Direct and inverse transient eddy current problems

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Abstract

This work considers direct and inverse transient eddy current problems.

Transient excitation currents generate electromagnetic fields, which in turn induce electric currents in proximal conductors. For slowly varying fields this can be described by the eddy current equation, an approximation to Maxwell's equations. It is a linear partial differential equation with non-smooth coefficients and of mixed parabolic-elliptic type.

The direct problem consists of determining the electric field as the distributional solution of the equation from knowledge of the excitation and the coefficients describing the considered medium. Conversely, the fields can be measured by measurement coils. The inverse problem is then to infer information about the coefficient describing the conductors from these measurements.

This work presents a variational solution theory and discusses if the equation is well-posed. Furthermore, the solution's behavior for vanishing conductivity coefficient is studied and a linearization of the equation without conducting object towards the appearance of a conducting object is given. Two modifications are proposed to regularize the equation, which lead to a fully parabolic, respectively, a fully elliptic problem. Both are verified by proving the convergence of the solutions. Finally, considering the inverse problem of locating the conductors surrounded by a homogeneous medium and using linear sampling and factorization methods, it is shown that their position and shape are uniquely determined by the measurements.

Zusammenfassung

Die vorliegende Arbeit behandelt Vorwärts- sowie Rückwärtstheorie transienter Wirbelstromprobleme.

Transiente Anregungsströme induzieren elektromagnetische Felder, welche sogenannte Wirbelströme in leitfähigen Objekten erzeugen. Im Falle von sich langsam ändernden Feldern kann diese Wechselwirkung durch die Wirbelstromgleichung, einer Approximation an die Maxwell-Gleichungen, beschrieben werden. Diese ist eine lineare partielle Differentialgleichung mit nicht-glatten Koeffizientenfunktionen von gemischt parabolisch-elliptischem Typ.

Das Vorwärtsproblem besteht darin, zu gegebener Anregung sowie den umgebungsbeschreibenden Koeffizientenfunktionen das elektrische Feld als distributionelle Lösung der Gleichung zu bestimmen. Umgekehrt können die Felder mit Messspulen gemessen werden. Das Ziel des Rückwärtsproblems ist es, aus diesen Messungen Informationen über leitfähige Objekte, also über die Koeffizientenfunktion, die diese beschreibt, zu gewinnen.

In dieser Arbeit wird eine variationelle Lösungstheorie vorgestellt und die Wohlgestelltheit der Gleichung diskutiert. Darauf aufbauend wird das Verhalten der Lösung für verschwindende Leitfähigkeit studiert und die Linearisierbarkeit der Gleichung ohne leitfähiges Objekt in Richtung des Auftauchens eines leitfähigen Objektes gezeigt. Zur Regularisierung der Gleichung werden Modifikationen vorgeschlagen, welche ein voll parabolisches bzw. elliptisches Problem liefern. Diese werden verifiziert, indem die Konvergenz der Lösungen gezeigt wird. Zuletzt wird gezeigt, dass unter der Annahme von sonst homogenen Umgebungsparametern leitfähige Objekte eindeutig durch die Messungen lokalisiert werden können. Hierzu werden die Linear Sampling Methode sowie die Faktorisierungsmethode angewendet.

Contents

Chapter 1

Introduction

Eddy currents are electric currents induced within conductors by a temporally changing (transient) magnetic field. The term *eddy current* comes from the fact, that the flow lines are closed as eddies without default paths. Mathematically, the interaction between the source inducing the magnetic field, the coefficients representing the considered medium and the resulting electric field can be described by the eddy current equation.

Various applications of direct and inverse eddy current applications are running across our daily life. To mention a few, we have eddy current brakes or induction heating. Inverse eddy current problems occur for instance in non-destructive testing and *magnetic induction tomography*. The latter is an imaging technique used to display electromagnetic properties of objects. Moreover, eddy current effects are used in metal detectors. Here, an important application is land mine detection, where a source current in an inductor coil is used to generate electromagnetic fields that, in turn, induce currents in a buried conductor. The resulting change in the magnetic field can then be measured by a receiver coil, so that one may try to reconstruct information about the buried object.

The subject of this work is the mathemathical analysis of direct and inverse problems for this equation. Besides questions like existence and uniqueness of solutions of the direct problem, we are concerned with the solution's dependence on the conductor. Beyond that, we study the inverse shape detection problem whether the conductor can be detected from electromagnetic measurements, that is, from partial knowledge of the solutions.

The transient eddy current equation

Let us start with a formulation of the transient eddy current problem. Transient excitation currents $J(x,t)$ generate electric and magnetic fields $E(x,t)$ and $H(x,t)$, which can be described by Maxwell's equations

$$\operatorname{curl} H = \epsilon \partial_t E + \sigma E + J,$$
$$\operatorname{curl} E = -\mu \partial_t H,$$

where the operator curl acts on the three spatial coordinates, ∂_t denotes the time-derivative, and (under the assumption of linear and isotropic time-independent material laws) $\sigma(x)$, $\epsilon(x)$ and $\mu(x)$ are the conductivity, permittivity and permeability of the considered domain, respectively, material.

For slowly varying electromagnetic fields, the displacement currents $\epsilon \frac{\partial E}{\partial t}$ can be neglected. This leads to

$$\operatorname{curl} H = \sigma E + J,$$
$$\operatorname{curl} E = -\mu \partial_t H,$$

and after eliminating H, to the *transient eddy current equation*

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu} \operatorname{curl} E\right) = -\partial_t J. \tag{1.1}$$

The eddy current model is well-established in the engineering literature, see for instance Albanese and Rubinacci in [AR90] or Dirks in [Dir96]. A rigorous mathematical justification has been derived by Alonso in [Alo99], Pepperl [Pep05] and Ammari et al. in [ABN00] in case of time-harmonic excitations. [ABN00, Section 8] also justifies the transient model when the excitation is composed of low-frequency components. While time-harmonic eddy current problems are well studied, see, for instance the book of Alonso-Rodríguez and Valli [RV10] and the references therein, we consider transient eddy current problems in this work.

The direct problem

The direct problem consists of determining the solution E of (1.1) from knowledge of the excitation J and the coefficients σ and μ describing the considered medium.

In a typical application the domain under consideration consists of both, conducting regions ($\sigma(x) > 0$) and non-conducting regions ($\sigma(x) = 0$). An interesting consequence is the fact that equation (1.1) is of parabolic-elliptic type. The physical interpretation is that the time-scale is different in the conducting and the insulating region. In the insulating regions, the field instantaneously adapts to the excitation (quasi stationary elliptic behavior), while in the conducting regions, due to eddy currents induced by the varying electromagnetic fields, this adaptation takes some time (parabolic behavior). A particular consequence is that equation (1.1) (together with meaningful initial values) does not determine its solution E uniquely. To be precise, the equation only determines $\operatorname{curl} E$ and σE. Beside the fact that the solution is not unique, several applications such as inverse problems, sensitivity considerations, or the regularization of the equation require a variational solution theory that should be somehow independent from σ, and in particular, independent from the conducting domain. It turns out to be mathematically challenging to derive such a varational solution theory and then to solve the direct problem of determining the (unique part of the) solution E of (1.1).

In this work, we derive a variational formulation for the eddy current equation that is unified with respect to σ. To be more precise, we present a variational formulation independent from the conducting domain, that is uniquely solvable, and whose solution represents all solutions of the equation. We then use our formulation to study the solution's sensitivity on the conductivity for $\sigma \to 0$. Moreover, we analyze the change of the solutions of the equation without conducting object with respect to the problem becoming parabolic in some parts.

In some applications, for instance for computational reasons, one tries to overcome the non-uniqueness of the solutions of (1.1). One natural possibility is to regularize the problem by setting the conductivity to a small value in the non-conducting region. In that way, the eddy current equation is made fully parabolic and uniquely solvable. Analogously, an elliptic regularization can be established. The aim of this work is to verify these regularizations. The main tool here is our unified variational formulation: It covers both, the original and the regularized equation and thus enables us to prove the convergence of the solutions.

The inverse problem

Conversely, the induced electromagnetic fields can be measured by sensing coils. The aim in several practical applications is to obtain information about the electromagnetic properties from such measurements. Mathematically, this is the inverse problem of reconstructing the coefficients σ and μ in (1.1) from knowledge of the excitations J and a part of the solutions E of (1.1).

In this work the focus is on locating the conductors surrounded by a non-conducting medium. More precisely, the aim is to detect the support of the conductivity coefficient σ in (1.1) from knowledge of the operator mapping the excitation currents to measurements of the corresponding electric fields. We show that the position and the shape of this support are uniquely determined by the mapping and to state an explicit criterion to decide whether a given point is inside the sought domain or not. This criterion might serve as a base for non-iterative numerical reconstruction strategies.

Overview

We start with a brief introduction of our notation in Chapter 2.

Chapter 3 treats the direct eddy current problem. In case of unbounded domains, we derive a variational formulation for the equation, that is unified with respect to the conductivity σ. We then use this formulation to study the case when the conductivity approaches zero, and linearize the eddy current equation around a non-conducting domain with respect to the introduction of a conducting object.

The subject of Chapter 4 is the inverse problem of locating conductors surrounded by a non-conducting medium from electromagnetic measurements. Based on our solution theory developed in Chapter 3 we show that the conductors are uniquely determined by these measurements, and give an explicit criterion to decide whether a given point is inside the conducting domain or not.

The aim of Chapter 5 is to justify two regularizations of the parabolic-elliptic eddy current equation. Therefore we carry over the results of Chapter 3 to the case of bounded domains. Then, the eddy current equation is made fully parabolic by setting the conductivity in the insulating region to a small positive value. We show that this leads to a well-posed problem whose solutions converge against the solution of the original parabolic-elliptic eddy current equation. We also consider an elliptic regularization and show an analogous result there.

Published results

All results of this work have been published or are accepted for publication. All these publications are joint work with my supervisor Prof. Dr. Bastian von Harrach.

The results of the third chapter have been published in the SIAM Journal of Applied Mathematics under the title "A unified variational formulation for the parabolic-elliptic eddy current equations" [AH12].

The results of the fourth chapter are accepted for publication in the journal Inverse Problems under the title "Unique shape detection in transient eddy current problems" [AH13b].

The results of the fifth chapter are accepted for publication in the Conference Proceedings of the 4^{th} International Symposium on Inverse Problems, Design and Optimization (IPDO-2013) under the title "Justification of regularizations for the parabolic-elliptic eddy current equation" [AH13a]. They are also submitted for publication in the Journal of Inverse Problems in Science and Engineering. The decision about the acceptance is still open.

Chapter 2

Assumptions and notations

Let us start with a short introduction to the assumptions, the frequently used function spaces, and some notations used throughout this work.

We fix $T > 0$ and $\mu \in L^\infty_+(\mathbb{R}^3)$, where we denote by $L^\infty_+(\mathbb{R}^3)$ the space of $L^\infty(\mathbb{R}^3)$-functions with positive (essential) infimum (denoted by $\inf \mu$). For the conductivity coefficient σ we assume that

$$\sigma \in L^\infty(\mathbb{R}^3)$$

is (essentially) non-negative and has bounded support.

2.1 Function spaces

Let $\mathcal{D}(\mathbb{R}^3)$, $\mathcal{D}(]0,T[)$ and $\mathcal{D}(\mathbb{R}^3\times]0,T[)$ denote the spaces of C^∞-functions in x, t and (x,t), which are compactly supported in $\mathbb{R}^3$, $]0,T[$ and $\mathbb{R}^3\times]0,T[$, respectively. We also use the notations $\mathcal{D}([0,T[)$ and $\mathcal{D}(\mathbb{R}^3 \times [0,T[)$ for the spaces of restrictions of functions from $\mathcal{D}(]-\infty,T[)$ and $\mathcal{D}(\mathbb{R}^3\times]-\infty,T[)$ to $[0,T]$ and $\mathbb{R}^3 \times [0,T]$, respectively.

$\mathcal{D}'(\mathbb{R}^3)$ denotes the space of distributions, i.e. the space of continuous linear mappings from $\mathcal{D}(\mathbb{R}^3)$ to $\mathbb{R}$. $\mathcal{D}'(\mathbb{R}^3)^3$ and $\mathcal{D}'(\mathbb{R}^3\times]0,T[)^3$ are defined analogously.

For a bounded domain or a finite union of bounded domains $\mathcal{O} \subset \mathbb{R}^3$, the space $\mathcal{D}(\mathcal{O})$ is defined as the space of C^∞-functions which are compactly supported in $\mathcal{O}$. In the same way, we also use the spaces $\mathcal{D}(\overline{\mathcal{O}})$, $\mathcal{D}(\mathcal{O}\times]0,T[)$, $\mathcal{D}(\mathcal{O} \times [0,T[)$ and the associated distributional spaces.

Let $L^2_\rho(\mathbb{R}^3)$ and $W(\mathrm{curl})$ denote the distributional spaces

$$\begin{aligned} L^2_\rho(\mathbb{R}^3) &:= \{e \in \mathcal{D}'(\mathbb{R}^3) \mid (1+|x|^2)^{-\frac{1}{2}} e \in L^2(\mathbb{R}^3)\}, \\ W(\mathrm{curl}) &:= \{E \in L^2_\rho(\mathbb{R}^3)^3 \mid \mathrm{curl}\, E \in L^2(\mathbb{R}^3)^3\}. \end{aligned}$$

$L^2_\rho(\mathbb{R}^3)^n$, $n = 1, 3$, and $W(\mathrm{curl})$ are Hilbert spaces with norms

$$\|\cdot\|_\rho := \|(1+|x|^2)^{-\frac{1}{2}} \cdot \|_{L^2(\mathbb{R}^3)^n}, \text{ and } \|\cdot\|^2_{W(\mathrm{curl})} := \|\cdot\|^2_\rho + \|\,\mathrm{curl}\cdot\|^2_{L^2(\mathbb{R}^3)^3}.$$

The space $W(\mathrm{curl}, \mathbb{R}^3 \setminus \overline{\mathcal{O}})$ is defined analogously, and the space $H(\mathrm{curl}, \mathcal{O})$ accordingly as the space of $L^2(\mathcal{O})^3$-functions having their curl in $L^2(\mathcal{O})^3$. We introduce the Beppo-Levi spaces

$$W^1(\mathbb{R}^3) := \{e \in L^2_\rho(\mathbb{R}^3) \,|\, \nabla e \in L^2(\mathbb{R}^3)^3\},$$
$$W^1(\mathbb{R}^3)^3 := \{E \in L^2_\rho(\mathbb{R}^3)^3 \,|\, \nabla E \in L^2(\mathbb{R}^3)^{3\times 3}\}.$$

In the latter space, ∇E denotes the Jacobian of E. If $\mathcal{O}$ is a bounded Lipschitz domain with connected complement, $W^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})$ is defined analogously. These spaces are Hilbert spaces with respect to the norms

$$\|\cdot\|_{W^1(\mathbb{R}^3)} := \|\nabla\cdot\|_{L^2(\mathbb{R}^3)^3},$$
$$\|\cdot\|_{W^1(\mathbb{R}^3)^3} := \|\nabla\cdot\|_{L^2(\mathbb{R}^3)^{3\times 3}},$$
$$\|\cdot\|_{W^1(\mathbb{R}^3\setminus\overline{\mathcal{O}})} := \|\nabla\cdot\|_{L^2(\mathbb{R}^3\setminus\overline{\mathcal{O}})^3},$$

cf., e.g., [DL00c, IX.A, §1, Remark 7] and [DL00d, XI.B, §1, Theorem 1 and Remark 2], where Theorem 1 also holds for bounded Lipschitz domains with connected complement, cf. [Gri85, Theorem 1.4.4.1]. Note that $\mathcal{D}(\mathbb{R}^3)$ is dense in $L^2_\rho(\mathbb{R}^3)$ and in $W^1(\mathbb{R}^3)$, and that $\mathcal{D}(\mathbb{R}^3)^3$ is dense in $L^2_\rho(\mathbb{R}^3)^3$, in $W(\mathrm{curl})$ and in $W^1(\mathbb{R}^3)^3$.

We also frequently use the space

$$W^1_\diamondsuit := \{E \in W^1(\mathbb{R}^3)^3 \,|\, \mathrm{div}\, E = 0\}, \quad \|\cdot\|_{W^1_\diamondsuit} := \|\,\mathrm{curl}\cdot\|_{L^2(\mathbb{R}^3)^3}.$$

On $W^1_\diamondsuit$ we have

$$\|\cdot\|_{W^1(\mathbb{R}^3)^3} = \|\nabla\cdot\|_{L^2(\mathbb{R}^3)^{3\times 3}} = \|\cdot\|_{W^1_\diamondsuit},$$

cf., e.g., the proof of [DL00c, IX.A, §1, Theorem 3], so that $W^1_\diamondsuit$ equipped with the norm $\|\cdot\|_{W^1_\diamondsuit}$ is a Hilbert space.

For a Banach space X, we denote by $C(0,T,X)$ and $L^2(0,T,X)$ the spaces of vector-valued functions

$$E: \ [0,T] \to X,$$

which are continuous on $[0,T]$, respectively, square integrable on $[0,T]$, cf., e.g., [DL00e, XVIII, §1]. Spaces of functions with vector-valued time-derivatives are introduced in detail in Subsection 3.2.1.

2.2 Notations

We denote the dual space of a space H by H' and the dual pairing on $H' \times H$ by $\langle \cdot, \cdot \rangle_H$. The inner product on an inner product space H is denoted by $(\cdot \, , \cdot)_H$. In case of real Hilbert spaces, the inner product and the dual pairing on $H' \times H$ are related by the isometry $\iota_H : H \to H'$, that identifies H with its dual:

$$\langle \iota_H u, \cdot \rangle_H := (u, \cdot)_H \quad \text{for all } u \in H.$$

We denote the dual operator of an operator $A \in \mathcal{L}(H_1, H_2)$ between real Hilbert spaces H_1, H_2 by A'. For $h_2' \in H_2'$, A' is defined by

$$\langle A' h_2', h_1 \rangle_{H_1} := \langle h_2', A h_1 \rangle_{H_2} \quad \text{for all } h_1 \in H_1.$$

We rigorously distinguish between the dual and the adjoint operator, the latter denoted by A^*. They satisfy the identity $A^* = \iota_{H_1}^{-1} A' \iota_{H_2}$.

In this work, we frequently use the dual pairing between $W(\text{curl})'$ and $W(\text{curl})$, hence in this case we write

$$\langle G, E \rangle := \langle G, E \rangle_{W(\text{curl})} \quad \text{for } G \in W(\text{curl})', \; E \in W(\text{curl}).$$

We also write $\mathbb{R}^3_T := \mathbb{R}^3 \times]0, T[$ and $L^2(\mathbb{R}^3_T)$ instead of $L^2(\mathbb{R}^3 \times]0, T[)$ and accordingly $L^2(\mathcal{O}_T)$, and usually omit the arguments x and t and only use them where we expect them to improve readability.

Chapter 3

A unified variational formulation for the parabolic-elliptic eddy current equation

In this chapter, we derive a unified variational formulation for the eddy current equation, that is uniformly coercive with respect to the conductivity and we discuss the solvability of the eddy current equation. We then use this formulation to study the case when the conductivity approaches zero. On top of that, we linearize the eddy current equation without conducting object with respect to the equation being parabolic in some parts.

The Sections 3.2–3.4 are the Sections 2–4 of the paper [AH12] up to minor changes.

3.1 Introduction

We consider the transient eddy current equation

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -\partial_t J, \tag{3.1}$$

with the three-dimensional time-dependent electric field $E(x,t)$ and the source current $J(x,t)$. The scalar coefficients $\sigma(x)$ and $\mu(x)$ denote the conductivity and the permeability of the considered domain.

We consider a domain that consists of conducting regions ($\sigma(x) > 0$) as well as non-conducting regions ($\sigma(x) = 0$), so that equation (3.1) is of parabolic-elliptic type. A particular consequence is that initial values are only meaningful in the conducting region. The second consequence is that equation (3.1) (together with meaningful initial values) does not uniquely determine its solution. It only determines E up to the addition of a *gauge field*, which is a curl-free field that vanishes inside the conductor. However, in many applications one is interested only in the unique parts σE and $\operatorname{curl} E$ of the solution.

For fixed, and in the most cases constant, conductivity, the transient eddy current equation has been studied many times. Several variational formulations have been proposed and used for the numerical solution, such as by Bossavit in [Bos99], Beck et al. in [BHHW00] and in [BDH$^+$99], and by Flemisch et al. in [FMRW04]. For a well-posed variational formulation of the H-based formulation of the transient eddy current model, that is obtained by eliminating E instead of the magnetic field H, let us refer to Meddahi and Selgas in [MS08]. For the E-based formulation and constant conductivity, rigorous theoretical results on the well-posedness of variational formulations can be found by Bachinger et al. in [BLS05], Hömberg and Sokolowski in [HS03], Jiang and Zheng in [JZ12] and Nicaise and Tröltzsch in [NT14]. Acevedo et al. in [AMR09] and Kolmbauer in [Kol11] allow also spatially varying conductivity. All these approaches concentrate on solving the eddy current equation with a fixed conducting region in which the conductivity is assumed to be bounded from below by some positive constant. The corresponding variational formulations, along with their underlying solution spaces and coercivity constants, depend in some form or another on this lower bound or on the support of the conductivity. Here, the usual approach is the following. To ensure uniqueness, one imposes a gauge condition, for instance $\operatorname{div} E = 0$ in the whole or the insulating part of the domain, where the solution is not unique. Then, one concentrates on showing the well-posedness of a proposed variational formulation and on how to solve it numerically. One point that is sometimes neglected here is the question, whether the solution of the variational equation also solves the eddy current equation. To the knowledge of the author, there is no completely rigorous variational solution theory for the eddy current equation (3.1) in the literature so far.

We consider the general case of spatially varying σ. Moreover, for our further analysis, such as the sensitivity considerations (see Section 3.4), the treatment of the inverse problem (see Chapter 4) and the regularization of the equation (see Chapter 5), it turns out to be valuable to have a variational formulation for the equation that is unified with respect to σ in the following sense: It should not depend on the support of σ and should be uniformly coercive with respect to σ and hence uniquely solvable. In particular, the coercivity and continuity constants should not depend on the lower bound of σ.

In this chapter we derive such a unified variational formulation for the eddy current equation posed on the whole $\mathbb{R}^3$. To be more precise, we present a variational formulation that is uniformly coercive (and hence uniquely solvable) in the space of divergence-free functions and whose solution agrees with the true solution up to the addition of a gradient field. At this point it should be stressed that, for spatially varying σ, the standard variational formulation of (3.1) restricted to divergence-free functions does not determine the solution up to a curl-free field. Although the solution of our variational formulation does not solve the eddy current equation, we can prove the solvability of the equation in this way: The unique solution of the variational formulation agrees with every solution of the eddy current equation up to the addition of a gradient field. In this sense, the unique solution of the variational formulation represents all solutions. In Chapter 5, we moreover extend our solution theory to bounded domains.

We use our variational formulation to study the solution's dependence on the conductivity. To the authors knowlegde, there are no rigorous results so far. We first study the limit of the solutions of (3.1) for $\sigma \to 0$ and prove convergence against their magnetostatic counterparts, which are the solutions of the equation with $\sigma \equiv 0$. Beyond that, we analyze the solution's sensitivity with respect to the equation changing from elliptic to parabolic type. The main question here is: How does the solution of the elliptic magnetostatic problem change if the problem becomes parabolic in a part of the domain? For a scalar analog, the heat equation, this question has been answered by Harrach in [Geb07]. In our case, we use an analogous approach and rigorously determine the directional derivative of the solutions of (3.1) with $\sigma \equiv 0$, with respect to σ, that is, we linearize the solutions of the elliptic (magnetostatic) problem with respect to the solutions of the parabolic-elliptic problem.

The first step towards our unified solution theory is the handling of initial values. We show that solutions of the equation have vector-valued time-derivatives and that, for every solution E, the term $\sqrt{\sigma}E$ is continuous in time. This enables us to formulate meaningful initial values independent from the conducting domain. Here, we follow the theory on the heat equation by Harrach in [Geb07], again.

This chapter is organized as follows: In Section 3.2 we characterize well-defined initial conditions, derive the standard variational formulation for equation (3.1), and prove the uniqueness of the solution up to gauge fields. Section 3.3 contains our main theoretical tool: a uniformly coercive variational formulation that determines the solution up to the addition of a gradient field. This also proves solvability of the eddy current equation. Finally, in Section 3.4 we use our variational formulation to study the behavior of the solutions when the conductivity approaches zero and linearize (3.1) without conducting domain with respect to the equation being parabolic in some parts.

3.2 Formulation of the equation in $\mathbb{R}^3$

We consider the space $L^2(0, T, W(\text{curl}))$ as the space to look for a solution of the eddy current equation (3.1).

Generally, it is not the case that every $E \in L^2(0, T, W(\text{curl}))$ has some well-defined initial values. However, in the following we show that at least every solution of (3.1) has well-defined initial values. Then, we derive a standard variational formulation and discuss, in what sense uniqueness can be expected.

Throughout this chapter, we assume that we are given the time derivative of the excitation currents

$$\begin{aligned} J_t \in L^2(0,T,W(\mathrm{curl})') \text{ with } \operatorname{div} J_t = 0 \quad \text{and} \\ E^0 \in L^2(\mathbb{R}^3)^3 \text{ with } \operatorname{div}(\sigma E^0) = 0. \end{aligned} \tag{3.2}$$

3.1 Theorem *Let $E \in L^2(0,T,W(\mathrm{curl}))$. The eddy current problem reads*

$$\partial_t(\sigma(x)E(x,t)) + \operatorname{curl}\left(\frac{1}{\mu(x)}\operatorname{curl} E(x,t)\right) = -J_t(x,t) \qquad \text{in } \mathbb{R}^3\times]0,T[, \tag{3.3}$$

$$\sqrt{\sigma(x)}E(x,0) = \sqrt{\sigma(x)}E^0(x) \quad \text{in } \mathbb{R}^3. \tag{3.4}$$

The following holds:

a) For every solution $E \in L^2(0,T,W(\mathrm{curl}))$ of (3.3) we have

$$\sqrt{\sigma}E \in C(0,T,L^2(\mathbb{R}^3)^3).$$

b) $E \in L^2(0,T,W(\mathrm{curl}))$ solves (3.3)–(3.4) if and only if E solves

$$\begin{aligned} -\int_0^T \int_{\mathbb{R}^3} \sigma E \cdot \partial_t \Phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \,\mathrm{d}x\,\mathrm{d}t \\ = -\int_0^T \langle J_t, \Phi \rangle \,\mathrm{d}t + \int_{\mathbb{R}^3} \sigma E^0 \cdot \Phi(0)\,\mathrm{d}x \end{aligned} \tag{3.5}$$

for all $\Phi \in \mathcal{D}(\mathbb{R}^3 \times [0,T[)^3$.

c) Equations (3.3)–(3.4) uniquely determine $\operatorname{curl} E$ and $\sqrt{\sigma}E$.
Moreover, if $E \in L^2(0,T,W(\mathrm{curl}))$ solves (3.3)–(3.4), then every function $F \in L^2(0,T,W(\mathrm{curl}))$ with $\operatorname{curl} F = \operatorname{curl} E$ and $\sqrt{\sigma}F = \sqrt{\sigma}E$ also solves (3.3)–(3.4).

Before we prove Theorem 3.1 in the following subsection, let us stress again the somewhat subtle point that the initial condition (3.4) is only meaningful for solutions of (3.3). When we speak of a solution $E \in L^2(0,T,W(\mathrm{curl}))$ of (3.3)–(3.4), then this is to be understood in the following order: First of all, $E \in L^2(0,T,W(\mathrm{curl}))$ has to solve (3.3), so that $\sqrt{\sigma}E \in C(0,T,L^2(\mathbb{R}^3)^3)$, and, second, this continuous function $\sqrt{\sigma}E$ has to fulfill the initial condition (3.4). Note that this is similar to the interpretation of Neumann boundary values for second-order elliptic equations.

The multiplication with $\sqrt{\sigma}$ in the initial condition (3.4) can be interpreted as stating that, wherever it makes sense to speak of initial values, they must agree with E^0. In $\operatorname{supp}\sigma$, the equation is parabolic and initial values are meaningful and necessary. Outside of $\operatorname{supp}\sigma$, where the equation is elliptic, initial conditions are meaningless and (3.4) does not contain any information.

Let us stress that, in this section, we only require that σ is nonnegative, bounded and has bounded support.

3.2.1 Initial values, a standard variational formulation and uniqueness

For $E \in L^2(0,T,W(\operatorname{curl}))$ we have that $E(t), \operatorname{curl} E(t) \in L^2(\mathbb{R}^3)^3$ for $t \in]0,T[$ a.e. and consequently the products

$$\frac{1}{\mu} \operatorname{curl} E(t), \ \sigma E(t) \in L^2(\mathbb{R}^3)^3$$

are well-defined. Moreover, the assumption $\operatorname{div}(\sigma E^0) = 0$ is well-defined in the sense of distributions since $E^0 \in L^2(\mathbb{R}^3)^3$. Since $\mathcal{D}(\mathbb{R}^3)^3$ is dense in $W(\operatorname{curl})$, we can regard $L^2(0,T,W(\operatorname{curl})')$ as a subspace of $\mathcal{D}'(\mathbb{R}^3 \times]0,T[)^3$. Hence, also $\operatorname{div} J_t$ is well-defined in the sense of distributions.

Now, the transient eddy current equation (3.3) is equivalent to:

$$-\int_0^T \int_{\mathbb{R}^3} \sigma E \cdot \partial_t \Phi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \, \mathrm{d}x \, \mathrm{d}t = -\int_0^T \langle J_t, \Phi \rangle \, \mathrm{d}t \quad \text{for all } \Phi \in \mathcal{D}(\mathbb{R}^3 \times]0,T[)^3. \tag{3.6}$$

In the rest of this subsection we continue along the lines in [Geb07, Section 2].

We first recall the definition of the time-derivative in the sense of vector-valued distributions: For two Banach spaces X, Y and a continuous injection $\iota : X \hookrightarrow Y$, $E \in L^2(0,T,X)$ has a time-derivative in $L^2(0,T,Y)$ in the sense of vector-valued distributions, if there exists $\dot{E} \in L^2(0,T,Y)$ which fulfills

$$\int_0^T \dot{E} \, \varphi \, \mathrm{d}t = -\int_0^T \iota E \, \partial_t \varphi \, \mathrm{d}t \quad \text{for all } \varphi \in \mathcal{D}(]0,T[)$$

(cf., e.g., [DL00e, XVIII, §1]). For a Gelfand triple $\mathcal{V} \overset{\iota}{\hookrightarrow} \mathcal{H} \overset{\iota'}{\hookrightarrow} \mathcal{V}'$ of real separable Hilbert spaces $\mathcal{V}$ and $\mathcal{H}$, the space

$$\mathcal{W}(0,T,\mathcal{V},\mathcal{V}') := \left\{ E \in L^2(0,T,\mathcal{V}) \,|\, \dot{E} \in L^2(0,T,\mathcal{V}') \right\}$$

is defined by taking the time-derivative with respect to the injection $\iota'\iota : \mathcal{V} \hookrightarrow \mathcal{V}'$. The image of the space $\mathcal{W}(0,T,\mathcal{V},\mathcal{V}')$ under ι is continuously imbedded in $C(0,T,\mathcal{H})$ and, for $E, F \in \mathcal{W}(0,T,\mathcal{V},\mathcal{V}')$, the following integration by parts formula holds:

$$\int_0^T \left[\langle \dot{E}(t), F(t) \rangle_{\mathcal{V}} + \langle \dot{F}(t), E(t) \rangle_{\mathcal{V}} \right] \mathrm{d}t = (\iota E(T), \iota F(T))_{\mathcal{H}} - (\iota E(0), \iota F(0))_{\mathcal{H}},$$

cf., e.g., [DL00e, XVIII, §1, Theorems 1 and 2]. As a special case we have

$$H^1(0,T,\mathcal{V}) = \mathcal{W}(0,T,\mathcal{V},\mathcal{V})$$

where $\mathcal{V} = \mathcal{H}$ is identified with its dual and ι is the identity mapping.

In view of (3.3), we introduce the space

$$\mathcal{W}_\sigma := \left\{ E \in L^2(0,T,W(\mathrm{curl})) \,|\, (\sigma E)^\cdot \in L^2(0,T,W(\mathrm{curl})') \right\},$$

where $(\sigma E)^\cdot$ denotes the time-derivative of $\sigma E \in L^2(\mathbb{R}^3_T)^3$ in the sense of vector-valued distributions with respect to the canonical injection $L^2(\mathbb{R}^3)^3 \hookrightarrow W(\mathrm{curl})'$. Note that for every $E \in H^1(0,T,W(\mathrm{curl}))$, $\sigma E \in L^2(\mathbb{R}^3_T)^3$ and, in that sense, $E \in \mathcal{W}_\sigma$ with $(\sigma E)^\cdot = \sigma \dot{E}$.

3.2 Lemma *If $E \in \mathcal{W}_\sigma$, then $\sqrt{\sigma}E \in C(0,T,L^2(\mathbb{R}^3)^3)$. Additionally, for two fields $E, F \in \mathcal{W}_\sigma$ the following integration by parts formula holds:*

$$\int_0^T \left[\langle (\sigma E)^\cdot, F \rangle + \langle (\sigma F)^\cdot, E \rangle \right] \mathrm{d}t = \int_{\mathbb{R}^3} \sigma \left[E(T) \cdot F(T) - E(0) \cdot F(0) \right] \mathrm{d}x. \quad (3.7)$$

Proof In [Geb07, Section 2] this lemma is proven for a scalar analog. We repeat the proof for the convenience of the reader.

We define the space L^2_σ by taking the closure of

$$\left\{ \sqrt{\sigma}E \,|\, E \in L^2(\mathbb{R}^3)^3 \right\} \subseteq L^2(\mathbb{R}^3)^3$$

with respect to the $L^2(\mathbb{R}^3)^3$-norm. L^2_σ is a separable Hilbert space equipped with the standard $L^2(\mathbb{R}^3)^3$-inner product.

Then we define a mapping I by

$$I: \; W(\mathrm{curl}) \to L^2_\sigma, \quad E \mapsto \sqrt{\sigma}E,$$

which is continuous and has dense range. We identify the Hilbert space L^2_σ with its dual. Then, after factoring out the null space N of I we obtain, that

$$\iota: \; W(\mathrm{curl})/N \to L^2_\sigma, \quad E + N \mapsto IE$$

defines an injective, continuous mapping and hence a Gelfand triple

$$W(\mathrm{curl})/N \overset{\iota}{\hookrightarrow} L^2_\sigma \overset{\iota'}{\hookrightarrow} (W(\mathrm{curl})/N)'.$$

For all $G \in L^2_\sigma$ the dual mapping i' is given by

$$\langle \iota' G, F + N \rangle_{W(\mathrm{curl})/N} = \int_{\mathbb{R}^3} G \cdot \sqrt{\sigma} F \,\mathrm{d}x \qquad \text{for all } F \in W(\mathrm{curl}). \quad (3.8)$$

Let $E \in \mathcal{W}_\sigma$ and $G = (\sigma E)^\cdot \in L^2(0,T,W(\mathrm{curl})')$ be the time-derivative of $\sigma E \in L^2(\mathbb{R}^3_T)^3$ with respect to the canonical injection $L^2(\mathbb{R}^3)^3 \hookrightarrow W(\mathrm{curl})'$. Now we show that G is the time derivative of $E+N \in L^2(0,T,W(\mathrm{curl})/N)$ with respect to $\iota'\iota$. For $\varphi \in \mathcal{D}(]0,T[)$ and $F \in N$ we have

$$\int_0^T \langle G(t), F\rangle \varphi(t)\,\mathrm{d}t = -\int_0^T \int_{\mathbb{R}^3} \sigma E(t)\cdot F\,\mathrm{d}x\,\partial_t\varphi(t)\,\mathrm{d}t = 0$$

and thus $\langle G(t), F\rangle = 0$ for $t \in]0,T[$ a.e. Hence, $G(t) \in N^\perp$ and we can identify G with an element of $L^2(0,T,(W(\mathrm{curl})/N)')$. Then, for $F+N \in W(\mathrm{curl})/N$ it follows that

$$\begin{aligned}\int_0^T \langle G(t), F+N\rangle_{W(\mathrm{curl})/N}\varphi(t)\,\mathrm{d}t &= \int_0^T \langle G(t), F\rangle\varphi(t)\,\mathrm{d}t \\ &= -\int_0^T \int_{\mathbb{R}^3} \sigma E(t)\cdot F\,\mathrm{d}x\,\partial_t\varphi(t)\,\mathrm{d}t \\ &= -\int_0^T \langle \iota'\iota(E(t)+N), F+N\rangle_{W(\mathrm{curl})/N}\partial_t\varphi(t)\,\mathrm{d}t\end{aligned}$$

and, accordingly, $G = (E+N)^\cdot$ and

$$E+N \in \mathcal{W}(0,T,W(\mathrm{curl})/N,(W(\mathrm{curl})/N)').$$

Now, it follows that $\sqrt{\sigma}E = \iota(E+N) \in C(0,T,L^2_\sigma) \subseteq C(0,T,L^2(\mathbb{R}^3)^3)$ and using (3.8) we obtain the integration by parts formula (3.7). □

For the next lemma recall that for $E \in L^2(0,T,W(\mathrm{curl}))$ the equation (3.1) is to be understood in the sense of distributions, cf. the beginning of this subsection.

3.3 Lemma *Every solution $E \in L^2(0,T,W(\mathrm{curl}))$ of (3.1) is in $\mathcal{W}_\sigma$ and thus has well-defined initial values*

$$\sqrt{\sigma(x)}E(x,0) \in L^2(\mathbb{R}^3)^3.$$

For $t \in]0,T[$ a.e., $(\sigma E)^\cdot(t) \in W(\mathrm{curl})'$ is given by

$$\langle(\sigma E)^\cdot(t), F\rangle = -\langle J_t(t), F\rangle - \int_{\mathbb{R}^3} \frac{1}{\mu}\operatorname{curl} E(t)\cdot\operatorname{curl} F\,\mathrm{d}x \quad \textit{for all } F \in W(\mathrm{curl}). \tag{3.9}$$

Proof Let E be a solution of (3.1). Define $G(t) \in W(\mathrm{curl})'$ by

$$\langle G(t), \Psi\rangle := -\langle J_t(t), \Psi\rangle - \int_{\mathbb{R}^3} \frac{1}{\mu}\operatorname{curl} E(t)\cdot\operatorname{curl}\Psi\,\mathrm{d}x \ \text{ for all } \Psi \in W(\mathrm{curl}).$$

Then $G \in L^2(0,T,W(\mathrm{curl})')$, and, due to the fact that E solves (3.6) with $\Phi = \Psi\varphi$ for all $\varphi \in \mathcal{D}(]0,T[)$ and all $\Psi \in \mathcal{D}(\mathbb{R}^3)^3$, it holds that

$$\begin{aligned}\int_0^T \langle G(t), \Psi\rangle\varphi(t)\,\mathrm{d}t &= -\int_0^T\int_{\mathbb{R}^3} \sigma E\cdot\Psi\,\mathrm{d}x\,\partial_t\varphi\,\mathrm{d}t \\ &= -\int_0^T \langle\sigma E(t), \Psi\rangle\partial_t\varphi(t)\,\mathrm{d}t.\end{aligned} \tag{3.10}$$

Since $\mathcal{D}(\mathbb{R}^3)^3$ is dense in $W(\mathrm{curl})$ and both sides depend continuously on Ψ, we obtain that equation (3.10) holds for all $\Psi \in W(\mathrm{curl})$. Now it follows from the fact, that $W(\mathrm{curl}) \otimes \mathcal{D}(]0,T[)$ is dense in $L^2(0,T,W(\mathrm{curl}))$, that $G = (\sigma E)^\cdot$ with respect to the canonical injection $L^2(\mathbb{R}^3)^3 \hookrightarrow W(\mathrm{curl})'$. This shows that $E \in \mathcal{W}_\sigma$.
□

Lemma 3.3 shows, that the initial condition (3.4) makes sense for solutions of equation (3.3), and, in that sense, we can speak of solutions $E \in L^2(0,T,W(\mathrm{curl}))$ of (3.3)–(3.4). Now, we give an equivalent variational formulation:

3.4 Lemma *The following problems are equivalent:*

a) Find $E \in L^2(0,T,W(\mathrm{curl}))$ that solves (3.3) and (3.4).

b) Find $E \in \mathcal{W}_\sigma$ that solves (3.4) and

$$\int_0^T \langle (\sigma E)^\cdot, F\rangle \,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} F \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \langle J_t, F\rangle \,\mathrm{d}t \quad (3.11)$$

for all $F \in L^2(0,T,W(\mathrm{curl}))$.

c) Find $E \in L^2(0,T,W(\mathrm{curl}))$ that solves

$$-\int_0^T \langle (\sigma F)^\cdot, E\rangle \,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} F \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \langle J_t, F\rangle \,\mathrm{d}t + \int_{\mathbb{R}^3} \sigma E^0 \cdot F(0)\,\mathrm{d}x$$

for all $F \in \mathcal{W}_\sigma$ with $\sqrt{\sigma}F(T) = 0$.

d) Find $E \in L^2(0,T,W(\mathrm{curl}))$ that solves

$$-\int_0^T \int_{\mathbb{R}^3} \sigma E \cdot \partial_t \Phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \langle J_t, \Phi\rangle \,\mathrm{d}t + \int_{\mathbb{R}^3} \sigma E^0 \cdot \Phi(0)\,\mathrm{d}x$$

for all $\Phi \in \mathcal{D}(\mathbb{R}^3 \times [0,T[)^3$.

Proof We start by showing "a) $\Longrightarrow$ b)". If $E \in L^2(0,T,W(\mathrm{curl}))$ solves equations (3.3)–(3.4) it follows from Lemma 3.3 that $E \in \mathcal{W}_\sigma$ and (3.11) holds for all $F(x,t) = G(x)\varphi(t)$ with $G \in W(\mathrm{curl})$ and $\varphi \in \mathcal{D}(]0,T[)$. Since $W(\mathrm{curl}) \otimes \mathcal{D}(]0,T[)$ is dense in $L^2(0,T,W(\mathrm{curl}))$, and both sides of (3.11) depend continuously on $F \in L^2(0,T,W(\mathrm{curl}))$, b) follows.

"b) $\Longrightarrow$ c)" follows from the integration by parts formula (3.7).

"c) $\Longrightarrow$ d)" follows from the fact that for $\Phi \in \mathcal{D}(\mathbb{R}^3 \times [0,T[)^3$ the time-derivative $(\sigma\Phi)^{\cdot} \in L^2(0,T,W(\mathrm{curl})')$ of $\sigma\Phi \in L^2(\mathbb{R}^3_T)^3$ with respect to the canonical injection $L^2(\mathbb{R}^3)^3 \hookrightarrow W(\mathrm{curl})'$ is the image of the classical time-derivative $\sigma\partial_t\Phi(t)$ under this injection, i.e.

$$\langle (\sigma\Phi)^{\cdot}(t), E(t)\rangle = \int_{\mathbb{R}^3} \sigma\partial_t\Phi(t)\cdot E(t)\,\mathrm{d}x \qquad \text{for } t \in]0,T[\text{ a.e.}$$

Finally, to show the implication "d) $\Longrightarrow$ a)" we use the equation in d) applied on $\Phi \in \mathcal{D}(\mathbb{R}^3 \times]0,T[)^3$. Then $E \in L^2(0,T,W(\mathrm{curl}))$ solves (3.4) and Lemma 3.3 yields $E \in \mathcal{W}_\sigma$. Now, the integration by parts formula (3.7) applied on d) with $\Phi = \Psi\varphi$, $\Psi \in \mathcal{D}(\mathbb{R}^3)^3$, $\varphi \in \mathcal{D}([0,T[)$ with $\varphi(0) = 1$, and using Lemma 3.3, implies that $\sqrt{\sigma}E^0 = \sqrt{\sigma}E(0)$. □

Now, the proof of Theorem 3.1 reads:

Proof of Theorem 3.1

a) This follows from Lemma 3.2 and Lemma 3.3.

b) This is the equivalence of a) and d) in Lemma 3.4.

c) Assume that $E \in \mathcal{W}_\sigma$ is a solution of (3.3)–(3.4) with $\sqrt{\sigma}E(0) = 0$ and $J_t = 0$. Using Lemma 3.4 b) and the integration by parts formula (3.7) implies

$$\begin{aligned} 0 &= \int_0^T \langle (\sigma E)^{\cdot}, E\rangle \,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} E \,\mathrm{d}x\,\mathrm{d}t \\ &\geq \frac{1}{2}\|\sqrt{\sigma}E(T)\|^2_{L^2(\mathbb{R}^3)^3} + \frac{1}{\|\mu\|_\infty}\|\operatorname{curl} E\|^2_{L^2(\mathbb{R}^3_T)^3}. \end{aligned}$$

We obtain $\operatorname{curl} E = 0$ and $\sqrt{\sigma}E = 0$. The second assertion is obvious. □

3.3 A unified variational formulation

In this section we present a new, uniquely solvable and uniformly coercive variational formulation that determines the solution of the eddy current problem, (3.3) and (3.4), up to the addition of a gradient field. From this we obtain solvability of (3.3) and (3.4), and a continuity result that is uniform with respect to the conductivity σ.

Our general approach is as follows. We write

$$E = \tilde{E} + \nabla u$$

with a divergence-free field $\tilde{E}$, and a gradient field ∇u. Note that this is very similar to the classical (A,φ)-formulation with Coulomb gauge, cf., e.g., [DL00a, I.A, §4, Section 3], where A is a divergence-free magnetic vector potential and φ a scalar function with

$$E = -\partial_t(A + \nabla\varphi).$$

The crucial point is to consider $\nabla u = \nabla u_{\tilde{E}}$ as a continuous linear function of $\tilde{E}$, cf. Lemma 3.5. This allows us to rewrite the eddy current problem (3.3)–(3.4) as a variational equation for $\tilde{E}$, which is uniformly coercive on the space of divergence-free functions and thus uniquely determines the field $\tilde{E}$. Note that $\tilde{E}$ does not solve the eddy current equation. Our new variational formulation enables us to study the asymptotic behavior of $\tilde{E}$ for $\sigma \to 0$. From this we can then deduce properties of the asymptotic behavior of any solution E of the eddy current problem.

For our results we need stronger assumptions on σ. Let $R > 0$ and let B_R denote the open ball with radius R centered at the origin. For the rest of this chapter, we assume that

$$\begin{aligned}\sigma \in L^\infty_R(\mathbb{R}^3) := \{\sigma \in L^\infty(\mathbb{R}^3) \,|\, \exists\, \Omega \subset B_R :\ &\sigma|_\Omega \in L^\infty_+(\Omega),\ \Omega = \cup_{i=1}^s \Omega_i,\ s \in \mathbb{N},\\ &\text{with bounded Lipschitz domains } \Omega_i,\ \overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset,\ i \neq j,\\ &\text{such that } \mathbb{R}^3 \setminus \overline{\Omega} \text{ is connected and } \overline{\Omega} = \operatorname{supp}\sigma\}. \qquad (3.12)\end{aligned}$$

Note that our continuity results do not depend on the lower bound of σ.

The case of $\sigma \equiv 0$ is treated separately.

3.5 Lemma *There is a continuous linear map*

$$L^2_\rho(\mathbb{R}^3)^3 \to H(\operatorname{curl} 0, \mathbb{R}^3) := \{E \in L^2(\mathbb{R}^3)^3 \mid \operatorname{curl} E = 0\}, \quad E \mapsto \nabla u_E,$$

with

$$\operatorname{div}(\sigma(E + \nabla u_E)) = 0 \quad \text{in } \mathbb{R}^3, \qquad (3.13)$$

and which extends (by setting $\nabla u_E(t) := \nabla u_{E(t)}$ for $t \in]0, T[$ a.e.) to a continuous linear map

$$L^2(0, T, L^2_\rho(\mathbb{R}^3)^3) \to L^2(0, T, H(\operatorname{curl} 0, \mathbb{R}^3)), \quad E \mapsto \nabla u_E,$$

for which $E \in H^1(0, T, L^2_\rho(\mathbb{R}^3)^3)$ implies

$$\nabla u_E \in H^1(0, T, H(\operatorname{curl} 0, \mathbb{R}^3)) \quad \text{and} \quad (\nabla u_E)^\cdot = \nabla u_{\dot{E}}.$$

Proof Let $E \in L^2_\rho(\mathbb{R}^3)^3$. Due to Poincare's inequality (cf., e.g., [DL00b, IV, §7, Prop. 2]), the fact, that σ is positively bounded from below on Ω, and Lax-Milgram's Theorem (cf., e.g., [RR04, §8, Theorem 8.14]), there exists a unique $u_E \in H^1_\square(\Omega)$ that solves

$$\int_\Omega \sigma \nabla u \cdot \nabla v \,\mathrm{d}x = -\int_\Omega \sigma E \cdot \nabla v \,\mathrm{d}x \quad \text{for all } v \in H^1(\Omega). \qquad (3.14)$$

Here, $H^1_\square(\Omega) := \left\{ v \in H^1(\Omega) \mid \int_{\Omega_i} v \,\mathrm{d}x = 0,\, i = 1, \ldots, s \right\}$. Furthermore, u_E depends continuously on $E|_\Omega \in L^2(\Omega)^3$.

We extend u_E to an element of $W^1(\mathbb{R}^3)$ by solving $\Delta u = 0$ on $\mathbb{R}^3 \setminus \overline{\Omega}$ with $u|_{\partial\Omega} = u_E|_{\partial\Omega}$ for $u \in W^1(\mathbb{R}^3 \setminus \overline{\Omega})$. Again, Lax-Milgram's Theorem provides a unique solution, which depends continuously on $u_E|_{\partial\Omega}$ and thus on E.

Let u_E, again, denote its extension. Then, $u_E \in W^1(\mathbb{R}^3)$, and the mapping $E \mapsto \nabla u_E$ is well-defined, linear and continuous with a continuity constant that depends on the lower and upper bounds of σ. Moreover, (3.13) is fulfilled.

The remaining assertions follow from standard time regularity arguments, cf., e.g., the proof of Lemma 3.11a), below. □

For the rest of this paper, let ∇u_E denote the image of E under this mapping. Note that there are different possibilities to construct this map, but $\sqrt{\sigma}\nabla u_E$ is uniquely determined by the condition (3.13). Moreover, it holds that

$$\|\sqrt{\sigma}\nabla u_E\|_{L^2(\mathbb{R}^3)^3} \leq \|\sqrt{\sigma}E\|_{L^2(\mathbb{R}^3)^3}, \tag{3.15}$$

and, obviously, for all $E \in W^1(\mathbb{R}^3)^3$, we have $E + \nabla u_E \in W(\mathrm{curl})$.

The fact that the curl of a solution is unique, but not the solution itself, leads to the idea to work with spaces where $\|\operatorname{curl} \cdot\|_{L^2(\mathbb{R}^3)^3}$ defines a norm. Therefore, we recall the Hilbert space

$$W^1_\diamondsuit := \{E \in W^1(\mathbb{R}^3)^3 \mid \operatorname{div} E = 0\}, \quad \|\cdot\|_{W^1_\diamondsuit} := \|\operatorname{curl} \cdot\|_{L^2(\mathbb{R}^3)^3}.$$

We define the bilinear form a by

$$a : L^2(0,T,W^1(\mathbb{R}^3)^3) \times H^1(0,T,W^1(\mathbb{R}^3)^3) \to \mathbb{R}$$
$$a(E,\Phi) := -\int_0^T \int_{\mathbb{R}^3} \sigma(E + \nabla u_E) \cdot \dot{\Phi} \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \,\mathrm{d}x \,\mathrm{d}t, \tag{3.16}$$

and, motivated by Lemma 3.4 d), the linear form $l : H^1(0,T,W^1(\mathbb{R}^3)^3) \to \mathbb{R}$:

$$l(\Phi) := -\int_0^T \langle J_t, \Phi \rangle \,\mathrm{d}t + \int_{\mathbb{R}^3} \sigma E^0 \cdot \Phi(0) \,\mathrm{d}x.$$

Now we can state the main result of this section. Let

$$H^1_{T0}(0,T,W^1_\diamondsuit) := \{\Psi \in H^1(0,T,W^1_\diamondsuit) \mid \Psi(T) = 0\}.$$

3.6 Theorem (Unified variational formulation)

a) *If* $\tilde{E} \in L^2(0,T,W^1_\diamondsuit)$ *solves*

$$a(\tilde{E},\Phi) = l(\Phi) \quad \text{for all } \Phi \in H^1_{T0}(0,T,W^1_\diamondsuit), \tag{3.17}$$

then $\tilde{E} + \nabla u_{\tilde{E}} \in L^2(0,T,W(\mathrm{curl}))$ *solves* (3.3)–(3.4).

$a|_{H^1_{T0}(0,T,W^1_\diamondsuit)^2}$ *is uniformly coercive with respect to* $\|\cdot\|_{L^2(0,T,W^1_\diamondsuit)}$*:*

$$a(\Phi,\Phi) \geq \frac{1}{\|\mu\|_\infty}\|\Phi\|^2_{L^2(0,T,W^1_\diamondsuit)} \quad \text{for all } \Phi \in H^1_{T0}(0,T,W^1_\diamondsuit).$$

b) *There is a unique solution* $\tilde{E} \in L^2(0,T,W^1_\diamondsuit)$ *of* (3.17)*.* $\tilde{E}$ *depends continuously on* J_t *and* $\sqrt{\sigma}E^0$*:*

$$\|\tilde{E}\|_{L^2(0,T,W^1_\diamondsuit)} \leq \sqrt{2}\max(\|\mu\|_\infty, 2)\max(\sqrt{5}\|J_t\|_{L^2(0,T,W(\mathrm{curl})')}, \|\sqrt{\sigma}E^0\|_{L^2(\mathbb{R}^3)^3}). \tag{3.18}$$

$\tilde{E} + \nabla u_{\tilde{E}}$ *solves the eddy current equation* (3.3) *and* (3.4) *and any other solution* $E \in L^2(0,T,W(\mathrm{curl}))$ *of* (3.3)–(3.4) *fulfills*

$$\mathrm{curl}\, E = \mathrm{curl}\, \tilde{E}, \quad \sqrt{\sigma}E = \sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E}}). \tag{3.19}$$

$\mathrm{curl}\, E$ *and* $\sqrt{\sigma}E$ *depend continuously on* J_t *and* $\sqrt{\sigma}E^0$*:*

$$\|\mathrm{curl}\, E\|_{L^2(\mathbb{R}^3_T)^3} \leq \sqrt{2}\max(\|\mu\|_\infty, 2)\max(\sqrt{5}\|J_t\|_{L^2(0,T,W(\mathrm{curl})')}, \|\sqrt{\sigma}E^0\|_{L^2(\mathbb{R}^3)^3})$$

$$\|\sqrt{\sigma}E\|_{L^2(\mathbb{R}^3_T)^3} \leq 4\sqrt{1+R^2}\|\sqrt{\sigma}\|_\infty\|\mathrm{curl}\, E\|_{L^2(\mathbb{R}^3_T)^3}.$$

If σ equals zero, we have the following result:

3.7 Theorem *For* $\sigma \equiv 0$, $E \in L^2(0,T,W^1_\diamondsuit)$ *is a solution of* (3.3) *if and only if* E *solves*

$$a_0(E,F) = l_0(F) \quad \text{for all } F \in L^2(0,T,W^1_\diamondsuit), \tag{3.20}$$

where a_0 *and* l_0 *denote* $a(\cdot,\cdot)$ *and* $l(\cdot)$ *with* $\sigma \equiv 0$*. There exists a unique solution* $E \in L^2(0,T,W^1_\diamondsuit)$ *and this solution depends continuously on* J_t*:*

$$\|E\|_{L^2(0,T,W^1_\diamondsuit)} \leq \sqrt{5}\|\mu\|_\infty\|J_t\|_{L^2(0,T,W(\mathrm{curl})')}.$$

The proofs can be found in the following subsection.

3.8 Corollary *Let* $(\sigma_n)_{n\in\mathbb{N}} \subset L^\infty_R(\mathbb{R}^3)$ *be a bounded sequence and* $\tilde{E}_n$, $n \in \mathbb{N}$, *be the corresponding unique solutions of* (3.17)*. Then the sequences*

$$(\tilde{E}_n)_{n\in\mathbb{N}} \subset L^2(0,T,W^1_\diamondsuit) \quad \text{and} \quad (\sqrt{\sigma_n}\tilde{E}_n)_{n\in\mathbb{N}},\ (\sqrt{\sigma_n}\nabla u_{\tilde{E}_n})_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^3_T)^3$$

are bounded. The bounds depend on the bound of $(\sigma_n)_{n\in\mathbb{N}}$*.*

In particular, for any sequence $(E_n)_{n\in\mathbb{N}} \subset L^2(0,T,W(\mathrm{curl}))$ *of corresponding solutions of* (3.3)–(3.4) *the sequences*

$$(\mathrm{curl}\, E_n)_{n\in\mathbb{N}},\ (\sqrt{\sigma}_n E_n)_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^3_T)^3$$

are bounded.

3.3.1 Solution theory

To show the first part of Theorem 3.6a), we use of the following simple decomposition.

3.9 Lemma

a) *Every $\Phi \in \mathcal{D}(\mathbb{R}^3)^3$ can be written as*

$$\Phi = \Psi + \nabla\varphi,$$

with $\Psi \in W^1_\diamondsuit$, $\varphi \in W^1(\mathbb{R}^3)$, and $\nabla\varphi \in W^1(\mathbb{R}^3)^3$.

b) *Every $\Phi \in \mathcal{D}(\mathbb{R}^3 \times [0,T[)^3$ can be written as*

$$\Phi = \Psi + \nabla\varphi,$$

with $\Psi \in H^1_{T0}(0,T,W^1_\diamondsuit)$, $\varphi \in H^1(0,T,W^1(\mathbb{R}^3))$, $\nabla\varphi \in H^1(0,T,W^1(\mathbb{R}^3)^3)$, and $\nabla\varphi(T) = 0$.

Proof Let $\Phi \in \mathcal{D}(\mathbb{R}^3)^3$. Then Lax-Milgram's Theorem yields a unique solution $\varphi \in W^1(\mathbb{R}^3)$ of

$$\Delta\varphi = \operatorname{div}\Phi \quad \text{in } \mathbb{R}^3.$$

By standard regularity results $\varphi \in C^\infty(\mathbb{R}^3)$. For a centered ball $B \subset \mathbb{R}^3$ containing the support of Φ, φ solves the exterior Dirichlet problem

$$\Delta\varphi = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B}, \quad \varphi|_{\partial B} \in H^{3/2}(\partial B)$$

so that it follows from, e.g., [Néd01, Theorem 2.5.1] that $\nabla\varphi \in W^1(\mathbb{R}^3 \setminus \overline{B})^3$, and hence $\nabla\varphi \in W^1(\mathbb{R}^3)^3$. With $\Psi := \Phi - \nabla\varphi \in W^1_\diamondsuit$ we obtain assertion a).

Assertion b) follows from standard time regularity arguments, cf., e.g., the proof of Lemma 3.11a), below. □

We prove the existence result in Theorem 3.6b) using the Lions-Lax-Milgram Theorem.

3.10 Lemma (Lions-Lax-Milgram Theorem) *Let $\mathcal{H}$ be a Hilbert space and V be a normed (not necessarily complete) vector space. Let $a : \mathcal{H} \times V \to \mathbb{R}$ be a bilinear form satisfying the following properties:*

a) *For every $\Phi \in V$, the linear form $E \mapsto a(E, \Phi)$ is continuous on $\mathcal{H}$.*

b) *There exists $\alpha > 0$ such that*

$$\inf_{\|\Phi\|_V = 1} \sup_{\|E\|_{\mathcal{H}} \leq 1} |a(E, \Phi)| \geq \frac{1}{\alpha}.$$

Then for each continuous linear form $l \in V'$, there exists $E_l \in \mathcal{H}$ such that

$$a(E_l, \Phi) = \langle l, \Phi \rangle \text{ for all } \Phi \in V \quad \text{and} \quad \|E_l\|_{\mathcal{H}} \leq \alpha \|l\|_{V'}.$$

The proof of Lemma 3.10 can be found, for example, in [Sho97, §3, Theorem 2.1 and Corollary 2.1].

Proof of Theorem 3.6

a) It is obvious, that for gradient fields $\nabla\varphi \in H^1(0, T, W^1(\mathbb{R}^3)^3)$ with $\varphi \in H^1(0, T, W^1(\mathbb{R}^3))$, $a(\cdot, \nabla\varphi)$ as well as $l(\nabla\varphi)$ vanish. (For the latter, recall that $\operatorname{div} J_t = 0$ and $\operatorname{div}(\sigma E^0) = 0$.) Hence, it follows from the decomposition in Lemma 3.9, and from the linearity of a and l, that (for any $\tilde{E} \in L^2(0, T, W^1_\diamondsuit)$)

$$a(\tilde{E}, \Phi) = l(\Phi)$$

holds for all $\Phi \in \mathcal{D}(\mathbb{R}^3 \times [0, T[)^3$, if it holds for all $\Phi \in H^1_{T0}(0, T, W^1_\diamondsuit)$. Lemma 3.4 yields the first assertion.

For $\Phi \in H^1_{T0}(0, T, W^1_\diamondsuit)$, Lemma 3.5 and the integration by parts formula (3.7) yield that

$$\begin{aligned} a(\Phi, \Phi) &= -\int_0^T \int_{\mathbb{R}^3} \sigma(\Phi + \nabla u_\Phi) \cdot \dot{\Phi} \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} |\operatorname{curl} \Phi|^2 \, dx \, dt \\ &\geq \frac{1}{2} \|\sqrt{\sigma}(\Phi + \nabla u_\Phi)(0)\|^2_{L^2(\mathbb{R}^3)^3} + \frac{1}{\|\mu\|_\infty} \|\Phi\|^2_{L^2(0,T,W^1_\diamondsuit)} \end{aligned} \tag{3.21}$$

and thus the second assertion.

b) We apply the Lions-Lax-Milgram Theorem. We use the Hilbert space $\mathcal{H} := L^2(0, T, W^1_\diamondsuit)$ and equip its subspace $V := H^1_{T0}(0, T, W^1_\diamondsuit)$ with the norm

$$\|\Phi\|^2_V := \|\Phi\|^2_{L^2(0,T,W^1_\diamondsuit)} + \|\sqrt{\sigma}(\Phi + \nabla u_\Phi)(0)\|^2_{L^2(\mathbb{R}^3)^3}.$$

Then equation (3.21) implies that

$$\inf_{\|\Phi\|_V = 1} \sup_{\|E\|_{\mathcal{H}} \leq 1} |a(E, \Phi)| \geq \inf_{\|\Phi\|_V = 1} |a(\Phi, \Phi)| \geq \frac{1}{\max(\|\mu\|_\infty, 2)}.$$

Given $\Phi \in V$ we set $C := \max(\|\Phi\|_{L^2(0,T,W^1_\diamondsuit)}, \|\dot{\Phi}\|_{L^2(0,T,L^2(B_R)^3)})$. Then it follows from (3.15) and $\mu \in L^\infty_+(\mathbb{R}^3)$, that for all $E \in \mathcal{H}$

$$\begin{aligned} |a(E, \Phi)| &= \left| \int_0^T \int_{\mathbb{R}^3} \left[-\sigma(E + \nabla u_E) \cdot \dot{\Phi} + \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \right] dx \, dt \right| \\ &\leq C \left[2\|\sqrt{\sigma}\|_\infty \|\sqrt{\sigma} E\|_{L^2(\mathbb{R}^3_T)^3} + \frac{1}{\inf \mu} \|E\|_{L^2(0,T,W^1_\diamondsuit)} \right] \\ &\leq C \left[\|\sigma\|_\infty 2 \|E\|_{L^2(0,T,L^2(B_R)^3)} + \frac{1}{\inf \mu} \|E\|_{L^2(0,T,W^1_\diamondsuit)} \right] \\ &\leq C \left[\|\sigma\|_\infty 2\sqrt{1 + R^2} \|E\|_{L^2(0,T,L^2_\rho(\mathbb{R}^3)^3)} + \frac{1}{\inf \mu} \|E\|_{L^2(0,T,W^1_\diamondsuit)} \right]. \end{aligned}$$

Similarly to the proof of [DL00d, XI.B, §1, Lemma 1], it holds that

$$\|F\|_\rho \leq 2\|\nabla F\|_{L^2(\mathbb{R}^3)^3} = 2\|F\|_{W^1_\diamondsuit} \quad \text{for all } F \in W^1_\diamondsuit, \tag{3.22}$$

and thus

$$|a(E,\Phi)| \leq C\left[4\|\sigma\|_\infty\sqrt{1+R^2} + \frac{1}{\inf\mu}\right]\|E\|_{\mathcal{H}}.$$

Hence, for fixed $\Phi \in V$, $a(\cdot,\Phi)$ is continuous on $\mathcal{H}$.

Equation (3.22) also yields

$$\|F\|^2_{W(\mathrm{curl})} = \|F\|^2_\rho + \|\operatorname{curl} F\|^2_{L^2(\mathbb{R}^3)^3} \leq 5\|F\|^2_{W^1_\diamondsuit} \quad \text{for all } F \in W^1_\diamondsuit, \tag{3.23}$$

so that we obtain for all $\Phi \in V$,

$$\begin{aligned}
|l(\Phi)| &= \left|-\int_0^T \langle J_t, \Phi\rangle \,\mathrm{d}t + \int_{\mathbb{R}^3} \sigma E^0 \cdot \Phi(0)\,\mathrm{d}x\right| \\
&\leq \|J_t\|_{L^2(0,T,W(\mathrm{curl})')}\|\Phi\|_{L^2(0,T,W(\mathrm{curl}))} \\
&\quad + \|\sqrt{\sigma}E^0\|_{L^2(\mathbb{R}^3)^3}\|\sqrt{\sigma}(\Phi + \nabla u_\Phi)(0)\|_{L^2(\mathbb{R}^3)^3} \\
&\leq \sqrt{2}\max(\sqrt{5}\|J_t\|_{L^2(0,T,W(\mathrm{curl})')}, \|\sqrt{\sigma}E^0\|_{L^2(\mathbb{R}^3)^3})\|\Phi\|_V.
\end{aligned}$$

Hence, $l \in V'$ and

$$\|l\|_{V'} \leq \sqrt{2}\max(\sqrt{5}\|J_t\|_{L^2(0,T,W(\mathrm{curl})')}, \|\sqrt{\sigma}E^0\|_{L^2(\mathbb{R}^3)^3}).$$

Now, Lemma 3.10 yields the existence of an $\tilde{E} \in \mathcal{H} = L^2(0,T,W^1_\diamondsuit)$ that fulfills (3.17) and depends continuously on l, i.e.

$$\|\tilde{E}\|_{L^2(0,T,W^1_\diamondsuit)} \leq \sqrt{2}\max(\|\mu\|_\infty, 2)\max(\sqrt{5}\|J_t\|_{L^2(0,T,W(\mathrm{curl})')}, \|\sqrt{\sigma}E^0\|_{L^2(\mathbb{R}^3)^3}).$$

Part a) yields that $\tilde{E} + \nabla u_{\tilde{E}} \in L^2(0,T,W(\mathrm{curl}))$ is a solution of the eddy current problem (3.3)–(3.4).

To show uniqueness, let $\tilde{E}_1, \tilde{E}_2 \in L^2(0,T,W^1_\diamondsuit)$ be two solutions of (3.17). Then, $\tilde{E}_1 + \nabla u_{\tilde{E}_1}, \tilde{E}_2 + \nabla u_{\tilde{E}_2} \in L^2(0,T,W(\mathrm{curl}))$ both solve the eddy current equation (3.3) and (3.4). Now, Theorem 3.1c) implies

$$\operatorname{curl}\tilde{E}_1 = \operatorname{curl}(\tilde{E}_1 + \nabla u_{\tilde{E}_1}) = \operatorname{curl}(\tilde{E}_2 + \nabla u_{\tilde{E}_2}) = \operatorname{curl}\tilde{E}_2$$

and it follows, that

$$0 = \|\operatorname{curl}(\tilde{E}_1 - \tilde{E}_2)\|_{L^2(\mathbb{R}^3)^3} = \|\tilde{E}_1 - \tilde{E}_2\|_{W^1_\diamondsuit}.$$

The remaining assertions of b) follow similarly from Theorem 3.1c). □

Proof of Theorem 3.7 Theorem 3.7 follows from $\mu \in L^\infty_+(\mathbb{R}^3)$, (3.23), and Lax-Milgram's Theorem. □

3.3.2 On time regularity

We close this section by showing a result on time regularity of the solutions.

3.11 Lemma *Let $J_t \in H^1(0,T,W(\mathrm{curl})')$ and $E^0 \in W(\mathrm{curl})$ such that*

$$\mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\,E^0\right) = -J_t(0)$$

in addition to the general assumptions (3.2). *Let $\tilde{E} \in L^2(0,T,W^1_\diamondsuit)$ be the solution of* (3.17). *Then, the following holds:*

a) *$\tilde{E} \in H^1(0,T,W^1_\diamondsuit)$ and $\tilde{F} = (\tilde{E})^{\cdot}$ is the solution of*

$$a(\tilde{F},\Phi) = -\int_0^T \langle (J_t)^{\cdot},\Phi\rangle\,\mathrm{d}t \qquad \textit{for all } \Phi \in H^1_{T0}(0,T,W^1_\diamondsuit). \tag{3.24}$$

$F = \tilde{F} + \nabla u_{\tilde{F}} \in L^2(0,T,W(\mathrm{curl}))$ solves

$$\partial_t(\sigma F) + \mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\,F\right) = -(J_t)^{\cdot} \qquad \textit{in } \mathbb{R}^3\times]0,T[$$

with zero initial conditions.

b) *For any solution $E \in L^2(0,T,W(\mathrm{curl}))$ of the eddy current problem* (3.3)–(3.4) *we have that*

$$\begin{aligned} E|_\Omega &\in H^1(0,T,L^2(\Omega)^3), \quad (E|_\Omega)^{\cdot} = F|_\Omega,\\ \mathrm{curl}\,E &\in H^1(0,T,L^2(\mathbb{R}^3)^3), \quad (\mathrm{curl}\,E)^{\cdot} = \mathrm{curl}\,F = \mathrm{curl}\,\tilde{F}. \end{aligned}$$

Proof

a) Theorem 3.6 yields that (3.24) has a unique solution $\tilde{F} \in L^2(0,T,W^1_\diamondsuit)$, so it only remains to show that $\tilde{F} = (\tilde{E})^{\cdot}$, which, in turn, follows if

$$Z(t) = \int_0^t \tilde{F}(s)\,\mathrm{d}s + E^0 + \nabla v_{E^0} \in H^1(0,T,W^1_\diamondsuit)$$

solves (3.17). Here, $v_{E^0} \in W^1(\mathbb{R}^3)$ is the unique solution of

$$\Delta v_{E^0} = -\,\mathrm{div}\,E^0 \qquad \text{in } \mathbb{R}^3.$$

Let $\Phi \in H^1_{T0}(0,T,W^1_\diamondsuit)$. We define

$$\Psi(t) = \int_0^t \Phi(s)\,\mathrm{d}s - \int_0^T \Phi(s)\,\mathrm{d}s \in H^1_{T0}(0,T,W^1_\diamondsuit).$$

Note that the assumption $\operatorname{div}(\sigma E^0) = 0$ together with Lemma 3.5 implies that

$$\sigma \nabla u_{Z(0)} = \sigma \nabla u_{(E^0 + \nabla v_{E^0})} = -\sigma \nabla v_{E^0},$$

so that we obtain

$$\begin{aligned}
a(Z,\Phi) &= \int_0^T \int_{\mathbb{R}^3} \left(-\sigma (Z + \nabla u_Z) \cdot \dot{\Phi} + \frac{1}{\mu} \operatorname{curl} Z \cdot \operatorname{curl} \Phi \right) \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}^3} \left(\sigma (Z + \nabla u_Z)^{\cdot} \cdot \dot{\Psi} - \frac{1}{\mu} \operatorname{curl} \dot{Z} \cdot \operatorname{curl} \Psi \right) \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_{\mathbb{R}^3} \left(\sigma \left(Z(0) + \nabla u_{Z(0)} \right) \cdot \dot{\Psi}(0) - \frac{1}{\mu} \operatorname{curl} Z(0) \cdot \operatorname{curl} \Psi(0) \right) \mathrm{d}x \\
&= - a(\dot{Z}, \Psi) + \int_{\mathbb{R}^3} \left(\sigma E^0 \cdot \Phi(0) - \frac{1}{\mu} \operatorname{curl} E^0 \cdot \operatorname{curl} \Psi(0) \right) \mathrm{d}x \\
&= \int_0^T \langle (J_t)^{\cdot}, \Psi \rangle \, \mathrm{d}t + \int_{\mathbb{R}^3} \sigma E^0 \cdot \Phi(0) \, \mathrm{d}x + \langle J_t(0), \Psi(0) \rangle \\
&= - \int_0^T \langle J_t, \dot{\Psi} \rangle \, \mathrm{d}t + \int_{\mathbb{R}^3} \sigma E^0 \cdot \Phi(0) \, \mathrm{d}x \\
&= l(\Phi).
\end{aligned}$$

b) follows immediately from a) and Theorem 3.1c). □

The analogous assertion holds for $\sigma \equiv 0$:

3.12 Lemma *Let $\sigma \equiv 0$ and let $J_t \in H^1(0,T,W(\operatorname{curl})')$ in addition to the general assumptions* (3.2) *on J_t.*

If $\tilde{E} \in L^2(0,T,W^1_\diamondsuit)$ is the solution of (3.20)*, then $\tilde{E} \in H^1(0,T,W^1_\diamondsuit)$ and $F = (\tilde{E})^{\cdot}$ is the solution of*

$$\operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} F \right) = -(J_t)^{\cdot} \quad \text{in } \mathbb{R}^3 \times]0,T[.$$

The proof is analogously to the proof of Lemma 3.11a).

3.4 Sensitivity Analysis

In this section we keep E^0 and J_t fixed and analyze the solution(s) behavior if σ approaches zero. To this end, let $(\sigma_n)_{n\in\mathbb{N}} \subset L^\infty_R(\mathbb{R}^3)$ be a sequence such that

$$\lim_{n\to\infty} \sigma_n = 0 \quad \text{in } L^\infty(\mathbb{R}^3).$$

Corresponding to $(\sigma_n)_{n\in\mathbb{N}}$, let $(E_n)_{n\in\mathbb{N}} \subset L^2(0,T,W(\mathrm{curl}))$ denote any sequence of solutions of (3.3)–(3.4) and let $(\tilde{E}_n)_{n\in\mathbb{N}} \subset L^2(0,T,W^1_\diamondsuit)$ denote the sequence of unique solutions of (3.17). For $\sigma \equiv 0$, let $E \in L^2(0,T,W(\mathrm{curl}))$ denote any solution of (3.3) and let $\tilde{E} \in L^2(0,T,W^1_\diamondsuit)$ denote the solution of (3.20).

Our first result is that the solutions converge:

3.13 Theorem (Convergence) *It holds, that*

$$\mathrm{curl}\, E_n \to \mathrm{curl}\, E, \ \sqrt{\sigma_n} E_n \to 0 \ \text{in } L^2(\mathbb{R}^3_T)^3$$
$$\text{and} \ \ (\sigma_n E_n)^{\cdot} \to 0 \ \ \text{in } L^2(0,T,W(\mathrm{curl})').$$

Moreover we show that (under some regularity assumptions) the directional derivative of E with respect to σ exists and can be characterized in the following way:

3.14 Theorem (Linearization) *Let $J_t \in H^1(0,T,W(\mathrm{curl})')$, and $E^0 \in W(\mathrm{curl})$ such that*

$$\mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\, E^0\right) = -J_t(0)$$

in addition to our general assumptions (3.2) *on J_t and E^0. Let $d \in L^\infty_R(\mathbb{R}^3)$ and $h > 0$. Let $E_d \in H^1(0,T,W(\mathrm{curl}))$ be a solution of* (3.3) *with $\sigma \equiv 0$ that fulfills* $\mathrm{div}(dE_d) = 0$ *and $F \in L^2(0,T,W(\mathrm{curl}))$ be a solution of*

$$\mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\, F\right) = -d\dot{E}_d \quad \text{in } \mathbb{R}^3\times]0,T[.$$

Let $E_h \in L^2(0,T,W(\mathrm{curl}))$ be a solution of (3.3)–(3.4) *with $\sigma = hd$. Then*

$$\frac{1}{h}(\mathrm{curl}\, E_h - \mathrm{curl}\, E) \to \mathrm{curl}\, F \quad \text{in } L^2(\mathbb{R}^3_T)^3 \qquad (h \to 0^+).$$

Let us first comment on the existence of E_d and F. For instance we can choose $E_d = \tilde{E} + \nabla u_{\tilde{E}}$, where $\nabla u_{\tilde{E}}$ is the image of $\tilde{E}$ under the mapping defined in Lemma 3.5 with $\sigma = d$. Then the time regularity of E_d and the existence of F follow from Lemma 3.12, Lemma 3.5, and Theorem 3.7. Note that E_d, F, and also E_h are not unique. Theorem 3.14 holds for every choice of E_h, E_d and F.

The two theorems are proved in the following two subsections.

3.15 Remark More general meaningful initial conditions that obey $\mathrm{div}(\sigma_n E^0) = 0$ for every n can be obtained, for instance, by replacing the initial condition (3.4) by $\sqrt{\sigma_n} E_n(0) = \sqrt{\sigma_n}(E^0 + \nabla u_{E^0})$ for some fixed $E^0 \in L^2(\mathbb{R}^3)^3$. Here, ∇u_{E^0} is taken with respect to σ_n. The assertions of this section as well as Corollary 3.8 hold for this particular choice.

3.4.1 Convergence

Obviously, $\sqrt{\sigma_n} E^0 \to 0$ in $L^2(\mathbb{R}^3)^3$.

3.16 Lemma *It holds, that*

$$\tilde{E}_n \to \tilde{E} \quad \text{in } L^2(0,T,W^1_\diamondsuit), \quad \text{and} \quad \sqrt{\sigma_n}\tilde{E}_n,\, \sqrt{\sigma_n}\nabla u_{\tilde{E}_n} \to 0 \quad \text{in } L^2(\mathbb{R}^3_T)^3.$$

Proof First, we show that $\tilde{E}_n \rightharpoonup \tilde{E}$. To prove this it suffices to show that every subsequence of $(\tilde{E}_n)_{n\in\mathbb{N}}$ has a subsequence that converges weakly against $\tilde{E}$. From Corollary 3.8 we know that $(\tilde{E}_n)_{n\in\mathbb{N}} \subset L^2(0,T,W^1_\diamondsuit)$ is bounded. Using that $\operatorname{supp}\sigma_n \subset \overline{B_R}$ and Lemma 3.5 we obtain the second part of the assertion,

$$\sqrt{\sigma_n}\tilde{E}_n, \sqrt{\sigma_n}\nabla u_{\tilde{E}_n} \to 0 \quad \text{in } L^2(\mathbb{R}^3_T)^3.$$

Alaoglu's Theorem, cf., e.g., [RR04, Theorem 6.62], yields that every subsequence of $(\tilde{E}_n)_{n\in\mathbb{N}}$ contains a subsequence (that we still denote by $(\tilde{E}_n)_{n\in\mathbb{N}}$ for ease of notation) that converges weakly against some $\tilde{E}' \in L^2(0,T,W^1_\diamondsuit)$. We show that all these weak limits are identical to $\tilde{E}$:

$\tilde{E}_n \rightharpoonup \tilde{E}'$ in $L^2(0,T,W^1_\diamondsuit)$ implies that $\operatorname{curl}\tilde{E}_n \rightharpoonup \operatorname{curl}\tilde{E}'$ in $L^2(\mathbb{R}^3_T)^3$, so that for every $\Phi \in H^1_{T0}(0,T,W^1_\diamondsuit)$ the left hand side $a(\tilde{E}_n,\Phi)$ of (3.17) with $\sigma = \sigma_n$ converges against $a_0(\tilde{E}',\Phi)$. Clearly, the right hand side of (3.17) with $\sigma = \sigma_n$ converges against $l_0(\Phi)$. Hence, $\tilde{E}'$ solves (3.20) and thus uniqueness provides $\tilde{E} = \tilde{E}'$, and hence

$$\tilde{E}_n \rightharpoonup \tilde{E} \quad \text{in } L^2(0,T,W^1_\diamondsuit).$$

Since $\tilde{E}_n + \nabla u_{\tilde{E}_n}$ solves the eddy current problem (3.3)–(3.4) with $\sigma = \sigma_n$, we obtain using Lemma 3.4b)

$$\begin{aligned}
\|\mu^{-\frac{1}{2}}\operatorname{curl}\tilde{E}_n\|^2_{L^2(\mathbb{R}^3_T)^3} &= -\int_0^T \langle(\sigma_n(\tilde{E}_n + \nabla u_{\tilde{E}_n}))^\cdot, \tilde{E}_n + \nabla u_{\tilde{E}_n}\rangle\,\mathrm{d}t - \int_0^T \langle J_t, \tilde{E}_n\rangle\,\mathrm{d}t\\
&\le \frac{1}{2}\|\sqrt{\sigma_n}E^0\|^2_{L^2(\mathbb{R}^3)^3} - \int_0^T \langle J_t, \tilde{E}_n\rangle\,\mathrm{d}t\\
&= \frac{1}{2}\|\sqrt{\sigma_n}E^0\|^2_{L^2(\mathbb{R}^3)^3} + \int_0^T\int_{\mathbb{R}^3}\frac{1}{\mu}\operatorname{curl}\tilde{E}\cdot\operatorname{curl}\tilde{E}_n\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

and hence

$$\limsup_{n\to\infty}\|\mu^{-\frac{1}{2}}\operatorname{curl}\tilde{E}_n\|_{L^2(\mathbb{R}^3_T)^3} \le \|\mu^{-\frac{1}{2}}\operatorname{curl}\tilde{E}\|_{L^2(\mathbb{R}^3_T)^3}$$

which, together with $\tilde{E}_n \rightharpoonup \tilde{E}$, yields $\tilde{E}_n \to \tilde{E}$. □

Proof of Theorem 3.13 For any solutions E_n, respectively, E of (3.3)–(3.4) with $\sigma = \sigma_n$, respectively, $\sigma \equiv 0$, we have that

$$\sqrt{\sigma_n} E_n = \sqrt{\sigma_n}(\tilde{E}_n + \nabla u_{\tilde{E}_n}), \quad \operatorname{curl} E_n = \operatorname{curl} \tilde{E}_n, \quad \text{and} \quad \operatorname{curl} E = \operatorname{curl} \tilde{E},$$

so that Lemma 3.16 provides $\operatorname{curl} E_n \to \operatorname{curl} E$ and $\sqrt{\sigma_n} E_n \to 0$.

From the explicit form (3.9) of $(\sigma_n E_n)^\cdot$ given in Lemma 3.3, we obtain for all $F \in L^2(0, T, W(\operatorname{curl}))$

$$\left| \int_0^T \langle (\sigma_n E_n)^\cdot, F \rangle \, \mathrm{d}t \right| \leq \frac{1}{\inf \mu} \| \operatorname{curl}(E - E_n) \|_{L^2(\mathbb{R}^3_T)^3} \| \operatorname{curl} F \|_{L^2(\mathbb{R}^3_T)^3},$$

and hence $(\sigma_n E_n)^\cdot \to 0$. □

3.4.2 Linearization results

To characterize the directional derivative of E with respect to σ, some more time regularity is needed. To this end, we assume in addition to (3.2), that $J_t \in H^1(0, T, W(\operatorname{curl})')$, and $E^0 \in W(\operatorname{curl})$ such that

$$\operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} E^0 \right) = -J_t(0).$$

3.17 Lemma *For every $n \in \mathbb{N}$, $E_n - E \in L^2(0, T, W(\operatorname{curl}))$ solves*

$$\operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl}(E_n - E) \right) = -\sigma_n \dot{E}_n \quad \text{in } \mathbb{R}^3 \times]0, T[.$$

Moreover, there is a constant C so that

$$\limsup_{n \to \infty} \frac{\| \tilde{E}_n - \tilde{E} \|_{L^2(0,T,W^1_\diamond)}}{\| \sigma_n \|_\infty} \leq C.$$

Proof From Lemma 3.11 and Lemma 3.12 we know that the time derivatives of $\tilde{E}_n$, $\tilde{E}$, $u_{\tilde{E}_n}$ and $E_n|_{\Omega_n}$ exist. Then, it is easily checked, that $\tilde{E}_n - \tilde{E}$ solves

$$a_0(\tilde{E}_n - \tilde{E}, \Phi) = - \int_0^T \int_{\mathbb{R}^3} \sigma_n(\dot{\tilde{E}}_n + \nabla u_{\dot{\tilde{E}}_n}) \cdot \Phi \, \mathrm{d}x \, \mathrm{d}t$$

for all $\Phi \in H^1_{T0}(0, T, W^1_\diamond)$ and thus also for all $\Phi \in L^2(0, T, W^1_\diamond)$. So the first assertion follows from the identity $(\tilde{E}_n + \nabla u_{\tilde{E}_n})|_{\Omega_n} = E_n|_{\Omega_n}$.

From Theorem 3.7 and (3.15) we now obtain a constant $C' > 0$ (depending on μ and R) so that

$$\begin{aligned} \| \tilde{E}_n - \tilde{E} \|_{L^2(0,T,W^1_\diamond)} &\leq C' \| \sqrt{\sigma_n} \|_\infty \| \sqrt{\sigma_n} \dot{E}_n \|_{L^2(\mathbb{R}^3_T)^3} \\ &\leq 2C' \| \sigma_n \|_\infty \| (\tilde{E}_n)^\cdot \|_{L^2(B_R \times (0,T))^3} \end{aligned}$$

As every $(\tilde{E}_n)^\cdot$ solves (3.24) with $\sigma = \sigma_n$, Corollary 3.8 yields that $((\tilde{E}_n)^\cdot)_{n \in \mathbb{N}}$ is a bounded sequence in $L^2(0, T, W^1_\diamond)$ and thus the second assertion follows. □

3.18 Lemma *Let $d \in L^\infty_R(\mathbb{R}^3)$, and $\tilde{E}_d = \tilde{E} + \nabla u_{\tilde{E}}$, where $\nabla u_{\tilde{E}}$ is the image of $\tilde{E}$ under the mapping defined in Lemma 3.5 with $\sigma = d$. Let $\tilde{F} \in L^2(0, T, W^1_\diamondsuit)$ be the solution of*

$$a_0(\tilde{F}, \Phi) = -\int_0^T \int_{\mathbb{R}^3} d(\tilde{E}_d)^\cdot \cdot \Phi \,\mathrm{d}x \,\mathrm{d}t \quad \text{for all } \Phi \in L^2(0, T, W^1_\diamondsuit). \tag{3.25}$$

Furthermore, for $h > 0$ let $\tilde{E}_h \in L^2(0, T, W^1_\diamondsuit)$ be the solution of (3.17) corresponding to $\sigma = hd$. Then for $h \to 0^+$

$$\frac{1}{h}(\tilde{E}_h - \tilde{E}) \to \tilde{F} \quad \text{in } L^2(0, T, W^1_\diamondsuit).$$

Proof Lemma 3.11, Lemma 3.12 and Lemma 3.16 yield that $(\tilde{E}_h)^\cdot \to (\tilde{E})^\cdot$ in $L^2(0, T, W^1_\diamondsuit)$. The mapping defined in Lemma 3.5 does not change if we take $\sigma = d$ instead of $\sigma = hd$. Hence, as d is fixed, the continuity of this mapping implies that $\nabla u_{\dot{\tilde{E}}_h} \to \nabla u_{\dot{\tilde{E}}}$ in $L^2(\mathbb{R}^3_T)^3$.

From Lemma 3.17 we obtain that for all $\Phi \in L^2(0, T, W^1_\diamondsuit)$

$$a_0\left(\frac{1}{h}(\tilde{E}_h - \tilde{E}) - \tilde{F}, \Phi\right) = -\int_0^T \int_{\mathbb{R}^3} d(\tilde{E}_h + \nabla u_{\tilde{E}_h} - \tilde{E} - \nabla u_{\tilde{E}})^\cdot \cdot \Phi \,\mathrm{d}x \,\mathrm{d}t$$

The assertion now follows from setting $\Phi := \frac{1}{h}(\tilde{E}_h - \tilde{E}) - \tilde{F}$ and using the coercivity of a_0. □

Proof of Theorem 3.14 Let $E_d \in H^1(0, T, W(\mathrm{curl}))$ be a solution of (3.3) with $\sigma \equiv 0$ that fulfills $\mathrm{div}(dE_d) = 0$ and $F \in L^2(0, T, W(\mathrm{curl}))$ be a solution of

$$\mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\, F\right) = -d\dot{E}_d \quad \text{in } \mathbb{R}^3 \times]0, T[.$$

Let $\tilde{E}_d$ and $\tilde{F}$ be as in Lemma 3.18.

Since both, E_d and $\tilde{E}_d$, solve (3.3) with $\sigma \equiv 0$, we have $\mathrm{curl}\, E_d = \mathrm{curl}\, \tilde{E}_d$. Hence, for $t \in]0, T[$ a.e., using the Poincaré Lemma on B_R, cf., e.g., [DL00c, IX.A, §1, Lemma 4], we obtain a $p \in H^1(B_R)$ with $(E_d(t) - \tilde{E}_d(t))|_{B_R} = \nabla p$. Now $\mathrm{div}(d(E_d - \tilde{E}_d)) = 0$ implies that

$$\int_{\mathbb{R}^3} d\nabla p \cdot \nabla \varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^3),$$

so that $\sqrt{d}\nabla p = 0$. It follows that $d(E_d)^\cdot = d(\tilde{E}_d)^\cdot$ and hence $\mathrm{curl}\, F = \mathrm{curl}\, \tilde{F}$. Since also $\mathrm{curl}\, E_h = \mathrm{curl}\, \tilde{E}_h$, and $\mathrm{curl}\, E = \mathrm{curl}\, \tilde{E}$, the assertion follows from Lemma 3.18. □

Chapter 4

Unique shape detection in transient eddy current problems

The subject of this chapter is the inverse problem of locating conductors surrounded by a non-conducting medium from electromagnetic measurements, i.e. from knowledge of the operator mapping the excitation currents to measurements of the corresponding electric fields. We show that the conductors are uniquely determined by the measurements, and give an explicit criterion to decide whether a given point is inside the conducting domain or not.

The Sections 4.3–4.6 and 4.8 are the Sections 4–8 of [AH13b] up to minor changes.

4.1 Introduction

Inferring information about the electromagnetic properties from knowledge of the excitation currents and the corresponding measured fields in eddy current applications corresponds to the inverse problem of reconstructing the coefficients σ and μ in

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -\partial_t J. \tag{4.1}$$

from knowledge of the excitations $\partial_t J$ and a part of the solutions E of (4.1).

Various inverse eddy current problems have been studied in the engineering literature. Reconstruction of electromagnetic properties in time harmonic eddy current problems is the aim of magnetic induction tomography (MIT) which is used for medical and industrial imaging (see for example Griffiths in [Gri01] or Scharfetter et al. in [SCR03] and the references therein). An overview about non-destructive evaluation is given by Auld and Moulder in [AM99], see also Krause et al. in [KPZ03] and Tian et al. in [TSTR05]. Inverse problems in transient eddy current problems are considered, for instance, by Fu and Bowler in [FB06] and by Cheng and Komura in [CK08].

In the mathematical literature, inverse problems for time harmonic eddy current problems are treated, for instance, by Ammari et al. in [ACC+14], Alonso Rodríguez et al. in [ARCnV12], Wei et al. in [WMS12] and Soleimani in [Sol07]. To the knowledge of the author, no mathematical results exist on inverse problems for transient eddy current problems.

We now concentrate on detecting the position and the shape of conductors surrounded by a non-conducting medium in transient eddy current problems. Mathematically this corresponds to detecting the support of the conductivity coefficient σ in (4.1).

For the modelling of the measurements we follow Harrach et al. in [GHK+05] and in [GHS08]: Transient excitation currents through an idealized measurement instrument given by a two-dimensional sheet S (representing infinitely many infinitesimal excitation coils and measurement coils) are used to generate the fields. Then, the induced voltages in sensing coils on S are detected. Mathematically, this is encoded in a measurement operator Λ, that maps I (the negative time-derivative of the transient excitation current J, i.e. $I := -\partial_t J$) on the electric field E that solves (4.1) restricted to S:

$$\Lambda : I \mapsto E|_S.$$

A proper definition of Λ is given in Section 4.3. The aim of this work is then to show that the conducting domains are uniquely determined by Λ and to propose a strategy for shape reconstruction.

A well-established non-iterative method for shape reconstruction is the *factorization method* invented by Kirsch in [Kir98] in the context of inverse scattering. Based on a factorization of the measurement operator, an explicit criterion is developed, which determines whether a given point is inside the domain of interest or not. The factorization method has been extended and widely used for shape detection in several inverse problems, see, for instance, Kirsch and Grinberg in [KG07] and the references therein. For an overview on the application in *electrical impedance tomography* see Brühl and Hanke in [BH03] and the recent work of Harrach [Har13]. In [Kir04], Kirsch applies this method to an inverse problem involving the time harmonic Maxwell system. In the context of land mine detection, the magnetostatic limit of Maxwell's equations is treated by Harrach et al. in [GHS08]. Results on the heat equation, a scalar parabolic-elliptic analog of the eddy current equation, can be found in Frühauf et al. [FGS07]. Another approach are *linear sampling methods* originated by Colton and Kirsch in [CK96]. Like the factorization method, a sufficient (but not necessary) condition on a point to be inside the domain of interest is produced.

In this chapter we show that both methods can be applied for shape detection in transient eddy current problems. First, we use the linear sampling method to detect a subset of the conducting domain. On top of that, considering diamagnetic materials, we show that the unknown domain is uniquely determined by the measurement operator Λ. Here, the key is to control Λ from above and from below with constraining operators which determine a subset and a superset of the sought domain, as proposed by Harrach in [Har13]. Then, an explicit criterion can be stated to determine whether a given point is inside or outside the domain. This criterion also serves as a base for non-iterative numerical reconstruction strategies. Despite the fact that we do not provide any factorization of Λ, we finally show that this criterion is equivalent to the one used in the factorization method. We also reformulate it in in terms of the *Picard criterion*. The latter has been used for numerical implementation of shape reconstruction algorithms in electrical impedance tomography and in three dimensional related problems, cf., e.g., Harrach et al. in [GHK$^+$05, GHS08] for numerical results. Analogously, we expect our criterion to serve as a base for non-iterative reconstruction algorithms for transient eddy current problems.

This chapter is organized as follows: Section 4.2 summarizes our variational solution theory from Sections 3.2 and 3.3 for the direct problem. The setting for the inverse problem and the definition of the measurement operator is provided in Section 4.3. In Section 4.4 we show that the linear sampling method can be applied to detect a subset of the conducting domain. Our main result is presented in Section 4.5: In case of diamagnetic materials, the conductor is uniquely located by the measurement operator. Here we also present the explicit criterion for detecting the conducting domain and show its equivalence to the factorization method. Section 4.6 contains the proof of our main result. Finally, in Section 4.7 we rewrite our criterion in terms of the Picard criterion. A conclusion can be found in Section 4.8.

4.2 The direct problem

This section briefly summarizes the most important results of Chapter 3 on the solution theory of the direct problem.

Throughout this chapter we assume for the conductivity σ, that there is some $\Omega \subset \mathbb{R}^3$ such that

$$\sigma|_\Omega \in L^\infty_+(\Omega).$$

For this Ω we assume that it is the finite union of smoothly bounded domains Ω_i with $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ if $i \neq j$, that $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected and that $\overline{\Omega} = \operatorname{supp} \sigma$. Let Γ denote the union of the boundaries of Ω_i and ν denote the outer normal unit vector on Γ. We call Ω the conductor.

The permeability $\mu \in L^\infty_+(\mathbb{R}^3)$ is assumed to be constant outside of Ω, for simplicity we assume

$$\mu|_{\mathbb{R}^3 \setminus \overline{\Omega}} \equiv 1.$$

We assume that we are given some right hand side $J_t \in L^2(0,T,W(\mathrm{curl})')$, that obeys $\operatorname{div} J_t = 0$ and initial values $\sqrt{\sigma}E^0$ with $E^0 \in L^2(\mathbb{R}^3)^3$, that fulfill $\operatorname{div}(\sigma E^0) = 0$.

Then, for $E \in L^2(0,T,W(\mathrm{curl}))$, the eddy current problem reads

$$\partial_t(\sigma(x)E(x,t)) + \operatorname{curl}\left(\frac{1}{\mu(x)}\operatorname{curl} E(x,t)\right) = -J_t(x,t) \qquad \text{in } \mathbb{R}^3\times]0,T[, \tag{4.2}$$

$$\sqrt{\sigma(x)}E(x,0) = \sqrt{\sigma(x)}E^0(x) \quad \text{in } \mathbb{R}^3. \tag{4.3}$$

Recall the mapping

$$L^2_\rho(\mathbb{R}^3)^3 \to H(\mathrm{curl}\,0,\mathbb{R}^3) := \{E \in L^2(\mathbb{R}^3)^3 \mid \operatorname{curl} E = 0\}, \quad E \mapsto \nabla u_E,$$

with $\operatorname{div}(\sigma(E+\nabla u_E)) = 0$ in $\mathbb{R}^3$ from Lemma 3.5, the bilinear form

$$a_\sigma : L^2(0,T,W^1(\mathbb{R}^3)^3) \times H^1(0,T,W^1(\mathbb{R}^3)^3) \to \mathbb{R},$$
$$a_\sigma(E,\Phi) := -\int_0^T\int_{\mathbb{R}^3} \sigma(E+\nabla u_E)\cdot\dot{\Phi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^3}\frac{1}{\mu}\operatorname{curl} E\cdot\operatorname{curl}\Phi\,\mathrm{d}x\,\mathrm{d}t$$

and the Hilbert space

$$W^1_\diamondsuit := \{E \in W^1(\mathbb{R}^3)^3 \mid \operatorname{div} E = 0\}.$$

Then, the solution theory on the eddy current problem is summarized in the following theorem.

4.1 Theorem *(cf. Theorem 3.6)*

a) *If $E \in L^2(0,T,W^1_\diamondsuit)$ solves*

$$a_\sigma(E,\Phi) = -\int_0^T \langle J_t,\Phi\rangle\,\mathrm{d}t + \int_{\mathbb{R}^3}\sigma E^0\cdot\Phi(0)\,\mathrm{d}x \quad \textit{for all } \Phi \in H^1_{T0}(0,T,W^1_\diamondsuit), \tag{4.4}$$

then $E + \nabla u_E \in L^2(0,T,W(\mathrm{curl}))$ solves (4.2)–(4.3), where

$$H^1_{T0}(0,T,W^1_\diamondsuit) := \{\Psi \in H^1(0,T,W^1_\diamondsuit) \mid \Psi(T) = 0\}.$$

b) *There is a unique solution $E \in L^2(0,T,W^1_\diamondsuit)$ of (4.4). E depends continuously on J_t and $\sqrt{\sigma}E^0$. $E+\nabla u_E$ solves the eddy current problem (4.2)–(4.3) and any other solution $F \in L^2(0,T,W(\mathrm{curl}))$ of (4.2)–(4.3) fulfills*

$$\operatorname{curl} F = \operatorname{curl} E, \quad \sqrt{\sigma}F = \sqrt{\sigma}(E+\nabla u_E).$$

$\operatorname{curl} F$ *and* $\sqrt{\sigma}F$ *depend continuously on* J_t *and* $\sqrt{\sigma}E^0$.

We also consider the case $\sigma \equiv 0$ and $\mu \equiv 1$, that we call the *reference problem*. This case corresponds to the eddy current problem without any conducting medium. Then, the solution theory on the reference problem reduces to

4.2 Theorem *(cf. Theorem 3.7) Let* $E \in L^2(0,T,W(\mathrm{curl}))$.

a) *The reference problem reads*

$$\mathrm{curl}\,\mathrm{curl}\,E(x,t) = -J_t(x,t) \qquad \text{in } \mathbb{R}^3 \times]0,T[. \tag{4.5}$$

b) *E solves* (4.5) *if and only if E solves*

$$\begin{aligned} a_0(E,\Phi) &:= \int_0^T \int_{\mathbb{R}^3} \mathrm{curl}\,E \cdot \mathrm{curl}\,\Phi \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T \langle J_t, \Phi \rangle \,\mathrm{d}t \quad \text{for all } \Phi \in L^2(0,T,W^1_\diamondsuit), \end{aligned} \tag{4.6}$$

where $a_0 : L^2(0,T,W(\mathrm{curl}))^2 \to \mathbb{R}$.

c) *There exists a unique solution* $E \in L^2(0,T,W^1_\diamondsuit)$ *of* (4.6) *and this solution depends continuously on* J_t. *Any other solution* $F \in L^2(0,T,W(\mathrm{curl}))$ *fulfills*

$$\mathrm{curl}\,F = \mathrm{curl}\,E$$

and $\mathrm{curl}\,F$ *depends continuously on* J_t.

4.3 Electromagnetic measurements

We now turn to the description of our idealized measurement instrument. As in, e.g., [GHK+05, GHS08], we assume that the electric field E is generated by transient surface currents on a two-dimensional sheet S. In this way we assume that we can apply every divergence-free tangential function I (that corresponds to $-J_t$) supported in S as excitation on the right hand side of (4.2). Our idealized measurement instrument also measures the tangential component of the electric field on S.

Mathematically, the setting is as follows. We assume that

$$S \subset \mathbb{R}^3_0 := \{(x_1,x_2,0)^T \in \mathbb{R}^3\}$$

is (as a subset of $\mathbb{R}^2$) a bounded Lipschitz domain. Let n be the outer normal unit vector on S, i.e. $n = (0,0,1)^T$. We assume that Ω is placed below S and that $\overline{\Omega} \cap \overline{S} = \emptyset$, i.e.

$$\overline{\Omega} \subset \{(x_1,x_2,x_3)^T \in \mathbb{R}^3 \,|\, x_3 < 0\}.$$

We consider the excitation I as an element of the space $L^2(0,T,TL^2_\diamondsuit(S))$. Here, the space $TL^2_\diamondsuit(S)$ denotes the subspace of the space $TL^2(S)$ of elements with vanishing divergence, where

$$TL^2(S) := \{u \in L^2(S)^3 \,|\, n \cdot u = 0\}$$

is the space of tangential functions. Using the continuous extension of the identification of an element $I \in TL^2(S)$ with the distribution

$$\Phi \mapsto \int_S I \cdot \Phi \, \mathrm{d}S = \int_S I \cdot ((n \times \Phi|_S) \times n) \, \mathrm{d}S \quad \text{for all } \Phi \in \mathcal{D}(\mathbb{R}^3)^3$$

to $W(\mathrm{curl})$, we consider the spaces $TL^2(S)$ and $TL^2_\diamondsuit(S)$ as subspaces of $W(\mathrm{curl})'$. Both, $TL^2(S)$ and $TL^2_\diamondsuit(S)$ are Hilbert spaces equipped with the usual $L^2(S)^3$-inner product. Hence, every $I \in L^2(0,T,TL^2_\diamondsuit(S))$ defines an element of the space $L^2(0,T,W(\mathrm{curl})')$ that satisfies $\operatorname{div} I = 0$. In this sense we can consider the surface current $I \in L^2(0,T,TL^2_\diamondsuit(S))$ as a source term for the eddy current equation (4.2), respectively, the reference problem (4.5). In the following, we do not distinguish between $I \in L^2(0,T,TL^2_\diamondsuit(S))$ and the corresponding element of $L^2(0,T,W(\mathrm{curl})')$ and still write the dual pairing as a $L^2(S)^3$-product.

To define the measurement operator we first remark, that the mapping

$$W^1(\mathbb{R}^3)^3 \to TL^2(S), \; E \mapsto \gamma_S E := (n \times E|_S) \times n$$

is linear and continuous. Moreover, let

$$N_S := \overline{\mathcal{R}\left(\gamma_S \nabla D(\mathbb{R}^3)\right)} \subset TL^2(S).$$

It can easily be verified, that $N_S \oplus^\perp TL^2_\diamondsuit(S) = TL^2(S)$ and

$$TL^2(S)/N_S \cong TL^2_\diamondsuit(S)'. \tag{4.7}$$

Together with the identification of $TL^2_\diamondsuit(S)$ with its dual, we consider the measurements as elements of $L^2(0,T,TL^2_\diamondsuit(S))$. This can be interpreted as measuring the electric field, such that it is adequately gauged to be divergence-free on S. Now, Theorems 4.1 and 4.2 yield the following linear continuous operators.

4.3 Definition (Measurement operator) We define the measurement operator

$$\Lambda := \Lambda_0 - \Lambda_\sigma : L^2(0,T,TL^2_\diamondsuit(S)) \to L^2(0,T,TL^2_\diamondsuit(S)).$$

Here, Λ_0 and Λ_σ are the mappings

$$\begin{aligned} \Lambda_0, \Lambda_\sigma : L^2(0,T,TL^2_\diamondsuit(S)) &\to L^2(0,T,TL^2_\diamondsuit(S)), \\ I &\mapsto \gamma_S E_0, \quad \text{respectively,} \quad \gamma_S E_\sigma, \end{aligned} \tag{4.8}$$

where $E_0, E_\sigma \in L^2(0,T,W^1_\diamondsuit)$ are the unique solutions of

$$a_0(E_0, F) = \int_0^T (\gamma_S F, I)_{L^2(S)^3} \, \mathrm{d}t \qquad \text{for all } F \in L^2(0,T,W^1_\diamondsuit), \tag{4.9}$$

$$a_\sigma(E_\sigma, F) = \int_0^T (\gamma_S F, I)_{L^2(S)^3} \, \mathrm{d}t \qquad \text{for all } F \in H^1_{T0}(0,T,W^1_\diamondsuit). \tag{4.10}$$

Note that if E_0 and E_σ solve (4.9) and (4.10), then they are the unique solutions of (4.6) and (4.4) with right hand side I. This means that $E_\sigma + \nabla u_{E_\sigma} \in L^2(0,T,W(\mathrm{curl}))$ solves (4.2) with right hand side I and zero initial condition, cf. Theorem 4.1 b). Especially, the above defined operators do not match the tangential value of the "real" electric field but just the tangential value of its divergence-free part.

Let us stress, that even if (4.2)–(4.3) does not determine the solution uniquely, in the measurement space, the measurements of different solutions still coincide. This is up to (4.7) and the fact, that, in a neighborhood of S, all solutions $E \in L^2(0,T,W(\mathrm{curl}))$ of (4.2)–(4.3) equal up to gradient fields. Hence, the evaluation of $\gamma_S E$ in $L^2(0,T,TL^2_\diamondsuit(S))$ is also well-defined, linear and continuous and defines the same element as $\gamma_S E_\sigma$. Therefore, we understand Λ as a gauged measurement operator, where $\gamma_S E_0$, $\gamma_S E_\sigma$ actually represent equivalence classes, cf. (4.7).

Before we start with the inverse problem, we introduce the time-integral operator

$$\Xi : L^2(0,T,TL^2_\diamondsuit(S)) \to TL^2_\diamondsuit(S), \qquad h \mapsto \int_0^T h(t)\,\mathrm{d}t.$$

Its adjoint operator maps a time-independent function $I \in TL^2_\diamondsuit(S)$ on its counterpart in $L^2(0,T,TL^2_\diamondsuit(S))$ that is constant in time, i.e.

$$(\Xi^* I)(t) = I,\ t \in (0,T).$$

To maintain lucidity, we usually omit Ξ^*.

In the following three sections, we use of the space $TH^{-1/2}(\mathrm{curl}_\Gamma)$ and its dual space $TH^{-1/2}(\mathrm{div}_\Gamma)$, cf., e.g., [Ces96, Chp. 2], and the surjective trace mappings

$$\begin{aligned} H(\mathrm{curl},\Omega) &\to TH^{-1/2}(\mathrm{curl}_\Gamma), \qquad & E &\mapsto \gamma_\Gamma E := (\nu \times E|_\Gamma) \times \nu, \\ H(\mathrm{curl},\Omega) &\to TH^{-1/2}(\mathrm{div}_\Gamma), \qquad & E &\mapsto \nu \times E|_\Gamma. \end{aligned}$$

4.4 Linear sampling method

In this section we show that a subset of Ω is determined by the measurements. Therefore, we factorize the measurement operator into

$$\Lambda = LN,$$

where N maps an excitation on S to its effect on the conductor, and L measures then the induced electric field on S. In linear sampling or factorization method context, L is often called the *virtual measurement operator*. Its range contains information needed to detect Ω.

We start with this operator. Let $H(\mathrm{curl},\Omega)'_\diamondsuit$ denote the subspace of $H(\mathrm{curl},\Omega)'$ of elements with vanishing divergence,

$$H(\mathrm{curl},\Omega)'_\diamondsuit := \{g \in H(\mathrm{curl},\Omega)' \,|\, \langle g, \nabla\varphi\rangle_{H(\mathrm{curl},\Omega)} = 0 \ \text{ for all } \varphi \in \mathcal{D}(\overline{\Omega})\}.$$

Then, $H(\mathrm{curl},\Omega)'_\diamondsuit$ is a Hilbert space and the following operator is linear and continuous:

$$L : L^2(0,T,H(\mathrm{curl},\Omega)'_\diamondsuit) \to L^2(0,T,TL^2_\diamondsuit(S)),\ \ B \mapsto \gamma_S H,$$

where $H \in L^2(0,T,W^1_\diamondsuit)$ solves

$$a_0(H,F) = \int_0^T \langle B, F|_\Omega\rangle_{H(\mathrm{curl},\Omega)}\,\mathrm{d}t \qquad \text{for all } F \in L^2(0,T,W^1_\diamondsuit). \tag{4.11}$$

We show the following relation between L and Λ:

4.4 Lemma *It holds that $\mathcal{R}(\Lambda) \subset \mathcal{R}(L)$.*

Proof We show that $\Lambda = LN$ with an appropriate operator N.

The assumption $\overline{\Omega} \cap \overline{S} = \emptyset$ ensures, that for solutions $E \in L^2(0,T,W(\mathrm{curl}))$ of (4.2) the evaluation $\nu \times \mathrm{curl}\,E|^+_\Gamma \in L^2(0,T,TH^{-\frac{1}{2}}(\mathrm{div}_\Gamma))$ is linear and continuous, where we denote by the $+$-sign the value from the outside of Ω. Moreover, for $t \in (0,T)$ a.e. we have, that

$$F \mapsto \langle \nu \times \mathrm{curl}\,E(t)|^+_\Gamma, \gamma_\Gamma F\rangle_{TH^{-\frac{1}{2}}(\mathrm{curl}_T)} \quad \text{for all } F \in H(\mathrm{curl},\Omega)$$

defines an element of $H(\mathrm{curl},\Omega)'_\diamondsuit$. Hence, the following operator is linear and continuous:

$$N :\ L^2(0,T,TL^2_\diamondsuit(S)) \to L^2(0,T,H(\mathrm{curl},\Omega)'_\diamondsuit), \quad I \mapsto h,$$

with

$$h : F \mapsto \int_0^T \int_\Omega \mathrm{curl}\,E_\sigma \cdot \mathrm{curl}\,F\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle \nu \times \mathrm{curl}\,E_\sigma|^+_\Gamma, \gamma_\Gamma F\rangle_{TH^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)}\,\mathrm{d}t$$

for all $F \in L^2(0,T,H(\mathrm{curl},\Omega))$, and where E_σ solves (4.10) with source I.

To show that $\Lambda = LN$, let $I \in L^2(0,T,TL^2_\diamondsuit(S))$ and E_0 and E_σ denote the solutions of (4.9) and (4.10) with source I. For $t \in (0,T)$ a.e. a short computation using (3.11) shows, that for every $\Phi \in \mathcal{D}(\mathbb{R}^3)^3$

$$\begin{aligned}&\langle (\sigma(E_\sigma + \nabla u_{E_\sigma}))^{\cdot}(t), \Phi\rangle \\ &\qquad = -\int_\Omega \frac{1}{\mu}\,\mathrm{curl}\,E_\sigma(t)\cdot \mathrm{curl}\,\Phi\,\mathrm{d}x - \langle \nu \times \mathrm{curl}\,E_\sigma(t)|^+_\Gamma, \gamma_\Gamma\Phi\rangle_{TH^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)}.\end{aligned}$$

The right hand side depends continuously on $\Phi|_\Omega \in \mathcal{D}(\overline{\Omega})^3 \subset H(\mathrm{curl},\Omega)$, thus, due to the denseness, it defines an element of $H(\mathrm{curl},\Omega)'$. Using this, (3.11) and integration by parts (3.7), we obtain for every $\Phi \in \mathcal{D}(\mathbb{R}^3\times]0,T[)^3$, that

$$\begin{aligned}
&a_0(E_0 - E_\sigma, \Phi) = a_\sigma(E_\sigma, \Phi) - a_0(E_\sigma, \Phi) \\
&= \int_0^T \langle (\sigma(E_\sigma + \nabla u_{E_\sigma}))^\cdot, \Phi\rangle \,\mathrm{d}t + \int_0^T \int_\Omega \frac{1}{\mu} \operatorname{curl} E_\sigma \cdot \operatorname{curl} \Phi \,\mathrm{d}x\,\mathrm{d}t - a_0(E_\sigma, \Phi) \\
&= -\int_0^T \langle \nu \times \operatorname{curl} E_\sigma|_\Gamma^+, \gamma_\Gamma \varphi\rangle_{TH^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)} \,\mathrm{d}t - \int_0^T \int_\Omega \operatorname{curl} E_\sigma \cdot \operatorname{curl} \Phi \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \langle NI, \Phi|_\Omega\rangle_{H(\mathrm{curl},\Omega)} \,\mathrm{d}t.
\end{aligned}$$

On the other hand, let $H \in L^2(0,T,W^1_\diamondsuit)$ be the solution of (4.11) with $B = NI$. Again, denseness implies

$$a_0(E_0 - E_\sigma, \Phi) = a_0(H, \Phi) \quad \text{for all } \Phi \in L^2(0,T,W^1_\diamondsuit),$$

and then uniqueness implies $H = E_0 - E_\sigma$, cf. Theorem 4.2 c). It follows

$$\Lambda I = \gamma_S(E_0 - E_\sigma) = \gamma_S H = LNI.$$

□

To characterize the conductor, we introduce for an arbitrary direction $d \in \mathbb{R}^3$, $|d| = 1$, the functions

$$G_{z,d} : \mathbb{R}^3 \setminus \{z\} \to \mathbb{R}^3, \quad x \mapsto \operatorname{curl} \frac{d}{|x-z|},$$

that have a dipole in $z \in \mathbb{R}^3$. In $\mathbb{R}^3 \setminus \{z\}$, every component of $G_{z,d}$ solves the homogeneous Laplace equation. Therefore, $G_{z,d}$ is analytic in $\mathbb{R}^3 \setminus \{z\}$.

The following theorem shows, that a subset of Ω is determined by Λ.

4.5 Theorem (Linear sampling method) *For every direction $d \in \mathbb{R}^3$, $|d| = 1$, and every point $z \in \mathbb{R}^3$ below S, $z \notin \Gamma$,*

$$\gamma_S G_{z,d} \in \mathcal{R}(\Xi\Lambda) \quad \textit{implies} \quad z \in \Omega.$$

Proof Let $\gamma_S G_{z,d} \in \mathcal{R}(\Xi\Lambda)$. Lemma 4.4 yields $\mathcal{R}(\Lambda) \subset \mathcal{R}(L)$, hence there is a preimage $B \in L^2(0,T,H(\mathrm{curl},\Omega)'_\diamondsuit)$ and some $H \in L^2(0,T,W^1_\diamondsuit)$, that solves (4.11) and that fulfills

$$\Xi\gamma_S H = \gamma_S G_{z,d}.$$

We consider $E := \int_0^T H(t)\,\mathrm{d}t \in W^1_\diamondsuit$ and obtain $\gamma_S E = \gamma_S G_{z,d}$, i.e.

$$\gamma_S(E - G_{z,d}) \in N_S \quad \text{and}$$
$$\operatorname{curl}\operatorname{curl} E = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \qquad \operatorname{div} E = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}.$$

Thus E is analytic in $\mathbb{R}^3 \setminus \overline{\Omega}$. Moreover, $G_{z,d}$ is analytic in $\mathbb{R}^3 \setminus \{z\}$, and it follows that $\operatorname{curl}(E - G_{z,d})$ is analytic in $\mathbb{R}^3 \setminus \big(\overline{\Omega} \cup \{z\}\big)$. Now, following [GHS08], we obtain by unique continuation of analytic functions, that

$$\operatorname{curl} E = \operatorname{curl} G_{z,d} \quad \text{in } \mathbb{R}^3 \setminus \{z\}.$$

The fact, that $\operatorname{curl} E \in L^2(\mathbb{R}^3 \setminus \overline{\Omega})$ but $\operatorname{curl} G_{z,d} \in L^2(\mathbb{R}^3 \setminus \overline{\Omega})$ only if $z \in \Omega$, yields the assertion. □

Further results on unique characterization can be obtained if we assume some additional feature on the permeability μ. This is done in the following sections.

4.5 Unique shape identification

For the rest of this paper we assume in addition, that the permeability is smaller on the conductor than on the background:

$$1 - \mu|_\Omega \in L^\infty_+(\Omega).$$

This is the case, for instance, for diamagnetic materials.

We moreover assume that the connected components of Ω are simply connected. This is only due to technical reasons, we expect our theory also to hold for multiply connected domains, that fulfill [DL00c, IX, Part A, §3, (1.45)], for instance, if Ω has the form of a torus.

Now we formulate our main result. The proof is postponed to Section 4.6.

4.6 Theorem (Unique shape identification) *It holds for every direction $d \in \mathbb{R}^3$, $|d| = 1$, and every point $z \in \mathbb{R}^3$ below S, $z \notin \Gamma$, that*

$$z \in \Omega \qquad \textit{if and only if}$$
$$\exists C > 0 : \quad (G_{z,d}, I)^2_{L^2(S)^3} \leq C \int_0^T (\Lambda I, I)_{L^2(S)^3}\,\mathrm{d}t \quad \textit{for all } I \in TL^2_\diamondsuit(S) \tag{4.12}$$

with

$$G_{z,d}(x) = \operatorname{curl} \frac{d}{|x - z|}.$$

In particular, Λ uniquely determines Ω. Let us stress, that therefore only time-independent I are needed. This means, that the applied source currents J on S (recall that I denotes the time-derivative of J) only depend linearly on time.

To formulate an equivalent formulation of Theorem 4.6, we make the following observation. Let $I \in L^2(0,T,TL^2_\diamond(S))$ and E_0 and E_σ be the solutions of (4.9) and (4.10) with source I. Then, integrating E_σ by parts in time (3.7), and using the fact, that E_0 minimizes the functional

$$L^2(0,T,W^1_\diamond) \to \mathbb{R}, \quad E \mapsto \frac{1}{2}a_0(E,E) - \int_0^T (\gamma_S E, I)_{L^2(S)^3}\,\mathrm{d}t,$$

leads to

$$\begin{aligned}\int_0^T (\Lambda I, I)_{L^2(S)^3}\,\mathrm{d}t &\geq \int_0^T (\gamma_S E_\sigma, I)_{L^2(S)^3}\,\mathrm{d}t - a_0(E_\sigma, E_\sigma) \\ &\geq \frac{1}{2}\|\sqrt{\sigma}(E_\sigma + \nabla u_{E_\sigma})(T)\|^2_{L^2(\Omega)^3} + \inf_\Omega \left[\frac{1}{\mu} - 1\right] \|\operatorname{curl} E_\sigma\|^2_{L^2(\Omega_T)^3} \geq 0. \end{aligned} \tag{4.13}$$

An immediate consequence is the following. The linear continuous and (by construction) self adjoint operator

$$\tilde{\Lambda} := \Xi(\Lambda + \Lambda^*)\Xi^* : TL^2_\diamond(S) \to TL^2_\diamond(S)$$

is positive, as for every $I \in TL^2_\diamond(S)$ it holds

$$\begin{aligned}(\tilde{\Lambda} I, I)_{L^2(S)^3} = (\Xi(\Lambda + \Lambda^*)\Xi^* I, I)_{L^2(S)^3} &= \int_0^T ((\Lambda + \Lambda^*)\Xi^* I, \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t \\ &= 2\int_0^T (\Lambda \Xi^* I, \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t \geq 0.\end{aligned}$$

Hence, the square root $\tilde{\Lambda}^{\frac{1}{2}}$ exists.

We use the following result on the relation between the norm of an operator and the range of its dual. In this form it is called the “14th important property of Banach spaces“ in Bourbaki [Bou87]:

4.7 Lemma *Let X, Y be two Banach spaces. Let $A \in \mathcal{L}(X,Y)$ and $x' \in X'$. Then*

$$x' \in \mathcal{R}(A') \quad \textit{if and only if} \quad \exists C > 0 : \ |\langle x', x\rangle_X| \leq C\|Ax\|_Y \ \textit{for all } x \in X.$$

An elementary proof can be found, for instance, in [FGS07, Lemma 3.4].

4.8 Corollary (Factorization method) *It holds for every direction $d \in \mathbb{R}^3$, $|d| = 1$, and every point $z \in \mathbb{R}^3$ below S, $z \notin \Gamma$, that*

$$z \in \Omega \quad \textit{if and only if} \quad \gamma_S G_{z,d} \in \mathcal{R}\left(\tilde{\Lambda}^{1/2}\right). \tag{4.14}$$

Proof Theorem 4.6 yields that $z \in \Omega$ if and only if

$$\exists C > 0: \ (\gamma_S G_{z,d}, I)^2_{L^2(S)^3} \leq C \int_0^T (\Lambda \Xi^* I, \Xi^* I)_{L^2(S)^3} \,\mathrm{d}t \quad \text{for all } I \in TL^2_\diamond(S). \tag{4.15}$$

For every $I \in TL^2_\diamond(S)$, (4.15) equals

$$(\gamma_S G_{z,d}, I)^2_{L^2(S)^3} \leq C \int_0^T (\Lambda \Xi^* I, \Xi^* I)_{L^2(S)^3} \,\mathrm{d}t = \frac{C}{2} (\tilde{\Lambda} I, I)_{L^2(S)^3} = \frac{C}{2} \|\tilde{\Lambda}^{1/2} I\|^2_{L^2(S)^3}.$$

A reformulation of Lemma 4.7 in the case of Hilbert spaces yields immediately that this is equivalent to

$$\gamma_S G_{z,d} \in \mathcal{R}\left(\tilde{\Lambda}^{1/2}\right).$$

□

4.6 Constraining operators for Λ

The key of the proof of Theorem 4.6 is to find adequate operators that control the measurement operator from below and from above, cf. [Har13]. To be more precise, we are looking for operators R_1 and R_2 mapping into particular Hilbert spaces, that fulfill

$$c\|R_1 I\|^2 \leq \int_0^T (\Lambda I, I)_{L^2(S)^3} \,\mathrm{d}t \leq c' \|R_2 I\|^2$$

for all $I \in L^2(0, T, TL^2_\diamond(S))$ with some positive constants c, c'. These Hilbert spaces will depend on Ω, so that the operators can be used to determine Ω uniquely.

In this section we introduce the operators R_1 and R_2 and show how they can be used to characterize Ω. At the end of this section we give a proof of Theorem 4.6.

4.6.1 Lower bound

For the lower bound, an appropriate candidate for R_1 can be found easily. Let $I \in L^2(0, T, TL^2_\diamond(S))$ and E_0 and E_σ be the solutions of (4.9) and (4.10) with source I. Then, (4.13) yields

$$\begin{aligned} \int_0^T (\Lambda I, I)_{L^2(S)^3} \,\mathrm{d}t &\geq \frac{1}{2} \|\sqrt{\sigma}(E_\sigma + \nabla u_{E_\sigma})(T)\|^2_{L^2(\Omega)^3} + \inf_\Omega \left[\frac{1}{\mu} - 1\right] \|\operatorname{curl} E_\sigma\|^2_{L^2(\Omega_T)^3} \\ &\geq c \left[\|\sigma(E_\sigma + \nabla u_{E_\sigma})(T)\|^2_{L^2(\Omega)^3} + \|\operatorname{curl} E_\sigma\|^2_{L^2(\Omega_T)^3}\right] \\ &=: c\, \|R_1 I\|^2 \end{aligned} \tag{4.16}$$

with the constant

$$c = \min\left\{\frac{1}{2\|\sigma\|_\infty}, \inf_\Omega \left[\frac{1}{\mu} - 1\right]\right\}.$$

To define R_1 rigorously, let us first introduce the following factor space

$$X := H(\mathrm{curl}, \Omega)/\mathcal{N}, \qquad \text{where } \mathcal{N} := \ker \mathrm{curl} = \nabla H^1(\Omega),$$

cf. [DL00c, IX, Part A, §1, Proposition 2 and Remark 6]. X is a Hilbert space with respect to the induced norm

$$\|u + \mathcal{N}\|_X := \inf_{m\in\mathcal{N}} \|u - m\|_{H(\mathrm{curl},\Omega)}.$$

4.9 Lemma *An equivalent norm on X is given by*

$$u + \mathcal{N} \mapsto \|\mathrm{curl}\, u\|_{L^2(\Omega)}.$$

Proof We consider $u + \mathcal{N} \in X$. Then we have

$$\|u + \mathcal{N}\|_X^2 = \inf_{m\in\mathcal{N}} \|u - m\|^2_{H(\mathrm{curl},\Omega)} \geq \|\mathrm{curl}\, u\|^2_{L^2(\Omega)}.$$

Moreover, [DL00c, IX, Part A, §1, Corollary 5 and Remark 6] yields that every u has a unique orthogonal decomposition

$$u = \nabla p + \mathrm{curl}\, w$$

where $p \in H^1(\Omega)$ and $w \in H^1(\Omega)^3$ with $\nu \cdot \mathrm{curl}\, w|_\Gamma = 0$ (w must not be unique, but $\mathrm{curl}\, w$ is). A short computation shows

$$\|u + \mathcal{N}\|_X^2 = \|\mathrm{curl}\, w\|^2_{L^2(\Omega)} + \|\mathrm{curl}\, u\|^2_{L^2(\Omega)}.$$

Now, [DL00c, IX, Part A, §1, Remarks 4 and 6] yields that

$$\mathrm{curl} : \{a \in H^1(\Omega)^3 \mid \mathrm{div}\, a = 0, \nu \cdot a|_\Gamma = 0\} \to \mathrm{curl}\, H^1(\Omega)^3$$

is an isomorphism and therefore has a continuous linear inverse. Since $\mathrm{curl}\, w$ is an element of that space, it follows

$$\begin{aligned}\|u + \mathcal{N}\|_X^2 &= \|\mathrm{curl}\, w\|^2_{L^2(\Omega)} + \|\mathrm{curl}\, u\|^2_{L^2(\Omega)} \\ &\leq c''\|\mathrm{curl}\,\mathrm{curl}\, w\|^2_{L^2(\Omega)} + \|\mathrm{curl}\, u\|^2_{L^2(\Omega)} = (c''+1)\|\mathrm{curl}\, u\|^2_{L^2(\Omega)}\end{aligned}$$

with a constant c'' independent of u (or its decomposition). □

Let $L^2(\Omega)^3_\diamondsuit$ be the space of $L^2(\Omega)^3$-functions with vanishing divergence. Obviously, $L^2(\Omega)^3_\diamondsuit$ is a Hilbert space.

4.10 Corollary *The following mapping is linear and continuous:*

$$R_1 : L^2(0,T,TL^2_\diamondsuit(S)) \to L^2(\Omega)^3_\diamondsuit \times L^2(0,T,X),$$
$$I \mapsto (\sigma(E_\sigma + \nabla u_{E_\sigma})(T)|_\Omega, E_\sigma|_\Omega + \mathcal{N}),$$

where E_σ solves (4.10) *with source I. Its dual mapping is given by*

$$R_1' : \left(L^2(\Omega)^3_\diamondsuit\right)' \times L^2(0,T,X') \to L^2(0,T,TL^2_\diamondsuit(S)), \quad (v,w) \mapsto h,$$

where h obeys for every $I \in L^2(0,T,TL^2_\diamondsuit(S))$

$$\int_0^T (h,I)_{L^2(S)^3}\,\mathrm{d}t = \int_0^T (R_1'(v,w),I)_{L^2(S)^3}\,\mathrm{d}t$$
$$= \langle v, \sigma(E_\sigma + \nabla u_{E_\sigma})(T)|_\Omega\rangle_{L^2(\Omega)^3_\diamondsuit} + \int_0^T \langle w, E_\sigma|_\Omega + \mathcal{N}\rangle_X\,\mathrm{d}t,$$

where E_σ denotes the solution of (4.10) *with source I, again.*

Now, the inequality (4.16) reads: There is a positive constant c so that

$$c\,\|R_1 I\|^2_{L^2(\Omega)^3_\diamondsuit \times L^2(0,T,X)} \leq \int_0^T (\Lambda I, I)_{L^2(S)^3}\,\mathrm{d}t \quad \text{for all } I \in L^2(0,T,TL^2_\diamondsuit(S)). \tag{4.17}$$

The following lemma shows, that the range of R_1' determines a superset of Ω: Whenever a point z is inside Ω, then $\gamma_S G_{z,d}$ is contained in the range of the dual operator of R_1.

4.11 Lemma *Let $z \in \Omega$. For every direction $d \in \mathbb{R}^3$, $|d| = 1$, there is a preimage $(v,w) \in \left(L^2(\Omega)^3_\diamondsuit\right)' \times L^2(0,T,X')$ of $\Xi R_1'$ with*

$$\gamma_S G_{z,d} = \Xi R_1'(v,w).$$

Proof For every $z \in \Omega$ there is an $\varepsilon > 0$ such that for the open ball $B_\varepsilon(z)$ it holds $\overline{B_\varepsilon(z)} \subset \Omega$. Now we choose a smooth cutoff function $\varphi \in C^\infty(\mathbb{R}^3)$ with $\varphi \equiv 1$ outside of $B_\varepsilon(z)$ and $\varphi \equiv 0$ in $B_{\frac{\varepsilon}{2}}(z)$. We obtain

$$\tilde{G}_{z,d}(x) := \operatorname{curl}\left(\frac{\varphi(x)d}{|x-z|}\right) \in H(\operatorname{curl},\mathbb{R}^3)$$
$$\text{and we have} \quad \tilde{G}_{z,d} = G_{z,d} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}.$$

Let $\tilde{G}_{z,d}(t) := \tilde{G}_{z,d}$. Then, it holds

$$\tilde{G}_{z,d} \in L^2(0,T,W^1_\diamondsuit), \quad \operatorname{curl}\tilde{G}_{z,d} \in L^2(0,T,H(\operatorname{curl},\mathbb{R}^3))$$

and $\operatorname{curl}\operatorname{curl}\tilde{G}_{z,d} = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$.

We define $v \in \left(L^2(\Omega)^3_\diamond\right)'$ and $w \in L^2(0,T,X')$ by

$$v : H \mapsto \int_\Omega H \cdot \tilde{G}_{z,d} \,\mathrm{d}x,$$
$$w : F + \mathcal{N} \mapsto \int_0^T \int_\Omega \left[\operatorname{curl}\operatorname{curl}\tilde{G}_{z,d} \cdot F + \left(\frac{1}{\mu} - 1\right)\operatorname{curl}\tilde{G}_{z,d} \cdot \operatorname{curl} F\right] \mathrm{d}x\,\mathrm{d}t.$$

We use the fact, that for all $F \in L^2(0,T,W^1_\diamond)$ it holds

$$\int_0^T \int_{\mathbb{R}^3\setminus\overline{\Omega}} \operatorname{curl}\tilde{G}_{z,d} \cdot \operatorname{curl} F \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_\Omega \left[\operatorname{curl}\operatorname{curl}\tilde{G}_{z,d} \cdot F - \operatorname{curl}\tilde{G}_{z,d} \cdot \operatorname{curl} F\right] \mathrm{d}x\,\mathrm{d}t,$$

the identity (3.11) and the integration by parts formula (3.7) and obtain, that for every $I \in TL^2_\diamond(S)$ it holds

$$\begin{aligned}
&(\Xi R_1'(v,w), I)_{L^2(S)^3} = \int_0^T (R_1'(v,w), \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t\\
&= \int_\Omega \sigma(E_\sigma + \nabla u_{E_\sigma})(T) \cdot \tilde{G}_{z,d}\,\mathrm{d}x\\
&\quad + \int_0^T \int_\Omega \left[\operatorname{curl}\operatorname{curl}\tilde{G}_{z,d} \cdot E_\sigma + \left(\frac{1}{\mu} - 1\right)\operatorname{curl}\tilde{G}_{z,d} \cdot \operatorname{curl} E_\sigma\right] \mathrm{d}x\,\mathrm{d}t\\
&= \int_0^T \int_{\mathbb{R}^3} \langle (\sigma(E_\sigma + \nabla u_{E_\sigma}))^\cdot, \tilde{G}_{z,d}\rangle\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu}\operatorname{curl}\tilde{G}_{z,d} \cdot \operatorname{curl} E_\sigma\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^T (\gamma_S \tilde{G}_{z,d}, \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t\\
&= (\gamma_S G_{z,d}, I)_{L^2(S)^3},
\end{aligned}$$

where E_σ denotes the solution of (4.10) with source $\Xi^* I$. □

4.6.2 Upper bound

To define R_2, we consider the subspace of elements of $TH^{-1/2}(\operatorname{div}_\Gamma)$ with vanishing divergence,

$$TH^{-1/2}_\diamond(\Gamma) := \{g \in TH^{-1/2}(\operatorname{div}_\Gamma) \mid \operatorname{div} g = 0\},$$

where we understand $TH^{-1/2}(\operatorname{div}_\Gamma)$ as a subspace of $W(\operatorname{curl})'$ by

$$E \mapsto \langle g, \gamma_\Gamma E\rangle_{TH^{-1/2}(\operatorname{curl}_\Gamma)} \quad \text{for all } E \in W(\operatorname{curl}).$$

Clearly, $TH_\diamondsuit^{-1/2}(\Gamma)$ is a Hilbert space with respect to $\|\cdot\|_{TH^{-1/2}(\mathrm{div}_\Gamma)}$. As the tangential components of elements of $W(\mathrm{curl})$ are in $TH^{-1/2}(\mathrm{curl}_\Gamma)$, every $E \in W(\mathrm{curl})$ defines an element of $TH_\diamondsuit^{-1/2}(\Gamma)'$ by

$$g \mapsto \langle g, \gamma_\Gamma E\rangle_{TH^{-1/2}(\mathrm{curl}_\Gamma)} \quad \text{for all } g \in TH_\diamondsuit^{-1/2}(\Gamma).$$

Now, Theorems 4.1 and 4.2 yield the following corollary.

4.12 Corollary *For $i = 0, \sigma$, linear continuous mappings are given by*

$$K_i : L^2(0,T, TL^2_\diamondsuit(S)) \to L^2(0,T, TH_\diamondsuit^{-1/2}(\Gamma)'), \quad I \mapsto d,$$
$$\text{with} \quad d : g \mapsto \int_0^T \langle g, \gamma_\Gamma E_i\rangle_{TH^{-1/2}(\mathrm{curl}_\Gamma)} \,\mathrm{d}t,$$

and where $E_0, E_\sigma \in L^2(0,T,W^1_\diamondsuit)$ are the solutions of (4.9) *and* (4.10) *with source I.*

Their dual operators are given by

$$K_i' : L^2(0,T, TH_\diamondsuit^{-1/2}(\Gamma)) \to L^2(0,T, TL^2_\diamondsuit(S)), \quad g \mapsto \gamma_S H_i,$$

where $H_0 \in L^2(0,T,W^1_\diamondsuit)$ solves the variational problem

$$a_0(H_0, \Phi) = \int_0^T \langle g, \gamma_\Gamma \Phi\rangle_{TH^{-1/2}(\mathrm{curl}_\Gamma)} \,\mathrm{d}t$$

for all $\Phi \in L^2(0,T,W^1_\diamondsuit)$, and $H_\sigma \in L^2(0,T,W^1_\diamondsuit)$ solves

$$a_\sigma(H_\sigma, \Phi) = \int_0^T \langle g, \gamma_\Gamma \Phi\rangle_{TH^{-1/2}(\mathrm{curl}_\Gamma)} \,\mathrm{d}t$$

for all $\Phi \in H^1(0,T,W^1_\diamondsuit)$ with $\Phi(0) = 0$.

We need two more operators and their duals:

4.13 Lemma *For $i = 0, \sigma$, linear continuous mappings are given by*

$$M_i : L^2(0,T, TL^2_\diamondsuit(S)) \to L^2(0,T, TH^{-1/2}(\mathrm{div}_\Gamma)), \quad I \mapsto \nu \times \mathrm{curl}\, E_i|_\Gamma^+,$$

where $E_0, E_\sigma \in L^2(0,T,W^1_\diamondsuit)$ are the solutions of (4.9) *and* (4.10) *with source I.*

Their dual operators obey

$$M_i' : L^2(0,T, TH^{-1/2}(\mathrm{curl}_\Gamma)) \to L^2(0,T, TL^2_\diamondsuit(S)), \; f \mapsto -\gamma_S G_i$$

for some $G_i \in L^2(0,T,W(\mathrm{curl}, \mathbb{R}^3 \setminus \Gamma))$ that fulfill

$$\gamma_\Gamma G_i^+ - \gamma_\Gamma G_i^- = f \qquad \text{in } \Gamma \times (0,T),$$
$$\mathrm{curl}\,\mathrm{curl}\, G_i = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{\Omega} \times (0,T).$$

Proof Again, the first assertion follows from Theorem 4.1, Theorem 4.2, and the fact, that the evaluation of $\nu \times \operatorname{curl} E|_\Gamma^+$ for solutions of (4.9) or (4.10) in $TH^{-1/2}(\operatorname{div}_\Gamma)$ is linear and continuous.

For the second assertion, let γ_Γ^{-1} be a linear continuous right inverse of

$$\gamma_\Gamma : W(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{\Omega}) \to TH^{-1/2}(\operatorname{curl}_\Gamma).$$

For $f \in L^2(0, T, TH^{-1/2}(\operatorname{curl}_\Gamma))$ we denote

$$U^f := \gamma_\Gamma^{-1} f \in L^2(0, T, W(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{\Omega})).$$

Let $U_0 \in L^2(0, T, W^1_\diamond)$ be the solution of

$$\int_0^T \int_{\mathbb{R}^3} \operatorname{curl} U_0 \cdot \operatorname{curl} F \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_{\mathbb{R}^3\setminus\overline{\Omega}} \operatorname{curl} U^f \cdot \operatorname{curl} F \,\mathrm{d}x\,\mathrm{d}t$$

for all $F \in L^2(0, T, W^1_\diamond)$. Then, for every $I \in L^2(0, T, TL^2_\diamond(S))$ we obtain

$$\begin{aligned}
\int_0^T \langle M_0' f, I\rangle_{L^2(S)^3} \,\mathrm{d}t &= \int_0^T \langle M_0 I, f\rangle_{TH^{-1/2}(\operatorname{curl}_\Gamma)} \,\mathrm{d}t \\
&= \int_0^T \langle \nu \times \operatorname{curl} E_0|_\Gamma^+, \gamma_\Gamma U^f\rangle_{TH^{-1/2}(\operatorname{curl}_\Gamma)} \,\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}^3\setminus\overline{\Omega}} \operatorname{curl} E_0 \cdot \operatorname{curl} U^f \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle \gamma_S U^f + N_S, I\rangle_{L^2(S)^3} \,\mathrm{d}t \\
&= -\int_0^T \int_{\mathbb{R}^3} \operatorname{curl} E_0 \cdot \operatorname{curl} U_0 \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle \gamma_S U^f + N_S, I\rangle_{L^2(S)^3} \,\mathrm{d}t \\
&= -\int_0^T (\gamma_S(U_0 + U^f), I)_{L^2(S)^3} \,\mathrm{d}t,
\end{aligned}$$

where $E_0 \in L^2(0, T, W^1_\diamond)$ is the solution of (4.9) with source I. The assertion for M_0' follows now by the choice

$$G_0 := \begin{cases} U_0 + U^f & \text{in } \mathbb{R}^3 \setminus \overline{\Omega} \times (0, T), \\ U_0 & \text{in } \Omega \times (0, T). \end{cases}$$

The assertion for M_σ' follows similarly by replacing U_0 with the solution $U \in L^2(0, T, W^1_\diamond)$ of

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^3} \left[\sigma(U + \nabla u_U) \cdot \dot{F} + \frac{1}{\mu} \operatorname{curl} U \cdot \operatorname{curl} F\right] \mathrm{d}x\,\mathrm{d}t& \\
= -\int_0^T \int_{\mathbb{R}^3\setminus\overline{\Omega}} \operatorname{curl} F \cdot \operatorname{curl} U^f \,\mathrm{d}x\,\mathrm{d}t&
\end{aligned}$$

for all $F \in H^1(0, T, W^1_\diamond)$ with $F(0) = 0$. □

Now we are prepared to define the operator R_2:

$$R_2 : L^2(0,T,TL^2_\diamondsuit(S)) \to L^2(0,T,TH^{-\frac{1}{2}}(\mathrm{div}_\Gamma))^2 \times L^2(0,T,TH^{-1/2}_\diamondsuit(\Gamma)')^2,$$
$$I \mapsto (M_0 I, M_\sigma I, K_0 I, K_\sigma I).$$

Obviously, its dual is given by

$$R_2' : L^2(0,T,TH^{-\frac{1}{2}}(\mathrm{curl}_\Gamma))^2 \times L^2(0,T,TH^{-1/2}_\diamondsuit(\Gamma))^2 \to L^2(0,T,TL^2_\diamondsuit(S)),$$
$$(e,f,g,h) \mapsto M_0' e + M_\sigma' f + K_0' g + K_\sigma' h.$$

A reformulation of the measurement operator in terms of M_0, M_σ, K_0, K_σ yields the estimation

$$\int_0^T (\Lambda I, I)_{L^2(S)^3}\,\mathrm{d}t = \left| \int_0^T \left[\langle M_0 I, K_\sigma I\rangle_{TH^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)} - \langle M_\sigma I, K_0 I\rangle_{TH^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)} \right] \mathrm{d}t \right|$$
$$\leq \frac{1}{2} \|R_2 I\|^2_{L^2(0,T,TH^{-\frac{1}{2}}(\mathrm{div}_\Gamma))^2 \times L^2(0,T,TH^{-1/2}_\diamondsuit(\Gamma)')^2}. \tag{4.18}$$

In the following lemma we show likewise to Theorem 4.4, that the dual of R_2 determines a subset of Ω.

4.14 Lemma *For every direction $d \in \mathbb{R}^3$, $|d| = 1$, and every point $z \in \mathbb{R}^3$ below S, $z \notin \Gamma$,*

$$\gamma_S G_{z,d} \in \mathcal{R}(\Xi R_2') \quad \textit{implies} \quad z \in \Omega.$$

Proof Assume $\gamma_S G_{z,d} \in \mathcal{R}(\Xi R_2')$. Then, there are

$$g_0, g_\sigma \in L^2(0,T,TH^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)) \text{ and } f_0, f_\sigma \in L^2(0,T,TH^{-1/2}_\diamondsuit(\Gamma))$$

such that

$$\gamma_S G_{z,d} = \Xi(M_0' g_0 + M_\sigma' g_\sigma + K_0' f_0 + K_\sigma' f_\sigma)$$
$$= \Xi\left(\gamma_S H_0 + \gamma_S H_\sigma + \gamma_S G_0 + \gamma_S G_\sigma\right).$$

Here, the functions $H_0, H_\sigma \in L^2(0,T,W^1_\diamondsuit)$ are such as in Corollary 4.12 and $G_0, G_\sigma \in L^2(0,T,W(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)$ are such as in Lemma 4.13. Let $V_i = \int_0^T H_i(t)\,\mathrm{d}t \in W^1_\diamondsuit$ and $P_i = \int_0^T G_i(t)\,\mathrm{d}t \in W(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)$ for $i = 0,\sigma$ and consider

$$E := (V_0 + V_\sigma + P_0 + P_\sigma)|_{\mathbb{R}^3\setminus\overline{\Omega}}.$$

Then, we have $E \in W(\mathrm{curl},\mathbb{R}^3\setminus\overline{\Omega})$ and $\mathrm{curl}\,\mathrm{curl}\,E = 0$ in $\mathbb{R}^3\setminus\overline{\Omega}$, moreover it holds $\gamma_S E = \gamma_S G_{z,d}$ and especially $\gamma_S(E - G_{z,d}) \in N_S$.

Now we study the function

$$Z := \mathrm{curl}(E - G_{z,d}).$$

As a start, Z is analytic in $\mathbb{R}^3 \setminus \big(\overline{\Omega} \cup \{z\}\big)$, as $\operatorname{curl} G_{z,d}$ is analytic in $\mathbb{R}^3 \setminus \{z\}$ and $\operatorname{curl} E$ is analytic in $\mathbb{R}^3 \setminus \overline{\Omega}$. Further, the third component of Z (denoted by Z_3) vanishes on $\mathbb{R}^3_0$. To see this we add a gradient field ∇a that fulfills $\operatorname{div}(E+\nabla a) = 0$ in a neighborhood of S and we obtain that $E + \nabla a - G_{z,d}$ is analytic in this neighborhood. Beyond that,

$$\gamma_S(E + \nabla a - G_{z,d}) \in N_S$$

implies that there is a sequence $(\varphi_n) \in \mathcal{D}(\mathbb{R}^3)$ with

$$\gamma_S \nabla \varphi_n \to \gamma_S(E + \nabla a - G_{z,d}) \quad \text{in } TL^2(S)$$

and hence, as $\gamma_S F = n \times (F|_S \times n) = (F_1|_S, F_2|_S, 0)^T$ for every $F \in W(\operatorname{curl})$, we have

$$(\nabla \varphi_n)_1|_S \to (E + \nabla a - G_{z,d})_1|_S,\ (\nabla \varphi_n)_2|_S \to (E + \nabla a - G_{z,d})_2|_S \text{ in } L^2(S).$$

Because of $\partial_2(\nabla\varphi_n)_1 = \partial_1(\nabla\varphi_n)_2$ it follows in a distributional sense, that

$$\partial_2(E + \nabla a - G_{z,d})_1 - \partial_1(E + \nabla a - G_{z,d})_2 = 0 \text{ on } S.$$

Moreover, as $E + \nabla a$ and $G_{z,d}$ are analytic on S, the classical derivatives exist and are equal to the distributional ones. It follows that

$$\operatorname{curl}(E + \nabla a - G_{z,d})_3 = \partial_1(E + \nabla a - G_{z,d})_2 - \partial_2(E + \nabla a - G_{z,d})_1 = 0 \text{ on } S$$

and hence, that $Z_3 = \operatorname{curl}(E - G_{z,d})_3 = \operatorname{curl}(E + \nabla a - G_{z,d})_3 = 0$ on S. As Z_3 is analytic in $\mathbb{R}^3_0$ and vanishes on S, unique continuation implies that

$$Z_3 = 0 \quad \text{in } \mathbb{R}^3_0.$$

The next step is to conclude, that Z vanishes in

$$\mathbb{R}^3_{x_3>0} := \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 | x_3 > 0\}.$$

By choosing a transformation $\alpha : \mathbb{R}^3 \to \mathbb{R}^3,\ x \mapsto x - 2x_3(0,0,1)^T$ and analyzing the function

$$\tilde{Z}(x) := \begin{cases} Z(x) & x_3 \geq 0 \\ \alpha(Z(\alpha(x))) & x_3 < 0 \end{cases},$$

one ends up with

$$\tilde{Z} \in L^2(\mathbb{R}^3)^3 \qquad \text{and} \qquad \operatorname{div} \tilde{Z} = \operatorname{curl} \tilde{Z} = 0 \quad \text{in } \mathbb{R}^3.$$

Hence, there is some $U \in W^1_\diamond$ with $\operatorname{curl} U = \tilde{Z}$. This U also solves

$$\operatorname{curl}\operatorname{curl} U = 0 \quad \text{in } \mathbb{R}^3.$$

It follows $U = 0$ and thus $Z|_{\mathbb{R}^3_{x_3>0}} = 0$. Again, unique continuation of analytic functions yields $Z = 0$ in $\mathbb{R}^3 \setminus \big(\overline{\Omega} \cup \{z\}\big)$. It follows

$$\operatorname{curl} G_{z,d} = \operatorname{curl} E \quad \text{in } \mathbb{R}^3 \setminus \big(\overline{\Omega} \cup \{z\}\big).$$

If $z \notin \mathbb{R}^3 \setminus \overline{\Omega}$, then $\operatorname{curl} G_{z,d} \notin L^2(\mathbb{R}^3 \setminus \overline{\Omega})$, which contradicts to the fact that $\operatorname{curl} E \in L^2(\mathbb{R}^3 \setminus \overline{\Omega})^3$. It follows $z \in \Omega$. □

4.6.3 Proof of the main result

Proof of Theorem 4.6 "$\Longrightarrow$": Assume $z \in \Omega$. Lemma 4.11 yields that there is a preimage (v, w) of $\gamma_S G_{z,d}$ under $\Xi R_1'$, i.e.

$$\Xi R_1'(v, w) = \gamma_S G_{z,d}.$$

We use inequality (4.17) and conclude for all $I \in TL^2_\diamond(S)$ that

$$\begin{aligned}(\gamma_S G_{z,d}, I)_{L^2(S)^3} &= (\Xi R_1'(v,w), I)_{L^2(S)^3} = \int_0^T (R_1'(v,w), \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t \\ &= \langle (v,w), R_1 \Xi^* I \rangle_{L^2(\Omega)^3_\diamond \times L^2(0,T,X)} \\ &\leq \|(v,w)\|_{\left(L^2(\Omega)^3_\diamond\right)' \times L^2(0,T,X')} \|R_1 \Xi^* I\|_{L^2(\Omega)^3_\diamond \times L^2(0,T,X)} \\ &\leq C \left[\int_0^T (\Lambda \Xi^* I, \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t\right]^{1/2}\end{aligned}$$

with a constant C independent of I. The inequality (4.12), i.e.

$$\exists C > 0: \; (\gamma_S G_{z,d}, I)^2_{L^2(S)^3} \leq C \int_0^T (\Lambda \Xi^* I, \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t \quad \text{for all } I \in TL^2_\diamond(S),$$

follows immediately.

"$\Longleftarrow$": Assume (4.12) holds. Then, equation (4.18) yields for all $I \in TL^2_\diamond(S)$, that

$$\begin{aligned}(\gamma_S G_{z,d}, I)^2_{L^2(S)^3} &\leq C \int_0^T (\Lambda \Xi^* I, \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t \\ &\leq \frac{C}{2} \|R_2 \Xi^* I\|^2_{L^2(0,T,TH^{-\frac{1}{2}}(\mathrm{div}_\Gamma))^2 \times L^2(0,T,TH_\diamond^{-1/2}(\Gamma)')^2}\end{aligned}$$

with a constant C independent of I. We use Lemma 4.7, again, and conclude

$$\gamma_S G_{z,d} \in \mathcal{R}\left(\Xi R_2'\right).$$

Lemma 4.14 shows that $z \in \Omega$. □

4.7 An explicit criterion for shape reconstruction

Finally we show that the criterion (4.14) used in the factorization method can be rewritten in terms of the Picard criterion.

Let us first remark that Λ can be written as the composition of linear continuous mappings, and one of them is the compact embedding from the space of trace values of W^1-functions, $H^{1/2}(S)$, into $L^2(S)$, cf. the assumptions on S and, e.g., [Gri85, Theorem 1.4.4.1]. Hence $\tilde{\Lambda}$ is a positively definite self adjoint linear continuous compact mapping. Then for instance [Wer95, Theorem VI.3.2] yields a unique eigenvalue decomposition of $\tilde{\Lambda}$, i.e. a null sequence $(\alpha_n) \subset \mathbb{R}_{\geq 0}$ of eigenvalues and an orthonormal system $(\Psi_n) \subset TL^2_\diamondsuit(S)$ of eigenfunctions that builds a basis of $\ker(\tilde{\Lambda})^\perp$. Moreover, we have for all $I \in TL^2_\diamondsuit(S)$ that

$$\tilde{\Lambda}^{1/2} I = \sum_{n=1}^{\infty} \alpha_n^{1/2} (\Psi_n, I)_{L^2(S)^3} \Psi_n.$$

Finally, we deduce with the Picard criterion, cf., e.g., [EHN00, Theorem 2.8]:

4.15 Corollary *For all $I \in TL^2_\diamondsuit(S)$ we have*

$$I \in \mathcal{R}\left(\tilde{\Lambda}^{1/2}\right) \oplus \mathcal{R}\left(\tilde{\Lambda}^{1/2}\right)^\perp \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} \frac{(\Psi_n, I)^2_{L^2(S)^3}}{\alpha_n} < \infty.$$

4.16 Lemma *$\tilde{\Lambda}^{1/2}$ is injective.*

Proof Let $I \in TL^2_\diamondsuit(S)$ with $\tilde{\Lambda}^{1/2} I = 0$. Then inequality (4.13) yields

$$\begin{aligned} 0 = \|\tilde{\Lambda}^{1/2} I\|^2_{L^2(S)^3} &= (\tilde{\Lambda} I, I)_{L^2(S)^3} = 2 \int_0^T (\Lambda \Xi^* I, \Xi^* I)_{L^2(S)^3} \, \mathrm{d}t \\ &\geq \|\sqrt{\sigma}(E_\sigma + \nabla u_{E_\sigma})(T)\|^2_{L^2(\Omega)^3} + 2 \inf_\Omega \left[\frac{1}{\mu} - 1\right] \|\operatorname{curl} E_\sigma\|^2_{L^2(\Omega_T)^3}, \end{aligned}$$

where E_σ is the solution of (4.10) with source $\Xi^* I$ and zero initial values. Hence $E := E_\sigma + \nabla u_{E_\sigma} \in \mathcal{W}_\sigma$ solves (4.2) and

$$\begin{aligned} \operatorname{curl} E &= 0 && \text{in } \Omega \times (0,T), \\ \sqrt{\sigma} E(0) = \sqrt{\sigma} E(T) &= 0 && \text{in } \Omega. \end{aligned}$$

This implies $\partial_t(\sigma E) = 0$ and it follows $E = 0$ in $\Omega \times [0,T]$. This and the fact that $\mu_{\mathbb{R}^3 \setminus \overline{\Omega}} \equiv 1$ yields

$$\operatorname{curl}\operatorname{curl} E = \Xi^* I \qquad \text{in } \mathbb{R}^3 \times (0,T). \tag{4.19}$$

We consider the function

$$A := \operatorname{curl} E.$$

Then A solves the homogeneous Laplace equation in the open set $\mathbb{R}^3 \setminus \overline{S} \times (0,T)$ and is thus an analytic function that vanishes in $\Omega \times (0,T)$. Unique continuation of analytic functions implies $A = 0$ in $\mathbb{R}^3 \setminus \overline{S} \times (0,T)$, i.e. $\operatorname{curl} E = 0$ on $\mathbb{R}^3 \setminus \overline{S} \times (0,T)$. This together with (4.19) implies that $I = 0$. □

The precedent Lemma yields that $\mathcal{R}\left(\tilde{\Lambda}^{1/2}\right)^\perp = \emptyset$. Altogether, we conclude:

4.17 Corollary *It holds for every direction $d \in \mathbb{R}^3$, $|d| = 1$, and every point $z \in \mathbb{R}^3$ below S, $z \notin \Gamma$, that*

$$z \in \Omega \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} \frac{(\Psi_n, \gamma_S G_{z,d})^2_{L^2(S)^3}}{\alpha_n} < \infty.$$

4.8 Concluding remarks

We have extended the ideas of the factorization method to the problem of localizing conducting objects by electromagnetic measurements in the eddy-current regime. We have shown that the position and shape of conducting (diamagnetic) objects are uniquely determined by such measurements. We also showed how a subset of the object can be characterized using a linear sampling approach.

The criteria derived in this work are constructive and may be implemented as in the previous works on factorization and sampling methods, cf., e.g., [GHK$^+$05, GHS08] for numerical results for the time-harmonic Maxwell equations and [FGS07] for results on the scalar parabolic-elliptic analogue of the eddy current equation.

The linear sampling method in Theorem 4.5 is closely related to the MUSIC-type imaging (introduced in [Dev00]). This is shown in [AGH07] for electrical impedance tomography in case of small conductors, where the measurement operator is expanded in terms of the size of the conductor. In [AKK$^+$08], MUSIC-type imaging is used for corrosion detection. It might be interesting to apply the results of the paper to the problem of corrosion detection using eddy currents.

Let us remark, that our theoretical results in Section 4.5 require only excitations, that are linear in time and only time integral measurements. Moreover, our results hold for every final time T. In practice, this final time might play an important role. For instance in thermal imaging, the imaging functional is quite sensitive to the final time T, as pointed out in [AIKK05].

Chapter 5

Justification of regularizations for the parabolic-elliptic eddy current equation

In this chapter we consider the parabolic-elliptic eddy current equation in a bounded domain. We first extend our variational solution theory to the bounded setting and then apply it to show two regularizations for the equation: A parabolic one and an elliptic one. Both lead to well-posed and thus uniquely solvable problems. The aim of this chapter is to rigorously justify these regularizations by proving the convergence of the solutions against the solution of the original equation.

The Sections 5.2–5.6 are the Sections 2-6 of the paper [AH13a] up to minor changes. Moreover, in Section 5.5, Theorem 5.15 is added.

5.1 Introduction

Let us recall that the parabolic-elliptic eddy current equation

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -\partial_t J \tag{5.1}$$

does not uniquely determine its solutions in the insulating part of the domain, i.e. where $\sigma = 0$. Indeed, only σE and $\operatorname{curl} E$ are determined uniquely.

To overcome this non-uniqueness and also for computational reasons (cf., e.g., Lang and Teleaga in [LT08] or Bachinger et al. in [BLS05]), it seems natural to regularize the problem by setting the conductivity to a small value $\varepsilon > 0$ in the non-conducting region: Setting

$$\sigma_\varepsilon = \begin{cases} \sigma(x) & \text{if } \sigma(x) > 0, \\ \varepsilon & \text{if } \sigma(x) = 0, \end{cases}$$

the eddy current equation is made fully parabolic

$$\partial_t(\sigma_\varepsilon E_\varepsilon) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E_\varepsilon\right) = -\partial_t J \tag{5.2}$$

and uniquely solvable. An aim of this chapter is to rigorously justify this regularization: We show that

$$\sigma_\varepsilon E_\varepsilon \to \sigma E \quad \text{and} \quad \operatorname{curl} E_\varepsilon \to \operatorname{curl} E$$

as ε approaches zero, where E denotes any solution of (5.1) and E_ε the solution of (5.2). Note that for the scalar parabolic-elliptic analogue, the heat equation, this result was shown by Harrach in [Geb07].

Unfortunately, our solution theory developed in Chapter 3 only holds for conductivity coefficients with bounded support. This is not the case for σ_ε. In cases of interest, for instance in computational applications, the equation is considered in a bounded domain, anyway. Therefore we start this chapter by carrying over the results of Chapter 3 to bounded domains: We restrict the solutions to a comparatively large domain, so that we can assume the fields to be small at its boundary far away from the source and the conductors. Hence we consider the solutions of (5.1) to have vanishing tangential components at the boundary of the domain. Our solution theory restricted to a bounded domain might be of interest on its own, since up to the author's knowledge in the literature there cannot be found any complete solution theory for the bounded setting that holds for spatially varying conductivity coefficient, cf. Chapter 3.

In the bounded setting, the conductivity is allowed to be non-zero in the whole considered domain, also. It is shown in Sections 5.3 and 5.4 that the regularized equation (5.2) is uniquely solvable. Then, the fact that our variational formulation is unified with respect to the conductivity enables us to prove the convergence of the solutions if ε approaches zero.

A second possibility to regularize the parabolic-elliptic eddy current equation is to add a regularization term $\varepsilon E_\varepsilon$ as proposed by Nicaise and Tröltzsch in [NT14]:

$$\partial_t(\sigma E_\varepsilon) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E_\varepsilon\right) + \varepsilon E_\varepsilon = -\partial_t J. \tag{5.3}$$

This equation is coercive on the whole solution space and thus uniquely solvable, as we show in Theorem 5.15. In contrast to the eddy current equation (5.1) and its parabolic regularization (5.2), the standard variational formulation of (5.3) yields unique solvability and continuous dependence on the right hand side and on the coefficients, especially on $1/\varepsilon$. Here, our unified variational solution theory does not help to analyze the solution's behavior if ε tends to zero. Especially, our appropriately regularized unified variational formulation is not equivalent to the equation: Its solution does not yield a solution of (5.3) (cf. Section 5.5), as it is the case for (5.1) and (5.2).

However, in some applications one might be interested in regularizing the variational problem on itself. Hence, the second aim is to establish an elliptic regularization of our variational formulation of (5.1) that is indeed motivated by, but not equivalent to equation (5.3).

This chapter is organized as follows: In Section 5.2 we formulate the eddy current problem in a bounded domain and carry over the results of Section 3.2 about the well-definedness of (5.1). Section 5.3 then contains our variational formulation and the solvability of (5.1). In Section 5.4 we justify the parabolic regularization: We prove the convergence of the solutions when the fully positive conductivity approaches zero in a part of the domain. We finish this chapter by presenting a similar result for an elliptic regularization in Section 5.5. This chapter ends with a conclusion in Section 5.6.

5.2 Formulation of the eddy current problem in a bounded domain

Let $\mathcal{O} \subset \mathbb{R}^3$ be a simply connected bounded domain with Lipschitz boundary Σ and outer normal unit vector ν.

We consider the space $L^2(0,T,H_0(\mathrm{curl}))$ as a proper space to look for a solution of the eddy current equation (5.1). Here, the Hilbert space $H_0(\mathrm{curl})$ is defined as

$$H_0(\mathrm{curl}) := \{E \in H(\mathrm{curl},\mathcal{O}) \,|\, \nu \times E|_\Sigma = 0\}.$$

Let us assume that $\mu \in L^\infty_+(\mathcal{O})$ and either

$$\sigma \in L^\infty_+(\mathcal{O})$$

or (cf. Chapter 3)

$$\begin{aligned}\sigma \in L_C := \{\sigma \in L^\infty(\mathcal{O}) \,|\, \exists\, \Omega \subset \mathcal{O}:\ & \sigma|_\Omega \in L^\infty_+(\Omega),\ \Omega = \cup_{i=1}^s \Omega_i,\ s \in \mathbb{N},\\ & \text{with bounded Lipschitz domains } \Omega_i,\ \overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset,\ i \neq j,\\ & \text{such that } \mathcal{O} \setminus \overline{\Omega} \text{ is connected and } \overline{\Omega} = \operatorname{supp}\sigma \subsetneq \mathcal{O}\}.\end{aligned}$$

We assume that we are given $E^0 \in L^2(\mathcal{O})^3$ with $\mathrm{div}(\sigma E^0) = 0$ and the excitation

$$J_t \in L^2(0,T,H(\mathrm{curl},\mathcal{O})') \quad \text{with} \quad \mathrm{div}\, J_t = 0.$$

Then, for $E \in L^2(0,T,H_0(\mathrm{curl}))$, equation (5.1) posed on $\mathcal{O}\times]0,T[$ is well-defined in a distributional sense and equivalent to

$$\begin{aligned} -\int_0^T \int_{\mathcal{O}} \sigma E \cdot \partial_t \Phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathcal{O}} \frac{1}{\mu} \mathrm{curl}\, E \cdot \mathrm{curl}\, \Phi \,\mathrm{d}x\,\mathrm{d}t \\ = -\int_0^T \langle J_t, \Phi \rangle_{H(\mathrm{curl},\mathcal{O})}\,\mathrm{d}t \quad \text{for all } \Phi \in \mathcal{D}(\mathcal{O}\times]0,T[)^3. \end{aligned} \tag{5.4}$$

The assertions of this section are proven in Section 3.2 for unbounded domains. The proofs are analogously.

We first establish, that every solution of (5.1) has well-defined initial values. Therefore we introduce the space

$$\mathcal{W}_{\sigma,\mathcal{O}} := \left\{ E \in L^2(0,T,H_0(\mathrm{curl})) \,|\, (\sigma E)^\cdot \in L^2(0,T,H_0(\mathrm{curl})') \right\},$$

where $(\sigma E)^\cdot$ denotes the time-derivative of $\sigma E \in L^2(\mathcal{O}_T)^3$ in the sense of vector-valued distributions with respect to the canonical injection $L^2(\mathcal{O})^3 \hookrightarrow H_0(\mathrm{curl})'$.

5.1 Lemma *(cf. Lemma 3.2) If $E \in \mathcal{W}_{\sigma,\mathcal{O}}$, then $\sqrt{\sigma}E \in C(0,T,L^2(\mathcal{O})^3)$. Additionally, for $E, F \in \mathcal{W}_{\sigma,\mathcal{O}}$ the following integration by parts formula holds:*

$$\begin{aligned}\int_0^T \langle (\sigma E)^\cdot, F\rangle_{H_0(\mathrm{curl})}\,\mathrm{d}t + \int_0^T \langle (\sigma F)^\cdot, E\rangle_{H_0(\mathrm{curl})}\,\mathrm{d}t \\ = \int_{\mathcal{O}} \sigma\left(E(T)\cdot F(T) - E(0)\cdot F(0)\right)\,\mathrm{d}x. \end{aligned} \tag{5.5}$$

5.2 Lemma *(cf. Lemma 3.3) If $E \in L^2(0,T,H_0(\mathrm{curl}))$ solves (5.1), then $E \in \mathcal{W}_{\sigma,\mathcal{O}}$ and thus has well-defined initial values $\sqrt{\sigma}E(0) \in L^2(O)^3$.*

For $t \in]0,T[$ a.e., $(\sigma E)^\cdot(t) \in H_0(\mathrm{curl})'$ is given by

$$\langle (\sigma E)^\cdot(t), F\rangle_{H_0(\mathrm{curl})} = -\langle J_t(t), F\rangle_{H(\mathrm{curl},\mathcal{O})} - \int_{\mathcal{O}} \frac{1}{\mu}\,\mathrm{curl}\,E(t)\cdot \mathrm{curl}\,F\,\mathrm{d}x \tag{5.6}$$

for all $F \in H_0(\mathrm{curl})$.

5.3 Corollary *The following problem is well-defined: Find $E \in L^2(0,T,H_0(\mathrm{curl}))$ that solves*

$$\partial_t(\sigma(x)E(x,t)) + \mathrm{curl}\left(\frac{1}{\mu(x)}\,\mathrm{curl}\,E(x,t)\right) = -J_t(x,t) \qquad \text{in } \mathcal{O}\times]0,T[, \tag{5.7}$$

$$\sqrt{\sigma(x)}E(x,0) = \sqrt{\sigma(x)}E^0(x) \quad \text{in } \mathcal{O}. \tag{5.8}$$

Now, we give an equivalent variational formulation:

5.4 Lemma *(cf. Lemma 3.4) The following problems are well-defined and equivalent:*

a) Find $E \in L^2(0,T,H_0(\mathrm{curl}))$ that solves (5.7)–(5.8).

b) Find $E \in \mathcal{W}_{\sigma,\mathcal{O}}$ that solves (5.8) and

$$\begin{aligned}\int_0^T \langle (\sigma E)^\cdot, F\rangle_{H_0(\mathrm{curl})}\,\mathrm{d}t + \int_0^T \int_{\mathcal{O}} \frac{1}{\mu}\,\mathrm{curl}\,E\cdot\mathrm{curl}\,F\,\mathrm{d}x\,\mathrm{d}t \\ = -\int_0^T \langle J_t, F\rangle_{H(\mathrm{curl},\mathcal{O})}\,\mathrm{d}t \end{aligned} \tag{5.9}$$

for all $F \in L^2(0,T,H_0(\mathrm{curl}))$.

c) Find $E \in L^2(0,T,H_0(\mathrm{curl}))$ *that solves*

$$-\int_0^T \int_{\mathcal{O}} \sigma E \cdot \partial_t \Phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathcal{O}} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \langle J_t, \Phi \rangle_{H(\mathrm{curl},\mathcal{O})} \,\mathrm{d}t + \int_{\mathcal{O}} \sigma E^0 \cdot \Phi(0) \,\mathrm{d}x$$

for all $\Phi \in \mathcal{D}(\mathcal{O} \times [0,T[)^3$.

5.5 Theorem *(cf. Theorem 3.1c)) Equations* (5.7)–(5.8) *uniquely determine* $\sqrt{\sigma}E$ *and* $\operatorname{curl} E$.
Moreover, if $E \in L^2(0,T,H_0(\mathrm{curl}))$ *solves* (5.7)–(5.8), *then every function* $F \in L^2(0,T,H_0(\mathrm{curl}))$ *with* $\operatorname{curl} F = \operatorname{curl} E$ *and* $\sqrt{\sigma}F = \sqrt{\sigma}E$ *also solves* (5.7)–(5.8).

5.3 A variational solution theory for bounded domains

Unfortunately, the result on the non-uniqueness implies, that none of the variational formulations in Lemma 5.4 is well-posed. Our approach is as follows. We keep this non-uniqueness and try to determine the unique part of the solutions - that is the divergence-free part. Therefore, we write

$$E = \tilde{E} + \nabla u$$

with a divergence-free field $\tilde{E}$, and a gradient field ∇u. The crucial point is to consider $\nabla u = \nabla u_{\tilde{E}}$ as a continuous linear function of $\tilde{E}$, cf. Lemma 5.6. This allows us to rewrite the eddy current problem (5.7)–(5.8) as a variational equation for $\tilde{E}$, which is uniformly coercive on the space of divergence-free functions and thus uniquely determines the field $\tilde{E}$. Note that $\tilde{E}$ does not solve the eddy current equation.

This section is similar to Section 3.3 for the case of unbounded domains.

5.6 Lemma *(cf. Lemma 3.5) There is a continuous linear map*

$$L^2(\mathcal{O})^3 \to H_0(\operatorname{curl} 0) := \{E \in H_0(\mathrm{curl}) \mid \operatorname{curl} E = 0\},$$
$$E \mapsto \nabla u_E,$$

with

$$\operatorname{div}(\sigma(E + \nabla u_E)) = 0 \quad \text{in } \mathcal{O}. \tag{5.10}$$

Proof Let $E \in L^2(\mathcal{O})^3$.

We first consider the case $\Omega = \mathcal{O}$. Due to Poincare's inequality (cf., e.g., [DL00b, IV, §7, Proposition 2]), the fact, that σ is positively bounded from below on $\mathcal{O}$, and Lax-Milgram's Theorem (cf., e.g., [RR04, §8, Theorem 8.14]), there exists a unique $u_E \in H_0^1(\mathcal{O})$ that solves

$$\int_{\mathcal{O}} \sigma \nabla u \cdot \nabla v \, \mathrm{d}x = - \int_{\mathcal{O}} \sigma E \cdot \nabla v \, \mathrm{d}x \quad \text{for all } v \in H_0^1(\mathcal{O}),$$

and u_E depends continuously on $E \in L^2(\mathcal{O})^3$.

Now, let $\Omega \subsetneq \mathcal{O}$. Again, since σ is positively bounded from below on Ω, we obtain as above a unique $u_E \in H^1_\Box(\Omega)$ that solves

$$\int_{\Omega} \sigma \nabla u \cdot \nabla v \, \mathrm{d}x = - \int_{\Omega} \sigma E \cdot \nabla v \, \mathrm{d}x \quad \text{for all } v \in H^1(\Omega),$$

where $H^1_\Box(\Omega) := \left\{ v \in H^1(\Omega) \mid \int_{\Omega_i} v \, \mathrm{d}x = 0,\, i = 1, \ldots, s \right\}$, and u_E depends continuously on $E|_\Omega$. We extend u_E to an element of $H_0^1(\mathcal{O})$ by solving $\Delta u = 0$ on $\mathcal{O} \setminus \overline{\Omega}$ with $u|_{\partial\Omega} = u_E|_{\partial\Omega}$ for $u \in H^1(\mathcal{O} \setminus \overline{\Omega})$ with $u|_\Sigma = 0$. Again, Lax-Milgram's Theorem provides a unique solution, that depends continuously on $u_E|_{\partial\Omega}$ and thus on E. Let u_E, again, denote its extension.

In both cases $u_E \in H_0^1(\mathcal{O})$, $\nabla u_E \in H_0(\operatorname{curl} 0)$ and the mapping $E \mapsto \nabla u_E$ is well-defined, linear and continuous with a continuity constant that depends on the lower and upper bounds of σ. Moreover, (5.10) is fulfilled. □

We refer to Section 3.3 for the mapping's extension to time-dependent functions.

For the rest of this chapter, let ∇u_E denote the image of E under this mapping. Obviously, there are different possibilities to construct this map, but $\sqrt{\sigma} \nabla u_E$ is uniquely determined by the condition (5.10). Moreover, it holds that

$$\|\sqrt{\sigma} \nabla u_E\|_{L^2(\mathcal{O})^3} \leq \|\sqrt{\sigma} E\|_{L^2(\mathcal{O})^3}. \tag{5.11}$$

Note that ∇u_E depends nonlinearly on σ. Also continuous dependence on σ for fixed E must not be true. A special case will be discussed in Section 5.4.

Now we use this Lemma to show a variational formulation for (5.7)–(5.8). We define the bilinear form

$$\begin{aligned} a &: L^2(0, T, H_0(\operatorname{curl})) \times H^1(0, T, H_0(\operatorname{curl})) \to \mathbb{R} : \\ a(E, \Phi) &:= - \int_0^T \int_{\mathcal{O}} \sigma (E + \nabla u_E) \cdot \dot{\Phi} \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathcal{O}} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \, \mathrm{d}x \, \mathrm{d}t, \end{aligned} \tag{5.12}$$

and, motivated by Lemma 5.4c), the linear form $l : H^1(0, T, H_0(\operatorname{curl})) \to \mathbb{R}$:

$$l(\Phi) := - \int_0^T \langle J_t, \Phi \rangle_{H(\operatorname{curl}, \mathcal{O})} \, \mathrm{d}t + \int_{\mathcal{O}} \sigma E^0 \cdot \Phi(0) \, \mathrm{d}x.$$

To get around the non-uniqueness, cf. Theorem 5.5, we consider the Hilbert space

$$W_0 := \{ E \in H_0(\operatorname{curl}) \mid \operatorname{div} E = 0 \}$$

equipped with the norm $\|\operatorname{curl}\cdot\|_{L^2(\mathcal{O})^3}$, that is equivalent to the graph norm, cf. [GR86, Lemma 3.4]. Especially, there is a constant $C_{\mathcal{O}}$ only depending on $\mathcal{O}$ such that

$$\|E\|_{L^2(\mathcal{O})^3} \le C_{\mathcal{O}} \|\operatorname{curl} E\|_{L^2(\mathcal{O})^3}.$$

Let $H^1_{T0}(0,T,W_0) := \{\Psi \in H^1(0,T,W_0) \,|\, \Psi(T) = 0\}$.

5.7 Theorem *(cf. Theorem 3.6a)) If $\tilde{E} \in L^2(0,T,W_0)$ solves*

$$a(\tilde{E},\Phi) = l(\Phi) \quad \text{for all } \Phi \in H^1_{T0}(0,T,W_0), \tag{5.13}$$

then $\tilde{E} + \nabla u_{\tilde{E}} \in L^2(0,T,H_0(\operatorname{curl}))$ solves (5.7)–(5.8).

Proof Obviously, for fields $\nabla\varphi \in H^1(0,T,H_0(\operatorname{curl}))$ with $\varphi \in H^1(0,T,H^1(\mathcal{O}))$, $a(\cdot,\nabla\varphi)$ as well as $l(\nabla\varphi)$ vanish. (For the latter, recall that $\operatorname{div} J_t = 0$ and $\operatorname{div}(\sigma E^0) = 0$.) Now we use the following simple decomposition (cf. Lemma 3.9): Every $\Phi \in \mathcal{D}(\mathcal{O})^3$ can be written as

$$\Phi = \Psi + \nabla\varphi, \tag{5.14}$$

with $\Psi \in W_0$, $\varphi \in H^1_0(\mathcal{O})$. From that and the linearity of a and l it follows, that (for any $\tilde{E} \in L^2(0,T,W_0)$)

$$a(\tilde{E},\Phi) = l(\Phi)$$

holds for all $\Phi \in \mathcal{D}(\mathcal{O} \times [0,T[)^3$, if it holds for all $\Phi \in H^1_{T0}(0,T,W_0)$. Lemma 5.4 yields the assertion. □

We now show that (5.13) is well-posed. We use the Lions-Lax-Milgram Theorem 3.10.

5.8 Theorem *(cf. Theorem 3.6b)) There is a unique solution $\tilde{E} \in L^2(0,T,W_0)$ of (5.13). $\tilde{E}$ depends continuously on J_t and $\sqrt{\sigma}E^0$ and with $\alpha = \max(\|\mu\|_\infty, 2)$ it holds, that*

$$\|\tilde{E}\|_{L^2(0,T,W_0)} \le \alpha\sqrt{2}\max\left((C^2_{\mathcal{O}}+1)^{1/2}\|J_t\|_{L^2(0,T,H(\operatorname{curl},\mathcal{O})')}, \|\sqrt{\sigma}E^0\|_{L^2(\mathcal{O})^3}\right). \tag{5.15}$$

$\tilde{E} + \nabla u_{\tilde{E}}$ solves the eddy current problem (5.7)–(5.8) and any other solution $E \in L^2(0,T,H_0(\operatorname{curl}))$ of (5.7)–(5.8) fulfills

$$\operatorname{curl} E = \operatorname{curl}\tilde{E}, \quad \sqrt{\sigma}E = \sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E}}). \tag{5.16}$$

$\operatorname{curl} E$ and $\sqrt{\sigma}E$ depend continuously on J_t and $\sqrt{\sigma}E^0$:

$$\|\operatorname{curl} E\|_{L^2(\mathcal{O}_T)^3} \le \alpha\sqrt{2}\max\left((C^2_{\mathcal{O}}+1)^{1/2}\|J_t\|_{L^2(0,T,H(\operatorname{curl},\mathcal{O})')}, \|\sqrt{\sigma}E^0\|_{L^2(\mathcal{O})^3}\right),$$
$$\|\sqrt{\sigma}E\|_{L^2(\mathcal{O}_T)^3} \le 2C_{\mathcal{O}}\|\sqrt{\sigma}\|_\infty \|\operatorname{curl} E\|_{L^2(\mathcal{O}_T)^3}.$$

Proof To apply the Lions-Lax-Milgram Theorem we use the Hilbert space $\mathcal{H} := L^2(0,T,W_0)$ and equip its subspace $V := H^1_{T0}(0,T,W_0)$ with the norm

$$\|\Phi\|_V^2 := \|\Phi\|^2_{L^2(0,T,W_0)} + \|\sqrt{\sigma}(\Phi + \nabla u_\Phi)(0)\|^2_{L^2(\mathcal{O})^3}.$$

Then, it is straightforward to show that for fixed $\Phi \in V$ the linear form $E \mapsto a(E,\Phi)$ is continuous on $\mathcal{H}$ and that $l \in V'$ with

$$\|l\|_{V'} \leq \sqrt{2} \max\left((C_\mathcal{O}^2 + 1)^{1/2} \|J_t\|_{L^2(0,T,H(\mathrm{curl},\mathcal{O})')}, \|\sqrt{\sigma}E^0\|_{L^2(\mathcal{O})^3}\right).$$

Moreover, for $\Phi \in V$, Lemma 5.6 and the integration by parts formula (5.5) yield that

$$a(\Phi,\Phi) \geq \frac{1}{2}\|\sqrt{\sigma}(\Phi + \nabla u_\Phi)(0)\|^2_{L^2(\mathcal{O})^3} + \frac{1}{\|\mu\|_\infty}\|\Phi\|^2_{L^2(0,T,W_0)}, \tag{5.17}$$

which implies, that

$$\inf_{\|\Phi\|_V=1} \sup_{\|E\|_\mathcal{H}\leq 1} |a(E,\Phi)| \geq \frac{1}{\alpha}.$$

Now, Lemma 3.10 yields the existence of an $\tilde{E} \in \mathcal{H}$ that fulfills (5.13) and depends continuously on l.

Theorem 5.7 yields that $\tilde{E} + \nabla u_{\tilde{E}} \in L^2(0,T,H_0(\mathrm{curl}))$ is a solution of the eddy current equation (5.7) and (5.8).

To show uniqueness, let $\tilde{E}_1, \tilde{E}_2 \in L^2(0,T,W_0)$ be two solutions of (5.13). Then, $\tilde{E}_1 + \nabla u_{\tilde{E}_1}, \tilde{E}_2 + \nabla u_{\tilde{E}_2} \in L^2(0,T,H_0(\mathrm{curl}))$ both solve equations (5.7)–(5.8) and Theorem 5.5 implies $\tilde{E}_1 = \tilde{E}_2$.

The remaining assertions follow similarly from Theorem 5.5. □

5.9 Corollary *Let $(\sigma_n)_{n\in\mathbb{N}} \subset L_C \cup L^\infty_+(\mathcal{O})$ be a bounded sequence and $\tilde{E}_n$, $n \in \mathbb{N}$, be the corresponding unique solutions of* (5.13). *Then the sequences*

$$(\tilde{E}_n)_{n\in\mathbb{N}} \subset L^2(0,T,W_0),\ (\sqrt{\sigma_n}\tilde{E}_n)_{n\in\mathbb{N}},\ (\sqrt{\sigma_n}\nabla u_{\tilde{E}_n})_{n\in\mathbb{N}} \subset L^2(\mathcal{O}_T)^3$$

are bounded. The bounds depend on the bound of $(\sigma_n)_{n\in\mathbb{N}}$.

In particular, for any sequence $(E_n)_{n\in\mathbb{N}} \subset L^2(0,T,H_0(\mathrm{curl}))$ of corresponding solutions of (5.7)–(5.8) *the sequences*

$$(\mathrm{curl}\, E_n)_{n\in\mathbb{N}},\ (\sqrt{\sigma}_n E_n)_{n\in\mathbb{N}} \subset L^2(\mathcal{O}_T)^3$$

are bounded.

5.10 Remark The results from Section 3.4 on the dependence of the solution on the conductivity, in particular the solution's sensitivity with respect to the eddy current equation changing from elliptic to parabolic type, can be directly carried over to the bounded setting.

5.4 Parabolic regularization

In this section we keep $\sigma \in L_C$, $E^0 \in L^2(\mathcal{O})^3$ with $\operatorname{div}(\sigma E^0) = 0$ and J_t as in Section 5.2 fixed and analyze the solution(s) behavior corresponding to

$$\sigma_\varepsilon = \begin{cases} \sigma, & x \in \Omega, \\ \varepsilon, & x \in \mathcal{O} \setminus \overline{\Omega}, \end{cases}$$

if the positive real number ε approaches zero. Obviously, we have $\lim_{\varepsilon \to 0} \sigma_\varepsilon = \sigma$ in $L^\infty(\mathcal{O})$. In that way, the eddy current equation is made fully parabolic:

$$\partial_t(\sigma_\varepsilon E_\varepsilon) + \operatorname{curl}\left(\frac{1}{\mu} \operatorname{curl} E_\varepsilon\right) = -J_t. \tag{5.18}$$

Our main result is Theorem 5.14, where we show that the relevant parts of the solutions of (5.18), i.e. $\operatorname{curl} E_\varepsilon$ and $\sigma_\varepsilon E_\varepsilon$, converge against the corresponding unique parts of the solutions of the eddy current equation

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu} \operatorname{curl} E\right) = -J_t$$

if ε tends to zero. Therefore, we use the variational formulation (5.13) and show that its (unique) solutions converge (cf. Theorem 5.13).

Let us first remark, that, since $\sigma_\varepsilon \in L^\infty_+(\mathcal{O})$, the theory of Sections 5.2 and 5.3 (with appropriate initial conditions) holds. Especially, (5.18) is uniquely solvable, and the unique solution is given by $\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon}$, where $\tilde{E}_\varepsilon \in L^2(0,T,W_0)$ is the unique solution of (5.13) with $\sigma = \sigma_\varepsilon$ and $\nabla u_{\tilde{E}_\varepsilon,\varepsilon}$ is its image under the mapping from Lemma 5.6 with $\sigma = \sigma_\varepsilon$.

We start with the analysis of the mapping from Lemma 5.6,

$$L^2(\mathcal{O})^3 \to H_0(\operatorname{curl} 0), \quad E \mapsto \nabla u_{E,\varepsilon}$$

such that $\operatorname{div}(\sigma_\varepsilon(E + \nabla u_{E,\varepsilon})) = 0$, as $\varepsilon \to 0$. Here, we indicate the nonlinear dependence of u_E on σ_ε by $u_{E,\varepsilon}$.

5.11 Lemma *Let $(F_\varepsilon) \subset L^2(\mathcal{O})^3$ with $F_\varepsilon \rightharpoonup F \in L^2(\mathcal{O})^3$ as $\varepsilon \to 0$. Let $(u_{F_\varepsilon,\varepsilon}) \subset H^1_0(\mathcal{O})$ denote the corresponding unique elements from Lemma 5.6, that solve*

$$\int_{\mathcal{O}} \sigma_\varepsilon \nabla u_{F_\varepsilon,\varepsilon} \cdot \nabla v \,\mathrm{d}x = -\int_{\mathcal{O}} \sigma_\varepsilon F_\varepsilon \cdot \nabla v \,\mathrm{d}x \quad \text{for all } v \in H^1_0(\mathcal{O})$$

and let $u_{F,\sigma} \in H^1_0(\mathcal{O})$ be the corresponding element from Lemma 5.6 (that is unique by construction). Then

a) $\|\sqrt{\sigma_\varepsilon} F_\varepsilon\|_{L^2(\mathcal{O}\setminus\overline{\Omega})^3} \to 0$, $\sqrt{\sigma_\varepsilon} F_\varepsilon \rightharpoonup \sqrt{\sigma} F$ in $L^2(\mathcal{O})^3$ and $(\sqrt{\sigma_\varepsilon} \nabla u_{F_\varepsilon,\varepsilon}) \subset L^2(\mathcal{O})^3$ is bounded, *b)* $\sigma_\varepsilon \nabla u_{F_\varepsilon,\varepsilon} \rightharpoonup \sigma \nabla u_{F,\sigma} \in L^2(\mathcal{O})^3$

as $\varepsilon \to 0$. *Especially, for fixed* $F \in L^2(\mathcal{O})^3$ *it holds that* $\sqrt{\sigma_\varepsilon} F \to \sqrt{\sigma} F$ *in* $L^2(\mathcal{O})^3$ *and* $\sqrt{\sigma_\varepsilon}\nabla u_{F,\varepsilon} \to \sqrt{\sigma}\nabla u_{F,\sigma}$ *in* $L^2(\mathcal{O})^3$.

Proof Let $\varphi \in L^2(\mathcal{O})^3$.

a) Obviously, it holds that $\|\sqrt{\sigma_\varepsilon} F_\varepsilon\|_{L^2(\mathcal{O}\setminus\overline{\Omega})^3} = \sqrt{\varepsilon}\|F_\varepsilon\|_{L^2(\mathcal{O}\setminus\overline{\Omega})^3} \to 0$,

$$(\sqrt{\sigma_\varepsilon} F_\varepsilon - \sqrt{\sigma} F, \varphi)_{L^2(\mathcal{O})^3} = \sqrt{\varepsilon}(F_\varepsilon, \varphi)_{L^2(\mathcal{O}\setminus\overline{\Omega})^3} + (F_\varepsilon - F, \sqrt{\sigma}\varphi)_{L^2(\Omega)^3} \to 0,$$

and since

$$\|\sqrt{\sigma_\varepsilon}\nabla u_{F_\varepsilon,\varepsilon}\|_{L^2(\mathcal{O})^3} \leq \|\sqrt{\sigma_\varepsilon} F_\varepsilon\|_{L^2(\mathcal{O})^3}$$

we obtain, that $(\sqrt{\sigma_\varepsilon}\nabla u_{F_\varepsilon,\varepsilon})$ is bounded in $L^2(\mathcal{O})^3$.

b) First we show that every subsequence of $(\sqrt{\sigma_\varepsilon}\nabla u_{F_\varepsilon,\varepsilon})$ has a subsequence that converges weakly against $\sqrt{\sigma}\nabla h$ for some $h \in H^1_0(\mathcal{O})$. In a second step we show that all these weak limits coincide.

Since $(\sqrt{\sigma_\varepsilon}\nabla u_{F_\varepsilon,\varepsilon}) \subset L^2(\mathcal{O})^3$ is bounded, every subsequence is bounded, and Alaoglu's Theorem, cf., e.g., [RR04, Theorem 6.62], yields that every subsequence contains subsequence (that we still indicate by ε for the ease of notation), again, that converges weakly against some $a \in L^2(\mathcal{O})^3$:

$$\sqrt{\sigma_\varepsilon}\nabla u_{F_\varepsilon,\varepsilon} \rightharpoonup a \in L^2(\mathcal{O})^3.$$

We then also have

$$\sqrt{\sigma_\varepsilon}\nabla u_{F_\varepsilon,\varepsilon}|_\Omega = \sqrt{\sigma}\nabla u_{F_\varepsilon,\varepsilon}|_\Omega \rightharpoonup a|_\Omega \in L^2(\Omega)^3$$

and therefore

$$\nabla u_{F_\varepsilon,\varepsilon}|_\Omega \rightharpoonup \frac{a|_\Omega}{\sqrt{\sigma}} \in L^2(\Omega)^3.$$

The orthogonal decomposition

$$\nabla H^1(\Omega) \oplus^\perp H_0(\operatorname{div} 0, \Omega) = L^2(\Omega)^3,$$

cf. [DL00c, IX, §3, Proposition 1], where

$$H_0(\operatorname{div} 0, \mathcal{O}) = \{E \in L^2(\mathcal{O})^3 \mid \operatorname{div} E = 0,\ \nu \cdot E|_\Sigma = 0\},$$

yields then $\frac{a|_\Omega}{\sqrt{\sigma}} \in \nabla H^1(\Omega)$ and hence there is some $h \in H^1(\Omega)$ with

$$\frac{a|_\Omega}{\sqrt{\sigma}} = \nabla h.$$

Obviously, ∇h is uniquely determined, but h is not. To overcome this, we fix h by the choice $h \in H^1_\square(\Omega)$ as in Lemma 5.6 and extend it to an element of $H^1_0(\mathcal{O})$ by solving $\Delta h = 0$ on $\mathcal{O} \setminus \overline{\Omega}$. Then it still holds that

$$\sqrt{\sigma}\nabla u_{F_\varepsilon,\varepsilon} \rightharpoonup \sqrt{\sigma}\nabla h \ \text{ in } \ L^2(\mathcal{O})^3$$

and hence

$$(\sigma_\varepsilon \nabla u_{F_\varepsilon,\varepsilon} - \sigma\nabla h, \varphi)_{L^2(\mathcal{O})^3} = \\ (\sigma\nabla u_{F_\varepsilon,\varepsilon} - \sigma\nabla h, \varphi)_{L^2(\Omega)^3} + \sqrt{\varepsilon}(\sqrt{\varepsilon}\nabla u_{F_\varepsilon,\varepsilon}, \varphi)_{L^2(\mathcal{O}\setminus\overline{\Omega})^3} \to 0,$$

i.e. $\sigma_\varepsilon \nabla u_{F_\varepsilon,\varepsilon} \rightharpoonup \sigma\nabla h$ in $L^2(\mathcal{O})^3$.

To conclude, that all these weak limits are identical, we show

$$\sigma\nabla h = \sigma\nabla u_{F,\sigma}.$$

For every $v \in H^1_0(\mathcal{O})$, a) yields

$$\begin{aligned} 0 &= \int_{\mathcal{O}} \sigma_\varepsilon \nabla u_{F_\varepsilon,\varepsilon} \cdot \nabla v \,\mathrm{d}x + \int_{\mathcal{O}} \sigma_\varepsilon F_\varepsilon \cdot \nabla v \,\mathrm{d}x \\ &\to \int_{\Omega} \sigma\nabla h \cdot \nabla v \,\mathrm{d}x + \int_{\Omega} \sigma F \cdot \nabla v \,\mathrm{d}x \end{aligned}$$

and therefore also the right hand side vanishes for every $v \in H^1_0(\mathcal{O})$. Accordingly, $\sigma\nabla h = \sigma\nabla u_{F,\sigma}$ and $\nabla h|_\Omega = \nabla u_{F,\sigma}|_\Omega$.

Altogether, the second assertion follows. $\square$

The next step is to show that the sequence of solutions of the variational equation (5.13) converge.

To obtain meaningful initial values for (5.18), we modify the initial value $E^0 \in L^2(\mathcal{O})^3$ to make its product with σ_ε divergence-free by $E^0 + \nabla u_{E^0,\varepsilon}$. The precedent Lemma then yields $\sqrt{\sigma_\varepsilon}(E^0 + \nabla u_{E^0,\varepsilon}) \to \sqrt{\sigma}E^0$ in $L^2(\mathcal{O})^3$ and the right hand side of (5.13), $l_\varepsilon : H^1(0,T,H_0(\mathrm{curl})) \to \mathbb{R}$, obviously fulfills

$$\begin{aligned} l_\varepsilon(\Phi) &:= -\int_0^T \langle J_t, \Phi\rangle_{H(\mathrm{curl},\mathcal{O})} \,\mathrm{d}t + \int_{\mathcal{O}} \sigma_\varepsilon(E^0 + \nabla u_{E^0,\varepsilon}) \cdot \Phi(0) \,\mathrm{d}x \\ &\to -\int_0^T \langle J_t, \Phi\rangle_{H(\mathrm{curl},\mathcal{O})} \,\mathrm{d}t + \int_{\mathcal{O}} \sigma E^0 \cdot \Phi(0) \,\mathrm{d}x = l(\Phi) \quad \text{as } \varepsilon \to 0 \end{aligned}$$

for every $\Phi \in H^1(0,T,H_0(\mathrm{curl}))$.

Corresponding to σ_ε let $\tilde{E}_\varepsilon \in L^2(0,T,W_0)$ denote the unique solution of

$$a_\varepsilon(\tilde{E}_\varepsilon, \Phi) = l_\varepsilon(\Phi) \qquad \text{for all } \Phi \in H^1_{T0}(0,T,W_0), \tag{5.19}$$

that is (5.13) with $\sigma = \sigma_\varepsilon$. The bilinear form a_ε is then given by

$$a_\varepsilon : L^2(0,T,H_0(\mathrm{curl})) \times H^1(0,T,H_0(\mathrm{curl})) \to \mathbb{R} :$$
$$a_\varepsilon(E,\Phi) := -\int_0^T \int_{\mathcal{O}} \sigma_\varepsilon(E + \nabla u_{E,\varepsilon}) \cdot \dot{\Phi} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathcal{O}} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \,\mathrm{d}x\,\mathrm{d}t.$$

The next lemma shows that the solutions converge weakly towards the solution $\tilde{E} \in L^2(0,T,W_0)$ of (5.13) (that corresponds to $\varepsilon = 0$).

5.12 Lemma *It holds, that*

$$\tilde{E}_\varepsilon \rightharpoonup \tilde{E} \text{ in } L^2(0,T,W_0) \text{ and } \sqrt{\sigma_\varepsilon}\tilde{E}_\varepsilon \rightharpoonup \sqrt{\sigma}\tilde{E},\ \ \sigma_\varepsilon \nabla u_{\tilde{E}_\varepsilon,\varepsilon} \rightharpoonup \sigma \nabla u_{\tilde{E},\sigma} \text{ in } L^2(\mathcal{O}_T)^3$$

as $\varepsilon \to 0$.

Proof The precedent Lemma yields that it suffices to show that $\tilde{E}_\varepsilon \rightharpoonup \tilde{E}$. To show this, we use the same technique: From Corollary 5.9 we know that $(\tilde{E}_\varepsilon) \subset L^2(0,T,W_0)$ is bounded. Again, Alaoglu's Theorem yields that every subsequence contains a subsequence (that we still denote by $(\tilde{E}_\varepsilon)$ for ease of notation) that converges weakly against some $\tilde{E}' \in L^2(0,T,W_0)$. In the following we show that all these weak limits are identical to $\tilde{E}$.

The previous Lemma yields

$$\sqrt{\sigma_\varepsilon}\tilde{E}_\varepsilon \rightharpoonup \sqrt{\sigma}\tilde{E}' \text{ in } L^2(\mathcal{O}_T)^3$$

and

$$\sigma_\varepsilon \nabla u_{\varepsilon,\tilde{E}_\varepsilon} \rightharpoonup \sigma \nabla u_{\tilde{E}',\sigma} \in L^2(\mathcal{O}_T)^3.$$

Moreover, $\tilde{E}_\varepsilon \rightharpoonup \tilde{E}'$ in $L^2(0,T,W_0)$ implies that $\operatorname{curl} \tilde{E}_\varepsilon \rightharpoonup \operatorname{curl} \tilde{E}'$ in $L^2(\mathcal{O}_T)^3$, so that for every $\Phi \in H^1_{T0}(0,T,W_0)$ the left hand side $a_\varepsilon(\tilde{E}_\varepsilon,\Phi)$ of (5.13) with $\sigma = \sigma_\varepsilon$ converges against $a(\tilde{E}',\Phi)$:

$$a_\varepsilon(\tilde{E}_\varepsilon,\Phi) = -\int_0^T \int_{\mathcal{O}} \sigma_\varepsilon(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon}) \cdot \dot{\Phi} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathcal{O}} \frac{1}{\mu} \operatorname{curl} \tilde{E}_\varepsilon \cdot \operatorname{curl} \Phi \,\mathrm{d}x\,\mathrm{d}t$$
$$\to a(\tilde{E}',\Phi).$$

Since $l_\varepsilon(\Phi) \to l(\Phi)$, $\tilde{E}'$ solves (5.13) and thus uniqueness provides $\tilde{E} = \tilde{E}'$. □

5.13 Theorem *It holds, that* $\tilde{E}_\varepsilon \to \tilde{E}$ *in* $L^2(0,T,W_0)$, $\sqrt{\sigma_\varepsilon}\tilde{E}_\varepsilon \to \sqrt{\sigma}\tilde{E}$ *and* $\sqrt{\sigma_\varepsilon}\nabla u_{\tilde{E}_\varepsilon,\varepsilon} \to \sqrt{\sigma}\nabla u_{\tilde{E},\sigma}$ *in* $L^2(\mathcal{O}_T)^3$ *as* $\varepsilon \to 0$.

Proof Using the fact, that $\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon}$ solves (5.18) with initial values $\sqrt{\sigma_\varepsilon}(E^0 + \nabla u_{E^0,\varepsilon})$, the integration by parts formula (5.5) and Lemma 5.4b) we obtain for every ε, that

$$\begin{aligned}
&\|\mu^{-\frac{1}{2}} \operatorname{curl}\tilde{E}_\varepsilon\|^2_{L^2(\mathcal{O}_T)^3} + \frac{1}{2}\|\sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})(T)\|^2_{L^2(\mathcal{O})^3} \\
&\quad = -\int_0^T \langle J_t, \tilde{E}_\varepsilon\rangle_{H(\operatorname{curl},\mathcal{O})}\,\mathrm{d}t - \int_0^T \langle (\sigma(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\sigma}))^\cdot, \tilde{E}_\varepsilon\rangle_{H_0(\operatorname{curl},\mathcal{O})}\,\mathrm{d}t \\
&\qquad + \frac{1}{2}\|\sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})(T)\|^2_{L^2(\mathcal{O})^3} \\
&\quad = -\int_0^T \langle J_t, \tilde{E}_\varepsilon\rangle_{H(\operatorname{curl},\mathcal{O})}\,\mathrm{d}t + \frac{1}{2}\|\sqrt{\sigma_\varepsilon}(E^0 + \nabla u_{E^0,\varepsilon})\|^2_{L^2(\mathcal{O})^3}.
\end{aligned} \tag{5.20}$$

The precedent lemma and the fact, that $\tilde{E} + \nabla u_{\tilde{E},\sigma}$ solves (5.18) with initial values $\sqrt{\sigma}E^0$, analogously yields

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Big[\|\mu^{-\frac{1}{2}} \operatorname{curl}\tilde{E}_\varepsilon\|^2_{L^2(\mathcal{O}_T)^3} + \frac{1}{2}\|\sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})(T)\|^2_{L^2(\mathcal{O})^3}\Big] \\
&\quad = \lim_{\varepsilon\to 0}\Big[-\int_0^T \langle J_t, \tilde{E}_\varepsilon\rangle_{H(\operatorname{curl},\mathcal{O})}\,\mathrm{d}t + \frac{1}{2}\|\sqrt{\sigma_\varepsilon}(E^0 + \nabla u_{E^0,\varepsilon})\|^2_{L^2(\mathcal{O})^3}\Big] \\
&\quad = -\int_0^T \langle J_t, \tilde{E}\rangle_{H(\operatorname{curl},\mathcal{O})}\,\mathrm{d}t + \frac{1}{2}\|\sqrt{\sigma}E^0\|^2_{L^2(\mathcal{O})^3} \\
&\quad = \|\mu^{-\frac{1}{2}} \operatorname{curl}\tilde{E}\|^2_{L^2(\mathcal{O}_T)^3} + \frac{1}{2}\|\sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E},\sigma})(T)\|^2_{L^2(\mathcal{O})^3}.
\end{aligned} \tag{5.21}$$

This yields that $(\sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})(T)) \subset L^2(\mathcal{O})^3$ is bounded, hence every subsequence has a subsequence that converges weakly against some $H \in L^2(\mathcal{O})^3$. It follows for every $A \in \mathcal{D}(\mathcal{O}\times(0,T])$ that

$$\begin{aligned}
&(\sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})(T), \sqrt{\sigma_\varepsilon}A(T))_{L^2(\mathcal{O})^3} \\
&= \int_0^T \langle (\sigma(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon}))^\cdot, A\rangle_{H_0(\operatorname{curl},\mathcal{O})}\,\mathrm{d}t + \int_0^T \langle (\sigma_\varepsilon A)^\cdot, \tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon}\rangle_{H_0(\operatorname{curl},\mathcal{O})}\,\mathrm{d}t \\
&= -\int_0^T \langle J_t, \tilde{E}_\varepsilon\rangle_{H(\operatorname{curl},\mathcal{O})}\,\mathrm{d}t - \int_0^T\int_{\mathcal{O}} \frac{1}{\mu}\operatorname{curl}\tilde{E}_\varepsilon\cdot\operatorname{curl}A\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T\int_{\mathcal{O}} \sigma_\varepsilon(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})\cdot\partial_t A\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

As before we obtain

$$\begin{aligned}
(H, \sqrt{\sigma}A(T))_{L^2(\mathcal{O})^3} &= \lim_{\varepsilon\to 0}(\sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})(T), \sqrt{\sigma_\varepsilon}A(T))_{L^2(\mathcal{O})^3} \\
&= \lim_{\varepsilon\to 0}\Big[-\int_0^T \langle J_t, \tilde{E}_\varepsilon\rangle_{H(\operatorname{curl},\mathcal{O})}\,\mathrm{d}t - \int_0^T\int_{\mathcal{O}} \frac{1}{\mu}\operatorname{curl}\tilde{E}_\varepsilon\cdot\operatorname{curl}A\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_0^T\int_{\mathcal{O}} \sigma_\varepsilon(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})\cdot\partial_t A\,\mathrm{d}x\,\mathrm{d}t\Big] \\
&= (\sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E},\sigma})(T), \sqrt{\sigma}A(T))_{L^2(\mathcal{O})^3},
\end{aligned}$$

so that the denseness of $\sqrt{\sigma}\mathcal{D}(\Omega) \subset L^2(\Omega)$ implies $H|_\Omega = \sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E},\sigma})(T)|_\Omega$. It follows for the full sequence, that $\sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})(T) \rightharpoonup \sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E},\sigma})(T)$ in $L^2(\Omega)^3$. Now equation (5.21) yields

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\Big[\|\mu^{-\frac{1}{2}}\operatorname{curl}(\tilde{E}_\varepsilon - \tilde{E})\|^2_{L^2(\mathcal{O}_T)^3} \\
&\quad + \frac{1}{2}\|\sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})(T) - \sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E},\sigma})(T)\|^2_{L^2(\mathcal{O})^3}\Big] \\
&= \lim_{\varepsilon\to 0}\Big[\|\mu^{-\frac{1}{2}}\operatorname{curl}\tilde{E}_\varepsilon\|^2_{L^2(\mathcal{O}_T)^3} + \|\mu^{-\frac{1}{2}}\operatorname{curl}\tilde{E}\|^2_{L^2(\mathcal{O}_T)^3} \\
&\quad - 2(\mu^{-\frac{1}{2}}\operatorname{curl}\tilde{E}_\varepsilon, \mu^{-\frac{1}{2}}\operatorname{curl}\tilde{E})_{L^2(\mathcal{O}_T)^3} + \frac{1}{2}\|\sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})(T)\|^2_{L^2(\mathcal{O})^3} \\
&\quad + \frac{1}{2}\|\sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E},\sigma})(T)\|^2_{L^2(\mathcal{O})^3} \\
&\quad - (\sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})(T), \sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E},\sigma})(T))_{L^2(\Omega)^3}\Big] \\
&= 0.
\end{aligned}
\tag{5.22}
$$

Hence the first and the second assertion follow immediately. For the third assertion note that equation (5.22) holds for almost every $t \in (0, T)$ and that

$$\|\sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})(t) - \sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E},\sigma})(t)\|^2_{L^2(\mathcal{O})^3}$$

is uniformly bounded with respect to ε and t. Consequently we have

$$\lim_{\varepsilon\to 0}\|\sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon}) - \sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E},\sigma})\|^2_{L^2(\mathcal{O}_T)^3} = 0$$

so that the third assertion follows from the second assertion. □

Now we can formulate our main result. Corresponding to σ_ε, we denote by $E_\varepsilon \in L^2(0, T, H_0(\operatorname{curl}))$ the unique solution of (5.18) with initial values $\sqrt{\sigma_\varepsilon}(E^0 + \nabla u_{E^0,\varepsilon})$. For $\varepsilon = 0$, let $E \in L^2(0, T, H_0(\operatorname{curl}))$ denote any solution of (5.7)–(5.8).

5.14 Theorem *It holds, that* $\operatorname{curl} E_\varepsilon \to \operatorname{curl} E$ *and* $\sqrt{\sigma_\varepsilon}E_\varepsilon \to \sqrt{\sigma}E$ *in* $L^2(\mathcal{O}_T)^3$ *and* $(\sigma_\varepsilon E_\varepsilon)^\cdot \to (\sigma E)^\cdot$ *in* $L^2(0, T, H_0(\operatorname{curl})')$ *as* $\varepsilon \to 0$.

Proof It holds $\sqrt{\sigma_\varepsilon}E_\varepsilon = \sqrt{\sigma_\varepsilon}(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon,\varepsilon})$, $\operatorname{curl} E_\varepsilon = \operatorname{curl}\tilde{E}_\varepsilon$ and $\operatorname{curl} E = \operatorname{curl}\tilde{E}$, so that the precedent Lemma provides the first and the second assertion.

From the explicit form (5.6) of $(\sigma_\varepsilon E_\varepsilon)^\cdot$ given in Lemma 5.2, we obtain for all $F \in L^2(0, T, H_0(\operatorname{curl}))$

$$\left|\int_0^T \langle(\sigma_\varepsilon E_\varepsilon)^\cdot - (\sigma E)^\cdot, F\rangle_{H_0(\operatorname{curl})}\,\mathrm{d}t\right| = \left|\int_0^T\int_{\mathcal{O}}\frac{1}{\mu}\operatorname{curl}(E - E_\varepsilon)\cdot\operatorname{curl} F\,\mathrm{d}x\,\mathrm{d}t\right| \to 0.$$

This yields $(\sigma_\varepsilon E_\varepsilon)^\cdot \to (\sigma E)^\cdot$ in $L^2(0, T, H_0(\operatorname{curl})')$. □

5.5 Elliptic regularization

We finish this chapter by justifying an elliptic regularization. We keep $E^0 = 0$, J_t and $\sigma \in L_C \cup L_+^\infty(\mathcal{O})$ fixed and add the regularization term $\varepsilon E_\varepsilon$ to the left hand side of equation (5.1). This is a natural way to make the problem fully coercive and hence leads to a well-posed problem:

5.15 Theorem *For $E_\varepsilon \in L^2(0, T, H_0(\mathrm{curl}))$, the equations*

$$\partial_t(\sigma E_\varepsilon) + \mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\, E_\varepsilon\right) + \varepsilon E_\varepsilon = -J_t \qquad \text{in } \mathcal{O} \times (0, T), \tag{5.23}$$
$$\sqrt{\sigma} E_\varepsilon(0) = 0 \qquad \text{in } \mathcal{O}$$

are well-defined and equivalent to

$$-\int_0^T \int_{\mathcal{O}} \sigma E_\varepsilon \cdot \dot{\Phi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathcal{O}} \left[\frac{1}{\mu}\,\mathrm{curl}\, E_\varepsilon \cdot \mathrm{curl}\,\Phi + \varepsilon E_\varepsilon \cdot \Phi\right] \mathrm{d}x\,\mathrm{d}t$$
$$= -\int_0^T \langle J_t, \Phi\rangle_{H(\mathrm{curl},\mathcal{O})}\,\mathrm{d}t \qquad \text{for all } \Phi \in H^1_{T0}(0, T, H_0(\mathrm{curl})). \tag{5.24}$$

The variational problem (5.24) *is uniquely solvable. The solution depends continuously on ε and J_t:*

$$\|E_\varepsilon\|_{L^2(0,T,H_0(\mathrm{curl}))} \le \max\left(2, \|\mu\|_\infty, \frac{1}{\varepsilon}\right) \|J_t\|_{L^2(0,T,H(\mathrm{curl},\mathcal{O})')}.$$

Proof Well-definedness, equivalence and uniqueness follow as in Section 5.2. Moreover, the left hand side of equation (5.24) defines a bilinear form posed on $L^2(0, T, H_0(\mathrm{curl})) \times H^1(0, T, H_0(\mathrm{curl}))$, and the right hand side a linear form on $H^1(0, T, H_0(\mathrm{curl}))$. Then, the Lions-Lax-Milgram Theorem 3.10 (applied like in Theorem 5.8) yields a unique solution $E_\varepsilon \in L^2(0, T, H_0(\mathrm{curl}))$ that depends continuously on J_t. □

Unfortunately, we can not provide any assertion about the solutions behaviour if ε tends to zero. First of all, the precedent theorem does not contain any information about the boundedness of the regularized solutions. Beyond that, the variational formulation (5.24) of the regularized equation is not equivalent to our variational formulation of the eddy current equation (5.13) (appropriately regularized).

Anyway, in some applications, one might be interested in the variational formulation on itself. Therefore, we finish this chapter by justifying an elliptic regularization of the variational problem (5.13).

We modify the variational equation (5.13) in the following way. Let the left hand side $a_\varepsilon : L^2(0,T,H_0(\mathrm{curl})) \times H^1(0,T,H_0(\mathrm{curl})) \to \mathbb{R}$ be defined by

$$\begin{aligned} a_\varepsilon(E,\Phi) &:= a(E,\Phi) + \varepsilon(E,\Phi)_{L^2(\mathcal{O}_T)^3} \\ &= -\int_0^T \int_{\mathcal{O}} \sigma(E+\nabla u_E)\cdot \dot{\Phi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathcal{O}} \frac{1}{\mu}\,\mathrm{curl}\,E\cdot \mathrm{curl}\,\Phi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T \int_{\mathcal{O}} \varepsilon E\cdot\Phi\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

for some $\varepsilon > 0$. Then, a_ε is (with respect to the space variable) coercive on the whole space $H_0(\mathrm{curl})$.

We consider the variational problem of finding $\tilde{E}_\varepsilon \in L^2(0,T,W_0)$ that solves

$$a_\varepsilon(\tilde{E}_\varepsilon,\Phi) = l(\Phi) \quad \text{for all } \Phi \in H^1_{T0}(0,T,W_0) \tag{5.25}$$

and study the solutions behavior if ε tends to zero.

In the following we show that the solutions of (5.25) converge against the solution of (5.13), if ε tends to zero. Therefore, let us shortly answer the question of well-posedness of (5.25). Obviously, the problem of finding $\tilde{E}_\varepsilon \in L^2(0,T,W_0)$ that solves (5.25) for all $\Phi \in H^1_{T0}(0,T,W_0)$ still fits into the framework of the proof of the first part of Theorem 5.8 and hence there is a solution. Moreover, it can be shown that if $\tilde{E}_\varepsilon \in L^2(0,T,W_0)$ is such a solution, then $\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon} \in \mathcal{W}_{\sigma,\mathcal{O}}$ (cf. Lemma 5.2 and the proof of Lemma 3.3). Therefore, the integration by parts formula (5.5) holds and a result similar to Lemma 5.4. Using this, one easily sees that $\tilde{E}_\varepsilon$ is unique.

5.16 Theorem *Let $\tilde{E} \in L^2(0,T,W_0)$ denote the unique solution of* (5.13) *and $\tilde{E}_\varepsilon \in L^2(0,T,W_0)$ denote the unique solution of* (5.25). *Then we have $\tilde{E}_\varepsilon \to \tilde{E}$ in $L^2(0,T,W_0)$ as $\varepsilon \to 0$.*

Proof First of all the coercivity and continuity constants in Theorem 5.8 are the same for both, the regularized and the original problem. Therefore, Theorem 5.9 yields that $\tilde{E}_\varepsilon$ is bounded. Moreover, it obviously holds for all $F \in L^2(0,T,W_0)$ that

$$0 = l(F) - l(F) = a_\varepsilon(\tilde{E}_\varepsilon,F) - a(\tilde{E},F) = a(\tilde{E}_\varepsilon - \tilde{E},F) + \varepsilon(\tilde{E}_\varepsilon,F)_{L^2(\mathcal{O}_T)^3}.$$

By use of a similar equivalent formulation as in Lemma 5.4b), we obtain with $\alpha = \max(\|\mu\|_\infty, 2)$ that

$$\begin{aligned} \|\tilde{E}_\varepsilon - \tilde{E}\|^2_{L^2(0,T,W_0)} &\le \alpha\varepsilon(\tilde{E}_\varepsilon, \tilde{E}_\varepsilon - \tilde{E})_{L^2(\mathcal{O}_T)^3} \\ &\le \alpha\varepsilon C^2_{\mathcal{O}}\|\tilde{E}_\varepsilon\|_{L^2(0,T,W_0)}\|\tilde{E}_\varepsilon - \tilde{E}\|_{L^2(0,T,W_0)} \end{aligned}$$

and hence

$$\|\tilde{E}_\varepsilon - \tilde{E}\|_{L^2(0,T,W_0)} \le \alpha\varepsilon C^2_{\mathcal{O}}\|\tilde{E}_\varepsilon\|_{L^2(0,T,W_0)}.$$

The assertion follows from the fact, that $\|\tilde{E}_\varepsilon\|_{L^2(0,T,W_0)}$ is bounded. □

In addition, one can show as in Section 5.4 that

$$\sigma(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon}) \to \sigma(\tilde{E} + \nabla u_{\tilde{E}}) \quad \text{and} \quad (\sigma(\tilde{E}_\varepsilon + \nabla u_{\tilde{E}_\varepsilon}))^{\cdot} \to (\sigma(\tilde{E} + \nabla u_{\tilde{E}}))^{\cdot}$$

as $\varepsilon \to 0$.

Let us stress again that, in contrast to the parabolic regularization, we do not have any assertion about the solutions of the related (but not equivalent) regularized eddy current problem (5.23). This is due to the fact that a solution of (5.25) does not naturally imply a solution of (5.23), as it is the case for the original problem, cf. Theorem 5.7 and the parabolic regularization in Section 5.4.

5.6 Conclusion

We have considered the transient eddy current equation in a bounded domain consisting of a conducting and a non-conducting part, which are described by the conductivity coefficient. A consequence is, that the equation is of parabolic-elliptic type and does not determine its solutions uniquely in the non-conducting part.

We have presented a variational solution theory, that is uniquely solvable and whose solution represents all solutions of the eddy current equation. This solution theory treats the conductivity merely as a parameter, especially it does not depend on the conducting region. We have used this theory to show a parabolic and an elliptic regularization for the equation.

A natural way to regularize the equation is to set the conductivity to a small positive value ε in the non-conducting part. Then the resulting equation is fully parabolic and leads to a well-posed problem. We have justified this regularization by proving the convergence of its solutions against the solution of the original parabolic-elliptic equation if ε tends to zero.

We have also showed an adequate result for an elliptic regularization.

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