Modern Aspects of Scattering Amplitudes in Quantum Chromodynamics and Gravity

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Abstract

Gauge-theory scattering amplitudes are a necessary ingredient to describe collision experiments. The method based on Feynman diagrams becomes computationally difficult to use in practice when the number of particles involved increase or when more precision is required. The search for new methods of computation of scattering amplitudes for gauge theories involves several ideas, which lead to improvement of the current techniques and to establish new ones. The new techniques and concepts lead to a better understanding of perturbative quantum field theory. In this thesis, the Cachazo-He-Yuan (CHY) formalism based on the scattering equations and the Bern-Carrasco-Johansson (BCJ) duality are used to compute amplitudes in Quantum Chromodynamics (QCD) at tree-level. These formalisms can naturally be utilized to explore gravity amplitudes by the BCJ double copy mechanism. This mechanism is used to study relations between QCD amplitudes and gravity. One of the main results of this thesis is the proof of the CHY representation of QCD, towards finding a closed integrand of the CHY representation. The second result is the proposal of a new gravitational theory built from QCD amplitudes, which may be relevant for discussions about dark matter. Finally, with the aid of the techniques introduced in this thesis, relations among Einstein-Yang-Mills and Yang-Mills amplitudes are explored.

Zusammenfassung

Streuamplituden in Eichtheorien sind grundlegende Bausteine, um Kollisionsexperimente zu beschreiben. Die auf Feynman-Diagrammen basierende Methode wird rechnerisch aufwendig, sobald viele Teilchen involviert sind oder mehr Präzision benötigt wird. Die Suche nach neuen Methoden zur Berechnung von Streuamplituden in Eichtheorien schließt mehrere Ansätze ein, welche zur Verbesserung der aktuellen Methoden, aber auch zur Entwicklung von neuen Methoden führen. Die neuen Methoden und Konzepte führen zu einem besseren Verständis der pertubativen Quantenfeldtheorie. In dieser Dissertation werden der auf Streugleichungen basierende Cachazo-He-Yuan(CHY)-Formalismus sowie die Bern-Carrasco-Johansson(BCJ)-Dualität benutzt, um Streuamplituden in der Quantenchromodynamik (QCD) auf Tree-Level zu berechnen. Diese Formalismen können des Weiteren zur Untersuchung von Gravitationsamplituden durch den BCJ-Double-Copy-Mechanismus herangezogen werden. Diese Prozedur wird zur Diskussion von Beziehungen von Amplituden in QCD und Gravitation verwendet. Ein Hauptresultat dieser Dissertation ist der Beweis der CHY-Darstellung von QCD-Amplituden mit dem Ziel eine geschlossene Form des Integranden in der CHY-Darstellung zu finden. Ein weiteres Ergebnis besteht im Vorschlag einer neuen Theorie für Gravitation, die aus QCD-Amplituden aufgebaut wird, die auch für Überlegungen hinsichtlich Dunkler Materie relevant sein könnte. Abschließend werden, mit Hilfe der in dieser Dissertation eingeführten Methoden, Beziehungen zwischen Einstein-Yang-Mills und Yang-Mills-Amplituden untersucht.

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Notations and conventions

We use the mostly minus signature $\eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ for the Minkowsky metric.

In the computation of amplitudes all particles are considered to be outgoing.

Calligraphic letters $\mathcal{A}_n$ and $\mathcal{M}_n$ are used for n-point full amplitudes. Uppercase letters A_n and M_n are used for n-point primitive amplitudes, where the color and coupling information has been extracted.

The set of external momenta, polarization vectors, and helicities is denoted by $p = (p_1, p_2, \ldots, p_n)$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, and $h = (h_1, \ldots, h_n)$, respectively.

Kinematic invariants are normalized according to

$$\epsilon_{ab} = \varepsilon_a \cdot \varepsilon_b, \qquad \rho_{ab} = \sqrt{2}\varepsilon_a \cdot p_b, \qquad s_{ab} = 2(p_a \cdot p_b).$$

Orderings of primitive amplitudes are indicated by the labels w, v. The ordering $\bar{w}$ is obtained from the ordering $w = l_1 \ldots l_{n-1} l_n$ by exchanging the last two letters, i.e., $\bar{w} = l_1 \ldots l_n l_{n-1}$.

Unless stated otherwise z is the tuple $z = (z_1, z_2, \ldots, z_n)$ for an amplitude in the CHY representation. The upper index in $z^{(j)}$ denotes the j-th solution of the scattering equations.

Parke-Taylor factors are denoted by $C(w, z)$, where w is some ordering. The differences $z_i - z_j$ in in these factors are denoted by z_{ij}.

The Kawai-Lewellen-Tye kernel $S[u|\bar{v}]$ is usually indexed by the orderings u and $\bar{v}$ as $S_{u\bar{v}}$.

Generators of the Lie algebra are normalized according to

$$[T^a, T^b] = \mathrm{i}\sqrt{2} f^{abc} T^c, \qquad \mathrm{Tr}(T^a T^b) = \delta^{ab}.$$

Chapter 1

Introduction

The quantum theory of fields (QFT) is the theoretical framework that describes the fundamental constituents of matter. A particular model in this framework, known as the standard model of particle physics, describes accurately processes at distances of 1×10^{-18} m. Recently, experiments at the Large Hadron Collider (LHC) have discovered the missing piece of the standard model of particle physics—the Higgs boson. However, this model is incomplete since it does not include gravitational interactions—among other things. The reason is that gravitational interactions seem to be relevant at distances of 10^{-35} m—the Plank scale—or equivalently, at extremely high energies. The Standard Model is then an effective description of the interactions far below the Planck scale, which is composed by a set of quantum field theories known as gauge theories, e.g., the strong interactions of quarks and gluons are described by quantum chromodynamics (QCD).

At very large distances, we have the *standard cosmological model*, which describes gravitational interactions and explains the geometry and evolution of the universe. This model is based on the Einstein's classical field equations of general relativity. One of the biggest challenges in contemporary theoretical (high energy) physics is to unify these models into a single framework. One approach to reach such *unification* is to develop a complete theory able to describe gravitational interactions at arbitrary high energies, hence requiring a method of quantization of gravity. Several attempts to formulate such a theory have been made over the years, one of them being string theory. Interestingly, it might be that the problem is unsolvable because gravity emerges from other phenomena, e.g., from entanglement as has been recently suggested [1].

Although we do not have the ultimate quantum theory of gravity, we can still perturbatively quantize gravity using ordinary methods in quantum field theory (QFT). The main obstacle is that the theory inevitably has to be treated as an effective field theory due to the well-known problem that the resulting theory is non-renormalizable. Perturbative quantum gravity then will be a theory valid far below the Planck scale. The quantum theory of fields is then able to describe effectively perturbative quantum gravity and the standard model of particles in a single framework. This rather conservative point of view is the one we will adopt in this work.

We will be mainly concerned with observables coming from scattering experiments, which theoretically can be computed from S-matrix elements—amplitudes of probability. The S-matrix elements are formed by objects called scattering amplitudes, which in the framework of perturbative quantum field theory can be computed via Feynman diagrams. QFT is based on the principles of quantum mechanics and special relativity plus the requirement of the cluster decomposition principle [2]. This principle reflects the fact that separated experiments in space will have uncorrelated results. In QFT, the Feynman rules follow from these principles. They appear as a recipe for keeping track of the perturbative expansion of an exponential function—the Dyson series. Quantum field theory then provides a method to compute observables perturbatively in a Feynman diagrammatic expansion. It is well-known that the number of these diagrams increases rapidly as the number of particles involved increases. In theories like QCD or gravity, it is also well-known that the number of terms in the diagrammatic expansion grows very fast. In the case of gravity, even the simplest cases are computationally prohibitive due to the huge number of terms in the expansion. Fortunately, for current phenomenological applications in scattering experiments, the quantum effects of gravity can be neglected. In contrast, for QCD applications we require as many terms as possible in order to increase the precision of the theoretical predictions.

In principle, it is possible to sort out the technical difficulty by using computational techniques but it turns out that in some cases the task is out of reach even for modern computers.

The main motivation for the modern approach to the S-matrix is the computation of the scattering amplitudes, avoiding the issues generated by the traditional QFT method, i.e., too many diagrams, too many terms, gauge redundancy, etc. It is a challenge to improve our understanding of quantum field theory methods to deal with these issues and to give alternative methods in the cases where the Feynman diagrammatic approach becomes unpractical. Another motivation for the search of new methods and possibly a new framework is the fact that many times the final results in the diagrammatic expansion are extremely simple, thus indicating a hidden simplicity at the level of amplitudes.

1.1 Ingredients for a modern approach to amplitudes

Historically, one can arguably state that the study of scattering amplitudes and its properties in its modern form comes from the beginning of the study of dispersion in electrodynamics via the Kramer-Krönig relation—dispersion relations are inspired on this relation. This points to the *first* element on which the modern description of scattering amplitudes is based, i.e., the use of the complex analytic properties of amplitudes and in general complex numbers. This is, of course, not specific to the study of scattering amplitudes, but it is a fundamental tool in modern approaches to amplitudes. Actually, the importance of complex quantities and the analytical properties of the amplitudes led to the S-matrix program, which aimed at computing the matrix elements using only the analytical properties of the S-matrix and the postulates of quantum mechanics and special relativity, and considering all interactions to be short-range [3].

The S-matrix program developed in the study of the dual resonant models and then transformed in what is known generically as string theory. String theory methods are the *second* main ingredient in the current research of scattering amplitudes. In particular, we use the idea from string theory methods, that scattering amplitudes can be recasted into an integral which encodes the analytical properties—poles and branch cuts—and contains all physical information. This is not unique to string theory methods, but it is a source of inspiration for finding new methods to obtain amplitudes.

In string theory the S-matrix is obtained from the amplitudes, which can be computed via the recipe of inserting vertex operators V on the string world-sheet. For example, at tree-level the n-particle amplitude reads

$$\mathcal{A}_n^{\text{strings}} = g^{n-2} \left\langle \psi_1 | V_2 \Delta V_3 \cdots \Delta V_{n-1} | \psi_n \right\rangle , \tag{1.1}$$

where Δ are the analogous to propagators in QFT and the vertex insertions V_i depend on the particle type [4]. Once the insertions have been made this formula transforms into a standard integral, i.e., we have an integral representation of the scattering amplitude for strings. For example, a typical amplitude for open strings schematically becomes

$$\mathcal{A}_n^{\text{strings}} = g^{n-2} \int \mathrm{d}\Omega f(x_1, \ldots, x_n) \prod_{1 \le i < j \le n-1} (x_i - x_j)^{2\alpha' p_i \cdot p_j}, \tag{1.2}$$

where the measure $\mathrm{d}\Omega$ results from a gauge fixing procedure[1]. This amplitude describes extended objects (strings) which in the infinite tension limit ($\alpha' \to 0$)—also called field theory limit—correspond to QFT amplitudes. In this sense, the amplitudes for particles are “represented” by the amplitudes of strings such that

$$\mathcal{A}_n^{\text{particles}} = \lim_{\alpha' \to 0} \mathcal{A}_n^{\text{strings}}. \tag{1.3}$$

It is also possible to write amplitudes in quantum field theory as in string theory by doing first quantization and writing an expression analogous to Eq.(1.1) for particles, but with insertions of operators in a world-line. This is not the point of view we are interested here.

[1] Going from Eq.(1.1) to a simplified integral like Eq.(1.2) is a highly non-trivial task which requires the whole machinery of string methods.

What we would like is a representation of *quantum field theory* amplitudes as integrals in a certain auxiliary space, i.e., amplitudes of particles expressed as an integral instead of a sum over Feynman diagrams.

Additionally, from relations between open and closed string amplitudes it is possible to deduce a relation between gauge theory amplitudes and gravity amplitudes—the Kawai-Lewellen-Tye relations [5]. Schematically these relations are represented as the "equation"

$$\text{gravity} = \text{gauge} \times \text{gauge}. \tag{1.4}$$

The *third* ingredient is the study of gauge amplitudes, from which it is possible to obtain gravitational amplitudes by squaring gauge theory ones. In this sense, we have access to a simplified method for computing gravitational amplitudes without Feynman diagrams whenever we have access to amplitudes in gauge theories.

On the side of quantum field theory developments, the introduction of spinor products and the use of helicity amplitudes is another ingredient in current approaches to amplitudes. When the amplitude involves large sets of Feynman diagrams, it is more convenient to calculate matrix elements with external definite helicities. Therefore, using a well suited set of variables that describe particles is the *fourth* ingredient in the current approach to amplitudes. In particular helicity methods are relevant for QCD amplitudes involving only gluons, also known as pure Yang-Mills amplitudes. In fact, one of the most important results obtained in the field of amplitudes was obtained for the scattering of gluons, where two of them have certain helicity configurations. The Parke-Taylor formula [6] gives the amplitude squared for the scattering of n gluons—with particle 1 and 2 having negative helicity—expressed in terms of the invariant products $p_i \cdot p_j$ as

$$|\mathcal{A}_n(1^-2^-3^+\cdots n^+)|^2 = c_n(g,N)\left[(p_1\cdot p_2)^4\sum_P \frac{1}{(p_1\cdot p_2)(p_2\cdot p_3)\cdots(p_n\cdot p_1)}\right], \tag{1.5}$$

where the coefficient $c_n(g,N)$ has a dependence on the number of colors N and the coupling constant g. The sum runs over all permutations P of $1,\dots,n$. This compact result corresponds to the sum of 220 Feynman diagrams for the case of 6 gluons and 559405 diagrams for the case of 9 gluons. It serves to illustrate that the quantum field theory approach based on Feynman diagrams becomes unpractical to handle this problem, although the result is extremely simple. This formula has been the classical example for the methods in scattering amplitudes and can be generalized in many ways. For instance, a similar formula appears for the amplitudes of the maximally supersymmetric Yang-Mills theory and in the field limit of amplitudes in string theory.

Another use of the spinor helicity amplitudes came with the introduction of twistor variables as the central objects in the amplitude. This was emphasized by Witten in 2003 [7], who found that topological string theory and the Fourier transform into twistor space of scattering amplitudes are connected. For example, Witten found that the color independent Parke-Taylor amplitude—also known as the Maximally Helicity Violating (MHV) amplitude—can be written as

$$A_n(\lambda_i, \mu_i) = g^{n-2} \int \mathrm{d}^4 x \prod_{i=1}^{n} \delta^2(\mu_{i\dot{a}} + x_{a\dot{a}} \lambda_i^a) f(\lambda_i) \tag{1.6}$$

where $f(\lambda_i)$ is an integrand which only depends on the spinors λ_i. This approach was the starting point of representations of amplitudes supported on the solutions of a particular set of equations. In the original Witten's formulation these equations are given by

$$0 = \mu_{i\dot{a}} + x_{a\dot{a}} \lambda_i^a, \tag{1.7}$$

which localize the Parke-Taylor amplitude on a zero degree curve. If x is complex, the Parke-Taylor amplitude localizes in a Riemann sphere ($\mathbb{CP}^1$).

The fact that we can separate the amplitude in a color independent part—for example in Eq.(1.6)—was found almost simultaneously using QFT methods and string theory inspired methods. This simplification of the computation of gluon amplitudes is known as the color decomposition of the amplitude [8, 9] and we take it as the *fifth* ingredient. It amounts to the separation of the information about the color and the kinematics of the amplitude into a color dependent part and a gauge invariant kinematic part known as the primitive (partial) amplitude. The color decomposition for the scattering of n gluons was originally written as

$$\mathcal{A}_n = \sum_{\{1,2,\ldots,n\}'} \mathrm{Tr}(T^{a_1} \ldots T^{a_n}) m(p_1, \epsilon_1; p_2, \epsilon_2; \ldots; p_n, \epsilon_n), \tag{1.8}$$

where the sum is over $(n-1)!$ noncyclic permutations of $\{1, \ldots, n\}$ and the T's are the matrices of the symmetry group in the fundamental representation. The problem then reduces to the calculation of the cyclic gauge invariant primitive amplitudes m, which we consider as a basis of the full amplitude $\mathcal{A}_n$. The $(n-1)!$ primitive amplitudes are not independent because there are some relations among them. First, there are Kleiss-Kuijf relations [10], which relate amplitudes with different color orderings allowing us to reduce the number of basis amplitudes to $(n-2)!$. These results were obtained in the search for a simplified computation of gluon amplitudes. Some time had to pass until this basis could be further reduced. The Bern-Carrasco-Johansson (BCJ) relations [11] which relate amplitudes with different orderings and different coefficients allow us to reduce the basis amplitudes to $(n-3)!$.

The BCJ relations were proposed by invoking a procedure which resembles the color decomposition with an additional requirement on the construction of the kinematic and color dependent parts. The scattering amplitude of gluons is represented as a sum of color dependent part c_i and a kinematic part n_i by

$$\mathcal{A}_n(1,2,\ldots,n) = \mathrm{i}\, g^{n-2} \sum_i \frac{n_i c_i}{(\prod_j p_j^2)_i}, \tag{1.9}$$

where the sum runs over color ordered trivalent diagrams and both c_i and n_i satisfy Jacobi-like identities. This is the Bern-Carrasco-Johansson duality—also known as the color-kinematics duality. We consider the color-kinematics duality as the *sixth* ingredient in the modern approach to scattering amplitudes. The BCJ construction gives a gravitational theory by invoking the idea gravity = gauge × gauge. This is the double copy procedure, which tells us that the numerators n_i obtained for the gauge theory can be recycled to build a gravitational amplitude as

$$\mathcal{M}_n = \mathrm{i}\left(\frac{\kappa}{2}\right)^{n-2} \sum_i \frac{n_i \widetilde{n}_i}{(\prod_j p_j^2)_i}, \tag{1.10}$$

where κ is the gravitational coupling constant. The sum runs over the same set of diagrams of the gauge theory and we are allowed to use a second set of numerators $\widetilde{n}$ from another gauge theory [12].

The six ingredients mentioned so far appear in the formulation of a representation of tree-level scattering amplitudes which involves an auxiliary space that encodes the kinematics—a Riemann sphere. In analogy to Witten's formula, the integral representation of amplitudes is located at solutions of a set of equations called the scattering equations

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{s_{ij}}{z_i - z_j} = 0, \quad i = 1,\ldots,n, \tag{1.11}$$

where $s_{ij} = (2p_i \cdot p_j)$ are the usual Mandelstam invariants. On the support of the scattering equations, Cachazo, He, and Yuan (CHY) proposed an elegant representation—valid in D-dimensions—of amplitudes for scalar, gluons, and gravitons [13, 14]

$$\mathcal{M}_n^{(s)} = \int \frac{\mathrm{d}^n z}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})} \prod_i{}' \delta\left(\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{s_{ij}}{(z_i - z_j)}\right)\left(\frac{\mathrm{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})}{(z_1-z_2)\cdots(z_n-z_1)} + \dots\right)^{2-s} (\mathrm{Pf}'\Psi)^s, \tag{1.12}$$

where $s = 0$, $s = 1$ and, $s = 2$ indicates scalars, gluons, and gravitons, respectively.[2] Details of this formula are one of the main subjects of this work and we delay their precise definition. The delta functions under the integral sign in the above equation completely localizes the integral which means that there are no integrations to be done. The result is then a sum over the evaluations of the solutions of the scattering equations. This is the CHY representation of amplitudes and we can consider it as a combination of several ingredients in the modern approach of scattering amplitudes. This formula has the following properties:

1. It uses complex variables in order to reproduce the analytical properties of tree-level amplitudes (poles).
2. It is an integral representation similar to a string amplitude but describing particles.
3. The gravitational amplitude ($s = 2$) can be thought as a realization of the idea "gravity as the square of a gauge theory".
4. It uses a well suited set of variables, which are directly relevant to the process, i.e., Mandelstam variables. However, we can also use spinor variables and compute specific helicities.
5. The color decomposition can be embedded in the CHY representation as can be seen from the traces in Eq.(1.12).
6. Although it is a nontrivial fact, there is an equivalence between the color kinematics duality and the CHY representation.

Hence, the CHY representation is an extraordinary tool to explore contemporary aspects of scattering amplitudes at tree-level[3].

In this brief description of the developments in scattering amplitudes we have pointed out how each ingredient simplifies the task of the computation of amplitudes. Starting with the simple fact that complex variables are at the heart of the analytic properties of the S-matrix, we culminated with the CHY representation which nicely combines in a single formula several ingredients of the modern approach of scattering amplitudes. These

[2] This formula can be considered as a generalization of the Roiban-Spradlin-Volovich [15], which is valid in 4-dimensions.

[3] The CHY representation has been extended at loop level.

are the ingredients we will consider throughout this work and use them for our modern approach to amplitudes in QCD and gravity.

Conceptually, the sixth ingredient is very interesting as it reveals aspects of amplitudes that cannot be formally deduced—as far as we know—from quantum field theory. It also relevant for discussions regarding locality and unitarity which many times are not manifest in modern formulations of the S-matrix.

1.2 About this work

In this work, we will focus on amplitudes in QCD and pure Yang-Mills. We will explore their relations with perturbative quantum gravity in light of the recent methods developed to obtain gravity amplitudes from gauge theory, specifically the color-kinematics duality and the CHY representation. As we will see, these are equivalent methods to obtain gravity amplitudes from the point of view of gravity as the square of a gauge theory.

This work is organized as follows. In Chapter 2, we set the playground by defining the basic objects of interest: the S-matrix, observables, Feynman rules for theories of various spins, etc. In particular we give a brief introduction to QFT in the approach by Weinberg and briefly mention other methods of field quantization. We also introduce the canonical quantization of perturbative gravity via Feynman diagrams. The conventions we follow are summarized in Appendix A. Lastly, we present some ingredients in current approaches to scattering amplitudes. In particular the color-kinematics duality and the double copy procedure by Bern-Carraco-Johansson. In Chapter 3, we deal with the CHY formalism. We present the scattering equations and the general features of the formalism without specifying integrands. We then give explicit integrands and present the methods for computation of residues, which we are fully developed in Appendix B. In Chapter 4, we deal with gauge theories and introduce one of the main results of this work, i.e., the CHY representation of primitive QCD amplitudes. An important tool for the proof of this representation is the construction of a basis of amplitudes which we present in detail in Appendix C. In Chapter 5, inspired on the CHY formalism, we describe a gravitational theory built from QCD amplitudes by generalizing the Kawai-Lewellen-Tye kernel, which describes double copies of massless or massive fermions and briefly discuss its relevance for dark matter. Also in Chapter 5, we prove the recently proposed Stieberger-Taylor relations among Einstein-Yang-Mills amplitudes and Yang-Mills amplitudes generalizing them for the case of several gravitons. The summary and conclusions are presented in Chapter 6.

Chapter 2

Review of S-matrix theory

The *scattering matrix* or S-matrix is the quantity that allow us to obtain probability amplitudes of the transition from an initial state to a final state. This is the quantity that ultimately can be measured in a laboratory in the form of cross sections and decay rates. In order to compute the S-matrix, the principles of quantum mechanics and special relativity have to be satisfied, namely unitarity and Lorentz invariance. In QFT, these properties are manifest and therefore the principles are automatically satisfied. Actually, it is possible to start with the most general features of the S-matrix that come from quantum mechanics and special relativity to obtain the matrix elements without reference to a specific Lagrangian. The old "S-matrix" program is based on this idea. Research in this area was performed during the 1960s but partially abandoned in view of the success of QCD as the theory of strong interactions.

One of the features of the old S-matrix program that remains, in what can be considered as the *current* S-matrix program, is the study of the analytic properties of amplitudes. Over the years, some of the current ingredients for computing amplitudes have been developed in the frameworks of QFT and string theory, e.g., the color-decomposition of gauge amplitudes, Berends-Giele recursions, and unitarity methods. In the last two decades, several ideas have appeared that revitalized the old S-matrix program, e.g., the Britto-Cachazo-Feng-Witten recursion relations, generalized unitarity, and representations of the S-matrix supported on solutions of equations. This work is a follow-up of these ideas.

In this Chapter, we revisit the approach based on Feynman diagrams from QFT and present several theories relevant for later Chapters. We present various methods which are at the heart of the current approach of amplitudes. This review aims to be comprehensive but it is not exhaustive.

2.1 The scattering matrix

This Section is inspired on the classical treatments on the subject. For general features of the S-matrix we follow R. J. Eden et'al [3], while for the development of Feynman rules we follow S. Weinberg [2]. Original sources can be found in these references.

In a typical experiment we are interested in the physical process where particles interact in a small region for a short period of time. The particle content is known long before the particles interact and long after they have interacted. This situation corresponds to a collision of particles or to a decay depending on the number of initial particles. Schematically, we have the reaction

$$\alpha_1 + \alpha_2 + \cdots \longrightarrow \beta_1 + \beta_2 + \ldots, \tag{2.1}$$

where α_i labels the initial particles and β_i the final particles. These are multi-labels containing the information about the four momenta p_i, the spin z component σ_i, and the particle type n_i. We can enumerate the particles because at both ends of the process the particles do not interact. We can then pack all multi-states of one particle into a single multi-particle state, which describes initial (final) free states in the far past (future) of the process. We label these states by α and β, respectively.[1] These states are known as "in" and "out" and we write

$$|\alpha, \mathrm{in}\rangle\,, |\beta, \mathrm{out}\rangle \tag{2.2}$$

to describe asymptotic initial states and final states, respectively. They build a complete set of orthonormal states, i.e.,

$$1 = \int \mathrm{d}\beta\, |\beta, \mathrm{out}\rangle\, \langle\beta, \mathrm{out}|\,, \qquad 1 = \int \mathrm{d}\alpha\, |\alpha, \mathrm{in}\rangle\, \langle\alpha, \mathrm{in}|\,. \tag{2.3}$$

The integration measure is defined as the sum over all the discrete quantum numbers such as spin and particle type and integration over continuous quantum numbers.

We can use these states to obtain the probability amplitude of the transition between the "in" state to the "out" state, which from quantum mechanics reads

$$\langle\beta, \mathrm{out}|\alpha, \mathrm{in}\rangle\,. \tag{2.4}$$

Since both states correspond to measured states with definite quantum numbers, they should be related somehow. Formally, we say that they are isomorphic—since they

[1] Here α collects all the labels for each state, i.e., $\alpha = \alpha_1; \alpha_2; \ldots$ and similarly for β.

only differ by how they are labeled—and we define the operator S as the operator that transforms an "out" state into an "in" state

$$|\alpha, \text{in}\rangle \equiv S\,|\alpha, \text{out}\rangle\,, \tag{2.5}$$

which trivially leads to the recognition of the coefficients $\langle\beta, \text{out}|\alpha, \text{in}\rangle$, as the matrix elements of a unitary operator S defined through

$$S = \int \mathrm{d}\gamma\,|\gamma, \text{in}\rangle\,\langle\gamma, \text{out}|\,. \tag{2.6}$$

Therefore, the matrix elements of the S-operator are given by

$$S_{\beta\alpha} = \langle\beta, \text{out}|\alpha, \text{in}\rangle = \langle\beta, \text{in}|S|\alpha, \text{in}\rangle = \langle\beta, \text{out}|S|\alpha, \text{out}\rangle\,, \tag{2.7}$$

where we used the unitarity of S, i.e.,

$$SS^\dagger = S^\dagger S = 1. \tag{2.8}$$

The unitarity of the S-operator reflects the fact that probabilities are conserved—a constraint that has to be present in calculations based on quantum mechanics. In fact, this was one of the postulates of the old S-matrix program.

The next step is to introduce special relativity, i.e., to make the "in" states and "out" states to furnish a representation of the inhomogeneous Lorentz group. This requires to express the "in" and "out" states to be asymptotic states of free multi-particle states $|\alpha\rangle$ and $|\beta\rangle$, respectively. For example, we have for the "in" state

$$|\alpha, \text{in}\rangle = \lim_{\tau\to\infty} U^{-\tau}\,|\alpha\rangle\,, \tag{2.9}$$

where the operator $U^{-\tau}$ takes the free state from the long past. We have a similar expression with the opposite sign for the out state. The states $|\alpha\rangle$, $|\beta\rangle$ are time independent state eigenvectors of the free Hamiltonian of the system H_0. These states are classified according to its transformation under the inhomogeneous Lorentz group. The one particle states are normalized by

$$\langle p|p'\rangle \equiv \delta(p' - p) = (2\pi)^3 2p^0 \delta(\mathbf{p}' - \mathbf{p}). \tag{2.10}$$

Thus, we can redefine the S-matrix elements of (2.7) in terms only of free states through

$$S_{\beta\alpha} = \langle\beta|S|\alpha\rangle \equiv \delta(\beta-\alpha) + (2\pi)^4\delta^4(p_\beta - p_\alpha)\mathcal{A}_{\beta\alpha}, \tag{2.11}$$

where we have denoted by $\mathcal{A}$ the part of the S-matrix that represents the actual interactions. The quantity $\mathcal{A}_{\beta\alpha}$ is know as the scattering amplitude.[2]

In addition to Lorentz invariance of the S-matrix—defined in terms of the transformation rules of the asymptotic states—states could have symmetries acting by unitary transformations on the particle species labels, leading to internal symmetries which will commute with the S-matrix.

2.1.1 Observables

In order to make contact with experiments, we have to relate the scattering amplitude with the quantities that we can measure, i.e., observables. Quantum mechanics tells us that in order to obtain observables, we have to compute the probability amplitude of the transition $\alpha \to \beta$ (recall that α and β may contain several particles). Thus, the probability of the transition is simply the square of the S-matrix. We can idealize the system as being in a box of volume V interacting for a time T such that the probability per unit time—the transition rate—is given by the master formula

$$\mathrm{d}\Gamma \equiv \frac{\mathrm{d}P}{T} = \frac{(2\pi)^{3N_\alpha+4}V^{1-N_\alpha}}{\prod\limits_\alpha(2\pi)^3 2E_\alpha}\delta^4(p_\beta - p_\alpha)|\mathcal{A}_{\beta\alpha}|^2\frac{\mathrm{d}\beta}{\prod\limits_\beta(2\pi)^3 2E_\beta}, \tag{2.12}$$

where we have defined $\mathrm{d}\beta$ as the product $\mathrm{d}^3\mathbf{p}$ for each final particle. This quantity becomes a decay rate for $N_\alpha = 1$ and the case $N_\alpha = 2$ corresponds to the differential cross section after dividing by the incoming flux. However, the situation can be more complicated when the particles under consideration are composed by others, such as the collisions of protons at the LHC. In these cases we have to take into account the inner structure of the proton by using the parton model. Furthermore, in such complex situations we do not detect all particles in the final state and in some cases we simply cannot record all the information about these final states. Thus, the expected value of an observable O in scattering experiments involving protons can be written as a convolution of the parton distribution functions f and its perturbative calculation, i.e.,

$$\langle O\rangle = \frac{1}{N} f \otimes \int O(p_\alpha, p_\beta)\mathrm{d}\Gamma \tag{2.13}$$

[2]Typically the S-matrix is defined as $\delta(\beta-\alpha) + \mathrm{i}(2\pi)^4\delta^4(p_\beta - p_\alpha)\mathcal{T}_{\beta\alpha}$, where $\mathcal{T}_{\beta\alpha}$ is the "transition matrix" element. The amplitude is then $\mathrm{i}\mathcal{T}_{\beta\alpha}$.

where the $\otimes$ represents the convolution and N is a normalization factor which include the information about the flux and multiplicity of the outgoing particles. As indicated, the observable O depends in general of the initial and final sets of momenta, i.e., p_α and p_β, respectively. This formula is valid for a large set of observables which are infrared safe, that is, observables which have a continuous limit when one of more particles become unresolved, in other words they are insensitive to the emission of soft or collinear particles. Some examples of infrared observables include the so-called "jet shape" characteristics of hadronic events. These observables measure some property of the final hadronic states, e.g., the thrust, spherocity and the C-parameters can be used to identify pencil-like events and spherical events. Special cases of Eq.(2.13) include:

Elementary particles: In this case the initial states do not involve protons, hence eliminating the convolution functions. The observable then simplifies to

$$\langle O \rangle = \frac{1}{N} \int O(p_\beta) \mathrm{d}\Gamma. \tag{2.14}$$

A typical example of this type of situation is the electron-positron scattering.

Composite particles: In general, one has to calculate the observable as an integral over n_α parton distribution functions

$$\langle O \rangle = \frac{1}{N} \int \prod_{i=1}^{n_\alpha} f(x_i) \mathrm{d}x_i \int O(p_\beta) \mathrm{d}\Gamma. \tag{2.15}$$

Notice that the special case of $O = 1$—choosing an appropriate normalization factors—corresponds to the total cross section σ and the total decay rate Γ for $N_\alpha = 2$ and $N_\alpha = 1$, respectively. For future reference we present Lorentz invariant versions of these formulas. The total cross section[3] reads

$$\sigma = \frac{1}{\prod\limits_{i=1} n_i!} \frac{(2\pi)^4}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int \delta^4(p_\beta - p_\alpha) |\mathcal{A}_{\beta\alpha}|^2 \frac{\mathrm{d}\beta}{\prod\limits_{\beta} (2\pi)^3 2E_\beta}, \tag{2.16}$$

[3]Here we divide (2.12) by the flux factor $j = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}/(V E_1 E_2)$, which in an arbitrary frame of reference reads

$$j = \frac{\sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2}}{V},$$

where $\mathbf{v}_1, \mathbf{v}_2$ are the velocities of the colliding particles. This coincides with the usual definition of flux density whenever $\mathbf{v}_1$ is parallel to $\mathbf{v}_2$, since in that case $j = |\mathbf{v}_1 - \mathbf{v}_2|/V$ [16].

where we have included a symmetry factor counting the number n_i of identical particles of type i in the final state.

For the decay rate we have

$$\Gamma = \frac{1}{\prod\limits_{i=1} n_i!} \frac{(2\pi)^4}{2E_1} \int \delta^4(p_\beta - p_\alpha) |\mathcal{A}_{\beta\alpha}|^2 \frac{\mathrm{d}\beta}{\prod\limits_{\beta} (2\pi)^3 2E_\beta}. \tag{2.17}$$

Additional factors appear whenever we sum over spins.

2.2 Clusters and Quantum Fields: the Hamiltonian approach

Up to this point, we have not discussed how to obtain the scattering amplitudes $\mathcal{A}_{\beta\alpha}$ that appear in our expressions for observables. In this Section, we discuss how to compute these amplitudes using perturbation theory via the Feynman diagrammatic approach. Here, we will follow the approach by Weinberg [2] using the cluster decomposition principle as the main ingredient to introduce quantum fields.[4]

There is a logical requirement which can be taken as the starting point of our discussion: Experiments far away from each other should not be correlated. In other words, the matrix elements obtained by spatially separated laboratories should *factorize*. Let us denote by A (B) the multi-particle state formed by combining N initial (final) states in N different laboratories far away from each other.[5]The factorization property indicates that the S-matrix satisfies

$$S_{BA} \longrightarrow S_{B_1A_1} S_{B_2A_2} \cdots S_{B_NA_N}, \tag{2.18}$$

if for $i \neq j$, all particles in the states A_i and B_i are at great spatial distance from all particles in A_j and B_j. The factorization property of Eq.(2.18) can be written in terms of the so called connected S-matrix. Let us define recursively the connected part of the operator S by

$$\langle\beta| S |\alpha\rangle = \sum_{\bigcup\limits_i \beta_i = \beta,\, \bigcup\limits_i \alpha_i = \alpha} \prod_{i,j} \langle\alpha_i| S^C |\beta_j\rangle, \tag{2.19}$$

[4]Following this approach is not free of caveats. Nevertheless, it allows a systematic way of constructing the S-matrix starting from the basic principles of Quantum Mechanics and special relativity. Discussion about the possible caveats can be found in e.g., Ref. [17].

[5]This means that we have $A_i \longrightarrow B_i$ for $i = 1, \ldots, N$, $A = A_1 + A_2 + \cdots + A_N$ and $B = B_1 + B_2 + \cdots + B_N$

where the sum runs over all partitions into clusters of the set of initial and final particles.[6] The start of the recursion occurs when there is a single particle with momentum p and p' in the initial and final state, respectively. We define the start of the recursion by

$$S^C_{p'p} \equiv S_{p'p} = \langle p'|p\rangle \,, \tag{2.21}$$

In Fig.2.1 we represent graphically the definition of the connected part of the S-matrix.

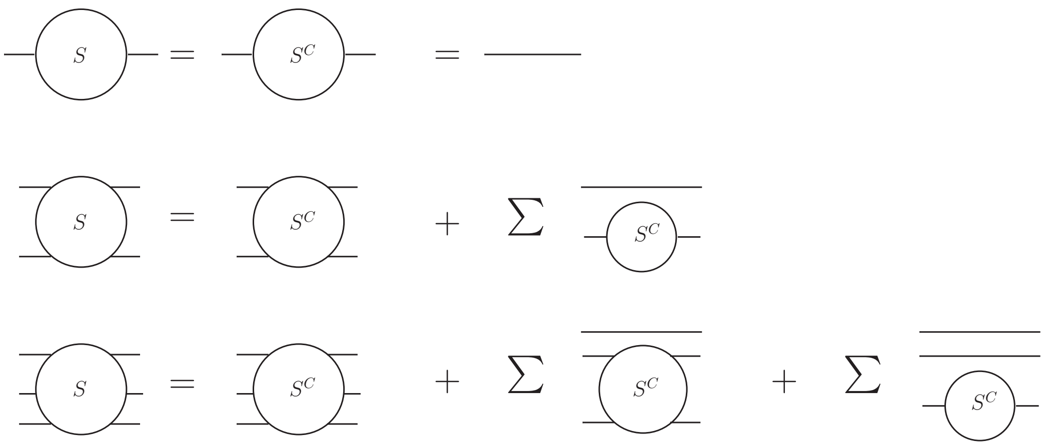

Figure 2.1: Graphical representation of the connected part of the S-matrix. The sum runs over the permutations of the labels of the states.

From the Figure we can see the recursive structure of this definition[7]. The definition of connected amplitudes then allows us to state the cluster decomposition principle as the requirement: *The connected part of the* S*-matrix, which is denoted by* S_{BA} *must vanish when any one or more particles in the states* B *and/or* A *are far away in space from the others* [2]. This requirement implies that the connected part of the S-matrix will contain a single Dirac delta function which impose 3-momentum conservation and energy conservation. This is the reason behind Eq.(2.11). Hence, the connected part S^C of the S-matrix is given by

$$S^C_{\beta\alpha} = (2\pi)^4\delta^4(p_\beta - p_\alpha)\mathcal{A}_{\beta\alpha}. \tag{2.23}$$

[6] For example, in the case of two bosonic particles in the initial and final states, we have

$$\langle\beta_1\beta_2|\,S\,|\alpha_1\alpha_2\rangle = \langle\beta_1\beta_2|\,S^C\,|\alpha_1\alpha_2\rangle + \langle\beta_1|\,S^C\,|\alpha_1\rangle\,\langle\beta_2|\,S^C\,|\alpha_2\rangle + \langle\beta_1|\,S^C\,|\alpha_2\rangle\,\langle\beta_2|\,S^C\,|\alpha_1\rangle\,. \tag{2.20}$$

[7] For example, in the case of two bosonic particles we have

$$\langle\beta_1\beta_2|\,S\,|\alpha_1\alpha_2\rangle = \langle\beta_1\beta_2|\,S^C\,|\alpha_1\alpha_2\rangle + \delta(p_{\beta_1} - p_{\alpha_1})\delta(p_{\beta_2} - p_{\alpha_2}) + \delta(p_{\beta_1} - p_{\alpha_2})\delta(p_{\beta_2} - p_{\alpha_1}). \tag{2.22}$$

In addition, the cluster decomposition principle implies that $\mathcal{A}_{\beta\alpha}$ should be "smooth". The smoothness means that it has to allow poles and branch cuts, but no other singularities such as Dirac delta functions.

The interaction will satisfy the cluster decomposition principle if the Hamiltonian operator can be expressed as a sum of products involving creation and annihilation operators, i.e., if

$$H = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \int \mathcal{D}q' \mathcal{D}q \prod_{n=1}^{N} a'^{\dagger}_{n} \prod_{m=1}^{M} a_m h_{NM}(q, q'), \tag{2.24}$$

where $\mathcal{D}q' = \mathrm{d}q'_1 \cdots \mathrm{d}q'_N$, $\mathcal{D}q = \mathrm{d}q_1 \cdots \mathrm{d}q_M$, and the coefficients h_{NM} contain a single Dirac delta function. Notice that we are already considering that the Hamiltonian is normal ordered. Now, for a Hamiltonian with an interaction term V we have that the complete Hamiltonian is given by

$$H = H_0 + V, \tag{2.25}$$

which by standard time-dependent perturbation theory[8] leads to the Dyson series for the S-operator, i.e,

$$S = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{-\infty}^{\infty} \prod_{j=1}^{n} \mathrm{d}t_j T\{V(t_j)\}, \tag{2.26}$$

where T denotes time ordering. For $n = 0$ we define the time ordered product $T\{V(t_j)\}$ as the unit operator[9]. To make Lorenz invariance manifest, we write the perturbation potential as

$$V(t) = \int \mathrm{d}^3x \mathcal{H}_I(x, t), \tag{2.27}$$

which allows us to write the matrix elements as

$$S_{\beta\alpha} = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{-\infty}^{\infty} \mathrm{d}^4x_1 \cdots \mathrm{d}^4x_n \langle\beta| T\{\mathcal{H}_I(x_1) \cdots \mathcal{H}_I(x_n)\} |\alpha\rangle . \tag{2.28}$$

[8] For a recent treatment see Chapter 8 of Ref. [18].
[9] Equivalently, we extend the usual definition of a product from 1 to 0 to hold for operators as well.

This will be Lorentz invariant if $\mathcal{H}_I(x)$ commutes at space-like or light-like separations, i.e., if

$$\left[\mathcal{H}_I(x), \mathcal{H}_I(x')\right] = 0, \qquad \text{for } (x-x')^2 \leq 0. \tag{2.29}$$

Quantum field theory then emerges as the necessity of writing the interaction Hamiltonian constructed out of creation and annihilation operators—to satisfy the cluster decomposition principle—and the requirement to be a scalar under Lorentz transformations. In order to fulfill these requirements we are forced to write $\mathcal{H}_I(x)$ out of *local field operators* $\phi_l(x)$ defined through

$$\phi_l(x) \equiv \sum_{\sigma,n} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 E_p} \left(u_l(\mathbf{p},\sigma,n) a(\mathbf{p},\sigma,n) e^{-\mathrm{i}p\cdot x} + v_l(\mathbf{p},\sigma,n) a^\dagger(\mathbf{p},\sigma,n^c) e^{\mathrm{i}p\cdot x} \right), \tag{2.30}$$

where l indicates the type of particle and the representation of the homogeneous Lorentz group by which the field transforms. The creation and annihilation operators satisfy commutation/anti-commutation relations

$$\left[a(\mathbf{p},\sigma,n), a^\dagger(\mathbf{p}',\sigma,n)\right]_{\mp} = (2\pi)^3 2E_p \delta^3(\mathbf{p}-\mathbf{p}'), \tag{2.31}$$

with $\mp$ the commutation and anti-commutation relations for bosons and fermions, respectively. From this point of view, the Lorentz-invariant differential equations that the fields satisfy are nothing more than a reminder of the conventions used to construct irreducible representations of the homogeneous Lorentz group. Therefore, we do not need a classical Lagrangian density to back up the quantum description of particles. However, the easiest way of getting an interaction which satisfies Lorentz invariance and other symmetries is to start with a classical Lagrangian density.

Once we convince ourselves that fields are necessary, we continue by writing the general interaction term using fields and their adjoints. In general, we have

$$\mathcal{H}_I(x) = \sum_i g_i \mathcal{H}_{Ii}(\phi_l, \phi_l^\dagger). \tag{2.32}$$

The S-matrix elements can be computed from Eqs.(2.28), (2.30), (2.32). Explicitly, the Dyson series[10] reads

[10] Interactions with derivatives are also taken into account, they are just fields with different coefficients in Eq.(2.30)

$$S_{\beta\alpha} = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int \mathrm{d}^4x_1 \cdots \mathrm{d}^4x_n \, \langle 0| \cdots a_{\beta_2} a_{\beta_1} T\{\mathcal{H}_I(x_1) \cdots \mathcal{H}_I(x_n)\} a^\dagger_{\alpha_1} a^\dagger_{\alpha_2} \cdots |0\rangle \,. \tag{2.33}$$

This formula—after some manipulations—is what we need to obtain the S-matrix in any theory with a given interaction Hamiltonian. Using the standard commutation relations of creation and annihilation operators, we can compute terms in Eq.(2.33) at a given order in $\mathcal{H}_I$. The final result is computed by summing over all the possible ways of pairing creation and annihilation operators in Eq.(2.33), which can be systematized with the aid of the Wick's theorem [19]. This procedure results into diagrammatic rules to compute the terms in the expansion—the Feynman rules in position space or momentum space. Notice that because we have constructed the theory using the cluster decomposition principle—see Eq.(2.23)— we have only to consider connected diagrams.

Summarizing, this point of view focus on first principles of quantum mechanics and special relativity including the cluster decomposition principle, without assuming previous knowledge of a classical field theory. As we have mentioned, the equations of motion of the field operators are a consequence of our conventions on the representations of the homogeneous Lorentz group. Furthermore, this approach makes the unitarity of the S-matrix explicit by construction. In addition, we have included locality by introducing the cluster decomposition principle since it allows the S-matrix to have poles and branch cuts.[11]

2.2.1 Example

Before presenting the Feynman rules to obtain matrix elements, let us give an example of the formalism using creation and annihilation operators. We illustrate the procedure using Eqs.(2.24), (2.30), and (2.33). One of the "simplest" examples—from the QFT point of view—consists of an interaction term with three real bosonic fields ϕ of a single specie. This method does not require a classical Lagrangian density because the bosonic fields ϕ already satisfy the Klein-Gordon equations. Nevertheless, if we were using canonical quantization, the interaction we would like to study corresponds to the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \mathcal{L}_I = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda\phi^3. \tag{2.34}$$

Hence, the interaction Hamiltonian reads

$$\mathcal{H}_I = -\mathcal{L}_I = \lambda\phi^3. \tag{2.35}$$

[11] A nice modern discussion of this can be found in Chapter 24 of Ref. [20]

The bosonic interaction corresponds to the case $n = 1$, $\sigma = 0$ in Eq.(2.30). In our normalization $u_l = v_l = 1$, thus the field operator simplifies to

$$\phi(x) = \int \widetilde{\mathrm{d}p} \left(a(\mathbf{p}) e^{-\mathrm{i}p\cdot x} + a^\dagger(\mathbf{p}) e^{\mathrm{i}p\cdot x} \right), \tag{2.36}$$

where the Lorentz invariant differential form is defined through

$$\widetilde{\mathrm{d}p} \equiv \frac{\mathrm{d}^3 p}{(2\pi)^3 E_p}. \tag{2.37}$$

In normal ordered form, the expansion (2.24) of the interaction Hamiltonian (see Eq.(2.27)) becomes

$$\begin{aligned} \mathcal{H}_I = \lambda \int \widetilde{\mathrm{d}q_1}\widetilde{\mathrm{d}q_2}\widetilde{\mathrm{d}q_3} \Big[& e^{\mathrm{i}(-q_1-q_2-q_3)\cdot x} a_1^\dagger a_2^\dagger a_3^\dagger + e^{\mathrm{i}(q_1-q_2-q_3)\cdot x} a_2^\dagger a_3^\dagger a_1 + e^{\mathrm{i}(-q_1+q_2-q_3)\cdot x} a_1^\dagger a_3^\dagger a_2 \\ & + e^{\mathrm{i}(q_1+q_2-q_3)\cdot x} a_3^\dagger a_1 a_2 + e^{\mathrm{i}(-q_1-q_2+q_3)\cdot x} a_1^\dagger a_2^\dagger a_3 + e^{\mathrm{i}(q_1-q_2+q_3)\cdot x} a_2^\dagger a_1 a_3 \\ & + e^{\mathrm{i}(-q_1+q_2+q_3)\cdot x} a_1^\dagger a_2 a_3 + e^{\mathrm{i}(q_1+q_2+q_3)\cdot x} a_1 a_2 a_3 \Big], \end{aligned}$$

where $a_i \equiv a(\mathbf{q}_i)$. After relabeling, we obtain

$$\begin{aligned} \mathcal{H}_I = \lambda \int \widetilde{\mathrm{d}q_1}\widetilde{\mathrm{d}q_2}\widetilde{\mathrm{d}q_3} \big[& e^{-\mathrm{i}(q_1+q_2+q_3)\cdot x} a_1^\dagger a_2^\dagger a_3^\dagger + 3 e^{-\mathrm{i}(-q_1+q_2+q_3)\cdot x} a_2^\dagger a_3^\dagger a_1 \\ & + 3 e^{-\mathrm{i}(q_1-q_2-q_3)\cdot x} a_1^\dagger a_2 a_3 + e^{\mathrm{i}(q_1+q_2+q_3)\cdot x} a_1 a_2 a_3 \big]. \end{aligned} \tag{2.38}$$

Let us use this interaction Hamiltonian to calculate the trivial matrix element for the process $1 \to 2 + 3$. This is the lowest order in the Dyson series, that is

$$S_{\mathbf{p}_2\mathbf{p}_3|\mathbf{p}_1} = -\mathrm{i} \int \mathrm{d}^4 x \, \langle 0 | a(\mathbf{p}_3) a(\mathbf{p}_2) \mathcal{H}_I(x) a^\dagger(\mathbf{p}_1) | 0 \rangle \, . \tag{2.39}$$

We shall use the first Wick's theorem applied to the creation and annihilation operators to evaluate this expression. This theorem states that the product of creation and annihilation operators is the sum of all possible paired normal products:

$$\begin{aligned} A_1 A_2 \cdots A_n = \, : A_1 A_2 \ldots A_n : \, & + \sum_{k \neq l} C_{kl} : A_1 \ldots A_n : \\ & + \sum_{i \neq j} C_{ij} \left(\sum_{k \neq l} C_{kl} : A_1 \ldots A_n : \right) + \ldots , \end{aligned} \tag{2.40}$$

where

$$C_{kl}\colon A_1\ldots A_n\colon\ \equiv C(A_kA_l)\colon A_1\ldots A_{k-1}A_{k+1}\ldots A_{l-1}A_{l+1}\ldots A_n\colon, \tag{2.41}$$

where the dots indicate that we have to sum over all possible 3-contractions, 4-contractions, etc. The contraction of operators is defined by

$$C(A_kA_l) = \langle 0|\, A_kA_l\, |0\rangle\,, \tag{2.42}$$

which is usually denoted by $C(A_kA_l) = \underbrace{A_kA_l}$.

The vacuum expectation value of a normal ordered product vanishes, therefore we have to consider products of the maximum number of contractions which in our example is 3. Then Eq.(2.39) yields

$$\begin{aligned} S_{\mathbf{p}_2\mathbf{p}_3|\mathbf{p}_1} = -\mathrm{i}\lambda \int \mathrm{d}^4x \int \mathrm{d}^3q_1\mathrm{d}^3q_2\mathrm{d}^3q_3\Big[3e^{i(q_1-q_2-q_3)\cdot x}\big(\delta(\mathbf{p}_3-\mathbf{q}_3)\delta(\mathbf{p}_2-\mathbf{q}_2)\delta(\mathbf{p}_1-\mathbf{q}_1) \\ +\,\delta(\mathbf{p}_3-\mathbf{q}_2)\delta(\mathbf{p}_2-\mathbf{q}_3)\delta(\mathbf{p}_1-\mathbf{q}_1)\big)\Big], \end{aligned} \tag{2.43}$$

where we have used the commutation relations of creation and annihilation operators. After integration of the Dirac delta functions, we obtain

$$S_{\mathbf{p}_2\mathbf{p}_3|\mathbf{p}_1} = -6\mathrm{i}\lambda \int \mathrm{d}^4x\Big[e^{i(p_1-p_2-p_3)\cdot x}\Big] = -6\mathrm{i}\lambda(2\pi)^4\delta(p_1-p_2-p_3). \tag{2.44}$$

Therefore, from Eq.(2.23) the “amplitude” for this process reads

$$\mathcal{A}_{\mathbf{p}_2\mathbf{p}_3|\mathbf{p}_1} = -6\mathrm{i}\lambda. \tag{2.45}$$

At lowest order in λ, we do not have propagators thus we only have to contract creation and annihilation operator with fields at the same point. This gives six possible pairings and simplifies our result. The appearance of a numerical factor motivates the introduction of a factor of $1/m!$ for an interaction containing m identical fields.

In the general case, we also have pairings of fields at different times. Hence, we require the main Wick's theorem and the formulas

$$\langle 0|a(\mathbf{p})\phi(x)|0\rangle = e^{\mathrm{i}p\cdot x}, \quad \langle 0|\phi(x)a^\dagger(\mathbf{p})|0\rangle = e^{-\mathrm{i}p\cdot x}, \quad \langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \mathrm{i}\Delta(x-y), \tag{2.46}$$

where the Feynman propagator is defined by

$$\Delta(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{-\mathrm{i}k\cdot(x-y)}}{k^2-m^2+\mathrm{i}\epsilon}. \tag{2.47}$$

In order to deal with time ordered products of operators, we replace the contractions in Eq.(2.40) by "time contractions", i.e.,

$$C(A_kA_l) \to \langle 0|\, T\{A_kA_l\}\, |0\rangle \,, \tag{2.48}$$

in particular $C(\phi(x)\phi(y)) = \mathrm{i}\Delta(x-y)$.

Let us see how these rules work for the process $1+2 \to 3+4$. At the lowest order in λ, we have

$$S_{\mathbf{p}_3\mathbf{p}_4|\mathbf{p}_1\mathbf{p}_2} = \frac{(-\mathrm{i}\lambda)^2}{2} \int \mathrm{d}^4x \int \mathrm{d}^4x' \,\langle 0|a(\mathbf{p}_4)a(\mathbf{p}_3)T(\phi\phi\phi\phi'\phi'\phi')a^\dagger(\mathbf{p}_2)a^\dagger(\mathbf{p}_1)|0\rangle\,. \tag{2.49}$$

Here, we should have 4 operators to be contracted with a and $a^\dagger$. Hence a possible term that contributes reads

$$g(x,x') = \mathrm{i}\,\langle 0|\, a(\mathbf{p}_4)a(\mathbf{p}_3) :\phi\phi\phi'\phi': a^\dagger(\mathbf{p}_2)a^\dagger(\mathbf{p}_1)\,|0\rangle\, \Delta(x-x'). \tag{2.50}$$

This term can be conveniently represented by the first Feynman diagram in Fig.2.2. Inserting the contraction formulas (2.46) and the definition of the propagator (2.47), we obtain

$$g(x,x') = \mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{-\mathrm{i}(x-x')\cdot k}e^{\mathrm{i}(p_3+p_4)\cdot x}e^{-\mathrm{i}(p_1+p_2)\cdot x'}}{k^2-m^2+\mathrm{i}\epsilon}. \tag{2.51}$$

The diagrams can be used to obtain the symmetry factor, which corresponds to the 4! possible permutations of the legs times 3 different types of contractions. Therefore,

$$S_{\mathbf{p}_3\mathbf{p}_4|\mathbf{p}_1\mathbf{p}_2} = \mathrm{i}\frac{(-\mathrm{i}\lambda)^2}{2}(3\times 4!)\left[\int \frac{\mathrm{d}^4k}{(2\pi)^4}\int \mathrm{d}^4x \int \mathrm{d}^4x' \frac{e^{-\mathrm{i}(x-x')\cdot k}e^{\mathrm{i}(p_3+p_4)\cdot x}e^{-\mathrm{i}(p_1+p_2)\cdot x'}}{k^2-m^2+\mathrm{i}\epsilon} + \ldots\right], \tag{2.52}$$

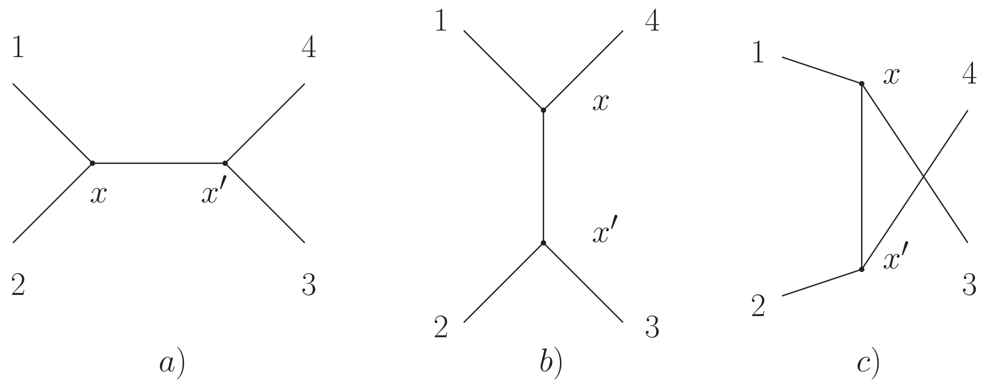

Figure 2.2: Contributing 4-point diagrams for the process $1+2 \to 3+4$ in position space. The term in Eq.(2.50) is represented by diagram a).

where the dots indicate the remaining diagrams—see b) and c) in Fig.2.2. After some algebra the above expression simplifies to

$$S_{\mathbf{p}_3\mathbf{p}_4|\mathbf{p}_1\mathbf{p}_2} = -\,36\mathrm{i}\lambda^2(2\pi)^4\delta^4(p_1+p_2-p_3-p_4) \times \left[\frac{1}{(p_1+p_2)^2-m^2+\mathrm{i}\epsilon}+\frac{1}{(p_1-p_3)^2-m^2+\mathrm{i}\epsilon}+\frac{1}{(p_1-p_4)^2-m^2+\mathrm{i}\epsilon}\right], \tag{2.53}$$

where we see again the convenience of introducing the factor 1/3! at the beginning. Ignoring this factor we see that the amplitude (2.23) for this process reads

$$\mathcal{A}_{\mathbf{p}_3\mathbf{p}_4|\mathbf{p}_1\mathbf{p}_2} = -\,\mathrm{i}\lambda^2\left[\frac{1}{(p_1+p_2)^2-m^2+\mathrm{i}\epsilon}+\frac{1}{(p_1-p_3)^2-m^2+\mathrm{i}\epsilon}+\frac{1}{(p_1-p_4)^2-m^2+\mathrm{i}\epsilon}\right]. \tag{2.54}$$

We proceed with the discussion of the general rules for constructing the S-matrix using Feynman rules. These rules are derived from the Hamiltonian interaction.

2.2.2 Feynman rules

In this Section we summarize the rules for outgoing particles in momentum space. The reason of considering only outgoing particles is that we assume crossing symmetry, i.e., we consider the *process*

$$0 \to \beta_1+\beta_2+\cdots+\bar{\alpha}_1+\bar{\alpha}_2+\dots, \tag{2.55}$$

which amounts to make the transformation $(-p_i) \to p_i$ for the momenta of the incoming particles. This is a formal procedure that extends the domain of the analytic function which describes the amplitudes. It exchanges an incoming particle P to its outgoing antiparticle $\bar{P}$. The S-matrix always contains a global four-momentum conservation imposed by the cluster decomposition principle (See Eq.(2.23)). In addition, momentum conservation is imposed at each vertex in a Feynman diagram. Therefore, for a process involving n particles, the following rules in momentum space tells us how to construct the amplitude:

$$\mathcal{A}_{\bar{\alpha}\beta|0} \equiv \mathcal{A}(p_1, p_2, \ldots, p_n), \tag{2.56}$$

where we only specify the dependence on the momenta p_i, $i = 1, \ldots, n$, but in general it depends on the quantities $u_l^*(\mathbf{p}_i, \sigma_i, k)$ and $v_l(\mathbf{p}_i', \sigma_i', k)$ as well. Consider a general Hamiltonian interaction

$$\mathcal{H}_I = -\mathcal{L}_I = \sum_i g_i \mathcal{H}_{Ii}(\phi_l, \phi_l^\dagger), \tag{2.57}$$

which may depend on the adjoint of the fields. The Feynman rules in momentum space for outgoing particles contain factors:

(i) For each outgoing particle of type k: $u_l^*(\mathbf{p}', \sigma', k)$.

(ii) For each outgoing anti-particle of type k: $v_l(\mathbf{p}', \sigma', k)$.

(iii) For each vertex of type i:

$$V^{\mu_1 \cdots \mu_k} = -\mathrm{i} g_i f^{\mu_1 \cdots \mu_k}(\mathrm{i}p). \tag{2.58}$$

The function $f^{\mu_1 \cdots \mu_k}(\mathrm{i}p)$ accounts for the possibility of derivatives in the interaction and in general it will be a matrix with additional Lorentz indices[12].

(iv) For each internal line with momentum k:

$$\mathrm{i}\frac{P_{lm}(k)}{k^2 - m^2 + \mathrm{i}\epsilon}. \tag{2.59}$$

The polynomial P_{lm} depends on the representation of the inhomogeneous Lorentz group, e.g., for scalars $P_{lm} = 1$. It may contain additional group factors in the case of gauge theories.

[12] In the next section we consider several cases.

(v) For each loop an integration:

$$\int \frac{d^4 l}{(2\pi)^4}. \tag{2.60}$$

(vi) Additional combinatoric factors coming from the symmetry under relabelings of the vertices and a factor (-1) for a closed fermion loop.

In this list, the only item that is not completely specified is the vertex, which depends on the interaction polynomial (2.57).

2.2.3 Other methods to derive the Feynman rules

The canonical approach to the S-matrix assumes that we have a classical field theory which satisfies a definite set of field equations, e.g., Maxwell equations, Klein-Gordon equations, or Dirac equations. We can then construct a Lagrangian density which leads to these equations and use it to obtain the Hamiltonian of the theory and follow the procedure of Section 2.2.2. Alternatively, we can postulate a Lagrangian density to obtain a new theory following the same steps. The Hamiltonian is then quantized by replacing the fields by operators and imposing canonical commutation relations. In this formalism the S-matrix is obtained in terms of Green's functions of the interacting theory through the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula [21]. The Green's functions are the vacuum expectation values of the time-ordered products of the fields in the non-interacting theory. The Feynman rules are then obtained by manipulating the time ordered products using the Wick's theorem.

Equivalently, we can obtain the Green's functions by using the Feynman path integral approach. In this approach, the Green's functions can be computed as functionals of classical fields. With these functions it is possible to obtain an arbitrary Green function from a generating functional, which gives the connected contributions of the S-matrix.

A standard source for the canonical formalism is C. Itzykson et'al [22]. Modern treatments of canonical quantization, which include the spinor helicity formalism are Refs. [20, 23].

2.3 From scalar particles to gravitons

Let us give the interactions and Feynman rules for several theories which are relevant for this work. In each case we give the classical Lagrangian density, then identify the Hamiltonian interactions and the interaction vertices in order to obtain the Feynman rules.

2.3.1 Spin 0: Bi-adjoint scalar

In this Section, we introduce a rather sophisticated version of the cubic interaction (2.34). Let $\phi^{aa'}$ a massless scalar field transforming under the group $U(N)\times U(N)$. This field is known as the bi-adjoint scalar [24]. It is defined by the interacting Lagrangian density

$$\mathcal{L}=\frac{1}{2}\partial^{\mu}\phi^{ab}\partial_{\mu}\phi^{ab}-\frac{1}{3!}\lambda f^{a_1a_2a_3}f^{b_1b_2b_3}\phi^{a_1b_1}\phi^{a_2b_2}\phi^{a_3b_3}, \tag{2.61}$$

where $\phi^{aa'}$ can be thought as the coefficients of the tensor valued field $\phi\equiv\phi^{aa'}T^a\otimes T^{a'}$. In general the groups do not have to be the same, but here we concentrate in this case. The fields transform under $U(N)\times U(N)$ as

$$\delta\phi^{ab}=\epsilon^{a_1}f^{a_1a_2a}\phi^{a_2b}, \tag{2.62}$$

and similarly for the second $U(N)$ index. These fields satisfy the equations of motion

$$\partial^2\phi^{ab}+\frac{\lambda}{2}f^{aa_2a_3}f^{bb_2b_3}\phi^{a_2a_3}\phi^{b_2b_3}=0. \tag{2.63}$$

From the Lagrangian, we can read off the interaction potential

$$\mathcal{H}_I=\lambda\frac{1}{3!}f^{a_1a_2a_3}f^{b_1b_2b_3}\phi^{a_1b_1}\phi^{a_2b_2}\phi^{a_3b_3}, \tag{2.64}$$

which after integration by parts yields the kinetic term

$$-\frac{1}{2}\phi^{aa_1}\left(\delta^{a_1b}\delta^{aa_2}\partial^2\right)\phi^{a_2b}. \tag{2.65}$$

Hence the vertex and the polynomial P_{lm} for the Feynman rules (See Eqs.(2.58)-(2.59)) read

$$V=-\mathrm{i}\lambda f^{a_1a_2a_3}f^{b_1b_2b_3} \tag{2.66}$$
$$P=\delta^{a_1b_1}\delta^{a_2b_2} \tag{2.67}$$

In Chapter 3, we shall compute some amplitudes of this theory. This theory is an ingredient for the computation of gravity amplitudes, as we shall study in Chapter 5.

2.3.2 Spin 1 and 1/2: Gauge Theories

The $SU(N)$ Yang-Mills Lagrangian density reads

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}), \tag{2.68}$$

where the field strength $\mathsf{F}_{\mu\nu}$ is a $N \times N$ matrix field defined by

$$\mathsf{F}_{\mu\nu} \equiv \frac{\mathrm{i}\sqrt{2}}{g}\left[D_\mu, D_\nu\right] = \partial_\mu \mathsf{A}_\nu - \partial_\nu \mathsf{A}_\mu - \mathrm{i}\frac{g}{\sqrt{2}}\left[\mathsf{A}_\mu, \mathsf{A}_\nu\right]. \tag{2.69}$$

The covariant derivative is defined by

$$D_\mu = \partial_\mu - \mathrm{i}\frac{g}{\sqrt{2}}\mathsf{A}_\mu(x), \tag{2.70}$$

where the is an implicit unit matrix multiplying the partial derivative. The factors of $\sqrt{2}$ appear due to our normalizations for the generators of the Lie group

$$[T^a, T^b] = \mathrm{i}\sqrt{2}f^{abc}T^c, \qquad \mathrm{Tr}(T^aT^b) = \delta^{ab}. \tag{2.71}$$

In order to derive the Feynman rules, we have to add a gauge fixing term and a ghost term in the Lagrangian. The latter is only needed for diagrams involving loops. We will be mainly interested in tree level amplitudes, hence we will ignore the ghost term. The Feynman rules depend on the choice of a gauge fixing term. A typical procedure is to decompose the traceless hermitian matrix fields A_μ and $\mathsf{F}_{\mu\nu}$ as

$$\mathsf{A}_\mu = A^a_\mu T^a, \qquad \mathsf{F}_{\mu\nu} = F^a_{\mu\nu}T^a, \tag{2.72}$$

and add to the Lagrangian a the R_ξ gauge fixing term

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2}\xi^{-1}\partial^\mu A^a_\mu \partial^\nu A^a_\nu. \tag{2.73}$$

In this gauge, the Lagrangian plus gauge fixing reads

$$\mathcal{L} + \mathcal{L}_{\rm gf} = -\frac{1}{2}\left(\partial_\mu A^a_\nu \partial^\mu A^{\nu a} - \partial_\mu A^a_\nu \partial^\nu A^{\mu a} + \xi^{-1}\partial^\mu A^a_\mu \partial^\nu A^a_\nu\right)$$
$$- g f^{abc}\partial_\mu A^a_\nu A^{\mu b} A^{\nu c} - \frac{1}{4} g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e}, \tag{2.74}$$

which gives the interaction Hamiltonian:

$$-\mathcal{L}_I = g f^{abc}\partial_\mu A^a_\nu A^{\mu b} A^{\nu c} + \frac{1}{4} g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e}, \tag{2.75}$$

and a quadratic term

$$\frac{1}{2} A^{\mu a}\left(\delta^{ab}\eta_{\mu\nu}\partial^2 - \delta^{ab}\partial_\mu\partial_\nu + \delta^{ab}\xi^{-1}\partial_\mu\partial_\nu\right) A^{\nu b}. \tag{2.76}$$

The quadratic terms yield a polynomial P_{lm} for the propagator (2.59)

$$P_{\mu a;\nu b} = \delta_{ab}(-\eta_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{k^2}). \tag{2.77}$$

In this case we have two types of interactions: a three-gluon vertex and a four-gluon vertex. The three gluon vertex reads

$$V^{a,b,c}_{\mu\nu\rho}(p,q,r) = - g f^{abc}\left[\eta_{\mu\nu}(p-q)_\rho + \eta_{\nu\rho}(q-r)_\mu + \eta_{\rho\mu}(r-p)_\nu\right]. \tag{2.78}$$

We can similarly obtain the four-gluon vertex, but we shall not need this vertex for our calculations. These rules can be straightforward used to compute tree-level amplitudes but they contain many terms even for the simplest processes. For this reason, we will turn our attention to color-ordered Feynman rules, which reduce the number of terms to compute.

Let us consider the Lagrangian density (2.68) and let us work with the matrix fields without the decomposition (2.72). A convenient gauge choice which simplifies tree-level computations is the *Gervais-Neveu gauge* [25]. In this gauge we add to the Lagrangian[13] the term

$$\mathcal{L}_{\rm gf} = -\frac{1}{2}\mathrm{Tr}\left(\partial^\mu \mathsf{A}_\mu - \mathrm{i}\frac{g}{\sqrt{2}}\mathsf{A}^\mu \mathsf{A}_\mu\right)^2. \tag{2.79}$$

[13] See e.g., Chapter VI.B of [26]

Thus, the Lagrangian in Eq.(2.68) and the gauge fixing term can be written explicitly in terms of the matrix field A_μ as

$$\mathcal{L} + \mathcal{L}_{\mathrm{gf}} = \frac{1}{2}\mathrm{Tr}\left(\mathsf{A}^\mu\partial^2\mathsf{A}_\mu + \mathrm{i}\sqrt{2}g\left(\partial_\mu\mathsf{A}_\nu\mathsf{A}^\mu\mathsf{A}^\nu - \partial_\mu\mathsf{A}_\nu\mathsf{A}^\nu\mathsf{A}^\mu + \partial^\mu\mathsf{A}_\mu\mathsf{A}^2\right) + \frac{g^2}{2}\mathsf{A}_\mu\mathsf{A}_\nu\mathsf{A}^\mu\mathsf{A}^\nu\right), \tag{2.80}$$

where we have used integration by parts and properties of the trace. Integrating by parts, we can rewrite this Lagrangian as

$$\mathcal{L} + \mathcal{L}_{\mathrm{gf}} = \mathrm{Tr}\left(\frac{1}{2}\mathsf{A}_\mu g^{\mu\nu}\partial^2\mathsf{A}_\nu - \mathrm{i}\sqrt{2}g\partial_\mu\mathsf{A}_\nu\mathsf{A}^\nu\mathsf{A}^\mu + \frac{g^2}{4}\mathsf{A}_\mu\mathsf{A}_\nu\mathsf{A}^\mu\mathsf{A}^\nu\right), \tag{2.81}$$

which can be used to derive the color-ordered Feynman rules for $N \times N$ matrix fields[14]. From (2.81), the three-gluon-vertex and the four-gluon-vertex read

$$V_{\mu,\nu,\rho}(p,q,r) = -\,\mathrm{i}\sqrt{2}[\eta_{\mu\nu}(p)_\rho + \eta_{\nu\rho}(q)_\mu + \eta_{\rho\mu}(r)_\nu], \tag{2.82}$$

$$V_{\mu,\nu,\rho,\sigma} = \mathrm{i}\eta_{\mu\nu}\eta_{\rho\sigma}, \tag{2.83}$$

respectively. The polynomial for the propagator (2.59) reads

$$P_{\mu\nu} = -\,\eta_{\mu\nu}. \tag{2.84}$$

Notice that we have absorbed the dependence of the coupling in these rules, which will reappear as a general factor in the amplitude.

Those rules yield color-ordered primitive amplitudes, i.e., amplitudes with a fixed order in the legs. This can be seen as follows. Consider the 3-gluon interaction in Eq.(2.81)

$$\mathcal{L}_{ggg} = -\mathrm{i}\sqrt{2}g\mathrm{Tr}(T^aT^bT^c)\partial_\mu A^a_\nu A^{\nu b}A^{\mu c}, \tag{2.85}$$

where we have used (2.72). In combination with the Fierz identity

$$(T^a)_i^j(T^b)_l^m = \delta_i^l\delta_m^j - \frac{1}{N}\delta_i^j\delta_l^m, \tag{2.86}$$

the amplitude can be written as

[14]See Chapters 79-80 of Ref. [23] and Ref. [27]

$$\mathcal{A}(p_1, p_2, \ldots, p_n) = g^{n-2} \sum_{\substack{\text{noncyclic} \\ \text{permutations}}} \text{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n}) A(p_1, p_2, \ldots, p_n), \tag{2.87}$$

where the *primitive amplitudes* $A(p_1, p_2, \ldots, p_n)$ are cyclic ordered, gauge invariant objects. The sum runs over noncyclic permutations of $\{1, 2, \ldots, n\}$. Notice the overall coupling factor.

We can also include quarks in the fundamental representation of $SU(N)$ by adding to the Lagrangian

$$\mathcal{L}_M = -\mathrm{i}\bar{\Psi}(\not{D} - m)\Psi, \tag{2.88}$$

which can be used to derive the color-ordered rule for the fermion-anti-fermion-gluon vertex and the corresponding fermion polynomial

$$V_\mu = \mathrm{i}g\frac{\gamma_\mu}{\sqrt{2}}, \tag{2.89}$$

$$P = \not{p} + m. \tag{2.90}$$

The inclusion of quarks modify the trace structure and the decomposition (2.87), however the general structure is preserved. We postpone the details about this decomposition for Section 2.4.3.

2.3.3 Spin 2: Gravity using Feynman diagrams

The procedure we have followed requires a supporting classical Lagrangian density to derive the Feynman rules. We have access to such a Lagrangian density in the case of gravitation—the Einstein-Hilbert Lagrangian. However, there are well-known problems regarding the renormalization of the theory and the complexity of the Feynman rules that we obtain in doing so. The conservative point of view is to consider the theory in the language of effective field theory, i.e., a theory which is valid at low energies in comparison with the Planck scale [28]. In this way we do not worry about the issues of renormalization and we treat the theory using ordinary methods of quantization [29].

The Feynman rules for gravity are known at least since the sixties thanks to the works of DeWitt and Feynman [30–33]. As in any QFT, we make a perturbative expansion of the Einstein-Hilbert action[15] in powers of the coupling constant $\kappa^2 = 32\pi G_N$ (in the Gauss unit system). The Einstein-Hilbert action is given by

[15] Perturbative gravity is not renormalizable since we have a coupling $\kappa^2 = 32\pi G_N$ that carries dimensions of length.

$$S_{\text{EH}} = \frac{2}{\kappa^2} \int \mathrm{d}^4x \sqrt{-g} R \tag{2.91}$$

with the metric tensor $g^{\mu\nu}$, and $g = \det g_{\mu\nu}$. R is the Ricci scalar $g^{\mu\nu} R_{\mu\nu}$, with

$$R_{\mu\nu} = \partial_\nu \Gamma^\lambda{}_{\mu\lambda} - \partial_\lambda \Gamma^\lambda{}_{\mu\nu} + \Gamma^\sigma{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\sigma{}_{\mu\nu}\Gamma^\lambda{}_{\lambda\sigma}, \tag{2.92}$$

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}). \tag{2.93}$$

Performing the expansion around flat space $\eta^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ such that

$$g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}, \tag{2.94}$$

leads to the Feynman rules of gravity. In the de Donder gauge—also known as harmonic gauge, where $\partial^\beta h_{\alpha\beta} = 1/2\partial_\alpha h$—the propagator reads

$$D_{\mu\nu;\rho\sigma}(q) = \frac{\mathrm{i}}{2} \frac{1}{q^2 + \epsilon} \left(-\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma} \right). \tag{2.95}$$

Now, the three-graviton vertex in DeWitt notation reads [30–32]

$$\tau_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1, k_2, k_3) = \text{sym}\left(\frac{1}{2} P_3(k_1 \cdot k_2 \eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + \text{many terms} \right), \tag{2.96}$$

which in total has around 100 terms. If we want to compute the scattering amplitude of, say 4 gravitons, we need also to use the four-point vertex which we will not show here. These expressions can be found in a simplified notation in Ref. [34]. We can add interaction with matter by adding the minimal coupling

$$\mathcal{L}_{\text{matter}} = -\frac{\kappa}{2} h_{\mu\nu} T^{\mu\nu}, \tag{2.97}$$

where

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g} \mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{2.98}$$

For example, we have for scalar particles

$$\sqrt{-g}\mathcal{L}_{\text{matter}} = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}D_\mu\phi D_\nu\phi - \frac{1}{2}m^2\phi^2\right). \tag{2.99}$$

After expansion in powers of κ, this Lagrangian yields the interaction vertex [35]

$$V^{\mu\nu}(p_1, p_2) = \mathrm{i}\frac{\kappa}{2}\left[-p_2^\mu p_1^\nu - p_2^\nu p_1^\mu + \eta^{\mu\nu}(p_2 \cdot p_1 + m^2)\right]. \tag{2.100}$$

The straightforward quantization is conceptually simple but very difficult to use in practice. Tractable problems appear where matter is involved and a few particles are involved, e.g., in Chapter 5 we will consider a tractable four-point example. More simplification occurs when we specify the helicities, because many terms vanish. As we will see in Section 2.4, there is an alternative method which involves a relation between gauge and gravity at the level of amplitudes. With this method, we avoid computing amplitudes from the Feynman rules of gravity but instead we obtain them from gauge theory.

2.4 Modern techniques

After revisiting the standard approach to the S-matrix, we are ready to introduce the relevant improvement to this approach. Most of this material is available in the literature—except some recent material, e.g., the color-kinematics duality for QCD amplitudes. Some parts of this review are based on the recent book by Elvang and Huang [36, 37] and the recent review [38]. The main differences from those references are the conventions for the generators of the group and the metric[16]. Other helpful reviews include [39, 40] and references therein.

2.4.1 Massless amplitudes in four dimensions

Amplitudes in four space-time dimensions for massless particles can be better described in terms of spinor variables. These are a set of kinematic variables which exploit the fact that the complexified Lorentz group $\mathrm{SO}(3, 1, \mathbb{C})$ is locally isomorphic to two copies of the complex special linear group $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$. This motivates the realization of the four momentum vector p_μ as a bi-spinor $p_{a\dot{a}}$, where $p_{a\dot{a}}$ labels the $(\frac{1}{2}, \frac{1}{2})$ *representation* of the Lorentz group[17]. The map from p^μ to $p_{a\dot{a}}$ can be made using the chiral part of the Dirac matrices, i.e.,

[16] We use the mostly plus signature of the Minkowsky metric and include a factor of $\sqrt{2}$ in the normalization of the generators.

[17] This is a finite dimensional spinor representation, not to be confused with the infinite dimensional representation for spinor fields.

$$p_{a\dot{a}} \equiv p_\mu(\sigma^\mu)_{a\dot{b}} = \begin{pmatrix} p^0 - p^3 & -p^1 + \mathrm{i}p^2 \\ -p^1 - \mathrm{i}p^2 & p^0 + p^3 \end{pmatrix}, \tag{2.101}$$

where a and $\dot{a}$ label the spinor index for each chirality. For massless particles $p^2 = 0$, which implies that the momenta can be written as the product

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}, \tag{2.102}$$

where λ_a and $\tilde{\lambda}_{\dot{a}}$ are two-vectors. We will use Dirac notation for the two-vectors, i.e.,

$$\lambda_a \to |p]_a \qquad \tilde{\lambda}_{\dot{a}} \to \langle p|_{\dot{a}}\,. \tag{2.103}$$

Using spinors, the relevant quantities and properties for computing n-point amplitudes can be summarized as follows:

$$\text{Antisymmetry}: \quad \langle ij\rangle = -\langle ji\rangle\,, \quad [ij] = -[ji], \tag{2.104}$$

$$\text{Momentum conservation}: \quad \sum_{i=1}^{n} [ji]\,\langle ik\rangle = 0, \tag{2.105}$$

$$\text{Lorentz invariants}: \quad s_{ij} = (p_i + p_j)^2 = \langle ij\rangle\,[ji], \tag{2.106}$$

$$\text{Polarization Vectors}: \quad \epsilon^+_\mu(p,q) = \frac{[p|\gamma_\mu\,|q\rangle}{\sqrt{2}\,\langle qp\rangle}, \qquad \epsilon^-_\mu(p,q) = -\frac{\langle p|\,\gamma_\mu|q]}{\sqrt{2}[qp]}, \tag{2.107}$$

$$\text{Schouten identity}: \quad \langle ri\rangle\,\langle jk\rangle + \langle rj\rangle\,\langle ki\rangle + \langle rk\rangle\,\langle ij\rangle = 0. \tag{2.108}$$

Other relevant properties are summarized in Appendix A.1. These can be used to compute amplitudes with definite helicities in terms of spinors products[18].

It is interesting to note that the spinor representations are unique up to scaling. The reason is that the momentum is invariant under the rescalings

$$|p\rangle \to t\,|p\rangle\,, \qquad |p] \to t^{-1}|p], \tag{2.109}$$

which is called *little group scaling*[19]. This property can be used to establish a general feature of amplitudes of massless amplitudes

[18] For a full treatment of the spinor-helicity formalism see Ref. [36]

[19] The litte group is the subgroup of the homogeneous Lorentz group which leaves invariant a standard momentum of an on-shell particle. See p.66 of Ref. [2]

$$A\left(\{|1\rangle,|1],h_1\},\ldots,\{t_i\,|i\rangle,t_i^{-1}|i],h_i\},\ldots,\{|n\rangle,|n],h_n\}\right)= \\ t_i^{-2h_i}A\left(\{|1\rangle,|1],h_1\},\ldots,\{|i\rangle,|i],h_i\},\ldots,\{|n\rangle,|n],h_n\}\right). \tag{2.110}$$

Together with general kinematic properties of the 3-particle momenta, this can be used to determine the full structure of the 3-point amplitudes [41]. In combination with a recursive procedure, this can be used to obtain the n-point amplitude with fixed helicities and hence all amplitudes. Theories with this property are called *constructible*. The kinematics tell us that a 3-point amplitude can only depend either on angle or square spinor products of the external momenta[20]. Suppose the amplitude only depends on angle spinors, i.e.,

$$A_3(1^{h_1}2^{h_2}3^{h_3})=\mathrm{i}\,c\,\langle 12\rangle^{x_{12}}\,\langle 13\rangle^{x_{13}}\,\langle 23\rangle^{x_{23}}\,, \tag{2.111}$$

where c is constant to be determined and x_{ij} have to be determined using the litte group scaling. Eq.(2.110) gives the system of equations

$$-2h_1=x_{12}+x_{13},\quad -2h_2=x_{12}+x_{23},\quad -2h_3=x_{13}+x_{23}, \tag{2.112}$$

therefore

$$A_3(1^{h_1}2^{h_2}3^{h_3})=\mathrm{i}\,c\,\langle 12\rangle^{h_3-h_1-h_2}\,\langle 13\rangle^{h_2-h_1-h_3}\,\langle 23\rangle^{h_1-h_2-h_3}\,. \tag{2.113}$$

For example, the three-gluon amplitude where $h_1=h_2=-1$ and $h_3=+1$ yields

$$A_3(1^-2^-3^+)=\mathrm{i}\frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 31\rangle}, \tag{2.114}$$

where we have multiplied and divided by $\langle 12\rangle$ and used dimensional analysis to set the constant equal to the one.

2.4.2 BCFW recursion relations

The constructibility of amplitudes requires a recursive method to build up higher point amplitudes from lower ones. Factorization properties of amplitudes in combination with analytic properties of amplitudes were used by Britto, Cachazo, Feng, and Witten [42, 43]

[20] To determine which type of spinor products we should use dimensional analysis.

to construct a recursion based on the complexification of a subset of the involved momenta. Consider the complexified version of the amplitude $A_n(1,2,\ldots,n)$, which we denoted by $A_n(z)$. At tree-level, the singularities of the amplitudes are poles, hence by considering the quantity $A(z)/z$ we have

$$A_n(0) + \sum_{\text{Poles of } A_n(z)} \text{Res}\left(\frac{A_n(z)}{z}\right) + A_n(\infty) = 0, \tag{2.115}$$

which follows from the Cauchy's theorem. The last term corresponds to the residue of the pole at infinity. If $A(z)$ vanishes in the limit $z \to \infty$, then

$$A_n \equiv A_n(0) = - \sum_{\text{Poles of } A_n(z)} \text{Res}\left(\frac{A_n(z)}{z}\right). \tag{2.116}$$

The point is that the poles of $A_n(z)$ occur when the propagators are on-shell, thus the amplitude factorizes into two on-shell parts

$$A_n = -\sum_I A_L(z)\frac{\mathrm{i}}{P_I^2}A_R(z), \tag{2.117}$$

where we sum over all possible factorization channels I as shown in Fig.2.3. This gives a gauge invariant formula to recursively construct higher point amplitudes.

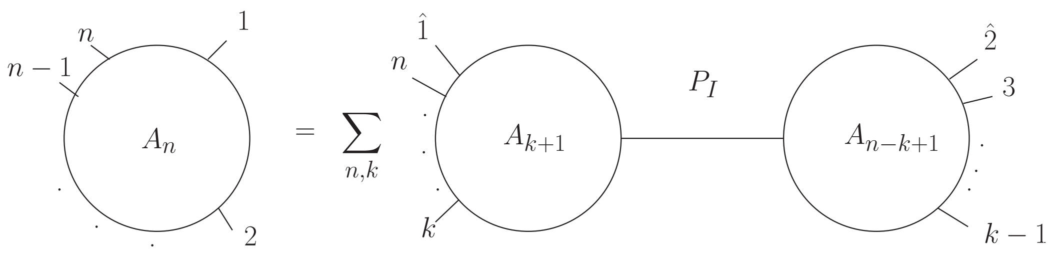

Figure 2.3: The BCFW recursion. The complexification of legs 1 and 2 have been indicated with a hat. The sum over factorization channels is implemented as a sum over amplitudes containing k legs times the amplitude containing the remaining points. This sum also runs over the possible helicity configurations indicated by h. Note that the "propagator" P_I is on-shell.

Since we have access to the 3-point amplitudes in various theories, we can construct n-point amplitudes recursively whenever the amplitude satisfies

$$A_n(z) \to 0, \qquad \text{for} \qquad z \to \infty. \tag{2.118}$$

This behavior can be estimated by inspecting the contributing Feynman diagrams [36]. In conclusion, we have a method to determine constructible amplitudes, which is particularly useful in combination with the spinor-helicity formalism. In this case, the complexification of momenta is achieved through shifts on the spinor variables which preserve momentum conservation, while keeping shifted particles on-shell. A BCFW-shift of the legs i, j—called a $[i, j\rangle$-shift—reads

$$|\hat{i}] = |i] + z|j], \quad |\hat{j}] = |j], \quad |\hat{i}\rangle = |i\rangle\,, \quad |\hat{j}\rangle = |j\rangle - z\,|i\rangle\,. \tag{2.119}$$

Let us illustrate the concept of BCFW-constructibility for the case of the 4-point amplitude in scalar QED[21]. Let us consider the amplitude for two scalar and two gauge bosons

$$A_4(\phi\phi^*\gamma^+\gamma^-) \equiv A_4(123^+4^-), \tag{2.120}$$

where the gauge bosons have different helicities. The first ingredients are the 3-point amplitudes, which for scalar QED corresponds to $h_1 = h_2 = 0$, and $h_3 = \pm 1$ in Eq. (2.113), hence the 3-point amplitudes for the helicities $\pm$ of the photons read

$$A_3(\phi, \phi^*, \gamma^-) \equiv A_3(123^-) = \mathrm{i}\frac{\langle 13\rangle\,\langle 23\rangle}{\langle 12\rangle}, \tag{2.121}$$

$$A_3(\phi, \phi^*, \gamma^+) \equiv A_3(123^+) = \mathrm{i}\frac{[13][23]}{[12]}, \tag{2.122}$$

where the second line is obtained by using complex conjugation. These results can be easily verified using Feynman diagrams. Now, let us consider a $[4, 3\rangle$ BCFW-shift:

$$|\hat{4}] = |4] + z|3], \quad |\hat{3}] = |3], \quad |\hat{4}\rangle = |4\rangle\,, \quad |\hat{3}\rangle = |3\rangle - z\,|4\rangle\,. \tag{2.123}$$

We have two factorization channels as shown in Fig.2.4, which we label as $\mathcal{D}_1$ and $\mathcal{D}_2$.

[21]Scalar QED is defined by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi|^2 - \frac{1}{4}\lambda|\phi|^4,$$

where $D_\mu\phi = \partial_\mu\phi + \mathrm{i}eA_\mu\phi$.

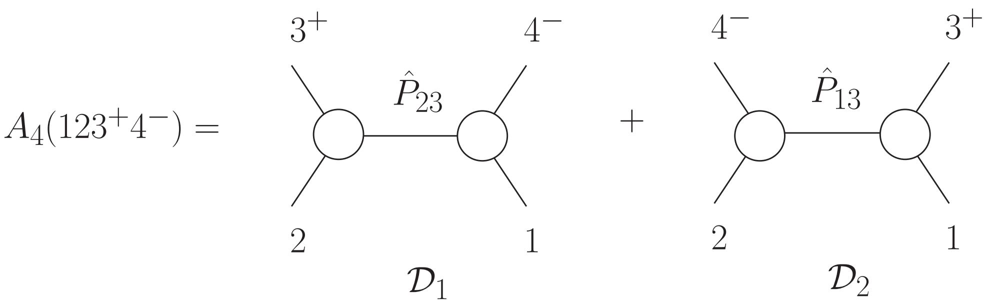

Figure 2.4: Scalar QED 4-point amplitude as a sum over factorizations channels. The blobs indicate on-shell amplitudes.

Then, the amplitude is given by the sum of the factorization channels

$$\begin{aligned} A_4(123^+4^-) &\equiv \mathcal{D}_1 + \mathcal{D}_2 \\ &= -\hat{A}_3(2\hat{P}_{23}\hat{3}^+)\frac{\mathrm{i}}{\hat{P}_{23}^2}\hat{A}_3(-\hat{P}_{23}1\hat{4}^-) - \hat{A}_3(1\hat{P}_{13}\hat{3}^+)\frac{\mathrm{i}}{\hat{P}_{13}^2}\hat{A}_3(-\hat{P}_{13}2\hat{4}^-), \end{aligned} \tag{2.124}$$

which using Eqs.(2.121), (2.122) yields

$$\mathcal{D}_1 = -\mathrm{i}\frac{\langle 4\hat{P}_{23}\rangle \langle 14\rangle [\hat{P}_{23}3]}{\langle 1\hat{P}_{23}\rangle [\hat{P}_{23}2] \langle 23\rangle}. \tag{2.125}$$

Using $\langle i|\, P|j] = \langle ip\rangle\, [pj]$ and properties of the spinor products summarized in the Appendix A, we obtain

$$\mathcal{D}_1 = \mathrm{i}\frac{\langle 14\rangle \langle 42\rangle}{\langle 23\rangle \langle 1\hat{3}\rangle}. \tag{2.126}$$

The next step is to find the location of the pole. Since $0 = \hat{P}_{23}^2 = \langle 23\rangle - z_{23}\langle 24\rangle$, then $z_{23} = \langle 23\rangle / \langle 24\rangle$. Therefore,

$$\langle 1\hat{3}\rangle = \frac{\langle 24\rangle \langle 13\rangle - \langle 23\rangle \langle 14\rangle}{\langle 24\rangle}, \tag{2.127}$$

which leads to

$$\mathcal{D}_1 = \mathrm{i}\frac{\langle 14\rangle \langle 24\rangle^2}{\langle 23\rangle \langle 21\rangle \langle 34\rangle}. \tag{2.128}$$

For the second diagram, similar steps lead to

$$\mathcal{D}_2 = \mathrm{i}\frac{\langle 24\rangle \langle 14\rangle^2}{\langle 13\rangle \langle 21\rangle \langle 43\rangle}. \tag{2.129}$$

Using the properties (2.104)-(2.108), we obtain

$$A_4(123^+4^-) = \mathrm{i}\frac{\langle 14\rangle \langle 24\rangle}{\langle 21\rangle \langle 43\rangle}\left(-\frac{\langle 24\rangle}{\langle 23\rangle} + \frac{\langle 14\rangle}{\langle 13\rangle}\right). \tag{2.130}$$

Finally, with the aid of the Schouten identity (2.108) and massaging (2.130), we obtain

$$A_4(123^+4^-) = -\mathrm{i}\frac{\langle 14\rangle \langle 24\rangle}{\langle 23\rangle \langle 13\rangle}. \tag{2.131}$$

In this example, we have shown that using only the little group scaling and the BCFW recursion relations, we can recover the 4-point amplitude $A_4(123^+4^-)$ without using Feynman diagrams. However, this process will fail for the general amplitude in scalar QED due to the existence of a boundary term—for example it fails for the 4-point amplitude $A_4(\phi, \phi^*, \phi, \phi^*)$, where the BCFW recursion will give an amplitude for a specific value of the scalar coupling [36].

Gravity and pure Yang-Mills are examples of BCFW constructible theories due to the good large z behavior. BCFW recursions work very well for theories which fall off as $1/z$ under a BCFW-shift. Sometimes the behavior is $1/z^2$, which leads to relations among amplitudes [44].

Let us end this section by introducing the most famous formula in amplitude theory. The 3-point Parke-Taylor formula, where two gluons i, and j have negative helicity, can be derived from the little group scaling (see Eq.(2.114)). Then, using BCFW recursion relations we can show that the Parke-Taylor formula (MHV amplitude) for n gluon scattering is given by

$$A_n(1^+2^+ \dots i^- \dots j^- \dots n^+) = \mathrm{i}\frac{\langle ij\rangle^4}{\langle 12\rangle \langle 23\rangle \cdots \langle n1\rangle}, \tag{2.132}$$

which up to color factors and coupling and squaring yields Eq.(1.5).

2.4.3 Color decomposition and primitive amplitudes

In our first encounter with gauge theories, we have seen that information about the gauge group can be encoded only in terms containing single traces. This was the result (2.87), which can be obtained by introducing the Gervais-Neveu gauge and use the Fierz identity.

Alternatively, this result may be obtained from color-dependent Feynman rules (2.78) after reorganizing the terms and by using the gauge group identity:

$$f^{abc} = -\frac{\mathrm{i}}{\sqrt{2}}\mathrm{Tr}(T^a T^b T^c - T^c T^b T^a), \tag{2.133}$$

where the generators T^a and the structure constant f^{abc} are normalized according to Eq.(2.71). In this way, the amplitude can be written in terms of the single trace factors

$$\mathrm{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n}), \tag{2.134}$$

which then yields Eq.(2.87).

In QCD we have quark interactions (see Eq.(2.88)). These appear in the color-dependent Feynman rule

$$V_{ij}^{\mu,a}(p,q,r) = \mathrm{i}g\frac{T_{ij}^a}{\sqrt{2}}\gamma^\mu, \tag{2.135}$$

which is the color-dependent version of Eq.(2.89). This interaction contains indices of the fundamental representation of the gauge group. Hence, the color decomposition will have indices from the *fundamental representation* of $SU(N)$. For example, an amplitude containing one quark pair contains single trace terms of the form

$$(T^{a_1} T^{a_2} \cdots T^{a_n})_{ij}. \tag{2.136}$$

Our aim is to write the amplitude only in terms of single traces of the generators of the gauge group. We can achieve this by the methods discussed above. The bottom line is that the color structure of the amplitude can be separated by writing it as sum over cyclic ordered gauge invariant *primitive amplitudes*[22]

$$A_n(p_1, p_2, \ldots, p_n) \equiv A_n(12\ldots n). \tag{2.137}$$

Here we have introduced the usual notation for primitive amplitudes by labeling only the external particles and the fixed order. Primitive amplitudes are gauge invariant objects with a fixed order indicated by the string $(12\cdots n)$. In this context, the ordering $(12\cdots n)$

[22] Primitive amplitudes are also known as partial amplitudes for pure Yang-Mills where are equivalent concepts. They differ when fermions are added or when we increase the number of loops.

is different from, say, $(134\ldots n)$. For pure Yang-Mills at tree-level, the separation into primitive amplitudes and color factors reads

$$\mathcal{A}_n = g^{n-2} \sum_{\sigma\in S_n/\mathbb{Z}_n} \mathrm{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})A_n(\sigma(1)\ldots\sigma(n)), \tag{2.138}$$

where the sum runs over noncyclic permutations of $\{1,2,\ldots,n\}$, or equivalently over the elements of the set $S_n/\mathbb{Z}_n$. The procedure of writing the amplitudes separating the color information from the kinematic information was found by Berends-Giele [8] using QFT methods[23] and by Mangano-Parke [9] inspired on string theory gauge amplitudes—color factors correspond to the Chan-Paton factors [4]. The color decomposition is independent of the level in perturbation theory—tree or loop level. It is always possible to perform the color decomposition procedure at loop-level.[24] Schematically, the color decomposition at L-loops reads

$$\mathcal{A}_n^{L-\mathrm{loop}} = g^{n+2(L-1)} \sum_k C_{n,k}^{L-\mathrm{loop}} P_{n,k}^{L-\mathrm{loop}}, \tag{2.139}$$

where $P_{k,n}$ labels the kth primitive amplitude and $C_{n,k}$ are the color factors. In this thesis we will focus on tree-level amplitudes and consequently we will use Eq. (2.138). In Appendix A.1, we summarize the Feynman rules to compute primitive amplitudes.

2.4.4 Basis of pure Yang-Mills primitive amplitudes

The color decomposition of the amplitude runs over $(n-1)!$ primitive amplitudes, which can be further reduced by invoking relations among the primitives as we will see in the next Section. We frequently say that these amplitudes form a *basis* and that the relations among amplitudes shrink the basis. In order to introduce these relations we introduce some notation. First, let us define the *alphabet* $\mathbb{A}_{\mathrm{gluons}}$ as the particle content of the pure Yang-Mills primitive amplitude and their corresponding labels[25], i.e.,

$$\mathbb{A}_{\mathrm{gluons}} = \{g_1, g_2, \ldots, g_n\} = \{1,\ldots,n\}, \tag{2.140}$$

where we have associated gluons to labels in the set $\{1,\ldots,n\}$. The elements of the alphabet are called *letters* l_i, which we can associate to a given gluon. Notice that the label itself is, of course, not relevant but the relative order of the letters. Ordered sequences of letters are called *words*

[23] The decomposition into gauge invariant and color parts was studied in Ref. [45]

[24] It a highly nontrivial task as shown e.g. in [46, 47]

[25] The full treatment of this technology is discussed in the Appendix C.

$$w = l_1 l_2 \dots l_n. \tag{2.141}$$

We also define the reversed word w^T by

$$w^T = l_n l_{n-1} \dots l_1, \tag{2.142}$$

and the no-word or zero-length word by e. The words in an alphabet form an algebra with the product operation being the shuffle[26]. The shuffle product of two words $w_1 = l_1 l_2 \dots l_k$ and $w_2 = l_{k+1} \dots l_r$ is defined by

$$w_1 \sqcup\!\sqcup\, w_2 = l_1 l_2 \dots l_k \sqcup\!\sqcup\, l_{k+1} \dots l_r = \sum_{\text{shuffles } \sigma} l_{\sigma(1)} l_{\sigma(2)} \dots l_{\sigma(r)}, \tag{2.143}$$

where the sum runs over all permutations σ that preserve the relative order of $l_1, l_2, \dots, l_k$ and $l_{k+1}, \dots, l_r$. Words w in the alphabet $\mathbb{A}_{\text{gluons}}$ will encode the particle content of the amplitude and we will write simply $A_n(w)$ when the focus is on the ordering. In this sense, we can think of the amplitude A_n as linear operator on the vector space of orderings. Let λ_1, λ_2 be two numbers and let w_1, w_2 be two orderings in a basis W_0, then we have[27]

$$A_n(\lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 A_n(w_1) + \lambda_2 A_n(w_2). \tag{2.144}$$

2.4.4.1 Relations among primitive gauge amplitudes

Primitive amplitudes are not independent of each other. Actually the trace structure already tells us to sum over noncyclic permutations, i.e., over $(n-1)!$ primitive amplitudes. The trace structure gives us the *cyclic relations*, i.e.,

$$A_n(12 \dots n) = A_n(2 \dots n1). \tag{2.145}$$

Studying the sum over Feynman diagrams contributing to each primitive amplitude gives us the *reflexion property*[28]

[26] It is a vector space over a number field k with an additional operation $\sqcup\!\sqcup$ called the vector multiplication. The vector operation is commutative and associative, and the unit element is e.

[27] Notice that this depends on the proper definition of a sum of words, which is guaranteed because the set of words form an algebra.

[28] Interestingly, these relations follow immediately from the properties of the Koba-Nielsen amplitudes in string theory(See e.g., [48]).

$$A_n(12\dots n) = (-1)^n A_n(n\, n-1\dots 1). \tag{2.146}$$

If we take one of the particles as a photon—equivalently by setting one of the generators $T^a = 1$—implies that Eq. (2.138) vanishes, thus giving the $U(1)$ *decoupling identify*

$$A_n(12\dots n) + A_n(213\dots n-1n) + A_n(231\dots n-1n)\cdots + A_n(234\dots 1n) = 0. \tag{2.147}$$

The reflexion identity and the $U(1)$ decoupling identity can be understood as special cases of the Kleiss-Kuijf (KK) relations [10]. Let us use words and let use think on the amplitude as a linear operator. Let

$$w_1 = l_{\alpha_1} l_{\alpha_2}\dots l_{\alpha_j}, \qquad w_2 = l_{\beta_1} l_{\beta_2}\dots l_{\beta_{n-2-j}}, \tag{2.148}$$

be two sub-words such that

$$\{l_1\}\cup\{w_1\}\cup\{w_2\}\cup\{l_n\} = \{l_1 l_2\dots l_n\}. \tag{2.149}$$

Then the Kleiss-Kuijf relations read

$$A_n(1w_1 l_n w_2) = (-1)^{n-2-j} A_n(l_1(w_1 \sqcup\!\sqcup\, w_2^T) l_n). \tag{2.150}$$

Then the reflexion property and the $U(1)$ decoupling identity correspond to the cases where the $w_1 = e$ and $w_2 = l_{\beta_1}$, respectively.

Thus far the relations do involve only relations where the coefficients of the amplitudes are unity. In Ref. [11], Bern, Carrasco, and Johansson (BCJ) found that there are relations among primitive amplitudes which involve kinematic coefficients. These relations are called the fundamental BCJ relations

$$\sum_{i=2}^{n-1}\left(\sum_{j=i+1}^{n} 2p_2\cdot p_j\right) A_n(13\dots i\,2\,i+1\dots n-1\,n) = 0. \tag{2.151}$$

They allow us to reduce the number of quantities to compute and consequently shrink the basis. In fact, by using the BCJ relations the number of independent amplitudes to compute becomes $(n-3)!$. As we will see in Section 2.4.5, these relations are implied by a duality between color and kinematics.

2.4.4.2 Basis for pure Yang-Mills amplitudes

In Section 2.4.4.1, we saw that primitive amplitudes in pure Yang-Mills are cyclic invariant, and satisfy KK and BCJ relations. Let us now define the basis of amplitudes characterized by words in the alphabet $\mathbb{A}_{\text{gluons}}$, which results after imposing these relations. The most general set of words for the alphabet (2.140) is given by the set of words containing each letter once, i.e.,

$$W_0 = \{l_1, l_2, \ldots, l_n | l_i \in \mathbb{A}_{\text{gluons}},\ l_i \neq l_j,\ \text{for } i \neq j\}, \tag{2.152}$$

which contains $n!$ elements. Starting with words in W_0 we use the *cyclic relations* to fix one of the legs in w. By fixing the first letter to $l_1 = 1$, we have the basis

$$W_1 = \{l_1, l_2, \ldots, l_n | l_i \in \mathbb{A}_{\text{gluons}}, l_i \neq l_j,\ \text{for } i \neq j, \quad l_1 = 1\}, \tag{2.153}$$

which has $(n-1)!$ elements. KK relations can be used to fix a second letter, which we choose to be the last letter l_n. Then, the basis

$$W_2 = \{l_1, l_2, \ldots, l_n | l_i \in \mathbb{A}_{\text{gluons}}, l_i \neq l_j,\ \text{for } i \neq j, \quad l_1 = 1,\ l_n = n\} \tag{2.154}$$

describes amplitudes in the KK basis containing $(n-2)!$ elements. Finally, we can impose the BCJ relations to fix a third letter. We choose the letter l_{n-1} and thus the BCJ basis reads

$$B \equiv W_3 = \{l_1, l_2, \ldots, l_n | l_i \in \mathbb{A}_{\text{gluons}}, l_i \neq l_j,\ \text{for } i \neq j, \quad l_1 = 1,\ l_{n-1} = n-1,\ l_n = n\}, \tag{2.155}$$

which contains $(n-3)!$ elements.

2.4.5 Color-kinematics duality

We have seen that relations among amplitudes shrink the basis, hence it is interesting to study their origin. While cyclic relations and KK relations do not involve coefficients with a dependence on the kinematic invariants, the BCJ relations involve kinematic coefficients. Let us see how these coefficients appear in relations among amplitudes, i.e, how BCJ relations are derived [11]. The four point-relation can be easily derived from the photon decoupling identity, i.e.,

$$A_4(1234) + A_4(2134) + A_4(2314) = 0. \tag{2.156}$$

This identity follows from the dependence on the Mandelstam variables of the amplitudes, thus implying that these amplitudes should be proportional to each other [11]. This leads to

$$\begin{aligned} tA_4(1234) &= uA_4(1243), \\ sA_4(1234) &= uA_4(1324), \\ tA_4(1324) &= sA_4(1342), \end{aligned} \tag{2.157}$$

which tells us that we can shrink the basis of amplitudes from $(4-2)!$ to $(4-3)!$. Here as usual $s = (p_1 + p_2)^2$, $t = (p_1 + p_4)^2$, and $u = (p_1 + p_3)^2$.

Let us illustrate how these relations can be derived for the case of 4-points. The photon decoupling identity—special case of the KK-relations—tell us that only two of these amplitudes are independent. We will fix the legs 1 and 4 in the color decomposition, then the KK basis is given by the following amplitudes

$$A_4(1234), \qquad A_4(1324). \tag{2.158}$$

Suppose we can write the 4-point amplitudes as a sum over the poles appearing in the corresponding Feynman diagrams, i.e.,

$$\begin{aligned} A_4(1234) &= \frac{n(12;34)}{s} + \frac{n(23;41)}{t}, \\ A_4(1342) &= \frac{n(13;42)}{u} + \frac{n(34;21)}{s}, \\ A_4(1324) &= \frac{n(13;24)}{u} + \frac{n(32;41)}{t}. \end{aligned} \tag{2.159}$$

We are labeling the numerators $n(ij;kl)$ by the corresponding poles—the quartic contact terms have been reabsorbed in the diagrams with cubic interaction. Furthermore, we assume that the numerators $n(ij;kl)$ satisfy antisymmetry and the Jacobi-like identity [11, 49], i.e.,

$$n(ij;kl) = -n(ji;kl), \qquad n(ij;kl) + n(jk;il) + n(ki;jl) = 0, \qquad n(ij;kl) = n(lk;ji). \tag{2.160}$$

Using these constraints for the amplitudes in the KK-basis, we have a system of equations with coefficient matrix

$$\Theta = \begin{pmatrix} \frac{1}{s} & \frac{1}{t} \\ \\ \frac{1}{u} & -\frac{1}{t} - \frac{1}{u} \end{pmatrix}. \tag{2.161}$$

The rank of this matrix is 1 as can be easily seen by multiplying the second row by u/s. This implies that there are relations among the primitive amplitudes. For instance, we have

$$A_4(1234) = \frac{n(12;34)}{s} + \frac{n(23;41)}{t} = \frac{u}{s} A_4(1324) + \left(\frac{u}{s} \left(\frac{1}{t} + \frac{1}{u} \right) + \frac{1}{t} \right) n(23;41), \tag{2.162}$$

which gives the second line in Eq.(2.157), since the second term in Eq.(2.162) vanishes. Usually, the numerators are labeled by the pole they correspond, for instance it is customarily to define:

$$n_s \equiv n(12;34), \qquad n_t \equiv n(23;41), \qquad n_u \equiv n(13;24). \tag{2.163}$$

Then the Jacobi-like identity in (2.160) reads

$$n_u - n_s + n_t = 0, \tag{2.164}$$

in clear analogy to the color factor accompanying the corresponding diagrams—see Fig(2.5).

The generalization of this procedure is known as color-kinematics duality—also known as BCJ duality—which states that the n-point gluon amplitude can be written as

$$\mathcal{A}_n^{\text{tree}} = \mathrm{i}\, g^{n-2} \sum_{i \in \text{trivalent}} \frac{n_i c_i}{\prod\limits_{\alpha_i} s_{\alpha_i}}, \tag{2.165}$$

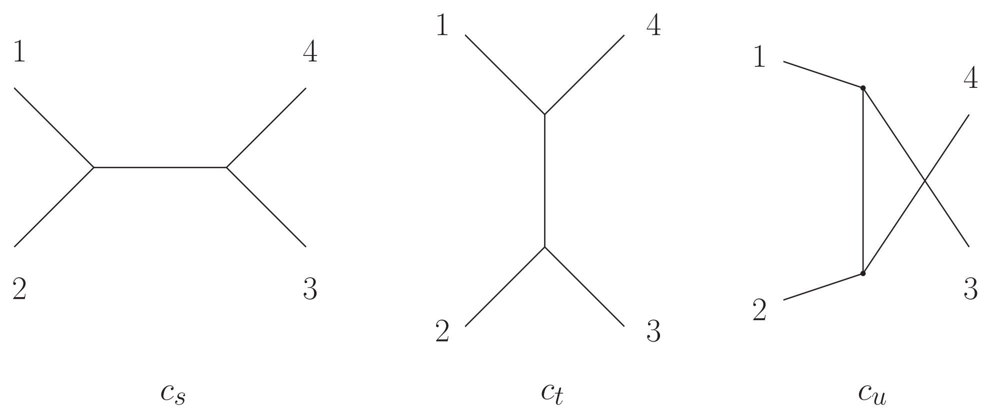

Figure 2.5: Trivalent diagrams for the amplitude $\mathcal{A}_4$. The color factors satisfy the Jacobi identity $c_u - c_s + c_t = 0$.

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where n_i satisfy Jacobi identities whenever c_i does. The sum runs over the set of $(2n-5)!!$ trivalent color ordered diagrams. In practice, one consider the numerators as being unknowns and consider the set of equations for the primitives [11]

$$0 = n_\alpha - n_\beta + n_\gamma, \tag{2.166}$$

$$A_n = \mathrm{i} \sum_{i \in \text{trivalent}} \frac{n_i}{\prod_{\alpha_i} s_{\alpha_i}}. \tag{2.167}$$

Then we construct $(n-3)!$ equations for the numerators in terms of an amplitude basis (independent of the KK relations). There will be $(n-2)!$ equations for the color factors, hence the same number of Jacobi-like identities for the numerators. This leaves $(n-2)! - (n-3)!$ unspecified numerators which can be set to zero. The color-kinematics duality is the concept behind relations among amplitudes involving kinematic coefficients, i.e., the fundamental BCJ relations (2.151).

2.4.6 Two approaches to obtain gravity amplitudes

2.4.6.1 KLT relations

Primitive amplitudes in the color decomposition (2.138) can be related to gravity amplitudes thanks to the Kawai-Lewellen-Tye (KLT) relations [5]. These relations were originally found in the context of string theory methods and they reflect the fact that a n point amplitude of a closed string (gravity) can be obtained from n-point amplitudes of open strings (Yang-Mills). In the context of string theory these relations depend on kinematic variables and the string tension $(1/2\pi\alpha')$. In the infinite tension limit—quantum field theory limit—corresponding to $\alpha' \to 0$ they become a relation between gravity M_n

and primitive gauge theory amplitudes A_n. This is the origin of the concept of squaring a gauge theory to obtain gravity, which is usually written as

$$\text{gravity} = \text{gauge} \times \text{gauge}. \tag{2.168}$$

Let us denote the full gravitational amplitudes in terms of the *primitives* by

$$\mathcal{M}_n(p_1, p_2, \ldots, p_n) = \left(\frac{\kappa}{2}\right)^{n-2} M_n(p_1, p_2, \ldots, p_n). \tag{2.169}$$

At 3-points the gravity amplitude can be obtained from the little group scaling property. For $n = 3$, the amplitude for gravitons may be written as a product of angle brackets, i.e.,

$$M_3(1^{-2}2^{-2}3^{+2}) = \mathrm{i}\,\langle 12\rangle^{x_{12}}\,\langle 23\rangle^{x_{23}}\,\langle 13\rangle^{x_{13}}\,, \tag{2.170}$$

where ± 2 represents the helicity. Using the little group scaling (2.110) we have three equations and three unknowns:

$$2 = x_{12} + x_{13}, \qquad 2 = x_{12} + x_{23}, \qquad -2 = x_{13} + x_{23}, \tag{2.171}$$

which can be easily solved giving the 3-point amplitude

$$M_3(1^{-2}2^{-2}3^{+2}) = \mathrm{i}\frac{\langle 12\rangle^6}{\langle 23\rangle^2\,\langle 13\rangle^2}. \tag{2.172}$$

Alternatively, we can square the Parke-Taylor formula for $n = 3$

$$M_3(1^{-2}2^{-2}3^{+2}) = -A_3(1^{-1}2^{-1}3^{+1})^2 = -\left(\frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 31\rangle}\right)^2, \tag{2.173}$$

where up to a factor of $(-\mathrm{i})$ gives (2.172). If we increase the number of points, we would acquire double poles which have to be canceled by inserting kinematic factors. These kinematic factors were obtained by KLT and are the heart of the KLT relations. Using the gluon alphabet (2.140), up to five points the KLT relations read [5]

$$M_4 = -\mathrm{i}s_{23}A_4(1234)A_4(1324), \tag{2.174}$$

$$M_5 = -\mathrm{i}s_{23}s_{45}A_5(12345)A_5(13254) - \mathrm{i}s_{24}s_{35}A_5(12435)A_5(14253), \tag{2.175}$$

where the amplitudes A_n in the right hand side are primitive amplitudes obtained from the decomposition of full amplitudes in the color trace basis (2.138). The cancellation of double poles become more involved as the number of particles increases. This behavior can be traced to the fact that in the right hand side of the KLT relations we have a product of fixed ordered amplitudes while in the left-hand side there is no fixed ordering[29].

We can work out the 4-point MHV gravity amplitude $M_4(1^-2^-3^+4^+)$ using the spinor helicity formalism. Starting with the 4-point MHV gluon amplitude

$$A_4(1^-,2^-,3^+,4^+) = \mathrm{i}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = \mathrm{i}\frac{[34]^4}{[12][23][34][41]}, \tag{2.176}$$

we obtain

$$M_4 = \mathrm{i}\,t\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{[34]^4}{[13][32][24][41]} = \mathrm{i}\frac{\langle 12\rangle^4[34]^4}{stu}, \tag{2.177}$$

where $s = (p_1+p_2)^2 = \langle 12\rangle[12]$. In this example we have avoided mixing the helicities of the particles, but this is not mandatory—helicities of particles in both gauge amplitudes are the same. Actually, by taking linear combination of polarization vectors, we may obtain amplitudes for dilatons and axions.

In principle, if we have access to a recursive method to obtain gauge amplitudes to arbitrary n, then we can obtain all gravity amplitudes provided we have access to the kinematic factors of the KLT relations. These factors for general n were obtained in Ref. [51]. Using QFT methods, the kinematic factors have been obtained in Refs. [52,53]. The kinematic factors are expressed in terms of the so called momentum kernel $S[w_1|w_2]$ [54]—also known as KLT kernel. For two words $w_1 = l_1\ldots l_n$ and $\bar{w}_2 = k_1\ldots k_n k_{n-1}$, the momentum kernel is defined by

$$S[w_1|\bar{w}_2] = (-1)^n\prod_{i=2}^{n-2}\left[s_{l_1l_i} + \sum_{j=2}^{i-1}\theta_{\bar{w}_2}(l_j,l_i)s_{l_jl_i}\right], \tag{2.178}$$

with

[29]A nice review on the KLT relations is Ref. [50].

$$\theta_{\bar{w}_2}(l_j, l_i) = \begin{cases} 1 & \text{if } l_j \text{ comes before } l_i \text{ in the sequence } k_2, k_3, \dots k_{n-2}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.179}$$

If we use the BCJ basis (2.155) for the gauge amplitudes, then the KLT relations for n-point gravity amplitudes read

$$M_n(12\dots n) = -\mathrm{i} \sum_{w_1, w_2 \in B} A_n(w_1) S[w_1|\bar{w}_2] A_n(\bar{w}_2). \tag{2.180}$$

An important feature of the momentum kernel is its dependence on the kinematic invariants and the order of the amplitudes indicated by w_1, w_2. Notice that the complicated expression for the momentum kernel is needed since gravity amplitudes are not color ordered, hence any combination of external momenta can appear as poles.

As an example, let us consider the 5-point gravity amplitude. In this case the basis is given by $B = \{12345, 13245\}$ and the momentum kernel in matrix form reads

$$\mathsf{S}_5 = \begin{pmatrix} -s_{12}\,(s_{13} + s_{23}) & -s_{12}s_{13} \\ -s_{12}s_{13} & -s_{13}\,(s_{12} + s_{23}) \end{pmatrix}, \tag{2.181}$$

where rows and columns are labeled by 12345 and 13245. The original form of the KLT relations can be recovered back using BCJ relations. This kernel is an important object in this work and we postpone its general treatment for later chapters. In Chapter 3, we use it to describe the KLT orthogonality, which is a property of the scattering equations. In Chapters 4 and 5, we generalize the KLT kernel in two directions: first by allowing masses in the particles and by including fermions.

2.4.6.2 Double copy procedure

In Ref. [11] Bern-Carrasco-Johansson argued that the numerators constructed for the gauge theory via the color-kinematics duality could be used to write gravity amplitudes at tree level by making a *double copy* of the numerators such that

$$\mathcal{M}_n^{\text{tree}} = \mathrm{i}\left(\frac{\kappa}{2}\right)^{n-2} \sum_{i \in \text{trivalent}} \frac{n_i \widetilde{n}_i}{\left(\prod_{\alpha_j} s_{\alpha_j}\right)}, \tag{2.182}$$

where the tilde indicates that the numerators could be from a different gauge theory which satisfies color-kinematics duality as well [12, 55]. The double copy procedure is another incarnation of gravity as the square of a gauge theory. The double copy is also connected with the CHY representation as we will explore in Chapter 5.

2.5 Loop techniques

Although this work is concerned with tree-level amplitudes, there is one method which allows us to recycle amplitudes at tree-level to compute loop-level amplitudes. This is the method of generalized unitarity, which is based on the unitarity of the S-matrix. Expressing $S = 1 + \mathrm{i}T$ and from Eq.(2.8), we have

$$T - T^{\dagger} = \mathrm{i}\, T^{\dagger} T. \tag{2.183}$$

This equation translates into a relation between tree-level and loop-level amplitudes after expanding T in powers of the coupling constant g

$$T = g^2 T^{(0)} + g^4 T^{(1)} + \dots, \tag{2.184}$$

where the superscript indicates tree-level (0), 1-loop (1), etc. This equation express the relations among discontinuities between a k-loop amplitude and a product of lower loop amplitudes. In terms of Feynman diagrams the information about discontinuities is obtained by the "Cutkosky" rules [56], which are obtained by cutting diagrams and integrating them as

$$\mathrm{Disc}(T) = \int \mathrm{d}\mu A_1 A_2 ... \tag{2.185}$$

where $\mathrm{d}\mu$ is an appropriate phase space and A_i are lower loop amplitudes. In the case of the maximal cuts, the factors A_i are tree-level amplitudes. The information about the cuts can be used effectively if we have access to a decomposition of the amplitude in terms of a basis of scalar integrals—for example, using the Passarino-Veltman reduction method [57]—meaning that we can express the amplitude as

$$A_n^L = \sum_i C_i I_n^{(L),i}, \tag{2.186}$$

where C_i are kinematic coefficients and i runs over the set of master integrals. Then the information about the discontinuities of the amplitude match the discontinuities of the Feynman integral, i.e.,

$$\text{Disc}\, A_n^{(L)} = \sum_i C_i\, \text{Disc}\, I_n^{(L),i}. \tag{2.187}$$

This yields the coefficients C_i as

$$C_i = \sum_{\text{states}} A_{(1)} A_{(2)} \cdots A_{(n)}, \tag{2.188}$$

where $A_{(i)}$ are tree level amplitudes—in the case of the generalized unitarity with maximal cuts. This method allows us to connect tree-level amplitudes and loop amplitudes whenever a basis of integrals is known. For example, the 1-loop amplitude may be written in terms of m-gon integrals as follows:

$$A^{\text{1-loop}} = \sum_i c_D^{(i)} I_D^{(i)} + \sum_j c_{D-1}^{(j)} I_{D-1}^{(j)} + \cdots + \sum_k c_2^{(k)} I_2^{(k)} + \text{rational terms}, \tag{2.189}$$

where $I^{(m)}$ are m-gon scalar integrals, which are illustrated in Fig.2.6.

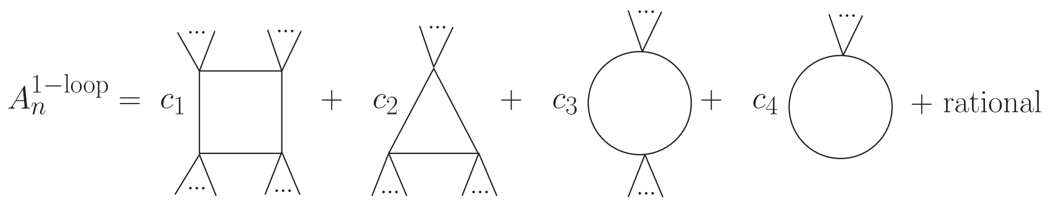


Figure 2.6: Representation of the unitarity method. The contributing integral topologies and the rational terms are shown. The coefficients can be obtained with the help of tree-level amplitudes.

The last term correspond to rational functions which do not have branch cuts, hence appearing as a consequence of the regularization procedure. This method has been systematized and used for a variety of theories, including QCD[30]. Finally, since the method of generalized unitarity allows us to recycle tree-level amplitudes, we could use the color-kinematics duality at tree-level to conclude that it is also possible to find numerators satisfying Jacobi-like identities whenever color factors do. Therefore, the color-kinematics duality and the double copy can be elevated at loop-level

[30] See e.g., [58] and references therein

$$\mathcal{A}_n^{(L)} = g^{n-2+2L} \sum_j \int \prod_{l=1}^{L} \frac{\mathrm{d}^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{c_j n_j}{\prod_{\alpha_j} s_{\alpha_j}}, \tag{2.190}$$

$$\mathcal{M}_n^{(L)} = \left(\frac{\kappa}{4}\right)^{n-2+2L} \sum_j \int \prod_{l=1}^{L} \frac{\mathrm{d}^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} s_{\alpha_j}}, \tag{2.191}$$

where we include symmetry factors S_j.

2.6 Comments

In this Chapter we have studied several techniques to compute amplitudes using the traditional methods of quantum field theory and later using the modern methods inspired in string theory. From the purely practical point of view, one of the aims of the current research in scattering amplitudes is to exploit generalized unitary to recycle tree-level amplitudes and use them to compute gauge and gravity amplitudes. To achieve this, a nontrivial fact is to find a suitable basis of integrals, which by itself is a full research area. In addition, one would like to extend these methods to theories which are relevant to describe the particle content of the standard model. Here we have focused on amplitudes involving gluons because they are relevant for the hard scattering part of the computation of cross sections. However, the real problem appears when we compute loop-level amplitudes with fermions and gluons, i.e., QCD amplitudes.

On the other hand, conceptually it is not clear how the new methods can be understood as a manifestation of unitarity, locality, Lorentz invariance, gauge invariance, etc. In other words, whether the old idea of formulating an S-matrix only using physical information is possible. In this conceptual front one may ask for example: what is the S-matrix as low-level object without locality or gauge invariance? Even more, what are the ultimate symmetries of the S-matrix that make evident the simple structure of the final result? Finally, is there a realization of the cluster decomposition principle which makes evident a construction such as the color-kinematics duality? These are questions that can be studied in a recent framework developed by Cachazo, He, and Yuan, which is the topic of the next Chapter. This method nicely connects with all the methods discussed in this Chapter, which motivated the review.

Chapter 3

Scattering Equations and the CHY formalism

The formalism developed by Cachazo, He, and Yuan (CHY) based on the scattering equations reflects very well the idea of "amplitudes without Feynman diagrams". It relies on the localization of the kinematic invariants of the amplitude in an auxiliary space—Riemann sphere—which encodes the factorization properties of the amplitude. This localization is realized by a map from the Riemann sphere to momentum space, keeping the momenta of the external particles on-shell. In order to determine this map, the scattering equations have to be satisfied. The amplitude is then written as an integral over all possible maps from the space of kinematics to the Riemann sphere—these maps imply the scattering equations. In other words, we localize an integral on the punctured Riemann sphere and the amplitude becomes the sum over the evaluations on the solutions of scattering equations.

The CHY formalism transforms the problem of summing over Feynman diagrams to a problem in algebra—that of finding the solutions of a set of equations—or to a problem in algebraic geometry, where it is not necessary to solve the equations but instead use residue calculation techniques. These views are obviously connected by the fact that the scattering equations are a set of polynomial equations in many variables.

This Chapter is organized as follows: In Section 3.1, we introduce the scattering equations and its properties. In Section 3.2, we introduce the general features of the CHY formalism as a contour integral and as a sum over solutions. We study some integrands and give some examples in Sections 3.3 and 3.4, respectively. We end this Chapter with a discussion of the CHY-formalism.

3.1 The scattering equations

Historically, the scattering equations were first formulated by Fairle and Roberts in 1972 as a minimization requirement of the Koba-Nielsen formula for the scattering of n open strings [59]. They also appeared as the Weierstrass condition for an isometric coordinate system in the minimization problem of a two-dimensional surface embedded in a four dimensional space [60]. Then, they appeared in the work by Gross and Mende on the high energy behavior of string theory, when they studied the saddle point approximation [61]. A generalized version valid in D-dimensions reappeared in the work by CHY in 2013 [62], where they were dubbed them as the *scattering equations*. We will follow their procedure to derive the scattering equations, which fits more in the modern approach to amplitudes.

The scattering equations in $D = 4$ were also known in the context of the connected formalism as pointed out by Cachazo in Ref. [63], where the scattering equations localize the integrals of the amplitudes. This approach was inspired in the framework introduced by Witten in 2003 [7], as we pointed out in the Introduction. In this framework, amplitudes are Fourier transformed from spinor variables to twistor variables. Then, amplitudes are supported on solutions of a set of equations in twistor space. These equations map points in twistor space to the Riemann sphere (see, e.g., [64]).

Remember that the formulation of scattering amplitudes in terms of spinor or twistor variables simplifies the description of amplitudes. By writing amplitudes in terms of spinor products we are changing the variables of interest to be $|p\rangle$, $|p]$, and the helicities h. However, spinor variables are tied to $D = 4$ dimensions and in principle they are valid only for massless particles[1]. Nevertheless, in the massless case, spinors become the right variables to describe the kinematics of the process and can also be used to recursively obtain higher point amplitudes as we have seen in Section 2.4.

According to the cluster decomposition principle, singularities appear in the form of poles or branch cuts for certain values of the momenta—usually as Lorentz invariant products. Therefore, a convenient way to write an amplitude would be in terms of spinor or momentum variables and an additional object (or space) which takes care of the singularities of the amplitude. The idea behind the scattering equations is to introduce an additional space which takes care of the singularities and embeds the factorization properties of the amplitude, together with a map which connects the kinematic space to the additional space [66]. The additional space turns out to be the space of all possible Riemann spheres with n marked points[2], in other words, the moduli space of n-punctured spheres $\mathcal{M}_{0,n}$. Thus, the scattering equations connect the singularities of the scattering amplitude and its factorization channels to the singularities of the moduli space of Riemann spheres with n marked points.

3.1.1 Finding the scattering equations

Let us describe the necessary ingredients for the scattering equations. In a physical process involving n particles, the momentum configuration space K_n consists of the collection

[1]The formalism can be used for massive particles but the results depend on reference vectors [65].
[2]A brief description of this space can be found in Appendix B.

of momenta p_i subject to momentum conservation and on-shell conditions[3]. We write formally

$$K_n = \left\{ (p_1, p_2, \ldots, p_n) \in (\mathbb{C}M)^n | \sum_{i=1}^{n} p_i = 0, \text{ and } p_i^2 = 0 \right\}, \tag{3.1}$$

where $p_i \in \mathbb{C}M$ is the momentum in complexified Minkowski space in D-dimensions. The n-tuple $p = (p_1, p_2, \ldots, p_n)$ of momentum vectors belongs to K_n. We define the kinematic invariants for the scattering of n massless particles formed by Lorentz invariant products of subsets of momenta as

$$s_A \equiv s_{l_1 l_2 \ldots l_a} = (p_{l_1} + \cdots + p_{l_a})^2, \tag{3.2}$$

where $l_1, l_2, \ldots, l_a \in A \subset \{1, 2, \ldots, n\}$. These are the Mandelstam variables which characterize the singularities of the amplitude. Now, let us consider the complex projective line $\mathbb{CP}^1$ with n marked points—denoted by z_i, $i = 1, \ldots, n$. The space $\mathbb{CP}^1$ is isomorphic to the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The map between points in Riemann sphere to momentum space is given by

$$p_i^\mu = \frac{1}{2\pi \mathrm{i}} \oint_{|z-z_i|=\delta} \mathrm{d}z \frac{P^\mu(z)}{\prod_{j=1}^{n} (z - z_j)}, \quad i = 1, \ldots, n, \tag{3.3}$$

where the integral is over a small counter-clockwise oriented circle $\{|z - z_0| = \delta\}$ around z_i, for any sufficiently small δ (see Fig. 3.1). The map $P^\mu(z)$ is a collection of D polynomials of degree $n-2$, which we have to determine. This object is a map from the Riemann sphere to the null cone in complexified Minkowski space, therefore the on-shell conditions $p_a^2 = 0$ impose [62]

$$P^\mu(z) P_\mu(z) = 0. \tag{3.4}$$

There is an alternative approach due to Fairle and Roberts [59, 60] to determine the scattering equations. In this approach, the scattering equations are obtained as a way to replace the Virasoro conditions for closed strings with the condition of the vanishing of $\omega(z)^2$, where $\omega(z)$ is a map from the Riemann sphere to complexified Minkowski space, i.e., $\omega : \mathbb{CP}^1 \to C\mathbb{M}$. Explicitly

[3]Here we will focus on the massless case but the massive case can be treated similarly [67].

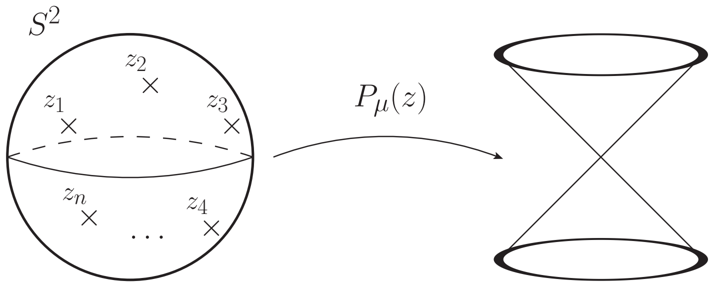


Figure 3.1: The map $P(z)^\mu$ has a simple pole for each marked point i.

$$\omega^\mu(z) = \sum_{j=1}^{n} \frac{p_j^\mu}{z - z_j}. \tag{3.5}$$

Using momentum conservation, we see that this map is equivalent to the integrand of (3.3), since

$$\omega^\mu(z) = \frac{c_0^\mu + c_1^\mu z + \cdots + c_{n-2}^\mu z^{n-2}}{\prod_{j=1}^{n}(z - z_j)}, \tag{3.6}$$

where c_i^μ are the coefficients of the polynomials $P^\mu(z)$. Using (3.5), we have the condition

$$\sum_{i,j=1}^{n} \frac{p_i \cdot p_j}{(z - z_i)(z - z_j)} = 2\sum_{i=1}^{n} \frac{1}{(z - z_j)} \sum_{j=1}^{n} \frac{p_i \cdot p_j}{(z_i - z_j)} = 0. \tag{3.7}$$

This equation must be valid for all z, in particular it should be free of double and simple poles, thus implying $p_i^2 = 0$ and the scattering equations, respectively. Therefore, the scattering equations for the momentum configuration space of n massless particles are given by

$$f_i(z, p) \equiv \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{2p_i \cdot p_j}{z_i - z_j} = 0, \quad i = 1, \ldots, n, \tag{3.8}$$

where z_i are the locations of the punctures and $z = (z_1, \ldots, z_n)$ denotes the dependence on all the punctures. This formula is Möbius invariant which reflects that we have a $\mathbb{CP}^1$ with n punctures. The solutions will correspond to n distinct points on $\mathbb{CP}^1$ modulo Möbius transformations, thus the solutions of the scattering equations are points in the moduli space of Riemann spheres with n marked points $\mathcal{M}_{0,n}$. This establish the connection between the space of kinematic invariants $p_i \cdot p_j$ and $\mathcal{M}_{0,n}$.

The scattering equations can also be derived directly from string theory [68]. They appear in the context of the ambitwistor string theory—loosely speaking, string theory in a target space made of null geodesics in complexified spacetime [69]. In this context, $\omega(z)$ is a 1-form, i.e.,

$$\omega_F^\mu(z) = \omega^\mu(z)\mathrm{d}z, \tag{3.9}$$

and the scattering equations enforce the vanishing of the residues of $\omega_F^2(z)$ at $(n-3)$ points [70], i.e.,

$$\mathrm{res}_{z_i}\omega_F^2(z) = f_i(z,p) = 0. \tag{3.10}$$

In this context, there is a natural loop-level generalization by increasing the genus of the Riemann surface, thus giving the scattering equations again as the vanishing of the residues of the 1-loop generalization of $\omega_F^2(z)$ plus an additional constraint on $\omega_F^2(z)$ [71–73].

3.1.2 Properties of the scattering equations

Möbius invariance. The scattering equations are invariant under Möbius transformations, which are defined through the mapping

$$g : \mathbb{CP}^1 \to \mathbb{CP}^1, \qquad g(z_i) = \frac{az_i + b}{cz_i + d}, \qquad i \in \{1, \ldots, n\}, \tag{3.11}$$

where a, b, c, and d are complex such that $ad - cb \neq 0$. These transformations form a group isomorphic to $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 = \mathrm{PSL}(2,\mathbb{C})$. The group operation of these transformations can be realized as function composition or as matrix multiplication. Möbius invariance implies the identities:

$$\sum_{j=1}^{n} f_j(z,p) = 0, \qquad \sum_{j=1}^{n} z_j f_j(z,p) = 0, \quad \sum_{j=1}^{n} z_j^2 f_j(z,p) = 0. \tag{3.12}$$

Therefore only $(n-3)$ equations are independent.

Factorization. The scattering equations factorize into two sets when one of the Mandelstam variables vanishes. Let us denote by s_A with $A \subset \{1, \ldots, n\}$ the vanishing

Mandelstam variable. Suppose that $z_a = z_A + \epsilon x_a + \mathcal{O}(\epsilon^2)$, as $\epsilon \to 0$, for $a \in A$, and $z_a \not\to z_A$ for $a \notin A$, then

$$\begin{aligned} \sum_{\substack{j \in A \\ j \neq i}}^{n} \frac{p_i p_j}{x_i - x_j} &= 0, \quad i \in A, \\ \frac{p_i p_A}{z_i - z_A} + \sum_{\substack{j \notin A \\ j \neq i}}^{n} \frac{p_i p_j}{x_i - x_j} &= 0, \quad i \notin A, \end{aligned} \tag{3.13}$$

with $p_A = \sum_{i \in A} p_i$. Hence, for $s_A = 0$, the scattering equations factorize in two sets:

1. The first set for the momenta $(p_i, i \in A; -p_A)$ with associated variables $(x_i, a \in A; \infty)$.
2. The second set for the momenta $(k_A; k_i, i \notin A)$ with associated variables $(z_A; z_i, i \notin A)$.

The factorization properties of the scattering equations mimic the factorization properties of the moduli spaces of Riemann spheres. This is a well-known fact in string amplitudes where the physical singularities of the amplitudes are connected with the behavior of moduli spaces at infinity (see e.g., Section 2.3 of [74]).

KLT orthogonality. Among the properties that were presented in the seminal work by CHY [62], there is an orthogonality-like relation on the support of the solutions of the scattering equations. There are $(n-3)!$ solutions of Eq.(3.8), which we denote by

$$z^{(j)} = (z_1^{(j)}, \dots, z_n^{(j)}), \quad j = 1, \dots, (n-3)! \tag{3.14}$$

Let consider a solution $z^{(j)}$ of the scattering equations and define the $(n-3)!$ dimensional *vectors*

$$C^j \equiv C(w, z^{(j)}) = \frac{1}{z_{l_1 l_2}^{(j)} \cdots z_{l_n l_1}^{(j)}}, \qquad \bar{C}^j \equiv C(\bar{v}, z^{(j)}) = \frac{1}{z_{l_1 l_2}^{(j)} \cdots z_{l_n l_{n-1}}^{(j)} z_{l_{n-1} l_1}^{(j)}}, \tag{3.15}$$

with orderings $w = l_1 \dots l_{n-1} l_n$ and $\bar{v} = l_1 \dots l_n l_{n-1}$, respectively. Here we have used the notation $z_{l_i} - z_{l_j} = z_{l_i l_j}$. The KLT orthogonality states that these vectors satisfy

$$\frac{\langle C^i, \bar{C}^j \rangle}{\langle C^i, \bar{C}^i \rangle^{\frac{1}{2}} \langle C^j, \bar{C}^j \rangle^{\frac{1}{2}}} = \delta^{ij}, \tag{3.16}$$

where the inner product $\langle \bullet, \bullet \rangle$ is defined by

$$\langle C^i, \bar{C}^j \rangle = \sum_{w \in B} \sum_{v \in B} C(w, z^{(i)}) S[w|\bar{v}] C(\bar{v}, z^{(j)}), \tag{3.17}$$

where B is a basis of $(n-3)!$ orderings of the legs $l_1, \ldots, l_n$. The size of the basis is such that three legs are fixed[4]. The KLT momentum kernel $S[w|\bar{v}]$ was introduced in Section 2.4.6. Choosing $\{1, 2, \ldots, n\}$ as the set of labels, the basis is constructed by setting $l_1 = 1$, $l_{n-1} = n-1$, and $l_n = n$.

3.1.3 Polynomial form of the scattering equations

The scattering equations become a system of $(n-3)$ rational equations after removing the redundancy due to Möbious invariance. They can be transformed to system of polynomial equations as was shown by Dolan and Goddard in Ref. [75]. This form is suitable for finding its solutions employing usual techniques available in the literature, for example via the Gröbner basis. In addition, tools from algebraic geometry can be used to simplify the contour integrals in the CHY formalism to avoid finding the solutions—a short summary of Gröbner basis and some relevant tools from algebraic geometry can be found in the Appendix B.

In order to introduce the polynomial form of the scattering equations let us consider a subset

$$S \subseteq \{2, \ldots, n-1\}. \tag{3.18}$$

We also define

$$p_S = \sum_{i \in S} p_i, \qquad z_S = \prod_{i \in S} z_i, \tag{3.19}$$

where by definition $p_\emptyset = 0$ and $z_\emptyset = 1$. In addition, we define the homogeneous polynomials in $(n-3)$ variables as

[4]See Appendix C for details about basis of amplitudes.

$$h_m(z,p) = \sum_{\substack{S\subset\{2,\ldots,n-1\}\\ |S|=m}} p_S^2 z_S. \tag{3.20}$$

The scattering equations are then equivalent to set of equations

$$h_m(z,p) = 0, \qquad 1 \leq m \leq n-3. \tag{3.21}$$

This set of equations defines a zero dimensional algebraic variety—in other words the solutions of the equations are a set of points—dubbed as the scattering variety [76]

$$V_n(p) = \{z \in \mathbb{C}^{n-3} | h_m(p,z) = 0; \quad 1 \leq m \leq n-3\}. \tag{3.22}$$

Bezout's theorem tell us that $V_n(p)$ consists of $(n-3)!$ points as was found empirically by CHY. Since the scattering equations have finitely many common zeros in $\mathbb{C}^{n-3}$, we can focus on the ideal I generated by the polynomials h_m, i.e.,

$$I = \langle h_1, \ldots, h_{n-3} \rangle . \tag{3.23}$$

Therefore, if we want to find the zeros of the ideal I we may use the Gröbner basis method. This requires the introduction of a Gröbner basis for the ideal I [77]. However, the computation of Gröbner basis is computationally challenging when the number of equations increase. However, Bosma, Søgaard, and Zhang showed that the polynomials h_m form a simpler basis called the Macaulay H-basis for the ideal I [78]. Let us briefly review this notion. Let $R = \mathbb{C}[z_1, \ldots, z_n]$ be a ring of polynomials in n variables $z_1, z_2, \ldots, z_n$ over the field of complex numbers[5]. A set of polynomials $\{b_1, \ldots, b_k\} \subset I$ is an H-basis for $I \subseteq R$ if for all $P \in I$ if there exists polynomials $q_1, \ldots, q_k \in R$ such that

$$P = \sum_{j=1}^{k} q_j b_j, \tag{3.24}$$

with $\deg q_j \leq \deg P - \deg b_j$. Alternatively, for any polynomial $P \in R$ we define the initial form of P "in(P)" as the homogeneous part of P of degree $\deg P$. Thus, the condition for a set of polynomials $\{b_1, \ldots, b_k\} \in R$ for being an H basis reads

[5] We can choose an arbitrary number field.

$$\langle \text{in } (I)\rangle = \langle \text{in } (b_1), \ldots, \text{in } (b_k)\rangle \,. \tag{3.25}$$

The scattering polynomials form an H-basis and thus any polynomial $P \in R$ of degree d can be reduced with respect to the scattering polynomials $\{h_1, \ldots, h_{n-3}\}$ as

$$P = \sum_{j=1}^{n-3} q_j h_j + \tilde{P}, \tag{3.26}$$

where the remainder $\tilde{P}$ is bounded by $d^* = (n-3)(n-4)/2$, i.e., $d > d^*$.

In this work, we concentrate on the scattering polynomials seen as the ideal (3.23) and hence we concentrate in the Gröbner basis method, instead of the H-basis method.

3.1.4 Solutions of the scattering equations

We are ready to review the solutions of the scattering equations. The solutions have been studied extensively in the literature. Let us mention some selected works.

First, in the original form of the scattering equations (3.8) the numerical solutions were studied by CHY finding that the number of solutions is $(n-3)!$ [62]. Solutions for special kinematics were studied in Ref. [79], where they were associated with the zeros of Jacobi polynomials. In 4D the solutions can be found in terms of spinor products and momentum twistor variables [80]. In 4D at four points a solution was also known since the works of Fairle and Roberts [60]. Finally, 4D solutions were studied in Refs. [75, 81]. Second, with the introduction of the polynomial form of the scattering equations by Dolan and Goddard [75] the problem of finding the solutions became a well-defined problem in algebraic geometry, therefore many techniques were proposed, e.g., using elimination theory [82, 83]. Finally, Cachazo, Mizera, and Zhang found that in certain regions of the space of kinematic invariants the solutions are real[6].

Let us review the solutions for the simplest cases.

3.1.4.1 Three points

The trivial case correspond to $n = 3$ external particles, where the solutions are completely fixed by Möbius invariance. Therefore, in this case there is only one solution, namely

$$z^{(1)} = (\infty, 1, 0), \tag{3.27}$$

[6] This may be used to find a CHY representation only using real numbers [84].

which also sets our convention for the fixed values of z_i. In general, we fix:

$$z_1 = \infty, \qquad z_2 = 1, \qquad z_n = 0. \tag{3.28}$$

3.1.4.2 Four points

In this case Eqs. (3.20)-(3.21) give

$$0 = s_{12} z_2 + s_{13} z_3, \tag{3.29}$$

therefore

$$z^{(1)} = (\infty, 1, -\frac{s_{12}}{s_{13}}, 0). \tag{3.30}$$

3.1.4.3 Five points

Things start to become a bit more exciting with $n = 5$ particles where we have the equations:

$$\begin{aligned} 0 &= s_{12} z_2 + s_{13} z_3 + s_{14} z_4, \\ 0 &= s_{123} z_2 z_3 + s_{134} z_4 z_3 + s_{124} z_2 z_4. \end{aligned} \tag{3.31}$$

This system of equations can be easily solved using the '`Solve`' command of Mathematica. However, for illustration purposes we shall use the method of elimination via Gröbner bases. The scattering polynomials generate the ideal I_5, i.e.,

$$I_5 = \langle s_{12} z_2 + s_{13} z_3 + s_{14} z_4, s_{123} z_2 z_3 + s_{134} z_4 z_3 + s_{124} z_2 z_4 \rangle \,, \tag{3.32}$$

which in lexicographic order (See Appendix B.2) leads to the Gröbner basis:

$$\begin{aligned} g_1 &= s_{14} s_{134} z_4^2 + (s_{14} s_{123} - s_{13} s_{124}) z_4 + s_{12} s_{134} z_4 + s_{12} s_{123}, \\ g_2 &= s_{13} z_3 + s_{14} z_4 + s_{12}, \\ g_3 &= s_{123} z_3 + s_{134} z_4 z_3 + s_{124} z_4, \end{aligned} \tag{3.33}$$

where the set of equations $g_i = 0$, $i = 1, 2, 3$ have the same solutions as (3.31). However, g_1 is a polynomial in one variable[7] which can be easily solved and inserted in the remaining equations. Therefore, the solutions read

$$z^{(1)} = \tag{3.34}$$
$$\left(\infty, 1, -\frac{s_{12}}{2s_{13}} - \frac{s_{124}}{2s_{134}} + \frac{s_{14}s_{123}}{2s_{13}s_{134}} - \frac{\sqrt{Q}}{2s_{13}s_{134}} - \frac{s_{12}}{2s_{14}} - \frac{s_{123}}{2s_{134}} + \frac{s_{13}s_{124}}{2s_{14}s_{134}} + \frac{\sqrt{Q}}{2s_{14}s_{134}}, 0\right),$$

$$z^{(2)} = \tag{3.35}$$
$$\left(\infty, 1, -\frac{s_{12}}{2s_{13}} - \frac{s_{124}}{2s_{134}} + \frac{s_{14}s_{123}}{2s_{13}s_{134}} + \frac{\sqrt{Q}}{2s_{13}s_{134}}, -\frac{s_{12}}{2s_{14}} - \frac{s_{123}}{2s_{134}} + \frac{s_{13}s_{124}}{2s_{14}s_{134}} - \frac{\sqrt{Q}}{2s_{14}s_{134}}, 0\right),$$

where $Q = (-s_{14}s_{123} + s_{13}s_{124} + s_{12}s_{134})^2 - 4s_{12}s_{13}s_{124}s_{134}$.

3.1.4.4 Six points

At six points, the problem becomes very difficult to handle through the MATHEMATICA command "`Solve`". The equations in this case are

$$\begin{aligned}
0 &= s_{12}z_2 + s_{13}z_3 + s_{14}z_4 + s_{15}z_5, \\
0 &= s_{123}z_2z_3 + s_{134}z_4z_3 + s_{135}z_5z_3 + s_{124}z_2z_4 + s_{125}z_2z_5 + s_{145}z_4z_5, \\
0 &= s_{1234}z_2z_3z_4 + s_{1245}z_2z_5z_4 + s_{1345}z_3z_5z_4 + s_{1235}z_2z_3z_5,
\end{aligned} \tag{3.36}$$

where now the Gröbner basis is formed by 25 elements. The first element in the basis depends only on z_5 and is a sextic polynomial

$$g_1 = (s_{15}^2 s_{135}s_{145}s_{1345}^2)z_5^6 + (s_{124}s_{135}s_{1345}^2 s_{15}^2 + s_{123}s_{145}s_{1345}^2 s_{15}^2 + 12 \text{ terms})z_5^5 + \ldots \tag{3.37}$$

The rather lengthy size of the basis and coefficients is a very well-known problem of the lexicographic monomial order, but it illustrates the technical problem in this approach. We can do better by choosing another monomial ordering or by using another technique for elimination. A general procedure for solving these equations using resultants is worked out in Ref. [83].

It should be clear that despite its apparent simplicity, it is a nontrivial task to solve analytically the scattering equations for higher number of particles. One can do a bit

[7] This is a general result from elimination theory, see e.g., Ref. [85], Chapter 3.

better numerically, e.g., using generic kinematics it is possible to find the solutions up to 10 points using the `Nsolve` command of MATHEMATICA in some hours.

However, as we shall see in the following Sections we may use the polynomial form of the scattering equations to compute global residues and thus we shall not need to solve the scattering equations—we may use that the polynomials form a H-basis to compute residues as well. Nevertheless, the *concept* of evaluating over solutions—but not necessarily the explicit evaluation—will be important whenever we use the KLT orthogonality (3.17), which has a dependence on the solutions.

3.2 CHY-formalism

The scattering equations can be used to write an integral representation of amplitudes for scalars, gluons, and gravitons in the same spirit as the Roiban-Spradlin-Volovich connected formula for amplitudes in $\mathcal{N}=4$ SYM [15]. The idea is to write an integral of $I(z,p,\varepsilon)\,\mathrm{d}\Omega_{\mathrm{CHY}}$, where $I(z,p,\varepsilon)$ may depend on the polarization vectors ε. The rational function I satisfies the properties of an amplitude and $\mathrm{d}\Omega_{\mathrm{CHY}}$ represents the integration over maps from the space of kinematic invariants to $\mathcal{M}_{0,n}$ [86]. The CHY proposal for the n-point amplitude reads

$$\mathcal{A}_n(p,\varepsilon)=\int I(z,p,\varepsilon)\mathrm{d}\Omega_{\mathrm{CHY}}, \tag{3.38}$$

where the covariant and permutation invariant "measure"

$$\mathrm{d}\Omega_{\mathrm{CHY}}=\frac{\mathrm{d}z_1\mathrm{d}z_2\cdots\mathrm{d}z_n}{\text{vol PSL}(2,\mathbb{C})}\prod_a{}'\,\delta\left(f_a\right) \tag{3.39}$$

restricts the integration over z_a to the solutions of the scattering equations. The modified product is defined by

$$\prod_a{}'\,X_a=(-1)^{(i+j+k)}z_{ij}z_{jk}z_{kl}\prod_{a\neq i,j,k}X_a, \tag{3.40}$$

where X_a is an arbitrary expression. This product restricts the integration to the $(n-3)$ independent equations (See Section 3.1.2). The $\mathrm{PSL}(2,\mathbb{C})$ invariant measure is constructed by fixing three of the values of z_i, say, p, q, and r such that

$$\frac{\mathrm{d}z_1\mathrm{d}z_2\cdots\mathrm{d}z_n}{\text{vol PSL}(2,\mathbb{C})}=(-1)^{(p+q+r)}z_{pq}z_{qr}z_{rp}\prod_{a\neq p,q,r}\mathrm{d}z_a, \tag{3.41}$$

where the sign makes the quantity $\mathrm{PSL}(2,\mathbb{C})$ invariant. Explicitly, the measure reads

$$\mathrm{d}\Omega_{\mathrm{CHY}} = (-1)^{(p+q+r)} z_{pq} z_{qr} z_{rp} \prod_{a\neq p,q,r} \mathrm{d}z_a \prod_a{}' \delta\left(f_a\right). \tag{3.42}$$

Thus, the task is reduced to finding the rational integrand $I(z,p,\varepsilon)$. In order to find a $\mathrm{PSL}(2,\mathbb{C})$ invariant integrand $I(z,p,\varepsilon)$ we have to ask which properties should the amplitude satisfy for a given theory such that it reproduces the S-matrix. A systematic procedure to obtain integrands is yet unknown, however a classification and the relations among theories with a known representation can be found in Ref. [87]. These theories are also studied from the point of view of the ambitwistor models in Ref. [88].

The Dirac delta functions completely localize the integral on the solutions of the scattering equations, hence there are no integrations to do. This can be easily shown by invoking a property of multi-residues, which allows us to rewrite Eq.(3.38) as a contour integral using a known prescription for writing a residue as a Dirac delta[8]. This prescription amounts to the substitution

$$\mathrm{d}\Omega_{\mathrm{CHY}} = \frac{\mathrm{d}z_1 \mathrm{d}z_2 \cdots \mathrm{d}z_n}{\mathrm{vol}\ \mathrm{PSL}(2,\mathbb{C})} \prod_a{}' \frac{1}{f_a} \tag{3.43}$$

in Eq. (3.39). Thus, the explicit contour integral reads

$$\mathcal{A}_n(p,\varepsilon) = \frac{(-1)^{(i+j+k+p+q+r)}}{(2\pi \mathrm{i})^{n-3}} \oint_{\mathcal{O}} I(z,p,\varepsilon)(z_{ij}z_{jk}z_{ki})(z_{pq}z_{qr}z_{rp}) \prod_{a\neq i,j,k} \frac{1}{f_a} \prod_{a\neq p,q,r} \mathrm{d}z_a, \tag{3.44}$$

where the contour $\mathcal{O}$ encloses the inequivalent solutions of the scattering equations. This is the form used for the proof of the CHY-formula by Dolan and Goddard [89]. In order to solve this integral, let us consider the computation of the Grothendieck residue[9]

$$\mathrm{res}_{z^{(j)}}\left(\frac{g(z)}{f_1(z)\cdots f_n(z)}\right) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\Gamma_{\mathbf{f}}(\delta)} \frac{g(z)}{f_1(z)\cdots f_n(z)} dz, \qquad dz = dz_1 \wedge \cdots \wedge dz_n, \tag{3.45}$$

where we recall that $z = (z_1, z_2, \ldots, z_n)$. If $z^{(j)}$ is solution of $f_1 = 0, f_2 = 0, \ldots, f_n = 0$, then the Jacobian of $\mathbf{f} = (f_1, \ldots, f_n)$ [90], i.e.,

[8] This particular prescription is given in Eq. (B.11) of Appendix B, where some important features of multi-residues are briefly discussed.

[9] Details of this formula are described in Appendix B.3.

$$J_{\mathbf{f}}(z^{(j)}) = \det\left(\frac{\partial f_a}{\partial z_b}(z^{(j)})\right) \tag{3.46}$$

is nonzero and

$$\operatorname{res}_{z^{(j)}}\left(\frac{g(z)}{f_1(z)\cdots f_n(z)}\right) = \frac{g(z^{(j)})}{J_{\mathbf{f}}(z^{(j)})}. \tag{3.47}$$

Then, for each solution $z^{(j)}$ of $\mathbf{f}$ we have the result (3.47). The full result is then the global residue[10]

$$\operatorname{Res}_{\{\mathbf{f}\}}(g(z)) = \sum_{z^{(j)}\in\text{solutions}} \operatorname{res}_{\{z^{(j)}\}}(\omega). \tag{3.48}$$

If we want to use this property for the whole set of the scattering equations, we notice that the determinant of the Jacobian vanishes. Furthermore, in Eq.(3.44) we are integrating over $(n-3)$ variables and we have $(n-3)$ equations. Therefore, we have to consider only $(n-3)$ equations. The Jacobian of the transformation (3.46) can be obtained from the $n\times n$ matrix Φ given by

$$\Phi_{ab} = \frac{\partial f_a}{\partial z_b} = \begin{cases} \dfrac{s_{ab}}{(z_a-z_b)^2}, & a\neq b, \\ \\ -\displaystyle\sum_{\substack{j=1\\ j\neq a}} \dfrac{s_{aj}}{(z_a-z_j)^2}, & a=b. \end{cases} \tag{3.49}$$

We then delete the rows $\{i,j,k\}$ and the columns $\{p,q,r\}$. We denote the resulting $(n-3)\times(n-3)$ matrix by Φ^{ijk}_{pqr}. Therefore, using Eq. (3.48) for the integral (3.44) we obtain

$$\mathcal{A}_n(p,\varepsilon) = (-1)^{i+j+k+p+q+r} \sum_{z^{(j)}\in\text{ solutions}} (z_{ij}z_{jk}z_{ki})(z_{pq}z_{qr}z_{rp}) \left.\frac{I(z,p,\varepsilon)}{\det(\Phi^{ijk}_{pqr})}\right|_{z=z^{(j)}}, \tag{3.50}$$

where "solutions" is the set of $(n-3)!$ solutions. It is customary to define the reduced determinant of the matrix Φ^{ijk}_{rst} as

[10] Ibid.

$$\det{}'(\Phi) = (-1)^{i+j+k+r+s+t} \frac{\det(\Phi^{ijk}_{rst})}{(z_{ij}z_{jk}z_{ki})(z_{rs}z_{st}z_{tr})}, \tag{3.51}$$

where we have introduced again a sign to make the reduced determinant independent of the choice of $\{i, j, k\}$ and $\{r, s, t\}$. Let us now define

$$J(z, p) = \frac{1}{\det{}'(\Phi)}. \tag{3.52}$$

Therefore, we can write the integral as a sum over all the evaluations of the solutions of the scattering equations, i.e.,

$$\mathcal{A}_n(p, \varepsilon) = \sum_{\text{solutions } j} J(z^{(j)}, p) I(z^{(j)}, p, \varepsilon). \tag{3.53}$$

where the $z^{(j)}$ indicates the evaluation of the j-th solution in the expression and the sum runs over $(n-3)!$ elements. Notice that we write "solutions j" to indicate "$z^{(j)} \in$ solutions". The Jacobian transforms as

$$J(g(z), p) = \left(\prod_{j=1}^{n} \frac{1}{(cz_j + d)^4} \right) J(z, p) \tag{3.54}$$

under $\mathrm{PSL}(2, \mathbb{C})$. Therefore, the integrand must transform as

$$I(g(z), p, \varepsilon) = \left(\prod_{j=1}^{n} (cz_j + d)^4 \right) I(z, p, \varepsilon). \tag{3.55}$$

Of course, we may use the polynomial form of the scattering equations to derive these results following a similar procedure. Taking $z_1 \to \infty$, $z_2 = 1$, and $z_n \to 0$ in the polynomial form of the scattering equations, we can write the contour integral as [75, 77]

$$\mathcal{A}_n(p, \varepsilon) = \frac{1}{(2\pi \mathrm{i})^{n-3}} \oint_{\mathcal{O}} \tilde{I}(z, p, \varepsilon) \, \mathrm{d}\tilde{\Omega}_{\text{CHY}}. \tag{3.56}$$

The integrand $\tilde{I}(z, p, \varepsilon)$ is defined by

$$\tilde{I}(z,p,\varepsilon) \equiv I(z,p,\varepsilon)\prod_{i=1}^{n}(z_i - z_{i+1})^2, \tag{3.57}$$

where by definition $z_{n+1} = z_1$, i.e., we have an inverse Parke-Taylor factor squared. Similarly, $\mathrm{d}\tilde{\Omega}_{\mathrm{CHY}}$ is given by

$$\mathrm{d}\tilde{\Omega}_{\mathrm{CHY}} \equiv \frac{z_2}{z_{n-1}}\prod_{m=1}^{n-3}\frac{1}{h_m}\prod_{2\le j<k\le n-1}(z_j - z_k)\prod_{i=2}^{n-2}\frac{z_i}{(z_i - z_{i+1})^2}\mathrm{d}z_{i+1}. \tag{3.58}$$

In this work we will be interested in the CHY representation as a sum over solutions (Eqs.(3.51)-(3.53)) and as a contour integral based on the polynomial form of the scattering equations (Eqs. (3.56)-(3.58)).

The contour integrals just introduced can be systematically solved using multi-residues and using the fact that the scattering polynomials form an H-basis [77,78]. An iterative procedure to solve these integrals was proposed in Ref. [91]. In addition, integration rules can be used to perform these integrals as shown in Refs. [92,93].

3.3 Integrands in the CHY-formula

We are ready to discuss the integrands appearing in (3.53) and (3.57) modulo inverse Parke-Taylor factors. These integrands are required to have the properties of an amplitude, i.e., on the support of the scattering equations they factorize, and they have the correct soft behavior. Let us introduce the integrands of the original CHY proposal, which using our conventions reads

$$\mathcal{A}_n^{(s)}(p,\varepsilon) = \mathrm{i}\int \mathrm{d}\Omega_{\mathrm{CHY}}\left(\frac{\mathrm{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})}{(z_1 - z_2)(z_2 - z_3)\cdots(z_n - z_1)} + \ldots\right)^{2-s}(\mathrm{Pf}'\Psi)^s \tag{3.59}$$

for $s = 0, 1, 2$. Here we identify three fundamental integrands for each value of s:

$$I_s(z,p,\varepsilon) = \mathrm{i}\left(\frac{\mathrm{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})}{(z_1 - z_2)(z_2 - z_3)\cdots(z_n - z_1)} + \ldots\right)^{2-s}(\mathrm{Pf}'\Psi)^s, \tag{3.60}$$

for scalars $s = 0$, gauge bosons $s = 1$, and gravitons $s = 2$. From these integrands, we identify a Parke-Taylor factor (also called a cyclic factor) given by

$$C(w,z) = \frac{1}{(z_{l_1} - z_{l_2})(z_{l_2} - z_{l_3})\cdots(z_{l_n} - z_{l_1})}, \tag{3.61}$$

where z denotes the dependence on the z_{l_i}'s. Here we are using the ordering $w = l_1 \dots l_n$ with $l_i \in \{1, 2, \dots, n\}$, and $l_i \neq l_j$ for $i \neq j$. These factors are the unevaluated *vectors* of the KLT orthogonality defined in Eq.(3.15). We also identify a permutation invariant factor $E(z, p, \varepsilon)$ corresponding to a reduced Pfaffian[11]

$$E(z, p, \varepsilon) \equiv \mathrm{Pf}'\Psi = \frac{(-1)^{i+j}}{z_{ij}} \mathrm{Pf}\Psi^{ij}_{ij}(z, p, \varepsilon), \tag{3.63}$$

where the $2n \times 2n$ antisymmetric matrix $\Psi(z, p, \varepsilon)$ is defined as follows:

$$\Psi(z, p, \varepsilon) = \begin{pmatrix} \mathsf{A} & -\mathsf{C}^T \\ \mathsf{C} & \mathsf{B} \end{pmatrix}, \tag{3.64}$$

with the $n \times n$ matrices A, B, and C given by[12]

$$A_{ab} = \begin{cases} \frac{s_{ab}}{z_{ab}}, & a \neq b, \\ 0, & a = b, \end{cases}, \qquad B_{ab} = \begin{cases} \frac{\epsilon_{ab}}{z_{ab}}, & a \neq b, \\ 0, & a = b, \end{cases}, \qquad C_{ab} = \begin{cases} \frac{\rho_{ab}}{z_{ab}}, & a \neq b, \\ -\sum\limits_{j=1,\, j\neq a} \frac{\rho_{ab}}{z_{aj}}, & a = b, \end{cases}, \tag{3.65}$$

where we define $\epsilon_{ab} = \varepsilon_a \cdot \varepsilon_b$ and $\rho_{ab} = \sqrt{2}\varepsilon_a \cdot p_b$. The matrix $\Psi^{ij}_{ij}(z, p, \varepsilon)$ denotes the $(2n-2) \times (2n-2)$ matrix, where the rows and columns i and j of Ψ have been removed $(1 \leq i < j \leq n)$.

Henceforth, we will drop the information about the color and focus on primitive amplitudes, therefore we write for scalars, gluons and gravitons, respectively

$$A^{(0)}_n(p, w, \tilde{w}) \equiv m(w|\tilde{w}) = \frac{\mathrm{i}}{(2\pi \mathrm{i})^{n-3}} \oint_{\mathcal{O}} \mathrm{d}\tilde{\Omega}_{\mathrm{CHY}} \prod_{i=1}^{n} (z_i - z_{i+1})^2 C(w, z) C(\tilde{w}, z), \tag{3.66}$$

$$A^{(1)}_n(p, w) \equiv A_n(p, w) = \frac{\mathrm{i}(-1)^n}{(2\pi \mathrm{i})^{n-3}} \oint_{\mathcal{O}} \mathrm{d}\tilde{\Omega}_{\mathrm{CHY}} \prod_{i=1}^{n} (z_i - z_{i+1})^2 C(w, z) E(z, p, \varepsilon), \tag{3.67}$$

$$A^{(2)}_n(p) \equiv M_n(p) = \frac{\mathrm{i}}{(2\pi \mathrm{i})^{n-3}} \oint_{\mathcal{O}} \mathrm{d}\tilde{\Omega}_{\mathrm{CHY}} \prod_{i=1}^{n} (z_i - z_{i+1})^2 E(z, p, \varepsilon)^2, \tag{3.68}$$

[11]The Pfaffian of a $2n \times 2n$ skew symmetric matrix satisfy $\det(\mathsf{A}) = \mathrm{Pf}(\mathsf{A})^2$. It is defined by

$$\mathrm{Pf}(\mathsf{A}) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sign}(\sigma) \prod_{i=1}^{n} A_{\sigma(2i-1), \sigma(2i)}, \tag{3.62}$$

where $\mathrm{sign}(\sigma)$ is the signature of the permutation.

[12]A refined version which makes manifest the Möbius invariance of the matrix C was proposed in Ref. [94].

where in the scalar case we have allowed the cyclic factors to have distinct orderings w and $\tilde{w}$. The minus sign in Eq.(3.67) was introduced to find amplitudes in agreement with the Feynman rules in Appendix A.1. The fact that gravity amplitudes are not ordered is manifest in Eq.(3.68), where we see that there is no dependence on the cyclic factor C—this is implicit in (3.59), where the term involving the cyclic factor becomes one. The factors $C(w,z)$ and $E(z,p,\varepsilon)$ are the basic ingredients of the known theories that admit a CHY representation. Notice that they transform as expected under Möbius transformations, i.e.,

$$C(w,g(z)) = \left(\prod_{j=1}^{n}(cz_j+d)^2\right) C(w,z), \tag{3.69}$$

$$E(g(z),p,\varepsilon) = \left(\prod_{j=1}^{n}(cz_j+d)^2\right) E(z,p,\varepsilon), \tag{3.70}$$

in agreement with Eq. (3.55). These transformations under $\mathrm{PSL}(2,\mathbb{C})$ constrain the space of theories that admit a CHY representation, but it is possible to generalize these factors to "CHY represent" other theories. The known representations are obtained by generalizing those ingredients. Furthermore, theories like Einstein-Yang-Mills, Einstein-Maxwell, the nonlinear sigma model, etc, can be obtained by *compactifying*, *squeezing*, and *generalizing* the factor $E(z,p,\varepsilon)$. These operations and the theories which result from them are discussed in Ref. [87]. This ends our review of the integrands of the CHY representation. In the following Section, we give examples of computations using these formulas for the scalars, gluons and gravitons for simple cases.

3.4 Elementary examples

For simplicity, in this Section we give some worked examples up to four points. The main point of these examples is to illustrate the algebraic approach using computer algebraic geometry techniques. In the Appendix B, we give an example up to $n=6$ for the scalar theory. Details of residues and the generalities of the algorithm are presented in Appendix B.3.

We call the *algebraic approach* the approach based on the computation of the multivariate residues from the point of view of algebraic geometry. Hence, we are interested in the master formula for the global residues. i.e.,

$$\mathrm{Res}\left(\frac{N(z)}{f_1(z)\cdots f_n(z)}\right) = \mathrm{Res}_{\{\mathbf{f}\}}(N(z)) \equiv \langle N(z),1\rangle\,, \tag{3.71}$$

where the last equality we write the sum over residues as an inner product. This approach makes the computation of residues a computer algebra problem.

3.4.1 Scalar theory

Let us start with Eq.(3.66) which computes amplitudes for the bi-adjoint scalar. Its Lagrangian was introduced in Chapter 2, where the Feynman rules were also introduced.

3.4.1.1 Sum over solutions

Let us start with an example of Eq.(3.53). The integrand is given by

$$I = \mathrm{i}\ C(w,z)C(\tilde{w},z). \tag{3.72}$$

For $n=4$ there is one solution of the scattering equations (see Section 3.1.4). The Jacobian reads

$$J(z^{(1)},p) = -\frac{{s_{12}}^2 x^2 \left(\frac{s_{12}}{s_{13}}+x\right)^2}{{s_{13}}^2 \left(\frac{s_{12}}{(x-1)^2}+\frac{s_{23}}{\left(\frac{s_{12}}{s_{13}}+1\right)^2}+s_{24}\right)}, \tag{3.73}$$

where we have chosen $\{i,j,k\}$ and $\{r,s,t\}$ to be $\{1,3,4\}$. Here x is a placeholder for the third component of $z^{(1)}$. With the orderings $w=1234$ and $\tilde{w}=1324$, the integrand reads

$$I(z^{(1)},p) = \mathrm{i}\ C(1234,z^{(1)})C(1324,z^{(1)}) = \mathrm{i}\frac{{s_{13}}^4}{s_{12}\left(s_{12}+s_{13}\right)^2(x-1)x^2\left(s_{13}x+s_{12}\right)}, \tag{3.74}$$

hence taking the limit $x\to\infty$, we obtain

$$\begin{aligned} m(1234|1324) =& \lim_{x\to\infty} J(z^{(1)},p)I(z^{(1)},p) \\ =& -\mathrm{i}\frac{s_{12}s_{13}}{s_{24}{s_{12}}^2+2s_{13}s_{24}s_{12}+{s_{13}}^2\left(s_{23}+s_{24}\right)} \\ =& -\mathrm{i}\frac{1}{s_{12}+2s_{24}-s_{13}} \\ =& \mathrm{i}\frac{1}{s_{14}}. \end{aligned}$$

We can verify that this is the correct result in many ways, for example using Feynman diagrams[13].

3.4.1.2 Algebraic approach

This approach is based on Eq. (3.66). For $n = 4$ we have

$$\mathrm{d}\tilde{\Omega}_{\mathrm{CHY}} = \frac{z_2}{z_3}\frac{1}{h_1} z_{23}\frac{z_2}{z_{23}^2}\mathrm{d}z_3, \tag{3.75}$$

therefore

$$m(1234|1324) = \\ \frac{\mathrm{i}}{(2\pi\mathrm{i})}\oint_{\mathcal{O}}\left(\frac{1}{z_{12}z_{23}z_{34}z_{41}}\frac{1}{z_{13}z_{32}z_{24}z_{41}}\right)(z_{12}z_{23}z_{34}z_{41})^2\Big|_{\substack{z_1\to\infty\\ z_4=0}}\left(\frac{z_2}{z_3}\frac{z_{23}}{h_1}\frac{z_2}{z_{23}^2}\mathrm{d}z_3\right), \tag{3.76}$$

where we have grouped the integrand and the measure separately. Inserting the scattering equation h_1 and taking the limits, the integral becomes

$$m(1234|1324) = \frac{\mathrm{i}}{(2\pi\mathrm{i})}\oint_{\mathcal{O}}\left(-\frac{z_3}{z_2}\right)\left(\frac{z_2^2}{z_{23}}\frac{1}{z_3(s_{12}z_2+s_{13}z_3)}\mathrm{d}z_3\right).$$

Setting $z_2 = 1$ and performing the contour integral, we obtain

$$m(1234|1324) = \frac{\mathrm{i}}{(2\pi\mathrm{i})}\oint_{\mathcal{O}} -\frac{\mathrm{d}z_3}{1-z_3}\frac{1}{s_{12}+s_{13}z_3} \tag{3.77}$$

$$= \mathrm{i}\frac{1}{s_{14}}, \tag{3.78}$$

which coincides with the result of the sum over solutions. Here, the procedure is straightforward because we have only one integral to perform. In the case where more integrals are needed, we can use the global residue theorem (See. e.g., [89]). However, the method proposed by Søgaard and Zhang based on the Bezoutian matrix in algebraic geometry is more convenient. Let us follow the algorithm based on the Bezoutian matrix[14].

1. Calculate the basis. The aim of this algorithm is to use Eq.(3.71) as an inner product. Therefore we need a basis and its dual. For the ideal $I = \langle s_{12} + s_{13}z_3\rangle$, the Gröbner basis in Degree reverse lexicographic monomial order reads

[13] An interesting approach to verify these results is to use the recursive algorithm by Mafra [95] based on the perturbiner approach [96].

[14] Details can be found in Appendix B.3.

$$g_1 = s_{13} z_3 + s_{12}. \tag{3.79}$$

A canonical linear basis—all monomials of degree lower than z_3—in the quotient ring $\mathbb{C}[z_3]/I$ is given by $\{e_i\} = \{1\}$.

2. Calculate the dual basis Δ_i.

   (a) Compute the 1×1 Bezoutian matrix:

$$\mathsf{B} = \frac{(s_{12} + s_{13} z_3) - (s_{12} + s_{13} y_1)}{z_3 - y_1} = s_{13}. \tag{3.80}$$

   (b) Define the associated Gröbner basis by setting $z_i \to y_i$:

$$\tilde{g}_1 = s_{13} y_1 + s_{12}. \tag{3.81}$$

   (c) Calculate the polynomial division of $\det(B)$ over $G \otimes G$:

$$\det(\mathsf{B}) = 0 \times (s_{12} + s_{13} z_3) + 0 \times (s_{13} y_1 + s_{12}) + s_{13} \times 1. \tag{3.82}$$

   (d) The remainder of the above division gives the dual basis as the coefficients of the basis (i.e., the coefficients of $e_1 = 1$), hence $\Delta_i = \{s_{13}\}$.

3. Compute the inverse of $(1 - z_3)$ with respect to the ideal $J = \langle s_{12} + s_{13} z_3, 1 - z_3 \rangle$. This inverse is obtained by computing the Gröbner basis of J and writing

$$1 = (s_{13} z_3 + s_{12}) \frac{1}{s_{12} + s_{13}} + (1 - z_3) \frac{s_{13}}{s_{12} + s_{13}}, \tag{3.83}$$

   hence the polynomial inverse of $(1 - z_3)$ is $(-s_{13}/s_{14})$.

4. The global residue is then given by

$$\text{Res}\left(-\frac{1}{(1 - z_3)}\right) = \text{Res}\left(\frac{s_{13}}{s_{14}}\right). \tag{3.84}$$

   The numerator can be expressed in terms of the basis e_i and the unity can be decomposed in terms of the dual basis, i.e.,

$$\frac{s_{13}}{s_{14}} = \left(\frac{s_{13}}{s_{14}}\right) e_1,$$
$$1 = \frac{1}{s_{13}}\Delta_1, \tag{3.85}$$

respectively. Finally,

$$\begin{aligned} m(1234|1324) =& \mathrm{i}\ \mathrm{Res}\left(-\frac{1}{(1-z_3)}\right) \\ =& \mathrm{i}\ \langle \frac{s_{13}}{s_{14}}, 1\rangle \\ =& \mathrm{i}\frac{s_{13}}{s_{14}}\frac{1}{s_{13}}\ \langle e_1, \Delta_1\rangle \\ =& \mathrm{i}\frac{1}{s_{14}}, \end{aligned} \tag{3.86}$$

as expected.

It may look involved, but this algorithm can handle more complex cases once a computer algebra implementation is performed[15]. Notice that the power of this method is that we have successfully computed the contour integral without *solving* the scattering equations.

3.4.2 Yang-Mills

The simplest example is the 3-point amplitude, where of course there are no integrations to do. In this case, the amplitude is fixed and is given by

$$A_3(123) = \tilde{I}(z,p,\varepsilon)\Big|_{\substack{z_1\to\infty \\ z_4\to 0 \\ z_2\to 1}} = -\mathrm{i}\sqrt{2}(\epsilon_{12}\rho_{31} + \epsilon_{23}\rho_{12} + \epsilon_{13}\rho_{23}), \tag{3.87}$$

where we remind the reader that $\rho_{ij} = \sqrt{2}\varepsilon_i \cdot p_j$ and $\epsilon_{ij} = \varepsilon_i \cdot \varepsilon_j$. Let us now work with Yang-Mills for $n = 4$ with the ordering $w = 1234$. In this case the integrand reads

$$\tilde{I}(z,p,\varepsilon) = \mathrm{i}(z_{12}z_{23}z_{34}z_{41})^2 C(1234,z)E(z,p,\varepsilon) = \mathrm{i}(z_{12}z_{23}z_{34}z_{41})E(z,p,\varepsilon), \tag{3.88}$$

where explicitly

[15]For this example we have used MATHEMATICA and the Package MathematicaM2 [97] which integrates MACAULAY2 [98].

$$
\begin{aligned}
E(z,p,\varepsilon) = & \frac{\rho_{13}\epsilon_{23}\rho_{41}}{z_{12}z_{13}z_{14}z_{23}} - \frac{\epsilon_{13}\rho_{23}\rho_{41}}{z_{12}z_{13}z_{14}z_{23}} + \frac{\rho_{14}\epsilon_{24}\rho_{31}}{z_{12}z_{13}z_{14}z_{24}} - \frac{\epsilon_{14}\rho_{24}\rho_{31}}{z_{12}z_{13}z_{14}z_{24}} + \frac{\epsilon_{12}\rho_{31}\rho_{41}}{z_{12}^2z_{13}z_{14}} \\
& + \frac{\rho_{13}\epsilon_{23}\rho_{42}}{z_{12}z_{13}z_{23}z_{24}} - \frac{\epsilon_{13}\rho_{23}\rho_{42}}{z_{12}z_{13}z_{23}z_{24}} + \frac{\rho_{13}\epsilon_{23}\rho_{43}}{z_{12}z_{13}z_{23}z_{34}} - \frac{\epsilon_{13}\rho_{23}\rho_{43}}{z_{12}z_{13}z_{23}z_{34}} + \frac{\epsilon_{12}\rho_{31}\rho_{42}}{z_{12}^2z_{13}z_{24}} \\
& - \frac{\epsilon_{13}\rho_{24}\rho_{43}}{z_{12}z_{13}z_{24}z_{34}} + \frac{s_{34}\epsilon_{13}\epsilon_{24}}{z_{12}z_{13}z_{24}z_{34}} + \frac{\rho_{13}\rho_{24}\epsilon_{34}}{z_{12}z_{13}z_{24}z_{34}} - \frac{\rho_{13}\epsilon_{24}\rho_{34}}{z_{12}z_{13}z_{24}z_{34}} + \frac{\epsilon_{12}\rho_{31}\rho_{43}}{z_{12}^2z_{13}z_{34}} \\
& + \frac{\rho_{14}\epsilon_{24}\rho_{32}}{z_{12}z_{14}z_{23}z_{24}} - \frac{\epsilon_{14}\rho_{24}\rho_{32}}{z_{12}z_{14}z_{23}z_{24}} + \frac{\epsilon_{12}\rho_{32}\rho_{41}}{z_{12}^2z_{14}z_{23}} + \frac{\rho_{14}\epsilon_{23}\rho_{43}}{z_{12}z_{14}z_{23}z_{34}} - \frac{s_{34}\epsilon_{14}\epsilon_{23}}{z_{12}z_{14}z_{23}z_{34}} \\
& + \frac{\epsilon_{14}\rho_{23}\rho_{34}}{z_{12}z_{14}z_{23}z_{34}} - \frac{\rho_{14}\rho_{23}\epsilon_{34}}{z_{12}z_{14}z_{23}z_{34}} + \frac{\epsilon_{14}\rho_{24}\rho_{34}}{z_{12}z_{14}z_{24}z_{34}} - \frac{\rho_{14}\epsilon_{24}\rho_{34}}{z_{12}z_{14}z_{24}z_{34}} - \frac{\epsilon_{12}\rho_{34}\rho_{41}}{z_{12}^2z_{14}z_{34}} \\
& + \frac{\epsilon_{12}\rho_{32}\rho_{42}}{z_{12}^2z_{23}z_{24}} + \frac{\epsilon_{12}\rho_{32}\rho_{43}}{z_{12}^2z_{23}z_{34}} - \frac{\epsilon_{12}\rho_{34}\rho_{42}}{z_{12}^2z_{24}z_{34}} - \frac{s_{34}\epsilon_{12}\epsilon_{34}}{z_{12}^2z_{34}^2}.
\end{aligned}
\tag{3.89}
$$

The measure is the same as in the scalar case, i.e.,

$$
\mathrm{d}\tilde{\Omega}_{\text{CHY}} = \frac{\mathrm{d}z_3}{(1-z_3)z_3(s_{12}+s_{13}z_3)}. \tag{3.90}
$$

For concreteness, let us consider the MHV amplitude $A_4(1^-2^-3^+4^+)$. Using the spinor-helicity formalism[16], and choosing the reference vectors to be $q_1 = q_2 = p_3$ and $q_3 = q_4 = p_2$, we have

$$
\epsilon_{1^-3^+} = \epsilon_{1^-2^-} = \epsilon_{2^-4^+} = \epsilon_{3^+4^+} = \epsilon_{2^-3^+} = \rho_{2^-3^+} = \rho_{3^+2^-} = 0, \tag{3.91}
$$

where we have indicated the helicities by the superscripts $\pm$. We have the integrand

$$
\tilde{I}(z,p,\varepsilon)\Big|_{\substack{z_1\to\infty \\ z_4\to 0 \\ z_2\to 1}} = -2\mathrm{i}\,\rho_{2^-4^+}\rho_{3^+4^+}\epsilon_{1^-4^+}(1-z_3). \tag{3.92}
$$

In principle, we need to generate the inverse of $(1-z_3)$ and (z_3) which are given by

$$
-\frac{s_{13}}{s_{14}}, \quad -\frac{s_{13}}{s_{12}}, \tag{3.93}
$$

respectively, but in this example we only require the inverse of z_3. This can be seen as follows:

[16] See Appendix A

$$
\begin{aligned}
A_4(1^-2^-3^+4^+) =& -2\mathrm{i}\rho_{2^-4^+}\rho_{3^+4^+}\epsilon_{1^-4^+}\mathrm{Res}\left(\frac{(1-z_3)}{(1-z_3)z_3}\right) \\
=& -2\mathrm{i}\rho_{2^-4^+}\rho_{3^+4^+}\epsilon_{1^-4^+}\left\langle -\frac{s_{13}}{s_{12}}, 1\right\rangle \\
=& -2\mathrm{i}\rho_{2^-4^+}\rho_{3^+4^+}\epsilon_{1^-4^+}\left\langle \frac{-s_{13}}{s_{12}}, 1\right\rangle \\
=& 2\mathrm{i}\rho_{2^-4^+}\rho_{3^+4^+}\epsilon_{1^-4^+}\frac{1}{s_{13}}\frac{s_{13}}{s_{12}}\langle e_1, \Delta_1\rangle \\
=& 2\mathrm{i}\rho_{2^-4^+}\rho_{3^+4^+}\epsilon_{1^-4^+}\frac{1}{s_{12}}.
\end{aligned}
\tag{3.94}
$$

Finally, inserting the spinor products we obtain

$$
A_4(1^-2^-3^+4^+) = 2\mathrm{i}\left(-\frac{\langle 24\rangle [43]}{\sqrt{2}[32]}\right)\left(\frac{[34]\langle 42\rangle}{\sqrt{2}\langle 23\rangle}\right)\left(-\frac{\langle 12\rangle [43]}{\langle 24\rangle [31]}\right)\frac{1}{\langle 12\rangle [12]},
\tag{3.95}
$$

which yields

$$
A_4(1^-2^-3^+4^+) = \mathrm{i}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}.
\tag{3.96}
$$

3.4.3 Gravity

In this example, we will compute the MHV amplitude for gravity. This amplitude was introduced in Chapter 2 in the context of the KLT relations. Here, we recover this amplitude using the CHY formalism and the Bezoutian matrix method. The contour integral reads

$$
M_4(1^{--}2^{--}3^{++}4^{++}) = \frac{\mathrm{i}}{2\pi\mathrm{i}}\oint_{\mathcal{O}} 4\epsilon^2_{1^-4^+}\rho^2_{2^-4^+}\rho^2_{3^+4^+}(z_3-1)^2\frac{\mathrm{d}z_3}{(1-z_3)z_3(s_{12}+s_{13}z_3)}.
\tag{3.97}
$$

We have all the necessary ingredients to compute this amplitude from the examples above. The residue computation gives

$$\begin{aligned} M_4(1^{--}2^{--}3^{++}4^{++}) =& 4\mathrm{i}\epsilon^2_{1^-4^+}\rho^2_{2^-4^+}\rho^2_{3^+4^+}\mathrm{Res}\left(\frac{1-z_3}{z_3}\right) \\ =& 4\mathrm{i}\epsilon^2_{1^-4^+}\rho^2_{2^-4}\rho^2_{3^+4^+}\langle -\frac{s_{13}}{s_{12}}(1-z_3),1\rangle \\ =& 4\mathrm{i}\epsilon^2_{1^-4^+}\rho^2_{2^-4}\rho^2_{3^+4^+}\langle -\frac{s_{13}}{s_{12}}-1,1\rangle \\ =& 4\mathrm{i}\epsilon^2_{1^-4^+}\rho^2_{2^-4}\rho^2_{3^+4^+}\frac{1}{s_{13}}\left(-\frac{s_{13}}{s_{12}}-1\right) \\ =& 4\mathrm{i}\epsilon^2_{1^-4^+}\rho^2_{2^-4}\rho^2_{3^+4^+}\left(\frac{s_{23}}{s_{12}s_{24}}\right). \end{aligned} \tag{3.98}$$

In the third line we have used that the residue only depends on the class of the numerator. In other words, we have considered the remainder of $N(z) = q(z) + r(z)$ with respect to the grevlex monomial order, which gave a representative of $[N] \in C[z_3]/I$[17]. Using the result of the MHV amplitudes for gluons, we have

$$\begin{aligned} M_4(1^{--}2^{--}3^{++}4^{++}) = & 2s_{23}A_4(1^-2^-3^+4^+) \\ & \times\left(-\frac{\langle 24\rangle\,[43]}{\sqrt{2}[32]}\right)\left(\frac{[34]\,\langle 42\rangle}{\sqrt{2}\,\langle 23\rangle}\right)\left(-\frac{\langle 12\rangle\,[43]}{\langle 24\rangle\,[31]}\right)\frac{1}{\langle 24\rangle\,[24]}, \end{aligned} \tag{3.99}$$

which after some algebra simplifies to the KLT relation (2.175) for $n = 4$, i.e.,

$$\begin{aligned} M_4(1^{--}2^{--}3^{++}4^{++}) =& s_{23}A_4(1^-2^-3^+4^+)\left(\frac{[34]^4}{[13][32][24][41]}\right) \\ =& -\mathrm{i}A_4(1^-2^-3^+4^+)\; s_{23}\; A_4(1^-3^-2^+4^+). \end{aligned} \tag{3.100}$$

3.5 Comments on the CHY-formalism

In this section we have presented the main features of the CHY-formalism based on the scattering equations. We introduced the different equivalent flavors of the formalism to write amplitudes.

First, we can use it as a recipe for localizing integrals on the solutions of the scattering equations, in analogy with Witten's formalism. Actually this makes clear that the CHY-formula can be understood as the generalization of the *connected formalism* by Roiban-Spradlin-Volovich-Witten [15]. In this way, we can think of the CHY-formula as an abstract "Fourier transform" from $\mathcal{M}_{0,n}$ to the kinematic space. The integrand playing the role of Fourier transformed amplitude. The relation of the CHY-formula with string

[17]See details in the Appendix B.3

integrals is, of course, no coincidence. The precise connection was shown in Ref. [68]. In Ref. [69] the CHY formulas were understood from the point of view of the ambitwistor string.

Second, we can view amplitudes as the result of performing the contour integral such that the result is a sum over evaluations of the scattering equations. Formally, this solves the problem if we have at our disposal the solutions of the scattering equations, which of course is the big elephant in the room[18]. Nevertheless, as a sum over solutions, we can use the KLT orthogonality and study the CHY formula as a problem in linear algebra. For instance, we may investigate under which circumstances one can *invert* the CHY formula and investigate the space of CHY integrands which reproduce known amplitudes. In the next Chapter, we will use this approach to prove that QCD admits a CHY representation and in Chapter 5 to show relations among amplitudes in Einstein-Yang-Mills Theory.

Third, we can view amplitudes in the CHY formalism as a well-defined problem in algebraic geometry, i.e., as a contour integral with the contour enclosing the solutions of the scattering equations. This transforms the problem into a problem of computational residue calculation. Indeed, in the examples presented in this Chapter, we have seen that the amplitudes ultimately can be computed as an inner product, i.e.,

$$A_n^{(s)}(p,w,\varepsilon) = \langle N_{\text{CHY}}^{(s)}, 1\rangle \,, \tag{3.101}$$

where the numerator is theory dependent. This amounts to decompose the numerator in an algebraic basis $\{e_i\}$ and the unity in a dual basis $\{\Delta_i\}$. The versatility of the CHY-formalism makes it a conceptual tool for the understanding of amplitudes.

Finally, let us think what does the CHY-formula tell us about amplitudes at tree-level. In all flavors of the CHY formula, we compute the connected part of the S-matrix without computing an explicit integral, drawing a Feynman diagram, using Hilbert space, using creation and annihilation operators, and most important we did not use the Lagrangian at any point in the description. Of course, we may rephrase this sentence and say that we have computed a QFT amplitude without actually using QFT. It is very interesting then to ask: where are the elements of special relativity and quantum mechanics encoded in the CHY-formula? By construction, the formula contains Lorentz invariant products and it also encodes unitarity and locality through factorization—in analogy to BCFW recursions. It is also very interesting to ask whether one can deduce from QFT—where unitarity and locality (local poles) arise from first principles and the cluster decomposition principle—a formalism like the CHY[19]. Concretely, we may pose the question of how a formula like the Dyson series (Eq.(2.33)), which makes explicit the ingredients of QFT, can be mapped to the CHY-formula. Of course one can pose the same question for other approaches to

[18] As we have seen, solving the scattering equations is a highly nontrivial problem.
[19] An equivalence to Feynman diagrams was studied in Ref. [99].

QFT, e.g., Feynman path integrals. One of the main obstacles to address those questions is the lack of a guiding physical principle such as the cluster decomposition principle in QFT, which tells us that singularities occur in the form of poles or branch cuts.

In the next Chapters, we will first find another theory which fits into the CHY formalism and then show the interplay with the color-kinematics duality. The connection to this duality may be used as a tool to make the connection with QFT.

Chapter 4

QCD in the CHY formalism

In this Chapter we continue our study of the CHY representation but now we focus on pure Yang Mills and QCD. We discuss the necessary ingredients for a CHY representation of QCD to exist and how the building blocks should be redefined—namely the permutation invariant factor $E(z, p, \varepsilon)$ and the cyclic factor $C(w, z)$. The known theories that admit a CHY representation in one way or another can be related to modifications of those fundamental building blocks. We give the necessary modifications to these building blocks and prove that we can represent QCD as a sum over evaluations of the solutions of the scattering equations. We end with a brief account of the search for a closed integrand for the CHY representation of QCD.

4.1 Pure Yang-Mills amplitudes

Before studying QCD, let us study the simpler case of pure Yang-Mills amplitudes, i.e., amplitudes containing only gluons. The color decomposition for pure Yang-Mills which we reviewed in Section 2.4.3 separates the color information from the kinematic information. The kinematic information is contained in the gauge invariant and color independent primitive amplitudes. These amplitudes have a fixed ordering and depend on a set of polarization vectors[1] ε and momentum vectors p. The color decomposition of the full gauge amplitude in terms of these primitives reads

$$\mathcal{A}_n(p, \varepsilon) = \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) A(p_{\sigma(1)}, \varepsilon_{\sigma(1)}; p_{\sigma(2)}, \varepsilon_{\sigma(2)}; \ldots; p_{\sigma(n)}, \varepsilon_{\sigma(n)}), \quad (4.1)$$

where the ordering of the legs is specified by the permutations σ and the sum runs over noncyclic permutations of the set $\{1, 2, \ldots, n\}$. We use the notation

[1] Recall that ε stands for the whole collection of polarization vectors and similarly for the whole collection of momentum vectors p.

$$A_n(\sigma, p, \varepsilon) \equiv A(p_{\sigma(1)}, \varepsilon_{\sigma(1)}; p_{\sigma(2)}, \varepsilon_{\sigma(2)}; \ldots; p_{\sigma(n)}, \varepsilon_{\sigma(n)}), \tag{4.2}$$

indicating that we are interested in the ordering specified by σ. Furthermore, we set an *alphabet*

$$\mathbb{A} = \{g_1, \ldots, g_n\} = \{1, 2, \ldots, n\}, \tag{4.3}$$

indicating the particle content and their corresponding labels. Of course, we can choose other labels for the *alphabet* $\mathbb{A}$ since we are interested on the relative ordering of the *letters* in this alphabet and not on their labels[2]. The alphabet was defined so we can work with sequences of letters

$$w = l_1 l_2 \ldots l_n, \tag{4.4}$$

which are called *words*. The letters l_i specify the particle content as well—in pure Yang-Mills all particles are gluons so this information is not particularly useful. The space of *words* inside and *alphabet* form a vector space V of orderings[3] and therefore we can define a basis B of the vector space. In order to find the basis of the space, consider the most general set of words such that ever letter of the alphabet occurs exactly once, i.e.,

$$W_0 = \{l_1, l_2, \ldots, l_n | l_i \in \mathbb{A}, l_i \neq l_j, \quad \text{for} \quad i \neq j\}. \tag{4.5}$$

The sum in Eq.(4.1) is equivalent to a sum of $(n-1)!$ elements, where one of the letters is fixed and consider all permutations of the remaining $(n-1)$ letters [37,100]. A possible basis is then given by

$$W_1 = \{l_1 l_2 \ldots l_n \in W_0 | l_1 = g_1\}. \tag{4.6}$$

In this sense the amplitude A_n can be taken as a linear operator in the space of words w with basis W_1 (See Section 2.4.4). The color decomposition in this language reads

$$\mathcal{A}_n(p, \varepsilon) = \sum_{w \in W_1} \mathrm{Tr}(T^{a_{l_1}} T^{a_{l_2}} \cdots T^{a_{l_n}}) A_n(w, p, \epsilon). \tag{4.7}$$

[2] See Appendix C.
[3] Ibid.

In the CHY representation of pure Yang-Mills we also have one fixed letter in w. Consequently, the basis B will have $(n-1)!$ orderings [14] as well. Expressing the primitive amplitudes $A_n(w,p,\varepsilon)$ in the CHY representation (3.38), we have

$$\mathcal{A}_n(p,\varepsilon) = \mathrm{i} \sum_{w \in W_1} \int \mathrm{d}\Omega_{\mathrm{CHY}} \, \mathrm{Tr}(T^{a_{l_1}} \cdots T^{a_{l_n}}) C(w,z) E(z,p,\varepsilon). \tag{4.8}$$

It should be clear that we can work with the primitive amplitudes exclusively. Although in Eq.(4.8) we wrote the CHY-formula using only orderings in W_1, the formula is valid for all primitive amplitudes, i.e., for the set of orderings W_0. Following the rules outlined in Chapter 3 (See Eq.(3.53)), the CHY representation of pure Yang-Mills primitive amplitudes reads

$$A_w \equiv A_n(w,p,\epsilon) = \mathrm{i} \sum_{\text{solutions } j} J(z^{(j)},p) C(w,z^{(j)}) E(z^{(j)},p,\varepsilon), \qquad \forall w \in W_0, \tag{4.9}$$

where w stands for orderings and j labels the $j-$th solution of the scattering equations. It is remarkable that the CHY representation allows us to separate the information of the scattering amplitude even more than the original color decomposition. The color decomposition separates the color structure of Yang-Mills from the gauge invariant part, which is the primitive amplitude. Furthermore, the CHY-formula permits the factorization of the information: the ordering is encoded in the Parke-Taylor factor C and the permutation invariant factor E which is gauge invariant and permutation invariant. This has similarities with the color-kinematics duality where a similar factorization occurs but in that case the factorization is encoded in the algebra of numerators—for instance see Section 2.4.5. We add to our list of “worded equations” the CHY representation of gauge theories as factorization of information, i.e,

$$\text{full gauge} = \text{color} \times \text{cyclic} \times \text{permutation}. \tag{4.10}$$

We would like to determine the additional conditions that the building blocks of the CHY representation should satisfy besides the properties imposed by the scattering equations (e.g., $\mathrm{PSL}(2,\mathbb{C})$ invariance). In order to study these conditions, let us first assume that we are in the situation where we can compute the primitive amplitudes in pure Yang-Mill by other means, e.g., Feynman diagrams, BCFW, Berends-Giele, etc. Then, we want deduce the conditions that allow the factorization of information in the CHY representation. In other words, we are first concerned with the problem of existence.

The first step is to set is a basis of amplitudes and determine if amplitudes in this basis have a CHY representation. The basis can be set by shrinking the set W_1 after imposing KK and BCJ relations[4] giving $N_{\text{basis}} = (n-3)!$ orderings. We choose the basis to be

$$B = \{l_1 l_2 \dots l_n \in W_0 | l_1 = g_1, l_{n-1} = g_{n-1}, l_n = g_n\}. \tag{4.11}$$

Thinking of the amplitude as an operator in the space of words—as we did in Section 2.4.4—such that A_w in Eq.(4.9) is the N_{basis}-dimensional vector

$$A_w \equiv \mathrm{i} M_{wj} E_j, \qquad w \in B, \tag{4.12}$$

where there is an implicit sum over repeated indices. We have defined the $N_{\text{basis}} \times N_{\text{solutions}}$ matrix as

$$M_{wj} = J(z^{(j)}, p) C(w, z^{(j)}), \qquad w \in B. \tag{4.13}$$

Notice that Eq.(4.12) is satisfied if we can invert it, in other words if we find a permutation invariant factor E_j such that

$$E_j = -\mathrm{i} \sum_{v \in B} N_{jv} A_v = -\mathrm{i} N_{jv} A_v, \tag{4.14}$$

where the matrices N and M satisfy

$$M_{wj} N_{jv} = \delta_{wv}. \tag{4.15}$$

Thinking ahead, we have avoided the use of the inverse matrix since we are thinking in the general case where the matrices may be rectangular. The sum runs over the $N_{\text{solutions}} = (n-3)!$ solutions, which is the same number of elements in the basis. Therefore, in this case we have square matrices.

In Section 3.1.2, we met the key property of the scattering equations that can be used to prove Eq.(4.14)—the KLT orthogonality—i.e., the property which relates cyclic factors and the KLT momentum kernel when evaluated at solutions of the scattering equations [14, 62]. The KLT orthogonality reads

[4] In pure Yang-Mills, due to relations among primitive amplitudes, the size of the basis is $N_{\text{basis}} = (n-3)!$ as we reviewed in Section 2.4.4.1

$$\frac{\langle C^i, \bar{C}^j \rangle}{\langle C^i, \bar{C}^i \rangle^{\frac{1}{2}} \langle C^j, \bar{C}^j \rangle^{\frac{1}{2}}} = \delta^{ij}, \tag{4.16}$$

where

$$\langle C^i, \bar{C}^j \rangle = \sum_{w\in B} \sum_{v\in B} C(w, z^{(i)}) S_{w\bar{v}} C(\bar{v}, z^{(j)}). \tag{4.17}$$

Here we emphasize that the momentum kernel $S_{w\bar{v}}$ can be thought as a matrix in the space of words w, where the bar in $\overline{v}$ indicates that we modify the last two letters in the ordering of the Parke-Taylor factor. We can relate to the momentum kernel to the CHY formula because the Jacobian in (4.9) is given by

$$J(z^{(j)}, p) = \frac{1}{\langle C^j, \overline{C}^j \rangle}, \tag{4.18}$$

which can be used to find the inverse of $S_{w\bar{v}}$

$$\left(S^{-1}\right)_{w\bar{v}} = \sum_{j=1}^{(n-3)!} C(w, z^{(j)}) J(z^{(j)}, p) C(\overline{v}, z^{(j)}). \tag{4.19}$$

where we have used matrix notation in the space of orderings for $S[w|\bar{v}]$[5]. Thus, the unit matrix in the space of orderings can be recovered as the product

$$\sum_{u\in B} S_{w\bar{u}} \left(S^{-1}\right)_{\bar{u}v} = \sum_{j=1}^{(n-3)!} \sum_{u\in B} S_{w\bar{u}} C^j(\bar{u}, z^{(j)}) J(z^{(j)}, p) C(v, z^{(j)}) = \delta_{wv}. \tag{4.20}$$

We write explicitly the dependence on the j'th solution to indicate that we do not evaluate the momentum kernel. Therefore, setting

$$N_{jv} = \sum_{u\in B} S_{v\bar{u}} C(\bar{u}, z^{(j)}), \tag{4.21}$$

gives the desired N matrix in Eq.(4.14). Then the permutation invariant factor is given by $E_j = -\mathrm{i} N_{jv} A_v$. Explicitly

[5] Notice the importance of the fact that $(n-3)!$ is the dimension of the space of solutions and the dimension of B

$$E_j = -\mathrm{i} \sum_{v \in B} \sum_{u \in B} S_{v\bar{u}} C(\bar{u}, z^{(j)}) A_v. \tag{4.22}$$

The factor E satisfies (4.12) since

$$A_w = M_{wj} N_{jv} A_v = \delta_{wv} A_v = A_w. \tag{4.23}$$

Notice the importance of the KLT orthogonality for this result. It is also clear that we obtain another equivalent representation, if we multiply the matrices N, M by an arbitrary nonzero number and its inverse, respectively. Hence, the representation (4.23) is not unique.

In conclusion, we have proved that the amplitudes in the basis B have a CHY representation—which in this case we already knew—but what about the set $W_0 \backslash B$, or the whole W_0? Notice that the information about the orderings w enters through $C(w, z^{(j)})$ in M_{wj}, therefore the question is if the amplitude defined as

$$\hat{A}_w \stackrel{?}{=} \mathrm{i} M_{wj} E_j, \qquad \forall w \in W_0, \tag{4.24}$$

gives all possible amplitudes provided E_j is given by Eq.(4.22). Keeping in mind that in this case the matrix M_{wj} has dimension $n! \times N_{\text{solutions}}$. If this is the case, then $\hat{A}_w$ is an arbitrary primitive Yang-Mills amplitude and it *must* be cyclic, satisfy KK relations, and satisfy BCJ relations [101]. In other words we should prove that Eq.(4.24) satisfy these relations. The first requirement is met by recognizing that we have been calling the Parke-Taylor factors *cyclic factors* $C(w, z^{(j)})$ because of course they are cyclic. Similarly, the cyclic factors satisfy KK relations and on the support of the scattering equations they also satisfy BCJ relations [62]. We conclude that the permutation invariant factor E_j defined through Eq.(4.22) can be used for all amplitudes.

Once we have shown that the representation exist, one would like to find a permutation invariant factor which does not depend on other amplitudes—in pure Yang-Mills this factor is the reduced Pfaffian. This procedure gives an integrand of the CHY-formalism, which in view of the procedure outlined in Section 3.2, is given by

$$I(z, p, \varepsilon) = -\mathrm{i}\; C(w, z) \sum_{v \in B} \sum_{u \in B} S_{v\bar{u}} C(\bar{u}, z) A_v, \qquad \forall w \in W_0, \tag{4.25}$$

which, of course, we knew. Of course, this integrand is not very useful for applications since it is not closed in the sense that it depends on A_v. Nevertheless, this procedure is useful when we want to show that such formula exists, for example with QCD. Another way of thinking is to consider the amplitudes in the integrand as given—computed with other method e.g., Feynman diagrams—and then show that using the contour integral we return to the same amplitude [101]. Before proceeding with the case of QCD, let us consider an example for the construction of a basis and the connection between the integrand and the amplitude.

4.1.1 Example

Consider the $n = 4$ primitive Yang-Mills amplitude. This amplitude contains only gluons, hence the alphabet reads

$$\mathbb{A}_4 = \{g_1, g_2, g_3, g_4\} = \{1, 2, 3, 4\}. \tag{4.26}$$

The basis B contains one element, which we choose to be $B = \{w_1 = 1234\}$. Recalling that the cyclic factor for a given ordering $w = l_1 \dots l_n$ reads

$$C(w, z) = \frac{1}{(z_{l_1 l_2})(z_{l_2 l_3}) \cdots (z_{l_n l_{n-1}})}, \tag{4.27}$$

where as before $z_{l_i} - z_{l_j} = z_{l_i l_j}$. The permutation invariant factor was introduced in Chapter 2, which explicitly reads

$$E(z, p, \varepsilon) = \text{Pf}' \ \Psi = \frac{(-1)^{i+j}}{z_{ij}} \text{Pf} \ \Psi^{ij}_{ij}(z, p, \varepsilon), \tag{4.28}$$

where the elements of the matrix $\Psi^{ij}_{ij}(z, p, \varepsilon)$ are defined in Eqs.(3.64)-(3.65). For $n = 4$ we have one independent solution of the scattering equations (Eq.(3.30)) and therefore we have 1×1 matrices. We would like to check that Eq.(4.22) and Eq.(4.28) give the same results evaluated at the solutions of the scattering equations. Therefore, the cyclic factor is simply

$$C(1234, z) = \frac{1}{(z_{12})(z_{23})(z_{34})(z_{41})}, \tag{4.29}$$

while the cyclic factor with the ordering $\bar{w}_1$ reads

$$C(1243, z) = \frac{1}{(z_{12})(z_{24})(z_{43})(z_{31})}. \tag{4.30}$$

For the reduced Pfaffian, we delete the first and second row as well as the first and second column. We obtain

$$E(z,p,\varepsilon) = -\frac{1}{z_{12}}\mathrm{Pf}\begin{pmatrix} 0 & \frac{s_{34}}{z_{34}} & -\frac{\rho_{13}}{z_{13}} & -\frac{\rho_{23}}{z_{23}} & \sum\limits_{\substack{i=1\\ i\neq 3}}^{4}\frac{\rho_{3i}}{z_{3i}} & \frac{\rho_{43}}{z_{34}} \\ -\frac{s_{34}}{z_{34}} & 0 & -\frac{\rho_{14}}{z_{14}} & -\frac{\rho_{24}}{z_{24}} & -\frac{\rho_{34}}{z_{34}} & \sum\limits_{\substack{i=1\\ i\neq 4}}^{4}\frac{\rho_{4i}}{z_{4i}} \\ \frac{\rho_{13}}{z_{13}} & \frac{\rho_{14}}{z_{14}} & 0 & \frac{\epsilon_{12}}{z_{12}} & \frac{\epsilon_{13}}{z_{13}} & \frac{\epsilon_{14}}{z_{14}} \\ \frac{\rho_{23}}{z_{23}} & \frac{\rho_{24}}{z_{24}} & -\frac{\epsilon_{21}}{z_{12}} & 0 & \frac{\epsilon_{23}}{z_{23}} & \frac{\epsilon_{24}}{z_{24}} \\ -\sum\limits_{\substack{i=1\\ i\neq 3}}^{4}\frac{\rho_{3i}}{z_{3i}} & \frac{\rho_{34}}{z_{34}} & -\frac{\epsilon_{31}}{z_{13}} & -\frac{\epsilon_{32}}{z_{23}} & 0 & \frac{\epsilon_{34}}{z_{34}} \\ -\frac{\rho_{43}}{z_{34}} & -\sum\limits_{\substack{i=1\\ i\neq 4}}^{4}\frac{\rho_{41}}{z_{41}} & -\frac{\epsilon_{41}}{z_{14}} & -\frac{\epsilon_{42}}{z_{24}} & -\frac{\epsilon_{43}}{z_{34}} & 0 \end{pmatrix}, \tag{4.31}$$

where we have used the conventions of Chapter 3, i.e., $\epsilon_{ab} = \varepsilon_a \cdot \varepsilon_b$ and $\rho_{ab} = \sqrt{2}\varepsilon_a \cdot p_b$. In order to compare Eq.(4.22) and Eq.(4.28), we will use the freedom to multiply and divide for a nonzero constant. The reason behind this normalization is our choice of fixed values for the solutions which contain infinity. Equivalently, we normalize the Pfaffian and the Parke-Taylor factors[6] by the square root of the Jacobian in agreement with Eq.(4.16). Thus, we want to show that

$$\left(\sqrt{J(z,p)}\mathrm{Pf}'\Psi\right)\Big|_{z=z^{(j)}} = \sqrt{J(z^{(j)},p)}\sum_{v\in B}\sum_{u\in B} S_{v\bar{u}}C(\bar{u},z^{(j)})A_v, \tag{4.32}$$

for $n = 4$. The full expressions for the reduced Pfaffian and the Jacobian have been calculated in Chapter 3 (See Eq.(3.89) and Eq.(3.73)). Evaluating the LHS of Eq.(4.32), we have

$$\begin{aligned}\mathrm{LHS} = s_{13}\sqrt{\frac{s_{12}}{s_{13}}}\Big[&- \epsilon_{13}\epsilon_{24} - \frac{s_{13}}{s_{14}}\epsilon_{14}\epsilon_{23} - \frac{s_{13}}{s_{12}}\left(\epsilon_{12}\epsilon_{34}\right) + \frac{1}{s_{14}}\Big(- \epsilon_{12}\rho_{32}\rho_{42} - \rho_{13}\epsilon_{23}\rho_{42} \\ &+ \epsilon_{13}\rho_{23}\rho_{42} + \rho_{14}\left(-\epsilon_{24}\right)\rho_{32} + \epsilon_{14}\rho_{24}\rho_{32}\Big) + \frac{s_{13}}{s_{12}s_{14}}\Big(\epsilon_{12}\rho_{32}\rho_{43} - \epsilon_{13}\rho_{23}\rho_{43} \\ &+ \rho_{13}\epsilon_{23}\rho_{43} + \rho_{14}\rho_{23}\left(-\epsilon_{34}\right) + \epsilon_{14}\rho_{23}\rho_{34} + \rho_{14}\epsilon_{23}\rho_{43}\Big) + \frac{1}{s_{12}}\Big(\epsilon_{12}\rho_{34}\rho_{42} \\ &+ \rho_{13}\rho_{24}\left(-\epsilon_{34}\right) + \rho_{13}\epsilon_{24}\rho_{34} + \epsilon_{13}\rho_{24}\rho_{43} - \epsilon_{14}\rho_{24}\rho_{34} + \rho_{14}\epsilon_{24}\rho_{34}\Big)\Big].\end{aligned} \tag{4.33}$$

The momentum kernel is given by $S[1234|1243] = s_{12}$, hence the RHS becomes

[6] See Sections 3-4 of Ref. [14]

$$\text{RHS} = s_{13}\sqrt{\frac{s_{12}}{s_{13}}}\Big[-A_4(1234)\Big]. \tag{4.34}$$

Inspecting the LHS we observe that it has the correct pole structure. For instance, the first term in the LHS corresponds to the 4-gluon vertex diagram. Similarly, we have poles for the remaining diagrams for the primitive amplitude $A_4(1234)$. Therefore the LHS term in brackets is in fact the 4-gluon amplitude. If we specialize in the helicities $h_1 = h_2 = -1$ and $h_3 = h_4 = +$, we can easily see that the term in brackets is the MHV amplitude. Using the reference vectors $q_1 = q_2 = p_3$ and $q_3 = q_4 = p_2$, we have (see Example in Section 3.4.2)

$$\epsilon_{1^-3^+} = \epsilon_{1^-2^-} = \epsilon_{2^-4^+} = \epsilon_{3^+4^+} = \epsilon_{2^-3^+} = \rho_{2^-3^+} = \rho_{3^+2^-} = 0. \tag{4.35}$$

Therefore, the LHS reduces to

$$\text{LHS} = s_{13}\sqrt{\frac{s_{12}}{s_{13}}}\Big[-\frac{\rho_{2^-4^+}\rho_{3^+4^+}\epsilon_{1^-4^+}}{s_{12}}\Big], \tag{4.36}$$

which up to the normalization factor is the MHV amplitude (3.94). In this example, we have seen two of the ingredients we need for other theories, i.e., we need a basis of amplitudes B and a proper definition of the permutation invariant factor E.

4.2 CHY representation of QCD

In the first Section we have seen how to prove that a certain gauge amplitude admits a CHY representation. The main requirement is the invertibility of Eq.(4.12) in the sense of Eq.(4.23). We would like to test if a formula similar to Eq.(4.12) exists for a theory which includes fermions or more generally QCD. This part of the thesis is based on Ref. [102] by the author, Alexander Kniss, and Stefan Weinzierl.

There are several reasons to suspect that such representation exists. First, there is a correspondence between the color-kinematics duality and the CHY representations for pure Yang-Mills[7] and it was found that the duality extends to QCD amplitudes [103], thus indicating that QCD may also have a CHY representation. Second, there is a formula for $\mathcal{N} = 4$ SYM which employs the 4-dimensional version of the scattering equations known as the Roiban-Spradin-Volovich (RSV) formula [15] which contains the pure gluonic sector. In addition, it is possible to obtain massless QCD amplitudes from $\mathcal{N} = 4$ SYM by taking subsets of the amplitudes in the supersymmetric theory, associating gluinos with quarks,

[7] We will explore this correspondence in the next Chapter.

and to avoid the $\mathcal{N}=4$ scalar sector [104]. Furthermore, for amplitudes containing only external gluons or a single quark line it is well known that in fact the amplitudes in both theories are equivalent. In third place, QCD amplitudes can be obtained by the generalized version of the BCFW recursions[8] that we studied in Chapter 2. The CHY formalism has been established for theories which accept BCFW recursions or generalizations of them[9]. These observations represent indirect evidence of the existence of such a formula. In this Section we will show that this formula exists and give a generalized version of Eq.(4.12) for QCD primitive amplitudes.

Let us proceed with the issue of existence of a CHY representation of QCD based on the scattering equations. We will proceed in analogy with the pure Yang-Mills case. Let us summarize the main differences between the pure Yang-Mills case and QCD:

- Quarks can be massive which implies we need to use the massive version of the scattering equations.
- The basis of amplitudes B depends on the number of fermions, thus in general the matrices in the generalization Eq.(4.12) will not be square matrices.
- The KLT orthogonality holds only for the subset of QCD amplitudes with only gluons.
- Pairs of quarks can have an arbitrary orientation. Then we have to set a standard orientation.

We will address each point independently. Our goal is to show that QCD primitive amplitudes can be written as

$$A_n(w,p,\epsilon) = \mathrm{i} \sum_{\text{solutions } j} J(z^{(j)},p)\hat{C}(w,z^{(j)})\hat{E}(z^{(j)},p,\varepsilon), \tag{4.37}$$

where the hats indicate the generalization of the ingredients of the CHY representation. Equivalently, let $N_{\text{permutations}} = n!$ and define the $N_{\text{permutations}} \times N_{\text{solutions}}$ matrix $\hat{M}_{wj}$ as

$$\hat{M}_{wj} = J(z^{(j)},p)\hat{C}(w,z^{(j)}), \tag{4.38}$$

and the $N_{\text{solutions}}$ vector $\hat{E}_j$ as

$$\hat{E}_j = \hat{E}(z^{(j)},p,\varepsilon). \tag{4.39}$$

[8] BCFW with massive particles, see. e.g., [105]

[9] For example, effective theories like the non linear sigma model admit a CHY representation but the usual BCFW shifts do not work and need to be generalized [106, 107].

With this notation we can write Eq.(4.37) in analogy to Eq.(4.12) as follows

$$A_w = \mathrm{i}\hat{M}_{wj}\hat{E}_j. \tag{4.40}$$

Notice that the dimension of the matrix $\hat{\mathsf{M}}$ indicate that our goal is to show that Eq.(4.40) is valid for all $w \in W_0$.

4.2.1 Massive scattering equations

QCD amplitudes contain two types of particles: massless gauge bosons (gluons) g and massive or massless fermions (quarks) q. The configuration space of n external particles with n_g gluons and $2n_q$ quarks reads

$$K_n = \left\{ (p_1, p_2, \ldots, p_n) \in (\mathbb{C}M)^n | \sum_{i=1}^{n} p_i = 0,\ \ p_{g_j}^2 = 0,\ \text{and}\ p_{q_j}^2 = p_{\bar{q}_j}^2 = m_{q_j}^2 \right\}, \tag{4.41}$$

where $p_i \in \mathbb{C}M$ is the momenta in complexified Minkowski space in D dimensions. The n-tuple $p = (p_1, p_2, \ldots, p_n)$ of momentum vectors belong to K_n. The massive scattering equations amount to the replacement

$$s_{ij} \to 2p_i \cdot p_j + 2\Delta_{ij}, \tag{4.42}$$

in the massless scattering equations. Hence

$$f_i(z, p) \equiv \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{2p_i \cdot p_j + 2\Delta_{ij}}{z_i - z_j} = 0, \quad i = 1, \ldots, n. \tag{4.43}$$

In order to preserve $\mathrm{PSL}(2, \mathbb{C})$ invariance we must impose the conditions [67]

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \Delta_{ij} = m_i^2, \qquad \Delta_{ij} = \Delta_{ji}. \tag{4.44}$$

Assuming that all n_q quarks have different flavors, we have that to every external quark q_a corresponds an external anti-quark $\bar{q}_a$ with the same mass m_a. In this case, we impose the conditions (4.44) by setting

$$\Delta_{q_a\bar{q}_a} = \Delta_{\bar{q}_a q_a} = m_{q_a}^2, \tag{4.45}$$

and $\Delta_{ij} = 0$ in all other cases. These constraints can be easily understood if we consider massive particles as massless particles in higher dimensional space [108]. Since the scattering equations are valid in any space-time dimension, consider a theory in $D + n_q$ space-time dimensions. In this theory, the a-th quark carries—in the a-th extra dimension—a momentum component m_{q_a} and the anti-quark of flavor a carries —in the a-th extra dimension—the momentum component $(-m_{q_a})$. Suppose we have

$$P_i = (p_j|\kappa_j), \tag{4.46}$$

in $D + n_q$ space-time dimensions. The scattering equations then read

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{2P_i \cdot P_j}{z_i - z_j} = 0, \quad i = 1, \dots, n, \tag{4.47}$$

which imply Eq.(4.43) with $\Delta_{ij} = -\kappa_i\kappa_j$. We take the signature of the metric to be $(+, -, \dots)$. Thus, the Jacobian in Eq.(3.49) is given in terms of the modified matrix

$$\Phi_{ab} = \frac{\partial f_a}{\partial z_b} = \begin{cases} \frac{2p_a\cdot p_b + 2\Delta_{ab}}{(z_a - z_b)^2} & a \neq b, \\ -\sum\limits_{j=1, j\neq a} \frac{2p_a\cdot p_b + 2\Delta_{ab}}{(z_a - z_j)^2} & a = b. \end{cases} \tag{4.48}$$

as

$$J(z, p) = \frac{1}{\det'(\Phi)}, \tag{4.49}$$

in analogy with the massless case as we discussed in Chapter 3.

4.2.2 Basis of primitive amplitudes

Unlike pure Yang-Mills, the basis of amplitudes in QCD will depend on the number of quarks. In total, the basis for pure Yang-Mills has $(n-3)!$ elements after using KK and BCJ relations (See Section 2.4.4). In contrast, only a subset of QCD primitive amplitudes satisfy BCJ relations [103, 109], therefore the basis depends on this subset. For a n-point

QCD primitive amplitude containing $2n_q$ quarks and n_g gluons, we have $n = n_g + 2n_q$. The size of the basis is given by

$$N_{\text{basis}} = \begin{cases} (n-3)!, & n_q \in \{0,1\}, \\ (n-3)!\frac{2(n_q-1)}{n_q!}, & n_q \geq 2. \end{cases} \tag{4.50}$$

Notice that due to

$$\frac{2(n_q-1)}{n_q!} = \frac{2}{n_q}\frac{1}{(n_q-2)!} \leq 1, \qquad \text{for} \qquad n_q \geq 2, \tag{4.51}$$

we have

$$N_{\text{basis}} \leq N_{\text{solutions}}, \tag{4.52}$$

where $N_{\text{solutions}} = (n-3)!$ is the number of inequivalent solutions of the scattering equations. This condition is essential for finding a CHY representation for tree-level primitive QCD amplitudes.

In analogy with our procedure for pure Yang-Mills, the basis of amplitudes consists of amplitudes with orderings specified by *words* w in a given *alphabet* $\mathbb{A}$ of labeled particles which can be quarks q_a, anti-quarks $\bar{q}_a$ and gluons g_i.[10] We will assume without loss of generality that the quarks have different flavors, hence in general the alphabet reads

$$\mathbb{A} = \{q_1, q_2, \ldots, q_{n_q}, \bar{q}_1, \bar{q}_2, \ldots, \bar{q}_{n_q}, g_1, g_2, \ldots, g_n\}. \tag{4.53}$$

Remember that the most general basis is a set of words with n letters, such that every letter from this alphabet occurs exactly once. This basis is given by all possible words corresponding to all permutations of the letters, i.e.,

$$W_0 = \{l_1, l_2, \ldots, l_n | l_i \in \mathbb{A}, l_i \neq l_j \text{ for } i \neq j\}, \tag{4.54}$$

which has $n!$ elements. The inclusion of quarks requires a mechanism which takes into account that primitive amplitudes with crossed fermions lines vanish. This is done by

[10] For example, if we have an amplitude of 2 pairs of quark anti-quark and one gluon the alphabet and a possible choice of labels reads

$$\mathbb{A}_5 = \{q_1, q_2, \bar{q}_1, \bar{q}_2, g_1\} = \{l_1, l_2, \ldots, l_5\} = \{1, 2, 3, 4, 5\}.$$

introducing Dyck words [110, 111]. Dyck words can be easily understood as all possible ways of matching opening and closing brackets[11]. Non-matching brackets correspond to crossed quark lines and therefore we avoid these configurations by using Dyck words. Generalizing this procedure, we introduce n_q distinct opening and closing brackets

$$(_i, \quad)_i, \tag{4.55}$$

respectively. These brackets only match if they have the same index (flavor). There are

$$N_{\text{Dyck}} = \frac{(2n_q)!}{(n_q+1)!} \tag{4.56}$$

"generalized Dyck words" of length $2n_q$. In addition, we define as the *standard orientation* for a quark line as follows. A quark of flavor i is associated with an opening bracket $(_i$ and the corresponding anti-quark of flavor i is associated with a closing bracket $)_i$[12]. We define a projector P on the letters of the alphabet such that

$$P(q_i) = (_i, \qquad P(g_i) = \text{empty}, \qquad P(\bar{q}_i) =)_i, \tag{4.57}$$

where "empty" denotes that the inclusion of gluons does not give brackets. We then set,

$$\text{Dyck}_{n_q} = \{w \in W_0 | P(w) \text{ is a generalized Dyck word.}\}, \tag{4.58}$$

which contains all words, without crossed fermions lines and where fermions lines have the standard orientation[13]. For amplitudes in an arbitrary orientation we can use cyclic invariance, KK relations, and "no-crossed-fermion-lines" relations to reduce them to the standard orientation [110, 111]—these relations are studied in Appendix C.

We set the basis of primitive QCD amplitudes depending on the number of quarks as follows

$$B = \begin{cases} \{l_1 l_2 \dots l_n \in W_0 | l_1 = g_1, l_{n-1} = g_{n-1}, l_n = g_n\}, & n_q = 0, \\ \{l_1 l_2 \dots l_n \in W_0 | l_1 = q_1, l_{n-1} = g_{n-2}, l_n = \bar{q}_1\}, & n_q = 1, \\ \{l_1 l_2 \dots l_n \in \text{Dyck}_{n_q} | l_1 = q_1, l_{n-1} \in \{\bar{q}_2, \dots, \bar{q}_{n_q}\}, l_n = \bar{q}_1\}, & n_q \geq 2. \end{cases} \tag{4.59}$$

[11] For example if we have two types of brackets $(), \{\}$, $w_1 = ()\{\}$, $w_2 = \{()\}$ are valid (generalized) Dyck words, while $w_3 = \{)(\}$ is not valid.

[12] This definition is not cyclic invariant. Cyclic invariance is recovered by using KK relations to fix particle 1 to be q_1 and particle n to be $\bar{q}_1$

[13] For example, for the alphabet $\mathbb{A} = \{q_1, q_2, g_1, \bar{q}_1, \bar{q}_2\} = \{1, 2, 3, 4, 5\}$, the generalized Dyck words with the standard orientation are $31245, 13245, 12345, \dots, 31524, 13524, 15324, \dots$, etc.

This basis fixes three letters in the alphabet, in analogy with the basis of pure Yang-Mills in Eq.(4.59) which was obtained by using cyclic invariance, KK relations and BCJ relations. In QCD, we can use cyclic invariance and KK relations but only a set of primitive amplitudes satisfy BCJ relations[14].

Let us review how to build this basis starting with an arbitrary amplitude $A_n(w)$ with $w \in W_0$ and express it as a linear combination of amplitudes $A_n(w_j)$ with $w_j \in B$. Cyclic invariance can be used to fix particle 1 to be g_1 (case $n_q = 0$) or to be q_1 (case $n_q \geq 1$). This defines a subset of W_0 given by

$$W_1 = \begin{cases} \{l_1 l_2 \dots l_n \in W_0 | l_1 = g_1\}, & n_q = 0, \\ \{l_1 l_2 \dots l_n \in W_0 | l_1 = q_1\}, & n_q \geq 1. \end{cases} \tag{4.60}$$

We can fix an additional letter in W_0—equivalently we fix a second letter in W_1—by using KK relations. We set the particle n to be g_n (case $n_q = 0$) or to be $\bar{q}_1$ (case $n_q \geq 1$). Then, we define the subset

$$W_2 = \begin{cases} \{l_1 l_2 \dots l_n \in W_1 | l_n = g_n\}, & n_q = 0, \\ \{l_1 l_2 \dots l_n \in W_1 | l_n = \bar{q}_1\}, & n_q \geq 1. \end{cases} \tag{4.61}$$

Amplitudes with crossed lines vanish, and fermions with a non-standard orientation can be expressed in terms of amplitudes with the standard orientation[15]. Using these "relations", we can define a subset of W_3 where the quark lines are standard oriented and there are no crossed lines. Recalling Eq.(4.58), we have

$$W_3 = \begin{cases} W_2, & n_q \leq 1, \\ \left\{ w \in W_2 | w \in \text{Dyck}_{n_q} \right\}, & n_q \geq 2. \end{cases} \tag{4.62}$$

Finally, we use the fundamental BCJ relations for QCD to fix the position of particle $(n-1)$ to be g_{n-1} (case $n_q = 0$), to be g_{n-2} (case $n_q = 1$) or to remove any gluon from position $(n-1)$ (case $n_q > 1$). The latter means that we must have an anti-quark in position $(n-2)$. The basis is then given by

$$B = \begin{cases} \{l_1 l_2 \dots l_n \in W_3 | l_{n-1} = g_{n-1}\}, & n_q = 0, \\ \{l_1 l_2 \dots l_n \in W_3 | l_{n-1} = g_{n-2}\}, & n_q = 1, \\ \{l_1 l_2 \dots l_n \in W_3 | l_{n-1} \in \{\bar{q}_2, \dots, \bar{q}_{n_q}\}\}, & n_q \geq 2, \end{cases} \tag{4.63}$$

[14] See Appendix C.2.1.
[15] See Appendix C.2.2.

which contains all words with three fixed letters. We have the inclusions

$$W_0 \supseteq W_1 \supseteq W_2 \supseteq W_3 \supseteq B. \tag{4.64}$$

This chain of inclusions is crucial for the proof of the existence of the CHY representation of QCD.

Recall that in pure Yang-Mills, once we have found a valid permutation invariant factor E in the basis of primitive Yang-Mills amplitudes, we wanted to show that the same factor could be used for all amplitudes. In order to do this, the amplitude (4.24) had to satisfy cyclic invariance, KK, and BCJ relations. This idea can be used for QCD as well. The concept behind this procedure is that we may view the amplitude A_n as a *linear operator* on the $n!$-dimensional vector space of words V with basis W_0. Assuming there is another operator $\hat{A}_n$ in V, the question we want to address is under which conditions these two operators are identical. This is the case if and only if they agree on all basis vectors of V, i.e.,

$$\hat{A}(w) = A_n(w), \qquad \forall w \in W_0. \tag{4.65}$$

Like in pure Yang-Mills we know that amplitudes in QCD $A_n(w_i)$ satisfy some relations. If A_n and $\hat{A}_n$ are identical operators, we must have the same relations among the hatted amplitudes $\hat{A}(w_i)$. Therefore, we first check that the amplitudes agree on the basis B and then check that the images of $\hat{A}_n$ satisfy the same relations as QCD amplitudes. In other words, it is sufficient to check that:

1. $\hat{A}_n$ is cyclic invariant for all $w \in W_0$.
2. $\hat{A}_n$ satisfies the KK relations for all $w \in W_1$.
3. $\hat{A}_n$ does not have crossed fermion lines and can be brought[16] to the standard orientation for all $w \in W_2$.
4. $\hat{A}_n$ satisfies the fundamental BCJ relations for all $w \in W_3$.
5. $\hat{A}_n$ agrees with A_n for all $w \in B$, i.e., $\hat{A}_n(w) = A_n(w), \quad \forall w \in B$.

Starting with words $w \in B$—recall the chain of inclusions (4.64)—we can show that these conditions are sufficient to prove Eq.(4.65) as follows:

- Condition 5 guarantees that $A_n(w) = \hat{A}_n(w)$ for all $w \in B$

[16] Via "orientation relations". See Eq.(C.25)

- Condition 4 guarantees that $\hat{A}_n(w)$, $w \in W_3 \backslash B$ may be expressed as a linear combination of $\hat{A}_n(w')$, $w' \in B$. Therefore it agrees with $A_n(w)$ in $W_3 \backslash B$—the same relation holds for $A_n(w)$, $w \in W_3 \backslash B$ and $A_n(w')$, $w \in B$—and since it also agrees in B, it agrees in the complete set W_3.
- Condition 3 guarantees that $\hat{A}_n(w)$, $w \in W_2 \backslash W_3$ can be expressed as a combination of amplitudes $A_n(w')$, $w' \in W_3$, which coincides with the relations that $A_n(w)$, $w \in W_2 \backslash W_3$ and $A_n(w')$, $w' \in W_3$ satisfy[17]. Therefore, $\hat{A}_n(w) = A_n(w)$ in W_2.
- Condition 2 tells us that $\hat{A}_n(w)$, $w \in W_1 \backslash W_2$ satisfy KK relations, which $A(w)$ satisfy as well. Therefore, they agree in the complete W_1.
- Condition 1 ensures that $\hat{A}_n(w)$ and $A_n(w)$ agree on W_0.

4.2.2.1 Example of a basis

Before proceeding with the ingredients of the CHY representation let us give an example of the basis of amplitudes. Consider alphabet

$$\mathbb{A}_6 = \{q_1, q_2, q_3, \bar{q}_3, \bar{q}_2, \bar{q}_1\} = \{1, 2, 3, 4, 5, 6\}, \tag{4.66}$$

for primitive QCD amplitudes with $n = 6$ particles. Since we have only quarks, we have $n_q > 2$ and therefore the basis is built by fixing the position of the particle 1 to be q_1, particle 6 to be the corresponding anti-quark $\bar{q}_1$, and particle 5 to be, either $\bar{q}_2$ or $\bar{q}_3$. Thus, in terms of brackets we have to consider the structures

$$({}_1\bullet_i\bullet_j\bullet_k)_2)_1, \qquad ({}_1\bullet_i\bullet_j\bullet_k)_3)_1, \tag{4.67}$$

where the bullets indicate either a opening bracket or a closing bracket and the subscripts indicate the flavor. The remaining flavor in each case corresponds to the brackets $({}_i,)_i$, $i = 2, 3$, and may be used to construct only one type of generalized Dyck word, i.e., $({}_i)_i$, $i = 2, 3$. Therefore, the allowed Dyck words are

$$({}_1({}_2({}_3)_3)_2)_1, \qquad ({}_1({}_3)_3({}_2)_2)_1, \qquad ({}_1({}_3({}_2)_2)_3)_1, \qquad ({}_1({}_2)_2({}_3)_3)_1, \tag{4.68}$$

and the basis reads

$$B = \{123456, 134256, 132546, 125346\}. \tag{4.69}$$

[17] Eq.(C.25).

4.2.3 Generalized cyclic factor

Let us start with the definition of the building blocks of the CHY representation. QCD contains pure gluonic amplitudes, which we know can be represented by the standard cyclic factor (See e.g., Eq.(4.27)). Therefore, a natural choice is that the *generalized cyclic factor* $\hat{C}(w,p)$ agrees with the *standard cyclic factor*(Parke-Taylor factor) $C(w,p)$ for $(n_q = 0)$. In addition, for amplitudes with a single quark line $(n_q = 1)$ and for amplitudes with the standard orientation and $(n_q \geq 2)$, we choose standard Parke-Taylor factors as well.

We define our alphabet as

$$\mathbb{A} = \{1, 2, \ldots, n\}, \tag{4.70}$$

corresponding to the external legs of the primitive amplitude, and associate to each external leg j the complex variable z_j. The standard cyclic factor is then given by

$$C(w,z) = \frac{1}{(z_{l_1 l_2})(z_{l_2 l_3}) \cdots (z_{l_n l_{n-1}})}, \tag{4.71}$$

where $w = l_1 l_2 \ldots l_n \in W_0$. The standard cyclic factor satisfies cyclic invariance, KK relations, and BCJ relations. The latter is only satisfied on the support of the scattering equations. In analogy with the amplitudes, it is convenient to view $C(w,z)$ and $\hat{C}(w,z)$ as linear operators on the vector space of words with basis W_0, i.e.,

$$C(\lambda_1 w_1 + \lambda_2 w_2, z) = \lambda_1 C(w_1, z) + \lambda_2 C(w_2, z), \tag{4.72}$$

$$\hat{C}(\lambda_1 w_1 + \lambda_2 w_2, z) = \lambda_1 C(w_1, z) + \lambda_2 C(w_2, z). \tag{4.73}$$

We proceed by defining the generalized cyclic factor $\hat{C}$, in each set of words:

1. For $w \in B$, the orientation is standard for $n_q \geq 2$, therefore $\hat{C}(w,z) = C(w,z)$, as we said at the beginning.

2. For $w \in W_3$, we set $\hat{C}(w,z) = C(w,z)$, since in this set amplitudes are also standard oriented.

3. For $w \in W_2 \backslash W_3$—the subset of words which give nonstandard oriented amplitudes but which satisfy KK relations—we first define

$$\hat{C}(w,z) = 0, \tag{4.74}$$

for all words corresponding to crossed fermions lines. For words with no crossed fermion lines we relate $\hat{C}(w,z)$ to a linear combination of $\hat{C}(w_i,z)$'s with $w_j \in W_3$—in analogy with the amplitudes, see Eq.(C.25). Let x_k and y_k be sub-words defined by

$$x_k = l_{i_1} l_{i_2} \dots l_{i_r}, \qquad y_k = l_{j_1} l_{j_2} \dots l_{i_s}. \tag{4.75}$$

For these words we have

$$\begin{aligned}&\hat{C}(x_{k-1} q_i x_k \bar{q}_j w_{k+1} q_j y_k \bar{q}_i y_{k-1}, z) = \\ &\qquad (-1)^{|w_{k+1}|+1} \sum_{a=0}^{r} \sum_{b=0}^{s} \hat{C}(x_{k-1} q_i l_{i_1} \dots l_{i_a} q_j w'_{k+1} \bar{q}_j l_{j_{b+1}} \dots l_{j_s} \bar{q}_i y_{k-1}, z),\end{aligned} \tag{4.76}$$

with

$$w'_{k+1} = (l_{i_a} \dots l_{i_r}) ⧢ w^T_{k+1} ⧢ (l_{j_1} \dots l_{j_b}). \tag{4.77}$$

This relation allows us to define recursively the generalized cyclic factor for words with $w \in W_2 \backslash W_3$ in terms of generalized cyclic factors $\hat{C}(w,z)$ of words with $w \in W_3$. Therefore, $\hat{C}(w,z)$ with $w \in W_2 \backslash W_3$ is a linear combination of standard cyclic factors.

4. For $w \in W_1 \backslash W_2$—words which are cyclic invariant but do not have two fixed legs—we set

$$\hat{C}(l_1 w_1 l_n w_2, z) = (-1)^{|w_2|} \hat{C}(l_1 (w_1 ⧢ w_2^T) l_n, z), \tag{4.78}$$

which defines the generalized cyclic factor for words, where the letter l_n is not fixed in the last place in terms of generalized cyclic factors for words where l_n occurs in the last place. These are the KK relations (2.150).

5. For $w \in W_0 \backslash W_1$—words which do not have fixed any leg—we set

$$\hat{C}(w_1 l_1 w_2, z) = \hat{C}(l_1 w_2 w_1, z), \tag{4.79}$$

which defines the generalized factor where the letter l_1 does not appear in the first place in terms of generalized cyclic factors for words, where the letter l_1 occurs in the first place. This is simply cyclic invariance.

4.2.4 Generalized permutation invariant factor

We proceed with the definition of the *generalized permutation invariant factor* $\hat{E}(z, p, \varepsilon)$. Recall that in Yang-Mills we had KLT orthogonality, which helped us to find two matrices such that $M_{wj}N_{jv} = \delta_{wv}$, thus giving the permutation invariant factor $E(z, p, \varepsilon)$ in Eq.(4.14). In QCD, we only have KLT orthogonality in a subset of words, corresponding to $n_q \leq 2$ since in these cases the amplitude basis has $(n-3)!$ elements (See Eq.(4.50)). We would like to use KLT orthogonality in this subset.

Consider the restriction of $w \in B$ (for $n_q \geq 2$) and the associated $N_{\text{basis}} \times N_{\text{solutions}}$ sub-matrix of (4.38) which we denote by $\hat{M}^{\text{red}}_{wj}$. This is the (rectangular) matrix[18], which we are interested to invert. In addition, this matrix will be defined in terms of a standard cyclic factor $C(w, z)$ since $w \in B$. For $w \in B$, the generalized cyclic factors coincides with the standard one, hence the entries of $\hat{M}^{\text{red}}_{wj}$ are given by

$$\hat{M}^{\text{red}}_{wj} = J(z^{(j)}, p)C(w, z^{(j)}), \qquad w \in B. \tag{4.80}$$

Therefore, we require that $\hat{M}^{\text{red}}_{wj}$ and $\hat{N}^{\text{red}}_{vj}$ satisfy

$$\hat{M}^{\text{red}}_{wj}\hat{N}^{\text{red}}_{jv} = \delta_{wv}, \tag{4.81}$$

where $\hat{N}^{\text{red}}_{jv}$ is the right inverse of $\hat{M}^{\text{red}}_{wj}$. If this condition is met, the generalized permutation invariant factors reads

$$\hat{E}_j = -\mathrm{i}\hat{N}^{\text{red}}_{jv}A_v, \qquad v \in B. \tag{4.82}$$

in analogy with the pure Yang-Mills case (see Eq.(4.14))

The information of the flavors does not enter in the definition of the individual entries of the matrix $\hat{M}^{\text{red}}_{wj}$, i.e., it only affects the set B, giving all possible indices $w \in B$ of $\hat{M}^{\text{red}}_{wj}$. As the flavor information is to a large extent irrelevant, we will consider the alphabet

$$\mathbb{A} = \{1, 2, \ldots, n\}, \tag{4.83}$$

[18] $(N_{\text{basis}} \leq N_{\text{solutions}})$

with the implicit understanding that we may recover the information about the flavor of the particles if needed. For this alphabet, the basis W_2 obtained by imposing KK relations is given by

$$W_2 = \{l_1 l_2 \dots l_n \in W_0 | l_1 = 1, l_n = n\}. \tag{4.84}$$

As we pointed out at the beginning of this Section, the case $n_q \leq 2$ reduces to pure Yang-Mills —keeping in mind that there is an additional flavor structure—and therefore we give a name to this basis. Let

$$B_{n_q \leq 2} = \{l_1 l_2 \dots l_n \in W_0 | l_1 = 1, l_{n-1} = n-1, l_n = n\}, \tag{4.85}$$

be the basis of words which describes amplitudes with ($n_q \leq 2$) which is obtained after using cyclic, KK, and BCJ relations. In this basis $\hat{M}^{\text{red}}_{wj}$ is a square matrix which we identify by removing the hat:

$$M^{\text{red}}_{wj} = J(z^{(j)}, p) C(w, z^{(j)}), \qquad w \in B_{n_q \leq 2}, \tag{4.86}$$

which satisfies

$$M^{\text{red}}_{wj} N^{\text{red}}_{jv} = \delta_{wv}, \qquad N^{\text{red}}_{iw} M^{\text{red}}_{wj} = \delta_{ij} \tag{4.87}$$

with

$$N^{\text{red}}_{jv} = \sum_{u \in B_{n_q \leq 2}} S_{v\bar{u}} C(\bar{u}, z^{(j)}). \tag{4.88}$$

Notice that here we extend the definition of the KLT kernel to account for the masses of the quarks. For $w_1 = l_1 \dots l_n \in B_{n_q \leq 2}$ and $w_2 = k_1 \dots k_n \in B_{n_q \leq 2}$, we define

$$S_{w_1 \bar{w}_2} \equiv S[w_1 | \bar{w}_2] = (-1)^n \prod_{i=2}^{n-2} \left[2p_{l_1} p_{l_i} + 2\Delta_{l_1 l_i} + \sum_{j=2}^{i-1} \theta_{\bar{w}_2}(l_j, l_i) \left(2p_{l_j} p_{l_i} + 2\Delta_{l_j l_i} \right) \right], \tag{4.89}$$

with

$$\theta_{\bar{w}_2}(l_j, l_i) = \begin{cases} 1 & \text{if } l_j \text{ comes before } l_i \text{ in the sequence } k_2, k_3, \dots k_{n-2}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.90}$$

This in agreement with the pure Yang-Mills case (see Eqs.(4.20), (4.21)).

Let us study the case $n_q > 2$, where $N_{\text{basis}} < N_{\text{solutions}}$. The matrix $\hat{M}^{\text{red}}_{wj}$ is rectangular of dimensions $N_{\text{basis}} \times N_{\text{solutions}}$ with the first index given by $w \in B$. If $\hat{M}^{\text{red}}_{wj}$ has full row rank,i.e.,

$$\text{rank } \hat{M}^{\text{red}}_{wj} = N_{\text{basis}}, \tag{4.91}$$

then a right inverse $\hat{N}^{\text{red}}_{jv}$ exists and we can establish Eq.(4.81). The right inverse may not be unique. We are interested in a right-inverse such that the entries in the j-th row of $\hat{N}^{\text{red}}_{jv}$ depends only on $z^{(j)}$, but not on other solutions of the scattering equations. From Eq.(4.59), we know that B does not have a fixed letter in position $(n-1)$, therefore in general

$$B \nsubseteq B_{n_q \le 2}, \tag{4.92}$$

which makes the determination of the rank difficult. In order to sort out this, we recall that the standard cyclic factors satisfy BCJ relations on the support of the scattering equations, i.e.,

$$C(w, z^{(j)}) = F_{ww'} C(w', z^{(j)}), \qquad w \in B, \qquad w' \in B_{n_q \le 2}, \tag{4.93}$$

where a sum over w' is understood. The entries of the $N_{\text{basis}} \times N_{\text{solutions}}$ matrix $F_{ww'}$ are defined in Eq.(C.42) of Appendix C. The entries of $F_{ww'}$ depend only o the scalar products $2p_ip_j$ but not on $z^{(j)}$. Inserting this equation in Eq. (4.80) and using (4.86), we have

$$\hat{M}^{\text{red}}_{wj} = J(z^{(j)}, p) F_{ww'} C(w', z^{(j)}) = F_{ww'} M^{\text{red}}_{w'j}, \qquad w \in B, \qquad w' \in B_{n_q \le 2}. \tag{4.94}$$

Notice that the case $n_q \le 2$ is trivially included in Eq.(4.94) by taking $F_{ww'}$ to be the $N_{\text{solutions}} \times N_{\text{solutions}}$ identity matrix. The matrix $M^{\text{red}}_{w'j}$ has rank $N_{\text{solutions}}$ and it is invertible. It follows that $\hat{M}^{\text{red}}_{w'j}$ has rank $N_{\text{solutions}}$ if and only if the matrix $F_{ww'}$ has rank

N_{basis}. This can be checked for generic kinematics [19]. We will assume that $F_{ww'}$ has rank N_{basis}, i.e.,

$$\text{rank } F_{ww'} = N_{\text{basis}}. \tag{4.95}$$

Notice that Eq.(4.95) is a pure kinematical statement, in other words it is independent of the solutions of the scattering equations. Assuming, from now on that the matrix $F_{ww'}$ has maximal row rank, the $N_{\text{basis}} \times N_{\text{basis}}$-dimensional matrix FF^T is invertible and the $N_{\text{solutions}} \times N_{\text{basis}}$ matrix

$$\mathsf{G} = \mathsf{F}^T(\mathsf{FF}^T)^{-1} \tag{4.96}$$

defines a right inverse to F:

$$F_{uv'}G_{v'w} = \delta_{uw}. \tag{4.97}$$

Setting $\hat{\mathsf{N}}^{\text{red}} = \mathsf{N}^{\text{red}}\mathsf{G}$ gives the desired $\hat{\mathsf{N}}^{\text{red}}$ in Eq.(4.81) since

$$\hat{M}^{\text{red}}_{wj}\hat{N}^{\text{red}}_{jv} = F_{ww'}M^{\text{red}}_{w'j}N^{\text{red}}_{ju}G_{uv} = \delta_{wv}. \tag{4.98}$$

Therefore, we set

$$\hat{E}_j = -\mathrm{i}\hat{N}^{\text{red}}_{jw}A_w, \tag{4.99}$$

where a sum over $w \in B$ is understood. This equation is QCD version of Eq.(4.14), which was set for pure Yang-Mills. Putting everything together we obtain

$$\hat{E}(z^{(j)}, p, \varepsilon) = -\mathrm{i} \sum_{u,v \in B_{n_q \leq 2}} \sum_{w \in B} S_{u\bar{v}}G_{uw}C(\bar{v}, z^{(j)})A_n(w, p, \varepsilon) \equiv -\mathrm{i}\,\mathsf{CS}^T\mathsf{GA}_n. \tag{4.100}$$

A few comments are in order: First, notice that the dependence on the solutions of the scattering equations is isolated on the factor $C(\bar{v}, z^{(j)})$, which actually allows us to take this factor to $\mathbb{C}^n$. Therefore, the unevaluated generalized factor

[19] We verified for all cases with $n \leq 10$ that $F_{ww'}$ has rank N_{basis}. Furthermore, by a suitable ordering of the bases B and $B_{n_q \leq 2}$ the matrix can be brought into an upper triangular block structure. It is therefore sufficient to show that all (square) matrices on the main diagonal have full rank.

$$\hat{E}(z,p,\varepsilon) = -\mathrm{i}\sum_{u,v\in B_{n_q\leq 2}}\sum_{w\in B} S_{u\bar{v}}G_{uw}C(\bar{v},z)A_n(w,p,\varepsilon), \tag{4.101}$$

is one of the building blocks of the CHY-integrand for QCD. Second, for $n_q > 2$ notice that Eq.(4.100) is not unique due to the non-uniqueness of the right inverse of $F_{ww'}$. However, this fact does not affect the primitive amplitudes as we can show as follows. Let us parametrize the general form of the right-inverses as

$$G_{w'w} + \left(\delta_{w'w'_2} - G_{w'w_1}F_{w_1w'_2}\right)X_{w'_2w}, \tag{4.102}$$

with an arbitrary $N_{\text{solutions}}\times N_{\text{basis}}$-dimensional matrix $X_{w'w}$. Plugging this into Eq.(4.100) we find

$$\hat{E}(z^{(j)},p,\varepsilon) \to \hat{E}(z^{(j)},p,\varepsilon) - \mathrm{i}N^{\text{red}}_{jw'}\left(\delta_{w'w'_2} - G_{w'w_1}F_{w_1w'_2}\right)x_{w'_2}, \tag{4.103}$$

with some arbitrary $N_{\text{solutions}}$-dimensional vector $x_{w'}$. This arbitrariness does not affect expressions of the form

$$\mathrm{i}\sum_{\text{solutions } j} J(z^{(j)},p)\hat{Y}(z^{(j)})\hat{E}(z^{(j)},p,\varepsilon), \tag{4.104}$$

as long as $\hat{Y}(z^{(j)})$ has an expansion in $\hat{C}(w,z^{(j)})$, i.e.,

$$\hat{Y}(z^{(j)}) = \sum_{w\in B} c_w\hat{C}(w,z^{(j)}). \tag{4.105}$$

Then, we may write

$$J(z^{(j)},p)\hat{Y}(z^{(j)}) = \sum_{w\in B} c_w\hat{M}^{\text{red}}_{wj}, \tag{4.106}$$

and then we have

$$\mathrm{i}\sum_{\text{solutions } j}\sum_{w\in B} c_w \hat{M}^{\text{red}}_{wj}\left[\hat{E}_j - \mathrm{i}N^{\text{red}}_{jw'}\left(\delta_{w'w'_2} - G_{w'w_1}F_{w_1w'_2}\right)x_{w'_2}\right] = \mathrm{i}\sum_{\text{solutions } j}\sum_{w\in B} c_w \hat{M}^{\text{red}}_{wj}\hat{E}_j, \tag{4.107}$$

since

$$\hat{M}^{\text{red}}_{wj}N^{\text{red}}_{jw'} = F_{wu'}M^{\text{red}}_{u'j}N^{\text{red}}_{jw'} = F_{ww'} \quad \text{and} \quad F_{ww'}\left(\delta_{w'w'_2} - G_{w'w_1}F_{w_1w'_2}\right) = 0. \tag{4.108}$$

For tree-level QCD amplitudes, we will always have that the factor $\hat{Y}$ in Eq.(4.104) is in the form (4.105). Therefore, we have shown that the non-uniqueness of the right-inverse does not affect tree-level QCD amplitudes.

4.2.5 Proof of the CHY representation

The proof consists on checking the conditions at the end of Section 4.2.2. Recall that we want to show Eq.(4.40), i.e.,

$$\hat{A}_n(w) = A_n(w), \qquad \forall w \in W_0,$$

where

$$\hat{A}_n(w) = \mathrm{i}\sum_{\text{solutions } j} J(z^{(j)}, p)\hat{C}(w, z^{(j)})\hat{E}(z^{(j)}, p, \varepsilon). \tag{4.109}$$

We proceed in the order of inclusions (4.64).

1. For $w \in B$, we have

$$\hat{A}_n(w) = \hat{M}_{wj}\hat{N}^{\text{red}}_{jw'}A_{w'} = \hat{M}^{\text{red}}_{wj}\hat{N}^{\text{red}}_{jw'}A_{w'} = A_w. \tag{4.110}$$

   Therefore $\hat{A}_n(w) = A_n(w)$.

2. For $w \in W_3\backslash B$, we have to verify the fundamental BCJ relation, i.e.,

$$\sum_{i=2}^{n-1}\left(\sum_{k=i+1}^{n} 2p_2\cdot p_k\right)\hat{A}(l_1l_3\dots l_il_2l_{i+1}\dots l_{n-1}l_n) = 0, \tag{4.111}$$

which translates into a condition for the generalized cyclic factor $\hat{C}$. We must have

$$\sum_{i=2}^{n-1}\left(\sum_{k=i+1}^{n} 2p_2p_k\right)\hat{C}(l_1l_3\dots,l_il_2l_{i+1}\dots l_{n-1}l_n,z^{(j)})=0 \tag{4.112}$$

for all solutions of the scattering equations. For $w\in W_3$, the generalized cyclic factor coincides wit the standard Parke-Taylor factor and therefore satisfies BCJ relations on the support of the scattering equations.

3. For $W_2\backslash W_3$, we have to consider only the cases with no-crossed fermion lines. In this case we have that the amplitudes $\hat{A}_n$ have to satisfy the orientation relations. However, this information is encoded in the generalized cyclic factor $\hat{C}$ which in this case satisfy the orientation relations (4.76). Hence, they are also satisfied by $\hat{A}_n$.
4. We repeat this argumentation for $w\in W_1\backslash W_2$. The definition of $\hat{C}$ in this subset is such that it satisfies KK relations and therefore $\hat{A}_n(w)$ satisfies them as well.
5. For $w\in W_0\backslash W_1$, the Parke-Taylor factors satisfy cyclic invariance and therefore $\hat{A}(w)$ satisfies them as well.

This completes the proof of Eq.(4.40). We have shown that any tree-level primitive QCD amplitude has a CHY representation in the form of Eq.(4.37) with $\hat{C}$ defined in Section 4.2.3 and $\hat{E}$ defined in Section 4.2.4. The generalized cyclic factor can always be expressed as a linear combination of standard Parke-Taylor factors. By construction all elements transform properly under $\mathrm{PSL}(2,\mathbb{C})$. The QCD integrand depends on the basis through $\hat{C}$, hence for $w\in B$, we have

$$I^{\mathrm{QCD}}(z,p,\varepsilon)=-\mathrm{i}\; C(w,z)\sum_{u,v\in B_{n_q\le 2}}\sum_{w\in B} S_{u\bar{v}}G_{uw}C(\bar{v},z)A_n(w,p,\varepsilon),\qquad w\in B. \tag{4.113}$$

where we have used the fact that the generalized cyclic factor for $w\in B$ coincides with the Parke-Taylor factor. Of course, we can give the integrand for any $w\in W_0$ using the generalized cyclic factor. Notice that $\hat{E}$ is the same for all $w\in W_0$, just as in the case of pure Yang-Mills.

4.3 Examples

Example 4.3.1. A rather trivial but illustrative example is the case of two pairs of massless quarks. The alphabet reads

$$\mathbb{A}_4 = \{q_1, q_2, \bar{q}_2, \bar{q}_1\} = \{1, 2, 3, 4\}, \tag{4.114}$$

and the relevant bases are given by

$$B = \{1234\}, \quad B_{n_q \leq 2} = \{1234\}, \quad \bar{B}_{n_q \leq 2} = \{1243\}. \tag{4.115}$$

The matrix 1×1 matrix G is trivial since $\mathsf{F} = 1$. Then, the permutation invariant factor reads

$$\hat{E}(z, p, \varepsilon) = -\,\mathrm{i} \sum_{v \in \bar{B}_{n_q \leq 2}} \sum_{w \in B} c_{\bar{v}w} C(\bar{v}, z) A_4(w). \tag{4.116}$$

where $c_{\bar{v}w} = \sum_u S[u|\bar{v}]G$. With these ingredients, the permutation invariant factor reads,

$$\hat{E}(z, p, \varepsilon) = -\mathrm{i}\Big(s_{12} C(1243, z) A_4(1234)\Big), \tag{4.117}$$

Suppose we are interested in the non-standard oriented amplitude $A_4(1324)$, then the generalized cyclic factor is given by

$$\hat{C}(1324, z) = -\hat{C}(1234, z) = -C(1234, z). \tag{4.118}$$

Therefore the integrand of the CHY representation for $A_4(1324)$ reads

$$I^{\mathrm{QCD}}(z, p, \varepsilon) = \mathrm{i}\; C(1234, z) s_{12} C(1243, z) A_4(1234). \tag{4.119}$$

Using this integrand in the general expression for the CHY at 4-points (Eqs.(3.56)-(3.58)), we have

$$\tilde{I}^{\mathrm{QCD}}(z, p, \varepsilon) = \mathrm{i}(z_{12} z_{23} z_{34} z_{41})^2 C(1234, z) s_{12} C(1243, z) A_4(1234) = \mathrm{i}\frac{z_{23} z_{41}}{z_{24} z_{31}} s_{12} A_4(1234). \tag{4.120}$$

This equation requires that

$$\text{Res}\left(\frac{z_{23}z_{41}}{z_{24}z_{31}}s_{12}\frac{1}{z_3(1-z_3)}\right) = -1, \tag{4.121}$$

where the factor $1/(z_3(1-z_3))$ comes from the measure. Using our conventions for the fixed values of z_1, z_2 and z_4 and Eqs.(3.92)-(3.94), we can happily check that

$$\text{Res}\left(\frac{z_{23}z_{41}}{z_{24}z_{31}}s_{12}\frac{1}{z_3(1-z_3)}\right) = s_{12}\text{Res}\left(\frac{1}{z_3}\right) = s_{12}\,\langle -\frac{s_{13}}{s_{12}}, 1\rangle = -s_{12}\frac{1}{s_{12}} = -1, \tag{4.122}$$

as expected.

It is interesting to see that in this simple case we can make an educated guess about the integrand without referring to the amplitude $A_4(1234)$. Notice that

$$\text{Res}\left(-\frac{1}{1-z_3}\right) = \langle \frac{s_{13}}{s_{14}}, 1\rangle = \frac{1}{s_{14}}\,\langle e_1, \Delta_1\rangle = \frac{1}{s_{14}}, \tag{4.123}$$

which is pole associated with the amplitude $A_4(1234)$. Hence, the integrand for $A_4(1324)$ after normalization should reduce to

$$\tilde{I}^{\text{QCD}}(z,p,\bar{u},v) = -\frac{1}{1-z_3}\,(\bar{u}_1\gamma^\mu v_4)\,(\bar{u}_2\gamma_\mu v_3)\,. \tag{4.124}$$

Therefore our educated guess for the full integrand reads

$$\tilde{I}^{\text{QCD}}(z,p,\bar{u},v) = (z_{12}z_{23}z_{34}z_{41})^2\,(C(1234,z)C(1324,z)\,(\bar{u}_1\gamma^\mu v_4)\,(\bar{u}_2\gamma_\mu v_3))\,. \tag{4.125}$$

In other words, we use the integrand of the double ordered scalar amplitude $m(1234|1324)$ in Eq.(3.76) times the Lorenz invariant products involved in the amplitude. Of course, the point is that we have a method to generate the required poles for any tree-level amplitude but we would like a compact closed form of the integrand.

Example 4.3.2. Let us consider now a 6-point example. Consider the alphabet

$$\mathbb{A}_6 = \{q_1, q_2, q_3, \bar{q}_3, \bar{q}_2, \bar{q}_1\} = \{1,2,3,4,5,6\}. \tag{4.126}$$

The relevant bases are given by

$$\begin{aligned}
B =& \{123456, 134256, 132546, 125346\}, && (4.127)\\
B_{n_q \leq 2} =& \{123456, 124356, 132456, 134256, 142356, 143256\}, && (4.128)\\
\bar{B}_{n_q \leq 2} =& \{123465, 124365, 132465, 134265, 142365, 143265\}. && (4.129)
\end{aligned}$$

In order to compute the matrix G we need to compute the corresponding matrix F and use

$$\mathsf{G} = \mathsf{F}^T (\mathsf{F}\mathsf{F}^T)^{-1}. \tag{4.130}$$

The matrix F of rank 4 is obtained in Eq.(C.53) in Appendix C. Then,

$$\hat{E}(z, p, \varepsilon) = -\mathrm{i} \sum_{v \in B_{n_q \leq 2}} \sum_{w \in B} c_{\bar{v}w} C(\bar{v}, z^{(j)}) A_6(w, p, \varepsilon), \tag{4.131}$$

where

$$c_{\bar{v}w} = \sum_{u \in B_{n_q \leq 2}} S_{u\bar{v}} G_{uw}. \tag{4.132}$$

The matrix S_6 reads

$$\mathsf{S}_6 = \theta_{12}\theta_{13}\theta_{14} \begin{pmatrix}
\frac{a_{12,3}a_{123,4}}{\theta_{13}\theta_{14}} & \frac{a_{12,3}a_{12,4}}{\theta_{13}\theta_{14}} & \frac{a_{123,4}}{\theta_{14}} & \frac{a_{13,4}}{\theta_{14}} & \frac{a_{12,3}}{\theta_{13}} & 1 \\
\frac{a_{12,3}a_{12,4}}{\theta_{13}\theta_{14}} & \frac{a_{12,4}a_{124,3}}{\theta_{13}\theta_{14}} & \frac{a_{12,4}}{\theta_{14}} & 1 & \frac{a_{124,3}}{\theta_{13}} & \frac{a_{14,3}}{\theta_{13}} \\
\frac{a_{123,4}}{\theta_{14}} & \frac{a_{12,4}}{\theta_{14}} & \frac{a_{13,2}a_{123,4}}{\theta_{12}\theta_{14}} & \frac{a_{13,2}a_{13,4}}{\theta_{12}\theta_{14}} & 1 & \frac{a_{13,2}}{\theta_{12}} \\
\frac{a_{13,4}}{\theta_{14}} & 1 & \frac{a_{13,2}a_{13,4}}{\theta_{12}\theta_{14}} & \frac{a_{13,4}a_{134,2}}{\theta_{12}\theta_{14}} & \frac{a_{14,2}}{\theta_{12}} & \frac{a_{134,2}}{\theta_{12}} \\
\frac{a_{12,3}}{\theta_{13}} & \frac{a_{124,3}}{\theta_{13}} & 1 & \frac{a_{14,2}}{\theta_{12}} & \frac{a_{14,2}a_{124,3}}{\theta_{12}\theta_{13}} & \frac{a_{14,2}a_{14,3}}{\theta_{12}\theta_{13}} \\
1 & \frac{a_{14,3}}{\theta_{13}} & \frac{a_{13,2}}{\theta_{12}} & \frac{a_{134,2}}{\theta_{12}} & \frac{a_{14,2}a_{14,3}}{\theta_{12}\theta_{13}} & \frac{a_{14,3}a_{134,2}}{\theta_{12}\theta_{13}}
\end{pmatrix}, \tag{4.133}$$

where we have used the notation

$$\begin{aligned}
\theta_{l_i l_j} =& 2p_{l_i l_j} + 2\Delta_{l_i, l_j}, && (4.134)\\
a_{l_1 l_2 \ldots l_k, l_i} =& \theta_{l_1 l_i} + \theta_{l_2 l_i} + \cdots + \theta_{l_k l_i}. && (4.135)
\end{aligned}$$

The next step is to compute the right inverse and the coefficients $c_{\bar{v}w}$. The results are lengthy due to the inverse matrix in Eq.(4.130) and can be checked numerically.

Now, suppose we are interested in the amplitude $A_6(153426)$ which corresponds to a non-standard oriented amplitude. The first task is then to orient the generalized cyclic factor using Eq.(4.76). Performing this procedure gives

$$\hat{C}(153426, z) = -\hat{C}(124356, z) = \hat{C}(123456, z) = C(123456, z). \tag{4.136}$$

Finally, using matrix notation, we have that the integrand for $A_6(153426)$ reads

$$I^{\text{QCD}}(z, p, \varepsilon) = -\mathrm{i}\; C(123456, z)\left(\mathsf{C}^T \mathsf{S} \mathsf{G}\right) \mathsf{A}_6, \tag{4.137}$$

where C, and A_6 are vectors of orderings $\bar{v}$ and w, respectively.

4.4 Outlook

The main result in this Chapter is the construction of all the building blocks of the CHY representation for QCD amplitudes. These building blocks allow us to write an integrand of the CHY representation which depends on other QCD amplitudes. The generalized permutation invariant factor $\hat{E}$ in Eq.(4.100) is valid for any ordering of the primitive amplitude. In contrast, the generalized cyclic factor $\hat{C}$ contains the information about the external ordering and it depends on the primitive amplitude of interest. In general, $\hat{C}$ will be a combination of standard Parke-Taylor factors. The fact that we can give all the ingredients proves the existence of the CHY representation of QCD, which is valid in D dimensions.

However, it would be desirable to have a closed form of the generalized permutation invariant factor like in the pure gluonic case, where the reduced Pffafian computes all the necessary poles and Lorenz products (See Section 4.1.1). Most of the hints that made us suspect that a CHY representation for QCD exist may be invoked again to suspect that a closed formula exists as well. In particular, the hint for massless QCD in 4D, which is the argument that we can use $\mathcal{N} = 4$ SYM to construct QCD amplitudes and therefore a CHY representation is strong due to the closed formula introduced by Dixon, et'al [104]. In Ref. [112] He and Zhang followed this path to construct a closed formula for massless QCD amplitudes in 4D in the connected formalism. The connected formalism can be seen as a 4D version of the CHY formalism, thus requiring a translation from the D-scattering equations to the 4D-scattering-equations [113]. The search for this formula is still an open problem.

Chapter 5

QCD and Gravity

QCD and gravity seem to be unrelated theories at various levels. At the level of the Lagrangian they contain different types of interactions, which generate a limited number of Feynman rules in the case of QCD and an infinite number of them in the case of gravity. They also differ in the fact that QCD is renormalizable in the traditional sense—meaning that we can redefine a finite number of parameters such that the quantities computed are finite. In contrast, gravity is not renormalizable and necessarily has to be regarded as an effective field theory.

However, at the level of amplitudes we have seen that for pure gluonic amplitudes the KLT relations connect gauge and gravity. These relations, originally introduced in the context of string theory, represented a new way of studying perturbative quantum gravity. In this paradigm, one invokes that the square of a gauge theory amplitude corresponds to a gravity amplitude. This paradigm adopts various forms. First, the original formulation based on the KLT relations in the field theory limit—or in terms of the KLT kernel. We have seen in Chapter 2 that these relations contain kinematic factors that cancel double poles and deal with the fact that gravity has not fixed ordering. Second, In Chapter 2 we also have seen that using the color kinematics duality and the double copy procedure—which gives a recipe of constructing gravity amplitudes as a sum over trivalent graphs—constitutes another implementation of the paradigm. Finally, the CHY formalism of Chapters 3 and 4 gives a third way of squaring gravity by simple substitution of the cyclic factor C in Yang-Mills for a permutation invariant factor E.

In this Chapter, we study the squaring of a QCD primitive amplitude and consider the resulting gravitational theory. We also study the relations between amplitudes of gravitons and gluons (Einstein-Yang-Mills) and pure gluonic amplitudes proposed by Stieberger and Taylor. In the former case we use the KLT kernel and the double copy procedure and in the latter we use the CHY to prove these relations.

5.1 Squaring a gauge theory

In Section 2.4.6, we have seen that the KLT momentum kernel and the color kinematics duality are two methods to square a gauge theory. Actually, we can recover the KLT kernel from the numerators and vice versa as was shown in Ref. [11]. This connection was made explicit in Ref. [54] where the numerators were expressed in terms of the KLT kernel[1]. In the case of the CHY representation, the color kinematics duality and the double copy procedure is equivalent to the transformation [62]

$$C(w,z) \to E(z,p,\varepsilon) \tag{5.1}$$

in Eq.(3.67). Hence, we have three flavors of the *equation*

$$\text{gravity} = \text{gauge} \times \text{gauge}. \tag{5.2}$$

1. Using the KLT momentum kernel $S[w_1|\bar{w}_2]$ (Eq.(2.178))

$$M_n \equiv M_n(1,2,\ldots,n) = -\mathrm{i} \sum_{w_1,w_2\in B} A_n(w_1) S[w_1|\bar{w}_2] A_n(\bar{w}_2), \tag{5.3}$$

   where the gauge amplitudes are in the BCJ basis[2].

2. By constructing BCJ numerators n_i and $\tilde{n}_i$ for amplitudes in two gauge theories (which may be the same), the gravitational amplitude reads

$$M_n = \mathrm{i}(-1)^{n-3} \sum_i \frac{n_i \widetilde{n}_i}{\prod_{\alpha_i} s_{\alpha_i}}, \tag{5.4}$$

   where the sum runs over $(2n-5)!!$ trivalent diagrams.

3. By making the replacement $C(w,z) \to E(z,p,\varepsilon)$ in

$$A_n(p,w) = \frac{\mathrm{i}(-1)^n}{(2\pi \mathrm{i})^{n-3}} \oint_{\mathcal{O}} \mathrm{d}\tilde{\Omega}_{\mathrm{CHY}} \prod_{i=1}^{n} (z_i - z_{i+1})^2 C(w,z) E(z,p,\varepsilon). \tag{5.5}$$

[1] In this case the numerators correspond to the kinematic factors appearing in the multi-peripheral expansion of gravity (See e.g., Chapter 13 in Ref. [36])

[2] This form is convenient for our purposes, this relation can be presented in several ways as discussed in Refs. [52, 114]

It is interesting to see how these approaches are equivalent by finding their connection with the KLT momentum kernel.

5.1.1 The KLT matrix and color-kinematics duality

The color-kinematics duality instructs us to decompose the full gauge amplitude in terms of color factors c_i and numerators n_i which satisfy Jacobi relations and can be computed from trivalent graphs. An alternative formulation of this duality appears by first considering the color decomposition by Del Duca, Dixon, and Maltoni (DDM) [115, 116]. Using the KK basis (2.154), this decomposition reads

$$\mathcal{A}_n(p, \varepsilon) = \sum_{w \in W_2} c_w A_n(w), \tag{5.6}$$

where the color factors are given by

$$c_w = f^{a_1 a_{l_2} a_{b_1}} f^{a_{b_1} a_{l_3} a_{b_2}} \cdots f^{a_{b_{n-3}} a_{l_{n-1}} a_n}. \tag{5.7}$$

These color factors can be computed from multi-peripheral diagrams (ladder type diagrams) shown in Fig.5.1.

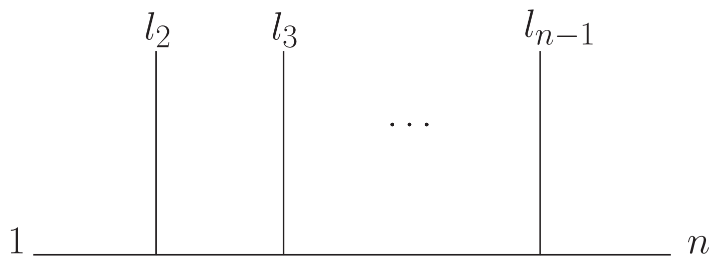


Figure 5.1: Multi-peripheral diagrams for words $w = 1l_2 \ldots l_{n-1}n$. The words are in the set W_2, where two of the letters are fixed, i.e., $l_1 = 1$, $l_n = n$. The set W_2 contains $(n-2)!$ elements.

An observation made in Ref. [55] is that when the numerators satisfy the color kinematics duality they have the same structure as color factors[3], and therefore there is a dual formula for gauge amplitudes given by

$$\mathcal{A}_n(p, \varepsilon) = \sum_{w \in W_2} n_w A_n^{\text{dual}}(w), \tag{5.8}$$

[3] An interesting issue is then to determine the kinematic algebra of the numerators [117].

where the kinematic factors n_w are numerators of Fig.5.1 and the dual amplitudes A_n^{dual} can be computed from primitive amplitudes by replacing kinematic numerators by color factors [118] of a cubic theory. Then the dual form (5.8) implies that the gravitational amplitude has the dual form

$$M_n = \sum_{w\in W_2} n_w M_n^{\text{dual}}(w) = \sum_{w\in W_2} n_w A_n(w), \tag{5.9}$$

where the corresponding dual amplitude is a gauge primitive amplitude. Now, these numerators have an explicit form in terms of the KLT momentum kernel [54]. For $w = 1l_2 \dots l_{n-1}l_n$, we have

$$n_w = \begin{cases} -\mathrm{i} \sum\limits_{v\in B} S[w|\bar{v}]A_n(\bar{v}), & l_{n-1} = n-1, \\ 0 & \text{else}\ , \end{cases} \tag{5.10}$$

where B is the BCJ basis[4]. This formula is what connects the KLT kernel to the color-kinematics duality and the double copy procedure (See also Ref. [119]).

Furthermore, if we restrict the amplitude to the BCJ basis, we found in Chapter 4 that the permutation invariant factor for pure Yang-Mills is given by

$$E(z,p,\varepsilon) = -\mathrm{i} \sum_{v\in B}\sum_{u\in B} S_{v\bar{u}} C(\bar{u},z) A_v, \tag{5.11}$$

hence

$$E(z,p,\varepsilon) = \sum_{w_2\in W_2} n_{w_2} C(\bar{w}_2, z), \tag{5.12}$$

where the numerators have to be restricted to the BCJ basis [14]. Similarly, the decomposition (5.6) for the full amplitude can be expressed in terms of the color factors (5.7) by defining

$$\mathcal{C}(z) = \sum_{w\in W_2} c_w C(w,z), \tag{5.13}$$

where $\mathcal{C}$ is the color dependent factor for the full amplitude. Using this factor the CHY-representation reads

[4]See Section 2.4.4

$$\mathcal{A}_n(p,\varepsilon)=\frac{\mathrm{i}(-1)^n}{(2\pi\mathrm{i})^{n-3}}\oint_{\mathcal{O}}\mathrm{d}\tilde{\Omega}_{\text{CHY}}\prod_{i=1}^{n}(z_i-z_{i+1})^2\mathcal{C}(z)E(z,p,\varepsilon). \tag{5.14}$$

Using Eq.(5.12) and the substitution $\mathcal{C}\to E$—including color for the case of the full amplitude—transforms

$$\begin{aligned}\mathcal{A}_n(p,\varepsilon)\to&\left(\sum_{w_2\in W_2}n_{w_2}\right)\frac{\mathrm{i}(-1)^n}{(2\pi\mathrm{i})^{n-3}}\oint_{\mathcal{O}}\mathrm{d}\tilde{\Omega}_{\text{CHY}}\prod_{i=1}^{n}(z_i-z_{i+1})^2C(w_2,z)E(z,p,\varepsilon)\\=&\left(\sum_{w_2\in W_2}n_{w_2}\right)A_n(w_2),\end{aligned} \tag{5.15}$$

which coincides with the dual form of the gravitational amplitude in Eq.(5.9). This rather simple observation can be made explicit by using a expansion in terms of trivalent diagrams and thus showing that indeed the numerators and color factors satisfy Jacobi relations and that the double copy procedure is equivalent to the substitution $\mathcal{C}\to E$ [14].

The bottom line of this discussion is to realize that we can move from one realization (color-kinematics-duality) of *gravity* $=$ *gauge* $\times$ *gauge* to CHY by using the KLT momentum kernel. We can take for example Eq.(5.3) as the starting point, then use Eq.(5.10) to move to the dual form of gravity (5.9) and finally insert the CHY representation (5.14) to obtain a squared permutation invariant factor. Given the importance of the KLT kernel, we would like to explore all its flavors, and more important to relate it with the quantities we already know.

5.1.2 The KLT matrix and the CHY representation

The KLT momentum kernel was found and proved using recursive methods in QFT (via BCFW) and properties of the S-matrix [52] and then studied in the context of string theory in Ref. [54]. We have defined the elements of the KLT kernel $S[w_1|w_2]$ in Eq.(2.178). Now, from the double copy procedure it is clear that the KLT relations arise from the substitution of the numerators in Eq.(5.4). This fact becomes obvious after dualizing the gravity amplitude.

Then, the remaining task is to identify the KLT kernel from the CHY representation. This was done by CHY using the KLT orthogonality of the scattering equations (See Section 3.1.2). An easy way to find the connection is to consider again the substitution $\mathcal{C}\to E$, which we know generates gravity, i.e.,

$$M_n = \left(\sum_{w_2 \in W_2} n_{w_2} \right) \frac{\mathrm{i}(-1)^n}{(2\pi \mathrm{i})^{n-3}} \oint_{\mathcal{O}} \mathrm{d}\tilde{\Omega}_{\mathrm{CHY}} \prod_{i=1}^{n} (z_i - z_{i+1})^2 C(w_2, z) E(z, p, \varepsilon). \tag{5.16}$$

Now, the permutation invariant factor in Eq.(5.11) was obtained by inverting the CHY formula using KLT orthogonality (See Chapter 3). Inserting the permutation factor E in Eq.(5.16), we obtain

$$M_n = \sum_{w_2 \in W_2} n_{w_2} \sum_{v \in B} \sum_{u \in B} S_{v\bar{u}} \left(\frac{1}{(2\pi \mathrm{i})^{n-3}} \oint_{\mathcal{O}} \mathrm{d}\tilde{\Omega}_{\mathrm{CHY}} \prod_{i=1}^{n} (z_i - z_{i+1})^2 C(w_2, z) C(\bar{u}, z) \right) A_v. \tag{5.17}$$

According to Eq.(5.9), Eq.(5.17) implies that the inverse of the KLT matrix is given by the double ordered scalar amplitudes (Eq.(3.66)), i.e.,

$$(S^{-1})_{u\bar{v}} = -\mathrm{i}\ m(u|\bar{v}) \equiv -\mathrm{i}\ m_{uv}, \tag{5.18}$$

where now the rows and columns of the matrices S and m in (5.18) correspond to orderings. The distinction between v and $\bar{v}$ is not relevant since this relation hold within the BCJ basis. Remember that this basis contains already $(n-3)!$ elements, i.e, the same number of solutions of the scattering equations. This fact was considered in Eq.(4.19) where we used the solution space to obtain the inverse of the KLT kernel. In conclusion, the KLT kernel can be computed as the inverse of the matrix $m(w_1|w_2)$, i.e.,

$$\mathsf{S} = \mathrm{i}\ (\mathsf{m})^{-1}. \tag{5.19}$$

Finally, the double ordered amplitudes $m(w_1|w_2)$ can be understood as a sum over trivalent graphs G, which are consistent with the orderings w_1 and w_2 [14]

$$m(w_1|w_2) = \mathrm{i}\ (-1)^{n-3+n_{\mathrm{flip}}(w_1,w_2)} \sum_{G \in \mathcal{T}(w_1) \cap \mathcal{T}(w_2)} \prod_{e \in E(G)} \frac{1}{s_e}. \tag{5.20}$$

where $\mathcal{T}(w_1) \cap \mathcal{T}(w_2)$ is the set of compatible diagrams with external orderings. The quantity $n_{\mathrm{flip}}(w_1, w_2)$ is the number of flips needed to transform any diagram from $\mathcal{T}(w_1) \cap \mathcal{T}(w_2)$ with ordering w_1 into a diagram with external ordering w_2. The set of internal edges e for each graph G is denoted by $E(G)$ (See Fig.5.2).

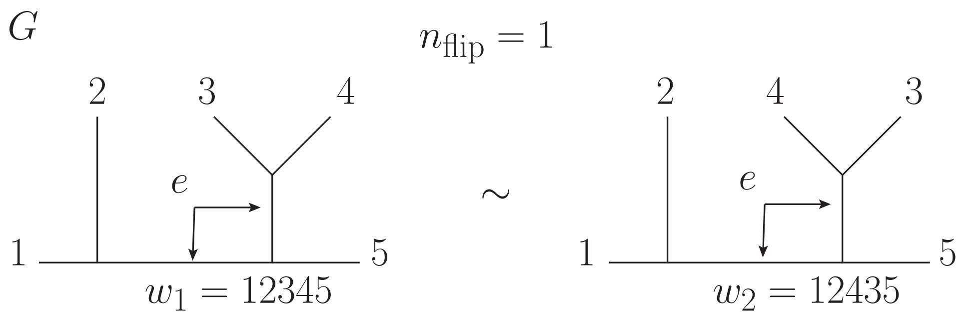


Figure 5.2: Flip operation. Two diagrams G with orderings w_1 and w_2 are equivalent after exchanging edges 3 and 4. We associate invariants s_e to the internal edges e.

In conclusion, we can compute the components of the KLT matrix using the momentum kernel definition, as the limit of the string momentum kernel, as the inverse of the matrix made of double ordered amplitudes or equivalently, as a sum over trivalent graphs.

5.1.3 Double copies of gluons

The quantized gravitational field have polarizations tensors $\varepsilon_{\mu\nu}$ multiplying creation an annihilation operators. These functions are reminders of the representation of the Lorentz group, which in the case of gravity is the tensor representation. There are two physically significant polarization tensors for gravity, which in the helicity representation of states correspond to the helicities ± 2. Now, for each helicity the polarization tensors of gravity satisfy

$$\varepsilon^{+}_{\mu\nu} \equiv \varepsilon^{++}_{\mu\nu} = \varepsilon^{+}_{\mu}\varepsilon^{+}_{\nu}, \qquad \varepsilon^{-}_{\mu\nu} \equiv \varepsilon^{--}_{\mu\nu} = \varepsilon^{-}_{\mu}\varepsilon^{-}_{\nu}, \tag{5.21}$$

where $\varepsilon^{\pm}_{\mu}$ are ordinary spin 1 polarization vectors. Defined in this way, the polarization tensors are traceless and transverse. In Chapter 2, we used implicitly this property when we used the spinor helicity formalism in e.g., Eq.(2.177). Now, on the RHS of Eq.(5.3) there is no instruction to assign specific helicities to the amplitudes and therefore we have the freedom to choose

$$\varepsilon^{+-}_{\mu\nu} = \varepsilon^{+}_{\mu}\varepsilon^{-}_{\nu}, \qquad \varepsilon^{-+}_{\mu\nu} = \varepsilon^{-}_{\mu}\varepsilon^{+}_{\nu}, \tag{5.22}$$

which can be used to construct linear combinations of scalar states—the symmetric combination corresponds to a dilaton and the antisymmetric to an axion. That these correspond to scalar states can be seen from the little group scaling (See Section 2.4.1). In this sense, the squaring of amplitudes in gauge theories produces gravity coupled to dilatons and axions.

The same discussion holds from the BCJ double copy procedure and the CHY formalism where all quantities involved depend only on gluon polarizations. In this sense, we say that gravitons are double copies of gluons.

5.1.4 Examples

Example 5.1.1. Let us start with $n = 4$. The relevant basis reads $W_2 = \{1234, 1324\}$ and from Eq.(5.10) the numerators are given by

$$n_{1234} = -\mathrm{i}s_{12}A_4(1243), \qquad n_{1324} = 0 \tag{5.23}$$

therefore

$$M_4 = \begin{pmatrix} -\mathrm{i}s_{12}A_4(1243) & 0 \end{pmatrix} \begin{pmatrix} A_4(1234) \\ A_4(1324) \end{pmatrix} = -\mathrm{i}s_{12}A_4(1234)A_4(1243), \tag{5.24}$$

which trivially recovers the KLT relation (2.174) at 4-points (after using BCJ relations). The KLT matrix in this case contains one element which corresponds to inverse of $m(1234|1243)$, i.e,

$$\mathsf{S} = \mathrm{i}\ \mathsf{m}^{-1} = s_{12}, \tag{5.25}$$

which can be easily checked with the methods of Chapter 3. Let us now use Eq.(5.20). For the orderings $w_1 = 1234$, $w_2 = 1243$, we have the trees[5] shown in Fig. 5.3.

Therefore,

$$n_{\text{flip}} = 1, \quad \mathcal{T}(w_1) \cap \mathcal{T}(w_2) = \{[2, [3, 4]]\}, \tag{5.26}$$

and

$$m(1234|1243) = \mathrm{i}\frac{1}{s_{12}}. \tag{5.27}$$

[5] Here we use the leg 1 as the root of the three and use the notation $[R, L]$ to denote the right and left tree.

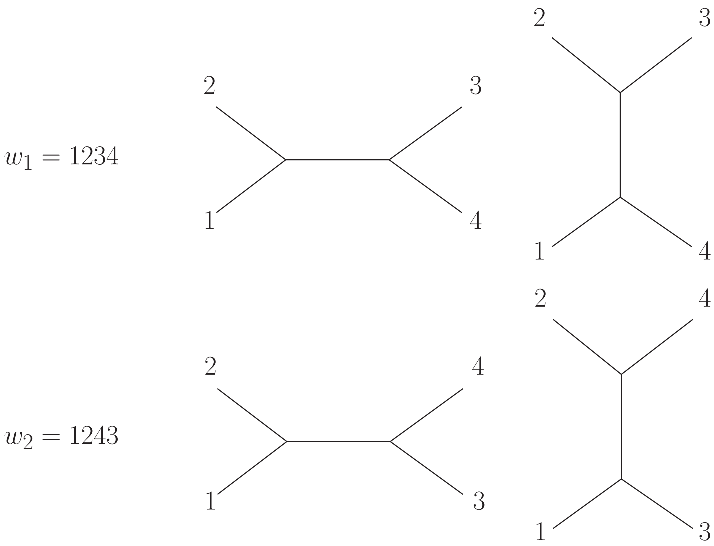


Figure 5.3: Trivalent graphs for the orderings 1234, 1243. For 1234, the threes are $[2,[3,4]]$, $[[2,3],4]$ and similarly for 1243. There is only one tree compatible with these orderings, namely $[2,[3,4]]$.

Example 5.1.2. For $n = 5$, the relevant bases are

$$\begin{aligned} W_2 =& \{12345, 12435, 13245, 13425, 14235, 14325\}, \\ B =& \{12345, 13245\}, \quad \bar{B} = \{12354, 13254\}. \end{aligned} \tag{5.28}$$

The numerator matrix reads

$$\mathsf{n}^T = \begin{pmatrix} \mathrm{i}s_{12}\left(s_{13}+s_{23}\right)A_5(12354) + \mathrm{i}s_{12}s_{13}A_5(13254) \\ 0 \\ \mathrm{i}s_{12}s_{13}A_5(13254) + \mathrm{i}s_{13}\left(s_{12}+s_{23}\right)A_5(13254) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.29}$$

which reproduces Eq.(2.175). The relevant ordered trivalent diagrams for $m(12345|12354)$ are shown in Fig. 5.4.

Then, only the first and second diagrams are compatible with both orderings, hence

$$\mathcal{T}(w_1) \cap \mathcal{T}(w_2) = \{[2,[3,[4,5]]], [[2,3],[4,5]]\}, \tag{5.30}$$

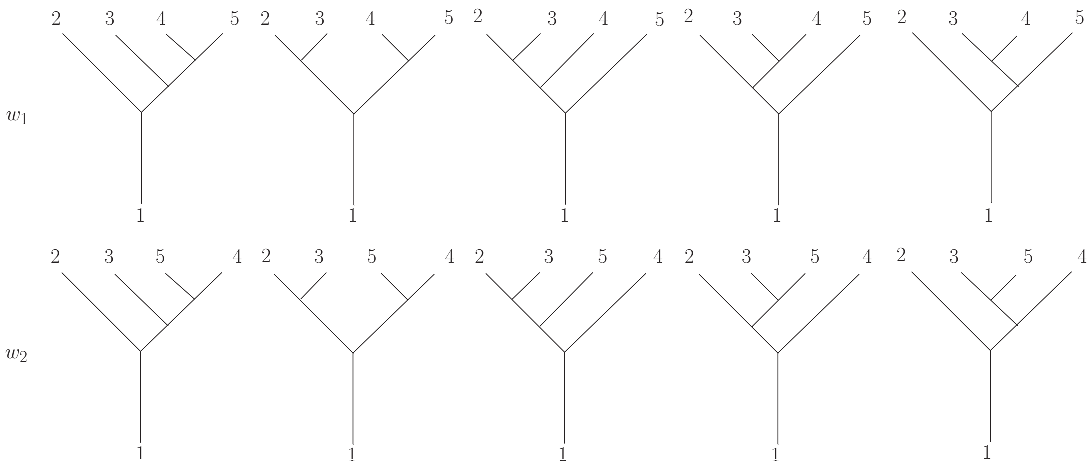


Figure 5.4: Trivalent graphs for $n = 5$ for the orderings $w_1 = 12345$, $w_2 = 12354$. Using the leg 1 as the root, we have 5 trivalent graphs and two compatible with both orderings, namely [2, [3, [4, 5]]] and [[2, 3], [4, 5]].

and

$$m(12345|12354) = \mathrm{i}\left(\frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{45}}\right). \tag{5.31}$$

We can also calculate the other contributing double ordered amplitudes, giving

$$\mathsf{m} = \mathrm{i}\begin{pmatrix} -\left(\frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{123}}\right) & \frac{1}{s_{32}s_{123}} \\ \frac{1}{s_{32}s_{123}} & -\left(\frac{1}{s_{13}s_{123}} + \frac{1}{s_{32}s_{123}}\right) \end{pmatrix}, \tag{5.32}$$

which has the inverse

$$\mathsf{m}^{-1} = -\mathrm{i}\; s_{123}\begin{pmatrix} -\frac{s_{12}(s_{13}+s_{23})}{s_{12}+s_{13}+s_{23}} & -\frac{s_{12}s_{13}}{s_{12}+s_{13}+s_{23}} \\ -\frac{s_{12}s_{13}}{s_{12}+s_{13}+s_{23}} & -\frac{s_{13}(s_{12}+s_{23})}{s_{12}+s_{13}+s_{23}} \end{pmatrix} \tag{5.33}$$

hence

$$\mathsf{S}^{-1} = \mathrm{i}\;\mathsf{m}^{-1} = -\begin{pmatrix} s_{12}\left(s_{13}+s_{23}\right) & s_{12}s_{13} \\ s_{12}s_{13} & s_{13}\left(s_{12}+s_{23}\right) \end{pmatrix} \tag{5.34}$$

in agreement with Eq.(2.181).

5.2 Double copies of gluons and fermions

This section is based on Ref. [120] by the author, Alexander Kniss, and Stefan Weinzierl.

A natural question to ask is whether the idea of squaring a gauge theory to obtain gravity extends for other theories besides pure Yang-Mills. Notice that the double copy construction by BCJ gives already an affirmative answer to this question. As we pointed out, the numerators in Eq.(5.4) can be of two different gauge theories and therefore not only holds for pure Yang-Mills. Furthermore, unlike the KLT kernel based approach[6], the double copy procedure has been conjectured to hold for loop level amplitudes as well [12]. In this sense, the double copy procedure is a more general realization of the concept of squaring a gauge theory. In particular, it can be applied to a wider range of gravitational theories, specially but not exclusively those including supersymmetry[7]. The idea of the double copy has also been considered at the classical level by studying solutions to Einsteins equations and their relations to Yang-Mills solutions, in particular for black hole solutions [125–128].

A possible route in the construction of gravitational amplitudes—which results from the square of gauge theory amplitudes—is first investigate whether the associated gauge theory satisfies color-kinematics duality. The second step is to study the resulting gravitational theory and determine which amplitudes are computed by the double copy procedure.

In the case of QCD, Johansson and Ochirov found that QCD amplitudes satisfy color kinematics duality [103]. Hence, a natural question is what is the gravitational theory obtained by squaring QCD primitive amplitudes. In addition, one would like to know if we have three equivalent approaches to compute the amplitudes of such gravitational theory as in Section 5.1.

5.2.1 Color kinematics for QCD amplitudes

Recall that the idea of the color kinematics duality is to expand the full amplitude in terms only of cubic diagrams. In the case of QCD, these cubic diagrams contain flavor conserving vertices and unflavored vertices. Therefore, the number of diagrams depend on the number of quark lines. For a n point QCD amplitude with n_g gluon and $2n_q$ quarks, we denote the set of $(2n-5)!!/(2n_q-1)!!$ trivalent graphs by $\mathcal{U}$. For each diagram G we denote by $E(G)$ the set of internal edges and by s_e and m_e the Lorentz invariants and the mass of the corresponding internal edge e. In Ref. [103] it was established that QCD exhibit color-kinematics duality, i.e., that we can write QCD amplitudes as

$$\mathcal{A}_n^{\mathrm{QCD}} = \mathrm{i} \sum_{G\in\mathcal{U}} \frac{C(G)N(G)}{D(G)}, \tag{5.35}$$

where the denominators D are the propagators of the graph G, i.e.,

[6] A recent proposal for KLT at loop level has been presented in Ref. [121].

[7] For reviews see Refs. [122, 123]. Pure gravities are considered in Ref. [124].

$$D(G) = \prod_{e\in E(G)} \left(s_e - m_e^2\right). \tag{5.36}$$

The color kinematics duality then states that for graphs G_1, G_2, G_3

$$C(G_1) + C(G_2) + C(G_3) = 0 \Rightarrow N(G_1) + N(G_2) + N(G_3) = 0, \tag{5.37}$$

Notice that there is an arbitrariness in the sign of the numerators N. The usual convention is trivially recover by making $N(G_2) \to -N(G_2)$. This modifications do not change the residues that the numerators correspond to [11]. This can be easily seen in the following Example.

Example 5.2.1. Consider the case $n = 5$ with two pairs of quarks and one gluon. The alphabet reads $\mathbb{A} = \{q_1, q_2, g_1, \bar{q}_2, \bar{q}_1\} = \{1, 2, 3, 4, 5\}$. The set of diagrams is shown in Fig.5.5. The decomposition (5.35) reads

$$\mathcal{A}_5^{\mathrm{QCD}} = \tag{5.38}$$
$$\mathrm{i}\left(\frac{C_1N_1}{s_{24}s_{15}} + \frac{C_2N_2}{s_{24}(s_{13}-m_1^2)} + \frac{C_3N_3}{s_{24}(s_{35}-m_1^2)} + \frac{C_4N_4}{s_{15}(s_{23}-m_2^2)} + \frac{C_5N_5}{s_{15}(s_{34}-m_2^2)}\right),$$

where

$$C_1 = -\,\mathrm{i}\sqrt{2} f^{a_3ab}(T^a)_{i_2j_4}(T^b)_{i_1j_5}, \tag{5.39}$$
$$C_2 = (T^{a_3})_{i_1j}(T^b)_{jj_5}(T^b)_{i_2j_4}, \quad C_3 = -(T^b)_{i_1j}(T^{a_3})_{jj_5}(T^b)_{i_2j_4}, \tag{5.40}$$
$$C_4 = -\,(T^b)_{i_ij_5}(T^{a_3})_{i_2j}(T^b)_{jj_4}, \quad C_5 = (T^{a_3})_{j_4i}(T^b)_{ii_2}(T^b)_{i_ij_5}. \tag{5.41}$$

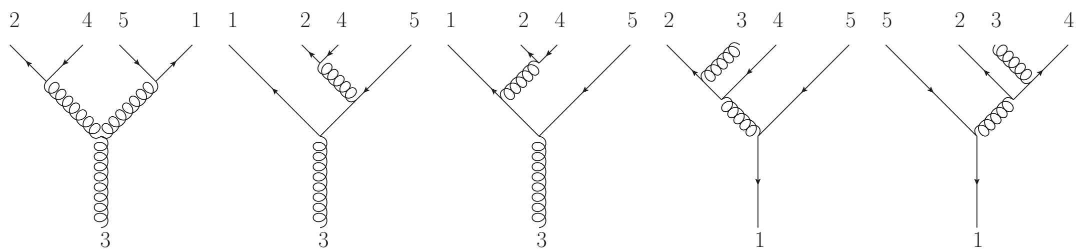


Figure 5.5: Feynman diagrams for the full amplitude $\mathcal{A}_5^{\mathrm{QCD}}(1, 2, 3, 4, 5)$.

Hence, the color factors satisfy

$$C_1 + C_2 + C_3 = 0, \quad C_1 + C_3 + C_4 = 0, \tag{5.42}$$

therefore from the color-kinematics duality, the numerators satisfy

$$N_1 + N_2 + N_3 = 0, \quad N_1 + N_3 + N_4 = 0, \tag{5.43}$$

which can be checked from Feynman diagrams. From (5.38), we see e.g. that we may redefine $N_2 \to -N_2$ and obtain the signs of Eqs.(2.166).

We can also write the color-kinematics duality for primitive amplitudes. For an amplitude with the ordering dictated by a word $w \in B$, where B is some suitable basis, we have

$$A_n(p, w, \varepsilon) = \mathrm{i} \sum_{G \in \mathcal{T}(w)} \frac{N(G)}{D(G)}, \tag{5.44}$$

where the sum runs over the set of ordered trivalent graphs denoted by $\mathcal{T}(w)$. Also, the numerators $N(G)$ satisfy Jacobi-like identities whenever the corresponding color factors do.

Since we are considering bases for amplitudes in QCD, we will use the BCJ basis that we used in Chapter 4. An account of the basis and the relations that are used at each point to obtain the basis is reviewed in Appendix C. For the benefit of the reader, we rewrite the basis. For an amplitude with $n = n_g + 2n_q$ particles the size of the basis is given by

$$N_{\text{basis}} = \begin{cases} (n-3)!, & n_q \in \{0, 1\}, \\ (n-3)!\frac{2(n_q-1)}{n_q!}, & n_q \geq 2. \end{cases} \tag{5.45}$$

A possible basis is given by

$$B = \begin{cases} \{l_1 l_2 \dots l_n \in W_0 | l_1 = g_1, l_{n-1} = g_{n-1}, l_n = g_n\}, & n_q = 0, \\ \{l_1 l_2 \dots l_n \in W_0 | l_1 = q_1, l_{n-1} = g_{n-2}, l_n = \bar{q}_1\}, & n_q = 1, \\ \{l_1 l_2 \dots l_n \in \text{Dyck}_{n_q} | l_1 = q_1, l_{n-1} \in \{\bar{q}_2, \dots, \bar{q}_{n_q}\}, l_n = \bar{q}_1\}, & n_q \geq 2. \end{cases} . \tag{5.46}$$

Examples of the basis are given in Sections 4.2.2.1, 4.3.

5.2.2 Double copies of fermions

Since we have color kinematics duality for QCD amplitudes, we would like to use the double copy procedure and determine the resulting gravitational theory. In the same sense as gravitons can be thought as double copies of gluons, we propose that the resulting gravitational theory contains particles with wavefunction coefficients [120]

$$\bar{u}^{\lambda\tilde{\lambda}}_{\alpha\beta} \equiv \bar{u}^{\lambda}_{\alpha}\bar{u}^{\tilde{\lambda}}_{\beta}, \qquad v^{\lambda\tilde{\lambda}}_{\alpha\beta} \equiv v^{\lambda}_{\alpha}v^{\tilde{\lambda}}_{\beta}, \tag{5.47}$$

where $\bar{u}^{\lambda\tilde{\lambda}}_{\alpha\beta}$ corresponds to the double copy of a fermion and $v^{\lambda\tilde{\lambda}}_{\alpha\beta}$ to the double copy of an anti-fermion. With the numerators of Eq.(5.44), we can then compute the gravitational theory, i.e.,

$$M_n = \mathrm{i}(-1)^{n-3}\sum_{G\in\mathcal{U}}\frac{N(G)N(G)}{D(G)}, \tag{5.48}$$

where the sum runs over all trivalent diagrams with and without flavor.

Example 5.2.2. $n = 4$. Let us consider first the alphabet $\mathbb{A} = \{q_1, q_2, \bar{q}_2, \bar{q}_1\} = \{1, 2, 3, 4\}$. The set of flavor conserving trivalent diagrams $\mathcal{U}$ contains $3!!/3!!$ elements, therefore we need to find one numerator, which can be read from

$$A_4(1234) = \mathrm{i}\frac{N(G_1)}{s_{23}} \Rightarrow N(G_1) = -\mathrm{i}A_4(1234)s_{23}. \tag{5.49}$$

The gravitational amplitude in this case is simply

$$M_4 = \mathrm{i}\frac{A_4(1234)s_{23}^2A_4(1234)}{s_{23}} = \mathrm{i}A_4(1234)s_{23}A_4(1234). \tag{5.50}$$

Next, consider the alphabet $\mathbb{A} = \{q_1, g_1, g_2, \bar{q}_1\} = \{1, 2, 3, 4\}$, then we have $3!!/1!!$ diagrams (Fig.5.6). Here, the basis has a single element, namely $B = \{1234\}$. Hence,

$$A_4(1234) = \mathrm{i}\left(\frac{N(G_1)}{s_{23}} + \frac{N(G_2)}{s_{12} - m^2}\right). \tag{5.51}$$

From the color-kinematics duality

$$C(G_1) - C(G_2) + C(G_3) = 0 \Rightarrow N(G_1) - N(G_2) + N(G_3) = 0, \tag{5.52}$$

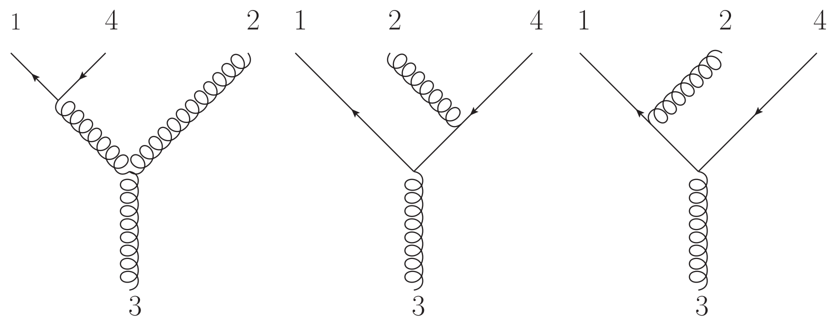


Figure 5.6: Feynman diagrams with for the full amplitude $\mathcal{A}_4^{\mathrm{QCD}}(1,2,3,4)$
.

and therefore

$$A_4(1234) = \mathrm{i}N(G_1)\left(\frac{1}{s_{23}} + \frac{1}{s_{12}-m^2}\right) + \mathrm{i}\frac{N(G_3)}{s_{12}-m^2}, \tag{5.53}$$

which leaves $(4-2)! - (3-1)!$ numerators unspecified. We set them to zero. In particular, for $N(G_3) = 0$, we obtain

$$N(G_1) = -\mathrm{i}A_4(1234)\left(\frac{s_{23}(s_{12}-m^2)}{s_{23}+s_{12}-m^2}\right) = N(G_2). \tag{5.54}$$

Therefore, the gravity amplitude reads

$$M_4 = \mathrm{i}A_4(1234)^2\left(\frac{1}{s_{23}} + \frac{1}{s_{12}-m^2}\right)\left(\frac{s_{23}(s_{12}-m^2)}{s_{23}+s_{12}-m^2}\right)^2, \tag{5.55}$$

which simplifies to

$$M_4 = \mathrm{i}A_4(1234)\left(\frac{2p_2\cdot p_3(2p_1\cdot p_2)}{-2p_2\cdot p_4}\right)A_4(1234). \tag{5.56}$$

The pure gluonic case can be treated in the same way. Actually, the pure gluonic case gives the original double copy procedure.

5.2.3 Generalized KLT kernel

We have already constructed a gravitational theory which contains double copies of fermions and gravitons—it also contains dilatons and axions due to the helicity combinations $\pm$. A natural question to ask is whether the KLT matrix of Sec.5.1.2 or a generalization of it can be used to describe this new theory. For $n = 4$, the KLT kernel contains a single element, namely $S[1234, 1243] = s_{12}$, which does not correspond to the required value to cancel the double poles of the amplitude with two quark lines (See Eq.(5.50)). Therefore, we need a new KLT kernel that cancel the appropriate poles for such amplitude. In addition, it must lead to the standard KLT matrix for the pure gluonic case.

From our comments at the beginning of the Chapter, we know that the KLT kernel was found in the context of string theory by taking the field theory limit of the relation between open and closed strings. In this new situation we do not have access to such supporting theory and therefore we cannot generalize the KLT kernel in this way. However, we do have an alternative procedure to obtain the KLT kernel in CHY formalism—as the inverse of the matrix formed by double ordered scalar amplitudes. In this case, we do not have access to the CHY representation of this new theory, then the last option is to generalize the KLT kernel in terms of ordered trees in analogy with Eq.(5.20).

Since QCD amplitudes can be decomposed in terms on only diagrams with two types of trivalent vertices, namely the three-gluon vertex and the quark-gluon vertex, it is natural to consider ordered trees with two types of vertices—in the same sense that we decompose gluonic ordered amplitudes in terms only of trivalent vertices with one type of vertex.

Consider the basis B in Eq.(5.46). Let use then denote by $\tilde{\mathcal{T}}(w)$ the set of all ordered tree diagrams with trivalent flavor-conserving vertices and external ordering $w \in B$. These diagrams have two types of vertices: a vertex with three unflavored particles (gluon-vertex) and a vertex with two flavored particles and one unflavored particle (quark-gluon vertex). The number of diagrams in this set depends on the number of quarks n_q. In order to obtain the KLT kernel let us first consider the double ordered flavored amplitude $m_n(p, w, \tilde{w})$. For two orderings $w_2, w_2 \in B$, we define the $N_{\text{basis}} \times N_{\text{basis}}$ dimensional matrix m by

$$m_{w_1 w_2} = \mathrm{i}(-1)^{n-3+n_{\text{flip}}(w_1,w_2)} \sum_{G \in \tilde{\mathcal{T}}(w_1) \cap \tilde{\mathcal{T}}(w_2)} \prod_{e \in E(G)} \frac{1}{s_e - m_e^2}, \tag{5.57}$$

where the sum runs over the set of all diagrams compatible with both orderings w_1 and w_2 denoted by $\tilde{\mathcal{T}}(w_1) \cap \tilde{\mathcal{T}}(w_2)$. The masses of the internal edges e are denoted by m_e. The number of flips to transform any diagram from $\tilde{\mathcal{T}}(w_1) \cap \tilde{\mathcal{T}}(w_2)$ with external ordering w_1 into a diagram with external ordering w_2 is denoted by n_{flip}(See Fig.5.2). We define the momentum kernel as the inverse of the matrix m, i.e.,

$$\mathsf{S} = \mathrm{i}\ \mathsf{m}^{-1}. \tag{5.58}$$

Finally, the gravitational amplitude in terms of the generalized KLT kernel is given by

$$M_n(p,\epsilon,\widetilde{\epsilon}) = -\mathrm{i} \sum_{w,\tilde{w}\in B} A_n(p,w,\epsilon) S_{w\tilde{w}} A_n(p,\tilde{w},\epsilon). \tag{5.59}$$

Notice that we use a product of color ordered amplitudes $A_n(p,w,\epsilon)$ to get an unordered gravitational amplitude $M_n(p,\epsilon,\widetilde{\epsilon})$. This is the proper generalization of the KLT kernel when one wants to add fermions, i.e., with $n_q > 0$. The essential ingredient is the restriction to diagrams with flavor-conserving vertices. We have checked that this is consistent with the color-kinematic-duality for $n \le 8$.

5.2.3.1 Examples

Example 5.2.3. Let us consider the cases with $n = 4$ for the alphabets in Example 5.2.2.

(a) For two quark lines of masses m, m', there is a single ordered flavor conserving diagram and one elements in the basis, therefore

$$\mathsf{S} = \mathrm{i}\,\mathsf{m}^{-1} = \mathrm{i}\, m(1234|1234)^{-1} = \frac{\mathrm{i}}{-\mathrm{i}} \left(\frac{1}{s_{23}}\right)^{-1} = -s_{23}. \tag{5.60}$$

(b) For a single quark line of mass m, there are two ordered flavor conserving diagrams and the basis contains a single element, namely $B = \{1234\}$. Hence,

$$\mathsf{S} = \mathrm{i}\,\mathsf{m}^{-1} = \mathrm{i}\, m(1234|1234)^{-1} = \frac{\mathrm{i}}{-\mathrm{i}} \left(\frac{1}{s_{23}} + \frac{1}{s_{12}-m^2}\right)^{-1} = \frac{2p_1\cdot p_2 2p_2\cdot p_3}{2p_2\cdot p_4}. \tag{5.61}$$

(c) The unflavored case corresponds to the usual scalar double ordered amplitude $m(1234|1234)$ which gives

$$\mathsf{S}^{-1} = -(\frac{1}{s} + \frac{1}{t}) = \frac{st}{u} \tag{5.62}$$

Example 5.2.4. If we have $n = 5$ and only gluons, things get more interesting and we have

$$B = \{12345, 13245\}. \tag{5.63}$$

Consider the double copy of and amplitude containing two unflavored vertices. Here we have $n_q = 0$, and we have two possible diagrams with a single ordering. The 1×1 matrix is

$$\begin{aligned} \mathsf{m} &= -(\frac{1}{s} + \frac{1}{t}) = \frac{u}{st}, \\ &\Rightarrow \\ M_4 &= -\mathrm{i}\frac{st}{u} A_4(1234)^2. \end{aligned} \tag{5.64}$$

If we want to recover the conventional KLT relations we have to remember the BCJ relations [11]

$$sA_4(1234) = uA_4(1324), \tag{5.65}$$

thus recovering the usual KLT relation.

The procedure is the same if we increase the number of points, so it becomes only technically more challenging. In Eq. (5.64) we square the amplitude producing double poles and we have to be sure that these double poles are canceled with the aid of the momentum kernel. Starting with $n = 8$, $n_q = 4$ we have a non-factorizable polynomial in the denominator, which zeros are spurious singularities. To solve the issue we have to use a generalized gauge transformation.

5.2.4 Amplitudes

After computing several examples for the generalized KLT kernel, we are ready to consider the resulting gravitational amplitudes. The full gravitational amplitude reads

$$\mathcal{M}_n = \left(\frac{\kappa}{2}\right)^{n-2} M_n(p, \epsilon, \widetilde{\epsilon}), \tag{5.66}$$

in the Gauss unit system with $\kappa = \sqrt{32\pi G_N}$. We would like to compute 4-point amplitudes involving double copies of fermions, which are the most relevant for phenomenological applications. In Examples 5.2.3, we have calculated the necessary kernels. The gravitational amplitudes are constructed from 4-point primitive amplitudes. We consider the amplitude

$$A_4 = A_4\left(q_1^{\lambda_1}, g_2^{\lambda_2}, g_3^{\lambda_3}, \overline{q}_4^{\lambda_4}\right), \tag{5.67}$$

which corresponds to an amplitude involving a single quark line of mass m. Particles are labeled by their helicities. Similarly, the amplitude involving two quark lines of masses m and m' reads

$$A'_4 = A'_4\left(q_1^{\lambda_1}, q_2'^{\lambda_2}, \overline{q}_3'^{\lambda_3}, \overline{q}_4^{\lambda_4}\right). \tag{5.68}$$

The Feynman diagrams required to calculate these amplitudes are shown in Fig.5.7. Using the Feynman rules of Chapter 2, we have

$$A_4 = \frac{\mathrm{i}}{t}\bar{u}_1\left[\varepsilon_2\cdot\varepsilon_3\not{p}_2 + p_3\cdot\varepsilon_2\not{\varepsilon}_3 - p_2\cdot\varepsilon_3\not{\varepsilon}_2\right]v_4 - \frac{\mathrm{i}}{2(s-m^2)}\bar{u}_1\left[\not{\varepsilon}_2(\not{p}_1+\not{p}_2+m)\not{\varepsilon}_3\right]v_4, \tag{5.69}$$

and

$$A'_4 = -\frac{\mathrm{i}}{2}(\bar{u}_1\gamma^\mu v_4)\frac{1}{s_{23}}(\bar{u}_2\gamma_\mu v_4). \tag{5.70}$$

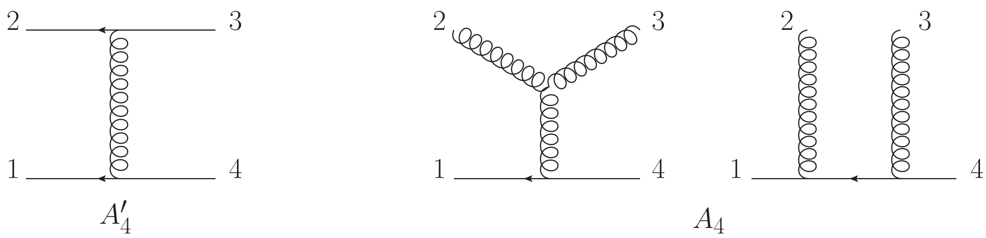


Figure 5.7: Feynman diagrams for the amplitude A'_4 (left) and for A_4 (right).

Using these primitives and the KLT kernels (Section 5.2.3), we obtain

$$\mathcal{M}_4(d_1^{\lambda_1\tilde{\lambda}_1}, h_2^{\lambda_2\tilde{\lambda}_2}, h_3^{\lambda_3\tilde{\lambda}_3}, \overline{d}_4^{\lambda_4\tilde{\lambda}_4}) = -\mathrm{i}\left(\frac{\kappa^2}{4}\right)\frac{2p_1p_2 2p_2p_3}{2p_1p_4} \times A_4\left(q_1^{\lambda_1}, g_2^{\lambda_2}, g_3^{\lambda_3}, \overline{q}_4^{\lambda_4}\right)A_4\left(q_1^{\tilde{\lambda}_1}, g_2^{\tilde{\lambda}_2}, g_3^{\tilde{\lambda}_3}, \overline{q}_4^{\tilde{\lambda}_4}\right), \tag{5.71}$$

where $d(\tilde{d})$ labels the double copy of a fermion(anti-fermion) and h labels a graviton. Similarly the amplitude involving only double copies of fermions reads

$$\mathcal{M}_4(d_1^{\lambda_1\widetilde{\lambda}_1}, d_2'^{\lambda_2\widetilde{\lambda}_2}, \overline{d}_3'^{\lambda_3\widetilde{\lambda}_3}, \overline{d}_4^{\lambda_4\widetilde{\lambda}_4}) = \mathrm{i}\left(\frac{\kappa^2}{4}\right)(2p_2p_3 + 2m'^2) \times A_4\left(q_1^{\lambda_1}, q_2'^{\lambda_2}, \overline{q}_3'^{\lambda_3}, \overline{q}_4^{\lambda_4}\right) A_4\left(q_1^{\widetilde{\lambda}_1}, q_2'^{\widetilde{\lambda}_2}, \overline{q}_3'^{\widetilde{\lambda}_3}, \overline{q}_4^{\widetilde{\lambda}_4}\right). \tag{5.72}$$

Notice the minus sign in Eq.(5.72) due to Eq.(5.60). This sign is responsible for an attractive $1/r$-potential in the classical limit.

5.2.5 Cross sections and dark matter

In this section, we will compute cross sections for various processes. For the computation, we will consider all possible spins states for the double copies, i.e., $\lambda, \widetilde{\lambda} = \pm$. In this way, the spin sums for $\mathcal{M}_4$ and $\mathcal{M}_4'$ factorize into two individual spin sums for A_4 and A_4', respectively. First, let us consider the amplitude

$$|\mathcal{M}_4|^2 = \frac{\kappa^4}{16}\frac{(2p_1p_2)^2(2p_2p_3)^2}{(2p_1p_3)^2}\left(|A_4|^2\right)^2. \tag{5.73}$$

From Eq.(5.69) we can compute the spin summed amplitude

$$|A_4|^2 = \frac{1}{2}\left[3 + \frac{4s}{t} + \frac{2(s-m^2)^2}{t^2} + \frac{t+4m^2}{s-m^2} + \frac{4m^4}{(s-m^2)^2}\right], \tag{5.74}$$

which we can use to compute the cross section for the annihilation process

$$\bar{d}_1 d_4 \to h_2 h_3. \tag{5.75}$$

The cross section corresponds to the case[8] $N_\alpha = 2$ in Eq.(2.12), leading to Eq.(2.16). The cross section reads

$$\sigma = \frac{(2\pi)^4}{4\sqrt{(p_1\cdot p_4)^2 - m^4}}\int \delta^4(p_2+p_3-p_1-p_4)|\mathcal{M}_4|^2 \frac{\mathrm{d}^3\mathbf{p}_2\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3 2E_2(2\pi)^3 2E_3}. \tag{5.76}$$

In the center of momentum frame, this integral becomes

[8] After normalizing with flux $\sqrt{(p_1\cdot p_4)^2 - m^4}/(VE_1E_4)$. See comments Section 2.1.1.

$$\sigma = \frac{16\pi G_N^2}{2E_1 2E_4|\frac{\mathbf{p}_1}{E_1} - \frac{\mathbf{p}_4}{E_4}|} \int_{-\frac{t}{2}+\frac{\chi}{2}}^{-\frac{t}{2}-\frac{\chi}{2}} \mathrm{d}x \frac{1}{\chi} \frac{x^2 t^2}{(x+t)^2} \left(|A_4|^2\right)^2, \tag{5.77}$$

where $x = s - m^2$ and $\chi = \sqrt{t(t-4m^2)}$. After integration, the cross section reads

$$\sigma = \frac{4\pi^2 G_N^2}{2E_1 2E_4|\frac{\mathbf{p}_1}{E_1} - \frac{\mathbf{p}_4}{E_4}|} \tag{5.78}$$
$$\times \left[\frac{7}{60}t^2 + \frac{16}{15}m^2 t + \frac{103}{15}m^4 + 8\frac{m^6}{t} - 4\frac{m^4}{\chi t}\left(t^2 + 4m^2 t - 4m^4\right) \ln\left(\frac{t+\chi}{t-\chi}\right)\right],$$

where the quantity

$$2E_1 2E_4|\frac{\mathbf{p}_1}{E_1} - \frac{\mathbf{p}_4}{E_4}| \tag{5.79}$$

represent the incoming flux of particles $\bar{d}_1$ and d_4 in the center of momentum frame. Let us now consider the gravitational amplitude (5.72)

$$\left|\mathcal{M}_4'\right|^2 = \frac{\kappa^4}{16}\left(2p_2 p_3 + 2m'^2\right)^2 \left(|A_4'|^2\right)^2. \tag{5.80}$$

Using Eq. (5.70), the spin summed amplitude reads

$$|A_4'|^2 = \frac{1}{2}\left[\frac{2(u - m^2 - m'^2)^2}{t^2} + \frac{2u}{t} + 1\right]. \tag{5.81}$$

Let us analyze the process

$$\bar{d}_1 d_4 \to d_2' \bar{d}_3', \tag{5.82}$$

i.e., the annihilation of the pair $\bar{d}_1 d_4$ followed by the creation of the pair $d_2' \bar{d}_3'$. In this case, we have the cross section

$$\sigma = \frac{4\pi^2 G_N^2}{2E_1 2E_4 |\frac{\mathbf{p}_1}{E_1} - \frac{\mathbf{p}_4}{E_4}|} \sqrt{\frac{t-4m'^2}{t}} \left[\frac{7}{30}t^2 + \frac{4}{5}(m^2+m'^2)t + \frac{16}{15}(m^4+m'^4) + \frac{64}{15}m^2m'^2 \right. \\ \left. + \frac{32}{15}\frac{m^2m'^2(m^2+m'^2)}{t} + \frac{32}{5}\frac{m^4m'^4}{t^2} \right]. \quad (5.83)$$

The amplitude (5.70) is relevant for the graviton exchange process as well, i.e., for $d_4 d'_3 \to d_1 d'_2$. Using the replacements

$$t = -\frac{P^2(1+z)}{2}, \qquad u = 2m^2 + 2m'^2 + \frac{1}{2}P^2(1+z) - s \quad (5.84)$$

where

$$P^2 = \frac{(s-m^2-m'^2)^2 - 4m^2m'^2}{s}, \quad (5.85)$$

the differential cross section reads

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{2\pi G_N^2}{s} \left[4\frac{(s-m^2-m')^2}{P^4(z+1)^2} - 4\frac{s(s-m^2-m'^2)^2}{P^2(z+1)} \right. \\ \left. + (s-m^2-m'^2)^2 + \frac{1}{16}(4s - P^2(z+1))^2 \right]. \quad (5.86)$$

In the non-relativistic limit we have $s - (m+m')^2 \ll (m+m')^2$, or equivalently $\mathbf{p} \ll M$. Hence, expanding in powers of

$$\epsilon = \frac{s-(m+m')^2}{(m+m')^2} \simeq \frac{|\mathbf{p}|^2}{M^2} = 2\frac{E}{M}, \quad (5.87)$$

we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{8\pi G_N^2 m^2 m'^2 (m+m')^2}{|\mathbf{p}|^4 (z+1)^2} = \frac{2\pi G_N^2 m^2 m'^2}{E^2(z+1)^2}, \quad (5.88)$$

where $|\mathbf{p}|$ is the magnitude of the momentum of the scattered double copies d, and E is the total kinetic energy. It is interesting to compare this result with the Rutherford cross section. In order to compare, let us consider a related gravity amplitude, i.e., the cross section for the exchange of a graviton by two scalar particles. The relevant Feynman rule for this interaction is given by [35, 129]

$$\langle p_2|V^{(1)\mu\nu}|p_1\rangle_{S=0} = \mathrm{i}\frac{\kappa}{2}[-p_2^{\mu}p_1^{\nu} - p_2^{\nu}p_1^{\mu} + \eta^{\mu\nu}(p_2\cdot p_1 + m^2)]. \tag{5.89}$$

For scalar particles of masses m, m', we obtain

$$\mathcal{M}_4^{S=0}(1,2,3,4) = \mathrm{i}\frac{\kappa^2}{16}\left[\frac{(m^2+m'^2)(4s-2t+4u) - 4(m^4+m'^4) - 2s^2 + t^2 - 2u^2}{t}\right], \tag{5.90}$$

which in the nonrelativistic limit leads the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{2\pi {G_N}^2 m^2 m'^2 (m+m')^2}{Z^2\left(s-(m+m')^2\right)^2} = \frac{2\pi {G_N}^2 m^2 m'^2}{4E^2(z+1)^2}. \tag{5.91}$$

The cross section in Eq.(5.88) is four times larger than the usual Rutherford cross section (Eq.(5.91)). The reason is that in the nonrelativistic limit, the internal propagator is almost on-shell and we have to sum over all possible polarizations, i.e, $++$, $+-$, $-+$, $--$, therefore the amplitude is two times larger than an amplitude which only exchanges $++$ and $--$.

In this model, we have double copies of fermions which interact *only* gravitationally. Since these particles can be massive and thus non-relativistic, the model we propose may be relevant for the discussion of dark matter. This is because so far all evidence of dark matter is gravitational.[9] As the double copies only interact gravitationally they are dissipationless, compatible with the dark matter halos around galaxies. Since they only interact gravitationally, the cross sections are tiny and the explanation of dark matter relic abundance would require a non-thermal mechanism.

Finally, a realistic analysis should restrict the analysis to states of equal spin, thus removing the mixed polarization states $+-$, $-+$, which can be achieved by considering these states as ghosts [124]. These degrees of freedom are common, but not exclusive, to string theories, where they correspond to dilatons and axions. Alternatively, we may describe these states in the context of scalar-tensor theories of gravity. However, these theories are strongly constrained by experiments in the solar system[10].

[9] See e.g., Section 25 in Ref. [130].
[10] See Section 21 of Ref. [130].

5.3 Einstein-Yang-Mill relations

In the past Section we have seen that gauge theories (including QCD) and gravitational theories are connected in several equivalent ways: the KLT kernel, the double copy, and the CHY formalism. We also learn in Section 5.1.2 that we can move from one to another realization of the concept "gravity as the square of gauge theory". For example, in the last Section, starting with the color kinematics duality for QCD primitives, we generalized the KLT kernel by invoking a CHY-like recipe to build the KLT kernel. In this Section, we proceed with another example of the relations between gauge theories and gravity theories that can be easily understood from the CHY representation. This Section is based on Ref. [131] by the author, Alexander Kniss, and Stefan Weinzierl.

Consider a theory of gravitons and gluons, i.e., Einstein-Yang-Mills theory. Although we will not perform any computation with it, let us introduce the Lagrangian density of the theory. In the Gauss unit system it reads

$$\mathcal{L}_{\text{EYM}} = \frac{2}{\kappa^2}\sqrt{-\det \mathsf{g}}\, R - \frac{1}{4}\sqrt{-\det \mathsf{g}}\, g^{\mu\nu} g^{\rho\lambda} F^a_{\mu\rho} F^a_{\nu\lambda}, \tag{5.92}$$

where $g^{\mu\nu}$ is the metric tensor, and R is the usual Ricci scalar (See Section 2.3.3). In this theory, the Feynman rules are obtained by straightforward use of the methods in Chapter 2 by expanding the metric tensor

$$g_{\mu\nu} = \eta^{\mu\nu} + \kappa h^{(1)}_{\mu\nu} + \dots \tag{5.93}$$

and collecting the relevant terms in the expanded Lagrangian density.

In the case of the KLT relations, the connection between gravity and gauge amplitudes was first studied in the context of string theory by relating closed and open strings. These relations then were shown to hold in the field theory limit. Similarly, the relations between Einstein-Yang-Mills (closed string plus open string) and gauge theory (only open strings) have been studied in the context of string theory using the so called "disk relations" [132–135]. In Ref. [136], Stieberger and Taylor found a relation between single trace field theory amplitudes involving a *single graviton* and an arbitrary number of gluons and pure gauge amplitudes. Schematically they found that for an amplitude in Einstein-Yang-Mills with a single graviton we have

$$\text{gravity} \oplus \text{gauge} = \alpha(p, \varepsilon) \otimes \text{gauge}, \tag{5.94}$$

where $\alpha(p, \varepsilon)$ is a purely kinematic coefficient. Let us introduce some notation for the amplitudes. We denote by

$$A^{\text{EYM}}_{n,n-1}(p^{\lambda_1}_{\sigma(1)},\ldots,p^{\lambda_{n-1}}_{\sigma(n-)},p^{\lambda_n\lambda_n}_n)\equiv A^{\text{EYM}}_{n,n-1}(\sigma,p,\varepsilon,\tilde{\varepsilon}),\tag{5.95}$$

the primitive EYM amplitude with a single graviton and $(n-1)$ bosons of helicities $\lambda_j\in\{+,-\}$. The graviton is labeled by n as can be inferred from the helicity labels. These primitives are gauge invariant objects with a fixed order specified by σ. Similarly, we denote by

$$A^{\text{YM}}_n(p^{\lambda_1}_{\sigma(1)},\ldots,p^{\lambda_n}_{\sigma n})\equiv A^{\text{YM}}_n(\sigma,p,\varepsilon),\tag{5.96}$$

the primitive YM amplitude with ordering σ. Using this notation, the Stieberger-Taylor relations read

$$A^{\text{EYM}}_{n,n-1}(p^{\lambda_1}_1,\ldots,p^{\lambda_{n-1}}_{n-1},p^{\lambda_n\lambda_n}_n)=-\sum_{j=1}^{n-2}(\sqrt{2}q_j\cdot\varepsilon^{\lambda_n}_n)A^{\text{YM}}_n(p^{\lambda_1}_1,\ldots p^{\lambda_j}_j,p^{\lambda_n}_n,p^{\lambda_{j+1}}_{j+1},\ldots p^{\lambda_{n-1}}_{n-1})\tag{5.97}$$

where

$$q_j=\sum_{k=1}^{j}p_k.\tag{5.98}$$

The minus sign in Eq.(5.97) comes from our conventions of for the field strength tensor $F_{\mu\nu}$ (See Sec. 2.3.2). The Stieberger-Taylor relations were found by using the low energy limits of string theory amplitudes and by considering appropriate soft and collinear limits. In Ref. [136] the question of whether Eq.(5.97) could be derived withing the framework of the CHY formalism was posed. In Ref. [131] we gave an affirmative answer to this question. Notice that these relations resembles the KLT relations with a "kernel" which now also depends on the polarizations. let us explore the ingredients to find Eq.(5.97) within the CHY.

5.3.1 CHY integrand for Einstein-Yang-Mills

In Ref. [137], CHY introduced the integrand for EYM primitives. The integrand is composed by the usual ingredients, i.e., the standard Parke-Taylor factor C and a specialization of the permutation invariant factor E. We will be interested in the standard building blocks but specialized for a subset of particles. For an amplitude with r gravitons and $n-r$ gauge bosons, we define the Parke-Taylor factor by

$$C_r(\sigma, z) = \begin{cases} 1, & r = 0, \\ 0, & r = 1, \\ \frac{1}{z_{\sigma(1)\sigma(2)} z_{\sigma(2)\sigma(3)} \cdots z_{\sigma(n)\sigma(1)}}, & 2 \leq r \leq n, \end{cases} \tag{5.99}$$

where $\sigma \in r$ with $2 \leq r \leq n$. Similarly, for a subset $\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, n\}$ with $1 \leq r \leq n$, we denote the corresponding polarization vectors by $\varepsilon' = (\varepsilon_{i_1}^{\lambda_{i_1}}, \ldots, \varepsilon_{i_r}^{\lambda_{i_r}})$. For this subset, we define a $(2r) \times (2r)$ antisymmetric matrix $\Psi(z, p, \varepsilon')$ through

$$\Psi(z, p, \varepsilon') = \begin{pmatrix} \mathsf{A} & -\mathsf{C}^T \\ \mathsf{C} & \mathsf{B} \end{pmatrix}, \tag{5.100}$$

with

$$A_{ab} = \begin{cases} \frac{s_{i_a i_b}}{z_{i_a i_b}}, & a \neq b \\ 0, & a = b \end{cases}, \qquad B_{ab} = \begin{cases} \frac{\epsilon_{i_a i_b}}{z_{i_a i_b}}, & a \neq b \\ 0, & a = b \end{cases}, \qquad C_{ab} = \begin{cases} \frac{\rho_{i_a i_b}}{z_{i_a i_b}}, & a \neq b \\ -\sum\limits_{j=1,\, j \neq a} \frac{\rho_{i_a i_b}}{z_{i_a j}}, & a = b \end{cases}, \tag{5.101}$$

where we used our conventions, i.e., $\epsilon_{i_a i_b} = \varepsilon_{i_a} \cdot \varepsilon_{i_b}$ and $\rho_{i_a i_b} = \sqrt{2}\varepsilon_{i_a} \cdot p_{i_b}$. Notice that the sum in the diagonal entries of C_{ab} is over $(n-1)$ terms and not just $(r-1)$ terms. This definition of the matrix Ψ is a slight modification of Eqs.(3.64)-(3.65). It has chosen to be valid for the subset of labels under consideration. Now, the factor E_r is defined as follows

$$E_r(z, p, \varepsilon') = \begin{cases} 1, & r = 0, \\ \text{Pf } \Psi(z, p, \varepsilon'), & 1 \leq r \leq (n-2), \\ 0, & r = n-1, \\ \text{Pf}'\Psi(z, p, \varepsilon), & r = n, \end{cases} \tag{5.102}$$

where the reduced Pfaffian $\text{Pf}'\Psi$ is defined in Eq.(3.63). With these ingredients we can then write the CHY representation of three types of amplitudes in Einstein-Yang-Mills theory. Without loss of generality we consider the first r particles to be gauge bosons, while the particles labeled from $(r+1)$ to n are gravitons. We then set the polarization tuples as follows

$$
\begin{aligned}
\varepsilon &= (\varepsilon_1^{\lambda_1}, \ldots, \varepsilon_r^{\lambda_r}, \varepsilon_{r+1}^{\lambda_{r+1}}, \ldots, \varepsilon_n^{\lambda_n}), \\
\tilde{\varepsilon} &= (\varepsilon_{r+1}^{\lambda_{r+1}}, \ldots, \varepsilon_n^{\lambda_n}),
\end{aligned} \tag{5.103}
$$

Hence, the CHY integrand for Einstein-Yang-Mills reads

$$
I^{\text{EYM}}(z, \sigma, \varepsilon, \varepsilon') = C_r(\sigma, z) E_n(z, p, \varepsilon) E_{n-r}(z, p, \varepsilon). \tag{5.104}
$$

This integrand can be used in any of the flavors of the CHY representation that we have studied so far: in Eq.(3.38) as a multivariate contour integral supported on the scattering equations, in Eq.(3.53) as a sum over the solutions of the scattering equations, or as a multivariate contour integral supported on the polynomial form of the scattering equations in Eq.(3.56). By construction the cases $r = n$ and $r = 0$ recover pure Yang-Mills amplitudes and pure graviton amplitudes, respectively.

5.3.2 One graviton

The CHY representation for EYM corresponds to the case $r = n - 1$ in Eq.(5.104), hence with the definitions of Eq.(3.56)-(3.58) , we have

$$
A_{n,n-1}^{\text{EYM}}(\sigma, \varepsilon, \tilde{\varepsilon}) = \frac{\mathrm{i}(-1)^n}{(2\pi \mathrm{i})^{n-3}} \oint_{\mathcal{O}} \mathrm{d}\tilde{\Omega}_{\text{CHY}} \prod_{i=1}^{n} (z_i - z_{i+1})^2 C_n(\sigma, z) E_n(z, p, \varepsilon) E_1(z, p, \tilde{\varepsilon}), \tag{5.105}
$$

and for pure Yang-Mills

$$
A_n^{\text{YM}}(\sigma, \varepsilon, \tilde{\varepsilon}) = \frac{\mathrm{i}(-1)^n}{(2\pi \mathrm{i})^{n-3}} \oint_{\mathcal{O}} \mathrm{d}\tilde{\Omega}_{\text{CHY}} \prod_{i=1}^{n} (z_i - z_{i+1})^2 C_n(\sigma, z) E_n(z, p, \varepsilon). \tag{5.106}
$$

Now, the proof of Eq.(5.97) is based on the “Eikonal” identify for Parke-Taylor factors, i.e.,

$$
\sum_{l=k}^{n-2} \frac{z_l - z_{l+1}}{(z_l - z_n)(z_n - z_{l+1})} = \frac{z_k - z_{n-1}}{(z_k - z_n)(z_n - z_{n-1})}, \tag{5.107}
$$

which can be proved by repeated use of the identity

$$\frac{z_k - z_{l+1}}{(z_k - z_n)(z_n - z_{l+1})} + \frac{z_{l+1} - z_{l+2}}{(z_{l+1} - z_n)(z_n - z_{l+2})} = \frac{z_k - z_{l+2}}{(z_k - z_n)(z_n - z_{l+2})}. \tag{5.108}$$

Consider the integrand of the LHS of Eq.(5.97). In terms of the blocks of the CHY representation, we have

$$\text{LHS} = C_{n-1}(1, \ldots, n-1) E_n E_1, \tag{5.109}$$

where we have used the shorthand notation $C_r(1, 2, \ldots, r) = C_r(1, 2, \ldots, r, z)$ and we have excluded common factors inside the integrand. The polarization factor E_1 is the Pfaffian of a 2×2 matrix which simplifies to

$$E_1 = -\sum_{k=1}^{n-1} \frac{\rho_{kn}}{z_k - z_n} \tag{5.110}$$

$$= \frac{\rho_{nn}}{z_{n-1} - z_n} + \sum_{k=1}^{n-2} \frac{\rho_{kn}}{z_{n-1} - z_n} - \sum_{k=1}^{n-2} \frac{\rho_{kn}}{z_k - z_n} \tag{5.111}$$

$$= -\sum_{k=1}^{n-2} \rho_{kn} \frac{z_k - z_{n-1}}{(z_k - z_n)(z_n - z_{n-1})}. \tag{5.112}$$

In the second line we have separated the $n-1$ element in the sum and used momentum conservation. In the third line we used the fact that $p_n \cdot \varepsilon_n = 0$. Then, we have

$$\text{LHS} = -C_{n-1}(1, \ldots, n-1) E_n \sum_{k=1}^{n-2} \rho_{kn} \frac{z_k - z_{n-1}}{(z_k - z_n)(z_n - z_{n-1})}. \tag{5.113}$$

Let us now focus on the RHS of Eq.(5.97), at the level of the integrand we have

$$\text{RHS} = -\sum_{l=1}^{n-2} \sqrt{2} q_l \cdot \varepsilon_n E_n C_n(1, \ldots, l, n, l+1, \ldots, n-1) \tag{5.114}$$

$$= -E_n \sum_{l=1}^{n-2} \sum_{k=1}^{l} \sqrt{2} p_k \cdot \varepsilon_n C_n(1, \ldots, l, n, l+1, \ldots, n-1). \tag{5.115}$$

Exchanging the order of the summations and using

$$C_n(1, \ldots, l, n, l+1, \ldots, n-1) = \frac{z_l - z_{l+1}}{(z_l - z_n)(z_n - z_{l+1})} C_{n-1}(1, \ldots, n-1), \tag{5.116}$$

we obtain

$$\text{RHS} = C_{n-1}(1,\ldots,n-1)E_n \sum_{k=1}^{n-2} \sqrt{2} p_k \cdot \varepsilon_n \sum_{l=k}^{n-2} \frac{z_l - z_{l+1}}{(z_l - z_n)(z_n - z_{l+1})}. \tag{5.117}$$

Eq.(5.113) and Eq.(5.117) are equal if for all $k \le (n-2)$ and $n \ge 3$, we have

$$\sum_{l=k}^{n-2} \frac{z_l - z_{l+1}}{(z_l - z_n)(z_n - z_{l+1})} = \frac{z_k - z_{n-1}}{(z_k - z_n)(z_n - z_{n-1})}. \tag{5.118}$$

This is precisely the Eikonal identity of Eq.(5.107) and completes the proof of Eq.(5.97).

5.3.3 More than one graviton

We would like to discuss generalizations of Eq.(5.107) towards tree-level single trace amplitudes with r gauge bosons and $(n-r)$ gravitons[11]. In Chapter 3, we learned that for words w in the BCJ basis B we have

$$A_w^{\text{YM}} = \mathrm{i} M_{wj} E_j, \qquad w \in B, \tag{5.119}$$

where M_j is defined in Eq.(4.13). Equivalently, after solving the contour integral the EYM amplitude becomes

$$A^{\text{EYM}} = \mathrm{i} G_j E_j = J(z^{(j)}, p) C_r(\sigma, z^{(j)}) E_{n-r}(z^{(j)}, p, \varepsilon) E_n(z^{(j)}, p, \varepsilon), \tag{5.120}$$

where

$$G_j \equiv J(z^{(j)}, p) C_r(\sigma, z^{(j)}) E_{n-r}(z^{(j)}, p, \varepsilon). \tag{5.121}$$

Thus, we may express Einstein-Yang-Mills amplitudes with $(n-r)$ gravitons as a linear combination of Yang-Mills amplitudes with cyclic order $w \in B$ if we find coefficients $\alpha_w(\sigma, p, \tilde{\varepsilon})$ such that

$$G_j = \alpha_w M_{wj}, \quad w \in B, \tag{5.122}$$

[11] In Ref. [138] an explicit generalization up to three gravitons was given.

where $\alpha_w \equiv \alpha_w(\sigma, p, \tilde{\varepsilon})$. In other words, we need to find the inverse of M_{wj} in order to find the coefficients α_w. In Chapter 3, we computed this inverse (see Eqs.(4.14)-(4.21)), which is given by

$$N_{jv} = \sum_{w \in B} S[w|\bar{v}] C_n(\bar{v}, z^{(j)}). \tag{5.123}$$

Since $M_{wj} N_{jv} = \delta_{wv}$, the coefficients α_w can be easily found to be

$$\alpha_w = G_j N_{jw}. \tag{5.124}$$

Therefore

$$A_{n,r}^{\mathrm{EYM}}(\sigma, \varepsilon, \tilde{\varepsilon}) = \sum_{w \in B} \alpha_w(\sigma, p, \tilde{\varepsilon}) A_n^{\mathrm{YM}}(w, p, \varepsilon). \tag{5.125}$$

In general, the coefficients depend on the basic building blocks of the CHY representation and the Jacobian, hence we can write the coefficients as a contour integral following the general rules of Chapter 3, i.e.,

$$\alpha_w(\sigma, p, \tilde{\varepsilon}) = \frac{\mathrm{i}}{(2\pi\mathrm{i})^{n-3}} \sum_{v \in B} S[w|\bar{v}] \oint_{\mathcal{O}} \mathrm{d}\Omega_{\mathrm{CHY}} I_\alpha, \tag{5.126}$$

where

$$I_\alpha \equiv C_r(\sigma, z) C_n(\bar{v}, z) E_{n-r}(z, p, \varepsilon), \tag{5.127}$$

or the integrand for the contour formula based on the polynomial scattering equations

$$\tilde{I}_\alpha = \prod_{i=1}^{n} (z_i - z_{i+1})^2 I_\alpha. \tag{5.128}$$

The techniques to solve these contour integrals have been studied in Chapter 3. In particular, we know that we do not require to solve the scattering equations if we use the Bezoutian matrix method for the computations of the residues (See Appendix B.3). The construction of a basis is discussed in Chapter 2 and a summary is given in Appendix C.

Very recently, several methods for the computation of the coefficients α_w have been proposed. A recursive approach based on gauge invariance was introduced in Ref. [139], while from the color-kinematics duality and the double copy, a semi-recursive approach was developed in Ref. [140]. In the CHY formalism, another recursive approach based on the expansions of the Pfaffian was proposed in Ref. [141].

5.3.4 Example

Let us give an example for the computations of the coefficients in Eq.(5.125). Let us consider the case $n = 4$ for the amplitude of $r = 2$ gauge bosons and 2 gravitons. The basis B of gauge amplitudes contains a single element, namely $B = \{1234\}$. For the permutation of two gauge bosons we take $\sigma = (1, 2)$. We will follow the algebraic approach based on the Bezoutian matrix (See Examples in Section 3.4 and Appendix B.3). The measure reads

$$\mathrm{d}\tilde{\Omega}_{\mathrm{CHY}} = \frac{\mathrm{d}z_3}{(1-z_3)z_3(s_{12}+s_{13}z_3)}, \tag{5.129}$$

and after using our conventions for z_1, z_2, and z_4, we have

$$\begin{aligned}\tilde{I}_\alpha\Big|_{\substack{z_1\to\infty\\ z_4\to 0\\ z_2\to 1}} =& - z_3^2\rho_{32}\rho_{42} + z_3\rho_{32}\rho_{42} + \rho_{32}\rho_{43} - z_3\rho_{32}\rho_{43} - \rho_{34}\rho_{42} - z_3^2\rho_{34}\rho_{42} + 2z_3\rho_{34}\rho_{42} \\ & - s_{34}z_3\epsilon_{34} - \frac{s_{34}\epsilon_{34}}{z_3} + 2s_{34}\epsilon_{34}. \end{aligned} \tag{5.130}$$

Therefore, on the support of the scattering equations we have

$$\begin{aligned}\alpha_w(\sigma,p,\tilde{\varepsilon}) =& s_{12}\mathrm{Res}\left(\rho_{32}\rho_{42} + \frac{\rho_{32}\rho_{43}}{z_3} + \rho_{34}\rho_{42} - \frac{\rho_{34}\rho_{42}}{z_3} + \frac{s_{34}\epsilon_{34}}{z_3} - \frac{s_{34}\epsilon_{34}}{z_3^2}\right) \\ =& s_{12}\Big[s_{34}\epsilon_{34}\left(\langle -\frac{s_{13}}{s_{12}},1\rangle - \langle\frac{s_{13}^2}{s_{12}^2},1\rangle\right) + (\rho_{32}\rho_{43} - \rho_{34}\rho_{42})\langle -\frac{s_{13}}{s_{12}},1\rangle \\ & + \rho_{32}\rho_{42} + \rho_{34}\rho_{42}\Big] \\ =& s_{12}\left[s_{34}\epsilon_{34}\left(-\frac{1}{s_{12}} - \frac{s_{13}}{s_{12}^2}\right) - (\rho_{32}\rho_{43} - \rho_{34}\rho_{42})\frac{1}{s_{12}} + (\rho_{32}\rho_{42} + \rho_{34}\rho_{42})\frac{1}{s_{13}}\right],\end{aligned}$$

where we have used that the inverse of $1/z_3$ is given by $(-s_{13}/s_{12})$. Using momentum conservation and the fact that $\rho_{ii} = 0$, we obtain

$$A_{4,2}^{\mathrm{EYM}} = t\left[\epsilon_{34} + \frac{\rho_{32}\rho_{41}}{t} + \frac{\rho_{31}\rho_{42}}{u}\right] A_4^{\mathrm{YM}}(1234), \tag{5.131}$$

where the polarizations in ρ_{ij} are the graviton polarizations, i.e., $\rho_{ij} = \sqrt{2}\tilde{\varepsilon}_i \cdot p_j$ and similarly for ϵ_{ij}.

5.4 Remarks

In this Chapter, we have studied three equivalent approaches to relate perturbative quantum gravity and Yang-Mills based on the concept of gravity as the square of gauge theory: the KLT kernel approach, the CHY representations, and the double copy. We have seen that this concept can be used also to relate gravity and QCD and to relate Einstein-Yang-Mills to pure Yang-Mills. Except in the case of QCD, the relations between gravity and gauge have been first found in the context of string theory methods and then shown in the context of the S-matrix to be valid in D-dimensions. An important tool to establish results in D dimensions is the CHY representation based on the scattering equations.

A problem that can be posed from the results in this Chapter goes as follows. For the double copy construction of QCD based on the color kinematics duality, we have established the corresponding generalized KLT kernel. In analogy with pure Yang-Mills, we can write Eq.(5.59) as

$$M_n = \sum_{\tilde{w}\in B} n_{\tilde{w}} A^{\text{QCD}}_{\tilde{w}}, \tag{5.132}$$

where $n_{\tilde{w}} \equiv \sum_{w\in B} A^{\text{QCD}}_{w} S_{w\tilde{w}}$. Since the numerators in the pure gravity amplitude (Eq.(5.9)) can be thought as the dual of the color factors of the decomposition by Del Duca et al., it would be interesting to explore how the numerators in Eq.(5.132) can be associated with the color factors in the decomposition by Johansson and Ochirov in Ref. [103]. This would give insight about the CHY representation of the gravitational theory proposed in Section 5.2.

This problem is an example of the more general question of how to make the color kinematics duality manifest in the CHY representation. We know from Ref. [14] that we can always define numerators associated with trivalent graphs and that they satisfy Jacobi-like identities. However, an open problem is how to make this duality manifest on the factor E of the CHY representation. For example, for gluons we know that the permutation invariant factor can be expanded in terms of numerators as we did in (5.12). An algorithm for the construction of these numerators explicitly for pure Yang-Mills, was introduced in Refs. [142, 143]. A similar problem can be posed for single trace Einstein-Yang-Mills amplitudes, which have a CHY representation and satisfy color kinematics duality.

Chapter 6

Summary and outlook

In this thesis, we have explored several aspects of the modern developments of scattering amplitudes aimed towards its application to QCD. To this end, in Chapter 2 we have first explored the foundations of the theory of scattering amplitudes in the framework of quantum field theory. We emphasized the approach by Weinberg that leads to the diagrammatic Feynman rules of quantum field theory. In this approach, the Feynman diagrams to calculate S-matrix elements arise as a logical consequence of the basic principles of quantum mechanics, i.e., special relativity, and the cluster decomposition principle. In particular, the analytic properties of the S-matrix arise as a physical restriction on the connected part of the S-matrix, namely the cluster decomposition principle, which leads to the fact that we have poles and branch cuts as singularities. These are the basic analytic ingredients of any S-matrix approach, including the ones we have studied in this thesis.

We then presented some recent techniques of computation of amplitudes. We briefly reviewed the BCFW method for tree-level amplitudes, which uses the basic analytic properties, the complexification of momenta, and the factorization properties of the amplitude to recursively compute higher point amplitudes involving lower point amplitudes. In combination with the little group property it is a powerful method for computation. The BCFW method illustrates another main point in the philosophy of current methods in scattering amplitudes, i.e., the use of only physical on-shell information in order to obtain the full amplitude. The color decomposition of amplitudes is a key element for the treatment of amplitudes in gauge theories. In particular, we have seen that the color decomposition provides a useful concept in the treatment of amplitudes—we can use a basis of amplitudes characterized by orderings (words) and that we can treat amplitudes as operators in the space of orderings. This was a key element for subsequent developments in the thesis. Another relevant concept is the color-kinematics duality and the double copy which relate gauge theories and gravity. We treated perturbative gravity in this sense, i.e., as connected to a gauge theory through the concept of gravity as the square of a gauge theory. We presented two equivalent methods to square a gauge theory: the KLT kernel based method and the double copy by BCJ. Although tree level amplitudes can be computed using recursive techniques—hence the problem is ultimately solved—tree level amplitudes are an important ingredient for loop-level amplitudes through the method of generalized unitarity, which we briefly reviewed. The techniques outlined in Chapter 2 are at the heart of current approaches towards amplitudes for gauge theories and in particular QCD.

In Chapter 3, we reviewed the recently proposed CHY representation based on the scattering equations for amplitudes in several theories. We emphasized that we can think of the CHY representation in basically two ways: as a sum over the solutions and as a contour integral. The later is very useful for explicit computations due to the complexity of finding the solutions of the scattering equations, as we have shown in Sec.3.1. It allows us to skip the step of solving the scattering equations and obtain the amplitudes using e.g., methods in computational algebraic geometry. Nevertheless, amplitudes as a sum over solutions and the concept of basis of amplitudes can be combined to study existence of CHY representations. We have presented the general features of the CHY representations and emphasized that on the support on the scattering equations they allow us to give several properties of the amplitudes without having a closed integrand. Therefore, an important issue in the CHY representation is to find such an integrand. Once the integrand is known, we could use the techniques outlined in Sec.3.2, e.g., using multi-residues, or numerically solving the scattering equations. In particular, we used the method based on the Bezoutian matrix to compute the integral. Using this method, the amplitude can be though as an inner product of polynomials with support on the algebraic variety formed by the scattering equations as was proposed by Søgaard and Zhang. Thus, the CHY formalism can be thought as a well-defined problem in algebraic geometry where the physical input comes from the integrand. It would be desirable to find a method to obtain this integrand within the framework of algebraic geometry on the grounds of the general features of the S-matrix. A possible route to follow is to make explicit the color kinematics duality on the building blocks of the CHY formalism, hence finding a relation to trivalent graphs and ultimately to the Feynman diagrammatic approach[1]. Furthermore, if these numerators are local (i.e., not rational functions) then the pole structure of the amplitude would be encoded in a collection of polynomials instead of rational functions—The reason is that we can invert the denominators with respect to the ideal generated by the scattering equations as we studied in Section 3.4.

In Chapter 4, one of the main results of this work was introduced. We have proved that QCD primitive amplitudes admit a CHY representations by giving the explicit blocks of the integrand of the CHY representation. This new result adds a theory to the list of theories compatible with the CHY formalism. This representation describes QCD amplitudes in D-dimensions with the full particle content of QCD at tree-level, thus describing massive or massless quarks. Hence, it shows that the CHY representations

[1] A different route to reach the Feynman diagrammatic expansion was followed in Ref. [99] using graph expansions.

could also be used to describe realistic tree level amplitudes with fermions, which are relevant for current experiments. However, this result is only a partial success since we lack a closed formula for the building blocks of the CHY representation—as we discussed in Chapter 3, it would be desirable to obtain a closed expression for the generalized permutation invariant factor which does not depend on any amplitude. The main result of this Chapter is the formulation of the generalized cyclic factors and the generalized permutation invariant factor given in Sections 4.2.3 and 4.2.4, respectively. With the notation of Chapter 4, the main result is given by the CHY integrand

$$I^{\mathrm{QCD}}(z,p,\varepsilon) = -\mathrm{i}\; C(w,z) \sum_{u,v\in B_{n_q\leq 2}} \sum_{w\in B} S_{u\bar{v}} G_{uw} C(\bar{v},z) A_n(w,p,\varepsilon), \qquad w \in B. \tag{6.1}$$

An important element for the proof of the CHY representation was the concept of a basis B of amplitudes and to consider amplitudes as operators in the space of orderings. In order to construct the basis of amplitudes the fact the QCD primitives satisfy BCJ relations was fundamental. That QCD satisfy these relations was conjectured in Ref. [103] and we proved to be valid for an arbitrary number of particles in Ref. [109].

The problem of the closed CHY formula for the QCD integrand is related to the understanding of the properties that a CHY integrand should satisfy—besides Möbius invariance—and to the space of theories which admit a CHY representation. One of the problems is that, unlike the known closed formulas for CHY representations, QCD mixes spin one and spin one-half and we lack a description of only fermions in the CHY representation. In this direction some progress has been made from the ambitwistor formulation of the CHY formalism [69,70,88]. Finding such formula for QCD is certainly a big challenge and it would reveal a new face of realistic quantum field theories. Combined with the fact that integrands of the known CHY representations can be written as polynomials, it would reveal a new structure for realistic gauge theories, i.e., that the amplitude is completely determined by the scattering variety and the ideal formed by the scattering polynomials.

In Chapter 5, we introduced the second main result of this work. We introduced amplitudes for a new gravitational theory which is built from QCD primitive amplitudes using two different methods: a generalized version of the KLT matrix and via the color kinematics duality and double copy procedure by BCJ. This theory contains double copies of fermions which may be massive or massless. With the notation of Chapter 5, the main result of this section is the introduction of the generalized KLT matrix such that the gravitational amplitude reads

$$M_n(p,\epsilon,\widetilde{\epsilon}) = -\mathrm{i} \sum_{w,\tilde{w}\in B} A_n(p,w,\epsilon) S_{w\tilde{w}} A_n(p,\tilde{w},\epsilon), \tag{6.2}$$

where the procedure to compute the generalized KLT kernel is given in Section 5.2.3. An interesting aspect of this theory is that it contains massive particles which interact only gravitationally, i.e., there an interaction vertex of the double copy and a graviton. Thus, we can study the non-relativistic limit of the graviton exchange between two double copies of fermions. Given that the evidence of dark matter so far is gravitational, we argued that these particles could be relevant to the discussion of dark matter. In fact, considering that gravity and gauge amplitudes are closely related—via the concept gravity as the square of a gauge theory—one can argue that conceptually perturbative gravity arises from gauge theory. Of course, classically this is not obvious at all since one must explain how the equivalence principle arises from gauge theory. However, at the level of amplitudes the concept of gravity as the square of gauge theory can be used to explain how the symmetries of (super)gravity arise from the symmetries of (super)Yang-Mills [144]. Furthermore, the recent developments on the classical double copy [125–128] represent a new perspective of the relationship between gravity and gauge theories. The gravitational theory proposed in this Chapter could then serve for further explorations of how gravitational symmetries arise from gauge theories. It would also be interesting to explore the classical double copy version of this theory.

Finally, in Chapter 5 we explored another aspect of the relationship between gauge theories and gravity. The Stieberger-Taylor relations between Eintein-Yang-Mills theory with one graviton and pure Yang-Mills were shown in the context of the CHY representation. This showed a new role of the CHY representation, i.e., as a tool to find relations among amplitudes. We proved that these relations can be extended for an arbitrary number of gravitons. With the notation of Section 5.3.3, we have proved

$$A^{\text{EYM}}_{n,r}(\sigma,\varepsilon,\tilde{\varepsilon}) = \sum_{w\in B} \alpha_w(\sigma,p,\tilde{\varepsilon}) A^{\text{YM}}_n(w,p,\varepsilon). \tag{6.3}$$

The coefficients in this expansion can be obtained using the methods of Chapter 3 for contour integrals and thus we do not require to solve the scattering equations to find them. Furthermore, those coefficients are given as inner products on the support of the ideal generated by the scattering polynomials, thus these coefficients are computed from polynomials instead of rational functions.

The obvious continuation of this thesis is to generalize these methods for computation at loop-level. The straightforward path follows from generalized unitarity as mentioned at the end of Chapter 3. Basically by using the CHY formula at tree level as an input in the computation of the coefficients for the expansion of amplitudes in terms of master integrals. A more ambitious goal may be to generalize the CHY formalism at loop level. There are several proposals to generalize the CHY construction at loop level. First, using the ambitwistor string there is an obvious generalization by promoting the Riemann sphere to a torus as in any string theory formulation of scattering amplitudes. However, in

Refs. [72, 73], it was shown that it is possible to write loop integrands for massless particles by pinching the torus and getting a Riemann surface with two extra punctures. A similar approach was followed for the case of two loop integrands [145]. Another alternative is to consider higher dimensional tree level amplitudes with $n+2$ particles and obtain from it a loop amplitude with n particles by integrating out two of the particles and additionally dimensional reduction. This was the approach used in Ref. [146] and shown to be equivalent to the construction based on ambitwistor strings. A third approach emerges by considering the so called elliptic scattering equations, i.e., a generalization of the scattering equations in a genus one surface. This leads to a prescription for loop amplitudes analogous to the CHY formula [147]. Therefore, at loop level there are equivalent approaches that can be used to obtain loop amplitudes, but still there is not a systematic method to compute loop integrands. As in the case of tree-level, the guiding concept for the construction of loop amplitudes in the CHY representation is the color kinematics duality which is known to hold at one loop and two loops and it was conjectured to be valid at all loops [12, 148]. For example, it would be desirable to relate loop numerators for gauge theories and the known integrands for gauge theories in, say, the ambitwistor construction of loop amplitudes.

In loop calculations based on generalized unitarity, once the set of master integrals is known, the problem is (in principle) solved. The set of master integrals are known to be described in the language of algebraic geometry. In particular, integrals as periods of algebraic curves. It is well-known that a class of Feynman integrals can be computed as iterated integrals in the moduli space of Riemann spheres with n marked points (See e.g., [149]). Hence, the same language is applicable to describe the main ingredients of loop amplitudes and tree level amplitudes, i.e., algebraic geometry. In the string method tree-level and loop level are characterized precisely by the genus of algebraic curves, hence providing a direct connection between algebraic geometry and the S-matrix. However, there are well-known conceptual difficulties that arise in the string framework as a physical theory. Thus, it would be interesting to find an alternative version of the S-matrix that connects loop level and tree level using the language of algebraic geometry. For example, amplitudes for SYM $\mathcal{N}=4$ were found to be volumes of a mathematical object called the amplituhedron, which is a generalization of a Grassmannian manifold [150]. This is an example of what could be an alternative to the S-matrix that neither requires a Feynman diagrammatic expansion nor the postulation of strings.

In this work, we presented various representations of the S-matrix that could lead to such alternative version of the S-matrix. In particular, the color-kinematics duality and the CHY representation are two formalisms that encode unitarity and locality in different ways, in comparison with the Feynman approach. The alternative version of the S-matrix should also encode these properties, but it may be that they are not manifest or that they arise from unexpected restrictions in the amplitudes (See e.g., [151]). In any case, it is the opinion of the author that the CHY representations and the color-kinematics duality are two faces of an algebraic geometric formulation of the S-matrix.

Appendix **A**

Feynman rules and spinors

A.1 Summary of Feynman rules for Yang-Mills

Here we summarize the color-ordered Feynman rules for gauge amplitudes. In 4D, for outgoing particles we have the factors:

- External outgoing fermion q_i, helicity $\pm$

$$[i|, \langle i| \,. \tag{A.1}$$

- External outgoing anti-fermion $\overline{q}_i$, helicity $\pm$

$$|i], |i\rangle \,. \tag{A.2}$$

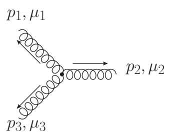
- Polarization vectors with reference momenta q

$$\epsilon^{+}_{\mu}(p,q) = \frac{[p|\gamma_\mu\, |q\rangle}{\sqrt{2}\,\langle qp\rangle}, \tag{A.3}$$

$$\epsilon^{-}_{\mu}(p,q) = -\frac{\langle p|\,\gamma_\mu|q]}{\sqrt{2}[qp]}. \tag{A.4}$$

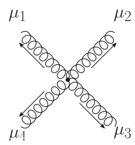
In the Gervais-Neveu gauge [25] the color-ordered Feynman rules for the three-gluon and four-gluon vertices are given by [23, 36, 65]

$$V_{\mu_1,\mu_2,\mu_3}(p_1,p_2,p_3) =$$



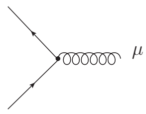
$$= -\,\mathrm{i}\sqrt{2}[\eta_{\mu_1\mu_2}(p_1)_{\mu_3} + \eta_{\mu_2\mu_3}(p_2)_{\mu_1} + \eta_{\mu_3\mu_1}(p_3)_{\mu_2}], \tag{A.5}$$

$$V_{\mu_1,\mu_2,\mu_3,\mu_4} =$$



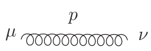
$$= \mathrm{i}(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}). \tag{A.6}$$

Additionally, we have the fermion-gluon-antifermion vertex

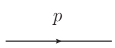


$$= \mathrm{i}\frac{\gamma_\mu}{\sqrt{2}}, \tag{A.7}$$

with a minus sign for the anti-fermion-gluon-fermion vertex. The massless propagators are given by



$$= -\,\mathrm{i}\frac{\eta_{\mu\nu}}{p^2}, \tag{A.8}$$



$$= \mathrm{i}\frac{\not{p}}{p^2}. \tag{A.9}$$

A.2 Useful identities

$$\langle a|\,\gamma_\mu|b]\,\langle c|\,\gamma^\mu||d] = 2\,\langle ac\rangle\,[db], \tag{A.10}$$

$$|p\rangle\,[p| + |p]\,\langle p| \equiv \not{p}, \tag{A.11}$$

$$\langle a|\,\gamma_\mu|b]p^\mu = \langle ap\rangle\,[pb]. \tag{A.12}$$

Using these conventions and identities, the $n = 4$ MHV-amplitude reads

$$A[1^-,2^-,3^+,4^+] = \mathrm{i}\frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 41\rangle}. \tag{A.13}$$

Appendix **B**

Mathematical tools from algebraic geometry

B.1 Moduli spaces

Formally, a moduli space is an equivalence class of a given mathematical structure. Conceptually, a moduli space can be understood as a "space of possibilities". Suppose we want to describe the space of all possible lines passing through the origin in a plane. We can characterize the space by introducing a circle as shown in Fig. B.1, where we see that this space is topologically homeomorphic to a circle since every line meets the circle in exactly two points except the horizontal line through the x-axis which is associated with the origin. Thus, the moduli space of all lines through the origin is the projective line $\mathbb{P}^1 \simeq S^1$ [152]. In this example, the mathematical structure we are considering is the affine line $\mathbb{A}$ and resulting moduli space is the circle S^1. We want to study the algebraic properties of these spaces, which means that we would like to think on both spaces as *algebraic curves* and use the tools of Algebraic Geometry.

In physics, moduli spaces appear in different contexts being the application to string theory amplitudes a well-known example. They are also connected to the computation of Feynman path integrals where we integrate over all possible states to obtain correlation functions. For example, in gauge theories we integrate over the space of inequivalent gauge potentials A—the moduli space of gauge potentials $\mathrm{Mod}(A)$—via the path integrals

$$\int_{[A]\in \mathrm{Mod}(A)} e^{\mathrm{i}S([A])}\mathcal{D}([A]), \tag{B.1}$$

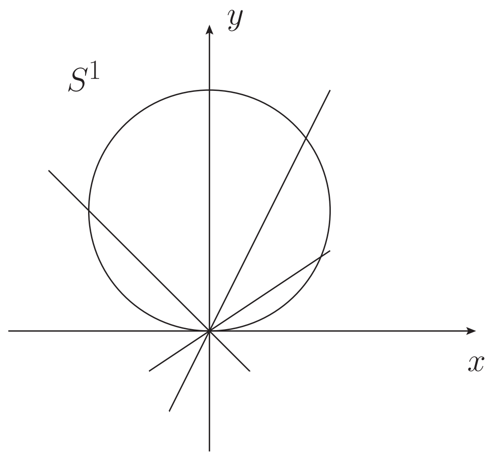


Figure B.1: The moduli space of lines passing through the origin

where are $[A]$ are all physical states, i.e., an equivalence class of connections modulo gauge transformations [153]. Another example of the use of moduli spaces is in the computation of periods in [149].

B.1.1 Important example

The moduli space of Riemann spheres—genus 0 curves—with $n \geq 3$ marked points is defined through

$$\mathcal{M}_{0,n} = \{(z_1, z_2, \ldots, z_n) \in \mathbb{CP}^n | z_i \quad \text{distinct}\}/\mathrm{PSL}(2, \mathbb{C}), \tag{B.2}$$

where $\mathbb{CP}^1 \simeq \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere.

B.2 Algebraic concepts

Polynomials are algebraic objects that we can add and multiply but not necessarily divide[1]. The algebraic structure with these features is called a *ring* usually denoted by R. Polynomials have coefficients in a given *number field*[2] k. The ring of polynomials in n variables with coefficients in this field is denoted by $k[x_1, \ldots, x_n]$. Given any collection of polynomials $f_1, \ldots, f_n$, we can *generate* an ideal I by considering all polynomials that can be built up from them by multiplication by an arbitrary polynomial and by taking sums. In the following paragraphs we give some useful definitions.

[1] The material that follows is standard. We follow the note [154] and Ref. [155].
[2] Most of the time this field is the field of complex numbers $\mathbb{C}$.

Ideal. Let $f_1, \dots, f_s \in k[x_1, \dots, x_n]$. Let $\langle f_1, \dots f_n \rangle$ denote the ideal I generated by f_1, $\dots$, f_n, i.e.,

$$I = \langle f_1, \dots, f_n \rangle = \{p_1 f_1 + \cdots + p_s f_s | p_i \in k[x_1, \dots, x_n] \text{ for } i = 1, \dots, s\}$$

The generating set of polynomials is also called a basis of the ideal. This basis is finite due to the Hilbert Basis Theorem.

Algebraic variety. Computationally the notion of the ideal emerges as the necessity of solving a system of polynomials equations defined by the generators of the ideal I, i.e, the problem of finding the zeros of the set of equations $f_i = 0$. The set of zeros of the system of polynomial equations defines a variety $\mathcal{V}$ which formally reads

$$\mathcal{V}(I) = \{(z_1, \dots, z_n) \in \mathbb{C}^n | f_i(z_1, \dots, z_n) = 0, \quad \text{for all } f_i \in I\}. \tag{B.3}$$

Gröbner basis. The variety does not change if we make a *change of basis* of the ideal I in $k[x_1, \dots, x_n]$. In particular, we can compute a Gröbner basis G with elements g_i, $i = 1, \dots, r$. Before defining the Gröbner bases, let us introduce some useful definitions.

We say that a Gröbner basis generates the same ideal I, i.e.,

$$I = \langle f_1, \dots, f_n \rangle = \langle g_1, \dots, g_r \rangle . \tag{B.4}$$

Then, the varieties satisfy $\mathcal{V}(f) = \mathcal{V}(g)$, in other words they define the same variety. The Gröbner basis depends on the definition of a monomial order. A monomial in the variables $x_1, \dots, x_n$ is the product $x^a \equiv x_1^{a_1} \cdots x_n^{a_n}$ for a_i nonnegative integers. A monomial order $\prec$ is a *total* (linear) order, *compatible with multiplication* ($x^a \prec x^b \Rightarrow x^{a+c} \prec x^{b+c}$), and it is a *well ordering*, meaning that for a nonempty collection of monomials there is a smallest element with respect to $\prec$. In particular, the constant polynomial is the smallest, i.e., $1 \prec x^a$ for all nonnegative integers a. We set the ordering in the variables such that $x_1 \succ x_2 \succ \cdots \succ x_n$. There are three usual monomial orderings:

Lexicographic. $x^a \prec_{\text{lex}} x^b$ if the first elements a_i, b_i in a, b from the *left*, which are different, satisfy $a_i < b_i$. In the case of two variables x_1, x_2, we have

$$1 \prec_{\text{lex}} x_2 \prec_{\text{lex}} x_2^2 \prec_{\text{lex}} x_2^3 \prec_{\text{lex}} \cdots \prec_{\text{lex}} x_1 \prec_{\text{lex}} x_2 x_1 \prec_{\text{lex}} x_2^2 x_1 \prec_{\text{lex}} \dots \tag{B.5}$$

Degree lexicographic. $x^a \prec_{\text{grlex}} x^b$ if we have

$$\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} b_i, \quad \text{or if} \quad \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i, \quad \text{and } x^a \prec_{\text{lex}} x^b.$$

Equivalently, we compare the total degree and break ties by the lexicographic order. In the case of two variables x_1, x_2

$$1 \prec_{\text{grlex}} x_2 \prec_{\text{grlex}} x_1 \prec_{\text{grlex}} x_2^2 \prec_{\text{grlex}} x_1 x_2 \prec_{\text{grlex}} x_1^2 \prec_{\text{lex}} x_2^3 \prec_{\text{grlex}} x_1 x_2^2 \prec_{\text{grlex}} \dots \tag{B.6}$$

Degree reverse lexicographic. $x^a \prec_{\text{grevlex}} x^b$ if we have $\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} b_i$, or if $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and the first elements a_i, b_i in a, b from the *right*, which are different, satisfy $a_i > b_i$. In the case of two variables this order is the same as the "grlex" order. However, in the case of three variables, for example we have

$$x_1^2 x_2 x_3 \prec_{\text{grevlex}} x_1 x_2^3. \tag{B.7}$$

Let us now define the Gröbner basis. Given a monomial order we have a leading term $\text{lt}(f)$, a leading power $\text{lp}(f)$, and a leading coefficient $\text{lc}(f)$ in any polynomial $f \in k[x_1, \dots, x_n]$. A set of nonzero polynomials $G = \{g_1, \dots, g_r\}$ contained in the ideal I, is called Gröbner basis for the ideal I, if and only if for all $f \in I$ such that $f \neq 0$, there exists $i \in 1, \dots, r$ such that $\text{lp}(g_i)$ divides $\text{lp}(f)$ [156].

Example B.2.1. For instance, consider the ideal

$$I = \langle x^2 + 2xy - x - 1, x^2 - 8x + y^2 \rangle$$

with Gröbner basis

$$G = \left\{ 2xy + 7x - y^2 - 1, x^2 - 8x + y^2, 53x + 10y^3 - 11y^2 + 2y - 39 \right\} \tag{B.8}$$

in "grlex" order. The leading polynomials of I and G are given by

$$\text{lp}(x^2 + 2xy - x - 1) = \text{lp}(x^2 - 8x + y^2) = x^2, \tag{B.9}$$

which can be divided by $\mathrm{lp}(x^2 - 8x + y^2) = x^2$. The reduced Gröbner basis can be computed from any generating set of I by a method introduces by Buchberger in 1965 [157]. In practice we can use a computer algebra system.

B.3 Multivariate residues and the CHY representation

The CHY formula can be written as a contour integral in many variables as was shown by Dolan and Goddard [89]. For this reason we would like to introduce some definitions of multivariate residues. This Section has as prerequisite Section B.2.

B.3.1 Generalities

In one variable, the residue of a holomorphic function h with an isolated singularity $z_0 \in \mathbb{C}$ is defined by [90]

$$\mathrm{res}_{z_0}(h) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|z-z_0|=\delta} h(z)\mathrm{d}z \equiv \frac{1}{2\pi \mathrm{i}} \oint\limits_{\mathcal{O}} h(z)\mathrm{d}z, \tag{B.10}$$

where we integrate over a small counter-clockwise oriented circle $\{|z - z_0| = \delta\}$ around z_0 for any sufficiently small δ. We will specialize in rational functions $h(z) = g(z)/f(z)$, where g and f are polynomials in z. We can use Dirac delta functions as a prescription to write the residue of a meromorphic 1-form[3] $\omega = g(z)/f(z)\mathrm{d}z$ by defining

$$\mathrm{res}_f(\omega) \equiv \int \mathrm{d}z\, g(z)\delta(f(z)). \tag{B.11}$$

This prescription is based on the property of the Dirac delta function

$$\delta(f(x)) = \frac{\delta(x - x_i)}{|f'(x_i)|}, \tag{B.12}$$

which is valid for an arbitrary function with a simple zero at $x = x_i$. Thus,

$$\int \mathrm{d}z\, g(z)\delta(f(z)) = \int \mathrm{d}z\, g(z)\frac{\delta(z - z_i)}{|f'(z_i)|} = \frac{g(z_i)}{|f'(z_i)|}. \tag{B.13}$$

For multiple zeros of f we have the global residue, i.e., we sum over of local residues. This sum is identified with the symbol in uppercase "Res", thus

[3] The definition as a 1-form makes it invariants under local change of coordinates.

$$\mathrm{Res}_f(g) = \sum_{z_i \in Z_f} \mathrm{res}_{z_i} \frac{g(z)}{f(z)}, \tag{B.14}$$

where Z_f is the set of poles $Z_f = \{z_i \in \mathbb{C} : f(z_i) = 0\}$. Let us define the property of duality in one variable, which can be generalized to the multivariate case.

Duality. *If we write $h = f/q$, with $Z_f \cap Z_q = \emptyset$, then by the Nullstellensatz, there exists polynomials r, s, such that $1 = rf + sq$. It follows that the sum of local residues*

$$\sum_{z_i \in Z_f} \mathrm{res}_{z_i} \frac{g(z)}{f(z)} = \mathrm{Res}_f(gs), \tag{B.15}$$

coincides with the global residue of the polynomial gs.

In many variables, we consider a meromorphic n-form ω depending on the variables

$$\mathbf{z} = (z_1, \ldots, z_n), \qquad \mathbf{z} \in \mathbb{C}^n, \tag{B.16}$$

given by

$$\omega = \frac{g(\mathbf{z})}{f_1(\mathbf{z}) \cdots f_n(\mathbf{z})} d\mathbf{z}, \qquad d\mathbf{z} = dz_1 \wedge \cdots \wedge dz_n, \tag{B.17}$$

where f_i, g are holomorphic functions such that $f_1, \ldots, f_n$ share a single zero at $\mathbf{z} = \mathbf{z}_0$. The local *Grothendieck* residue is defined through the contour integral [158, 159]

$$\mathrm{res}_{\{\mathbf{f},\mathbf{z}_0\}}(g(\mathbf{z})) = \mathrm{res}_{\{\mathbf{z}_0\}}(\omega) \equiv \frac{1}{(2\pi \mathrm{i})^n} \int_{\Gamma_{\mathbf{f}}(\delta)} \omega, \tag{B.18}$$

where $\Gamma_{\mathbf{f}}(\delta)$ is the real n cycle $\Gamma_{\mathbf{f}}(\delta) = \{\mathbf{z} \in \mathbb{C}^n | \quad |f_i(\mathbf{z})| = \delta_i\}$, oriented by the n-form such that $d(\arg(f_1)) \wedge \cdots \wedge d(\arg(f_n)) \geq 0$, where δ_i are sufficiently small positive real numbers. In analogy with the single variable case, we define the *global residue* as

$$\mathrm{Res}(\omega) \equiv \mathrm{Res}_{\{\mathbf{f}\}}(g(\mathbf{z})) = \sum_{\mathbf{z}_i \in Z(\mathbf{f})} \mathrm{res}_{\{\mathbf{z}_i\}}(\omega), \tag{B.19}$$

where $Z(\mathbf{f}) \subset \mathbb{C}^n$ denotes the nonempty zero set of the polynomials $f_1, \ldots, f_n$—in other words the associated algebraic variety. We also have the duality property and the transformation property which we reproduce.

Transformation law. *Let $s_1, \ldots, s_n \in \mathbb{C}[\mathbf{z}]$ have finitely common roots such that*

$$s_i = \sum_{j=1}^{n} A_{ij} f_j, \qquad A_{ij} \in \mathbb{C}[\mathbf{z}], \qquad i = 1, \ldots, n. \tag{B.20}$$

Then, for $g \in \mathbb{C}[\mathbf{z}]$,

$$\text{Res}\left(\frac{g\; d\mathbf{z}}{f_1 \cdots f_n}\right) = \text{Res}\left(\frac{g\; \det(A_{ij})\; d\mathbf{z}}{s_1 \cdots s_n}\right). \tag{B.21}$$

Duality. *A polynomial $g \in \mathbb{C}[\mathbf{z}]$ lies in the ideal $I = \langle s_1, \ldots, s_n \rangle$ if and only if*

$$\text{Res}_{\mathbf{f}}(sg) = 0, \qquad \forall s_i \in \mathbb{C}[\mathbf{z}]. \tag{B.22}$$

B.3.2 Global residues from the Bezoutian matrix

We will summarize the algorithm by Søgaard and Zhang to compute residues based on the scattering equations [77]. The mathematical statements and proofs can be found in [90, 159]. For additional examples and exercises see the lectures by Zhang [160] and by Weinzierl [38]. Recently, a MATHEMATICA package which automatize some these methods has been released [161].

The orderings of the Gröbner basis will be taken as "grlex" unless stated otherwise. Consider the following problem in multivariate residue calculus. For the rational polynomial $N = P/Q$, $P, Q \in R$ such that $\{f_1, \ldots, f_n, Q\}$ have no common zeros, i.e., we want to calculate

$$\text{Res}_{\{\mathbf{f}\}}(N) = \text{Res}_{\{\mathbf{f}\}}\left(\frac{P}{Q}\right). \tag{B.23}$$

The first step in the algorithm is to invert the polynomial Q with respect to $\mathbf{f}$—for nondegenerate residues—such that

$$\text{Res}_{\{\mathbf{f}\}}(N) = \text{Res}_{\{\mathbf{f}\}}\left(P\tilde{Q}\right), \tag{B.24}$$

where the inverse is denoted by $\tilde{Q}$. The inverse is in the ring R.

Polynomial inverse. Compute the Gröbner bases of the ideal $I = \langle f_1, \ldots, f_n, Q \rangle$ in some monomial order and record the converting matrix[4] such that

[4] This algorithm depends on the extraction of this matrix which cannot be obtained by conventional computer algebra software like MATHEMATICA, see Ref. [77] for an alternative algorithm

$$1 = a_1 f_1 + \cdots + a_1 f_1 + Q\tilde{Q}, \tag{B.25}$$

where $a_1, \ldots a_n, Q \in R$. The inverse is given by $\tilde{Q}$.

Computing the residue. The residue is obtained as the inner product

$$\mathrm{Res}_{\mathbf{f}}(N) \equiv \langle N, 1 \rangle \,, \tag{B.26}$$

in R/I. In order to compute we have to find a basis $\{e_i\}$ and a dual basis $\{\Delta_i\}$ of R/I such that

$$\begin{aligned} [N] &= \sum_i \lambda_i e_i, \\ 1 &= \sum_i \mu_i \Delta_i, \end{aligned} \tag{B.27}$$

where $\lambda_i, \mu_i \in \mathbb{C}$. The notation $[N]$ means that we have to care only about a representative of the class of numerators $[N]$ of N in R/I. The residue in then computed as

$$\langle N, 1 \rangle = \sum_i \lambda_i \mu_i. \tag{B.28}$$

In particular if one of the coefficients in the dual basis is a constant Δ_r, then [5]

$$\mathrm{Res}_{\mathbf{f}}(N) = \langle \frac{N}{\Delta_r}, \frac{\Delta_r}{\Delta_r} \rangle = \frac{\lambda_r}{\Delta_r}, \tag{B.29}$$

if not, proceed by performing the polynomial division with respect to a Gröbner basis in some monomial order such that $N(z) = q(z) + r(z)$, the remainder $r(z) \in R$ is the representative of $[N]$.

1. Find the basis. This can be achieved by computing the Gröbner basis G of I in grlex and drop the leading terms of each element of G with respect to "grlex".

[5]This can be understood as the fact that the scattering polynomials build an H-basis [78].

2. Find the dual basis. Let us first compute the associated $\tilde{G}$ by setting $x_i \to y_i$. For example, if the variables are $\{x, y\}$ the associated basis $\tilde{G}$ could be, say in variables $\{u, v\}$. Second, we compute the $n \times n$ Bezoutian matrix

$$B_{ij} \equiv \frac{f_i(y_1, \dots, y_{j-1}, z_j, \dots, z_n) - f_i(y_1, \dots, y_j, z_{j+1}, \dots, z_n)}{z_j - y_j} \tag{B.30}$$

,

where f_i are the members of the ideal I.

Compute the $G \otimes G$ and perform the polynomial reduction of $\det B$ with respect to $G \otimes G$ and obtain the remainder.

$$r = \sum_i a_i(y) e_i(z). \tag{B.31}$$

The dual basis is obtained by the replacement $y_i \to z_i$ in $a_i(y)$, thus giving $\Delta_i = a_i(z)$.

3. Compute the residue. The residue can be calculated using Eqs.(B.27)-(B.28) or simply using (B.29) if there is a constant term in the dual basis.

Example B.3.1. Let $I = \langle x^2 + 2xy - x - 1, x^2 - 8x + y^2 \rangle$ and let $N = 1/(1 - x)$. Calculating Gröbner basis in degree lexicographic monomial order, results into

$$\begin{aligned} 1 =& \frac{-4}{27}\left(x^2 - 8x + y^2\right) + \frac{(2y+1)}{27}\left(x^2 + 2xy - x - 1\right) \\ &+ \frac{(1-x)}{27}\left(2xy - 3x + 4y^2 + 2y + 28\right), \end{aligned} \tag{B.32}$$

therefore the polynomial inverse of $(1 - x)$ can be obtained from Eq.(B.25), i.e.,

$$\tilde{Q} = (2xy - 3x + 4y^2 + 2y + 28)/27. \tag{B.33}$$

For this ideal, the Gröbner basis is given by

$$G = \left\{2xy + 7x - y^2 - 1, x^2 - 8x + y^2, 53x + 10y^3 - 11y^2 + 2y - 39\right\}. \tag{B.34}$$

Then the basis can be read off from this equation

$$e_i = \left\{y^2, x, y, 1\right\}. \tag{B.35}$$

We proceed with the computation of the dual basis. We need the Bezoutian matrix

$$\mathsf{B} = \begin{pmatrix} u+x+2y-1 & 2u \\ u+x-8 & v+y \end{pmatrix}, \tag{B.36}$$

and

$$\begin{aligned} G \otimes \tilde{G} =& \{2xy+7x-y^2-1, x^2-8x+y^2, 53x+10y^3-11y^2+2y-39, \\ & 2uv+7u-v^2-1, u^2-8u+v^2, 53u+10v^3-11v^2+2v-39\}, \end{aligned} \tag{B.37}$$

which produces the remainder

$$\begin{aligned} r =& -2ux+uy-\frac{7u}{2}+\frac{5v^2}{2}+vx+2vy-v-\frac{7x}{2}+\frac{5y^2}{2}-y+1 \\ =& \frac{5}{2}(y^2)+\left(\frac{1}{2}(-4u+2v-7)\right)x+\left(\frac{1}{2}(2u+4v-2)\right)y+\frac{1}{2}\left(-7u+5v^2-2v+2\right). \end{aligned} \tag{B.38}$$

Hence the dual basis reads

$$\{\Delta_i\} = \left\{\frac{5}{2}, \frac{1}{2}(-4x+2y-7), \frac{1}{2}(2x+4y-2), \frac{1}{2}\left(-7x+5y^2-2y+2\right)\right\}. \tag{B.39}$$

Putting all together, we have

$$\begin{aligned} \mathrm{Res}_{\{\mathbf{f}\}}\left(\frac{1}{(1-x)}\right) =& \langle \frac{2xy}{27}-\frac{x}{9}+\frac{4y^2}{27}+\frac{2y}{27}+\frac{28}{27}, 1\rangle \\ =& \langle -\frac{10x}{27}+\frac{5y^2}{27}+\frac{2y}{27}+\frac{29}{27}, 1\rangle \\ =& \langle \frac{5}{27}e_1-\frac{10}{27}e_2+\frac{2}{27}e_3+\frac{29}{27}, \frac{2}{5}\Delta_1\rangle \\ =& \frac{2}{27}. \end{aligned} \tag{B.40}$$

B.3.3 Six-point example for the scalar bi-adjoint

In this section, we give a $n = 6$ example using the Bezoutian matrix method. Consider the double ordered amplitude $m(123456|153246)$. The amplitude from the CHY representation reads

$$m(123456|153246) = \frac{\mathrm{i}}{(2\pi\mathrm{i})^{n-3}} \oint_{\mathcal{O}} \mathrm{d}\tilde{\Omega}_{\mathrm{CHY}} \prod_{i=1}^{6} (z_i - z_{i+1})^2 C(123456, z) C(153246, z) \Bigg|_{\substack{z_1 \to \infty \\ z_2 = 1 \\ z_4 = 0}} \tag{B.41}$$

with

$$\mathrm{d}\tilde{\Omega}_{\mathrm{CHY}} = \frac{\mathrm{d}z_3 \mathrm{d}z_4 \mathrm{d}z_5 \; z_3 \,(1 - z_4)\, z_4 \,(1 - z_5)\,(z_3 - z_5)}{h_1 h_2 h_3 \,(1 - z_3)\, z_5 \,(z_3 - z_4)\,(z_4 - z_5)}. \tag{B.42}$$

Taking the limit for z_1, we obtain

$$\begin{aligned} m(123456|153246) = \frac{\mathrm{i}}{(2\pi\mathrm{i})^{n-3}} \oint_{\mathcal{O}} & \frac{\mathrm{d}z_3 \mathrm{d}z_4 \mathrm{d}z_5 \; z_3 \,(1 - z_4)\, z_4 \,(1 - z_5)\,(z_3 - z_5)}{h_1 h_2 h_3 \,(1 - z_3)\, z_5 \,(z_3 - z_4)\,(z_4 - z_5)} \\ & \times \left(- \frac{z_5 \,(z_3 - z_4)\,(z_4 - z_5)}{(z_4 - 1)\, z_4 \,(z_3 - z_5)} \right). \end{aligned} \tag{B.43}$$

After some simplifications the amplitude—in terms of the inner product (B.26)—becomes

$$m(123456|153246) = \mathrm{i} \left\langle \frac{z_3 \,(z_5 - 1)}{(1 - z_3)}, 1 \right\rangle . \tag{B.44}$$

First, we will calculate the dual basis and determine if it contains a constant term, thus simplifying the computation due to Eq. (B.29). The basis can be easily calculated giving

$$\{e_i\} = \left\{ z_5^3, z_4 z_5, z_5^2, z_4, z_5, 1 \right\} . \tag{B.45}$$

For the dual basis we find the term

$$\Delta_1 = - \frac{{s_{15}}^2 s_{135} s_{145} s_{1345}}{s_{13} s_{145} - s_{14} s_{135}}. \tag{B.46}$$

The next step is to determine the inverse of $Q = (1 - z_3)$ with respect to $I = \langle h_1, h_2, h_3 \rangle$. Using the Gröbner basis method we obtain

$$\tilde{Q} = A + B z_4 + C z_5 + D z_3 z_5 + E z_4 z_5 + F z_5^2 + G z_3 z_5^2 + H z_5^3, \tag{B.47}$$

which unfortunately generates huge coefficients of the Mandelstam variables. The next step is to compute the remainder of $(z_3\,(z_5 - 1))\,\tilde{Q}$ with respect to $\mathbb{C}[z_3, z_4, z_5] / \langle h_1, h_2, h_3 \rangle$ leading to

$$[(z_3\,(z_5-1))\,\tilde{Q}] = \lambda_1 z_5^3 + \ldots, \tag{B.48}$$

where again the coefficients of this decomposition are huge rational functions of the Mandelstam variables. Since we have a constant term in the dual basis, we can use Eq.(B.29) and our result is simply the coefficient of z_5^3 in the last equation. Therefore

$$m(123456|153246) = -\mathrm{i}\frac{(s_{13}s_{145} - s_{14}s_{135})\,\lambda_1}{{s_{15}}^2 s_{135}s_{145}s_{1345}}, \tag{B.49}$$

which we can check numerically. It reproduces the desired result, i.e.,

$$m(123456|153246) = \mathrm{i}\frac{1}{s_{23}s_{234}s_{2345}}. \tag{B.50}$$

In this example we see that the bottleneck of this method is the computation of the inverse with respect to R/I. This is tied to the computation of the Gröbner basis, which is computationally time-consuming. For these reasons, a more convenient method would be the use of the H-basis which is a less refined version of the Gröbner basis. In combination with the transformation law, it can be used to simplify calculations.

Appendix C

Relations and bases for gauge amplitudes

In this Appendix, we review the construction of bases for QCD primitive amplitudes at tree-level. First we review notation. We consider the special case of pure Yang-Mills and proceed with QCD.

C.1 Primitive QCD amplitudes at tree level

Let us consider a primitive amplitude with n external particles containing n_g gluons and n_q quark-antiquark pairs [162], i.e., we have

$$n = n_g + 2n_q. \tag{C.1}$$

Without loss of generality we assume that the quarks have different flavors. The quarks may be massless or massive. We label the quarks by $q_1, q_2, \ldots, q_{n_q}$, the corresponding anti-quarks by $\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_{n_q}$, and the gluons by $g_1, g_2, \ldots, g_{n_g}$. Then, the alphabet reads

$$\mathbb{A} = \{q_1, q_2, \ldots, q_{n_q}, \bar{q}_1, \bar{q}_2, \ldots, \bar{q}_{n_q}, g_1, g_2, \ldots, g_n\}. \tag{C.2}$$

Sequences of letters are called *words* and they form an algebra. We denote these words by $w = l_1 l_2 \ldots l_n$ and by $w^T = l_n l_{n-1} \ldots l_1$ the reversed words. The set of words form an algebra over a number field k, where the unit element corresponds to the empty word e. The product operation corresponds to the shuffle product defined by

$$l_1 l_2 \dots l_k \sqcup\!\sqcup\, l_{k+1} \dots l_r = \sum_{\text{shuffles } \sigma} l_{\sigma(1)} l_{\sigma(2)} \dots l_{\sigma(r)}, \tag{C.3}$$

where the sum runs over all permutations σ that preserve the relative order of $l_1, l_2, \dots, l_k$ and $l_{k+1} \dots l_r$. The product is commutative and associative. Let w_1, w_2 be two words, then we have:

$$e \sqcup\!\sqcup\, w_1 = w_1, \tag{C.4}$$
$$w_1 \sqcup\!\sqcup\, w_2 = w_2 \sqcup\!\sqcup\, w_1, \tag{C.5}$$
$$(w_1 \sqcup\!\sqcup\, w_2) \sqcup\!\sqcup\, w_3 = (w_1 \sqcup\!\sqcup\, w_2) \sqcup\!\sqcup\, w_3. \tag{C.6}$$

The most general set of words with n letters, is the set such that each letter occurs exactly once. This set is given by

$$W_0 = \{l_1, l_2, \dots, l_n | l_i \in \mathbb{A},\ l_i \neq l_j, \quad \text{for} \quad i \neq j\}, \tag{C.7}$$

which contains $n!$ elements. For $w \in W_0$, we write

$$A_n(w) \qquad \text{or} \qquad A_n(l_1 l_2 \dots l_n) \tag{C.8}$$

to encode the information about the external ordering. We will take A_n as a linear operator in the space of words, i.e., for $\lambda_1, \lambda_2 \in k$ and $w_1, w_2 \in W_0$. we have

$$A_n(\lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 A_n(w_1) + \lambda_2 A_n(w_2), \tag{C.9}$$

which is a convenient way to think when we consider relations among primitive amplitudes.

C.1.1 Special case: pure Yang-Mills

As we showed in Chapter 2.4.4, the alphabet for n point amplitude in pure Yang-Mills is given by

$$\mathbb{A}_{\text{gluons}} = \{g_1, g_2, \dots, g_n\} = \{1, \dots, n\}, \tag{C.10}$$

formed by letters l_i that we associate to a given gluon. In Section 2.4.4.1, we have seen that primitive amplitudes in pure Yang-Mills are cyclic invariant and satisfy KK and BCJ relations. Therefore, starting with the set W_0—all words where each letter occurs once—the associated bases after imposing the relations are given by

$$W_1 = \{l_1 l_2 \dots l_n \in W_0 | l_1 = 1\}, \quad \text{(C.11)}$$
$$W_2 = \{l_1 l_2 \dots l_n \in W_0 | l_1 = 1, l_n = n\}, \quad \text{(C.12)}$$
$$B = \{l_1 l_2 \dots l_n \in W_0 | l_1 = 1, l_{n-1} = n-1, l_n = n\}, \quad \text{(C.13)}$$

where the number of elements in the bases is $(n-1)!$, $(n-2)!$, and $(n-3)!$, respectively. We refer to W_2 as the KK basis and to B as the BCJ basis.

C.2 QCD

In general, the alphabet consists of quark and gluon labels as in Eq.(C.2). The basis has to take into account the especial case where all particle are gluons. Therefore, Eqs.(C.11)–(C.13) will be part of the definition of the general basis. On the other hand, we have the case where all particles are quarks and therefore we have to set a new basis, which also takes into account this situation. Therefore, in general the basis has specific legs fixed. The particle content (the letter) in each leg corresponds to either a quark or a gluon depending on the total number of quarks.

C.2.1 Relations for QCD amplitudes

Primitive QCD amplitudes are cyclic invariant, i.e.,

$$A_n(l_1 l_2 \dots l_n) = A_n(l_2 \dots l_n l_1), \quad \text{(C.14)}$$

and satisfy the KK relations introduced in Eq.(2.150). This can be understood from the fact that primitive QCD amplitudes can be expanded in terms of diagrams with only antisymmetric cubic vertices [11]. For the alphabet (C.2), let

$$w_1 = l_{\alpha_1} l_{\alpha_2} \dots l_{\alpha_j}, \qquad w_1 = l_{\beta_1} l_{\beta_2} \dots l_{\beta_{n-2-j}}, \quad \text{(C.15)}$$

be two sub-words such that

$$\{l_1\} \cup \{w_1\} \cup \{w_2\} \cup \{l_n\} = \{l_1 l_2 \dots l_n\}. \quad \text{(C.16)}$$

Then the KK relations for QCD amplitudes mimic the pure gluon case

$$A_n(l_1 w_1 l_n w_2) = (-1)^{n-2-j} A_n(l_1(w_1 \sqcup\!\sqcup w_2^T) l_n). \tag{C.17}$$

Primitive QCD amplitudes with ($n_q > 1$) vanish when two or more fermions lines cross, in other words non-planar configurations of the amplitudes vanish. The reason is that crossed fermion lines can only be drawn in planar way with flavor-changing currents, which are not allowed in QCD and therefore the amplitudes vanish. Thus, we have

$$A_n(\ldots q_i \ldots q_j \ldots \bar{q}_i \ldots \bar{q}_j \ldots) = A_n(\ldots q_i \ldots \bar{q}_j \ldots \bar{q}_i \ldots q_j \ldots) = 0. \tag{C.18}$$

For amplitudes with at least one gluon there are further relations. Assuming that particle 2 is a gluon, i.e.,

$$l_2 = g_\alpha, \qquad \alpha \in \{1, \ldots, n_g\}. \tag{C.19}$$

Then, the fundamental Bern-Carrasco-Johansson relations for QCD mimic the pure gluon case, i.e.,

$$\sum_{i=2}^{n-1} \left(\sum_{j=i+1}^{n} 2p_2 \cdot p_j \right) A_n(l_1 l_3 \ldots l_i \, l_2 \, l_{i+1} \ldots l_{n-1} \, l_n) = 0. \tag{C.20}$$

These relations were conjectured for QCD in Ref. [103] and then we proved them for the general case in Ref. [109]. A proof based on a symmetry of the amplitudes was found for pure Yang-Mills and then for QCD in Ref. [163] and Ref. [164], respectively.

C.2.2 Orientation of fermion lines

Non-crossed fermion lines can have an arbitrary orientation. For example, consider the 6 point amplitude with 3 fermion lines and the alphabet

$$\mathbb{A}_6 = \{q_1, q_2, q_3, \bar{q}_3, \bar{q}_2, \bar{q}_1\} = \{1, 2, 3, 4, 5, 6\}. \tag{C.21}$$

Consider the fermion line configuration in Fig.C.1, where we have drawn the fermion lines in a disk. The quark line 3-4 can have two different orientations .

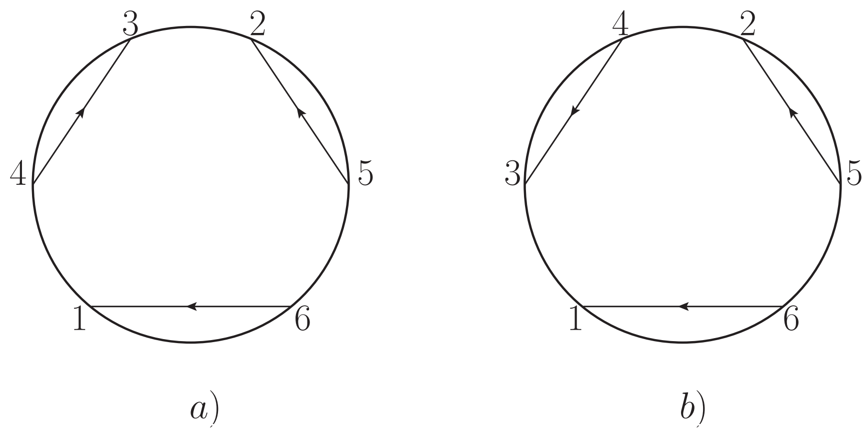


Figure C.1: Two orientation of the fermion line 3-4.

For an arbitrary configuration of quark lines let us define the standard orientation as the orientation where for each quark flavor, each quark appears before its corresponding anti-quark when reading clockwise starting at the quark 1. Let us consider for simplicity the alphabet

$$\mathbb{A} = \{1, 2, \ldots, 2n_q\}, \tag{C.22}$$

which describes amplitudes with only quarks. Cyclic invariance and the KK relations allow us to fix two letters, which we choose to be the first and last, i.e., $l_1 = q_1$, $l_n = \bar{q}_1$. In the example of Fig.C.1, case b) has already the standard orientation.

In the general case, we have the following algorithm [110, 111]:

Assigning levels.

(i) The quark line q_1, $\bar{q}_1$ is already in the standard orientation. Assign to this fermion line the level 0.

(ii) Assign the level 1 to quark lines not separated by another fermion line from the fermion line of level 0.

(iii) Iterate this procedure and assign the level k to all fermion lines, which are not separated by another fermion line of level $(k-1)$, and which have not been assigned any level before.

Orienting.

(i) Bring all fermion lines of level 1 into the standard orientation

(ii) Iterate for the level 2, 3, etc.

(iii) At level k: Consider the amplitude

$$A_n(x_{k-1}q_i x_k \bar{q}_j w_{k+1} q_j y_k \bar{q}_i y_{k-1}), \tag{C.23}$$

where x_{k-1}, x_k, w_{k+1}, y_k, and y_{k-1} are sub-words. We assume that the fermion line q_i-$\bar{q}_i$ is of level $(k-1)$ and that all fermion lines contained in the sub-words x_{k-1} and y_{k-1} have already been oriented. The sub-words x_k and y_k may contain further fermion lines of level k and higher level. The sub-word w_{k+1} may contain fermion lines of level $(k+1)$ and higher (See Fig.C.2).

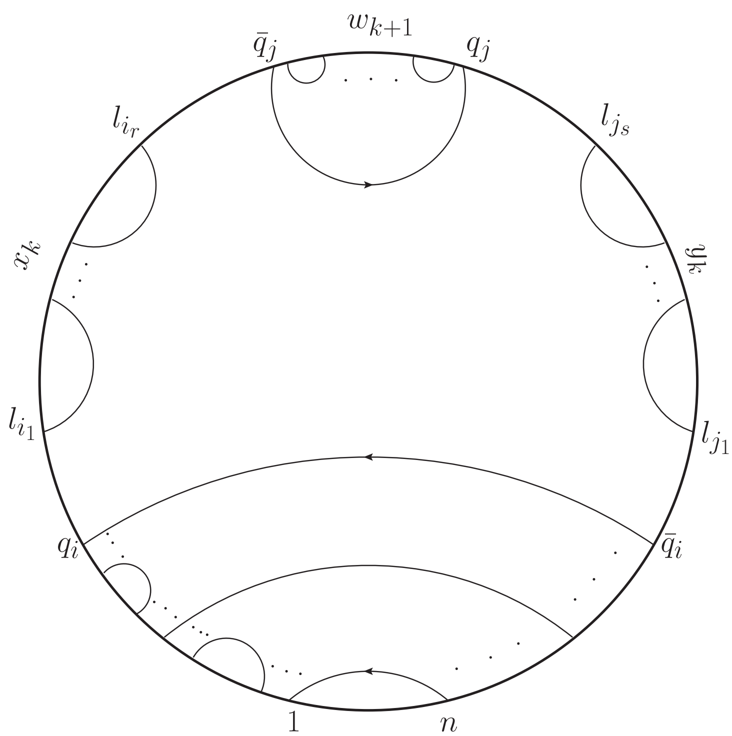


Figure C.2: Quark line graph for the orientation of the fermion line $\bar{q}_j$-q_j. Quark lines of level $(k-1)$ bounded by the quark line q_i-$\bar{q}_i$ are already oriented. Sub-words o level k may contain fermion lines of level k or higher.

(iv) Orient the fermion line q_i-$\bar{q}_i$, respecting the orientations of all fermion lines with level $\leq k$. Let us write

$$x_k = l_{i_1} l_{i_2} \dots l_{i_r}, \qquad y_k = l_{j_1} l_{j_2} \dots l_{j_s}. \tag{C.24}$$

Then

$$A_n(x_{k-1}q_ix_k\bar{q}_jw_{k+1}q_jy_k\bar{q}_iy_{k-1}) =$$
$$(-1)^{|w_{k+1}|+1}\sum_{a=0}^{r}\sum_{b=0}^{s}A_n(x_{k-1}q_il_{i_1}\dots l_{i_a}q_jw'_{k+1}\bar{q}_jl_{j_{b+1}}\dots l_{j_s}\bar{q}_iy_{k-1}), \tag{C.25}$$

with

$$w'_{k+1} = (l_{i_a}\dots l_{i_r}) \sqcup\!\sqcup\, w^T_{k+1} \sqcup\!\sqcup\, (l_{j_1}\dots l_{j_b}) \tag{C.26}$$

Comments:

- All fermions lines of w'_{k+1} are of level $(k+1)$ or higher.
- Some amplitudes in Eq.(C.25) may be zero due to the crossed fermion lines. This is either the case if a quark-anti-quark pair from x_k is split between $l_{i_1}\dots l_{i_a}$ and w'_{k+1}, or if a quark-anti-quark pair from y_k is split between w'_{k+1} and $l_{j_{b+1}}\dots l_{j_s}$.
- The inclusion of gluons does not modify the general structure of the orientation, and is already considered in Eq.(C.25).

Eq.(C.25) is referred to the "fermion orientation relations".

Example C.2.1. Consider the alphabet

$$\mathbb{A}_7 = \{q_1, q_2, q_3, g_1, \bar{q}_3, \bar{q}_2, \bar{q}_1\} = \{1, 2, 3, 4, 5, 6, 7\}. \tag{C.27}$$

Suppose we have the non-standard oriented amplitude $A_7(1543267)$, which can be drawn in a disk as shown in Fig.(C.3). The quark line $1-7$ is already oriented and has level 0. At level 1, the quark line $3-5$ has the wrong orientation and the line $2-6$ is already oriented. Therefore, we need only one iteration. For this amplitude we have $x_0 = y_0 = e$ and

$$x_1 = e, \quad w_2 = 4, \quad y_1 = 26, \tag{C.28}$$

which means that $r = 0$, $s = 2$. Therefore, we have

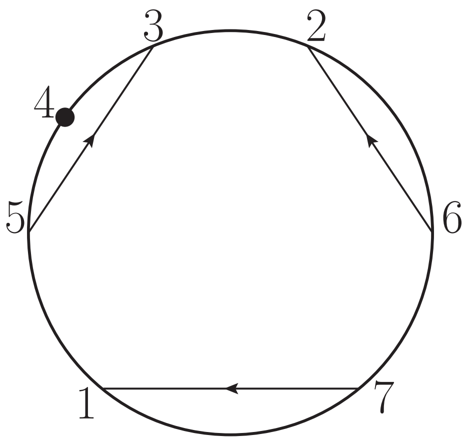


Figure C.3: Disk diagram of an amplitude with non-standard orientation in the line q_3-$\bar{q}_3$.

$$\begin{aligned} A_7(1543267) =& \sum_{b=0}^{2} A_7(13w'_{2,b}5l_{j_b+1}\ldots l_{j_s}7) \\ =& A_7(13w'_{2,1}5267) + A_7(13w'_{2,2}567) \end{aligned} \tag{C.29}$$

where b labels the two shuffles. In the first term $w'_{2,1} = e \sqcup\!\sqcup 4 \sqcup\!\sqcup e = 4$ and in the second $w'_{2,2} = e \sqcup\!\sqcup 4 \sqcup\!\sqcup 2 = 24 + 42$. Therefore,

$$\begin{aligned} A_7(1543267) =& A_7(1345267) + A_7(1342567) + A_7(1324567) \\ =& A_7(1345267) + A_7(1324567). \end{aligned} \tag{C.30}$$

In Fig.C.4, we have drawn the orientation diagram of each term in Eq.(C.30).

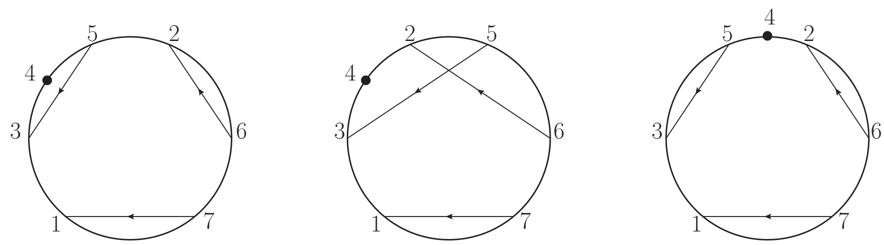


Figure C.4: Resulting amplitudes with the correct orientation. The second diagram vanishes due to the no-crossed fermion lines relations.

C.2.3 The QCD basis

We are ready to construct the basis of amplitudes based on the relations among primitive QCD amplitudes. These relations are

1. Cyclic invariance (Eq.(C.14)).
2. KK relations. (Eq.(C.17)).
3. No-crossed fermion lines. (Eq.(C.18)).
4. Orientation relations. (Eq.(C.25)).
5. BCJ relations. (Eq.(C.20)).

The first two relations are easily handled by fixing two letters in the basis and they must include the pure Yang-Mills case. Therefore, the KK basis reads

$$W_2 = \begin{cases} \{l_1 l_2 \ldots l_n \in W_0 | l_1 = g_1, l_n = g_n\}, & n_q = 0, \\ \{l_1 l_2 \ldots l_n \in W_0 | l_1 = q_1, l_n = \bar{q}_1\}, & n_q \geq 1. \end{cases} \tag{C.31}$$

Next we have to construct a set of words such that the crossed-fermion lines are not considered. This is done by associating quark lines of flavor i, with opening $(_i$ and closing brackets $)_i$. There are two possible orientations for each fermion line, either

$$q_i \to (_i, \qquad \bar{q}_i \to)_i, \tag{C.32}$$

or

$$\bar{q}_i \to (_i, \qquad q_i \to)_i. \tag{C.33}$$

We define the standard orientation of fermions lines by requiring that every quark corresponds to an opening braket and every anti-quark corresponds to a closing bracket, i.e., the standard orientation is given by Eq.(C.32). In agreement with the discussion of orientation relations in the last section.

Consider the alphabet $\mathbb{A}$ in Eq.(C.22). A *generalized Dyck word* is any word in this alphabet with properly matched brackets. Defining the projector P, such that

$$P(q_i) = (_i, \qquad P(g_i) = e, \qquad P(\bar{q}_i) =)_i, \tag{C.34}$$

we set

$$\text{Dyck}_{n_q} = \{w \in W_0 | P(w) \ \text{ is a generalized Dyck word}\}. \tag{C.35}$$

Words which do are not oriented according to Eq.(C.32) can be brought to the standard orientation with the aid of Eq.(C.25). Therefore, the basis obtained after imposing the no-crossed relations and the orientation relations reads

$$W_3 = \begin{cases} \{l_1 l_2 \dots l_n \in W_0 | l_1 = g_1, l_n = g_n\}, & n_q \leq 1, \\ \left\{l_1 l_2 \dots l_n \in W_0 | l_1 = q_1, l_n = \bar{q}_1, l_1 l_2 \dots l_n \in \text{Dyck}_{n_q}\right\}, & n_q \geq 2. \end{cases} \tag{C.36}$$

Finally, imposing the fundamental BCJ relations, we can fix the letter l_{n-1} to be a gluon whenever we have up to 1 quark pair. Otherwise, we choose the letter l_{n-1} to be one of the remaining anti-quarks by removing any gluon from position $(n-1)$. Therefore, the BCJ basis reads

$$B = \begin{cases} \{l_1 l_2 \dots l_n \in W_0 | l_1 = g_1, l_{n-1} = g_{n-1}, l_n = g_n\}, & n_q = 0, \\ \{l_1 l_2 \dots l_n \in W_0 | l_1 = q_1, l_{n-1} = g_{n-2}, l_n = \bar{q}_1\}, & n_q = 1, \\ \{l_1 l_2 \dots l_n \in \text{Dyck}_{n_q} | l_1 = q_1, l_{n-1} \in \{\bar{q}_2, \dots, \bar{q}_{n_q}\}, l_n = \bar{q}_1\}, & n_q \geq 2. \end{cases} \tag{C.37}$$

C.2.4 The matrix F

In this section of the appendix we define the entries of the matrix $F_{ww'}$ occurring in Eq.(4.93). The flavor information can be neglected and therefore we consider the alphabet

$$\mathbb{A} = \{1, 2, \dots, n\}. \tag{C.38}$$

This alphabet coincides with the pure Yang-Mills case, but it contains massive particles. Making the formal substitution $\mathbb{A}_{\text{gluons}} \to \mathbb{A}$ in Eqs.(C.11)-(C.13), we can use the bases W_1, W_2, and B. For a sub-word $w = l_1 l_2 \dots l_k$, we set

$$S(w) = \sum_{\sigma \in S_k} l_{\sigma(1)} l_{\sigma(2)} \dots l_{\sigma(k)}. \tag{C.39}$$

Let $w_1 = l_1 l_2 \dots l_j$ and $w_2 = l_{j+1} l_{j+2} \dots l_{n-3}$ be two sub-words, such that

$$w = 1 w_1 (n-1) w_2 n$$

is a word in W_2. For convenience, we set $l_{n-2} = n - 1$. The standard cyclic factors satisfy the BCJ relations and we have

$$C(w, z^{(j)}) = F_{ww'} C(w', z^{(j)}). \tag{C.40}$$

The sum runs over all words occurring in

$$1(w_1 \sqcup\!\sqcup S(w_2))(n-1)n. \tag{C.41}$$

For a given w we define $F_{ww'} = 0$ whenever w' does not appear in the sum of Eq.(C.40). Otherwise, the coefficients are given for $w' = 1\sigma_1\sigma_2 \ldots \sigma_{n-3}(n-1)n = 1\sigma(n-1)n$ by the elements [11]

$$F_{ww'} = \prod_{k=j+1}^{n-3} \frac{\mathcal{F}(1\sigma(n-1)|l_k)}{\hat{s}_{n,l_k,\ldots,l_{n-3}}}, \tag{C.42}$$

where for $\rho = 1\sigma(n-1)$ the function $\mathcal{F}(\rho|l_k)$ is given by

$$\mathcal{F}(\rho|l_k) = \tag{C.43}$$
$$\left\{ \begin{array}{ll} \sum_{r=1}^{t_{l_k}-1} \mathcal{G}(l_k, \rho_r) & \text{if } t_{l_k} < t_{l_{k+1}} \\ -\sum_{r=t_{l_k}+1}^{n-1} \mathcal{G}(l_k, \rho_r) & \text{if } t_{l_k} > t_{l_{k+1}} \end{array} \right\} + \left\{ \begin{array}{ll} \hat{s}_{n,l_k,\ldots,l_{n-3}} & \text{if } t_{l_{k-1}} < t_{l_k} < t_{l_{k+1}} \\ -\hat{s}_{n,l_k,\ldots,l_{n-3}} & \text{if } t_{l_{k-1}} > t_{l_k} > t_{l_{k+1}} \\ 0 & \text{else} \end{array} \right\}.$$

t_a denotes the position of leg a in the string ρ, except for $t_{l_{n-2}}$ and t_{l_j}, which are always defined to be

$$t_{l_{n-2}} = t_{l_{n-4}}, \qquad t_{l_j} = n. \tag{C.44}$$

For $j = n-4$ this implies

$$t_{l_{n-2}} = t_{l_{n-4}} = n. \tag{C.45}$$

The function $\mathcal{G}$ is given by

$$\mathcal{G}(l_k,\rho_r)=\left\{\begin{array}{ll}2p_{l_k}p_{\rho_r}+2\Delta_{l_k\rho_r} & \text{if } \rho_r=1,(n-1)\\ 2p_{l_k}p_{\rho_r}+2\Delta_{l_k\rho_r} & \text{if } \rho_r=l_t,\ \text{and } t<k\\ 0 & \text{else}\end{array}\right\}. \tag{C.46}$$

We used the notation

$$\hat{s}_{\alpha_1,\ldots,\alpha_k}=\sum_{i<j}(2p_{\alpha_i}p_{\alpha_j}+2\Delta_{\alpha_i\alpha_j}). \tag{C.47}$$

This formula for $F_{ww'}$ generalizes the matrix $F_{ww'}$ appearing in the general BCJ relations for QCD [11, 103], since in this form it holds for massive particles as well. The general form of the BCJ relations reads

$$A_n(w)=\sum_{w'}F_{ww'}A_n(w'), \tag{C.48}$$

where $w=1w_1(n-1)w_2n\in W_2$ and where any type of particle may occurs in the subwords w_1 and w_2. The general BCJ relations of Eq.(C.48) follow from the fundamental BCJ relations, while the latter arise from the behavior at infinity of the BCFW-deformed amplitudes [44].

Example C.2.2. Let us see how to use Eq.(C.48) in practice. Consider the 6-point amplitude $A_6(132546)$, then $w_1=32$ and $w_2=4$. The sum runs over

$$1(32\sqcup S(4))56=132456+134256+143256, \tag{C.49}$$

and the decomposition (C.48) reads

$$A_6(132546)=F_{132546,132456}A_6(132456)+F_{132546,134256}A_6(134256)+F_{132546,143256}A_6(143256) \tag{C.50}$$

$$=-\frac{(s_{45}+s_{46})\,A_6(132456)}{s_{46}}+\frac{s_{14}A_6(143256)}{s_{46}}+\frac{(s_{14}+s_{34})\,A_6(134256)}{s_{46}}. \tag{C.51}$$

Notice that for the trivial case $w_2=e$, the shuffle product has only one term and necessarily $A_6(1w56)=A_6(1w56)$, therefore these coefficients are set to one. Suppose we have the amplitudes $A_6(123456)$, $A_6(125346)$ and $A_6(134256)$, hence the amplitude matrix A reads

$$\mathsf{A} = \begin{pmatrix} A_6(123456) \\ A_6(125346) \\ A_6(132546) \\ A_6(134256) \end{pmatrix}. \tag{C.52}$$

We have already calculated one of the rows of the matrix F and two of their coefficients are one. The last and second row can be easily calculated giving

$$\mathsf{F} = \tag{C.53}$$
$$-\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ \frac{s_{13}s_{45}}{s_{46}s_{346}} & \frac{s_{13}(s_{24}+s_{45})}{s_{46}s_{346}} & \frac{(s_{13}+s_{23})s_{45}}{s_{46}s_{346}} & \frac{(s_{14}+s_{24})(s_{35}s_{346})}{s_{46}s_{346}} & \frac{s_{14}(s_{23}+s_{35}+s_{346})}{s_{46}s_{346}} & \frac{s_{14}(s_{35}+s_{346})}{s_{46}s_{346}} \\ \frac{s_{45}+s_{46}}{s_{46}} & -\frac{s_{14}+s_{34}}{s_{46}} & 0 & 0 & -\frac{s_{14}}{s_{46}} & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with the amplitude matrix in the RHS of (C.48) given by

$$\mathsf{A}' = \begin{pmatrix} A_6(132456) \\ A_6(134256) \\ A_6(123456) \\ A_6(124356) \\ A_6(143256) \\ A_6(142356) \end{pmatrix}. \tag{C.54}$$

Notice that the order in the rows and columns is arbitrary. Once obtained, can write this matrix showing an upper triangle block structure. Actually, this observation may be used as a sufficient condition for the proof of the row rank of the matrix F, which we conjectured in Section 4.2.4. We use this to give evidence of the full row rank of F, up to $n \leq 10$ [102].

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