Monodromy Calculations for some Differential Equations

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1 Introduction

Linear homogeneous differential equations of order n with polynomial coefficients

$$L := p_n(x)y^{(n)} + \ldots + p_1(x)y' + p_0 y = 0$$

play a prominent role in many areas of mathematics. The set of singular points of L, denoted by Σ, consists of the roots of $p_n(x)$ and possibly ∞; points of $\mathbb{P}^1 \setminus \Sigma$ are called ordinary. If for $0 \leq i \leq n-1$ the pole order of $p_i(x)/p_n(x)$ at a singular point $p \in \mathbb{C}$ is less than $n - i + 1$, the singular point p is called a regular singular point. The point at ∞ is called regular singular, if the same is true for 0 after a coordinate change $z = 1/x$. A linear homogeneous differential equation is called Fuchsian, if all its singular points are regular singular. Fuchsian differential equations are of special interest, since differential equations appearing naturally in algebraic and complex geometry are usually of this type. Locally near any ordinary point p of $\mathbb{P}^1$ there are n over $\mathbb{C}$ linear independent solutions of L. Their behavior under analytic continuation along a loop in $\mathbb{P}^1 \setminus \Sigma$ is encoded in the monodromy representation, whose image in $GL_n(\mathbb{C})$ is called monodromy group. The analytic techniques to determine this group are very limited. For order greater than three our knowledge of generators of the monodromy group is restricted to rigid cases, where the global monodromy is fixed by local data. This is the case for generalized hypergeometric differential equations, which were investigated by A. H. M. Levelt and F. Beukers, and G. Heckman [BH89, Lev61]. For more general rigid cases generators of the monodromy group can be computed with an algorithm developed by N. M. Katz in the formulation of M. Dettweiler and S. Reiter [Kat82, DR07]. To handle general non-rigid cases a method that computes approximations of generators of the monodromy group of a general Fuchsian differential equation is desirable. In several papers [CC86, CC87a, CC87b] D. V. and G. V. Chudnovsky outline an efficient computer implementable approach based on the classical Frobenius method. In Chapter 2 we give the necessary definitions and statements from the theory of differential algebra and differential equations. Furthermore we recall in detail the approach of D. V. and G. V. Chudnovsky and describe our implementations. Throughout this thesis these implementations are tested for two types of differential equations. Second order differential equations from the theory of Riemann uniformization and fourth order differential equations occurring in mirror symmetry for Calabi-Yau threefolds. In the first case the crucial observation is that we are able to compute approximations of matrices generating the monodromy groups with precision as high as necessary to get plausible guesses of their traces as algebraic numbers. In the second case with respect to a special basis we are able the identify the entries as elements of $\mathbb{Q}(\zeta(3)/(2\pi i)^3)$.

A Fuchsian group Γ is a discrete subgroup of $PSL_2(\mathbb{R})$ which acts on the upper half plane $\mathbb{H}$ as Möbius transformations. The quotient $X = \mathbb{H}/\Gamma$ can be equipped with a complex structure and thus is a Riemann surface. Two generator Fuchsian groups with compact quotient X are certain triangle groups and groups with signature $(1; e)$ or $(0; 2, 2, 2, q)$. The first entry in the signature is the genus of X and the remaining numbers describe the branching data of the uniformizing map $\phi : \mathbb{H} \to \mathbb{H}/\Gamma$. Up to conjugation a Fuchsian group with two generators α and β can be described completely by the trace triple $(\mathrm{tr}(\alpha), \mathrm{tr}(\beta), \mathrm{tr}(\alpha\beta))$. Arithmetic Fuchsian groups are those which are commensurable with the embedding into $PSL_2(\mathbb{R})$ of the norm one elements of an order of a quaternion algebra. Arithmetic triangle groups and arithmetic Fuchsian groups of signature $(1; e)$ were classified in terms of the their trace triples by T. Takeuchi [Tak83, Tak77a]. Later P. Ackermann, M. Näätänen, and G. Rosenberger also classified arithmetic Fuchsian groups with signature $(0; 2, 2, 2, q)$ [ANR03]. In the first two sections of Chapter 3 we recall this classification and present the foundations of hyperbolic geometry. If one chooses coordinates on $\mathbb{H}$ and $\mathbb{H}/\Gamma$, there is no general method to determine the uniformizing map ϕ explicitly. The multivalued inverse ϕ^{-1} is the quotient of two solutions of a Fuchsian differential equation D of degree two, called uniformizing differential equation. In Section 3.3 we explain which information about D can be extracted from the signature of Γ directly. In particular, for triangle groups D can be immediately determined from the signature of Γ as a hypergeometric differential equation. For Fuchsian groups with signature $(1; e)$ or $(0; 2, 2, 2, q)$ either D itself or a pullback of D to $\mathbb{P}^1$ has four singular points. If three of these points are assumed to be $0, 1, \infty$, the signature of Γ determines D up to the location of the fourth singular point A and an additional accessory parameter C. Section 3.5 describes how we used the algorithms obtained in Chapter 2 to determine candidates for A and C. We use that the monodromy group of D divided out by its center coincides with Γ. Hence, to find the uniformizing differential equation for a given arithmetic Fuchsian group with trace triple T, one has to determine A and C such that the trace triple of the monodromy group of $D(A, C)$ matches T. This approach was used by D. V. and G. V. Chudnovsky in [CC89] for $(1; e)$-groups, where they also list some A and C for some cases. We are able to apply the numerical monodromy method to get approximations of A and C, such that both parameters coincide with algebraic numbers of plausible height and degree up to high precision. Section 3.4 reviews algorithms designed for the recognition of algebraic numbers given by rational approximations. In the case of groups with signature $(1; e)$ we profited from the work of J. Sijsling, who used the theory of algebraic models of Shimura curves to compute A in many cases [Sij13]. We found our results to be true whenever the uniformizing differential equation is listed in the literature and use the theory of Belyi maps to prove one further case.

Mirror symmetry is a physical theory that entered mathematics several years ago and the process of turning it into a mathematical theory is still ongoing. One of the first constructions was done by P. Candelas and his collaborators for a generic quintic threefold in $\mathbb{P}^4$ with Hodge numbers $h^{1,1}(X) = 1$ and $h^{1,2}(X) = 101$. By an orbifold construction they found a topological mirror Calabi-Yau threefold X' with $h^{1,1}(X') = 101$ and $h^{2,1}(X') = 1$. Hence X' varies in a deformation family $\mathcal{X}$ over a one-dimensional base, whose Picard-Fuchs equation can be computed explicitly. It is a generalized hypergeometric differential equation of order four with singular points $\{0, 1/3125, \infty\}$. In a special basis, the monodromy along a loop encircling only $1/3125$ once in counterclockwise direction is a symplectic reflection

$$x \mapsto x - \langle x, C\rangle C,$$

with $C = (H^3, 0, c_2(X)H/24, c_3(X)\lambda)$, where the complex number λ is defined as $\zeta(3)/(2\pi i)^3$ and H is the first Chern class of the ample generator of $\mathrm{Pic}(X)$. Furthermore to X one can assign three characteristic numbers, namely the degree H^3, the second Chern number $c_2(X)H$ and $c_3(X)$, the Euler number of X. For all known topological mirror partners (X, X'), that is Calabi-Yau threefolds with $h^{1,1}(X) = h^{2,1}(X')$ and $h^{2,1}(X) = h^{1,1}(X')$, with $h^{1,1}(X') = 1$ we are able make the same observation as for the quintic example using approximations of generators of the monodromy group. The right formulation to understand this observations seems to be M. Kontsevich's categorical approach to mirror symmetry. Following this approach, certain projective Calabi-Yau varieties should come in pairs (X, X'), where $D^b(X)$ is equivalent to $DFuk(X')$ as well as $D^b(X')$ is equivalent to $DFuk(X)$. The category $\mathcal{D}^b(X)$ is the well studied bounded derived category of coherent sheaves. The second category $DFuk^b(X)$ depends on the symplectic structure of X and is called derived Fukaya category. To explain the occurrence of H^3, c_2H, c_3 and λ we can reformulate an approach of L. Katzarkov, M. Kontsevich, and T. Pantev [KKP08] and H. Iritani [Iri09, Iri11]. They introduced the characteristic class $\widehat{\Gamma}_X$ as a square root of the Todd class. If δ_i are the Chern roots of T_X, it is defined by

$$\widehat{\Gamma}_X := \widehat{\Gamma}(T_X) = \prod_i \Gamma(1+\delta_i),$$

where $\Gamma(1+\delta_i)$ is defined by the series expansion of the classical Γ-function

$$\Gamma(1+z) = \exp\left(-\gamma z + \sum_{l=2}^{\infty} \frac{\zeta(l)}{l}(-z)^l\right).$$

If $h^{1,1}(X^{'}) = 1$ the geometric monodromy transformation induces an autoequivalence

$$\rho : \pi_1(B \setminus \Sigma, b) \to \mathrm{Auteq}(DFuk(X^{'})).$$

Under mirror symmetry autoequivalences of $DFuk(X^{'})$ should correspond to autoequivalences of $\mathcal{D}^b(X)$. Especially a spherical twist $\Phi_{\mathcal{O}_X}$ along $\mathcal{O}_X$ is expected to correspond to a generalized Dehn twist with respect to a Lagrangian sphere [ST01]. On the level of cohomology such a Dehn twist is described by the Picard-Lefschetz formula. On the other hand the passage from $\mathcal{D}^b(X)$ to cohomology via the Γ-character

$$\mathcal{E}^{\bullet} \mapsto (2\pi i)^{deg/2}\,\mathrm{ch}([\mathcal{E}^{\bullet}]) \cup \widehat{\Gamma}_X$$

yields that $\Phi^H_{\mathcal{O}_X}$ with respect to a distinguished basis and a suitable bilinear form is a symplectic reflection with reflection vector $(H^3, 0, c_2H/24, c_3\lambda)$. The local system $\mathcal{V}_{\mathbb{C}}$ with fiber $\mathcal{V}_{\mathbb{C},b} = H^3(X^{'}_b, \mathbb{C})$ can be described by solutions of its Picard-Fuchs equation L . And the monodromy of L coincides with the monodromy of $\mathcal{V}_{\mathbb{C}}$. Fuchsian differential equations that share certain algebraic and arithmetic properties of Picard-Fuchs equations of Calabi-Yau threefolds are called CY(3)-equations. Finitely many CY(3)-equations are known and collected in [AESZ10, Str12]. We expect that many of these equations actually are Picard-Fuchs equations and that it is possible to recover topological invariants of certain Calabi-Yau threefolds with $h^{1,1} = 1$. In Chapter 4 we recall the characterization of $CY(3)$-equations and introduce the Γ-character. We furthermore recall how H. Iritani used the Γ-character to define mirror equivalent lattices in two variations of Hodge structure attached to X and $X^{'}$. Finally, in Section 4.6.3 we apply the algorithms of Chapter 2 to produce a list of triples (H^3, c_2H, c_3), we find many more such triples than Calabi-Yau threefolds with $h^{1,1}$ are known.

2 Monodromy of Linear Differential Equations

2.1 Differential Operators and Differential Modules

2.1.1 Basic Notions

In this section we will recall basic definitions and properties of scalar differential equations, matrix differential equations, differential operators, and differential modules and explain how this four objects are related. A complete reference is the book [PS03] by M. van der Put and M. F. Singer. For a ring R a *derivation* is an additive map

$$(\cdot)^{'} : R \to R$$

that satisfies the Leibniz rule

$$(rs)^{'} = rs^{'} + rs^{'}, \text{ for } r, s \in R.$$

A ring R with a derivation is called a *differential ring*, if R is in addition a field, then R is called a *differential field*. A *differential extension* of the differential ring R is a differential ring S such that the derivation of S restricts to the derivation of R. An element $c \in R$ is called a *constant* of R, if $c^{'} = 0$. The most prominent examples of differential rings are the *rational functions* $\mathbb{C}(x)$, the ring of formal *Laurent series* $\mathbb{C}((x))$ and its algebraic closure, the ring of *Puiseux series*. In this cases the constants are the algebraically closed field $\mathbb{C}$. For the rest of this section k will always denote a differential field and C its field of constants. At first we define differential modules.

Definition 2.1.1 (Differential module) *For a differential field k, a differential module is a finite dimensional k-vector space (M, ∂) equipped with an additive map $\partial : M \to M$, that has the property*

$$\partial(fm) = f^{'}m + f\partial(m),$$

for all $f \in k$ and $m \in M$.

With respect to a basis $e_1, \ldots, e_n$ of a differential module M define the matrix $A = (a_{i,j})_{i,j} \in Mat_n(k)$ by $\partial(e_i) = -\sum_{j=1}^{n} a_{j,i}e_j$. For a general element $m = \sum f_i e_i$ of M the element $\partial(m)$ has the form

$$\sum_{i=1}^{n} f_i^{'} e_i - \sum_{i=1}^{n} \left(\sum_j a_{i,j} f_j \right) e_i$$

and the equation $\partial(m) = 0$ translates to $(y_1^{'}, \ldots, y_n^{'})^t = A(y_1, \ldots, y_n)^t$. Extending the derivation component-wise to k^n this leads to the definition of a *matrix differential equation* or a *system of first order differential equations.*

Definition 2.1.2 (Matrix differential equation) *An equation $y^{'} = Ay$ with $A \in Mat_n(k)$ and $y \in k^n$ is called a* linear matrix differential equation *or a* system of linear differential equations.

A different choice of the basis of M results in a different matrix differential equation, if y is replaced by $B\tilde{y}$ for $B \in GL_n(k)$ the matrix equation changes to

$$\tilde{y}' = \left(B^{-1}AB - B'B\right)\tilde{y}.$$

We say that two matrix differential equations are *equivalent*, if they correspond to the same differential module, that is if they can be translated into each other by a differential change of basis. Given any matrix differential equation $y' = Ay,\ A \in GL_n(k)$ it is possible to reconstruct a differential module by setting $M = k^n$ with standard basis $e_1, \ldots, e_n$ and derivation given by $\partial e_i = \sum_{j=1}^n a_{j,i} e_j$. A matrix $F \in GL_n(R)$, where R is an differential extension of k with the same set of constants, is called a *fundamental matrix* for the equation $y' = Ay$, if $F' = AF$ holds. The third notion we introduce is that of a scalar linear differential equation.

Definition 2.1.3 (Linear scalar differential equation) *A* linear (scalar) differential equation *over the differential field k is an equation $L(y) = b, b \in k$, with $L(y)$ defined by*

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y,$$

where $(\cdot)^{(j)}$ denotes the j-times application of the derivation $(\cdot)'$. If $b = 0$ the equation $L(y) = b$ is called homogeneous*, otherwise it is called* inhomogeneous*. The natural number n is called the* degree *or the* order *of L.*

A solution of $L(y) = b$ is an element f of a differential extension of k such that $L(f) = b$. The solutions of $L(y) = 0$ build a C-vector space. A set of n C-linear independent solutions of an order n scalar linear differential equation is called a *fundamental system*. The *companion matrix* of the equation $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 = 0$ is defined as

$$A_L := \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \ldots & -a_{n-1} \end{pmatrix}.$$

For any differential extension $k \subset R$ the solution space

$$\{y \in R | L(y) = 0\}$$

of $L(y) = 0$ is isomorphic to the vector space of solutions $y' = A_L y$ defined as

$$\left\{Y \in R^n | Y' - A_L Y = 0\right\},$$

where the isomorphism is given as $y \mapsto \left(y, y', \ldots, y^{(n-1)}\right)$. In the next section it is explained how a scalar differential equation can be assigned to a differential module.

2.1.2 Differential Operators and Dual Modules

In this section we add the assumption that the constants C of k are an algebraically closed field of characteristic 0. To be able to speak about scalar differential equations in a more algebraic way we introduce the ring of differential operators.

Definition 2.1.4 *The ring $\mathcal{D} := k[\partial]$ of differential operators with coefficients in k consists of all expressions of the form $a_n\partial^n + \ldots + a_1\partial + a_0$, $n \in \mathbb{N}_0$, $a_i \in k$, where the addition is defined in the obvious way and the multiplication is uniquely determined by $\partial a := a\partial + a'$.*

The *degree* of L is the natural number m such that $a_m \neq 0$ and $a_i = 0$ for $i > m$, the degree of 0 is defined to be $-\infty$. The action of a differential operator on a differential extension of k is defined by $\partial^i(y) = y^{(i)}$, thus the equation $L(y) = 0$ for $L \in \mathcal{D}$ has the same meaning as a scalar differential equation. The *greatest common left divisor* of two differential operators L_1, L_2 (GCLD) is the unique monic generator of the left ideal $L_1\mathcal{D} + L_2\mathcal{D}$, the same construction done with the right ideal $\mathcal{D}L_1 + \mathcal{D}L_2$ yields the *greatest common right divisor* (GCRD). A differential module M over k can be described as a left $\mathcal{D}$*-module* such that $\dim_k M < \infty$. The structure of left $\mathcal{D}$-modules is described by Proposition 2.9 of [PS03].

Proposition 2.1.5 *Every finitely generated left $\mathcal{D}$-module has the form $\mathcal{D}^n$ or $\mathcal{D}^n \oplus \mathcal{D}/\mathcal{D}L$ with $n \geq 0$ and $L \in \mathcal{D}$. Especially every $\mathcal{D}$-module of finite k dimension n is isomorphic to $\mathcal{D}/\mathcal{D}L$.*

To describe L in the above proposition we introduce cyclic vectors.

Definition 2.1.6 *An element $e \in M$ of a differential module M is called a* cyclic vector *if $e, \partial e, \ldots$ generate M as k-vector space.*

The existence of a cyclic vector e of a differential module is for example assured by [Kat87]. For a differential module $\mathcal{D}/\mathcal{D}L$ the operator L is the minimal monic operator of degree n annihilating the cyclic vector e. To be more precise, if e is a cyclic vector of $M \cong \mathcal{D}/\mathcal{D}L$, where L is of order n, the $n+1$ vectors $B = \{e, \partial e, \ldots, \partial^{n-1}e, \partial^n e\}$ satisfy a relation

$$a_n\partial^n e + a_{n-1}\partial^{n-1}e + \ldots + a_1\partial e + a_0 e,\ a_j \in k.$$

Hence, any choice of a cyclic vector assigns a differential operator to a differential module. The transpose of the matrix representing ∂ in the basis B is a companion matrix. From this observation it follows that any matrix differential equation is equivalent to a matrix differential equation in companion form. Two differential operators L_1 and L_2 are of the *same type* if the differential modules $M_{L_1} = \mathcal{D}/\mathcal{D}L_1$ and $M_{L_2} = \mathcal{D}/\mathcal{D}L_2$ are isomorphic. This property can be formulated purely in terms of differential operators.

Proposition 2.1.7 *For two monic differential operators L_1 and L_2 the differential modules M_{L_1} and M_{L_2} are isomorphic exactly if there are elements $R, S \in \mathcal{D}$ of degree smaller than n such that $L_1 R = S L_2$ and $GCDR(R, L_2) = 1$.*

Proof The desired isomorphism is given by $[1] \mapsto [R]$. □

As expected, the *dual differential module* M^* of M is defined as the module of k-module homomorphisms from M to the trivial differential module of dimension one denoted $\mathrm{Hom}_k(M, k)$. In general the k-vector space $\mathrm{Hom}_k((M_1, \partial_1), (M_2, \partial_2))$ for two differential k-modules is turned into a differential k-module itself by setting

$$\partial(l(m_1)) = l(\partial_1(m_1)) - \partial_2(l(m_1)) \text{ for } m_1 \in M_1.$$

Suppose that $M \cong \mathcal{D}/\mathcal{D}L$, then M^* is isomorphic $\mathcal{D}/\mathcal{D}L^*$, where L^* is the *formal adjoint* of L defined as the image $i(L)$ of the involution $i : \mathcal{D} \to \mathcal{D}$, $i(\sum a_i\partial^i) = \sum(-1)^i\partial^i a_i$. This can be proven by checking that the element of M^* defined by $e^*(\partial^i e) = \delta_{i(n-1)}$ is a cyclic vector with $L^*e^* = 0$.

If for a differential module M of dimension n over k there is a non-degenerate $(-1)^n$ symmetric bilinear form $\langle\cdot,\cdot\rangle : M \times M \to k$ with

$$(\langle\cdot,\cdot\rangle)' = \langle\partial\cdot,\cdot\rangle + \langle\cdot,\partial\cdot\rangle,$$

the adjunction map $M \to M^*,\ a \mapsto \langle a,\cdot\rangle$ gives an isomorphism of M and its dual M^*. Hence if $M \cong \mathcal{D}/\mathcal{D}L$, then L and L^* are of the same type. The following proposition will be of special importance in Chapter 4.

Proposition 2.1.8 *Let M be a differential $\mathbb{C}(z)$-module of rank n with non-degenerate $(-1)^n$-symmetric pairing $\langle\cdot,\cdot\rangle : M \times M \to k$ with the above compatibility property and let e be a cyclic vector with minimal operator L such that*

$$\langle e, \partial^i e\rangle = \begin{cases} \alpha, & i = n-1 \\ 0, & else \end{cases}, \ \alpha \neq 0$$

then there is $\alpha \in \mathbb{C}(z)$ such that $L\alpha = \alpha L^*$.

Proof Let $e_1 := e, \ldots, e_n := \partial^{n-1}e$ be a basis of M and let $e_1^*, \ldots, e_n^*$ be the dual basis of M^* as k-vector space with $e_i^*(e_j) = \partial_{ij}$. Then $\langle e,\cdot\rangle = \alpha(\partial^{n-1}e)^*, \alpha = \langle e, \partial e^{n-1}\rangle$ by the property of the bilinear form and $(\partial^{n-1}e)^*$ is cyclic vector of M^* with minimal operator L^*. Since $\langle\partial a,\cdot\rangle = \partial^*\langle a,\cdot\rangle$ the equality

$$0 = \langle L(e),\cdot\rangle = L(\langle e,\cdot\rangle) = (L\alpha)((\partial^{n-1}e)^*)$$

is true. But L^* is the minimal monic operator that annihilates $(\partial^{n-1}e)^*$, thus $L\alpha = \beta L^*$. Together with the fact that $\deg(L) = \deg(L^*)$ and that the leading coefficients of L and L^* are 1 the equality of α and β follows. ☐

For $n = 4$ this can be expressed in terms of the coefficients of

$$L = \delta^n + \sum_{i=0}^{3} a_i \partial^i,$$

as

$$a_1 = \frac{1}{2}a_2a_3 + a_2' - \frac{3}{4}a_3a_3' - \frac{1}{2}a_3''.$$

2.2 Series Solutions of Linear Differential Equations

Consider a differential operator

$$D := a_n(x)\partial_x^n + a_{n-1}(x)\partial_x^{n-1} + \ldots + a_0(x)$$

with coefficients analytic throughout the neighborhood of a point $p \in \mathbb{P}^1$ and denote the set of points where some of the a_i have poles by $\Sigma = \{x_1, \ldots, x_r\}$. A point of Σ is called a *singular* and a point in $\mathbb{P}^1 \setminus \Sigma$ is called an *ordinary point* of L. As described on page four of [Poo60] Cauchy's theorem says that the solutions of this differential equation build a $\mathbb{C}$-local system of rank n on $\mathbb{P}^1 \setminus \Sigma$. In particular, their exist n solutions linear independent over $\mathbb{C}$ locally at an ordinary point p. In this section we will show that a similar result can be obtained if p is a regular singular point of the differential equation and introduce the necessary definitions. Following the pages from 396 of E. L. Ince' book [Inc56] and the articles [CC86,CC87b] written by G. V. and D. V. Chudnovsky we collect what is classically known about the construction of the solution space of a linear homogeneous differential equation in a neighborhood of a point $p \in \mathbb{P}^1$, with special emphasis on the Frobenius method.

2.2.1 Indicial Equation, Spectrum and the Riemann Scheme

From now on we restrict to differential operators with regular singular points.

Definition 2.2.1 (Fuchsian differential operator) *A point $p \in \mathbb{C}$ is called a* regular singular *point of the linear homogeneous differential operator L with analytic coefficients if the pole order of $a_j(x)/a_n(x)$ is less than $n-j+1$ for $j = 1,\dots,n-1$. The point at infinity is called regular singular if 0 is a regular singular point of the differential equation one obtaines after the change of coordinates $z = 1/x$. L is called* Fuchsian *if all singular points of L are regular singular.*

The property of p being a regular singular point of L can also be formulated as a growth condition of the solutions of L locally at p. For simplicity we assume $p = 0$, this is not a constraint, because any point on the projective line can be mapped to zero by $x \mapsto x - p$ or $x \mapsto 1/x$. We define an open sector $S(a,b,r)$ as the set of nonzero complex numbers with absolute value smaller than r and argument between a and b. A function $f(x)$ on $S(a,b,r)$ is said to be of *moderate growth* on $S(a,b,r)$, if there is $\epsilon > 0$, $N \in \mathbb{Z}_{>0}$ and $c \in \mathbb{R}$ such that

$$|f(x)| < c\frac{1}{|x|^N}$$

for $x \in S(a,b,r)$ and $|x| < \epsilon$. The claim, known as *Fuchsian criterion*, says that 0 is a regular singular point of a linear homogeneous differential equation exactly if it has a fundamental system B, where all elements of B are of moderate growth on any open sector $S(a,b,\rho)$ with $|a-b| < 2\pi$ and ρ sufficiently small, see Theorem 5.4 of [PS03]. In the rest of this text we concentrate on differential operators $L \in \mathbb{C}[x][\partial_x]$ with polynomial coefficients q_i. If q_n is the coefficient of ∂_x^n, the singular points of L are the roots of q_n, and possibly ∞. If p is a regular singular point, L can always be written as

$$L = p_n(z)z^n\partial_z^n + p_{n-1}(z)z^{n-1}\partial_z^{n-1} + \ldots + p_0(z)$$

for $z = x - p$, polynomials $p_i(z)$ and $p_n(0) \neq 0$. Denote the i-th coefficient of the polynomial $p_j(z)$ by $p_{ji}, i = 0,\dots,m$, where m is the highest degree of any of the polynomials $p_i(z)$. To find a solution of L set up a function in the variable x with an additional parameter σ

$$f = \sum_{j=0}^{\infty} f_j(\sigma)z^{j+\sigma}$$

and try to determine σ and $f_j(\sigma)$ for $n \geq 0$ such that $L(f) = 0$ holds. With the notation $(\sigma)_k := \sigma(\sigma-1)\cdot\ldots\cdot(\sigma-k+1)$ for the *Pochhammer symbol* and

$$F_l(\sigma) := \begin{cases} \sum_{k=0}^{n} p_{kl}(\sigma)_k & 0 \leq l \leq m \\ 0, & else \end{cases},$$

one has

$$L(f) = \sum_{t=0}^{\infty}\left(\sum_{l=0}^{m} f_{t-l}F_l(t-l+\sigma)\right)z^{t+\sigma}.$$

Solving $L(f) = 0$ with the assumption $f_t = 0$ for $t < 0$ coefficientwise for $f_t(\sigma)$ leads to the linear equations

$$f_0F_0(\sigma) = 0$$

and
$$\sum_{l=0}^{m} f_{t-l}F_l(t-l+\sigma) = 0.$$
This means that the sequence $f_t(\sigma),\ t > 0$ of rational functions has to be a solution of a *homogeneous linear difference equation* of rank $m+1$. If the coefficients $f_t(\sigma)$ are computed from to the recurrence relation above,
$$L(f) = f_0F_0(\sigma)$$
holds and for $f_0 \neq 0$ the series f is a solution of L exactly if σ is substituted by a root ν of $F_0(\sigma)$.

Definition 2.2.2 (Indicial equation, local exponents, spectrum) *The polynomial $F_0(\sigma)$ of degree n is called* indicial equation *of L at p. Its roots are called* local exponents *and the n-tuple of all local exponents counted with multiplicities is called* spectrum.

At an ordinary point $F_0(\sigma)$ simplifies to $p_{n0}(\sigma)_n$ which has roots $0, \ldots, n-1$. The local data of L, that is the set of singular points and the corresponding exponents can organized as the so called Riemann scheme.

Definition 2.2.3 (Riemann scheme) *If the singular points of a Fuchsian differential operator L of degree n are $x_1, \ldots, x_r$ with corresponding local exponents $(e_1^1, \ldots, e_n^1), \ldots, (e_1^r, \ldots, e_n^r)$, these data can be conveniently displayed as* Riemann scheme
$$\mathcal{R}(L) := \left\{\begin{array}{cccc} x_1 & x_2 & \ldots & x_r \\ \hline e_1^1 & e_1^2 & \ldots & e_1^r \\ e_2^1 & e_2^2 & \ldots & e_2^r \\ \vdots & \vdots & & \vdots \\ e_n^1 & e_n^2 & \ldots & e_n^r \end{array}\right\}.$$

The calculations to obtain the indicial equation can be tedious, but the following two propositions explain how to simplify them.

Proposition 2.2.4 *If $p \in \mathbb{C}$ is an ordinary or regular singular point of*
$$L := \partial^n + q_1(x)\partial^{n-1} + \ldots + q_n(x),\ q_i \in \mathbb{C}(x)$$
the indicial equation of L at p is
$$\sigma(\sigma-1)\cdot\ldots\cdot(\sigma-n+1) + c_1\sigma(\sigma-1)\cdot\ldots\cdot(\sigma-n+2) + \ldots + c_{n-1}\sigma + c_n = 0,$$
where $c_i := \lim_{x\to p}(x-p)^i q_i(x)$. Similarly, with $b_i = \lim_{x\to\infty} x^i q_i(x)$ at ∞ one has
$$\sigma(\sigma+1)\cdot\ldots\cdot(\sigma+n-1) - b_1\sigma(\sigma+1)\cdot\ldots\cdot(\sigma+n-2) + \ldots + (-1)^{n-1}b_{n-1}\sigma + (-1)^n b_n = 0.$$
At $p = 0, \infty$ it is even easier if the differential operator is written as
$$L = \delta^n + a_{n-1}\delta^{n-1} + \ldots + a_n,$$
with $\delta = x\partial_x$ and rational functions a_j.

Proposition 2.2.5 *The indicial equation of*
$$L = \sum_{j=0}^{n} a_j(x)\delta^j = \sum_{j=0}^{m} \tilde{a}_i(\delta)x^j, \tilde{a}_m \neq 0$$
at 0 is $\tilde{a}_0(\sigma) = 0$ and at ∞ it reads $\tilde{a}_m(\sigma) = 0$.

There is a relation between all local exponents of a Fuchsian differential operator.

Proposition 2.2.6 (Fuchs relation) *Let L be a Fuchsian differential operator of degree n with rational function coefficients. Define s_p as the sum over the local exponents at p. Then the* Fuchs relation

$$\sum_{p\in\mathbb{P}^1}\left(s_p-\binom{n}{2}\right)+2\binom{n}{2}=0$$

holds.

Proof Note that $s_p=\binom{n}{2}$ for regular points p and the series in the lemma is a finite sum. By Proposition 2.2.4 we have

$$s_p=\binom{n}{2}-\operatorname{Res}_p(q_1(x)), p\in\mathbb{C} \text{ and } s_\infty=-\binom{n}{2}-\operatorname{Res}_\infty(q_1(x)).$$

Subtracting $\binom{n}{2}$ on both sides and adding up yields the claim. □

This proposition stays true for differential operators with algebraic coefficients, see [Sai58].

2.2.2 The Method of Frobenius

We return to the construction of a fundamental system of

$$L=p_n(x)x^n\partial_x^n+p_{n-1}(x)x^{n-1}\partial_x^{n-1}+\ldots+p_0(x) \text{ with } p_n(0)\neq 0,$$

but this time we restrict to a neighborhood of $p=0$. By Cramers rule the solutions $f_t(\sigma)$ of the recurrence

$$\sum_{l=0}^{m} f_{t-l}F_l(t-l+\sigma)=0$$

are

$$f_t=\frac{f_0}{F_0(\sigma+1)\cdot\ldots\cdot F_0(\sigma+t)}P_t(\sigma),\ t\geq 1$$

for a polynomial

$$P_t(\sigma):=(-1)^t\begin{vmatrix} F_0(\sigma+1) & 0 & \ldots & 0 & 0 & F_1(\sigma)\\ F_1(\sigma+1) & F_0(\sigma+2) & & 0 & 0 & F_2(\sigma)\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ F_{t-2}(\sigma+1) & F_{t-3}(\sigma+2) & & F_1(\sigma+t-2) & 0 & F_{t-1}(\sigma)\\ F_{t-1}(\sigma+1) & F_{t-2}(\sigma+2) & \ldots & F_2(\sigma+t-2) & F_1(\sigma+t-1) & F_t(\sigma)\end{vmatrix}$$

independent of f_0. If the spectrum consists of n distinct local exponents ν_i with multiplicity one, whose pairwise differences are not integers, the above procedure yields n solutions of L. Inspection of the leading powers proves that these solutions are $\mathbb{C}$-linear independent. If for a local exponent ν_i the sum ν_i+n for a natural number n is again a local exponent, $F_0(\nu_i+n)$ will vanish and the above recurrence cannot be used to associate a solution of L to the local exponent ν_i. This can be fixed if f_0 is replaced by

$$c_0F_0^s(\sigma):=c_0F_0(\sigma+1)\cdot\ldots\cdot F_0(\sigma+s),$$

where s is the maximal integer distance between ν_i and any other local exponent ν_j with $\Re(\nu_i) \leq \Re(\nu_j)$ and c_0 is a nonzero complex constant. Again, if σ is chosen as a local exponent ν the right hand side of

$$L(f) = c_0 F_0^s(\sigma)$$

vanishes and $f(x, \nu)$ is a solution of L. If there are local exponents with multiplicity greater than one or with integer difference, the discussed procedure is in general not adequate to produce n linear independent solutions and we have to employ a method going back to F. G. Frobenius [Fro68]. Up to now $f(x, \sigma)$ is a formal expression and the reasoning that $f(x, \sigma)$ is convergent is postponed until Section 2.2.4. Indeed, the radius of convergence coincides with the smallest absolute value of one of the nonzero singular points of L and the convergence is uniform in σ, if σ varies in a vicinity of one of the local exponents. This allows us to differentiate $f(x, \sigma)$ with respect to σ. Two local exponents with non-integer difference can be treated independently, hence we may assume without loss of generality that the difference of any two local exponents is an integer. Then $S := (\nu_0, \ldots, \nu_k)$ ordered descending according to the real parts of the local exponents can be written as

$$S = (\nu_{i_0}, \ldots, \nu_{i_1-1}, \nu_{i_1}, \ldots, \nu_{i_2-1}, \ldots, \nu_{i_l}, \ldots, \nu_{i_{l+1}-1}),$$

with $\nu_{i_j} = \nu_{i_j+1} = \ldots = \nu_{i_{j+1}-1}$, $j = 0, \ldots, l$ and $\nu_{i_0} = \nu_0$, $\nu_{i_l} = \nu_k$. Set $s = \nu_0 - \nu_k$ and consider the group $v_{i_0} = \ldots = \nu_{i_1-1}$ of local exponents. Let ∂_σ act on x^σ by $\partial_\sigma(x^\sigma) := x^\sigma \log(\sigma)$ and extend this action to $\mathbb{C}[\![x^\sigma[\log(x)]]\!]$. Since ∂_σ commutes with ∂_x the equality

$$L(\partial_\sigma^r f) = \partial_\sigma^r (L(f)) = c_0 \partial_\sigma^r F_0^s(\sigma),$$

where $\partial_\sigma^r f$ has the explicit expression

$$\partial_\sigma^r f(x, \sigma) = \sum_{i=0}^{r} \binom{r}{i} \partial_\sigma^i (f(x, \sigma)) \log(x)^{r-i},$$

is true. Thus $\partial_\sigma^i f(x, \sigma)\big|_{\sigma=\nu_0}$ is a solution of L for $i = 0, \ldots, i_1 - 1$, because by definition ν_0 is a i_1-fold root of $F_0(\sigma)$. Equally, for $j = 1, \ldots, l$ the polynomial $F_0^s(\sigma)$ vanishes with order i_j at $\nu_{i_{j-1}}$ and

$$\partial_\sigma^t f(x, \sigma)\big|_{\sigma=\nu_{i_{j-1}}}, \; t = i_j, \ldots, i_{j+1} - 1$$

also solve L. We summarize the method of Frobenius in the following theorem.

Theorem 2.2.7 *Let L be a differential operator of degree n with 0 a regular singular point. The vector space of multivalued solutions of L in a vicinity of 0 has dimension n. In particular, if*

$$S = \left(\nu_{i_0} = \ldots = \nu_{i_1-1}, \ldots, \nu_{i_{l-1}} = \ldots = \nu_k\right)$$

is as above, $k + 1$ linear independent solutions of L are

$$y_l(x) = \sum_{j=0}^{l} \binom{l}{j} f_j(x, \sigma)\big|_{\sigma=\nu_l} \log(x)^{l-j}$$

with analytic functions $f_j(x, \sigma)$.

We will refer to this basis as *Frobenius basis*, it is unique up to the choice of c_0, we fix $c_0 = 1$.

2.2.3 Explicit Recurrences

Once the method of Frobenius is established we see that in general it is unsuited for a computer implementation, because a huge amount of computations in the field of rational functions have to be carried out. The goal of this section is to give some slight modifications to replace, whenever possible, the computations with rational functions by computations with complex numbers, which are usually cheaper in both time and memory. To start we fix conventions for fundamental systems at ordinary points. If 0 is an ordinary point the spectrum of L at 0 is $(n-1,\ldots,0)$. To gain a solution y_j corresponding to the local exponent j one may choose initial conditions

$$f_t(j) = \begin{cases} 1, & t = j \\ 0, & t < 0 \text{ or } 0 < t < n-1-j \end{cases}$$

and the linear difference equation for the remaining coefficients reads

$$f_t(j) = \sum_{l=0}^{m} f_{t-l} \left(\sum_{k=0}^{d_1} p_{kl}(t-l+j)_k \right),\ t \geq n-1-j.$$

An advantage of this choices is that for the fundamental matrix $(F_{ij})_{i,j\leq n}$ with $F_{ij} = \partial_x^{j-1} y_{i-1}$ the identity $F_{ij}\big|_{x=0} = Id_n$ holds. In the next step we explain how the coefficients of solutions of L at a regular singular point can be computed as solution of an inhomogeneous difference equation. Assume again that L has the spectrum

$$S = (\nu_0, \ldots, \nu_{i_1-1}, \nu_{i_1}, \ldots, \nu_{i_2-1}, \ldots, v_{i_l}, \ldots, \nu_k),$$

at 0, where S is exactly as in the last section. The solution corresponding to $\nu := \nu_r$ is

$$f = \sum_{l=0}^{r} \binom{r}{l} f_l \log(x)^{r-l} = f_r + R \text{ with } f_l \in \mathbb{C}\{x^\nu\}$$

If the series $f_l,\ l = 0,\ldots,r-1$ are already known, the second summand of

$$0 = L(f) = L(f_r) + L\left(\sum_{l=0}^{r-1} \binom{r}{l} f_l \log(x)^{r-l} \right)$$

is an element of $\mathbb{C}\{x^\nu\}$. The coefficients of f_r are subject to an inhomogeneous difference equation

$$\sum_{j=0}^{m} f_{rt-l} F_l(t-l+\nu) = R_t,$$

where f_{kl} is the coefficient of f_k at $x^{l+\nu}$ and R_t is the coefficient of R at $x^{t+\nu}$. To give R_t explicitly we introduce some new notation. For an integer n define

$$\lfloor n \rceil := \begin{cases} n, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

and

$$n!^* := \begin{cases} n!, & n > 0 \\ \frac{(-1)^{-n-1}}{(-n-1)!}, & n \leq 0 \end{cases}.$$

By an elementary calculation or in [Rom93] one can find a convenient way to express $\partial_x^n(\log(x)^k), k \in \mathbb{Z}$ explicitly as

$$\partial_x^n(\log(x)^k) = \frac{1}{-n!^*}\left(\sum_{j=0}^{k}(-1)^j(k)_j c_{-n}^j \log(x)^{k-j}\right)x^{-n},$$

where the rational numbers c_n^j for $n \in \mathbb{Z}$ and $j \in \mathbb{Z}_{>0}$ are recursively defined by

$$nc_n^j = c_n^{j-1} + \lfloor n \rceil c_{n-1}^j$$

with initial conditions

$$c_n^0 = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad \text{and} \quad c_0^j = \begin{cases} 1, & j = 0 \\ 0, & j \neq 0 \end{cases}.$$

This gives a proposition, which enables us to write down R_t explicitly.

Proposition 2.2.8 *For a complex number m and a natural number k*

$$\partial_x^n\left(x^m \log(x)^k\right) = \sum_{j=0}^{n}\binom{n}{j}(m)_j\left(\sum_{l=0}^{k}(-1)^l(k)_l c_{j-n}^l \log(x)^{k-l}\right)\frac{x^{m-n}}{j-n!^*}$$

holds.

Hence, the inhomogeneous part can be computed as

$$R_t = \sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{l=0}^{r-1}\binom{r}{l}q_{kj}f_{lt-j}\sum_{s=0}^{k}\binom{k}{s}\frac{(t-j+\nu)_s(-1)^{r-l}(r-l)!c_{s-k}^{r-l}}{s-k!^*}.$$

Suitable initial values to recover the solution in the Frobenius basis are $f_{r0} = 1$ and

$$f_{rt} = \left(\frac{F_0(\sigma)\cdot\ldots\cdot F_0(\sigma+\nu_0-\nu_k)}{F_0(\sigma)\cdot\ldots\cdot F_0(\sigma+t)}P_t(\sigma)\right)^{(r)}\Bigg|_{\sigma=\nu} \quad \text{for } t = 1,\ldots,m.$$

We compare three different implementations whose functionality include the computation of N coefficients of a full Frobenius basis at an ordinary or regular singular point of a linear homogeneous differential equation L. The first one is offered by the `DEtools` package included in the standard distribution of `Maple 12` (A), the second (B) and third (C) one are implemented by the author following the discussion in this section. In one case the implementation is done in `Maple` (B) in the other case the programming language C with the library `gmc` [EGTZ12] was used for time consuming computations and the facilities of `Maple` were used to manage the input and output. The code is available at [Hof12b]. In all case the computations were done using floating point arithmetic with an accuracy of 300 digits. The two example equations are relevant in this text, one of degree two is

$$L_1 := \left(x(x-1)(x+\frac{2}{27})\right)\partial_x^2 + \left(\frac{1}{2}(x-1)(x+\frac{2}{27}) + x(x+\frac{2}{27}) + x(x-1)\right)\partial + \frac{3}{64}x - \frac{1}{144}$$

and another one of degree four is

$$L_2 := (x^4 - 256x^5)\partial_x^4 + (6x^3 - 2048x^4)\partial_x^3 + (7x^2 - 3712x^3)\partial_x^2 - (1280x^2 + x)\partial_x - 16x.$$

We compute the Frobenius basis of L_1 and L_2 at two different points, the first point $p_a := \frac{1}{256}$ is an ordinary point of L_1 and a singular point of L_2 and the second point $p_b := \frac{1}{81}$ is vice versa an ordinary point of L_2 and a singular point of L_1. The Tables 2.2.3 and 2.2.3 compare the performance of the implementations (A), (B) and (C). The reason that implementation (C) performs slower than the `maple` routines probably stems from the slow data transfer when passing the input, a disadvantage that becomes less important if several solutions are computed in one program fetch.

	L_1			L_2		
	A	B	C	A	B	C
Intel Core2 CPU @ 2.13GHz, 1 GB Ram	181s	0.8s	0.6s	1206s	2.5s	-
23 x Intel Xeon CPU @ 2.67GHz, 128 GB Ram	186s	0.3s	0.5s	878s	14.5s	-

Table 2.1: Runtime in seconds for $N = 500$ for L_1 and L_2 and base point p_a

	L_1			L_2		
	A	B	C	A	B	C
Intel Core2 CPU @ 2.13GHz, 1 GB Ram	197s	1.7s	-	446s	0.8s	0.7s
23 x Intel Xeon CPU @ 2.67GHz, 128 GB Ram	187s	0.6s	-	460s	0.8s	1.4s

Table 2.2: Runtime in seconds for $N = 500$ for L_1 and L_2 and base point p_b

2.2.4 Convergence and Error Bounds

As claimed in Section 2.2.2 we will show the convergence of a series solution

$$f = \sum_{i=0}^{\infty} f_i(\sigma) x^{i+\sigma}$$

obtained as explained above. Additionally we gain error bounds $|f(x_0, \nu) - f^N(x_0, \nu)|$ for truncations $f^N(x, \sigma) := \sum_{i=0}^{N} f_i(\sigma) x^{i+\sigma}$ of those solutions. If zero is an ordinary or regular singular point of the differential operator L, we already know that there are polynomials $p_k(x)$ such that L can be written as

$$L = p_n(x) x^n \partial_x^n + \ldots + p_0(x) \in \mathbb{C}[x][\partial_x],$$

After multiplication by $1/p_n(x)$ this differential operator reads

$$E = \sum_{k=0}^{n} R_k(x) x^k \partial_x^k \text{ with } R_n(x) = 1.$$

with rational function coefficients. Let d be the smallest absolute value of any of the zeros of $q_n(x)$ and $R = d - \epsilon$ for $\epsilon > 0$ arbitrary small. If E is applied to $f(x, \sigma)$ one finds

$$E(f(x,\sigma)) = \sum_{i=0}^{\infty} \sum_{k=0}^{n} f_i R_k(x) (i+\sigma)_k x^{i+\sigma}$$

and since in the series expansion of

$$c(x, i+\sigma) := (\sigma+i)_{d_1} + (\sigma+i)_{d_1-1} R_{d_1-1}(x) + \ldots + R_0(x)$$

no negative powers of x occur it can be written as

$$\sum_{j \geq 0} c_j(\sigma+i) x^j.$$

Cauchy's integral theorem allows the estimation

$$|jc_j(i+\sigma)| \leq \frac{1}{2\pi}\left|\int_{|x|=R}\frac{c'(x,i+\sigma)}{x^j}dx\right| \leq \frac{M(i+\sigma)}{R^{j-1}},$$

where $M(i+\sigma)$ is the upper bound for $c'(x,i+\sigma)$ on the boundary of $U_R(0)$. If i is sufficiently big, the recursion for $f_i(\sigma)$ gives

$$|f_i(\sigma)| = \left|-\frac{1}{c_0(i+\sigma)}\left(\sum_{j=0}^{i-1} f_j(\sigma)c_{i-j}(j+\sigma)\right)\right| \leq \frac{1}{|c_0(i+\sigma)|}\left(\sum_{j=0}^{i-1}\left|f_j(\sigma)\frac{M(j+\sigma)}{R^{i-1-j}}\right|\right).$$

If the last term of this inequality is called F_i, then

$$F_i = \frac{|f_{i-1}|M(i-1+\sigma)}{|c_0(i+\sigma)|} + \frac{|c_0(i-1+\sigma)|}{|c_0(i+\sigma)|}\frac{1}{R}F_{i-1} \leq \left(\frac{M(i-1+\sigma)}{|c_0(i+\sigma)|} + \frac{|c_0(i-1+\sigma)|}{|c_0(i+\sigma)|}\frac{1}{R}\right)F_{i-1}.$$

holds. By choosing $A_i = F_i,\ i > M$ (M big enough) and computing a_i as solution of the first order recursion

$$a_i = \left(Q_1(i+\sigma) + Q_2(i+\sigma)\frac{1}{R}\right)a_{i-1} = \left(\frac{M(i-1+\sigma)}{|c_0(i+\sigma)|} + \frac{|c_0(i-1+\sigma)|}{|c_0(i+\sigma)|}\frac{1}{R}\right)a_{i-1}$$

one finally gets

$$|f_i(\sigma)| \leq F_i \leq a_i,\ i > N.$$

With the notation M_i for the upper bound of $|R_i'(x)|$ on $\partial U_R(0)$ the number $M(i+\sigma)$ is bounded by $\sum_{j=0}^{n-1}(i+\sigma)_j M_j,\ |x| \leq R$ and therefore

$$\lim_{i\to\infty}\frac{M(i-1+\sigma)}{c_0(i+\sigma)} = 0,$$

what gives

$$\lim_{i\to\infty}\frac{a_i}{a_{i-1}} = \frac{1}{R},$$

hence the radius of convergence of $f(x,\sigma)$ equals R. All the examples we are interested in have rational local exponents only, hence we may assume that σ is real. Now choose a point $x_0 \in U_R(0)$ and $\delta > 0$ and compute $N \gg 0$, such that

$$r := |f(x_0,\sigma) - f^N(x_0,\sigma)| \leq \sum_{i=N}^{\infty}|a_i x_0|^i \leq \delta.$$

Notice that

$$a_i = C\prod_{j=M}^{i} Q_1(j+\sigma) + Q_2(j+\sigma)\frac{1}{R},\ C = \prod_{j=0}^{M} F_j,$$

since $Q_1(i+\sigma) + Q_2(i+\sigma)\frac{|x_0|}{R}$ is monotonous for $i \gg 0$ and converges to $\frac{|x_0|}{R}$ for $i \to \infty$, there is an i_0 such that $C\prod_{i=M}^{N} Q_1(i_0+\sigma) + Q_2(i_0+\sigma)\frac{|x_0|}{R} = \epsilon \leq 1$. This yields

$$r \leq \sum_{i=N+1}^{\infty}\epsilon^i < \frac{1}{1-\epsilon} - \sum_{i=0}^{N-1}\epsilon^i = \frac{\epsilon^N}{1-\epsilon}$$

and explains how N has to be chosen. If one takes r as $10^{-\mu}$ and remembers the definition of ϵ, to obtain an error smaller than r one has to compute more than

$$\left(\log_{10}\left(\frac{|x_0|}{R}\right)\right)^{-1}\mu(1+O(1))$$

coefficients of the solution $f(x,\sigma)$ and furthermore all the constants depend on the differential equation and can be made explicit.

2.3 Analytic Continuation and the Monodromy Group

2.3.1 Towards Optimal Approximated Analytic Continuation

Again let L be a Fuchsian differential equation of degree n with singular set Σ. For a fundamental system F_p we define the *local monodromy* of L at p by the equation $MF_p(x) = F_p(x\exp(2\pi i))$. Although the local monodromy is easily computed, its global equivalent is harder to control. For a representative of $\gamma \in \mathbb{P}^1 \setminus \Sigma$ with base point p, analytic continuation along γ transforms a fundamental matrix F_p into $\tilde{F}_p$ and since F_p and $\tilde{F}_p$ are two bases of the same vector space, the solution space of L, analytic continuation yields an element $M_\gamma \in GL_n(\mathbb{C})$ as

$$\tilde{F}_p = F_p M_\gamma.$$

Since analytic continuation on $\mathbb{P}^1 \setminus \Sigma$ only depends on the homotopy class of a path, there is the well defined notion of the monodromy representation.

Definition 2.3.1 (Monodromy representation) *The representation*

$$\rho : \pi_1(\mathbb{P}^1 \setminus \Sigma, p) \to GL_n(\mathbb{C}),\ \gamma \mapsto M_\gamma.$$

is called monodromy representation. *Its image $\rho\left(\pi_1(\mathbb{P} \setminus \Sigma, p)\right)$ is called monodromy group.*

If we fix the composition of loops as

$$\sigma\gamma(t) = \begin{cases} \gamma(2t), & 0 \le t \le \frac{1}{2} \\ \sigma(2t-1), & \frac{1}{2} \le t \le 1 \end{cases},$$

the map ρ is a homeomorphism. Sometimes in the literature some other conventions like $\tilde{F}_p = M_\gamma F_p$ are used. If $\Sigma = \{p_1, \ldots, p_s\} \cup \{\infty\}$, the fundamental group $\pi_1(\mathbb{P} \setminus \Sigma, p)$ is generated by loops $\gamma_1, \gamma_2, \ldots, \gamma_s$ starting at p and encircling exactly one of the finite points in Σ in counterclockwise direction, with the additional property that composition of paths $\gamma_s\gamma_{s-1}\ldots\gamma_1$ is homotopic to a path γ_∞ encircling ∞ once in clockwise direction. The images of theses paths under ρ are called $M_{\gamma_1}, M_{\gamma_2}, \ldots, M_{\gamma_s}$ and M_{γ_∞}. This situation can be depicted in familiar way as shown in Figure 2.3.1. For two regular points x_1 and x_2, with $F_{x_2}(x_2) = Id_n$

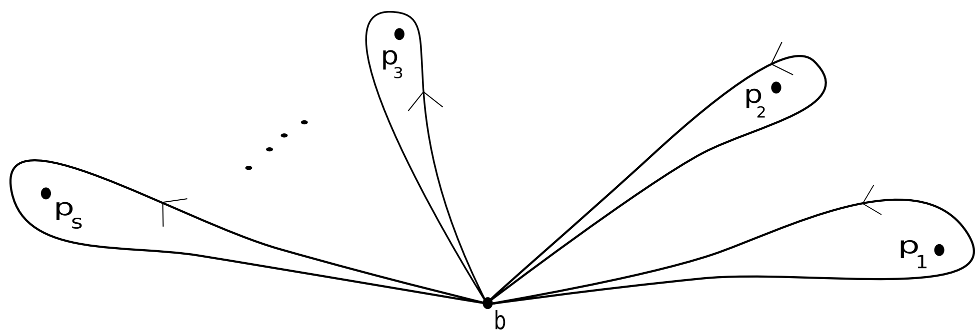

Figure 2.1: Generators of the fundamental group of $\mathbb{P}^1 \setminus \{p_1, \ldots, p_s\}$

and x_2 lying in the disc of convergence of the entries of F_{x_1}, a matrix $M \in GL_n(\mathbb{C})$ such that

$$F_{x_1}(x) = F_{x_2}(x).M$$

holds, is determined by the equality

$$F_{x_1}(x_2) = F_{x_2}(x_2).M = M.$$

To do the analytic continuation of $F_p(x)$ along a path γ, one has to choose points $x_0, \ldots, x_m$, $x_{m+1} = x_0$, which constitute a polygon P_γ with vertices x_i, $i = 0, \ldots, m+1$ homotopic to γ, such that $F_{x_i}(x)$ converges at x_{i+1}. Repeated usage of $F_{x_i}(x_{i+1}) = F_{x_{i+1}}(x_{i+1})M_i = M_i$ gives

$$M_\gamma^{-1} = \prod_{i=1}^{m+1} M_{m+1-i}.$$

Sometimes the choice of a base point x_0 is not only crucial with regard to the complexity of the calculations, but has a deeper meaning, see Chapter 4. Especially a tangential basepoint at a regular singular point should be allowed. If γ is not a loop but a path with different start and endpoint the computations above can be made equally, the resulting matrix M_γ is called *connection matrix*. In general it is not possible to compute closed expressions for the elements of F_{x_i} and the evaluations $F_{x_i}(x_{i+1})$. The most famous exception is the generalized hypergeometric differential equation

$$D(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) := (\theta + \beta_1 - 1) \cdot \ldots \cdot (\theta + \beta_n - 1) - z(\theta + \alpha_1) \cdot \ldots \cdot (\theta + \alpha_n),$$

where $\alpha_i, \beta_i \in \mathbb{C}$, $1 \leq i \leq n$. $D(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ has Riemann scheme

$$\mathcal{R}(D) := \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \hline 1-\beta_1 & 0 & \alpha_1 \\ 1-\beta_2 & 1 & \alpha_2 \\ \vdots & \vdots & \vdots \\ 1-\beta_{n-1} & n-1 & \alpha_{n-1} \\ 1-\beta_n & \gamma & \alpha_n \end{array} \right\},$$

with $\gamma = \sum \beta_j - \sum \alpha_j$. Suppose $\alpha_j \neq \beta_k$, $j, k = 1, \ldots, n$ and let M_0, M_1 and M_∞ be standard generators of the monodromy group of D. There exists a non-zero element

$$v \in \bigcap_{j=0}^{n-2} M_\infty^{-j} \left(\ker(M_0^{-1} - M_\infty) \right)$$

and $M_\infty^j v$, $j = 0, \ldots, n-1$ build the so called *Levelt basis* [Lev61]. Define complex numbers A_i, B_i by

$$\prod_{i=1}^{n} (z - \alpha_j) = z^n + A_1 z^{n-1} + \ldots + A_n.$$

and

$$\prod_{i=1}^{n} (z - \beta_j) = z^n + B_1 z^{n-1} + \ldots + B_n,$$

then again by Levelt's work [Lev61] with respect to the Levelt basis generators of the monodromy group of $D(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ are expressed as

$$M_\infty = \begin{pmatrix} 0 & 0 & \ldots & 0 & -A_n \\ 1 & 0 & \ldots & 0 & -A_{n-1} \\ 0 & 1 & \ldots & 0 & -A_{n-2} \\ & & \ldots & & \\ 0 & 0 & \ldots & 1 & -A_1 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 0 & 0 & \ldots & 0 & -B_n \\ 1 & 0 & \ldots & 0 & -B_{n-1} \\ 0 & 1 & \ldots & 0 & -B_{n-2} \\ & & \ldots & & \\ 0 & 0 & \ldots & 1 & -B_1 \end{pmatrix}^{-1}.$$

In various papers during the 1980's they described their vision of a computer algebra system which, amongst other things, is capable to compute a high precision approximation of the above monodromy matrices, see for example [CC87a]. Additionally they did some implementations and included it in IBMs SCRATCHPAD II, but it seems as if these implementations are not available. Some of the ingredients needed are already implemented [Pfl97]. When we started our implementations there was no complete package available which takes a Fuchsian linear differential equation as input and returns approximations of the generators of its monodromy group. By now the package `NumGfun` by M. Mezzarobba described in [Mez10] and based on theoretical considerations of J. van der Hoeven [Hoe07] parallels our approach in some aspects. Nevertheless we introduce the package `MonodromyApproximation` that is designed for the applications in the subsequent chapters. If

$$f = \sum_{i=0}^{n} \binom{n}{i} \tilde{f}_i \log(x)^{n-i} = \sum_{i=0}^{n} \sum_{j=0}^{\infty} f_{ij} (x-p)^{\sigma+j} \log(x)^{n-i}$$

is a solution of a Fuchsian differential equation on $\mathbb{P}^1$ we denote its truncation after N coefficients by

$$f^N := \sum_{i=0}^{n} \sum_{j=0}^{N} f_{ij} (x-p)^{\sigma+j} \log(x)^{n-i}.$$

Additionally denote a fundamental matrix of L locally at p, which is build from truncations of solutions and derivatives thereof by F_p^N. At first we will focus on the case where the base point x_0 chosen in the monodromy representation is an ordinary point of L. As pointed out in Section 2.3.1 for ordinary points p of L there is always a unique fundamental matrix which evaluates to the identity at p. If a polygon with vertices $x_0, x_1, \ldots, x_m, x_{m+1} = x_0$ in the class of $\gamma \in \pi_1(\mathbb{P}^1 \setminus \Sigma, p_0)$ is chosen such that x_i lies in the region of convergence of $F_{x_{i+1}}$ denote $M_{\gamma,i}^N = F_{x_i}^N(x_{i+1})$. Then an approximation of the monodromy along γ is obtained as

$$(M_\gamma^N)^{-1} = \prod_{i=1}^{m+1} M_{\gamma,m+1-i}^N.$$

Sometimes a practical drawback of this method is that the entries of the approximations of the monodromy matrices get big if the entries of $F_{p_i}(x)|_{x=p_{i-1}}$ are big. Often this problem can be avoided by a slight modification. Denote again the vertices of a polygon P homotopic to the γ by $p_0, p-1, \ldots, p_s = p_0$, and fundamental matrices of L at p_i by F_{p_i} but this time chose $p_i, i = 0, \ldots, s$ such that the region of convergence U_i and U_{i+1} of F_{p_i} resp. $F_{p_{i+1}}$ overlap for $0 \leq i \leq s$. As another choice pick a point $e_{i+1} \in U_i \cap U_{i+1}$ and find an invertible matrix M_i such that

$$F_{p_i}(x)|_{x=e_i} = F_{p_{i+1}}(x)|_{x=e_i}.M_i.$$

Again, the monodromy of L along γ is

$$M_\gamma^{-1} = \prod_{i=1}^{m+1} M_{m+1-i}.$$

and it is clear how to obtain an approximation M_γ^N. This is also the way one should choose if at least one of the expansion points is singular. Analytic continuation depends only on the homotopy type of the path γ, but the choice of a polygon p with vertices p_i in the homotopy class of γ is crucial with respect to the efficiency of the computation of M_γ^N with a given error $\|M_\gamma^N - M_\gamma\| < r$. We already analyzed the error r that occurs if instead of a solution f_0 of a linear differential equation the truncation f_0^N is evaluated at a point x_1. It was not hard to obtain this asymptotics, but to find the best choice of P is very hard. In some special cases a solution is claimed as the following proposition that can be found as Propositions 4.1 and 4.2 of [CC87a].

Proposition 2.3.2 *1. Let $l \subset \mathbb{C}$ be the line from 1 to $p_n \in \mathbb{R}^+$, where 1 is an ordinary point. Assume that for each two points p_1, p_2 on l the singular points where the minima $\min\{p_i - \sigma | \sigma \in \sigma\}, i = 1, 2$ are achieved coincide and equals 0, i.e. there is exactly one closest singular point for all points on the line l, then the points p_i have to be chosen as $p_i = p_{fin}^{i/(m+1)}$, where m is defined as $[\log(p_{fin}) / \log(\gamma)]$ and γ is the solution of*

$$\frac{\log(t-1)}{\log(t)} + \frac{t}{t-1} = 0.$$

2. If C is a circle whose center is a singular point q and whose radius is smaller than the half of the distance from q to any other singular point, the vertices p_i have to build a regular 17-gon on C.

In Appendix A a manual and an example calculation of `MonodromyApproximation` can be found, the whole package is available at [Hof12b].

3 Uniformizing Differential Equations

A discrete subgroup Γ of $PSL_2(\mathbb{R})$ is called *Fuchsian group* and its elements act on the upper half plane $\mathbb{H}$ as Möbius transformation. The quotient $X(\Gamma) = \mathbb{H}/\Gamma$ can be equipped with a complex structure and is therefore a Riemann surface. An important class of Fuchsian groups are congruence subgroups of $SL_2(\mathbb{Z})$ which lead to modular curves, that can be compactified by adding finitely many cusps. An immediate generalization of congruence subgroups are *arithmetic Fuchsian groups*. A group is called *arithmetic* if it is commensurable with the embedding into $Mat_2(\mathbb{R})$ of the group of elements of norm one of an order of a quaternion algebra. If Γ is a an arithmetic Fuchsian group, then $X(\Gamma)$ can be realized as a projective algebraic curve defined by equations with coefficients in a number field. In Section 3.1.3 we briefly review the basic facts on arithmetic Fuchsian groups. Special attention is paid to the case where Γ has signature $(1; e)$ or $(0; 2, 2, 2, q)$. In the first case, the quotient $X(\Gamma)$ has genus one and the projection $\phi : \mathbb{H} \to \mathbb{H}/\Gamma$ branches at exactly one point with branching index e. In the second case, $X(\Gamma)$ has genus zero and branching data $(2, 2, 2, q)$. Arithmetic Fuchsian groups of signature $(1; e)$ were classified by K. Takeuchi in [Tak83] and arithmetic Fuchsian groups with signature $(0; 2, 2, 2, q)$, where q is odd, were classified in [ANR03] and [MR83]. As we explain in Section 3.3 the inverse ω of ϕ can be realized as the quotient of two linearly independent solutions of a differential equation L, called the *uniformizing differential equation*. Recently, J. Sijsling gave a list of equations for projective curves that yield models for several of the Riemann surfaces associated to the groups Γ from Takeuchi's list [Sij13]. If such a model is known, it determines the local data of the uniformizing equation. In the special cases when Γ is commensurable with a triangle group the theory of Belyi maps can be applied to construct the uniformizing differential equation as pullback of a hypergeometric differential equation. In general, one complex parameter, called the *accessory parameter*, remains undetermined. In the case of $(0; 2, 2, 2, q)$ groups the local data are not known completely and the problem of finding uniformizing differential equations varies with two dimensional complex moduli. In Section 3.5, we apply the method to compute high precision approximations of the generators of the monodromy group of a linear Fuchsian differential equation from Chapter 2 to approach the determination of the accessory parameter and the missing local data. Finally, with one exception, we obtain a complete list of candidates for uniformizing differential equations for arithmetic Fuchsian groups with signature $(1; e)$ and $(0; 2, 2, 2, q)$.

3.1 Hyperbolic Geometry and Fuchsian Groups

An accessible introduction to hyperbolic geometry and Fuchsian groups is the book [Kat92] by S. Katok. The missing proofs of most claims made below can be found there. The standard references for basic statements from algebraic number theory include the book [Neu99] by J. Neukirch. Finally, we recommend M.-F. Vignéras' text [Vig80] for facts on quaternion algebras.

3.1.1 Shortcut through Hyperbolic Geometry

The main goal of this section is to introduce the notion of the signature of a Fuchsian group and to have a closer look at Fuchsian groups with two generators. An element of the set of isometries of the upper half plane

$$T \in PSL_2(\mathbb{R}) = \left\{ z \mapsto T(z) = \frac{az+b}{bz+d} \mid ad - bc \neq 0 \right\} \cong SL_2(\mathbb{R}) / \{\pm 1\}$$

is called *Möbius transformation* and has two lifts to $SL_2(\mathbb{R})$, namely $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -a & -b \\ -c & -d \end{smallmatrix}\right)$. The *trace* of T is defined as the absolute value of the trace of one of its lifts. The extension of the action of T to the *extended upper half plane* $\mathbb{H}^* = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ is as expected. A transformation T is said to be *elliptic* if $\operatorname{tr}(T) < 2$, *parabolic* if $\operatorname{tr}(T) = 2$ and *hyperbolic* if $\operatorname{tr}(T) > 2$. By solving $T(z) = z$, one finds that elliptic transformations have two complex conjugated fixed points, hyperbolic transformations have two fixed points in $\mathbb{R} \cup \{\infty\}$ and parabolic transformations have one fixed point in $\mathbb{R} \cup \{\infty\}$. The names come from the fact that invariant curves of elliptic (hyperbolic) transformations are ellipses (hyperbolas). An elliptic element is always conjugated to

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \text{ with } \theta \in [0, 2\pi),$$

where parabolic resp. hyperbolic elements are conjugated to

$$\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ resp. } \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \ \lambda \in \mathbb{R}.$$

Note that a transformation that fixes ∞ is necessarily of the form $z \mapsto \frac{az+b}{d}$ with $a > 0$ and either hyperbolic or parabolic. In the first case its other fixed point is $\frac{b}{1-a}$. Points in $\mathbb{H}$ that are fixed by an elliptic transformation are called *elliptic points*, these which are fixed by hyperbolic transformations are called *hyperbolic points*, and the points which are fixed by parabolic transformations are called *cusps*. The group $PSL_2(\mathbb{R})$ can be identified with the quotient of a subset of $\mathbb{R}^4$ and therefore carries a natural topology. Discrete subgroups are of special importance.

Definition 3.1.1 (Fuchsian group) *A discrete subgroup of $PSL_2(\mathbb{R})$ is called* Fuchsian group.

For a Fuchsian group Γ two points of $\mathbb{H}^*$ are said to be Γ- *congruent* or *equivalent* if there exists an element $T \in \Gamma$ such that $T(u) = v$, i.e. u and v are in the same Γ-orbit.

Definition 3.1.2 (Elementary Fuchsian group) *A Fuchsian group Γ which acts on $\mathbb{H}^*$ with at least one finite orbit is called* elementary.

This is equivalent to the equality $\operatorname{tr}([A, B]) = 2$, whenever $A, B \in \Gamma$ have infinite order. The equivalence class of an elliptic point is called an *elliptic cycle* and the order of an elliptic cycle is the order of the stabilizer of any of its representatives. The closure F of an nonempty open subset F° of $\mathbb{H}$ is called a fundamental domain of the action of a Fuchsian group Γ on $\mathbb{H}^*$ if

- $\bigcup_{T \in \Gamma} T(F) = \mathbb{H}$
- $F^\circ \cap T(F^\circ) = \emptyset$ for all $T \in \Gamma$ except the identity

holds. The set $\{T(F) | T \in \Gamma\}$ is called a *tessellation* of $\mathbb{H}$. To give a fundamental domain that has nice properties for any Fuchsian group, we have to recall that the upper half plane can be equipped with a metric

$$ds = \sqrt{dx^2 + dy^2)}/y.$$

The *hyperbolic length* $h(\gamma)$ of a piecewise differentiable path $\gamma : [0,1] \to \mathbb{H},\ t \mapsto x(t) + iy(t)$ is given by

$$h(\gamma) = \int_0^1 dt \left(\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \right) / y(t)$$

and the hyperbolic distance $d(z,w)$ between the points $z, w \in \mathbb{H}$ is defined as

$$d(z,w) = \inf_\gamma h(\gamma),$$

where γ is any piecewise differentiable path joining z and w. The *geodesics* are semicircles and straight lines orthogonal to the real line. Fundamental domains with some nice features are

$$D_p(\Gamma) = \{z \in \mathbb{H} | d(z,p) \leq d(T(z),p) \text{ for all } T \in \Gamma\},$$

called *Dirichlet fundamental domains*.

Theorem 3.1.3 *For a Fuchsian group Γ and $p \in \mathbb{H}$ not fixed by any nontrivial transformation from Γ the set $D_p(\Gamma)$ is a connected fundamental domain of Γ.*

The boundary of $D_p(\Gamma)$ is a countable union of closed *sides* S_i which are closed segments of geodesics or closed intervals of the real line, that are called *free edges*. The intersections of two distinct sides is either a point or empty. Such intersection points and elliptic fixed points of order two are called a *vertices*.
The limit set $\Lambda(\Gamma)$ of a Fuchsian group is the set of all accumulation points of orbits of the action of Γ on $\mathbb{H}$, it is a subset of $\mathbb{R} \cup \{\infty\}$. Groups where $\Lambda(\Gamma) = \mathbb{R} \cup \{\infty\}$ are of special interest, since they are related to Fuchsian groups with nice fundamental domains.

Definition 3.1.4 (Fuchsian group of the first and second kind) *If the limit $\Lambda(\Gamma)$ of a Fuchsian group Γ coincides with $\mathbb{R} \cup \{\infty\}$ then Γ is called of the* first kind *otherwise it is called of the* second kind.

The *hyperbolic area* for a subset $A \subset \mathbb{H}$ is

$$\mu(A) = \int_A \frac{dxdy}{y^2},$$

for any two fundamental domains F_1, F_2 of the same Fuchsian group Γ one has $\mu(F_1) = \mu(F_2)$ and a fundamental invariant is found. It is reasonable to define the *covolume* $\mathrm{vol}(\Gamma)$ of a Fuchsian group as the hyperbolic volume of its fundamental domains. Theorems 4.5.1 and 4.5.2 of [Kat92] relate Fuchsian group of the first kind and Fuchsian groups with finite covolume.

Theorem 3.1.5 *If a Fuchsian group Γ has a convex fundamental domain with finitely many sides and is of the first kind, then Γ has a fundamental region of finite hyperbolic volume. If a Fuchsian group Γ has a fundamental region of finite hyperbolic area then Γ is of the first kind.*

Hence, if $D_p(\Gamma)$ has only finitely many sides and no free edges, Γ has finite covolume and is therefor of the first kind. Each equivalence class of a cusp or an elliptic point of order bigger than two shows up as a vertex of a Dirichlet domain. These two types of vertices can be distinguished, because cusps are by definition in $\mathbb{R} \cup \{\infty\}$. The elliptic points of order two are not visible immediately in $D_p(\Gamma)$, because the corresponding elliptic transformations are rotations by π. The discussion of side pairings will resolve this problem. That is the one-to-one correspondence between the set

$$\Gamma^* = \{T \in \Gamma | T(D_p(\Gamma)) \cap D_p(\Gamma) \neq \emptyset\}$$

and the sides of $D_p(\Gamma)$, especially sides on which lies a representative of an elliptic cycle correspond to elements of Γ^* of order two.
The quotient $\mathbb{H}/\Gamma$ for a Fuchsian group Γ of the first kind is an oriented surface of genus g that becomes compact after the addition of the finitely many points corresponding to the cusps and marked points corresponding to elliptic cycles. Now we have collected enough information to give the definition of the signature of a Fuchsian group of the first kind.

Definition 3.1.6 (Signature) *If the Fuchsian group of the first kind Γ has n elliptic cycles of orders $e_1, \dots, e_n$ and r cusps and the quotient $\mathbb{H}^*/\Gamma$ has genus g, then the* signature *of Γ is defined as*

$$(g; e_1, \dots, e_n; r).$$

A group G is said to admit a presentation $\langle s \mid r \rangle = \langle s_1, \dots, s_n \mid r_1, \dots, r_t \rangle := F_s/N_r$, where N_r is the smallest normal subgroup of the free group F_s on the symbols $s_1, \dots, s_n$, that contains the words $r_1, \dots, r_t \in F_s$, if there is an isomorphism of groups $G \cong \langle s \mid r \rangle$. If this isomorphism ϕ is realized by extending $S_i \mapsto s_i$ for $i = 1, \dots, n$, we use

$$\langle S_1, \dots, S_n | \, R_1, \dots, R_t \rangle, \; R_j := \phi(r_j)$$

to denote G. Poincaré's theorem states that the signature determines a presentation.

Theorem 3.1.7 *A Fuchsian group with signature $(g; e_1, \dots, e_n, r)$ admits a presentation*

$$\langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_n, \rho_1, \dots, \rho_r | \gamma_i^{e_i} = 1, \prod_{i=1}^{g} [\alpha_i, \beta_i] \cdot \gamma_1 \cdot \ldots \cdot \gamma_n \cdot \rho_1 \cdot \ldots \cdot \rho_r \rangle.$$

The covolume of Γ can be expressed in terms of this presentation.

Theorem 3.1.8 *The covolume of a Fuchsian group Γ with signature $(g; e_1, \dots, e_n; r)$ is*

$$\operatorname{vol}(\Gamma) = 2\pi \left(2g - 2 + r + \sum_{i=1}^{n} \left(1 - \frac{1}{e_i} \right) \right).$$

Conversely, if $g, r \geq 0$ and $e_i \geq 2$ for i from 1 to n with $2g - 2 + r + \sum_{i=1}^{n} \left(1 - \frac{1}{e_i}\right) > 0$, there is a Fuchsian group of signature $(g; m_1, \dots, m_n, r)$.

Proof The first complete proof was given by B. Maskit [Mas71]. □

The covolume of a finite index subgroup Δ of a Fuchsian group Γ is determined by the index $[\Gamma : \Delta]$.

Proposition 3.1.9 *Given a finite index subgroup Δ of the Fuchsian group Γ, the index $[\Gamma : \Delta]$ and the covolumes are related by*

$$[\Gamma : \Delta] = \frac{\operatorname{vol}(\Delta)}{\operatorname{vol}(\Gamma)}.$$

Thus the covering map $\mathbb{H}/\Delta \to \mathbb{H}/\Gamma$ has degree $\operatorname{vol}(\Delta)/\operatorname{vol}(\Gamma)$.

3.1.2 Two-Generator Fuchsian Groups

Amongst Fuchsian groups of the first kind the simplest are groups with two generators. Non-elementary two-generator Fuchsian groups were considered by R. Fricke and F. Klein and these authors were aware of a classification. We will rely on a series of papers mainly from N. Purzitzky and G. Rosenberger, to state this classification, that for example in terms of presentations can be found in [PRZ75]. The next definition is intended to simplify the notation.

Definition 3.1.10 ((1;e)-group, (0;2,2,2,q)-group) *We will call a Fuchsian group with signature* $(1;e)$ *resp.* $(0;2,2,2,q)$ *a* $(1;e)$*-group resp. a* $(0;2,2,2,q)$*-group.*

Before passing to the classification theorem of two-generator Fuchsian groups, recall that $[a,b]$ denotes the commutator $aba^{-1}b^{-1}$.

Theorem 3.1.11 *A non-elementary two-generator Fuchsian group has one and only one of the following presentations.*

$(1;e)$-groups	triangle groups
$(1;e,0),\ e \geq 2$	$(0;r,s,t;0),\ \frac{1}{r}+\frac{1}{s}+\frac{1}{t}<1$
$(1;-;1)$	$(0;r,s;1),\ \frac{1}{r}+\frac{1}{s}<1$
	$(0;r,2),\ r \geq 2$
$(0,2,2,2,q)$-groups	$(0,-,3)$
$(0;2,2,2,q;0),\ gcd(q,2)=1$	

Often a slightly different notation is used, that is the occurrence of a cusp is denoted by ∞, for example $(2,3,\infty)$ denotes the triangle group with signature $(0;2,3;1)$. The triangle groups are named after the shape of their fundamental regions and they are widely used to depict spectacular tessellations of the unit disc as shown in Figure 3.1 for the $(3,3,5)$-triangle group [1]. This picture also illustrates the origin of the name triangle group.

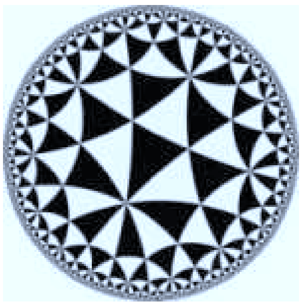

Figure 3.1: Tessellation of the unit disc by a (3,3,5)-group

In Section 3.3 we explain that the uniformizing differential equations of triangle groups can be obtained easily, hence our special interest lies in $(1;e)$ and $(0;2,2,2,q)$-groups. If q is odd, a Fuchsian group $\Gamma = \langle s_1,s_2,s_3,s_4\rangle$ with signature $(0;2,2,2,q)$ is indeed generated by two elements. Denote $x_1 = s_1s_2$ and $x_2 = s_2s_3$ and let H be the subgroup of Γ generated by x_1 and x_2. The equality

$$x_1(x_1x_2)^{-1}x_2 = s_1s_2(s_1s_2s_2s_3)^{-1}s_2s_3 = (s_1s_2s_3)^2 = s_4^{-2}$$

[1] This picture is available under public domain from `http://en.wikipedia.org/wiki/File:H2checkers_335.png`

yields that even powers of s_4 are elements of H and s_4 is contained in H because

$$1 = s_4^q = s_4^{2k+1} = s_4^{2k} s_4 \text{ for } k \geq 0.$$

From

$$x_1 x_2 = s_1 s_2 s_3 s_1 = (s_4)^{-1} s_1$$

we get $s_1 \in H$ and finally one concludes that also the remaining generators s_2, s_3 are elements of H. The above is not true if q is even.

Lemma 3.1.12 *If q is even the group Δ generated by x_1 and x_2 is a normal subgroup of Γ of index 2 and H has the presentation $\langle x_1, x_2 | [x_1, x_2]^{e/2} = 1 \rangle$.*

Proof That $[x_1, x_2]^{p/2} = 1$ holds in Δ is clear from the fact that

$$[s_1 s_2, s_2 s_3] = (s_1 s_2 s_2 s_3 s_2^{-1} s_1^{-1} s_3^{-1} s_2^{-1})$$

can be conjugated to

$$s_1 (s_1 s_2 s_2 s_3 s_2^{-1} s_1^{-1} s_3^{-1} s_2^{-1}) s_1^{-1} = (s_1 s_2 s_3)^{-1}.$$

As explained in Chapter 4 of [LS77] a Reidemeister-Schreier computation can be used to determine the index and a presentation, see also Corollary 3.3 in [PK78]. □

A statement converse to Lemma 3.1.12 is also true.

Lemma 3.1.13 *Each $(1; e)$-group is contained in a $(0; 2, 2, 2, 2e)$-group with index two.*

This is well known and we give a topological proof later. By arguments of N. Purzitzky and G. Rosenberger [PR72] and [Pur76] given two hyperbolic transformations there is a simple criterion when two hyperbolic elements of $PSL_2(\mathbb{R})$ generate a $(1; e)$-group or a $(0; 2, 2, 2, q)$-group.

Proposition 3.1.14
1. *Two hyperbolic transformations $A, B \in PSL_2(\mathbb{R})$ with $[A, B]^e = 1$ generate a Fuchsian group exactly if $\mathrm{tr}([A, B]) = -2\cos(\pi/e)$.*
2. *Two hyperbolic transformations $A, B \in PSL_2(\mathbb{R})$ generate a $(0; 2, 2, 2, q)$-group exactly if $\mathrm{tr}([A, B]) = -2\cos(2\pi/q)$ and $(2, q) = 1$.*

Moreover, the $GL_2(\mathbb{R})$ conjugacy classes of the last mentioned types of groups can be characterized by the following theorem whose parts can be found as Theorem 2 in [PR72] and as Theorem 3.6 in [PK78].

Theorem 3.1.15 *If Γ is presented as*

$$\langle A, B \mid [A, B]^p \rangle \subset PGL_2(\mathbb{R}),$$

with $p \geq 2$ or

$$\langle S_1, S_2, S_3, S_4 \mid S_1^2, S_2^2, S_3^2, S_4^p, (S_1 S_2 S_3 S_4), \rangle \subset PGL_2(\mathbb{R})$$

with $(2, p) = 1,\ p \geq 3$, then G is conjugated over $GL_2(\mathbb{R})$ to one and only one group

$$G(\lambda, \mu, \rho) = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix}, \begin{pmatrix} 0 & \rho \\ -1/\rho & \mu \end{pmatrix} \right\rangle,$$

where

$$\lambda^2 + \mu^2 + (\rho + 1/\rho)^2 - \lambda\mu(\rho + 1/\rho) = 2 - 2\cos(c\pi/p), \lambda \leq \mu \leq (\rho + \frac{1}{\rho}) \leq \frac{1}{2}\mu\lambda,\ 2 < \lambda < 3,$$

with $c = 1$ in the first case and $c = 2$ in the second case.

As indicated above, the investigation of $(1;e)$ and $(0;2,2,2,q)$-groups as groups generated by side pairings of certain polygons has a very long tradition. It was already considered by H. Poincaré and R. Fricke and F. Klein [FK97]. For instance if $A, B \in PSL_2(\mathbb{R})$ generate a $(1;e)$-group Γ and x is the fixed point of the elliptic element $[A, B]$, the maps A and B are the side pairings of the hyperbolic polygon P with vertices x, Ax, Bx, ABx and angle sum $2\frac{\pi}{e}$. The theory of side pairings shows that P is a fundamental domain for Γ. After suitable choices the fundamental domain of a $(1;e)$-group in the unit disc can be depicted as a quadrilateral as in Figure 3.2. If $x = \mathrm{tr}(A)$, $y = \mathrm{tr}(B)$, and $z = \mathrm{tr}(AB)$ for hyperbolic transformations

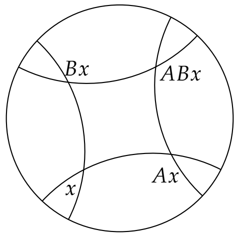

Figure 3.2: Fundamental domain of a $(1;e)$ group

A, B are given, the triple (x, y, z) determines the group $\langle A, B\rangle$ up to $GL_2(\mathbb{R})$ conjugation. The generators A, B fulfilling the conditions from Proposition 3.1.14 and 3.1.2 are clearly not fixed by the signature. For example the *elementary transformations*

$$\text{i) } A_1 = B,\, B_1 = A \quad \text{ii) } A_2 = AB,\, B_2 = A^{-1} \quad \text{iii) } A_3 = A^{-1},\, B_3 = ABA^{-1}$$

do not change the generated group i.e $\langle A, B\rangle = \langle A_1, B_1\rangle$. The elementary transformations affect the trace triple (x, y, z) as

$$\text{i) } (x_1, y_1, z_1) = (y, x, z) \quad \text{ii) } (x_2, y_2, z_2) = (z, x, y) \quad \text{iii) } (x_3, y_3, z_3) = (x, y, xy - z).$$

By Proposition 3.2 of [PK78] finitely many of the operations i)-iii) will transfer (A, B) to a *minimal* generating tuple with trace triple $(\lambda, \mu, \rho + 1/\rho)$, that by definition satisfies

$$\lambda^2 + \mu^2 + (\rho + 1/\rho)^2 - \lambda\mu(\rho + 1/\rho) = 2 - 2\cos(c\pi/p), \lambda \leq \mu \leq (\rho + \frac{1}{\rho}) \leq \frac{1}{2}\mu\lambda,\ 2 < \lambda < 3,$$

where $c = 1$ in case of a $(1, e)$-group and $c = 2$ in the case of a $(0;2,2,2,p)$-group. Hence, to check if a Fuchsian $(1;e)$-group or a $(0;2,2,2,p)$-group $\langle A, B\rangle$ generated by two hyperbolic transformations is a member of the conjugation class of a Fuchsian group with a minimal triple (x, y, z) it is enough to check if $(\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB))$ can be transformed into (x, y, z) with finitely many elementary transformations. In the next chapter, we face the problem that the matrices A and B will have complex entries and a priori we do not know if it is possible to find a $G \in GL_2(\mathbb{C})$ such that A^G and B^G have real entries. Because of the following lemma from page 115 of [MR02] it is possible to decide if a discrete subgroup Γ of $PSL_2(\mathbb{C})$ i.e. a *Kleinian* group is conjugated to a Fuchsian group in terms of the *trace field* $\mathbb{Q}(\mathrm{tr}\,\Gamma) := \mathbb{Q}(\mathrm{tr}\,\widetilde{\gamma} | \gamma \in \Gamma)$, where $\widetilde{\gamma}$ is a lift of $\gamma \in PSL_2(\mathbb{C})$ to $SL_2(\mathbb{C})$.

Lemma 3.1.16 *A non-elementary Kleinian group $\Gamma \subset PSL_2(\mathbb{C})$ is conjugated to a subgroup of $PSL_2(\mathbb{R})$ exactly if $\mathbb{Q}(\mathrm{tr}\,\Gamma) \subset \mathbb{R}$.*

Further by Lemma 3.5.2 of [MR02] we know that in the case of two generator groups it is easy to compute the trace field.

Lemma 3.1.17 *If $\Gamma \subset PSL_2(\mathbb{C})$ is generated by two elements A, B the trace field is*

$$\mathbb{Q}(\mathrm{tr}(\widetilde{A}), \mathrm{tr}(\widetilde{B}), \mathrm{tr}(\widetilde{AB})).$$

3.1.3 Arithmetic Fuchsian groups

There are infinitely many conjugacy classes of $(1;e)$-groups and $(0;2,2,2,q)$-groups, but only finitely many of these are arithmetic. To define arithmetic Fuchsian groups, quaternion algebras are crucial.

Definition 3.1.18 (Quaternion-algebra) *For two elements a,b of a field F of characteristic not equal to 2, the algebra $Fi \oplus Fj \oplus Fk \oplus F1$ with multiplication given by $i^2 = a$, $j^2 = b$ and $ij = ji = k$ is denoted by $(\frac{a,b}{F})$ and called the* quaternion algebra *determined by a and b.*

Two familiar examples of quaternion algebras are the Hamiltonian quaternion algebra $(\frac{-1,-1}{\mathbb{R}})$ and the matrix algebra $(\frac{1,1}{F}) \cong M_2(F)$. We will see that after passing to a completion of F, quaternion algebras are governed by the above two examples. The equivalence classes of Arichmedean values are called *infinite places*, their non-Arichmedean counterparts are called *finite places*. Ostrowski's theorem for number fields classifies places. Every infinite place is equivalent to a place of the form $x \mapsto |\iota(x)|$, where $\iota : F \hookrightarrow \mathbb{C}$ is an embedding of F into the complex numbers and $|\cdot|$ is the usual absolute value. If $\iota(F) \subset \mathbb{R}$ the corresponding infinite place is called *real place*. Every finite place is equivalent to a p-adic valuation $|\cdot|_p$ for a unique prime ideal $p \subset O_F$. After passing to the completion with respect to any real place or finite place, there are only two choices for a quaternion algebra up to isomorphism.

Lemma 3.1.19 *Given an embedding $\nu : F \hookrightarrow \mathbb{R}$ the quaternion algebra $(\frac{a,b}{F}) \otimes_{F,\nu} \mathbb{R}$ is either isomorphic to $M_2(\mathbb{R})$ or $(\frac{-1,-1}{\mathbb{R}})$. The same dichotomy is true for $(\frac{a,b}{F}) \otimes_F F_\nu$ for a finite place ν with $\mathbb{R}$ replaced by F_ν.*

In the first case, $(\frac{a,b}{F})$ is said to be *split* or *unramified* and *non-split* or *ramified* otherwise. Given a totally real number field, the set of ramified places determines the quaternion algebra up to isomorphism.

Theorem 3.1.20 *The cardinality of the set of ramified places of a quaternion algebra over a number field F is finite and even. Conversely, given a finite set S of places of even cardinality, there is a quaternion algebra that ramifies exactly at the places in S. Furthermore two quaternion algebras are isomorphic exactly if their sets of ramifying places coincide.*

We will only consider quaternion algebras which are split at the identity ϕ_1 and non-split at all other infinite places. By this restriction we can assure, that the corresponding Shimura variety is a curve [Mil05]. Denote the isomorphism $(\frac{a,b}{F}) \otimes_{F,1} \mathbb{R} \to M_2(\mathbb{R})$ by j. The discriminant $D(A)$ of a quaternion algebra A is the set of finite ramifying places of A. Together with the assumption on the ramifying behavior at infinity above, the discriminant determines A up to isomorphism. Finally, to give the definition of arithmetic Fuchsian groups we have to introduce orders in quaternion algebras.

Definition 3.1.21 (Order) *A finitely generated O_F submodule of $(\frac{a,b}{F})$ is said to be an* order *if it is a subring of $(\frac{a,b}{F})$.*

An element x of a quaternion algebra $(\frac{a,b}{F})$ can be written as $x = f_0 i + f_1 j + f_2 k + f_3$, $f_i \in F$ and

$$|x| := f_0^2 - a f_1^2 - b f_2^2 + ab f_3^2$$

introduces a norm. Denote all elements of norm 1 of an order $\mathcal{O}$ by $\mathcal{O}^1$.

Definition 3.1.22 (Arithmetic Fuchsian group) *A Fuchsian group Γ is called* arithmetic *if there is a quaternion algebra Q with order $\mathcal{O}$ and $\sigma \in PSL_2(\mathbb{R})$, such that $\sigma^{-1}\Gamma\sigma$ is commensurable with $P\left(j(\mathcal{O}^1)\right)$. Since Q is uniquely determined up to isomorphism, it is called the* quaternion algebra *associated to Γ.*

Recall, that two groups G and H are commensurable if their intersection $G \cap H$ is of finite index in both G and H. It is natural to ask which of the two generator groups of the last section are arithmetic. For triangle groups the answer was given by K. Takeuchi in [Tak77a]. Two Fuchsian triangle groups with the same signature are conjugated over $GL_2(\mathbb{R})$, hence if a Fuchsian triangle group is arithmetic can be read off from its signature. The finite list of signatures of arithmetic triangle groups are given as Theorem 3 in [Tak77a]. The classification of $PGL_2(\mathbb{R})$ conjugacy classes of Fuchsian groups of signature $(1;e)$ and $(0;2,2,2,q)$ heavily relies on Takeuchi's result [Tak75] that the arithmeticy of a Fuchsian group is implied by properties of its trace field.

Theorem 3.1.23 *A Fuchsian group of the first kind is an arithmetic Fuchsian group if and only if*

- $k_1 = \mathbb{Q}(\operatorname{tr}\gamma \mid \gamma \in \Gamma) = \mathbb{Q}(x,y,z)$ *is a totally real number field and* $\operatorname{tr}\Gamma \subset O_{k_1}$
- *for every* $\mathbb{Q}$*-isomorphism* $\sigma : k_1 \to \mathbb{R}$ *such that* σ *does not restrict to the identity on* $k_2 = \mathbb{Q}((\operatorname{tr}\gamma)^2 | \gamma \in \Gamma)$ *the norm* $|\sigma(\operatorname{tr}(\gamma))|$ *is smaller than two for all* $\gamma \neq Id$

In the case of $(1;e)$ and $(0;2,2,2,q)$-groups with minimal trace triple (x,y,z), it is furthermore possible to give the corresponding quaternion algebra as

$$\left(\frac{a,b}{k_2}\right), \text{ with } a = x^2(x^2-4),\ b = -x^2y^2(2+2\cos(\pi/e)) \text{ resp. } b = -x^2y^2(2+2\cos(2\pi/q)).$$

This was used by Takeuchi in [Tak83] to obtain all minimal trace triples of generators of the conjugacy classes of arithmetic Fuchsian groups with signature $(1,e)$. The analogous list for $(0;2,2,2,q)$-groups was compiled in two steps. First in [MR83] and [MR92] C. Maclachlan and G. Rosenberger determined possible values for q, the degree of k_1 over $\mathbb{Q}$ and the discriminant d_{k_1}. In a second step R. Ackermann, M. Näätänen and G. Rosenberger listed minimal trace triples for each conjugation class of arithmetic $(0;2,2,2,q)$-groups [ANR03].

Example 3.2 Consider the hyperbolic transformations represented by the matrices

$$A = \begin{pmatrix} 1 & 2 \\ \sqrt{7}/2 - 1 & \sqrt{7}-1 \end{pmatrix},\ B = \begin{pmatrix} (3-\sqrt{5})/2 & 3-\sqrt{5} \\ (\sqrt{7}-3)/2 & (2\sqrt{7}+\sqrt{5}-3)/2 \end{pmatrix}.$$

As $(x,y,z) = (\operatorname{tr}(A), \operatorname{tr}(B), \operatorname{tr}(AB)) = (\sqrt{7}, \sqrt{7}, 3)$ and $\operatorname{tr}([A,B]) = 0 = -2\cos(\pi/2)$ and

$$[A,B]^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

the group generated by A and B is a $(1;2)$- group. The corresponding quaternion algebra is

$$Q := \left(\frac{x^2(x^2-4), -2(x^2y^2)}{k_2}\right) = \left(\frac{21,-98}{k_2}\right),$$

where $k_2 = \mathbb{Q}(x^2, y^2, xyz) = \mathbb{Q}$ and the discriminant of Q is $(2)(7)$.

3.3 Orbifold Uniformization and Differential Equations

3.3.1 Schwarzian Differential Equations, Orbifolds and Belyi Maps

In this section we recall shortly the classical theory of orbifold uniformization. A reference which explains the mathematical as well as the historical aspect of the uniformization of Riemann surfaces is the book [dSG11] by the writers H. P. de Saint-Gervais. For the proofs of the claims below see the Chapters four and five of M. Yoshida's book [Yos87].

Definition 3.3.1 (Orbifold) *Let X be a complex manifold and $Y \subset X$ a hypersurface, which splits as $Y = \cup_j Y_j$ in irreducible components. Furthermore, associate to every Y_j a natural number $b_j \geq 2$ or ∞. The triple $(X, Y, (b_j)_j)$ is called an* orbifold *if for every point in $X \setminus \cup_j \{Y_j | b_j = \infty\}$ there is an open neighborhood U and a covering manifold which ramifies along $U \cap Y$ with branching indices given by $(b_j)_j$.*

If the above local coverings can be realized globally, this definition can be subsumed as:

Definition 3.3.2 (Uniformization) *If there is a complex manifold M and a map $\phi : M \to X$ which ramifies exactly along the hypersurfaces Y_j with the given branching indices b_j, then the orbifold $(X, Y, (b_j)_j)$ is called uniformizable and M is called a* uniformization.

An important special case are Belyi maps.

Definition 3.3.3 (Belyi map) *A* Belyi map *is a holomorphic map $\phi(x) : M \to \mathbb{P}^1$ from a compact Riemann surface M to the projective line that branches only above $\{0, 1, \infty\}$.*

Because of its relation to a certain differential equation usually the multivalued inverse $\phi(x)^{-1}$ is considered.

Definition 3.3.4 (Developing map) *If the uniformization M of the orbifold $(X, B, (b_j)_j)$ is simply connected, the multivalued inverse of the projection $\phi : M \to X$ is called* developing map.

From now on, we will restrict to the case where X is the quotient of the upper half plane by a Fuchsian group and therefore M coincides with the upper half plane itself. Then the hypersurface Y is just a finite set of points. To the developing map to a linear differential equation we need the following notion.

Definition 3.3.5 (Schwarzian derivative) *The* Schwarzian derivative *$S(w(x))$ of a non-constant smooth function $w(x)$ of one complex variable x is*

$$(S(w(x)) := \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2$$

The crucial property of the Schwarzian derivative is its $PGL_2(\mathbb{C})$ invariance.

Proposition 3.3.6 *For an element of $PGL_2(\mathbb{C})$ represented as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ the equality*

$$S(w(x)) = S\left(\frac{aw(x) + b}{cw(x) + d}\right)$$

holds.

Proof See Proposition 4.1.1 of [Yos87]. □

For a local coordinate x on X, the inverse map $\omega = \phi^{-1}(x)$ is $PGL_2(\mathbb{C})$-multivalued, consequently the Schwarzian derivative is a single-valued map on X. By the next proposition the developing map can be described as ratio of to solutions of a differential equation with single valued coefficients.

Proposition 3.3.7 *Let $\omega(x)$ be a non-constant $PGL_2(\mathbb{C})$-multivalued map, then there are two $\mathbb{C}$-linearly independent solutions $y_0(x), y_1(x)$ of the differential equation*

$$L_{uni} := u'' + \frac{1}{2} S(\omega(x)) u = 0,$$

such that $\omega(x) = \frac{y_0(x)}{y_1(x)}$.

Noting that this proposition goes back to H. A. Schwarz makes the notions of the next definition obvious.

Definition 3.3.8 (Uniformizing differential equation) *If $\omega(x)$ is the developing map of an orbifold $(S, Y, (b_j)_j)$, the differential equation L_{uni} from Proposition 3.3.7 is called* uniformizing differential equation *or* Schwarzian differential equation *of $(S, Y, (b_j)_j)$.*

The quotient of two linearly independent solutions $\omega_0(x), \omega_1(x)$ of a linear differential equation

$$u'' + p_1(x)u' + p_2(x)u$$

of degree two is determined up to the action of $PSL_2(\mathbb{C})$. The class of $\omega_1(x)/\omega_0(x)$ under this action is called *projective solution*. Two differential equations that have the same projective solutions are called *projectively equivalent*. If f is an algebraic function and ω a solution of L, then the differential equation $\tilde{L}$ whose solution is $f\omega$ is projectively equivalent to L. The function f is a solution of the differential equation

$$Lf = u' - \frac{f'}{f}u = 0,$$

the local exponents of L_f at x_i are $\mathrm{ord}_{x=x_i} f$. If L_f has Riemann scheme

$$\left\{ \begin{array}{ccc} x_1 & \dots & x_{m+1} = \infty \\ \hline e_1 & \dots & e_{m+1} \end{array} \right\},$$

the Riemann scheme $\mathcal{R}(\tilde{L})$ of $\tilde{L}$ can be obtained from $\mathcal{R}(L)$ by adding e_i to each local exponent of L at x_i. This possibly introduces new singularities, but the projectivization of the monodromy group does not change, since the monodromy matrices at the new singularities are multiples of the identity. Among all projectively equivalent differential equations there is always one with vanishing coefficient at u' and this can be obtained from L by replacing a solution u of L by $\exp(-1/2 \int p_1 dx)u$.

Definition 3.3.9 (Projective normal form) *A linear differential equation*

$$u'' + p_1(x)u' + p_2(x)u = 0$$

of degree two is said to be in projective normal form *if $p_1 = 0$.*

Note, that the uniformizing differential equation is in projective normal form. It is seen from the local behavior of the developing map that the Riemann scheme of L_{uni} is

$$\left\{ \begin{array}{cccc} \alpha_1 & \dots & \alpha_m & \alpha_{m+1} = \infty \\ \hline \frac{1-1/b_1}{2} & \dots & \frac{1-1/b_m}{2} & \frac{-1-1/b_{m+1}}{2} \\ \frac{1+1/b_1}{2} & \dots & \frac{1+1/b_m}{2} & \frac{-1+1/b_{m+1}}{2} \end{array} \right\}.$$

if all the b_j are finite. If $b_j = \infty$ the spectrum at the corresponding singular point α_i is $\{0\}$. This happens if the point α_i is a cusp of $\mathbb{H}/\Gamma$. If Γ is a hyperbolic triangle group than $m = 2$ and the uniformizing differential is projectively equivalent to the hypergeometric differential equation

$$x(x-1)u'' + (c - (a+b+1)x)u' - abu = 0.$$

with Riemann scheme

$$\left\{ \begin{array}{ccc} 0 & 1 & c \\ \hline 0 & 0 & a \\ 1-c & c-a-b & b \end{array} \right\}.$$

Assume that the triangle group Γ is presented as

$$\Gamma = \langle A, B|\ A^q = B^q = (AB)^r \rangle,$$

then the projection $\phi : \mathbb{H} \to \mathbb{H}/\Gamma$ branches at three points with branching data (q, p, r). If these points are chosen to be $0, 1, \infty$ the uniformizing differential equation has Riemann scheme

$$\left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline \frac{1+1/p}{2} & \frac{1+1/q}{2} & \frac{-1+1/r}{2} \\ \frac{1-1/p}{2} & \frac{1-1/q}{2} & \frac{-1-1/r}{2} \end{array}\right\}.$$

By comparison of this two Riemann schemes the parameters a, b, c can be determined by

$$\frac{1}{p} = |1-c|,\ \frac{1}{q} = |c-a-b|,\ \frac{1}{r} = |a-b|.$$

It was possible to read off the differential equation, because it is determined by its Riemann scheme alone. In general the uniformizing differential equation cannot be computed from Γ directly. In the case of a $(0; 2, 2, 2, q)$ group the uniformizing differential equation is projectively equivalent to

$$L = P(x)u'' + \frac{1}{2}P(x)'u' + (n(n+1)x + C)y = 0,\ n = \frac{1}{q} - \frac{1}{2}$$

with $P(x) = 4(x-x_1)(x-x_2)(x-x_3)$ and a complex constant C. This differential equation is called *algebraic Lamé equation* and the complex number C is referred to as *accessory parameter*. The accessory parameter of a Lamé equation does not affect the local exponents and the location of the singularities of L. That L is closely related to a uniformizing differential equation related to a $(0; 2, 2, 2, q)$-group can be explained by a closer look at the Riemann schemes of L_{uni} and L

$$\mathcal{R}(L) = \left\{\begin{array}{cccc} x_1 & x_2 & x_3 & \infty \\ \hline 0 & 0 & 0 & -\frac{n}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{n+1}{2} \end{array}\right\} \text{ and } \mathcal{R}(L_{uni}) = \left\{\begin{array}{cccc} x_1 & x_2 & x_3 & \infty \\ \hline \frac{1-1/2}{2} & \frac{1-1/2}{2} & \frac{1-1/2}{2} & -\frac{n}{2} + 3/4 \\ \frac{1+1/2}{2} & \frac{1+1/2}{2} & \frac{1+1/2}{2} & \frac{n+1}{2} + 3/4 \end{array}\right\}.$$

Then L_{uni} can be obtained from L by changing the solution y to $(x-x_1)^{-1/4}(x-x_2)^{-1/4}(x-x_3)^{-1/4}y$. Lamé equations are very classical objects in mathematics, they were first investigated in connection with ellipsoidal harmonics, see Chapter IX of [Poo60]. They are special cases of Heun equations, that is general differential equations on $\mathbb{P}^1$ of order two with four singular points at $0, 1, A$ and ∞. The Riemann scheme of a Heun equation is

$$\left\{\begin{array}{cccc} 0 & 1 & A & \infty \\ \hline 0 & 0 & 0 & \alpha \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta \end{array}\right\},$$

and it can be written as

$$u'' + (\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-A})u' + \frac{\alpha\beta x - C}{x(x-1)(x-A)}u = 0.$$

These kind of equations reentered the focus of number theorists at the latest when the connection between the sequences showing up in Apéry's proof [Ape79] of the irrationality of

$\zeta(3)$ with a certain *Heun equation* became clear [Dwo81]. The equation associated to Apéry's proof is

$$L_3 := (x^4 - 34x^3 + x)y''' + (6x^3 - 153x^2 + 3x)y'' + (7x^2 - 112x + 1)y' + (x-5)y = 0$$

its solution space is spanned by squares of solutions of

$$L_2 := (x^3 - 34x^2 + x)y'' + (2x^2 - 51x + 1)y' + 1/4(x-10)y = 0,$$

which indeed is up to coordinate change a Heun equation. There are two specific power series related to L_3

$$A(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^n$$

and

$$B(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \left\{ \sum_{m=1}^{k} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right\}.$$

The series $A(x)$ is a solution of L_3 since its coefficients A_n are solutions of the recurrence equation

$$(n+1)u_{n+1} = (34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1}$$

with initial values $A_0 = 1$ and $A_1 = 5$ and the coefficients of $B(x)$ satisfy the same recurrence with initial values $B_0 = 0$ and $B_1 = 6$. Furthermore the coefficients A_n and B_n satisfy $|A_n - \zeta(3)B_n| < (\sqrt{2}-1)^{4N}$ which implies the irrationality of $\zeta(3)$ as explained in [Ape79], see also [SB85] for a link with Chapter 4.

Sometimes the theory of Belyi maps helps to determine the accessory parameter C. If a $(0;2,2,2,q)-$group Γ is contained in a triangle group Δ the uniformizing differential equation $L_2(u(x)) = 0$ for $\mathbb{H}/\Gamma$ is a pull-back of the uniformizing differential equation $L_1(y(z)) = 0$ for $\mathbb{H}/\Delta$, that is $L_2(u(x)) = 0$ can be obtained from $L_1(y(z)) = 0$ by substituting the unknown function $y(z)$ with

$$y(z) \mapsto f(x)y(\phi(x)),$$

where ϕ is a rational function and f is a product of powers of rational functions. To explain which pullback transformations along finite coverings $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ are of hypergeometric to Heun type it is necessary to describe the behavior of the Riemann scheme under such pullbacks. The Frobenius method introduced in Chapter 2 allows us to compute a basis B of the solution space of a linear Fuchsian differential equation in a vicinity of a regular singular point s, if B does not contain any logarithmic solution then s is called a *non-logarithmic singularity* of L. A non-logarithmic singularity with local exponent difference 1 is called *irrelevant*. An irrelevant singularity s can always be turned into an ordinary point by conjugating L with $(x-s)^k$, where k is the smallest local exponent of L at s.

Proposition 3.3.10 *Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a finite covering and let $L_1(x)$ be a second order Fuchsian differential equation. Denote the branching order of ϕ at $p \in \mathbb{P}^1$ by d_p and denote by $L_2(z)$ the pullback of $L_1(x)$ along ϕ.*

1. *If $\phi(p)$ is a singular point of L_1, then p is an ordinary point or an irrelevant singularity of L_2 exactly if either $d_p > 1$ and the exponent difference at $\phi(p)$ equals $1/d_p$, or $d_p = 1$ and $\phi(p)$ is irrelevant.*

2. *If $\phi(p)$ is an ordinary point of L_1, then p is an ordinary point or an irrelevant singularity of L_2 exactly if ϕ does not branch at p.*

3. *If e_1 and e_2 are the local exponents of L_1 at $\phi(p)$ then the local exponents of L_2 at p equal $de_1 + m$ and $de_2 + m$ for a an integer m.*

Proof This claim is Lemma 2.4 of [Vid09]. □

If less than two of the three exponent differences of L_1 are equal $\frac{1}{2}$ and if L_1 has no basis of algebraic solutions, the hypergeometric to Heun pullbacks ϕ can be classified up to Möbius transformations. It turns out that in all possible cases the rational function ϕ is a Belyi map. This is done in a paper by M. van Hoeji and R. Vidunas [HV12] and in a second paper [VF12] by G. Filipuk and R. Vidunas. The limitations on L_1 mentioned above are made to exclude degenerated cases with finite or infinite dihedral monodromy groups. The first paper lists all (klm)-minus-n-hyperbolic Belyi maps.

Definition 3.3.11 ((klm)-minus-n-hyperbolic Belyi maps) *If $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$ then a Belyi map $\phi(x) : \mathbb{P}^1 \to \mathbb{P}^1$ is called (klm)-minus-n-hyperbolic, if there is at least one branching point of order k, l, m above $0, 1, \infty$ and all but n_0 points above 0 have branching order k, all but n_1 points above 1 have branching order l and all but n_∞ points above ∞ have branching order m, where $n_0 + n_1 + n_\infty = n$.*

The (klm)-minus-4-hyperbolic Belyi maps are exactly those finite coverings $\phi(x) : \mathbb{P}^1 \to \mathbb{P}^1$ that transform a hypergeometric differential equation with fixed local exponent differences $1/k, 1/l, 1/m$ to Heun differential equations. The assumption on the existence of points with special branching index in each fiber is made to exclude some parametric pullbacks. That is pullbacks with less than three fixed local exponent difference, since these parametric hypergeometric to Heun transformations are listed the second mentioned paper. This and similar techniques were used by J. Sijsling in the case of $(1;e)$-groups in [Sij12a].
If none of the methods discussed above works another approach is to recover the uniformizing differential equation L_{uni} from its monodromy group M. Denote the set of points above which the projection $\mathbb{H} \to X$ branches by Σ and let b a point outside of Σ. The representation

$$\rho_0 : \pi_1(X \setminus \Sigma, b) \to \Gamma \subset PSL_2(\mathbb{C})$$

associated to the covering $\mathbb{H} \to X$ describes the change of branches of the developing map under analytic continuation along loops in $\pi_1(X \setminus \Sigma, b)$. Hence ρ_0 is conjugated to the projectivized monodromy representation

$$\tilde{\rho} : \pi_1(X \setminus \Sigma, b) \to GL_2(\mathbb{C}) \to PGL_2(\mathbb{C}).$$

of L_{uni}. In the case of a $(1;e)$-group Γ_1 the genus of the X equals one, a claim in the group theoretic discussion in the last section was that this group is contained in a $(0;2,2,2,2e)$-group Γ_2 with index 2. Hence, we can restrict to uniformizing differential equations on the projective line. We will give a topological explanation for the relation between $(1;e)$-groups and $(0;2,2,2,2e)$-groups. Since we deal with arithmetic Fuchsian groups whose associated quaternion algebra is not $Mat_2(\mathbb{Q})$ only, the Riemann surface $X = \mathbb{H}/\Gamma_1$ will have no cusps and can assumed to be compact. Hence it can be realized as an elliptic curve, that in an affine chart can be given in the form

$$E : y^2 = 4x^3 + ax + b, \ a, b \in \mathbb{C}.$$

The map $\pi : E \to \mathbb{P}^1 \setminus \Sigma, \ (x, y) \mapsto x$ is a twofold cover of $\mathbb{P}^1 \setminus \{x_1, x_2, x_3, \infty\}$, branching of index two at the points of $\Sigma := \{x_1, x_2, x_3, \infty\}$, where x_i are the roots of $P(x) := 4x^3 + ax + b$. Thus the uniformizing differential equation $L_{uni}^{\Gamma_1}$ of Γ_1 is the pullback of the uniformizing differential equation $L_{uni}^{\Gamma_2}$ of Γ_2. And $L_{uni}^{\Gamma_1}$ is defined on the projective line furnished with a local coordinate x. To relate the monodromy representation of $L_{uni}^{\Gamma_2}$ with our initial group Γ_1, it is enough to understand how the fundamental group of $\mathbb{P}^1 \setminus \Sigma$ lifts to the fundamental group of X. We recall the standard construction of the fundamental group of an pointed elliptic curve as branched two fold covering of the four times punctured sphere. Cut the Riemann sphere from x_1 to x_2 and from x_2 to ∞, expand the cuts a little and glue two copies of this cut sphere along the cuts with opposite orientation. The fundamental group of the pointed elliptic curve is generated by lifts of the loops δ and γ as in Figure 3.3. Hence the

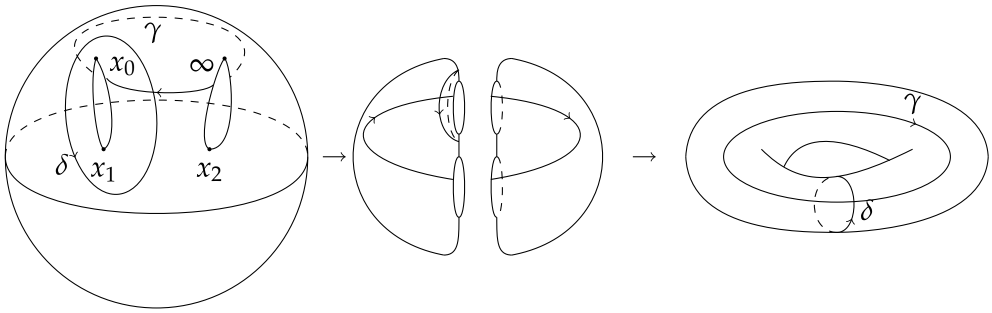

Figure 3.3: Connection of $\pi_1(E)$ and $\pi_1(\mathbb{P}^1 \setminus \Sigma)$

monodromy of $L_{uni}^{\Gamma_1}$ along δ coincides with the monodromy of $L_{uni}^{\Gamma_2}$ along a loop $\gamma_1\gamma_2$ and the monodromy of $L_{uni}^{\Gamma_1}$ along γ coincides with the monodromy of $L_{uni}^{\Gamma_2}$ along $\gamma_\infty\gamma_2^{-1} \sim \gamma_3\gamma_1$. Thus we can identify Γ with the group generated by M_3M_1 and M_1M_2 up to multiplication by scalars. Note that $M_3M_1M_1M_2 = M_3M_2$. This relation between the monodromy of $L_{uni}^{\Gamma_1}$ and the Fuchsian group Γ_1 can also been established, in the spirit of F. Klein and H. Poincaré, by a look at the image of a quotient $y = \frac{y_1}{y_0}$ of the upper half. Then suitable analytic continuation of y and the Schwarz reflection principle would exhibit how certain elements of the monodromy group of $L_{uni}^{\Gamma_1}$ coincide with side pairings of the fundamental domain of Γ_1. This is exactly the way how the monodromy of the hypergeometric function is related to triangle groups. As explained in Chapter 2 a basis of the solution space of the algebraic Lamé equation L at a regular point can be given by two power series and in a vicinity of any of the finite singular points it is given by

$$y_0(x) = \sum_{n=0}^{\infty} a_n(x - x_i)^n \text{ and}$$

$$y_1(x) = (x - x_i)^{\frac{1}{2}} \sum_{n=0}^{\infty} b_n(x - x_i)^n$$

with $a_0 = b_0 = 1$, whose quotient is $\frac{y_0(x)}{y_1(x)} = (x - x_i)^{\frac{1}{2}} \sum_{n=0}^{\infty} c_n(x - x_0)^n$. Similarly since $-\frac{n}{2} - \frac{n-1}{2} = -\frac{1}{2e}$ the shape of this quotient at infinity is

$$\left(\frac{1}{x}\right)^{\frac{1}{2e}} \sum_{n=0}^{\infty} c_n \left(\frac{1}{x}\right)^n.$$

Again from Chapter 2 we know, that the local exponents at an ordinary point p of a Fuchsian equation L of order two are 0 and 1. The ansatz

$$y_\sigma(t) = \sum_{i=0}^{\infty} a_n(\sigma)(t - p)^{n+\sigma} = 0, \ \sigma \in \{0, 1\}$$

provides a basis of the solution space of L locally at p consisting of convergent power series. Adding derivatives a fundamental matrix can be found as

$$F_p = \begin{pmatrix} y_1(t) & y_1'(t) \\ y_0(t) & y_0'(t) \end{pmatrix}.$$

The radius of convergence equals $[\Sigma, p]$, the minimal distance from p to any of the singular points of L. Especially if

$$L = (4x^3 + ax + b)u'' + \frac{1}{2}(4x^3 + ax + b)'u' + (n(n+1)x + C)y = 0, \; n = \frac{1}{e} - \frac{1}{2}$$

the coefficients are computed recursively as

$$\begin{aligned} a_n(\sigma) = &-\frac{a((j-1+\sigma)(j-2+\sigma) + (j-1+\sigma))a_{j-1}}{b(n+\sigma)(n-1+\sigma)} \\ &+\frac{(n(n+1) + C - 4(j-2+\sigma)(j-3+\sigma))a_{j-2}}{b(n+\sigma)(n-1+\sigma)} \\ &-\frac{(6(j-3+\sigma))a_{j-3}}{b(n+\sigma)(n-1+\sigma)}, \; n \geq 3. \end{aligned}$$

The initial conditions, i.e. the values of a_0, a_1 and a_2 could be chosen arbitrarily with at least one $a_i \neq 0$, but we will fix them as $a_0 = 1, a_1 = 0$ and $a_2 = 0$. With the method from Chapter 2 it is possible to compute approximations of generators of the monodromy group. Four points on the Riemann sphere can always be mapped to $0, 1, A, \infty$ by a Möbius transformation. That means given a $(1; e)$-group or a $(0; 2, 2, 2, q)$-group the uniformizing differential equation depends on two parameters, A and the accessory parameter C. It is explained in [Iha74] that A and C have to be elements of $\overline{\mathbb{Q}}$ and moreover in a suitable coordinate they have to be elements of a number field that can be determined from Γ directly. An explanation of this would need the of discussion of class field theory and canonical models of Shimura curves. We avoid this highly advanced theory and try to approximation A and C to high precision to be able to identify them as algebraic numbers. In general the identification is not easy, thus we have to review known methods that accomplish this task.

3.4 Identification of Algebraic Numbers given by Rational Approximations

3.4.1 Roots of Polynomials with Integer Coefficients

To describe an algebraic number α the *degree* and the *height* are essential. If $g \in \mathbb{Z}[x]$ is the minimal polynomial, the degree of α is defined as the degree of g and the height of α is $|g|_\infty$ the biggest absolute values of any of the coefficients of g. Assume that $\tilde{\alpha}$ is a rational approximation of an algebraic number α of degree n and height bounded by N, that is a complex number $\tilde{\alpha}$ such that $|\alpha - \tilde{\alpha}| < \epsilon$, $\epsilon > 0$. In this section we will answer the question which assumptions on ϵ, n, N and $g(\tilde{\alpha})$ one has to impose to be able to conclude that alpha is a root of the irreducible polynomial $g \in \mathbb{Z}[x]$. At first we give a lower bound for the value of a polynomial $g \in \mathbb{Z}[x]$ at an algebraic number α.

Lemma 3.4.1 *If $\alpha \in \mathbb{C}$ is a root of the irreducible non-zero polynomial $h \in \mathbb{Z}[x]$ of degree m with $|\alpha| < 1$ and if $g \in \mathbb{Z}[x]$ is a nonzero polynomial of degree n with $g(\alpha) \neq 0$, then*

$$|g(\alpha)| \geq n^{-1}|h|_2^{-m}|g|_2^{1-n},$$

where $\left|\sum_i^n a_i x^i\right|_2 := \sqrt{\sum_i^n |a_i|^2}$.

Proof See Proposition 1.6 of [KLL88]. □

This lemma yields the desired result.

Corollary 3.4.2 *For two nonzero polynomials* $h, g \in \mathbb{Z}[x]$ *of degree* n *resp.* m*, where* h *is irreducible and* $\alpha \in \mathbb{C}, |\alpha| < 1$ *is a root of* h *such that* $g(\alpha) \neq 0$*, the inequality*

$$|g(\alpha)| \geq n^{-1}(n+1)^{-m/2}(m+1)^{(1-n)/2}|h|_\infty^{-m}|g|_\infty^{1-n}$$

holds.

Proof Notice that for a polynomial of degree n one has $|f|_2^2 \leq (n+1)|f|_\infty$ and combine this inequality with Lemma 3.4.1. □

Lemma 3.4.3 *If* α *is an algebraic number of degree* n *and height bounded by* N *together with a complex number* $\tilde{\alpha}$ *such that* $|\alpha - \tilde{\alpha}| < \epsilon$ *and if* $g \in \mathbb{Z}[x]$ *is polynomial of degree* n*, then*

$$|g(\tilde{\alpha})| < n^{-1}(n+1)^{-n+1/2}|g|_\infty^{-n}N^{1-n} - n\epsilon|g|_\infty$$

implies the upper bound

$$|g(\alpha)| < n^{-1}(n+1)^{-1+1/2}|g|_\infty^{-n}N^{1-n}.$$

Proof Notice that for complex α with $|\alpha| < 1$ the difference $|g(\alpha) - g(\tilde{\alpha})|$ is strictly smaller than $n\epsilon|g|_\infty$ and that the triangle inequality yields $|g(\alpha)| - |g(\tilde{\alpha})| < |g(\alpha) - g(\tilde{\alpha})|$. The claim follows immediately.

Combining Lemma 3.4.1 and the previous Lemma 3.4.3 we obtain the corollary below.

Corollary 3.4.4 *If an algebraic number* α *of degree* n *and height bounded by* N *and a polynomial* $g \in \mathbb{Z}[x]$ *of degree* n *satisfy*

$$|g(\alpha)| < n^{-1}(n+1)^{-n+1/2}|g|_\infty^{-n}N^{1-n}$$

then α *is a root of* g*.*

Hence to identify an algebraic number α we need the data $(N, n, \tilde{\alpha}, g)$, where N is the height of α, the degree of α is n, the rational number $\tilde{\alpha}$ is an approximation of α with $|\alpha - \tilde{\alpha}| < \epsilon$ and $g \in \mathbb{Z}[x]$ is a polynomial such $g(\tilde{\alpha})$ fulfills the estimation in Lemma 3.4.3. If only N, n and $\tilde{\alpha}$ are given in general it is not easy to find a suitable polynomial g, but some useful algorithms are available.

3.4.2 Integer Relation Finding and Lattice Reduction Algorithms

Given a real number a rational number α known up to a given precision of n decimal digits, we are interested in testing if there is an algebraic number α with minimal polynomial with low degree and small height N which coincides with x up to the specified precision. One possibility is to use a direct integer relation algorithm that tries to find a relation $c_0 + c_1 x + \ldots + c_n x^n = 0$ with coefficients $c_i \in \mathbb{Z}$ and therefore a candidate for the minimal polynomial of α. Several such algorithms as `PSLQ` or the algorithms presented in [QFCZ12] and [HJLS86] are available. Usually an algorithm of this type will either find an integer relation or will return an upper bound B such that no integer relation between the x^i exists with coefficient vector $(c_0,\dots,c_n)$ of norm smaller then B. A second approach makes use of lattice reduction algorithms like the one presented by Lenstra-Lenstra-Lovasz [LLL82] named `LLL` to find a sequence $(c_0,c_1,\dots,c_n)$ of elements of $\mathbb{Z}$ such that $c_0+c_1x+\ldots+c_nx^n$ is small. We will shortly review both approaches. The `PSLQ`-algorithm was developed and investigated by H. R. P. Ferguson and D. H. Bailey, the abbreviation `PSLQ` refers to partial sum and the lower trapezoidal orthogonal decomposition. Given a vector $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ an integer relation algorithm tries to find a relation $c_1x_1+\ldots+c_nx_n=0$ with $r=(c_1,\dots,c_n)\in\mathbb{Z}^n, r\neq 0$. That `PSLQ` provides such a relation is content of the following theorem from [FBA96]. Note that the `PSLQ` algorithm depends on a real parameter λ.

Theorem 3.4.5 *Assume that $n\geq 2$ and that there is an integer relation for $x=(x_1,\dots,x_n)$ exists, fix $\gamma>2/\sqrt{3}$ and $\rho=2$ in the real case and $\gamma>\sqrt{2}$ and $\rho=\sqrt{2}$ in the complex case and determine τ as solution of*

$$\frac{1}{\tau^2}=\frac{1}{\rho^2}+\frac{1}{\gamma^2}.$$

If x is normalized such that $|x|=1$ and r_x is the integer relation of x with smallest norm and if $1<\tau<2$, the `PSLQ` algorithm returns an integer relation r for x in no more than

$$\binom{n}{2}\frac{\log(\gamma^{n-1}r_x)}{\log\tau}$$

steps. Moreover, in this case the discrepancy between $|r|$ and $|r_x|$ can be estimated by

$$|r|\leq\gamma^{n-2}r_x.$$

Before explaining the lattice reduction approach, the notion of a reduced lattice basis has to be introduced. Given a basis $b_1,\dots,b_n$ of $\mathbb{R}^n$ equipped with a positive definite scalar product $\langle\cdot,\cdot\rangle$ and the induced norm $|\cdot|$ the Gram-Schmidt process turns this basis into an orthogonal basis $b_1^*,\dots,b_n^*$, defined by

$$b_i^*=b_i-\sum_{j=1..i-1}\mu_{i,j}b_j^*,\ \text{where}\ \mu_{i,j}=\frac{\langle b_i,b_i^*\rangle}{\langle b_j^*,b_j^*\rangle}.$$

Since several divisions have to be done in the Gram-Schmidt process, it will not work if $\mathbb{R}^n$ is replaced by a lattice and the notion of an orthogonal basis has to be replaced.

Definition 3.4.6 (LLL-reduced basis) *The basis $b_1,\dots,b_n$ of a lattice $L\subset\mathbb{R}^m$, $m\geq n$ is called* `LLL` reduced *if*

$$|\mu_{i,j}|\leq\frac{1}{2},\quad 1\leq i,j\leq n$$

and

$$|b_i^*|^2\geq\left(\frac{3}{4}-\mu_{i,i-1}^2\right)|b_{i-1}^*|^2,\quad i\leq 2$$

where b_i^* is the result of the Gram-Schmidt process as above.

Such a basis approximates an orthogonal one and its vectors are short, the precise formulation of this claim is the following theorem that can be found in any textbook on computational number theory, for example Chapter 2 of [Coh93] or in the original paper [LLL82].

Proposition 3.4.7 *For a `LLL` reduced basis $b_1,\dots,b_n$ of a lattice $L\subset\mathbb{R}^n$ let $\det(L)$ be the absolute value of the determinant of the matrix $(b_i)_{i=1,\dots,n}$, then L has the following four properties*

- $\det(L) \leq \prod_{i=1}^{n} |b_i| \leq 2^{n(n-1)/4} \det(L)$
- $|b_j| \leq 2^{i-1}/2 |b_i^*|, \quad 1 \leq j \leq i \leq n$
- $|b_1| \leq 2^{(n-1)/4} \det(L)^{1/n}$ *and*
- *for any set of linearly independent vectors* $x_1, \ldots, x_m$ *we have* $|b_j| \leq 2^{(n-1)/2} \max(|x_1|, \ldots, |x_m|), \quad 1 \leq j \leq m$

Proof This is Proposition 2.6.1 of [Coh93]. □

Note, that in such a basis b_1 is short compared to other vectors x from L i.e $|b_1| \leq 2^{(n-1)/2}|x|$.

Theorem 3.4.8 *Given any basis of a lattice L there exists an algorithm that returns a* `LLL`*-reduced basis in polynomial time in the number of digits in the input. In addition the algorithm outputs a integer matrix that describes the reduced basis in terms of the initial basis.*

Proof Such an algorithm was first constructed in [LLL82] by A. K. Lenstra, H. W. Lenstra, and L. Lovasz. □

The algorithm in the proof is usually referred to as `LLL`-algorithm. Suppose that $\tilde{\alpha}$ is a decimal approximation of the algebraic number $\alpha \in \overline{\mathbb{Q}}$ and that a natural number $n \geq 1$ and an integer K are given. Then a polynomial $f \in \mathbb{Z}[x]$ of degree n whose value at α is small can be found using the `LLL`-algorithm as follows. Consider the lattice L in $\mathbb{R}^{n+2}$ spanned by the rows of the $(n+1) \times (n+2)$ matrix

$$A := \begin{pmatrix} 1 & 0 & \ldots & 0 & K \\ 0 & 1 & \ldots & 0 & \lfloor K\tilde{\alpha} \rfloor \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & \lfloor K\tilde{\alpha}^n \rfloor \end{pmatrix} \in \mathbb{R}^{n+1,n+2}.$$

If B is build row by row from the vectors that constitute a reduced basis of the lattice spanned by the rows of A, there exists a matrix $M \in Gl_{n+1,n+1}(\mathbb{Z})$ such that $B = MA$. For the first row $(c_0, \ldots, c_{n+1})$ of B the special shape of A yields

$$c_{n+1} = c_0 K + c_1 \lfloor K\tilde{\alpha} \rfloor + \ldots + c_n \lfloor K\tilde{\alpha}^n \rfloor.$$

The size of the value of the polynomial

$$f(x) = c_0 + c_1 x + \ldots + c_n x^n$$

at α is relatively close to c_{n+1}/K and c_{n+1} is expected to be small, because it is one component of the first vector of a `LLL`-reduced basis. Hence, f is a good candidate for the minimal polynomial of α. For our purpose the crucial characteristic of an algorithms that tries to find minimal polynomials is rather the input precision needed than the running time. To get an indication of the numerical stability of the three algorithms we compiled Table 3.1. The data listed in the first column is a chosen algebraic number α. To obtain the entries of the remaining columns, we computed numbers $\alpha_i = \lfloor \alpha 10^i \rfloor / 10^i$. Then we ran all three algorithms for increasing $i > 1$ and marked the i where the correct minimal polynomial occurred the first time and the i at which the algorithms started to produce a continuous flow of correct output. One input parameter is the expected degree of the minimal polynomial, if this degree is lower than the actual degree the algorithms cannot succeed. If it is greater than the actual degree

	n	LLL	PSLQ(1.99)	PSLQ($\sqrt{4/3}$)		LLL	PSLQ(1.99)	PSLQ($\sqrt{4/3}$)
α	4	6/6	13/23	12/12	β	37/37	44/44	44/44
	5	6/6	12/23	11/11		42/44	46/46	46/46
	6	8/8	11/13	11/11		49/49	69/65	60/60
	10	11/11	8/12	14/14		77/77	97/97	97/97

Table 3.1: Input precision needed for PSLQ and LLL for $\alpha = \sqrt{2} + \sqrt{3}$ and $\beta = \frac{(\sqrt{2}+\sqrt{3})}{2^2 3^2 17}$

the minimal polynomial can occur as an factor of the output. We used the values $\gamma = 1.99$ and $\gamma = \sqrt{4/3}$ as parameter for PSLQ. Various implementations are available for both PSLQ and LLL, we used the LLL implementation of `Maple 12` and the PSLQ implementation provided by P. Zimmermann at Loria [Zim04]. In recent years, slight algorithmic improvements were made [QFCZ09]. Moreover, there are estimates on the precision of the input depending on the degree and the height of the algebraic number α under consideration needed to guarantee the success of the various algorithms [QFCZ12]. Since our experiments did not show substantial improvements compared to LLL and PSLQ and since most of the estimations are too rough to yield practical benefit, we will not touch these topics.

3.5 Approximation of Uniformizing Differential Equations

3.5.1 The Case of (1;e)-Groups

Recently, J. Sijsling [Sij13] constructed Weierstraß-equations for 56 members of isomorphism classes of the elliptic curves associated to the 73 conjugacy classes of arithmetic $(1;e)$-groups. In the remaining cases he is able to give isogeny classes. If the elliptic curve is known, the local data of the uniformizing differential equation i.e. the Riemann scheme can be read off immediately. Whenever Γ is commensurable with a triangle group in [Sij12a] he additionally used Belyi maps to determine the accessory parameters of the uniformizing differential equations. We will use his results and numerical methods in the next section to tackle the problem of the determination of accessory parameters by investigating the monodromy of the uniformizing differential equations. This approach was also used by G. V. and D. V. Chudnovsky, as they list only 14 cases in [CC89], it is not clear how many cases they handled successfully. In a talk in Banff in 2010 [Beu10] F. Beukers stated that J. Sijsling used similar numerical methods and was again successful in a few cases. But there is no complete list of accessory parameters available. Once we gained the insights from the last sections a possible numerical approach is straight forward. Namely, we have to determine

$$L := P(x)u'' + \frac{1}{2}P(x)'u' + (n(n+1)x + C)u = 0,\ C \in \mathbb{C},\ n = \frac{1}{2e} - \frac{1}{2}$$

with set of singular points Σ such that the conjugacy class of the projectivization of the monodromy group of L coincides with a prescribed group from Takeuchi's list. Recall that this equality can be read off from the trace triple

$$\mathbf{M} := (\mathrm{tr}(M_3M_2), \mathrm{tr}(M_1M_2), \mathrm{tr}(M_3M_1)),$$

where the M_i are generators of the monodromy group associated to standard generators γ_i of $\pi_1(\mathbb{P}^1 \setminus \Sigma)$. If L is a uniformizing differential equation, the entries of $\mathbf{M}$ will be real, hence it is reasonable to consider the real valued functions $f, g : \mathbb{C}^3 \to \mathbb{R}$ given by

$$f(x,y,z) := \max\{\Im(x), \Im(y), \Im(z)\} \quad \text{and} \quad g(x,y,z) := |\Im(x)| + |\Im(y)| + |\Im(z)|.$$

If we assume P as $x(x-1)(x-A)$, then in the 56 cases where Weierstraß-equations are known from J. Sijsling's work A and thereby the Riemann schemes are known. In the remaining cases where only the isogeny class of E is accessible we have to compute a list of isogenous elliptic curves first. We start with one member of the isogeny class of E and computed all elliptic curves $\tilde{E}$, such that there is an isogeny $\rho : E \mapsto \tilde{E}$ of degree bounded by a given $n \in \mathbb{N}$ to obtain a list of possible values for A. The tools provided by J. Sijsling [Sij12b] and the functionality of the computer algebra system `magma` [BCP97] are very helpful. For a fixed complex number C, truncations of elements y_0, y_1 of a fundamental system of L at an ordinary point p can be computed explicitly as

$$y_\sigma^N(x) = \sum_{i=0}^{N} a_n(\sigma)(x-p)^{n+\sigma}, \ \sigma \in \{0,1\}$$

and completed to a truncated fundamental matrix F_p by

$$F_p^N = \begin{pmatrix} y_1^N(t) & y_1^{N'}(t) \\ y_0^N(t) & y_0^{N'}(t) \end{pmatrix}.$$

An approximation of the monodromy with respect to the basis y_0, y_1 along a loop γ of $\pi_1(\mathbb{P} \setminus \Sigma)$ is

$$A_\gamma(C) = \prod_{j=0}^{m-1} F_{p_{m-j}}^N(p_{m-j-1}),$$

where the p_i are m suitable points on a representative of γ. The goal is to find a value for C such that for the triple

$$\mathbf{A}(C) := (\mathrm{tr}(A_3(C)A_2(C)), \mathrm{tr}(A_1(C)A_2(C)), \mathrm{tr}(A_3(C)A_1(C))), \ A_i(C) := A_{\gamma_i}(C)$$

the real numbers $f(\mathbf{A}(C))$ and $g(\mathbf{A}(C))$ are small and such that $\mathbf{A}(C)$ is close to a triple in the list given by Takeuchi. Indeed, it is not necessary to match the listed triple exactly, but only up to elementary transformations, since this does not change the projectivization of the generated group. It is the content of a theorem of E. Hilb, see [Hil08], that for a fixed Riemann scheme of an algebraic Lamé equation corresponding to the uniformizing differential equation of a quotient of $\mathbb{H}$ by a $(1;e)$-group there is an infinite but discrete subset of accessory parameters in the complex plane such that $f(M) = 0$ and $g(M) = 0$. This oscillatory phenomenon can

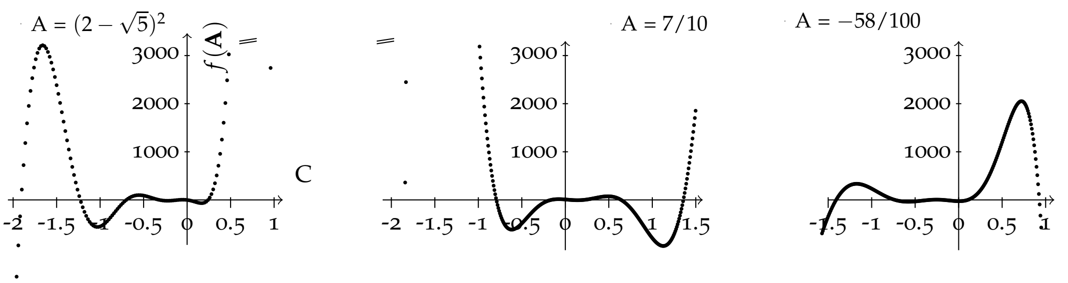

Figure 3.4: $f(\mathbf{A})$ for $n = -1/4$ and three different values of A

be made visible by computing $f(\mathbf{A}(C))$ for some combinations of A and C and $n = -\frac{1}{4}$. The

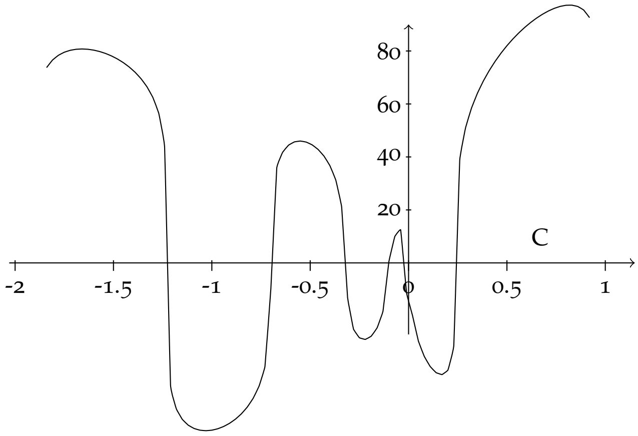

Figure 3.5: $f(\mathbf{A}(C))$ for $n = -1/4$ and $A = (2-\sqrt{5})^2$ on a logarithmic scale

results are shown in the Figures 3.4 and 3.5. In Figure 3.4 the values of the singular point A are $(2-\sqrt{5})^2, \frac{7}{10}$ and $-\frac{58}{100}$. In Figure 3.5 we have chosen a logarithmic scale on the y-axis and interpolated the graph for $C \mapsto f(\mathbf{A}(C))$.

A list that contains $f(\mathbf{A}(C))$ for many values of A and C and all values of n that occur in [Tak83] and [ANR03] is provided in [Hof12a]. A priori we do not have any assumption on the magnitude or the argument of the uniformizing C, but luckily numerical investigations suggest, that it has small absolute value. This leads to the following two algorithms, that try to find a good candidate for the accessory parameter. If the coefficients of the elliptic curve E are real, we use Algorithm 1 to approximate C. All computations are done in multi precision arithmetic as provided by the C library `mpc` [EGTZ12] or the computer algebra system `Maple`.

Algorithm 1: Approximation of the accessory parameter for $(1;e)$-groups (real coefficients)

input : S
if $|f(A(C_1)| < ex$ **then**
  return C_1
else if $|f(A(0)| < ex$ **then**
  return 0
else
  $t = C_1/2$
  while $|f(A(t)| > ex$ **do**
    $t = (S1+S2)/2$
    if $\operatorname{sgn}(f(A(t))) = \operatorname{sgn}(f(A(S1)))$ **then**
      S=(t,S2)
    else
      S=(S1,t)
    end
    adjust N and the length of the mantissa
  end
  return t
end

Start by choosing a small number ex and C_1, such that the signs of $f(\mathbf{A}(0))$ and $f(\mathbf{A}(C_1))$ differ and build the pair $S = (S1, S2) = (f(\mathbf{A}(0)), f(\mathbf{A}(C_1)))$ which is used as input of Algorithm 1. The output is a real number C_a such that $f(\mathbf{A}(C_a)) < ex$. Adjust N and the mantissa means that in every step the number of coefficients in the power series expansion of the solutions and the length of the mantissa are chosen as short as possible to guarantee that the errors made by the truncation and by cutting off the involved floating point numbers is smaller than the precision we have already achieved in $\mathbf{A}(C)$ compared to the target traces $M(C)$. If the accessory parameter is expected not to be real, we use Algorithm 2 with input parameter $r \ll 1$ and $C = 0$. The running time of the Algorithms 1 and 2 depends mainly

Algorithm 2: Approximation of the accessory parameter for $(1; e)$-groups (complex coefficients)

```
input : t,C
if g(A(C)) < 0 then
|  return C;
else
|  while g(A(C)) > ex do
|  |  cont = false;
|  |  for k = 0 to 7 do
|  |  |  CC = C + r exp(2πi/k)
|  |  |  if g(A(CC) < g(A(C)) then
|  |  |  |  C = CC; cont = true; adjust N and the length of the mantissa; break;
|  |  |  end
|  |  |  if cont = false then
|  |  |  |  r = r/2;
|  |  |  end
|  |  end
|  end
|  return C;
end
```

on two factors. The first factor is the value chosen for ex and this choice again is governed by the expected height and the expected degree of the accessory parameter of the uniformizing differential equation. This two quantities govern the precision needed to identify ex as algebraic number as explained in Section 3.4.1. The second factor is the position of the singular points as the configuration of the singular points mainly determines the number of expansion points p_i needed in the approximation of the monodromy. Hence, in the simplest cases after a view minutes we obtained a promising candidate for the accessory parameter, but in the more complex cases the computer had to run for several hours. Once we obtained an algebraic number C_{alg} as candidate we checked that $\mathbf{A}(C_{alg})$ coincides with the trace triple of a set of generators of the groups in Takeuchi's list up to at least 300 digits. The elliptic curves and accessory parameters found in this way are listed in the Tables 3.2-3.9 below. If the elliptic curve under consideration is a model of $\mathbb{H}/\Gamma$, where Γ is an arithmetic Fuchsian group associated to $(\frac{a,b}{F})$ we use the label $n_d/n_D r$, where

- n_d: the discriminant of the coefficient field F of $(\frac{a,b}{F})$
- n_D: the norm of the discriminant of the associated $(\frac{a,b}{F})$
- r: roman number used to distinguish cases with equal n_d and n_D.

to encode $\mathbb{H}/\Gamma$. This labels are chosen in accordance with [Sij13]. The data in the tables below is the label $n_d/n_D r$ an equation for the corresponding elliptic curve

$$E: \; y^2 + x^3 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

the candidate for the accessory parameter C and the the squares of the entries of the corresponding trace triple. The polynomial P which specifies the coefficients of the differential equation L can be recovered from E by the substitution of y by $\frac{1}{2}(y - a_1x - a_3)$ and the elimination of the x^2 term. If an algebraic number γ is involved in the coefficients of E or the given trace triples has a complicated radical expression, we give some digits of its floating point expansion and its minimal polynomial f_γ. Whenever all nonzero singular points of L have non-vanishing real part the base point p is chosen as the imaginary unit i, elsewise it is chosen as 0. The singular points t_i, $i = 1, \ldots, 3$, which are the roots of P are ordered according to the argument of the complex number $t_i - p$. This fixes the loops γ_i and hence the corresponding trace triple.

n_d/n_Dr	elliptic curve / accessory parameter / trace triple
1/6i	$y^2 + xy + y = x^3 - 334x - 2368$ $C = -\frac{79}{64}$ $(5, 12, 15)$
1/6ii	$y^2 = x^3 - x^2 - 4x + 4$ $C = \frac{1}{16}$ $(8, 6, 12)$
1/14 5/4i	$y^2 = x^3 + 23220x - 2285712$ $C = \frac{9}{8}$ $(7, 7, 9)$ $y^2 = x^3 - \left(-\frac{1}{2}\sqrt{5} + \frac{1}{2}\right) x^2 - \left(8 + 3\sqrt{5}\right) x - \frac{25}{2} - \frac{11}{2}\sqrt{5}$ $C = -\frac{5}{32} - \frac{1}{32}\sqrt{5}$ $\left(3 + \sqrt{5}, 6 + 2\sqrt{5}, 7 + 3\sqrt{5}\right)$
5/4ii	$y^2 + \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) y = x^3 - \left(-\frac{1}{2}\sqrt{5} + \frac{1}{2}\right) x^2 - \left(-\frac{329}{2} + \frac{111}{2}\sqrt{5}\right) x + \frac{769}{2} - \frac{287}{2}\sqrt{5}$ $C = -\frac{5}{128} + \frac{1}{128}\sqrt{5}$ $\left(\frac{7}{2} + \frac{3}{2}\sqrt{5}, 6 + 2\sqrt{5}, \frac{7}{2} + \frac{3}{2}\sqrt{5}\right)$
5/4iii	$y^2 + xy + y = x^3 - \left(-\frac{1}{2}\sqrt{5} + \frac{1}{2}\right) x^2 - \left(\frac{101}{2} + \frac{45}{2}\sqrt{5}\right) x - \frac{1895}{2} - \frac{847}{2}\sqrt{5}$ $C = -\frac{5}{64} - \frac{1}{32}\sqrt{5}$ $\left(\frac{9}{2} + \frac{3}{2}\sqrt{5}, \frac{9}{2} + \frac{3}{2}\sqrt{5}, \frac{7}{2} + \frac{3}{2}\sqrt{5}\right)$
8/7i	$y^2 = x^3 + \sqrt{2}x^2 - \left(142\sqrt{2} + 202\right) x - 1170\sqrt{2} - 1655$ $C = -\frac{15}{16} - \frac{5}{8}\sqrt{2}$ $\left(3 + \sqrt{2}, 12 + 8\sqrt{2}, 13 + 9\sqrt{2}\right)$
8/7ii 8/2	$y^2 = x^3 + \sqrt{2}x^2 - \left(142\sqrt{2} + 202\right) x - 1170\sqrt{2} - 1655$ $C = -\frac{15}{16} + \frac{5}{8}\sqrt{2}$ $\left(3 + 2\sqrt{2}, 5 + 3\sqrt{2}, 5 + 3\sqrt{2}\right)$ $y^2 = x^3 - x$ $C = 0$ $\left(3 + 2\sqrt{2}, 5 + 3\sqrt{2}, 5 + 3\sqrt{2}\right)$

12/3	$y^2 = x^3 - \left(\sqrt{3}-1\right)x^2 - \left(65\sqrt{3}+111\right)x + 348\sqrt{3}+603$ $C = \frac{5}{8} + \frac{5}{16}\sqrt{3}$ $\left(8+4\sqrt{3}, 3+\sqrt{3}, 9+5\sqrt{3}\right)$
12/2	$y^2 = x^3 + ax^2 + x + 3\sqrt{3} - 5$ $C = 0$ $\left(4+2\sqrt{3}, 4+2\sqrt{3}, 4+2\sqrt{3}\right)$
13/36	$y^2 + \left(\frac{3+\sqrt{13}}{2}\right)y = x^3 - \frac{5601845}{2} - \left(\frac{60077-16383\sqrt{13}}{2}\right)x + \frac{1551027}{2}\sqrt{13}$ $C = -\frac{1625}{128} + \frac{375}{128}\sqrt{13}$ $\left(\frac{5}{2}+\frac{1}{2}\sqrt{13}, 16+4\sqrt{13}, \frac{33}{2}+\frac{9}{2}\sqrt{13}\right)$
13/4	$y^2 + \left(\frac{3+\sqrt{13}}{2}\right)y = x^3 - \left(\frac{3+\sqrt{13}}{2}\right)x^2 - \left(\frac{275+75\sqrt{13}}{2}\right)x - \frac{1565-433\sqrt{13}}{2}$ $C = -\frac{65}{128} - \frac{15}{128}\sqrt{13}$ $\left(\frac{11+3\sqrt{13}}{2}, \frac{11+3\sqrt{13}}{2}, \frac{7+\sqrt{13}}{2}\right)$
17/2i	$y^2 + xy + (\gamma+1)y = x^3 + \gamma x^2 - (-61\gamma+157)x + 348\gamma - 896$ $\gamma = \frac{1-\sqrt{17}}{2}$ $C = \frac{-55+20\gamma}{64}$ $\left(\frac{5+\sqrt{17}}{2}, 10+2\sqrt{17}, \frac{21+5\sqrt{17}}{2}\right)$
17/2ii	$y^2 + xy + (a+1)y = x^3 + ax^2 - (-61\gamma+157)x + 348\gamma - 896$ $\gamma = \frac{1+\sqrt{17}}{2}$ $C = \frac{-55+20\gamma}{64}$ $\left(\frac{7+\sqrt{17}}{2}, 5+\sqrt{17}, \frac{13+3\sqrt{17}}{2}\right)$
21/4	$y^2 + \left(\frac{\gamma^2+19\gamma+15}{14}\right)y = x^3 - \left(\frac{\gamma^2-9\gamma+15}{14}\right)x^2 - \left(\frac{99\gamma^3-144\gamma^2-618\gamma-1451}{14}\right)x$ $-\frac{88\gamma^3-505\gamma^2-777\gamma-555}{14}$ $f_\gamma = x^4 - 4x^3 - x^2 + 10x + 43, \quad \gamma = -1.292 - 1.323i$ $C = \frac{15-5\sqrt{21}+56i\sqrt{3}-24i\sqrt{7}}{128}$ $\left(\frac{15+3\sqrt{21}}{2}, \frac{5+\sqrt{7}\sqrt{3}}{2}, \frac{17+3\sqrt{7}\sqrt{3}}{2}\right)$
24/3	$y^2 = x^3 - \left(-\frac{1}{16}\gamma^3 - \frac{1}{8}\gamma^2 - \frac{1}{2}\gamma + 1\right)x^2$ $-\left(\frac{17295}{4}\gamma^3 - \frac{14243}{8}\gamma^2 - 60459\gamma + 155218\right)x$ $+\frac{8148639}{8}\gamma^3 - 697026\gamma^2 - \frac{25765125}{2}\gamma + 34343808$ $f_\gamma = x^4 - 8x^2 + 64, \gamma = -2.450 - 1.414i$ $C = \frac{4027}{288} - \frac{137}{24}\sqrt{6} + \frac{17}{36}i\sqrt{2} - \frac{37}{96}i\sqrt{3}$ $\left(5+2\sqrt{2}\sqrt{3}, 3+\sqrt{6}, 9+3\sqrt{6}\right)$
33/12	$y^2 + xy = x^3 - \left(\frac{-3-\sqrt{33}}{2}\right)x^2 - \left(\frac{141}{2} - \frac{27}{2}\sqrt{33}\right)x + \frac{369}{2} - \frac{63}{2}\sqrt{33}$

	$C = \frac{55}{576} - \frac{5}{288}\sqrt{33}$ $\left(\frac{7+\sqrt{11}\sqrt{3}}{2}, \frac{9+\sqrt{33}}{2}, 6+\sqrt{33}\right)$
49/56	$y^2 + xy + y = x^3 - 874 - 171x$ $C = -\frac{55}{64}$ $(3\rho^2 + 2\rho - 1, 3\rho^2 + 2\rho - 1, \rho^2 + \rho)$ $\rho = 2\cos(\frac{\pi}{7})$
81/1	$y^2 + xy + y = x^3 - x^2 - 95x - 697$ $C = -\frac{15}{64}$ $(\rho^2 + \rho + 1, (\rho+1)^2, (\rho+1)^2)$ $f_\rho = x^3 - 3x - 1, \quad \rho = 1.879$
148/1i	$y^2 = x^3 - \left(464\gamma^2 - 320\gamma - 1490\right) x^2 + x$ $f_\gamma = x^3 - x^2 - 3x + 1,\ \gamma = 2.170$ $C = \frac{1363+292\gamma-424\gamma^2}{16}$ $(\gamma^2 + \gamma, \gamma^2 + 2\gamma + 1, \gamma^2 + \gamma)$
148/1ii	$y^2 = x^3 - \left(464\gamma^2 - 320\gamma - 1490\right) x^2 + x$ $f_\gamma = x^3 - x^2 - 3x + 1,\ \gamma = 0.311$ $C = \frac{1363+292\gamma-424\gamma^2}{16}$ $(-12\gamma^2 + 8\gamma + 40, -\gamma^2 + \gamma + 4, -13\gamma^2 + 9\gamma + 42)$
148/1iii	$y^2 = x^3 - \left(464\gamma^2 - 320\gamma - 1490\right) x^2 + x$ $f_\gamma = x^3 - x^2 - 3x + 1, \quad \gamma = 1.481$ $C = \frac{1363+292\gamma-424\gamma^2}{16}$ $(\gamma^2 - 3\gamma + 2, \gamma^2 - 2\gamma + 1, \gamma^2 - 3\gamma + 2)$
229/8i	$y^2 = x^3 - \left(663\gamma^2 - 219\gamma - 2485\right) x^2 - \left(-30778\gamma^2 + 13227\gamma + 109691\right) x$ $f_\gamma = x^3 - 4x - 1, \quad \gamma = 2.115$ $C = \frac{205}{2} + \frac{105}{16}\gamma - \frac{105}{4}\gamma^2$ $(\gamma + 2, 8\gamma^2 + 16\gamma + 4, 8\gamma^2 + 17\gamma + 4)$
229/8ii	$y^2 = x^3 - \left(663\gamma^2 - 219\gamma - 2485\right) x^2 - \left(-30778\gamma^2 + 13227\gamma + 109691\right) x$ $f_\gamma = x^3 - 4x - 1, \quad \gamma = -0.254$ $C = \frac{205}{2} + \frac{105}{16}\gamma - \frac{105}{4}\gamma^2$ $(-3\gamma^2 + \gamma + 13, -\gamma^2 + 5, -4\gamma^2 + \gamma + 16)$
229/8iii	$y^2 = x^3 - \left(663\gamma^2 - 219\gamma - 2485\right) x^2 - \left(-30778\gamma^2 + 13227\gamma + 109691\right) x$ $f_\gamma = x^3 - 4x - 1, \quad \gamma = -1.860$ $C = \frac{205}{2} + \frac{105}{16}\gamma - \frac{105}{4}\gamma^2$

	$(\gamma^2 - 2\gamma, \gamma^2 - 2\gamma + 1, \gamma^2 - 2\gamma)$
725/16i	$y^2 + xy + \gamma y = x^3 + x^2 - (-447\gamma + 4152)\,x - 85116\gamma + 59004$ $f_\gamma = x^2 - x - 1, \quad \gamma = 1.618$ $C = \frac{205-300\gamma}{64}$ $(-2\rho^3 + 5\rho^2 - \rho - 1, -2\rho^3 + 5\rho^2 - \rho - 1, -\rho^3 + 3\rho^2 + \rho)$ $f_\rho = x^4 - x^3 - 3x^2 + x + 1, \quad \rho = -1.355$
725/16ii	$y^2 + xy + \gamma y = x^3 + x^2 - (-447\gamma + 4152)\,x - 85116\gamma + 59004$ $f_\gamma = x^2 - x - 1, \quad \gamma = -0.618$ $C = \frac{205-300\gamma}{64}$ $(9\rho^3 - 13\rho^2 - 21\rho + 19, \rho^3 - 2\rho^2 - 2\rho + 4, 9\rho^3 - 13\rho^2 - 21\rho + 19)$ $f_\rho = x^4 - x^3 - 3x^2 + x + 1, \quad \rho = -0.477$
1125/16	$y^2 + 1/15\left(\gamma^5 + \gamma^4 + 17\gamma^3 + 14\gamma^2 + 13\gamma + 19\right) xy$ $+\frac{1}{4629075}\left(\gamma^7 + 211776\gamma^6 + 471599\gamma^5 + 182985\gamma^4 + 3251185\gamma^3 + 8290968\gamma^2\right.$ $\left.+9151653\gamma + 7962897\right) y - x^3 - \frac{1}{4629075}\left(\gamma^7 + 520381\gamma^6 + 162994\gamma^5 - 125620\gamma^4\right.$ $\left.-1995100\gamma^3 + 3970498\gamma^2 - 3192547\gamma + 5185452\right) x^2$ $-\frac{1}{149325}\left(-12692863\gamma^7 + 86428787\gamma^6 - 164116067\gamma^5 + 518100715\gamma^4\right.$ $\left.-967426690\gamma^3 + 3757504646\gamma^2 - 3892822254\gamma + 10486471269\right) x$ $+\frac{10639239397}{925815}\gamma^7 - \frac{26847806027}{925815}\gamma^6 + \frac{17109768152}{185163}\gamma^5 - \frac{4243037037}{28055}\gamma^4 + \frac{595858143338}{925815}\gamma^3$ $-\frac{316402671731}{308605}\gamma^2 + \frac{2018798518646}{925815}\gamma - \frac{2030456532221}{925815}$ $f_\gamma = x^8 - 6x^7 + 22x^6 - 48x^5 + 135x^4 - 312x^3 + 757x^2 - 999x + 1471$ $\gamma \sim 2.327 + 1.936i$ $C = -\frac{2129}{1234420}a^7 - \frac{42497}{29626080}a^6 - \frac{50863}{7406520}a^5 - \frac{173}{16833}a^4 - \frac{746393}{11850432}a^3 - \frac{9461581}{59252160}a^2$ $-\frac{11911217}{59252160}a - \frac{4076357}{9875360}$ $(-\rho^3 + \rho^2 + \rho + 1, \gamma^2 - \gamma, \rho^2 - 2\rho + 1)$ $f_\rho = x^4 - x^3 - 4x^2 + 4x + 1, \quad \rho = -1.956$

Table 3.2: Ramification index 2

$n_d/n_D r$	elliptic curve / accessory parameter / trace triple
1/15	$y^2 + xy + y = x^3 + x^2 - 135x - 660$ $C = -\frac{55}{54}$ $(5, 16, 20)$
1/10	$y^2 + xy + y = x^3 + 26 - 19x$ $C = \frac{95}{432}$ $(10, 6, 15)$

1/6i	$y^2+xy+y=x^3+x^2-104x+101$ $C=\frac{67}{432}$ $(8,7,14)$
1/6ii	$y^2=x^3-x^2+16x-180$ $C=1/27$ $(8,8,9)$
5/9	$y^2+\left(\frac{1+\sqrt{5}}{2}\right)y=x^3+\left(\frac{1\sqrt{5}}{2}\right)x^2-\left(\frac{495+165\sqrt{5}}{2}\right)x-\frac{4125+1683\sqrt{5}}{2}$ $C=-\frac{245-49\sqrt{5}}{216}$ $\left(\sqrt{5}+3,7+3\sqrt{5},9+4\sqrt{5}\right)$
5/5	$y^2+xy+y=x^3+x^2-110x-880$ $C=-\frac{35}{108}$ $\left(5+2\sqrt{5},5+2\sqrt{5},\frac{7+3\sqrt{5}}{2}\right)$
8/9	$y^2=x^3-\left(2\sqrt{2}+4\right)x^2-\left(154\sqrt{2}+231\right)x-1064\sqrt{2}-1520$ $C=-\frac{20+10\sqrt{2}}{27}$ $\left(4\sqrt{2}+6,4\sqrt{2}+6,3+2\sqrt{2}\right)$
12/3	$y^2+\sqrt{3}y=x^3-\left(-\sqrt{3}+1\right)x^2$ $C=0$ $\left(4+2\sqrt{3},4+2\sqrt{3},7+4\sqrt{3}\right)$
13/3i	$y^2+xy+y=x^3+x^2-(-495\gamma-637)\,x+9261\gamma+12053$ $f_\gamma=x^2-x-3,\quad \gamma=2.302$ $C=-\frac{35}{108}$ $\left(4+\sqrt{13},\frac{11+3\sqrt{13}}{2},4+\sqrt{13}\right)$
13/3ii	$y^2+xy+y=x^3+x^2-(-495\gamma-637)\,x+9261\gamma+12053$ $f_\gamma=x^2-x-3,\quad \gamma=-1.303$ $C=-\frac{35}{108}$ $\left(\frac{5+\sqrt{13}}{2},22+6\sqrt{13},\frac{47+13\sqrt{13}}{2}\right)$
17/36	$y^2+xy+\gamma y=x^3-\gamma x^2-(19694\gamma+30770)\,x-2145537\gamma-3350412$ $\gamma=\frac{1+\sqrt{17}}{2}$ $C=-\frac{6545+1540\sqrt{17}}{432}$ $\left(\frac{5+1\sqrt{17}}{2},13+3\sqrt{17},\frac{29+7\sqrt{17}}{2}\right)$
21/3	$y^2=x^3-\left(-\frac{6\gamma^7+7\gamma^6-30\gamma^5+4\gamma^4-108\gamma^3+90\gamma^2+42\gamma-75}{8}\right)x^2$ $-\left(\frac{7\gamma^7-41\gamma^6-20\gamma^5-126\gamma^4+180\gamma^3-108\gamma^2+51\gamma-63}{192}\right)x$

	$C = \frac{75-42\gamma-90\gamma^2+36\gamma^3-4\gamma^4+30\gamma^5-7\gamma^6+6\gamma^7}{144}$ $f_\gamma = x^8+3x^6+12x^4-9x^2+9, \quad \gamma = 0.770+0.445$ $\left(\frac{5+\sqrt{7}\sqrt{3}}{2}, 10+2\sqrt{7}\sqrt{3}, \frac{23+5\sqrt{21}}{2}\right)$
28/18	$y^2+\left(\sqrt{7}+1\right)y = x^3-\left(\sqrt{7}+1\right)x^2-\left(944\sqrt{7}+2496\right)x+25532\sqrt{7}+67552$ $C = \frac{1295}{576}+\frac{185}{216}\sqrt{7}$ $\left(6+2\sqrt{7}, 3+\sqrt{7}, 8+3\sqrt{7}\right)$
49/1	$y^2+\left(\gamma^2+1\right)y = x^3-\left(-\gamma^2-\gamma-1\right)x^2-\left(649\gamma^2+910\gamma+131\right)x$ $-21451\gamma^2-21320\gamma+6760$ $f_\gamma = x^3-x^2-2x+1, \quad \gamma = 1.802$ $C = -\frac{10+40\gamma+40\gamma^2}{27}$ $(4\rho^2+3\rho-1, 4\rho^2+3\rho-1, \rho^2+\rho)$ $\rho = 2\cos(\frac{\pi}{7})$
81/1	$y^2+y = x^3-7$ $C = 0$ (ρ^2, ρ^2, ρ^2) $\rho = -1/\left(2\cos(\frac{5\pi}{9})\right)$

Table 3.3: Ramification index 3

n_d/n_Dr	elliptic curve / accessory parameter / trace triple
8/98	$y^2+xy+y = x^3-55146-2731x$ $C = -\frac{1575}{256}$ $\left(3+\sqrt{2}, 20+12\sqrt{2}, 21+14\sqrt{2}\right)$
8/7i	$y^2 = x^3-\left(4\sqrt{2}-14\right)x^2-\left(32\sqrt{2}-48\right)x$ $C = \frac{3+6\sqrt{2}}{16}$ $\left(8+4\sqrt{2}, 4+\sqrt{2}, 10+6\sqrt{2}\right)$
8/2i	$y^2+xy = x^3-\left(1-\sqrt{2}\right)x^2-\left(38\sqrt{2}+51\right)x-160\sqrt{2}-227$ $C = -\frac{87}{256}-\frac{15}{64}\sqrt{2}$ $\left(3+2\sqrt{2}, 7+4\sqrt{2}, 7+4\sqrt{2}\right)$
8/2ii	$y^2 = 4x^3-(1116\sqrt{-2}+147)x-(6966\sqrt{-2}-6859)$ $C = \frac{(-78\sqrt{-2}-123)}{2^7}$, taken from [Sij12a] $\left(3+2\sqrt{2}, 9+4\sqrt{2}, 6+4\sqrt{2}\right)$
8/7ii	$y^2 = x^3-\left(-4\sqrt{2}-14\right)x^2-\left(-32\sqrt{2}-48\right)x$

	$C=\frac{3-6\sqrt{2}}{16}$ $\left(4+2\sqrt{2},6+2\sqrt{2},8+5\sqrt{2}\right)$
8/2iii	$y^2+xy=x^3-\left(\sqrt{2}+1\right)x^2-\left(-38\sqrt{2}+51\right)x+160\sqrt{2}-227$ $C=-\frac{87}{256}+\frac{15}{64}\sqrt{2}$ $f_\alpha=5184x^4+59616x^3+171252x^2+10404x+248113, \quad \alpha=0.1891+1.1341i$ $\left(5+2\sqrt{2},6+4\sqrt{2},5+2\sqrt{2}\right)$
2624/4ii	$y^2+xy+y=x^3-\left(1+\sqrt{2}\right)x^2-\left(-391\sqrt{2}+448\right)x+4342\sqrt{2}-6267$ $C=-\frac{387}{256}+\frac{69}{64}\sqrt{2}$ $(\rho,\rho,\rho+2\sqrt{\rho}+1)$ $f_\rho=x^4-10x^3+19x^2-10x+1, \quad \rho=7.698$
2624/4i	$y^2+xy+y=x^3-\left(1-\sqrt{2}\right)x^2-\left(391\sqrt{2}+448\right)x-4342\sqrt{2}-6267$ $C=-\frac{387}{256}-\frac{69}{64}\sqrt{2}$ (ρ_1,ρ_2,ρ_2) $f_{\rho_1}=x^4-10x^3+31x^2-30x+1, \quad \rho_1=4.965$ $f_{\rho_2}=x^5-41x^4+473x^2-1063x^2+343x-19, \quad \rho_2=19.181$
2304/2	$y^2-\sqrt{3}y=x^3-1$ $C=0$ (ρ,ρ,ρ) $\rho=3+\sqrt{2}\sqrt{3}+\sqrt{2}+\sqrt{3}$

Table 3.4: Ramification index 4

$n_d/n_D r$	elliptic curve / accessory parameter / trace triple
5/5i	$y^2+\gamma y=x^3-\gamma x^2-(4217\gamma+2611)x-157816\gamma-97533$ $f_\gamma=x^2-x-1, \ \gamma=1.6180$ $C=-\frac{1083+495\sqrt{5}}{200}$ $\left(\frac{7+1\sqrt{5}}{2},14+6\sqrt{5},16+7\sqrt{5}\right)$
5/180	$y^2+xy+y=x^3-2368-334x$ $C=-\frac{651}{400}$ $\left(\sqrt{5}+3,9+3\sqrt{5},\frac{21+9\sqrt{5}}{2}\right)$
5/5ii	$y^2+\gamma y=x^3-\gamma x^2-(4217\gamma+2611)x-157816\gamma-97533$ $f_\gamma=x^2-x-1, \quad \gamma=-0.618$ $C=-\frac{1083+495\sqrt{5}}{200}$ $\left(6+2\sqrt{5},4+\sqrt{5},\frac{17+7\sqrt{5}}{2}\right)$

5/5iii	$y^2 = x^3 + x^2 - 36x - 140$ $C = -\frac{6}{25}$ $\left(6 + 2\sqrt{5}, 6 + 2\sqrt{5}, \frac{7+3\sqrt{5}}{2}\right)$
5/9	$y^2 + xy + y = x^3 + x^2 + 35x - 28$ $C = \frac{3}{100}$ $\left(\frac{9+3\sqrt{5}}{2}, \frac{9+3\sqrt{5}}{2}, \frac{15+5\sqrt{5}}{2}\right)$
725/25i	$y^2 + (\gamma^2 + \gamma) y = x^3 - (-\gamma^3 - \gamma^2 + 1) x^2 - (135\gamma^3 + 316\gamma^2 - 136\gamma + 2) x$ $-4089\gamma^3 - 6001\gamma^2 + 3228\gamma + 1965$ $f_\gamma = x^4 - x^3 - 3x^2 + x + 1,\ \gamma = 0.738$ $C = \frac{-24+18\gamma-12\gamma^2-12\gamma^3}{25}$ (ρ_1, ρ_2, ρ_2) $f_{\rho_1} = x^4 - 3x^3 + 4x - 1,\ \rho_1 = 2.356,\ f_{\rho_2} = x^4 - 4x^3 + 3x - 1,\ \rho_2 = 3.811$
725/25ii	$y^2 + (\gamma^2 + \gamma) y = x^3 - (-\gamma^3 - \gamma^2 + 1) x^2 - (135\gamma^3 + 316\gamma^2 - 136\gamma + 2) x$ $-4089\gamma^3 - 6001\gamma^2 + 3228\gamma + 1965$ $f_\gamma = x^4 - x^3 - 3x^2 + x + 1,\ \gamma = 2.0953$ $C = \frac{-24+18\gamma-12\gamma^2-12\gamma^3}{25}$ (ρ_1, ρ_2, ρ_2) $f_{\rho_1} = x^4 - x^3 - 3x^2 + x + 1,\ \rho_1 = 2.095$ $f_{\rho_2} = x^4 - 8x^3 + 10x^2 - x - 1,\ \rho_2 = 6.486$
1125/5	$y^2 + (\gamma + 1) y = x^3 - (-\gamma^3 + \gamma^2 + 1) x^2 - (2\gamma^3 - 7\gamma^2 + 5\gamma + 1) x$ $+6\gamma^3 - 14\gamma^2 - 2\gamma + 12$ $f_\gamma = x^4 - x^3 - 4x^2 + 4x + 1,\ \gamma = 1.338$ $C = 0$ (ρ, ρ, ρ) $f_\rho = x^4 - 3x^3 - x^2 + 3x + 1,\ \rho = 2.956$

Table 3.5: Ramification index 5

$n_d/n_D r$	elliptic curve / accessory parameter / trace triple
12/66i	$y^2 + xy + \left(1 - \sqrt{3}\right) y = x^3 - \left(\sqrt{3} + 1\right) x^2$ $- \left(836 - 405\sqrt{3}\right) x - 4739\sqrt{3} + 7704$ $C = \frac{53-387\sqrt{3}}{54}$ $\left(3 + \sqrt{3}, 14 + 6\sqrt{3}, 15 + 8\sqrt{3}\right)$

12/66ii	$y^2+xy+\left(\sqrt{3}+1\right)y=x^3-\left(1-\sqrt{3}\right)x^2$
	$-\left(405\sqrt{3}+836\right)x+4739\sqrt{3}+7704$ $C=\frac{53+387\gamma}{54}$ $\left(6+2\sqrt{3},5+\sqrt{3},9+4\sqrt{3}\right)$

Table 3.6: Ramification index 6

n_d/n_Dr	elliptic curve / accessory parameter / trace triple
49/91i	$y^2+xy+ay=x^3+x^2-\left(10825\gamma^2-24436\gamma+8746\right)x$ $-995392\gamma^2+2235406\gamma-797729$ $C=-\frac{815}{196}+\frac{495}{49}\gamma-\frac{30}{7}\gamma^2$ $f_\gamma=x^3-x^2-2x+1,\ \gamma=0.445$ $\left(\rho^2+1,16\rho^2+12\rho-8,17\rho^2+13\rho-9\right)$ $\rho=2\cos(\frac{\pi}{7})$
49/91ii	$y^2+xy+ay=x^3+x^2-\left(10825\gamma^2-24436\gamma+8746\right)x$ $-995392\gamma^2+2235406\gamma-797729$ $C=-\frac{815}{196}+\frac{495}{49}\gamma-\frac{30}{7}\gamma^2$ $f_\gamma=x^3-x^2-2x+1,\quad \gamma=-1.247$ $\left(\rho^2+\rho,5\rho^2+3\rho-2,5\rho^2+3\rho-2\right)$ $\rho=2\cos\left(\frac{\pi}{7}\right)$
49/91iii	$y^2+xy+ay=x^3+x^2-\left(10825\gamma^2-24436\gamma+8746\right)x$ $-995392\gamma^2+2235406\gamma-797729$ $f_\gamma=x^3-x^2-2x+1,\quad \gamma=1.802$ $C=-\frac{815}{196}+\frac{495}{49}\gamma-\frac{30}{7}\gamma^2$ $\left(2\rho^2+\rho,2\rho^2+\rho,3\rho^2+\rho-1\right)$ $\rho=2\cos\left(\frac{\pi}{7}\right)$
49/1	$y^2+xy+y=x^3-70-36x$ $C=-\frac{55}{196}$ $\left(2\rho^2,2\rho^2+2\rho,4\rho^2+3\rho-2\right)$ $\rho=2\cos\left(\frac{\pi}{7}\right)$

Table 3.7: Ramification index 7

n_d/n_Dr	elliptic curve / accessory parameter / trace triple
81/51i	$y^2=x^3-\left(-446\gamma^2-836\gamma-214\right)x^2-\left(-375921\gamma^2-706401\gamma-199989\right)x$

	$C=-\frac{1309}{243}-\frac{3206}{243}\gamma-\frac{1529}{243}\gamma^2$ $f_\gamma=x^3-3x-1,\quad \gamma=1.879$ $\left(\rho^2+1,4\rho^2+8\rho+4,5\rho^2+9\rho+3\right),\ \rho=2\cos(\frac{\pi}{9})$
81/51ii	$y^2=x^3-\left(-446\gamma^2-836\gamma-214\right)x^2-\left(-375921\gamma^2-706401\gamma-199989\right)x$ $-\frac{1309}{243}-\frac{3206}{243}\gamma-\frac{1529}{243}\gamma^2$ $f_\gamma=x^3-3x-1,\quad \gamma=1.879$ $\left(\rho^2+2\rho+1,\rho^2+2\rho+1,\rho^2+\rho+1\right),\ \rho=2\cos(\frac{\pi}{9})$
81/51iii	$y^2=x^3-\left(-446\gamma^2-836\gamma-214\right)x^2-\left(-375921\gamma^2-706401\gamma-199989\right)x$ $C=-\frac{1309}{243}-\frac{3206}{243}\gamma-\frac{1529}{243}\gamma^2$ $f_\gamma=x^3-3x-1,\quad \gamma=-0.607+1.450i$ $\left(\rho^2+2\rho+2,\rho^2+2\rho+1,\rho^2+2\rho+2\right),\ \rho=2\cos(\frac{\pi}{9})$

Table 3.8: Ramification index 9

$n_d/n_D r$	elliptic curve / accessory parameter / trace triple
14641/1	$y^2+y=x^3-x^2-10x-20$ $C=-\frac{14}{121}$ $\left(\rho^3-2\rho,\rho^3-2\rho,\rho^2-1\right),\ \rho=2\cos(\frac{\pi}{11})$

Table 3.9: Ramification index 11

The correctness of the suggested accessory parameters is proven whenever the Fuchsian group is commensurable with a triangle group, i.e. the uniformizing differential equation is a pullback of a hypergeometric differential equation, these 25 cases can be found in [Sij12a]. The cases $1/15$ with $e=3$ can be found in D. Krammer's article [Kra96], the case $1/6i$ was found by N. D. Elkies in [Elk98]. Whenever the quaternion algebra is defined over the rational numbers, S. Reiter determined the accessory parameter in [Rei11]. In principal it should be possible to adapt Krammer's method to prove the correctness at least a few more cases. The Tables 3.2 - 3.9 emphasize the equations for the elliptic curves, for convenience we added a list of A and $\tilde{C}$ such that the pullback of the uniformizing differential equation to $\mathbb{P}^1$ is

$$P(z)u''+\frac{1}{2}P(z)'u'+\left(\tilde{C}-\frac{(n(n-1))}{4}z\right)u=0$$

with

$$P(z)=z(z-1)(z-A)\text{ and }\tilde{C}=C/4.$$

and $n=\frac{1}{2}-\frac{1}{2e}$ in Appendix B.

3.5.2 The Case of (0;2,2,2,q)-Groups

The ad hoc method of the last section worked because only one complex parameter had to be determined. In the case of $(0;2,2,2,q)$-groups the Riemann scheme is not known completely and two complex parameters have to be found. Hence, we must take a closer look on methods for multidimensional root finding. Given a real valued continuous function $f : \mathbb{R} \to \mathbb{R}$ using the intermediate value theorem, it is not hard to decide if there is a real number x_0 such that $f(x_0) = 0$. In general it is not possible to compute x_0 explicitly. The bisection method in combination with Newton approximation and modifications for the case of multiple or clustered roots yield useful tools for the approximation of zeros of f. But to increase the rate of convergence usually a close inspection of the function f is needed. If the derivative of f is not available or the computation of its value at a given point up to a given accuracy is to expensive, the tangent in the Newton approximation method can be approximated by a suitable secant. The situation gets much more involved, if f is replaced by a system of maps

$$f_l(x_1, \dots, x_n),\ l = 1, \dots, n,$$

that can also be written as the vector valued problem

$$F(x) = (f_1, \dots, f_n) = 0.$$

We will follow the account in Chapter 14 of [Act70] and Section 2 of [MC79] and restrict to elementary local methods. Keeping our goal in mind , that is to compute uniformizing differential equations, only methods that work if the partial derivatives of F are unknown are useful. One instance of such a method is the *discretized multidimensional Newton method*. For the vector $x \in \mathbb{R}$ define the Matrix $A(x)$ as an approximation of the Jacobi-Matrix of F at x by

$$A(x)e_i = \frac{(F(x + h_i e_i) - F(x))}{h_i},\ 1 \leq n,$$

where e_i is the i-th column of the n-dimensional identity matrix and the parameter $h_i = h_i(x)$ has to be chosen as

$$h_i = \begin{cases} \epsilon|x_i|, & x_i \neq 0 \\ h_i = \epsilon, & \text{otherwise} \end{cases}.$$

The positive real constant ϵ is the square root of the precision available and with the above notation the discretized multidimensional Newton method can be implemented as in Algorithm 3.

The major problem is to find a suitable starting value x_{in} and one has to put high effort

Algorithm 3: The discretized multidimensional Newton method (dN)

Input: A procedure that evaluates $F(x)$ at $x_0 \in \mathbb{R}^n$ and $\delta > \epsilon$ and an initial value x_{in}
Output: An approximation of a zero of F or an error
Set $x_0 = x_{in}$ and $i = 0$.
while $F(x_i) > \delta$ **do**
  Construct the approximated Jacobi Matrix $A(x_i)$.
  Solve the system of linear equations $A(x_i)\delta_{x_i} - F(x_i)$.
  Set $x_{i+1} := x_i + \delta_{x_i}$ and $i := i + 1$.
end

in developing global methods. Since we are not interested in pushing forward the theory of numerical root finding methods and since we have some heuristics where the zeros of the functions under consideration should be located from Section 3.5 a naive problem-oriented approach seems to be adequate. Hence, we try the vertices of an equidistant lattice

$L = [c_1, \ldots, c_n] \times \ldots \times [c_1, \ldots, c_n]$ as values for x_{in}. Let $MonApp(A, C, n)$ be a procedure that given (A, C, n) returns the trace triple

$$\mathbf{A}(A, C) = \mathbf{M}(\Re(A), \Im(A), \Re(C), \Im(C))$$

of a the Lamé equation

$$L := P(x)u'' + \frac{1}{2}P(x)'u' + (n(n+1)x + C)u = 0,$$

with $P(x) = 4x(x-1)(x-A)$. Combining $MonApp$ with the discretized Newton method (dN) and a permutation σ on $\{1, 2, 3\}$ gives the following Algorithm 4, where A_0 and C_0 are two suitable initial values.

With one exception, we are able to give combinations of algebraic numbers A and C of plausi-

Algorithm 4: The algorithm for $(0; 2, 2, 2, q)$-groups

Input: (x, y, z) trace triple of a $(0; 2, 2, 2, q)$-group, a natural number maxit, real numbers δ, $\Re(A_0)$, $\Im(A_0)$,$\Re(C_0)$, $\Im(C_0)$, $\sigma \in S_3$.
Output: Two complex floating point numbers A and C.
Start algorithm dN(F, δ , maxit , $x_i n$), with
$F : \mathbb{R}^4 \to \mathbb{R}^4$, $(u, v, w, t) \mapsto (\sigma(\mathbf{A}(u, v, w, t))_1 - x, \sigma(\mathbf{A}(u, v, w, t))_2 - y, \sigma(\mathbf{A}(u, v, w, t))_3 - z)$ and
$x_{in} = (u_{in}, v_{in}, w_{in}, t_{in}) = (\Re(A_0), \Im(A_0), \Re(C_0), \Im(C_0))$

ble height and degree such that $\mathbf{A}(A, C)$ coincides with the traces triples describing arithmetic Fuchsian $(0; 2, 2, 2, q)$-groups up to at least 300 digits. Our results are listed in Appendix B. There seem to be two mistakes in the list of trace triples of arithmetic $(0; 2, 2, 2, q)$-groups in [ANR03]. In the case $7/49/1i$ we assume that $(\rho + 3, 4\rho^2 + 4\rho, 5\rho^2 + 4\rho - 2)$ has to be replaced by $(\rho + 3, 4\rho^2 + 4\rho, \rho^2 + 5\rho - 1)$ and in the case $3/49/1i$ instead of $(\rho^2 + \rho, 4\rho^2 + 4\rho, 2\rho^2 + \rho + 1)$ it should be $(\rho^2 + \rho, \rho^2 + 2\rho + 1, 2\rho^2 + \rho + 1)$, where $\rho = 2\cos(\pi/7)$. In both cases the version in [ANR03] is not minimal and it is easily checked that our suggestion yields the correct associated quaternion algebra.

We discuss the case $2/49/1ii$, where the correctness of our suggested differential equation can be proven easily using the theory of Belyi maps as explained in Section 3.3. The associated Quaternion algebra with discriminant (1) defined over the field $F = \mathbb{Q}(\cos(\pi/7))$ with discriminant 49 is $Q = (\frac{a,b}{F})$, where $a = 2\cos\left(\frac{1}{7}\pi\right) + 4\left(\cos\left(\frac{1}{7}\pi\right)\right)^2$ and $b = -2\cos\left(\frac{1}{7}\pi\right) - 12\cos\left(\frac{1}{7}\pi\right)^2 - 24\cos\left(\frac{1}{7}\pi\right)^3 - 16\cos\left(\frac{1}{7}\pi\right)^4$. The order $O_F[\Gamma]$ is maximal, hence by line 10 of Table 3 of [Tak77b] the embedded Fuchsian group $\Delta_{2,3,7} := \rho(O_F[\Gamma]^1)$ is the triangle group with signature $(0; 2, 3, 7)$ and clearly $\Gamma \subset O_F[\Gamma]^1$. The covolume of Γ is $-2 + \frac{3}{2} + \frac{2}{3} = \frac{1}{3}$ and $\mathrm{vol}(\Delta_{2,3,7}) = -2 + \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{1}{42}$, thus the covering $\mathbb{H}/\Gamma \to \mathbb{H}/\Delta_{2,3,7}$ has degree 7 by Proposition 3.1.9. The monodromy group of the uniformizing differential equation for $\mathbb{H}/\Delta_{2,3,7}$ is neither finite nor infinite dihedral, hence the uniformizing differential equation for $\mathbb{H}/\Gamma$ must be one of the finite coverings listed in [HV12] and [VF12]. There is no suitable parametric case and there occurs only one (237)-minus-4-hyperbolic Belyi map $\phi(x) : \mathbb{P}^1 \to \mathbb{P}^1$ such that the pullback of

$$u'' + \left(\frac{1}{2}x + \frac{2}{3(x-1)}\right)u' + \frac{13}{7056x(x-1)u}$$

along $\phi(x)$ is a Lamé equation with exponent differences $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}$. This Belyi map is called $G35$ in [HV12] and reads

$$\phi(x) = -\frac{(47 + 45a)\left(28x^2 - 35x - 7ax + 8 + 4a\right)^3}{27\,(4\,x - 3 - a)^7}.$$

Applying this pullback it is easy to check that the uniformizing differential equation we obtained with our numerical method is the correct one.

4 Monodromy of Calabi-Yau Differential Equations

In this Chapter we explain why the following conjectures are plausible.

Conjecture 1 *With respect to the normalized Frobenius basis the monodromy group of a Picard-Fuchs equation of a one parameter family of Calabi-Yau manifolds of dimension n is a subgroup of*

$$GL_n\left(\mathbb{Q}\left(\frac{\zeta(3)}{(2\pi i)^3},\frac{\zeta(5)}{(2\pi i)^5},\ldots,\frac{\zeta(m)}{(2\pi i)^m}\right)\right),$$

where m is the biggest odd number smaller than or equal to n. Moreover, if with respect to the normalized Frobenius basis there is a reflection with reflection vector

$$(a_0,0,a_2,\ldots,a_n)$$

the numbers a_i have an interpretation as polynomial expressions in numerical invariants of a Calabi-Yau variety with $h^{j,j}=1, j=1,\ldots,\lfloor n/2\rfloor$.

At least for dimension three we can formulate a version of this conjecture purely in terms of differential equations.

Conjecture 2 *If L is CY(3)-operator with N-integral instanton numbers, with respect to the normalized Frobenius basis the monodromy group is contained in*

$$GL_4\left(\mathbb{Q}\left(\frac{\zeta(3)}{(2\pi i)^3}\right)\right).$$

If there is a symplectic reflection with reflection vector $(a,0,b,c\lambda)$ with $\lambda=\frac{\zeta(3)}{(2\pi i)^3}$, the rational numbers a,b and c have an interpretation as numerical topological invariants $a=H^3$, $b=\frac{c_2(X)H}{24}$, $c=\chi(X)$ of a Calabi-Yau threefold X with $h^{1,1}=1$.

In our explanation mirror symmetry is crucial, the book [CK99] by D. A. Cox and S. Katz collects most of the early approaches to mirror symmetry made during the outgoing 20th century. Whenever a claim is not proven or no citation is given, a proof can be found in this book.

4.1 The Famous Mirror Quintic

The attention of the mathematical community was brought to a physical phenomenon called mirror symmetry by an article of P. Candelas, X. C. della Ossa, P. S. Green and L. Parkes. There they stated an explicit conjecture on the number of rational curves on a general quintic threefold $X\subset\mathbb{P}^4$ [COGP91]. This is an example of a three-dimensional Calabi-Yau manifold. For the first Chern class H of an ample generator of the Picard group of the quintic the *degree* $d:=H^3$, the second Chern number $c_2(X)H$ and the Euler characteristic $c_3(X)$ are $5,50$ and -200. They are important due to the classification theorem of Wall [Wal66] stating that the diffeomorphism type of a Calabi-Yau threefold is completely determined by $c_2(X)H$,H^3 and the Hodge numbers. To define Calabi-Yau manifolds we recall the basic facts from Hodge theory. A *pure Hodge structure of weight k* is a finitely generated $\mathbb{Z}$-module $V_{\mathbb{Z}}$ with the additional data of a *Hodge decomposition*

$$V := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q},$$

where $V^{p,q} = \overline{V^{q,p}}$. The *Hodge numbers* $h^{p,q}$ are the dimensions of the $\mathbb{C}$-vector spaces $V^{p,q}$. Another way of presenting the data of a Hodge structure is the finite decreasing *Hodge filtration* $F^{\bullet}$ of complex subspaces F^p of V, satisfying the condition $F^0 = V$ and

$$V = F^p \oplus \overline{F^{k-p+1}}.$$

We can construct the filtration from the decomposition by

$$F^p := V^{k,0} \oplus \ldots \oplus V^{p,k-p}$$

and vice versa the Hodge decomposition is obtained from the Hodge filtration by setting

$$V^{p,q} := F^p \oplus \overline{F^q}.$$

The *Weil operator* of a Hodge Structure $(V_{\mathbb{Z}}, F^{\bullet})$ is defined as the automorphism

$$C : V \to V, v \mapsto i^{p-q} v \text{ for } v \in V^{p,q}.$$

A polarization of Hodge structure is a $(-1)^k$ symmetric bilinear form

$$Q : V_{\mathbb{Z}} \to V_{\mathbb{Z}}$$

such that the $\mathbb{C}$-linear extension of Q to V satisfies

- The orthogonal complement of F^p is F^{k-p+1}
- The hermitian form $Q(Cu, \overline{v})$ is positive definite.

If everywhere in the above discussion $\mathbb{Z}$ is replaced by $\mathbb{Q}$, we speak of a $\mathbb{Q}$-Hodge structure. A *morphism of Hodge structure* is a morphism of $\mathbb{Z}$-modules whose complexification respects the Hodge decomposition. If V is the $\mathbb{Z}$-module $(2\pi i)^k \mathbb{Z}$, $k \in \mathbb{Z}$, then $V_{\mathbb{Z}} \otimes \mathbb{C} = V^{-k,-k}$ and V becomes a Hodge structure of weight $-2k$, called $\mathbb{Z}(k)$. Indeed, $(V, V^{i,j})$ is up to morphism the unique Hodge structure of weight $-2k$ with a underlying rank one $\mathbb{Z}$-module, it is usually called the *Tate Hodge structure*.
The prototype of a Hodge structure is the decomposition

$$H^k(X, \mathbb{C}) := H^k(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{k=p+q} H^{p,q}(X)$$

of the k-th cohomology of a compact Kähler manifold X of dimension n in classes of closed (p,q)-forms. To get a polarization we have to choose a Kähler form ω and restrict to *primitive cohomology groups*

$$H_0^{n-k}(X, \mathbb{C}) := \ker\left(\cdot \wedge \omega^{k+1} : H^{n-k}(X, \mathbb{C}) \to H^{n+k+2}(X, \mathbb{C})\right).$$

These vector spaces decompose as

$$H_0^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H_0^{p,q}(X),$$

where

$$H_0^{p,q}(X) := H^{p,q}(X) \cap H_0^{p+q}(X,\mathbb{C}).$$

If we define the bilinear form $Q : H^{n-k}(X,\mathbb{C}) \times H^{n-k}(X,\mathbb{C}) \to \mathbb{C}$ as

$$Q(\alpha,\beta) = \int_X \alpha \wedge \beta \wedge \omega^k,\ \alpha,\beta \in H_0^{n-k}(X,\mathbb{C}),$$

then $H_0^k(X,\mathbb{C})$ together with $(-1)^{k(k-1)/2}Q$ builds a polarized variation of Hodge structure. In this geometric situation the Hodge numbers are subject to special symmetries since complex conjugation yields $\overline{H^{p,q}}(X) = H^{q,p}(X)$, the Hodge $*$-operator induces $H^{p,q}(X) \cong H^{n-q,n-p}(X)$, and $H^{p,q}(X) \cong H^{n-p,n-q}(X)$ is true by Serre duality.

Definition 4.1.1 (Calabi-Yau manifold) *A Calabi-Yau manifold is a compact Kähler manifold X of complex dimension n which has trivial canonical bundle, and vanishing Hodge numbers $h^{k,0}(X)$ for $0 < k < n$.*

A one dimensional Calabi-Yau manifold is an *elliptic curve*, a Calabi-Yau manifold of dimension 2 is a *K3 surface*, and Calabi-Yau manifolds of dimension three are often called *Calabi-Yau threefolds*. For later use we remark that for Calabi-Yau threefolds the vanishing of $H^5(X,\mathbb{C})$ implies, that the primitive part $H_0^3(X,\mathbb{C})$ is all of $H^3(X,\mathbb{C})$. The Hodge numbers can be displayed as a "Hodge diamond". In the case of a Calabi-Yau threefold it looks like

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 1 & h^{2,1} & & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}.$$

The unobstructedness theorem for Calabi-Yau manifolds of dimension n due to F. A. Bogomolov [Bog78], G. Tian [Tia87] and A. N. Todorov [Tod89] tells that the moduli space of complex structures of a n-dimensional Calabi-Yau manifold is a manifold of dimension $h^{n-1,1}$. Candelas considered the special pencil that can be defined by the equation

$$X_t = \left\{[x_0 : \ldots : x_4] \in \mathbb{P}^4 | f_t = 0\right\},$$

where

$$f_t = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5t x_0 x_1 x_2 x_3 x_4.$$

For $t \notin \{0, t^5 = 1\}$ the variety X_t has Hodge numbers $h^{1,1}(X_t) = 1$ and $h^{1,1}(X_t) = 101$. The group $G := \left\{(a_0,\ldots,a_4) \in (\mathbb{Z}/5\mathbb{Z})^5 | \sum a_i = 0\right\}$ acts on $\mathbb{P}^4$ diagonally by multiplication with a fixed fifth root of unity ζ

$$(x_0,\ldots,x_4) \mapsto (\zeta^{a_0}x_0,\ldots,\zeta^{a_4}x_4).$$

The quotient

$$X_t' = X_t/G$$

is singular but for $t \neq 0, \infty$ and $t^5 \neq 1$ this singularities can be resolved to obtain a Calabi-Yau manifold $\check{X}_t'$ with $h^{2,1}(\check{X}_t') = 1$ and $h^{2,1}(\check{X}_t') = 101$. One observes that the Hodge diamonds

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & 101 & & & 101 & 1 & \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \longrightarrow \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & 1 & & & 1 & 1 & \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}.$$

of X_v and $\check{X}_t'$ are related by a ninety degree rotation. This phenomenon is called *topological mirror symmetry* for Calabi-Yau threefolds, it is easily generalized to higher dimension as $h^{p,q}(X) = h^{\dim(X')-p,q}(X')$ and $h^{p,q}(X') = h^{\dim(X)-p,q}(X)$. Despite the fact that such pairs cannot exist for rigid Calabi-Yau manifolds, since for a Kähler manifold $h^{1,1}$ is always nonzero it appears that this phenomenon is true for a big class of Calabi-Yau threefolds. Coming back to the quintic mirror using the Griffiths-Dwork method and the coordinate change $z = (5t)^{-5}$ one can compute the differential equation

$$L = (\theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4))y = 0,\ \theta = z\frac{d}{dz},$$

that is solved by the periods of X'. The differential equation L has singularities at 0, $\frac{1}{3125}$ and ∞. Since L is a generalized hypergeometric differential equation its solutions can be expressed as Mellin-Barnes integrals, this was for example used in [CYYE08] to express generators M_0 and $M_{1/3125}$ of the monodromy group of L with respect the Frobenius basis with a certain normalization.

Definition 4.1.2 *If $y_0, \dots y_{n-1}$ is the Frobenius basis of a differential operator of order n, the basis*

$$\tilde{y}_j = \frac{y_j}{(2\pi i)^j j!} \text{ for } j = 0, \dots, n-1$$

is called normalized Frobenius basis.

At a MUM-point of a linear differential equation of order four the monodromy with respect to the normalized Frobenius basis is always equal to

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & 1 & 1 \end{pmatrix}.$$

The monodromy along a loop starting at 0 and encircling the single singular point $1/3125$ turns out to be

$$M_{\frac{1}{3125}} = \begin{pmatrix} 1-200\lambda & -25/12 & 0 & -5 \\ 0 & 1 & 0 & 0 \\ -250/3\lambda & -125/144 & 1 & -25/12 \\ 8000\lambda^2 & 250/3\lambda & 0 & 1+200\lambda \end{pmatrix},\ \lambda = \frac{\zeta(3)}{(2\pi i)^3}.$$

The real number $\zeta(3)$ is the value of the Riemann ζ-function $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ at three. One observes that the entries of $M_{1/3125}$ are in the field $\mathbb{Q}(\lambda), \lambda = \frac{\zeta(3)}{(2\pi i)^3}$ and that the geometric invariants introduced above can be read off from the last row. In our specific example the occurrence of H^3, c_2H and c_3 was expected, because the analytic continuation of a special solution at $1/3125$ to 0 as computed in [COGP91] expressed in a special coordinate already involved these invariants. The local exponents of L at $\frac{1}{3125}$ are $0, 1, 1, 2$ and the rank of $M_{1/3125} - Id_4$ equals one. We call such points conifold points.

Definition 4.1.3 (Conifold point) *A singular point p of a Fuchsian differential equation L of order four is called* conifold point *if the spectrum of L at p is $[0,1,1,2]$ and if the local monodromy has the Jordan form*

$$M_p = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The choice of the name conifold is justified by the observation that in the quintic example at $\frac{1}{3125}$ the mirror $\check{X}_t'$ acquires a double point. The Yukawa coupling $K(q)$ of a differential equation with a MUM-point at 0 can be expanded as Lambert series

$$K(q) = n_0^0 + \sum_{n=0}^{\infty} C_n q^n = n_0^0 + \sum_{l=1}^{\infty} \frac{n_0^l l^3 q^l}{1-q^l},$$

with coefficients given as

$$n_0^0 = C_0 \text{ and } n_l^0 = \frac{1}{l^3} \sum_{d|l} \mu\left(\frac{l}{d}\right) C_d \in \text{ for } l > 0,$$

where μ is the Möbius function. For L one observes that the numbers n_l^0 are integral.

Definition 4.1.4 (Genus 0 - instanton numbers) *The coefficients n_l^0 of the Lambert series expansion of $K(q)$ are called* genus 0-instanton numbers*.*

These numbers are related to the Clemens conjecture, that a general quintic hypersurface in $\mathbb{P}^4$ contains only finitely many rational curves of any degree d. The work of various authors [Kat86, JK96, Cot05, Cot12] confirms the Clemens conjecture for $d \leq 11$. Attempts to solve this conjecture and if it is true to compute the number of rational curves of degree d have a long tradition going back to H. C. H. Schubert [Sch79], who calculated the number of lines as 2875. For conics S. Katz found the number 609250 [Kat86]. In [COGP91] P. Candelas and his collaborators discovered that with $n_l^0 = 5$ for $l = 1, 2$ the instanton numbers n_l^0 of the Picard-Fuchs equation of the mirror quintic coincide with these numbers computed before. G. Ellingsrud and S. A. Strømme confirmed that the number of twisted cubics coincides with the value 317206375 of n_3^0. A discussion of the numbers n_l^0 and actual curve counting can be found in [Pan98]. If L is a Picard-Fuchs equation of a family of Calabi-Yau threefolds with topological data c_2H, H^3 and c_3, with the notation of [BCOV93, SE06] the *genus one instanton numbers* are computed as solution of

$$\frac{c_2 H}{24} + \sum_{d=1}^{\infty} \sum_{k=1}^{\infty} \left(\frac{1}{12} n_d^0 + n_d^1\right) dq^{kd} = \partial_t \log\left(\frac{z(q)^{1+c_2H/12} f(z)}{y_0^{4-c_3/12} \frac{\partial_t}{\partial_z}}\right)$$

The function $f(z)$ is called *holomorphic anomaly* and conjectured to have the form

$$f(z) := \prod_i (d_i(z))^{p_i},$$

where d_i are the over $\mathbb{R}$ irreducible factors of the leading coefficient $z^4 \prod_i (d_i(z))^{k_i}$ of L. Furthermore the rational numbers p_i conjecturally equal $-\frac{1}{6}$, if the roots of $d_i(z)$ are conifold points, in general it is an open question how to choose values for p_i. In the physical literature this process of obtaining n_1 from n_0 is known as topological recursion relation. For the quintic example the first three numbers n_0^1, n_1^1, n_2^1 coincide with the number of elliptic curves of degree $0, 1$ and 2 on a general quintic, but as in the genus zero case the connection with the number of curves is more subtle, for example $n_4^1 = 3721431625$ but the number of elliptic curves on a general quintic is 3718024750 [ES96]. If X_t is considered as a hypersurface in the toric variety $\mathbb{P}^4$, there is another well known construction to obtain a mirror $\check{X}_t$. For the quintic threefold X_t it states that the mirror of $\check{X}_t'$ is the resolution of the compactification of an affine variety $X_t' \subset \mathbb{T} \cong (\mathbb{C}^*)^4$ defined as zero locus of

$$F_t(x) := 1 - tf(x)$$

where f is the Laurent polynomial

$$f := x_1 + x_2 + x_3 + x_4 + \frac{1}{(x_1 x_2 x_3 x_4)} \in \mathbb{Z}((x_1, x_2, x_3, x_4)).$$

A holomorphic 3-form on X_t' is

$$\omega_t := \mathrm{Res}_{X_t'}\left(\frac{\Omega}{F_t}\right)$$

with

$$\Omega = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{x_1 x_2 x_3 x_4}$$

and on X_t' there exists a 3-cycle γ_t whose Leray coboundary is homologous to

$$\Gamma = \{x_1, \ldots, x_4 | \, |x_i| = \epsilon_i\}, \; \epsilon_i > 0.$$

Hence, if $[f^k]_0$ denotes the constant term of f^k the *fundamental period*

$$\phi_f(t) := \int_{\gamma(t)} \omega_t = \left(\frac{1}{2\pi i}\right)^4 \int_\Gamma \frac{\Omega}{F_t}$$

can be expressed as generating series of f_0^k by the equality

$$\begin{aligned}\phi_f(t) &= \left(\frac{1}{2\pi i}\right)^4 \int_\Gamma \frac{\Omega}{1 - tf} = \left(\frac{1}{2\pi i}\right)^4 \int_\Gamma \sum_{k=0}^{\infty} f^k t^k \Omega = \left(\frac{1}{2\pi i}\right)^4 \sum_{k=0}^{\infty} \int_\Gamma f^k t^k \Omega \\ &= \sum_{k=0}^{\infty} \int_\Gamma f^k t^k = \sum_{k=0}^{\infty} [f^k]_0 t^k.\end{aligned}$$

After a change $z = t^5$ of coordinates

$$\phi_f(z) = \sum_{k=0}^{\infty} \frac{(5k)!}{(k!)^5} z^k$$

is a solution of the Picard-Fuchs equation L obtained above.

4.2 Characteristics of Picard-Fuchs Equations of Calabi-Yau Threefolds

The aim of this section is to collect properties of Picard-Fuchs differential equations and to investigate characteristics that a Picard-Fuchs operator of certain families of three dimensional Calabi-Yau manifolds have. Such an operator will be called CY(3)-operator. Necessarily, this definition will be preliminary, since it is not known which conditions attached to a differential equation assure its origin as Picard-Fuchs equation. Some of the properties of a Picard-Fuchs equations are related to variations of Hodge structure, hence it is necessary to recall the basic definitions thereof. There is plenty of literature on this topic, see for example [PS08].

4.2.1 Variations of Hodge Structure and Picard-Fuchs Equations

Let B be a complex manifold, a variation of Hodge structure (VHS) of weight k on B consists of a local system $\mathcal{V}_{\mathbb{Z}}$ of finitely generated $\mathbb{Z}$-modules and a finite decreasing filtration $\mathcal{F}^\bullet$ of holomorphic subbundles of the holomorphic vector bundle $\mathcal{V} := \mathcal{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_B$. These two items have to be subject to the following two properties. For each $b \in B$ the filtration $\mathcal{F}^\bullet_b$ of $\mathcal{V}_{\mathbb{Z},b} \otimes_{\mathbb{Z}} \mathbb{C}$ defines a pure Hodge structure of weight k on $\mathcal{V}_{\mathbb{Z},b}$. The flat connection

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_B} \Omega^1_B,\ \nabla(sf) = s \otimes df,$$

whose sheaf of horizontal sections is $\mathcal{V}_{\mathbb{C}} := \mathcal{Z}_{\mathbb{V}} \otimes_{\mathbb{Z}} \mathbb{C}$ satisfies *Griffiths transversality*

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_B.$$

The data of a variation of Hodge structure can be presented as the quadruple $(\mathcal{V}, \nabla, \mathcal{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$. The trivial variation of Hodge structure, that induces the Hodge structure $\mathbb{Z}(k)$ on each fiber is denoted by $\mathbb{Z}(k)_B$.
A *polarization* of a variation of Hodge structure of weight k on B is a morphism of variations

$$\mathcal{Q} : \mathcal{V} \otimes \mathcal{V} \to \mathbb{Z}(-k)_B,$$

which induces on each fiber a polarization of the corresponding Hodge structure of weight k. The definition of a variation of Hodge structure is inspired by results on the behavior of the cohomology groups $H^k(X_b, \mathbb{Z})$ in a family of complex manifolds as pointed out by P. Griffith in [Gri68a, Gri68b]. A family of compact Kähler manifolds is a proper submersive holomorphic mapping $\pi : X \to B$ from a Kähler manifold X to the complex base manifold B. Each fiber $X_b = \pi^{-1}(b)$ is a smooth n-dimensional manifold. The local system

$$R^k\pi_*\mathbb{Z} = \left\{ H^k(X_b, \mathbb{Z}) \right\}_{b\in B}$$

on B determines the holomorphic vector bundle $\mathcal{H}^k$ defined by

$$\mathcal{H}^k := R^k\pi_*\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_B.$$

In this geometric situation the connection ∇ is called *Gauss-Manin connection*. The subbundles

$$\mathcal{F}^p = \left\{ F^p H^k(X_b) \right\}_{b\in B} \subset \mathcal{H}^k$$

together with ∇ satisfy Griffiths transversality and the resulting variation of Hodge structure is said to be *geometric*. Assume the base B to be one-dimensional with a local coordinate t centered at a $p \in B$ and consider the variation of Hodge structure $(\mathcal{H}^n, \nabla, R^n\pi_*\mathbb{Z}, \mathcal{F}^\bullet)$. An element $L = \sum_j^n a_j(t)\partial_t^j$ of the stalk $\mathbb{C}\{t\}[\partial_t]$ at t of the sheaf of differential operators acts on $\mathcal{F}^0$ by extending $\partial_t\omega(t) := \nabla_{\frac{\partial}{\partial_t}}\omega(t)$ for a fixed local section $\omega(t)$ of $\mathcal{F}^n$. If L annihilates $\omega(t)$, the differential equation $L(y) = 0$ associated to the differential operator L is called *Picard-Fuchs equation*. For a locally constant n-cycles γ_t the equality

$$\frac{d}{dt}\int_{\gamma_t} \omega(t) = \int_{\gamma_t} \nabla_{\frac{\partial}{\partial t}}\omega(t)$$

implies that the *periods* $\int_\gamma \omega$ are solutions of $L(y)$. In the case of families of Calabi-Yau threefolds with $h^{2,1} = 1$ the $\mathbb{C}$-vector space $H^3(X, \mathbb{C})$ has dimension four and ω , $\nabla_{\partial/\partial t}$, $\nabla^2_{\partial/\partial t}$, $\nabla^3_{\partial/\partial t}$, $\nabla^4_{\partial/\partial t}$ must be linear dependent over $\mathbb{C}\{t\}$, hence the order of the Picard-Fuchs differential equation is ≤ 4.

Example 4.3 An elliptic surface $\pi : X \to \mathbb{P}^1$ is said to be in Weierstraß-normalform if the smooth fibers at t are given by

$$y^2 = 4x^3 - g_2(t)x - g_3(t)$$

with non-constant J-function

$$J(t) := \frac{g_2(t)^3}{\Delta(t)},\ \Delta = g_2(t)^3 - 27g_3(t)^2.$$

The periods of differentials of the first kind $\int_\gamma \frac{dx}{y}$ satisfy the differential equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0,$$

of degree two, where the coefficients are given by

$$p = \frac{g_3'}{g_3} - \frac{g_2'}{g_2} + \frac{J'}{J} - \frac{J''}{J'}$$

and

$$q = \frac{J'}{144J(J-1)} + \frac{\Delta'}{12\Delta}\left(p + \frac{\Delta''}{\Delta} - \frac{13\Delta'}{12\Delta}\right).$$

This was already known to F. Klein [KF90], a nice account together with a classification of elliptic surfaces with three singular fibers can be found in U. Schmickler-Hirzebruch's diploma thesis [SH85].

4.3.1 Monodromy and Limiting Mixed Hodge Structure

Given a ring A and a local system of A-modules $\mathcal{V}$ on a topological space S, which is locally and globally path connected, the *monodromy representation* of $\mathcal{V}$ is a map

$$\rho : \pi_1(S, s) \to \mathrm{Aut}_A(\mathcal{V}_s),\ s \in S.$$

For any loop $\gamma : [0,1] \to S$ with base point s the pullback $\gamma^*\mathcal{V}$ is again a local system and $\gamma^*\mathcal{V}$ is trivial since $[0,1]$ is simply connected. This yields the identification of $\gamma^*\mathcal{V}_0 \cong \mathcal{V}_{\gamma(0)}$ with $\gamma^*\mathcal{V}_0 \cong \mathcal{V}_{\gamma(1)}$ and hence the map ρ.

Definition 4.3.1 (Monodromy representation of a local system) *The map*

$$\rho : \pi_1(S, s) \to \mathrm{Aut}_A(\mathcal{V}_s),\ s \in S.$$

is called the monodromy representation *of the local system $\mathcal{V}$, its image is called monodromy group.*

The monodromy of a variation of Hodge structure is the monodromy of the underlying local system. Note that for a geometric variation of Hodge structure the action of $\rho(\gamma)$ is compatible with the Poincaré pairing. In the following we restrict to the case of a variation of Hodge structure $\mathcal{V}$ over the punctured disc $\Delta^* := \Delta \setminus \{0\}$ and describe its limiting behavior over Δ. Denote the image of $\rho(\gamma)$ of a simple loop γ generating the fundamental group $\pi_1(\Delta^*, p)$ by T. The local monodromy theorem going back to A. Borel and A. Landmann, which can be found as Theorem 6.1 of [Sch73], says that T is quasi-unipotent and that its eigenvalues are roots of unity. An endomorphism is called quasi-unipotent, if there are a natural number e and n such that

$$(T^e - Id)^n = 0.$$

Equivalently, on the level of Picard-Fuchs equations this translates to the property that the local exponents at 0 are all rational. After eventually pulling back $\mathcal{V}$ along an e-fold cover one can assume $e = 1$. Construct the nilpotent endomorphism N as $\frac{1}{2\pi i}$ times the logarithm of T, where the logarithm is defined as

$$\log(T) = (T - Id) - \frac{(T - Id)^2}{2} + \ldots + (-1)^{n+1}\frac{(T - Id)^n}{n}.$$

In [Del70] P. Deligne describes how N is used to obtain a natural extension of $\mathcal{V}$ and ∇ to Δ. Choose a flat multivalued frame $\phi_1(t), \ldots, \phi_k(t)$, then

$$e_j(t) := \phi_j(t)t^{-N}, j = 1, \ldots, k$$

is single valued and $F := \langle e_1(t), \ldots, e_k(t)\rangle$ determines the extension of $\mathcal{V}$ to $\tilde{\mathcal{V}}$ over Δ. With respect to F the connection reads

$$\tilde{\nabla} = d - N\frac{dt}{t}.$$

This connection on $\tilde{\mathcal{V}}$ has a simple pole at the origin and the residue of the connection at $t = 0$ is $-N$. This two properties determine the canonical extension up to isomorphism. The connection ∇ is an instance of regular singular connection.

Definition 4.3.2 (Regular singular connection) *A* regular singular connection *on the Riemann surface B with singular locus $\Sigma = \{b_1, \ldots, b_r\} \subset B$ is a pair $(\mathcal{V}, \nabla)$ of a vector bundle $\mathcal{V}$ and a morphism of sheaves of groups $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_B(\log(\Sigma))$ that satisfies the Leibniz rule*

$$\nabla(fm) = df \otimes m + f\nabla(m).$$

for local sections f of $\mathcal{O}_B$ and m of $\mathcal{V}_{\mathbb{C}}$.

The sheaf $\Omega^1_B(\log(\Sigma))$ is the sheaf of differential one forms that possess poles at most of order one at points of Σ. In particular, it follows that Picard-Fuchs equations are Fuchsian equations. W. Schmid [Sch73] showed that the Hodge-subbundles $\mathcal{F}^p$ of $\mathcal{V}$ extend to holomorphic subbundles of the canonical extension $\tilde{\mathcal{V}}$, hence $F^\bullet_{\lim} := \mathcal{F}^\bullet_0$ is a filtration on the fiber V_0 of the canonical extension $\tilde{\mathcal{V}}$. The canonical isomorphism between the $\mathbb{C}$-vector spaces $\tilde{\mathcal{V}}_0$ and V_b induces an action of N on $F^0_{\lim}$, which satisfies $N(F^p_{\lim}) \subset F^{p-1}_{\lim}$. Thus N induces a map

$$N : F^p_{\lim}/F^{p+1}_{\lim} \to F^{p-1}_{\lim}/F^p_{\lim}.$$

If $\delta = t\partial_t$, Griffith transversality $\tilde{\nabla}_\delta(\tilde{\mathcal{F}}^p) \subset \tilde{\mathcal{F}}^{p-1}$ is also true for the extended connection and we get a second linear map

$$\tilde{\nabla}_\delta : F^p_{\lim}/F^{p+1}_{\lim} \to F^{p-1}_{\lim}/F^p_{\lim}$$

and these two maps are related by

$$\tilde{\nabla}_\delta = -N.$$

Denote the $\mathbb{Z}$-module generated by $\langle e_1(0), \ldots, e_k(0)\rangle$ by $V^0_{\mathbb{Z}}$, then in general $(V^0_{\mathbb{Z}}, \mathcal{F}^\bullet_0)$ is not a Hodge structure. Again P. Deligne found the right notion to fix this misfortune, he introduced mixed Hodge structures.

Definition 4.3.3 (Mixed Hodge structure) *A* mixed Hodge structure *(MHS) V consists of a $\mathbb{Z}$-module $V_{\mathbb{Z}}$, the increasing* weight filtration

$$\ldots \subset W_{m-1} \subset W_m \subset W_{m+1} \subset \ldots$$

on $V_{\mathbb{Q}}$ and the decreasing Hodge filtration

$$\ldots \supset F^{p-1} \supset F^{p} \supset F^{p+1} \supset \ldots$$

on $V_{\mathbb{C}}$. for each m, these two filtrations are required to satisfy the condition, that the induced filtration

$$F^p Gr_m^W := \mathrm{Im}\left(F^p \cap W_m \to Gr_m^W\right)$$

on

$$Gr_m^W := W_m / W_{m-1}$$

is a $\mathbb{Q}$-Hodge structure of weight m.

For a geometric variation of Hodge structure of the punctured disc the weight filtration interacting with $F^{\bullet}_{\lim}$ can be defined using the nilpotent logarithm of the monodromy N.

Definition 4.3.4 (Monodromy weight filtration) *Suppose that V is a finite dimensional vector space over a field of characteristic zero, and suppose that N is nilpotent endomorphism on V. Then there is a unique increasing filtration $W^{\bullet}(N)$ of V, called the* monodromy weight filtration *such that*

1. $N(W_l(N)) \subset W_{l-2}(N)$ *and*
2. $N^l := Gr_{k+l}^W \to Gr_{k-l}^W$, $l \geq 0$ *is an isomorphism.*

After the shift $W_n V_0 = W_{k-n}$, the following theorem holds.

Theorem 4.3.5 *The logarithm of the monodromy acts on V_0 and builds together with $F^{\bullet}_{\lim}$ a mixed Hodge structure.*

If the indicial equation of a differential equation on $\mathbb{P}^1$ of order n at a point x_0 equals $(\sigma - l)^n$ then by the discussion of Section 2.2.2 the monodromy at x_0 is maximal unipotent. Thanks to the tools from Hodge theory we are ready to prove that for Picard-Fuchs equations of families of Calabi-Yau manifolds the converse is also true.

Proposition 4.3.6 *If the Picard-Fuchs equation of a family of Calabi-Yau threefolds*

$$f : \mathcal{X} \to B \setminus \{b_1, \ldots, b_r\}$$

over a one dimensional base B has maximal unipotent monodromy at b_j, the indicial equation at b_j is $(\lambda - l)^4 = 0$

Proof Our proof is guided by the proof on Page 79 in [CK99]. All we need to is to show that for a local generator $\omega(z)$ of $\tilde{F}^3$ the set $\omega, \nabla_\delta \omega, \nabla_\delta^2 \omega, \nabla_\delta^3 \omega$ are linear independent near $z = 0$ and then read of the indicial equation of the corresponding differential operator. The first step is to verify the decomposition

$$F^3_{\lim} \oplus W_5 = H^3(X, \mathbb{C}).$$

Note $W_6 = H^3(X, \mathbb{C})$ and $W_5 = ker(N^3)$ by definition. The linear map N is nilpotent with $N^4 = 0$ but $N^3 \neq 0$, this implies $\dim(W_6/W_5) = 1$. Because $F^i_{\lim} = 0, i = 4, 5, 6$ their quotients $(W_6 \cap F^i_{\lim})/W_5, i = 4, \ldots, 6$ are zero, too. Since F^p induces a Hodge structure of weight 6 this implies $(W_6 \cap F^3_{\lim})/W_5 \neq \{0\}$ and therefor $F^3 \cap W_5 \neq \{0\} = \{0\}$. The decomposition above of $H^3(X, \mathbb{C})$ shows that a local basis ω of $\tilde{\mathcal{F}}^3$ evaluated at 0 is not contained in the kernel of N^3 and

$$\omega(0), N(\omega(0)), N^2(\omega(0)), N^3(\omega(0))$$

constitute a basis of $H^3(X, \mathbb{C})$.
Recall that the action of $\tilde{\nabla}_\delta$ and $-N$ coincide on $F_{\lim}^{p+1}/F_{\lim}^{p}$. In other words $\tilde{\nabla}^j(\omega)(0)$ and $(-N)^j(\omega(0))$ differ only by elements of $F_{\lim}^{4-j}, j = 1, \ldots, 3$. Together with $N(F_{\lim}^p) \subset F_{\lim}^{p-1}$ and $\dim(F_{\lim}^p) = 4 - p$ this yields the linear independence of $\omega, \nabla_\delta(\omega), \nabla_\delta^2(\omega), \nabla_\delta^3(\omega)$ at $z = 0$. In this basis the Picard-Fuchs equation is

$$\left(\tilde{\nabla}_\delta^4 + f_3(z)\tilde{\nabla}_\delta^3 + f_2(z)\tilde{\nabla}_\delta^2 + f_1(z)\tilde{\nabla}_\delta + f_0(z)\right)\omega = 0$$

with indicial equation

$$f(\lambda) = \lambda^4 + f_3(0)\lambda^3 + f_2(0)\lambda^2 + f_1(0)\lambda + f_0(0).$$

Since we were able to write the connection $\tilde{\nabla}$ in a local basis of $\tilde{\mathcal{F}}$ at $z = 0$, we can conclude that the residue of the connection, which reads

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -f_0(0) & -f_1(0) & -f_2(0) & -f_3(0) \end{pmatrix},$$

is $-1/2\pi i \log(T) = -N$. Using the nilpotency of N it follows that $f_i(0) = 0, i = 1, \ldots, 3$ and $f(\lambda) = \lambda^4$. $\square$

This leads to the definition of a MUM-point.

Definition 4.3.7 (MUM-point) *If $p \in \mathbb{P}^1$ is a singular point of a Fuchsian differential equation L of order n and the indicial equation of L at p is $\sigma^n = 0$, the point p is called a* MUM-point.

Families of Calabi-Yau manifolds do not need to have a point of maximal unipotent monodromy. Three-dimensional families without this property were constructed by J. C. Rohde in [Roh09] and also considered by A. Garbagnati and B. van Geemen in [GG10,Gar13]. In this case the Picard Fuchs equation is of order two. Further examples with Picard-Fuchs equations of order four were found by S. Cynk and D. van Straten [CS13]. Up to now we obtained Picard-Fuchs equations locally from holomorphic vector bundles on a Riemann surface B with a connection, hence the coefficients are holomorphic functions. Since Serre's GAGA principle is true for vector bundles with regular singular points over the projective line and passing to a generic point turns an algebraic vector bundles with regular singular connections into a $\mathbb{C}(z)$-module, we can assume that the coefficients of the Picard-Fuchs equations are rational functions. Furthermore we can employ $\mathcal{D}$-module techniques if we investigate the variation of cohomology of families of projective varieties over $\mathbb{P}^1$. Details can be found in Chapter six of [PS03] by M. van der Put and M. F. Singer and in N. M.Katz' article [Kat82]. For the vector bundle $\mathcal{H}^{\dim(X)}$ the role of a cyclic vector is played by the holomorphic n-form $\omega^{n,0}$, that by Griffiths transversality satisfies

$$\int_X \omega^{n,0} \wedge \nabla\omega^{n,0} \begin{cases} = 0, & i \neq \dim(X) \\ \neq 0, & i = \dim(X) \end{cases}.$$

Hence we can apply Proposition 2.1.8 to conclude that the differential module M_L associated to a Picard Fuchs operator of a one-parameter family of Calabi-Yau varieties L is self dual.

4.3.2 Arithmetic Properties

In this section we follow the books by Y. André [And89] and by B. Dwork, G. Gerotto, and F. J. Sullivan [DGS94] to explain the behavior of the denominators of coefficients of power series expansion of monodromy invariant periods of a family of projective manifolds over a smooth curve.

Definition 4.3.8 (G-function) *A power series $f = \sum_{i=0}^{\infty} f_i x^i \in \overline{\mathbb{Q}}[\![x]\!]$, where all coefficients are in a fixed number field K with ring of integers O_K, is called* G-function *if*

1. *for every embedding $j_v : K \to \mathbb{C}$, $\sum j_v(f_i)x^i$ is analytic at 0.*
2. *there is a sequence $(d_n)_{n\in\mathbb{N}}$ with $sub_i \left\{\frac{1}{i}\log(d_i)\right\} < \infty$ such that $d_n y_m \in O_K$, $m = 0, \dots, n$ for all n.*
3. *there is a linear homogeneous differential operator $D \in K(x)[dx]$ with $D(f) = 0$.*

If we do not demand that f solves a differential equation and strengthen the condition on the coefficients of f, we get the notion of globally boundedness.

Definition 4.3.9 (Globally bounded) *A Laurent series $f \in K((x)) \subset \overline{\mathbb{Q}}((x))$ is called* globally bounded*, if f has the following two properties.*

1. *For all places v of K the radius of convergence of the embedded series is nonzero.*
2. *There exists a nonzero natural number N such that $f \in O_K[\frac{1}{N}]((x))$.*

If $f \in \mathbb{Q}[\frac{1}{x}]\{x\}$ is a convergent Laurent series, the statement that f is a globally bounded G-function can be rephrased in terms of N-integrality.

Definition 4.3.10 (N-integral) *A formal power series $f = \sum_{i=0}^{\infty} a_i x^i \in \mathbb{Q}[\![x]\!]$ is called* N-integral *if a natural number $N \in \mathbb{N}^*$ exists such that such $N^i a_i \in \mathbb{Z}$.*

For convergent power series with $\mathbb{Q}$-coefficients that solve differential equations the coincidence of globally boundedness and N-integrality is an immediate consequence from the definitions.

Proposition 4.3.11 *A convergent power series f with coefficients in $\mathbb{Q}$ is a globally bounded G-function exactly if it is N-integral and if there exist a linear homogeneous differential equation L such that f solves L.*

The diagonalization operator is a map $\delta : K((x_1, \dots, x_n)) \to K((x))$ which sends a Laurent series $\sum a_{i_1,\dots,i_n} x^{i_1} \cdot \dots \cdot x^{i_n}$ to $\sum a_{k,\dots,k} x^k$. Via series expansion this map δ can also be interpreted as a map on the rational functions in n-variables. A series in the image of δ is called a *diagonal*. On page 26 of [And89] it is explained that diagonals and globally bounded G-functions are closely related.

Lemma 4.3.12 *The diagonal of a rational function $f \in K[\![x]\!]$ is a globally bounded G-function.*

Extending results of G. Christol and H. Fürstenberg in the one variable case, J. Denef and L. Lipschitz described in Section 6 of [DL87] the relation between diagonals of algebraic functions in n variables and that of rational functions in $2n$ variables.

Proposition 4.3.13 *Any diagonal of an algebraic function in n variables can be written as the diagonal of a rational function in $2n$ variables.*

This is used on page 185 of [And89] to describe periods of certain families of projective varieties.

Theorem 4.3.14 *Let $f : X \to B \setminus \{b_0, \ldots, b_s\}$ be a family of projective smooth varieties of dimension n over a smooth curve B defined over $K \subset \overline{\mathbb{Q}}$ with local coordinate x near b_0. Assume that f extends to a projective morphism $f^{'} : X^{'} \to B$ such that $(f^{'})^{-1}(b_0)$ is a simple normal crossing divisor. Then there is a basis ω_i of logarithmic n-forms such that for any n-cycle γ_b invariant under the local monodromy at b_0 the Taylor expansion of $\frac{1}{(2\pi i)^n} \int_{\gamma_b} (\omega_i)_b$ in x is the diagonal of an algebraic function in $n+1$ variables and therefor a globally bounded G-function.*

Hence the unique holomorphic solution y_0 of a Picard-Fuchs equation of a family of projective Calabi-Yau threefolds X_z with maximal unipotent monodromy defined over $\mathbb{Q}$ is N-integral. As explained in Chapter 2 and in Proposition 4.3.6. If the corresponding Picard-Fuchs equation has a MUM-point, there are two further solutions y_1, y_2

$$\begin{aligned} y_0 &:= 1 + a_1 z + a_2 z + \ldots \\ y_1 &:= \log(z) y_0 + f_1,\ f_1 \in z\mathbb{C}\{z\} \\ y_2 &:= \log(z)^2 y_0 + 2\log(z) f_1 + f_2,\ f_2 \in z\mathbb{C}\{z\} \end{aligned}$$

The theory of elliptic curves suggests to consider the exponential of the quotient $\tau = y_1/y_0$.

Definition 4.3.15 (q-coordinate and mirror map) *If L is a Fuchsian linear differential equation of order greater than one in the coordinate z with maximal unipotent monodromy at 0 the* q-coordinate *is defined as*

$$q(z) := \exp\left(2\pi i \frac{y_0(z) ln(z) + y_1(z)}{y_0(z)}\right) = z \exp\left(2\pi i \frac{y_1(z)}{y_0(z)}\right) \in z\mathbb{C}[\![z]\!].$$

Its inverse $z(q)$ is called mirror map.

Analytic continuation along a small loop around zero carries $\tau = y_1/y_0$ to $\tau + 1$, hence q is single valued locally at 0. In the case of Picard-Fuchs equations of families of elliptic curves we directly touch Chapter 3. For example in [Dor00] it is proven that the q-series $z(q)$ of a degree two Picard-Fuchs equation L of a family of elliptic curves is a modular form exactly if L is a uniformizing differential equation. More general for Picard-Fuchs equations of families of elliptic curves $q(z)$ of the corresponding Picard-Fuchs equation is N-integral. The same is true for families of $K3$ surfaces whose Picard-Fuchs equation can be written as a symmetric square of a second order Fuchsian equation. Now we restrict to the case of Picard-Fuchs differential equations of Calabi-Yau threefolds with a MUM-point. For certain hypergeometric Fuchsian differential equations including the famous quintic example, that are mostly known to be Picard-Fuchs equations of families of Calabi-Yau threefolds, Krattenthaler and Rivoal prove the N-integrality of the Taylor coefficients of $q(z)$ in [KR10]. More cases can be found in the work of E. Delaygue [Del12]. In [Vol07] V. Vologodsky announces that the N-integrality of $q(z)$ is true for a wide range of families of Calabi-Yau threefolds. Using the coordinate q the Picard Fuchs differential equation has a special local normal form.

Proposition 4.3.16 *Consider a self-dual linear homogeneous differential equation L of order 4 with 0 a MUM-point and q-coordinate as above, then L can be written as*

$$L = \theta^2 \frac{1}{K(q)} \theta^2,\ \theta = q\frac{d}{dq},\ K(q) = N_0 \left((q\frac{d}{dq})^2 \left(y_2(q)/y_0(q) \right) \right) \in \mathbb{C}[\![q]\!].$$

Proof This equality can be found by a direct computation as in Section 5.21 of [Inc56] combined with the relation of the coefficients of L induced by self-duality of L. Another complete account is Section 3.4 of [Bog12].

The above local invariant multiplied by an integer $N_0 K(q)$ is called *Yukawa coupling* and it has a geometric interpretation. If L is the Picard Fuchs equation of a family of Calabi-Yau threefolds with a MUM-point at 0, then in Chapter 5.6 of [CK99] it is explained that for a suitable multiplicative constant $K(q)$ coincides with

$$\int_{X_z} \omega(z) \wedge \frac{d^n}{dz^n} \omega(z),$$

where $\omega(z)$ is a top dimensional holomorphic form. A discussion of $K(q)$ in terms of extension data of Hodge structure is P. Deligne's article [Del96]. The Yukawa coupling is a local invariant of a differential equation and can be used to check if two differential equations are related by a formal change of coordinates. The discussion of the sections above enables us to give a characterization of Picard-Fuchs differential equations of families of three dimensional Calabi-Yau threefolds with a MUM-point.

- The differential operator L is irreducible, Fuchsian and of order four.
- The differential module M_L is self-dual and monodromy group is conjugated to a subgroup of $SP_4(\mathbb{Z})$.
- 0 is a MUM-point.
- The unique holomorphic solution $1 + O(z)$ near 0 is N-integral and all occurring local exponents are rational.
- The q-coordinate $q(z)$ is N-integral.

Characterization (CY(3)-equation) *A differential equation that matches the properties listed above is called CY(3)-equation.*

It will become clear in Section 4.4.2 why we emphasized the existence of a MUM-point The N-integrality of $q(z)$ implies the N-integrality of $z(q)$ and by Formula (2.3) of [AZ06] also the N-integrality $K(q)$. Given a Fuchsian equation it is in general not possible to determine if it is a CY(3)-equation, since usually the N-integrality properties can only check for finitely many coefficients. As indicated by example 4.8 it seems to be necessary to add at least the N-integrality of the genus 0 instanton numbers to the characterization of $CY(3)$-equations. Several years ago G. Almkvist, C. van Enckevort, D. van Straten, and W. Zudilin started to compile a still growing list of CY(3)-equations [AESZ10] that is also available as an online database [Str12].

4.4 Reconstruction of Topological Data

4.4.1 Homological Mirror Symmetry and the $\widehat{\Gamma}$-Class

Homological mirror symmetry (HMS) tries to formulate the mirror phenomenon in categorical terms. The necessary material on derived categories is covered in Huybrecht's book [Huy06] and the book [GM10] by S. I. Gelfand and Y. I.Manin . To get an impression how (HMS) should look like we start with the case of elliptic curves studied by E. Zaslow and A. Polischuk in [ZP98]. For an elliptic curve $M = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ and its dual $\tilde{M} = (\mathbb{R}^2/(\mathbb{Z} \oplus \mathbb{Z}))$ they proved an equivalence $\mathcal{D}^b(M) \approx \mathcal{F}^0(\tilde{M})$, where $\mathcal{F}^0(\tilde{M})$ is obtained from $\mathcal{F}(\tilde{M})$. The objects of $\mathcal{F}(\tilde{M})$ are pairs $\mathcal{U}_i = (\mathcal{L}_i, \mathcal{E}_i)$ of special Lagrangian submanifolds $\mathcal{L}_i$ equipped with certain line bundles $\mathcal{E}_i$.

Definition 4.4.1 (Special Lagrangian submanifold) *For a Calabi-Yau manifold X of complex dimension n a special Lagrangian submanifold L is a Lagrangian submanifold of X with $\Im(\Omega|L) = 0$, where Ω is a nowhere vanishing holomorphic n-form on X.*

By Proposition 7.1 of [GHJ03], special Lagrangian manifolds are volume minimizing in their homology class, for the elliptic curve $\tilde{M}$ they correspond to lines with rational slope. The morphisms of $\mathcal{F}(\tilde{M})$ are defined as

$$\mathrm{Hom}(\mathcal{U}_i, \mathcal{U}_j) = \mathbb{C}^{\#\{\mathcal{L}_i \cap \mathcal{L}_j\}} \times \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j).$$

The construction for elliptic curves was inspired by Kontsevich's proposal to call two smooth algebraic complex varieties X and $X^{'}$ mirror equivalent if for $D^b(X)$ and the derived Fukaya category $DFuk(X^{'})$ there are equivalences

$$Mir : D^b(X) \xrightarrow{\approx} DFuk(X^{'}),\ D^b(X^{'}) \xrightarrow{\approx} DFuk(X).$$

This definition comes with a conjecture that Calabi-Yau threefolds come in pairs satisfying the above equivalences [Kon95]. Note, that $D^b(X)$ depends only on the complex structure of X and $DFuk(X^{'})$ depends on the symplectic structure of $X^{'}$. We do not try to survey the constructions of $DFuk(X^{'})$ and refer to [FOOO10] for an extensive discussion. Some slightly different versions of this conjecture are (partially) proven in special cases. Results are due to K. Fukaya (Abelian varieties) [Fuk02], M. Kontsevich and Y. Soibelmann (torus fibrations) [KS01], P. Seidel (quartic K3 surfaces, genus two curve) [Sei03, Sei11], D. Auroux, L. Katzarkov, and D. Orlov (weighted projective planes) [AKO08], M. Abouzaid and I. Smith (4-torus) [AS10], A. Efimov (curves of higher genus) [Efi11], K. Ueda (toric del Pezzo surfaces) [Ued06], Y. Nohara and K. Ueda (quintic threefold) [NU11] and N. Sheridan (Calabi-Yau hypersurfaces in projective space) [She11]. If homological mirror symmetry is true, a natural question is how to build a mirror $X^{'}$ from X. The points on $X^{'}$ parameterize the sheaves $\mathcal{O}_p$ and by homological mirror symmetry also certain objects $\mathcal{L}_p$ of $DFuk(X)$. If these are true Lagrangian subvarieties an inspection of the Ext-groups of $\mathcal{O}_p$ suggests that $\mathcal{L}_p$ should be a Lagrangian torus together with a local system on it. The following theorem by R. C. McLean [McL98] explains why the local systems necessarily have to be added.

Theorem 4.4.2 *The moduli space of special Lagrangian submanifolds L is a smooth manifold B with $T_L B \cong H^1(L, \mathbb{R}) \cong \mathbb{R}^n$.*

Hence, there are not enough Lagrangian submanifolds to get a matching of those with $\mathcal{O}_p$. This suggested to consider special Lagrangian tori L with an $U(1)$-flat $\mathbb{C} \to L$-connection ∇. Finally, we get an alternative formulation of the mirror symmetry phenomenon, the Strominger-Yau-Zaslow (SYZ) [SYZ96] picture. It formulates mirror symmetry as the existence of dual special Lagrangian torus fibrations

$$\begin{tikzcd} X \arrow[dr, "\pi"] & & X^{'} \arrow[dl, "\pi^{'}"'] \\ & B & \end{tikzcd},$$

where the preimage of a generic point $\pi^{-1}(p)$ is a special Lagrangian torus $\mathbb{T}$ and $(\pi')^{-1}$ is a dual torus $\check{\mathbb{T}}$ defined by $\mathrm{Hom}(\pi_1(\mathbb{T}), U(1))$. Hence, the mirror X' is constructed as $\{(L, \nabla),\ L \text{ is a fibre of } \pi, \nabla \in \mathrm{Hom}(\pi_1(\mathbb{T}), U(1))\}$. The way back from (SYZ) to topological mirror symmetry for Calabi-Yau threefolds is indicated in the first chapter of [Gro09]. By a similar argument using Ext as above for $\mathcal{O}_p$, the structure sheaf $\mathcal{O}_X$ is expected to correspond to a Lagrangian sphere.

Recall that the group freely generated by isomorphism classes $[X]$ of coherent sheaves on X divided out by the relations $[X] + [Z] = [Y]$, if X, Y, and Z fit in a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is called Grothendieck K-group $K(X)$. A passage from $D^b(X)$ to the K-group $K(X)$ is given as the map

$$D^b(X) \to K(X),\ \mathcal{F}^\bullet \mapsto [\mathcal{F}^\bullet] := \sum (-1)^i [\mathcal{F}^i].$$

The common passage from $K(X)$ to $H^*(X, \mathbb{Q})$ is via the Chern character

$$\mathrm{ch} : K(X) \to H^*(X, \mathbb{Q}).$$

In various situations the *Mukai map* $v = \mathrm{ch}(-)\sqrt{\mathrm{td}(X)}$ is relevant. Recently, L. Karzakov, M. Kontsevich, and T. Pantev [KKP08] introduced the $\widehat{\Gamma}$-character to define a variant of the Mukai map.

Definition 4.4.3 ($\widehat{\Gamma}$-character) *For $\mathcal{E} \in K(X)$ denote the r-th Chern class by $c_r(\mathcal{E})$ and the Chern roots by δ_i and define the* Euler-Mascheroni constant *as*

$$\gamma := \lim_{n\to\infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log(n)\right).$$

For a projective smooth variety the map $\widehat{\Gamma} : K(X) \to H^*(X, \mathbb{C})$ from the K-group to complex cohomology defined by

$$\widehat{\Gamma}(\mathcal{E}) := \prod_i \Gamma(1 + \delta_j)$$

is called $\widehat{\Gamma}$-character. *The map $\widehat{\Gamma}(1 + \delta)$ is defined as* $\exp\left(-\gamma\delta + \sum_{l=2}^{\infty} \frac{\zeta(l)}{l}(-\delta)^l\right)$ *and $\widehat{\Gamma}_X := \widehat{\Gamma}(T_X)$ is called* Γ-class.

The $\widehat{\Gamma}$-class can be seen as an almost square root of the Todd class

$$\mathrm{td}(\mathcal{E}) := \prod_{i=1}^{r} \frac{1}{1 - \exp(\delta_i)} = 1 + \frac{c_1(\mathcal{E})}{2} + \frac{c_1^2(\mathcal{E}) - 2c_2(\mathcal{E})}{12} + \frac{c_1(\mathcal{E})c_2(\mathcal{E})}{24} + \ldots .$$

The key to understand this is Euler's reflection formula for the classical Γ-function

$$\Gamma(1 + z)\Gamma(1 - z) = z\Gamma(z)\Gamma(1 - z) = \frac{\pi z}{\sin(\pi z)} = \frac{2\pi i z}{\exp(\pi i z)(1 - \exp(-2\pi i z))},$$

and the series expansion

$$\log(\Gamma(z + 1)) = -\gamma z + \sum_{l=2}^{\infty} \frac{\zeta(l)}{l}(-z)^l,\ |z| < 1.$$

That yields

$$\widehat{\Gamma}_X^\vee \widehat{\Gamma}_X e^{\pi i c_1(X)} = (2\pi i)^{\frac{\deg}{2}}\, \mathrm{td}(T_X),$$

where $(2\pi i)^{\deg/2}$ acts on $H^{2k}(X)$ by multiplication with $(2\pi i)^k$. We also define the map $\psi : K(X) \to H^*(X,\mathbb{C})$ by

$$\psi : K(X) \to H^*(X,\mathbb{C}),\ \mathcal{E} \mapsto (2\pi i)^{-n}\left((2\pi i)^{deg/2}\,\mathrm{ch}(\mathcal{E}) \cup \widehat{\Gamma}_X\right),$$

that can be extended to $\mathcal{D}^b(X)$ by $\psi(\mathcal{E}^\bullet) := \psi([\mathcal{E}^\bullet])$. The map ψ is compatible with the Euler-bilinear form χ and the pairing

$$Q_M : H^*(X,\mathbb{C}) \times H^*(X,\mathbb{C}) \to \mathbb{C},\ (\alpha,\beta) \mapsto (2\pi i)^n \int_X \alpha^\vee \cup \beta \cup \exp(\pi i c_1(X)/2).$$

To be more precise let

$$\chi(\mathcal{F},\mathcal{E}) := \sum (-1)^i \dim \mathrm{Hom}(\mathcal{F},\mathcal{E}[i])$$

the Euler pairing for complexes $\mathcal{F},\mathcal{E}$, then

$$\begin{aligned}
\chi(\mathcal{E},\mathcal{F}) &= \chi(X,\mathcal{E}^\vee \otimes \mathrm{ch}(\mathcal{F})) = (2\pi i)^{-n}\int_X (2\pi i)^{deg/2}\left(\mathrm{ch}(\mathcal{E}^\vee)\cup \mathrm{ch}(\mathcal{F}) \cup \mathrm{td}(X)\right)\\
&= (2\pi i)^n \int_X (2\pi i)^{-2n}\left((2\pi i)^{\frac{deg}{2}}\,\mathrm{ch}(\mathcal{E}^\vee)\cup \widehat{\Gamma}_X^\vee \cup (2\pi i)^{\frac{deg}{2}}\,\mathrm{ch}(\mathcal{F})\cup \widehat{\Gamma}_X\right)\exp\left(\frac{\pi i c_1(X)}{2}\right)\\
&= Q_M(\psi(\mathcal{E}),\psi(\mathcal{F})),
\end{aligned}$$

where we applied the Riemann-Roch-Hirzebruch theorem in the first row. For Calabi-Yau manifolds the paring Q_M coincides with Q_A introduced in Section 4.4.2. H. Iritani used the $\widehat{\Gamma}$-class to prove a mirror correspondence for Calabi-Yau hypersurfaces in toric varieties. To explain his account we have to introduce quantum cohomology and Landau-Ginzburg models.

4.4.2 Gromov-Witten Invariants and the Dubrovin Connection

In this section let X be a projective variety of dimension n with vanishing odd cohomology, set $H^*(X) := \bigoplus_{k=0}^n H^{2k}(X,\mathbb{Q})$. Assume that $H^2(X,\mathbb{Z})$ and $H_2(X,\mathbb{Z})$ are torsion free and choose two basis $\{T_a\}$ and $\{T^b\}$, $a,b = 0,\ldots,N$ of $H^*(X,\mathbb{Q})$ with

$$\langle T_a, T^b\rangle = \delta_{ab}.$$

Furthermore, put $\tau = t_0 1 + \sum_{i=1}^N t_i T^i$ and $\tau = \tau' + \tau_2$, where $\tau' \in \bigoplus_{k\neq 2} H^{2k}(X)$ and $\tau_2 = \sum_{i=1}^l t_i T^i$ and $\tau_{0,2} = t_0 T_0 + \tau_2$. Frequently we will consider the zero locus $Y \hookrightarrow X$ of a generic section of a decomposable rank r vector bundle

$$\mathcal{E} = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r,$$

where we assume $\mathcal{L}_i$, $i = 1,\ldots,r$ and $c_1(T_X) - \sum_{i=1}^r \mathcal{L}_i$ to be nef. To define quantum multiplication it is necessary to state what Gromov-Witten invariants and stable maps are.

Definition 4.4.4 (n- pointed stable map) *For a projective algebraic variety a genus g n-*pointed stable map *is a connected complex curve C of genus g with n marked points $(C,p_1,\ldots,p_n)$ and a morphism $f : C \to X$, such that*

1. *the curve C posses at worst ordinary double points.*
2. *the points $p_1,\ldots,p_n$ are distinct.*

3. *if C_i is a rational component and if f is constant on C_i, then C_i contains at least three nodal or marked points.*

4. *if the arithmetic genus equals one and there are no marked points, then f is not constant.*

The third and fourth property assures that there are only finitely many automorphisms of an n-pointed stable map $(f, C, p_1, \ldots, p_n)$. Denote the moduli stack of genus zero, n-pointed stable maps with $f_*(C) = d \in H_2(X, \mathbb{Z})$ by $\overline{\mathcal{M}}_{0,n}(X, d)$ and its coarse moduli space by $\overline{M}_{0,n}(X, d)$ [MB96]. If the evaluation maps at the i-th point are denoted by ev_i, Gromov-Witten invariants can be defined as follows.

Definition 4.4.5 (Gromov-Witten invariants with gravitational descendents) *For classes γ_i in $H^*(X)$, nonnegative integers d_i, and $d \in H_2(X, \mathbb{Z})$ the n-pointed* Gromov-Witten invariants with gravitational descendents *are defined as*

$$\langle \psi_{d_1}\gamma_1, \ldots, \psi_{d_n}\gamma_n \rangle_{0,n,d} := \int_{[\overline{M}_{0,n}(X,d)]^{virt}} \psi_1^{d_1} ev_1^*(\gamma_1) \cup \ldots \cup \psi_n^{d_n} ev_n^*(\gamma_n).$$

where $[\overline{M}_{0,n}(X, d)]^{virt}$ is the virtual fundamental class of $\overline{M}_{0,n}$ and ψ_i is the first Chern class of the line bundle over $\overline{M}_{0,n}(X, d)$, whose fiber at a stable map is the cotangent space of the coarse curve at the i-th marked point.

The virtual fundamental class was introduced exactly to be able to define the above integral in general. Sometimes we can avoid the use of this advanced technology, for example if X is smooth and convex, that is if for all maps $f : \mathbb{P}^1 \to X$, we have $H^1(\mathbb{P}^1, f^*(T_X)) = 0$. Then by [FP95] $\overline{M}_{0,n}(X, d)$ is a projective orbifold of dimension $n + \dim(X) - \int_d \omega_X - 3$ and we can drop the label virt in the above definition. Here an orbifold of dimension m is a variety locally equivalent to a quotient U/G, where $G \subset GL_m(\mathbb{C})$ is a finite group which contains no non-trivial complex reflections and $U \subset \mathbb{C}^m$ is a G-stable neighborhood of the origin. If $d_i = 0$ for all i, then

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{0,n,d} := \langle \psi_0\gamma_1, \ldots, \psi_0\gamma_n \rangle_{0,n,d}$$

are called *Gromov-Witten invariants*. By the so called *effectivity axiom* for a smooth projective variety X the genus zero Gromov-Witten invariants $\langle \cdot, \ldots, \cdot \rangle_{0,l,d}$ are nonzero only if d lies in the integral *Mori cone*

$$M_{\mathbb{Z}}(X) := \left\{ \sum a_i[C_i] \mid a_i \in \mathbb{N}_0, C_i \text{ effective} \right\}$$

of effective curves or if $m \leq 2$ and $d = 2$. The *degree axiom* furthermore implies that for homogeneous classes γ_i the genus 0 Gromov-Witten invariants with gravitational descendents are nonzero only if

$$\sum_{i=1}^{n} (\deg \gamma_i + 2d_i) = 2 \dim X + 2\langle \omega_X, d \rangle + 2n - 6.$$

Gromov-Witten invariants are used to define the big quantum ring $QH^*(X)$, that is a deformation of the usual ring structure on $H^*(X, \mathbb{C})$. It depends on an element of

$$K_{\mathbb{C}}(X) := \left\{ \omega \in H^2(X, \mathbb{C}) \mid \mathrm{Im}(\omega) \text{ is Kähler} \right\} / \, \mathrm{Im}(H^2(X, \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{C})),$$

which is called a *complexified Kähler class*.

Definition 4.4.6 (Big quantum product) *For $\alpha, \beta \in H^*(X, \mathbb{C})$ the* big quantum product *is defined as*

$$\alpha *_\tau^{big} \beta = \sum_{l \geq 0} \sum_{d \in M_{\mathbb{Z}}(X)} \sum_{k=0}^{N} \frac{1}{l!} \langle \alpha, \beta, \tau', \ldots, \tau', T_k \rangle_{0,l+3,d} T^k e^{\langle d, \tau_2 \rangle} q^d,$$

where $q^d = e^{2\pi i \int_d \omega}$, $\langle d, \tau_2 \rangle = \int_d \tau_2$, $\tau = \tau_2 + \tau' \in H^*(X)$, and $\omega \in H^2(X, \mathbb{C})$ is a complexified Kähler class.

Remark that the dependence on ω and τ_2 is redundant in this context since $2\pi i \omega + \tau_2$ is a class in $H^2(X, \mathbb{C})$, nevertheless sometimes the explicit dependence on the complexified Kähler moduli space is important. To be more precise, notice that the dependency on τ_2 is exponential and the big quantum product actually varies over

$$H^0(X) \oplus H^2(X)/2\pi i H^2(X, \mathbb{Z}) \oplus H^{>2}(X, \mathbb{C}),$$

where $2\pi i H^2(X, \mathbb{Z})$ acts on $H^2(X, \mathbb{C})$ by translation. Assuming the convergence of $*_\tau^{big}$ the so called *WDVV equations* guarantee its associativity. In general it is not known if the above product converges, one possibility to overcome this problem is to consider the formal ring $\mathcal{R}[\![t_0, \ldots, t_N]\!]$ with $\mathcal{R} := \mathbb{Q}[\![q^d, d \in M_{\mathbb{Z}}(X)]\!]$ and interpret $*_\tau^{big}$ as product on $\mathcal{R}$. If X is not projective the effectivity axiom does not hold and the even more general *Novikov ring* has to be introduced, see Section 5 of [Man99]. Note that in the *large structure limit* $\tau = q = 0$ or equivalently $q = 1$, $\tau' = 0$, and $\Re(e^{\langle \tau_2, d \rangle}) \to -\infty$ the quantum product coincides with the usual cup product. Whenever we consider $*_\tau^{big}$ not as a formal product, we assume that $*_\tau^{big}$ converges for τ in

$$U := \left\{ \tau \in H^*(X) | \Re(\langle \tau_2, d \rangle) \leq -M \text{ for all } d \in M_{\mathbb{Z}}(X),\ \|\tau'\| < e^{-M} \right\},$$

for a constant $M > 0$ and a norm $\|\cdot\|$ on $H^*(X)$. The big quantum product involves all genus zero Gromov-Witten invariants, it is useful to consider a restriction that only involves three point genus zero Gromov-Witten invariants. Two similar ways to restrict to a product, which involves three pointed Gromov-Witten invariants only, are to set $\tau = 0$ or $\tau' = 0$ and $q = 1$. In the first case the resulting family of products will vary over $K_{\mathbb{C}}(X)$ and in the second case over $H^2(X, \mathbb{C})/2\pi i H^2(X, \mathbb{Z})$. As formal products on $\mathcal{R}[\![t_0, \ldots, t_N]\!]$ these two restrictions coincide if we set $q^d = e^{\langle d, \tau_2 \rangle}$.

Definition 4.4.7 (Small quantum product) *For fixed ω the abelian group $\bigoplus_k H^k(X, \mathbb{C})$ together with the product*

$$\alpha *_{\tau_2} \beta := \sum_{d \in M_{\mathbb{Z}}(X)} \sum_{k=1}^{N} \langle \alpha, \beta, T_k \rangle_{0,3,d} T^k q^d$$

is called small quantum cohomology ring.

If X is a Fano variety by Proposition 8.1.3 in [CK99] the small quantum product is a finite sum and hence defines a proper product on $H^*(X)$. If odd cohomology vanishes both ring structures introduced above are equipped with a grading by setting $\deg(q^d) := \int_d \omega_X$ and $deg(T_i)$ as half of the usual degree. The small quantum ring is well understood for homogeneous varieties, for example quantum analogues of the classical Giambelli and Pieri formulas are known, see the survey [Tam07] for references. We will follow [Buc03] and [Ber97] to review the case of the Grassmanian of planes in four space $X = \mathrm{Gr}(2, 4) \hookrightarrow \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$. The smooth projective variety X has dimension four and an additive basis B of $\bigoplus_k H^{2k}(X, \mathbb{Z})$ can be given in terms of Schubert classes as

H^8	H^6	H^4	H^2	H^0
$\omega_{2,2}$	$\omega_{2,1}$	$\omega_{1,1}$ $\omega_{2,0}$	$\omega_{1,0}$	$\omega_{0,0}$

.

Since $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$ generates the ring $H^*(X,\mathbb{Z})$ and is itself additively generated by a single element $\omega_{1,0}$ the small quantum product can be considered as a product on $H^*(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[q]$. Denote the image of the canonical embedding of $\omega_\lambda \in H^*(X,\mathbb{Z})$ into $H^*(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[q]$ by σ_λ. For $\mathrm{Gr}(2,4)$ the small quantum product is completely determined by giving the multiplication with $\omega_{1,0}$. From Section 8 of [Buc03] we know that the small quantum product with σ_{10} expressed in the basis $\{\sigma_{00}, \sigma_{10}, \sigma_{11}, \sigma_{20}, \sigma_{21}\sigma_{22}\}$ is given by the matrix

$$M_{\mathrm{Gr}(2,4)} := \begin{pmatrix} 0 & 0 & 0 & 0 & q & 0 \\ 1 & 0 & 0 & 0 & 0 & q \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We recalled this construction to exhibit an prototypical example of a situation we assume to be in, that is the odd cohomology vanishes, there is no torsion and $H^2(X,\mathbb{Z})$ is generated by a single element. The small quantum product can be used to define a connection on the trivial holomorphic vector bundle $F : H^*(X) \times (U \times \mathbb{C}) \to (U \times \mathbb{C})$ over $U \times \mathbb{C}$ with fibers $H^*(X,\mathbb{C})$.

Definition 4.4.8 (Extended Dubrovin connection) *The meromorphic flat* extended Dubrovin connection ∇ *on F is defined by*

$$\nabla_i := \frac{\partial}{\partial t_i} - \frac{1}{z} T_i *_\tau^{big}, \qquad \nabla_z := z\frac{\partial}{\partial z} - \frac{1}{z} E *_\tau^{big} + \mu,$$

where

$$E := c_1(T_X) + \sum_{i=1}^{N} \left(1 - \frac{\deg(T_k)}{2}\right) t_k T_k$$

is the Euler vector field *and $\mu \in \mathrm{End}(H^*(X))$ is the* Hodge grading operator *defined by*

$$\mu_{|H^k(X,\mathbb{C})} := \frac{1}{2}(k-n) Id_{H^k(X,\mathbb{C})}$$

and z is a coordinate on $\mathbb{C}$. For $z = 1$ the extended Dubrovin connection reduces to the Dubrovin connection.

The bundle F together with the action of ∇ on the sections $\mathcal{O}_F$ can also be considered as a $\mathcal{D}$-module denoted $QDM(X)$, where the action of the ring of differential operators

$$\mathcal{D} := \mathcal{O}_U[z]\langle z\partial_z, \partial_0, \dots, \partial_N \rangle$$

on sections σ of $\mathcal{O}_F$ is defined by extending

$$\partial_i \sigma = \nabla_i(\sigma) \text{ and } z\partial_z \sigma = z\nabla_z(\sigma).$$

The horizontal sections of the connection ∇ are solutions of a system of differential equations in several variables

$$\nabla_i(\sigma) = 0, \; i = 0, \dots, N$$
$$\nabla_z(\sigma) = 0.$$

The partial differential equation $L(t_1,\dots,t_N,\partial_1,\dots,\partial_N,z)(y)=0$ associated to the above system is called *extended quantum differential equation*. If $z=1$, the differential equation

$$L(t_1,\dots,t_N,\partial_1,\dots,\partial_N,1)(y)=0$$

is called *quantum differential equation*. If we furthermore restrict F to $H^2(X,\mathbb{C})$ and $H^2(X,\mathbb{Z})$ isomorphic to $\mathbb{Z}$, the quantum differential equation $L(t)$ is a linear homogeneous differential equation in one variable.

Example 4.5 For $\mathrm{Gr}(2,4)$ the Dubrovin connection restricted to $H^2(X,\mathbb{C})$ with $z=1$ is

$$\left(q\frac{d}{dq}-M_{\mathrm{Gr}(2,4)}\right)y(q)=0.$$

After choosing the cyclic vector $v:=(1,0,0,0,0,0)$ this is equivalent to the differential equation

$$\theta_q^5-2q(2\theta_q+1),\ \theta_q=q\frac{d}{dq}.$$

Note that this differential equation is not Fuchsian, since the singular point at ∞ is not regular singular. The index of $\mathrm{Gr}(k,n)$ is n, hence the first order differential system $\nabla_z(\sigma)=0$ with respect to the basis B is

$$z\frac{\partial}{\partial z}+\frac{4}{z}M_{\mathrm{Gr}(2,4)}+\begin{pmatrix}-\frac{1}{2}&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&\frac{1}{2}\end{pmatrix}.$$

With the same cyclic vector v as above this is equivalent to the differential equation

$$z^4(2\theta_z+7)(2\theta_z+1)(\theta_z+2)^3-4096q\theta_z,\ \theta_z=z\frac{d}{dz}.$$

If Y is a Calabi-Yau manifold the Dubrovin connection induces the natural *A-model variation of Hodge structure*, see Section 8.5 of [CK99]. We restrict to the case where $\iota: Y\hookrightarrow X$ is a Calabi-Yau hypersurface in the n-dimensional ambient variety X. Let $\tilde{U}=\iota^*(U)\cap H^2(Y)$ then $\alpha\in\iota^*(H^2(X,\mathbb{Z}))$ acts on $\tilde{U}$ by translation with $(2\pi i)\alpha$ and we denote $\tilde{U}'$ by U'. Define the ambient cohomology classes as $H^*_{amb}(Y):=\mathrm{Im}(\iota^*:H^*(Y)\to H^*(X))$. By Corollary 2.5 of [Iri09] the small quantum product respects ambient classes, hence it is possible to restrict the Dubrovin connection to $F_{amb}:=H^*_{amb}(Y)\times U'\to U'$ as

$$\nabla^A_{amb}:H^*_{amb}(Y)\otimes\mathcal{O}_{U'}\to H^*_{amb}(Y)\otimes\Omega^1_{U'}.$$

The connection ∇^A_{amb} induces on $\mathcal{H}^A_{amb}$ a variation of Hodge structure.

Definition 4.5.1 (Ambient A-VHS) *The quadruple $(\mathcal{H}^A_{amb},\nabla^A,\mathcal{F}^\bullet,Q_A)$ is called* ambient A-variation of Hodge structure, *it consists of the locally free sheaf $\mathcal{H}^A_{amb}:=H^*_{amb}\otimes\mathcal{O}_{U'}$, the flat Dubrovin connection ∇^A, the Hodge filtration $\mathcal{F}^p_A:=\bigoplus_{j\le 2(n-1-p)}H^j_{amb}(Y)$ and the pairing*

$$Q_A(\alpha,\beta)=(2\pi i)^{n-1}\int_Y\alpha^\vee\cup\beta,$$

where $\alpha^\vee$ is defined as $\sum(i)^k\alpha_k$, if $\alpha=\sum_k\alpha_k\in H^*(X,\mathbb{C})=\bigoplus_k H^k(X,\mathbb{C})$.

An important feature of the (extended) Dubrovin connection ∇ is that a fundamental system of horizontal sections of ∇ can be given explicitly in terms of gravitational descendents. Let $z^{\deg/2}$ act on $\alpha \in H^*(X,\mathbb{C})$ by $z^{deg/2}_{|H^k(X,\mathbb{C})} = z^{k/2} Id_{H^k(X,\mathbb{C})}$.

Proposition 4.5.2 *The* $\mathrm{End}(H^*(X))$ *valued function*

$$S(\tau,z)\alpha = e^{\frac{-\tau_{0,2}}{z}}\alpha - \sum_{\substack{(d,l)\neq(0,0)\\ d\in M_{\mathbb{Z}}(X)}} \sum_{k=1}^{N} \frac{T^k}{l!} \left\langle \frac{e^{-\tau_{0,2}/z}\alpha}{z-\psi_1}, T_k, \tau', \dots, \tau' \right\rangle_{0,l+2,d} e^{\langle \tau_2, d\rangle}$$

implies multivalued maps

$$s_i(\tau,z) := S(\tau,z) z^{-\frac{deg}{2}} z^{\rho} T_i,\ i = 1,\dots,N$$

that build a fundamental system of flat sections of ∇, where ρ denotes the first Chern class of the tangent bundle $\rho = c_1(T_X)$. Each s_i is characterized by its asymptotic behavior $s_i(\tau,z) \sim z^{-\deg/2} z^{\rho} e^{-\tau_2} T_i$ in the large radius limit.

Notice that after the expansion

$$\frac{1}{z-\psi_1} = \sum_{k=0}^{\infty} (1/z)^{k+1} (\psi_1)^k$$

the symbol $\left\langle \frac{e^{-\tau_{0,2}/z}\alpha}{z+\psi_1}, T_k, \tau', \dots, \tau' \right\rangle_{0,l+2,d}$ is explained by

$$\sum_l \left\langle \frac{\alpha}{z-\psi_1}, \beta, \gamma, \dots, \gamma \right\rangle_{0,l+2,\beta} = \sum_{k=0}^{\infty} (1/z)^{k+1} \sum_l \langle \psi_1^k \alpha, \beta, \gamma, \dots \gamma \rangle_{0,l+2,\beta}.$$

Proof The idea of the proof goes back to A. Givental. The horizontality of s_i in the direction t_i is worked out as Proposition 2 of [Pan98]. The rest is Proposition 2.4 of [Iri09].

By Lemma 10.3.3 of [CK99] the generating function of $\langle s_i, 1\rangle$ can be defined as follows.

Definition 4.5.3 ($\mathcal{J}_X$-function) *The* $H^*(X,\mathbb{C})$ *valued function*

$$\mathcal{J}_X(\tau,z) = e^{\tau_{0,2}/z} \left(1 + \sum_{d\in M_{\mathbb{Z}}(X)} \sum_{k=0}^{N} q^d \left\langle \frac{T_k}{z-\psi_1}, 1 \right\rangle_{0,2,d} T^k \right) = e^{\tau_{0,2}} (1 + O(z^{-1})),$$

where $q^d = e^{\langle d,\tau_2\rangle}$ is called Givental $\mathcal{J}$-function.

To define an analogue for $\iota : Y \hookrightarrow X$ given as the zero set of a global section of the decomposable vector bundle $\mathcal{E}$, it is necessary to introduce a twisted version of the Gromov-Witten invariants. The vector bundle over $\overline{M}_{0,k}(X,d)$ whose fiber over f is $H^0(C, f^*(\mathcal{E}))$ is denoted by $\mathcal{E}_{d,k}$ and the vector bundle whose fibers over f consist of sections $s \in H^0(C, f^*(\mathcal{E}))$ that vanish at the i-th marked point is denoted by $\mathcal{E}'_{(d,k,i)}$. These vector bundles fit into the exact sequence

$$0 \longrightarrow \mathcal{E}'_{d,k,i} \longrightarrow \mathcal{E}_{d,k} \longrightarrow e_i^*(\mathcal{E}) \longrightarrow 0\,.$$

and twisting with $c_n(\mathcal{E}'_{d,n,n})$ yields the twisted Gromov-Witten invariants and the function $\mathcal{J}_{\mathcal{E}}$.

Definition 4.5.4 (Twisted Gromov-Witten invariants and $\mathcal{J}_{\mathcal{E}}$) *With assumptions as in Definition 4.4.5 of the untwisted Gromov-Witten invariants with gravitational descendents we define twisted Gromov Witten-invariants with gravitational descendents for the vector bundle $\mathcal{E}$ as*

$$\langle \gamma_1 \psi_{d_1}, \ldots, \gamma_n \psi_{d_n} \rangle^{\mathcal{E}}_{0,n,d} := \int_{[\overline{M}_{0,n}(X,d)]^{virt}} ev_1^*(\gamma_1) \cup \psi_1^{d_1} \cup \ldots \cup ev_n^*(\gamma_n) \cup \psi_1^{d_n} c_{top}(\mathcal{E}'_{d,n,n})$$

and a corresponding function $\mathcal{J}_{\mathcal{E}}$ by

$$\mathcal{J}_{\mathcal{E}}(\tau, z) := e^{\tau_{0,2}} \left(1 + \sum_{d \in M_{\mathbb{Z}}(X)} \sum_{k=0}^{N} q^d \left\langle \frac{T_k}{z - \psi_1}, 1 \right\rangle^{\mathcal{E}}_{0,2,d} T^k \right)$$

In the next section we see that the $\mathcal{J}$-function associated to a toric variety is closely related to the $\mathcal{I}$-function.

4.5.1 Giventals Approach and Landau-Ginzburg Models

The necessary material on toric manifolds can be found in W. Fulton's book [Ful93]. Let $M \cong \mathbb{Z}^n$ be a lattice and $N = \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice. Denote $M_{\mathbb{R}} = M \otimes \mathbb{R}$, $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and define the dual pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{R}$. A *fan* Σ is a collection of cones $\{\sigma \subset N_{\mathbb{R}}\}$ with $\sigma \cap -\sigma = \{0\}$ and $\sigma = \sum \mathbb{R}_{\geq 0} b_i$, $b_i \in N$, such that any face of a cone in Σ is again a cone in Σ. A cone with the first mentioned property is called *strongly convex* and a cone with the second mentioned property is called a *polyhedral cone*. There is a standard procedure that associates a toric variety X_Σ to any fan. The toric variety X_Σ is projective exactly if Σ admits a strictly convex piecewise linear *support function* $\phi : N_{\mathbb{R}} \to \mathbb{R}$. The set of l-dimensional cones of Σ is denoted by $\Sigma(l)$. Much information of X_Σ is encoded as combinatorial data of Σ, for example X is compact exactly if $\bigcup_{\sigma \in \Sigma} \sigma = N_{\mathbb{R}}$ and X_Σ is smooth if and only if for every $\sigma \in \Sigma$ the generators of σ can be extended to a basis of the $\mathbb{Z}$-module N. In addition, X_Σ is an orbifold exactly if the generators of all cones of Σ are $\mathbb{R}$-linear independent. A fan with this property is called *simplicial*. The set of irreducible subvarieties of dimension one $D_i, i = 1, \ldots, m$ of X_Σ that are mapped to themselves under the torus action correspond to the elements of $\Sigma(1)$. Denote the line bundles associated to D_i by $\mathcal{O}(D_i)$ and their first Chern classes by ξ_{D_i}. A *toric divisor* is a sum $\sum a_i D_i$ with integer coefficients a_i. Especially the canonical bundle ω_{X_Σ} can be given in terms of the divisors D_i as $\mathcal{O}\left(-\sum D_i\right)$. The primitive generators $w_1, \ldots, w_m \in N$ of the elements of $\Sigma(1)$ define a map $\beta : \mathbb{Z}^m \to \mathbb{Z}^n$ mapping the standard basis vectors e_i to w_i. In toric mirror symmetry as introduced in [Bat94] another approach using polytopes is used. Given an n-dimensional polytope $\Delta \subset M_{\mathbb{R}}$ with integral vertices its supporting cone C_Δ is defined as

$$C_\Delta := 0 \cup \{(x_0, x_1, \ldots, x_n) \in \mathbb{R} \oplus M_{\mathbb{R}} \mid (x_1/x_0, \ldots, x_n/x_0) \in \Delta,\ x_0 > 0\}.$$

Then a projective toric variety X_Δ can be defined as the projective spectrum $\mathrm{Proj}\, S_\Delta$ of the subring S_Δ of $\mathbb{C}((x_0, \ldots, x_n))$ whose $\mathbb{C}$-basis are monomials $x_0^{m_0} \cdot \ldots \cdot x_n^{m_n}$ with $(m_0, \ldots, m_n) \in C_\Delta$. That for toric projective varieties these two approaches are equivalent is a standard fact in toric geometry and is explained in Section 2 of [Bat94]. Starting with a polytope Δ for every l-dimensional face $\tau \subset \Delta$ define the cone $\sigma(\tau) \subset M_{\mathbb{R}}$ consisting of the vectors $\lambda(p - p')$, where $\lambda \in \mathbb{R}_{\geq 0}$, $p \in \Delta$, $p' \in \tau$, and the dual cone $\check{\sigma}(\tau)$. The fan $\Sigma(\Delta)$ with $X_{\Sigma(\Delta)} = X_\Delta$ consists of the cones $\check{\sigma}(\tau)$, where τ runs over all faces of Δ. Conversely, the construction of a polytope $\Delta \subset M_{\mathbb{R}}$ from a fan $\Sigma \subset N_{\mathbb{R}}$ involves the choice of the strictly convex piecewise linear function ϕ on Σ. The convex polytope $\Delta(\Sigma, \phi)$ is constructed as

$$\Delta(\Sigma, \phi) := \bigcap_{\sigma \in \Sigma(n)} (-h_{|\sigma} + \check{\sigma}),$$

where the restriction $h_{|\sigma}$ is considered as an element of M. A lattice polytope $\Delta \subset M$ of dimension n with interior lattice point 0 is called *reflexive* if the dual polytope

$$\Delta^\circ = \{v \in N_{\mathbb{R}} \mid \langle m, v\rangle \geq 1, m \in \Delta\}.$$

is a again lattice polytope. This construction is involutional ($\Delta^\circ = \Delta$). In [Bat94] V. Batyrev used this construction to associate to a family of Calabi-Yau hypersurfaces in a toric variety X_Δ another family of Calabi-Yau varieties in X_{Δ°. The prelude on toric varieties allows us to define the $\mathcal{I}_{\mathcal{E}}$-function associated to a subvariety $Y \hookrightarrow X = X_\Sigma$ defined by the vector bundle $\mathcal{E} = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r$. This function depends only on combinatorial data of Σ and the first Chern classes $\xi_{\mathcal{L}_i}$ of $\mathcal{L}_i$.

Definition 4.5.5 ($\mathcal{I}$-function) *The $H^*(X)$ valued function*

$$\mathcal{I}_{\mathcal{E}}(q,z) := e^{\langle d,\tau_2\rangle/z}\left(\sum_{d\in M_{\mathbb{Z}}(X)} q^d \prod_{a=1}^{r} \frac{\prod_{v=-\infty}^{\langle d,\xi_{\mathcal{E}_a}\rangle}(\xi_{\mathcal{L}_a}+vz)}{\prod_{v=-\infty}^{0}(\xi_{\mathcal{L}_a}+vz)} \prod_{j=1}^{v} \frac{\prod_{v=-\infty}^{0}\left(\xi_{D_j}+vz\right)}{\prod_{v=-\infty}^{\langle\xi_{D_j},d\rangle}\left(\xi_{D_j}+vz\right)}\right)$$

is called Givental $\mathcal{I}$-function. *If $\mathcal{E} = 0$ we write $\mathcal{I}_X := \mathcal{I}_0$.*

One of A. Giventals insights in [Giv98] was that the function $\mathcal{J}_{\mathcal{E}}$ and $\mathcal{I}_{\mathcal{E}}$ are identical up to change of coordinates. Such a direct relation cannot exist if we consider $\mathcal{I}_{\mathcal{E}}$ of a complete intersection Y, since $\mathcal{J}_Y$ takes values in $H^*(Y)$ but the image of $\mathcal{I}_Y$ lies in $H^*(X)$, indeed this was the reason why we introduced twisted Gromov-Witten invariants in the last section. The next theorem clarifies why the additional variable z was needed.

Theorem 4.5.6 *Let X denote a non-singular compact Kähler toric variety and $Y \subset X$ be a non-singular complete intersection with positive first Chern class $c_1(T_Y)$. The function $\mathcal{I}_{\mathcal{E}}(q,z)$ has an expansion*

$$\mathcal{I}_{\mathcal{E}} = f(q) + g(q)/z + O(z^{-2})$$

and the equality

$$f(q)\mathcal{J}_{\mathcal{E}}(\eta(q),z) = \mathcal{I}_{\mathcal{E}}(q,z),\ \eta = \frac{g(q)}{f(q)}$$

holds.

If $\mathcal{E} = 0$ the asymptotic expansion simplifies to

$$\mathcal{I}_X = 1 + g(q)/z + O(z^{-2}),$$

and $\mathcal{I}_X(q,z) = \mathcal{J}_X(g(q),z)$ holds. If moreover X is Fano then $g(q) = Id$. The map $\eta(q)$: $\{(q_1,\ldots,q_l) \mid 0 < |q_i| < \epsilon\} \to H^{\leq 2}(X,\mathbb{C})/2\pi i H^2(X,\mathbb{Z})$ is called *mirror map*. Much work has been done to transfer Theorem 4.5.6 to more general situations, see for example [Kim99, Lee08]. In view of mirror symmetry the crucial fact about $\mathcal{I}_{\mathcal{E}}(\tau,z)$ is that it solves a partial differential equation $\mathcal{P}_d$.

Proposition 4.5.7 *With $\theta_{q_j} := q_j\partial_{q_j} = \partial_{t_j}$ the function $\mathcal{I}_{\mathcal{E}}$ is a solution of the partial differential equation*

$$\begin{aligned}\mathcal{P}_d := &\prod_{\rho:\langle\zeta_{D_\rho},d\rangle>0}\ \prod_{k=0}^{\langle\zeta_{D_\rho},d\rangle-1}\left(\sum_{j=1}^{N}(\langle\zeta_{D_\rho},d\rangle\theta_{q_j})-kz\right)\prod_{i:\langle\zeta_{\mathcal{E}_i},d\rangle<0}\ \prod_{k=0}^{-\langle\zeta_{\mathcal{L}_i},d\rangle+1}\left(-\sum_{j=1}^{r}(\langle\zeta_{\mathcal{L}_i},d\rangle\theta_{q_j})-kz\right)\\ &-q^d\prod_{\rho:\langle\zeta_{D_\rho},d\rangle<0}\ \prod_{k=0}^{\langle\zeta_{D_\rho},d\rangle+1}\left(\sum_{j=1}^{N}(\langle\zeta_{D_\rho},d\rangle\theta_{q_j})-kz\right)\prod_{i:\langle\zeta_{\mathcal{E}_i},d\rangle>0}\ \prod_{k=0}^{\langle\zeta_{\mathcal{L}_i},d\rangle-1}\left(-\sum_{j=1}^{r}(\langle\zeta_{\mathcal{L}_i},d\rangle\theta_{q_j})-kz\right).\end{aligned}$$

It was again A. Givental who observed that another solution of $\mathcal{P}_d$ is given by an integral of the form $\int_\Gamma e^{W_q^0/z}\omega$ related to a Landau-Ginzburg model that is constructed from the combinatorial data of the fan Σ.

Definition 4.5.8 (Landau-Ginzburg model) *A* Landau - Ginzburg model *is a not necessary compact quasi projective variety X together with a smooth map $W : X \to \mathbb{C}$ called* potential.

The most important special case is $X = (\mathbb{C}^*)^n$ and W a *Laurent polynomial*. A partition $I_0 \sqcup \ldots \sqcup I_r$ of $\{1,\ldots,m\}$ such that $\sum_{i\in I_j} D_i$ is nef for $1 \le j \le r$ is called *nef partition*. If X_Σ is smooth, $\beta : M \to \mathbb{Z}^m, u \mapsto (u(w_1),\ldots,u(w_m))^t$ for integral primitive vectors $w_1,\ldots,w_m$ of $\Sigma(1)$ and $D : \mathbb{Z}^m \to \mathrm{Pic}(X_\Sigma), v \mapsto \sum v_i\mathcal{O}(D_i)$ build the exact *divisor sequence*

$$0 \longrightarrow M \xrightarrow{\beta} \mathbb{Z}^m \xrightarrow{D} \mathrm{Pic}(X_\Sigma) \longrightarrow 0\,.$$

If $\mathrm{Pic}(X_\Sigma)$ is isomorphic to $H^2(X,\mathbb{Z})$ tensoring with $\mathbb{C}^*$ yields the exact sequence

$$0 \longrightarrow \mathbb{T} := \mathrm{Hom}(N,\mathbb{C}^*) \longrightarrow (\mathbb{C}^*)^m \xrightarrow{pr} H^2(X,\mathbb{C}^*) \longrightarrow 0\,.$$

The exact divisor sequence splits, hence there is a section l of $D : \mathbb{Z}^m \to H^2(X,\mathbb{Z})$. After choosing an isomorphism $H^2(X,\mathbb{C}^*) \cong (\mathbb{C}^*)^{m-n}$ and a basis $\{q_1,\ldots,q_{m-n}\}$ of $(\mathbb{C}^*)^{m-n}$ the section l can be given as a matrix $(l_{ia})_{1\le i\le m, 1\le a\le m-n}$. The mirror of the zero locus $Y \hookrightarrow X$ of a general section of $\mathcal{E}$ defined by a nef partition $I_0 \sqcup \ldots \sqcup I_r$ is constructed as

$$Y_q' := \left\{t \in \mathbb{T}|\ W_q^1(t) = 1,\ldots,W_q^r(t) = 1\right\},$$

where

$$W_q^j := \sum_{i\in I_j}\prod_{a=1}^{m-n}(q_a^{l_{ia}})\prod_{k=1}^{n} t_k^{(w_i)_k}, j = 1,\ldots,r$$

and $t_1,\ldots,t_{m-n}$ are coordinates for $\mathbb{T}$. The set I_0 defines the potential on $Y'(q)$ as

$$W_q^0 := \sum_{i\in I_0}\prod_{a=1}^{m-n}(q_a^{l_{ia}})\prod_{k=1}^{n} t_k^{(w_i)_k}.$$

Two special cases are $I_0 = \emptyset$ and $I_0 = \{1,\ldots,m\}$, in the first case $Y'(q)$ is Calabi-Yau and in the second case $X_q' = Y_q'$ is a mirror family for the toric variety X itself. The potential W_q^0 is sometimes called *Hori-Vafa potential* since it has been considered by K. Hori and C. Vafa in [HV00].

Example 4.6 The rays $\Sigma(1)$ of the fan defining $\mathbb{P}^4$ are the standard vectors of $\mathbb{Z}^4$ and the vector $\sum_{i=1}^4 -e_i$, thus the divisor sequence is

$$0 \longrightarrow \mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}} \mathbb{Z}^5 \xrightarrow{(1\,1\,1\,1\,1)} \mathbb{Z} \longrightarrow 0\,.$$

If $I_0 = \emptyset$ and $I_1 = \{1,2,3,4,5\}$, then Y is the zero locus of a general section of the line bundle associated to the anticanonical divisor $-K_{\mathbb{P}^4}$ and hence Y is Calabi-Yau. The section l can simply be taken as $l(c) = (0,\ldots,0,c)^t \in \mathbb{Z}^{n+1}$, $c \in \mathbb{Z}$ and the Laurent-polynomial defining the mirror Y_q' inside $(\mathbb{C}^*)^4$ is

$$W_q^1(t) = t_1 + t_2 + t_3 + t_4 + \frac{q}{t_1\cdot\ldots\cdot t_4}.$$

If there is a resolution of singularities of the compactification $\overline{Y}_q'$ of Y_q', it is denoted by $\check{Y}_q'$ and we call $(Y_q, \check{Y}_q')$ a *Batyrev mirror pair*. Especially if Y_q' is a three-dimensional Calabi-Yau hypersurface by [Bat94] there is a Zariski open subset of $\mathcal{B}_{reg} \subset \mathcal{B} = H^2(X, \mathbb{C}^*)$, such that for $q \in \mathcal{B}_{reg}$ a crepant resolution always exists. Actually the resolution is constructed from a resolution $X' \to \overline{\mathbb{T}}$ of a toric compactification of $\mathbb{T}$. Recall that crepant resolutions preserve the triviality of the canonical bundle. In general such a resolution does not exist and orbifold singularities cannot be avoided. Once the construction of the the $(k-1)$-dimensional hypersurface $\check{Y}_q'$ is established, it is straightforward to define a second variation of Hodge structure. The Poincaré residue $Res(\omega) \in H^{k-1}(\check{Y}_q')$ for $\omega \in H^{k-1}(\check{X}' \setminus \check{Y}_q')$ is defined by the equality

$$\int_\gamma Res(\omega) = \frac{1}{2\pi i} \int_\Gamma \omega,$$

where Γ is a tube of a $(k-1)$-cycle γ of $\check{Y}_q'$.

Definition 4.6.1 (Residue part) *The* residue part $H^{k-1}_{res}(\check{Y}_q')$ *is the* $\mathbb{Q}$*-subspace of* $H^{k-1}(\check{Y}_q')$ *defined as the image of the residue map*

$$Res : H^0(\check{X}', \Omega^k_{\check{X}'}(*\check{Y}_q')) \to H^{k-1}(\check{Y}_q'),$$

where $\Omega^k_{\check{X}}(*\check{Y}_q')$ *is the space of algebraic k-forms on* $\check{X}'$ *with arbitrary poles along* $\check{Y}_q'$.

Denote the subbundle of $R^{k-1}pr_*\mathbb{C}_{\mathcal{B}}$ with fibers at $q \in \mathcal{B}_{reg}$ given by $H^{k-1}_{res}(\check{Y}_q')$ by $H_{B,\mathbb{Q}}$ and consider the locally free sheaf

$$\mathcal{H}_B := H_{B,\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{B}_{reg}}.$$

Together with the Gauss-Manin connection ∇_B, the filtration

$$\mathcal{F}^p_B := \bigoplus_{j \geq p} H^{j,k-1-j}_{res}(\check{Y}_q')$$

and the polarization $Q_B(\alpha, \beta) := (-1)^{\frac{(k-2)(k-1)}{2}} \int_{\check{Y}_q'} \alpha \cup \beta$ build a polarized variation of Hodge structure.

Definition 4.6.2 (residual B-model VHS) *The quadruple* $(\mathcal{H}^B, \nabla^B, \mathcal{F}^\bullet_B, Q_B)$ *is called* residual B-model VHS.

4.6.1 Toric Mirror Correspondence with $\widehat{\Gamma}$-Integral Structure

We follow the treatment of H. Iritani in [Iri09,Iri11], but instead of working with orbifolds we restrict to the case of compact toric manifolds. A similar but more algebraic strategy can be found in the article [RS10] by S. Reichelt and C. Sevenheck. Both approaches try to formulate mirror symmetry as isomorphism of certain $\mathcal{D}$-modules with matching integral structures. As before we restrain to the case of hypersurfaces, where Y_q' is given by a single equation $W_q^1(t)$. And we give a natural $\mathbb{Z}$-structure inside $H_{B,\mathbb{Q}}$ spanned by *vanishing cycles*. If C denotes the set of critical points of $W_q := W_q^1$ the relative homology group $H_k(\mathbb{T}, Y_q', \mathbb{Z})$ is isomorphic to

$$R_{\mathbb{Z},q} := H_k(\mathbb{T}, \{\Re(W_q(t)) \gg 0\}, \mathbb{Z}) \cong \mathbb{Z}^{|C|}.$$

The group $R_{\mathbb{Z},q}$ is generated by *Lefschetz thimbles* collapsing at the points of C. Furthermore it fits into the exact sequence

$$0 \longrightarrow H_k(\mathbb{T}) \longrightarrow H_k(\mathbb{T}, Y_q') \xrightarrow{\partial} H_{k-1}(Y_q') \longrightarrow H_{k-1}(\mathbb{T}) \longrightarrow 0,$$

where the boundary map ∂ maps a Lefschetz thimble to the corresponding vanishing cycle in Y_q'. The vanishing cycle map

$$vc : H_k(\mathbb{T}, Y_q') \to H^{k-1}(\check{Y}_q')$$

is defined as the composition of ∂ with the passage to the smooth manifold $\check{Y}_q'$ and the Poincaré duality map

$$H_{k-1}(Y_q') \xrightarrow{\partial} H_{k-1}(Y_q') \longrightarrow H_{k-1}(\check{Y}_q') \xrightarrow{PD} H^{k-1}(\check{Y}_q') \,.$$

The image of vc is described in Lemma 6.6 of [Iri11].

Lemma 4.6.3 *The image of the map vc coincides with $H^{k-1}_{res}(\check{Y}_q')$.*

Hence the residual B-model variation of Hodge structure is equipped with a natural lattice.

Definition 4.6.4 *The* vanishing cycle structure $H^{vc}_{B,\mathbb{Z}}$ *is defined as the image of $H_k(T, Y_q', \mathbb{Z})$ under the map vc.*

The crucial point of Iritani's work is that he is able to specify a lattices in the A-model via the Γ-character from K-theory to cohomology that matches with $H^{vc}(B, \mathbb{Z})$ under the mirror correspondence. For a Calabi-Yau hypersurface $\iota : Y \hookrightarrow X$ denote the subset of elements of $K(Y)$ such that $\mathrm{ch}(\mathcal{E}) \in H^*_{amb}(Y)$ by $K^{amb}(Y)$.

Definition 4.6.5 (A-model lattice) *For $\mathcal{E} \in K(Y)$ satisfying $\mathrm{ch}(\mathcal{E}) \in H^*_{amb}(Y)$ define a ∇^A-flat section by*

$$sol(\mathcal{E})(\tau) := (2\pi i)^{-(k-1)} S(\tau, 1)(\widehat{\Gamma}_Y \cup (2\pi i)^{deg/2}\, \mathrm{ch}(\mathcal{E})).$$

Then the local system $H^{amb}_{A,\mathbb{Z}}$ in $H_A = ker\nabla^A \subset \mathcal{H}_A$ is defined by

$$H^{amb}_{A,\mathbb{Z}} := \{sol(\iota^*(\mathcal{E})) | \mathcal{E} \in K(X)\}\,.$$

We have collected all definitions needed to state H. Iritani's theorems in the suitable context.

Theorem 4.6.6 *Let $\iota : Y \hookrightarrow X$ a subvariety of the toric variety X and $(Y, \check{Y}')$ be a Batyrev mirror pair, then there is an isomorphism*

$$Mir_Y : (\mathcal{H}^A_{amb}, \nabla^A, \mathcal{F}^\bullet_A, Q_A) \to (\mathcal{H}_B, \nabla^B, \mathcal{F}^\bullet_B, Q_B)$$

in a neighborhood of 0 with an induced isomorphism of $\mathbb{Z}$-local systems

$$Mir^{\mathbb{Z}}_Y : H^{amb}_{A,\mathbb{Z}} \cong H^{vc}_{B,\mathbb{Z}}.$$

of local systems. Moreover, if $\mathcal{E} \in \iota^* K(X)$ or $\mathcal{E} = \mathcal{O}_p$, the A-periods identify with B-periods as

$$Q_A(\phi, sol(\mathcal{E})(\eta(q))) = \int_{C_\mathcal{E}} Res\left(\frac{dt_1 \wedge \ldots \wedge t_n}{(W_q(t) - 1)t_1 \cdot \ldots \cdot dt_n}\right)$$

for any section ϕ of $\mathcal{H}_A$ and some integral $(n-1)$-cycle $C_\mathcal{E}$.

4.6.2 Special Autoequivalences

For a homological mirror pair $(X, X^{'})$ of n-dimensional Calabi-Yau manifolds autoequivalences of $\mathrm{Auteq}(DFuk(X^{'}))$ should correspond to autoequivalences of $D^b(X)$. We are concerned with autoequivalences of $D^b(X)$ induced by tensoring with a line bundle $\mathcal{L}$ and Seidel-Thomas twists. Recall that the map ψ was defined in Section 4.4.1 as

$$\psi : K(X) \to H^*(X, \mathbb{C}),\ \mathcal{E} \mapsto (2\pi i)^{-n} \left((2\pi i)^{deg/2} \operatorname{ch}(\mathcal{E}) \cup \widehat{\Gamma}_X \right).$$

If the first Chern class of the ample generator $\mathcal{L} \in \mathrm{Pic}(X) \cong \mathbb{Z}$ is H, from the multiplicativity of the Chern character we get

$$\psi(\zeta \otimes \mathcal{L}) = (2\pi i)^{-n} \left((2\pi i)^{\frac{deg}{2}} (\operatorname{ch}(\zeta) \cup \operatorname{ch}(\mathcal{L})) \cup \widehat{\Gamma}_X \right) = \psi(\zeta) \cup (2\pi i)^{\frac{deg}{2}} e^H.$$

It follows that the tensor product with $\mathcal{L}$ on the level of cohomology is

$$\alpha \mapsto \alpha (2\pi i)^{\frac{deg}{2}} e^H.$$

With respect to the basis $\frac{H^{n-j}}{(2\pi i)^j}, j = 0, \ldots, n$ this is described by the matrix $M_{\otimes L}$ with entries

$$(M_{\otimes L})_{i,j} = \begin{cases} \frac{1}{(i-j)!}, & i \geq j \\ 0, & else \end{cases}.$$

If furthermore X has dimension three the matrix $M_{\otimes L}$ simplifies to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & 1 & 1 \end{pmatrix}.$$

The matrix $M_{\otimes L}$ coincides with the local monodromy of a Fuchsian differential equation at a MUM-point with respect to the normalized Frobenius basis at this point. In addition consider $X^{'}$ as generic fiber of a family $p : \mathcal{X} \to \mathbb{P}^1$ that posses an A_1-singularity in a specific fiber X_c. The geometric monodromy M_γ along a loop γ in $\mathbb{P}^1$, that encircles c exactly once and avoids encircling further special basepoints, is a symplectic Dehn twist corresponding to a vanishing sphere S. As explained in [Sei00] to any Lagrangian sphere S in $X^{'}$ it is possible to associate a symplectic automorphism τ_S called *generalized Dehn twist*, that induces an autoequivalence of the symplectic invariant $DFuk(X)$. On the level of homology $H_n(X, \mathbb{Q})$ the map M_γ is described by the Picard-Lefschetz formula

$$\alpha \mapsto \alpha + (-1)^{(n-1)n/2} \langle \alpha, \delta \rangle \delta,$$

where δ is the homology class of S and $n = \dim(X)$. R. P. Thomas and P. Seidel proposed certain autoequivalences mirroring generalized Dehn twist. We have to introduced Fourier-Mukai transformations to understand their claim.

Definition 4.6.7 (Fourier-Mukai transformation) *For two smooth complex projective varieties X, Y and projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ the* Fourier-Mukai transform *for $\mathcal{P} \in D^b(X \times Y)$ is the exact functor*

$$\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y),\ \phi_{\mathcal{P}}(\mathcal{G}) = R\pi_{2*}(\pi_1^* \mathcal{G} \otimes^L \mathcal{P}).$$

A cohomological analog of $\Phi_{\mathcal{P}}$ can be defined as

$$\Phi^H_\alpha : H^*(X,\mathbb{C}) \to H^*(Y,\mathbb{C}),\ \beta \mapsto \pi_{2*}(\alpha \cup \pi_1^*(\beta)),\ \alpha \in H^*(X \times Y).$$

Using the Riemann-Roch-Grothendieck formula and the projection formula, analogue to Section 5.2 of [Huy06], one sees that via ψ the Fourier-Mukai transforms $\Phi_{\mathcal{P}}$ and $\Phi^H_{\psi(\mathcal{P})}$ are compatible, that is the diagram

$$\begin{tikzcd} D^b(X) \arrow[r, "\phi_{\mathcal{P}}"] \arrow[d, "\psi([\cdot])"'] & D^b(Y) \arrow[d, "\psi([\cdot])"] \\ H^*(X,\mathbb{C}) \arrow[r, "\phi^H_{\psi([\mathcal{P}])}"] & H^*(Y,\mathbb{C}) \end{tikzcd}$$

is commutative. An instance of Fourier-Mukai transforms of special interest is induced by a spherical object.

Definition 4.6.8 (Spherical object) *A complex $\mathcal{E}^\bullet \in D^b(X)$ is called* spherical *if*

- $Ext^i_X(E,E) \cong \begin{cases} \mathbb{C}, & \text{if } i = 0, \dim X \\ 0, & \text{otherwise} \end{cases}$
- $E \otimes \omega_X \cong E$

that is if $Ext^*(E,E) \cong H^*(S^n,\mathbb{C})$ for the n-sphere S^n.

For Calabi-Yau varieties the second condition is trivially true since $\omega_X = \mathcal{O}_X$. Two examples of spherical objects are the structure sheaf $\mathcal{O}_X$ and line bundles on X. Using spherical objects P. Seidel and R. P. Thomas introduced special autoequivalences of $D^b(X)$ [ST01].

Definition 4.6.9 (Spherical twist) *The Fourier-Mukai transform $T_{\mathcal{E}} : D^b(X) \to D^b(X)$ associated to*

$$\mathcal{P}_{\mathcal{E}} = Cone(\pi_2^*\mathcal{E}^\vee \otimes \pi_1^*\mathcal{E} \to \Delta^*(\mathcal{O}_X)),$$

where $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding is called a spherical twist.

The spherical twist $T_{\mathcal{E}}$ is an autoequivalence

$$T_{\mathcal{E}} : D^b(X) \to D^b(X).$$

The descent $T^H_{\psi(\mathcal{E}^\bullet)}$ of $T_{\mathcal{E}^\bullet}$ to the level of cohomology spherical twist can be described fairly easy.

Lemma 4.6.10 *Let $\mathcal{E}^\bullet$ be a spherical object and X a Calabi-Yau manifold. Then $T^H_{\psi(\mathcal{E}^\bullet)} : H^*(X,\mathbb{C}) \to H^*(X,\mathbb{C})$ is given as*

$$T^H_{\mathcal{E}}(\alpha) = \alpha - Q_M\left(\alpha, \psi([\mathcal{E}^\bullet])^\vee\right)\psi([\mathcal{E}^\bullet]).$$

Proof Similar to the proof of Lemma 8.12 in [Huy06]. □

For a Calabi-Yau variety X the Chern class $c_i(X)$ for $i = 1$ and $i > \dim(X)$ vanish, hence $\exp(\pi i c_1(X)/2)$ equals one. Moreover, the Chern character of $\mathcal{O}_X$ is trivial, thus $\psi(\mathcal{O}_X) = (2\pi i)^{-n}\widehat{\Gamma}_X$. To express $\Phi^H_{\mathcal{O}_X}$ with respect to the basis $1/(2\pi i)^n, \ldots, H^n/(2\pi i), H^3$, we have to expand

$$\widehat{\Gamma}(T_X) = \exp\left(-\gamma \sum_{i=1}^n \delta_i + \sum_{l\geq 2} \frac{(-1)^l \zeta(l)}{l} \sum_{i=1}^n \delta_i^l\right).$$

as series in the Chern classes $c_1(X), \dots, c_{\dim(X)}(X)$. We use Newton's identities, which express $\delta_1^l + \dots + \delta_n^l$ as polynomial in the elementary symmetric polynomials in δ_i. For a Calabi-Yau manifold of dimension smaller than seven the $\widehat{\Gamma}$-class simplifies to

$$\begin{aligned}
\widehat{\Gamma}(T_X) =&1 - \zeta(2)c_2^{'} - \zeta(3)c_3^{'} + \left(\frac{1}{2}\zeta(2)^2 + \frac{1}{2}\zeta(4)\right)(c_2^{'})^2 - \zeta(4)c_4^{'} \\
&+ (\zeta(5) + \zeta(2)\zeta(3))\, c_2^{'}c_3^{'} - \zeta(5)c_5^{'} + (\zeta(2)\zeta(4) + \zeta(6))\, c_2^{'}c_4^{'} \\
&- \left(\frac{1}{3}\zeta(6)(c_2^{'})^3 + \frac{1}{6}\zeta(2)^3\right)(c_2^{'})^3 + \left(\frac{1}{2}\zeta(3)^2 + \frac{1}{2}\zeta(6)\right)(c_3^{'})^2 - \zeta(6)c_6^{'} \\
&+ (\zeta(7) + \zeta(3)\zeta(4))\, c_3^{'}c_4^{'} - \left(\frac{1}{2}\zeta(3)\zeta(4) + \zeta(2)\zeta(5) + \frac{1}{2}\zeta(2)^2\zeta(3) + \zeta(7)\right)(c_2^{'})^2c_3^{'} \\
&+ (\zeta(2)\zeta(5) + \zeta(7))\, c_5^{'}c_2^{'} - \zeta(7)c_7^{'},
\end{aligned}$$

where

$$c_j^{'} = \begin{cases} c_j(X), & j \le \dim(X) \\ 0, & else \end{cases}.$$

Recall that the Bernoulli numbers can be used to express the values of the ζ-function at even positive integers $\zeta(2n) = (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n}$. Especially for dimension three the $\widehat{\Gamma}$-class is

$$\widehat{\Gamma}(T_X) = 1 - \zeta(2)c_2(X) - \zeta(3)c_3(X) = 1 - \frac{\pi^2 c_2(X)}{6} - \zeta(3)c_3(X).$$

Hence with respect to the basis $\frac{1}{(2\pi i)^n}, \frac{H}{(2\pi i)^{n-1}}, \dots, H^n$ the spherical twist $\Phi^H_{\mathcal{O}_X}$ reads

$$\alpha \mapsto \alpha - Q_M(\alpha, C)C.$$

where in dimension $n \le 7$ the vector $C = (C_1, \dots C_{n+1})$ is given by

$$\begin{aligned}
&C_1 = H^n,\ C_2 = 0,\ C_3 = \frac{1}{24}c_2H^{n-2},\ C_4 = c_3H^{n-3}\lambda_3,\ C_5 = \left(-\frac{1}{2^5 3^2 5}c_4 + \frac{1}{2^7 3^2 5}c_2^2\right)H^{n-4}, \\
&C_6 = \left(\frac{1}{2^3 3}c_2c_3\lambda_3 - \frac{1}{32}(c_2c_3 - c_5)\lambda_5\right)H^{n-5}, \\
&C_7 = \left(\frac{1}{2^6 3^3 5^1 7}c_6 - \frac{1}{2^7 3^3 5^1 7}c_3^2 + \frac{31}{2^{10} 3^3 5^1 7}c_2^3 - \frac{11}{2^8 3^3 5^1 7}c_2c_4 + \frac{1}{2}c_3^2\lambda_3^2\right)H^{n-6} \\
&C_8 = \left(\left(-\frac{1}{2^5 3^2 5}c_4c_3 + \frac{1}{2^7 3^2 5}c_2^2c_3\right)\lambda_3 - \frac{1}{3^2 2}c_2(c_2c_3 - c_5)\lambda_5\right)H^{n-7} \\
&- \left((c_4c_3 + c_5c_2 - c_7 - c_2^2c_3)\lambda_7\right)H^{n-7}
\end{aligned}$$

and $\lambda_n := \frac{\zeta(n)}{(2\pi i)^n}$. For $\dim(X) = 3$ the vector C reads

$$C = (H^3, 0, c_2H/24, c_3\lambda_3).$$

Coming back to mirror symmetry it is crucial to reobserve M. Kontsevich's discovery that for the quintic threefold the matrices for $\otimes\psi(\mathcal{L})$ and $\Phi^H_{\mathcal{O}_X}$ coincide with the generators of the monodromy group of the Picard-Fuchs equation of the mirror quintic written with respect to the normalized Frobenius basis. The monodromy should be considered as an autoequivalence of the category $DFuk(X^{'})$. Remember that we assumed $X^{'}$ to be a a generic fiber of family of Calabi-Yau manifolds of dimension n. Let L be its Picard-Fuchs equation, where L

has a MUM-point 0 and a further singular point c corresponding to a fiber with an A_1 singularity. Then by the discussion above we have two candidates mirroring the autoequivalences induced by the monodromy transformations M_0 and M_c. The monodromy at the MUM-point is expected to correspond to the map given by tensoring with $\mathcal{L}$ and the reflection M_c should correspond to the spherical twist $\phi^{H}_{\mathcal{O}_X}$. Because $\mathcal{O}_p \otimes \mathcal{L} = \mathcal{O}_p$, the torus mirror equivalent to $\mathcal{O}_p$ should be invariant under the action of M_0. Thus with respect to the Frobenius basis of L at 0, the monodromy invariant torus is given up to a scalar as the holomorphic solution. The monodromy M_c is a reflection

$$x \mapsto x + (-1)^{(n-1)n/2}\langle x, C\rangle C$$

at a Lagrangian vanishing sphere S represented by C. The object of $D^b(X)$ mirroring S is expected to be $\mathcal{O}_X$. Since we have $\chi(\mathcal{O}_X, \mathcal{O}_p) = 1$ the monodromy invariant symplectic/symmetric form $\langle\cdot,\cdot\rangle$ on the solution space of L at 0 should be normalized such that $\langle T, S\rangle = 1$. This can be made explicit with respect to the normalized Frobenius basis, where the form $\langle\cdot,\cdot\rangle$ is represented by the matrix

$$\begin{pmatrix} & & & s \\ & \mathbf{0} & -s & \\ & & \iddots & \\ & \mp s & \mathbf{0} & \\ \pm s & & & \end{pmatrix}$$

for a complex number s. The monodromy invariant torus is described by $v_1 = (0, 0, \dots, 0, 1)$ and δ is represented by a vector $C = (C_1, \dots, C_{n+1})$, the constant s is fixed by $\langle v_1, C\rangle = C_1 s = 1$. One observes that with this choices the observation for the quintic can be extended to the 14 hypergeometric CY(3)-equations listed in [AESZ10]

$$C = (a, 0, b/24, c\lambda)\,.$$

Since in this cases mirror constructions are known $a = H^3$, $b = c_2 H$, and $c = c_3$ can be checked.

4.6.3 Numerical Experiments and Statistics from the Database

Content of this section is to review the numerical monodromy calculations done for all CY(3)-equations from [AESZ10,Str12] and some further examples. To apply the method from Chapter 2 we have to fix some choices. To any Fuchsian differential equation of order n with a MUM-point the Frobenius method introduced in Section 2.2 associates the normalized Frobenius basis

$$y_i = \frac{1}{(2\pi i)^i i!}\left(\sum_{k=0}^{i}\binom{i}{k} f_k(z)\log(z)^{i-k}\right), i = 0, \dots, n-1,$$

in neighborhood of 0. If $\Sigma = \{\sigma_1, \dots, \sigma_r\}$ denotes the set of singular points of L the goal is to find approximations of the images of representatives of standard generators of the fundamental group $\pi_1(\mathbb{P}^1 \setminus \Sigma)$ under the monodromy representation

$$\pi_1(\mathbb{P}^1 \setminus \Sigma, b) \to GL(V_0),$$

where V_0 is the vector space spanned by the normalized Frobenius basis. Therefor choose a point $b \in \mathbb{P}^1 \setminus \Sigma$ such that for $i \neq j$ the distance

$$d_{ij} := \inf\{p - \sigma_j | p \in l_i, i \neq j\}$$

of the lines l_i that connect b with any of the σ_j is not zero. The numbers r_j defined as

$$r_j = (\min \{\sigma_i - \sigma_j, d_{ij} | i, j = 1 \ldots, r \text{ for } i \neq j, \} \cup \{|\sigma_j - b|\})/3,$$

the smallest among them will be denoted by r_{min}. Denote the arguments of $b - \sigma_j$ by

$$\tau_j = \arg(\sigma_j - b)$$

and compute p_{jfin} and n_j as

$$p_{jfin} = \sigma_j - r_j \exp(2\pi i \tau_j) \text{ and } n_j = \lceil |p_{jfin} - b| / r_{min} \rceil.$$

Then points $\tilde{p}_{jk}$ are computed as

$$\tilde{p}_{jk} = b + k(p_{jfin} - b)/n \text{ for } k = 0, \ldots, n_j.$$

and among $\tilde{p}_{jk}$ the points p_{jk} are determined algorithmically as follows. Set $p_{j0} = \tilde{p}_{j0}$ and set $k, l = 1$, as long as $\tilde{p}_{jk} \neq \tilde{p}_{jn_j}$ determine $r_{jl} = \min \{p_{jl} - \sigma_m, m = 1, \ldots, r\}$. If $|\tilde{p}_{jk} - p_{jl}| > r_j l/3$, set $p_{jl} = \tilde{p}_{jk}$ and increase l and k by 1, otherwise increase k by 1. Finally set the value of $m_j = l$. Now it easy to determine the remaining p_{jk} as

$$p_{jm_j+l} = \sigma_j - r_j \exp(2\pi i(\tau_j + l/17)), l = 1, \ldots, 18$$

and

$$p_{jm_j+17+l} = p_{jm_j-l}, l = 1, \ldots m_j.$$

Points p_{0k} on the line connecting 0 and b are computed by the same equivariant choice with $\tilde{p}_{00} = 0$ and $\tilde{p}_{0sfin}$ and with the same selection rule as above. These choices are not optimal but they work fine and they assure convergence and contrary to Chapter 3 only a comparatively small amount of computations has to be done. Figure 4.6.3 indicates how paths and expansion points are chosen. Consider the series of hypergeometric differential operators

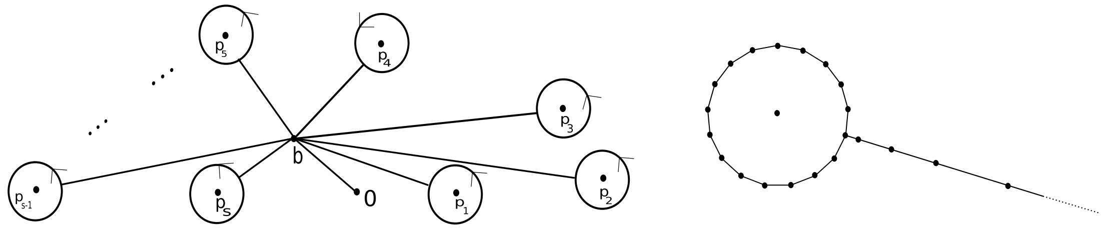

Figure 4.1: Choice of pathes for CY(3)-equations

$$L_k := \theta^k - 2^k z(2\theta + 1)^k, 2 \leq k \leq 8.$$

At 0 the monodromy is maximally unipotent and the monodromy entries of M_0 in the normalized Frobenius basis are

$$(M_0)_{i,j} = \begin{cases} \frac{1}{(i-j)!}, & i \geq j \\ 0, & else \end{cases}.$$

Using the methods from Chapter 2 with the choices fixed above or exploiting the rigidity of the monodromy tuple of L_k, we find the second generator of the monodromy group M_c^k. It is in each case M_c^k is a reflection $x \mapsto x + (-1)^{(k-1)(k-2)/2}\langle x, C_k\rangle_k C_k$ with

$$
\begin{aligned}
C_2 &= (4,0),\ C_3 = (8,0,1)\\
C_4 &= (16,0,8/3,-128\lambda_3)\\
C_5 &= \left(32,0,\frac{20}{3},-320\lambda_3,\frac{11}{36}\right)\\
C_6 &= \left(64,0,16,-768\lambda_3,\frac{16}{15},-192\lambda_3+2304\lambda_5\right)\\
C_7 &= \left(128,0,\frac{112}{3},-1792\lambda_3,\frac{49}{15},\frac{1568}{3}\lambda_3-5376\lambda_5,,-\frac{703}{360}+12544\lambda_3^2\right)\\
C_8 &= \left(256,0,\frac{256}{3},-4096\lambda_3,\frac{416}{45},-\frac{4096}{3}\lambda_3-12288\lambda_5,\right.\\
&\qquad\left.\frac{256}{945}+32768\lambda_3^2,-\frac{6656}{45}\lambda_3-4096\lambda_5-36864\lambda_7\right).
\end{aligned}
$$

From now we restrict to CY(3)-operators and set $\lambda := \lambda_3$. We call a conifold point of a CY(3)-equation *fruitful* if the corresponding reflection vector with respect to the normalized Frobenius basis is $(a, 0, b/24, c\lambda)$ for rational numbers a, b and c. Originally this observation was our reason to start the development of computer tools that are able to approximate generators of the monodromy group.

Example 4.7 An example of an operator with fruitful conifold points is

$$L_1 := \theta^4 - 4z\,(2\theta+1)^2\left(11\theta^2+11\theta+3\right) - 16z\,(2\theta+1)^2\,(2\theta+3)^2\,,$$

with Riemann scheme

$$\mathcal{R}(L_1) = \left\{\begin{array}{cccc} -\frac{11}{32}-\frac{5}{32}\sqrt{5} & 0 & -\frac{11}{32}+\frac{5}{32}\sqrt{5} & \infty\\ \hline 0 & 0 & 0 & \frac{1}{2}\\ 1 & 0 & 1 & \frac{1}{2}\\ 1 & 0 & 1 & \frac{3}{2}\\ 2 & 0 & 2 & \frac{3}{2} \end{array}\right\}.$$

It was computed as Picard-Fuchs equation of a mirror of a complete intersection $X = X(1,2,2) \subset \mathrm{Gr}(2,5)$ using toric degenerations in [BCFKS98]. The monodromy group is generated by

$$M_0 \approx \begin{pmatrix} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1/2 & 1 & 1 & 0\\ 1/6 & 1/2 & 1 & 1 \end{pmatrix},\ M_{-\frac{11}{32}+\frac{5}{32}\sqrt{5}} \approx \begin{pmatrix} 1-120\lambda & -17/6 & 0 & -20\\ 0 & 1 & 0 & 0\\ -17\lambda & -289/720 & 1 & -17/6\\ 720\lambda^2 & 17\lambda & 0 & 1+120\lambda \end{pmatrix},$$

$$M_{-\frac{11}{32}-\frac{5}{32}\sqrt{5}} \approx \begin{pmatrix} -480\lambda-10/3 & -46/3 & -40 & -80\\ 240\lambda+13/6 & 26/3 & 20 & 40\\ -92\lambda-299/360 & -529/180 & -20/3 & -46/3\\ 2880\lambda^2+52\lambda+169/720 & 92\lambda+299/360 & +240\lambda+13/6 & +480\lambda+16/3 \end{pmatrix}.$$

The linear map $M_{-\frac{11}{32}+\frac{5}{32}\sqrt{5}}$ is a symplectic reflection with reflection vector

$$C_1 = (20,0,17/6,-120\lambda)\,.$$

The vector corresponding to $M_{-\frac{11}{32}-\frac{5}{32}\sqrt{5}}$ is

$$C_2 = \left(40, -20, \frac{23}{3}, -\frac{13}{6} - 240\lambda\right).$$

Indeed, it can be checked that the numbers 20, 68, and -120 coincide with H^3, c_2H and c_3 of X. Recall also the identity $\int_X c_3 = \chi^{top}$.

We present some CY(3)-operators with unusual properties.

Example 4.8 The operator

$$L_1 = \theta^4 - 8z(2\theta+1)^2(5\theta^2+5\theta+2) + 192z^2(2\theta+1)(3\theta+2)(3\theta+4)(2\theta+3)$$

was constructed by M. Bogner and S. Reiter in [BR12], it has Riemann scheme

$$\mathcal{R}(L_2) = \left\{\begin{array}{cccc} 0 & x_1 & x_2 & \infty \\ \hline 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & \frac{2}{3} \\ 0 & 1 & 1 & \frac{4}{3} \\ 0 & 2 & 2 & \frac{3}{2} \end{array}\right\},$$

where the singular points $x_1 \approx 0.0115 - 0.0032i$ and $x_2 \approx 0.0115 + 0.0032i$ are the roots of $6912x^2 - 160X + 1$. Up to degree 200 the Yukawa coupling

$$K(q) = 1 + 8q - 5632q^3 - 456064q^4 - 17708032q^5 - 435290112q^6 - 1114963968q^7 + O\left(q^8\right)$$

has integral coefficients but the genus 0 instanton numbers seem not to be N-integral. Applying the monodromy approximation method we were not able to identify all entries of the monodromy matrices at the conifold points. Some of them seem to be integers and some conjecturally involve $\log(2)$. The traces of products of our approximations of M_0, M_{x_1} and M_{x_2} are very close to integers. This indicates that further points should be added to the characterization of CY(3)-equations, for example one should insist on the N-integrality of the genus 0-instanton numbers.

The next example shows that more than one fruitful conifold point can occur.

Example 4.9 The operator

$$\begin{aligned} L_3 :=& \theta^4 - z\left(2000\theta^4 + 3904\theta^3 + 2708\theta^2 + 756\theta + 76\right) \\ &+ z^2\left(634880\theta^4 + 634880\theta^3 - 213760\theta^2 - 186240\theta - 2832\right) \\ &- z^3\left(5120000\theta^4 + 245760\theta^3 - 378880\theta^2 + 61440\theta + 3072\right) + 4096z^4(2\theta+1)^4 \end{aligned}$$

associated to the degree 5 Pfaffian constructed by A. Kanazawa [Kan12] has Riemann scheme

$$\mathcal{R}(L_3) = \left\{\begin{array}{ccccc} 0 & x_1 & \frac{1}{16} & x_2 & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & 1 & \frac{1}{2} \\ 0 & 1 & 3 & 1 & \frac{1}{2} \\ 0 & 2 & 4 & 2 & \frac{1}{2} \end{array}\right\},$$

where x_i are the roots of $256x^2 - 1968x + 1$ and the monodromy at the two conifold points is described by the two reflection vectors

$$C_{x_1} = \left(5, 0, \frac{19}{12}, -100\lambda\right) \text{ and } C_{x_2} = \left(20, 0, \frac{1}{3}, -400\lambda\right).$$

Only x_1 is the closest nonzero singular point of L_3, but x_1 as well as x_2 are fruitful.

Another oddity occurs for the following operator, the potential Euler number is positive.

Example 4.10 Let

$$\begin{aligned}L_4 :=&\theta^4 - z\left(756\theta^4 + 1080\theta^3 + 810\theta^2 + 270\theta + 36\right)\\&+ z^2\left(174960\theta^4 + 419904\theta^3 + 440316\theta^2 + 215784\theta + 38880\right)\\&- 314928z^3(2\theta+1)^2(13\theta^2 + 29\theta + 20) + 34012224z^4(2\theta+1)^2(2\theta+3)^2.\end{aligned}$$

It has Riemann scheme

$$\mathcal{R}(L_4) := \left\{\begin{array}{cccc} 0 & \frac{1}{432} & \frac{1}{108} & \infty \\ \hline 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & 2 & 2 & \frac{3}{2} \end{array}\right\}$$

and one fruitful conifold point at $\frac{1}{432}$. The monodromy $M_{\frac{1}{432}}$ in the normalized Frobenius basis is a reflection in $(9, 0, \frac{5}{4}, 12\lambda)$. That suggest that c_3 equals 12, but it follows from the shape of the Hodge diamond and the definition of the Euler number, that this is not possible for Calabi-Yau manifolds with $h^{1,1} = 1$. For other examples of this kind, see the Tables 4.2 and C.2.

A similar incompatibility happens also for c_2 and H^3.

Example 4.11 The CY(3)-operator

$$\begin{aligned}L_5 :=&\theta^4 - z\left(73\theta^4 + 578\theta^3 + 493\theta^2 + 204\theta + 36\right)\\&- z^2\left(10440\theta^4 - 20880\theta^3 - 99864\theta^2 - 77184\theta - 21600\right)\\&+ z^3\left(751680\theta^4 + 4510080\theta^3 + 1829952\theta^2 - 1306368\theta - 933120\right)\\&+ z^4\left(272471040\theta^4 - 1067489280\theta^3 - 2997181440\theta^2 - 2463436800\theta - 67184640\right)\\&- z^5\left(1934917632(\theta+1)^4\right)\end{aligned}$$

has Riemann scheme

$$\mathcal{R}(L_5) = \left\{\begin{array}{cccccc} -\frac{1}{72} & 0 & \frac{1}{81} & \frac{1}{72} & \frac{1}{64} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 3 & 0 & 1 & 1 & 1 & 1 \\ 4 & 0 & 2 & 2 & 2 & 1 \end{array}\right\}.$$

There are three reflection vectors associated to L_5

$$C_{\frac{1}{81}} = (144, 0, 0, 1056\lambda), C_{\frac{1}{72}} = (432, -72, 0, 3168\lambda), C_{\frac{1}{64}} = (324, -108, 0, 2376\lambda),$$

and the monodromy at $M_{-\frac{1}{72}}$ is trivial. The vector $C_{\frac{1}{81}}$ belongs to a fruitful conifold point, but there is no Calabi-Yau manifold X with $h^{1,1} = 1$ and $c_3(X) = 1956$ and $c_2(X).H = 0$.

Extending the work of C. van Enckevort and D. van Straten [SE06], we were able to confirm the first part of Conjecture 2 for all differential equations in [Str12, AESZ10].
If a monodromy representation $\rho : \pi_1(\mathbb{P}^1 \setminus \Sigma) \to GL_n(\mathbb{C})$ comes from a geometric variation of Hodge structure, there must be a monodromy invariant lattice corresponding to integral cohomology of the underlying family of varieties and an integral representation

$$\rho_{\mathbb{Z}} : \pi_1(\mathbb{P}^1 \setminus \Sigma, t) \to \mathrm{Aut}(H^n(X_t, \mathbb{Z})).$$

In [DM06] C. Doran and J. Morgan pointed out recalling constructions of B. Green and M. Plesser [GP90], that from the viewpoint of mirror symmetry the difference between the ρ and $\rho_{\mathbb{Z}}$ is crucial. Namely, parallel to the construction of the mirror X' as resolution of the quotient X/G of the quintic

$$x_0^5 + \dots x_4^5 + 5\psi x_0 \cdot \dots \cdot x_4 = 0$$

as explained above two further families of Calabi-Yau manifolds can be constructed. Compare the quotients of the quintic by the two actions

$$\begin{aligned} g1 :& (x_0 : \ldots : x_4) \mapsto (x_0 : \mu x_1 : \mu^2 x_2 : \mu^3 x_3 : \mu^4 x_4) \\ g2 :& (x_0 : \ldots : x_4) \mapsto (x_0 : \mu x_1 : \mu^3 x_2 : \mu^1 x_3 : x_4), \end{aligned}$$

the first action is free and the quotient X_1 is smooth with $h^{1,1} = 1$ and $h^{2,1} = 21$, after a suitable resolution of singularities the second quotient X_2' has Hodge numbers $h^{1,1} = 21$ and $h^{2,1} = 1$. Therefor X_2' varies in a family over a one dimensional base, that can determined to be $\mathbb{P}^1 \setminus \Sigma$ and the corresponding Picard-Fuchs equation coincides with the Picard-Fuchs equation of X'. The difference is reflected by the lattices spanned by the periods. Stimulated by this observation they classified integral variations of Hodge structure which can underly families of Calabi-Yau threefolds over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Their classification is based on a special lattice. Namely, if the monodromy T_0 at 0 is maximal unipotent and if the monodromy T_1 at 1 is unipotent of rank 1 then set $N_i = T_i - Id$ choose a nonzero vector $v \in \ker N_0$. Furthermore define $m \in \mathbb{R}$ by $(N_0)^3(N_1 v) = -mv$ and if $m \neq 0$

$$B := \left\{ N_1(v), N_0(N_1(-v)), \frac{N_0^2(N_1(-v))}{m}, v \right\}.$$

Definition 4.11.1 (Doran-Morgan lattice) *The $\mathbb{Z}$-span of B is called* Doran-Morgan lattice.

In Appendix C we give a table of the form, that collects the data of CY(3)-equations with monodromy invariant Doran-Morgan lattice. The entries in reflection vector are labeled e_r^c, where r is the minimal positive integer such that the corresponding reflection is of the form $x \mapsto cx - \langle x, e_r^c \rangle e_r^c$.

<table>
<tr><td colspan="2">differential operator</td></tr>
<tr><td>H^3</td><td rowspan="5">Riemann symbol</td></tr>
<tr><td>c_2H</td></tr>
<tr><td>c_3</td></tr>
<tr><td>$|H|$</td></tr>
<tr><td>cr</td></tr>
</table>

Yukawa coupling $K(q)$
genus 0 instanton numbers
exponents in the holomorphic anomaly
genus 1 instanton numbers
geometric description
integral monodromy with respect to the Doran-Morgan lattice
reflection vectors

Table 4.1: Presentation of the data attached to a CY(3)-equation

In the tables below we omitted any reference to a differential equation since a searchable extended version of [AESZ10] is available online at

`http://www.mathematik.uni-mainz.de/CYequations.`

and some of the data is covered in Appendix C. The label *BKn* stands for number n from [BK10]. Sometimes it happens that the entries of the reflection vector corresponding to a reflection at a conifold point c has entries in $\mathbb{Q}(\lambda)$ only after an additional factor n is introduced as

$$x \mapsto x - n\langle x, C\rangle C.$$

For an interpretation of the factor n see Appendix A of [SE06]. In Table C.1 of Appendix C we denote the vector C with the symbol v_c^n. If we ignore the Doran-Morgan lattice and concentrate on $CY(3)$-equations with a fruitful conifold point, we find about 230 such equations in [Str12]. The corresponding values for H^3, c_2H, and c_3 are listed in Table 4.2. If a Calabi-Yau threefold with $h^{1,1}$ and matching topological invariants is known, a description is given in the column origin. To identify the corresponding CY(3)-equations and find references for the geometric realization check the column L. An $*$ indicates that the differential operator producing the triple (H^3, c_2H, c_3) has monodromy invariant Doran-Morgan lattices and hence appears in Appendix C. Otherwise the corresponding operator can be found in [Str12]. Remark that the dimension of the linear system $|H|$ equals $c_2H/12 + H^3/6$.

H^3	c_2H	c_3	L	origin	H^3	c_2H	c_3	L	origin
1	22	-120	$*$	**X(6,6)** $\subset\mathbb{P}^5$**(1,1,2,2,3,3)**	1	34	-288	$*$	**X(10)** $\subset\mathbb{P}^4$**(1,1,1,2,5)**
1	46	-484	$*$	**X(2,12)** $\subset\mathbb{P}^5$**(1,1,1,1,4,6)**	2	-4	432	$*$	—
2	20	-16	$*$	—	2	32	-156	$*$	**X(3,4)** $\subset\mathbb{P}^5$**(1,1,1,1,1,2)**
2	44	-296	$*$	**X(8)** $\subset\mathbb{P}^4$**(1,1,1,1,4)**	3	30	-92	$*$	—
3	42	-204	$*$	**X(6)** $\subset\mathbb{P}^4$**(1,1,1,1,2)**	4	-8	640		—
4	16	136		—	4	28	-60	$*$	—
4	28	-32	$*$	—	4	28	-18		—
4	28	24		—	4	40	-144	$*$	**X(4,4)** $\subset\mathbb{P}^5$**(1,1,1,1,2,2)**
4	52	-256	$*$	**X(2,6)** $\subset\mathbb{P}^5$**(1,1,1,1,1,3)**	5	38	-100	$*$	**X_5 Pfaffian**
5	38	-102	$*$	—	5	50	-200	$*$	**X(5)** $\subset\mathbb{P}^4$
5	62	-310	$*$	—	6	36	-72	$*$	—
6	36	-64	$*$	—	6	36	-56		—
6	48	-156	$*$	**X(4,6)** $\subset\mathbb{P}^5$**(1,1,1,2,2,3)**	6	72	-366		—
6	72	-364		—	7	34	-36		—
7	46	-120	$*$	**X_7 Pfaffian**	8	8	216		—
8	32	-20	$*$	—	8	32	-8	$*$	—
8	32	6		—	8	32	48		—
8	44	-92	$*$	—	8	44	-78		—
8	56	-176	$*$	**X(2,4)** $\subset\mathbb{P}^5$	8	80	-400		—
8	92	-470		—	9	30	12	$*$	—
9	54	-144	$*$	**X(3,3)** $\subset\mathbb{P}^5$	10	40	-50	$*$	—

H^3	c_2H	c_3	L	origin	H^3	c_2H	c_3	L	origin
10	40	-32	$*$	—	10	52	-120		—
10	52	-116	$*$	**X_{10} Pfaffian**	10	64	-200	$*$	**$X \to B_5$ [2:1]**
11	50	-92	$*$	—	11	50	-90	$*$	—
11	50	-88	$*$	—	12	24	-144		—
12	24	80		—	12	24	304		—
12	36	52	$*$	—	12	48	-88		—
12	48	-68	$*$	—	12	48	-60	$*$	—
12	60	-144	$*$	**$X(2,2,3) \subset \mathbb{P}^6$**	12	72	-224		—
12	132	-676		—	13	58	-120	$*$	**5×5 Pfaffian $\subset \mathbb{P}^6$**
14	8	-100		—	14	56	-100	$*$	—
14	56	-98	$*$	—	14	56	-96	$*$	**$X \to B$ [2:1]**
15	54	-80	$*$	—	15	54	-76	$*$	**$X \to B$ [2:1]**
15	66	-150	$*$	**$X(1,1,3) \subset Gr(2,5)$**	16	16	-40		—
16	16	320		—	16	40	40		—
16	40	68		—	16	52	-72	$*$	**$X \to B$ [2:1]**
16	52	-58	$*$	—	16	52	-44	$*$	—
16	64	-128	$*$	**$X(2,2,2,2) \subset \mathbb{P}^7$**	16	64	-72		—
16	88	-268		—	17	62	-108	$*$	**$X \to B$ [2:1]**
18	-36	960		—	18	12	-92		**$X \to B$ [2:1]**
18	48	-52		—	18	60	-92	$*$	—
18	60	-90	$*$	**$X'(2,2,3) \subset \mathbb{P}^6$**	18	60	-88	$*$	**$X'(2,2,3) \subset \mathbb{P}^6$**
18	60	-64		—	18	72	-162		—
18	72	-158	$*$	—	18	72	-156		—
19	58	-76		**BK 40**	20	44	12		—
20	56	-72		—	20	56	-64	$*$	—
20	68	-128		—	20	68	-120	$*$	**$X(1,2,2) \subset Gr(2,5)$**
21	66	-104	$*$	**$X \to B$ [2:1]**	21	66	-102	$*$	—
21	66	-100	$*$	**$X \to B$ [2:1]**	22	16	-92	$*$	—
22	64	-92		**BK 56**	22	64	-86	$*$	—
22	64	-84	$*$	—	23	62	-74	$*$	—
24	48	48		—	24	48	160		—
24	72	-120		—	24	72	-116	$*$	**$X(1,1,1,1,1,1,2) \subset X_{10}$**
24	72	-112		—	24	84	-162		—
24	120	-344		—	25	70	-100	$*$	—
26	20	-100		—	26	68	-86	$*$	—
26	68	-72		—	28	40	-144		—
28	64	-88		—	28	76	-116	$*$	**$X(1,1,1,1,2) \subset Gr(2,6)$**
29	74	-104	$*$	**BK 50**	29	74	-100	$*$	—
30	72	-90		—	30	72	-86	$*$	—
30	72	-80		—	30	72	-76		—
32	20	-74		—	32	32	192		—
32	32	304		—	32	56	80		—
32	80	-116	$*$	**$X(1,1,2) \subset LGr(3,6)$**	32	80	-88		—
33	6	-276		—	33	78	-108		—
34	76	-90	$*$	—	34	76	-88	$*$	—
36	0	-204		—	36	60	-6		—
36	60	36		—	36	72	-128		—
36	72	-72	$*$	—	36	72	-60		—
36	72	-48		—	36	84	-128		—
36	84	-120	$*$	**$X(1,2) \subset X_5$**	36	84	-16		—
36	108	-156		—	38	80	-92	$*$	—
40	52	180		—	40	64	-116		—
40	64	-16		—	40	76	-30		—
40	88	-128		—	42	84	-112		—
42	84	-108		—	42	84	-98	$*$	—
42	84	-96	$*$	**$X(1,1,1,1,1,1) \subset Gr(3,6)$**	42	84	-84		—
44	92	-128	$*$	**BK 58**	46	88	-106		—
47	86	-90	$*$	—	48	72	12		—
48	72	28		—	48	84	-92		—
48	84	-86		—	48	96	-128		—

H^3	c_2H	c_3	L	origin	H^3	c_2H	c_3	L	origin
48	96	-16		—	52	-8	-172		—
52	88	-144		—	52	100	-144		—
54	72	-18		—	56	80	-64	$*$	—
56	92	-92	$*$	**X(1,1,1,1) $\subset$ F$_1$(Q$_5$)**	56	92	-78		—
57	90	-84	$*$	**Tjøtta's example**	60	84	100		—
60	96	-132		—	60	96	-110		—
61	94	-86	$*$	—	64	64	160		—
64	112	-36		—	68	20	880		—
70	100	-100		—	72	0	816		—
72	72	72		—	72	72	192		—
72	72	216		—	72	72	264		—
72	120	-128		—	80	104	-88		—
80	128	-176		—	84	108	-108		—
90	108	-90		—	91	106	-78	$*$	**BK 41**
96	48	472		—	96	96	-32		—
96	108	-76		**BK 39**	97	106	-64	$*$	**BK 24**
102	108	-74		**BK 38**	112	160	-464		—
112	160	-296		—	116	116	-80	$*$	**BK 43**
117	114	-72	$*$	**BK 37**	120	120	-80		—
128	128	-128		—	128	176	-296		—
132	132	-408		—	132	132	-96		—
144	0	-288		—	144	0	1056		—
144	192	-400		—	150	120	-50		—
153	90	-126		—	160	160	-128		—
162	108	216		—	162	132	-88		—
208	160	-128		—	216	48	-328		—
216	144	-72		—	230	140	-80		
252	72	-160		—	288	216	-216		—
324	276	-528		—	350	-280	-2450		**7×7 Pfaffian $\subset \mathbb{P}^6$**
378	-252	-882		**X(1,1,1,1,1,1,1) $\subset$ Gr(2,7)**	396	120	-464		—
400	-320	-1600		**X$_{25}$ Pfaffian**	432	288	-1152		—
456	312	-1104		—	460	40	-160	$*$	—
500	140	-1100		—	504	240	-304		—

Table 4.2: Potential topological data

In all cases the fruitful conifold point that yielded the numbers in Table 4.2 was positive, real and the non-zero singular point closest to zero. Some Calabi-Yau 3-folds with numerical invariants (c_2H, H^3, c_3) and $h^{1,1} = 1$ that do not appear in the tables above are known. We list this numbers and the places where the associated 3-folds were constructed in Table 4.3, compare it with the list by G. Kapustka [Kap13].

H^3	c_2H	c_3	source	H^3	c_2H	c_3	source
7	58	-156	[Kap09]	10	62	-116	[Kap09]
12	84	-120	[Kap09]	13	82	-102	[Kap09]
14	80	-84	[Kap09]	14	80	-86	[Kap09]
14	68	-120	[Kap09]	15	78	-68	[Kap09]
15	54	-78	[Ton04]	15	54	-84	[Lee08]
16	52	-60	[Kap13, Ton04]	17	50	-44	[Kap13, Ton04]
17	50	-64	[Lee08]	18	60	-84	[Lee08]
19	58	-74	[Kap13]	19	58	-76	[Lee08]
20	56	-60	[Kap13]	22	64	-92	[Lee08]
24	60	-50	[Kap13]	29	74	-96	[Lee08]
30	72	-96	[Lee08]	34	76	-96	[Kap09]
34	76	-98	[Kap09]	35	?	-50	[KK10]

Table 4.3: Numerical invariants that do not appear in the database

Remark that in the list by M. Kreuzer and V. Batyrev [BK10] more cases then these displayed in Table 4.2 are, but it includes some subtleties we do not fully understand, hence its entries are omitted in Table 4.3.

A MonodromyApproximation Manual

```
MonodromyApproximation[Monodromy]  - approximation of generators of the monodromy
                                     group of a Fuchsian differential operator with
                                     polynomial coefficients in a fixed basis
Calling Sequence
   Monodromy(dop, basept, singularities, soi, nterms,
   NumberOfDigits,IntermediateBasePoint)

Parameters
   dop                    - Fuchsian differential operator with polynomial
                            coefficients or a CYeqn
   basept                 - an ordinary or regular singular point of dop
   singularities          - (optional) a list containing the finite regular
                            singular points of dop
   soi                    - (optional) a list of finite regular singular points
                            around which the monodromy shall be computed
   nterms                 - (optional) degree up to which elements of the
                            Frobenius basis are approximated, default = 50
   NumberOfDigits         - (optional) Number of Digits used in the involved
                            floating point computations, default = 100
   IntermediateBasePoint  - (optional) Replaces basept by IntermediateBasePoint
                            in the choice of paths and computes finally a
                            base change from Intermediate Basepoint to basept,
                            default = none

  Description
    Given a Fuchsian differential operator of order n i.e.  an element of C[x]<dx>
    with highest dx degree n this procedure computes an approximation of the
    monodromy along standard loops gamma_i, i=1,..,nops(soi) with basepoint
    basept, encircling the i-th of the singular points in soi.  If soi is not
    given it is set to singularities, if moreover singularities and soi are not
    passed to this procedure both values are set to a list that contains all
    finite singular points of dop.  The monodromy matrices depend on a choice of a
    basis, here all the matrices are expressed with respect to a Frobenius basis
    (f_1,..,f_n) at basept.  If basept is an ordinary point, the Frobenius basis
    is made unique by the requirement that diff(f_i(x) ,x , j)|x=basept =
    delta_ij,i,j=0..n-1.  At a regular singular point it coincides with the basis
    obtained in [1] on Page 403.  The leading coefficient of the solution
    corresponding to the local exponent at basept, whose real part has biggest
    absolute value amongst the absolute values of all real parts of local
    exponents, is chosen as one.  The loops gamma_i encircle exactly the i-th
    entry of soi, and they are homotopic to loops composed of l_i, 1/l_i and c_i,
    where l_i is a straight line from l_i to (l_i + soi[i])*(1-eps), 1/l_i is l_i
    with opposed direction and c_i is a circle with center soi[i] and radius eps.
    The real number eps is smaller than half the distance between any two elements
    of singularities.  If an entry soi[j] of soi lies on gamma_i, chosen as above,
    l_i and hence 1/l_i are modified such that l_i passes soi[j] on the right.
    The output out is an Array, where every entry is a list that contains an
    element soi[j] of soi as first entry and the approximated monodromy along
    gamma_j as second entry.
```

The monodromy around infinity can be obtained as the product out[n][2]*...*out[1][2], where n=ArrayTools[Size](out)[2]. Notice that NumberOfDigits influences the speed of the computations.

[1] E.L.Ince, Ordinary Differential Equations, Dover Publications.

Examples

```
>L:=(z^2-16*z^3) *dz^2+(z-32* z^2) *dz-4*z:
>singularpoints(L);

              [0, 1/16, infinity]

>Monodromy(L,0,NumberOfDigits=10);
 part1
 appr.50.000000% finished
 appr.100.000000% finished
 part2:finished
 [[   [1.0000000035+0.1394273099*10^-9I -1.0000000029-0.1787021475*10^-8I          ]
 [[0, [                                                                             ],
 [[   [-0.1562919793*10^-8-0.60349839708*10^-9I 1.00000000125+0.1374597546*10^-8I   ]

 [       [1.0000000086+0.4572612526*10^-8I -0.1538629501*10 ^-9-0.491967522*10^-9I   ]]]
 [[1/16, [                                                                           ]]]
 [       [4.0000000035-0.2936308422*10^-7I 0.9999999896-0.5788312051*10^-8I          ]]]
```

See Also

MonodromyApproximation[FrobeniusBasis], MonodromyApproximation[singularpoints]

B List of Accessory Parameters C and Location of the Singular Point A for $(1;e)$ and $(0;2,2,2,q)$-Type Equations

In the tables below for an algebraic number σ we denote is minimal polynomial by f_σ, we also use the notation

$$w_d = \begin{cases} (1+\sqrt{d})/2 & d \equiv_4 1 \\ \sqrt{d}/2 & d \equiv_4 0 \end{cases}.$$

If with our choices for A and C and generators of the monodromy group, we do not match the trace triples from [Tak83, ANR03] exactly, but only up to elementary transformation this is indicated with a superscript star $(\cdot,\cdot,\cdot)^*$.

B.1 (1;e)-Type

2/1/6/i	$(5,12,25)$ $A = \frac{3}{128}$, $C = -\frac{13}{2^{11}}$
2/1/6/ii	$(6,8,12)$ $A = \frac{1}{4}$, $C = -\frac{1}{64}$
2/1/14	$(7,7,9)$ $A = \frac{1}{2} + \frac{13}{98}i\sqrt{7}$, $C = -\frac{3}{128} - \frac{15}{6272}i\sqrt{7}$
2/5/4i	$(2w_5+2, 4w_5+4, 6w_5+4)$ $A = (2-\sqrt{5})^2$, $C = \frac{-5+2\sqrt{5}}{2^6}$
2/5/4ii	$(3w_5+2, 3w_5+2, 4w_5+4)$ $f_A = 8388608x^6 - 25165824x^5 + 3361993x^4 + 35219054x^3 + 3361993x^2 - 25165824x + 8388608,\ A \sim 0.5 - 0.1258i$ $f_C = 219902325555200x^6 + 30923764531200x^5 + 1624023040000x^4 + 39005952000x^3 + 433866640x^2 + 2367480x + 11881,\ C \sim -0.0234 + 0.0023i$
2/5/4iii	$(3w_5+2, 3w_5+3, 3w_5+3)$ $A = \frac{243}{1024} + \frac{171}{1024}i\sqrt{15}$, $C = -\frac{293}{16384} - \frac{45}{16384}i\sqrt{15}$
2/8/7i	$(w_8+3, 8w_8+12, 9w_8+13)$ $f_A = x^4 - 7058x^3 + 13771x^2 - 7058x + 1,\ A \sim 0.0001$ $f_C = 16384x^4 + 294912x^3 + 22144x^2 + 448x + 1,\ C \sim -0.0025$
2/8/7ii	$(2w_8+3, 3w_8+5, 3w_8+5)$ $f_A = x^4 + 7054x^3 - 7397x^2 + 686x - 343,\ A \sim 0.02431 + 0.2192i$ $f_C = 16777216x^4 - 298844160x^3 - 19570688x^2 - 316672x - 1679,$ $C \sim -0.0104 - 0.00404i$

2/8/2	$(2w_8+4, 2w_8+4, 4w_8+6)$ $A=-1,\ C=0$
2/12/3	$(w_{12}+3, 4w_{12}+8, 5w_{12}+9)$ $A=-56\sqrt{3}+97,\ C=\frac{1}{4}\sqrt{3}-\frac{7}{16}$
2/12/2	$(2w_{12}+4, 2w_{12}+4, 2w_{12}+4)$ $A=\frac{1}{2}+\frac{1}{2}i\sqrt{3},\ C=-\frac{3}{128}-\frac{1}{128}i\sqrt{3}$
2/13/36	$(w_{13}+2, 8w_{13}+12, 9w_{13}+12)$ $A=\frac{1}{2}-\frac{71}{512}\sqrt{13},\ C=-\frac{3}{128}+\frac{49}{8192}\sqrt{13}$
2/13/4	$(w_{13}+3, 3w_{13}+4, 3w_{13}+4)$ $f_A=8388608x^6-25165824x^5+1731245917x^4-3420548794x^3+1731245917x^2$ $-25165824x+8388608,\ A\sim 0.0024+0.07008i$ $f_C=8796093022208x^6+1236950581248x^5+167777206272x^4+11199264768x^3$ $+304328016x^2+2956824x+9477,\ C\sim -0.0075-0.0013i$
2/17/2i	$(w_{17}+2, 4w_{17}+8, 5w_{17}+8)$ $A=\frac{897}{2048}-\frac{217}{2048}\sqrt{17}\ ,\ C=-\frac{5}{256}+\frac{1}{256}\sqrt{17}$
2/17/2ii	$(w_{17}+3, 2w_{17}+4, 3w_{17}+5)$ $A=\frac{1151}{2048}-\frac{217}{2048}\sqrt{17}\ ,\ C=-\frac{7}{256}+\frac{1}{256}\sqrt{17}$
2/21/4	$(w_{21}+2, 3w_{21}+6, 3w_{21}+7)$ $f_A=512x^4-72577x^3+8532738x^2-72577x+512,\ A\sim 0.004253+0.006475i$ $f_C=536870912x^4+386334720x^3+210057216x^2+1990512x+4761,$ $C\sim -0.0048-0.00013i$
2/24/3	$(w_{24}+3, 2w_{24}+5, 2w_{24}+6)$ $f_A=x^4-10x^3+99x^2-10x+1,\ A\sim 4.9492+8.5731i$ $f_C=1327104x^4+184320x^3+14464x^2+224x+1,\ C\sim -0.0602-0.0704i$
2/33/12	$(w_{33}+3, w_{33}+4, 2w_{33}+5)$ $A=-\frac{27}{256}+\frac{21}{256}\sqrt{33},\ C=-\frac{419}{36864}-\frac{17}{12288}\sqrt{33}$
2/49/56	$(\rho^2+\rho, 3\rho^2+2\rho-1, 3\rho^2+2\rho-1),\ \rho=2\cos(\pi/7)$ $A=\frac{7}{16384}+\frac{181}{16384}i\sqrt{7},\ C=-\frac{1609}{262144}-\frac{59}{262144}i\sqrt{7}$
2/81/1	$(\rho^2+\rho+1, \rho+1, \rho+1),\ f_\rho=x^2+3x+1,\ \rho\sim 1.8749$ $f_A=536870912x^6-1610612736x^5+4380314097x^4-6076273634x^3+4380314097x^2$ $-1610612736x+536870912,\ A\sim 0.1099+0.4559i$ $f_C=562949953421312x^6+79164837199872x^5+4990992187392x^4$ $+3705530256x^2+41846328x+205209,\ C\sim -0.0143-0.0078i$
2/148/1i	$(\rho^2+\rho, \rho^2+\rho, \rho^2+2\rho+1),\ f_\rho=x^3-x^2-3x+1,\ \rho\sim 2.1700$ $f_A=1048576x^6-3145728x^5+11894784x^4-18546688x^3+11134976x^2$ $-2385920x-1,\ A\sim 0.5000+0.1796i$ $f_C=1099511627776x^6+154618822656x^5+9663676416x^4+339738624x^3$ $+6422528x^2+52224x+59,\ C\sim -0.0234-0.0033i$

2/148/1ii	$(-\rho^2+\rho+4, -12\rho^2+8\rho+40, -13\rho^2+9\rho+42),\ f_\rho = x^3-x^2-3x+1,\ \rho \sim 0.3111$ $f_A = x^6-2385926x^5+794639x^4+2133996x^3+794639x^2-2385926x+1$ $A \sim 0.2386\ 10^7$ $f_C = 68719476736x^6+218667522457600x^5+16703513690112x^4+403826540544x^3$ $+3574460416x^2+28113024x+32041,\ C \sim -3181.9549$
2/148/1iii	$(\rho^2-2\rho+1, \rho^2-38\rho+1, \rho^2-3\rho+1),\ f_\rho = x^3-x^2-3x+1,\ \rho \sim -1.4811$ $f_A = x^6+2385920x^5-11134976x^4+18546688x^3-11894784x^2+3145728x$ $-1048576,\ A \sim 0.06202+0.3467i$ $f_C = 1048576x^6-3336306688x^5-527106048x^4-31684608x^3-888320x^2$ $-11472x-59,\ C \sim -0.01256-0.006139i$
2/229/8i-iii	$f_A = 70368744177664x^6-211106232532992x^5+251740951674880x^4$ $-151638182461440x^3+41307815036519x^2-673095894631x+128$ $f_C = 18446744073709551616x^6+2594073385365405696x^5+138442257832345600x^4$ $+3479181207797760x^3+40391778507776x^2+158376087888x+94437631$ $f_\rho = x^3-4x-3$ $(\rho+2, 8\rho^2+16\rho+4, 8\rho^2+17\rho+4),\ \rho \sim 2.1149$ $A = 0.5258\ 10^{10},\ C = -0.3790\ 10^7$ $(-\rho^2+5, -3\rho^2+\rho+13, -4\rho^2+\rho+16),\ \rho \sim -0.2541$ $A \sim 0.0173,\ C \sim -0.0058$ $(\rho^2-2\rho, \rho^2-2\rho, \rho^2-2\rho+1),\ \rho \sim -1.8608$ $A \sim 0.5000+0.5571i,\ C \sim -0.0234-0.0094i$
2/725/16i-ii	$f_A = 1125899906842624x^{12}-6755399441055744x^{11}+13682101105429390242 6112x^{10}$ $-684104993346974635786240x^9+1757642980655831621406585093x^8$ $-7026467292737554031663236116x^7+10538743192140415080955291166x^6$ $-7026467292737554031663236116x^5+1757642980655831621406585093x^4$ $-684104993346974635786240x^3+136821011054293902426112x^2$ $-6755399441055744x+1125899906842624$ $f_C = 1237940039285380274899124224x^{12}+348170636049013202315378688x^{11}$ $+86315309702998394672527704064 0x^{10}+20229449440217542609059446784 0x^9$ $+2777934318122908292672787578 88x^8+494193185232996914384150200 32x^7$ $+35931156922477161162746101 76x^6+129328085927179185372463104x^5$ $+2340174744709681011356928x^4+19846988436629919916800x^3$ $+82407159639508548064x^2+162154973676768816x+120863826843029$ $f_\rho = x^4-x^3-3x^2+x-1$ $(-\rho^3+2\rho^2+\rho, -2\rho^3+5\rho^2-\rho+1, -2\rho^3+5\rho^2-\rho+1),\ \rho \sim -1.3356$ $A \sim 0.00003+0.0088i,\ C \sim -0.0048-0.00018i$ $(9\rho^3-13\rho^2-21\rho+19, \rho^3-2\rho^2-2\rho+4, 9\rho^3-13\rho^2-21\rho+19)^*,\ \rho \sim -0.4772$ $A \sim 0.9999-0.00009i\ ,\ C \sim -0.0444+0.2017i$
2/1125/16	$(\rho^2-\rho, 9\rho^2-2\rho+1, -\rho^3+\rho^2+\rho+1),\ f_\rho = x^4-x^3-4x^2+4x+1,\ \rho \sim -1.9562$

	$f_A = 2097152x^4 - 16009797x^3 + 61379722x^2 - 16009797x + 2097152$ $A \sim 3.6819 + 3.6993i$ $f_C = 2199023255552x^4 + 297931571200x^3 + 16036157440x^2 + 278544560x + 1551245$ $C \sim -0.0553 - 0.0363i$
3/1/15	$(5, 16, 20)$ $A = \frac{1}{81}$, $C = -\frac{1}{2^{13}4}$
3/1/10	$(6, 10, 15)$ $A = \frac{5}{32}$, $C = -\frac{67}{2^9 3^2}$
3/1/6ii	$(7, 8, 14)$ $A = \frac{32}{81}$, $C = -\frac{31}{1296}$
3/1/6i	$(8, 8, 9)$ $A = 1/2 + \frac{7}{16}\, i\sqrt{2}$, $C = -1/36 - \frac{5}{576}\, i\sqrt{2}$
3/5/9	$(2w_5 + 2, 6w_5 + 4, 8w_5 + 5)$ $A = \frac{1}{2} - \frac{19}{128}\sqrt{10}$, $C = -\frac{1}{36} + \frac{29}{4608}\sqrt{10}$
3/5/5	$(3w_5 + 2, 4w_5 + 3, 4w_5 + 3)$ $A = \frac{640}{6561} + \frac{1264}{6561}\, i\sqrt{5}$, $C = -\frac{431}{26244} - \frac{26}{6561}\, i\sqrt{5}$
3/8/9	$(2w_8 + 3, 4w_8 + 6, 4w_8 + 6)$ $A = \frac{2}{243} + \frac{22}{243}\, i\sqrt{2}$, $C = -\frac{5}{486} - \frac{1}{486}\, i\sqrt{2}$
3/12/3	$(2w_{12} + 4, 2w_{12} + 4, 4w_{12} + 7)$ $A = \frac{1}{2}$, $C = -\frac{1}{36}$
3/13/13i	$(2w_{13} + 3, 2w_{13} + 3, 3w_{13} + 4)$ $f_A = 47775744x^4 - 95551488x^3 + 63231232x^2 - 15455488x - 81$ $A \sim 0.5000 - 0.2710i$ $f_C = 12230590464x^4 + 1358954496x^3 + 48844800x^2 + 616448x + 991$ $C \sim -0.0277 + 0.0059i$
3/13/13ii	$(w_{13} + 2, 12w_{13} + 16, 13w_{13} + 17)$ $f_A = 481x^4 - 15455812x^3 - 16864282x^2 - 15455812x + 81,\ A \sim 90813.585$ $f_C = 1679616x^4 + 599664384x^3 + 10297440x^2 + 171632x + 289,\ C \sim -357.00751$
3/17/36	$(w_{17} + 2, 6w_{17} + 10, 7w_{17} + 11)$ $A = \frac{59113}{64} - \frac{14337}{64}\sqrt{17}$, $C = -\frac{29087}{9216} + \frac{783}{1024}\sqrt{17}$
3/21/3	$(w_{21} + 2, 4w_{21} + 8, 5w_{21} + 9)$ $A = \frac{8-3\sqrt{7}}{2^4}$, $C = \frac{5\sqrt{7}-16}{2^6 3^2}$
3/28/18	$(w_{28} + 3, 2w_{28} + 6, 3w_{28} + 8)$ $A = \frac{1}{2} - \frac{5}{32}\sqrt{7},\ C = -1/36 + \frac{115}{18432}\sqrt{7}$
3/49/1	$(\rho^2 + \rho, 4\rho^2 + 3\rho - 2, 4\rho^2 + 3\rho - 2),\ \rho = 2\cos(\pi/7)$

<table>
<tr><td></td><td>$f_A = 1594323x^6 - 4782969x^5 + 6704530690x^4 - 13401089765x^3 + 6704530690x^2$
$- 4782969x + 1594323,\ A \sim 0.0001 + 0.0154i$
$f_C = 8264970432x^6 + 1377495072x^5 + 1393200000x^4 + 147714120x^3 + 4860160x^2$
$+ 44242x + 121,\ C \sim -0.0061 - 0.0003i$</td></tr>
<tr><td>3/81/1</td><td>$(\rho^2, \rho^2, \rho^2),\ \rho = \frac{-1}{2\cos(5\pi/9)}$
$A = \frac{1}{2} + \frac{1}{2}i\sqrt{3}\ , C = -\frac{1}{36} - \frac{1}{108}\, i\sqrt{3}$</td></tr>
<tr><td>4/8/98</td><td>$(3 + \sqrt{2}, 20 + 12\sqrt{2}, 21 + 14\sqrt{2})$
$A = \frac{1}{2} - \frac{181}{512}\sqrt{2}\ , C = -\frac{15}{512} + \frac{155}{8192}\sqrt{2}$</td></tr>
<tr><td>4/8/7i</td><td>$(4 + \sqrt{2}, 8 + 4\sqrt{2}, 10 + 6\sqrt{2})$
$A = \frac{3}{4} - \frac{1}{2}\sqrt{2}\ , C = -\frac{1}{32} + \frac{1}{64}\sqrt{2}$</td></tr>
<tr><td>4/8/2i+iii</td><td>$f_A = 131072x^4 - 420864x^3 + 425777x^2 - 9826x + 4913$
$f_C = 2199023255552x^4 + 271656681472x^3 + 11168448512x^2 + 154082816x + 702961$

$(3 + 2\sqrt{2}, 7 + 4\sqrt{2}, 7 + 4\sqrt{2})$
$A \sim 0.0058582 + 0.10808i,\ C \sim -0.010254 - 0.0026200i$
$(5 + 2\sqrt{2}, 5 + 2\sqrt{2}, 6 + 4\sqrt{2})$
$A \sim 1.5996 + 0.8002i,\ C \sim -0.0515 + 0.0141i$</td></tr>
<tr><td>4/8/2ii</td><td>$(3 + 2\sqrt{2}, 6 + 4\sqrt{2}, 9 + 4\sqrt{2})$
$f_A = 531441x^8 - 64796436x^7 + 3009036438x^6 - 6026830596x^5 + 3361388930x^4$
$- 423440384x^3 + 150933504x^2 - 8388608x + 2097152,\ A \sim -0.0278 - 0.1643I$
$f_C = 9814425580665536312770 56x^8 + 602610274368410534019072x^7$
$+ 14693232753461671074 2016x^6 + 10415653268576658259968x^5$
$+ 321388066379823906816x^4 + 4937075259269087232x^3 + 40344446803654656x^2$
$+ 169641680184576x + 292036636097,\ C \sim -0.0066 + 0.0022i$</td></tr>
<tr><td>4/8/7ii</td><td>$(4 + 2\sqrt{2}, 6 + 2\sqrt{2}, 8 + 5\sqrt{2})$
$A = -11 + 8\sqrt{2}\ , C = \frac{17}{256} - \frac{1}{16}\sqrt{2}$</td></tr>
<tr><td>4/2624/4i-ii</td><td>$f_A = 2251799813685248x^{12} - 13510798882111488x^{11} + 30030912007767588864x^{10}$
$- 15003071104908525568 0x^9 + 1906092905276826495 77x^8$
$+ 1375984853960777094 04x^7 - 4163911839669348466 02x^6$
$+ 1375984853960777094 04x^5 + 1906092905276826495 77x^4$
$- 15003071104908525568 0x^3 + 30030912007767588864x^2$
$- 13510798882111488x + 2251799813685248$
$f_C = 633825300114114700748351602688x^{12} + 222829207071368449481842360320x^{11}$
$+ 30627469077907573090759055769 6x^{10} + 82716199087120919896994611200x^9$
$+ 9914106848589680637422600192x^8 + 648609089172833816197201920x^7$
$+ 24732547659337126861864960x^6 + 549364377983268991795200x^5$
$+ 6738722599361240434944x^4 + 41208401069204094720x^3$
$+ 124531476866368864x^2 + 245447096786160x + 446471492969$

$(\rho + 2, (1 + 2\sqrt{2})\rho + 5 + 2\sqrt{2}, (1 + 2\sqrt{2})\rho + 5 + 2\sqrt{2}),\ \rho = (1 + \sqrt{13 + 8\sqrt{2}}/2)$
$A \sim 0.00004 + 0.0086i, C \sim -0.0057 - 0.0002i$
$(\rho^2, \rho^2, (\rho + 1)^2),\ \rho = (1 + \sqrt{2} + \sqrt{7 + 2\sqrt{2}})/2$</td></tr>
</table>

	$A \sim 0.5000 + 0.0359i,\ C \sim -0.0292 - 0.00087i$
4/2304/2	$(\rho^2, \rho^2, \rho^2),\ \rho = (2 + \sqrt{2} + \sqrt{6})/2$ $A = \frac{1}{2} - \frac{1}{2}i\sqrt{3}\ , C = -\frac{15}{512} + \frac{5}{512}\,i\sqrt{3}$
5/5/5i-ii	$f_A = 15625x^4 - 21726500x^3 + 130650166x^2 - 21726500x + 15625$ $f_C = 4000000x^4 + 21216000x^3 + 2406496x^2 + 41008x + 121$ $(w_5 + 3, 12w_5 + 8, 14w_5 + 9)$ $A \sim 0.0007,\ C \sim -0.0037$ $(2w_5 + 3, 4w_5 + 4, 7w_5 + 5)$ $A \sim 0.1704,\ C \sim -0.01628$
5/5/180	$(2w_5 + 2, 6w_5 + 6, 9w_5 + 6)$ $A = \frac{3}{128}\ , C = -\frac{397}{51200}$
5/5/5iii	$(3w_5 + 2, 4w_5 + 4, 4w_5 + 5)$ $A = \frac{8}{125} + \frac{44}{125}\,i\ , C = -\frac{2}{125} - \frac{1}{125}\,i$
5/5/9	$(3w_5 + 3, 3w_5 + 3, 5w_5 + 5)$ $A = 1/2 + \frac{7}{48}\,i\ , C = -\frac{3}{100} - \frac{17}{4800}\,i$
5/725/25i-ii	$f_A = 1220703125x^{12} - 7324218750x^{11} + 12831285035060156 25x^{10}$ $- 6415642450391406250x^9 + 41130860183010641803 4x^8$ $- 1606740552698643640886x^7 + 2401128929644272961329x^6$ $- 1606740552698643640886x^5 + 411308601830106418034x^4$ $- 6415642450391406250x^3 + 12831285035060156 25x^2$ $- 7324218750x + 1220703125$ $f_C = 19531250000000000000x^{12} + 70312500000000000000x^{11}$ $+ 1177522415625000000000x^{10} + 353254404375000000000x^9$ $+ 727714526413400000000x^8 + 98348635231716000000x^7$ $+ 7673023770098440000x^6 + 334234551232497600x^5$ $+ 7831619538672864x^4 + 91790301677904x^3$ $+ 522279444004x^2 + 1301037324x + 1148429$ $(\rho + 2, (w_5 + 1)\rho + 2w_5 + 2, (w_5 + 1)\rho + 2w_5 + 2),\ \rho = (w_5 + \sqrt{13w_5 + 9})/2$ $A \sim 0.0016 + 0.0567iC \sim -0.0087847 - 0.0014537i$ $((5w_5 + 2)\rho - 2w_5 + 1, \rho, (5w_5 + 2)\rho - 2w_5 + 1)^*,\ \rho = (w_5 + 3 + \sqrt{7w_5 + 6})/2$ $A \sim 0.9999 + 0.00003i,\ C \sim -0.0576 - 0.8\ 10^{-6}i$
5/1125/5	$(\rho^2, \rho^2, \rho^2),\ \rho = (1 + w_5 + (2 - w_5)\sqrt{33w_5 + 21})/2$ $A = \frac{1}{2} + \frac{1}{2}i\sqrt{3}\ , C = -\frac{3}{100} - \frac{1}{100}\,i\sqrt{3}$
6/12/66i	$(3 + \sqrt{3}, 14 + 6\sqrt{3}, 15 + 8\sqrt{3})$ $A = \frac{47}{128} - \frac{27}{128}\,\sqrt{3}\ , C = -\frac{409}{18432} + \frac{21}{2048}\,\sqrt{3}$
6/12/66ii	$(5 + \sqrt{3}, 6 + 2\sqrt{3}, 9 + 4\sqrt{3})$ $A = \frac{81}{128} - \frac{27}{128}\,\sqrt{3}\ , C = -\frac{79}{2048} + \frac{21}{2048}\,\sqrt{3}$

7/49/91i-ii	$f_A = 5764801x^6 - 10990807933574x^5 + 19901483769167x^4 - 17824198550292x^3 + 19901483769167x^2 - 10990807933574x + 5764801$ $f_C = 23612624896x^6 + 69010305710080x^5 + 10098549495552x^4 + 533695914240x^3 + 13079230960x^2 + 157473912x + 212521$ $(\rho^2+1, 16\rho^2+12\rho-8, 17\rho^2+13\rho-9),\ \rho = 2\cos(\pi/7)$ $A \sim 0.19065\ 10^7,\ C \sim -2922.45570$ $(5\rho^2+3\rho-2, \rho^2+\rho, 5\rho^2+3\rho-2)^*,\ \rho = 2\cos(\pi/7)$ $A \sim 0.9999 + 0.0108i,\ C \sim -0.0550 - 0.0002i$ $(2\rho^2+\rho, 2\rho^2+\rho, 3\rho^2+\rho-1),\ \rho = 2\cos(\pi/7)$ $A \sim -0.0945 - 0.9955i,\ C \sim -0.0173 + 0.0190i$
7/49/1	$\rho = 2\cos(\pi/7),\ (2\rho^2, 2\rho^2+2\rho, 4\rho^2+3\rho-2)$ $A = 1/2 - \frac{13}{64}\sqrt{2}\,,\ C = -\frac{3}{98} + \frac{107}{12544}\sqrt{2}$
9/81/51ii-iii	$f_A = 43046721x^6 + 170877413376x^5 - 286160283648x^4 + 235414355968x^3 - 129828716544x^2 + 14545846272x - 4848615424$ $f_C = 93703341895520256x^6 - 1186330581652543488x^5 - 163434666011132160x^4 - 8465909889680640x^3 - 216226514722320x^2 - 2661536591640x - 13023828271$ $(4\rho^2+8\rho+4, \rho^2+1, 5\rho^2+9\rho+3)^*,\ \rho = 2\cos(\pi/9)$ $A \sim 0.3147 + 0.7282i,\ C \sim -0.0258 - 0.0155i$ $(\rho^2+\rho+1, 2\rho^2+2\rho+1, 2\rho^2+2\rho+1),\ \rho = 2\cos(\pi/9)$ $A \sim 0.0225 + 0.2111i,\ C \sim -0.0133 - 0.0051i$ $(\rho^2+2\rho+1, \rho^2+2\rho+2, \rho^2+2\rho+2),\ \rho = 2\cos(\pi/9)$ $A \sim 0.3147 - 0.7282i,\ C \sim -0.0258 + 0.0155i$
$11/11^4/1$	$((\rho^2-1)^2, (\rho^3-2\rho)^2, (\rho^3-2\rho)^2),\ \rho = 2\cos(\pi/11)$ $f_A = 43046721x^6 + 170877413376x^5 - 286160283648x^4 + 235414355968x^3 - 129828716544x^2 + 14545846272x - 4848615424,\ A \sim 0.31474 + 0.77830i$ $f_C = 93703341895520256x^6 - 1186330581652543488x^5 - 163434666011132160x^4 - 8465909889680640x^3 - 216226514722320x^2 - 2661536591640x - 13023828271$ $C \sim -0.025847 - 0.015507i$

Table B.1: A and C for (1;e)-groups

B.2 (0;2,2,2,q)-Type

3/1/10	$(5,8,10)$ $A=\frac{2}{27}$, $C=-\frac{1}{144}$
3/1/6	$(6,6,9)$ $A=\frac{1}{2}$, $C=-\frac{5}{288}$
3/5/11i	$(\frac{7+\sqrt{5}}{2},6+2\sqrt{5},\frac{13+5\sqrt{5}}{2})$ $f_A=+729x^4-93393x^3+24277x^2-93393x+729,\ A\sim 0.0078$ $f_C=47775744x^4+24883200x^3+693760x^2+8800x+25,\ C\sim -0.0038$
3/5/5	$(3+\sqrt{5},5+\sqrt{5},5+2\sqrt{5})$ $A=\frac{5}{32}$, $C=-\frac{43}{4608}$
3/5/11ii	$(4+\sqrt{5},4+\sqrt{5},\frac{7+3\sqrt{5}}{2})$ $f_A=161051x^4-322102x^3+251528x^2-90477x-729,\ A\sim 0.5000+0.5654i$ $f_C=854925705216x^4+59369840640x^3+1427256320x^2+13768400x+33725$ $C\sim -0.017361-0.0070870i$
3/8/2i	$(3+\sqrt{2},8+4\sqrt{2},8+5\sqrt{2})$ $A=\frac{58}{27}-\frac{41}{27}\sqrt{2}$, $C=-\frac{11}{432}+\frac{7}{432}\sqrt{2}$
3/8/2ii	$(4+\sqrt{2},4+2\sqrt{2},5+3\sqrt{2})$ $A=783-\frac{1107}{2}\sqrt{2}$, $C=-2+\frac{45}{32}\sqrt{2}$
3/8/2iii	$(3+2\sqrt{2},4+2\sqrt{2},4+2\sqrt{2})$ $A=\frac{4}{27}-\frac{10}{27}i\sqrt{2}$, $C=\frac{1}{216}i\sqrt{2}-\frac{5}{432}$
3/12/2	$(3+\sqrt{3},6+2\sqrt{3},6+3\sqrt{3})$ $A=1/2-\frac{5}{18}\sqrt{3}$, $C=-\frac{5}{288}+\frac{19}{2592}\sqrt{3}$
3/13/3i-ii	$f_A=729x^4-1458x^3+18472x^2-17743x+1$ $f_C=429981696x^4+29859840x^3+1244160x^2+25200x+43$ $(\frac{5+\sqrt{13}}{2},10+2\sqrt{13},\frac{19+5\sqrt{13}}{2})$ $A\sim 0.00005,\ C\sim -0.001871$ $(\frac{7+\sqrt{13}}{2},4+\sqrt{13},4+\sqrt{13})$ $A\sim 0.5-4.9080i,\ C\sim -0.0173+0.0364i$
3/17/2i	$(\frac{5+\sqrt{17}}{2},7+\sqrt{17},\frac{13+3\sqrt{17}}{2})$ $A=-\frac{109}{512}+\frac{27}{512}\sqrt{17}$, $C=-\frac{1}{288}$
3/17/2ii	$(\frac{7+\sqrt{17}}{2},\frac{9+\sqrt{17}}{2},5+\sqrt{17})$ $A=\frac{4}{27}+1/27\sqrt{17}$, $C=-\frac{7}{864}-\frac{1}{864}\sqrt{17}$
3/28/2	$(3+\sqrt{7},4+\sqrt{7},5+\sqrt{7})$ $A=\frac{59}{54}-\frac{17}{54}\sqrt{7}+i\left(\frac{59}{54}-\frac{17}{54}\sqrt{7}\right)$, $C=-\frac{25}{1152}+\frac{1}{288}\sqrt{7}+i\left(-\frac{1}{96}+\frac{1}{384}\sqrt{7}\right)$

3/49/1 i+iii+iv	$f_A = 19683x^6 - 2171256516x^5 + 1397790000x^4 + 1484145149x^3 + 1397790000x^2 - 2171256516x + 19683$ $f_C = 330225942528x^6 + 55864750768128x^5 + 3657093414912x^4 + 76970233856x^3 + 652291584x^2 + 4120272x + 5041$ $(\rho^2+1, 4\rho^2+4\rho, 5\rho^2+4\rho-2),\ \rho = 2\cos(\pi/7)$ $A \sim 0.9065\,10^{-5},\ C \sim -0.00153$ $(\rho^2+\rho+1, \rho^2+\rho+1, 2\rho^2+2\rho+1),\ \rho = 2\cos(\pi/7)$ $A \sim -0.6737 - 0.7389i,\ C \sim -0.0034 + 0.0077i$ $(\rho^2+\rho, 2\rho^2+\rho, 2\rho^2+\rho)^*,\ \rho = 2\cos(\pi/7)$ $A \sim 0.9956 + 0.0928i, -0.0285 - 0.0013i$
3/49/1ii	$(\rho^2+\rho, \rho^2+2\rho+1, 2\rho^2+\rho+1),\ \rho = 2\cos(\pi/7)$ $f_A = 729x^4 - 15039x^3 + 94156x^2 - 15039x + 729,\ A \sim 0.0819 + 0.0368i$ $f_C = 334430208x^4 + 56512512x^3 + 3028480x^2 + 35728x + 121$ $C \sim -0.00731 - 0.00053i$
3/81/1	$(\rho^2+\rho+1, \rho^2+\rho+1, \rho^2+\rho+1),\ \rho = 2\cos(\pi/9)$ $A = 1/2 + 1/2\, i\sqrt{3}\,,\ C = -\frac{5}{288} - \frac{5}{864}\, i\sqrt{3}$
3/148/10i	$(-\rho+1, 2\rho^2+2, 2\rho^2-\rho),\ f_\rho = x^3 + 3x^2 - x - 1,\ \rho \sim -3.2143$ $f_A = 25x^3 - 11187282x^2 + 11206938x + 2,\ A \sim 447490.2782$ $f_C = 74649600x^3 + 44553431808x^2 + 1464840720x + 1505621,\ C \sim -596.8012$
3/148/1i-iii	$f_A = 256x^6 - 768x^5 + 968448x^4 - 1935616x^3 + 1434240x^2 - 466560x + 19683$ $f_C = 195689447424x^6 + 20384317440x^5 + 14212399104x^4 + 946012160x^3 + 23083008x^2 + 236160x + 775$ $f_\rho = x^3 + 3x^2 - x - 1$ $(\rho^2+2\rho+1, -2\rho+2, \rho^2), \rho \sim -3.2143$ $A \sim -0.0516,\ C \sim -0.0045$ $(-3\rho^2-8\rho+7, -\rho^2-2\rho+4, -3\rho^2-8\rho+7), \rho \sim -0.4608$ $A \sim 0.9998 - 0.0162i,\ C \sim -0.03047 + 0.0002i$ $(\rho^2+4\rho+3, \rho^2+4\rho+3, \rho^2+4\rho+4),\ \rho \sim 0.6751$ $A \sim -0.1481 - 0.9889i,\ C \sim -0.0092 + 0.0107i$
3/148/10ii+iii	$f_A = 2x^3 - 11206944x^2 + 11226600x - 19683$ $f_C = 8192x^3 + 48757248x^2 + 1772256x + 4819$ $f_\rho = x^3 + 3x^2 - x - 1$ $(-\rho^2-3\rho+3, -8\rho^2-20\rho+20, -9\rho^2-23\rho+20),\ \rho \sim -0.4608$ $A \sim 0.5603\,10^7,\ C \sim -5951.7761$ $(\rho^2+3\rho+2, 2\rho^2+8\rho+6, 3\rho^2+11\rho+5),\ \rho \sim 0.6751$ $A \sim 0.0017,\ C \sim -0.0029$
3/169/1	$(\rho^2+2\rho+1, \rho^2+3\rho+1, \rho^2+3\rho+1),\ f_\rho = x^3 + x^2 - 4x + 1,\ \rho \sim 1.3772$

	$f_A = 47775744x^6 - 143327232x^5 + 442771321x^4 - 646663922x^3 + 442771321x^2$ $- 143327232x + 47775744,\ A \sim 0.0865 + 0.4068i$ $f_C = 253613523861504x^6 + 26418075402240x^5 + 1247751475200x^4$ $+ 33565248000x^3 + 524410000x^2 + 4375000x + 15625,\ C \sim -0.0101 - 0.0052i$
3/229/1i-iv	$f_A = x^6 - 891800283x^5 - 323471580x^4 - 69682060x^3 - 4875984x^2$ $- 95499x - 19683$ $f_C = 12230590464x^6 + 7861915024883712x^5 + 217846257352704x^4$ $+ 2464230920192x^3 + 14182917888x^2 + 41406720x + 48889$ $f_\rho = x^3 - 4x + 1$ $(-\rho + 2, 4\rho^2 - 8\rho + 4, 4\rho^2 - 9\rho + 3),\ \rho \sim -2.1149$ $A \sim 0.89180\ 10^9,\ C \sim -642807.45145$ $(-\rho + 3, \rho^2 - \rho + 1, \rho^2 - 2\rho + 1),\ \rho \sim -2.1149$ $A \sim -0.1244\ C \sim -0.0047$ $(-\rho^2 + 5, -\rho^2 - \rho + 9, -2\rho^2 + 9),\ \rho \sim 0.2541$ $A \sim 0.0139 + 0.0564i, C \sim -0.0057 - 0.0008i$ $(\rho^2 + \rho, \rho^2 + 2\rho, \rho^2 + 2\rho + 1),\ \rho \sim 1.8608$ $A \sim -0.1330 - 0.1865i,\ C \sim -0.0057 + 0.0024i$
3/257/1i-iv	$f_A = 531441x^{12} + 525251601918x^{11} - 475583068628109x^{10}$ $+ 145091741406011523x^9 - 3841484122217531779x^8$ $+ 4387863979361988759x^7 + 6922379838810398979 0x^6$ $+ 4387863979361988759x^5 - 3841484122217531779x^4$ $+ 145091741406011523x^3 - 475583068628109x^2$ $+ 525251601918x + 531441$ $f_C = 98144255806655363127705 6x^{12} - 1199077445317811899012743168x^{11}$ $- 344017438006596816697137561 6x^10 - 34435312950871145167705866 24x^9$ $- 6400272832999031050170531 84x^8 - 5126842683296326258104729 6x^7$ $- 104754717565789437060710 4x^6 - 1152755289516433578393 6x^5$ $- 8245494539647005491 2x^4 - 379096792445173760x^3$ $- 1022891518176000x^2 - 1414076126000x - 738480625$ $f_\rho = x^3 + 3x^2 - 2x - 1$ $(-\rho + 1, \rho^2, \rho^2 + 1),\ \rho \sim -3.4909$ $A \sim 438.48055 + 242.68545i,\ C \sim -1.3264 - 0.77538i$ $\rho \sim -0.3434, (-\rho^2 - 3\rho + 4, -2\rho^2 - 5\rho + 7, -2\rho^2 - 5\rho + 8)$ $A \sim 0.0390 + 0.0336i, C \sim -0.0059 - 0.0004i$ $\rho \sim 0.8342,\ (4\rho^2 + 16\rho + 8, \rho^2 + 3\rho + 1, 5\rho^2 + 19\rho + 6)^*$ $A \sim -989258.4449, C \sim 1224.6146$ $\rho \sim 0.8342,\ (\rho^2 + 4\rho + 2, \rho^2 + 4\rho + 2, \rho^2 + 4\rho + 4)$ $A \sim -0.7732 - 0.6341i, C \sim -0.0023 + 0.0066i$
3/316/4i-iii	$f_A = 1024x^3 + 31488x^2 - 12831x + 2$ $f_C = 48922361856x^3 - 5796790272x^2 - 113303664x - 213223$ $f_\rho = x^3 + x^2 - 4x - 2$

	$(-\rho+2, 2\rho^2-2\rho, 2\rho^2-3\rho-1),\ \rho \sim -2.3429$ $A \sim 0.0002,\ C \sim -0.0021$ $\rho \sim -0.4707,\ (-\rho^2+6, -\rho^2-\rho+6, -2\rho^2-\rho+9)$ $A \sim 0.4020, C \sim -0.0151$ $\rho \sim 1.8136,\ (\rho^2+2\rho+2, \rho+3, -\rho^2+3\rho+2)^*$ $A \sim -31.1522, C \sim 0.1357$
3/725/11i-iv	$f_A = 531441x^8 - 7891391272196808x^7 + 46388145761677494x^6$ $- 114530990676764355x^5 + 152068472373343405x^4 - 114530990676764355x^3$ $+ 46388145761677494x^2 - 7891391272196808x + 531441$ $f_C = 2282521714753536x^8 + 20292714330884747034624x^7$ $+ 35667128139628570214440x^6 + 26146241330309863836x^5$ $+ 10245236898074460160x^4 + 226912926838304768x^3 + 2712430618683648x^2$ $+ 14116763723456x + 7527124081$ $f_\rho = x^4 + x^3 - 3x^2 - x + 1$ $(-\rho+2, -4\rho^3+4\rho^2+4\rho, -4\rho^3+4\rho^2+3\rho-1), \rho \sim -2.0953$ $A \sim 0.14849\ 10^{11},\ C \sim -0.88905\ 10^7$ $(-\rho^3+\rho^2+1, \rho^2, -\rho^3+\rho^2+1)^*, \rho \sim -2.0953$ $A \sim 0.9999 + 0.0004i,\ C \sim -0.0323 - 0.6730\ 10^{-5}$ $(\rho^3+2\rho^2+1, \rho^2+2\rho+1, \rho^3+2\rho^2+1)^*,\ \rho \sim 1.3557$ $A \sim 0.9396 - 0.3421i,\ C \sim -0.0253 + 0.0044i$ $(\rho^3+3\rho^2+\rho, \rho^3+2\rho-\rho, \rho^3+3\rho^2+\rho)^*,\ \rho \sim 1.3557$ $A \sim 0.9995 - 0.0312i,\ C \sim -0.0298 + 0.0004i$
3/1957/3i-iv	$f_A = x^8 - 34334530121767x^7 + 154100700344272x^6 - 283384691457345x^5$ $+ 251323610679999x^4 - 131558602718961x^3 + 43854675535269x^2$ $- 1549681956x + 387420489$ $f_C = 347892350976x^8 + 46281772619158121349 12x^7$ $+ 5822930761254845808 64x^6 + 296193452508614492 16x^5$ $+ 776123466778607616x^4 + 1108399853422 1824x^3$ $+ 84089886272256x^2 + 284269141584x + 339773323$ $f_\rho = x^4 - 4x^2 + x + 1$ $(-\rho+2, -4\rho^3+8\rho^2, -4\rho^3+8\rho^2-\rho-1), \rho \sim -2.0615$ $A \sim 0.34335\ 10^{14},\ C \sim -0.13303\ 10^{11}$ $(\rho^3-4\rho+3, 3\rho^3-\rho^2-11\rho+8, 3\rho^3-\rho^2-11\rho+8),\ \rho \sim -0.3963$ $A \sim 0.4417\ 10^{-5} - 0.0029i,\ C \sim -0.0031 + 0.00004i$ $(-\rho^3-\rho^2+4\rho+4, -2\rho^3-\rho^2+7\rho+3, -2\rho^3-\rho^2+7\rho+3),\ \rho \sim 0.6938$ $A \sim 0.2078 - 0.6103i, -0.0127 + 0.0074i$ $(\rho^3+\rho^2-2\rho+1, \rho^3+\rho^2-2\rho+1, \rho^2+2\rho+1),\ \rho \sim 1.7640$

	$A \sim 1.5362 - 0.8440i,\ C \sim -0.0298 + 0.0089i$
3/2000/5	$(\rho^2 - \rho, \rho^3 - 2\rho - \rho + 2, \rho^3 - \rho^2 - 3\rho + 1),\ f_\rho = x^4 - 4x^3 + x^2 + 6x + 1,\ \rho \sim 2.9021$ $f_A = 729x^4 - 6480x^3 + 22105x^2 - 31250x + 15625,\ A \sim 3.1332 + 1.5177i$ $f_C = 2418647040000x^4 + 341297971200x^3 + 18345533184x^2$ $+ 440949888x + 3978349,\ C \sim -0.0404 - 0.0128i$
3/4352/2i-ii	$f_A = 531441x^8 - 64796436x^7 + 3009036438x^6 - 6026830596x^5 + 3361388930x^4$ $- 423440384x^3 + 150933504x^2 - 8388608x + 2097152$ $f_C = 981442558066553631277056x^8 + 602610274368410534019072x^7$ $+ 146932327534616710742016x^6 + 10415653268576658259968x^5$ $+ 321388066379823906816x^4 + 4937075259269087232x^3$ $+ 40344446803654656x^2 + 169641680184576x + 292036636097$ $f_\rho = x^4 - 8x^2 + 8x + 1$ $(-2\rho^3 - 3\rho^2 + 11\rho + 2, -4\rho^3 - 6\rho^2 + 23\rho + 3, -4\rho^3 - 6\rho^2 + 23\rho + 4),\ \rho \sim 1.5266$ $A \sim 59.9283 + 42.6784i,\ C \sim -0.26185 - 0.17555i$ $(\rho^3 + 2\rho^2 - 4\rho, 2\rho^3 + 4\rho^2 - 9\rho - 1, 3\rho^3 + 5\rho^2 - 15\rho + 1),\ \rho \sim 1.8085$ $A \sim 0.0513 + 0.1517i,\ C \sim -0.0077 - 0.0021i$
3/24217/1i-v	The polynomials f_A and f_C of degree 30 can be found in [Hof12a]. $f_\rho = x^5 - 5x^3 - x^2 + 3x + 1$ $(\rho^4 - \rho^3 - 4\rho^2 + 2\rho + 3, \rho^4 - \rho^3 - 4\rho^2 + 2\rho + 3, \rho^2 - 2\rho + 1),\ \rho \sim -1.9600$ $A \sim -0.9877 - 0.1558i, C \sim -0.0001 + 0.0016i$ $(tr_1, tr_2, tr_3),\ tr_1 = -2\rho^4 + \rho^3 + 9\rho^2 - 2\rho - 1, tr_2 = -4\rho^4 + 3\rho^3 + 18\rho^2 - 9\rho - 5,$ $tr_3 = -4\rho^4 + 3\rho^3 + 18\rho^2 - 9\rho - 5,\ \rho \sim -0.7728$ $A \sim 0.0018 + 0.0615i, C \sim -0.0055 - 0.0008i$ $(3\rho^4 - \rho^3 - 14\rho^2 + 2\rho + 8, 3\rho^4 - \rho^3 - 15\rho^2 + 2\rho + 10, 3\rho^4 - \rho^3 - 15\rho^2 + 2\rho + 10),$ $\rho \sim -0.3697$ $A \sim 0.0413 + 0.2845i, -0.0086 - 0.0037i$ $(\rho^4 - \rho^3 - 5\rho^2 + 4\rho + 5, -2\rho^4 - \rho^3 + 9\rho^2 + 6\rho + 1, -2\rho^4 - \rho^3 + 9\rho^2 + 6\rho + 1),$ $\rho \sim 0.8781$ $A \sim 0.00001i + 0.0051i, C \sim -0.0034 - 0.00008i$ $(2\rho^3 + 3\rho^2 - 3\rho - 1, 2\rho^3 + 3\rho^2 - 3\rho - 1, \rho + 2)^*,\ \rho \sim 2.1744$ $A \sim 0.5 - 0.0392i, C \sim -0.0173 + 0.0005i$
3/38569/1 i | ii | iv | vi	The minimal polynomials f_A and f_C of degree 20 can be found in [Hof12a]. $f_\rho = x^5 + x^4 - 5x^3 - x^2 + 4x - 1$ $(-\rho^4 - 2\rho^3 + 4\rho^2 - 1, \rho^2, \rho^4 + \rho^3 - 4\rho^2 - 2\rho + 3),\ \rho \sim -2.5441$ $A \sim -0.2714 - 0.5108i, C \sim -0.0062 + 0.0059i$ $(-\rho^4 - \rho^3 + 4\rho^2 + 1, \rho^3 + \rho^2 - 4\rho + 2, \rho^3 - 5\rho + 4),\ \rho \sim -1.1101$ $A \sim 2.2678 + 0.8703i, C \sim -0.0355 - 0.0082i$ $(tr_1, tr_2, tr_3),\ tr_1 = -2\rho^4 - 3\rho^3 + 8\rho^2 + 6\rho - 2$

	$tr_2 = -49\rho^4 - 7\rho^3 + 15\rho^2 + 15\rho - 4$ $tr_3 = -4\rho^4 - 7\rho^3 + 15\rho^2 + 16\rho + 4,\ \rho \sim 0.7015$ $A = 0.0053 + 0.0064i\ , C = -0.0038 - 0.00009i$ $(\rho^4 + 2\rho^3 - 3\rho^2 - 3\rho + 3, \rho^4 + 2\rho^3 - 3\rho^2 - 3\rho + 3, \rho^2 + 2\rho + 1),\ \rho \sim 1.6460$ $A \sim 0.5 + 0.4948i\ , C \sim -0.0173 - 0.0063$
3/38569/1iii	The minimal polynomials f_A and f_C of degree 20 can be found in [Hof12a]. $(tr_1, tr_2, tr_2),\ tr_1 = \rho^4 + \rho^3 - 5\rho^2 - \rho + 5,\ tr_2 = 9\rho^4 + 12\rho^3 - 41\rho^2 - 23\rho + 30$ $f_\rho = x^5 + x^4 - 5x^3 - x^2 + 4x - 1,\ \rho \sim 0.3067$ $A = 1.000006 - 0.00001i\ , C = -0.0331 + 0.2309\ 10^{-6}$
3/38569/1v	$f_\rho = x^5 + x^4 - 5x^3 - x^2 + 4x - 1,\ \rho \sim 1.6460$ $(tr_1, tr_2, tr_3),\ tr_1 = -\rho^4 - \rho^3 + 5\rho^2 + 2\rho - 1$ $tr_2 = 8\rho^4 + 20\rho^3 - 8\rho^2 - 16\rho + 8,\ tr_3 = 7\rho^4 + 19\rho^3 - 3\rho^2 - 14\rho + 4$ no candidates for A and C known
5/5/2i+ii	$f_A = 3125x^2 - 3000x - 121$ $f_C = 200000x^2 + 11000x + 61$ $(\frac{7+\sqrt{5}}{2}, 12 + 4\sqrt{5}, 13 + 5\sqrt{5})^*$ $A \sim 0.9987, C \sim -0.0487$ $(3 + \sqrt{5}, 8 + 2\sqrt{5}, \frac{17+7\sqrt{5}}{2})$ $A \sim -0.0387, C \sim -0.0062$
5/5/9	$(\frac{9+\sqrt{5}}{2}, 6 + 6\sqrt{5}, 8 + 3\sqrt{5})$ $f_A = 3125x^4 - 37625x^3 + 62141x^2 - 37625x + 3125,\ A \sim 0.097978$ $f_C = 204800000x^4 + 40448000x^3 + 2387456x^2 + 54112x + 361,\ C \sim -0.011256$
5/5/2iii	$(3 + \sqrt{5}, \frac{15+5\sqrt{5}}{2}, 7 + 3\sqrt{5})$ $f_A = 3125x^4 - 46000x^3 + 2182902x^2 - 46000x + 3125, A \sim 7.34947 + 25.3812i$ $f_C = 12800000x^4 + 2112000x^3 + 520272x^2 + 7344x + 27, C \sim -0.075132 - 0.18092i$
5/5/2iv	$(5 + \sqrt{5}, 5 + 5\sqrt{5}, \frac{15+5\sqrt{5}}{2})$ $A = -1\ , C = 0$
5/5/19	$(\frac{7+3\sqrt{5}}{2}, \frac{11+3\sqrt{5}}{2}, \frac{11+3\sqrt{5}}{2})$ $f_A = 3125x^4 + 25125x^3 - 31984x^2 + 13718x - 6859,\ A \sim 0.1321 + 0.4968i$ $f_C = 128000000000x^4 + 1600000000x^3 - 372640000x^2 - 12089200x - 122849,$ $C \sim -0.0168 - 0.0094i$
5/725/19i+iii	$f_A = 9765625x^8 + 121056848828125x^7 - 419346380575000x^6$ $+ 598919028196875x^5 - 305309425863109x^4 + 7019894136811x^3$ $- 2339964744279x^2 + 27436x - 6859$ $f_C = 16384000000000000000000x^8 - 244987125760000000000000000x^7$ $- 376357760819200000000000000x^6 - 221938701824000000000000000x^5$ $- 615099167945984000000000x^4 - 786470380331264000000x^3$ $- 471635923737104000x^2 - 120899322378044000x - 10850689723109$ $f_\rho = x^4 + x^3 - 3x^2 - x + 1$

	$(\rho^2-\rho+1,\rho^2-\rho+1,\rho^2-2\rho+1),\ \rho\sim-2.0953$ $A\sim 1.2280+0.9736i, C\sim-0.0399-0.0158i$ $(8\rho^3+20\rho^2+4\rho-4,\rho^2+\rho+1,9\rho^3+22\rho^2+3\rho-4)^*,\ \rho\sim 1.3557$ $A\sim-0.1239\ 10^8, C\sim 14952.9811$
5/725/19ii+iv	$f_A=6859x^8-27436x^7+2339964744279x^6-7019894136811x^5$ $+305309425863109x^4-598919028196875x^3+419346380575000x^2$ $-121056848828125x-9765625$ $f_C=115074924544000000000x^8+2416573415424000000x^7$ $+22516905758047600640000x^6+354638934451083018240 0x^5$ $+4377639758769670717 44x^4+296740128339855728 64x^3$ $+884263559008833024x^2+10019598658093584x+10850689723109$ $f_\rho=x^4+x^3-3x^2-x+1$ $(-2\rho^3+3\rho^2,-2\rho^3+3\rho^2,\rho^2)^*,\ \rho\sim-2.0953$ $A\sim 0.5-18470.3094i, C\sim-0.0262+44.2344i$ $(\rho^2+2\rho+1,\rho^3+3\rho^2+2\rho+1,\rho^3+3\rho^2+2\rho+1),\ \rho\sim 1.3557$ $A\sim 0.5+11.26139, C\sim-0.02625-0.09833i$
5/2225/2i-iv	$f_A=9765625x^8+77448046875x^7+10966824884375x^6-37076606993375x^5$ $+51789770686056x^4-37076606993375x^3+10966824884375x^2$ $+77448046875x+9765625$ $f_C=41943040000000000x^8-89768591360000000x^7+48776651735040000 0x^6$ $+86206751178752000x^5+5330548124942336x^4+155797232070656x^3$ $+2126369278976x^2+10323832736x+15124321$ $f_\rho=x^4+5x^3+4x^2-5x-1$ $(-\rho+1,-\rho^3+4\rho+1,-\frac{3}{2}\rho^3-2\rho^2+2\rho+\frac{1}{2}),\ \rho\sim-3.4383$ $A\sim-0.00013, C\sim-0.0026$ $(\rho^2+2\rho+1,\rho^2+2\rho+1,-\frac{1}{2}\rho^3+2\rho+\frac{1}{2})^*,\ \rho\sim-3.4383$ $A\sim 1.20494,\ C\sim-0.0459$ $(\rho^3+6\rho^2+10\rho+5,\rho^2+3\rho+2,\frac{3}{2}\rho^3+9\rho^2+14\rho+\frac{7}{2}),\ \rho\sim 0.7566$ $A\sim-0.0067, C\sim-0.0048$ $(4\rho^2+4\rho+4,\frac{5}{2}\rho^3+5\rho^2+3\rho+\frac{1}{2},\frac{5}{2}\rho^3+5\rho^2+3\rho+\frac{1}{2}),\ \rho\sim 0.7566$ $A\sim 0.6535+0.7569i, C\sim-0.0297-0.0136i$
5/3600/2	$f_A=3125x^4+750x^3+25018x^2+750x+3125$ $f_C=1638400000x^4+100352000x^3+4425344x^2+76912x+529$ $f_\rho=x^4-4x^3+3x^2+14x+1$ $(\frac{1}{3}\rho^2+\frac{1}{3}\rho+\frac{1}{3},\frac{1}{3}\rho^3-\frac{2}{3}\rho^2-\frac{2}{3}\rho+3,\frac{1}{3}\rho^3-\frac{1}{3}\rho^2-\frac{4}{3}\rho+\frac{4}{3}),\rho\sim 3.8025$ $A\sim-0.0135-0.3561i,\ C\sim-0.0124+0.0069i$
7/49/1i+iii+v	

	$f_A = 823543x^6 + 219801946x^5 - 349963243x^4 + 260346983x^3 - 130234464$
	$x^2 + 73167x - 24389$ $f_C = 1625527855624486912x^6 - 1857746120713699328x^5$ $- 220782954203578368x^4 - 9004835937341440x^3 - 167836041028096x^2$ $- 1233603358848x - 3039746921$ $(4\rho^2 + 3\rho - 1, \rho^2 + \rho, 4\rho^2 + 3\rho - 1)^*,\ \rho = 2\cos(\pi/7)$ $A \sim -268.4851\ C \sim 1.25462$ $(\rho + 3, 4\rho^2 + 4\rho, \rho^2 + 5\rho - 1),\ 2\cos(\pi/7)$ $A \sim 0.00009 + 0.01369i\ C \sim -0.0061 - 0.0003i$ $(\rho^2 + 2\rho + 1, \rho^2 + 2\rho + 2, \rho^2 + 2\rho + 2),\ \rho = 2\cos(\pi/7)$ $A \sim 0.2953 - 0.7095i,\ C \sim -0.0235 + 0.0141i$
7/49/1ii+iv	The polynomials F_A and F_C of degree 18 can be found in [Hof12a]. $(\rho^2 + \rho, 4\rho^2 + 3\rho - 2, 4\rho^2 + 4\rho),\ \rho = 2\cos(\pi/7)$ $A \sim 0.0134 + 0.0022i,\ C \sim -0.0064 - 0.00005i$ $(\rho^2 + 2\rho + 1, \rho^2 + 2\rho + 1, 3\rho^2 + \rho - 1)$ $A \sim 0.5 + 0.3624i,\ C \sim -0.0286 - 0.0080i$
7/81/6	$(2\rho + 4, 2\rho + 4, \rho^2 + 4\rho + 4),\ \rho = 2\cos(\pi/9)$ $A = -1,\ C = 0$

Table B.2: A and C (0;2,2,2,q)-groups

C CY(3)-Equations with Monodromy Invariant Doran-Morgan Lattice

In this Appendix we list the data of CY(3)-equations with monodromy invariant Doran-Morgan lattice as announced in Chapter 4.

<table>
<tr><td colspan="2">Operator</td></tr>
<tr><td>H^3</td><td rowspan="5">$$\left\{\begin{array}{ccc} 0 & a & \infty \\ \hline 0 & a_1 & \infty_1 \\ 0 & a_2 & \infty_2 \\ 0 & a_3 & \infty_3 \\ 0 & a_4 & \infty_4 \end{array}\right\}$$</td></tr>
<tr><td>c_2H</td></tr>
<tr><td>c_3</td></tr>
<tr><td>$|H|$</td></tr>
<tr><td>$1/a$</td></tr>
<tr><td colspan="2">Yukawa coupling $K(q)$</td></tr>
<tr><td colspan="2">g_0 instanton number</td></tr>
<tr><td colspan="2">exponents in holomorphic anomaly</td></tr>
<tr><td colspan="2">g_1 instanton numbers</td></tr>
<tr><td colspan="2">geometric description</td></tr>
<tr><td colspan="2">monodromy with respect to the Doran-Morgan lattice</td></tr>
<tr><td colspan="2">reflection vectors</td></tr>
</table>

<table>
<tr><td colspan="2">$\theta^4-144z(6\theta+1)^2(6\theta+5)^2$</td></tr>
<tr><td>1</td><td rowspan="5">$$\left\{\begin{array}{ccc} 0 & \frac{1}{186624} & \infty \\ \hline 0 & 0 & \frac{1}{6} \\ 0 & 1 & \frac{1}{6} \\ 0 & 1 & \frac{5}{6} \\ 0 & 2 & \frac{5}{6} \end{array}\right\}$$</td></tr>
<tr><td>22</td></tr>
<tr><td>-120</td></tr>
<tr><td>2</td></tr>
<tr><td>186624</td></tr>
<tr><td colspan="2">$1+67104q+6778372896q^2+771747702257664q^3+91640648916794239776q^4+O(q^5)$</td></tr>
<tr><td colspan="2">$n_1^0=67104$, $n_2^0=847288224$, $n_3^0=28583248229280$, $n_4^0=1431885139218997920$,
$n_5^0=88985016340513371957600$, $n_6^0=6335857687001825322018363168$</td></tr>
<tr><td colspan="2">$e_{\frac{1}{3125}}=-\frac{1}{6}$</td></tr>
<tr><td colspan="2">$n_1^1=360$, $n_2^1=40691736$, $n_3^1=4956204918240$, $n_4^1=616199133057629184$,
$n_5^1=76418673008601139234704$, $n_6^1=9453033616452720472691303904$</td></tr>
<tr><td colspan="2">$X(6,6)\subset\mathbb{P}^5(1,1,2,2,3,3)$ [KT93]</td></tr>
</table>

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{\frac{1}{186624}} = \begin{pmatrix} 1 & -2 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{\frac{1}{186624}} = (1, 0, \frac{11}{12}, -120\lambda)$

$\theta^4 - 80z(10\theta+1)(10\theta+3)(10\theta+7)(10\theta+9)$

1	$\left\{\begin{array}{ccc} 0 & \frac{1}{800000} & \infty \\ \hline 0 & 0 & \frac{1}{10} \\ 0 & 1 & \frac{3}{10} \\ 0 & 1 & \frac{7}{10} \\ 0 & 2 & \frac{9}{10} \end{array}\right\}$
34	
-288	
3	
800000	

$1+231200q+97726516000q^2+45924147895040000q^3+22409898166582504500000q^4+O(q^5)$

$n_1^0 = 231200$, $n_2^0 = 12215785600$, $n_3^0 = 1700894366474400$, $n_4^0 = 350154658851324656000$,
$n_5^0 = 89338191421813572850115680$, $n_6^0 = 26107067114407746641915631734400$

$e_{\frac{1}{800000}} = -\frac{1}{6}$

$n_1^1 = 280$, $n_2^1 = 207680680$, $n_3^1 = 161279120326560$, $n_4^1 = 103038403740690105440$,
$n_5^1 = 59221844124053623534386928$, $n_6^1 = 32373322447001008788291066015840$

$X(10) \subset \mathbb{P}^4(1,1,1,2,5)$ [Mor92]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{\frac{1}{800000}} = \begin{pmatrix} 1 & -3 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{\frac{1}{800000}} = (1, 0, \frac{17}{12}, -288\lambda)$

$\theta^4 - 144z(12\theta+1)(12\theta+5)(12\theta+7)(12\theta+11)$

1	$\left\{\begin{array}{ccc} 0 & \frac{1}{2985984} & \infty \\ \hline 0 & 0 & \frac{1}{12} \\ 0 & 1 & \frac{5}{12} \\ 0 & 1 & \frac{7}{12} \\ 0 & 2 & \frac{11}{12} \end{array}\right\}$
46	
-484	
4	
2985984	

$1+678816q+1101481164576q^2+1865163478016858112q^3+O(q^4)$

$n_1^0 = 678816$, $n_2^0 = 137685060720$, $n_3^0 = 69080128815414048$, $n_4^0 = 51172489466251340674608$,
$n_5^0 = 46928387692914781844159094240$, $n_6^0 = 49273154611117098740758501416679344$

$e_{\frac{1}{2985984}} = -\frac{1}{3}$

$n_1^1 = 249312$, $n_2^1 = 188143760304$, $n_3^1 = 239409825319593792$,
$n_4^1 = 345368875891797981326112$, $n_5^1 = 534346826439271590322258006656$,
$n_6^1 = 861937915820071377248390457113456544$

$X(2,12) \subset \mathbb{P}^5(1,1,1,1,4,6)$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{\frac{1}{2985984}} = \begin{pmatrix} 1 & -4 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{\frac{1}{2985984}} = (1, 0, \frac{23}{12}, -484\lambda)$

$$\theta^4 - z\left(528 + 62272\theta + 280320\theta^2 + 435520\theta^3 + 6400\theta^4\right)$$
$$+z^2\left(5799936 + 486277712\theta + 1043333312\theta^2 - 2060451184\theta^3 - 1399848896\theta^4\right)$$
$$+z^3\left(2264924160 + 281857228880\theta + 1160983347720\theta^2 + 1288490188880\theta^3 - 3932579430400\theta^4\right)$$
$$+1717986918400z^4(4\theta+1)(2\theta+1)^2(4\theta+3)$$

2	$\left\{\begin{array}{ccccc} -\frac{1}{5120} & 0 & \frac{1}{16384} & \frac{1}{256} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{4} \\ 1 & 0 & 1 & 1 & \frac{1}{2} \\ 3 & 0 & 1 & 1 & \frac{1}{2} \\ 4 & 0 & 2 & 2 & \frac{3}{4} \end{array}\right\}$
20	
-16	
2	
16384	

$1 + 8224q + 27267872q^2 + 419644487680q^3 + 3494992695847712q^4 + O(q^5)$

$n^0_1 = 16448$, $n^0_2 = 6814912$, $n^0_3 = 31084776256$, $n^0_4 = 109218520893120$,
$n^0_5 = 564955143278513856$, $n^0_6 = 3186807897019572948416$

$e_{\frac{1}{16384}} = -\frac{1}{6}$, $e_{\frac{1}{256}} = -\frac{1}{3}$

$n^1_1 = 16$, $n^1_2 = -130544$, $n^1_3 = 689912000$, $n^1_4 = 8769686911936$, $n^1_5 = 113420969633710496$,
$n^1_6 = 1293678778019568775232$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{-\frac{1}{5120}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{\frac{1}{256}} = \begin{pmatrix} -3 & -6 & -4 & -2 \\ 4 & 7 & 4 & 2 \\ -4 & -6 & -3 & -2 \\ 4 & 6 & 4 & 3 \end{pmatrix}, M_{\frac{1}{16384}} = \begin{pmatrix} 1 & -2 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{\frac{1}{16384}} = (2, 0, \frac{5}{6}, -16\lambda)$, $v^2_{\frac{1}{256}} = (4, -4, \frac{5}{3}, -32\lambda - \frac{7}{3})$

$\theta^4 - 48z(6\theta+1)(4\theta+1)(4\theta+3)(6\theta+5)$

2	$\left\{\begin{array}{ccc} 0 & \frac{1}{27648} & \infty \\ \hline 0 & 0 & \frac{1}{6} \\ 0 & 1 & \frac{1}{4} \\ 0 & 1 & \frac{3}{4} \\ 0 & 2 & \frac{5}{6} \end{array}\right\}$
32	
-156	
3	
27648	

$1 + 7776q + 111624480q^2 + 1807441496064q^3 + 30421658654563104q^4 + O(q^5)$

$n^0_1 = 15552$, $n^0_2 = 27904176$, $n^0_3 = 133884554688$, $n^0_4 = 950676829466832$,
$n^0_5 = 8369111295497240640$, $n^0_6 = 84404875193344648810224$

$e_{\frac{1}{27648}} = -\frac{1}{6}$

$n^1_1 = 8$, $n^1_2 = 258336$, $n^1_3 = 5966034464$, $n^1_4 = 126729436130280$, $n^1_5 = 2512147219945401744$,
$n^1_6 = 47773385611506721364256$

$X(3,4) \subset \mathbb{P}^5(1,1,1,1,1,2)$ [KT93]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{\frac{1}{27648}} = \begin{pmatrix} 1 & -3 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{\frac{1}{27648}}=(2,0,\frac{4}{3},-156\lambda)$

$\theta^4 - 16z(8\theta+1)(8\theta+3)(8\theta+5)(8\theta+7)$

2	$\left\{\begin{array}{ccc} 0 & \frac{1}{65536} & \infty \\ \hline 0 & 0 & \frac{1}{8} \\ 0 & 1 & \frac{3}{8} \\ 0 & 1 & \frac{5}{8} \\ 0 & 2 & \frac{7}{8} \end{array}\right\}$
44	
-296	
4	
65536	

$1+14752q+515354400q^2+19220227397632q^3+742177793308740384q^4+O(q^5)$

$n^0_1 = 29504$, $n^0_2 = 128834912$, $n^0_3 = 1423720546880$, $n^0_4 = 23193056024793312$,
$n^0_5 = 467876474625249316800$, $n^0_6 = 10807872280363954752338400$

$n^1_1 = 0$, $n^1_2 = 41312$, $n^1_3 = 21464350592$, $n^1_4 = 1805292092664544$,
$n^1_5 = 101424054914016355712$, $n^1_6 = 4922755386485603458880576$

$e_{\frac{1}{65536}} = -\frac{1}{6}$

$X(8) \subset \mathbb{P}^4(1,1,1,1,4)$ [Mor92]

$$M_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{\frac{1}{65536}} = \begin{pmatrix} 0 & -4 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{\frac{1}{65536}}=(2,0,\frac{11}{6},-296\lambda)$

$\theta^4 - z\left(168 + 1844\theta + 21388/3\theta^2 + 31712/3\theta^3 + 18544/3\theta^4\right)$
$-z^2\left(125440 + 2871808/3\theta + 13438976/9\theta^2 - 26054656/9\theta^3 - 30232576/9\theta^4\right)$
$+z^3\left(1376256 + 25690112\theta + 127991808\theta^2 + 117440512\theta^3 - 4130603008/9\theta^4\right)$
$-822083584/9z^4(4\theta+1)(2\theta+1)^2(4\theta+3)$

3	$\left\{\begin{array}{ccccc} -\frac{349}{8192} - \frac{85}{8192}\sqrt{17} & 0 & -\frac{349}{8192} + \frac{85}{8192}\sqrt{17} & \frac{3}{896} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{4} \\ 1 & 0 & 1 & 1 & \frac{1}{2} \\ 1 & 0 & 1 & 3 & \frac{1}{2} \\ 2 & 0 & 2 & 4 & \frac{3}{4} \end{array}\right\}$
30	
-92	
3	
$2792 + 680\sqrt{17}$	

$1+\frac{4300}{3}q+\frac{14150636}{3}q^2+\frac{45949064056}{3}q^3+\frac{159011444150764}{3}q^4+\frac{562830048549808300}{3}q^5+O(q^6)$

$n^0_1 = 4300$, $n^0_2 = 1768292$, $n^0_3 = 1701817028$, $n^0_4 = 2484553593752$, $n^0_5 = 4502640388398432$,
$n^0_6 = 9350482662153015820$

$e_{-\frac{349}{8192}-\frac{85}{8192}\sqrt{17}} = -\frac{1}{6}$, $e_{-\frac{349}{8192}+\frac{85}{8192}\sqrt{17}} = -\frac{1}{6}$

$n^1_1 = 6$, $n^1_2 = 14606$, $n^1_3 = 61126060$, $n^1_4 = 244952566097$, $n^1_5 = 957433739298066$,
$n^1_6 = 3670123673483108140$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{-\frac{349}{8192}-\frac{85}{8192}\sqrt{17}} = \begin{pmatrix} -3 & -14 & -6 & -4 \\ 2 & 8 & 3 & 2 \\ -2 & -7 & -2 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{\frac{3}{896}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_{-\frac{349}{8192}+\frac{85}{8192}\sqrt{17}} = \begin{pmatrix} 1 & -3 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{-\frac{349}{8192}-\frac{85}{8192}\sqrt{17}}=(6,-3,2,-184\lambda-\frac{3}{4})$, $v^1_{-\frac{349}{8192}+\frac{85}{8192}\sqrt{17}}=(3,0,\frac{5}{4},-92\lambda)$

3	$\left\{\begin{array}{ccc} 0 & \frac{1}{11664} & \infty \\ \hline 0 & 0 & \frac{1}{6} \\ 0 & 1 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 2 & \frac{5}{6} \end{array}\right\}$
42	
-204	
4	
11664	

$1+2628q+16078500q^2+107103757608q^3+738149392199844q^4+5191459763880422628q^5+O(q^6)$

$n^0_1=7884$, $n^0_2=6028452$, $n^0_3=11900417220$, $n^0_4=34600752005688$,
$n^0_5=124595034333130080$, $n^0_6=513797193321737210316$

$e_{\frac{1}{11664}}=-\frac{1}{6}$

$n^1_1=0$, $n^1_2=7884$, $n^1_3=145114704$, $n^1_4=1773044315001$, $n^1_5=17144900584158168$,
$n^1_6=147664736456807801016$

$X(6)\subset\mathbb{P}^4(1,1,1,1,2)$ [Mor92]

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&3&1&0\\0&0&1&1\end{pmatrix},\ M_{\frac{1}{11664}}=\begin{pmatrix}1&-4&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$\lambda_{\frac{1}{11664}}=1$

$\theta^4-z\left(2432\theta^4+2560\theta^3+1760\theta^2+480\theta+48\right)$
$+z^2\left(1314816\theta^4+540672\theta^3-918528\theta^2-522240\theta-76800\right)$
$-z^3\left(1604321280\theta^4-254803968\theta^3-2123366400\theta^2-601620480\theta-58982400\right)$
$-452984832z^4(3\theta+1)(2\theta+1)^2(3\theta+2)$

4	$\left\{\begin{array}{ccccc} -\frac{1}{64} & 0 & \frac{1}{1728} & \frac{1}{384} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & 1 & 1 & \frac{1}{2} \\ 1 & 0 & 1 & 3 & \frac{1}{2} \\ 2 & 0 & 2 & 4 & \frac{2}{3} \end{array}\right\}$
28	
-60	
3	
1728	

$1+320q+488992q^2+533221376q^3+603454817056q^4+702313679160320q^5+O(q^6)$

$n^0_1=1280$, $n^0_2=244336$, $n^0_3=78995712$, $n^0_4=37715895504$, $n^0_5=22474037733120$,
$n^0_6=15381013322524080$

$e_{-\frac{1}{64}}=-\frac{1}{6}$, $e_{\frac{1}{1728}}=-\frac{1}{6}$

$n^1_1=8$, $n^1_2=3976$, $n^1_3=4042656$, $n^1_4=4551484672$, $n^1_5=5317338497296$,
$n^1_6=6288978429686080$

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&4&1&0\\0&0&1&1\end{pmatrix},\ M_{-\frac{1}{64}}=\begin{pmatrix}-3&-16&-6&-4\\2&9&3&2\\-4&-16&-5&-4\\2&8&3&3\end{pmatrix},\ M_{\frac{1}{384}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ M_{\frac{1}{1728}}=$$
$$\begin{pmatrix}1&-3&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v^1_{-\frac{1}{64}}=(8,-4,\frac{7}{3},-120\lambda-\frac{5}{6})$, $v^1_{\frac{1}{1728}}=(4,0,\frac{7}{6},-60\lambda)$

$$\theta^4 - z\left(112 + 1152\theta + 4480\theta^2 + 6656\theta^3 + 1792\theta^4\right) + z^2\left(94208 + 6553600\theta + 8847360\theta^2 - 36700160\theta^3 - 26869760\theta^4\right) - z^3\left(8388608 + 503316480\theta + 1006632960\theta^2 + 2013265920\theta^3 + 8053063680\theta^4\right) - 1073741824z^4(2\theta+1)^4$$

4	$\left\{\begin{array}{ccccc} -\frac{11}{512}-\frac{5}{512}\sqrt{5} & -\frac{1}{512} & 0 & -\frac{11}{512}+\frac{5}{512}\sqrt{5} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 & \frac{1}{2} \\ 1 & 3 & 0 & 1 & \frac{1}{2} \\ 2 & 4 & 0 & 2 & \frac{1}{2} \end{array}\right\}$
28	
-32	
3	
$1408+640\sqrt{5}$	

$1+992q+791328q^2+1720469504q^3+2620368415520q^4+4515256579220992q^5+O(q^6)$

$n_1^0 = 3968$, $n_2^0 = 395168$, $n_3^0 = 254884224$, $n_4^0 = 163772976512$, $n_5^0 = 144488210535040$, $n_6^0 = 141987141593265952$

$e_{-\frac{11}{512}-\frac{5}{512}\sqrt{5}} = -\frac{1}{6}$, $e_{-\frac{11}{512}+\frac{5}{512}\sqrt{5}} = -\frac{1}{6}$

$n_1^1 = 0$, $n_2^1 = -312$, $n_3^1 = 1486080$, $n_4^1 = 5487324248$, $n_5^1 = 12342963136768$, $n_6^1 = 25799163366493536$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{-\frac{1}{512}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{-\frac{11}{512}-\frac{5}{512}\sqrt{5}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 5 & 1 & 0 \\ -12 & -16 & -3 & 0 \\ 12 & 16 & 4 & 1 \end{pmatrix},$$

$$M_{-\frac{11}{512}+\frac{5}{512}\sqrt{5}} = \begin{pmatrix} 1 & -3 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{-\frac{11}{512}-\frac{5}{512}\sqrt{5}}=(0,-4,2,-\frac{11}{6})$, $v^1_{-\frac{11}{512}+\frac{5}{512}\sqrt{5}}=(4,0,\frac{7}{6},-32\lambda)$

$$\theta^4 - 16z(4\theta+1)^2(4\theta+3)^2$$

4	$\left\{\begin{array}{ccc} 0 & \frac{1}{4096} & \infty \\ \hline 0 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 1 & \frac{3}{4} \\ 0 & 2 & \frac{3}{4} \end{array}\right\}$
40	
-144	
4	
4096	

$1+928q+1965856q^2+4613478400q^3+11199994153760q^4+27748101800636928q^5+O(q^6)$

$n_1^0 = 3712$, $n_2^0 = 982464$, $n_3^0 = 683478144$, $n_4^0 = 699999511744$, $n_5^0 = 887939257620352$, $n_6^0 = 1289954523115535040$

$e_{\frac{1}{4096}} = -\frac{1}{6}$

$n_1^1 = 0$, $n_2^1 = 1408$, $n_3^1 = 6953728$, $n_4^1 = 26841853280$, $n_5^1 = 88647278203648$, $n_6^1 = 266969312902302592$

$X(4,4) \subset \mathbb{P}^5(1,1,1,1,2,2)$ [KT93]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{\frac{1}{4096}} = \begin{pmatrix} 1 & -4 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{\frac{1}{4096}}=(4,0,\frac{5}{3},-144\lambda)$

$$\theta^4-48z(6\theta+1)(2\theta+1)^2(6\theta+5)$$

4	$$\left\{\begin{array}{ccc} 0 & \frac{1}{6912} & \infty \\ \hline 0 & 0 & \frac{1}{6} \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 2 & \frac{5}{6} \end{array}\right\}$$
52	
-256	
5	
6912	

$1{+}1248q{+}4778784q^2{+}18441406464q^3{+}73593869806368q^4{+}299366057720821248q^5{+}O(q^6)$

$n_1^0=4992$, $n_2^0=2388768$, $n_3^0=2732060032$, $n_4^0=4599616564224$, $n_5^0=9579713847066240$, $n_6^0=22839268002374163616$

$e_{\frac{1}{6912}}=-\frac{1}{6}$

$n_1^1=0$, $n_2^1=-504$, $n_3^1=1228032$, $n_4^1=79275665304$, $n_5^1=633074010435840$, $n_6^1=3666182351841110880$

$X(2,6)\subset\mathbb{P}^5(1,1,1,1,1,3)$ [KT93]

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&4&1&0\\0&0&1&1\end{pmatrix},\ M_{\frac{1}{6912}}=\begin{pmatrix}1&-5&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v^1_{\frac{1}{6912}}=(4,0,\frac{13}{6},-256\lambda)$

$$\theta^4-z\left(72+714\theta+12723/5\theta^2+18306/5\theta^3+10449/5\theta^4\right)$$
$$-z^2\left(42624/5+248832/5\theta-69984/25\theta^2-10240992/25\theta^3-9051264/25\theta^4\right)$$
$$-z^3\left(1741824/5+13436928/5\theta+186437376/25\theta^2+60466176/5\theta^3+458535168/25\theta^4\right)$$
$$+53747712/25z^4(3\theta+1)^2(3\theta+2)^2$$

5	$$\left\{\begin{array}{ccccc} 0 & \frac{71}{1728}-\frac{17}{1728}\sqrt{17} & \frac{5}{432} & \frac{71}{1728}+\frac{17}{1728}\sqrt{17} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 1 & 1 & \frac{1}{3} \\ 0 & 1 & 3 & 1 & \frac{2}{3} \\ 0 & 2 & 4 & 2 & \frac{2}{3} \end{array}\right\}$$
38	
-102	
4	
$\frac{1917}{2}+\frac{459}{2}\sqrt{17}$	

$1{+}\frac{2043}{5}q{+}\frac{2234187}{5}q^2{+}480506616q^3{+}\frac{2753509074507}{5}q^4{+}\frac{3221857487091168}{5}q^5{+}O(q^6)$

$n_1^0=2043$, $n_2^0=279018$, $n_3^0=88982631$, $n_4^0=43023544380$, $n_5^0=25774859896713$, $n_6^0=17683308060957072$

$e_{\frac{71}{1728}-\frac{17}{1728}\sqrt{17}}=-\frac{1}{6}$, $e_{\frac{71}{1728}+\frac{17}{1728}\sqrt{17}}=-\frac{1}{6}$

$n_1^1=-425$, $n_2^1=-62094$, $n_3^1=-26244689$, $n_4^1=-15374152562$, $n_5^1=-10137191467518$, $n_6^1=-6952996463048562$

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&5&1&0\\0&0&1&1\end{pmatrix},\ M_{\frac{71}{1728}+\frac{17}{1728}\sqrt{17}}=\begin{pmatrix}1&-3&-1&-1\\0&1&0&0\\0&3&2&1\\0&-3&-1&0\end{pmatrix},\ M_{\frac{5}{432}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},$$
$$M_{\frac{71}{1728}-\frac{17}{1728}\sqrt{17}}=\begin{pmatrix}1&-4&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v^1_{\frac{71}{1728}-\frac{17}{1728}\sqrt{17}}=(5,0,\frac{19}{12},-102\lambda)$, $v^1_{\frac{71}{1728}+\frac{17}{1728}\sqrt{17}}=(5,0,\frac{7}{12},-102\lambda)$

<table>
<tr><td colspan="2">$\theta^4 - z\left(2000\theta^4 + 3904\theta^3 + 2708\theta^2 + 756\theta + 76\right)$
$+z^2\left(634880\theta^4 + 634880\theta^3 - 213760\theta^2 - 186240\theta - 28320\right)$
$-z^3\left(5120000\theta^4 + 245760\theta^3 - 378880\theta^2 + 61440\theta + 30720\right) + 4096z^4(2\theta+1)^4$</td></tr>
<tr><td>5</td><td rowspan="5">$$\left\{\begin{array}{ccccc} 0 & \frac{123}{32}-\frac{55}{32}\sqrt{5} & \frac{1}{16} & \frac{123}{32}+\frac{55}{32}\sqrt{5} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & 1 & \frac{1}{2} \\ 0 & 1 & 3 & 1 & \frac{1}{2} \\ 0 & 2 & 4 & 2 & \frac{1}{2} \end{array}\right\}$$</td></tr>
<tr><td>38</td></tr>
<tr><td>-100</td></tr>
<tr><td>4</td></tr>
<tr><td>$984 + 440\sqrt{5}$</td></tr>
<tr><td colspan="2">$1+444q+457276q^2+514376472q^3+603706740028q^4+722659323926444q^5+O(q^6)$</td></tr>
<tr><td colspan="2">$n_1^0 = 2220$, $n_2^0 = 285520$, $n_3^0 = 95254820$, $n_4^0 = 47164553340$, $n_5^0 = 28906372957040$,
$n_6^0 = 20291945542090480$</td></tr>
<tr><td colspan="2">$e_{\frac{123}{32}-\frac{55}{32}\sqrt{5}} = -\frac{1}{6}$, $e_{\frac{123}{32}+\frac{55}{32}\sqrt{5}} = -\frac{1}{6}$</td></tr>
<tr><td colspan="2">$n_1^1 = 0$, $n_2^1 = 460$, $n_3^1 = 873240$, $n_4^1 = 1498922217$, $n_5^1 = 2306959237408$,
$n_6^1 = 3311397401576040$</td></tr>
<tr><td colspan="2">X_5 Pfaffian [Kan12]</td></tr>
<tr><td colspan="2">$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{\frac{123}{32}-\frac{55}{32}\sqrt{5}} = \begin{pmatrix} 1 & -4 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{\frac{1}{16}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$M_{\frac{123}{32}+\frac{55}{32}\sqrt{5}} = \begin{pmatrix} 1 & -10 & -4 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 15 & 7 & 6 \\ 0 & -15 & -6 & -5 \end{pmatrix}$$</td></tr>
<tr><td colspan="2">$v^1_{\frac{123}{32}-\frac{55}{32}\sqrt{5}}=(5,0,\frac{19}{12},-100\lambda)$, $v^1_{\frac{123}{32}+\frac{55}{32}\sqrt{5}}=(10,0,\frac{1}{6},-200\lambda)$</td></tr>
</table>

<table>
<tr><td colspan="2">$\theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)$</td></tr>
<tr><td>5</td><td rowspan="5">$$\left\{\begin{array}{ccc} 0 & \frac{1}{3125} & \infty \\ \hline 0 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{2}{5} \\ 0 & 1 & \frac{3}{5} \\ 0 & 2 & \frac{4}{5} \end{array}\right\}$$</td></tr>
<tr><td>50</td></tr>
<tr><td>-200</td></tr>
<tr><td>5</td></tr>
<tr><td>3125</td></tr>
<tr><td colspan="2">$1+575q+975375q^2+1712915000q^3+3103585359375q^4+5732647222191200q^5+O(q^6)$</td></tr>
<tr><td colspan="2">$n_1^0 = 2875$, $n_2^0 = 609250$, $n_3^0 = 317206375$, $n_4^0 = 242467530000$, $n_5^0 = 229305888887625$, $n_6^0 = 248249742118022000$</td></tr>
<tr><td colspan="2">$e_{\frac{1}{3125}} = -\frac{1}{6}$</td></tr>
<tr><td colspan="2">$n_1^1 = 0$, $n_2^1 = 0$, $n_3^1 = 609250$, $n_4^1 = 3721431625$, $n_5^1 = 12129909700200$, $n_6^1 = 31147299732677250$</td></tr>
<tr><td colspan="2">$X(5) \subset \mathbb{P}^4$ [COGP91, Mor92]</td></tr>
<tr><td colspan="2">$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{\frac{1}{3125}} = \begin{pmatrix} 1 & -5 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$</td></tr>
<tr><td colspan="2">$v^1_{\frac{1}{3125}}=(5,0,\frac{25}{12},-200\lambda)$</td></tr>
</table>

$$\theta^4-12z(6\theta+1)(6\theta+5)(11\theta^2+11\theta+3)\ -144z^2(6\theta+1)(6\theta+5)(6\theta+7)(6\theta+11)$$

5
62
-310
6
$2376+1080\sqrt{5}$

$$\left\{\begin{array}{cccc} -\frac{11}{864}-\frac{5}{864}\sqrt{5} & 0 & -\frac{11}{864}+\frac{5}{864}\sqrt{5} & \infty \\ \hline 0 & 0 & 0 & \frac{1}{6} \\ 1 & 0 & 1 & \frac{5}{6} \\ 1 & 0 & 1 & \frac{7}{6} \\ 2 & 0 & 2 & \frac{11}{6} \end{array}\right\}$$

$1+684q+2027196q^2+5282248824q^3+14402599055676q^4+39917806117038684q^5+O(q^6)$

$n^0_1=3420$, $n^0_2=1266570$, $n^0_3=978194100$, $n^0_4=1125202892850$, $n^0_5=1596712244681520$, $n^0_6=2593390973297156910$

$e_{-\frac{11}{864}-\frac{5}{864}\sqrt{5}}=-\frac{1}{6}$, $e_{-\frac{11}{864}+\frac{5}{864}\sqrt{5}}=-\frac{1}{6}$

$n^1_1=1$, $n^1_2=-612$, $n^1_3=-3437966$, $n^1_4=524538556$, $n^1_5=45789504955347$, $n^1_6=237013737988236480$

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&5&1&0\\0&0&1&1\end{pmatrix},\quad M_{-\frac{11}{864}-\frac{5}{864}\sqrt{5}}=\begin{pmatrix}-9&-30&-6&-4\\5&16&3&2\\-15&-45&-8&-6\\10&30&6&5\end{pmatrix},\quad M_{-\frac{11}{864}+\frac{5}{864}\sqrt{5}}=\begin{pmatrix}1&-6&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v^1_{-\frac{11}{864}-\frac{5}{864}\sqrt{5}}=(-10,5,-\frac{17}{3},620\lambda+\frac{29}{12})$, $v^1_{-\frac{11}{864}+\frac{5}{864}\sqrt{5}}=(5,0,\frac{31}{12},-310\lambda)$

$$\begin{aligned}&\theta^4-z\left(1296\theta^4+1824\theta^3+1284\theta^2+372\theta+40\right)\\&+z^2\left(2990080\theta^4+2990080\theta^3+2048\theta^2-363520\theta-6656\right)\\&-z^3\left(212336640\theta^4+125829120\theta^3+80609280\theta^2+31457280\theta+458752\right)\ +16777216z^4(2\theta+1)^4\end{aligned}$$

6
36
-72
4
784

$$\left\{\begin{array}{cccc} -\frac{1}{512} & 0 & \frac{1}{784} & \infty \\ \hline 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & 1 & \frac{3}{4} \\ 1 & 0 & 1 & \frac{7}{4} \\ \frac{3}{2} & 0 & 2 & \frac{9}{4} \end{array}\right\}$$

$1+196q+137892q^2+80832040q^3+50551083172q^4+32597782628196q^5+O(q^6)$

$n^0_1=2352$, $n^0_2=206544$, $n^0_3=35925264$, $n^0_4=9478302240$, $n^0_5=3129387132288$, $n^0_6=1184285394173424$

$e_{\frac{1}{1024}}=-\frac{1}{6}$, $e_{\frac{1}{16}}=-\frac{1}{3}$

$n^1_1=-322$, $n^1_2=-28415$, $n^1_3=-5756564$, $n^1_4=-1708233758$, $n^1_5=-595308482782$, $n^1_6=-218747404004982$

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&6&1&0\\0&0&1&1\end{pmatrix},\quad M_{\frac{1}{16}}=\begin{pmatrix}1&-6&-2&-2\\0&1&0&0\\0&6&3&2\\0&-6&-2&-1\end{pmatrix},\quad M_{\frac{1}{128}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad M_{\frac{1}{1024}}=\begin{pmatrix}1&-4&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

<table>
<tr><td>$v^1_{\frac{1}{1024}}$=$(6,0,\frac{3}{2},-72\lambda)$, $v^2_{\frac{1}{16}}$=$(12,0,1,-144\lambda)$</td></tr>
</table>

<table>
<tr><td colspan="2">$\theta^4-z\left(1164\theta^4+2328\theta^3+1630\theta^2+466\theta+50\right)+4z^2(2316\theta^2+4632\theta+1907)(\theta+1)^2$
$-4624z^3(2\theta+5)(2\theta+1)(\theta+1)(\theta+2)$</td></tr>
<tr><td>6</td><td rowspan="5">$\left\{\begin{array}{cccc} 0 & \frac{1}{1156} & \frac{1}{4} & \infty \\ \hline 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & \frac{5}{2} \end{array}\right\}$</td></tr>
<tr><td>36</td></tr>
<tr><td>-64</td></tr>
<tr><td>4</td></tr>
<tr><td>1156</td></tr>
<tr><td colspan="2">$1+266q+154826q^2+102898706q^3+70788595914q^4+49700857746766q^5+O(q^6)$</td></tr>
<tr><td colspan="2">$n^0_1=1596$, $n^0_2=115920$, $n^0_3=22866320$, $n^0_4=6636416352$, $n^0_5=2385641171832$,
$n^0_6=982250381786688$</td></tr>
<tr><td colspan="2">$e_{\frac{1}{1156}}=-\frac{1}{6}$, $e_{\frac{1}{4}}=-2$</td></tr>
<tr><td colspan="2">$n^1_1=0$, $n^1_2=138$, $n^1_3=174464$, $n^1_4=169718436$, $n^1_5=152047741248$, $n^1_6=127834237726406$</td></tr>
<tr><td colspan="2">$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&6&1&0\\0&0&1&1\end{pmatrix}$, $M_{\frac{1}{4}}=\begin{pmatrix}5&20&7&5\\-4&-13&-4&-2\\12&30&7&0\\-4&-2&2&5\end{pmatrix}$, $M_{\frac{1}{1156}}=\begin{pmatrix}1&-4&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$</td></tr>
<tr><td colspan="2">$v^1_{\frac{1}{1156}}$=$(6,0,\frac{3}{2},-64\lambda)$</td></tr>
</table>

<table>
<tr><td colspan="2">$\theta^4-12z(4\theta+1)(3\theta+1)(3\theta+2)(4\theta+3)$</td></tr>
<tr><td>6</td><td rowspan="5">$\left\{\begin{array}{ccc} 0 & \frac{1}{1728} & \infty \\ \hline 0 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 2 & \frac{3}{4} \end{array}\right\}$</td></tr>
<tr><td>48</td></tr>
<tr><td>-156</td></tr>
<tr><td>5</td></tr>
<tr><td>1728</td></tr>
<tr><td colspan="2">$1+324q+298404q^2+291395880q^3+293151164580q^4+300655662948324q^5+O(q^6)$</td></tr>
<tr><td colspan="2">$n^0_1=1944$, $n^0_2=223560$, $n^0_3=64754568$, $n^0_4=27482893704$, $n^0_5=14431471821504$,
$n^0_6=8675274727197720$</td></tr>
<tr><td colspan="2">$e_{\frac{1}{1728}}=-\frac{1}{6}$</td></tr>
<tr><td colspan="2">$n^1_1=0$, $n^1_2=27$, $n^1_3=161248$, $n^1_4=381704238$, $n^1_5=638555324400$, $n^1_6=891094220156286$</td></tr>
<tr><td colspan="2">$X(4,6)\subset\mathbb{P}^5(1,1,1,2,2,3)$ [KT93]</td></tr>
<tr><td colspan="2">$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&6&1&0\\0&0&1&1\end{pmatrix}$, $M_{\frac{1}{1728}}=\begin{pmatrix}1&-5&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$</td></tr>
<tr><td colspan="2">$v^1_{\frac{1}{1728}}$=$(6,0,2,-156\lambda)$</td></tr>
</table>

$$\begin{aligned}&49\theta^4 - z\left(53424\theta^4 + 105336\theta^3 + 74718\theta^2 + 22050\theta + 2352\right)\\&+z^2\left(5244448\theta^4 + 4570056\theta^3 - 3090084\theta^2 - 2472296\theta - 40320\right)\\&-z^3\left(11819520\theta^4 - 6531840\theta^3 - 7223040\theta^2 - 1542240\theta - 6048\right) - 15552z^4(3\theta+1)(2\theta+1)^2(3\theta+2)\end{aligned}$$

7	$$\left\{\begin{array}{ccccc} -5/4-\frac{13}{18}\sqrt{3} & 0 & -5/4+\frac{13}{18}\sqrt{3} & \frac{7}{36} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & 1 & 1 & \frac{1}{2} \\ 1 & 0 & 1 & 3 & \frac{1}{2} \\ 2 & 0 & 2 & 4 & \frac{2}{3} \end{array}\right\}$$
46	
-120	
5	
$540+312\sqrt{3}$	

$1+\frac{1434}{7}q+\frac{825642}{7}q^2+\frac{504268878}{7}q^3+\frac{192233419562}{7}q^4+\frac{205848453920184}{7}q^5+\frac{134452784879321598}{7}q^6+O(q^7)$

$n^0_1 = 1434$, $n^0_2 = 103026$, $n^0_3 = 18676572$, $n^0_4 = 4988009280$, $n^0_5 = 1646787631350$, $n^0_6 = 622466594325132$

$e_{-5/4-\frac{13}{18}\sqrt{3}} = -\frac{1}{6}$, $e_{-5/4+\frac{13}{18}\sqrt{3}} = -\frac{1}{6}$

$n^1_1 = 0$, $n^1_2 = 26$, $n^1_3 = 53076$, $n^1_4 = 65171037$, $n^1_5 = 63899034076$, $n^1_6 = 54697568345378$

X_7 Pfaffian [Kan12]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad M_{-5/4-\frac{13}{18}\sqrt{3}} = \begin{pmatrix} -5 & -26 & -6 & -4 \\ 3 & 14 & 3 & 2 \\ -9 & -39 & -8 & -6 \\ 3 & 13 & 3 & 3 \end{pmatrix}, \quad M_{\frac{7}{36}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_{-5/4+\frac{13}{18}\sqrt{3}} = \begin{pmatrix} 1 & -5 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{-5/4-\frac{13}{18}\sqrt{3}} = (-14, 7, -\frac{10}{3}, 240\lambda + \frac{13}{12})$, $v^1_{-5/4+\frac{13}{18}\sqrt{3}} = (7, 0, \frac{23}{12}, -120\lambda)$

$$\begin{aligned}&64\theta^4 - z\left(1344 + 113920\theta + 378320\theta^2 + 528800\theta^3 + 400480\theta^4\right)\\&+z^2\left(55080 + 6035040\theta + 29454570\theta^2 + 70543980\theta^3 + 49616010\theta^4\right)\\&-z^3\left(46661832 + 3103090560\theta + 7037219250\theta^2 + 3288504420\theta^3 + 2813794200\theta^4\right)\\&+z^4\left(1364563341 + 88246368540\theta + 196671551850\theta^2 + 70441914060\theta^3 - 137779031700\theta^4\right)\\&-z^5\left(54717165360 + 2505988777860\theta + 1776155538150\theta^2 - 8043136329780\theta^3 + 1513809688500\theta^4\right)\\&+z^6\left(409632597036 + 18851880994740\theta + 34578570044880\theta^2 + 47652720147000\theta^3\right.\\&\left.+73343864174220\theta^4\right) + z^7\left(11046132982368 + 517787483548500\theta + 980344302185160\theta^2\right.\\&\left.+925113637273320\theta^3 + 390066570939870\theta^4\right) + 102521921742603z^8(\theta+1)^4\end{aligned}$$

8	$$\left\{\begin{array}{cccccccc} x_1 & x_2 & -\frac{7}{594}-\frac{1}{1782}\sqrt{1497} & 0 & x_3 & -\frac{7}{594}+\frac{1}{1782}\sqrt{1497} & \frac{1}{27} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 0 & 1 & 3 & 1 & 1 \\ 2 & 2 & 4 & 0 & 2 & 4 & 2 & 1 \end{array}\right\}$$
32	
-20	
4	
$1/x_3$	

$f(x) = 6561x^3 + 2430x^2 + 457X - 1$, $x_1 = -0.1863 - 0.1891i$, $x_2 = -0.1863 + 0.1891i$, $x_3 = 0.0022$, $x_4 = -\frac{7}{594} - \frac{1}{1782}\sqrt{1497}$, $x_5 = -\frac{7}{594} + \frac{1}{1782}\sqrt{1497}$

$1+\frac{713}{8}q+\frac{210249}{8}q^2+\frac{61728653}{8}q^3+\frac{17129939529}{8}q^4+\frac{5011823039963}{8}q^5+\frac{1489022944012557}{8}q^6+O(q^7)$

$n^0_1 = 1426$, $n^0_2 = 52384$, $n^0_3 = 4572440$, $n^0_4 = 535304040$, $n^0_5 = 80189168628$, $n^0_6 = 13787248908096$

$e_{x_1} = -\frac{1}{6}$, $e_{x_2} = -\frac{1}{6}$, $e_{x_3} = -\frac{1}{6}$, $e_{\frac{1}{27}} = -\frac{1}{3}$

$n^1_1 = 2,\ n^1_2 = 10,\ n^1_3 = 21650,\ n^1_4 = 8692206,\ n^1_5 = 2823202414,\ n^1_6 = 905305771966$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{x_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{x_1} = \begin{pmatrix} -9 & -38 & -8 & -4 \\ 10 & 39 & 8 & 4 \\ -55 & -209 & -43 & -22 \\ 40 & 152 & 32 & 17 \end{pmatrix},$$

$$M_{\frac{1}{27}} = \begin{pmatrix} 1 & -6 & -2 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 6 & 3 & 2 \\ 0 & -6 & -2 & -1 \end{pmatrix},\ M_{x_5} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{x_3} = \begin{pmatrix} 1 & -4 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{x_2} =$$

$$\begin{pmatrix} 11 & -6 & 0 & -4 \\ 10 & -5 & 0 & -4 \\ -25 & 15 & 1 & 10 \\ 10 & -6 & 0 & -3 \end{pmatrix}$$

$v^1_{x_1}=(16,16,\frac{17}{3},-40\lambda+\frac{7}{3}),\ v^1_{x_2}=(16,-16,\frac{17}{3},-40\lambda-\frac{7}{3}),\ v^1_{x_3}=(8,0,\frac{4}{3},-20\lambda),$
$v^2_{\frac{1}{27}}=(16,0,\frac{2}{3},-40\lambda)$

$$\theta^4 - z\left(320\theta^4+896\theta^3+608\theta^2+160\theta+16\right)$$
$$-z^2\left(860160\theta^4+2457600\theta^3+2672640\theta^2+1372160\theta+256000\right)-65536z^3(2\theta+1)^2(23\theta^2+55\theta+39)$$
$$-8388608z^4(2\theta+1)^2(2\theta+3)^2$$

8	$\left\{\begin{array}{cccc} -\frac{1}{64} & 0 & \frac{1}{512} & \infty \\ \hline 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ \frac{3}{2} & 0 & 1 & \frac{3}{2} \\ 2 & 0 & 2 & \frac{3}{2} \end{array}\right\}$
32	
-8	
4	
512	

$1+128q+32800q^2+10330112q^3+3296727840q^4+1080607664128q^5+358443314231296q^6+O(q^7)$

$n^0_1 = 1024,\ n^0_2 = 32672,\ n^0_3 = 3060736,\ n^0_4 = 412086880,\ n^0_5 = 69158890496,$
$n^0_6 = 13275677921056$

$e_{-\frac{1}{64}} = -\frac{1}{2},\ e_{\frac{1}{512}} = -\frac{1}{6}$

$n^1_1 = -16,\ n^1_2 = 1968,\ n^1_3 = -15168,\ n^1_4 = 12312352,\ n^1_5 = 3892748768,\ n^1_6 = 1390131583872$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{-\frac{1}{64}} = \begin{pmatrix} -3 & -8 & -1 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 32 & 11 & 8 \\ -4 & -40 & -11 & -7 \end{pmatrix},\ M_{\frac{1}{512}} = \begin{pmatrix} 1 & -4 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{\frac{1}{512}}=(8,0,\frac{4}{3},-8\lambda)$

$$\theta^4 - z\left(32+288\theta+960\theta^2+1344\theta^3+768\theta^4\right)$$
$$-z^2\left(4352+256000\theta+368640\theta^2-266240\theta^3-419840\theta^4\right)$$
$$+z^3\left(16384+1966080\theta+7208960\theta^2+7864320\theta^3-4587520\theta^4\right)-262144z^4(2\theta+1)^4$$

8	$\left\{\begin{array}{ccccc} -\frac{11}{128}-\frac{5}{128}\sqrt{5} & 0 & -\frac{11}{128}+\frac{5}{128}\sqrt{5} & \frac{1}{32} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 1 & \frac{1}{2} \\ 1 & 0 & 1 & 3 & \frac{1}{2} \\ 2 & 0 & 2 & 4 & \frac{1}{2} \end{array}\right\}$
44	
-92	
5	
$352+160\sqrt{5}$	

$1+128q+54944q^2+21610496q^3+9167755040q^4+3961492400128q^5+O(q^6)$

$n_1^0 = 1024$, $n_2^0 = 54816$, $n_3^0 = 6403072$, $n_4^0 = 1145962512$, $n_5^0 = 253535513600$, $n_6^0 = 64206844371872$

$e_{-\frac{11}{128}-\frac{5}{128}\sqrt{5}} = -\frac{1}{6}$, $e_{-\frac{11}{128}+\frac{5}{128}\sqrt{5}} = -\frac{1}{6}$

$n_1^1 = 0$, $n_2^1 = 48$, $n_3^1 = 26624$, $n_4^1 = 17431712$, $n_5^1 = 10116397056$, $n_6^1 = 5458349643392$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{-\frac{11}{128}-\frac{5}{128}\sqrt{5}} = \begin{pmatrix} -5 & -28 & -6 & -4 \\ 3 & 15 & 3 & 2 \\ -12 & -56 & -11 & -8 \\ 6 & 28 & 6 & 5 \end{pmatrix}, M_{\frac{1}{32}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_{-\frac{11}{128}+\frac{5}{128}\sqrt{5}} = \begin{pmatrix} 1 & -5 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{-\frac{11}{128}-\frac{5}{128}\sqrt{5}}=(-16, 8, -\frac{11}{3}, 184\lambda + \frac{7}{6})$, $v^1_{-\frac{11}{128}+\frac{5}{128}\sqrt{5}}=(8, 0, \frac{11}{6}, -92\lambda)$

$\theta^4 - 16z(4\theta+1)(2\theta+1)^2(4\theta+3)$

8	$\left\{\begin{array}{ccc} 0 & \frac{1}{1024} & \infty \\ \hline 0 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 2 & \frac{3}{4} \end{array}\right\}$
56	
-176	
6	
1024	

$1+160q+92448q^2+52836352q^3+31071312672q^4+18608176284160q^5+O(q^6)$

$n_1^0 = 1280$, $n_2^0 = 92288$, $n_3^0 = 15655168$, $n_4^0 = 3883902528$, $n_5^0 = 1190923282176$, $n_6^0 = 417874605342336$

$e_{\frac{1}{1024}} = -\frac{1}{6}$

$n_1^1 = 0$, $n_2^1 = 0$, $n_3^1 = 2560$, $n_4^1 = 17407072$, $n_5^1 = 24834612736$, $n_6^1 = 23689021707008$

$X(2,4) \subset \mathbb{P}^5$ [LT93]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{\frac{1}{1024}} = \begin{pmatrix} 1 & -6 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{\frac{1}{1024}}=(8, 0, \frac{7}{3}, -176\lambda)$

$\theta^4 - 9z(3\theta+1)^2(3\theta+2)^2$

9	$\left\{\begin{array}{ccc} 0 & \frac{1}{729} & \infty \\ \hline 0 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 2 & \frac{2}{3} \end{array}\right\}$
54	
-144	
6	
729	

$1+117q+47061q^2+19273095q^3+8102791125q^4+3469276285992q^5+O(q^6)$

$n_1^0 = 1053$, $n_2^0 = 52812$, $n_3^0 = 6424326$, $n_4^0 = 1139448384$, $n_5^0 = 249787892583$, $n_6^0 = 62660964509532$

$e_{\frac{1}{729}} = -\frac{1}{6}$

$n^1_1 = 0,\ n^1_2 = 0,\ n^1_3 = 3402,\ n^1_4 = 5520393,\ n^1_5 = 4820744484,\ n^1_6 = 3163476678678$

$X(3,3) \subset \mathbb{P}^6$ [LT93]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{\frac{1}{729}} = \begin{pmatrix} 1 & -6 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{\frac{1}{729}}=(9, 0, \frac{9}{4}, -144\lambda)$

$$\begin{aligned}&\theta^4 - z\left(465\theta^4 + 594\theta^3 + 431\theta^2 + 134\theta + 16\right)\\ &+z^2\left(42000\theta^4 + 30576\theta^3 - 15136\theta^2 - 14144\theta - 2816\right)\\ &-z^3\left(1030720\theta^4 - 231936\theta^3 - 335424\theta^2 - 82176\theta - 2304\right)\\ &-z^4\left(2222080\theta^4 + 4788224\theta^3 + 4386816\theta^2 + 1992704\theta + 372736\right)+3211264z^5(\theta+1)^4\end{aligned}$$

10	$\left\{\begin{array}{cccccc} -\frac{11}{64} - \frac{5}{64}\sqrt{5} & 0 & -\frac{11}{64} + \frac{5}{64}\sqrt{5} & \frac{1}{56} & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 4 & 2 & 1 \end{array}\right\}$
40	
-50	
5	
$176 + 80\sqrt{5}$	

$1+55q+15375q^2+3147040q^3+682334735q^4+153499785680q^5+35067730548360q^6+O(q^7)$

$n^0_1 = 550,\ n^0_2 = 19150,\ n^0_3 = 1165550,\ n^0_4 = 106612400,\ n^0_5 = 12279982850,\ n^0_6 = 1623505897500$

$e_{-\frac{11}{64}-\frac{5}{64}\sqrt{5}} = -\frac{1}{6},\ e_{-\frac{11}{64}+\frac{5}{64}\sqrt{5}} = -\frac{1}{6},\ e_1 = -\frac{1}{3}$

$n^1_1 = 0,\ n^1_2 = 50,\ n^1_3 = 8800,\ n^1_4 = 2205000,\ n^1_5 = 571891188,\ n^1_6 = 145348448125$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 10 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{-\frac{11}{64}-\frac{5}{64}\sqrt{5}} = \begin{pmatrix} -19 & -120 & -24 & -16 \\ 10 & 61 & 12 & 8 \\ -50 & -300 & -59 & -40 \\ 25 & 150 & 30 & 21 \end{pmatrix},\ M_1 = \begin{pmatrix} 1 & -60 & -18 & -18 \\ 0 & 1 & 0 & 0 \\ 0 & 100 & 31 & 30 \\ 0 & -100 & -30 & -29 \end{pmatrix},\ M_{\frac{1}{56}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{-\frac{11}{64}+\frac{5}{64}\sqrt{5}} = \begin{pmatrix} 1 & -5 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{-\frac{11}{64}-\frac{5}{64}\sqrt{5}}=(40, -20, \frac{20}{3}, -200\lambda - \frac{5}{3}),\ v^1_{-\frac{11}{64}+\frac{5}{64}\sqrt{5}}=(10, 0, \frac{5}{3}, -50\lambda),\ v^2_1=(60, 0, 0, -300\lambda)$

$$\begin{aligned}&25\theta^4 - z\left(8080\theta^4 + 21920\theta^3 + 15460\theta^2 + 4500\theta + 500\right)\\ &-z^2\left(1070336\theta^4 + 2198528\theta^3 + 1617536\theta^2 + 844800\theta + 186000\right)\\ &-3840z^3(2\theta+1)(5672\theta^3 + 9500\theta^2 + 8422\theta + 2689)\\ &-294912z^4(2\theta+1)(1208\theta^3 + 2892\theta^2 + 2842\theta + 969)\\ &-28311552z^5(2\theta+1)(6\theta+5)(6\theta+7)(2\theta+3)\end{aligned}$$

10	$\left\{\begin{array}{ccccc} -\frac{1}{16} & -\frac{5}{192} & 0 & \frac{1}{432} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & 1 & 0 & 1 & \frac{5}{6} \\ \frac{3}{4} & 3 & 0 & 1 & \frac{7}{6} \\ 1 & 4 & 0 & 2 & \frac{3}{2} \end{array}\right\}$
40	
-32	
5	
432	

$1+\frac{492}{5}q+16700q^2+4709784q^3+\frac{5844318764}{5}q^4+\frac{522739598492}{5}q^5+\frac{399663654969976}{5}q^6+O(q^7)$

$n^0_1 = 984,\ n^0_2 = 20752,\ n^0_3 = 1744328,\ n^0_4 = 182632352,\ n^0_5 = 24363833568,\ n^0_6 = 3700589179056$

<table>
<tr><td colspan="2">$e_{-\frac{1}{16}} = -\frac{1}{2}, e_{\frac{1}{432}} = -\frac{1}{6}$</td></tr>
<tr><td colspan="2">$n^1_1 = 0, n^1_2 = -5, n^1_3 = 2608, n^1_4 = 1122718, n^1_5 = 508591872, n^1_6 = 180758602038$</td></tr>
<tr><td colspan="2">$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 10 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{-\frac{5}{192}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{-\frac{1}{16}} = \begin{pmatrix} -1 & 4 & 2 & 2 \\ -2 & -13 & -3 & -2 \\ 26 & 128 & 25 & 14 \\ -26 & -110 & -19 & -9 \end{pmatrix},$
$M_{\frac{1}{432}} = \begin{pmatrix} 1 & -5 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$</td></tr>
<tr><td colspan="2">$v^1_{\frac{1}{432}}=(10, 0, \frac{5}{3}, -32\lambda)$</td></tr>
</table>

<table>
<tr><td colspan="2">$25\theta^4 - z\left(700 + 6100\theta + 19620\theta^2 + 27040\theta^3 + 13760\theta^4\right)$
$-z^2\left(4240 + 20160\theta - 1536\theta^2 - 112128\theta^3 - 93696\theta^4\right)$
$-z^3\left(5120 + 30720\theta + 70656\theta^2 + 122880\theta^3 + 180224\theta^4\right)\ 4096z^4(2\theta+1)^4$</td></tr>
<tr><td>10</td><td rowspan="5">$\left\{\begin{array}{ccccc} 0 & \frac{17}{16} - 3/4\sqrt{2} & \frac{5}{16} & \frac{17}{16} + 3/4\sqrt{2} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & 1 & \frac{1}{2} \\ 0 & 1 & 3 & 1 & \frac{1}{2} \\ 0 & 2 & 4 & 2 & \frac{1}{2} \end{array}\right\}$</td></tr>
<tr><td>52</td></tr>
<tr><td>-116</td></tr>
<tr><td>6</td></tr>
<tr><td>512</td></tr>
<tr><td colspan="2">$1 + \frac{444}{5}q + 26556q^2 + 8110392\,q^3 + \frac{12797926956}{5}q^4 + \frac{4108561110444}{5}q^5 + O(q^6)$</td></tr>
<tr><td colspan="2">$n^0_1 = 888, n^0_2 = 33084, n^0_3 = 3003816, n^0_4 = 399931068, n^0_5 = 65736977760,$
$n^0_6 = 12365885835028$</td></tr>
<tr><td colspan="2">$e_{\frac{17}{16}-3/4\sqrt{2}} = -\frac{1}{6}, e_{\frac{17}{16}+3/4\sqrt{2}} = -\frac{1}{6}$</td></tr>
<tr><td colspan="2">$n^1_1 = 0, n^1_2 = 1, n^1_3 = 2496, n^1_4 = 2089392, n^1_5 = 1210006912, n^1_6 = 571433264626$</td></tr>
<tr><td colspan="2">X_{10} Pfaffian [Kan12]</td></tr>
<tr><td colspan="2">$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 10 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{\frac{17}{16}-3/4\sqrt{2}} = \begin{pmatrix} 1 & -4 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -8 & -1 & -2 \\ 0 & 8 & 2 & 3 \end{pmatrix}, M_{\frac{5}{16}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$
$M_{\frac{17}{16}+3/4\sqrt{2}} = \begin{pmatrix} 1 & -2 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$</td></tr>
<tr><td colspan="2">$v^1_{\frac{17}{16}-3/4\sqrt{2}}=(10, 0, \frac{13}{6}, -116\lambda), v^1_{\frac{17}{16}+3/4\sqrt{2}}=(10, 0, \frac{1}{6}, -116\lambda)$</td></tr>
</table>

<table>
<tr><td colspan="2">$\theta^4 - 4z(4\theta+1)(4\theta+3)(11\theta^2 + 11\theta + 3)\ -16z^2(4\theta+1)(4\theta+3)(4\theta+5)(4\theta+7)$</td></tr>
<tr><td>10</td><td rowspan="5">$\left\{\begin{array}{cccc} -\frac{11}{128} - \frac{5}{128}\sqrt{5} & 0 & -\frac{11}{128} + \frac{5}{128}\sqrt{5} & \infty \\ \hline 0 & 0 & 0 & \frac{1}{4} \\ 1 & 0 & 1 & \frac{3}{4} \\ 1 & 0 & 1 & \frac{5}{4} \\ 2 & 0 & 2 & \frac{7}{4} \end{array}\right\}$</td></tr>
<tr><td>64</td></tr>
<tr><td>-200</td></tr>
<tr><td>7</td></tr>
<tr><td>$352 + 160\sqrt{5}$</td></tr>
<tr><td colspan="2">$1{+}92q{+}40508q^2{+}15805784q^3{+}6398889276q^4{+}2631769894092q^5{+}1095690103233560q^6{+}O(q^7)$</td></tr>
</table>

$n_1^0 = 920$, $n_2^0 = 50520$, $n_3^0 = 5853960$, $n_4^0 = 999820120$, $n_5^0 = 210541591520$, $n_6^0 = 50726392934600$

$e_{-\frac{11}{128}-\frac{5}{128}\sqrt{5}} = -\frac{1}{6}$, $e_{-\frac{11}{128}+\frac{5}{128}\sqrt{5}} = -\frac{1}{6}$

$n_1^1 = 0$, $n_2^1 = 3$, $n_3^1 = -1840$, $n_4^1 = 340006$, $n_5^1 = 1947267584$, $n_6^1 = 1647668878150$

$$X \xrightarrow{(2:1)} B_5 \text{ [Bor96]}$$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 10 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{-\frac{11}{128}-\frac{5}{128}\sqrt{5}} = \begin{pmatrix} -9 & -40 & -6 & -4 \\ 5 & 21 & 3 & 2 \\ -30 & -120 & -17 & -12 \\ 20 & 80 & 12 & 9 \end{pmatrix},\ M_{-\frac{11}{128}+\frac{5}{128}\sqrt{5}} = \begin{pmatrix} 1 & -7 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{-\frac{11}{128}-\frac{5}{128}\sqrt{5}} = (20, -10, \frac{19}{3}, -400\lambda - \frac{7}{3})$, $v^1_{-\frac{11}{128}+\frac{5}{128}\sqrt{5}} = (10, 0, \frac{8}{3}, -200\lambda)$

$$121\theta^4 - z\left(2662 + 225506\theta + 715555\theta^2 + 980980\theta^3 + 532730\theta^4\right)$$
$$-z^2\left(59532 + 2957768\theta + 934840\theta^2 - 13738240\theta^3 - 12487360\theta^4\right)$$
$$-z^3\left(250272 + 12582240\theta + 23195040\theta^2 + 41015040\theta^3 + 81915520\theta^4\right)$$
$$z^4\left(2048(2\theta+1)(1964\theta^3 + 3078\theta^2 + 1853\theta + 419)\right) - 262144z^5(2\theta+1)(\theta+1)^2(2\theta+3)$$

11	$\left\{\begin{array}{cccccc} 0 & \frac{13}{4}-\frac{15}{8}\sqrt{3} & \frac{11}{128} & 1 & \frac{13}{4}+\frac{15}{8}\sqrt{3} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & 1 & 1 \\ 0 & 2 & 4 & 2 & 2 & \frac{3}{2} \end{array}\right\}$
50	
-92	
6	
$208 + 120\sqrt{3}$	

$1 + \frac{741}{11}q + \frac{178597}{11}q^2 + \frac{41208006}{11}q^3 + \frac{10069083813}{11}q^4 + \frac{2499999425491}{11}q^5 + \frac{628361122601062}{11}q^6 + O(q^7)$

$n_1^0 = 2223$, $n_2^0 = 66696$, $n_3^0 = 4578585$, $n_4^0 = 471979932$, $n_5^0 = 59999986194$, $n_6^0 = 8727237239100$

$e_{\frac{13}{4}-\frac{15}{8}\sqrt{3}} = -\frac{1}{6}$, $e_1 = -\frac{1}{6}$, $e_{\frac{13}{4}+\frac{15}{8}\sqrt{3}} = -\frac{1}{6}$

$n_1^1 = -472$, $n_2^1 = -11820$, $n_3^1 = -1031768$, $n_4^1 = -128119382$, $n_5^1 = -18826534176$, $n_6^1 = -3071067757512$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 11 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{\frac{13}{4}+\frac{15}{8}\sqrt{3}} = \begin{pmatrix} -35 & -246 & -48 & -36 \\ 12 & 83 & 16 & 12 \\ -30 & -205 & -39 & -30 \\ -6 & -41 & -8 & -5 \end{pmatrix},\ M_1 = \begin{pmatrix} 1 & -12 & -3 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 24 & 7 & 6 \\ 0 & -24 & -6 & -5 \end{pmatrix},\ M_{\frac{11}{128}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{\frac{13}{4}-\frac{15}{8}\sqrt{3}} = \begin{pmatrix} 1 & -6 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{\frac{13}{4}-\frac{15}{8}\sqrt{3}} = (11, 0, \frac{25}{12}, -92\lambda)$, $v^3_1 = (33, 0, \frac{1}{4}, -276\lambda)$, $v^1_{\frac{13}{4}+\frac{15}{8}\sqrt{3}} = (66, -22, \frac{13}{2}, -552\lambda - \frac{11}{6})$

$$121\theta^4 - z\left(2783+235595\theta+752295\theta^2+1034000\theta^3+508750\theta^4\right)$$
$$-z^2\left(4444+295130\theta+1603820\theta^2+4176880\theta^3+2775430\theta^4\right)$$
$$+z^3\left(72226+4234890\theta+5589260\theta^2-6794280\theta^3-8341630\theta^4\right)$$
$$+z^4\left(94818+4251550\theta+5557850\theta^2-5065720\theta^3-13954910\theta^4\right)$$
$$-z^5\left(49505+3071040\theta+3382550\theta^2-571180\theta^3+14388080\theta^4\right)$$
$$-z^6\left(33242+1464660\theta+2788750\theta^2+4533660\theta^3+6897170\theta^4\right)$$
$$-z^7\left(12958+792680\theta+1966500\theta^2+2347640\theta^3+1145320\theta^4\right)\ -1444z^8(\theta+1)^4$$

11	
50	
-90	
6	
$1/x_5$	

$$\left\{\begin{array}{cccccccc} x_1 & x_2 & x_3 & -\frac{25}{76}-\frac{1}{76}i\sqrt{1047} & -\frac{25}{76}+\frac{1}{76}i\sqrt{1047} & 0 & x_5 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 & 3 & 0 & 1 & 1 \\ 2 & 2 & 2 & 4 & 4 & 0 & 2 & 1 \end{array}\right\}$$

$f(x) = x^4+78x^3+374x^2+425x-1$, $x_1=-72.9532$, $x_2=-3.2563$, $x_3=-1.7927$, $x_5=0.0023$

$1+\frac{795}{11}q+\frac{177259}{11}q^2+\frac{43057371}{11}q^3+\frac{10641704107}{11}q^4+\frac{2680205182670}{11}q^5+\frac{683390292486259}{11}q^6+O(q^7)$

$n_1^0=795$, $n_2^0=22058$, $n_3^0=1594688$, $n_4^0=166273857$, $n_5^0=21441641455$, $n_6^0=3163843746539$

$e_{x_1}=-\frac{1}{6}$, $e_{x_2}=-\frac{1}{6}$, $e_{x_3}=-\frac{1}{6}$, $e_{x_5}=-\frac{1}{6}$

$n_1^1=0$, $n_2^1=0$, $n_3^1=1353$, $n_4^1=810927$, $n_5^1=355799260$, $n_6^1=129995248341$

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&11&1&0\\0&0&1&1\end{pmatrix},\quad M_{x_3}=\begin{pmatrix}-5&-34&-6&-4\\3&18&3&2\\-15&-85&-14&-10\\6&34&6&5\end{pmatrix},\quad M_{x_2}=\begin{pmatrix}-13&-56&-8&-4\\14&57&8&4\\-112&-448&-63&-32\\77&308&44&23\end{pmatrix},\quad M_{x_1}=\begin{pmatrix}33&12&-12&-16\\8&4&-3&-4\\-216&-81&82&108\\232&87&-87&-115\end{pmatrix},\quad M_{-\frac{25}{76}+\frac{1}{76}i\sqrt{1047}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad M_{x_5}=\begin{pmatrix}1&-6&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad M_{-\frac{25}{76}-\frac{1}{76}i\sqrt{1047}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v_{x_1}^1=(-44,-11,\frac{79}{6},360\lambda-\frac{71}{12})$, $v_{x_2}^1=(22,-22,\frac{55}{6},-180\lambda-\frac{17}{6})$, $v_{x_3}^1=(22,-11,\frac{11}{3},-180\lambda-\frac{11}{12})$, $v_{x_5}^1=(11,0,\frac{25}{12},-90\lambda)$

$$121\theta^4 - z\left(2904+246840\theta+790680\theta^2+1087680\theta^3+480480\theta^4\right)$$
$$+z^2\left(50688+2048640\theta-399968\theta^2-27038720\theta^3-19892480\theta^4\right)$$
$$-z^3\left(1297296+77679360\theta+176136960\theta^2+227589120\theta^3+232327680\theta^4\right)$$
$$-18432z^4(964\theta^2+1360\theta+669)(2\theta+1)^2\ -5308416z^5(2\theta+1)^2(2\theta+3)^2$$

11	
50	
-88	
6	
$1/x_3$	

$$\left\{\begin{array}{cccccc} x_1 & x_2 & -\frac{11}{192} & 0 & x_3 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 1 & 0 & 1 & \frac{1}{2} \\ 1 & 1 & 3 & 0 & 1 & \frac{3}{2} \\ 2 & 2 & 4 & 0 & 2 & \frac{3}{2} \end{array}\right\}$$

$f(x) = 2304\,x^3+1664\,x^2+432\,x-1$, $x_1 - 0.3623-0.2407i$, $x_2=-0.3623+0.2407i$, $x_3=0.0023$

$1+\frac{852}{11}q+\frac{173428}{11}q^2+\frac{45311064}{11}q^3+\frac{11202581876}{11}q^4+\frac{2869555773852}{11}q^5+67415462809928q^6+O(q^7)$

$n_1^0 = 852,\ n_2^0 = 21572,\ n_3^0 = 1678156,\ n_4^0 = 175037632,\ n_5^0 = 22956446184,\ n_6^0 = 3433194654748$

$e_{x_1} = -\frac{1}{6},\ e_{x_2} = -\frac{1}{6},\ e_{x_3} = -\frac{1}{6}$

$n_1^1 = 0,\ n_2^1 = 0,\ n_3^1 = 456,\ n_4^1 = 678687,\ n_5^1 = 331237816,\ n_6^1 = 129451529224$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 11 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{-\frac{11}{192}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{x_2} = \begin{pmatrix} -3 & -14 & -2 & -1 \\ 4 & 15 & 2 & 1 \\ -32 & -112 & -15 & -8 \\ 24 & 84 & 12 & 7 \end{pmatrix},$$
$$M_{x_3} = \begin{pmatrix} 1 & -6 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{x_1} = \begin{pmatrix} 5 & -3 & 0 & -1 \\ 4 & -2 & 0 & -1 \\ -12 & 9 & 1 & 3 \\ 4 & -3 & 0 & 0 \end{pmatrix}$$

$v^1_{x_1}=(11, 11, \frac{55}{12}, -88\lambda + \frac{23}{12}),\ v^1_{x_2}=(11, -11, \frac{55}{12}, -88\lambda - \frac{23}{12}),\ v^1_{x_3}=(11, 0, \frac{25}{12}, -88\lambda)$

$$\begin{aligned}
&16\theta^4 - z\left(288 + 24000\theta + 7636\theta^2 + 10472\theta^3 + 5128\theta^4\right)\\
&-z^2\left(26424 + 1434496\theta + 2857950\theta^2 + 2412900\theta^3 + 855270\theta^4\right)\\
&-z^3\left(784728 + 43137360\theta + 74043720\theta^2 + 14813280\theta^3 - 12657870\theta^4\right)\\
&-z^4\left(8612406 + 341798940\theta + 297738180\theta^2 - 295974000\theta^3 - 220245480\theta^4\right)\\
&-z^5\left(39234780 + 1214047440\theta - 672633720\theta^2 - 4791104640\theta^3 + 543775680\theta^4\right)\\
&-z^6\left(102509064 + 2402113320\theta - 663710760\theta^2 + 8503056000\theta^3 + 14653599840\theta^4\right)\\
&-17006112z^7(2\theta+1)(100\theta^3 + 162\theta^2 + 95\theta + 21)\ +1224440064z^8(2\theta+1)(\theta+1)^2(2\theta+3)
\end{aligned}$$

12	$\left\{\begin{array}{cccccccc} x_1 & x_2 & \frac{1}{144}-\frac{1}{144}\sqrt{129} & 0 & x_3 & \frac{1}{144}+\frac{1}{144}\sqrt{129} & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 0 & 1 & 3 & 1 & 1 \\ 2 & 2 & 4 & 0 & 2 & 4 & 2 & \frac{3}{2} \end{array}\right\}$
48	
-68	
6	
$108 + 90\sqrt[3]{2} + 72\, 2^{2/3}$	

$x_1 = -0.1682 - 0.0224i\ ,\ x_2 = -0.1682 + 0.0224i,\ x_3 = 1/18\sqrt[3]{2} + 1/36\, 2^{2/3} - 1/9$

$1+\frac{229}{4}q+\frac{41605}{4}q^2+\frac{4011113}{2}q^3+\frac{1574502533}{4}q^4+79377665526q^5+\frac{32327149677353}{2}q^6+O(q^7)$

$e_{x_1} = -\frac{1}{6},\ e_{x_2} = -\frac{1}{6},\ e_{1/18\sqrt[3]{2}+1/36\, 2^{2/3}-1/9} = -\frac{1}{6},\ e_1 = -\frac{1}{2}$

$n_1^0 = 687,\ n_2^0 = 15516,\ n_3^0 = 891333,\ n_4^0 = 73802856,\ n_5^0 = 7620255885,\ n_6^0 = 897976267932$

$n_1^1 = 0,\ n_2^1 = 3,\ n_3^1 = 816,\ n_4^1 = 419154,\ n_5^1 = 127865136,\ n_6^1 = 35697233591$

$X \xrightarrow{2:1} B(conj)$ [Lee08]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 12 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{\frac{1}{144}-\frac{1}{144}\sqrt{129}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{x_2} = \begin{pmatrix} -3 & -15 & -2 & -1 \\ 4 & 16 & 2 & 1 \\ -36 & -135 & -17 & -9 \\ 28 & 105 & 14 & 8 \end{pmatrix},$$
$$M_1 = \begin{pmatrix} 1 & -12 & -3 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 24 & 7 & 6 \\ 0 & -24 & -6 & -5 \end{pmatrix},\ M_{\frac{1}{144}+\frac{1}{144}\sqrt{129}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{x_3} = \begin{pmatrix} 1 & -6 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$M_{x_1} = \begin{pmatrix} 5 & -3 & 0 & -1 \\ 4 & -2 & 0 & -1 \\ -12 & 9 & 1 & 3 \\ 4 & -3 & 0 & 0 \end{pmatrix}$$

$v^1_{x_1}$=$(12,12,5,-68\lambda+2)$, $v^1_{-0.1682+0.0224i}$=$(12,-12,5,-68\lambda-2)$,
$v^1_{1/18\,\sqrt[3]{2}+1/36\,2^{2/3}-1/9}$=$(12,0,2,-68\lambda)$, v^3_1=$(36,0,0,-204\lambda)$

$$9\theta^4 - z\left(216+1764\theta+5556\theta^2+7584\theta^3+3216\theta^4\right)$$
$$+z^2\left(56448+309504\theta+547840\theta^2+206848\theta^3-57344\theta^4\right)$$
$$-z^3\left(552960+11796480\theta-3096576\theta^2-8257536\theta^3-1703936\theta^4\right)$$
$$+524288z^4(8\theta^2-28\theta-33)(2\theta+1)^2 \; -16777216z^5(2\theta+1)^2(2\theta+3)^2$$

12	$\left\{\begin{array}{cccccc} -\frac{3}{64} & 0 & \frac{3}{64}-1/32\sqrt{2} & \frac{1}{16} & \frac{3}{64}+1/32\sqrt{2} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 1 & 1 & \frac{1}{2} \\ 3 & 0 & 1 & 1 & 1 & \frac{3}{2} \\ 4 & 0 & 2 & 2 & 2 & \frac{3}{2} \end{array}\right\}$
48	
-60	
6	
$192+128\sqrt{2}$	

$1+\frac{220}{3}q+9508q^2+\frac{7226392}{3}q^3+479007908q^4+\frac{307427260220}{3}q^5+21981433037800q^6+O(q^7)$

$n^0_1=880$, $n^0_2=14152$, $n^0_3=1070544$, $n^0_4=89812200$, $n^0_5=9837672320$, $n^0_6=1221190589976$

$e_{\frac{3}{64}-1/32\sqrt{2}}=-\frac{1}{6}$, $e_{\frac{1}{16}}=-\frac{1}{6}$, $e_{\frac{3}{64}+1/32\sqrt{2}}=-\frac{1}{6}$

$n^1_1=2$, $n^1_2=-193$, $n^1_3=5844$, $n^1_4=-184904$, $n^1_5=88380254$, $n^1_6=32082363158$

$$M_0 = \begin{pmatrix}1&0&0&0\\1&1&0&0\\0&12&1&0\\0&0&1&1\end{pmatrix}, \quad M_{-\frac{3}{64}} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \quad M_{\frac{3}{64}+1/32\sqrt{2}} =$$
$$\begin{pmatrix}-3&-14&-2&-1\\4&15&2&1\\-32&-112&-15&-8\\24&84&12&7\end{pmatrix}, \quad M_{\frac{1}{16}} = \begin{pmatrix}-7&-36&-6&-4\\4&19&3&2\\-24&-108&-17&-12\\16&72&12&9\end{pmatrix}, \quad M_{\frac{3}{64}-1/32\sqrt{2}} =$$
$$\begin{pmatrix}1&-6&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v^1_{\frac{3}{64}-1/32\sqrt{2}}$=$(12,0,2,-60\lambda)$, $v^1_{\frac{1}{16}}$=$(24,-12,4,-120\lambda-2)$, $v^1_{\frac{3}{64}+1/32\sqrt{2}}$=$(12,-12,4,-60\lambda-2)$

$$\theta^4-12z(3\theta+1)(2\theta+1)^2(3\theta+2)$$

12	$\left\{\begin{array}{ccc} 0 & \frac{1}{432} & \infty \\ \hline 0 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 2 & \frac{2}{3} \end{array}\right\}$
60	
-144	
7	
432	

$1+60q+15012q^2+3625944q^3+897077412q^4+225801372060q^5+57520045923048q^6+O(q^7)$

$n^0_1=720$, $n^0_2=22428$, $n^0_3=1611504$, $n^0_4=168199200$, $n^0_5=21676931712$, $n^0_6=3195557904564$

$e_{\frac{1}{432}}=-\frac{1}{6}$

$n^1_1=0$, $n^1_2=0$, $n^1_3=64$, $n^1_4=265113$, $n^1_5=198087264$, $n^1_6=89191834992$

$X(2,2,3)\subset\mathbb{P}^6$ [LT93]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 12 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{\frac{1}{432}} = \begin{pmatrix} 1 & -7 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{\frac{1}{432}}=(12, 0, \frac{5}{2}, -144\lambda)$

$$169\theta^4 - z\left(593970\theta^4 + 1175460\theta^3 + 868270\theta^2 + 280540\theta + 3380\right)$$
$$+z^2\left(1021760\theta^4 - 283840\theta^3 - 2863680\theta^2 - 1855360\theta - 33904\right)$$
$$+z^3\left(171520\theta^4 + 3194880\theta^3 + 2792960\theta^2 + 798720\theta + 6656\right) -4096z^4(2\theta+1)^4$$

13
58
-120
7
$\frac{349}{2} + \frac{85}{2}\sqrt{17}$

$$\left\{\begin{array}{ccccc} -\frac{349}{512} - \frac{85}{512}\sqrt{17} & 0 & -\frac{349}{512} + \frac{85}{512}\sqrt{17} & \frac{13}{16} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 1 & \frac{1}{2} \\ 1 & 0 & 1 & 3 & \frac{1}{2} \\ 2 & 0 & 2 & 4 & \frac{1}{2} \end{array}\right\}$$

$1+\frac{647}{13}q+\frac{129975}{13}q^2+\frac{25451198}{13}q^3+\frac{5134100919}{13}q^4+\frac{1052276876647}{13}q^5+\frac{218297309057358}{13}q^6+O(q^7)$

$n^0_1 = 647, n^0_2 = 16166, n^0_3 = 942613, n^0_4 = 80218296, n^0_5 = 8418215008, n^0_6 = 1010635571652$

$e_{-\frac{349}{512}-\frac{85}{512}\sqrt{17}} = -\frac{1}{6}, e_{-\frac{349}{512}+\frac{85}{512}\sqrt{17}} = -\frac{1}{6}$

$n^1_1 = 0, n^1_2 = 0, n^1_3 = 176, n^1_4 = 164696, n^1_5 = 78309518, n^1_6 = 26889884220$

5×5 $Pfaffian \subset \mathbb{P}^6$ [Tono4]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 13 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{-\frac{349}{512}-\frac{85}{512}\sqrt{17}} = \begin{pmatrix} -7 & -42 & -6 & -4 \\ 4 & 22 & 3 & 2 \\ -28 & -147 & -20 & -14 \\ 16 & 84 & 12 & 9 \end{pmatrix}, M_{\frac{13}{16}} =$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{-\frac{349}{512}+\frac{85}{512}\sqrt{17}} = \begin{pmatrix} 1 & -7 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{-\frac{349}{512}-\frac{85}{512}\sqrt{17}}=(26, -13, \frac{16}{3}, -240\lambda - \frac{19}{12}), v^1_{-\frac{349}{512}+\frac{85}{512}\sqrt{17}}=(13, 0, \frac{29}{12}, -120\lambda)$

$$49\theta^4 - z\left(784 + 6370\theta + 19558\theta^2 + 26376\theta^3 + 14784\theta^4\right)$$
$$-z^2\left(43008 + 217392\theta + 270444\theta^2 - 164064\theta^3 - 273120\theta^4\right)$$
$$z^3\left(178080 + 1256976\theta + 3122752\theta^2 + 2593920\theta^3 - 852992\theta^4\right)$$
$$-1216z^4(1189\theta^2 + 2533\theta + 1646)(2\theta+1)^2\ 739328z^5(2\theta+1)^2(2\theta+3)^2$$

14
56
-100
7
$142 + 34\sqrt{17}$

$$\left\{\begin{array}{cccccc} -\frac{1}{4} & 0 & \frac{71}{256} - \frac{17}{256}\sqrt{17} & \frac{7}{76} & \frac{71}{256} + \frac{17}{256}\sqrt{17} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 1 & 1 & \frac{1}{2} \\ 1 & 0 & 1 & 3 & 1 & \frac{3}{2} \\ 2 & 0 & 2 & 4 & 2 & \frac{3}{2} \end{array}\right\}$$

$1+\frac{274}{7}q+\frac{49194}{7}q^2+\frac{7597750}{7}q^3+\frac{1261271338}{7}q^4+\frac{212395784024}{7}q^5+\frac{36184435325694}{7}q^6+O(q^7)$

$n^0_1 = 548, n^0_2 = 12230, n^0_3 = 562776, n^0_4 = 39413192, n^0_5 = 3398332540, n^0_6 = 335040997028$

$e_{-\frac{1}{4}} = -\frac{1}{6}, e_{\frac{71}{256}-\frac{17}{256}\sqrt{17}} = -\frac{1}{6}, e_{\frac{71}{256}+\frac{17}{256}\sqrt{17}} = -\frac{1}{6}$

$n^1_1 = -132$, $n^1_2 = -2699$, $n^1_3 = -159420$, $n^1_4 = -14367228$, $n^1_5 = -1477695728$, $n^1_6 = -166270901243$

$X \xrightarrow{2:1} (conj)$ [Lee08]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 14 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad M_{-\frac{1}{4}} = \begin{pmatrix} -7 & -44 & -6 & -4 \\ 4 & 23 & 3 & 2 \\ -32 & -176 & -23 & -16 \\ 20 & 110 & 15 & 11 \end{pmatrix}, \quad M_{\frac{71}{256}+\frac{17}{256}\sqrt{17}} =$$
$$\begin{pmatrix} 1 & -5 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 10 & 3 & 2 \\ 0 & -10 & -2 & -1 \end{pmatrix}, M_{\frac{7}{76}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{\frac{71}{256}-\frac{17}{256}\sqrt{17}} = \begin{pmatrix} 1 & -7 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{-\frac{1}{4}}=(28, -14, \frac{17}{3}, -200\lambda - \frac{5}{3})$, $v^1_{\frac{71}{256}-\frac{17}{256}\sqrt{17}}=(14, 0, \frac{7}{3}, -100\lambda)$, $v^1_{\frac{71}{256}+\frac{17}{256}\sqrt{17}}=(14, 0, \frac{1}{3}, -100\lambda)$

$$\theta^4 - z\left(295\theta^4 + 572\theta^3 + 424\theta^2 + 138\theta + 17\right) + z^2\left(1686\theta^4 + 1488\theta^3 - 946\theta^2 - 962\theta - 202\right)$$
$$-z^3\left(2258\theta^4 - 1032\theta^3 - 1450\theta^2 - 318\theta + 8\right) - z^4\left(519\theta^4 + 1056\theta^3 + 870\theta^2 + 342\theta + 54\right)$$
$$+9z^5(\theta+1)^4$$

14	$\left\{\begin{array}{cccccc} x_1 & 0 & x_2 & \frac{1}{3} & x_3 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 4 & 2 & 1 \end{array}\right\}$
56	
-98	
7	
$1/x_2$	

$f(x) = x^3 - 57x^2 - 289x + 1$, $x_1 = -4.6883$, $x_2 = 0.0035$, $x_3 = 61.6848$

$1+42q+6958q^2+1126104q^3+189077294q^4+32226733042q^5+5559518418328q^6+O(q^7)$

$n^0_1 = 588$, $n^0_2 = 12103$, $n^0_3 = 583884$, $n^0_4 = 41359136$, $n^0_5 = 3609394096$, $n^0_6 = 360339083307$

$e_{x_1} = -\frac{1}{6}$, $e_{x_2} = -\frac{1}{6}$, $e_{x_3} = -\frac{1}{6}$

$n^1_1 = 0$, $n^1_2 = 0$, $n^1_3 = 196$, $n^1_4 = 99960$, $n^1_5 = 34149668$, $n^1_6 = 9220666042$

$7 \times 7 - Pfaffian \subset \mathbb{P}^6$ [Rød00]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 14 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{x_3} = \begin{pmatrix} 1 & -105 & -25 & -25 \\ 0 & 1 & 0 & 0 \\ 0 & 294 & 71 & 70 \\ 0 & -294 & -70 & -69 \end{pmatrix}, M_{\frac{1}{3}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{x_2} =$$
$$\begin{pmatrix} 1 & -7 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{x_1} = \begin{pmatrix} -27 & -168 & -24 & -16 \\ 14 & 85 & 12 & 8 \\ -98 & -588 & -83 & -56 \\ 49 & 294 & 42 & 29 \end{pmatrix}$$

$v^1_{x_1}=(56, -28, \frac{28}{3}, -392\lambda - \frac{7}{3})$, $v^1_{x_2}=(14, 0, \frac{7}{3}, -98\lambda)$, $v^1_{x_3}=(-70, 0, \frac{7}{3}, 490\lambda)$

$$\theta^4 - z(\theta+1)(285\theta^3 + 321\theta^2 + 128\theta + 18) + z^2\left(480 + 2356\theta + 2674\theta^2 - 2644\theta^3 - 3280\theta^4\right)$$
$$+z^3\left(180 + 2880\theta + 10296\theta^2 + 9216\theta^3 - 7668\theta^4\right) + 216z^4(2\theta+1)(22\theta^3 + 37\theta^2 + 24\theta + 6)$$
$$+432z^5(2\theta+1)(\theta+1)^2(2\theta+3)$$

<table>
<tr><td>14</td><td rowspan="5">$$\left\{\begin{array}{cccccc} -\frac{37}{12}-7/6\sqrt{7} & -\frac{1}{6} & 0 & -\frac{37}{12}+7/6\sqrt{7} & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 1 & 1 \\ 2 & 4 & 0 & 2 & 2 & \frac{3}{2} \end{array}\right\}$$</td></tr>
<tr><td>56</td></tr>
<tr><td>-96</td></tr>
<tr><td>7</td></tr>
<tr><td>$148+56\sqrt{7}$</td></tr>
<tr><td colspan="2">$1{+}45q{+}6805q^2{+}1175382q^3{+}197837973q^4{+}34183942295q^5{+}5968573821982q^6{+}O(q^7)$</td></tr>
<tr><td colspan="2">$n^0_1=630,\ n^0_2=11830,\ n^0_3=609434,\ n^0_4=43275568,\ n^0_5=3828601532,\ n^0_6=386851930360$</td></tr>
<tr><td colspan="2">$e_{-\frac{37}{12}-7/6\sqrt{7}}=-\frac{1}{6},\ e_{-\frac{37}{12}+7/6\sqrt{7}}=-\frac{1}{6},\ e_1=-\frac{1}{3}$</td></tr>
<tr><td colspan="2">$n^1_1=0,\ n^1_2=0,\ n^1_3=28,\ n^1_4=74298,\ n^1_5=30490376,\ n^1_6=8955874396$</td></tr>
<tr><td colspan="2">$X \xrightarrow{2:1} B$ [Lee08]</td></tr>
<tr><td colspan="2">$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&14&1&0\\0&0&1&1\end{pmatrix},\ M_{-\frac{1}{6}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ M_{-\frac{37}{12}-7/6\sqrt{7}}=\begin{pmatrix}1&0&0&0\\5&15&1&0\\-70&-196&-13&0\\60&168&12&1\end{pmatrix},$$ $$M_1=\begin{pmatrix}-15&-84&-12&-8\\8&43&6&4\\-56&-294&-41&-28\\32&168&24&17\end{pmatrix},\ M_{-\frac{37}{12}+7/6\sqrt{7}}=\begin{pmatrix}1&-7&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$</td></tr>
<tr><td colspan="2">$v^1_{-\frac{37}{12}-7/6\sqrt{7}}{=}(0,14,-7,\frac{8}{3}),\ v^1_{-\frac{37}{12}+7/6\sqrt{7}}{=}(14,0,\frac{7}{3},-96\lambda),\ v^2_1{=}(56,-28,\frac{28}{3},-384\lambda-\frac{10}{3})$</td></tr>
</table>

<table>
<tr><td colspan="2">$9\theta^4-z\left(126+999\theta+3036\theta^2+4074\theta^3+2433\theta^4\right)$
$-z^2\left(2424+7494\theta-17551\theta^2-88948\theta^3-73291\theta^4\right)$
$-z^3\left(3360+414000\theta+232288\theta^2+647928\theta^3+759472\theta^4\right)$
$+z^4\left(11328+2526608\theta+12434408\theta^2+27839968\theta^3+2886416\theta^4\right)$
$+z^5\left(307712+10817728\theta+3481600\theta^2-31896320\theta^3-42629120\theta^4\right)$
$-z^6\left(243712+9384960\theta+11399680\theta^2-4638720\theta^3-22023680\theta^4\right)$
$+z^7\left(368640+13824000\theta+18186240\theta^2+8724480\theta^3+307200\theta^4\right)-147456z^8(\theta+1)^4$</td></tr>
<tr><td>15</td><td rowspan="5">$$\left\{\begin{array}{cccccccc} x_1 & 0 & x_2 & \frac{11}{24}-1/24\sqrt{85} & x_3 & \frac{11}{24}+1/24\sqrt{85} & x_4 & \infty \\ \hline 0&0&0&0&0&0&0&1\\ 1&0&1&1&1&1&1&1\\ 1&0&1&3&1&3&1&1\\ 2&0&2&4&2&4&2&1 \end{array}\right\}$$</td></tr>
<tr><td>54</td></tr>
<tr><td>-80</td></tr>
<tr><td>7</td></tr>
<tr><td>$1/x_2$</td></tr>
<tr><td colspan="2">$f(x)=x^4+104x^3-827x^2+241x-1,\ x_1=-4.6072,\ x_2=0.0042,\ x_3=0.3008,\ x_4=2.6771$</td></tr>
<tr><td colspan="2">$1{+}\frac{100}{3}q{+}\frac{15140}{3}q^2{+}\frac{1974097}{3}q^3{+}\frac{275879716}{3}q^4{+}\frac{39280630850}{3}q^5{+}\frac{5654754934913}{3}q^6+O(q^7)$</td></tr>
<tr><td colspan="2">$n^0_1=500,\ n^0_2=9400,\ n^0_3=365555,\ n^0_4=21551920,\ n^0_5=1571225230,\ n^0_6=130897058930$</td></tr>
<tr><td colspan="2">$e_{x_1}=-\frac{1}{6},\ e_{x_2}=-\frac{1}{6},\ e_{x_3}=-\frac{1}{6},\ e_{x_4}=-\frac{1}{6}$</td></tr>
<tr><td colspan="2">$n^1_1=-110,\ n^1_2=-1925,\ n^1_3=-99134,\ n^1_4=-7376990,\ n^1_5=-641829038,$
$n^1_6=-60993384877$</td></tr>
</table>

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 15 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad M_{x_1} = \begin{pmatrix} -59 & -390 & -54 & -36 \\ 30 & 196 & 27 & 18 \\ -230 & -1495 & -206 & -138 \\ 120 & 780 & 108 & 73 \end{pmatrix}, \quad M_{x_4} = \begin{pmatrix} 1 & -18 & -4 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 45 & 11 & 10 \\ 0 & -45 & -10 & -9 \end{pmatrix}, \quad M_{\frac{11}{24}+1/24\sqrt{85}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{x_3} = \begin{pmatrix} 1 & -5 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 10 & 3 & 2 \\ 0 & -10 & -2 & -1 \end{pmatrix},$$

$$M_{\frac{11}{24}-1/24\sqrt{85}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{x_2} = \begin{pmatrix} 1 & -7 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{x_1}=(90,-45,14,-480\lambda-\frac{13}{4})$, $v^1_{x_2}=(15,0,\frac{9}{4},-80\lambda)$, $v^1_{x_3}=(15,0,\frac{1}{4},-80\lambda)$, $v^1_{x_4}=(-30,0,\frac{1}{2},160\lambda)$

$$\begin{aligned}
&25\theta^4 - z\left(375+3000\theta+9200\theta^2+12400\theta^3+5975\theta^4\right)\\
&-z^2\left(7350+327000\theta+702300\theta^2+961800\theta^3+518100\theta^4\right)\\
&-z^3\left(76050+3753000\theta+6421950\theta^2+3302100\theta^3+683550\theta^4\right)\\
&-z^4\left(275670+10233000\theta+13436010\theta^2+4487940\theta^3-2796660\theta^4\right)\\
&-z^5\left(417555+12263400\theta+5495850\theta^2-14280300\theta^3-3734100\theta^4\right)\\
&-z^6\left(38880-4228200\theta-15651630\theta^2-14652900\theta^3+2962170\theta^4\right)\\
&+z^7\left(72900+1516320\theta-1895400\theta^2-6823440\theta^3-4723920\theta^4\right)\ -104976z^8(\theta+1)^4
\end{aligned}$$

15	
54	
-76	
7	
$1/x_3$	

$$\left\{\begin{array}{cccccccc} x_1 & -1 & x_2 & \frac{5}{24}-\frac{\sqrt{105}}{24} & 0 & x_3 & \frac{5}{24}+\frac{\sqrt{105}}{24} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 0 & 1 & 3 & 1 \\ 2 & 2 & 2 & 4 & 0 & 2 & 4 & 1 \end{array}\right\}$$

$f(x) = x^3 + 13/3x^2 + \frac{82}{27}x - \frac{1}{81}$, $x_1 = -3.4527$, $x_2 = -0.8847$, $x_3 = 0.004$

$n^0_1 = 570$, $n^0_2 = 9204$, $n^0_3 = 392025$, $n^0_4 = 23615994$, $n^0_5 = 1762778535$, $n^0_6 = 150397140660$

$e_{x_1} = -\frac{1}{6}$, $e_{-1} = -1$, $e_{x_2} = -\frac{1}{6}$, $e_{x_3} = -\frac{1}{6}$

$n^1_1 = 0$, $n^1_2 = 0$, $n^1_3 = 90$, $n^1_4 = 51513$, $n^1_5 = 14907408$, $n^1_6 = 3445286666$

$1+38q+\frac{24734}{5}q^2+705683q^3+\frac{503832606}{5}q^4+14689821163q^5+\frac{10828597680479}{5}q^6+O(q^7)$

$X \xrightarrow{2:1} B$ [Lee08]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 15 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad M_{\frac{5}{24}-1/24\sqrt{105}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{x_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 16 & 1 & 0 \\ -75 & -225 & -14 & 0 \\ 65 & 195 & 13 & 1 \end{pmatrix},$$

$$M_{-1} = \begin{pmatrix} 7 & -18 & -6 & -6 \\ 0 & 1 & 0 & 0 \\ -24 & 72 & 25 & 24 \\ 30 & -90 & -30 & -29 \end{pmatrix}, \quad M_{x_1} = \begin{pmatrix} -47 & -378 & -54 & -36 \\ 24 & 190 & 27 & 18 \\ -168 & -1323 & -188 & -126 \\ 64 & 504 & 72 & 49 \end{pmatrix},$$

$$M_{\frac{5}{24}+1/24\sqrt{105}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{x_3} = \begin{pmatrix} 1 & -7 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{x_1}=(90,-45,12,-456\lambda-\frac{5}{4})$, $v^6_{-1}=(-90,0,\frac{21}{2},456\lambda-6)$, $v^1_{x_2}=(0,15,-\frac{15}{2},\frac{11}{4})$,
$v^1_{x_3}=(15,0,\frac{9}{4},-76\lambda)$

$\theta^4-3z(3\theta+1)(3\theta+2)(11\theta^2+11\theta+3)\ -9z^2(3\theta+1)(3\theta+2)(3\theta+4)(3\theta+5)$	
15	$\left\{\begin{array}{cccc} -\frac{11}{54}-\frac{5}{54}\sqrt{5} & 0 & -\frac{11}{54}+\frac{5}{54}\sqrt{5} & \infty \\ \hline 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & 1 & \frac{2}{3} \\ 1 & 0 & 1 & \frac{4}{3} \\ 2 & 0 & 2 & \frac{5}{3} \end{array}\right\}$
66	
-150	
8	
$\frac{297}{2}+\frac{135}{2}\sqrt{5}$	
$1+36q+6732q^2+1118403q^3+191549196q^4+33295313286q^5+5856252011883q^6+O(q^7)$	
$n^0_1=540,\ n^0_2=12555,\ n^0_3=621315,\ n^0_4=44892765,\ n^0_5=3995437590,\ n^0_6=406684089360$	
$e_{-\frac{11}{54}-\frac{5}{54}\sqrt{5}}=-\frac{1}{6},\ e_{-\frac{11}{54}+\frac{5}{54}\sqrt{5}}=-\frac{1}{6}$	
$n^1_1=0,\ n^1_2=0,\ n^1_3=-1,\ n^1_4=13095,\ n^1_5=17230617,\ n^1_6=6648808836$	
$X(1,1,3)\subset \mathrm{Gr}(2,5)$ [BCFKS98]	
$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&15&1&0\\0&0&1&1\end{pmatrix},\ M_{-\frac{11}{54}-\frac{5}{54}\sqrt{5}}=\begin{pmatrix}-9&-50&-6&-4\\5&26&3&2\\-45&-225&-26&-18\\30&150&18&13\end{pmatrix},\ M_{-\frac{11}{54}+\frac{5}{54}\sqrt{5}}=\begin{pmatrix}1&-8&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	
$v^1_{-\frac{11}{54}-\frac{5}{54}\sqrt{5}}=(30,-15,7,-300\lambda-\frac{9}{4}),\ v^1_{-\frac{11}{54}+\frac{5}{54}\sqrt{5}}=(15,0,\frac{11}{4},-150\lambda)$	

$\theta^4-z\left(8+64\theta+196\theta^2+264\theta^3+216\theta^4\right)\ -z^2\left(2464+124416\theta+195040\theta^2+4896\theta^3-6672\theta^4\right)$ $+z^3\left(5888+110592\theta+406784\theta^2+491520\theta^3+424960\theta^4\right)$ $-28672z^4(2\theta+1)(38\theta^3+45\theta^2+12\theta-2)\ -802816z^5(2\theta+1)(\theta+1)^2(2\theta+3)$	
16	$\left\{\begin{array}{cccccc} -\frac{11}{32}-\frac{5}{32}\sqrt{5} & -\frac{1}{16} & 0 & -\frac{11}{32}+\frac{5}{32}\sqrt{5} & \frac{1}{28} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 3 & 1 \\ 2 & 2 & 0 & 2 & 4 & \frac{3}{2} \end{array}\right\}$
52	
-72	
7	
$88+40\sqrt{5}$	
$1+20q+3488q^2+356096q^3+39026464q^4+4439059520q^5+513809807360q^6+O(q^7)$	
$n^0_1=320,\ n^0_2=6936,\ n^0_3=211008,\ n^0_4=9755744,\ n^0_5=568199616,\ n^0_6=38059959096$	
$e_{-\frac{11}{32}-\frac{5}{32}\sqrt{5}}=-\frac{1}{6},\ e_{-\frac{1}{16}}=-\frac{1}{6},\ e_{-\frac{11}{32}+\frac{5}{32}\sqrt{5}}=-\frac{1}{6}$	
$n^1_1=0,\ n^1_2=6,\ n^1_3=640,\ n^1_4=79386,\ n^1_5=11424640,\ n^1_6=1561536328$	
$X\xrightarrow{2:1}B,\ (conj)$ [Lee08]	

$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 16 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $M_{-\frac{1}{16}} = \begin{pmatrix} -7 & -48 & -6 & -4 \\ 4 & 25 & 3 & 2 \\ -40 & -240 & -29 & -20 \\ 28 & 168 & 21 & 15 \end{pmatrix}$, $M_{-\frac{11}{32}-\frac{5}{32}\sqrt{5}} = \begin{pmatrix} -51 & -256 & -28 & -16 \\ 39 & 193 & 21 & 12 \\ -468 & -2304 & -251 & -144 \\ 364 & 1792 & 196 & 113 \end{pmatrix}$, $M_{\frac{1}{28}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $M_{-\frac{11}{32}+\frac{5}{32}\sqrt{5}} = \begin{pmatrix} 1 & -7 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$v^1_{-\frac{11}{32}-\frac{5}{32}\sqrt{5}}=(64, -48, \frac{62}{3}, -288\lambda - \frac{13}{2})$, $v^1_{-\frac{1}{16}}=(32, -16, \frac{19}{3}, -144\lambda - \frac{11}{6})$, $v^1_{-\frac{11}{32}+\frac{5}{32}\sqrt{5}}=(16, 0, \frac{13}{6}, -72\lambda)$

$\theta^4 - z\left(16 + 120\theta + 352\theta^2 + 464\theta^3 + 264\theta^4\right) + z^2\left(3328 + 167768\theta + 315552\theta^2 + 286720\theta^3 + 111360\theta^4\right) - 512z^3(2\theta+1)(1660\theta^3 + 465\theta^2 + 477\theta + 158) + 24576z^4(2\theta+1)(3\theta+2)(3\theta+4)(2\theta+3)$	
16	$\left\{\begin{array}{cccc} 0 & \frac{1}{216} & \frac{1}{16} & \infty \\ \hline 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{2}{3} \\ 0 & 1 & \frac{3}{2} & \frac{4}{3} \\ 0 & 2 & 2 & \frac{3}{2} \end{array}\right\}$
52	
-58	
7	
216	
$1+32q+3712q^2+439808q^3+53816352q^4+6701036032q^5+843992670208q^6+O(q^7)$	
$n^0_1 = 512$, $n^0_2 = 7360$, $n^0_3 = 260608$, $n^0_4 = 13453160$, $n^0_5 = 857732608$, $n^0_6 = 62517942720$	
$e_{\frac{1}{216}} = -\frac{1}{6}$, $e_{\frac{1}{16}} = -\frac{1}{2}$	
$n^1_1 = 2$, $n^1_2 = -103$, $n^1_3 = 1160$, $n^1_4 = 1389$, $n^1_5 = 6142532$, $n^1_6 = 1301356132$	
$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 16 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $M_{\frac{1}{16}} = \begin{pmatrix} -3 & -12 & -1 & 0 \\ 0 & -4 & -1 & -1 \\ 8 & 84 & 15 & 12 \\ -8 & -69 & -11 & -8 \end{pmatrix}$, $M_{\frac{1}{216}} = \begin{pmatrix} 1 & -7 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	
$v^1_{\frac{1}{216}}=(16, 0, \frac{13}{6}, -58\lambda)$	

$\theta^4 - z\left(16 + 128\theta + 400\theta^2 + 544\theta^3 + 128\theta^4\right) + z^2\left(256 + 512\theta - 8192\theta^2 - 38400\theta^3 - 22272\theta^4\right) - z^3\left(77824 + 417792\theta + 864256\theta^2 + 983040\theta^3 + 827392\theta^4\right) - 196608z^4(2\theta+1)(22\theta^3 + 45\theta^2 + 38\theta + 12) - 9437184z^5(2\theta+1)(\theta+1)^2(2\theta+3)$	
16	$\left\{\begin{array}{cccccc} x_1 & x_2 & -\frac{1}{48} & 0 & x_3 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 0 & 1 & 1 \\ 2 & 2 & 4 & 0 & 2 & \frac{3}{2} \end{array}\right\}$
52	
-44	
7	
237.2400	
$f(x) = x^3 + 3/16\,x^2 + \frac{7}{512}x - \frac{1}{16384}$, $x_1 = -0.0959 - 0.0727i$, $x_2 = -0.0959 + 0.0727i$, $x_3 = 0.0042$	
$1+48q+2208q^2+680448q^3+61958432q^4+10111586048q^5+1260786432000q^6+O(q^7)$	

$n_1^0=768,\ n_2^0=4320,\ n_3^0=403200,\ n_4^0=15489056,\ n_5^0=1294283008,\ n_6^0=93391536992$
$e_{x_1}=-\frac{1}{6},\ e_{x_2}=-\frac{1}{6},\ e_{x_3}=-\frac{1}{6}$
$n_1^1=0,\ n_2^1=0,\ n_3^1=0,\ n_4^1=-1932,\ n_5^1=2406912,\ n_6^1=1241264448$
$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&16&1&0\\0&0&1&1\end{pmatrix},\ M_{-\frac{1}{48}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ M_{x_2}=\begin{pmatrix}-4&-19&-2&-1\\5&20&2&1\\-60&-228&-23&-12\\50&190&20&11\end{pmatrix},$ $M_{x_3}=\begin{pmatrix}1&-7&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ M_{x_1}=\begin{pmatrix}6&-3&0&-1\\5&-2&0&-1\\-20&12&1&4\\10&-6&0&-1\end{pmatrix}$
$v_{x_1}^1=(16,16,\frac{37}{6},-44\lambda+\frac{17}{6}),\ v_{x_2}^1=(16,-16,\frac{37}{6},-44\lambda-\frac{17}{6}),\ v_{x_3}^1=(16,0,\frac{13}{6},-44\lambda)$

$\theta^4-16z(2\theta+1)^4$	
16	$\left\{\begin{array}{ccc}0&\frac{1}{256}&\infty\\\hline 0&0&\frac{1}{2}\\0&1&\frac{1}{2}\\0&1&\frac{1}{2}\\0&2&\frac{1}{2}\end{array}\right\}$
64	
-128	
8	
256	
$1+32q+4896q^2+702464q^3+102820640q^4+15296748032q^5+2302235670528q^6+O(q^7)$	
$n_1^0=512,\ n_2^0=9728,\ n_3^0=416256,\ n_4^0=25703936,\ n_5^0=1957983744,\ n_6^0=170535923200$	
$e_{\frac{1}{256}}=-\frac{1}{6}$	
$n_1^1=0,\ n_2^1=0,\ n_3^1=0,\ n_4^1=14752,\ n_5^1=8782848,\ n_6^1=2672004608$	
$X(2,2,2,2)\subset\mathbb{P}^7$ [LT93]	
$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&16&1&0\\0&0&1&1\end{pmatrix},\ M_{\frac{1}{256}}=\begin{pmatrix}1&-8&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	
$v_{\frac{1}{256}}^1=(16,0,\frac{8}{3},-128\lambda)$	

$289\theta^4-z\left(4046+31790\theta+94826\theta^2+126072\theta^3+64668\theta^4\right)$ $-z^2\left(22644+96424\theta+40116\theta^2-274304\theta^3-249632\theta^4\right)$ $-z^3\left(19176+71196\theta+83140\theta^2+132192\theta^3+264720\theta^4\right)$ $+z^4\left(128(2\theta+1)(196\theta^3+498\theta^2+487\theta+169)\right)-4096z^5(2\theta+1)(\theta+1)^2(2\theta+3)$	
17	$\left\{\begin{array}{cccccc}0&x_1&\frac{17}{32}&x_2&x_3&\infty\\\hline 0&0&0&0&0&\frac{1}{2}\\0&1&1&1&1&1\\0&1&3&1&1&1\\0&2&4&2&2&\frac{3}{2}\end{array}\right\}$
62	
-108	
8	
$1/x_1$	
$f(x)=x^3-2x^2+\frac{55}{4}x-1/16,\ x_1=0.0045,\ x_2=0.9977+3.5701i,\ x_3=0.9977-3.5701i$	
$1+\frac{478}{17}q+\frac{62750}{17}q^2+\frac{7715998}{17}q^3+\frac{977992734}{17}q^4+\frac{125750597228}{17}q^5+\frac{16359251957822}{17}q^6+O(q^7)$	
$n_1^0=478,\ n_2^0=7784,\ n_3^0=285760,\ n_4^0=15280156,\ n_5^0=1006004774,\ n_6^0=75737241572$	

$e_{x_1} = -\frac{1}{6}$, $e_{x_2} = -\frac{1}{6}$, $e_{x_3} = -\frac{1}{6}$

$n^1_1 = 0$, $n^1_2 = 0$, $n^1_3 = 16$, $n^1_4 = 14215$, $n^1_5 = 4980596$, $n^1_6 = 1184399584$

[Kap09] conj and $X \xrightarrow{2:1} B$, $(conj)$ [Lee08]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 17 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{x_2} = \begin{pmatrix} -11 & -90 & -12 & -9 \\ 4 & 31 & 4 & 3 \\ -24 & -180 & -23 & -18 \\ 8 & 60 & 8 & 7 \end{pmatrix}, M_{\frac{17}{32}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{x_1} =$$
$$\begin{pmatrix} 1 & -8 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{x_3} = \begin{pmatrix} 13 & -39 & -6 & -9 \\ 4 & -12 & -2 & -3 \\ -44 & 143 & 23 & 33 \\ 28 & -91 & -14 & -20 \end{pmatrix}$$

$v^1_{x_1}$=$(17, 0, \frac{31}{12}, -108\lambda)$, $v^1_{x_2}$=$(51, -17, \frac{21}{4}, -324\lambda - \frac{17}{12})$, $v^1_{x_3}$=$(51, 17, \frac{21}{4}, -324\lambda + \frac{17}{12})$

$$81\theta^4 - z\left(972 + 74520\theta + 219330\theta^2 + 289620\theta^3 + 179370\theta^4\right)$$
$$+z^2\left(9504 + 892800\theta + 3916480\theta^2 + 8058880\theta^3 + 5595520\theta^4\right)$$
$$-z^3\left(539136 + 31864320\theta + 73996800\theta^2 + 89026560\theta^3 + 60467200\theta^4\right)$$
$$+32768z^4(2\theta+1)(340\theta^3 + 618\theta^2 + 455\theta + 129) \; -4194304z^5(2\theta+1)(\theta+1)^2(2\theta+3)$$

<table>
<tr><td>18</td><td rowspan="5">$$\left\{\begin{array}{cccccc} 0 & \frac{3}{32} - 1/16\sqrt{2} & \frac{9}{128} & \frac{3}{32} + 1/16\sqrt{2} & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & 1 & 1 \\ 0 & 2 & 4 & 2 & 2 & \frac{3}{2} \end{array}\right\}$$</td></tr>
<tr><td>60</td></tr>
<tr><td>-92</td></tr>
<tr><td>8</td></tr>
<tr><td>$96 + 64\sqrt{2}$</td></tr>
</table>

$1+\frac{205}{9}q+\frac{25853}{9}q^2+\frac{2635999}{9}q^3+\frac{286186109}{9}q^4+\frac{31754637080}{9}q^5+\frac{3558892191935}{9}q^6 + O(q^7)$

$n^0_1 = 410$, $n^0_2 = 6412$, $n^0_3 = 195244$, $n^0_4 = 8942508$, $n^0_5 = 508074190$, $n^0_6 = 32952680836$

$e_{\frac{3}{32}-1/16\sqrt{2}} = -\frac{1}{6}$, $e_{\frac{3}{32}+1/16\sqrt{2}} = -\frac{1}{6}$, $e_1 = -\frac{1}{3}$

$n^1_1 = -88$, $n^1_2 = -1103$, $n^1_3 = -47248$, $n^1_4 = -2697812$, $n^1_5 = -183760064$, $n^1_6 = -13666811772$

$X \xrightarrow{2:1} B$ [Lee08]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 18 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} -23 & -186 & -24 & -18 \\ 8 & 63 & 8 & 6 \\ -56 & -434 & -55 & -42 \\ 24 & 186 & 24 & 19 \end{pmatrix}, \quad M_{\frac{3}{32}+1/16\sqrt{2}} =$$
$$\begin{pmatrix} 1 & -6 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 12 & 3 & 2 \\ 0 & -12 & -2 & -1 \end{pmatrix}, M_{\frac{9}{128}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{\frac{3}{32}-1/16\sqrt{2}} = \begin{pmatrix} 1 & -8 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{\frac{3}{32}-1/16\sqrt{2}}$=$(18, 0, \frac{5}{2}, -92\lambda)$, $v^1_{\frac{3}{32}+1/16\sqrt{2}}$=$(18, 0, \frac{1}{2}, -92\lambda)$, v^2_1=$(108, -36, 11, -552\lambda - 3)$

$$9\theta^4 - z\left(108 + 837\theta + 2487\theta^2 + 3300\theta^3 + 1776\theta^4\right)$$
$$-z^2\left(3312 + 14904\theta + 18041\theta^2 - 6652\theta^3 - 13801\theta^4\right)$$
$$-2z^3\theta(-7236 - 28365\theta - 29160\theta^2 + 8461\theta^3)$$
$$+z^4\left(1344 - 5376\theta - 40908\theta^2 - 72576\theta^3 - 43092\theta^4\right) \; -1176z^5(\theta+1)^2(3\theta+2)(3\theta+4)$$

18	$\left\{\begin{array}{cccccc} -7/4-\frac{11}{36}\sqrt{33} & -1 & 0 & -7/4+\frac{11}{36}\sqrt{33} & \frac{3}{14} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 3 & 1 \\ 2 & 2 & 0 & 2 & 4 & \frac{4}{3} \end{array}\right\}$
60	
-90	
8	
$\frac{189}{2}+\frac{33}{2}\sqrt{33}$	
$1+\frac{73}{3}q+\frac{8597}{3}q^2+\frac{900712}{3}q^3+\frac{99593429}{3}q^4+\frac{11156137948}{3}q^5+\frac{1263937305152}{3}q^6+\frac{144430573438735}{3}q^7+O(q^8)$	
$n^0_1=438,\ n^0_2=6393,\ n^0_3=200142,\ n^0_4=9336078,\ n^0_5=535494618,\ n^0_6=35109344331$	
$e_{-7/4-\frac{11}{36}\sqrt{33}}=-\frac{1}{6},\ e_{-1}=-\frac{1}{2},\ e_{-7/4+\frac{11}{36}\sqrt{33}}=-\frac{1}{6}$	
$n^1_1=0,\ n^1_2=0,\ n^1_3=38,\ n^1_4=12972,\ n^1_5=3064596,\ n^1_6=571253789$	
$X'(2,2,3)\subset\mathbb{P}^6$ [Kapo09]	
$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&18&1&0\\0&0&1&1\end{pmatrix},\ M_{-1}=\begin{pmatrix}-23&-156&-18&-12\\12&79&9&6\\-120&-780&-89&-60\\72&468&54&37\end{pmatrix},\ M_{-7/4-\frac{11}{36}\sqrt{33}}=\begin{pmatrix}-101&-576&-60&-36\\68&385&40&24\\-816&-4608&-479&-288\\561&3168&330&199\end{pmatrix},\ M_{\frac{3}{14}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ M_{-7/4+\frac{11}{36}\sqrt{33}}=\begin{pmatrix}1&-8&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	
$v^1_{-7/4-\frac{11}{36}\sqrt{33}}=(108,-72,27,-540\lambda-7),\ v^3_{-1}=(108,-54,18,-540\lambda-\frac{9}{2}),$ $v^1_{-7/4+\frac{11}{36}\sqrt{33}}=(18,0,\frac{5}{2},-90\lambda)$	

$\theta^4-z\left(12+94\theta+282\theta^2+376\theta^3+180\theta^4\right)-z^2\left(768+3736\theta+6820\theta^2+6080\theta^3+2256\theta^4\right)$ $-z^3\left(16(2\theta+1)(286\theta^3+813\theta^2+851\theta+294)\right)-192z^4(2\theta+1)(4\theta+3)(4\theta+5)(2\theta+3)$	
18	$\left\{\begin{array}{cccc} -\frac{1}{4} & 0 & \frac{1}{192} & \infty \\ \hline 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & \frac{3}{4} \\ \frac{3}{2} & 0 & 1 & \frac{5}{4} \\ 2 & 0 & 2 & \frac{3}{2} \end{array}\right\}$
60	
-88	
8	
192	
$1+26q+2810q^2+310586q^3+34547706q^4+3911285026q^5+447867658394q^6+O(q^7)$	
$n^0_1=468,\ n^0_2=6264,\ n^0_3=207040,\ n^0_4=9715752,\ n^0_5=563225040,\ n^0_6=37322278752$	
$e_{-\frac{1}{4}}=-\frac{1}{2},\ e_{\frac{1}{192}}=-\frac{1}{6}$	
$n^1_1=0,\ n^1_2=0,\ n^1_3=8,\ n^1_4=9144,\ n^1_5=2658096,\ n^1_6=543849616$	
$X'(2,2,3)\subset\mathbb{P}^6$ [Kapo09]	
$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&18&1&0\\0&0&1&1\end{pmatrix},\ M_{-\frac{1}{4}}=\begin{pmatrix}-5&-18&-1&0\\1&-3&-1&-1\\6&144&23&18\\-14&-154&-21&-15\end{pmatrix},\ M_{\frac{1}{192}}=\begin{pmatrix}1&-8&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	
$v^1_{\frac{1}{192}}=(18,0,\frac{5}{2},-88\lambda)$	

$$\theta^4 - z\left(12+100\theta+305\theta^2+410\theta^3+205\theta^4\right)$$
$$-z^2\left(3168+149760\theta+237440\theta^2+162560\theta^3+40640\theta^4\right)$$
$$-z^3\left(87480+3364920\theta+3916680\theta^2+1863360\theta^3+310560\theta^4\right)$$
$$-z^4\left(806400+27302400\theta+25626240\theta^2+9400320\theta^3+1175040\theta^4\right)$$
$$-3456z^5(16\theta^2+80\theta+35)(2\theta+5)^2\ -10368z^6(2\theta+11)(2\theta+7)(2\theta+5)(2\theta+1)$$

18	$\left\{\begin{array}{ccccccc} -\frac{1}{3} & x_1 & x_2 & -\frac{1}{4} & 0 & x_3 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 1 & \frac{1}{2} & 0 & 1 & \frac{5}{2} \\ 1 & 1 & 1 & \frac{1}{2} & 0 & 1 & \frac{7}{2} \\ 2 & 2 & 2 & 1 & 0 & 2 & \frac{11}{2} \end{array}\right\}$
72	
-158	
9	
$72+60\sqrt[3]{2}+48\,2^{2/3}$	

$f(x) = 3456x^3+1728x^2+216x-1$, $x_1 = -0.2522-0.0336i$, $x_2 = -0.2522+0.0336i$, $x_3 = 0.0044$

$1+25q+3593q^2+471229q^3+61337801q^4+8137269025q^5+1091984845133q^6+O(q^7)$

$n^0_1 = 450$, $n^0_2 = 8028$, $n^0_3 = 314136$, $n^0_4 = 17250246$, $n^0_5 = 1171766736$, $n^0_6 = 90998697528$

$e_{-\frac{1}{3}} = -\frac{1}{3}$, $e_{x_1} = -\frac{1}{6}$, $p_{x_2} = -\frac{1}{6}$, $p_{-\frac{1}{4}} = -\frac{1}{2}$, $p_{x_3} = -\frac{1}{6}$

$n^1_1 = 6$, $n^1_2 = -333$, $n^1_3 = 2862$, $n^1_4 = -56697$, $n^1_5 = -7376388$, $n^1_6 = 176313699$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 18 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\quad M_{-\frac{1}{4}} = \begin{pmatrix} 163 & 936 & 104 & 72 \\ -91 & -523 & -58 & -40 \\ 990 & 5688 & 629 & 432 \\ -621 & -3564 & -393 & -269 \end{pmatrix},\quad M_{-\frac{1}{3}} =$$
$$\begin{pmatrix} -863 & -4536 & -456 & -288 \\ 504 & 2647 & 266 & 168 \\ -5832 & -30618 & -3077 & -1944 \\ 3888 & 20412 & 2052 & 1297 \end{pmatrix},\quad M_{x_2} = \begin{pmatrix} -54 & -315 & -35 & -25 \\ 22 & 127 & 14 & 10 \\ -198 & -1134 & -125 & -90 \\ 121 & 693 & 77 & 56 \end{pmatrix},$$
$$M_{x_3} = \begin{pmatrix} 1 & -9 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{x_1} = \begin{pmatrix} 56 & -135 & -15 & -25 \\ 22 & -53 & -6 & -10 \\ -198 & 486 & 55 & 90 \\ 121 & -297 & -33 & -54 \end{pmatrix}$$

$v^2_{-\frac{1}{3}}=(432,-252,108,-3792\lambda-30)$, $v^1_{x_1}=(90,36,15,-790\lambda+5)$, $v^1_{x_2}=(90,-36,15,-790\lambda-5)$, $v^1_{x_3}=(18,0,3,-158\lambda)$

$$25\theta^4 - z\left(5840\theta^4+7360\theta^3+5780\theta^2+2100\theta+300\right)$$
$$+z^2\left(3957760\theta^4+6983680\theta^3+6229760\theta^2+3036800\theta+59760\right)$$
$$-z^3\left(115752960\theta^4+265420800\theta^3+291594240\theta^2+153446400\theta+3018240\right)$$
$$+z^4\left(331776(2\theta+1)(232\theta^3+588\theta^2+590\theta+207)\right)\ -191102976z^5(2\theta+1)(\theta+1)^2(2\theta+3)$$

20	$\left\{\begin{array}{ccccc} 0 & \frac{1}{144} & \frac{5}{144} & \frac{1}{16} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & \frac{1}{2} & 1 \\ 0 & 1 & 3 & \frac{1}{2} & 1 \\ 0 & 2 & 4 & 1 & \frac{3}{2} \end{array}\right\}$
56	
-64	
8	
144	

$1+\frac{76}{5}q+\frac{8748}{5}q^2+\frac{670648}{5}q^3+\frac{54245932}{5}q^4+\frac{4567806076}{5}q^5+\frac{390123944568}{5}q^6+\frac{33588669380104}{5}q^7+O(q)$

$n^0_1 = 304$, $n^0_2 = 4336$, $n^0_3 = 99344$, $n^0_4 = 3389824$, $n^0_5 = 146169792$, $n^0_6 = 7224504912$

$e_{\frac{1}{144}} = -\frac{1}{6}$, $e_{\frac{1}{16}} = -\frac{2}{3}$

$n^1_1 = 2$, $n^1_2 = -61$, $n^1_3 = 164$, $n^1_4 = 5660$, $n^1_5 = 1164454$, $n^1_6 = 145922950$

$\lambda_{\frac{1}{144}} = 1$

$v^1_{\frac{1}{144}} = (20, 0, \frac{7}{3}, -64\lambda)$

$\theta^4 - 4z(2\theta+1)^2(11\theta^2+11\theta+3) - 16z^2(2\theta+1)^2(2\theta+3)^2$

20	
68	
-120	
9	
$88+40\sqrt{5}$	

$$\left\{\begin{array}{cccc} -\frac{11}{32}-\frac{5}{32}\sqrt{5} & 0 & -\frac{11}{32}+\frac{5}{32}\sqrt{5} & \infty \\ \hline 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & \frac{1}{2} \\ 1 & 0 & 1 & \frac{3}{2} \\ 2 & 0 & 2 & \frac{3}{2} \end{array}\right\}$$

$1+20q+2236q^2+221960q^3+22593852q^4+2331442020q^5+243349644568q^6+O(q^7)$

$n^0_1 = 400,\ n^0_2 = 5540,\ n^0_3 = 164400,\ n^0_4 = 7059880,\ n^0_5 = 373030720,\ n^0_6 = 22532353740$

$e_{-\frac{11}{32}-\frac{5}{32}\sqrt{5}} = -\frac{1}{6},\ e_{-\frac{11}{32}+\frac{5}{32}\sqrt{5}} = -\frac{1}{6}$

$n^1_1 = 0,\ n^1_2 = 0,\ n^1_3 = 0,\ n^1_4 = 1537,\ n^1_5 = 882496,\ n^1_6 = 214941640$

$X(1,2,2) \subset \mathrm{Gr}(2,5)$ [BCFKS98] and [Kap09]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 20 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{-\frac{11}{32}-\frac{5}{32}\sqrt{5}} = \begin{pmatrix} -9 & -60 & -6 & -4 \\ 5 & 31 & 3 & 2 \\ -60 & -360 & -35 & -24 \\ 40 & 240 & 24 & 17 \end{pmatrix},\ M_{-\frac{11}{32}+\frac{5}{32}\sqrt{5}} = \begin{pmatrix} 1 & -9 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{-\frac{11}{32}-\frac{5}{32}\sqrt{5}} = (40, -20, \frac{23}{3}, -240\lambda - \frac{13}{6}),\ v^1_{-\frac{11}{32}+\frac{5}{32}\sqrt{5}} = (20, 0, \frac{17}{6}, -120\lambda)$

$441\theta^4 - z\left(4410+335160\theta+975450\theta^2+1280580\theta^3+690690\theta^4\right)$
$-z^2\left(272580+13482000\theta+21377000\theta^2+9233600\theta^3-1542400\theta^4\right)$
$z^3\left(97440+1861776\theta+6723376\theta^2+7894656\theta^3+1706176\theta^4\right)$
$-1280z^4(2\theta+1)(1916\theta^3+2622\theta^2+1077\theta+91) - 102400z^5(2\theta+1)(\theta+1)^2(2\theta+3)$

21	
66	
-104	
9	
$1/x_3$	

$$\left\{\begin{array}{cccccc} x_1 & x_2 & 0 & x_3 & \frac{21}{80} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 3 & 1 \\ 2 & 2 & 0 & 2 & 4 & \frac{3}{2} \end{array}\right\}$$

$f(x) = x^3 + \frac{25}{2}x^2 + \frac{149}{64}x - \frac{1}{64},\ x_1 = -12.3108,\ x_2 = -0.1957,\ x_3 = 0.0065$

$1+\frac{361}{21}q+\frac{12867}{7}q^2+\frac{3253105}{21}q^3+14062309q^4+\frac{27032665361}{21}q^5+\frac{834346677435}{7}q^6+O(q^7)$

$n^0_1 = 361,\ n^0_2 = 4780,\ n^0_3 = 120472,\ n^0_4 = 4613592,\ n^0_5 = 216261320,\ n^0_6 = 11588133060$

$e_{x_1} = -\frac{1}{6},\ e_{x_2} = -\frac{1}{6},\ e_{x_3} = -\frac{1}{6}$

$n^1_1 = 0,\ n^1_2 = 0,\ n^1_3 = 10,\ n^1_4 = 3240,\ n^1_5 = 745650,\ n^1_6 = 126956518$

$X \xrightarrow{2:1} B$ [Lee08]

$$M_0 = \begin{pmatrix}1&0&0&0\\1&1&0&0\\0&21&1&0\\0&0&1&1\end{pmatrix}, \quad M_{x_2} = \begin{pmatrix}-9&-62&-6&-4\\5&32&3&2\\-65&-403&-38&-26\\45&279&27&19\end{pmatrix}, \quad M_{x_1} =$$

$$\begin{pmatrix}-63&-332&-28&-16\\48&250&21&12\\-752&-3901&-328&-188\\576&2988&252&145\end{pmatrix}, M_{\frac{21}{80}} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, M_{x_3} = \begin{pmatrix}1&-9&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v^1_{x_1}$=$(84, -63, \frac{53}{2}, -416\lambda - \frac{31}{4})$, $v^1_{x_2}$=$(42, -21, 8, -208\lambda - \frac{9}{4})$, $v^1_{x_3}$=$(21, 0, \frac{11}{4}, -104\lambda)$

$$49\theta^4 - z\left(539 + 40667\theta + 117781\theta^2 + 154428\theta^3 + 79458\theta^4\right)$$
$$z^2\left(287233\theta^4 + 407088\theta^3 + 132608\theta^2 - 1337\theta - 896\right)$$
$$-z^3\left(321268\theta^4 + 385148\theta^3 + 265118\theta^2 + 107318\theta + 1806\right)$$
$$+z^4\left(100108\theta^4 + 195588\theta^3 + 147848\theta^2 + 50058\theta + 616\right) + 121z^5(\theta+1)^4$$

<table>
<tr><td>21</td><td rowspan="5">$\left\{\begin{array}{cccccc} x_1 & 0 & x_2 & \frac{7}{11} & x_3 & \infty \\ \hline 0&0&0&0&0&1\\ 1&0&1&1&1&1\\ 1&0&1&3&1&1\\ 2&0&2&4&2&1 \end{array}\right\}$</td></tr>
<tr><td>66</td></tr>
<tr><td>−102</td></tr>
<tr><td>9</td></tr>
<tr><td>$1/x_2$</td></tr>
</table>

$f(x) = x^3 + 84\,x^2 - 159x + 1$, $x_1 = -85.8522$, $x_2 = 0.0063$, $x_3 = 1.8458$

$1 + \frac{129}{7}q + \frac{12585}{7}q^2 + \frac{1119036}{7}q^3 + \frac{102049833}{7}q^4 + 1347689322q^5 + 125997790260q^6 + O(q^7)$

$n^0_1 = 387$, $n^0_2 = 4671$, $n^0_3 = 124323$, $n^0_4 = 4782996$, $n^0_5 = 226411803$, $n^0_6 = 12249769449$

$e_{x_1} = -\frac{1}{6}$, $e_{x_2} = -\frac{1}{6}$, $e_{x_3} = -\frac{1}{6}$

$n^1_1 = 0$, $n^1_2 = 0$, $n^1_3 = 1$, $n^1_4 = 1854$, $n^1_5 = 606294$, $n^1_6 = 117751416$

$$M_0 = \begin{pmatrix}1&0&0&0\\1&1&0&0\\0&21&1&0\\0&0&1&1\end{pmatrix}, M_{x_1} = \begin{pmatrix}-209&-1470&-150&-100\\105&736&75&50\\-1197&-8379&-854&-570\\693&4851&495&331\end{pmatrix},$$

$$M_{x_3} = \begin{pmatrix}1&-6&-1&-1\\0&1&0&0\\0&18&4&3\\0&-18&-3&-2\end{pmatrix}, M_{\frac{7}{11}} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, M_{x_2} = \begin{pmatrix}1&-9&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v^1_{x_1}$=$(210, -105, 32, -1020\lambda - \frac{29}{4})$, $v^1_{x_2}$=$(21, 0, \frac{11}{4}, -102\lambda)$, $v^1_{x_3}$=$(-21, 0, \frac{1}{4}, 102\lambda)$

$$49\theta^4 - z\left(588 + 4410\theta + 12726\theta^2 + 16632\theta^3 + 8064\theta^4\right)$$
$$+z^2\left(29232 + 145824\theta + 245172\theta^2 + 140832\theta^3 + 14256\theta^4\right)$$
$$-z^3\left(111888 + 413532\theta + 373140\theta^2 - 54432\theta^3 - 57456\theta^4\right)$$
$$-z^4\left(1296(2\theta+1)(36\theta^3 + 306\theta^2 + 421\theta + 156)\right) - 5184z^5(2\theta+1)(3\theta+2)(3\theta+4)(2\theta+3)$$

<table>
<tr><td>21</td><td rowspan="5">$\left\{\begin{array}{cccccc} -\frac{7}{12} & 0 & x_1 & x_2 & x_3 & \infty \\ \hline 0&0&0&0&0&\frac{1}{2}\\ 1&0&1&1&1&\frac{2}{3}\\ 3&0&1&1&1&\frac{4}{3}\\ 4&0&2&2&2&\frac{3}{2} \end{array}\right\}$</td></tr>
<tr><td>66</td></tr>
<tr><td>−100</td></tr>
<tr><td>9</td></tr>
<tr><td>$1/x_1$</td></tr>
</table>

$f(x) = x^3 - \frac{2}{3}x^2 + \frac{7}{54}x - \frac{1}{1296}$, $x_1 = 0.0061$, $x_2 = 0.3303 + 0.1284i$, $x_3 = 0.3303 - 0.1284i$

$$1+\frac{138}{7}q+\frac{12186}{7}q^2+\frac{1160706}{7}q^3+\frac{105559194}{7}q^4+\frac{9868791888}{7}q^5+\frac{931278113874}{7}q^6+\frac{88626516807792}{7}q^7+O(q^8)$$

$n_1^0=414,\ n_2^0=4518,\ n_3^0=128952,\ n_4^0=4947516,\ n_5^0=236851002,\ n_6^0=12934401960$

$e_{x_1}=-\frac{1}{6},\ e_{x_2}=-\frac{1}{6},\ e_{x_3}=-\frac{1}{6}$

$n_1^1=0,\ n_2^1=0,\ n_3^1=-2,\ n_4^1=441,\ n_5^1=466830,\ n_6^1=108083098$

$X \xrightarrow{2:1} B$ [Lee08]

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&21&1&0\\0&0&1&1\end{pmatrix},\ M_{-\frac{7}{12}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ M_{x_2}=\begin{pmatrix}-9&-60&-6&-4\\5&31&3&2\\-60&-360&-35&-24\\40&240&24&17\end{pmatrix},$$
$$M_{x_1}=\begin{pmatrix}1&-9&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ M_{x_3}=\begin{pmatrix}11&-18&-2&-4\\5&-8&-1&-2\\-45&81&10&18\\25&-45&-5&-9\end{pmatrix}$$

$v_{x_1}^1=(21,0,\frac{11}{4},-100\lambda),\ v_{x_2}^1=(42,-21,7,-200\lambda-\frac{9}{4}),\ v_{x_3}^1=(42,21,7,-200\lambda+\frac{9}{4})$

$$\begin{aligned}&121\theta^4-z\left(190008\theta^4+274560\theta^3+209880\theta^2+72600\theta+968\right)\\&-z^2\left(91520+4132480\theta+4957120\theta^2-1515520\theta^3-4142080\theta^4\right)\\&-z^3\left(8550400\theta^4-91238400\theta^3-127305600\theta^2-66422400\theta-1236400\right)\\&-z^4\left(51200(2\theta+1)(4\theta+3)(76\theta^2+189\theta+125)\right)\\&+z^5\left(2048000(2\theta+1)(4\theta+3)(4\theta+5)(2\theta+3)\right)\end{aligned}$$

22	$\left\{\begin{array}{cccccc}x_1&0&x_2&\frac{11}{160}&x_3&\infty\\\hline 0&0&0&0&0&\frac{1}{2}\\1&0&1&1&1&\frac{3}{4}\\1&0&1&3&1&\frac{5}{4}\\2&0&2&4&2&\frac{3}{2}\end{array}\right\}$
16	
-92	
5	
$1/x_2$	

$f(x)=5120x^3-512x^2-128x+1,\ x_1=-0.1206,\ x_2=0.0076,\ x_3=0.2130$

$$1+\frac{148}{11}p+1500p^2+\frac{1191064}{11}p^3+\frac{93433204}{11}p^4+\frac{7526589148}{11}p^5+\frac{613847321112}{11}p^6+O(p^7)$$

$n_1^0=296,\ n_2^0=4088,\ n_3^0=88216,\ n_4^0=2919272,\ n_5^0=120425424,\ n_6^0=5683760312$

$e_{x_1}=-\frac{1}{6},\ e_{x_2}=-\frac{1}{6},\ e_{x_3}=-\frac{1}{6}$

$n_1^1=0,\ n_2^1=1,\ n_3^1=8,\ n_4^1=5210,\ n_5^1=709632,\ n_6^1=83326750$

$$M_{x_1}=\begin{pmatrix}-5&-56&-6&-4\\3&29&3&2\\-54&-504&-53&-36\\48&448&48&33\end{pmatrix},\ M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&22&1&0\\0&0&1&1\end{pmatrix},\ M_{x_2}=$$
$$\begin{pmatrix}1&-7&-1&-1\\0&1&0&0\\0&-14&-1&-2\\0&14&2&3\end{pmatrix},\ M_{\frac{11}{160}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ M_{x_3}=\begin{pmatrix}1&-5&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v_{x_1}^1=(44,-22,\frac{25}{3},-184\lambda-\frac{7}{3}),\ v_{x_2}^1=(22,0,\frac{8}{3},-92\lambda),\ v_{x_3}^1=(22,0,\frac{2}{3},-92\lambda)$

$$\begin{aligned}&121\theta^4 - z\left(1210+8954\theta+25707\theta^2+33506\theta^3+18403\theta^4\right)\\&+z^2\left(6600+345540\theta+1328850\theta^2+2829800\theta^3+1924700\theta^4\right)\\&-z^3\left(91872+4617736\theta+9098040\theta^2+9839280\theta^3+6699880\theta^4\right)\\&+z^4\left(268000+12452000\theta+22854000\theta^2+20540000\theta^3+8158000\theta^4\right)\\&-20000z^5(4\theta+3)(\theta+1)^2(4\theta+5)\end{aligned}$$

22	
64	
-86	
9	
$71+17\sqrt{17}$	

$$\left\{\begin{array}{cccccc} 0 & \frac{71}{128}-\frac{17}{128}\sqrt{17} & \frac{11}{50} & 1 & \frac{71}{128}+\frac{17}{128}\sqrt{17} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & 1 & 1 \\ 0 & 2 & 4 & 2 & 2 & \frac{5}{4} \end{array}\right\}$$

$$1+\frac{181}{11}q+\frac{16325}{11}q^2+\frac{1273798}{11}q^3+9489423q^4+\frac{8664158931}{11}q^5+\frac{726894174086}{11}q^6+\frac{61499016073403}{11}q^7+O(q^8)$$

$n^0_1=362$, $n^0_2=4036$, $n^0_3=94342$, $n^0_4=3261479$, $n^0_5=138626540$, $n^0_6=6730489668$

$e_{\frac{71}{128}-\frac{17}{128}\sqrt{17}}=-\frac{1}{6}$, $e_1=-1$, $e_{\frac{71}{128}+\frac{17}{128}\sqrt{17}}=-\frac{1}{6}$

$n^1_1=0$, $n^1_2=0$, $n^1_3=6$, $n^1_4=2289$, $n^1_5=461850$, $n^1_6=69825441$

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&22&1&0\\0&0&1&1\end{pmatrix},\quad M_{\frac{71}{128}+\frac{17}{128}\sqrt{17}}=\begin{pmatrix}-19&-235&-30&-25\\4&48&6&5\\-8&-94&-11&-10\\-12&-141&-18&-14\end{pmatrix},\quad M_1=\begin{pmatrix}1&-36&-6&-6\\0&1&0&0\\0&108&19&18\\0&-108&-18&-17\end{pmatrix},\quad M_{\frac{11}{50}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad M_{\frac{71}{128}-\frac{17}{128}\sqrt{17}}=\begin{pmatrix}1&-9&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v^1_{\frac{71}{128}-\frac{17}{128}\sqrt{17}}=(22,0,\frac{8}{3},-86\lambda)$, $v^6_1=(-132,0,2,516\lambda)$, $v^1_{\frac{71}{128}+\frac{17}{128}\sqrt{17}}=(110,-22,\frac{13}{3},-430\lambda-\frac{4}{3})$

$$\begin{aligned}&121\theta^4 - z\left(1210+9075\theta+26323\theta^2+34496\theta^3+16687\theta^4\right)\\&-z^2\left(24266+106953\theta+202166\theta^2+207620\theta^3+90362\theta^4\right)\\&-z^3\left(53130+217437\theta+415082\theta^2+507996\theta^3+245714\theta^4\right)\\&-z^4\left(15226+183269\theta+564786\theta^2+785972\theta^3+407863\theta^4\right)\\&-z^5\left(25160+279826\theta+728323\theta^2+790148\theta^3+434831\theta^4\right)\\&-z^6\left(61152+303576\theta+586744\theta^2+562248\theta^3+290888\theta^4\right)\\&-z^7\left(39040+172960\theta+310160\theta^2+274400\theta^3+104560\theta^4\right)\ -12800z^8(\theta+1)^4\end{aligned}$$

22	
64	
-84	
9	
$\frac{143}{2}+19/2\sqrt{57}$	

$$\left\{\begin{array}{ccccccc} -\frac{143}{64}-\frac{19}{64}\sqrt{57} & -1 & x_1 & x_2 & 0 & -\frac{143}{64}+\frac{19}{64}\sqrt{57} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & \frac{1}{2} & 1 & 1 & 0 & 1 & 1 \\ 1 & \frac{1}{2} & 3 & 3 & 0 & 1 & 1 \\ 2 & 1 & 4 & 4 & 0 & 2 & 1 \end{array}\right\}$$

$x_1=-\frac{17}{40}-\frac{i\sqrt{591}}{40}$, $x_2=-\frac{17}{40}+\frac{i\sqrt{591}}{40}$

$$1+\frac{193}{11}q+\frac{15977}{11}q^2+\frac{1311583}{11}q^3+\frac{108130089}{11}q^4+\frac{9055523068}{11}q^5+\frac{766967495303}{11}q^6+\frac{65496406190814}{11}q^7+O(q^8)$$

$n^0_1=386$, $n^0_2=3946$, $n^0_3=97140$, $n^0_4=3378566$, $n^0_5=144888366$, $n^0_6=7101538592$

$e_{-\frac{143}{64}-\frac{19}{64}\sqrt{57}}=-\frac{1}{6}$, $e_{-1}=-\frac{7}{6}$, $e_{-\frac{17}{40}-1/40\,i\sqrt{591}}=-\frac{1}{6}$, $e_{-\frac{143}{64}+\frac{19}{64}\sqrt{57}}=-\frac{1}{6}$

$n^1_1=0$, $n^1_2=0$, $n^1_3=0$, $n^1_4=1383$, $n^1_5=382260$, $n^1_6=64967349$

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&22&1&0\\0&0&1&1\end{pmatrix},\quad M_{-1}=\begin{pmatrix}-3&16&4&4\\-2&-37&-5&-4\\64&756&91&68\\-62&-626&-71&-51\end{pmatrix},\quad M_{-\frac{143}{64}-\frac{19}{64}\sqrt{57}}=$$
$$\begin{pmatrix}1&0&0&0\\7&25&1&0\\-168&-576&-23&0\\154&528&22&1\end{pmatrix},\quad M_{-\frac{17}{40}+1/40\,i\sqrt{591}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad M_{-\frac{143}{64}+\frac{19}{64}\sqrt{57}}=$$
$$\begin{pmatrix}1&-9&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\; M_{-\frac{17}{40}-1/40\,i\sqrt{591}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v^1_{-\frac{143}{64}-\frac{19}{64}\sqrt{57}}=(0,22,-13,\frac{13}{3})$, $v^1_{-\frac{143}{64}+\frac{19}{64}\sqrt{57}}=(22,0,\frac{8}{3},-84\lambda)$

$$\begin{aligned}&529\theta^4-z\left(4232+312111\theta+896087\theta^2+1167940\theta^3+75233\theta^4\right)\\&-z^2\left(205712+8508862\theta+7873930\theta^2-9999240\theta^3-1357863\theta^4\right)\\&-z^3\left(801872+14136720\theta-17505680\theta^2-21798480\theta^3+6206392\theta^4\right)\\&+z^4\left(7128704+272674880\theta+354453920\theta^2+152783040\theta^3-980880\theta^4\right)\\&-1905152z^5(4\theta+3)(\theta+1)^2(4\theta+5)\end{aligned}$$

23	
62	
-74	
9	
$1/x_3$	

$$\left\{\begin{array}{cccccc}x_1&x_2&0&x_3&\frac{23}{244}&\infty\\\hline 0&0&0&0&0&\frac{3}{4}\\1&1&0&1&1&1\\1&1&0&1&3&1\\2&2&0&2&4&\frac{5}{4}\end{array}\right\}$$

$f(x)=x^3+\frac{113}{512}x^2+\frac{121}{512}x-\frac{1}{512}$, $x_1=-0.1145-0.4745i$, $x_2=-0.1145+0.4745i$, $x_3=0.0082$

$1+\frac{308}{23}q+\frac{28516}{23}q^2+\frac{1911881}{23}q^3+\frac{137544292}{23}q^4+\frac{10175353058}{23}q^5+\frac{759826639561}{23}q^6+\frac{57197214710139}{23}q^7+O(q^8)$

$n^0_1=308$, $n^0_2=3526$, $n^0_3=70799$, $n^0_4=2148684$, $n^0_5=81402822$, $n^0_6=3517706942$

$e_{x_1}=-\frac{1}{6}$, $e_{x_2}=-\frac{1}{6}$, $e_{x_3}=-\frac{1}{6}$

$n^1_1=0$, $n^1_2=0$, $n^1_3=30$, $n^1_4=3185$, $n^1_5=418076$, $n^1_6=46375234$

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&23&1&0\\0&0&1&1\end{pmatrix},\; M_{x_2}=\begin{pmatrix}-11&-111&-12&-9\\4&38&4&3\\-40&-370&-39&-30\\20&185&20&16\end{pmatrix},\; M_{\frac{23}{244}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\; M_{x_3}=$$
$$\begin{pmatrix}1&-9&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\; M_{x_1}=\begin{pmatrix}13&-42&-6&-9\\4&-13&-2&-3\\-52&182&27&39\\32&-112&-16&-23\end{pmatrix}$$

$v^1_{x_1}=(69,23,\frac{25}{4},-222\lambda+\frac{17}{12})$, $v^1_{x_2}=(69,-23,\frac{25}{4},-222\lambda-\frac{17}{12})$, $v^1_{x_3}=(23,0,\frac{31}{12},-74\lambda)$

$\theta^4-2z(2\theta+1)^2(17\theta^2+17\theta+5)+4z^2(2\theta+1)(\theta+1)^2(2\theta+3)$

24	
72	
-116	
10	
$68+48\sqrt{2}$	

$$\left\{\begin{array}{cccc}0&\frac{17}{4}-3\sqrt{2}&\frac{17}{4}+3\sqrt{2}&\infty\\\hline 0&0&0&\frac{1}{2}\\0&1&1&1\\0&1&1&1\\0&2&2&\frac{3}{2}\end{array}\right\}$$

$1+14q+1226q^2+93830q^3+7275722q^4+572383514q^5+45521513426q^6+3649815392986q^7+O(q^8)$
$n_1^0=336,\ n_2^0=3636,\ n_3^0=83392,\ n_4^0=2727936,\ n_5^0=109897632,\ n_6^0=5057935376$
$e_{\frac{17}{4}-3\sqrt{2}}=-\frac{1}{6},\ e_{\frac{17}{4}+3\sqrt{2}}=-\frac{1}{6}$
$n_1^1=0,\ n_2^1=0,\ n_3^1=0,\ n_4^1=66,\ n_5^1=121056,\ n_6^1=29099400$
$X(1,1,1,1,1,1,2)\subset X_{10}$ [Bor96]
$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&24&1&0\\0&0&1&1\end{pmatrix},\ M_{\frac{17}{4}+3\sqrt{2}}=\begin{pmatrix}-49&-370&-35&-25\\20&149&14&10\\-240&-1776&-167&-120\\140&1036&98&71\end{pmatrix},\ M_{\frac{17}{4}-3\sqrt{2}}=\begin{pmatrix}1&-10&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$
$v^1_{\frac{17}{4}-3\sqrt{2}}=(24,0,3,-116\lambda),\ v^1_{\frac{17}{4}+3\sqrt{2}}=(120,-48,15,-580\lambda-4)$

$\theta^4-z\left(124\theta^4+242\theta^3+187\theta^2+66\theta+9\right)+z^2\left(123\theta^4-246\theta^3-787\theta^2-554\theta-124\right)$ $+z^3\left(123\theta^4+738\theta^3+689\theta^2+210\theta+12\right)-z^4\left(124\theta^4+254\theta^3+205\theta^2+78\theta+12\right)$ $+z^5(\theta+1)^4$	
25	$\left\{\begin{array}{cccccc}-1&0&\frac{123}{2}-\frac{55}{2}\sqrt{5}&1&\frac{123}{2}+\frac{55}{2}\sqrt{5}&\infty\\\hline 0&0&0&0&0&1\\1&0&1&1&1&1\\1&0&1&3&1&1\\2&0&2&4&2&1\end{array}\right\}$
70	
-100	
10	
$\frac{123}{2}+\frac{55}{2}\sqrt{5}$	
$1+13q+1037q^2+71563q^3+5082637q^4+365174388q^5+26527553579q^6+1942800852474q^7+O(q^8)$	
$n_1^0=325,\ n_2^0=3200,\ n_3^0=66250,\ n_4^0=1985000,\ n_5^0=73034875,\ n_6^0=3070310300$	
$e_{-1}=-\frac{1}{3},\ e_{\frac{123}{2}-\frac{55}{2}\sqrt{5}}=-\frac{1}{6},\ e_{\frac{123}{2}+\frac{55}{2}\sqrt{5}}=-\frac{1}{6}$	
$n_1^1=0,\ n_2^1=0,\ n_3^1=0,\ n_4^1=325,\ n_5^1=109822,\ n_6^1=19018900$	
X_{25} Pfaffian [Kan12] and [Kap09] and $X\xrightarrow{2:1}B$ [Lee08]	
$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&25&1&0\\0&0&1&1\end{pmatrix},\ M_{-1}=\begin{pmatrix}-19&-140&-12&-8\\10&71&6&4\\-150&-1050&-89&-60\\100&700&60&41\end{pmatrix},\ M_{\frac{123}{2}+\frac{55}{2}\sqrt{5}}=\begin{pmatrix}1&-100&-16&-16\\0&1&0&0\\0&375&61&60\\0&-375&-60&-59\end{pmatrix},\ M_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ M_{\frac{123}{2}-\frac{55}{2}\sqrt{5}}=\begin{pmatrix}1&-10&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	
$v^2_{-1}=(100,-50,\frac{50}{3},-400\lambda-\frac{25}{6}),\ v^1_{\frac{123}{2}-\frac{55}{2}\sqrt{5}}=(25,0,\frac{35}{12},-100\lambda),\ v^1_{\frac{123}{2}+\frac{55}{2}\sqrt{5}}=(-100,0,\frac{10}{3},400\lambda)$	

$$169\theta^4 - z\left(1352 + 9802\theta + 27651\theta^2 + 35698\theta^3 + 19565\theta^4\right)$$
$$-z^2\left(37024 + 1662296\theta + 2229640\theta^2 + 83440\theta^3 - 918440\theta^4\right)$$
$$-z^3\left(1664 - 861120\theta - 5161920\theta^2 - 8991360\theta^3 - 2407680\theta^4\right)$$
$$-z^4\left(530688 + 19741440\theta + 22428160\theta^2 + 7953920\theta^3 + 9268480\theta^4\right)$$
$$+z^5\left(265216 + 9717760\theta + 9996800\theta^2 - 4623360\theta^3 - 15582720\theta^4\right)$$
$$+z^6\left(225280 + 17940480\theta + 45107200\theta^2 + 41594880\theta^3 + 5662720\theta^4\right)$$
$$+z^7\left(606208 + 26214400\theta + 45056000\theta^2 + 37683200\theta^3 + 13434880\theta^4\right) + 262144z^8(\theta+1)^4$$

26
68
-86
10
$1/x_3$

$$\left\{\begin{array}{cccccccc} x_1 & -\frac{11}{32}-\frac{\sqrt{329}}{32} & x_2 & 0 & x_3 & -\frac{11}{32}+\frac{\sqrt{329}}{32} & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 4 & 2 & 0 & 2 & 4 & 2 & 1 \end{array}\right\}$$

$x_1 = -4.3623,\ x_2 = -0.3967,\ x_3 = 0.0090$

$1+\frac{151}{13}q+\frac{11639}{13}q^2+\frac{710062}{13}q^3+\frac{46057399}{13}q^4+\frac{3023016151}{13}q^5+\frac{200434389518}{13}q^6+\frac{13398630711365}{13}q^7+O(q^8)$

$n_1^0 = 302,\ n_2^0 = 2872,\ n_3^0 = 52586,\ n_4^0 = 1438930,\ n_5^0 = 48368256,\ n_6^0 = 1855867296$

$e_{x_1} = 1,\ e_{x_2} = 1,\ e_{x_3} = 1,\ e_1 = 2$

$n_1^1 = 0,\ n_2^1 = 0,\ n_3^1 = 2,\ n_4^1 = 620,\ n_5^1 = 106358,\ n_6^1 = 13419013$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 26 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{x_2} = \begin{pmatrix} -9 & -72 & -6 & -4 \\ 5 & 37 & 3 & 2 \\ -80 & -576 & -47 & -32 \\ 55 & 396 & 33 & 23 \end{pmatrix},\ M_{-\frac{11}{32}-1/32\sqrt{329}} =$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_1 = \begin{pmatrix} 1 & -14 & -2 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 42 & 7 & 6 \\ 0 & -42 & -6 & -5 \end{pmatrix},\ M_{-\frac{11}{32}+1/32\sqrt{329}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{x_3} =$$
$$\begin{pmatrix} 1 & -10 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{x_1} = \begin{pmatrix} -35 & -280 & -24 & -16 \\ 18 & 141 & 12 & 8 \\ -270 & -2100 & -179 & -120 \\ 171 & 1330 & 114 & 77 \end{pmatrix}$$

$v_{x_1}^1=(104,-52,\frac{46}{3},-344\lambda-\frac{10}{3}),\ v_{x_2}^1=(52,-26,\frac{26}{3},-172\lambda-\frac{13}{6}),\ v_{x_3}^1=(26,0,\frac{17}{6},-86\lambda),$
$v_1^2=(-52,0,\frac{1}{3},172\lambda)$

$$\theta^4 - 2z(2\theta+1)^2(13\theta^2+13\theta+4) - 12z^2(2\theta+1)(3\theta+2)(3\theta+4)(2\theta+3)$$

28
76
-116
11
108

$$\left\{\begin{array}{cccc} -\frac{1}{4} & 0 & \frac{1}{108} & \infty \\ \hline 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & \frac{2}{3} \\ 1 & 0 & 1 & \frac{4}{3} \\ 2 & 0 & 2 & \frac{3}{2} \end{array}\right\}$$

$1+10q+774q^2+46558q^3+2924294q^4+184776760q^5+11815272594q^6+761379437052q^7+O(q^8)$

$n_1^0 = 280,\ n_2^0 = 2674,\ n_3^0 = 48272,\ n_4^0 = 1279040,\ n_5^0 = 41389992,\ n_6^0 = 1531603276$

$n_1^1 = 0,\ n_2^1 = 0,\ n_3^1 = 0,\ n_4^1 = 27,\ n_5^1 = 26208,\ n_6^1 = 5914124$

$X(1,1,1,1,2) \subset \mathrm{Gr}(2,6)$ [BCFKS98]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 28 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad M_{-\frac{1}{4}} = \begin{pmatrix} -11 & -80 & -6 & -4 \\ 6 & 41 & 3 & 2 \\ -108 & -720 & -53 & -36 \\ 78 & 520 & 39 & 27 \end{pmatrix}, \quad M_{\frac{1}{108}} = \begin{pmatrix} 1 & -11 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{-\frac{1}{4}}=(56, -28, \frac{31}{3}, -232\lambda - \frac{17}{6})$, $v^1_{\frac{1}{108}}=(28, 0, \frac{19}{6}, -116\lambda)$

$$\begin{aligned}
&841\theta^4 - z\left(5046 + 370004\theta + 1047480\theta^2 + 1354880\theta^3 + 764440\theta^4\right) \\
&-z^2\left(673380 + 31663360\theta + 50701040\theta^2 + 29775360\theta^3 + 3639840\theta^4\right) \\
&-z^3\left(3654000 + 110403000\theta + 46701000\theta^2 - 77256000\theta^3 - 34172000\theta^4\right) \\
&+10000z^4(2\theta+1)(68\theta^3 + 1842\theta^2 + 2899\theta + 1215) - 5000000z^5(2\theta+1)(\theta+1)^2(2\theta+3)
\end{aligned}$$

29	$\left\{\begin{array}{cccccc} x_1 & x_2 & 0 & x_3 & \frac{29}{100} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 3 & 1 \\ 2 & 2 & 0 & 2 & 4 & \frac{3}{2} \end{array}\right\}$
74	
-104	
11	
$1/x_3$	

$f(x) = x^3 + \frac{64}{125}x^2 + \frac{21}{500}x - \frac{1}{2000}$, $x_1 = -0.4053$, $x_2 = -0.1172$, $x_3 = 0.0105$

$1+\frac{248}{29}q+\frac{20056}{29}q^2+\frac{1045364}{29}q^3+\frac{60210776}{29}q^4+\frac{3490781248}{29}q^5+\frac{204636408340}{29}q^6+\frac{12085634303424}{29}q^7+O(q^8)$

$n^0_1 = 248$, $n^0_2 = 2476$, $n^0_3 = 38708$, $n^0_4 = 940480$, $n^0_5 = 27926248$, $n^0_6 = 947385848$

$e_{x_1} = -\frac{1}{6}$, $e_{x_2} = -\frac{1}{6}$, $e_{x_3} = -\frac{1}{6}$

$n^1_1 = 0$, $n^1_2 = 0$, $n^1_3 = 2$, $n^1_4 = 248$, $n^1_5 = 44576$, $n^1_6 = 5163998$

$X \xrightarrow{2:1} B$ [Lee08] , $\sharp 50$ [BK10]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 29 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad M_{x_2} = \begin{pmatrix} -11 & -82 & -6 & -4 \\ 6 & 42 & 3 & 2 \\ -114 & -779 & -56 & -38 \\ 84 & 574 & 42 & 29 \end{pmatrix}, \quad M_{\frac{29}{100}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_{x_3} = \begin{pmatrix} 1 & -11 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{x_1} = \begin{pmatrix} -38 & -228 & -15 & -9 \\ 26 & 153 & 10 & 6 \\ -559 & -3268 & -214 & -129 \\ 442 & 2584 & 170 & 103 \end{pmatrix}$$

$v^1_{x_1}=(87, -58, \frac{93}{4}, -312\lambda - \frac{41}{6})$, $v^1_{x_2}=(58, -29, \frac{32}{3}, -208\lambda - \frac{35}{12})$, $v^1_{x_3}=(29, 0, \frac{37}{12}, -104\lambda)$

$$\begin{aligned}
&841\theta^4 - z\left(6728 + 479377\theta + 1327330\theta^2 + 1695920\theta^3 + 877540\theta^4\right) \\
&z^2\left(5568 + 577680\theta + 2391590\theta^2 + 4242200\theta^3 + 2586470\theta^4\right) \\
&-z^3\left(76560 + 3368640\theta + 5816470\theta^2 + 5326140\theta^3 + 2727430\theta^4\right) \\
&z^4\left(75616 + 3327920\theta + 5522280\theta^2 + 4211240\theta^3 + 1306960\theta^4\right) - 3468z^5(\theta+1)^2(3\theta+2)(3\theta+4)
\end{aligned}$$

29	$\left\{\begin{array}{cccccc} 0 & x_1 & \frac{29}{34} & x_2 & x_3 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & 1 & 1 \\ 0 & 2 & 4 & 2 & 2 & \frac{4}{3} \end{array}\right\}$
74	
-100	
11	
$1/x_1$	

$f(x) = x^3 - \frac{67}{27}x^2 + \frac{34}{9}x - \frac{1}{27}$, $x_1 = 0.0099$, $x_2 = 1.2358 + 1.4920i$, $x_3 = 1.2358 - 1.4920i$

$1+\frac{285}{29}q+\frac{19181}{29}q^2+\frac{1097187}{29}q^3+\frac{63857645}{29}q^4+\frac{3762105410}{29}q^5+\frac{223978352483}{29}q^6+\frac{13441541586898}{29}q^7+O(q^8)$

$n_1^0 = 285$, $n_2^0 = 2362$, $n_3^0 = 40626$, $n_4^0 = 997476$, $n_5^0 = 30096841$, $n_6^0 = 1036931650$

$n_1^1 = 0$, $n_2^1 = 0$, $n_3^1 = 0$, $n_4^1 = 34$, $n_5^1 = 22154$, $n_6^1 = 4012456$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 29 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{x_2} = \begin{pmatrix} -14 & -138 & -12 & -9 \\ 5 & 47 & 4 & 3 \\ -65 & -598 & -51 & -39 \\ 35 & 322 & 28 & 22 \end{pmatrix}, M_{\frac{29}{34}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{x_1} =$$
$$\begin{pmatrix} 1 & -11 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{x_3} = \begin{pmatrix} 16 & -51 & -6 & -9 \\ 5 & -16 & -2 & -3 \\ -80 & 272 & 33 & 48 \\ 50 & -170 & -20 & -29 \end{pmatrix}$$

$v_{x_1}^1=(29, 0, \frac{37}{12}, -100\lambda)$, $v_{x_2}^1=(87, -29, \frac{31}{4}, -300\lambda - \frac{23}{12})$, $v_{x_3}^1=(87, 29, \frac{31}{4}, -300\lambda + \frac{23}{12})$

$$25\theta^4 - z\left(175 + 1250\theta + 3480\theta^2 + 4460\theta^3 + 2365\theta^4\right)$$
$$-z^2\left(5490 + 21790\theta + 28834\theta^2 + 9272\theta^3 - 3946\theta^4\right)$$
$$-z^3\left(5760 + 6210\theta - 20646\theta^2 - 34560\theta^3 - 6174\theta^4\right)$$
$$-z^4\left(9234 + 32886\theta + 37098\theta^2 + 8424\theta^3 - 6723\theta^4\right) - 6561z^5(\theta+1)^4$$

30	$\left\{\begin{array}{cccccc} -\frac{46}{81}-\frac{13}{81}\sqrt{13} & -1 & 0 & -\frac{46}{81}+\frac{13}{81}\sqrt{13} & \frac{5}{9} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 3 & 1 \\ 2 & 2 & 0 & 2 & 4 & 1 \end{array}\right\}$
72	
-86	
11	
$46 + 13\sqrt{13}$	

$1+\frac{46}{5}q+\frac{2914}{5}q^2+\frac{152056}{5}q^3+\frac{8239874}{5}q^4+\frac{451221546}{5}q^5+\frac{24960879448}{5}q^6+\frac{1391968205352}{5}q^7+O(q^8)$

$$M_{-\frac{46}{81}-\frac{13}{81}\sqrt{13}} = \begin{pmatrix} -97 & -875 & -70 & -49 \\ 42 & 376 & 30 & 21 \\ -672 & -6000 & -479 & -336 \\ 406 & 3625 & 290 & 204 \end{pmatrix}, M_{-1} = \begin{pmatrix} -69 & -560 & -42 & -28 \\ 35 & 281 & 21 & 14 \\ -630 & -5040 & -377 & -252 \\ 420 & 3360 & 252 & 169 \end{pmatrix},$$
$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 30 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{-\frac{46}{81}+\frac{13}{81}\sqrt{13}} = \begin{pmatrix} 1 & -11 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{\frac{5}{9}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{-\frac{46}{81}-\frac{13}{81}\sqrt{13}}=(210, -90, 24, -602\lambda - 5)$, $v^1_{-\frac{46}{81}+\frac{13}{81}\sqrt{13}}=(30, 0, 3, -86\lambda)$

$\theta^4 - 8z(2\theta+1)^2(3\theta^2+3\theta+1) + 64z^2(2\theta+1)(\theta+1)^2(2\theta+3)$

32	$\left\{\begin{array}{cccc} 0 & 3/16 - 1/8\sqrt{2} & 3/16 + 1/8\sqrt{2} & \infty \\ \hline 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & \frac{3}{2} \end{array}\right\}$
80	
-116	
12	
$48 + 32\sqrt{2}$	

$1+8q+512q^2+27008q^3+1420320q^4+75771008q^5+4083679232q^6+221804832768q^7+O(q^8)$

$n_1^0 = 256$, $n_2^0 = 2016$, $n_3^0 = 32000$, $n_4^0 = 709904$, $n_5^0 = 19397376$, $n_6^0 = 604985440$

$n_1^1 = 0$, $n_2^1 = 0$, $n_3^1 = 0$, $n_4^1 = -8$, $n_5^1 = 1024$, $n_6^1 = 1220032$

$X(1,1,2) \subset \mathrm{LGr}(3,6)$ [Bor96]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 32 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \; M_{3/16+1/8\sqrt{2}} = \begin{pmatrix} -17 & -156 & -12 & -9 \\ 6 & 53 & 4 & 3 \\ -96 & -832 & -63 & -48 \\ 60 & 520 & 40 & 31 \end{pmatrix}, \; M_{3/16-1/8\sqrt{2}} = \begin{pmatrix} 1 & -12 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{3/16-1/8\sqrt{2}}=(32, 0, \frac{10}{3}, -116\lambda)$, $v^1_{3/16+1/8\sqrt{2}}=(96, -32, 10, -348\lambda - \frac{8}{3})$

$$\begin{aligned}
&289\theta^4 - z\left(1734 + 121138\theta + 332218\theta^2 + 421600\theta^3 + 219998\theta^4\right) \\
&-z^2\left(81600 + 3684492\theta + 5909360\theta^2 + 3590008\theta^3 + 577480\theta^4\right) \\
&-z^3\left(209304 + 6738120\theta + 5037480\theta^2 - 3081600\theta^3 - 2584560\theta^4\right) \\
&-z^4\left(634896 + 17361360\theta + 6651920\theta^2 - 13310720\theta^3 - 4153600\theta^4\right) \\
&+z^5\left(167808 + 10543200\theta + 21030240\theta^2 + 12856320\theta^3 - 4492800\theta^4\right) \\
&+z^6\left(42048 + 5169600\theta + 10113600\theta^2 + 92160\theta^3 - 9753600\theta^4\right) \\
&-z^7\left(338688 + 14768640\theta + 24929280\theta^2 + 19906560\theta^3 + 6635520\theta^4\right) \\
&-18432z^8(4\theta+3)(\theta+1)^2(4\theta+5)
\end{aligned}$$

34
76
−90
12
$1/x_3$

$$\left\{\begin{array}{cccccccc} -\frac{3}{16}-\frac{\sqrt{897}}{48} & x_1 & x_2 & -\frac{1}{4} & 0 & x_3 & -\frac{3}{16}-\frac{\sqrt{897}}{48} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 0 & 1 & 3 & 1 \\ 4 & 2 & 2 & 2 & 0 & 2 & 4 & \frac{5}{4} \end{array}\right\}$$

$x_1 = -0.6314 - 1.4335i$, $x_2 = -0.6314 + 1.4335i$, $x_3 = 0.0127$

$$1+\tfrac{118}{17}q+\tfrac{7102}{17}q^2+\tfrac{308566}{17}q^3+\tfrac{14342686}{17}q^4+\tfrac{669855868}{17}q^5+\tfrac{31621056670}{17}q^6+\tfrac{1504072957540}{17}q^7+O(q^8)$$

$n^0_1 = 236$, $n^0_2 = 1746$, $n^0_3 = 22848$, $n^0_4 = 447987$, $n^0_5 = 10717692$, $n^0_6 = 292784640$

$n^1_1 = 0$, $n^1_2 = 0$, $n^1_3 = 0$, $n^1_4 = 44$, $n^1_5 = 8208$, $n^1_6 = 952663$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 34 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \; M_{-\frac{1}{4}} = \begin{pmatrix} -11 & -92 & -6 & -4 \\ 6 & 47 & 3 & 2 \\ -132 & -1012 & -65 & -44 \\ 96 & 736 & 48 & 33 \end{pmatrix}, \; M_{-3/16-1/48\sqrt{897}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; M_{x_2} = \begin{pmatrix} -14 & -156 & -12 & -9 \\ 5 & 53 & 4 & 3 \\ -80 & -832 & -63 & -48 \\ 45 & 468 & 36 & 28 \end{pmatrix}, \; M_{-3/16+1/48\sqrt{897}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_{x_3} = \begin{pmatrix} 1 & -12 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; M_{x_1} = \begin{pmatrix} 16 & -54 & -6 & -9 \\ 5 & -17 & -2 & -3 \\ -90 & 324 & 37 & 54 \\ 55 & -198 & -22 & -32 \end{pmatrix}$$

$v^1_{x_1}=(102, 34, \frac{17}{2}, -270\lambda + \frac{11}{6})$, $v^1_{x_2}=(102, -34, \frac{17}{2}, -270\lambda - \frac{11}{6})$, $v^1_{-\frac{1}{4}}=(68, -34, \frac{34}{3}, -180\lambda - \frac{17}{6})$, $v^1_{x_3}=(34, 0, \frac{19}{6}, -90\lambda)$

$$289\theta^4 - z\left(2023+138872\theta+374000\theta^2+470560\theta^3+249050\theta^4\right)$$
$$z^2\left(30600+1318520\theta+2500340\theta^2+2631640\theta^3+1240300\theta^4\right)$$
$$-z^3\left(918000+4020840\theta+6913620\theta^2+5893560\theta^3+2335500\theta^4\right)$$
$$z^4\left(122472+5103000\theta+8150220\theta^2+6094440\theta^3+1931850\theta^4\right)\ -59049z^5(\theta+1)^4$$

34	$\left\{\begin{array}{ccccc} 0 & \frac{1}{81} & \frac{17}{27} & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & \frac{1}{2} & 1 \\ 0 & 1 & 3 & \frac{1}{2} & 1 \\ 0 & 2 & 4 & 1 & 1 \end{array}\right\}$
76	
−88	
12	
81	

$$1+\tfrac{126}{17}q+\tfrac{6910}{17}q^2+\tfrac{316026}{17}q^3+865566q^4+\tfrac{692407376}{17}q^5+\tfrac{32912055322}{17}q^6+\tfrac{1576716644916}{17}q^7+O(q^8)$$

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&34&1&0\\0&0&1&1\end{pmatrix},\ M_1=\begin{pmatrix}-9&-46&-1&1\\0&-15&-2&-2\\40&616&59&50\\-40&-504&-44&-35\end{pmatrix},\ M_{\frac{17}{27}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ M_{\frac{1}{81}}=\begin{pmatrix}1&-12&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v^1_{\frac{1}{81}}=(34,0,\frac{19}{6},-88\lambda)$

$$\theta^4-z\left(73\theta^4+98\theta^3+77\theta^2+28\theta+4\right)+z^2\left(520\theta^4-1040\theta^3-2904\theta^2-2048\theta-480\right)$$
$$+z^3\left(4160\theta^4+24960\theta^3+26688\theta^2+11520\theta+1792\right)$$
$$-z^4\left(37376\theta^4+99328\theta^3+113152\theta^2+63488\theta+14336\right)+32768z^5(\theta+1)^4$$

36	$\left\{\begin{array}{cccccc} -\frac{1}{8} & 0 & \frac{1}{64} & \frac{1}{8} & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 4 & 2 & 1 \end{array}\right\}$
72	
−72	
12	
64	

$$1+5q+341q^2+12263q^3+480213q^4+19293880q^5+788899463q^6+32438437820q^7+O(q^8)$$

$n^0_1=180,\ n^0_2=1512,\ n^0_3=16344,\ n^0_4=269928,\ n^0_5=5556636,\ n^0_6=131481144$

$p_{-\frac{1}{8}}=-\frac{1}{6},\ p_{\frac{1}{64}}=-\frac{1}{6},\ p_1=-\frac{2}{3}$

$n^1_1=0,\ n^1_2=0,\ n^1_3=4,\ n^1_4=189,\ n^1_5=14112,\ n^1_6=848138$

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&36&1&0\\0&0&1&1\end{pmatrix},\ M_{-\frac{1}{8}}=\begin{pmatrix}-11&-96&-6&-4\\6&49&3&2\\-144&-1152&-71&-48\\108&864&54&37\end{pmatrix},\ M_1=\begin{pmatrix}1&-36&-4&-4\\0&1&0&0\\0&108&13&12\\0&-108&-12&-11\end{pmatrix},\ M_{\frac{1}{8}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ M_{\frac{1}{64}}=\begin{pmatrix}1&-12&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v^1_{-\frac{1}{8}}=(72,-36,12,-144\lambda-3),\ v^1_{\frac{1}{64}}=(36,0,3,-72\lambda),\ v^1_1=(72,0,0,-144\lambda)$

$\theta^4 - 6z(2\theta+1)^2(3\theta^2+3\theta+1) - 108z^2(2\theta+1)(\theta+1)^2(2\theta+3)$

36	$$\left\{\begin{array}{cccc} -1/12-1/18\sqrt{3} & 0 & -1/12+1/18\sqrt{3} & \infty \\ \hline 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 2 & \frac{3}{2} \end{array}\right\}$$
84	
-120	
13	
$36+24\sqrt{3}$	

$1+6q+378q^2+16422q^3+752634q^4+34419006q^5+1595351322q^6+74451566286q^7+O(q^8)$

$n_1^0=216,\ n_2^0=1674,\ n_3^0=21888,\ n_4^0=423144,\ n_5^0=9912672,\ n_6^0=265889088$

$n_1^1=0,\ n_2^1=0,\ n_3^1=0,\ n_4^1=0,\ n_5^1=432,\ n_6^1=374244$

$X(1,2) \subset X_5$ [Bor96]

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&36&1&0\\0&0&1&1\end{pmatrix},\ M_{-1/12-1/18\sqrt{3}}=\begin{pmatrix}-13&-100&-6&-4\\7&51&3&2\\-168&-1200&-71&-48\\126&900&54&37\end{pmatrix},\ M_{-1/12+1/18\sqrt{3}}=\begin{pmatrix}1&-13&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v^1_{-1/12-1/18\sqrt{3}}=(72,-36,13,-240\lambda-\frac{7}{2}),\ v^1_{-1/12+1/18\sqrt{3}}=(36,0,\frac{7}{2},-120\lambda)$

$361\theta^4 - z\left(26030\theta^4+49780\theta^3+39691\theta^2+14801\theta+2166\right)$
$+z^2\left(39521\theta^4-3916\theta^3-106779\theta^2-95266\theta-25384\right)$
$+z^3\left(31008+1409040\theta+2373360\theta^2+1582320\theta^3+131920\theta^4\right)-80z^4(\theta+1)(499\theta^3+1411\theta^2+1378\theta+456)+12800z^5(\theta+1)^4$

38	$$\left\{\begin{array}{cccccc} -1 & 0 & \frac{71}{64}-\frac{17}{64}\sqrt{17} & \frac{19}{20} & \frac{71}{64}+\frac{17}{64}\sqrt{17} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 4 & 2 & 1 \end{array}\right\}$$
80	
-92	
13	
$\frac{71}{2}+17/2\sqrt{17}$	

$1+\frac{113}{19}q+\frac{5729}{19}q^2+\frac{230018}{19}q^3+\frac{9405345}{19}q^4+\frac{388177863}{19}q^5+\frac{16178756810}{19}q^6+O(q^7)$

$n_1^0=226,\ n_2^0=1404,\ n_3^0=17030,\ n_4^0=293738,\ n_5^0=6210844,\ n_6^0=149801122$

$n_1^1=0,\ n_2^1=0,\ n_3^1=0,\ n_4^1=1,\ n_5^1=1444,\ n_6^1=258932$

$$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&38&1&0\\0&0&1&1\end{pmatrix},\ M_{-1}=\begin{pmatrix}-35&-300&-18&-12\\18&151&9&6\\-432&-3600&-215&-144\\306&2550&153&103\end{pmatrix},\ M_{\frac{71}{64}+\frac{17}{64}\sqrt{17}}=\begin{pmatrix}1&-9&-1&-1\\0&1&0&0\\0&36&5&4\\0&-36&-4&-3\end{pmatrix},\ M_{\frac{19}{20}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ M_{\frac{71}{64}-\frac{17}{64}\sqrt{17}}=\begin{pmatrix}1&-13&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$v^3_{-1}=(228,-114,35,-552\lambda-8),\ v^1_{\frac{71}{64}-\frac{17}{64}\sqrt{17}}=(38,0,\frac{10}{3},-92\lambda),\ v^1_{\frac{71}{64}+\frac{17}{64}\sqrt{17}}=(-38,0,\frac{2}{3},92\lambda)$

$$9\theta^4 - z\left(519\theta^4 + 1020\theta^3 + 816\theta^2 + 306\theta + 45\right)$$
$$-z^2\left(2258\theta^4 + 10064\theta^3 + 15194\theta^2 + 9546\theta + 2166\right)$$
$$+z^3\left(1686\theta^4 + 5256\theta^3 + 4706\theta^2 + 1350\theta + 12\right) -z^4\left(295\theta^4 + 608\theta^3 + 478\theta^2 + 174\theta + 26\right)$$
$$+z^5(\theta+1)^4$$

42	$\left\{\begin{array}{cccccc} x_1 & 0 & x_2 & 3 & x_3 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 1 & 1 \\ 2 & 0 & 2 & 4 & 2 & 1 \end{array}\right\}$
84	
-98	
14	
$1/x_2$	

$f(x) = x^3 - 289\,x^2 - 57\,x + 1$, $x_1 = -0.2133$, $x_2 = 0.0162$, $x_3 = 289.1971$

$1+\frac{14}{3}q+238q^2+\frac{24584}{3}q^3+\frac{906122}{3}q^4+11062338q^5+410527768q^6+\frac{46010329400}{3}q^7+O(q^8)$

$n^0_1 = 196$, $n^0_2 = 1225$, $n^0_3 = 12740$, $n^0_4 = 198058$, $n^0_5 = 3716944$, $n^0_6 = 79823205$

$n^1_1 = 0$, $n^1_2 = 0$, $n^1_3 = 0$, $n^1_4 = 0$, $n^1_5 = 588$, $n^1_6 = 99960$

$X(1,1,1,1,1,1,1) \subset \mathrm{Gr}(2,7)$ [BCFKS98]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 42 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, M_{x_3} = \begin{pmatrix} 1 & -84 & -9 & -9 \\ 0 & 1 & 0 & 0 \\ 0 & 392 & 43 & 42 \\ 0 & -392 & -42 & -41 \end{pmatrix}, M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{x_2} =$$
$$\begin{pmatrix} 1 & -14 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, M_{x_1} = \begin{pmatrix} -13 & -112 & -6 & -4 \\ 7 & 57 & 3 & 2 \\ -196 & -1568 & -83 & -56 \\ 147 & 1176 & 63 & 43 \end{pmatrix}$$

$v^1_{x_1}=(84,-42,14,-196\lambda-\frac{7}{2})$, $v^1_{x_2}=(42,0,\frac{7}{2},-98\lambda)$, $v^1_{x_3}=(-126,0,\frac{7}{2},294\lambda)$

$$\theta^4 - z\left(65\theta^4 + 130\theta^3 + 105\theta^2 + 40\theta + 6\right) 4z^2(4\theta+3)(\theta+1)^2(4\theta+5)$$

42	$\left\{\begin{array}{cccc} 0 & \frac{1}{64} & 1 & \infty \\ \hline 0 & 0 & 0 & \frac{3}{4} \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & \frac{5}{4} \end{array}\right\}$
84	
-96	
14	
64	

$1+5q+229q^2+8429q^3+307941q^4+11381005q^5+424644781q^6+15963252737q^7+O(q^8)$

$n^0_1 = 210$, $n^0_2 = 1176$, $n^0_3 = 13104$, $n^0_4 = 201936$, $n^0_5 = 3824016$, $n^0_6 = 82568136$

$n^1_1 = 0$, $n^1_2 = 0$, $n^1_3 = 0$, $n^1_4 = 0$, $n^1_5 = 84$, $n^1_6 = 74382$

$X(1,1,1,1,1,1) \subset \mathrm{Gr}(3,6)$ [BCFKS98]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 42 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} -35 & -378 & -24 & -18 \\ 12 & 127 & 8 & 6 \\ -252 & -2646 & -167 & -126 \\ 156 & 1638 & 104 & 79 \end{pmatrix}, \quad M_{\frac{1}{64}} =$$
$$\begin{pmatrix} 1 & -14 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{\frac{1}{64}}=(42,0,\frac{7}{2},-96\lambda)$, $v^2_{1}=(252,-84,21,-576\lambda-5)$

$$\theta^4-4z\theta(\theta+1)(2\theta+1)^2-32z^2(2\theta+1)(2\theta+3)(11\theta^2+22\theta+12)$$
$$-1200z^3(2\theta+1)(2\theta+3)^2(2\theta+5)-4864z^4(2\theta+1)(2\theta+3)(2\theta+5)(2\theta+7)$$

<table>
<tr><td>44</td><td rowspan="5">$$\left\{\begin{array}{cccccc} x_1 & x_2 & -\frac{1}{16} & 0 & x_3 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 1 & 0 & 1 & \frac{3}{2} \\ 1 & 1 & 1 & 0 & 1 & \frac{5}{2} \\ 2 & 2 & 2 & 0 & 2 & \frac{7}{2} \end{array}\right\}$$</td></tr>
<tr><td>92</td></tr>
<tr><td>-128</td></tr>
<tr><td>15</td></tr>
<tr><td>$1/x_3$</td></tr>
</table>

$f(x)=x^3+\frac{7}{38}x^2+\frac{1}{152}x-\frac{1}{4864}$, $x_1=-0.1018-0.0132i$, $x_2=-0.1018+0.0132i$, $x_3=0.0195$

$1+4q+220q^2+7672q^3+279772q^4+10121004q^5+372031816q^6+13748223416q^7+O(q^8)$

$n^0_1=176$, $n^0_2=1188$, $n^0_3=12496$, $n^0_4=192192$, $n^0_5=3562592$, $n^0_6=75782652$

$e_{x_1}=-\frac{1}{6}$, $e_{x_2}=-\frac{1}{6}$, $e_{-\frac{1}{16}}=-\frac{1}{6}$, $e_{x_3}=-\frac{1}{6}$

$n^1_1=0$, $n^1_2=0$, $n^1_3=0$, $n^1_4=3$, $n^1_5=-352$, $n^1_6=12776$

$X \xrightarrow{2:1} A^*_{22}$ [Bor96], ♯58 [BK10]

$M_{x_3}=\begin{pmatrix}1&-15&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$, $M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&44&1&0\\0&0&1&1\end{pmatrix}$, $M_{-\frac{1}{16}}=\begin{pmatrix}-15&-120&-6&-4\\8&61&3&2\\-240&-1800&-89&-60\\184&1380&69&47\end{pmatrix}$, $M_{x_2}=\begin{pmatrix}-20&-207&-12&-9\\7&70&4&3\\-168&-1656&-95&-72\\112&1104&64&49\end{pmatrix}$, $M_{x_1}=\begin{pmatrix}22&-75&-6&-9\\7&-24&-2&-3\\-140&500&41&60\\84&-300&-24&-35\end{pmatrix}$

$v^1_{x_1}=(132,44,\frac{27}{2},-384\lambda+\frac{19}{6})$, $v^1_{x_2}=(132,-44,\frac{27}{2},-384\lambda-\frac{19}{6})$, $v^1_{-\frac{1}{16}}=(88,-44,\frac{47}{3},-256\lambda-\frac{25}{6})$, $v^1_{x_3}=(44,0,\frac{23}{6},-128\lambda)$

$$2209\theta^4-z\left(11045+728897\theta+1900210\theta^2+2342480\theta^3+1169830\theta^4\right)$$
$$-z^2\left(161022+7018851\theta+11358480\theta^2+7900720\theta^3+2088670\theta^4\right)$$
$$+z^3\left(38352+1493190\theta+3839120\theta^2+6376440\theta^3+3708570\theta^4\right)$$
$$-z^4\left(1770676+51612830\theta+44240490\theta^2+5118200\theta^3-2911610\theta^4\right)$$
$$+z^5\left(2151-2609360\theta-7507550\theta^2-7494820\theta^3-4061920\theta^4\right)$$
$$+z^6\left(483678+24622380\theta+41188770\theta^2+24502500\theta^3+14323500\theta^4\right)$$
$$+z^7\left(21870+1195560\theta+2872260\theta^2+3353400\theta^3+1545480\theta^4\right)-236196z^8(\theta+1)^4$$

<table>
<tr><td>47</td><td rowspan="5">$$\left\{\begin{array}{cccccccc} \frac{1}{108}-\frac{1}{108}\sqrt{10153} & x_1 & 0 & x_2 & x_3 & x_4 & \frac{1}{108}+\frac{1}{108}\sqrt{10153} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 & 1 & 3 & 1 \\ 4 & 2 & 0 & 2 & 2 & 2 & 4 & 1 \end{array}\right\}$$</td></tr>
<tr><td>86</td></tr>
<tr><td>-90</td></tr>
<tr><td>15</td></tr>
<tr><td>$1/x_2$</td></tr>
</table>

$f(x)=81x^4-50x^3+90x^2+53x-1$, $x_1=-0.4351$, $x_2=0.0183$, $x_3=0.5170+1.133i$, $x_4=0.5170-1.133i$

$1+\frac{189}{47}q+\frac{8021}{47}q^2+\frac{250668}{47}q^3+\frac{7994901}{47}q^4+\frac{256127689}{47}q^5+\frac{8289702164}{47}q^6+\frac{270201445577}{47}q^7+O(q^8)$

$n^0_1 = 189,\ n^0_2 = 979,\ n^0_3 = 9277,\ n^0_4 = 124795,\ n^0_5 = 2049020,\ n^0_6 = 38377054$

$e_{x_1} = -\frac{1}{6},\ e_{x_2} = -\frac{1}{6},\ e_{x_3} = -\frac{1}{6},\ e_{x_4} = -\frac{1}{6}$

$n^1_1 = 0,\ n^1_2 = 0,\ n^1_3 = 0,\ n^1_4 = 0,\ n^1_5 = 103,\ n^1_6 = 28519$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 47 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{x_1} = \begin{pmatrix} -13 & -122 & -6 & -4 \\ 7 & 62 & 3 & 2 \\ -217 & -1891 & -92 & -62 \\ 161 & 1403 & 69 & 47 \end{pmatrix},\ M_{\frac{1}{108}-\frac{1}{108}\sqrt{10153}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{x_3} = \begin{pmatrix} -17 & -207 & -12 & -9 \\ 6 & 70 & 4 & 3 \\ -144 & -1656 & -95 & -72 \\ 90 & 1035 & 60 & 46 \end{pmatrix},\ M_{\frac{1}{108}+\frac{1}{108}\sqrt{10153}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_{x_2} = \begin{pmatrix} 1 & -15 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{x_4} = \begin{pmatrix} 19 & -66 & -6 & -9 \\ 6 & -21 & -2 & -3 \\ -138 & 506 & 47 & 69 \\ 84 & -308 & -28 & -41 \end{pmatrix}$$

$v^1_{x_1}=(94,-47,\frac{44}{3},-180\lambda-\frac{41}{12}),\ v^1_{x_2}=(47,0,\frac{43}{12},-90\lambda),\ v^1_{x_3}=(141,-47,\frac{45}{4},-270\lambda-\frac{29}{12}),$
$v^1_{x_4}=(141,47,\frac{45}{4},-270\lambda+\frac{29}{12})$

$$\begin{aligned}
&49\theta^4 - z\left(14\theta(46\theta^3+52\theta^2+33\theta+7)\right)\\
&-z^2\left(25088+106680\theta+170532\theta^2+115392\theta^3+29328\theta^4\right)\\
&-z^3\left(561568+1780464\theta+1927728\theta^2+716160\theta^3+45760\theta^4\right)\\
&-z^4\left(1561088+3332608\theta+379392\theta^2-3031040\theta^3-1141760\theta^4\right)\\
&+z^5\left(3112960+17940480\theta+32292864\theta^2+21626880\theta^3+2850816\theta^4\right)\\
&+z^6\left(3670016+11010048\theta+47185920\theta^2-125829120\theta^3-115343360\theta^4\right)\ -z^7\left(33554432(\theta+1)^4\right)
\end{aligned}$$

56	$\left\{\begin{array}{cccccc} -\frac{1}{4} & -\frac{1}{16} & 0 & \frac{1}{32} & \frac{7}{32} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 & 1 & 1 & 1 \\ \frac{3}{2} & 1 & 0 & 1 & 3 & 1 \\ 2 & 2 & 0 & 2 & 4 & 1 \end{array}\right\}$
80	
-64	
16	
32	

$v^2_{-\frac{1}{16}}=(112,-56,\frac{56}{3},-128\lambda-\frac{14}{3}),\ v^1_{\frac{1}{32}}=(56,0,\frac{10}{3},-64\lambda)$

$$\begin{aligned}
&49\theta^4 - z\left(196+1274\theta+3276\theta^2+4004\theta^3+1876\theta^4\right)\\
&-z^2\left(7672+32900\theta+54004\theta^2+41064\theta^3+12732\theta^4\right)\\
&-z^3\left(9520+45192\theta+82048\theta^2+67200\theta^3+20704\theta^4\right)\\
&-z^4\left(8352+34416\theta+54768\theta^2+40704\theta^3+12288\theta^4\right)\ -2304z^5(\theta+1)^4
\end{aligned}$$

56	$\left\{\begin{array}{cccccc} -\frac{11}{8}-5/8\sqrt{5} & -\frac{7}{6} & -\frac{1}{4} & 0 & -\frac{11}{8}+5/8\sqrt{5} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 3 & 1 & 0 & 1 & 1 \\ 2 & 4 & 2 & 0 & 2 & 1 \end{array}\right\}$
92	
-92	
17	
$22+10\sqrt{5}$	

$1+\frac{20}{7}q+\frac{778}{7}q^2+\frac{19676}{7}q^3+75014q^4+1973360q^5+\frac{368887198}{7}q^6+\frac{9904071512}{7}q^7+\frac{267539368234}{7}q^8+O(q^9)$

$n^0_1 = 160,\ n^0_2 = 758,\ n^0_3 = 5824,\ n^0_4 = 65540,\ n^0_5 = 884064,\ n^0_6 = 13661732$

$n^1_1 = 0,\ n^1_2 = 0,\ n^1_3 = 0,\ n^1_4 = 0,\ n^1_5 = 0,\ n^1_6 = 3916$

$X(1,1,1,1) \subset F_1(Q_5)$ [Bor96]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 56 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \; M_{-\frac{1}{4}} = \begin{pmatrix} -15 & -144 & -6 & -4 \\ 8 & 73 & 3 & 2 \\ -304 & -2736 & -113 & -76 \\ 232 & 2088 & 87 & 59 \end{pmatrix}, \; M_{-\frac{7}{6}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_{-\frac{11}{8}-5/8\sqrt{5}} = \begin{pmatrix} -13 & -140 & -6 & -4 \\ 7 & 71 & 3 & 2 \\ -252 & -2520 & -107 & -72 \\ 182 & 1820 & 78 & 53 \end{pmatrix}, \; M_{-\frac{11}{8}+5/8\sqrt{5}} = \begin{pmatrix} 1 & -17 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^1_{-\frac{11}{8}-5/8\sqrt{5}}=(112,-56,\frac{47}{3},-184\lambda-\frac{19}{6})$, $v^1_{-\frac{1}{4}}=(112,-56,\frac{53}{3},-184\lambda-\frac{25}{6})$,
$v^1_{-\frac{11}{8}+5/8\sqrt{5}}=(56,0,\frac{23}{6},-92\lambda)$

$$\begin{aligned} &361\theta^4 - z\left(13300\theta^4+23522\theta^3+18981\theta^2+7220\theta+1083\right) \\ &-z^2\left(647745\theta^4+368006\theta^3+609133\theta^2+4127756\theta+102258\right) \\ &z^3\left(1727190\theta^4+3293460\theta^3-3219210\theta^2-7387200\theta-304722\right) \\ &z^4\left(4665600\theta^4+8412660\theta^3+17678250\theta^2+13471920\theta+354294\right) - 177147z^5(\theta+1)^4 \end{aligned}$$

57	$\left\{\begin{array}{cccccc} -1 & -\frac{35}{486}-\frac{13}{486}\sqrt{13} & 0 & -\frac{35}{486}+\frac{13}{486}\sqrt{13} & \frac{19}{27} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 3 & 1 \\ 2 & 2 & 0 & 2 & 4 & 1 \end{array}\right\}$
90	
-84	
17	
$\frac{35}{2}+13/2\sqrt{13}$	

$1+\frac{49}{19}q+\frac{2065}{19}q^2+\frac{47596}{19}q^3+64075q^4+\frac{30769924}{19}q^5+\frac{789487276}{19}q^6+\frac{20341733041}{19}q^7+\frac{527558864785}{19}q^8+O(q^9)$

$n^0_1=147$, $n^0_2=756$, $n^0_3=5283$, $n^0_4=56970$, $n^0_5=738477$, $n^0_6=10964412$

$n^1_1=0$, $n^1_2=0$, $n^1_3=0$, $n^1_4=0$, $n^1_5=102$, $n^1_6=7884$

Tjøtta's example [Tjø01]

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 57 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \; M_{-\frac{35}{486}-\frac{13}{486}\sqrt{13}} = \begin{pmatrix} -15 & -146 & -6 & -4 \\ 8 & 74 & 3 & 2 \\ -312 & -2847 & -116 & -78 \\ 240 & 2190 & 90 & 61 \end{pmatrix},$$

$$M_{-1} = \begin{pmatrix} -35 & -486 & -24 & -18 \\ 12 & 163 & 8 & 6 \\ -360 & -4860 & -239 & -180 \\ 228 & 3078 & 152 & 115 \end{pmatrix}, \; M_{\frac{19}{27}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; M_{-\frac{35}{486}+\frac{13}{486}\sqrt{13}} =$$

$$\begin{pmatrix} 1 & -17 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$v^2_{-1}=(342,-114,\frac{51}{2},-504\lambda-\frac{9}{2})$, $v^1_{-\frac{35}{486}-\frac{13}{486}\sqrt{13}}=(114,-57,18,-168\lambda-\frac{17}{4})$,
$v^1_{-\frac{35}{486}+\frac{13}{486}\sqrt{13}}=(57,0,\frac{15}{4},-84\lambda)$

$$\begin{aligned} &3721\theta^4 - z\left(184769\theta^4+339892\theta^3+285297\theta^2+115351\theta+18605\right) \\ &+z^2\left(1215215\theta^4+3428132\theta^3+4267228\theta^2+2572675\theta+611586\right) \\ &-z^3\left(3188970\theta^4+11354418\theta^3+16751367\theta^2+11517471\theta+3023892\right) \\ &+z^4\left(3713526\theta^4+14631030\theta^3+22018716\theta^2+14703201\theta+3661038\right) - 1594323z^5(\theta+1)^4 \end{aligned}$$

61	$\left\{\begin{array}{cccccc} 0 & x_1 & x_2 & x_3 & \frac{61}{81} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 3 & 1 \\ 0 & 2 & 2 & 2 & 4 & 1 \end{array}\right\}$
94	
-86	
18	
$1/x_1$	

$f(x) = x^3 - \frac{200}{243}x^2 + \frac{47}{243}x - \frac{1}{243}$ $x_1 = 0.0236$, $x_2 = 0.3997 + 0.1216i$, $x_3 = 0.3997 - 0.1216i$

$1+\frac{163}{61}q+\frac{5203}{61}q^2+\frac{129628}{61}q^3+\frac{3104211}{61}q^4+\frac{74976288}{61}q^5+\frac{1827344980}{61}q^6+\frac{44837600887}{61}q^7+O(q^8)$

$n_1^0 = 163$, $n_2^0 = 630$, $n_3^0 = 4795$, $n_4^0 = 48422$, $n_5^0 = 599809$, $n_6^0 = 8459307$

$n_1^1 = 0$, $n_2^1 = 0$, $n_3^1 = 0$, $n_4^1 = 0$, $n_5^1 = -2$, $n_6^1 = 264$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 61 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{x_2} = \begin{pmatrix} -20 & -261 & -12 & -9 \\ 7 & 88 & 4 & 3 \\ -231 & -2871 & -131 & -99 \\ 154 & 1914 & 88 & 67 \end{pmatrix},\ M_{\frac{61}{81}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$M_{x_1} = \begin{pmatrix} 1 & -18 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{x_3} = \begin{pmatrix} 22 & -78 & -6 & -9 \\ 7 & -25 & -2 & -3 \\ -196 & 728 & 57 & 84 \\ 119 & -442 & -34 & -50 \end{pmatrix}$$

$v^1_{x_1}$=$(61, 0, \frac{47}{12}, -86\lambda)$, $v^1_{x_2}$=$(183, -61, \frac{57}{4}, -258\lambda - \frac{37}{12})$, $v^1_{x_3}$=$(183, 61, \frac{57}{4}, -258\lambda + \frac{37}{12})$

$8281\theta^4 + z\left(91\theta(782\theta^3 - 1874\theta^2 - 1210\theta - 273)\right)$
$-z^2\left(25157785\theta^4 + 116225522\theta^3 + 152279390\theta^2 + 99629530\theta + 2649920\right)$
$-z^3\left(598275970\theta^4 + 2586781260\theta^3 + 4326078680\theta^2 + 3488191980\theta + 110445426\right)$
$-z^4\left(6120430420\theta^4 + 29988812180\theta^3 + 59019958200\theta^2 + 54397323800\theta + 1915723890\right)$
$-z^5\left(37628403420\theta^4 + 212268290580\theta^3 + 475032428130\theta^2 + 485227005630\theta + 18479595006\right)$
$-z^6\left(152654873820\theta^4 + 982103090940\theta^3 + 2447536247410\theta^2 + 2719415453790\theta + 110147546634\right)$
$-z^7\left(421032720020\theta^4 + 3044876322820\theta^3 + 8319650571140\theta^2 + 9918294826020\theta + 422269162452\right)$
$-2z^8(\theta+1)(392534006260\theta^3 + 2751089630010\theta^2 + 6543324166780\theta + 521254338620)$
$-z^9(\theta+1)(\theta+2)(949873554170\theta^2 + 5453407101930\theta + 799002779040)$
$-1540z^10(\theta+1)(\theta+2)(\theta+3)(437651590\theta + 149264765)$
$-21292817700z^11(\theta+1)(\theta+2)(\theta+3)(\theta+4)$

91	$\left\{\begin{array}{ccccccccccc} x_1 & x_2 & x_3 & -\frac{1}{3} & x_4 & x_5 & x_6 & x_7 & 0 & x_8 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 \\ 3 & 1 & 1 & 1 & 1 & 1 & 3 & 1 & 0 & 1 & 3 \\ 4 & 2 & 2 & 2 & 2 & 2 & 4 & 2 & 0 & 2 & 4 \end{array}\right\}$
106	
-78	
24	
$1/x_8$	

$x_1 = -\frac{573}{1540} - \frac{1}{1540}\sqrt{48049}$, $x_2 = -0.3432 + 0.0271i$, $x_3 = -0.3432 - 0.0271i$, $x_4 = -0.2588 + 0.1605i$, $x_5 = -0.2588 - 0.1605i$, $x_6 = -\frac{573}{1540} + \frac{1}{1540}\sqrt{48049}$, $x_7 = -0.1815$, $x_8 = 0.0419$

$1+\frac{118}{91}q+\frac{3102}{91}q^2+\frac{50770}{91}q^3+\frac{116850}{13}q^4+\frac{1875374}{13}q^5+\frac{213286578}{91}q^6+\frac{3475499411}{91}q^7+\frac{8140409970}{13}q^8+O(q^9)$

$v^1_{x_2}$=$(-637, 273, -\frac{785}{12}, 546\lambda + \frac{47}{4})$, $v^1_{x_3}$=$(637, 273, \frac{785}{12}, -546\lambda + \frac{47}{4})$,
$v^{\frac{5}{2}}_{-\frac{1}{3}}$=$(910, -455, \frac{365}{3}, -780\lambda - \frac{275}{12})$, $v^1_{x_4}$=$(273, -91, \frac{79}{4}, -234\lambda - \frac{43}{12})$,
$v^1_{x_5}$=$(273, 91, \frac{79}{4}, -234\lambda + \frac{43}{12})$, $v^1_{x_7}$=$(182, -91, \frac{79}{3}, -156\lambda - \frac{67}{12})$, $v^1_{x_8}$=$(91, 0, \frac{53}{12}, -78\lambda)$

$$9409\theta^4 + z\left(97\theta(1727\theta^3 - 2018\theta^2 - 1300\theta - 291)\right)$$
$$-z^2\left(16521350\theta^4 + 134288812\theta^3 + 161743393\theta^2 + 10216234\theta + 2709792\right)$$
$$-z^3\left(817534350\theta^4 + 3641261940\theta^3 + 5686394970\theta^2 + 4431155940\theta + 138000348\right)$$
$$-z^4\left(11755028620\theta^4 + 54233944640\theta^3 + 100063785700\theta^2 + 88694155200\theta + 3049275024\right)$$
$$-z^5\left(97262503970\theta^4 + 505074292340\theta^3 + 1061080234510\theta^2 + 1039641023500\theta + 38537290992\right)$$
$$-z^6\left(308040167808 + 7815277788840\theta + 7330536601500\theta^2 + 3123744348240\theta^3 + 527629358940\theta^4\right)$$
$$-z^7\left(1954534339080\theta^4 + 13131992350800\theta^3 + 33995270620440\theta^2 + 39010933561680\theta + 1619360309088\right)$$
$$-144z^8(\theta+1)(34326474790\theta^3 + 224873637870\theta^2 + 508086147110\theta + 38959393614)$$
$$-432z^9(\theta+1)(\theta+2)(19034936290\theta^2 + 102628645550\theta + 14314039440)$$
$$-438048z^{10}(\theta+1)(\theta+2)(\theta+3)(18629870\theta + 5992902)$$
$$-368028363456z^{11}(\theta+1)(\theta+2)(\theta+3)(\theta+4)$$

97	$$\left\{\begin{array}{ccccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & -\frac{38}{169}+\frac{1}{1014}\sqrt{2805} & 0 & x_8 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 \\ 3 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 0 & 1 & 3 \\ 4 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 0 & 2 & 4 \end{array}\right\}$$
106	
-64	
25	
$1/x_8$	

$f(x) = 7457x^5 + 6100x^4 + 1929x^3 + 257x^2 + 7x - 1$, $x_1 = -\frac{38}{169} - \frac{1}{1014}\sqrt{2805}$, $x_2 = -1/4 + 1/12 i\sqrt{3}$, $x_3 = -1/12\, i\sqrt{3} - 1/4$, $x_4 = -0.24 + 0.1101i$, $x_5 = -0.24 - 0.1101i$, $x_6 = -0.1911 + 0.0840i$, $x_7 = -0.1911 - 0.0840i$, $x_8 = 0.0441$

$$1 + \frac{136}{97}q + \frac{2472}{97}q^2 + \frac{47872}{97}q^3 + \frac{650664}{97}q^4 + \frac{10423511}{97}q^5 + \frac{155806080}{97}q^6 + \frac{2415791668}{97}q^7 + \frac{37201788328}{97}q^8 + O(q^9)$$

$e_{-1/4+1/12\, i\sqrt{3}} = -\frac{1}{3}$, $e_{-1/12\, i\sqrt{3}-1/4} = -\frac{1}{3}$, $e_{x_4} = -\frac{1}{6}$, $e_{x_5} = -\frac{1}{6}$, $e_{x_6} = -\frac{1}{6}$, $e_{x_7} = -\frac{1}{6}$, $e_{x_8} = -\frac{1}{6}$

$n^0_1 = 136$, $n^0_2 = 292$, $n^0_3 = 1768$, $n^0_4 = 10128$, $n^0_5 = 83387$, $n^0_6 = 721092$, $n^0_7 = 7043124$, $n^0_8 = 72658472$, $n^0_9 = 792624984$

$n^1_1 = 0$, $n^1_2 = 0$, $n^1_3 = 0$, $n^1_4 = 0$, $n^1_5 = 0$, $n^1_6 = 0$, $n^1_7 = 292$, $n^1_8 = 59024$, $n^1_9 = 2373710$

[BK10] [Met12] $1/t + y + t/x + zt/x + zt/xy + 1/z + x/z + y/zt + x/zt + xy/zt + y/t + t/y + t/xy + y/x + 1/x + zt/y + t + 1/y + x + x/t + 1/zt + z + z/y$

$$M_{x_8} = \begin{pmatrix} 1 & -25 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 97 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{-\frac{38}{169}+\frac{1}{1014}\sqrt{2805}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_{-\frac{38}{169}-\frac{1}{1014}\sqrt{2805}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{x_6} = \begin{pmatrix} -19 & -232 & -6 & -4 \\ 10 & 117 & 3 & 2 \\ -660 & -7656 & -197 & -132 \\ 510 & 5916 & 153 & 103 \end{pmatrix},$$

$$M_{x_7} = \begin{pmatrix} 21 & -38 & -2 & -4 \\ 10 & -18 & -1 & -2 \\ -310 & 589 & 32 & 62 \\ 160 & -304 & -16 & -31 \end{pmatrix},\ M_{x_4} = \begin{pmatrix} -84 & -1220 & -35 & -25 \\ 34 & 489 & 14 & 10 \\ -2023 & -29036 & -832 & -595 \\ 1462 & 20984 & 602 & 431 \end{pmatrix},$$

$$M_{x_5} = \begin{pmatrix} 86 & -250 & -15 & -25 \\ 34 & -99 & -6 & -10 \\ -1275 & 3750 & 226 & 375 \\ 714 & -2100 & -126 & -209 \end{pmatrix},\ M_{x_2} = \begin{pmatrix} -47 & -780 & -24 & -18 \\ 16 & 261 & 8 & 6 \\ -880 & -14300 & -439 & -330 \\ 608 & 9880 & 304 & 229 \end{pmatrix},$$

$$M_{x_3} = \begin{pmatrix} 49 & -198 & -12 & -18 \\ 16 & -65 & -4 & -6 \\ -672 & 2772 & 169 & 252 \\ 400 & -1650 & -100 & -149 \end{pmatrix}$$

$v^2_{-1/4+1/12\,i\sqrt{3}}$=$(582, -194, \frac{79}{2}, -384\lambda - \frac{43}{6})$, $v^2_{-1/12\,i\sqrt{3}-1/4}$=$(582, 194, \frac{79}{2}, -384\lambda + \frac{43}{6})$,
$v^1_{x_4}$=$(-485, 194, -\frac{529}{12}, 320\lambda + \frac{49}{6})$, $v^1_{x_5}$=$(485, 194, \frac{529}{12}, -320\lambda + \frac{49}{6})$,
$v^1_{x_6}$=$(194, -97, \frac{79}{3}, -128\lambda - \frac{67}{12})$, $v^1_{x_7}$=$(194, 97, \frac{79}{3}, -128\lambda + \frac{67}{12})$, $v^1_{x_8}$=$(97, 0, \frac{53}{12}, -64\lambda)$

$841\theta^4+z\left(58\theta(24\theta^3-198\theta^2-128\theta-29)\right)$
$-z^2\left(1771136\theta^4+6918816\theta^3+9943568\theta^2+6882288\theta+188384\right)$
$-z^3\left(210283 2\theta^4+96590888\theta^3+177905 72\theta^2+153561196\theta+5102376\right)$
$-z^4\left(113249992\theta^4+632880332\theta^3+1391635592\theta^2+1391708872\theta+51970096\right)$
$-z^5\left(16(\theta+1)(21520400\theta^3+121866368\theta^2+241793736\theta+16560506)\right)$
$-32z^6(\theta+1)(\theta+2)(19122256\theta^2+91085440\theta+11349571)$
$-10496z^7(\theta+1)(\theta+2)(\theta+3)(5671\theta+16301)$ $-24529152z^8(\theta+1)(\theta+2)(\theta+3)(\theta+4)$

116	$$\left\{\begin{array}{cccccccc} -\frac{1}{2} & -\frac{29}{82} & x_1 & x_2 & -\frac{1}{6} & 0 & x_3 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 1 & 1 & 1 & 1 & 0 & 1 & 2 \\ \frac{1}{2} & 3 & 1 & 1 & 1 & 0 & 1 & 3 \\ 1 & 4 & 2 & 2 & 2 & 0 & 2 & 4 \end{array}\right\}$$
116	
-80	
29	
$1/x_3$	

$f(x) = x^3 + \frac{21}{38}x^2 + \frac{7}{76}x - \frac{1}{152}$, $x_1 = -0.3028 - 0.1803i$, $x_2 = -0.3028 + 0.1803i$, $x_3 = 0.0530$

$1+\frac{24}{29}p+\frac{600}{29}p^2+\frac{7692}{29}p^3+\frac{104024}{29}p^4+\frac{1329024}{29}p^5+\frac{17522196}{29}p^6+\frac{229724960}{29}p^7+\frac{3038654040}{29}p^8+O(p^9)$

♯43 [BK10]

$$M_{-\frac{1}{2}} = \begin{pmatrix} 13 & 336 & 10 & 8 \\ -16 & -303 & -8 & -6 \\ 1508 & 27248 & 703 & 520 \\ -1244 & -22016 & -562 & -413 \end{pmatrix}, \quad M_{-\frac{29}{82}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{-\frac{1}{6}} = \begin{pmatrix} -23 & -280 & -6 & -4 \\ 12 & 141 & 3 & 2 \\ -984 & -11480 & -245 & -164 \\ 780 & 9100 & 195 & 131 \end{pmatrix},$$
$$M_{x_1} = \begin{pmatrix} -89 & -813 & -15 & -9 \\ 60 & 543 & 10 & 6 \\ -5520 & -49864 & -919 & -552 \\ 4680 & 42276 & 780 & 469 \end{pmatrix},$$
$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 116 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad M_{x_3} = \begin{pmatrix} 1 & -29 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{x_2} = \begin{pmatrix} -26 & -465 & -12 & -9 \\ 9 & 156 & 4 & 3 \\ -612 & -10540 & -271 & -204 \\ 432 & 7440 & 192 & 145 \end{pmatrix}$$

$v^1_{x_1}$=$(348, -232, \frac{165}{2}, -240\lambda - \frac{61}{3})$, $v^1_{x_2}$=$(348, -116, \frac{49}{2}, -240\lambda - \frac{25}{6})$,
$v^1_{-\frac{1}{6}}$=$(232, -116, \frac{101}{3}, -160\lambda - \frac{43}{6})$, $v^1_{x_3}$=$(116, 0, \frac{29}{6}, -80\lambda)$

$169\theta^4 - z\left(13\theta(56\theta^3+178\theta^2+115\theta+26)\right)$
$-z^2\left(28466\theta^4+109442\theta^3+165603\theta^2+117338\theta+32448\right)$
$-z^3\left(233114\theta^4+1257906\theta^3+2622815\theta^2+2467842\theta+872352\right)$
$-z^4\left(989585\theta^4+6852298\theta^3+17737939\theta^2+19969754\theta+8108448\right)$
$-z^5(\theta+1)(2458967\theta^3+18007287\theta^2+44047582\theta+35386584)$
$-9z^6(\theta+1)(\theta+2)(393163\theta^2+2539029\theta+4164444)$
$-297z^7(\theta+1)(\theta+2)(\theta+3)(86839+34604)$ $-722007z^8(\theta+1)(\theta+2)(\theta+3)(\theta+4)$

117	$\left\{\begin{array}{ccccccccc} -\frac{13}{11} & -\frac{1}{3} & x_1 & x_2 & -\frac{5}{26}-\frac{3}{26}\,i\sqrt{3} & -\frac{5}{26}+\frac{3}{26}\,i\sqrt{3} & 0 & x_3 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 \\ 3 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 3 \\ 4 & 2 & 2 & 2 & 2 & 2 & 0 & 2 & 4 \end{array}\right\}$
114	
−72	
29	
$1/x_3$	

$f(x) = 153x^3 + 75x^2 + 14x - 1$, $x_1 = -0.2721 - 0.2165i$, $x_2 = -0.2721 + 0.2165i$, $x_3 = 0.0541$

$1+\frac{11}{13}p+\frac{251}{13}p^2+\frac{3476}{13}p^3+\frac{43259}{13}p^4+\frac{554636}{13}p^5+\frac{7159364}{13}p^6+\frac{92648427}{13}p^7+\frac{1205037307}{13}p^8+\frac{15738154598}{13}p^9$

$n^0_1 = 99$, $n^0_2 = 270$, $n^0_3 = 1155$, $n^0_4 = 6048$, $n^0_5 = 39933$, $n^0_6 = 298152$, $n^0_7 = 2431008$, $n^0_8 = 21181536$, $n^0_9 = 194298162$, $n^0_10 = 1856260278$

$e_{-\frac{1}{3}} = -\frac{2}{3}$, $e_{x_1} = -\frac{1}{6}$, $e_{x_2} = -\frac{1}{6}$, $e_{-\frac{5}{26}-\frac{3}{26}\,i\sqrt{3}} = -\frac{1}{6}$, $e_{-\frac{5}{26}+\frac{3}{26}\,i\sqrt{3}} = -\frac{1}{6}$, $e_{x_3} = -\frac{1}{6}$

$n^1_1 = 0$, $n^1_2 = 0$, $n^1_3 = 0$, $n^1_4 = 0$, $n^1_5 = 0$, $n^1_6 = 4$, $n^1_7 = 270$, $n^1_8 = 13608$, $n^1_9 = 412956$, $n^1_{10} = 9525240$

♯37 [BK10]

$M_{-\frac{13}{11}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $M_{-\frac{1}{3}} = \begin{pmatrix} -87 & -1112 & -24 & -16 \\ 44 & 557 & 12 & 8 \\ -3564 & -45036 & -971 & -648 \\ 2772 & 35028 & 756 & 505 \end{pmatrix}$, $M_{x_1} = \begin{pmatrix} -194 & -2045 & -40 & -25 \\ 117 & 1228 & 24 & 15 \\ -10296 & -107976 & -2111 & -1320 \\ 8424 & 88344 & 1728 & 1081 \end{pmatrix}$, $M_{-\frac{5}{26}-\frac{3}{26}\,i\sqrt{3}} = \begin{pmatrix} -89 & -819 & -15 & -9 \\ 60 & 547 & 10 & 6 \\ -5580 & -50778 & -929 & -558 \\ 4740 & 43134 & 790 & 475 \end{pmatrix}$, $M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 117 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $M_{x_3} = \begin{pmatrix} 1 & -29 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $M_{-\frac{5}{26}+\frac{3}{26}\,i\sqrt{3}} = \begin{pmatrix} -26 & -468 & -12 & -9 \\ 9 & 157 & 4 & 3 \\ -621 & -10764 & -275 & -207 \\ 441 & 7644 & 196 & 148 \end{pmatrix}$, $M_{x_2} = \begin{pmatrix} -94 & -1460 & -35 & -25 \\ 38 & 585 & 14 & 10 \\ -2793 & -42924 & -1028 & -735 \\ 2052 & 31536 & 756 & 541 \end{pmatrix}$

$v^1_{-\frac{1}{3}}=(468, -234, 64, -288\lambda - \frac{25}{2})$, $v^1_{x_1}=(585, -351, \frac{449}{4}, -360\lambda - \frac{99}{4})$,
$v^1_{x_2}=(585, -234, \frac{215}{4}, -360\lambda - \frac{19}{2})$, $v^1_{-\frac{5}{26}-\frac{3}{26}\,i\sqrt{3}}=(351, -234, \frac{333}{4}, -216\lambda - \frac{41}{2})$,
$v^1_{-\frac{5}{26}+\frac{3}{26}\,i\sqrt{3}}=(351, -117, \frac{99}{4}, -216\lambda - \frac{17}{4})$, $v^1_{x_3}=(117, 0, \frac{19}{4}, -72\lambda)$

$$\begin{aligned}
&529\theta^4 - z\left(46\theta^2(1+2\theta+136\theta^2)\right) - z^2\left(84640 + 279312\theta + 359900\theta^2 + 219704\theta^3 + 30356\theta^4\right) \\
&+z^3\left(1010160 + 3070914\theta + 3791849\theta^2 + 2342274\theta^3 + 573259\theta^4\right) \\
&+z^4\left(6697440 + 140448860\theta + 7955470\theta^2 - 622660\theta^3 - 1223510\theta^4\right) \\
&-z^5\left(600(\theta+1)(16105\theta^3 + 133047\theta^2 + 320040\theta + 245740)\right) \\
&+z^6\left(18000(\theta+2)(\theta+1)(3107\theta^2 + 16911\theta + 22834)\right) \\
&-z^7\left(810000(133\theta+404)(\theta+3)(\theta+2)(\theta+1)\right) + 72900000z^8(\theta+1)(\theta+2)(\theta+3)(\theta+4)
\end{aligned}$$

460	$\left\{\begin{array}{ccccccccc} -\frac{1}{6} & 1/10-1/10\sqrt{5} & 0 & \frac{1}{10} & \frac{23}{90} & 1/10+1/10\sqrt{5} & \frac{1}{3} & \frac{1}{2} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 3 & 1 & 1 & 1 & 3 \\ 2 & 2 & 0 & 2 & 4 & 2 & 2 & 2 & 4 \end{array}\right\}$
40	
−160	
80	
$-5/2+5/2\sqrt{5}$	

$1+\frac{2}{23}q+\frac{146}{23}q^2+\frac{731}{23}q^3+\frac{8850}{23}q^4+\frac{49377}{23}q^5+\frac{441947}{23}q^6+\frac{2779674}{23}q^7+\frac{23564946}{23}q^8+\frac{162177716}{23}q^9+O(q^{10})$
f=x+y+z+t+$xy/$z/t +$z*t/x/y$ +1/x +1/y+1/z+1/t
$v^5_{-\frac{1}{6}}$=$(1150,-460,\frac{655}{6},-400\lambda-\frac{55}{3})$, $v^2_{1/10-1/10\sqrt{5}}$=$(460,-230,\frac{200}{3},-160\lambda-\frac{85}{6})$, $v^1_{\frac{1}{10}}$=$(230,0,\frac{35}{6},-80\lambda)$, $v^2_{1/10+1/10\sqrt{5}}$=$(460,0,\frac{5}{3},-160\lambda)$, $v^{\frac{5}{2}}_{\frac{1}{3}}$=$(1150,-115,\frac{50}{3},-400\lambda-\frac{25}{12})$, $v^{\frac{5}{2}}_{\frac{1}{2}}$=$(1150,-230,\frac{205}{6},-400\lambda-\frac{25}{6})$

Table C.1: Data for CY(3)-equations

There are some CY(3)-equations, where the conjectural Euler number $c_3=\chi$ is positive. For a non-rigid Calabi-Yau threefold with $h^{1,1}=1$ the Euler number $\chi=\sum(-1)^k h^k(X)=2(h^{1,1}-h^{2,1})$ is always non-positive. How can this obeservation be explained?

$\theta^4-z\left(5136+60544\theta+368000\theta^2+614912\theta^3-356096\theta^4\right)$ $+z^2\left(1974730752+157129113600\theta+612095754240\theta^2-1305728778240\theta^3+121173442560\theta^4\right)$ $+z^3\left(49313740750848+450868486864896\theta+1308771024371712\theta^2+1803473947459584\theta^3\right.$ $\left.-3619472019554304\theta^4\right)$ $+4155203974946881536z^4(2\theta+1)^4$	
2	$\left\{\begin{array}{ccccc} -\frac{1}{221184} & 0 & \frac{1}{65536} & \frac{1}{20736} & \infty \\ \hline 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 1 & \frac{1}{2} \\ 3 & 0 & 1 & 1 & \frac{1}{2} \\ 4 & 0 & 2 & 2 & \frac{1}{2} \end{array}\right\}$
-4	
432	
0	
65536	
$1+177184q$ $-15680097504q^2$+$1220253371840512q^3$ $-8651437279804562 3520q^4$+$O(q^5)$	
$n^0_1=354368$, $n^0_2=-3920068672$, $n^0_3=90389138641728$, $n^0_4=-270357414944892 2688$, $n^0_5=949713345287525334 72960$, $n^0_6=-367801559789151938 8084890432$	
$e_{\frac{1}{65536}}=-\frac{1}{6}$, $e_{\frac{1}{20736}}=-\frac{1}{3}$	
$n^1_1=34896$, $n^1_2=-1124168368$, $n^1_3=39188280020416$, $n^1_4=-155080482207088 0448$, $n^1_5=660232301386685666 53472$, $n^1_6=-296342056542218110 3061855424$	
$M_0=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&2&1&0\\0&0&1&1\end{pmatrix}$, $M_{-\frac{1}{221184}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$, $M_{\frac{1}{20736}}=\begin{pmatrix}-3&-2&-4&-2\\4&3&4&2\\-4&-2&-3&-2\\12&6&12&7\end{pmatrix}$,$M_{\frac{1}{65536}}=\begin{pmatrix}1&0&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	
$\lambda_{\frac{1}{65536}}=1$, $\lambda_{\frac{1}{20736}}=2$	

$\theta^4-z\left(756\theta^4+1080\theta^3+810\theta^2+270\theta+36\right)$ $+z^2\left(174960\theta^4+419904\theta^3+440316\theta^2+215784\theta+38880\right)$ $-z^3\left(314928(2\theta+1)^2(13\theta^2+29\theta+20)\right)$ $+34012224z^4(2\theta+1)^2(2\theta+3)^2$	
9	$\left\{\begin{array}{cccc} 0 & \frac{1}{432} & \frac{1}{108} & \infty \\ \hline 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & 2 & 2 & \frac{3}{2} \end{array}\right\}$
30	
12	
4	
432	

$1{+}90q{+}17658q^2{+}4094586q^3{+}948074490q^4{+}227600757090q^5{+}55358315201178q^6{+}O(q^7)$
$n^0_1 = 810,\ n^0_2 = 19764,\ n^0_3 = 1364832,\ n^0_4 = 133320492,\ n^0_5 = 16387254504,\ n^0_6 = 2306596295376$
$e_{\frac{1}{432}} = -\frac{1}{6},\ e_{\frac{1}{108}} = -\frac{1}{2}$
$n^1_1 = 36,\ n^1_2 = -3132,\ n^1_3 = 45384,\ n^1_4 = -2170512,\ n^1_5 = 256245660,\ n^1_6 = 118464898680$
$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{\frac{1}{108}} = \begin{pmatrix} -2 & -6 & -1 & 0 \\ 0 & -2 & -1 & -1 \\ 3 & 24 & 8 & 6 \\ -3 & -21 & -6 & -4 \end{pmatrix},\ M_{\frac{1}{432}} = \begin{pmatrix} 1 & -4 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$\lambda_{\frac{1}{432}} = 1$

$\theta^4{-}z\left(40+316\theta+3076/3\theta^2+4256/3\theta^3+496/3\theta^4\right)$ $+z^2\left(117376/3+1972448\theta+2838400/9\theta^2-748544/9\theta^3-1035776/9\theta^4\right)$ $+z^3\left(131072/3+71065560\theta+276926464/9\theta^2+420413344\theta^3-7389184/3\theta^4\right)$ $+z^4\left(2228224/9(2\theta+1)(2242\theta^3+1419\theta^2-1047\theta-733)\right)$ $-z^5\left(2424307712/3(2\theta+1)(3\theta+2)(3\theta+4)(2\theta+3)\right)$	
12	$\left\{\begin{array}{cccccc} -\frac{3}{544} & 0 & \frac{1}{432} & \frac{1}{64} & \frac{1}{32} & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 1 & 1 & \frac{2}{3} \\ 3 & 0 & 1 & 1 & 1 & \frac{4}{3} \\ 4 & 0 & 2 & 2 & 2 & \frac{3}{2} \end{array}\right\}$
36	
52	
5	
432	
$1{+}\frac{484}{3}p-\frac{48788}{3}p^2{+}\frac{22124392}{3}p^3-311442268p^4{+}\frac{692669188484}{3}p^5{+}\frac{12207068987320}{3}p^6+O(q^7)$	

$n^0_1 = 1936,\ n^0_2 = -24636,\ n^0_3 = 3277616,\ n^0_4 = -58392376,\ n^0_5 = 22165414016,$ $n^0_6 = 226056424300$
$e_{\frac{1}{432}} = -\frac{1}{6},\ e_{\frac{1}{64}} = -\frac{1}{6},\ e_{\frac{1}{32}} = -\frac{1}{3}$
$n^1_1 = 48,\ n^1_2 = -6636,\ n^1_3 = 598912,\ n^1_4 = -64584123,\ n^1_5 = 6217443376,\ n^1_6 = -735479566808$
$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 12 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},\ M_{-\frac{3}{544}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ M_{\frac{1}{32}} = \begin{pmatrix} -7 & -26 & -4 & -2 \\ 8 & 27 & 4 & 2 \\ -64 & -208 & -31 & -16 \\ 56 & 182 & 28 & 15 \end{pmatrix},$ $M_{\frac{1}{64}} = \begin{pmatrix} -7 & -32 & -6 & -4 \\ 4 & 17 & 3 & 2 \\ -24 & -96 & -17 & -12 \\ 20 & 80 & 15 & 11 \end{pmatrix},\ M_{\frac{1}{432}} = \begin{pmatrix} 1 & -5 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$\lambda_{\frac{1}{432}} = 1,\ \lambda_{\frac{1}{64}} = 1,\ \lambda_{\frac{1}{32}} = 2$

Table C.2: Data for CY(3)-equations with positive c_3

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Zusammenfassung

In vielen Teilgebieten der Mathematik ist es wünschenswert, die Monodromiegruppe einer homogenen linearen Differenzialgleichung zu verstehen. Es sind nur wenige analytische Methoden zur Berechnung dieser Gruppe bekannt, daher entwickeln wir im ersten Teil dieser Arbeit eine numerische Methode zur Approximation ihrer Erzeuger. Im zweiten Abschnitt fassen wir die Grundlagen der Theorie der Uniformisierung Riemannscher Flächen und die der arithmetischen Fuchsschen Gruppen zusammen. Außerdem erklären wir, wie unsere numerische Methode bei der Bestimmung von uniformisierenden Differenzialgleichungen dienlich sein kann. Für arithmetische Fuchssche Gruppen mit zwei Erzeugern erhalten wir lokale Daten und freie Parameter von Lamé Gleichungen, welche die zugehörigen Riemannschen Flächen uniformisieren. Im dritten Teil geben wir einen kurzen Abriss zur homologischen Spiegelsymmetrie und führen die $\widehat{\Gamma}$-Klasse ein. Wir erklären wie diese genutzt werden kann, um eine Hodge-theoretische Version der Spiegelsymmetrie für torische Varitäten zu beweisen. Daraus gewinnen wir Vermutungen über die Monodromiegruppe M von Picard-Fuchs Gleichungen von gewissen Familien $f : \mathcal{X} \to \mathbb{P}^1$ von n-dimensionalen Calabi-Yau Varietäten. Diese besagen erstens, dass bezüglich einer natürlichen Basis die Monodromiematrizen in M Einträge aus dem Körper $\mathbb{Q}(\zeta(2j+1)/(2\pi i)^{2j+1}, j = 1, \ldots, \lfloor (n-1)/2 \rfloor)$ haben. Und zweitens, dass sich topologische Invarianten des Spiegelpartners einer generischen Faser von $f : \mathcal{X} \to \mathbb{P}^1$ aus einem speziellen Element von M rekonstruieren lassen. Schließlich benutzen wir die im ersten Teil entwickelten Methoden zur Verifizierung dieser Vermutungen, vornehmlich in Hinblick auf Dimension drei. Darüber hinaus erstellen wir eine Liste von Kandidaten topologischer Invarianten von vermutlich existierenden dreidimensionalen Calabi-Yau Varietäten mit $h^{1,1} = 1$.

Abstract

In many branches of mathematics it is eligible to understand the monodromy group of a homogeneous linear differential equation. But only few analytic methods to compute this group are known. Hence, in the first part of this thesis we develop a numerical method to approximate its generators. In the second part we summarize the basics of uniformization of Riemann surfaces and of arithmetic Fuchsian groups. Furthermore we explain how our numerical method can be useful when computing uniformizing differential equations. For arithmetic Fuchsian groups with two generators we obtain the local data and the accessory parameter of a Lamé equation uniformizing the associated Riemann surface. In the third part we briefly review homological mirror symmetry and introduce the $\widehat{\Gamma}$-class. We explain how it can be used to prove a Hodge-theoretic version of mirror symmetry for toric varieties. We gain conjectures on the monodromy group M of Picard-Fuchs equations of certain families $f : \mathcal{X} \to \mathbb{P}^1$ of n-dimensional Calabi-Yau varieties. The first part of these conjectures tells that with respect to a natural basis the entries of the matrices in M are contained in the field $\mathbb{Q}(\zeta(2j+1)/(2\pi i)^{2j+1}, j = 1, \ldots, \lfloor (n-1)/2 \rfloor)$. The second part of the conjectures is that topological invariants of the mirror partner of a generic fiber of $f : \mathcal{X} \to \mathbb{P}^1$ are reconstructible from a special element of M. Finally, we apply our numerical method to verify the conjecture mainly in dimension three. Additionally we compile a list of candidates of topological invariants of conjecturally existing three-dimensional Calabi-Yau varieties with $h^{1,1} = 1$.

Lebenslauf

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Ich wurde am 24.10.1981 in Wiesbaden geboren. Die Hochschulreife erwarb ich 2001 am Mons Tabor-Gymnasium in Montabaur. Nach dem Zivildienst im Malteser Hilfsdienst, begann ich im Oktober 2002 mein Studium an der Johannes Gutenberg-Universität Mainz. Ich belegte im Hauptfach Mathematik und im Nebenfach Volkswirtschaftslehre. Von 2005 bis 2008 war ich am Fachbereich Mathematik als wissenschaftliche Hilfskraft beschäftigt. Im Oktober 2008 schloss ich mein Studium mit der Diplomarbeit *Monodromieberechnungen für Calabi-Yau Operatoren vierter Ordnung* und Prüfungen in den Bereichen Algebra, Riemannsche Flächen, Stochastik und Volkswirtschaftslehre ab. Die Arbeit wurde von Prof. Dr. D. van Straten betreut. Seit November 2008 bin ich wissenschaftlicher Mitarbeiter an der Johannes Gutenberg-Universität Mainz. Während dieser Zeit reiste ich zu Forschungsaufenthalten unter anderem zum IMPA in Rio de Janeiro und zum MSRI in Berkeley.