2 - 4b_1)u_2 + a_1(a_2 + \overline{\alpha}_3u_2) + a_2\overline{\alpha}_3), a_3 - u_3\overline{\alpha}_3),\end{aligned}$$

where $\overline{\alpha}_3 = \frac{\alpha_3}{\beta_3}$. Observe, that the exceptional divisor has the equation $a_3 = 0$, respectively, $u_3 = 0$.

The strict transform of $H_{22,1}$ is given by the equations $a_2 = 0$ and $u_2 = 0$. Hence the second blow-up $\hat{H}_{2,\eta_1^{-1}}$ is given above $\hat{H}_{3,\eta_1^{-1},\alpha_3^{-1}}$ by

$$\begin{aligned}&\operatorname{Proj}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,\overline{\beta}_3][\alpha_2,\beta_2]/\\&\qquad(a_3((a_1^2-4b_1)u_2\overline{\beta}_3+a_1(a_2\overline{\beta}_3+u_2)+a_2),a_3\overline{\beta}_3-u_3,a_2\beta_2-u_2\alpha_2)\end{aligned}$$

which is covered by the two affine schemes

$$\begin{aligned}&\operatorname{Spec}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,\overline{\beta}_2,\overline{\beta}_3]/\\&\qquad(a_2a_3((a_1^2-4b_1)\overline{\beta}_2\overline{\beta}_3+a_1(\overline{\beta}_3+\overline{\beta}_2)+1),a_3\overline{\beta}_3-u_3,a_2\overline{\beta}_2-u_2),\end{aligned}$$

where $\overline{\beta}_2 = \frac{\beta_2}{\alpha_2}$, and

$$\begin{aligned}&\operatorname{Spec}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,\overline{\alpha}_2,\overline{\beta}_3]/\\&\qquad(a_3u_2((a_1^2-4b_1)\overline{\beta}_3+a_1(\overline{\alpha}_2\overline{\beta}_3+1)+\overline{\alpha}_2),a_3\overline{\beta}_3-u_3,a_2-u_2\overline{\alpha}_2),\end{aligned}$$

where $\overline{\alpha}_2 = \frac{\alpha_2}{\beta_2}$.

Above $\hat{H}_{3,\eta_1^{-1},\beta_3^{-1}}$ the second blow-up $\hat{H}_{2,\eta_1^{-1}}$ is given by the scheme

$$\begin{aligned}&\operatorname{Proj}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,\overline{\alpha}_3][\alpha_2,\beta_2]/\\&\qquad(u_3((a_1^2-4b_1)u_2+a_1(a_2+\overline{\alpha}_3u_2)+a_2\overline{\alpha}_3),a_3-u_3\overline{\alpha}_3,a_2\beta_2-u_2\alpha_2)\end{aligned}$$

which is covered by the two affine schemes

$$\begin{aligned}&\operatorname{Spec}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,\overline{\alpha}_3,\overline{\beta}_2]/\\&\qquad(a_2u_3((a_1^2-4b_1)\overline{\beta}_2+a_1(1+\overline{\alpha}_3\overline{\beta}_2)+\overline{\alpha}_3),a_3-u_3\overline{\alpha}_3,a_2\overline{\beta}_2-u_2),\end{aligned}$$

where $\overline{\beta}_2 = \frac{\beta_2}{\alpha_2}$, and

$$\begin{aligned}&\operatorname{Spec}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,\overline{\alpha}_2,\overline{\alpha}_3]/\\&\qquad(u_2u_3(a_1^2-4b_1+a_1(\overline{\alpha}_2+\overline{\alpha}_3)+\overline{\alpha}_2\overline{\alpha}_3),a_3-u_3\overline{\alpha}_3,a_2-u_2\overline{\alpha}_2),\end{aligned}$$

where $\overline{\alpha}_2 = \frac{\alpha_2}{\beta_2}$.

Observe that the exceptional divisor of this second blow-up has the equation $a_2 = 0$, respectively, $u_2 = 0$ and that, as it was remarked earlier, the third blow-up σ_1 is an isomorphism above $H_{\eta_1^{-1}}$.

In conclusion, the strict transform $\hat{\mathcal{Y}}$ of $\mathcal{Y}$ under σ is given above $H_{\eta_1^{-1}}$ by the following four regular affine charts:

$$\begin{aligned}
&\operatorname{Spec}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,\overline{\beta}_2,\overline{\beta}_3]/\\
&\qquad((a_1^2-4b_1)\overline{\beta}_2\overline{\beta}_3+a_1(\overline{\beta}_3+\overline{\beta}_2)+1,a_3\overline{\beta}_3-u_3,a_2\overline{\beta}_2-u_2),\\
&\operatorname{Spec}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,\overline{\alpha}_2,\overline{\beta}_3]/\\
&\qquad((a_1^2-4b_1)\overline{\beta}_3+a_1(\overline{\alpha}_2\overline{\beta}_3+1)+\overline{\alpha}_2,a_3\overline{\beta}_3-u_3,a_2-u_2\overline{\alpha}_2),\\
&\operatorname{Spec}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,\overline{\alpha}_3,\overline{\beta}_2]/\\
&\qquad((a_1^2-4b_1)\overline{\beta}_2+a_1(1+\overline{\alpha}_3\overline{\beta}_2)+\overline{\alpha}_3,a_3-u_3\overline{\alpha}_3,a_2\overline{\beta}_2-u_2)\text{ and}\\
&\operatorname{Spec}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,\overline{\alpha}_2,\overline{\alpha}_3]/\\
&\qquad(a_1^2-4b_1+a_1(\overline{\alpha}_2+\overline{\alpha}_3)+\overline{\alpha}_2\overline{\alpha}_3,a_3-u_3\overline{\alpha}_3,a_2-u_2\overline{\alpha}_2).
\end{aligned}$$

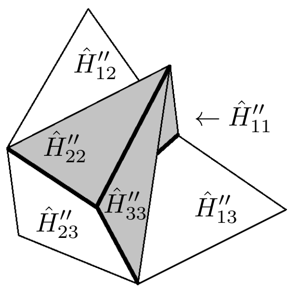
The flat family $\hat{\varrho}\colon \hat{\mathcal{Y}} \to \mathbb{A}^1$ is given over $H_{\eta_1^{-1}}$ by the mapping $t \mapsto -\frac{1}{2}b_1(a_2u_3+a_3u_2)$ in all of these charts and its central fibre $\mathcal{Y}_0$ by the following affine charts:

$$\begin{aligned}
&\operatorname{Spec}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,\overline{\beta}_2,\overline{\beta}_3]/\\
&\qquad((a_1^2-4b_1)\overline{\beta}_2\overline{\beta}_3+a_1(\overline{\beta}_3+\overline{\beta}_2)+1,b_1(a_2u_3+a_3u_2),a_3\overline{\beta}_3-u_3,a_2\overline{\beta}_2-u_2),\\
&\operatorname{Spec}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,\overline{\alpha}_2,\overline{\beta}_3,t]/\\
&\qquad((a_1^2-4b_1)\overline{\beta}_3+a_1(\overline{\alpha}_2\overline{\beta}_3+1)+\overline{\alpha}_2,b_1(a_2u_3+a_3u_2),a_3\overline{\beta}_3-u_3,a_2-u_2\overline{\alpha}_2),\\
&\operatorname{Spec}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,\overline{\alpha}_3,\overline{\beta}_2,t]/\\
&\qquad((a_1^2-4b_1)\overline{\beta}_2+a_1(1+\overline{\alpha}_3\overline{\beta}_2)+\overline{\alpha}_3,b_1(a_2u_3+a_3u_2),a_3-u_3\overline{\alpha}_3,a_2\overline{\beta}_2-u_2)\text{ and}\\
&\operatorname{Spec}\mathbb{C}[a_1,a_2,a_3,b_1,u_2,u_3,\overline{\alpha}_2,\overline{\alpha}_3,t]/\\
&\qquad(a_1^2-4b_1+a_1(\overline{\alpha}_2+\overline{\alpha}_3)+\overline{\alpha}_2\overline{\alpha}_3,b_1(a_2u_3+a_3u_2),a_3-u_3\overline{\alpha}_3,a_2-u_2\overline{\alpha}_2),
\end{aligned}$$

which are strict normal crossing schemes consisting of 5 (regular) components each. To check this, we used SINGULAR again ([5], using libraries [5.a], [5.b] and [5.c]).

This shows that $\hat{\varrho} := \varrho \circ \sigma|_{\hat{\mathcal{Y}}} : \hat{\mathcal{Y}} \to \mathbb{A}^1$ is a semi-stable family.

For the situation of the components of the central fibre $\hat{\mathcal{Y}}_0$ in relation to each other we may draw the following sketch in which all components are regular and where each line stands for three normal intersections of codimension 3:



,

where $\hat{H}''_{ii} := \sigma^{-1}(H''_{ii}) \cap \hat{\mathcal{Y}}$, $i = 1,2,3$ and $\hat{H}''_{ij}$, $i < j$, are the strict transforms under σ of the H''_{ij}.

The following diagram gives an overview of all objects and morphisms:

$$\begin{tikzcd}
\hat{H}_1 \arrow[r, "\sigma_1"] \arrow[rrr, bend left, "\sigma"] & \hat{H}_2 \arrow[r, "\sigma_2"] & \hat{H}_3 \arrow[r, "\sigma_3"] & \mathrm{Hilb}^2(\mathbb{A}^3) = H & \\
\hat{\mathcal{Y}} \arrow[u, hook] \arrow[rrr, "\sigma|_{\hat{\mathcal{Y}}}"] \arrow[rrrr, bend left, "\hat{\varrho}"] & & & \mathcal{Y} = \mathrm{Hilb}^2(\mathcal{Z}''/_p\mathbb{A}^1) \arrow[u, hook] \arrow[r, "\varrho"] & \mathbb{A}^1 \\
\hat{\mathcal{Y}}_0 \arrow[u, hook] \arrow[rrr] & & & \mathcal{Y}_0 = \mathrm{Hilb}^2(Z''/_p0) = H'' = \mathrm{Hilb}^2(Z'') \arrow[u, hook] \arrow[r] & 0. \arrow[u, hook]
\end{tikzcd}$$

6.3.2 Nagai's examples

A definition of a *good degeneration of a compact symplectic Kähler manifold* is given in [26, 4.2]:

6.3.2 Definition ([26, 4.2])

A *good degeneration of a compact symplectic Kähler manifold* is a degeneration $\pi\colon \mathcal{X} \to \Delta$, where Δ is a small complex disc of complex dimension 1, of relative dimension $2n$ satisfying

a) π is semi-stable

b) There exists a relative logarithmic 2-form $\varpi \in H^0(\mathcal{X}, \Omega^2_{\mathcal{X}/\Delta}(\log X)$ such that $\varpi^{\wedge n} \in H^0(\mathcal{X}, \mathcal{K}_{\mathcal{X}/\Delta})$ is nowhere vanishing.

For the algebraic setting, we replace the small disc Δ by the following log scheme C: Let the underlying scheme $\underline{C}$ of C be an open connected subvariety of a smooth curve, let $0 \in \underline{C}$ be a closed point and let the log structure α_C of C be the one associated to the open immersion of the complement of the point 0. Give 0 the log structure as a closed log subvariety of C, which makes it the standard log point $\operatorname{Spec} \mathbb{C}$.

6.3.3 Definition

Let $f_0\colon X \to \operatorname{Spec} \mathbb{C}$ be an SNC log variety. A *semi-stable log deformation of f_0 along C* is a log smooth deformation $f\colon \mathcal{X} \to C$ of f_0 along $i\colon \operatorname{Spec} \mathbb{C} \to C$ such that $\underline{\mathcal{X}}$ is smooth over $\operatorname{Spec} k$.

Observe, that since f_0 is log integral, the underlying morphism of schemes of f is a semi-stable deformation $\underline{f}$ of $\underline{X}$ along $\underline{C}$ in the usual sense.

6.3.4 Definition

A *projective good degeneration of symplectic varieties along C* is a projective log symplectic scheme $p\colon \mathcal{S} \to C$ such that its restriction $p_0\colon S_0 \to \operatorname{Spec} \mathbb{C}$ over the closed point 0 of C is an SNC log symplectic variety of non-twisted type and such that p is a semi-stable log deformation of p_0.

6.3.5 Theorem (cp. [26, 4.3])

Let $p\colon \mathcal{S} \to C$ be a projective type II degeneration of K3 surface, i. e. p is a projective good degeneration of a K3 surface along C with the singular fibre $S_0 = S_{[0]} \cup S_{[1]} \cup \ldots S_{[k-1]} \cup S_{[k]}$, where $S_{[0]}$ and $S_{[k]}$ are rational surfaces, $S_{[i]}$ $(0 < i < k)$ are elliptic ruled surfaces and $S_{[i]}$ meets only $S_{[i\pm 1]}$ in smooth elliptic curves $S_{[i]} \cap S_{[i+1]}$ $(i = 0, \ldots, k-1)$.
Consider the Hilbert scheme $\varrho\colon \mathcal{Y} = \mathrm{Hilb}^2(\mathcal{S}/C) \to C$ of relative subschemes of length 2. Then there exists a projective birational morphism $\sigma\colon \hat{\mathcal{Y}} \to \mathcal{Y}$, such that

$$\hat{\varrho} = \varrho \circ \sigma\colon \hat{\mathcal{Y}} \to \operatorname{Spec} \mathcal{T}$$

is a projective good degeneration of symplectic varieties. In particular, $\pi\colon \mathcal{X} \to C$ is an example of an SNC log symplectic variety of non-twisted type (of dimension 4).

Proof: Let D denote the diagonal in $\mathcal{S} \times \mathcal{S}$, let $\overline{D}$ be its image in $\mathrm{Sym}^2(\mathcal{S})$ and let C be as above. We have the following commutative diagram

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[ddd, "/\mathfrak{S}_2"] & \mathrm{Bl}_D(\mathcal{S}\times\mathcal{S}) \arrow[r, "\beta"] \arrow[d, "/\mathfrak{S}_2"] & \mathcal{S}\times\mathcal{S} \arrow[r, "p\times p"] \arrow[d, "/\mathfrak{S}_2"] & C \times C \arrow[d, "/\mathfrak{S}_2"] \\
 & \mathrm{Hilb}^2(\mathcal{S}) \arrow[r] & \mathrm{Sym}^2(\mathcal{S}) \arrow[r, "\mathrm{Sym}(p)"] & \mathrm{Sym}^2(C) \\
 & & & \\
\mathcal{Y} \arrow[r, equal] & \mathrm{Hilb}^2(\mathcal{S}/C) \arrow[uu, hook] \arrow[rr, "\varrho"] & & C \arrow[uu, hook]
\end{tikzcd}$$

where W denotes the strict transform in $\mathrm{Bl}_D(\mathcal{S} \times \mathcal{S})$ of $V := (p \times p)^{-1}(C)$.
The log structures of each of these scheme are the following:
To simplify the notation, we write $\alpha_{Z<X}$ for $\alpha_{X\setminus Z\subset X}$. The schemes $\mathcal{S}$ and C carry the log structures $\alpha_{S_0<\mathcal{S}}$ and $\alpha_{0<C}$, respectively. By lemma 6.1.2, this turns p canonically into a morphism of log schemes, because $S_0 = p^{-1}(0)$. The products $\mathcal{S} \times \mathcal{S}$ and $C \times C$ carry the log scheme structures $\alpha_{S_0<\mathcal{S}} \boxtimes \alpha_{S_0<\mathcal{S}}$, which by lemma 6.1.1 is isomorphic to $\alpha_{(S_0\times\mathcal{S})\cup(\mathcal{S}\times S_0)<\mathcal{S}\times\mathcal{S}}$, and $\alpha_{0<C} \boxtimes \alpha_{0<C} \cong \alpha_{0\times C\cup C\times 0<C\times C}$, respectively. By lemma 6.1.2, this turns $p \times p$ canonically into a morphism of log schemes. We write $S_0' := (S_0 \times \mathcal{S}) \cup (\mathcal{S} \times S_0) \subset \mathcal{S} \times \mathcal{S}$.
We give $\mathrm{Sym}^2(\mathcal{S})$ the log structure $\alpha_{\overline{D}<\mathrm{Sym}^2\mathcal{S}}$. If $q\colon \mathcal{S} \times \mathcal{S} \to \mathrm{Sym}^2(\mathcal{S})$ denotes the quotient morphism by the $\mathfrak{S}_2$-action, then by lemma 6.1.2 we have a canonical morphism of log structures $q^\times \alpha_{S_0<\mathrm{Sym}^2\mathcal{S}} \to \alpha_{S_0'<\mathcal{S}\times\mathcal{S}}$, turning q into a morphism of log schemes, because $S_0' = q^{-1}(S_0)$. Analogue for $\mathrm{Sym}^2(C)$.
We give $\mathrm{Bl}_D(\mathcal{S}\times\mathcal{S})$ and $\mathrm{Hilb}^2(\mathcal{S}) = \mathrm{Bl}_{\underline{D}}(\mathrm{Sym}^2(\mathcal{S}))$ the respective log structures as blow-ups, which by lemma 6.1.3 are isomorphic to $\alpha_{E\cup\hat{S}_0'<\mathrm{Bl}_D(\mathcal{S}\times\mathcal{S})}$ and $\alpha_{\overline{E}\cup\hat{S}_0<\mathrm{Bl}_{\overline{D}}(\mathrm{Sym}^2(\mathcal{S}))}$, respectively, where E and $\overline{E}$ are the respective exceptional divisors and where $\hat{S}_0'$ and $\hat{S}_0$ are the strict transforms of S_0' and S_0, respectively. Denoting the quotient morphism by the $\mathfrak{S}_2$-action by $q'\colon \mathrm{Bl}_D(\mathcal{S} \times \mathcal{S}) \to \mathrm{Hilb}^2(\mathcal{S})$, we have $q'^{-1}(\overline{E} \cup \hat{S}_0) = E \cup \hat{S}_0'$, so again by lemma 6.1.2 q' turns canonically into a morphism of log schemes.

Finally, $W \subset \mathrm{Bl}_D(\mathcal{S} \times \mathcal{S})$ and $\mathcal{Y} \subset \mathrm{Hilb}^2(\mathcal{S})$ carry the natural log structures as closed subschemes. Since $\mathcal{Y} \cap (\overline{E} \cup \hat{S}_0) = \mathcal{Y}_0$, as seen in the local calculations made in section 6.3.1, $\mathcal{Y}$ has the log structure $\alpha_{\mathcal{Y}_0 < \mathcal{Y}}$. Due to the fact, that $\mathcal{Y}_0 = \varrho^{-1}(0)$, also ϱ is compatible with the given log structures.

We may in addition conclude from these local calculations, that the central fibre $\mathcal{Y}_0$ consists of components $Y_{[ij]}$, where $Y_{[ii]} \cong \mathrm{Hilb}^2(S_{[i]})$ and where $Y_{[ij]}$ is a variety, the singular locus of which is supported in $Y_{[ii]} \cap Y_{[jj]}$. Due to the fact, that $S_{[i]} \cap S_{[j]} = \emptyset$ if $i+1 < j$, we have $H_{[ii]} \cap H_{[jj]} = \emptyset$ for $i+1 < j$, so the components $Y_{[ij]}$ for $i+1 < j$ are non-singular. Indeed, it is clear that $Y_{[ij]} \cong S_{[i]} \times S_{[j]}$ in this case. If $i+1 = j$, which is the case exactly when $S_{[i]} \cap S_{[j]} \neq \emptyset$, then $H_{[ii]} \cap H_{[jj]} \neq \emptyset$ and $Y_{[ij]}$ is singular, as shown by the local calculations in section 6.3.1.

Hence the situation of the components of the central fibre $\hat{\mathcal{Y}}_0$ in relation to each other may be sketched as:

$$\begin{array}{|c|c|c|c|}
\hline
Y_{00} & & & \\
\hline
Y_{01} & Y_{11} & & \\
\hline
Y_{02} & Y_{12} & Y_{22} & \\
\hline
\vdots & \vdots & \vdots & \ddots \\
\hline
\end{array}$$

where the points indicate the support of the singular loci of the components $Y_{[i,i+1]}$.

Nagai proposes blowing up the ambient variety $\mathcal{Y}$ along $\bigcup_{0 \le 2i \le 2j \le k} Y_{[2i,2j]}$, possibly due to thinking of possible singularities of components in all triple points of $\mathcal{Y}_0$. But in fact, as we have seen, the only singular components of $\mathcal{Y}_0$ are the $Y_{[i,i+1]}$, with their singularities supported in $Y_{[ii]} \cap Y_{[i+1,i+1]}$. Hence, we define $\sigma\colon \hat{\mathcal{Y}} \to \mathcal{Y}$ to be the blow-up of $\mathcal{Y}$ along $\bigcup_{0 \le 2i \le k} Y_{[2i,2i]}$ (with accordingly defined log structure).

Since the $Y_{[2i,2i]}$ do not intersect, we may split σ up into successive blow-ups along the $Y_{[2i,2i]}$ (and their strict transforms, respectively). Thus it is enough to consider the situation of $\mathcal{S}$ consisting only of two components $S_{[0]}$ and $S_{[1]}$. But this situation we have calculated étale-locally and seen already, that

$$\hat{\varrho} = \varrho \circ \sigma\colon \hat{\mathcal{Y}} \to C$$

is a semi-stable family.

Now we may continue as in the proof by Nagai to show the existence of a log symplectic form: By assumption, there exists a log symplectic form (of non-twisted type) $\omega \in \Gamma(\mathcal{S}, \Omega^2_p)$ on $p\colon \mathcal{S} \to C$. We denote its induced form on $\mathcal{S} \times \mathcal{S}$ by $\tilde{\omega} := pr_1^*\omega + pr_2^*\omega$. Its restriction to V $(= (p \times p)^{-1}(C))$ is invariant to the action of $\mathfrak{S}_2$, which makes $(\beta^*\tilde{\omega})|_W$

descend to a form $\varpi \in \Gamma(\mathcal{Y}, \Omega^2_\varrho)$. Put $\hat{\varpi} := \sigma^*\varpi \in \Gamma(\hat{\mathcal{Y}}, \Omega^2_{\hat{\varrho}})$. Then, because the subscheme of critical points of ϱ in $\mathcal{Y} = \mathrm{Hilb}^2(\mathcal{S}/C)$ is of codimension two and because the canonical divisor $K_\mathcal{Y}$ is trivial outside of this subscheme, it is globally trivial on $\mathcal{Y}$. Since $\sigma\colon \hat{\mathcal{Y}} \to \mathcal{Y}$ is a small resolution, we have $0 = \sigma^* K_\mathcal{Y} = K_{\hat{\mathcal{Y}}}$, but $\omega_\mathcal{Y} = \mathcal{O}_{\hat{\mathcal{Y}}}(K_{\hat{\mathcal{Y}}}) = \Omega^{2n}_\varrho$ is generated by its section $\hat{\varpi}^{\wedge n}$, so $\hat{\varpi}$ is a log symplectic form of non-twisted type, making $\hat{\varrho}$ a log symplectic scheme of non-twisted type (cp. [26, 4.3]). □

6.3.6 Theorem (cp. [26, 4.4])

Let $p\colon \mathcal{S} \to C$ be a projective type III degeneration of a K3 surface, i. e. p is a projective good degeneration of a K3 surface along C with the singular fibre $S_0 = \bigcup_i S_{[i]}$, where all $S_{[i]}$ are rational surfaces, the $S_{[i]} \cap S_{[j]}$ form a cycle of rational curves and the dual graph of S_0 is a triangulation of a sphere.

Consider the Hilbert scheme $\varrho\colon \mathcal{Y} = \mathrm{Hilb}^2(\mathcal{S}/C) \to C$ of relative subschemes of length 2. Then there exists a projective birational morphism $\sigma\colon \hat{\mathcal{Y}} \to \mathcal{Y}$, such that

$$\hat{\varrho} = \varrho \circ \sigma\colon \hat{\mathcal{Y}} \to \mathrm{Spec}\,\mathcal{T}$$

is a good degeneration of proper symplectic varieties. In particular, $\pi\colon \mathcal{X} \to C$ is an example of an SNC log symplectic variety of non-twisted type (of dimension 4).

Proof: The argument is analogue to that of the last proposition. It is clear from the étale-local calculations in section 6.3.1, that the small resolution $\sigma\colon \hat{\mathcal{Y}} \to \mathcal{Y}$ may be chosen to be the composition of the following series of blow-ups: Observing, that for each component $Y_{[ij]}$ its singular locus is supported in $H_{[ii]} \cap H_{[jj]}$ (such a component is singular if and only if $S_{[i]} \cap S_{[j]} \neq \emptyset$; otherwise $H_{[ii]} \cap H_{[jj]} = \emptyset$), we blow-up $\mathcal{Y}$ successively along the $Y_{[ii]}$ (and their strict transforms, respectively). Since the dual graph of S_0 is a triangulation of a sphere, every triple point is blown-up three times, as in the étale-local calculation. Those calculations show that the result is a regular scheme $\hat{\mathcal{Y}}$ together with a semi-stable family $\hat{\varrho} = \varrho \circ \sigma\colon \hat{\mathcal{Y}} \to C$. The rest of the argument is the same as before. □

7 Open questions and outlook

This thesis started with the idea, inspired by the paper “Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi-Yau varieties” by Y. Kawamata and Y. Namikawa ([21]), to try to construct new examples of proper symplectic varieties over $\operatorname{Spec}\mathbb{C}$ by glueing together two (or even more) varieties which are “almost symplectic” in some sense to form strict normal crossing varieties and then deforming these SNC varieties to smooth symplectic varieties. In what follows, we describe this idea a bit more concretely and look at answered, partially answered and unanswered questions.

Let $\underline{X}$ be a proper smooth variety over $\operatorname{Spec}\mathbb{C}$ of dimension $2n$ and denote the structure morphism by $\underline{f}\colon \underline{X} \to \operatorname{Spec}\mathbb{C}$. Let $\omega \in \Gamma(\underline{X}, \Omega^2_{\underline{f}})$ be a Kähler 2-form which is symplectic on an open subvariety $U \subset \underline{X}$ the reduced closed complement of which is a reduced divisor D. Hence, ω vanishes along a divisor supported in D.

We may endow $\underline{X}$ with the compactifying log structure $\alpha_X := \alpha_{U\subset X}$ associated to the open embedding $j\colon U \to X$ and $\operatorname{Spec}\mathbb{C}$ with the trivial log structure. Then there exists a canonical morphism of log schemes $f\colon X \to \operatorname{Spec}\mathbb{C}^\iota$ with underlying morphism of schemes $\underline{f}$ the morphism of log structures $f^\flat$ of which is the natural injective morphism $\underline{f}^\times \iota_{\mathbb{C}} = \iota_X \to \alpha_X$.

Question 1
Under which assumptions is it possible to interpret ω as a log symplectic form of general type $\omega \in \Gamma(X, \Omega^2_f \otimes \mathcal{O}(-nD))$ for some $n \in \mathbb{N}_0$ with regard to the log connection $\nabla\colon \mathcal{O}(-nD) \to \Omega^2_f \otimes \mathcal{O}(-nD)$ given by restricting the natural derivation $d\colon \mathcal{O}_X \to \Omega^1_{\underline{f}}$ to the subsheaf $\mathcal{O}_X(-nD) \subset \mathcal{O}_X$?

Put a little more general: Let $\omega \in \Gamma(U, \Omega^2_{f|_U})$ be a log symplectic form on U (which is just an ordinary symplectic form, because U carries the trivial log structure). Since $\Gamma(U, \Omega^2_{f|_U}) = \Gamma(X, j_*\Omega^2_{f|_U})$, ω may also be regarded as 2-form on X with zeros or poles along D, which is symplectic on U.

Question 2
Under which assumptions is it possible to interpret ω as a log symplectic form of general type $\omega \in \Gamma(X, \Omega^2_f \otimes \mathcal{O}(mD))$ for some $m \in \mathbb{Z}$ with regard to the log connection $\nabla\colon \mathcal{O}(mD) \to \Omega^2_f \otimes \mathcal{O}(mD)$ given by augmenting the natural derivation $d\colon \mathcal{O}_X \to \Omega^1_{\underline{f}}$ (by the quotient rule) to the subsheaf $\mathcal{O}_X(mD) \subset \mathcal{K}_X$?

Assume from now on that $\omega \in \Gamma(X, \Omega^2_f \otimes \mathcal{O}(mD))$ is a log symplectic form as in Question 1’ and assume that there is another such log variety $f'\colon X' \to \operatorname{Spec}\mathbb{C}^\iota$ with open subvariety U' the complement D' of which is a divisor and with a log symplectic form

$\omega' \in \Gamma(X', \Omega^2_{f'} \otimes \mathcal{O}(m'D))$ as in Question 1'. Assume moreover, that there is an isomorphism of varieties $D \to D'$ given.

Question 2

Under which assumptions may we glue X and X' together along $D \cong D'$ to form an SNC log variety $\hat{f}\colon \hat{X} \to \operatorname{Spec} \mathcal{C}$, where $\operatorname{Spec} \mathcal{C}$ is the standard log point?

Of course, D being isomorphic to D' is a necessary condition. If this is the case, as assumed, then, identifying D and D', the variety $\underline{\hat{X}} := \underline{X} \sqcup_D \underline{X}'$ exists as the pushout of the diagram of morphisms of $\mathbb{C}$-varieties $\underline{X} \leftarrow D \to \underline{X}'$ and hence comes with a natural morphism of schemes $\underline{\hat{f}}\colon \underline{\hat{X}} \to \operatorname{Spec} \mathbb{C}$. Since D and D' were assumed to be reduced, this is a variety with strict normal crossings. Yet, it does not admit the log structure of an SNC log variety, in general.

The condition necessary and sufficient condition for the existence of an SNC log variety $\hat{f}\colon \hat{X} \to \operatorname{Spec} \mathcal{C}$ the underlying morphism of schemes of which is $\underline{\hat{f}}\colon \underline{\hat{X}} \to \operatorname{Spec} \mathbb{C}$ is the following, which was already earlier included as 3.5.24:

7.0.7 Proposition ([17, 11.7], [29, III.1.8.8])

If $\underline{\hat{X}}$ is a strict normal crossing scheme over k, then giving $\underline{\hat{X}}$ a log structure of semi-stable type is equivalent to giving an isomorphism $\varphi\colon \mathcal{O}_D \to \mathcal{O}_D(-\underline{\hat{X}})$.

Hence, it is necessary and sufficient to ensure that

$$\mathcal{O}_D \cong \mathcal{O}_D(\underline{\hat{X}}) = \mathcal{O}_{\underline{X}}(D)\big|_D \otimes \mathcal{O}_{\underline{X}'}(D)\big|_D = \mathcal{N}_{D \subset \underline{X}} \otimes \mathcal{N}_{D \subset \underline{X}'}.$$

This means that the two line bundles $L := \mathcal{O}_{\underline{X}}(D)$ on $\underline{X}$ and $L' := \mathcal{O}_{\underline{X}'}(D)$ on $\underline{X}'$ agree on D. Hence, there exists a line bundle $\hat{L}$ on $\underline{\hat{X}}$ with $L|_{\underline{X}} = \mathcal{O}_{\underline{X}}(-D)$ and $L|_{\underline{X}'} = \mathcal{O}_{\underline{X}'}(-D)$ (cf. [11, 2.11], cp. examples 6.2.1 and 6.2.2).

In this case, the log structure of $\hat{X}$ is the one induced by the DF structure $l\colon L \to \mathcal{O}_{\hat{X}}$ which is the natural inclusion as a subsheaf. In étale-local coordinates, this log structure is just the one described in 1.2.1 in chapter 1, namely given by the chart

$$\begin{array}{ccc} \mathbb{N}_0^2 & \longrightarrow & \mathcal{O}_{\hat{X}} \\ {\scriptstyle \text{diag}}\uparrow & & \uparrow \\ \mathbb{N}_0 & \xrightarrow[n>0\mapsto 0]{0\mapsto 1} & \mathbb{C} \end{array}$$

Hence, question 2 seems to be completely answered for the case of two given varieties.

Question 3

Under which assumptions do the log symplectic forms ω and ω' on f and f' glue together to such a form $\hat{\omega}$ on $\hat{f}$?

First of all, the two line bundles $L^{\otimes m} = \mathcal{O}_X(mD)$ and $L'^{\otimes m'} = \mathcal{O}_X(m'D)$ have to glue together. But this, as we have just seen, is the case if and only if $m' = -m$. So one condition for the glueing of ω and ω' is certainly that the poles and zeros of the two forms cancel up to remaining logarithmic poles along D (the data of which is not encoded in the line bundles $\mathcal{O}_X(mD)$ and $\mathcal{O}_{X'}(m'D)$, respectively, but in the sheaves of logarithmic differentials Ω^2_f and $\Omega^2_{f'}$, respectively). The log connection $\hat{\nabla}$ on $\hat{f}$ is then given by $\mathcal{M}_{\hat{f}}(mD)$.

Question 4
Under which conditions does the log symplectic scheme $(\hat{f}, \hat{\nabla}, \hat{\omega})$ allow a smoothing to a symplectic variety in the usual sense?

This question is answered by the main theorems 5.4.7, 5.4.9, 5.4.13 and 5.4.14 and their corollaries.

Question 5
When are these conditions fulfilled?

If we assume X and D to be simply-connected, then the conditions $H^1(\hat{X}, \mathcal{O}_{\hat{X}}) = 0$ and $H^1(D, \mathcal{O}_D) = 0$ are fulfilled. Moreover, in the setting above, $\hat{\nabla}$ is automatically log Cartier. The meaning of the condition that $\mathbb{H}^1(\Omega^{\geq 2,\bullet}_{\hat{f}} \otimes L[2]) = 0$ stays mysterious, however. For example, it is not clear to us, whether this implies that for a smooth fibre $(\tilde{f}, \tilde{\omega})$ of a smoothing we have $\mathbb{H}^1(\Omega^{\geq 2,\bullet}_{\tilde{f}}[2]) = 0$ and whether this implies that the third Betti number $b_3(\tilde{X})$ and, hence, all third Hodge numbers $h^{p,q}(\tilde{X})$, $p + q = 3$, of this smooth fibre $\tilde{X}$ are zero.

Outlook

Within the scope of this thesis, it was not possible to find complete and satisfactory answers to all the questions formulated. In principle, it should, however, be possible to run a program consisting of the following steps:

- Identifying and classifying possible building blocks of proper SNC log symplectic varieties over the standard logarithmic point.
- Identifying and classifying smooth deformations of such proper SNC log symplectic varieties, i. e. proper smooth symplectic fibres in log smooth deformations of such SNC log symplectic varieties.
- Identifying certain characteristic invariants of such proper smooth deformations, like Betti or Euler numbers etc., and determining those only by knowing the corresponding invariants of the original building blocks.
- Comparing these invariants to those of the known examples of proper symplectic varieties to prove the existence of further examples.

A Appendix: Monoids

This appendix serves as a summary of basic facts about the category of monoids and of monoid sets. All statements collected here are taken from A. Ogus' lecture notes ([29, I]).

A.1 Monoids

A *monoid* P is a commutative semi-group with neutral element, whose binary operation will be denoted either multiplicatively or additively. In the first case we write 1 for the neutral element, in the latter case 0. As for groups, the neutral element is unique. A *monoid homomorphism* $\theta\colon P \to Q$ is a map $P \to Q$ such that $\theta(pp') = \theta(p)\theta(p')$ and $\theta(1) = 1$. We denote by $\underline{\mathrm{Mon}}$ the category of monoids and monoid homomorphisms and we write $\mathrm{Hom}_{\underline{\mathrm{Mon}}}(P, Q)$ for the set of all monoid homomorphisms $P \to Q$. We will usually use the multiplicative notation. Occasionally, we might regard non-commutative monoids; in this case, we will explicitly point out the non-commutativity.

A *submonoid* of P is a subset $Q \subset P$ such that the inclusion map $i\colon Q \to P$ is a monoid homomorphism. This is equivalent to $Q \subset P$ being a monoid with respect to the operation of P and having the same neutral element as P.

The category $\underline{\mathrm{Mon}}$ has products and direct sums: For a family $\{P_i\}_{i\in I}$ of monoids we may construct its *product* $\prod_{i\in I} P_i$ as the product of the underlying sets together with the operation $(p_i)_i \cdot (p'_i)_i := (p_i p'_i)_i$ and its *direct sum* $\bigoplus_{i\in I} P_i$ as the submonoid of $\prod_{i\in I} P_i$ of those elements $(p_i)_i$ with $p_i \neq 1$ for only finitely many i.

To each diagram of monoid homomorphisms $P \xrightarrow{\theta} Q \xleftarrow{\theta'} P'$ the *fibred product* exists, given by the submonoid $P \times_Q P' = \{(p, p') \in P \times P' \mid \theta(p) = \theta'(p')\}$ of $P \times P'$.

A diagram of monoid homomorphisms $1 \longrightarrow Q \overset{u}{\underset{v}{\rightrightarrows}} P \xrightarrow{w} R \longrightarrow 1$ is called *exact* if u and v are injective and if w is a coequaliser of u and v. A diagram of monoid homomorphisms $1 \longrightarrow Q \xrightarrow{u} P \xrightarrow{w} R \longrightarrow 1$ is called *exact*, if $1 \longrightarrow Q \overset{u}{\underset{1}{\rightrightarrows}} P \xrightarrow{w} R \longrightarrow 1$ is exact, where $1\colon Q \to P$ sends $q \mapsto 1$ for all q.

A submonoid E of $P \times P$ which is also an equivalence relation on P is called a *congruence* (or *congruence relation*) on P. For any congruence E on a monoid P the set P/E of equivalence classes with respect to the equivalence relation E is a monoid, with the monoid operation induced by representatives. Given two monoid homomorphisms $u, v\colon Q \to P$, their coequaliser may be constructed as the monoid P/E, where E is the congruence generated by all pairs $(u(q), v(q))$.

There is a natural bijection

$$\begin{aligned}&\{\text{congruences } E \subset P \times P\} \leftrightarrow\\ &\{\text{equivalence classes of surjective monoid homomorphisms } \theta\colon P \to P'\}\end{aligned}$$

given by $E \mapsto \theta_E\colon P \to P/E$ and $\theta \mapsto E_\theta := P \times_{P'} P$, respectively.

To each diagram of monoid homomorphisms $Q' \xleftarrow{\theta'} P \xrightarrow{\theta} Q$ its push-out is called the *amalgamed sum* $Q \oplus_P Q'$, which is the same as the coequaliser of the two maps

$$(\theta, 1), (1, \theta')\colon P \to Q \oplus Q'.$$

Hence, it may be given by $Q \oplus Q'/E$, where E is the congruence on $(Q \oplus Q')$ generated by all pairs $((\theta(p), 1), (1, \theta'(p)))$. In general, the description of E is complicated. We will always encounter the situation that at least one of P, Q and Q' is a group. In this case we have the following proposition:

A.1.1 Proposition ([29, I.1.1.5])
Let $Q' \xleftarrow{\theta'} P \xrightarrow{\theta} Q$ be a diagram of monoid homomorphisms. If any of P, Q or Q' is a group, then $Q \oplus_P Q' = Q \oplus Q'/E$, where

$$\begin{aligned} E = \big\{ &((q_1, q_1'), (q_2, q_2')) \in (Q \oplus Q') \times (Q \oplus Q') \;\big|\; \\ &\text{there exist } a, b \in P \text{ such that } q_1 + \theta(b) = q_2 + \theta(a) \text{ and } q_1' + \theta'(a) = q_2' + \theta'(b) \big\}. \end{aligned}$$

If all three, P, Q and Q', are Abelian groups, then $Q \oplus_P Q'$ is the amalgamed sum of Abelian groups.

For a monoid homomorphism $\theta\colon P \to Q$ the cokernel exists and may be given as $Q \oplus_P 1$, where the second homomorphism is $1\colon P \to 1$. This is the same as the coequaliser of $P \underset{1}{\overset{\theta}{\rightrightarrows}} Q$. If $Q \subset P$ is a submonoid, we write $P \to P/Q$ for this cokernel and call it the *quotient monoid* of P by Q. Two elements $p, p' \in P$ have the same image in P/Q if and only if there exist $q, q' \in Q$ such that $pq = p'q'$. If we have submonoids $Q \subset Q' \subset P$, then Q'/Q is a submonoid of P/Q and the natural map $(P/Q)/(Q'/Q) \to P/Q'$ is an isomorphism.

A.2 Monoid modules

We write $\underline{\mathrm{Set}}$ for the category of sets and maps. For a monoid P a set S together with a homomorphism of (possibly non-commutative) monoids $\varrho\colon P \to \mathrm{End}(S)$ is called a *P-set* (or *P-module*). For each P-set (S, ϱ) the homomorphism ϱ defines an action of P on S, written $ps := \varrho(p)(s)$ for $p \in P$ and $s \in S$. A *homomorphism of P-sets* $S \to T$ is a map $\phi\colon S \to T$ such that $\phi(ps) = p\phi(s)$; we call these maps *P-linear*. We write $\underline{\mathrm{Set}}_P$ for the category of P-modules and P-linear maps.

Let $\theta\colon P \to Q$ be a monoid homomorphism. Then Q has a natural P-set structure given by the action $pq := \theta(p)q$. We will call $\theta\colon P \to Q$ (or just Q) a *P-monoid*. Observe that being a P-monoid is a stronger property than being a P-set which is a monoid.

A *generating set* of a P-set S is a subset $B \subset S$ such that the map (of sets) $P \times B \to S$, $(p, b) \mapsto p \cdot b$, is surjective. S is called finitely generated if B can be chosen to be a finite set. We call B a *basis* of S if the above morphism is bijective; if a basis exists, then S is called a *free* P-set.

The category $\underline{\mathrm{Set}}_P$ has products and coproducts (direct sums) which may be given as follows. If $\{(S_i, \varrho_i)\}_{i\in I}$ is a family of P-modules, then its *product* is the set $\prod_{i\in I} S_i$ with P-action ϱ given by $\varrho(p)((s_i)_i) := (\varrho_i(p)s_i)_i$. Its *coproduct* is the disjoint union of sets $\coprod_{i\in I} S_i$ together with the P-action ϱ given by $\varrho(p)(s) := \varrho_i(p)(s)$ for $s \in S_i$.

Let S, T and W be P-monoids. A *(P-)bilinear map* $S \times T \to W$ is a map $\beta\colon S \times T \to W$ such that $\beta(ps, t) = \beta(s, pt) = p\beta(st)$ for $s \in S$, $t \in T$ and $p \in P$. The *tensor product* of S and T (over P) is the universal bilinear map $S \times T \to S \otimes_P T$. It may be constructed as the quotient $S \times T/{\sim}$ by the equivalence relation $\sim$ on $S \times T$ which is generated by all pairs $((ps, t), (s, pt)) \in (S \times T) \times (S \times T)$ with $s \in S$, $t \in T$ and $p \in P$, together with the canonical projection $S \times T \to S \times T/{\sim}$.

If S is a P-set and Q is a P-monoid, then the map $S \to S \otimes_P Q$, $s \mapsto s \otimes 1$, is P-linear and Q acts naturally on $S \otimes_P Q$ by $q(s \otimes q') = s \otimes (qq')$, making $S \otimes_P Q$ a Q-set.

If we have two P-monoids Q and Q', then $Q \otimes_P Q'$ with the operation $(q_1 \otimes q_1')(q_2 \otimes q_2') = (q_1q_2 \otimes q_1'q_2')$ is a monoid which is naturally isomorphic to $Q \oplus_P Q'$.

In this thesis, we prefer to write $Q \otimes_P Q'$ instead of $Q \oplus_P Q'$ for three monoids except in the case $P = 1$, where we will write $Q \oplus Q'$ (instead of $Q \otimes_1 Q'$). This is analogous to rather writing $A \otimes_R B$ (tensor product of R-modules) than $A \oplus_R B$ (amalgamed sum of rings) for a ring R and two R-algebras A and B.

A.3 Ideals and faces

Let P be a monoid. A *(monoid) ideal* of P is a subset $\mathfrak{w} \subset P$ which is a P-set by the natural action of P on $\mathfrak{w}$, i. e. it has the property that $pw \in \mathfrak{w}$ for all $w \in \mathfrak{w}$ and all $p \in P$. A monoid ideal may be empty; in fact, every monoid P contains at least the two ideals $\emptyset$ and P. As with rings, we denote by (S) the ideal generated by a subset $S \subset P$. A *prime ideal* of P is a proper ideal $\mathfrak{p} \subsetneq P$ such that $pp' \in \mathfrak{p}$ implies $p \in \mathfrak{p}$ or $p' \in \mathfrak{p}$. Observe that the empty set is the minimal prime ideal in any monoid. The complement $P \setminus \mathfrak{p}$ of a prime ideal is always a submonoid of P.

A *face* of P is a submonoid $F \subset P$ such that $pq \in F$ implies both $p, q \in F$. The complement $P \setminus F$ is always an ideal. In fact, faces and prime ideals are (set-theoretic) complements in P to each other, i. e. $F = P \setminus \mathfrak{p}$ is a face if and only if $\mathfrak{p} = P \setminus F$ is a prime ideal.

An element $p \in P$ is called a *unit* or *invertible* if there exists a $p' \in P$ with $pp' = 1$. We denote by $P^\times \subset P$ the set of all units in P, which is a face of P and, obviously, a subgroup. Its complement, the set of all non-invertible elements $\mathfrak{m}_P := P \setminus P^\times$, is the unique maximal ideal of P, which we also call its *characteristic ideal.*

For any ideal $\mathfrak{w} \subset P$ we define its *radical* as

$$\sqrt{\mathfrak{w}} := \{p \in P \mid \text{ there exists an } n \geq 1 \text{ such that } p^n \in \mathfrak{w}\}.$$

It is equal to the intersection of all prime ideals containing $\mathfrak{w}$.

The *spectrum* $\operatorname{Spec} P$ of a monoid P is defined as the set of all prime ideals (or equivalently, of all faces) of P together with the Zariski-topology. For each monoid P, $\operatorname{Spec} P$ is an irreducible topological space of a certain Krull-dimension with exactly one closed point, corresponding to the maximal ideal (or the face of units, respectively) and exactly one generic point, corresponding to the empty set (the full monoid, respectively). In this regard, monoids behave similarly to local rings. We will not proceed in this direction, but refer the reader to [29, I.14 ff.] for more details.

An element $a \in P \setminus \{1\}$ is called an *absorbent* (or, in multiplicative denotation, a *zero element*) if $ap = a$ for all $p \in P$. In multiplicative denotation we write 0 and in additive denotation ∞ for a. An absorbent is unique whenever it exists. We call an element $p \in P$ an *absorbent-divisor* (or *zero-divisor*) if there exists a $p' \in P$ with $pp' = 0$. The absorbent itself counts as an absorbent-divisor. The subset $\mathfrak{n}_P \subset P$ of all absorbent-divisors of P is a prime ideal in P. We call a monoid P *domainic* or a *(monoid) domain* if it has no absorbent-divisors; for any monoid P we define the *domainic face* of P to be $P^{\mathrm{dom}} := P \setminus \mathfrak{n}(P)$, which is the largest domainic submonoid of P.

Let P be a monoid and $Q \subset P$ a submonoid. We define the *localisation* $Q^{-1}P$ of P in Q as the monoid $P \times Q/{\sim}$, where $(p, q) \sim (p', q')$ if and only if there exists an $r \in Q$ such that $pq'r = p'qr$. The natural monoid homomorphism $P \to Q^{-1}P$, $p \mapsto [p, 1]$, has the universal property that any monoid homomorphism $P \to R$ mapping Q into $R^\times$ factors through it uniquely. Instead of $[p, q]$ we write $\frac{p}{q}$ or $q^{-1}p$ in multiplicative notation and $p - q$ in additive notation.

More generally, there is a notion of localising a monoid module, which we will omit here (cf. [29, I.14 ff.]).

We call $P^{\mathrm{grp}} := P^{-1}P$, which is in fact a group, the *group associated* to P or its *groupification.* The natural homomorphism $P \to P^{\mathrm{grp}}$ has the universal property that any monoid homomorphism $P \to G$ into a group G factors trough it uniquely. If P possesses an absorbent, then automatically $P^{\mathrm{grp}} = 1$.

We call $P^{\mathrm{rat}} := (P^{\mathrm{dom}})^{-1}P$ the *total fraction monoid* or the *rational monoid* of P. Its groupification $(P^{\mathrm{rat}})^{\mathrm{grp}}$ is equal to P^{grp}. Thus the groupification homomorphism naturally factors as $P \to P^{\mathrm{rat}} \to P^{\mathrm{grp}}$. If P possesses no absorbent, then $P^{\mathrm{rat}} = P^{\mathrm{grp}}$. Otherwise, P^{rat} is never a group.

A.3.1 Definition

A monoid P is called

a) *sharp* if $P^\times = 1$;

b) *unit-integral* if the natural homomorphism $P^\times \to P^{\mathrm{grp}}$ is injective. This means that for any elements $u \in P^\times$ and $p \in P$ we have $up = p \Leftrightarrow u = 1$;

c) *quasi-integral* if for any two elements $p, q \in P$ we have $pq = p \Leftrightarrow q = 1$;

d) *integral* if the natural homomorphism $P \to P^{\mathrm{grp}}$ is injective; this means that for any three elements $p, p', p'' \in P$ we have $pp' = pp'' \Leftrightarrow p' = p''$.

Between these properties the following implications are valid:

$$\begin{array}{ccccccc} \text{integral} & \Rightarrow & \text{quasi-integral} & \Rightarrow & \text{unit-integral} & \Rightarrow & \text{domainic} \\ & & & \text{sharp} \nearrow & & & \end{array}$$

We call the quotient monoid $\overline{P} := P/P^{\times}$ the *characteristic monoid* of P. It is a sharp monoid and the canonical monoid homomorphism $P \to \overline{P}$ is the universal homomorphism from P to a sharp monoid.

For any monoid P we denote its image in P^{grp} by P^{int} and call it the *integral monoid associated* to P or its *integralisation*. Hence P is integral if and only if $P \cong P^{\mathrm{int}}$. The amalgamed sum of integral monoids need not be integral. Indeed, the push-out in the category of integral monoids can be constructed as $P \oplus_Q^{\mathrm{int}} P' := (P \oplus_Q P')^{\mathrm{int}}$.

For an integral monoid P we define its saturation to be the monoid

$$P^{\mathrm{sat}} := \{p \in P^{\mathrm{grp}} \,|\, p^n \in P \text{ for some } n \geq 1\},$$

which is a monoid lying between P and P^{grp}.

A.3.2 Definition

A monoid P is called

a) *finitely generated* or *coherent* if there exists a surjective monoid homomorphism $\mathbb{N}_0^s \to P$ with $s \in \mathbb{N}_0$; this means that there exist finitely many elements $p_1, \ldots, p_s \in P$ such that any $p \in P$ can be written as $p = p_1^{n_1} \cdot \ldots \cdot p_s^{n_s}$ with numbers $n_1, \ldots, n_s \in \mathbb{N}_0$;

b) *saturated* if P is integral and $P = P^{\mathrm{sat}}$;

c) *fine* if P is finitely generated and integral;

d) *fs* (or *normal*) if P is fine and saturated; in this case P^{grp} can be viewed as the character group of an algebraic torus (cf. [29, p. 31]).

e) *toric* if P is fs and the group P^{grp} is torsion-free; this is equivalent to saying that P is finitely generated and torsion-free;

f) *free* of rank $s \in \mathbb{N}_0$, if there is an isomorphism $\mathbb{N}_0^s \xrightarrow{\cong} P$; in particular, free modules are toric and sharp.

We may regard the subcategories $\underline{\mathrm{Mon}}^{\mathrm{f}}$ and $\underline{\mathrm{Mon}}^{\mathrm{fs}}$ the objects of which are fine and fs monoids, respectively, and the morphism of which are homomorphisms of monoids.

A.3.3 Theorem ([29, 2.1.16])

a) Every submonoid of a fine (respectively, saturated, toric) monoid is fine (respectively, saturated, toric).

b) Every localisation of a fine (respectively, saturated, toric) is fine (respectively, saturated, toric).

c) The fibred product of two fine (respectively, saturated) monoids over an integral monoid is fine (respectively, saturated).

d) Let P be finitely generated. If Q is fine (respectively, saturated), then $\mathrm{Hom}_{\underline{\mathrm{Mon}}}(P, Q)$ is fine (respectively, saturated).

The amalgamed sum in $\underline{\mathrm{Mon}}^{\mathrm{f}}$ and $\underline{\mathrm{Mon}}^{\mathrm{fs}}$, however, is not the same as in the category of monoids $\underline{\mathrm{Mon}}$. In $\underline{\mathrm{Mon}}^{\mathrm{f}}$ it is given by the integralisation $(P \oplus_Q P')^{\mathrm{int}}$ of the usual amalgamed sum; in the category of fs monoids it is given by $((P \oplus_Q P')^{\mathrm{int}})^{\mathrm{sat}}$, the saturation of this integralisation.

A.4 Monoid algebras and toric affine schemes

We assume all rings in this thesis to be commutative with 1.

Let A be a ring. For each monoid P we define the A-algebra $A[P] := \bigoplus_{p \in P} A \cdot [p]$ with the multiplication induced by $[p] \cdot [p'] := [pp']$, which is called the *monoid algebra* of P over A and which comes with a natural inclusion homomorphism $P \to (A[P], \cdot)$ of monoids.

If $\theta \colon P \to Q$ is a monoid homomorphism, we get an induced homomorphism of A-algebras $A[\theta] \colon A[P] \to A[Q]$, which makes $A[\,\cdot\,]$ a functor $\underline{\mathrm{Mon}} \to \underline{\mathrm{Ring}}_A$.

We have a natural isomorphism $A[Q \otimes_P Q'] \cong A[Q] \otimes_{A[P]} A[Q']$.

If A is an integral domain and if P is torsion-free, then $A[P]$ is an integral domain. If P is a toric monoid, we call $A[P]$ a *toric A-algebra.*

For a P-set S we denote by $A[S]$ the A-module $\bigoplus_{s \in S} A \cdot [s]$. Then $A[S]$ is naturally an $A[P]$-module with the multiplication induced by the action of P on S.

A.5 Monoid homomorphisms

If $\theta \colon P \to Q$ is a monoid homomorphism, then obviously $P^\times \subset \theta^{-1}(Q^\times)$. Therefore, θ induces monoid homomorphisms $\theta^\times \colon P^\times \to Q^\times$ and $\overline{\theta} \colon \overline{P} \to \overline{Q}$ sitting inside a commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & P^\times \arrow[r, shift left, "i"] \arrow[r, shift right, "1"'] \arrow[d, "\theta^\times"] & P \arrow[r] \arrow[d, "\theta"] & \overline{P} \arrow[r] \arrow[d, "\overline{\theta}"] & 1 \\
1 \arrow[r] & Q^\times \arrow[r, shift left, "i"] \arrow[r, shift right, "1"'] & Q \arrow[r] & \overline{Q} \arrow[r] & 1.
\end{tikzcd}$$

A.5.1 Definition

A monoid homomorphism $\theta\colon P \to Q$ is called

a) *local* if $\theta^{-1}(\mathfrak{m}_Q) = \mathfrak{m}_P$, or equivalently, if $\theta^{-1}(Q^\times) = P^\times$;

b) *sharp* if the induced group homomorphism $\theta^\times : P^\times \to Q^\times$ is an isomorphism;

c) *logarithmic* if the induced homomorphism $\theta^{-1}(Q^\times) \to Q^\times$ is an isomorphism; equivalently, if θ is local and sharp;

d) *strict* if the induced homomorphism $\overline{P} \to \overline{Q}$ is an isomorphism.

A.5.2 Proposition ([29, I.4.1.2])

Let $\theta\colon P \to Q$ be a sharp and strict monoid homomorphism. Then θ is surjective, and if Q is unit-integral, then θ is bijective.

There is a notion of flatness of P-sets, given as follows: A P-set S is called *flat* if it satisfies one of the following equivalent conditions:

a) For every functor F from a finite connected category C to $\underline{\mathrm{Set}}_P$, the natural map $(\varprojlim F) \otimes_P S \to \varprojlim (F \otimes_P S)$ is an isomorphism.

b) S is a direct limit of free P-sets.

A criterion for flatness is the following: S is a flat P-set if and only

a) if, given $s_1, s_2 \in S$ and $p_1, p_2 \in P$ such that $p_1 s_1 = p_2 s_2$, there exist $s' \in S$ and $p'_1, p'_2 \in P$ such that $s_i = p'_i s'$ and $p_1 p'_1 = p_2 p'_2$ and

b) if, given $s \in S$ and $p_1, p_2 \in P$ such that $p_1 s = p_2 s$, there exist $s' \in S$ and $p' \in P$ such that $s = p' s'$ and $p_1 p' = p_2 p'$.

Any monoid P is flat considered as a P-set with the action of P given by the multiplication.

Useful for us will be the following equivalence: If P is an integral monoid, then a P-set S is flat if and only if $\mathbb{Z}[S]$ is flat as a $\mathbb{Z}[P]$-module.

We call a homomorphism of integral monoids $\theta\colon P \to Q$ *integral* if for every homomorphism $P \to P'$ of integral monoids the push-out (base change) $Q \otimes_P P'$ is an integral monoid. A criterion for the integrality of θ is the following: If, for all $q, q' \in Q$ and $p, p' \in P$ such that $\theta(q)p = \theta(q')p'$, there exist $r, r' \in Q$ and $\tilde{p} \in P$ such that $p = \theta(r)\tilde{p}$, $p' = \theta(r')\tilde{p}$ and $qr = q'r'$, then θ is integral.

The composition of two integral homomorphisms is integral and θ is integral if and only if $\overline{\theta}$ is. If $\theta\colon P \to Q$ is a local homomorphism of integral monoids with P sharp, then θ is integral if and only if it is flat.

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Glossary

$f_\infty\colon (\mathfrak{X}, \mathfrak{w}) \to \operatorname{Spec} \mathcal{T}$: formal log symplectic deformation of non-twisted type 130

$f_\infty\colon (\mathfrak{X}, \mathfrak{D}, \mathfrak{w}) \to \operatorname{Spec} \mathcal{T}$: formal log symplectic deformation of general type 130

$\beta\colon X^\beta \to X$: log blow-up 135

$\mathfrak{S}_2$: symmtric group of order two 147

Ξ: universal family of the Hilbert scheme $\operatorname{Hilb}^2(\mathbb{A}^3)$ 148

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Lebenslauf

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