Dissertation

Artin-Tate motives and cell modules

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Abstract

Spitzweck's representation theorem states that the triangulated category of mixed Tate motives, $\mathrm{DMT}(k)$, over a perfect field k is equivalent to the bounded homotopy category of finite $\mathcal{N}(k)$-cell modules, $\mathcal{KCM}^f_{\mathcal{N}(k)}$, where $\mathcal{N}(k)$ is the cycle algebra over k. The category $\mathrm{DMT}(k)$ is a full triangulated subcategory of the category of mixed Artin-Tate motives, $\mathrm{DMAT}(k)$. For a number field k, we construct a category of cell modules that is equivalent to $\mathrm{DMAT}(k)$ and restricts to the equivalence given by Spitzweck's representation theorem. Furthermore, $\mathrm{DMT}(k)$ and $\mathrm{DMAT}(k)$ carry non-degenerate t-structures whose hearts are the Tannakian categories $\mathrm{MT}(k)$ respectively $\mathrm{MAT}(k)$. We compute the Tannaka group of $\mathrm{MAT}(k)$ as the semi-direct product of the absolute Galois group of k and the Tannaka group of $\mathrm{MT}(\bar{k})$.

Contents

Introduction

The concept of motives was introduced by Grothendieck. The idea behind an abelian category $\mathcal{M}(k)$ of motives over a field k (opposed to the non-abelian categories Var_k and Sm_k of varieties respectively smooth schemes over k) is to provide a "universal" cohomology theory for algebraic varieties X over k that should contain the data of all reasonable cohomology theories of a variety (such as singular, Betti or de Rham cohomology). This means that for any algebraic variety X over k there should be an object $m(X) \in \mathcal{M}(k)$, the motive of X, such that any cohomology theory $H_?$ factors through the functor m, i.e. for any cohomology theory there should be a functor $\rho_?$ from $\mathcal{M}(k)$ to abelian groups, called the realization, with $H_?(X) = \rho_?(m(X))$.

There are several approaches to construct such an abelian category $\mathcal{M}(k)$ of motives, e.g. the category of Nori motives (see [HMS17]). However, we follow Voevodsky's construction of the triangulated category of geometric motives $\mathrm{DM}_{\mathrm{gm}}(k)$ over a perfect field k in [Voe00]. Here, $\mathrm{DM}_{\mathrm{gm}}(k)$ is not abelian and should be thought of as the derived category of a (conjectural) abelian category of mixed motives. Again, there are different approaches to construct this motivic triangulated category, such as by Hanamura, Huber or Levine (see [Lev05] for an overview) that turn out to be equivalent to Voevodsky's construction. Even though Voevodsky's category $\mathrm{DM}_{\mathrm{gm}}(k)$ is not abelian, it still carries more structures than the category Sm_k of smooth schemes over k, such as providing long exact sequences, certain isomorphisms (e.g. $X \times \mathbb{A}^1 \longrightarrow X$), the existence of pull-back maps and products.

Voevodsky constructs the category $\mathrm{DM}_{\mathrm{gm}}(k)$ in several steps. Starting with the category Sm_k of smooth schemes over k, one substitutes the morphisms $X \to Y$ of schemes over k by finite k-correspondences. These are free abelian groups of algebraic cycles, i.e. integral closed subschemes of

$X \times_k Y$. This yields the additive category $\mathrm{Cor}(k)$. Passing to the bounded homotopy category $\mathcal{K}^b(\mathrm{Cor}(k))$ gives a triangulated category and therefore provides long exact sequences. Localising this triangulated category by inverting $X \times \mathbb{A}^1 \to X$ and the Mayer-Vietoris sequence and taking the pseudo-abelian hull yields the triangulated category of effective geometric motives $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

Sending a scheme X to the same object concentrated in degree 0 and a morphism $f\colon X \to Y$ to its graph $f_* := \Gamma_f \subset X \times_k Y$ defines an embedding $\mathrm{Sm}_k \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$. We denote the image of X under this embedding by $[X]$ and call it the motive of X.

Inside $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ we consider the complex $[\operatorname{Spec} k] \xrightarrow{i_{\infty *}} [\mathbb{P}^1_k]$. We call this complex the Tate motive and denote it by $\mathbb{Z}_k(1)$. The Tate motive $\mathbb{Z}_k(1)$ and its tensor powers $\mathbb{Z}_k(q) := \mathbb{Z}_k(1)^{\otimes q}$, where the tensor product is given by the fibre product of schemes, are of particular interest (see below).

Formally inverting the Tate motive $\mathbb{Z}_k(1)$ in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$, we get the triangulated tensor category $\mathrm{DM}_{\mathrm{gm}}(k)$ of geometric motives over k.

An important application of the category $\mathrm{DM}_{\mathrm{gm}}(k)$ is the motivic cohomology of a scheme $X \in \mathrm{Sm}_k$. Voevodsky defines it using the tensor powers $\mathbb{Z}_k(q) := \mathbb{Z}_k(1)^{\otimes q}$, $q \in \mathbb{Z}$, of the Tate motive as

$$\mathrm{H}^p(X, \mathbb{Z}(q)) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}([X], \mathbb{Z}_k(q)[p])$$

and with coefficients in an arbitrary ring A as

$$\mathrm{H}^p(X, A(q)) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}([X], \mathbb{Z}_k(q)[p]) \otimes_{\mathbb{Z}} A.$$

In [Voe02], Voevodsky shows that the motivic cohomology groups are isomorphic to the higher Chow groups which in turn are related to algebraic K-theory. In the case of $\mathbb{Q}$-coefficients Levine proves in [Lev94] that there is an isomorphism between the higher Chow groups and the (graded pieces of the gamma filtration on) algebraic K-groups. More precisely:

$$\mathrm{H}^p(X, \mathbb{Q}(q)) \simeq \mathrm{CH}^q(X, 2q-p; \mathbb{Q}) \simeq K_{2q-p}(X)^{(q)}_{\mathbb{Q}}.$$

Furthermore, for $X = \operatorname{Spec} k$ there is an isomorphism of motivic cohomology and Milnor K-theory

$$\mathrm{H}^p(\operatorname{Spec} k, A(p)) \simeq K^M_p(k)$$

that has been used by Voevodsky in [Voe03] to prove the Milnor conjecture.

Another example of a motivic proof of an a priori "non-motivic" claim is Brown's proof of a conjecture by Hoffmann in [Hof97] stating that every multiple zeta value

$$\zeta(n_1,\ldots,n_r) := \sum_{0<k_1<\ldots<k_r} \frac{1}{n_1^{k_1}\ldots n_r^{k_r}}$$

is a $\mathbb{Q}$-linear combination of multiple zeta values $\zeta(n_1,\ldots,n_s)$, where $n_i \in \{2,3\}$ ([Bro12]). His proof uses the category of mixed Tate motives, $\mathrm{DMT}(k)$, which is defined as the full triangulated subcategory generated by the tensor powers of the Tate motive

$$\mathbb{Q}_k(1) := ([\operatorname{Spec} k] \xrightarrow{i_{\infty *}} [\mathbb{P}^1_k])$$

inside $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$, the $\mathbb{Q}$-linearisation of $\mathrm{DM}_{\mathrm{gm}}(k)$. Putting $\mathbb{Q}_k(0) := [\operatorname{Spec} k]$, the morphisms between its generators are given by the motivic cohomology of $\operatorname{Spec} k$ and hence by the rational K-groups of k:

$$\mathrm{Hom}_{\mathrm{DMT}(k)}(\mathbb{Q}_k(0), \mathbb{Q}_k(q)[p]) \cong K_{2q-p}(k)^{(q)}_{\mathbb{Q}},$$

where $\mathbb{Q}_k(q) := \mathbb{Q}_k(1)^{\otimes q}$ and $q,\ p \in \mathbb{Z}$. This makes the category $\mathrm{DMT}(k)$ more accessible for concrete computations than $\mathrm{DM}_{\mathrm{gm}}(k)$. If k is a number field, these K-groups are well-known. In this case, $\mathrm{DMT}(k)$ carries a non-degenerate t-structure that yields the Tannakian category $\mathrm{MT}(k)$ of mixed Tate motives as its heart, as shown by Levine in [Lev10]. This means, $\mathrm{DMT}(k)$ behaves like a classical derived category, e.g.

$$\mathrm{Ext}^p_{\mathrm{MT}(k)}(\mathbb{Q}_k(0), \mathbb{Q}_k(q)) \simeq \mathrm{Hom}_{\mathrm{DMT}(k)}(\mathbb{Q}_k(0), \mathbb{Q}_k(q)[p]).$$

This impression is confirmed by Spitzweck's representation theorem. It states the equivalence of $\mathrm{DMT}(k)$ with the derived category $\mathcal{D}^f_{\mathcal{N}(k)}$ of Adams graded dg modules of finite rank over the so-called cycle algebra. The cycle algebra $\mathcal{N}(k) = \oplus_{r,n\geq 0}\mathcal{N}(k)^n(r)$ is an Adams-graded cdga consisting of algebraic cycles on $\mathbb{A}^n_k \times_k (\mathbb{P}^1_k)^r$, where r denotes the Adams degree and n the cohomological degree. If k is a number field, the triangulated category $\mathcal{D}^f_{\mathcal{N}(k)}$ carries also a non-degenerate t-structure whose heart is the Tannakian category $\mathcal{H}^f_{\mathcal{N}(k)}$ and Spitzweck's representation theorem ensures an equivalence of $\mathcal{H}^f_{\mathcal{N}(k)}$ and the category of mixed Tate motives $\mathrm{MT}(k)$.

Theorem (Spitzweck's representation theorem, [Lev05, Theorem 5.23])
Let k be a perfect field. Then there is an equivalence of triangulated tensor categories

$$\Phi_k\colon \mathcal{D}^f_{\mathcal{N}(k)} \to \mathrm{DMT}(k).$$

If k is a number field, the functor Φ_k induces an equivalence of Tannakian categories

$$\Phi_k\colon \mathcal{H}^f_{\mathcal{N}(k)} \to \mathrm{MT}(k).$$

Under this equivalence the Tate motives $\mathbb{Q}_k(q)$, $q \in \mathbb{Z}$, correspond to the free rank 1 $\mathcal{N}(k)$-modules with generator b_q having Adams degree $-q$, cohomological degree 0 and $db_q = 0$. This motivates to denote these modules by $\mathbb{Q}_{\mathcal{N}(k)}(q)$.

Spitzweck's representation theorem allows us to describe Tate motives in terms of $\mathcal{N}(k)$-modules as the following example shows.

Example (Kummer motives)
Let k be a number field. By the known K-theory of k, we have

$$\mathrm{Ext}^1_{\mathrm{MT}(k)}(\mathbb{Q}_k(0), \mathbb{Q}_k(1)) \simeq K_1(k)^{(1)}_{\mathbb{Q}} \simeq k^* \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

On the other hand,

$$\mathrm{Ext}^1_{\mathcal{H}^f_{\mathcal{N}(k)}}(\mathbb{Q}_k(0), \mathbb{Q}_k(1)) \simeq \mathrm{Hom}_{\mathcal{D}^f_{\mathcal{N}(k)}}(\mathbb{Q}_{\mathcal{N}(k)}(0), \mathbb{Q}_{\mathcal{N}(k)}(1)[1]),$$

where the extension corresponding to a map $f\colon\ \mathbb{Q}_{\mathcal{N}(k)}(0) \to \mathbb{Q}_{\mathcal{N}(k)}(1)[1]$ is given as the module $\mathrm{Cone}(f)[-1] := \mathbb{Q}_{\mathcal{N}(k)}(0) \oplus \mathbb{Q}_{\mathcal{N}(k)}(1)$ with differential $(-d_{\mathcal{N}(k)}, f + d_{\mathcal{N}(k)})$.

This means that any extension E_a of $\mathbb{Q}_k(0)$ by $\mathbb{Q}_k(1)$ in $\mathrm{MT}(k)$ given by $a \in k^*$ can be expressed as the rank 2 module $\mathrm{Cone}(f_a)$, where $f_a\colon\ \mathbb{Q}_{\mathcal{N}(k)}(0) \to \mathbb{Q}_{\mathcal{N}(k)}(1)[1]$ is the map corresponding to a. Since

$$\mathrm{Hom}_{\mathcal{D}^f_{\mathcal{N}(k)}}(\mathbb{Q}_{\mathcal{N}(k)}(0), \mathbb{Q}_{\mathcal{N}(k)}(1)[1]) \simeq \mathrm{H}^1(\mathcal{N}(k)(1)),$$

$a \in k^*$ can be lifted to an element $\tilde{a} \in \mathcal{N}(k)^1(1)$ with differential $d\tilde{a} = 0$ and f_a is given by the multiplication with $\tilde{a}$. Then the module corresponding to E_a is the free $\mathcal{N}(k)$-module with generators b_0 and b_1, where the Adams degree of b_i is $-i$, the cohomological degree of b_i is 0 and $db_1 = 0$, $db_0 = \tilde{a} \cdot b_1$.

The proof of Spitzweck's representation theorem does not use the category $\mathcal{D}^f_{\mathcal{N}(k)}$ itself but rather the equivalent homotopy category $\mathcal{KCM}^f_{\mathcal{N}(k)}$ of finite $\mathcal{N}(k)$-cell modules. The latter category is the full triangulated subcategory of the homotopy category of finite $\mathcal{N}(k)$-modules generated by the modules $\mathbb{Q}_{\mathcal{N}(k)}(q)$, $q \in \mathbb{Z}$. Therefore, finite cell modules are free and finitely generated as bi-graded $\mathcal{N}(k)$-modules and admit a filtration on the set of generators that is compatible with the differential (cf. the module corresponding to E_a in the example above). The equivalence $\mathcal{KCM}^f_{\mathcal{N}(k)} \to \mathcal{D}^f_{\mathcal{N}(k)}$ was shown by Kriz and May in [KM95]. Being free modules, cell modules enable an easier construction of a functor $\mathcal{KCM}^f_{\mathcal{N}(k)} \to \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ that induces the equivalence $\mathcal{D}^f_{\mathcal{N}(k)} \to \mathrm{DMT}(k)$ in Spitzweck's representation theorem.

All these constructions have been generalised to the case of arbitrary separated smooth base schemes S over k that satisfy the Beilinson-Soulé vanishing conjectures, i.e.

$$\mathrm{H}^p(S, \mathbb{Q}(q)) = 0 \quad \text{for } p < 0, q \neq 0.$$

The construction of a triangulated category of motives over S was done by Ivorra in [Ivo07] and Cisinski and Déglise in [CD09]. The category of cell modules over k has been extended to arbitrary schemes S by Levine in [Lev10]. In *loc. cit.*, Levine also expanded Spitzweck's representation theorem to general base schemes S.

Beside the category of Tate motives, there is another well-understood subcategory of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$, the triangulated category of Artin motives $\mathrm{DMA}(k)$. It is the full triangulated subcategory of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ generated by the motives of smooth zero dimensional schemes X over $\operatorname{Spec} k$. It is equivalent to the bounded derived category of finite dimensional $\mathbb{Q}$-representations of the absolute Galois group $\mathrm{Gal}(\bar{k}|k)$ of k.

Considering the full triangulated subcategory of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ that is generated by the full subcategories $\mathrm{DMT}(k)$ and $\mathrm{DMA}(k)$ yields the triangulated category of mixed Artin-Tate motives, $\mathrm{DMAT}(k)$. This category has been studied by Wildeshaus in [Wil08]. He proves the existence of a non-degenerate t-structure on $\mathrm{DMAT}(k)$ whose heart is the Tannakian category of Artin-Tate motives, $\mathrm{MAT}(k)$, in the same fashion, as done by Levine for Tate motives.

The aim of this thesis is to compute the Tannaka group $\mathrm{G}(\mathrm{MAT}(k))$ of $\mathrm{MAT}(k)$ and obtain a better understanding of the category $\mathrm{DMAT}(k)$ in terms of cell modules, i.e. to construct a category of cell modules that is equivalent to $\mathrm{DMAT}(k)$ extending Spitzweck's representation theorem from Tate motives to Artin-Tate motives.

Structure of the thesis

We start by recalling some important definitions and results about categories in chapter 1. These are used throughout all chapters of the thesis. We do not give any proofs of the results in this chapter, with the exception of the sketch of the proof that the homotopy category of an abelian category is triangulated since we use the same arguments later on.

The aim of chapter 2 is to define the triangulated categories of geometric motives $\mathrm{DM}_{\mathrm{gm}}(S)$, motives $\mathrm{DM}(S)$ and Tate motives $\mathrm{DMT}(S)$ over a smooth scheme S over a perfect field k.

In the first section of the chapter we recall the definition of the group of algebraic cycles over a scheme. These allow us to define the category of finite S-correspondences $\mathrm{Cor}(S)$. In the second section we construct the category of geometric motives $\mathrm{DM}_{\mathrm{gm}}(S)$ out of $\mathrm{Cor}(S)$ following Voevodsky's approach.

The category of geometric motives can be identified with a full triangulated subcategory of a different category of motives $\mathrm{DM}(S)$. The latter category is defined using Nisnevich sheaves on Sm_S, as we illustrate in the third section. Furthermore, we state some basic properties of $\mathrm{DM}(S)$, most important the existence of a base change and a restriction functor with respect to the underlying base scheme. These are used to define Artin-Tate motives in chapter 4. The embedding $\mathrm{DM}_{\mathrm{gm}}(S) \to \mathrm{DM}(S)$ is the subject of section 2.4.

We conclude the second chapter by defining the triangulated category of Tate motives $\mathrm{DMT}(S)$ as a full triangulated subcategory of the $\mathbb{Q}$-linearisation of $\mathrm{DM}_{\mathrm{gm}}(S)$ that is generated by the tensor powers of the Tate motive. Furthermore, we define a t-structure on $\mathrm{DMT}(S)$ in case S satisfies the Beilinson-Soulé vanishing conjectures. The t-structure yields the Tannakian category $\mathrm{MT}(k)$ of mixed Tate motives as its heart.

In the third chapter we state and proof Spitzweck's representation theorem. To that end we define the cycle algebra $\mathcal{N}(S)$ associated to a smooth scheme S over k. This is done by defining a complex of Nisnevich sheaves $\mathcal{N}$ on Sm_k and evaluating it for the base scheme S. The elements of $\mathcal{N}(S)$ are algebraic cycles and the external product of cycles induces a product on $\mathcal{N}(S)$ making it an Adams graded cdga.

Section 3.2 gives the definition of the category $\mathcal{CM}_A^f$ of finite cell modules over an Adams graded cdga A. Moreover, we summarise the basic properties of its homotopy category $\mathcal{KCM}^f_A$, such as the existence of t-structure yielding a Tannakian category as its heart if A is cohomologically connected. Furthermore, we state the equivalence of $\mathcal{KCM}^f_A$ with the derived category of Adams graded dg A-modules $\mathcal{D}^f_A$ that is used to prove Spitzweck's representation theorem. As a triangulated category, $\mathcal{KCM}^f_A$ is generated by the free rank 1 modules which we call the Tate objects in $\mathcal{KCM}^f_A$.

In the following section we apply the results of section 3.2 to the cycle algebra $\mathcal{N}(S)$. We notice that $\mathcal{N}(S)$ is cohomologically connected if and only if S satisfies the Beilinson-Soulé vanishing conjectures and that the morphisms between the Tate objects in $\mathcal{KCM}^f_{\mathcal{N}(S)}$ are given as the same K-groups of the base scheme S as the morphism between the Tate objects in $\mathrm{DMT}(S)$.

We finish the chapter by stating and proving Spitzweck's representation theorem for smooth schemes S over a perfect field utilizing the results we collected in the first three chapters of the thesis.

Chapter 4 deals with the triangulated category of Artin-Tate motives $\mathrm{DMAT}(k)$ over a number field k and the construction of an equivalent category of cell modules. The first section recalls the definition of the triangulated category of Artin-Tate motives following [Wil08] and [Sch11].

We give Wildeshaus' theorem stating the existence of a non-degenerate t-structure on $\mathrm{DMAT}(k)$ yielding a Tannakian category $\mathrm{MAT}(k)$ as its heart. We conclude this section by computing the Tannaka group of $\mathrm{MAT}(k)$ in Theorem 4.24 as the semi-direct product of the absolute Galois group of k and the Tannaka group of the category of mixed Tate motives $\mathrm{MT}(\bar{k})$ over an algebraic closure $\bar{k}$ of k.

Theorem
Let k be a number field and let $\bar{k}$ denote its algebraic closure. Then there exists a split exact sequence

$$1 \to \mathrm{G}(\mathrm{MT}(\bar{k})) \to \mathrm{G}(\mathrm{MAT}(k)) \rightleftarrows \mathrm{Gal}(\bar{k}|k) \to 1.$$

This is done using the triangulated category of Artin-Tate motives over k that are trivialisable over L, $\mathrm{DMAT}(L|k)$, where L is an algebraic extension of k, and using the fact that $\mathrm{G}(\mathrm{MA}(L|k)) = \mathrm{Gal}(L|k)$.

The aim of section 4.2 is to construct a triangulated category $\mathcal{D}(k)$ of cell modules that is equivalent to $\mathrm{DMAT}(k)$. This category should also carry a non-degenerate t-structure whose heart is a Tannakian category $\mathcal{A}(k)$. The equivalence $\mathcal{D}(k) \to \mathrm{DMAT}(k)$ should be compatible with the t-structures yielding an equivalence of Tannakian categories $\mathcal{A}(k) \to \mathrm{MAT}(k)$. We achieve this by defining a triangulated category $\mathcal{D}(L|k)$ of $\mathcal{N}(L)$-cell modules with $\mathrm{Gal}(L|k)$-action for a fixed algebraic extension L of k. We show that $\mathcal{D}^f_{\mathcal{N}(k)}$ is a full triangulated subcategory of $\mathcal{D}(L|k)$ or more generally that $\mathcal{D}(K|k)$ for K an intermediate Galois extension $k \subset K \subset L$ is a full triangulated subcategory of $\mathcal{D}(L|k)$. We define $\mathcal{D}(k)$ as the union of the categories $\mathcal{D}(L|k)$ for $L \to \bar{k}$. We construct an equivalence $\mathcal{D}(L|k) \to \mathrm{DMAT}(L|k)$ for every finite Galois extension L of k. These functors induce the desired equivalence $\mathcal{D}(k) \to \mathrm{DMAT}(k)$. Again, the equivalence is compatible with the t-structure yielding an equivalence of Tannakian categories $\mathcal{A}(k) \to \mathrm{MAT}(k)$. Furthermore, $\mathcal{D}^f_{\mathcal{N}(k)}$ can be identified with a full triangulated subcategory of $\mathcal{D}(k)$ and the restriction of $\mathcal{D}(k) \to \mathrm{DMAT}(k)$ to that subcategory yields the statement of Spitzweck's representation theorem.

Theorem (see Theorem 4.41)
Let k be a number field. There is a natural exact tensor functor

$$\Phi\colon \mathcal{D}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$$

that induces an equivalence of triangulated tensor categories

$$\Phi\colon \mathcal{D}(k) \to \mathrm{DMAT}(k).$$

Furthermore, the functor Φ is compatible with the weight filtrations in $\mathcal{D}(k)$ and $\mathrm{DMAT}(k)$ and yields an equivalence of Tannakian categories

$$\Phi\colon \mathcal{A}(k) \to \mathrm{MAT}(k).$$

Restricted to the full subcategory $\mathcal{D}^f_{\mathcal{N}(k)}$ of $\mathcal{D}(k)$, the functor Φ agrees with the functor

$$\Phi_k\colon \mathcal{D}^f_{\mathcal{N}(k)} \to \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$$

in Spitzweck's representation theorem.

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1 Categories

The first chapter of this thesis recalls the definitions of some important types of categories and some basic facts about these categories. These results are used throughout the whole thesis.

We mainly follow [Lev06].

Let $\mathcal{A}$ be an additive category. Recall that a subcategory $\mathcal{C}$ of $\mathcal{A}$ is called full if $\mathrm{Hom}_{\mathcal{C}}(A,B) = \mathrm{Hom}_{\mathcal{A}}(A,B)$ for all $A, B \in \mathcal{C}$. $\mathcal{C}$ is called strictly full if it is full and closed under isomorphisms, i.e. if A is in $\mathcal{C}$ and there is an isomorphism $A \to B$ in $\mathcal{A}$, then B is in $\mathcal{C}$. Furthermore, a full subcategory $\mathcal{C}$ is called thick (or épaisse) if it is closed under taking direct summands, i.e. if A is in $\mathcal{C}$ and B is a direct summand of A in $\mathcal{A}$, then B is in $\mathcal{C}$.

An important type of categories are triangulated categories. These were introduced by Verdier in [Ver96]. To be able to define those we need the notion of a translation functor and a triangle in an additive category $\mathcal{A}$.

Definition 1.1
Let $\mathcal{A}$ be an additive category.

1. *A translation on $\mathcal{A}$ is an automorphism $T\colon \mathcal{A} \to \mathcal{A}$. If X is an object of $\mathcal{A}$, we usually write $X[1]$ for $T(X)$. Similarly, if f is a morphism in $\mathcal{A}$, we write $f[1]$ for $T(f)$.*

2. *A triangle (X, Y, Z, a, b, c) in an additive category $\mathcal{A}$ with a translation is a sequence of maps in $\mathcal{A}$ of the form*

$$X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} X[1].$$

A morphism of triangles in $\mathcal{A}$

$$(f, g, h) \colon (X, Y, Z, a, b, c) \to (X', Y', Z', a', b', c')$$

is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[r, "b"] \arrow[d, "g"'] & Z \arrow[r, "c"] \arrow[d, "h"'] & X[1] \arrow[d, "f[1]"'] \\ X' \arrow[r, "a'"] & Y' \arrow[r, "b'"] & Z' \arrow[r, "c'"] & X'[1] \end{tikzcd}$$

in $\mathcal{A}$*.*

Definition 1.2
A triangulated category is an additive category $\mathcal{D}$ *with a translation and a collection* $\mathcal{T}$ *of triangles in* $\mathcal{D}$*, called distinguished (or exact) triangles, satisfying:*

(TR1) $\mathcal{T}$ *is closed under isomorphisms of triangles.*

Any triangle of the form $A \xrightarrow{\mathrm{id}} A \to 0 \to A[1]$ *is distinguished.*

For any morphism $f \colon A \to B$ *in* $\mathcal{D}$ *there exists a distinguished triangle* $A \xrightarrow{f} B \to C \to A[1]$ *in* $\mathcal{D}$*.*

(TR2) A triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ *is distinguished if and only if the rotated triangle* $B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$ *is distinguished.*

(TR3) Given two distinguished triangles $A \xrightarrow{f} B \to C \to A[1]$ *and* $A' \xrightarrow{f'} B' \to C' \to A'[1]$ *and two maps* $\alpha \colon A \to A'$ *and* $\beta \colon B \to B'$ *such that* $f' \circ \alpha = \beta \circ f$*, there exists a map* $\gamma \colon C \to C'$ *in* $\mathcal{D}$ *giving a morphism of triangles, i.e. a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & A[1] \arrow[d, "\alpha[1]"'] \\ A' \arrow[r, "f'"] & B' \arrow[r] & C' \arrow[r] & A'[1]. \end{tikzcd}$$

(TR4) Given three distinguished triangles $A \xrightarrow{f} B \to C' \to A[1]$*,* $B \xrightarrow{g} C \to A' \to B[1]$ *and* $A \xrightarrow{g \circ f} C \to B' \to A[1]$*, there exist maps* $\alpha \colon B' \to A'$ *and* $\beta \colon C' \to B'$ *in* $\mathcal{D}$ *such that*

$C' \xrightarrow{\beta} B' \xrightarrow{\alpha} A' \to C'[1]$ is a distinguished triangle and the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & B \arrow[r] \arrow[d, "g"'] & C' \arrow[r] \arrow[d, "\beta"'] & A[1] \arrow[d, "\mathrm{id}[1]"'] \\
A \arrow[r, "g\circ f"] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "\mathrm{id}"'] & B' \arrow[r] \arrow[d, "\alpha"'] & A[1] \arrow[d, "f[1]"'] \\
B \arrow[r, "g"] & C \arrow[r] & A' \arrow[r] & B[1].
\end{tikzcd}$$

Triangles in a triangulated category $\mathcal{D}$ give rise to long exact sequences of abelian groups when applying the functor $\mathrm{Hom}_{\mathcal{D}}(X, -)$ for some object $X \in \mathcal{D}$ to them (or the contra-variant functor $\mathrm{Hom}_{\mathcal{D}}(-, X)$). As a consequence of this, a morphism of distinguished triangles (f, g, h) in a triangulated category is an isomorphism if two of the three corresponding morphisms f, g and h are isomorphisms (this follows by the Yoneda lemma and the five lemma for abelian groups). Therefore, any two objects C, C' completing a morphism $f\colon A \to B$ to distinguished triangles $A \xrightarrow{f} B \to C \to A[1]$ and $A \xrightarrow{f} B \to C' \to A[1]$ are isomorphic, hence C is unique up to (non-unique) isomorphism and we call it the mapping cone of f. We denote it by $\mathrm{Cone}(f)$.

If a triangulated category $\mathcal{D}$ is furthermore a tensor category, we call $\mathcal{D}$ a triangulated tensor category if the tensor product is compatible with the translation functor and the triangulated structure in the following sense.

Definition 1.3
Let $\mathcal{D}$ be an additive tensor category with a translation $T\colon \mathcal{D} \to \mathcal{D}$ and a collection of distinguished triangles such that $\mathcal{D}$ is a triangulated category. $\mathcal{D}$ is called a triangulated tensor category if

1. *$T \circ (- \otimes -) = T(-) \otimes -$ and $T^2(-) \otimes - = - \otimes T^2(-)$;*
2. *the natural isomorphisms $\tau_{X,Y}\colon X \otimes TY \to TX \otimes Y$ that are given as $X \otimes TY \simeq TY \otimes X = T(Y \otimes X) \simeq T(X \otimes Y) = TX \otimes Y$ satisfy: $T(\tau_{X,Y})\tau_{X,TY}\colon X \otimes T^2Y \to T^2(X \otimes Y) = X \otimes T^2Y$ is the identity;*
3. *for each distinguished triangle $A \to B \to C \to A[1]$ and every object $X \in \mathcal{D}$ the induced sequence $A \otimes X \to B \otimes X \to \mathrm{C} \otimes X \to A[1] \otimes X = (A \otimes X)[1]$ is a distinguished triangle.*

An additive functor $F\colon \mathcal{D} \to \mathcal{C}$ between two triangulated categories $\mathcal{D}$ and $\mathcal{C}$ is called triangulated (or exact) if it commutes with the translations on both categories and transforms distinguished triangles into distinguished triangles.

Similarly, two triangulated categories $\mathcal{D}$, $\mathcal{C}$ are equivalent as triangulated categories if there exists a triangulated functor $F\colon \mathcal{D} \to \mathcal{C}$ that is an equivalence of categories. Note that every quasi-inverse $G\colon \mathcal{C} \to \mathcal{D}$ of a triangulated functor F is automatically triangulated since every (left or right) adjoint of a triangulated functor is triangulated by [Nee01, Lemma 5.3.6].

A common example of a triangulated category is the homotopy category of an additive category which is defined as follows.

Let $\mathcal{A}$ be an additive category. Let $C(\mathcal{A})$ be the category of complexes of $\mathcal{A}$, i.e. the objects of $C(\mathcal{A})$ are sequences

$$\dots \longrightarrow A^{n-1} \xrightarrow{d_A^{n-1}} A^n \xrightarrow{d_A^n} A^{n+1} \longrightarrow \dots$$

in $\mathcal{A}$ such that $d_A^{n+1} \circ d_A^n = 0$ for all n. A morphism of complexes $f\colon A \to B$ is a family of maps $f^n\colon A^n \to B^n$ such that $d_B^n \circ f^n = f^{n+1} \circ d_A^n$. Two maps of complexes $f, g\colon A \to B$ are called homotopic if there exist maps $h^n\colon A^n \to B^{n-1}$ in $\mathcal{A}$ such that $f^n - g^n = d_B^{n-1} \circ h^n + h^{n+1} \circ d_A^n$. Homotopy defines an equivalence relation on the set of morphisms between two complexes.

Definition 1.4
The homotopy category $\mathcal{K}(\mathcal{A})$ of $\mathcal{A}$ is defined as the category consisting of the same objects as $C(\mathcal{A})$ and morphisms

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A, B) := \mathrm{Hom}_{C(\mathcal{A})}(A, B)/\{\textbf{\textit{homotopy}}\}.$$

On $\mathcal{K}(\mathcal{A})$ we have the translation functor $[1]\colon A \to A[1]$, where $A[1]^i := A^{i+1}$ and $d^i_{A[1]} = -d_A^{i+1}$. We call a triangle in $\mathcal{K}(\mathcal{A})$ distinguished if it is isomorphic to the image of a cone sequence $A \to B \to \mathrm{Cone}(f) \to A[1]$, where $\mathrm{Cone}(f) := A[1] \oplus B$ with differential $d := (-d_A, f + d_B)$.

Proposition 1.5
$\mathcal{K}(\mathcal{A})$ is a triangulated category, where the distinguished triangles are those triangles that are isomorphic to the image of a cone sequence.

There is a more general notion of the homotopy category of a differential graded category $\mathcal{C}$. These are additive categories, where $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a complex itself, i.e. it is given as a direct sum $\oplus_m \mathrm{Hom}^m_{\mathcal{C}}(A,B)$ and carries a differential $d^m\colon \mathrm{Hom}^m_{\mathcal{C}}(A,B) \to \mathrm{Hom}^{m+1}_{\mathcal{C}}(A,B)$ satisfying $d^{m+1} \circ d^m = 0$. The homotopy category $\mathcal{K}(\mathcal{C})$ is then defined as having the same objects as $\mathcal{C}$ and morphisms defined by

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{C})}(A,B) := \mathrm{H}^0(\mathrm{Hom}_{\mathcal{C}}(A,B)).$$

The category $C(\mathcal{A})$ of complexes of an additive category $\mathcal{A}$ can be made into a differential graded category by putting

$$\mathrm{Hom}^m_{C(\mathcal{A})}(A,B) := \mathrm{Hom}_{C(\mathcal{A})}(A, B[m])$$

and defining the differential of a map $f\colon A \to B[m]$ of degree m as

$$(df)^n := d^n_B \circ f^n + (-1)^{m+1} d^n_A \circ f^{n+1}.$$

Then it is easy to see that $df = 0$ is equivalent to the fact that f commutes with the differentials d_A and d_B on A and B respectively and two maps of degree 0, i.e. in $\mathrm{Hom}^0_{C(\mathcal{A})}(A,B)$, are homotopic if they differ by the differential of a map h of degree -1 (i.e. a homotopy in the sense as above). Hence,

$$\mathrm{Hom}_{C(\mathcal{A})}(A,B)/\{\textbf{homotopy}\} = \mathrm{H}^0(\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(A,B))$$

and both definitions of a homotopy category agree for a category of complexes $C(\mathcal{A})$ of an additive category $\mathcal{A}$.

Caution: The homotopy category of a differential graded category is not necessarily triangulated since in general cones and shifts do not need to exist in an arbitrary differential graded category. However, the examples we consider admit reasonable cones and shifts and hence, a triangulated structure can be defined on the homotopy category. We discuss this later on when they occur.

We give a sketch of the proof of Proposition 1.5 given in [Sos12]. From this it can be seen that the homotopy category of the differential graded categories we consider throughout this thesis are also triangulated.

The axiom (TR1) is straightforward.

Axiom (TR2) follows from the fact that for any morphism $f\colon A \to B$ there exists a morphism $\phi\colon A[1] \to C := \mathrm{Cone}(B \to \mathrm{Cone}(f))$ such that ϕ is an isomorphism in $\mathcal{K}(\mathcal{A})$ giving an isomorphism of triangles $(\mathrm{id}, \mathrm{id}, \phi)\colon (B, \mathrm{Cone}(f), A[1]) \to (B, \mathrm{Cone}(f), C)$. The maps $\phi^k\colon C^k \to A[1]^k$ are given as $(-f^{k+1}, \mathrm{id}^{k+1}_A, 0)$. See [Sos12, Lemma 2.6] for the proof that ϕ is an isomorphism in $\mathcal{K}(\mathcal{A})$ with (homotopy) inverse $\psi = (0, \mathrm{id}_{A[1]}, 0)$.

For (TR3): If $A \xrightarrow{f} B \to C \to A[1]$ and $A' \xrightarrow{f'} B' \to C' \to A'[1]$ are two distinguished triangles and there are two maps $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ such that $f' \circ \alpha = \beta \circ f$ in $\mathcal{K}(\mathcal{A})$, one needs to construct a map $\gamma\colon C \to C'$ giving a morphism of distinguished triangles. The map $\gamma\colon C \to C'$ can be defined as $\gamma^k = (\alpha^{k+1}, s^{k+1} + \beta^k)$, where $s^k\colon A^k \to B'^{k-1}$ are maps such that $\beta^k \circ f^k - f'^k \circ \alpha^k = s^{k+1} \circ d_A^k + d_B^{k-1} \circ s^k$ in $\mathcal{A}$. Again, we skip the proof that the map γ has indeed the desired properties.

Lastly, we are given three distinguished triangles $A \xrightarrow{f} B \to C' \to A[1]$, $B \xrightarrow{g} C \to A' \to B[1]$ and $A \xrightarrow{g \circ f} C \to B' \to A[1]$. We have to construct a distinguished triangle $\mathrm{Cone}(f) \to \mathrm{Cone}(g \circ f) \to \mathrm{Cone}(g)$ satisfying the properties stated in (TR4).
We define $\beta\colon \mathrm{Cone}(f) \to \mathrm{Cone}(g \circ f)$ and $\alpha\colon \mathrm{Cone}(g \circ f) \to \mathrm{Cone}(g)$ by $\beta^k = (\mathrm{id}_{A^{k+1}}, g^k)$ and $\alpha^k = (f^{k+1}, \mathrm{id}_{C^k})$. This gives indeed the desired commutative diagram of distinguished triangles in $\mathcal{K}(\mathcal{A})$. See [Sos12, Theorem 2.7] for details.

Throughout this thesis we encounter various examples of triangulated categories. Many of them arise as subcategories of triangulated categories that inherit the triangulated structure, those are called triangulated subcategories.

Definition 1.6
Let $\mathcal{C}$, $\mathcal{D}$ be two triangulated categories and $\mathcal{C} \subset \mathcal{D}$. $\mathcal{C}$ is called triangulated subcategory of $\mathcal{D}$ if the inclusion functor is triangulated.

If $\mathcal{C}$ is a full subcategory of a triangulated category $\mathcal{D}$, then $\mathcal{C}$ is a triangulated subcategory of $\mathcal{D}$ if $\mathcal{C}$ is closed under shifts and under distinguished triangles, i.e. if $A \to B \to C$ is a distinguished triangle in $\mathcal{D}$ such that A and B are in $\mathcal{C}$, then C is isomorphic to an object in $\mathcal{C}$.

Most of the triangulated subcategories that we consider are generated by a set of generating objects.

Definition 1.7
Let $\mathcal{D}$ be a triangulated category. Let A be an object of $\mathcal{D}$. We denote by $\langle A \rangle$ the smallest full (thick) triangulated subcategory of $\mathcal{D}$ that contains A and call it the full (thick) triangulated subcategory of $\mathcal{D}$ generated by A.

This definition can easily be extended to triangulated categories generated by a set of generators $\{A_i : i \in I\}$ in the obvious way.

Remark 1.8
The category $\langle A \rangle$ can be constructed as follows.
Denote by $\langle A \rangle_1$ the strictly full subcategory of $\mathcal{D}$ of objects isomorphic to (direct summands of) finite direct sums

$$\bigoplus_{i=1}^{r} A[n_i], \quad r \in \mathbb{N}, n_i \in \mathbb{Z}\,.$$

For $n > 1$ let $\langle A \rangle_n$ denote the full subcategory of $\mathcal{D}$ consisting of objects isomorphic to (direct summands of) objects X fitting into a distinguished triangle

$$B \to X \to C \to B[1],$$

where B is an object of $\langle A \rangle_1$ and C an object of $\langle A \rangle_{n-1}$. Now define

$$\langle A \rangle := \bigcup_n \langle A \rangle_n.$$

Ultimately, we are interested in certain abelian categories that are subcategories of triangulated categories. An important tool to construct these are t-structures on triangulated categories.

Definition 1.9
Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{D}_{\leq 0}$ and $\mathcal{D}_{\geq 0}$ be full subcategories of $\mathcal{D}$ satisfying

1. $\mathcal{D}_{\leq 0}[1] \subset \mathcal{D}_{\leq 0}$, $\mathcal{D}_{\geq 0}[-1] \subset \mathcal{D}_{\geq 0}$;
2. $\mathrm{Hom}_{\mathcal{D}}(X_{\leq 0}, X_{>0}) = 0$ *for all* $X_{\leq 0} \in \mathcal{D}_{\leq 0}$, $X_{>0} \in \mathcal{D}_{>0} := \mathcal{D}_{\geq 0}[-1]$;
3. *for every $X \in \mathcal{D}$ there is a distinguished triangle*

$$X_{\leq 0} \to X \to X_{>0},$$

   where $X_{\leq 0} \in \mathcal{D}_{\leq 0}$, $X_{>0} \in \mathcal{D}_{>0}$.

Then $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$ is called a t-structure on $\mathcal{D}$ with heart $\mathcal{D}_0 := \mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0}$. The t-structure is called non-degenerate if $A \in \bigcap_{n \leq 0} \mathcal{D}_{\leq n}$ and $B \in \bigcap_{n \geq 0} \mathcal{D}_{\geq n}$ imply $A \cong B \cong 0$, where $\mathcal{D}_{\leq n} := \mathcal{D}_{\leq 0}[n]$ and $\mathcal{D}_{\geq n} := \mathcal{D}_{\geq 0}[-n]$.

Note that the objects $X_{\leq 0}$ and $X_{>0}$ are (up to isomorphisms) uniquely determined by X.

The heart of a t-structure on a triangulated category $\mathcal{D}$ is always an abelian category that is closed under extension in $\mathcal{D}$, as shown by Beilinson, Bernstein and Deligne in [BBD82].

Recall that an additive category is called pre-abelian if every morphism has both a kernel and a cokernel. The coimage of a morphism is defined as the cokernel of its kernel, while the image of a morphism is the kernel of its cokernel. The universal properties of the kernel and cokernel induce a unique morphism between them. A pre-abelian category is called abelian if for every morphism f the induced unique morphism $\bar{f}\colon \mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism.

Beside the category R-Mod of modules over a commutative ring R, one of the most prominent examples of an abelian category is the category of representations of an algebraic group G over $\mathbb{Q}$ in finite dimensional $\mathbb{Q}$-vector spaces. We denote this category by $\mathrm{Rep}_{\mathbb{Q}}(G)$.

$\mathrm{Rep}_{\mathbb{Q}}(G)$ is a rigid tensor category, meaning for every object X in $\mathrm{Rep}_{\mathbb{Q}}(G)$ there exists an object Y and morphisms $\eta_X\colon \mathbb{1} \to X \otimes Y$ and $\epsilon_X\colon Y \otimes X \to \mathbb{1}$ in $\mathrm{Rep}_{\mathbb{Q}}(G)$ such that the compositions

$$X \xrightarrow{\eta_X \otimes \mathrm{id}_X} (X \otimes Y) \otimes X \xrightarrow{\simeq} X \otimes (Y \otimes X) \xrightarrow{\mathrm{id}_X \otimes \epsilon_X} X$$
$$Y \xrightarrow{\mathrm{id}_Y \otimes \eta_X} Y \otimes (X \otimes Y) \xrightarrow{\simeq} (Y \otimes X) \otimes Y \xrightarrow{\epsilon_X \otimes \mathrm{id}_Y} Y$$

are the identities. Hereby, $\mathbb{1}$ denotes the unit object for the tensor structure.

Furthermore, $\mathrm{Rep}_{\mathbb{Q}}(G)$ admits a forgetful functor ω_0 to the finite dimensional $\mathbb{Q}$-vector spaces. The functor ω_0 is in fact a faithful exact tensor functor.

Categories that satisfy these properties are called Tannakian categories.

Definition 1.10
A neutral Tannakian category over $\mathbb{Q}$ is a rigid abelian $\mathbb{Q}$-linear tensor category $\mathcal{A}$ that admits a fibre functor ω, i.e. a faithful exact tensor functor $\omega\colon \mathcal{A} \to \mathbb{Q}\text{-Vec}$, where $\mathbb{Q}$-Vec denotes the category of finite dimensional $\mathbb{Q}$-vector spaces.

The Tannaka group $G(\mathcal{A}, \omega)$ of a neutral Tannakian category $\mathcal{A}$ over $\mathbb{Q}$ with fibre functor ω is the algebraic group $\mathrm{Aut}(\omega)^{\otimes}_r$ over $\mathbb{Q}$. Here, $\mathrm{Aut}(\omega)^{\otimes}$ denotes the group of natural automorphisms of ω that are compatible with the tensor structures on $\mathcal{A}$ and $\mathbb{Q}$-Vec.

In the following we just write Tannakian category for a neutral Tannakian category over $\mathbb{Q}$.

The main theorem of Tannakian categories by Deligne states that every Tannakian category is (up to equivalence) of the form $\mathrm{Rep}_{\mathbb{Q}}(G)$ for some algebraic group G.

Theorem 1.11 ([Del90, Theorem 1.12])
Let $\mathcal{A}$ be a Tannakian category with fibre functor ω. Let $G := \mathrm{Aut}^{\otimes}(\omega)$. Then there is an equivalence $\mathcal{A} \to \mathrm{Rep}_{\mathbb{Q}}(G)$ transforming ω to ω_0.

A Tannakian category $\mathcal{A}$ is required to be $\mathbb{Q}$-linear, i.e. $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is a $\mathbb{Q}$-vector space for all A, $B \in \mathcal{A}$. Triangulated categories are additive but in general not $\mathbb{Q}$-linear. However, there is a way to produce a $\mathbb{Q}$-linear triangulated category, called the $\mathbb{Q}$-linearisation.

An object A in an additive category $\mathcal{A}$ is called compact if $\mathrm{Hom}_{\mathcal{A}}(A, \oplus_i B_i) \simeq \oplus_i \mathrm{Hom}_{\mathcal{A}}(A, B_i)$ for all families $\{B_i\}_i$ of objects whose sum exist. We call an object $A \in \mathcal{A}$ a torsion object if there exists an integer $n > 0$ such that $n \cdot \mathrm{id}_A = 0$. If $\mathcal{D}$ is a triangulated category that is generated by compact objects, we denote by $\mathcal{D}_{\mathrm{tor}}$ the localising subcategory of compact torsion objects and define the $\mathbb{Q}$-linearisation $\mathcal{D}_{\mathbb{Q}}$ of $\mathcal{D}$ as the Verdier localisation $\mathcal{D}/\mathcal{D}_{\mathrm{tor}}$. The localisation functor $\mathcal{D} \to \mathcal{D}_{\mathbb{Q}}$ is a triangulated functor.

If every object in $\mathcal{D}$ is compact, the $\mathbb{Q}$-linearisation of $\mathcal{D}$ is given in the following way by [Kel13, Corollary A.2.12]. The objects in $\mathcal{D}_{\mathbb{Q}}$ are the same as in $\mathcal{D}$ and

$$\mathrm{Hom}_{\mathcal{D}_{\mathbb{Q}}}(M, N) = \mathrm{Hom}_{\mathcal{D}}(M, N) \otimes \mathbb{Q}\,.$$

One last important categorical tool we need is the pseudo-abelian hull of an additive category.

An additive category is called pseudo-abelian (or Karoubian) if every idempotent endomorphism has a kernel (or equivalently a cokernel). In other words, for every idempotent endomorphism $\alpha\colon A \to A$ in $\mathcal{A}$ there exist objects A_0 and A_1 in $\mathcal{A}$ and an isomorphism $\phi\colon A \to A_0 \oplus A_1$ such that $\phi \circ \alpha \circ \phi^{-1} = 0_{A_0} \oplus \mathrm{id}_{A_1}$. ($A_0$ is then given as the kernel of α.)

As the name suggests, every abelian category is pseudo-abelian, in fact every pre-abelian category is pseudo-abelian as follows immediately from the definition.

Another example of a pseudo-abelian category (that is not necessarily pre-abelian) is the so-called pseudo-abelian hull (also called Karoubi envelope or idempotent completion) of an additive category $\mathcal{A}$.

Definition 1.12
Let $\mathcal{A}$ be an additive category. Then the pseudo-abelian hull of $\mathcal{A}$ consists of the objects (A, α), where $A \in \mathcal{A}$ and $\alpha\colon A \to A$ is an idempotent endomorphism on A in $\mathcal{A}$, and morphisms

$$\beta \circ f \circ \alpha\colon (A, \alpha) \to (B, \beta),$$

where $f \in \operatorname{Hom}_{\mathcal{A}}(A, B)$. The composition of two morphisms $(\gamma \circ g \circ \beta)$ and $(\beta \circ f \circ \alpha)$ is given by

$$(\gamma \circ g \circ \beta) \circ (\beta \circ f \circ \alpha) = \gamma \circ (g \circ \beta \circ f) \circ \alpha.$$

For a pair (A, α), the identity map on A gives an isomorphism

$$(A, \mathrm{id}) \simeq (A, \mathrm{id} - \alpha) \oplus (A, \alpha)$$

which can be used to show that the pseudo-abelian hull is indeed a pseudo-abelian category by the discussion above.

Sending an object A of an additive category $\mathcal{A}$ to the pair (A, id) defines a functor from $\mathcal{A}$ into its pseudo-abelian hull.

It can easily be seen from the definition that if $\mathcal{A}$ and $\mathcal{B}$ are two equivalent additive categories, then their respective pseudo-abelian hulls are also equivalent.

If $\mathcal{A}$ is a triangulated category, then Balmer and Schlichting have shown in [BS01] that the pseudo-abelian hull of $\mathcal{A}$ is again a triangulated category in a natural way.

2 Tate motives

The goal of this chapter is to define the Tannakian category of mixed Tate motives $\mathrm{MT}(S)$ over a smooth scheme S over a perfect field k. This is done in several steps.

First, we describe the construction of the category of geometric motives $\mathrm{DM}_{\mathrm{gm}}(S)$ over a scheme S and the sheaf theoretic construction of the category of motives $\mathrm{DM}(S)$ using the knowledge about triangulated tensor categories we acquired in chapter 1. Both constructions have been introduced by Voevodsky in [Voe00] for $S = \operatorname{Spec} k$, where k is a perfect field, and are linked by an embedding $\mathrm{DM}_{\mathrm{gm}}(S) \to \mathrm{DM}(S)$. This has been generalised to a regular noetherian base scheme S by Ivorra in [Ivo07] and Cisinski and Déglise in [CD09]. The reasoning behind defining a category of motives is to capture the fundamental properties and structures of a cohomology theory on smooth schemes over S. Like any reasonable cohomology theory, the category of motives should contain pull-back maps, products and long exact sequences. The pull-back maps are induced by the pull-back of algebraic cycles and correspondences. Long exact sequences can be produced by considering triangulated categories and the product comes from the tensor structure on the triangulated category.

While concrete computations in the category of motives $\mathrm{DM}_{\mathrm{gm}}(S)$ (respectively $\mathrm{DM}(S)$) for a scheme S are still quite difficult, there exists a full triangulated subcategory $\mathrm{DMT}(S)$, called triangulated category of Tate motives, that is much more accessible. In this subcategory the morphisms are (after tensoring with $\mathbb{Q}$) given by the rational K-theory of the underlying scheme (which for example in the case of $S = \operatorname{Spec} k$, k a number field, is well-known). The knowledge of these morphisms makes it possible to define a t-structure on $\mathrm{DMT}(S)$ which yields the desired Tannakian category $\mathrm{MT}(S)$ as its heart for schemes S that satisfy the Beilinson-Soulé vanishing conjectures.

In the first section we define the group of algebraic cycles on a scheme and sum up some useful properties such as the existence of pull-back and push-forward maps along morphisms of schemes. These cycles act as morphisms between schemes in the category of correspondences. Section 2.2 describes how to form the category of geometric motives $\mathrm{DM}_{\mathrm{gm}}(S)$ with the desired properties mentioned above out of the category of correspondences over S. $\mathrm{DM}_{\mathrm{gm}}(S)$ can be embedded into a bigger category of motives $\mathrm{DM}(S)$ that is constructed in section 2.3 using the derived category of Nisnevich sheaves with transfer on the category of smooth schemes. The embedding itself is the subject of section 2.4 and allows to consider $\mathrm{DM}_{\mathrm{gm}}(S)$ as a full subcategory of $\mathrm{DM}(S)$. The advantage of this is that it is easier to compute morphism in $\mathrm{DM}(S)$ than in $\mathrm{DM}_{\mathrm{gm}}(S)$. Section 2.5 gives the definition of the triangulated category of Tate motives $\mathrm{DMT}(S)$. It is the full triangulated subcategory of $\mathrm{DM}_{\mathrm{gm}}(S)$ that is generated by the tensor powers of the Tate motive $[S] \xrightarrow{i_{\infty *}} [\mathbb{P}^1_S]$. In case S satisfies the Beilinson-Soulé vanishing conjectures, we see that $\mathrm{DMT}(S)$ behaves similar to a classical derived category of an abelian category, namely its full Tannakian subcategory $\mathrm{MT}(S)$ that arises as the heart of a t-structure.

Throughout this chapter let S be a smooth separated scheme of finite type over a perfect field k. We denote the category of smooth separated schemes of finite type over S by Sm_S.

2.1 Cycles and Correspondences

This section contains the definition of the group of algebraic cycles on a scheme. These algebraic cycles are used to define the category of correspondences over a scheme which allows us to construct the category of geometric motives in the following section. Furthermore, the correspondences give essential examples of Nisnevich sheaves with transfer which are studied in section 2.3. In a similar fashion, correspondences are used in section 3.1 to define the cycle algebra $\mathcal{N}(S)$ over a scheme S.

We follow the notation of [Lev06], where he describes the case $S = \operatorname{Spec} k$, k a field. The generalisation to any base scheme $S \in \mathrm{Sm}_k$ holds true by [Dég07].

For $X \in \mathrm{Sm}_S$, we define $z_r(X)$ as the free abelian group on the closed integral subschemes $W \subset X$ of dimension r. We write $z_*(X) = \oplus_r z_r(X)$ and call it the group of algebraic cycles on X. For a cycle $Z = \sum n_i Z_i \in z_*(X)$ the support of Z is given as

$$\mathrm{supp}(Z) := \bigcup_i Z_i.$$

Conversely, if $W \subset X$ is a closed irreducible subscheme with irreducible components $W_1, \ldots, W_r$, we define the cycle associated to W by

$$|W| := \sum_{i=1}^{r} (l_{\mathcal{O}_{X,W_i}}(\mathcal{O}_{W,W_i})) \cdot W_i,$$

where $l_{\mathcal{O}_{X,W_i}}$ is the length as an $\mathcal{O}_{X,W_i}$-module. For example, if W is reduced, then $|W| = \sum_{i=1}^{r} W_i$.

We define the group $c_0(X/S)$ of finite relative cycles on X over S as the subgroup of $z_*(X)$ of cycles Z such that $\mathrm{supp}(Z)$ is finite equidimensional over S. Recall that a morphism $X \to S$ is called equidimensional if f is of finite type, the relative dimension of f is constant and every irreducible component of X dominates an irreducible component of S. By [Dég07, Lemma 1.2], Z being a finite relative cycle on X over S is equivalent to $\mathrm{supp}(Z)$ being finite over S and surjective over an irreducible component of S.

Note that if $S = S_1 \coprod S_2$, $c_0(X/S) = c_0(X/S_1) \oplus c_0(X/S_2)$. Therefore, we may assume S is irreducible.

For X, Y in Sm_S, we define the group of finite S-correspondences $c_S(X,Y) \subset z_*(X \times_S Y)$ as the free abelian group on the integral closed subschemes $W \subset X \times_S Y$ with $W \to X$ finite and surjective over an irreducible component of X, i.e. $c_S(X,Y) := c_0(X \times_S Y/X)$.

The groups $c_S(X,Y)$ act as morphism groups between two schemes X and Y in the category of finite S-correspondences. To be able to define a composition of two correspondences, we need the push-forward and pull-back of algebraic cycles.

Let $f\colon X \to Y$ a morphism in Sm_S. Let $W \subset X$ be a closed integral subscheme which is finite equidimensional over S. Then the push-forward of $|W|$ along f is defined as

$$f_*(|W|) := d \cdot |f(W)|,$$

where d is the degree of the extension of function fields induced by f. By [Dég07, Lemma 1.9], d is a finite number and $f(W)$ is a closed integral subscheme of Y that is finite and surjective over an irreducible component of S. Therefore, $f_*(|W|) \in c_0(Y/S)$ and by linearity, this defines a group homomorphism

$$f_*\colon c_0(X/S) \to c_0(Y/S).$$

This is functorial, i.e. $(g \circ f)_* = g_* \circ f_*$ for two morphisms $f\colon X \to Y$, $g\colon Y \to Z$ in Sm_S, by [Dég07, 1.11].

There is also a pull-back of cycles that is defined as follows. Let S, $T \in \mathrm{Sm}_k$. Suppose, we are given a cartesian square

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ T \arrow[r, "g"] & S \end{tikzcd}$$

such that $X \to S$ is smooth. Let $Z \in c_0(X/S)$. By [Dég07, section 1.12], $f^{-1}(\mathrm{supp}(Z))$ is a closed integral subscheme of Y that is finite over T and surjective over an irreducible component of T. If W is an irreducible component of $f^{-1}(\mathrm{supp}(Z))$, we define the multiplicity $m(Z, W; f)$ by Serre's formula (see [Ser65]):

$$m(Z, W; f) := \sum_{i \geq 0} (-1)^i l_{\mathcal{O}_{X,Z}}(\mathrm{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_Z, \mathcal{O}_{Y,W})).$$

The fact that $X \to S$ is smooth implies that the sum is finite. Now, the pull-back of Z is defined as

$$f^*(Z) := \sum_W m(Z, W; f) \cdot W \in c_0(Y/T),$$

where the sum is taken over all irreducible components of $f^{-1}(\mathrm{supp}(Z))$. By [Ser65, V.C.7, Exercise 1], the pull-back is functorial with respect to composition of cartesian squares.

Remark that if the morphism $g\colon T \to S$ is flat and if W is any closed integral subscheme of X that is finite over S and surjective over an irreducible component of S,

$$f^*(|W|) = |W \times_S T|$$

by [Dég07, Lemma 1.7].

The pull-back and push-forward of finite relative cycles allow us to define a composition of correspondences.

Let X, Y, Z in Sm_S. For $W \in c_S(X,Y)$, $W' \in c_S(Y,Z)$ we define the composition of W and W' as

$$W \circ W' := p_{XZ*}\,(p^*_{XY}(W) \cdot p^*_{YZ}(W')) \in c_S(X,Z),$$

where $\cdot$ is the intersection product on $X \times_S Y \times_S Z$ and p_{XY}, p_{XZ}, p_{YZ} are the evident projections from $X \times_S Y \times_S Z$. The composition $W \circ W'$ is a well-defined element of $c_S(X,Z)$ by [Dég07, Lemma 1.15].

Remark 2.1
Limiting ourselves to the subgroup $c_S(X,Y) \subset z_*(X \times_S Y)$ ensures that the pull-back maps used in the definition of the composition of correspondences are well-defined. For arbitrary algebraic cycles, there is in general only a well-defined push-forward along projective morphisms f. Furthermore, the pull-back is only partially defined.

Sending a morphism $f\colon X \to Y$ in Sm_S to its graph $\Gamma_f \subset X \times_S Y$ defines a map

$$\mathrm{Hom}_{\mathrm{Sm}_S}(X,Y) \to c_S(X,Y)$$

by [Dég07, Example 1.17] that is obviously injective. Sometimes we denote the graph Γ_f of f by f_*.

By [Dég07, Lemma 1.18], the composition in Sm_S agrees with the composition of correspondences under this map. Furthermore, the composition of correspondences is associative and for any $X \in \mathrm{Sm}_S$ the graph of the identity morphism $X \to X$ is the neutral element of this composition. Therefore, the following definition yields indeed a category.

Definition 2.2
We denote the category of S-correspondences by $\mathrm{Cor}(S)$. The objects are the same as in Sm_S and morphisms are given as

$$\mathrm{Hom}_{\mathrm{Cor}(S)}(X,Y) := c_S(X,Y)$$

with the composition law above.

The category $\mathrm{Cor}(S)$ is additive and the direct sum of two smooth S-schemes is given as its disjoint union.

As we have seen, sending a morphism $f\colon X \to Y$ in Sm_S to its graph $\Gamma_f \subset X \times_S Y$, defines an embedding $\mathrm{Sm}_S \to \mathrm{Cor}(S)$ that is the identity on objects. We denote the object in $\mathrm{Cor}(S)$ corresponding to a smooth S-scheme X by $[X]$.

The fibre product of schemes induces a tensor product on $\mathrm{Cor}(S)$.

Lemma 2.3 ([Dég07, Proposition 1.23])
$\mathrm{Cor}(S)$ *is a tensor category with tensor product*

$$[X] \otimes [Y] := [X \times_S Y]$$

for X, $Y \in \mathrm{Sm}_S$ *and the tensor product of finite correspondences* $W \in c_S(X,Y)$ *and* $W' \in c_S(X',Y')$ *is given by*

$$W \otimes W' := p^*_{XY}(W) \cdot p^*_{X'Y'}(W') \in c_S(X \times_S X', Y \times_S Y'),$$

where p_{XY} *and* $p_{X',Y'}$ *denote the canonical projections from* $X \times_S Y \times_S X' \times_S Y'$ *to* $X \times_S Y$ *and* $X' \times_S Y'$ *respectively.*

The embedding $\mathrm{Sm}_S \to \mathrm{Cor}(S)$ *is a tensor functor with the tensor product on* Sm_S *given by the fibre product over* S.

Proof. $W \otimes W'$ is a well-defined element of $c_S(X \times_S X', Y \times_S Y')$ by [Dég07, Lemma 1.20]. See [Dég07, Proposition 1.23] for further details. $\square$

Let $f\colon T \to S$ be a morphism in Sm_k. The pull-back and push-forward maps of cycles allow us to define a base change and restriction functor between the categories of correspondences.

Let $\alpha\colon X \to Y$ be a correspondence in $\mathrm{Cor}(S)$ between two smooth S-schemes X and Y. The morphism $f\colon T \to S$ induces a morphism $g\colon X \times_T Y \to X \times_S Y$. We define the pull-back of α along f by $\alpha_T := g^*(\alpha) \in c_S(X_T, Y_T)$, where $X_T := X \times_S T$, $Y_T := Y \times_S T$ and we identify $X_T \times_T Y_T$ with $(X \times_S Y)_T$. By [Dég07, Lemma 1.28], we have $\beta_T \circ \alpha_T = (\beta \circ \alpha)_T$ for $\alpha \in c_S(X,Y)$ and $\beta \in c_S(Y,Z)$, allowing us the following definition.

Definition 2.4
Let $f\colon T \to S$ *be a morphism in* Sm_k. *We define the base change functor* f^* *by*

$$\begin{aligned} f^*\colon \mathrm{Cor}(S) &\to \mathrm{Cor}(T) \\ X/S &\mapsto X_T/T \\ \alpha &\mapsto \alpha_T. \end{aligned}$$

Lemma 2.5 ([Dég07, Lemma 1.30])

1. *The functor* $f^*\colon \operatorname{Cor}(S) \to \operatorname{Cor}(T)$ *is a tensor functor.*
2. *If* f^* *also denotes the classical base change functor of schemes* $\mathrm{Sm}_T \to \mathrm{Sm}_S$*, base change is compatible with the embeddings* $\mathrm{Sm}_S \to \operatorname{Cor}(S)$ *and* $\mathrm{Sm}_T \to \operatorname{Cor}(T)$*, i.e. the following diagram commutes:*
$$\begin{tikzcd} \mathrm{Sm}_S \arrow[r] \arrow[d, "f^*"] & \operatorname{Cor}(S) \arrow[d, "f^*"] \\ \mathrm{Sm}_T \arrow[r] & \operatorname{Cor}(T). \end{tikzcd}$$
3. *If* $g\colon T' \to T$ *is another morphism in* Sm_k*, we have a canonical isomorphism of functors* $(f \circ g)^* \simeq g^* \circ f^*$*.*

Proof. See [Dég07, Lemma 1.30]. □

Now let $f\colon T \to S$ be a smooth morphism in Sm_k. It induces a restriction functor $\mathrm{Sm}_T \to \mathrm{Sm}_S$ by forgetting the base scheme: $X/T \mapsto X/S$. Again, this can be extended to a restriction functor $\operatorname{Cor}(T) \to \operatorname{Cor}(S)$.

Let X, Y be smooth T-schemes. The base change of $X \times_S Y$ along the diagonal $T \to T \times_S T$ gives a commutative diagram

$$\begin{tikzcd} X \times_T Y \arrow[r, "\delta_{XY}"] \arrow[d] & X \times_S Y \arrow[d] \\ T \arrow[r] & T \times_S T. \end{tikzcd}$$

The morphism δ_{XY} allows us to associate to any $\alpha \in c_T(X \times_T Y)$ a correspondence $\delta_{XY*}(\alpha) \in c_S(X, Y)$. By Lemma [Dég07, Lemma 1.31], we have $\delta_{XZ*}(\beta \circ \alpha) = (\delta_{YZ*}(\beta)) \circ (\delta_{XY*}(\alpha))$ for $\alpha \in c_T(X, Y)$, $\beta \in c_T(Y, Z)$.

Definition 2.6
Let $f\colon T \to S$ *be a smooth morphism in* Sm_k*. We define the restriction functor as*

$$\begin{aligned} f_\#\colon \operatorname{Cor}(T) &\to \operatorname{Cor}(S) \\ (X \to T) &\mapsto (X \to T \xrightarrow{f} S) \\ \alpha &\mapsto \delta_{XY*}(\alpha). \end{aligned}$$

By [Dég07, Lemma 1.31], $\delta_{XY*}(|\Gamma_g|) = |\Gamma_g|$ for all T-morphisms $g\colon X \to Y$. Therefore, $f_\#$ restricts to the classical forgetting the base functor on Sm_T. Furthermore, for smooth morphisms $f\colon T' \to T$, $g\colon T \to S$, we have $(g \circ f)_\# = g_\# \circ f_\#$.

The functors f^* and $f_\#$ are adjoint functors in case f is smooth and of finite type.

Lemma 2.7 ([Dég07, Proposition 1.34])
Let $f\colon T \to S$ be a smooth morphism of finite type in Sm_k. Then:

1. *The functor $f_\#$ is left adjoint to the functor f^*.*
2. *For every $X \in \mathrm{Sm}_S$ and $Y \in \mathrm{Sm}_T$ the obvious morphism obtained by adjunction*
$$f_\#(f^*X \times_T Y) \to X \times_S f_\# Y$$
is an isomorphism.

Proof. See [Dég07, Proposition 1.34]. □

2.2 Geometric motives

In the first section of this chapter we defined the additive category of S-correspondences $\mathrm{Cor}(S)$ with objects given as smooth schemes over S and morphisms given by finite S-correspondences. In this section we construct the triangulated category of geometric motives $\mathrm{DM}_{\mathrm{gm}}(S)$ over S.

We follow the notation of [Lev06], where this construction is done in detail for the case of $S = \operatorname{Spec} k$, the spectrum of a field. This construction was originally done by Voevodsky in [Voe00] and was generalised to the case of an arbitrary smooth base scheme S by Ivorra in [Ivo07, section 1.3] and Cisinski and Déglise in [CD09, section 11.1.b].

Since $\mathrm{Cor}(S)$ is an additive category, we can form the bounded homotopy category $\mathcal{K}^b(\mathrm{Cor}(S))$ of $\mathrm{Cor}(S)$. This is a triangulated category, where the distinguished triangles are the triangles that are isomorphic to a cone sequence. The tensor product $\otimes$ on $\mathrm{Cor}(S)$ makes $\mathcal{K}^b(\mathrm{Cor}(S))$ a triangulated tensor category and we denote the tensor product on $\mathcal{K}^b(\mathrm{Cor}(S))$ also by $\otimes$.

While we obtain a triangulated tensor category, our desired category of geometric motives should carry even more "nice" properties, such as being $\mathbb{A}^1$-homotopy invariant and having long exact sequences like the Mayer-Vietoris sequence for open covers. Therefore, we define the category $\widehat{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}}(S)$ as the localization of $\mathcal{K}^b(\mathrm{Cor}(S))$ with respect to these properties.

Definition 2.8
The category $\widehat{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}}(S)$ *is the localization of the triangulated tensor category* $\mathcal{K}^b(\mathrm{Cor}(S))$ *with respect to the thick triangulated subcategory generated by complexes of the following form:*

1. $[X \times_S \mathbb{A}^1_S] \xrightarrow{p_*} [X]$ *for* $X \in \mathrm{Sm}_S$*;*

2. $\mathrm{Cone}\Big([U \cap V] \xrightarrow{(j_{U*}, -j_{V*})} [U \oplus V]\Big) \xrightarrow{i_{U*}+i_{V*}} [X]$, *where we write* $X \in \mathrm{Sm}_S$ *as a union of Zariski open subschemes* U*,* V *and* $j_U \colon U \cap V \to U$*,* $j_V \colon U \cap V \to V$*,* $i_U \colon U \to X$*,* $i_V \colon V \to X$ *are the obvious inclusions.*

Since the morphisms inverted in the definition of $\widehat{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}}(S)$ are closed under $\otimes$, $\widehat{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}}(S)$ inherits the tensor structure from $\mathcal{K}^b(\mathrm{Cor}(S))$ making it a triangulated tensor category.

Definition 2.9
We define the category of effective geometric motives $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(S)$ *over* S *as the pseudo-abelian hull of* $\widehat{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}}(S)$*.*

$\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(S)$ is a triangulated tensor category by [BS01, Theorem 1.5].

The embedding $\mathrm{Sm}_S \to \mathrm{Cor}(S)$ extends canonically to a functor $\mathrm{m}^{\mathrm{eff}}_S \colon \mathrm{Sm}_S \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(S)$.

At the end of this chapter we study a subcategory of motives generated by the so-called Tate motives. In order to obtain a Tannakian category of Tate motives, we need a rigid tensor category. However, in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(S)$ the Tate motives are not rigid. Therefore, we construct the category $\mathrm{DM}_{\mathrm{gm}}(S)$ out of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(S)$ by formally inverting these objects.

Definition 2.10
We define $\mathbb{Z}_S(1) := \mathrm{Cone}([S] \xrightarrow{i_{\infty *}} [\mathbb{P}^1_S])[-2]$ *and call it the Tate motive. We put* $\mathbb{Z}_S(n) := \mathbb{Z}_S(1)^{\otimes n}$ *for* $n \geq 0$.

Now we define the triangulated category $\mathrm{DM}_{\mathrm{gm}}(S)$ *of geometric motives over* S *by inverting the functor* $- \otimes \mathbb{Z}_S(1)$ *on* $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(S)$, *i.e. one has objects* $M(n)$ *for* $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(S)$ *and* $n \in \mathbb{Z}$ *and*

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(S)}(X(n), Y(m))\\ &\qquad := \varinjlim_{N} \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(S)}(X \otimes \mathbb{Z}_S(n+N), Y \otimes \mathbb{Z}_S(m+N)).\end{aligned}$$

$\mathrm{DM}_{\mathrm{gm}}(S)$ is a triangulated tensor category by [Ivo07, section 1.3]. We denote the unit $[S]$ of the tensor structure $\otimes$ on $\mathrm{DM}_{\mathrm{gm}}(S)$ by $\mathbb{Z}_S(0)$. Furthermore, there are canonical isomorphisms $\mathbb{Z}_S(n) \otimes \mathbb{Z}_S(m) \simeq \mathbb{Z}_S(n+m)$ for all n, $m \in \mathbb{Z}$.

Sending X to $X(0)$ defines a functor $\iota_S \colon \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(S) \to \mathrm{DM}_{\mathrm{gm}}(S)$ and we denote the composition $\iota_S \circ \mathrm{m}^{\mathrm{eff}}_{\mathrm{S}}$ by $\mathrm{m}_S \colon \mathrm{Sm}_S \to \mathrm{DM}_{\mathrm{gm}}(S)$.

Remark 2.11
For $S = \mathrm{Spec}\, k$, k *a field, the functor* $\iota_k \colon \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)$ *is a full embedding by [Voe00, Chapter V, Theorem 3.4.1]. The analogous result for arbitrary schemes* $S \in \mathrm{Sm}_k$ *is not known.*

We define the motivic cohomology groups of a smooth scheme S over a field k with coefficients in $\mathbb{Z}$ respectively $\mathbb{Q}$ as:

$$\begin{aligned}\mathrm{H}^m(S, \mathbb{Z}(n)) &:= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(\mathrm{m}_k(S), \mathbb{Z}_k(n)[m]),\\ \mathrm{H}^m(S, \mathbb{Q}(n)) &:= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(\mathrm{m}_k(S), \mathbb{Z}_k(n)[m]) \otimes \mathbb{Q}\,.\end{aligned}$$

If $f \colon T \to S$ is a morphism in Sm_k, the functor $f^* \colon \mathrm{Cor}(S) \to \mathrm{Cor}(T)$ that was defined in Definition 2.4 induces a tensor functor $f^* \colon \mathrm{DM}_{\mathrm{gm}}(S) \to \mathrm{DM}_{\mathrm{gm}}(T)$ by [CD09, Remark 11.18].

Furthermore, if $f \colon T \to S$ is smooth, the functor $f_\# \colon \mathrm{Cor}(T) \to \mathrm{Cor}(S)$ (see Definition 2.6) induces a functor $f_\# \colon \mathrm{DM}_{\mathrm{gm}}(T) \to \mathrm{DM}_{\mathrm{gm}}(S)$ (again by [CD09, Remark 11.18]).

Example 2.12
Let $f \colon S \to T$ *be a morphism in* Sm_k. *Let* X *be a smooth scheme over*

T. As we have seen, the pull-back of $[X] \in \mathrm{DM}_{\mathrm{gm}}(T)$ via f is given by the base change $[X \times_T S]$.

Therefore, the pull-back of $\mathbb{Z}_T(1)[2] = \mathrm{Cone}([T] \to [\mathbb{P}^1_T])$ is

$$f^*\mathbb{Z}_T(1)[2] = \mathrm{Cone}([T \times_T S] \to [\mathbb{P}^1_T \times_T S]) = \mathrm{Cone}([S] \to [\mathbb{P}^1_S]) = \mathbb{Z}_S(1)[2].$$

Since f^* is a monoidal functor, we have $f^*\,\mathbb{Z}_T(q)[p] = \mathbb{Z}_S(q)[p]$ for all q, $p \in \mathbb{Z}$.

Example 2.13
Let k be a number field and L be a quadratic Galois extension with Galois group $G = \{\mathrm{id}, \sigma\}$. Thus, we have a finite, étale (hence smooth) morphism $\phi\colon \operatorname{Spec} L \to \operatorname{Spec} k$ in Sm_k. Therefore, $\phi_\#\,\mathbb{Z}_L(0) = \phi_\#[\operatorname{Spec} L]$ is given by $[\operatorname{Spec} L \xrightarrow{\phi} \operatorname{Spec} k]$.

We want to compute $\phi^*\phi_\#\,\mathbb{Z}_L(0)$ in $\mathrm{DM}_{\mathrm{gm}}(L)$:

$$\begin{aligned}\phi^*\phi_\#\,\mathbb{Z}_L(0) = \phi^*[\operatorname{Spec} L \xrightarrow{\phi} \operatorname{Spec} k] &= [\operatorname{Spec} L \times_k \operatorname{Spec} L] \\ &\simeq [\operatorname{Spec}(L \otimes_k L)] \simeq [\operatorname{Spec} L \amalg \operatorname{Spec} L] = [\operatorname{Spec} L] \oplus [\operatorname{Spec} L],\end{aligned}$$

where the isomorphism $L \otimes_k L \cong L \times L$ is given by $x \otimes y \mapsto (xy, x\sigma(y))$.

Similarly, for a finite Galois extension $L|k$ of degree n with Galois group G, we have

$$\phi^*\phi_\#\,\mathbb{Q}_L(0) \simeq \bigoplus_{\sigma \in G} [\operatorname{Spec} L].$$

2.3 Sheaves with transfer

This section covers the second construction by Voevodsky of a triangulated category of motives over a scheme S using Nisnevich sheaves with transfer. We follow [Lev10, Chapter 3] with additions from [Dég07].

Again, let S be a smooth separated scheme of finite type over a field k.

A Nisnevich cover of $X \in \mathrm{Sm}_S$ is a family of étale morphisms $p_i\colon Y_i \to X$ such that for any $x \in X$ there exists $y_i \in Y_i$ for some i with $p_i(y_i) = x_i$ such that the induced map between the residue fields $K(x) \to K(y_i)$ is an isomorphism. The Nisnevich covers define a Grothendieck pre-topology and we call the Grothendieck topology generated by it the Nisnevich topology.

We define the abelian category of presheaves with transfer over S, $\mathrm{PST}(S)$, as the category of additive presheaves of abelian groups on $\mathrm{Cor}(S)$, i.e. additive contravariant functors from $\mathrm{Cor}(S)$ to the category of abelian groups.

A Nisnevich sheaf with transfers over S is a presheaf with transfers P such that the restriction of P to a presheaf on Sm_S via the embedding $\mathrm{Sm}_S \to \mathrm{Cor}(S)$ is a sheaf for the Nisnevich topology. We denote by $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ the full subcategory of $\mathrm{PST}(S)$ of Nisnevich sheaves with transfer.

By $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$ we denote the category of (unbounded) complexes over $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$, and by $\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$ the derived category that is equivalent to the homotopy category of $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ by [Lev10, section 3.1].

For $Z \in \mathrm{Sm}_S$ we have the representable presheaf $\mathbb{Z}^{tr}_S(Z)\colon X \mapsto c_s(X, Z)$ and pull-back maps are given by composition of correspondences. $\mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(Z)$ is a Nisnevich sheaf by Lemma [Dég07, Lemma 2.4].

The operation $\mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(X) \otimes^{tr}_S \mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(Y) := \mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(X \times_S Y)$ extends to a tensor operation $\otimes^{tr}_S$ making $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ a tensor category since the sheaves $\mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(X)$, $X \in \mathrm{Sm}_S$, are generators for the Grothendieck abelian category $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ (see [Dég07, Proposition 2.8 and Lemma 2.11] for details). The identity object for the tensor structure is given by $\mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(S)$.

Thus, $\otimes^{tr}_S$ defines a left-derived tensor product

$$\otimes^L_S\colon \mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)) \times \mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)) \to \mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$$

which makes $\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$ a triangulated tensor category.

We define the category of effective motives on S, $\mathrm{DM}^{\mathrm{eff}}(S)$, as the localisation of the triangulated category $\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$ with respect to the localising category generated by the complexes $\mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(X \times \mathbb{A}^1) \to \mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(X)$, $X \in \mathrm{Sm}_S$. We denote by $\mathrm{m}^{\mathrm{eff}}_{\mathrm{S}}(X)$ the image of $\mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(X)$ in $\mathrm{DM}^{\mathrm{eff}}(S)$.

$\mathrm{DM}^{\mathrm{eff}}(S)$ is again a triangulated tensor category with tensor product $\otimes_S$ induced from $\otimes^L_S$ via the localization map by [CD07, Example 3.15]. Furthermore, $\mathrm{m}^{\mathrm{eff}}_{\mathrm{S}}(X) \otimes_S \mathrm{m}^{\mathrm{eff}}_{\mathrm{S}}(Y) = \mathrm{m}^{\mathrm{eff}}_{\mathrm{S}}(X \times_S Y)$.

We define the presheaf with transfer

$$\mathrm{T}^{\mathrm{tr}}_{\mathrm{S}} := \mathrm{coker}(\mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(S) \xrightarrow{i_{\infty *}} \mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(\mathbb{P}^1_S))$$

and denote by $\mathbb{Z}_S(1)$ the image of $\mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(1) := \mathrm{T}^{\mathrm{tr}}_{\mathrm{S}}[-2]$ in $\mathrm{DM}^{\mathrm{eff}}(S)$. Furthermore, we denote by $\mathbb{Z}_S(n)$ the image of $\mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(n) := (\mathrm{T}^{\mathrm{tr}}_{\mathrm{S}}[-2])^{\otimes^{tr}_S n}$, $n \geq 0$, in $\mathrm{DM}^{\mathrm{eff}}(S)$.

Again, we want to invert the motive $\mathbb{Z}_S(1)$ on $\mathrm{DM}^{\mathrm{eff}}(S)$ to define the triangulated category of motives $\mathrm{DM}(S)$. This is done via the category of symmetric $\mathrm{T}^{\mathrm{tr}}_{\mathrm{S}}$ spectra.

Before we are able to do so, we need to endow $\mathrm{DM}^{\mathrm{eff}}(S)$ with a model category structure. A model category structure on a category consists of three classes of morphisms – fibrations, cofibrations and weak equivalences – satisfying certain axioms. See [BG76, Definition 4.1] for the precise definition.

$\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$ has a model category structure $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathrm{Nis}}$ that is given by [CD07, Example 1.6, Theorem 1.7]. More explicitly, Levine describes the model structure in [Lev10] as follows:

The cofibrations are generated by maps of the form $\sigma_X[n]\colon \mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(X)[n] \to D_X[n]$, where $X \in \mathrm{Sm}_S$, $n \in \mathbb{Z}$, D_X is the cone of the identity map $\mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(X) \to \mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(X)$ and $\sigma_X[n]$ is the canonical map $\mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(X) \to D_X$. This means, that the class of cofibrations is the smallest class of morphisms in $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$ that contains the maps $\sigma_X[n]$ for all X and is closed under pushouts, transfinite compositions and retracts. The weak equivalences are the quasi-isomorphisms for the Nisnevich topology and the fibrations are the morphisms having the right lifting property with respect to acyclic cofibrations, i.e. maps that are cofibrations and weak equivalences. We denote this model category structure by $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathrm{Nis}}$. Applying the Bousfield localisation to $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathrm{Nis}}$ with respect to the complexes $\{\mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(X \times_k \mathbb{A}^1_k) \to \mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(X), X \in \mathrm{Sm}_S\}$ gives the model category structure $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$ on $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$. By [CD07, Proposition 3.5, Example 3.15], the homotopy category of $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$ is equivalent to $\mathrm{DM}^{\mathrm{eff}}(S)$, where the model category structure on $\mathrm{DM}^{\mathrm{eff}}(S)$ is induced by the one on $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathrm{Nis}}$.

Let $\mathrm{Spt}^{\mathfrak{S}}_{\mathrm{T}^{\mathrm{tr}}}(S)$ be the category of $\mathrm{T}^{\mathrm{tr}}_{\mathrm{S}}$ spectra in $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$. Its objects are sequences $E := (E_0, E_1, \ldots)$, where $E_n \in \mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$, with E_n endowed with an action of the symmetric group S_N and bonding maps $\epsilon_n\colon E_n \otimes^{tr}_S \mathrm{T}^{\mathrm{tr}}_{\mathrm{S}} \to E_{n+1}$. Furthermore, we require that for all $n \geq 0$ and $m \geq 1$ the iterated bonding map

$$E_n \otimes^{tr}_S (\mathrm{T}^{\mathrm{tr}}_{\mathrm{S}})^{\otimes m} \xrightarrow{\epsilon_n \otimes \mathrm{id}} E_{n+1} \otimes^{tr}_S (\mathrm{T}^{\mathrm{tr}}_{\mathrm{S}})^{\otimes m-1} \longrightarrow \ldots \longrightarrow E_{n+m}$$

is $S_n \times S_m$ equivariant, where we use the canonical inclusion $S_n \times S_m \subset S_{n+m}$. Morphisms are given by sequences of maps $f = \{f_n\}$ in $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$ that commute with the bonding maps and such that f_n is S_n-equivariant for all n. Note that this makes $\mathrm{Spt}^{\mathfrak{S}}_{\mathrm{T}^{\mathrm{tr}}}(S)$ a dg category.

Now, we define a model category structure on $\mathrm{Spt}^{\mathfrak{S}}_{\mathrm{T^{tr}}}(S)$ following the construction of Hovey in [Hov01]. For $A \in \mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$ and $i \geq 0$, we have the object $A\{-i\}^{\mathfrak{S}}$ in $\mathrm{Spt}^{\mathfrak{S}}_{\mathrm{T^{tr}}}(S)$ with $A\{-i\}^{\mathfrak{S}}_{i+n} := S_{i+n} \times_{S_n} A \otimes^{tr}_S (\mathrm{T}^{\mathrm{tr}}_{\mathrm{S}})^{\otimes n}$ and $A\{-i\}^{\mathfrak{S}}_n = 0$ for $n < i$. Sending A to $A\{-i\}^{\mathfrak{S}}$ defines a functor $(-)\{-i\}^{\mathfrak{S}}\colon \mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)) \to \mathrm{Spt}^{\mathfrak{S}}_{\mathrm{T^{tr}}}(S)$. The projective model structure on $\mathrm{Spt}^{\mathfrak{S}}_{\mathrm{T^{tr}}}(S)$ consists of weak equivalences and fibrations $f = \{f_n\}$ such that f_n is a weak equivalence respectively a fibration for all n. The class of cofibrations is the smallest class of morphisms containing the maps $f\{-i\}^{\mathfrak{S}}$ with f a cofibration in $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathrm{Nis}}$ that is closed under pushouts, transfinite compositions and retracts. We denote this model category by $\mathrm{Spt}^{\mathfrak{S}}_{\mathrm{T^{tr}}}(S)_{\mathrm{proj}}$.

A $\mathrm{T}^{\mathrm{tr}}_{\mathrm{S}}$-$\Omega$ spectrum is a $\mathrm{T}^{\mathrm{tr}}_{\mathrm{S}}$ spectrum $E = (E_0, E_1, \ldots)$ such that E_n is a fibrant object in $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}})_{\mathbb{A}^1}$, i.e. the unique map from E_n to the zero object is a fibration, and such that the map

$$E_n \to \mathrm{Hom}_{\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))}(\mathrm{T}^{\mathrm{tr}}_{\mathrm{S}}, E_{n+1})$$

adjoint to ϵ_n is a weak equivalence in $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$. We call a map $f\colon A \to B$ in $\mathrm{Spt}^{\mathfrak{S}}_{\mathrm{T^{tr}}}(S)$ a stable weak equivalence if the induced map

$$f^*\colon \mathrm{Hom}_{\mathrm{Spt}^{\mathfrak{S}}_{\mathrm{T^{tr}}}(S)_{\mathrm{proj}}}(B, E) \to \mathrm{Hom}_{\mathrm{Spt}^{\mathfrak{S}}_{\mathrm{T^{tr}}}(S)_{\mathrm{proj}}}(A, E)$$

is an isomorphism for all $\mathrm{T}^{\mathrm{tr}}_{\mathrm{S}}$-$\Omega$ spectra E. Then, the stable model category $\mathrm{Spt}^{\mathfrak{S}}_{\mathrm{T^{tr}}}(S)_s$ is the Bousfield localisation of the model category $\mathrm{Spt}^{\mathfrak{S}}_{\mathrm{T^{tr}}}(S)_{\mathrm{proj}}$ with respect to stable weak equivalences.

Definition 2.14
The triangulated category $\mathrm{DM}(S)$ *of motives over* S *is the homotopy category of* $\mathrm{Spt}^{\mathfrak{S}}_{\mathrm{T^{tr}}}(S)_s$.

$\mathrm{DM}(S)$ is indeed a triangulated category by [CD09, section 5.3.d]. Furthermore, sending $A \in \mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$ to the sequence $(A, A \otimes^{tr}_S \mathrm{T}^{\mathrm{tr}}_{\mathrm{S}}, A \otimes^{tr}_S (\mathrm{T}^{\mathrm{tr}}_{\mathrm{S}})^{\otimes 2}, \ldots)$ defines a functor

$$\textstyle\sum^{\infty}_T\colon \mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)) \to \mathrm{Spt}^{\mathfrak{S}}_{\mathrm{T^{tr}}}(S),$$

where the action of S_n is given by permutation on $(\mathrm{T}^{\mathrm{tr}}_{\mathrm{S}})^{\otimes n}$ and trivial action on A. This induces a triangulated functor between the homotopy categories

$$\textstyle\sum^{\infty}_T\colon \mathrm{DM}^{\mathrm{eff}}(S) \to \mathrm{DM}(S).$$

by [CD09, section 5.3].

We denote the image of $\mathbb{Z}_S(n)$, $n \geq 0$, under $\sum_T^\infty$ also by $\mathbb{Z}_S(n)$.

The tensor structure on $\mathrm{C}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$ extends to a tensor structure $\otimes$ on $\mathrm{DM}(S)$ via

$$(E \otimes_S^{tr} F)_n := \oplus_{p+q=n, \alpha \colon \{1,\dots,p\} \sqcup \{1,\dots,q\} \tilde{\to} \{1,\dots,n\}} E_p \otimes_S^{tr} F_q,$$

where α runs over all bijections of sets. This makes $\mathrm{DM}(S)$ a triangulated tensor category and $\sum_T^\infty \colon \mathrm{DM}^{\mathrm{eff}}(S) \to \mathrm{DM}(S)$ a triangulated tensor functor. We do not give the details here (an explicit description can be found in [Lev10, section 3.3] and [CD09, section 5.3]), but summarise the important facts in the following lemma:

Lemma 2.15
The functor $- \otimes \mathrm{T}^{\mathrm{tr}}_{\mathrm{S}} \colon \mathrm{DM}(S) \to \mathrm{DM}(S)$ *is an equivalence.*

The object $\mathbb{Z}_S(-1) := \mathbb{Z}_S(0)\{-1\}$ *is a tensor inverse to* $\mathbb{Z}_S(1)$ *in* $\mathrm{DM}(S)$.

Proof. The first statement is Proposition 3.3.1 in [Lev10]. For the second statement, see [CD09, section 5.3.23]. □

For $n \geq 0$, we define $\mathbb{Z}_S(-n) := \mathbb{Z}_S(-1)^{\otimes n}$.

Let $f \colon T \to S$ be a morphism in Sm_k. The base change functor $f^* \colon \mathrm{Cor}(S) \to \mathrm{Cor}(T)$ that we defined in Definition 2.4 yields a base change functor $f^* \colon \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S) \to \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(T)$ by [Dég07, section 2.5.2]. In particular, $f^* \mathbb{Z}^{\mathrm{tr}}_{\mathrm{S}}(Z) = \mathbb{Z}^{tr}_T(Z \times_S T)$ and therefore, $f^* \mathbb{Z}_S(1) = Z_T(1)$. Since $f^* \colon \mathrm{Cor}(S) \to \mathrm{Cor}(T)$ is a tensor functor, so is $f^* \colon \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S) \to \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(T)$.

This functor has a right adjoint $f_* \colon \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(T) \to \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ defined by

$$f_*(\mathcal{F})(X) := \mathcal{F}(X \times_S T),$$

where $\mathcal{F} \in \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(T)$ and $X \in \mathrm{Sm}_S$ (see [Dég07, section 2.5.1] for details).

If $f \colon T \to S$ is a smooth morphism of finite type, f^* also admits a left adjoint that is induced by the restriction functor $f_\# \colon \mathrm{Cor}(T) \to \mathrm{Cor}(S)$ defined in Definition 2.6. We denote the left adjoint of f^* also by $f_\# \colon \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(T) \to \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$. If $f \colon T \to S$ is furthermore finite and étale, the functors f_* and $f_\#$ agree by [Dég07, section 2.5.3]. In particular, in this case f_* and f^* are left and right adjoint to each other.

By [CD09], the functors f_*, f^* and $f_\#$ (if defined) yield well-defined adjoint derived functors on $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(-))$ and hence on $\mathrm{DM}(-)$.

For an arbitrary separated morphism $f\colon T \to S$ of finite type, there also exist the exceptional functors $f^!$ and $f_!$ on $\mathrm{DM}(-)$. We do not give a definition of these (see [CD09, section 2.2] for details) since $f_!$ and f_* agree if f is proper and f^* and $f^!$ agree if f is étale.

We summarise the important properties of these functors in the following lemma.

Lemma 2.16
Let $f\colon T \to S$ be a morphism in Sm_k.

1. *The functor $f_*\colon \mathrm{DM}(T) \to \mathrm{DM}(S)$ is a right adjoint to $f^*\colon \mathrm{DM}(S) \to \mathrm{DM}(T)$.*
2. *If $f\colon T \to S$ is smooth and of finite type, f^* also admits a left adjoint $f_\#\colon \mathrm{DM}(T) \to \mathrm{DM}(S)$.*
3. *If $f\colon T \to S$ is finite and étale, then $f_* = f_\#$, i.e. f^* and f_* are left and right adjoint to each other.*
4. *If $f\colon T \to S$ is finite and étale and $g\colon T' \to S$ is another finite and étale morphism, then there is a natural base-change isomorphism $f^*g_* \simeq g'_*f'^*$, where $S' := T \times_S T'$ and $f'\colon S' \to T'$ and $g'\colon S' \to T$ are induced by f and g via base-change.*

Proof. The first three statements follow by the previous discussion. The last claim follows by the more general base change isomorphism $f^*g_! \simeq g'_!f'^*$ using the exceptional functors $f^!$ and $f^!$ and the fact that $f_! = f_*$ for f proper and $f^! = f^*$ for f étale. For more details see [CD09, section 2.2]. Furthermore, see [CD09, A.5] for an overview of the relations between the functors f_*, f^*, $f_!$ and $f^!$. □

An important consequence of the adjointness of the functors f_* and f^* is the following statement.

Theorem 2.17 ([CD09, Example 11.2.3])
Let k be a field. Let $S \in \mathrm{Sm}_k$ and let $X \in \mathrm{Sm}_S$. Then there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{DM}(S)}(m_S(X), \mathbb{Z}_S(n)[m]) \simeq \mathrm{Hom}_{\mathrm{DM}(k)}(m_k(X), \mathbb{Z}_k(n)[m]).$$

Proof. Let $f\colon S \to \operatorname{Spec} k$.

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DM}(S)}(m_S(X), \mathbb{Z}_S(n)[m]) &\simeq \mathrm{Hom}_{\mathrm{DM}(S)}(m_S(X), f^* \mathbb{Z}_k(n)[m]) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}(k)}(f_\# m_S(X), \mathbb{Z}_k(n)[m]) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}(k)}(m_k(X), \mathbb{Z}_k(n)[m]).\end{aligned}$$

□

2.4 The embedding theorem

Sending $Y \in \mathrm{Sm}_S$ to the representable presheaf with transfers $\mathbb{Z}_{\mathrm{S}}^{\mathrm{tr}}(Y)$ gives an exact tensor functor

$$i_S^{\mathrm{eff}}\colon \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S) \to \mathrm{DM}^{\mathrm{eff}}(S).$$

Therefore, i_S^{eff} links both triangulated categories of motives that we have defined in the previous two sections.

Since the functor $- \otimes_S \mathbb{Z}_S(1)$ is invertible on $\mathrm{DM}(S)$ and $i_S^{\mathrm{eff}} \cong \mathbb{Z}_S(1)$, i_S^{eff} extends to an exact tensor functor

$$i_s\colon \mathrm{DM}_{\mathrm{gm}}(S) \to \mathrm{DM}(S)$$

giving us a commutative diagram of exact tensor functors:

$$\begin{tikzcd} \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S) \arrow[r, "i_S^{\mathrm{eff}}"] \arrow[d, "\iota_S"'] & \mathrm{DM}^{\mathrm{eff}}(S) \arrow[d] \\ \mathrm{DM}_{\mathrm{gm}}(S) \arrow[r, "i_S"'] & \mathrm{DM}(S). \end{tikzcd}$$

Theorem 2.18 ([Lev10, Theorem 3.5.3], [CD09, Theorem 11.1.13]) *The functors*

$$i_S^{\mathrm{eff}}\colon \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S) \to \mathrm{DM}^{\mathrm{eff}}(S)$$

and

$$i_s\colon \mathrm{DM}_{\mathrm{gm}}(S) \to \mathrm{DM}(S)$$

are fully faithful embeddings.

Proof. See [CD09, Theorem 11.1.13]. It identifies $\mathrm{DM}_{\mathrm{gm}}(S)$ with the full subcategory of compact objects of $\mathrm{DM}(S)$. □

Hence, we can identify the categories $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(S)$ and $\mathrm{DM}_{\mathrm{gm}}(S)$ with full subcategories of $\mathrm{DM}^{\mathrm{eff}}(S)$ and $\mathrm{DM}(S)$ respectively.

2.5 Tate Motives

In this section we define the triangulated category of Tate motives over a scheme S. It is the full triangulated subcategory generated by the motives $\mathbb{Z}_S(n)$ for all $n \in \mathbb{Z}$ in the $\mathbb{Q}$-linearisation of the category $\mathrm{DM}_{\mathrm{gm}}(S)$ (or $\mathrm{DM}(S)$ respectively).

The vanishing of certain morphism groups ensures the existence of a non-degenerate t-structure with heart $\mathrm{MT}(S)$. $\mathrm{MT}(S)$ is in fact a Tannakian category and its Tannaka group is given as the semidirect product of $\mathbb{G}_m$ with a unipotent group scheme.

This section follows [Lev10] and [Lev06].

As in the previous sections let S be a separated, smooth scheme of finite type over a field k. For simplicity we also assume S to be irreducible. The general case of a reducible base scheme S can be obtained by writing S as a direct sum of its irreducible components.

2.5.1 Definition

Before defining the triangulated category of Tate motives, recall that $\mathrm{DM}_{\mathrm{gm}}(S)$ consists of the compact objects of $\mathrm{DM}(S)$ by [CD09, Theorem 11.1.13]. Therefore, the $\mathbb{Q}$-linearisation $\mathrm{DM}_{\mathrm{gm}}(S)_{\mathbb{Q}}$ of $\mathrm{DM}_{\mathrm{gm}}(S)$ is given by the same objects as $\mathrm{DM}_{\mathrm{gm}}(S)$ and morphisms

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(S)_{\mathbb{Q}}}(X,Y) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(S)}(X,Y) \otimes \mathbb{Q}.$$

This follows by [Kel13, Corollary A.2.12]. We denote the image of the Tate motives $\mathbb{Z}_S(n)$ in $\mathrm{DM}_{\mathrm{gm}}(S)_{\mathbb{Q}}$ by $\mathbb{Q}_S(n)$. In particular, we have

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(S)_{\mathbb{Q}}}(\mathbb{Q}_S(0), \mathbb{Q}_S(n)[m]) \simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(S)}(\mathbb{Z}_S(0), \mathbb{Z}_S(n)[m]) \otimes \mathbb{Q}$$

for all n, $m \in \mathbb{Z}$.

By [CD09, Theorem 11.1.13], the category $\mathrm{DM}(S)$ is compactly generated. Hence, we can also define the $\mathbb{Q}$-linearisation $\mathrm{DM}(S)_{\mathbb{Q}}$ of $\mathrm{DM}(S)$ as

the Verdier localisation $\mathrm{DM}(S)/\,\mathrm{DM}(S)_{\mathrm{tor}}$, where $\mathrm{DM}(S)_{\mathrm{tor}}$ is the subcategory of compact torsion objects. Again, we denote the image of $\mathbb{Z}_S(n)$ in $\mathrm{DM}(S)_{\mathbb{Q}}$ by $\mathbb{Q}_S(n)$.
Note that for arbitrary objects $X, Y \in \mathrm{DM}(S)$, $\mathrm{Hom}_{\mathrm{DM}(S)_{\mathbb{Q}}}(X, Y)$ and $\mathrm{Hom}_{\mathrm{DM}(S)}(X, Y) \otimes \mathbb{Q}$ are not isomorphic in general. However, this is true if X is compact by [Kel13, Corollary A.2.13]. In particular, for the compact objects $\mathbb{Z}_S(0)$, $\mathbb{Z}_S(n)$ we have:

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(S)_{\mathbb{Q}}}(\mathbb{Q}_S(0), \mathbb{Q}_S(n)[m]) \simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(S)}(\mathbb{Z}_S(0), \mathbb{Z}_S(n)[m]) \otimes \mathbb{Q}\,.$$

Definition 2.19
The triangulated category of geometric mixed Tate motives $\mathrm{DMT}_{\mathrm{gm}}(S)$ *over* S *is the strictly full triangulated subcategory of* $\mathrm{DM}_{\mathrm{gm}}(S)_{\mathbb{Q}}$ *that is generated by the Tate objects* $\mathbb{Q}_S(n)$*,* $n \in \mathbb{Z}$*.*

The triangulated category of mixed Tate motives $\mathrm{DMT}(S)$ *over* S *is the strictly full triangulated subcategory of* $\mathrm{DM}(S)_{\mathbb{Q}}$ *that is generated by the Tate objects* $\mathbb{Q}_S(n)$*,* $n \in \mathbb{Z}$*.*

In Theorem 2.18 we have seen that the embedding $i_S\colon\ \mathrm{DM}_{\mathrm{gm}}(S) \to \mathrm{DM}(S)$ is fully faithful. Since $i_S(\mathbb{Z}_S(n)) \simeq \mathbb{Z}_S(n)$ for all $n \in \mathbb{Z}$, we have proven the following proposition.

Proposition 2.20 ([Lev10, Proposition 3.6.2])
The $\mathbb{Q}$*-linearisation of* $i_S\colon\ \mathrm{DM}_{\mathrm{gm}}(S) \to \mathrm{DM}(S)$ *restricted to* $\mathrm{DMT}_{\mathrm{gm}}(S)$ *defines an equivalence*

$$i_S\colon\ \mathrm{DMT}_{\mathrm{gm}}(S) \to \mathrm{DMT}(S)$$

of triangulated tensor categories.

Therefore, we identify the category $\mathrm{DMT}_{\mathrm{gm}}(S)$ with the subcategory $\mathrm{DMT}(S)$ of $\mathrm{DM}(S)_{\mathbb{Q}}$ and just write $\mathrm{DMT}(S)$ in the following.

Lemma 2.21 ([Lev10, Lemma 3.6.5])
$\mathrm{DMT}(S)$ *is a rigid tensor triangulated category.*

Proof. By [Lev98, Part I, IV. Theorem 1.2.5] it is enough to check that the generators $\mathbb{Q}_S(n)$, $n \in \mathbb{Z}$, of the triangulated category $\mathrm{DMT}(S)$ admit a dual. We put $\mathbb{Q}_S(n)^\vee := \mathbb{Q}_S(-n)$. Then we have the canonical isomorphisms $\mathbb{Q}_S(0) \simeq \mathbb{Q}_S(n) \otimes \mathbb{Q}_S(n)^\vee$ showing that $\mathbb{Q}_S(n)^\vee$ is indeed the dual of $\mathbb{Q}_S(n)$. $\square$

The advantage of the $\mathbb{Q}$-linearisation is that the groups of morphisms between the generators of $\mathbb{Q}(n)$ and $\mathbb{Q}(m)$, n, $m \in \mathbb{Z}$, are now given as the rational K-groups of the base scheme S. Therefore, the Tate motives can be used to compute the motivic cohomology of S in terms of K-groups or higher Chow groups. Furthermore, the $\mathbb{Q}$-linearisation gives us a $\mathbb{Q}$-linear category and therefore enables a Tannakian subcategory (since Tannakian categories are required to be $\mathbb{Q}$-linear). However, note that it is still possible to define a triangulated subcategory generated by the Tate motives $\mathbb{Z}(n)$, $n \in \mathbb{Z}$, of $\mathrm{DM}_{\mathrm{gm}}(S)$ or $\mathrm{DM}(S)$ (without passing to the $\mathbb{Q}$-linearisation).

Theorem 2.22 ([CD09, Coroallary 14.2.14])
For any regular scheme S and $p, q \in \mathbb{Z}$ we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{DMT}(S)}(\mathbb{Q}_S(0), \mathbb{Q}_S(q)[p]) \cong K_{2q-p}(S)^{(q)}_{\mathbb{Q}},$$

where $K_{2q-p}(S)^{(q)}$ denotes the q-th Adams eigenspace of the K-group K_{2q-p} tensored with $\mathbb{Q}$.

This theorem enables the computation of the morphisms $\mathbb{Q}_S(0) \to \mathbb{Q}_S(q)[p]$ for any smooth connected scheme S and any $q \leq 0$.

Corollary 2.23
Let S be an irreducible smooth separated scheme of finite type over k. Then:

$$\mathrm{Hom}_{\mathrm{DMT}(S)}(\mathbb{Q}_S(0), \mathbb{Q}_S(q)[p]) \simeq \begin{cases} 0 & \text{if } q < 0 \\ 0 & \text{if } q = 0, p \neq 0 \\ \mathbb{Q} & \text{if } q = 0, p = 0. \end{cases}$$

The case $S = \operatorname{Spec} k$, where k is a number field, is of particular interest for us. In this case we can even compute the morphisms $\mathbb{Q}_S(0) \to \mathbb{Q}_S(q)[p]$ for all p, $q \in \mathbb{Z}$.

Example 2.24
Let k be a number field. By the well-known K-theory for number fields (see

e.g. [Wei05]) we have

$$\operatorname{Hom}_{\mathrm{DMT}(k)}(\mathbb{Q}_k(0), \mathbb{Q}_k(q)[p]) \simeq \begin{cases} 0 & \textit{if } q < 0 \\ 0 & \textit{if } q = 0, p \neq 0 \\ \mathbb{Q} & \textit{if } q = 0, p = 0 \\ 0 & \textit{if } q > 0, p \leq 0 \\ 0 & \textit{if } q > 0 \textit{ even }, p = 1 \\ k^\times \otimes_{\mathbb{Z}} \mathbb{Q} & \textit{if } q = p = 1 \\ \mathbb{Q}^{r_1+r_2} & \textit{if } q > 1, q \equiv 1 \mod 4, p = 1 \\ \mathbb{Q}^{r_2} & \textit{if } q > 0, q \equiv 3 \mod 4, p = 1 \\ 0 & \textit{if } q > 0, p > 1, \end{cases}$$

where r_1 and r_2 are the numbers of real and pairs of complex embeddings of k, respectively.

Note that by Lemma 2.15 tensoring with $\mathbb{Q}_S(1)$ or more generally $\mathbb{Q}_S(n)$ defines an auto-equivalence on $\mathrm{DMT}(S)$. Therefore, it is enough to know the morphisms $\mathbb{Q}_S(0) \to \mathbb{Q}_S(q)[p]$ for any p, $q \in \mathbb{Z}$.

We have seen in Example 2.12 that for a morphism $\phi\colon S \to T$ the functor $\phi^*\colon \mathrm{DM}(T) \to \mathrm{DM}(S)$ preserves Tate-motives, while the functor ϕ_* does not. This is the origin of Artin-Tate motives that we define as the push-forwards of Tate motives under finite morphisms in the next chapter.

2.5.2 t-structure

The category $\mathrm{DMT}(S)$ carries a canonical weight filtration. For any $n \in \mathbb{Z}$ we define the category $W_{\leq n}\,\mathrm{DMT}(S)$ to be the full triangulated subcategory of $\mathrm{DMT}(S)$ generated by the objects $\mathbb{Q}_S(-m)$, $m \leq n$. Dually, we denote by $W_{>n}\,\mathrm{DMT}(S)$ the full triangulated subcategory generated by the objects $\mathbb{Q}_S(-m)$, $m > n$. These subcategories are used to define a t-structure on $\mathrm{DMT}(S)$. However, this is not yet the t-structure we are ultimately interested in.

Lemma 2.25 ([Lev10, Theorem 3.6.6])
$(W_{\leq n}\,\mathrm{DMT}(S), W_{>n}\,\mathrm{DMT}(S))$ *defines a t-structure on* $\mathrm{DMT}(S)$ *for every* $n \in \mathbb{Z}$.

Proof. This follows by [Lev93, Lemma 1.2]. The requirements for the lemma are fulfilled by Corollary 2.23. □

We denote the corresponding truncation functors by

$$W_{\leq n}\colon \mathrm{DMT}(S) \to W_{\leq n}\,\mathrm{DMT}(S)$$

and

$$W_{>n}\colon \mathrm{DMT}(S) \to W_{>n}\,\mathrm{DMT}(S).$$

For $a \leq b$ we denote by $W_{[a,b]}\,\mathrm{DMT}(S)$ the full triangulated subcategory generated by $\mathbb{Q}_S(-m)$, $a \leq m \leq b$. We write gr^W_a for the functor $W_{[a,a]}$ and $\mathrm{gr}^W_a\,\mathrm{DMT}(S)$ for the subcategory $W_{[a,a]}\,\mathrm{DMT}(S)$.

Since $\mathrm{Hom}_{\mathrm{DMT}(S)}(\mathbb{Q}(-n), \mathbb{Q}(-n)[m]) \simeq 0$ for $m \neq 0$ and $n \in \mathbb{Z}$ and $\mathrm{Hom}_{\mathrm{DMT}(S)}(\mathbb{Q}(-n), \mathbb{Q}(-n)) \simeq \mathbb{Q}$ the category $\mathrm{gr}^W_n\,\mathrm{DMT}(S)$ is equivalent to the bounded derived category of $\mathbb{Q}$-vector spaces $D^b(\mathbb{Q}\text{-}\mathrm{Vec}_{\mathbb{Q}})$. Thus, it makes sense to consider the $\mathbb{Q}$-vector spaces $\mathrm{H}^m(\mathrm{gr}^W_n M)$, $m, n \in Z$, for M in $\mathrm{DMT}(S)$.

Definition 2.26
We define $\mathrm{DMT}(S)^{\leq 0}$ *to be the full subcategory of* $\mathrm{DMT}(S)$ *with objects* M *such that* $\mathrm{H}^m(\mathrm{gr}^W_n M) = 0$ *for all* $m > 0$ *and* $n \in \mathbb{Z}$*. Dually we define* $\mathrm{DMT}(S)^{\geq 0}$ *as the full subcategory of* $\mathrm{DMT}(S)$ *with objects* M *such that* $\mathrm{H}^m(\mathrm{gr}^W_n M) = 0$ *for all* $m < 0$ *and* $n \in \mathbb{Z}$*. Let* $\mathrm{MT}(S) := \mathrm{DMT}(S)^{\leq 0} \cap \mathrm{DMT}(S)^{\geq 0}$.

Remark 2.27
Let $\mathrm{DMT}(S)^{\leq 0}_n$ *be the full subcategory of* $\mathrm{DMT}(S)$ *generated by the objects* $\mathbb{Q}_S(-n)[m]$*, where* $m \leq 0$*. Then one obtains* $\mathrm{DMT}(S)^{\leq 0}$ *as the full subcategory of* $\mathrm{DMT}(S)$ *generated by the objects* $M \in \mathrm{DMT}(S)$ *with* $\mathrm{gr}^W_n M \in \mathrm{DMT}(S)^{\leq 0}_n$*. Dually one defines the full subcategories* $\mathrm{DMT}(S)^{\geq 0}_n$ *and* $\mathrm{DMT}(S)^{\geq 0}$*.*
This notation is used to define the t-structure for Artin-Tate motives in section 4.1.

Recall that the motivic cohomology with $\mathbb{Q}$-coefficients of a smooth scheme S over k is defined as

$$\mathrm{H}^p(S, \mathbb{Q}(q)) := \mathrm{Hom}_{\mathrm{DM_{gm}}(k)}(\mathrm{m}_k(S), \mathbb{Z}_k(q)[p]) \otimes \mathbb{Q}\,.$$

Using the fully faithful embedding $i_S\colon \mathrm{DM}_{\mathrm{gm}}(S) \to \mathrm{DM}(S)$ and Theorem 2.17 we can rewrite this in the following way:

$$\begin{aligned}\mathrm{H}^q(S, \mathbb{Q}(p)) &:= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(\mathrm{m}_k(S), \mathbb{Z}_k(q)[p]) \otimes \mathbb{Q} \\ &\simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(S)}(\mathrm{m}_S(S), \mathbb{Z}_S(q)[p]) \otimes \mathbb{Q} \\ &\simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(S)_{\mathbb{Q}}}(\mathbb{Q}_S(0), \mathbb{Q}_S(q)[p]) \\ &\simeq \mathrm{Hom}_{\mathrm{DMT}(S)}(\mathbb{Q}_S(0), \mathbb{Q}_S(q)[p]).\end{aligned}$$

Thus, we have proven:

Lemma 2.28 ([Lev10, Lemma 3.6.4])
For $S \in \mathrm{Sm}_k$ there is a natural isomorphism

$$\mathrm{H}^q(S, \mathbb{Q}(p)) \simeq \mathrm{Hom}_{\mathrm{DMT}(S)}(\mathbb{Q}_S(0), \mathbb{Q}_S(q)[p]).$$

We say that S satisfies the Beilinson-Soulé vanishing conjectures if $\mathrm{H}^p(S, \mathbb{Q}(q)) = 0$ for $p \leq 0$ and $q \neq 0$. In particular, for k a number field, $S = \operatorname{Spec} k$ satisfies the Beilinson-Soulé vanishing conjectures (see Example 2.24). Further examples of schemes that satisfy the Beilinson-Soulé vanishing conjectures include $\mathbb{P}^1_k \setminus X$, where k is a number field and X a finite set of k-points of $\mathbb{P}^1$ (see [Lev10, Corollary 6.6.2]), or the spectrum of the ring of integers of a number field (see [Sch11, Lemma 3.2]).

Theorem 2.29 ([Lev10, Theorem 3.6.9])
Suppose that S satisfies the Beilinson-Soulé vanishing conjectures. Then:

1. $(\mathrm{DMT}(S)^{\leq 0}, \mathrm{DMT}(S)^{\geq 0})$ *is a non-degenerate t-structure on* $\mathrm{DMT}(S)$ *with heart* $\mathrm{MT}(S)$ *containing the Tate motives $\mathbb{Q}_S(n)$, $n \in \mathbb{Z}$.*
2. $\mathrm{MT}(S)$ *is equal to the smallest abelian subcategory of* $\mathrm{MT}(S)$ *which contains the Tate motives $\mathbb{Q}_S(n)$, $n \in \mathbb{Z}$, and is closed under extensions in* $\mathrm{MT}(S)$*.*
3. *The tensor operation in* $\mathrm{DMT}(S)$ *makes* $\mathrm{MT}(S)$ *a rigid $\mathbb{Q}$-linear abelian tensor category.*
4. *The functor $\mathrm{gr}^W_* = \oplus_n \mathrm{gr}^W_n \colon \mathrm{MT}(S) \to \mathbb{Q}\text{-}\mathrm{Vec}_{\mathbb{Q}}$ is a fibre functor making* $\mathrm{MT}(S)$ *a Tannakian category which we call the category of mixed Tate motives over S.*
5. *Each object M in* $\mathrm{MT}(S)$ *has a canonical weight filtration by subobjects*
$$0 \subset \ldots \subset W_{n-1}M \subset W_nM \subset \ldots \subset M.$$

This filtration is functorial and exact in M. It is uniquely characterized by the properties of being finite (i.e. $W_nM = 0$ for n small and $W_nM = M$ for n large), and of admitting subquotients $\mathrm{gr}_n^W M = W_nM/W_{n-1}M \in \mathrm{gr}_n^W \mathrm{MT}(S)$, $n \in \mathbb{Z}$.

6. *The natural maps*

$$\mathrm{Ext}^p_{\mathrm{MT}(k)}(M,N) \to \mathrm{Hom}^p_{\mathrm{DMT}(k)}(M,N)$$

are isomorphisms, for all p, and all M, $N \in \mathrm{MT}(k)$. Both sides are zero for $p \geq 2$.

Proof. This follows by [Lev93, Theorem 1.4 and Proposition 2.1], the necessary ingredients being:

The category $\mathrm{DMT}(S)$ is generated by the Tate objects $\mathbb{Q}_S(n)$, $n \in \mathbb{Z}$, with canonical isomorphisms $\mathbb{Q}_S(n) \otimes \mathbb{Q}_S(m) \to \mathbb{Q}_S(n+m)$ for all n, $m \in \mathbb{Z}$ and satisfying:

$$\mathrm{Hom}_{\mathrm{DMT}(S)}(\mathbb{Q}_S(0), \mathbb{Q}_S(q)[p]) \simeq \begin{cases} 0 & \text{if } q < 0 \\ 0 & \text{if } q = 0, p \neq 0 \\ \mathbb{Q} & \text{if } q = 0, p = 0 \\ 0 & \text{if } q \neq 0, p \leq 0, \end{cases}$$

where the first three isomorphisms follow by Corollary 2.23 and the last one is given by the Beilinson-Soulé vanishing for S. □

2.5.3 Tannaka formalism

Let S be a separated, smooth scheme of finite type over a field k. Furthermore, we assume that S satisfies the Beilinson-Soulé vanishing conjectures and therefore, the Tannakian category $\mathrm{MT}(S)$ exists.

We denote the Tannaka group of $\mathrm{MT}(S)$ with respect to the fibre functor gr_*^W by $\mathrm{G}(\mathrm{MT}(S))$. We have the following lemma describing the structure of $\mathrm{G}(\mathrm{MT}(S))$.

Lemma 2.30 ([Lev06, Lemma 13.3])
There is a split exact sequence

$$1 \to U \to \mathrm{G}(\mathrm{MT}(S)) \rightleftarrows \mathbb{G}_m \to 1,$$

where U is a unipotent group scheme.

Proof. $\mathbb{G}_m$ is the Tannaka group of the category $\mathrm{GrVec}_{\mathbb{Q}}$ of graded $\mathbb{Q}$-vector spaces and t acts via multiplication by t^m on the vector space in degree m. $\oplus_n V_n \to \oplus_n V_n \otimes_{\mathbb{Q}} \mathbb{Q}_S(-n)$ defines a rigid tensor functor $\mathrm{GrVec}_{\mathbb{Q}} \to \mathrm{MT}(S)$ and hence a map of the Tannaka groups $p\colon \mathrm{G}(\mathrm{MT}(S)) \to \mathbb{G}_m$.

Considering $\mathrm{gr}^W_* A = \oplus_n \mathrm{gr}^W_n A$ for $A \in \mathrm{MT}(S)$ as a graded vector space defines a rigid tensor functor $\mathrm{gr}^W_*\colon \mathrm{MT}(S) \to \mathrm{GrVec}_{\mathbb{Q}}$ that clearly gives a right inverse $s\colon \mathbb{G}_m \to \mathrm{G}(\mathrm{MT}(S))$ to p.

This gives us the split exact sequence

$$1 \to U \to \mathrm{G}(\mathrm{MT}(S)) \rightleftarrows \mathbb{G}_m \to 1.$$

So it only remains to check that $U = \ker(p)$ is unipotent.

Let $\phi\colon \mathrm{gr}^W_* \to \mathrm{gr}^W_*$ be an automorphism that restricts to the identity $\mathrm{gr}^W_n \to \mathrm{gr}^W_n$ for each n, i.e. $\phi \in U = \ker(p)$. For any $A \in \mathrm{MT}(S)$ we have the weight filtration

$$0 = W_{M-1}A \subset W_M A \subset \ldots \subset W_N A \subset W_{N+1}A = A$$

for some $M, N \in \mathbb{Z}$.

Since gr^W_* is exact and ϕ is natural, ϕ must preserve the weight filtration of $\mathrm{gr}^W_* A$ that is given by $W_{\leq n}(\mathrm{gr}^W_* A) = \oplus_{m \leq n} \mathrm{gr}^W_m A$. Thus, the (a, b) component $\phi_{a,b}\colon \mathrm{gr}^W_a \to \mathrm{gr}^W_b$ is zero for $b > a$. Since $\phi \in U = \ker(p)$, $\phi_{a,b}$ is the identity for $a = b$ and ϕ is unipotent. $\square$

If $S = \operatorname{Spec} k$, where k is a number field, then the unipotent algebraic group U is determined by its Lie algebra $\operatorname{Lie} U$ which is called the motivic Lie algebra over k. The splitting of the exact sequence in Lemma 2.30 makes $\operatorname{Lie} U$ a graded Lie algebra, concentrated in negative degrees. By [DG05, Proposition 2.3], $\operatorname{Lie} U$ is a free Lie algebra. See [DG05] for more details on the Lie algebra.

3 Cell modules

The aim of this chapter is to formulate and prove Spitzweck's representation theorem. It states that the triangulated category of mixed Tate motives $\mathrm{DMT}(S)$ is equivalent to the derived category of Adams-graded dg modules $\mathcal{D}^f_{\mathcal{N}(S)}$ of finite rank over the so-called cycle algebra $\mathcal{N}(S)$ over S. However, the proof uses the equivalent homotopy category of finite $\mathcal{N}(S)$-cell modules instead of the category $\mathcal{D}^f_{\mathcal{N}(S)}$.

We start by giving the definition of the cycle algebra in section 3.1. In section 3.2, we define the homotopy category of finite cell-modules, $\mathcal{KCM}^f_A$, over a cdga A. Furthermore, we study the structure of the triangulated category $\mathcal{KCM}^f_A$ and notice in section 3.3 that, for $A = \mathcal{N}(S)$, it resembles the triangulated category of Tate motives $\mathrm{DMT}(S)$ in some key properties, e.g. the existence of a t-structure whose heart is a Tannakian category. Lastly, we give the proof of Spitzweck's representation theorem (Theorem 3.31) in section 3.4 utilizing the knowledge about $\mathrm{DMT}(S)$ and $\mathcal{KCM}^f_{\mathcal{N}(S)}$ we acquired in section 2.5 and sections 3.2 and 3.3 respectively.

3.1 Cycle algebra

We give the definition of the cycle algebra $\mathcal{N}(S)$ of a smooth, separated scheme S of finite type over a field k. This is done by constructing a complex of Nisnevich sheaves $\mathcal{N}$ on Sm_k using the algebraic cycles we defined in section 2.1. This complex can be endowed with a product making it an Adams graded cdga object in the category of complexes of Nisnevich sheaves. Then the cycle algebra over S is defined as the evaluation of $\mathcal{N}$ at

S. Since $\mathcal{N}(S)$ is a cdga, we can consider the homotopy category of finite cell modules over $\mathcal{N}(S)$ that we define in the next subsection. It turns out to be equivalent to the triangulated category of Tate motives $\mathrm{DMT}(S)$ by Spitzweck's representation theorem that we discuss in section 3.4. The idea to use algebraic cycles to define a category of Tate motives goes back to Bloch (see e.g. [Blo89]). However, we will follow [Lev10, section 4]. His approach uses cubical complexes instead of Bloch's simplicial complexes.

3.1.1 Definition

We denote by $(\square^1, \delta\square^1)$ the pair $(\mathbb{A}^1_k, \{0,1\})$. We define $(\square^n, \delta\square^n)$ as the n-fold product of $(\square^1, \delta\square^1)$, i.e. $\square^n = \mathbb{A}^n_k$ and $\delta\square^n$ is the divisor $\sum_{i=1}^n (x_i = 0) + \sum_{i=1}^n (x_i = 1)$, where $x_1, \dots, x_n$ are the standard coordinates on $\mathbb{A}^n_k$. A face of $\square^n$ is a face of the normal crossing divisor $\delta\square^n$, i.e. it is a subscheme that is defined by equations of the form $x_{i_j} = \epsilon_j$, where $\epsilon_j \in \{0,1\}$.

For $\epsilon \in \{0,1\}$ and $j \in \{1,\dots,n\}$ we let $\iota_{j,\epsilon}\colon \square^{n-1} \to \square^n$ be the closed embedding defined by inserting ϵ in the jth coordinate. We let $\pi_j\colon \square^n \to \square^{n-1}$ be the projection which omits the jth factor.

Now we are able to define a cubical version of the Suslin-complex C_*^{Sus} from [Voe00].

Definition 3.1
Let X be in Sm_k and let $\mathcal{F}$ be a presheaf of abelian groups on Sm_k.

We refer to the subgroup $\sum_{j=1}^n \pi_j^*(\mathcal{F}(X \times_k \square^{n-1}))$ of $\mathcal{F}(X \times_k \square^n)$ as the degenerate elements, written $\mathrm{Deg_n}$. Let $C_n^{cb}(\mathcal{F})$ be the presheaf defined by

$$C_n^{cb}(\mathcal{F})(X) := \mathcal{F}(X \times_k \square^n)/\,\mathrm{Deg_n}$$

and let $C_*^{cb}(\mathcal{F})$ the complex with differential

$$d_n := \sum_{j=1}^n (-1)^{j-1}\mathcal{F}(\iota_{j,1}) - \sum_{j=1}^n (-1)^{j-1}\mathcal{F}(\iota_{j,0}).$$

If $\mathcal{F}$ is a Nisnevich sheaf, then $C_*^{cb}(\mathcal{F})$ is a complex of Nisnevich sheaves. We extend the construction to complexes of sheaves by taking the total complex of the evident double complex.

Definition 3.2
For a presheaf $\mathcal{F}$ *on* Sm_k *and a scheme* $X \in \mathrm{Sm}_k$ *let*

$$C_n^{\mathrm{Alt}}(\mathcal{F})(X) \subset C_n^{cb}(\mathcal{F})(X)_{\mathbb{Q}} = \mathcal{F}(X \times_k \square^n)_{\mathbb{Q}} / \operatorname{Deg_n}$$

be the $\mathbb{Q}$*-subspace consisting of alternating elements of* $\mathcal{F}(X \times_k \square^n)_{\mathbb{Q}}$ *with respect to the action of the symmetric group* S_n*, i.e. the elements satisfying* $(\mathrm{id} \times \sigma)^*(x) = \operatorname{sgn}(\sigma) \cdot x$ *for all* $\sigma \in S_n$*. Here,* S_n *acts on* $\square^n = \mathbb{A}_k^n$ *by permuting the coordinates.*

The subspaces $C_n^{\mathrm{Alt}}(\mathcal{F})(X)$, $n \geq 1$, form in fact a subcomplex of presheaves $C_*^{\mathrm{Alt}}(\mathcal{F}) \subset C_*^{cb}(\mathcal{F})_{\mathbb{Q}}$ for any presheaf (or complex of presheaves) $\mathcal{F}$ on Sm_k.

Furthermore, $C_*^{\mathrm{Alt}}(\mathcal{F})$ is quasi-isomorphic to the classical Suslin complex $C_*^{\mathrm{Sus}}(\mathcal{F})$ as the following lemma in [Lev10] shows.

Lemma 3.3 ([Lev10, Lemma 4.1.3])
Let $\mathcal{F}$ *be a complex of presheaves (with transfer) on* Sm_k*.*

There is a natural isomorphism $C_*^{\mathrm{Sus}}(\mathcal{F}) \simeq C_*^{cb}(\mathcal{F})$ *in the derived category of presheaves (with transfer) on* Sm_k*.*

Furthermore, the inclusion $C_n^{\mathrm{Alt}}(\mathcal{F})(X) \subset C_n^{cb}(\mathcal{F})(X)_{\mathbb{Q}}$ *is a quasi-isomorphism for all* $X \in \mathrm{Sm}_k$*.*

We denote by $\mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}(\mathbb{P}^1/\infty)$ the sheaf defined by the exactness of the split exact sequence

$$0 \to \mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}(\operatorname{Spec} k) \xrightarrow{i_{\infty *}} \mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}(\mathbb{P}_k^1) \to \mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}(\mathbb{P}^1/\infty) \to 0.$$

Therefore, by definition $\mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}(\mathbb{P}^1/\infty)$ agrees with $\mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}(1)[2]$ that we defined in section 2.3.

Similarly, let $\mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}((\mathbb{P}^1/\infty)^r)$ be the sheaf defined by the exactness of the split exact sequence

$$\oplus_{j=1}^r \mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}((\mathbb{P}_k^1)^{r-1}) \xrightarrow{\sum_j i_{j,\infty *}} \mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}((\mathbb{P}_k^1)^r) \to \mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}((\mathbb{P}^1/\infty)^r) \to 0,$$

where $i_{j,\infty *} \colon (\mathbb{P}_k^1)^{r-1} \to (\mathbb{P}_k^1)^r$ inserts ∞ at the jth coordinate. Again, $\mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}((\mathbb{P}^1/\infty)^r)$ agrees by definition with $\mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}(r)[2r] := \mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}(1)[2]^{\otimes r}$.

The symmetric group S_q acts on $\mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}((\mathbb{P}^1/\infty)^q)$ by permuting the coordinates in $(\mathbb{P}^1)^q$. Now we can apply the alternating cubical complex C_*^{Alt} to the sheaf $\mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}((\mathbb{P}^1)^q)$ and consider the subcomplex of symmetric section with respect to this action of S_q.

Definition 3.4
We define $\mathcal{N}(q) \subset C_*^{\mathrm{Alt}}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^q))$ as the subcomplex of sheaves consisting of symmetric sections with respect to the action of S_q induced by permuting the coordinates of $(\mathbb{P}^1_k)^q$. This defines $\mathcal{N}(q)$ as an object of $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k)_{\mathbb{Q}})$. By abuse of notation we write $\mathcal{N}(0)$ for the constant presheaf $\mathbb{Q}$.

We set $\mathcal{N} := \oplus_{q\geq 0}\mathcal{N}(q)$ and for $S \in \mathrm{Sm}_k$ we let $\mathcal{N}_S(q)$ denote the restriction of $\mathcal{N}(q)$ to $\mathrm{Cor}(S)$. Similarly, we define $\mathcal{N}_S := \mathbb{Q} \oplus \oplus_{q\geq 1}\mathcal{N}_S(q)$.

Lemma 3.5 ([Lev10, Lemma 4.2.1])
The inclusion $\mathcal{N}(q) \subset C_*^{\mathrm{Alt}}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^q))$ is a quasi-isomorphism of complexes of presheaves on Sm_k.

Therefore, the complex $\mathcal{N}(q)$ is quasi-isomorphic to the Suslin-complex $C_*^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}_{\mathrm{k}}((\mathbb{P}^1\ /\infty)^q))$ for all $q \in \mathbb{Z}$. Furthermore:

Lemma 3.6 ([Lev10, Lemma 4.3.3])
In $\mathrm{DM}^{\mathrm{eff}}(S)_{\mathbb{Q}}$ there is a canonical isomorphism $\mathbb{Q}_S(q) \to \mathcal{N}_S(q)$ for all $q \in \mathbb{N}$ giving a commutative diagram of isomorphisms

$$\begin{tikzcd}
\mathbb{Q}_S(q) \otimes \mathbb{Q}_S(p) \arrow[r, "\simeq"] \arrow[d] & \mathbb{Q}_S(q+p) \arrow[d] \\
\mathcal{N}_S(q) \otimes \mathcal{N}_S(p) \arrow[r, "\simeq"] & \mathcal{N}_S(q+p).
\end{tikzcd}$$

3.1.2 Algebra structure

In the previous subsection we defined $\mathcal{N} = \mathbb{Q} \oplus \oplus_{q\geq 1}\mathcal{N}(q)$ as an object in $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k)_{\mathbb{Q}})$. Our goal is to define a product on $\mathcal{N}$ to endow it with the structure of a cdga object in $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k)_{\mathbb{Q}})$ such that the evaluation $\mathcal{N}(S)$ for $S \in \mathrm{Sm}_k$ yields a cdga.

Recall that a commutative differential graded algebra (short: cdga) (A, d) over $\mathbb{Q}$ consists of a unital, graded-commutative $\mathbb{Q}$-algebra $A := \oplus_{n\in\mathbb{Z}}A^n$ together with a differential $d = \oplus_n d^n$, $d^n \colon A^n \to A^{n+1}$, such that $d^{n+1} \circ d^n = 0$ and d satisfies the Leibniz rule:

$$d^{n+m}(a \cdot b) = d^n a \cdot b + (-1)^n a \cdot d^m b,$$

where $a \in A^n$, $b \in A^m$.

A is called connected if $A^n = 0$ for all $n < 0$ and $A^0 = \mathbb{Q}\cdot 1$. A is called cohomologically connected if $\mathrm{H}^n(A) = 0$ for all $n < 0$ and $\mathrm{H}^0(A) = \mathbb{Q}\cdot 1$.

Furthermore, an Adams graded cdga is a cdga A together with a direct sum decomposition into subcomplexes $A = \oplus_{r\geq 0} A(r)$ such that $A(r)\cdot A(s) \subset A(r\cdot s)$ and $A(0) = \mathbb{Q}\cdot 1$. An Adams graded cdga is said to be (cohomologically) connected if the underlying cdga is (cohomologically) connected.

For $a \in A^n(r)$, we call n the cohomological degree of a, $\deg a := n$, and r the Adams degree of a, $|a| := r$.

Let X and $Y \in \mathrm{Sm}_k$. The external product of correspondences (see [Voe10, section 2])

$$\boxtimes\colon c_k(X,k) \times c_k(Y,k) \to c_k(X \times_k Y, k)$$

yields an associative external product

$$\begin{aligned}\boxtimes\colon\, & C^{cb}_n(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^q))(X) \otimes C^{cb}_m(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^p))(Y)\\ & \to C^{cb}_{m+n}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^{q+p}))(X\times_k Y).\end{aligned}$$

If we take $X = Y$, the pull-back via the diagonal morphism $X \to X \times_k X$ gives the associative cup product of complexes of sheaves

$$\cup\colon C^{cb}_n(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^q)) \otimes C^{cb}_m(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^p)) \to C^{cb}_{m+n}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^{q+p})).$$

Applying the alternating projection

$$\mathrm{Alt} := \frac{1}{(n+m)!}\sum_{\sigma\in S_{n+m}} \mathrm{sgn}(\sigma)\sigma$$

on $\square^{n+m}$ and the symmetric projection on $(\mathbb{P}^1)^{q+p}$ gives the associative, commutative product on $\mathcal{N}$

$$\cdot\colon \mathcal{N}(q)\otimes\mathcal{N}(p) \to \mathcal{N}(p+q).$$

This makes $\mathcal{N} = \oplus_{q\geq 0}\mathcal{N}(q)$ into an Adams-graded cdga object in $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k)_{\mathbb{Q}})$, i.e. $\mathcal{N}(X)$ is an Adams-graded cdga for any $X \in \mathrm{Sm}_k$, where $\mathcal{N}(q)(X)$ denotes the subcomplex of $\mathcal{N}(X)$ that is in Adams degree q.

For $S \in \mathrm{Sm}_k$ we denote by $\mathcal{N}_S(q)$ the restriction of $\mathcal{N}(q)$ to $\mathrm{Cor}(S)$ giving us the Adams graded cdga object $\mathcal{N}_S = \oplus_{q\geq 0}\mathcal{N}_S(q)$ in $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)_{\mathbb{Q}})$. Note that for $X \in \mathrm{Sm}_S$ we have $\mathcal{N}_S(X) = \mathcal{N}(X)$. Therefore, $\mathcal{N}_S$ is in fact a presheaf of Adams graded cdgas over $\mathcal{N}(S)$. In particular, For $f\colon X \to S$ in Sm_S, we have an algebra homomorphism $f^{\#}\colon \mathcal{N}(S) \to \mathcal{N}(X)$ induced by the pull-back of algebraic cycles.

3.2 Cell modules

The subject of this section is to give the definition of the category of finite cell modules $\mathcal{CM}^f_A$ over an Adams-graded commutative differential graded $\mathbb{Q}$-algebra A following [Lev10]. $\mathcal{CM}^f_A$ is a subcategory of $\mathcal{M}_A$, the category of dg A-modules. Subsequently we construct the homotopy category of $\mathcal{CM}^f_A$ which in fact is equivalent to the derived category of $\mathcal{M}_A$ and define a t-structure on these categories if the base algebra A is (cohomologically) connected. This discussion allows us to consider the homotopy category of cell modules $\mathcal{KCM}^f_{\mathcal{N}(S)}$ over the cycle algebra $\mathcal{N}(S)$. This category is used to prove Spitzweck's representation theorem in section 3.4.

3.2.1 Definition

Throughout this section let A be an Adams-graded cdga. Note that every Adams graded cdga A has a canonical augmentation $A \to \mathbb{Q}$ given by the projection onto $A(0) \simeq \mathbb{Q}$ with augmentation ideal $A^+ := \oplus_{r>0} A(r)$.

If $A = \oplus_{n,r} A^n(r)$ is an Adams graded cdga, we denote by $A\langle r\rangle[n]$ the (left) A-module which is $A^{m+n}(r+s)$ in bi-degree (m, s), with A-action given by left multiplication.

Definition 3.7
Let A be a cdga.

1. *A dg A-module (M, d) consists of a complex $M = \oplus_n M_n$ of $\mathbb{Q}$-vector spaces with differential d that satisfies the Leibniz rule, together with a graded, degree zero map $A \otimes M \to M$, $a \otimes m \mapsto a \cdot m$ which makes M into a graded A-module.*
2. *If $A = \oplus_{r\geq 0} A(r)$ is an Adams graded cdga, an Adams graded dg A-module is a dg A-module M together with a decomposition into subcomplexes $M = \oplus_s M(s)$ such that $A(r) \cdot M(s) \subset M(r+s)$.*
3. *An Adams graded dg A-module M is called A-cell module if*

(a) *M is free as a bi-graded A-module, that is, there is a set J and elements $b_j \in M^{n_j}(r_j)$, $j \in J$, such that the maps $a \mapsto a \cdot b_j$ induce an isomorphism of bi-graded A-modules*

$$\oplus_j A\langle -r_j\rangle[-n_j] \to M;$$

(b) *there is a filtration on the index set J of generators*

$$J_{-1} = \emptyset \subset J_0 \subset \ldots \subset J$$

such that $J = \cup_n J_n$ and $db_j = \sum_{i\in J_{n-1}} a_{ij} b_i$ for $j \in J_n$.

4. *A cell module with finite index set J is called finite A-cell module.*

5. *We denote the category of Adams graded dg A-modules by $\mathcal{M}_A$, the category of A-cell modules by $\mathcal{CM}_A$ and the category of finite cell modules by $\mathcal{CM}_A^f$.*

Let M, N be two Adams graded dg A-modules. Let $\mathcal{H}om_A(M,N)$ be the Adams graded dg A-module with $\mathcal{H}om_A(M,N)^n(r)$ the A-module consisting of maps $f\colon M \to N$ with $f(M^t(s)) \subset N^{t+n}(r+s)$, $f(am) = (-1)^{np} a f(m)$ for $a \in A^p$ and $m \in M$, and with differential d defined by $df(m) = d(f(m)) + (-1)^{n+1} f(dm)$ for $f \in \mathcal{H}om_A(M,N)^n(r)$.

This makes $\mathcal{M}_A$ into a differential graded category. Thus, we can define the homotopy category of $\mathcal{M}_A$.

Definition 3.8
Let A be an Adams graded cdga. We define $\mathcal{K}_A$ as the homotopy category of $\mathcal{M}_A$, i.e. the objects of $\mathcal{K}_A$ are the objects of $\mathcal{M}_A$ and

$$\mathrm{Hom}_{\mathcal{K}_A}(M,N) := \mathrm{H}^0(\mathcal{H}om_A(M,N)(0)).$$

We define the homotopy categories of A-cell modules respectively of finite A-cell modules as the full subcategories of $\mathcal{K}_A$ with objects in $\mathcal{CM}_A$ respectively in $\mathcal{CM}_A^f$ and denote them by $\mathcal{KCM}_A$ and $\mathcal{KCM}_A^f$ respectively.

On $\mathcal{K}_A$ there is the natural translation functor $[1]\colon M \mapsto M[1]$, where $M[1]^i := M^{i+1}$ and $d^i_{M[1]} := -d^{i+1}_M$. The cone of a map $f\colon M \to N$ of Adams graded dg modules is the well-defined object $\mathrm{Cone}(f) = M[1] \oplus N$ with differential $d(m,n) = (-d_M(m), f(m) + d_N(n))$ in $\mathcal{M}_A$. We call a triangle in $\mathcal{K}_A$ distinguished if it is isomorphic to a cone triangle.

Proposition 3.9
The homotopy category $\mathcal{K}_A$ of $\mathcal{M}_A$ is a triangulated category, with distinguished triangles being those triangles which are isomorphic in $\mathcal{K}_A$ to a cone sequence.

Proof. This can be shown by imitating the proof of Proposition 1.5. The definitions of the mapping cone agree in both cases and the maps needed to satisfy the axioms (TR1) - (TR4) can be constructed in the same way. □

If M and N are both (finite) cell-modules, then the cone $\mathrm{Cone}(f)$ of any map $f\colon M \to N$ is isomorphic to $M[1] \oplus N$ as a bi-graded A-module. Moreover, the filtrations on the sets of generators of M and N give a filtration on the set of generators of $\mathrm{Cone}(f\colon M \to N)$. More explicitly, let $(J_i^M)_{i=1}^m$ be the filtration on the index set J^M of generators of M and likewise $(J_i^N)_{i=1}^n$. Now, put $J_i = J_i^N$ for $i \leq n$ and $J_{n+i} = J^N \cup J_i^M$ for any $i \geq 1$. Thus, the cone $\mathrm{Cone}(f)$ of any map between (finite) cell modules is again a (finite) cell module making $\mathcal{KCM}_A$ and $\mathcal{KCM}_A^f$ triangulated subcategories of $\mathcal{K}_A$.

For two Adams-graded dg A-modules M, N let $M \otimes_A N$ be the Adams-graded dg A-module with underlying module $M \otimes_A N$ and differential $d(m \otimes n) = dm \otimes n + (-1)^{\deg m} m \otimes dn$. If furthermore, M and N are A-cell modules with index sets J' and J'' respectively of generators, then $J_i := J_i' \times J_i''$ defines a filtration on the index set of generators of $M \otimes_A N$ making $\mathcal{CM}_A$ and $\mathcal{CM}_A^f$ closed under tensor products. Therefore, $\mathcal{KCM}_A$ and $\mathcal{KCM}_A^f$ are triangulated tensor categories.

Example 3.10
For $n \in \mathbb{Z}$ we define the Tate object $\mathbb{Q}_A(n)$ as the object of $\mathcal{CM}_A^f$ which is the free rank one A-module with generator b_n having Adams degree $-n$, cohomological degree 0 and $db_n = 0$, i.e. $\mathbb{Q}_A(n) = A\langle n\rangle$. Then:

$$\mathrm{Hom}_{\mathcal{KCM}_A^f}(\mathbb{Q}_A(-a)[n], \mathbb{Q}_A(-b)[m]) = \mathrm{H}^{m-n}(A(a-b)).$$

In particular,

$$\mathrm{Hom}_{\mathcal{KCM}_A^f}(\mathbb{Q}_A(-a), \mathbb{Q}_A(-b)) = \mathrm{H}^0(A(a-b)) = \begin{cases} 0, & \text{if } a < b \\ \mathbb{Q} \cdot \mathrm{id}, & \text{if } a = b \end{cases}$$

since $A(a-b) = 0$ for $a < b$. By comparing this result to Corollary 2.23, we see that the Tate objects $\mathbb{Q}_A(n)$ in $\mathcal{KCM}_A$ behave like the Tate motives $\mathbb{Q}_S(n)$ in $\mathrm{DM}_{\mathrm{gm}}(S)$ which motivates the similar notation and name.

Furthermore, for $n \geq 0$ we have $\mathbb{Q}_A(\pm n) \cong \mathbb{Q}_A(\pm 1)^{\otimes n}$ and for all $n \in \mathbb{Z}$ we have $\mathbb{Q}(n)^\vee \cong \mathbb{Q}_A(-n)$.

These Tate objects are of particular interest for us since they generate $\mathcal{KCM}_A^f$ as a triangulated subcategory:

Lemma 3.11
$\mathcal{KCM}_A^f$ is the strictly full triangulated subcategory of $\mathcal{K}_A$ generated by the Tate objects $\mathbb{Q}_A(n)$, $n \in \mathbb{Z}$.

Proof. We denote the full triangulated subcategory of $\mathcal{K}_A$ generated by the Tate objects $\mathbb{Q}_A(n)$ by $\mathcal{A}$.

Obviously, $\mathcal{A} \subset \mathcal{KCM}_A^f$ since the generators $\mathbb{Q}_A(n)$, $n \in \mathbb{Z}$ are in $\mathcal{KCM}_A^f$.

Conversely, let $M \in \mathcal{KCM}_A^f$. We prove the result by induction on the number of generators of M.

Let M be a cell module that is of rank two as a bigraded module, i.e. there are two generators b_1, b_2 such that $M \simeq A \cdot b_1 \oplus A \cdot b_2$. Since M is a finite cell module, there is a filtration $\emptyset = J_{-1} \subset J_0 \subset J_1 \subset J$ on the index set $J = \{1, 2\}$ such that for any $j \in J_n$ the differential db_j only depends on those b_i with $i \in J_{n-1}$. For $J = \{1, 2\}$ there are only the possibilities $J_0 = J = \{1, 2\}$ or $J_0 = \{1\}$ and $J_1 = J = \{1, 2\}$.
If $J_0 = J$, i.e. $db_1 = db_2 = 0$, then M is isomorphic to the direct sum of two rank one modules, hence modules of the form $\mathbb{Q}_A(n)[m]$ and therefore clearly in $\mathcal{A}$.

Now, we assume $db_1 = 0$ and $db_2 = a \cdot b_1$ for some $a \in A$ with $\deg b_1 = p$, $\deg b_2 = q$, and $\deg a + \deg b_1 = q+1$. Since $0 = d(db_2) = d(a \cdot b_1) = da \cdot b_1$, $da = 0$ in A. Without loss of generality, we may assume that $q = \deg b_2 = 0$.

We want to show that M is isomorphic to the cone of a map in $\mathcal{A}$. To that end we define the Adams-graded A-modules $M_1 := A \cdot b_1$ and $M_2 := A \cdot b_2$ with differentials given by $d_{M_1} \colon c \cdot b_1 \mapsto dc \cdot b_1$ and $d_{M_2} \colon c \cdot b_2 \mapsto dc \cdot b_2$. Then $M_1 \simeq \mathbb{Q}_A(r)[-p]$ and $M_2 \simeq \mathbb{Q}_A(s)[-q]$, where $r := |b_1|$ and $s := |b_2|$. Therefore, M_1, M_2 are in $\mathcal{A}$.

We compute the differential on $M = \oplus_i M^i$. Let $a_1 b_1 + a_2 b_2 \in M^i$, i.e.

$\deg a_1 + \deg b_1 = i$ and $\deg a_2 = i$.

$$\begin{aligned} d_M(a_1b_1 + a_2b_2) &= d(a_1b_1) + d(a_2b_2) \\ &= da_1 \cdot b_1 + (-1)^{\deg a_1} a_1 db_1 + da_2 \cdot b_2 + (-1)^i a_2 db_2 \\ &= d_{M_1}(a_1 \cdot b_1) + d_{M_2}(a_2 \cdot b_2) + (-1)^i a_2 \cdot a \cdot b_1. \end{aligned}$$

Now, we put $f^i \colon M_2^i \to M_1^{i+1}, c \cdot b_2 \mapsto (-1)^i ca \cdot b_1$. This defines an dg A-module map $M_2 \to M_1[1]$ since $f = (f^i)$ commutes with the differentials $d_{M_1[1]} = -d_{M_1}$ and d_{M_2}. Indeed, let $a_2 \cdot b_2 \in M_2^i$.

$$\begin{aligned} f(d_{M_2}(a_2 \cdot b_2)) = f(da_2 \cdot b_2) &= ((-1)^{i+1} da_2 \cdot a) \cdot b_1 = (-1)^{i+1} d(a_2 a) b_1 \\ &= (-1)^i d_{M_1[-1]}(a_2 a \cdot b_1) = d_{M_1[-1]}(f(a_2 \cdot b_2)). \end{aligned}$$

The cone of f is given as the Adams graded A-module $M_2[1] \oplus M_1[1]$ with differential given by

$$\begin{aligned} d(a_2b_2 + a_1b_1) &= -d_{M_2}(a_2b_2) + f(a_2b_2) + d_{M_1[1]}(a_1b_1) \\ &= -d_{M_2}(a_2b_2) + (-1)^{i+1} a_2 a \cdot b_1 - d_{M_1}(a_1b_1) \\ &= -d_M(a_2b_2 + a_1b_1), \end{aligned}$$

where $a_2b_2 + a_1b_1 \in (M_2[1] \oplus M_1[1])^i \simeq (M_2^{i+1} \oplus M_1^{i+1})$.

Therefore, M is isomorphic to $\mathrm{Cone}(f)[-1]$.

More generally, if M is of rank $n+1$, then by the filtration of the set of generators, we can write $M \simeq M_0 \oplus M_{n+1}$ where M_{n+1} is of rank one with generator b_{n+1} in degree 0 and $db_{n+1} = \sum_{i=1}^n a_i b_i$. We define the differentials $d_{M_0} = d_M$ (restricted to M_0) and $d_{M_{n+1}} \colon c \cdot b_{n+1} \to dc \cdot b_{n+1}$. Computing the differential on M gives us in degree m:

$$\begin{aligned} & d_M(\sum_{j=1}^n c_j b_j + a_{n+1} \cdot b_{n+1}) \\ &= d_M(\sum_{j=1}^n c_j b_j) + d_M(a_{n+1} \cdot b_{n+1}) \\ &= d_{M_0}(\sum_{j=1}^n c_j b_j) + da \cdot b_{n+1} + (-1)^m \cdot a_{n+1} \cdot \sum_{i=1}^n a_i b_i. \end{aligned}$$

We define $f \colon M_{n+1}^m \to M_0^{m+1}$ by $a_{n+1} b_{n+1} \mapsto (-1)^m a_{n+1} \cdot \sum_{i=1}^n a_i b_i$ which defines an A-module map $f \colon M_{n+1} \to M_0[1]$. Again, this commutes with the differentials, hence is a map of Adams graded A-modules and M is isomorphic to $\mathrm{Cone}(f)[-1]$ as Adams-graded A-modules.

This shows that every finite cell module is given by iterated cones of maps between the Tate objects $\mathbb{Q}_A(n)[m]$, $n, m \in \mathbb{Z}$, proving the claim. $\square$

Since $\mathbb{Q}_A^\vee(n) \simeq \mathbb{Q}_A(-n)$ for all $n \in \mathbb{Z}$, it follows by Lemma 3.11 that every object in $\mathcal{KCM}_A^f$ admits a dual and is in particular rigid, proving the following corollary.

Corollary 3.12
$\mathcal{KCM}_A^f$ is a rigid triangulated tensor category.

Remark 3.13
In [Lev10, Proposition 1.7.1], Levine even proves that $\mathcal{KCM}_A^f$ is exactly the subcategory of rigid objects of $\mathcal{KCM}_A$, i.e. an object $M \in \mathcal{KCM}_A$ is rigid if and only if M is a finite cell module.

We want to construct a t-structure on $\mathcal{KCM}_A^f$. To be able to do so we define a weight structure on $\mathcal{CM}_A$ yielding a weight structure on $\mathcal{KCM}_A^f$.

Let $M \in \mathcal{CM}_A$. Thus, we can choose a basis $\mathcal{B} = \{b_j \colon j \in J\}$ of $M \simeq \oplus_j A \cdot b_j$. If we write $db_j = \sum_i a_{ij} b_i$, $a_{ij} \in A$, we see $|b_i| \leq |b_j|$ if $a_{ij} \neq 0$ since $|a_{ij}| \geq 0$ and d has Adams degree 0.

Therefore, $W_n^{\mathcal{B}} M = \{\oplus_{j, |b_j| \leq n} A \cdot b_j\}$ is in fact a subcomplex of M. The subcomplex does not depend on the choice of the base $\mathcal{B}$, thus we just write $W_n M$ instead of $W_n^{\mathcal{B}} M$. This defines an increasing filtration of A-cell modules

$$W_* M \colon \ldots \subset W_n M \subset W_{n+1} M \subset \ldots \subset M$$

such that $M = \cup_n W_n M$.

For $n \leq m$ we define $W_{n/m} M$ as the cokernel of the inclusion $W_m M \to W_n M$. We write $\mathrm{gr}_\mathrm{n}^\mathrm{W}$ for $W_{n/n-1}$ and $W^{>n}$ for $W_{\infty/n}$.

This filtration is functorial in M. In particular, if $f \colon M \to N$ is a homotopy equivalence in $\mathcal{CM}_A$ with inverse $g \colon N \to M$, then $W_n g \colon W_n N \to W_n M$ is a homotopy inverse to $W_n f \colon W_n M \to W_n N$. Hence the functors W_n respect homotopy equivalence of cell modules, so they yield a functorial tower of exact endo-functors on $\mathcal{KCM}_A$:

$$\ldots \to W_n \to W_{n+1} \to \ldots \to \mathrm{id}\,.$$

Remark 3.14 ([Lev10, Proposition 1.5.1])
The endo-functor W_n is exact for each n. Furthermore, for $m \leq n \leq \infty$, the sequence of endo-functors $W_m \to W_n \to W_{n/m}$ canonically extends to a distinguished triangle of endo-functors.

This filtration allows us to define the full subcategory $\mathcal{CM}_A^{+W}$ of $\mathcal{CM}_A$ with objects M such that $W_n M = 0$ for some n.

If we denote by $\mathcal{KCM}_A^{+W}$ the homotopy category of $\mathcal{CM}_A^{+W}$, then $\mathcal{KCM}_A^{+W}$ is exactly the full subcategory of $\mathcal{KCM}_A$ with objects M such that $W_n \simeq 0$ in $\mathcal{KCM}_A$ for some n.

Clearly, $\mathcal{KCM}_A^f$ is a subcategory of $\mathcal{KCM}_A^{+W}$. In subsection 3.2.4 we construct a t-structure on $\mathcal{KCM}_A^{+W}$ that induces the desired t-structure on $\mathcal{KCM}_A^f$.

3.2.2 The derived category

Another possibility to describe the category of A-cell modules is the derived category of dg A-modules as we will outline in this subsection following [Lev10].

Definition 3.15
The derived category $\mathcal{D}_A$ of dg A-modules is the localization of $\mathcal{K}_A$ with respect to morphisms $M \to N$ which are quasi-isomorphisms on the underlying complexes of vector spaces.

The derived category inherits the structure of a triangulated category from $\mathcal{K}_A$. This follows by the same arguments that show that the derived category of an abelian category is triangulated (see e.g. [Sos12]).

In [KM95], Kriz and May prove that the categories $\mathcal{D}_A$ and $\mathcal{KCM}_A$ are equivalent. We give the result as stated by Levine in [Lev10].

Theorem 3.16 ([Lev10, Proposition 1.4.3])
The evident functor $\mathcal{KCM}_A \to \mathcal{D}_A$ is an equivalence of triangulated categories.

Equivalently, let $f\colon M' \to M$ be a quasi-isomorphism in $\mathcal{M}_A$, $N \in \mathcal{CM}_A$. Then the induced map $f\colon \mathrm{Hom}_{\mathcal{K}_A}(N, M') \to \mathrm{Hom}_{\mathcal{K}_A}(N, M)$ is an isomorphism.

Proof. See [KM95, Construction 2.7]. □

We denote by $\mathcal{D}_A^f$ the full subcategory of $\mathcal{D}_A$ of those objects M that are isomorphic in $\mathcal{D}_A$ to a finite cell module. Then we get the following statement as an immediate consequence of Theorem 3.16:

Corollary 3.17 ([Lev10, Proposition 1.4.4])
The induced functor $\mathcal{KCM}_A^f \to \mathcal{D}_A^f$ is an equivalence of triangulated categories.

The equivalence $\mathcal{KCM}_A \to \mathcal{D}_A$ is a very powerful tool to impose additional structure on $\mathcal{D}_A$ as we see in the following.

The tensor functor on $\mathcal{KCM}_A$ defines a well-defined derived tensor product on $\mathcal{D}_A$ via the equivalence given in Theorem 3.16, making $\mathcal{D}_A$ a triangulated tensor category and $\mathcal{D}_A^f$ a triangulated tensor subcategory.

Similarly, the weight filtration W_n on $\mathcal{KCM}_A$ defines a weight filtration on $\mathcal{D}_A$ via the equivalence in Theorem 3.16.

Note that even though it is possible to define the weight filtration W_n for not just cell modules but any Adams graded dg A-module which is free as a bi-graded module, this definition does not work for modules that are not free. Thus, it is not possible to define the weight filtration W_n directly on $\mathcal{D}_A$. Furthermore, even for free modules M it is not clear that W_nM is invariant under quasi-isomorphisms in general.

Since cell modules are in particular free as bi-graded modules, they can be easily described by choosing a basis. Therefore, Theorem 3.16 allows us a better description of the objects of $\mathcal{D}_A$.

As in $\mathcal{KCM}_A$ we can consider the full subcategory $\mathcal{D}_A^{+W}$ of $\mathcal{D}_A$ of objects M such that $W_nM \simeq 0$ for some n.

The equivalence $\mathcal{KCM}_A \sim \mathcal{D}_A$ restricts in fact to an equivalence $\mathcal{KCM}_A^{+W} \sim \mathcal{D}_A^{+W}$ as was shown by Levine (see [Lev10, Lemma 1.5.6]).

3.2.3 Base change

To define the t-structure on $\mathcal{KCM}_A^f$, we need the base change of an A-cell module along the augmentation $A \to \mathbb{Q}$. Furthermore, the base change allows us to substitute a cohomologically connected cdga A by its minimal model, i.e. by a connected cdga (see Remark 3.18).

Let $\phi\colon A \to B$ be a homomorphism of Adams graded cdgas.

The functor

$$- \otimes_A B\colon \mathcal{M}_A \to \mathcal{M}_B$$

induces a functor on cell modules and on the homotopy category

$$\phi_*\colon \mathcal{KCM}_A \to \mathcal{KCM}_B$$

that restricts to an exact tensor functor

$$\phi_*\colon \mathcal{KCM}^f_A \to \mathcal{KCM}^f_B.$$

Via the equivalence $\mathcal{KCM}_A \sim \mathcal{D}_A$ and the equivalences of their respective subcategories, there are also exact tensor functors on the derived categories

$$\begin{aligned}\phi_*&\colon \mathcal{D}_A \to \mathcal{D}_B,\\ \phi_*&\colon \mathcal{D}^{+W}_A \to \mathcal{D}^{+W}_B,\\ \phi_*&\colon \mathcal{D}^f_A \to \mathcal{D}^f_B.\end{aligned}$$

Remark 3.18
In [KM95, Proposition 4.2] it is shown that if $\phi\colon A \to B$ is a quasi-isomorphism, then $\phi_*\colon \mathcal{D}_A \to \mathcal{D}_B$ is an equivalence of categories. Since ϕ_* is compatible with the weight filtrations, ϕ_* restricts to an equivalence $\phi_*\colon \mathcal{D}^{+W}_A \to \mathcal{D}^{+W}_B$. Because an equivalence of triangulated tensor categories induces an equivalence on the subcategory of rigid objects, ϕ_* also restricts by Remark 3.13 to an equivalence $\phi_*\colon \mathcal{D}^f_A \to \mathcal{D}^f_B$ as was pointed out by Levine in [Lev10, Corrallary 1.8.2 and Corollary 1.8.3]. These facts allow us in the following (see e.g. Theorem 3.21) to weaken the condition of A being a connected cdga to the condition of A being a cohomologically connected cdga by replacing A with its minimal model. See [Lev10, Section 1.11] for more details.

Recall that every cdga A admits an augmentation $\epsilon\colon A \to \mathbb{Q}$ that is given by the projection on $A(0) = \mathbb{Q}\cdot 1$. This induces the base change functor

$$q := \epsilon_*\colon \mathcal{CM}_A \to \mathcal{CM}_{\mathbb{Q}}, \quad qM := M \otimes_A \mathbb{Q}$$

and the exact tensor functors

$$
\begin{aligned}
q &\colon \mathcal{KCM}_A \to \mathcal{KCM}_{\mathbb{Q}},\\
q &\colon \mathcal{KCM}^f_A \to \mathcal{KCM}^f_{\mathbb{Q}},\\
q &\colon \mathcal{D}_A \to \mathcal{D}_{\mathbb{Q}},\\
q &\colon \mathcal{D}^{+W}_A \to \mathcal{D}^{+W}_{\mathbb{Q}},\\
q &\colon \mathcal{D}^f_A \to \mathcal{D}^f_{\mathbb{Q}}.
\end{aligned}
$$

The inclusion $\mathbb{Q} \to A$ splits the augmentation ϵ, identifying $\mathcal{D}_{\mathbb{Q}}$, $\mathcal{D}^{+w}_{\mathbb{Q}}$ and $\mathcal{D}^f_{\mathbb{Q}}$ with full subcategories of $\mathcal{D}_A$, $\mathcal{D}^{+w}_A$ and $\mathcal{D}^f_A$. Under this identification, the functor q is identified with the functor $\mathrm{gr}^W_* := \prod_{n\in\mathbb{Z}} \mathrm{gr}^{\mathrm{W}}_{\mathrm{n}}$.

We end this section by giving two lemmas that are needed for the definition of the t-structure.

Lemma 3.19 ([Lev10, Proposition 1.8.4])
Let $\phi\colon A \to B$ be a homomorphism of Adams graded cdgas.
Then $\phi_*\colon \mathcal{D}^{+W}_A \to \mathcal{D}^{+W}_B$ is conservative, i.e. $\phi_*(M) \cong 0$ implies $M \cong 0$, or equivalently, if $\phi_*(f)$ is an isomorphism, then f is an isomorphism.

Proof. We use the equivalence $\mathcal{KCM}^{+W}_A \to \mathcal{D}^{+W}_A$ and prove the claim for cell modules using the weight filtration. See [Lev10, Proposition 1.8.4] for further details. □

Lemma 3.20 ([Lev10, Proposition 1.9.2])
Let M be in $\mathcal{D}^{+W}_A$. Then M is in $\mathcal{D}^f_A$ if and only if

1. $\mathrm{gr}^{\mathrm{W}}_{\mathrm{n}} M$ *is in* $\mathrm{D}^b(\mathbb{Q}) \subset \mathrm{D}(\mathbb{Q})$ *for all n and*
2. $\mathrm{gr}^{\mathrm{W}}_{\mathrm{n}} M \cong 0$ *for all but finitely many n.*

Proof. It is clear that $M \in \mathcal{D}^f_A$ satisfies the conditions 1 and 2. For the proof of the converse, we use Lemma 3.19. See [Lev10, Proposition 1.9.2] for details. □

3.2.4 t-structure

With the help of the functor $q\colon \mathcal{D}^{+W}_A \to \mathcal{D}^{+W}_{\mathbb{Q}}$ we can construct a t-structure on the triangulated category $\mathcal{D}^{+W}_A$.

We define the full subcategories $\mathcal{D}_A^{\leq 0}$, $\mathcal{D}_A^{\geq 0}$ and $\mathcal{H}_A$ of $\mathcal{D}_A^{+W}$ by

$$\begin{aligned}\mathcal{D}_A^{\leq 0} &:= \{M \in \mathcal{D}_A^{+W} \colon \mathrm{H}^n(qM) = 0 \text{ for } n > 0\}\\ \mathcal{D}_A^{\geq 0} &:= \{M \in \mathcal{D}_A^{+W} \colon \mathrm{H}^n(qM) = 0 \text{ for } n < 0\}\\ \mathcal{H}_A &:= \{M \in \mathcal{D}_A^{+W} \colon \mathrm{H}^n(qM) = 0 \text{ for } n \neq 0\}.\end{aligned}$$

Theorem 3.21 ([Lev10, Theorem 1.12.6])
Let A be a cohomologically connected Adams graded cdga. Then $(\mathcal{D}_A^{\leq 0}, \mathcal{D}_A^{\geq 0})$ defines a non-degenerate t-structure on $\mathcal{D}_A^{+W}$ with heart $\mathcal{H}_A$.

For the proof of Theorem 3.21 we need the following lemma.

Lemma 3.22 ([Lev10, Lemma 1.12.3])
Let A be a connected Adams graded cdga.

1. *Let M be in $\mathcal{D}_A^{\leq 0}$. Then there exists an A-cell module $P \in \mathcal{CM}_A^{+W}$ with basis $\{e_\alpha\}$ such that $\deg(e_\alpha) \leq 0$ for all α and a quasi-isomorphism $P \to M$.*
2. *Let M be in $\mathcal{D}_A^{\geq 0}$. Then there exists an A-cell module $P \in \mathcal{CM}_A^{+W}$ with basis $\{e_\alpha\}$ such that $\deg(e_\alpha) \geq 0$ for all α and a quasi-isomorphism $P \to M$.*

Proof.

1. Choose a quasi-isomorphism $Q \to M$ with $Q \in \mathcal{CM}_A^{+W}$. Let $\{e_\alpha\}$ be a basis for Q. Decompose the differential d_Q as $d_Q = d_q^0 + d_Q^+$. After a $\mathbb{Q}$-linear change of basis, we may assume that the collection S_0 of e_α with $\deg e_\alpha$ und $d_q^0 e_\alpha = 0$ forms a basis of

   $$\ker\left(d^0 \colon \oplus_{\deg e_\alpha = 0} \mathbb{Q} \cdot e_\alpha \to \oplus_{\deg e_\alpha = 1} \mathbb{Q} \cdot e_\alpha\right).$$

   Let $\tau^{\leq 0} Q$ be the A-submodule of Q with basis $S_0 \cup \{e_\alpha | \deg e_\alpha < 0\}$. Then one can check that $\tau^{\leq 0} Q$ is in fact a subcomplex of Q.

   We claim that $\tau^{\leq 0} Q \to Q$ is a quasi-isomorphism. Applying Lemma 3.19 to the augmentation $A \to \mathbb{Q}$ the functor $q \colon \mathcal{D}_A^{+W} \to \mathcal{D}_{\mathbb{Q}}^{+W}$ is conservative, thus it suffices to show that $q\tau^{\leq 0} Q \to qQ$ is a quasi-isomorphism. Now, qQ represents qM in $\mathcal{D}_{\mathbb{Q}}$ and by assumption qM is in $\mathcal{D}_{\mathbb{Q}}^{\leq 0}$, hence qQ is in $\mathcal{D}_{\mathbb{Q}}^{\leq 0}$. By construction, $q\tau^{\leq 0} Q \to qQ$ is an isomorphism on H^n for all $n \leq 0$. Since $\mathrm{H}^n(q\tau^{\leq 0} Q) = 0$ for all $n > 0$, it follows that $q\tau^{\leq 0} Q \to qQ$ is a quasi-isomorphism as claimed.

2. This follows by [Lev10, Lemma 1.6.2].

□

Now we are able to give the proof of Theorem 3.21.

Proof of Theorem 3.21. By replacing A with its minimal model, we may assume that A is connected (see Remark 3.18).

The inclusions $\mathcal{D}_A^{\leq 0}[1] \subset \mathcal{D}_A^{\leq 0}$ and $\mathcal{D}_A^{\geq 0}[-1] \subset \mathcal{D}_A^{\geq 0}$ are obvious.

Let $M \in \mathcal{D}_A^{\leq 0}$ and $N \in \mathcal{D}_A^{\geq 0}$. We need to show $\mathrm{Hom}_{\mathcal{D}_A^{+W}}(M, N[-1]) = 0$. By Lemma 3.22 we can assume that M and $N[-1]$ are A-cell modules with bases $\{e_\alpha\}$ of M and $\{f_\beta\}$ of $N[-1]$, where $\deg e_\alpha \leq 0$ and $\deg f_\beta \geq 1$ for all α, β. Via the equivalence $\mathcal{D}_A^{+W} \to \mathcal{KCM}_A^{+W}$ we have

$$\mathrm{Hom}_{\mathcal{D}_A^{+W}}(M, N[-1]) = \mathrm{Hom}_{\mathcal{KCM}_A^{+W}}(M, N[-1]).$$

But if $\phi\colon M \to N[-1]$ is a map in $\mathcal{KCM}_A^{+W}$, then ϕ is given by a degree zero map of complexes and

$$\phi(e_\alpha) = \sum_\beta a_{\alpha\beta} f_\beta,$$

where $a_{\alpha\beta} \in A$. Since A is connected, $\deg(a_{\alpha\beta}) \geq 0$ for all α and β. Thus, we have

$$0 \geq \deg(e_\alpha) = \deg(a_{\alpha\beta}) + \deg(f_\beta) \geq 1$$

which is not possible. Therefore, $\mathrm{Hom}_{\mathcal{D}_A^{+W}}(M, N[-1]) = 0$.

For the third axiom of a t-structure we need to show the existence of a distinguished triangle for every $M \in \mathcal{D}_A^{+W}$.
We may assume $M \in \mathcal{KCM}_A^{+W}$. As in the proof of Lemma 3.22 we consider $\tau^{\leq 0}M$ which is by construction in $\mathcal{D}_A^{\leq 0}$. We choose $M^{\leq 0} = \tau^{\leq 0}M$ and $M^{>0} = \mathrm{Cone}(\tau^{\leq 0}M \to M)$. This gives us the distinguished triangle in $\mathcal{D}_A^{+W}$:

$$M^{\leq 0} \to M \to M^{>0} \to M^{\leq 0}[1].$$

Applying q to the distinguished triangle gives us a distinguished triangle in $\mathcal{D}_{\mathbb{Q}}^{+W}$:

$$qM^{\leq 0} \to qM \to qM^{>0} \to qM^{\leq 0}[1].$$

Since $\mathrm{H}^n(qM^{\leq 0}) \cong \mathrm{H}^n(qM)$ for all $n \leq 0$ and $\mathrm{H}^0(qM^{\leq 0}[1]) = \mathrm{H}^1(qM^{\leq 0}) = 0$, $\mathrm{H}^n(qM^{>0}) = 0$ for all $n \leq 0$. So $qM^{>0}$ is in $\mathcal{D}_{\mathbb{Q}}^{\geq 1}$ and hence $M^{>0}$ is in $\mathcal{D}_A^{\geq 1}$, as desired.

The t-structure is non-degenerate:
Let $M \in \cap_{n\leq 0}\mathcal{D}_A^{\leq n}$, i.e. $\mathrm{H}^n(qM) = 0$ for all n, hence $qM \cong 0$ in $\mathcal{D}_{\mathbb{Q}}^{+W}$ and since q is conservative by Lemma 3.19, $M \cong 0$ in $\mathcal{D}_A^{+W}$. Similarly, $M \in \cap_{n\geq 0}\mathcal{D}_A^{\geq n}$ implies $M \cong 0$. □

This t-structure on $\mathcal{D}_A^{+W}$ restricts to a t-structure on the full triangulated subcategory $\mathcal{D}_A^f$. Let $\mathcal{D}_A^{f,\leq 0} := \mathcal{D}_A^f \cap \mathcal{D}_A^{\leq 0}$, $\mathcal{D}_A^{f,\geq 0} := \mathcal{D}_A^f \cap \mathcal{D}_A^{\geq 0}$ and $\mathcal{H}_A^f := \mathcal{H}_A \cap \mathcal{D}_A^f = \mathcal{D}_A^{f,\leq 0} \cap \mathcal{D}_A^{f,\geq 0}$.

Corollary 3.23 ([Lev10, Corollary 1.12.8])
Let A be a cohomologically connected Adams graded cdga. Then $(\mathcal{D}_A^{f,\leq 0}, \mathcal{D}_A^{f,\geq 0})$ defines a non-degenerate t-structure on $\mathcal{D}_A^f$ with heart $\mathcal{H}_A^f$.

Proof. Since $\mathcal{D}_A^f$ is a full triangulated subcategory of $\mathcal{D}_A^{+W}$, closed under isomorphisms in $\mathcal{D}_A^{+W}$, all the properties of a non-degenerate t-structure are inherited from the non-degenerate t-structure on $\mathcal{D}_A^{+W}$, except for the existence of a distinguished triangle in $\mathcal{D}_A^f$.

Let $M \in \mathcal{D}_A^f$. Then there exists a distinguished triangle in $\mathcal{D}_A^{+W}$

$$M^{\leq 0} \to M \to M^{>0} \to M^{\leq 0}[1]$$

with $M^{\leq 0} \in \mathcal{D}_A^{\leq 0}$ and $M^{>0} \in \mathcal{D}_A^{\geq 1}$.

Applying the exact functor $\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}}$ gives the distinguished triangle

$$\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M^{\leq 0} \to \mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M \to \mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M^{>0} \to \mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M^{\leq 0}[1]$$

in $\mathrm{D}(\mathbb{Q})$ such that $\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M^{\leq 0} \in \mathrm{D}(\mathbb{Q})^{\leq 0}$ and $\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M^{>0} \in \mathrm{D}(\mathbb{Q})^{\geq 1}$, i.e. $\mathrm{H}^n(\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M^{\leq 0}) = 0$ for $n > 0$ and $\mathrm{H}^n(\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M^{>0}) = 0$ for $n \leq 0$.

Since M is in $\mathcal{D}_A^f$, it follows by Lemma 3.20 that $\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M$ is in $D^b(\mathbb{Q})$ for all n and $\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M \cong 0$ for all but finitely many n. The long exact cohomology sequence for a distinguished triangle in $\mathrm{D}(\mathbb{Q})$ shows that $\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M^{\leq 0}$ and $\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M^{>0}$ are in $\mathrm{D}^b(\mathbb{Q})$ as well for all n and that they are isomorphic to zero for all but finitely many n. Applying Lemma 3.20 again shows that $M^{\leq 0}$ and $M^{>0}$ are in $\mathcal{D}_A^f$. □

Remark 3.24
When identifying the functor q with $\oplus_n \mathrm{gr}_{\mathrm{n}}^{\mathrm{W}}$, we see that the t-structure $(\mathcal{D}_A^{f,\leq 0}, \mathcal{D}_A^{f,\geq 0})$ on $\mathcal{D}_A^f$ is defined in the same way as the t-structure $(\mathrm{DMT}(S)^{\leq 0}, \mathrm{DMT}(S)^{\geq 0})$ on $\mathrm{DMT}(S)$.

Indeed, the subcategory $\mathcal{D}_A^{f,\leq 0}$ is the category of those objects $M \in \mathcal{D}_A^f$ such that $\mathrm{H}^m(\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M) = 0$ for all $m > 0$ and all n. Similarly, $\mathcal{D}_A^{f,\geq 0}$ is the *category of objects* $M \in \mathcal{D}_A^f$ *such that* $\mathrm{H}^m(\mathrm{gr}_\mathrm{n}^\mathrm{W} M) = 0$ *for all* $m < 0$ *and all* n*, while* $\mathcal{H}_A^f$ *consists of the objects* $M \in \mathcal{D}_A^f$ *such that* $\mathrm{H}^m(\mathrm{gr}_\mathrm{n}^\mathrm{W} M) = 0$ *for all* $m \neq 0$ *and all* n*.*

The subcategory $W_n\mathcal{D}_A^f$ *of* $\mathcal{D}_A^f$ *is the strictly full triangulated subcategory generated by the objects* $\mathbb{Q}_A(-q)$*, where* $q \leq n$ *and dually* $W^{>n}\mathcal{KCM}_A^f$ *is the strictly full triangulated subcategory generated by the objects* $\mathbb{Q}_A(-q)$*, where* $q > n$*. Thus,* $\mathrm{gr}_n^W \mathcal{D}_A^f$ *is the full triangulated subcategory generated by the object* $\mathbb{Q}_A(-n)$*.*

If we define $\mathcal{D}_n^{f,\leq 0}$ *as the full subcategory of* $\mathcal{D}_A^f$ *generated by the objects* $\mathbb{Q}_A(-n)[m]$*, where* $m \leq 0$*, then* $\mathcal{D}_A^{f,\leq 0}$ *is given as the strictly full subcategory generated by objects* $M \in \mathcal{D}_A^f$ *such that* $\mathrm{gr}_\mathrm{n}^\mathrm{W} M \in \mathcal{D}_n^{f,\leq 0}$*. Dually, if we define* $\mathcal{D}_n^{f,\geq 0}$ *as the strictly full subcategory of* $\mathcal{D}_A^f$ *generated by the objects* $\mathbb{Q}_A(-n)[m]$*, where* $m \geq 0$*, then* $\mathcal{D}_A^{f,\geq 0}$ *is given as the strictly full subcategory generated by objects* $M \in \mathcal{D}_A^f$ *such that* $\mathrm{gr}_\mathrm{n}^\mathrm{W} M \in \mathcal{D}_n^{f,\geq 0}$*.*

The heart of a t-structure is always an abelian category, but as in the case of Tate motives the category $\mathcal{H}_A^f$ has even nicer properties.

Theorem 3.25 ([Lev10, Proposition 1.12.11 and Lemma 1.12.9])
Let A *be a cohomologically connected Adams graded cdga.* $\mathcal{H}_A^f$ *is neutral Tannakian category over* $\mathbb{Q}$ *with fibre functor* ω *given by the composition of* $q\colon \mathcal{H}_A^f \to \mathcal{H}_\mathbb{Q}^f$ *with the forgetful functor to* $\mathbb{Q}$*-vector spaces.*

Furthermore, $\mathcal{H}_A^f$ *is the smallest abelian subcategory of* $\mathcal{H}_A^f$ *containing the Tate objects* $\mathbb{Q}_A(n)$*,* $n \in \mathbb{Z}$ *and closed under extensions in* $\mathcal{H}_A^f$*.*

Before we prove this theorem, we give a lemma that is used in the proof.

Lemma 3.26
Let A *be a cohomologically connected cdga. Then:*

$$\mathrm{Hom}_{\mathcal{H}_A^f}(\mathbb{Q}_A(a), \mathbb{Q}_A(b)) \simeq \mathrm{H}^0(A(b-a)) = \begin{cases} 0 & \text{if } a \neq b \\ \mathbb{Q}\cdot\mathrm{id} & \text{if } a = b. \end{cases}$$

Proof. Since A is cohomologically connected, $\mathrm{H}^0(A) = \mathbb{Q}$. On the other hand, using the decomposition $A = A(0) \oplus A^+$ we have

$$\mathbb{Q} \simeq \mathrm{H}^0(A) = \mathrm{H}^0(A(0) \oplus A^+) \simeq \mathrm{H}^0(A(0)) \oplus \mathrm{H}^0(A^+),$$

showing that $\mathrm{H}^0(A^+) = 0$ and in particular $\mathrm{H}^0(A(r)) = 0$ for $r > 0$. This proves the case $a < b$.

The cases $a = b$ and $a > b$ follow from the facts that $\mathrm{H}^0(A(0)) = A(0) = \mathbb{Q}\cdot 1$ and $A(r) = 0$ for $r < 0$. □

Corollary 3.27 ([Lev10, Lemma 1.12.10])
Let A be a connected Adams graded cdga. Let M, $N \in \mathcal{H}_A^f$ and $n \leq m$. Then we have

$$\mathrm{Hom}_{\mathcal{H}_A^f}(W^{>m}M, W_n N) = 0.$$

Proof. Let $M = \mathbb{Q}_A(-a)$ and $N = \mathbb{Q}_A(-b)$ with $a > b$, then

$$\mathrm{Hom}_{\mathcal{H}_A^f}(M, N) = \mathrm{H}^0(\mathbb{Q}_A(a-b)) = 0$$

by the previous lemma. The general result follows by induction on the weight filtration. □

Proof of Theorem 3.25. To check that $\mathcal{H}_A^f$ is a Tannakian category we have to show that $\mathcal{H}_A^f$ is a rigid abelian $\mathbb{Q}$-linear tensor category and that the fibre functor is an exact and faithful tensor functor.

$\mathcal{H}_A^f$ is an abelian category since it is the heart of a t-structure. The restriction of the tensor product on the rigid tensor category $\mathcal{D}_A^f$ to $\mathcal{H}_A^f$ makes $\mathcal{D}_A^f$ a rigid tensor category. Clearly, $\mathcal{H}_A^f$ is $\mathbb{Q}$-linear.

The fibre functor $\mathcal{H}_A^f \to \mathbb{Q}\text{-Vec}_{\mathbb{Q}}$ is given by the composition of $q\colon \mathcal{H}_A^f \to \mathcal{H}_{\mathbb{Q}}^f$ with the forgetful functor since $\mathcal{H}_{\mathbb{Q}}^f$ is equivalent to the category of finite dimensional graded $\mathbb{Q}$-vector spaces. This functor is an exact tensor functor since q and the forgetful functor are. The forgetful functor is also faithful, so it remains to check that q is faithful as well.
Let $f\colon M \to N$ be a map in $\mathcal{H}_A^f$ such that $\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} f = 0$ for all n. By induction on the length of the weight structure, it follows that $W^{>m} f = 0$, where m is the minimal integer such that $W_m M \oplus W_m N \neq 0$. Since $M \to N \to W^{>m}N$ is therefore zero, f is given by a map $W^{>m}M \to \mathrm{gr}_m^W N$ which is zero by Corollary 3.27.

Finally, let $\mathcal{H}_A^T$ be a full abelian subcategory of $\mathcal{H}_A^f$ containing the objects $\mathbb{Q}_A(n)$, $n \in \mathbb{Z}$, and being closed under extensions in $\mathcal{H}_A^f$.

Let $M \in \mathcal{H}_A^f$. Then $\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M \simeq \mathbb{Q}(-n)^{r_n}$ for some $r_n \geq 0$, hence $\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M \in \mathcal{H}_A^T$. If N is the minimal n such that $W_n \neq 0$. Then we have the exact sequence $\mathrm{gr}_{\mathrm{n}}^{\mathrm{W}} M \to M \to W^{>N}M$ in $\mathcal{H}_A^f$. By induction on the weight filtration, also $W^{>N}$ is in $\mathcal{H}_A^T$. $\mathcal{H}_A^T$ is closed under extensions and therefore $M \in \mathcal{H}_A^T$ completing the proof. □

3.3 Cell modules over $\mathcal{N}(S)$

We now have associated to any smooth scheme $S \in \mathrm{Sm}_k$ an Adams-graded cdga $\mathcal{N}(S)$. Our goal in section 3.4 is to prove that the homotopy category of finite $\mathcal{N}(S)$-cell modules is equivalent to the triangulated category of Tate motives $\mathrm{DMT}(S)$. Recall that the homotopy category of cell-modules is generated by the modules $\mathbb{Q}_{\mathcal{N}(S)}(n)$, $n \in \mathbb{Z}$, by Lemma 3.11. Therefore, we are interested in the groups of morphisms between these generators. As we have seen in Example 3.10, these are determined by the homology groups of $\mathcal{N}(S)$:

$$\mathrm{Hom}_{\mathcal{KCM}^f_{\mathcal{N}(S)}}(\mathbb{Q}_{\mathcal{N}(S)}(0), \mathbb{Q}_{\mathcal{N}(S)}(q)[p]) = \mathrm{H}^p(\mathcal{N}(S)(q)).$$

Applying Lemma 3.3 and Lemma 3.5 we obtain an isomorphism

$$\mathrm{H}^p(\mathcal{N}(S)(q)) \simeq \mathrm{H}^p(C_*^{\mathrm{Sus}}(\mathbb{Z}_{\mathrm{S}}^{\mathrm{tr}}((\mathbb{P}^1/\infty)^q))(S)) \simeq \mathrm{H}^p(C_*^{\mathrm{Sus}}(\mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}(q)[2q])(S)).$$

By [Voe00, Theorem 4.2.2 and Proposition 4.2.3], we have an isomorphism to the motivic cohomology

$$\mathrm{H}^p(C_*^{\mathrm{Sus}}(\mathbb{Z}_{\mathrm{k}}^{\mathrm{tr}}(q)[2q])(S)) \simeq \mathrm{H}^p(S, \mathbb{Q}(q))$$

proving the following proposition.

Proposition 3.28
Let $S \in \mathrm{Sm}_k$. There is an isomorphism

$$\mathrm{H}^p(S, \mathbb{Q}(q)) \simeq \mathrm{H}^p(\mathcal{N}(S)(q)) = \mathrm{Hom}_{\mathcal{KCM}^f_{\mathcal{N}(S)}}(\mathbb{Q}_{\mathcal{N}(S)}(0), \mathbb{Q}_{\mathcal{N}(S)}(q)[p]).$$

Corollary 3.29
Let $S \in \mathrm{Sm}_k$. S satisfies the Beilinson-Soulé vanishing conjectures if and only if $\mathcal{N}(S)$ is cohomologically connected.

Therefore, if S satisfies the Beilinon-Soulé conjectures, we can apply the results of section 3.2 to $\mathcal{N}(S)$, especially we can utilize Theorem 3.21 and Theorem 3.25.

Theorem 3.30
Suppose S satisfies the Beilinson-Soulé vanishing conjectures. Then:

1. *$(\mathcal{D}^{f,\leq 0}_{\mathcal{N}(S)}, \mathcal{D}^{f,\geq 0}_{\mathcal{N}(S)})$ is a non-degenerate t-structure on $\mathcal{D}^{f}_{\mathcal{N}(S)}$ with heart $\mathcal{H}^{f}_{\mathcal{N}(S)}$ containing the Tate objects $\mathbb{Q}_{\mathcal{N}(S)}(q)$, $q \in \mathbb{Z}$.*
2. *$\mathcal{H}^{f}_{\mathcal{N}(S)}$ is equal to the smallest abelian subcategory of $\mathcal{H}^{f}_{\mathcal{N}(S)}$ which contains the Tate objects $\mathbb{Q}_{\mathcal{N}(S)}(q)$, $q \in \mathbb{Z}$, and is closed under extensions in $\mathcal{H}^{f}_{\mathcal{N}(S)}$.*
3. *The tensor product in $\mathcal{D}_{\mathcal{N}(S)}$ makes $\mathcal{H}^{f}_{\mathcal{N}(S)}$ a rigid $\mathbb{Q}$-linear abelian tensor category.*
4. *The functor $\omega\colon \mathcal{H}^{f}_{\mathcal{N}(S)} \to \mathbb{Q}\text{-Vec}$ is a fibre functor making $\mathcal{H}^{f}_{\mathcal{N}(S)}$ a Tannakian category.*

3.4 Spitzweck's representation theorem

We constructed the triangulated category of Tate motives $\mathrm{DMT}(S)$ in chapter 2 as well as the derived category $\mathcal{D}^{f}_{\mathcal{N}(S)}$ in this chapter. By comparing Theorem 2.29 and Theorem 3.30, we see that the structure of both categories is very similar. In fact, they are equivalent as we show in this section. Furthermore, the weight filtrations and therefore the t-structures on both categories are obtained in the same fashion out of the "base objects" $\mathbb{Q}_S(q)$ respectively $\mathbb{Q}_{\mathcal{N}(S)}(q)$, $q \in \mathbb{Z}$, such that the equivalence of triangulated tensor categories restricts to an equivalence of the Tannakian subcategories $\mathrm{MT}(S)$ and $\mathcal{H}^{f}_{\mathcal{N}(S)}$ in case S satisfies the Beilinson-Soulé vanishing conjectures.

This result is known as Spitzweck's representation. We formulate it as stated in [Lev05, Theorem 5.23], where it is proven for the case of S the spectrum of a field, but extend it to the case of smooth base schemes $S \in \mathrm{Sm}_k$ by [Lev10, Theorem 5.3.2].

Theorem 3.31 (Spitzweck's representation theorem)
Let S be a separated, smooth scheme of finite type over a field k. Then there is a natural exact tensor functor

$$\Phi_S\colon \mathcal{D}^{f}_{\mathcal{N}(S)} \to \mathrm{DM}_{\mathrm{gm}}(S)_{\mathbb{Q}}$$

that induces an equivalence of triangulated tensor categories

$$\Phi_S\colon \mathcal{D}^{f}_{\mathcal{N}(S)} \to \mathrm{DMT}(S).$$

The functor Φ_S is compatible with the weight filtrations in $\mathcal{D}^f_{\mathcal{N}(S)}$ and $\mathrm{DMT}(S)$*.*

If S satisfies the Beilinson-Soulé vanishing conjectures, then the functor Φ_S induces an equivalence of Tannakian categories

$$\Phi_S\colon \mathcal{H}^f_{\mathcal{N}(S)} \to \mathrm{MT}(S)$$

transforming the fibre functor ω on $\mathcal{H}^f_{\mathcal{N}(S)}$ into the fibre functor gr^W_* on $\mathrm{MT}(S)$*.*

For the proof of the theorem we use the following lemma.

Lemma 3.32
Let $\mathcal{KCM}^f_{\mathcal{N}(S),\geq 0}$ be the full triangulated subcategory of $\mathcal{KCM}^f_{\mathcal{N}(S)}$ that is generated by the objects $\mathbb{Q}_{\mathcal{N}(S)}(q)$, where $q \geq 0$. Then $\mathcal{KCM}^f_{\mathcal{N}(S)}$ is equivalent the category one obtains by inverting the functor $-\otimes \mathbb{Q}_{\mathcal{N}(S)}(1)$ on $\mathcal{KCM}^f_{\mathcal{N}(S),\geq 0}$.

Proof. We denote by $\mathcal{C}$ the category obtained by inverting the functor $-\otimes \mathbb{Q}_{\mathcal{N}(S)}(1)$ on $\mathcal{KCM}^f_{\mathcal{N}(S),\geq 0}$. The objects of $\mathcal{C}$ are given as $X(n)$, where $X \in \mathcal{KCM}^f_{\mathcal{N}(S),\geq 0}$ and $n \in \mathbb{Z}$ and

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{C}}(X(n), Y(m))\\ &\qquad := \varinjlim_{N} \mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(S)^f}}(X \otimes \mathbb{Q}_{\mathcal{N}(S)}(n+N), Y \otimes \mathbb{Q}_{\mathcal{N}(S)}(m+N)).\end{aligned}$$

We have the canonical isomorphisms $\mathbb{Q}_{\mathcal{N}(S)}(q)(n) \simeq \mathbb{Q}_{\mathcal{N}((S)}(q+n)(0)$ for $q > 0$ and $n \geq -q$.

Sending $X(n)$ to $X \otimes \mathbb{Q}_{\mathcal{N}(S)}(n)$ defines a tensor functor $F\colon \mathcal{C} \to \mathcal{KCM}^f_{\mathcal{N}(S)}$. This functor is clearly fully faithful since $-\otimes \mathbb{Q}_{\mathcal{N}(S)}(1)$ is a fully faithful endo-functor on $\mathcal{KCM}^f_{\mathcal{N}(S)}$.

Lastly, we need to check that F is essentially surjective. We do this by induction on the weight filtration. For the generators $\mathbb{Q}_{\mathcal{N}(S)}(q)$, $q \in \mathbb{Z}$, of the triangulated category $\mathcal{KCM}^f_{\mathcal{N}(S)}$ we have $F(\mathbb{Q}_{\mathcal{N}(S)}(0)(q)) = \mathbb{Q}_{\mathcal{N}(S)}(q)$. Now, let A', B' in $\mathcal{KCM}^f_{\mathcal{N}(S)}$ such that $A' \simeq F(A)$ and $B' \simeq F(B)$ for some objects A, B in $\mathcal{C}$. Let $f\colon A' \to B'$. We need to construct an objects $C \in \mathcal{C}$ such that $F(C) \simeq \mathrm{Cone}(f)$. Since A' and B' are finite cell modules, there exists an $N \geq 0$ such that $A' \otimes \mathbb{Q}_{\mathcal{N}(S)}(N)$, $B' \otimes \mathbb{Q}_{\mathcal{N}(S)}(N)$ and $\mathrm{Cone}(f) \otimes$

$\mathbb{Q}_{\mathcal{N}(S)}(N)$ are in $\mathcal{KCM}^f_{\mathcal{N}(S),\geq 0}$. Now define $A := (A' \otimes \mathbb{Q}_{\mathcal{N}(S)}(N))(-N)$ and $B := (B' \otimes \mathbb{Q}_{\mathcal{N}(S)}(N))(-N)$. The morphism $f\colon A' \to B'$ induces a morphism $f\langle N\rangle\colon A' \otimes \mathbb{Q}_{\mathcal{N}(S)}(N) \to B' \otimes \mathbb{Q}_{\mathcal{N}(S)}(N)$ in $\mathcal{KCM}^f_{\mathcal{N}(S),\geq 0}$ and hence an object $C := \mathrm{Cone}(f\langle N\rangle)(-N) \in \mathcal{C}$. Then:

$$F(C) = \mathrm{Cone}(f\langle N\rangle) \otimes \mathbb{Q}_{\mathcal{N}(S)}(-N) \simeq \mathrm{Cone}(f).$$

$\square$

Proof of Theorem 3.31. We actually define a tensor functor

$$\Phi_S\colon \mathcal{KCM}^{f,\prime}_{\mathcal{N}(S)} \to \mathrm{DM}_{\mathrm{gm}}(S)_{\mathbb{Q}},$$

where $\mathcal{KCM}^{f,\prime}_{\mathcal{N}(S)}$ is the homotopy category of $\mathcal{N}(S)$-cell modules with a choice of a basis. This is equivalent to $\mathcal{KCM}^f_{\mathcal{N}(S)}$ which in turn is equivalent to $\mathcal{D}^f_{\mathcal{N}(S)}$ by Corollary 3.17.

Such a functor Φ_S is in fact determined by giving a tensor functor $\mathcal{KCM}^{f,\prime}_{\mathcal{N}(S),\geq 0} \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(S)_{\mathbb{Q}}$ which extends canonically to a tensor functor $\mathcal{KCM}^{f,\prime}_{\mathcal{N}(S)} \to \mathrm{DM}_{\mathrm{gm}}(S)_{\mathbb{Q}}$ by Lemma 3.32.

Let $M = \oplus_j \mathcal{N}(S)m_j$ be a $\mathcal{N}(S)$-cell module in $\mathcal{KCM}^{f,\prime}_{\mathcal{N}(S),\geq 0}$ with basis $\{m_j\}$ and differential d given by

$$dm_j = \sum_i a_{ij} m_i.$$

Let $\Phi_S(M,d)$ be the complex of sheaves $\sum_j \mathcal{N}(r_j)[n_j]\mu_j$, where μ_j is a formal basis, $-r_j$ is the Adams-degree of m_j and $-n_j$ is the cohomological degree of m_j. Note that $r_j \geq 0$ since M is in $\mathcal{KCM}^{f,\prime}_{\mathcal{N}(S),\geq 0}$, hence $\mathcal{N}(r_j)$ is well-defined. The differential δ on $\Phi_S(M,d)$ is defined by

$$\delta\mu_j := \sum_i a_{ij}\mu_i$$

and the Leibniz rule. By $d^2 = 0$ it follows $\delta^2 = 0$, giving a well-defined object in $\mathrm{DM}^{\mathrm{eff}}(S)_{\mathbb{Q}}$.

If $f\colon M \to N$ is a morphism of $\mathcal{N}(S)$-cell modules, we choose bases $\{m_j\}$ and $\{n_j\}$ of M respectively N with corresponding bases $\{\mu_j\}$ and $\{\nu_j\}$ of $\Phi_S(M)$ respectively $\Phi_S(N)$. If $f(m_j) = \sum_i f_{ij} n_j$, we define $\Phi_S(f)(\mu_j) := \sum_i f_{ij}\nu_i$.

Since $\Phi_S(\mathbb{Q}_{\mathcal{N}(S)}(q))$, $q \geq 0$, is the object $\mathcal{N}_S(q)$ which isomorphic to $\mathbb{Q}_S(q)$ in $\mathrm{DM}^{\mathrm{eff}}(S)_{\mathbb{Q}}$ by Lemma 3.6, the image of the functor Φ_S is in fact in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(S)_{\mathbb{Q}}$.

One easily checks that Φ_S respects tensor products, the translation functor and cone sequences, so it yields a well-defined exact tensor functor

$$\Phi_S \colon \mathcal{KCM}^{f\prime}_{\mathcal{N}(S)} \to \mathrm{DM}_{\mathrm{gm}}(S)_{\mathbb{Q}}.$$

Furthermore, we have

$$\mathrm{Hom}_{\mathcal{KCM}^{f}_{\mathcal{N}(S)}}(\mathbb{Q}_{\mathcal{N}(S)}(0), \mathbb{Q}_{\mathcal{N}(S)}(n)[m]) \simeq \mathrm{Hom}_{\mathrm{DMT}(S)}(\mathbb{Q}_S(0), \mathbb{Q}_S(n)[m])$$

by Proposition 3.28 and Lemma 2.28 and Φ_S induces the identity maps between these Hom-groups.

Since the objects $\mathbb{Q}_{\mathcal{N}(S)}(q)$, $q \in \mathbb{Z}$, generate $\mathcal{KCM}^{f}_{\mathcal{N}(S)}$ as a triangulated category by Lemma 3.11, Φ_S is fully faithful. And since $\mathrm{DMT}(S)$ is generated by the objects $\mathbb{Q}_S(q)$, $q \in \mathbb{Z}$, as a triangulated subcategory of $\mathrm{DM}_{\mathrm{gm}}(S)_{\mathbb{Q}}$, the essential image of the functor Φ_S is $\mathrm{DMT}(S)$ and therefore, Φ_S induces an equivalence $\mathcal{D}^{f}_{\mathcal{N}(S)} \to \mathrm{DMT}(S)$ of triangulated tensor categories.

By definition the functor Φ_S respects the weight filtrations in $\mathcal{D}^{f}_{\mathcal{N}(S)}$ and $\mathrm{DMT}(S)$.

Now assume S satisfies the Beilinson-Soulé vanishing conjectures. Since the functor Φ_S is compatible with the weight filtration, it also respects the t-structures on $\mathcal{D}^{f}_{\mathcal{N}(S)}$ respectively $\mathrm{DMT}(S)$. Furthermore, identifying the fibre functor ω on $\mathcal{H}^{f}_{\mathcal{N}(S)}$ with $\oplus_n \mathrm{gr}^{\mathrm{W}}_{\mathrm{n}}$ (see Remark 3.24) shows that ω is indeed transformed into gr^{W}_{*} by Φ_S.

This completes the proof. $\square$

4 Artin-Tate motives and cell modules

In the previous chapter we constructed an equivalence between the triangulated category of Tate motives $\mathrm{DMT}(k)$ over a number field k and the derived category of Adams-graded dg modules $\mathcal{D}^f_{\mathcal{N}(k)}$ of finite rank over the cycle algebra $\mathcal{N}(k)$ over k. The category $\mathrm{DMT}(k)$ is a full triangulated subcategory of the triangulated category of Artin-Tate motives $\mathrm{DMAT}(k)$ over k which consists of the push-forwards of Tate motives under finite maps to $\operatorname{Spec} k$. So it is natural to ask whether a triangulated category of cell modules can be constructed that contains $\mathcal{D}^f_{\mathcal{N}(k)}$ as a full triangulated subcategory and is equivalent to $\mathrm{DMAT}(k)$. This question was the main motivation for this thesis and is answered in this chapter.

In the first section of this chapter we recall the definition of the triangulated category of Artin-Tate motives $\mathrm{DMAT}(k)$ over k and summarise some of its essential properties, first and foremost the existence of a non-degenerate t-structure that yields a Tannakian category as its heart that we denote by $\mathrm{MAT}(k)$. This was shown by Wildeshaus in [Wil08]. Furthermore, we define the triangulated category of Artin-Tate motives over k trivialisable over L, $\mathrm{DMAT}(L|k)$, that is used in the second section. The first section concludes with the computation of the Tannaka group of $\mathrm{MAT}(k)$ in Theorem 4.24 as the semi-direct product of the absolute Galois group $\mathrm{Gal}(\bar{k}|k)$ with the Tannaka group of $\mathrm{MT}(\bar{k})$.

Subject of the second section is the construction of a triangulated category of cell modules $\mathcal{D}(k)$ that is equivalent to $\mathrm{DMAT}(k)$ and contains the category $\mathcal{D}^f_{\mathcal{N}(k)}$ as a full triangulated subcategory. Furthermore, $\mathcal{D}(k)$ should carry a non-degenerate t-structure whose heart $\mathcal{A}(k)$ is a Tannakian category. This is done by constructing a triangulated category $\mathcal{D}(L|k)$, where L is a finite Galois extension of k, that is equivalent to $\mathrm{DMAT}(L|k)$ and carries the desired properties. Furthermore, there are canonical embedding $\mathcal{D}(K|k) \to \mathcal{D}(L|k)$ if K is an intermediate Galois extension $k \subset K \subset L$, that preserves the structures. Then, $\mathcal{D}(k)$ is the union of the categories $\mathcal{D}(L|k)$ over all finite Galois extensions L of k.

The chapter concludes with the main result of this thesis (Theorem 4.41). It states the desired equivalence of the triangulated tensor categories $\mathcal{D}(k) \to \mathrm{DMAT}(k)$ that, restricted to the full subcategory $\mathcal{D}^f_{\mathcal{N}(k)}$, yields the equivalence $\mathcal{D}^f_{\mathcal{N}(k)} \to \mathrm{DMT}(k)$ given in Spitzweck's representation theorem (Theorem 3.31). Moreover, the equivalence is compatible with the t-structures on $\mathcal{D}(k)$ and $\mathrm{DMAT}(k)$ respectively and hence gives an equivalence of the Tannakian categories $\mathcal{A}(k)$ and $\mathrm{MAT}(k)$.

Throughout this chapter let k be a number field.

4.1 Artin-Tate motives over number fields

In this section we construct the Tannakian category of mixed Artin-Tate motives over a number field k. There are several approaches to do this. We start by giving the construction of Scholbach in [Sch11] who defines the triangulated category of Artin-Tate motives $\mathrm{DMAT}(k)$ over k as the full triangulated subcategory of $\mathrm{DM}(S)_{\mathbb{Q}}$ generated by the push-forwards of the Tate motives under finite morphisms to $\operatorname{Spec} k$. For number fields, this is equivalent to the definition by Wildeshaus in [Wil08] who defines $\mathrm{DMAT}(k)$ as the triangulated category generated by the Tate motives over k and the motives of zero-dimensional schemes over k.

Similar to the case of Tate motives, a t-structure can be defined on $\mathrm{DMAT}(k)$ which again yields a Tannakian category $\mathrm{MAT}(k)$ as its heart. This was done by Wildeshaus in [Wil08] and we give his result in Theorem 4.13, but using the notations of [Sch11].

Furthermore, we state some properties of the category $\mathrm{DMAT}(k)$ that give us an indication of how to define an equivalent category of cell modules (and allow us to prove this equivalence). These results are also used to compute the Tannaka group $\mathrm{G}(\mathrm{MAT}(k))$ as the semi-direct product of the absolute Galois group $\mathrm{Gal}(\bar{k}|k)$ with the Tannaka group of $\mathrm{MT}(\bar{k})$ in Corollary 4.24.

4.1.1 Definition

Let X, $Y \in \mathrm{Sm}_k$ and $\phi\colon X \to Y$ a k-morphism. Recall that there exist adjoint functors $\phi^*\colon \mathrm{DM}_{\mathrm{gm}}(Y)_{\mathbb{Q}} \leftrightarrows \mathrm{DM}_{\mathrm{gm}}(X)_{\mathbb{Q}}\colon \phi_*$ and $\phi^*\colon \mathrm{DM}(Y)_{\mathbb{Q}} \leftrightarrows \mathrm{DM}(X)_{\mathbb{Q}}\colon \phi_*$. The functor ϕ^* respects the category of Tate motives, in particular $\phi^*\,\mathbb{Q}_Y(0) = \mathbb{Q}_X(0)$, whereas the functor ϕ_* does not.

So it makes sense to consider the full triangulated subcategory of $\mathrm{DM}(Y)_{\mathbb{Q}}$ generated by the objects $\phi_*\,\mathbb{Q}_X(0)$ or $\phi_*\,\mathbb{Q}_X(q)$, $q \in \mathbb{Z}$, respectively.

Definition 4.1 ([Sch11, Definition 2.1])
The triangulated category $\mathrm{DMA}(k)$ *of Artin motives over* k *is the full triangulated subcategory of* $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ *generated by direct summands of* $\phi_*\,\mathbb{Q}_X(0)$*, where* $\phi\colon X \to \operatorname{Spec} k$ *is a finite morphism.*

If S *is a scheme of the form* $S = \coprod_i \operatorname{Spec} k_i$*, a finite disjoint union of spectra of fields, we put* $\mathrm{DMA}(S) := \oplus_i \mathrm{DMA}(k_i)$*.*

Note that by Proposition 2.20, $\mathrm{DMA}(k)$ is equivalent to the full triangulated subcategory of $\mathrm{DM}(k)_{\mathbb{Q}}$ generated by direct summands of $\phi_*\,\mathbb{Q}_X(0)$, where $\phi\colon X \to \operatorname{Spec} k$ is a finite morphism.

Remark 4.2
Let $\phi\colon X \to \operatorname{Spec} k$ *be a finite morphism. In particular,* ϕ *is affine, hence* $X \simeq \operatorname{Spec} R$ *for some ring* R*. Since* ϕ *is surjective, we have* $\dim R = \dim X = \dim k = 0$*, so* R *is a finite product of finite field extensions* k_i *of* k*:* $X \simeq \operatorname{Spec} R \simeq \operatorname{Spec}(\prod k_i) \simeq \coprod \operatorname{Spec} k_i$ *and* $\phi_*\,\mathbb{Q}_X(0) \simeq \oplus \phi_{i*}\,\mathbb{Q}_{k_i}(0)$*. Therefore, one can also define* $\mathrm{DMA}(k)$ *to be the full triangulated subcategory of* $\mathrm{DM}(k)_{\mathbb{Q}}$ *that is generated by direct summands of* $\phi_*\,\mathbb{Q}_L(0)$*, where* L *is a finite field extension of* k*.*

Lemma 4.3
Every finite morphism $\phi\colon X \to \operatorname{Spec} k$*,* k *a number field, is étale, hence smooth. Conversely, every smooth zero-dimensional scheme* $X \xrightarrow{\phi} \operatorname{Spec} k$ *in* Sm_k *is finite over* k*.*

Proof. If $\phi\colon X \to \operatorname{Spec} k$ is finite, we have seen in the previous remark that X is isomorphic to $\operatorname{Spec} R$, where R is a finite product of finite field extensions. Since k is a number field, every such field extension is finite and separable. Therefore, R is an étale k-algebra and ϕ is an étale morphism. Conversely, let X be a smooth zero-dimensional scheme over k. Then

$\phi\colon X \to \operatorname{Spec} k$ is étale, hence, $X \simeq \operatorname{Spec} R$, where R is a finite product of finite separable field extensions of k. Therefore, $\phi\colon X \to \operatorname{Spec} k$ is finite. $\square$

Hence, the following definition by Wildeshaus in [Wil08] is equivalent to Definition 4.1 given above. Recall that for $\phi\colon X \to \operatorname{Spec} k$ finite (and hence étale) $\phi_* \mathbb{Q}_X(0) \simeq \phi_\# \mathbb{Q}_X(0) \simeq \mathrm{m}_k(X)_{\mathbb{Q}}$.

Definition 4.4 ([Wil08, Definition 1.2])
The triangulated category of Artin motives $\mathrm{DMA}(k)$ *is the full triangulated subcategory of* $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ *generated by the motives* $\mathrm{m}_k(X)_{\mathbb{Q}}$ *of smooth zero-dimensional schemes* X *over* $\operatorname{Spec} k$*.*

Remark 4.5 ([Wil08, Remark 1.4])
The triangulated category of Artin motives $\mathrm{DMA}(k)$ *is equivalent to the bounded derived category* $\mathrm{D}^b(\mathrm{MA}(k))$ *of the Tannakian category* $\mathrm{MA}(k)$ *of representations of the absolute Galois group* $\mathrm{Gal}(\bar{k}|k)$ *of* k *in finitely generated* $\mathbb{Q}$*-vector spaces, i.e.* $G(\mathrm{MA}(L(k))) = \mathrm{Gal}(\bar{k}|\underline{k})$*. More precisely, if* X *is a smooth zero-dimensional scheme over* k*,* $\mathrm{Gal}(\bar{k}/k)$ *acts canonically on the set of* $\bar{k}$*-valued points of* X*. Then the object in* $\mathrm{MA}(k)$ *corresponding to* $\mathrm{m}_k(X)_{\mathbb{Q}}$ *is just the formal* $\mathbb{Q}$*-linear envelope of this set with the induced Galois action. In particular,* $\mathrm{MA}(k)$ *is semi-simple and every object* $\mathrm{m}_k(X)_{\mathbb{Q}}$ *admits a dual.*
This follows by [Voe00, Remark 2 on page 33].

Now, we obtain the category of Artin-Tate motives if we consider the full triangulated subcategory of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ generated by the push-forwards of all Tate motives (not just of the trivial motive $\mathbb{Q}(0)$) under finite maps $X \to \operatorname{Spec} k$.

Definition 4.6 ([Sch11, Definition 2.1])
The triangulated category of mixed Artin-Tate motives $\mathrm{DMAT}(k)$ *over* k *is the full thick triangulated subcategory of* $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ *generated by the objects* $\phi_* \mathbb{Q}_X(q)$*, where* $\phi\colon X \to \operatorname{Spec} k$ *is a finite morphism of schemes and* $q \in \mathbb{Z}$*, i.e.* $\mathrm{DMAT}(k)$ *is the smallest full triangulated subcategory of* $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ *that contains the objects* $\phi_* \mathbb{Q}_X(q)$ *and is closed under direct summands.*

In [Wil08, Definition 1.3], $\mathrm{DMAT}(k)$ is defined as the full triangulated tensor subcategory of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ generated by $\mathrm{DMT}(k)$ and $\mathrm{DMA}(k)$. Since

$\phi_* \mathbb{Q}_X(q) \cong \phi_* \mathbb{Q}_X(0) \otimes \mathbb{Q}_k(q)$ for any finite map $X \to \operatorname{Spec} k$ this definition agrees with Definition 4.6. As a consequence the category $\mathrm{DMAT}(k)$ is a rigid tensor category since the generators $\mathbb{Q}_k(q)$, $q \in \mathbb{Z}$, and $\phi_* \mathbb{Q}_X(0)$ are rigid.

The same arguments as in Remark 4.2 show that $\mathrm{DMAT}(k)$ is the full thick triangulated subcategory of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ generated by direct summands of $\phi_* \mathbb{Q}_L(q)$, where L is a finite field extension of k, $\phi\colon \operatorname{Spec} L \to \operatorname{Spec} k$ and $q \in \mathbb{Z}$.

Remark 4.7
Like the category of Tate motives, one can define the category $\mathrm{DMAT}(S)$ *of Artin-Tate motives over an arbitrary smooth separated scheme S of finite type over k. However, in general it is not possible to define a t-structure on* $\mathrm{DMAT}(S)$*, even if S satisfies the Beilinson-Soulé vanishing conjectures. For the case $S = \operatorname{Spec} \mathcal{O}_k$, the spectrum of the ring of integers of a number field k, the construction of a t-structure was given by Scholbach in [Sch11].*

4.1.2 Hom-groups

In this subsection we compute the $\mathbb{Q}$-vector spaces of morphisms between the generators of $\mathrm{DMAT}(k)$. This serves two purposes. First we need the vanishing of certain morphism groups to ensure the existence of a t-structure on $\mathrm{DMAT}(k)$ (cf. Corollary 2.23 and the Beilinson-Soulé vanishing conjectures in the case of Tate motives) and secondly we use these results to show an equivalence of categories between $\mathrm{DMAT}(k)$ and the triangulated category of cell modules with Galois-action that we define in section 4.2.

Let L and K be two finite field extensions of k. We denote the structure morphisms over $\operatorname{Spec} k$ by $\phi\colon \operatorname{Spec} L \to \operatorname{Spec} k$ and $\psi\colon \operatorname{Spec} K \to \operatorname{Spec} k$ respectively. We define S as the fibre product $\operatorname{Spec} L \times_k \operatorname{Spec} K \simeq \operatorname{Spec}(L \otimes_k K)$:

$$\begin{array}{ccc} \operatorname{Spec}(L \otimes_k K) & \xrightarrow{\psi'} & \operatorname{Spec} L \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} \\ \operatorname{Spec} K & \xrightarrow{\psi} & \operatorname{Spec} k. \end{array}$$

We are interested in $\operatorname{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_* \mathbb{Q}_L(0), \psi_* \mathbb{Q}_K(q)[p])$, for all q,

$p \in \mathbb{Z}$. These $\mathbb{Q}$-vector spaces are given by

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_* \mathbb{Q}_L(0), \psi_* \mathbb{Q}_K(q)[p]) &\overset{(1)}{\simeq} \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \phi^*\psi_* \mathbb{Q}_K(q)[p]) \\
&\overset{(2)}{\simeq} \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \psi'_*\phi'^* \mathbb{Q}_K(q)[p]) \\
&\overset{(3)}{\simeq} \mathrm{Hom}_{\mathrm{DM}(S)_{\mathbb{Q}}}(\psi'^* \mathbb{Q}_L(0), \phi'^* \mathbb{Q}_K(q)[p]) \\
&\overset{(4)}{\simeq} \mathrm{Hom}_{\mathrm{DM}(S)_{\mathbb{Q}}}(\mathbb{Q}_S(0), \mathbb{Q}_S(q)[p])
\end{aligned}$$

using the following facts from Lemma 2.16:

(1) ϕ^* is right adjoint to ϕ_* since ϕ is finite and étale;

(2) $\phi^*\psi_* \simeq \psi'_*\phi'^*$ (base change);

(3) ψ'_* is right adjoint to ψ'^*;

(4) $\psi'^* \mathbb{Q}_L(0) \simeq \mathbb{Q}_S(0)$ and $\phi'^* \mathbb{Q}_K(0) \simeq \mathbb{Q}_S(0)$.

Now, $S = \mathrm{Spec}(L \otimes_k K) \simeq \mathrm{Spec}(\prod_{i=1}^r k_i) \simeq \bigsqcup_{i=1}^r \mathrm{Spec}(k_i)$ for some finite field extensions k_i over k and some $r \in \mathbb{Z}$, so

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{DM}(S)_{\mathbb{Q}}}(\mathbb{Q}_S(0), \mathbb{Q}_S(q)[p]) &\simeq \bigoplus_{i=1}^r \mathrm{Hom}_{\mathrm{DM}(k_i)_{\mathbb{Q}}}(\mathbb{Q}_{k_i}(0), \mathbb{Q}_{k_i}(q)[p]) \\
&\simeq \bigoplus_{i=1}^r K_{2q-p}(k_i)_{\mathbb{Q}}^{(q)}.
\end{aligned}$$

These computations allow us to prove the following lemma.

Lemma 4.8 (Variant of [Sch11, Lemma 3.2])
Let $\phi\colon X \to \mathrm{Spec}\, k$ and $\psi\colon Y \to \mathrm{Spec}\, k$ be two finite maps. Then:

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\mathbb{Q}_X(0), \psi_*\mathbb{Q}_Y(q)[p]) = \begin{cases} 0 & \text{if } q < 0 \\ 0 & \text{if } q = 0, p \neq 0 \\ \text{finite-dimensional} & \text{if } q = p = 0 \\ 0 & \text{if } q \neq 0, p \leq 0. \end{cases}$$

Proof. X and Y are given as the disjoint union of spectra of number fields over k by Remark 4.2. In particular, $\mathbb{Q}_X(0) \simeq \oplus_i \mathbb{Q}_{k_i}(0)$ for some number fields k_i and hence $\phi_* \mathbb{Q}_X(0) \simeq \oplus_i \phi_{i*} \mathbb{Q}_{k_i}(0)$, and similarly $\psi_* \mathbb{Q}_Y(q)[p] \simeq \oplus_j \psi_{j*} \mathbb{Q}_{l_j}(q)[p]$.

Since the functors $\mathrm{Hom}(M,-)$ and $\mathrm{Hom}(-,M)$ are additive, the $\mathbb{Q}$-vector spaces $\mathrm{Hom}_{\mathrm{DM_{gm}}(k)_{\mathbb{Q}}}(\phi_*\mathbb{Q}_X(0),\psi_*\mathbb{Q}_Y(n)[m])$ are given as direct sums of the vector spaces $\mathrm{Hom}_{\mathrm{DM_{gm}}(k)_{\mathbb{Q}}}(\phi_{i*}\mathbb{Q}_{k_i}(0),\psi_{j*}\mathbb{Q}_{l_j}(q)[p])$. On the other hand, by the computations preceding the lemma, these are given by direct sums of K-groups of number fields.
By the (known) K-theory for number fields (see Example 2.24), the statement follows. □

Example 4.9
Let L be a finite Galois extension of k of degree n with Galois group $G = \mathrm{Gal}(L|k)$ and let $\phi\colon \operatorname{Spec} L \to \operatorname{Spec} k$. Then $L\otimes_k L \simeq L^n$ and $\operatorname{Spec}(L\otimes_k L) \simeq \oplus_{i=1}^n \operatorname{Spec} L$. Therefore, for q, $p\in\mathbb{Z}$:

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(0),\phi_*\,\mathbb{Q}_L(q)[p]) \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\mathbb{Q}_L(q)[p])^n$$

in particular for $p=q=0$:

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(0),\phi_*\,\mathbb{Q}_L(0)) \simeq \mathbb{Q}^n\,.$$

Furthermore, since $L\otimes_k k\simeq L$ we have:

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\mathbb{Q}_k(0),\phi_*\,\mathbb{Q}_L(q)[p]) \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\mathbb{Q}_L(q)[p])$$

and

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(0),\mathbb{Q}_k(q)[p]) \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\mathbb{Q}_L(q)[p]).$$

In particular for $q=p=0$:

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\mathbb{Q}_k(0),\phi_*\,\mathbb{Q}_L(0)) \simeq \mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(0),\mathbb{Q}_k(0)) \simeq \mathbb{Q}\,.$$

Let K be an intermediate Galois field extension $k\subset K\subset L$ of degree m over k with Galois group $H=\mathrm{Gal}(K|k)$ and $\psi\colon \operatorname{Spec} K\to\operatorname{Spec} k$. Then $L\otimes_k K\simeq L^m$ and hence:

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(0),\psi_*\,\mathbb{Q}_K(q)[p]) \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\mathbb{Q}_L(q)[p])^m$$

and

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\psi_*\,\mathbb{Q}_K(0),\phi_*\,\mathbb{Q}_L(q)[p]) \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\mathbb{Q}_L(q)[p])^m.$$

We discuss these vector spaces in more detail in subsection 4.1.4.

Let L be a finite field extension of k and $\phi\colon \operatorname{Spec} L \to \operatorname{Spec} k$.
Using the adjointness of the functors ϕ_* and ϕ^* we see that the $\mathbb{Q}$-vector spaces $\operatorname{Hom}_{\mathrm{DM}(k)_\mathbb{Q}}(\phi_* \mathbb{Q}_L(0), \mathbb{Q}_k(0))$ and $\operatorname{Hom}_{\mathrm{DM}(k)_\mathbb{Q}}(\mathbb{Q}_k(0), \phi_* \mathbb{Q}_L(0))$ are isomorphic to the $\mathbb{Q}$-vector space $\operatorname{Hom}_{\mathrm{DM}(L)_\mathbb{Q}}(\mathbb{Q}_L(0), \mathbb{Q}_L(0)) \simeq \mathbb{Q}$. The first one is the $\mathbb{Q}$-vector space generated by the map $p\colon \phi_* \mathbb{Q}_L(0) \to \mathbb{Q}_k(0)$ in $\mathrm{DM}_{\mathrm{gm}}(k)$ that is induced by the morphism $\phi\colon \operatorname{Spec} L \to \operatorname{Spec} k$, whereas the latter $\mathbb{Q}$-vector space is generated by a map s that is not induced by a morphism of schemes. The composition $p \circ s$ is a non-zero element in $\operatorname{Hom}_{\mathrm{DM}(k)_\mathbb{Q}}(\mathbb{Q}_k(0), \mathbb{Q}_k(0)) = \mathbb{Q} \cdot \mathrm{id}_{\mathbb{Q}_k(0)}$, hence a $\mathbb{Q}$-multiple of $\mathrm{id}_{\mathbb{Q}_k(0)}$ showing that $\mathbb{Q}_k(0)$ is in fact a direct summand of $\phi_* \mathbb{Q}_L(0)$ in $\mathrm{DM}_{\mathrm{gm}}(k)_\mathbb{Q}$.

More generally, we have the following lemma:

Lemma 4.10
Let k be a number field. Let $k \subset K \subset L$ be a tower of finite field extensions of k. We denote the corresponding morphisms of schemes by $\psi\colon \operatorname{Spec} K \to \operatorname{Spec} k$ and $\phi\colon \operatorname{Spec} L \to \operatorname{Spec} K$.

Then $\psi_* \mathbb{Q}_K(n)$ is a direct summand of $(\psi \circ \phi)_* \mathbb{Q}_L(n)$ in $\mathrm{DMAT}(k)$. In particular, $\mathbb{Q}_k(n)$ is direct summand of $\psi_* \mathbb{Q}_K(n)$.

Proof. Since

$$\operatorname{Hom}_{\mathrm{DM}(K)_\mathbb{Q}}(\phi_* \mathbb{Q}_L(n), \mathbb{Q}_K(n)) \cong \operatorname{Hom}_{\mathrm{DM}(L)_\mathbb{Q}}(\mathbb{Q}_L(n), \mathbb{Q}_L(n)) \cong \mathbb{Q}$$

and similarly

$$\operatorname{Hom}_{\mathrm{DM}(K)_\mathbb{Q}}(\mathbb{Q}_K(n), \phi_* \mathbb{Q}_L(n)) \cong \mathbb{Q}$$

as well as

$$\operatorname{Hom}_{\mathrm{DM}(K)_\mathbb{Q}}(\mathbb{Q}_K(n), \mathbb{Q}_K(n)) \simeq \mathbb{Q},$$

there exist morphisms $p\colon \phi_* \mathbb{Q}_L(n) \to \mathbb{Q}_K(n)$ and $s\colon \mathbb{Q}_K(n) \to \phi_* \mathbb{Q}_L(n)$ in $\mathrm{DM}(K)_\mathbb{Q}$ such that $p \circ s = \mathrm{id}_{\mathbb{Q}_K(n)}$. (Choose $p \neq 0$ and $s \neq 0$. Then $p \circ s \neq 0$, so $p \circ s = \lambda \cdot \mathrm{id}_{\mathbb{Q}_K(n)}$ for some λ. Now scale accordingly.) Since $\mathrm{DMAT}(k)$ is pseudo-abelian by [Wil08, Corollary 2.6], this is equivalent to $\mathbb{Q}_K(n)$ being a direct summand of $\phi_* \mathbb{Q}_L(n)$ corresponding to the image of the idempotent endomorphism $s \circ p$ on $\phi_* \mathbb{Q}_L(n)$.
Because ϕ_* is an adjoint functor, it is additive and hence preserves direct summands. Therefore, $\psi_* \mathbb{Q}_K(n)$ is a direct summand of $(\psi \circ \phi)_* \mathbb{Q}_L(n)$ in $\mathrm{DMAT}(k)$. $\square$

4.1.3 t-structure

Now we are able to imitate the definition of the t-structure of the previous chapter to obtain the Tannakian category of mixed Artin-Tate motives over a number field k, as done in [Wil08] and [Sch11].

For any $q \in \mathbb{Z}$ we define the category $W_{\leq q}\,\mathrm{DMAT}(k)$ as the full thick triangulated subcategory of $\mathrm{DMAT}(k)$ generated by the objects $\phi_*\mathbb{Q}_X(-m)$, where $m \leq q$ and $\phi\colon X \to \operatorname{Spec} k$ a finite morphism. Dually, we define $W_{>q}\,\mathrm{DMAT}(k)$ to be the full thick triangulated subcategory generated by the objects $\phi_*\mathbb{Q}_X(-q)$, where $m > q$ and $\phi\colon X \to \operatorname{Spec} k$ a finite morphism. $(W_{\leq q}\,\mathrm{DMAT}(k), W_{>q}\,\mathrm{DMT}(k))$ defines a t-structure on $\mathrm{DMT}(k)$ for every $q \in \mathbb{Z}$ since

$$\mathrm{Hom}_{\mathrm{DMAT}(k)}(\phi_*\mathbb{Q}_X(a)[i], \psi_*\mathbb{Q}_Y(b)[j]) \simeq 0$$

for $b < a$ and two finite maps $\phi\colon \operatorname{Spec} X \to \operatorname{Spec} k$, $\psi\colon Y \to \operatorname{Spec} k$ by Lemma 4.8.

We denote the corresponding truncation functors by

$$W_{\leq q}\colon\ \mathrm{DMAT}(k) \to W_{\leq q}\,\mathrm{DMAT}(k)$$

and

$$W_{>q}\colon\ \mathrm{DMAT}(k) \to W_{>q}\,\mathrm{DMAT}(k).$$

For $a \leq b$ we denote by $W_{[a,b]}\,\mathrm{DMAT}(k)$ the full thick triangulated subcategory generated by the objects $\phi_*\mathbb{Q}_X(-m)$, where $a \leq m \leq b$ and $\phi\colon X \to \operatorname{Spec} k$ a finite morphism. We write gr_a^W for the functor $W_{[a,a]}$ and $\mathrm{gr}_a^W\,\mathrm{DMAT}(k)$ for the category $W_{[a,a]}\,\mathrm{DMAT}(k)$.

Let $\mathrm{DMAT}(k)_q^{\leq 0}$ be the smallest full additive subcategory of $\mathrm{DMAT}(k)$ containing the objects $\phi_*\mathbb{Q}_X(-q)[m]$, where $m \leq 0$ and $\phi\colon X \to \operatorname{Spec} k$ a finite morphism, and that is closed under direct summands and cones. Dually, let $\mathrm{DMAT}(k)_q^{\geq 0}$ be the smallest full subcategory of $\mathrm{DMAT}(k)$ containing the objects $\phi_*\,\mathbb{Q}_X(-q)[m]$, where $m \geq 0$, and that is closed under direct summands and fibres (i.e. if f is a morphism in $\mathrm{DMAT}(k)_q^{\geq 0}$, then $\mathrm{Cone}(f)[-1]$ is in $\mathrm{DMAT}(k)_q^{\geq 0}$).

Definition 4.11
We define $\mathrm{DMAT}(k)^{\leq 0}$ *as the full subcategory of* $\mathrm{DMAT}(k)$ *containing the objects* $M \in \mathrm{DMAT}(k)$ *with* $\mathrm{gr}_q^W M \in \mathrm{DMAT}(k)_q^{\leq 0}$ *and dually the full subcategory* $\mathrm{DMAT}(k)^{\geq 0}$*. Let* $\mathrm{MAT}(k) := \mathrm{DMAT}(k)^{\leq 0} \cap \mathrm{DMAT}(k)^{\geq 0}$*.*

Remark 4.12
Obviously $\mathrm{gr}_0^W \mathrm{DMAT}(k)$ *is equal to the category* $\mathrm{DMA}(k)$ *which is generated by direct summands of the objects* $\phi_* \mathbb{Q}_X(0)$*, where* $\phi\colon X \to \operatorname{Spec} k$ *is a finite morphism. Similarly tensoring with* $\mathbb{Q}_k(q)$ *gives an equivalence between* $\mathrm{gr}_q^W \mathrm{DMAT}(k)$ *and* $\mathrm{DMA}(k) \sim \mathrm{D}^b(\mathrm{MA}(k)) \sim \mathrm{Gr}_{\mathbb{Z}}(\mathrm{MA}(k))$*. This restricts to an equivalence of* $\mathrm{gr}_q^W \mathrm{MAT}(k)$ *and* $\mathrm{MA}(k)$*. Hereby denotes* $\mathrm{Gr}_{\mathbb{Z}}(\mathrm{MA}(k))$ *the* $\mathbb{Z}$*-graded category* $\oplus_{i\in\mathbb{Z}} \mathrm{MA}(k)$*.*

As in the case of Tate-motives (see Theorem 2.29), the full subcategories $\mathrm{DMAT}(k)^{\leq 0}$ and $\mathrm{DMAT}(k)^{\geq 0}$ define a non-degenerate t-structure on $\mathrm{DMAT}(k)$ as proven by Wildeshaus:

Theorem 4.13 ([Wil08, Theorem 3.1])
Let k *be a number field. Then:*

1. $(\mathrm{DMAT}(k)^{\leq 0}, \mathrm{DMAT}(k)^{\geq 0})$ *is a non-degenerate t-structure on* $\mathrm{DMAT}(k)$ *with heart* $\mathrm{MAT}(k)$ *containing the objects* $\phi_* \mathbb{Q}_X(q)$*, where* $q \in \mathbb{Z}$ *and* $\phi\colon X \to \operatorname{Spec} k$ *is a finite morphism.*

2. $\mathrm{MAT}(k)$ *is equal to the smallest abelian subcategory of* $\mathrm{MAT}(k)$ *which contains the objects* $\phi_* \mathbb{Q}_X(q)$*, where* $q \in \mathbb{Z}$*, and* $\phi\colon X \to \operatorname{Spec} k$ *finite, and is closed under extensions in* $\mathrm{MAT}(k)$*.*

3. *The tensor operation in* $\mathrm{DMAT}(k)$ *makes* $\mathrm{MAT}(k)$ *a rigid* $\mathbb{Q}$*-linear abelian tensor category.*

4. *The functor* $\mathrm{gr}_*^W\colon \mathrm{MAT}(k) \to \mathbb{Q}\text{-Vec}$ *which is defined by the composition of* $\oplus_q \mathrm{gr}_q^W\colon \mathrm{MAT}(k) \to \mathrm{Gr}_{\mathbb{Z}}(\mathrm{MA}(k))$ *and the forgetful functor to the category of (graded)* $\mathbb{Q}$*-vector spaces is an exact fibre functor, thus making* $\mathrm{MAT}(k)$ *a Tannakian category which we call the category of mixed Artin-Tate motives over* k*.*

5. *Each object* M *in* $\mathrm{MAT}(k)$ *has a canonical weight filtration by subobjects*
$$0 \subset \ldots \subset W_{q-1}M \subset W_q M \subset \ldots \subset M.$$
This filtration is functorial and exact in M*. It is uniquely characterized by the properties of being finite (i.e.* $W_q M = 0$ *for* q *small and* $W_q M = M$ *for* q *large), and of admitting subquotients* $\mathrm{gr}_q^W M = W_q M / W_{q-1} M \in \mathrm{gr}_q^W \mathrm{MAT}(k)$*,* $q \in \mathbb{Z}$*.*

6. *The natural maps*
$$\mathrm{Ext}^p_{\mathrm{MAT}(k)}(M,N) \to \mathrm{Hom}^p_{\mathrm{DMAT}(k)}(M,N)$$

are isomorphisms, for all p, and all M, $N \in \mathrm{MAT}(k)$. Both sides are zero for $p \geq 2$.

Proof. See [Wil08, Theorem 3.1 and Variant 3.2] for the proof. It uses the facts stated in Lemma 4.8 and the same arguments as in the proof of Theorem 2.29, namely [Lev93, Theorem 1.4]. □

Remark 4.14
A similar construction can be done for a finite field k. Then k is again a perfect field and every finite field extension is separable. Furthermore, the K-theory for finite fields gives a similar result to Lemma 4.8, so the corresponding t-structure with heart $\mathrm{MAT}(k)$ can be constructed. This is used in [Sch11] to define a t-structure on $\mathrm{DMAT}(\mathcal{O}_K)$, where K is a number field.

4.1.4 Fixing a Galois extension L

Throughout this subsection we fix a finite Galois extension L of k of degree n with Galois group $G = \mathrm{Gal}(L|k)$.

The goal of this subsection is to study the full thick triangulated subcategory $\mathrm{DMAT}(L|k)$ of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ that is generated by the objects $\psi_* \mathbb{Q}_K(q)[p]$, where K is any intermediate field extension $k \subset K \subset L$ (not necessarily Galois) and $\psi\colon \operatorname{Spec} K \to \operatorname{Spec} k$. This category is called the triangulated category of Artin-Tate motives over k trivialisable over L by Wildeshaus in [Wil08].

We have seen in Lemma 4.10 that $\psi_* \mathbb{Q}_K(q)[p]$ is a direct summand of $\phi_* \mathbb{Q}_L(q)[p]$, hence we can define $\mathrm{DMAT}(L|k)$ in the following way:

Definition 4.15
Let k be a number field. Let L be a finite Galois extension of k. We define the category $\mathrm{DMAT}(L|k)$ to be the full thick triangulated tensor subcategory of $\mathrm{DMAT}(k)$ generated by the objects $\phi_* \mathbb{Q}_L(q)$, where $q \in Z$ and $\phi\colon \operatorname{Spec} L \to \operatorname{Spec} k$.
Similarly, we define $\mathrm{MAT}(L|k)$ to be the full thick tensor subcategory of $\mathrm{MAT}(k)$ generated by the objects $\phi_* \mathbb{Q}_L(q)$, where $q \in \mathbb{Z}$ and $\phi\colon \operatorname{Spec} L \to \operatorname{Spec} k$.

This was inspired by Deligne's and Goncharov's definition of Artin-Tate motives in [DG05]. The major advantage of this approach is that it suffices to understand the morphisms $\phi_* \mathbb{Q}_L(0) \to \phi_* \mathbb{Q}_L(q)[p]$ since these already determine the morphisms between their direct summands, e.g. $\psi_* \mathbb{Q}_K(q)[p]$ for any intermediate field extension K. Limiting ourselves to Galois extensions L of k allows us to easily describe the morphisms $\phi_* \mathbb{Q}_L(0) \to \phi_* \mathbb{Q}_L(q)[p]$ in terms of morphisms $\mathbb{Q}_L(0) \to \mathbb{Q}_L(q)[p]$ in $\mathrm{DMT}(L)$ and the Galois group G as we have seen in Example 4.9.

Lemma 4.16
The category $\mathrm{DMAT}(L|k)$ *is exactly the full thick triangulated subcategory (without taking tensor products) of* $\mathrm{DMAT}(K)$ *generated by* $\phi_* \mathbb{Q}_L(q)$*, where* $\phi\colon \operatorname{Spec} L \to \operatorname{Spec} k$ *and* $q \in \mathbb{Z}$*.*

Proof. We have to show that the tensor products of any two objects in the full thick triangulated subcategory generated $\phi_* \mathbb{Q}_L(q)$ is still in this subcategory. We denote this category by $\mathcal{C}$.
For the generators we have:

$$\phi_* \mathbb{Q}_L(q) \otimes \phi_* \mathbb{Q}_L(p) \simeq \oplus_{i=1}^{n} \phi_* \mathbb{Q}_L(q+p) \in \mathcal{C}.$$

Similarly, tensor products of direct summands or direct sums of the generators are in $\mathcal{C}$. Let $f\colon A \to B$ be a map in $\mathcal{C}$ such that $A \otimes C$ and $B \otimes C$ are in $\mathcal{C}$ for some $C \in \mathcal{C}$. Then:

$$\mathrm{Cone}(f\colon A \to B) \otimes C \simeq \mathrm{Cone}(f \otimes \mathrm{id}_C\colon A \otimes C \to B \otimes C) \in \mathcal{C}.$$

It remains to show that the tensor product of two mapping cones $\mathrm{Cone}(f\colon A \to B) \otimes \mathrm{Cone}(g\colon C \to D)$ is in $\mathcal{C}$ if the tensor products $A \otimes C$, $A \otimes D$, $B \otimes C$ and $B \otimes D$ are in $\mathcal{C}$. Then:

$$\begin{aligned} & \mathrm{Cone}(f\colon A \to B) \otimes \mathrm{Cone}(g\colon C \to D) \\ \simeq\ & \mathrm{Cone}(\mathrm{id}_{\mathrm{Cone}(f)} \otimes g\colon\ \mathrm{Cone}(f) \otimes C \to \mathrm{Cone}(f) \otimes D) \\ \simeq\ & \mathrm{Cone}(\mathrm{Cone}(f \otimes \mathrm{id}_C) \to \mathrm{Cone}(f \otimes \mathrm{id}_D)) \in \mathcal{C} \end{aligned}$$

since all expressions are given as

$$(A \otimes C)[2] \oplus (B \otimes C)[1] \oplus (A \otimes D)[1] \oplus (B \otimes D)$$

and carry the same differential. This concludes the proof. $\square$

Remark 4.17
If we denote by $\mathrm{DMAT}(k)_L$ *the full triangulated tensor subcategory of* $\mathrm{DM}_{\mathrm{gm}}(k)$ *generated by the objects* $\phi_* \mathbb{Q}_L(q)$ *(without taking direct summands),* $\mathrm{DMAT}(L|k)$ *is the pseudo-abelian hull of* $\mathrm{DMAT}(k)_L$*. This follows from the fact that the category* $\mathrm{DMAT}(k)$ *and hence also* $\mathrm{DMAT}(L|k)$ *is pseudo-abelian and the uniqueness of the pseudo-abelian hull.*

The t-structure $(\mathrm{DMAT}^{\leq 0}(k), \mathrm{DMAT}^{\geq 0}(k))$ on $\mathrm{DMAT}(k)$ restricts to a t-structure on $\mathrm{DMAT}(L|k)$ with heart $\mathrm{MAT}(L|k)$.
Let $\mathrm{DMAT}(L|k)^{\leq 0} := \mathrm{DMAT}(L|k) \cap \mathrm{DMAT}(k)^{\leq 0}$ and $\mathrm{DMAT}(L|k)^{\geq 0} := \mathrm{DMAT}(L|k) \cap \mathrm{DMAT}(k)^{\geq 0}$. It is easy to see from the proof of Theorem 4.13 that $(\mathrm{DMAT}(L|k)^{\leq 0}, \mathrm{DMAT}(L|k)^{\geq 0})$ defines indeed a t-structure on $\mathrm{DMAT}(L|k)$ and its heart $\mathrm{MAT}(L|k)$ is exactly $\mathrm{MAT}(k) \cap \mathrm{DMAT}(L|k)$. Thus, $\mathrm{MAT}(L|k)$ is an abelian category and therefore a Tannakian subcategory of $\mathrm{MAT}(k)$.

For a tower of finite Galois extensions $L|K|k$ we have the evident embedding $\mathrm{DMAT}(K|k) \to \mathrm{DMAT}(L|k)$ that respects the triangulated structure as well as the t-structures.

In particular for $K = k$ we have an embedding of $\mathrm{DMT}(k)$ into $\mathrm{DMAT}(L|k)$ and $\mathrm{DMAT}(k)$ is the full subcategory of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ generated by the subcategories $\mathrm{DMAT}(L|k)$, L a finite Galois extension of k.

As we have seen in Example 4.9 we have a natural isomorphism of $\mathbb{Q}$-vector spaces

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_* \mathbb{Q}_L(0), \phi_* \mathbb{Q}_L(q)[p]) \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p])^n,$$

for all p, $q \in \mathbb{Z}$.

On the other hand, we have the evident map

$$\mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p])[G] \to \mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_* \mathbb{Q}_L(0), \phi_* \mathbb{Q}_L(q)[p]).$$

We claim this map is injective and therefore, for dimension reasons an isomorphism.

First, we consider the case $q = p = 0$. Let $\sigma \in G$. We also denote by σ the corresponding map $\sigma\colon \phi_* \mathbb{Q}_L(0) \to \phi_* \mathbb{Q}_L(0)$. The functor ϕ^* induces a map

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_* \mathbb{Q}_L(0), \phi_* \mathbb{Q}_L(0)) &\to \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\phi^*\phi_* \mathbb{Q}_L(0), \phi^*\phi_* \mathbb{Q}_L(0)) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\oplus_{\tau \in G} \mathbb{Q}_L(0), \oplus_{\tau \in G} \mathbb{Q}_L(0)).\end{aligned}$$

Under this map σ is mapped to the permutation $n \times n$-matrix that is corresponding to $\sigma \in \mathrm{End}(\mathbb{Q}[G]) \simeq \mathrm{End}(\mathbb{Q}^n)$. Since ϕ_* and ϕ^* are adjoint functors, we have a unit $\eta\colon \mathrm{id}_{\mathrm{DM}(L)_{\mathbb{Q}}} \to \phi^*\phi_*$ such that $\eta_{\mathbb{Q}_L(0)}\colon \mathbb{Q}_L(0) \to \oplus_{\tau \in G} \mathbb{Q}_L(0)$ is given by $(\mathrm{id}, 0, \ldots, 0)$. Composition with $\eta_{\mathbb{Q}_L(0)}$ yields a map

$$\mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\oplus_{\tau \in G} \mathbb{Q}_L(0), \oplus_{\tau \in G} \mathbb{Q}_L(0)) \to \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \oplus_{\tau \in G} \mathbb{Q}_L(0))$$

such that

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(0),\phi_*\,\mathbb{Q}_L(0)) &\to \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\phi^*\phi_*\,\mathbb{Q}_L(0),\phi^*\phi_*\,\mathbb{Q}_L(0))\\ &\simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\oplus_{\tau\in G}\,\mathbb{Q}_L(0),\oplus_{\tau\in G}\,\mathbb{Q}_L(0))\\ &\to \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\oplus_{\tau\in G}\,\mathbb{Q}_L(0))\\ &\simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\mathbb{Q}_L(0))^n\end{aligned}$$

is exactly the adjunction isomorphism

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(0),\phi_*\,\mathbb{Q}_L(0)) \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\mathbb{Q}_L(0))^n.$$

It is easy to see that the induced map

$$\mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\mathbb{Q}_L(0))[G] \to \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\oplus_{\tau\in G}\,\mathbb{Q}_L(0))$$

is injective and hence an isomorphism since $\sigma\in G$ is mapped to the morphism $\mathbb{Q}_L(0)\to\oplus_{\tau\in G}\,\mathbb{Q}_L(0)$ with id at the σ-component and 0 elsewhere, i.e. the images $\sigma\in G$ are linearly independent.

The general case for $q,\,p\in\mathbb{Z}$ is shown in a similar way. Again, we consider the map

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(0),\phi_*\,\mathbb{Q}_L(q)[p])\\ &\qquad\to \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\oplus_{\tau\in G}\,\mathbb{Q}_L(0),\oplus_{\tau\in G}\,\mathbb{Q}_L(q)[p])\end{aligned}$$

and note that a morphism $f\colon\ \mathbb{Q}_L(0)\to\mathbb{Q}_L(q)[p]$ that is defined over K is mapped to the diagonal matrix $f\cdot\mathbb{1}_n$. Again, it is easy to see that

$$\mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\mathbb{Q}_L(q)[p])[G] \to \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\oplus_{\tau\in G}\,\mathbb{Q}_L(q)[p])$$

is injective. Thus we have proven:

Lemma 4.18
Let L be a finite Galois extension of a number field k with Galois group $G=\mathrm{Gal}(L|k)$. Then there is a canonical isomorphism:

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(0),\phi_*\,\mathbb{Q}_L(q)[p]) \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\mathbb{Q}_L(q)[p])[G].$$

G acts canonically on $\mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\mathbb{Q}_L(q)[p])[G]$ by multiplication and hence on $\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(0),\phi_*\,\mathbb{Q}_L(q)[p])$ via composition with $\sigma\in G$. More generally, G acts on $\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(0),M)$ for $M\in\mathrm{DM}(k)_{\mathbb{Q}}$ in the same way. Putting $M=\mathbb{Q}_k(q)[p]$ we have the adjunction isomorphism

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(0),\mathbb{Q}_k(q)[p]) \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0),\mathbb{Q}_L(q)[p])$$

and we are interested in the behaviour of the G-action under this isomorphism.

Lemma 4.19
G acts via conjugation on $\mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p])$*. Then the isomorphism*

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_* \mathbb{Q}_L(0), \mathbb{Q}_k(q)[p]) \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p])$$

respects the G-action.
In particular,

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_* \mathbb{Q}_L(0), \mathbb{Q}_k(q)[p])^G \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p])^G.$$

Proof. A morphism $f\colon \phi_* \mathbb{Q}_L(0) \to \mathbb{Q}_k(q)[p]$ is mapped to the composition

$$\widetilde{f}\colon\ \mathbb{Q}_L(0) \xrightarrow{\Delta_L} \mathbb{Q}_L(0) \otimes_k \mathbb{Q}_L(0) \xrightarrow{f\otimes\mathrm{id}} \mathbb{Q}_k(q)[p] \otimes_k \mathbb{Q}_L(0) \simeq \mathbb{Q}_L(q)[p]$$

under the isomorphism

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_* \mathbb{Q}_L(0), \mathbb{Q}_k(q)[p]) \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p]).$$

Now, we claim that G acts on morphisms $g\colon\ \mathbb{Q}_L(0) \to \mathbb{Q}_L(q)[p]$ via conjugation, i.e. $g^\sigma := \sigma \circ g \circ \sigma^{-1}$, where $\sigma\colon\ \mathbb{Q}_L(0) \to \mathbb{Q}_L(0)$ and $\sigma^{-1}\colon\ \mathbb{Q}_k(q)[p] \otimes_k \mathbb{Q}_L(0) \to \mathbb{Q}_k(q)[p] \otimes_k \mathbb{Q}_L(0)$ denotes the morphism $(\mathrm{id} \otimes \sigma^{-1})$. It is easy to see that

$$\widetilde{(f \circ \sigma)} = \widetilde{f}^\sigma$$

proving that g^σ is again a morphism $\mathbb{Q}_L(0) \to \mathbb{Q}_L(q)[p]$ in $\mathrm{DM}(L)_{\mathbb{Q}}$. Clearly, $g^{\sigma\circ\tau} = (g^\sigma)^\tau$ and hence this defines indeed a G-action. By construction, the G-action is compatible with the isomorphism

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_* \mathbb{Q}_L(0), \mathbb{Q}_k(q)[p]) \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p]).$$

$\square$

We have seen that for any intermediate field extension $k \subset K \subset L$ the motive $\psi_* \mathbb{Q}_K(0)$, where $\psi\colon \operatorname{Spec} K \to \operatorname{Spec} k$, is a direct summand of $\phi_* \mathbb{Q}_L(0)$. Therefore, $\mathrm{DMAT}(K|k)$ is a subcategory of $\mathrm{DMAT}(L|k)$ and since both are full triangulated tensor subcategories of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$, $\mathrm{DMAT}(K|k)$ is also a full triangulated subcategory of $\mathrm{DMAT}(L|k)$.

Every direct summand of $\mathbb{Q}_L(0)$ can be expressed as the image of an idempotent endomorphism on $\mathbb{Q}_L(0)$. The endomorphism

$$\alpha \in \mathrm{Hom}_{\mathrm{DMAT}(L|k)}(\phi_* \mathbb{Q}_L(0), \phi_* \mathbb{Q}_L(0)) \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \mathbb{Q}_L(0))^n$$

corresponding to $\mathbb{Q}_k(0)$ is given by $\frac{1}{n}\sum_{\sigma\in G}\sigma$. Since

$$\begin{aligned}\mathbb{Q} &\simeq \mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\mathbb{Q}_k(0), \phi_*\,\mathbb{Q}_L(0))\\ &\simeq \{f\circ\alpha\colon f\in \mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(0), \phi_*\,\mathbb{Q}_L(0))\}\\ &\simeq \{f\circ\alpha\colon f\in \mathbb{Q}[G]\},\end{aligned}$$

α can be only of the form $\sum_{\sigma\in G}\lambda\cdot\sigma$ for some $\lambda\in\mathbb{Q}$ (otherwise the latter vector space is not one-dimensional). And the condition that α is idempotent implies that $\lambda = \frac{1}{n}$.

Similarly, let K be any intermediate field $k\subset K\subset L$ with Galois group $H = \mathrm{Gal}(K|k)$ and $\psi\colon \mathrm{Spec}\,K\to\mathrm{Spec}\,k$. We identify $\mathrm{Gal}(L|K)$ with a subgroup $N\subset G$ of order m. Now, $\psi_*\,\mathbb{Q}_K(0)$ is given as the image of the idempotent endomorphism $\alpha_H = \frac{1}{m}\sum_{\sigma\in N}\sigma$. This description of $\mathbb{Q}_k(0)$ allows us to prove the following fact in [MVW06].

Lemma 4.20 ([MVW06, Exercise 1.11])
Let L be a finite Galois extension of k with Galois group $G = \mathrm{Gal}(L|k)$ and $\phi\colon \mathrm{Spec}\,L\to\mathrm{Spec}\,k$. Then:

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\mathbb{Q}_k(0), \mathbb{Q}_k(q)[p]) \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p])^G,$$

where G acts via conjugation.

Proof. The functor $\phi^*\colon \mathrm{DMT}(k)\to\mathrm{DMT}(L)$ induces a map

$$\mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\mathbb{Q}_k(0), \mathbb{Q}_k(q)[p]) \to \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p]).$$

Let

$$\begin{aligned}\alpha &:= \frac{1}{n}\sum_{\sigma\in G}\sigma \in \mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(0), \phi_*\,\mathbb{Q}_L(0)) \text{ and}\\ \beta &:= \frac{1}{n}\sum_{\sigma\in G}\sigma \in \mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_*\,\mathbb{Q}_L(q)[p], \phi_*\,\mathbb{Q}_L(q)[p]).\end{aligned}$$

Let $f = \sum_{\sigma\in G} f_\sigma\circ\sigma\colon \phi_*\,\mathbb{Q}_L(0)\to\phi_*\,\mathbb{Q}_L(q)[p]$, where $f_\sigma\colon \mathbb{Q}_L(0)\to\mathbb{Q}_L(q)[p]$. Then:

$$\begin{aligned}\beta\circ f\circ\alpha &= \frac{1}{n^2}(\sum_\sigma \sigma)\circ(\sum_\sigma f_\sigma\circ\sigma)\circ(\sum_\sigma\sigma)\\ &= \frac{1}{n^2}(\sum_\sigma\sigma)\circ(\sum_\sigma(\sum_\tau f_\tau)\circ\sigma)\\ &= \frac{1}{n^2}\sum_\sigma(\sum_{\nu,\tau} f_\tau^\nu)\circ\sigma.\end{aligned}$$

Clearly, the morphism

$$\sum_{\nu,\tau} f_\tau^\nu\colon\ \mathbb{Q}_L(0) \to \mathbb{Q}_L(q)[p]$$

is G-equivariant and every G-equivariant morphism $\mathbb{Q}_L(0) \to \mathbb{Q}_L(q)[p]$ can be written this way. Therefore:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DMT}(k)}(\mathbb{Q}_k(0)\,\mathbb{Q}_k(q)[p]) &\simeq \{\beta \circ f \circ \alpha,\ f\colon \phi_*\,\mathbb{Q}_L(0) \to \phi_*\,\mathbb{Q}_L(q)[p]\}\\ &\simeq \left\{\frac{1}{n^2}\sum_\sigma (\sum_{\nu,\tau} f_\tau^\nu)\circ\sigma\right\}\\ &\simeq \mathrm{Hom}_{\mathrm{DM}(L)_\mathbb{Q}}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p])^G.\end{aligned}$$

□

4.1.5 Tannaka formalism

We compute the Tannaka group of $\mathrm{MAT}(L|k)$ for a finite Galois extension L of k utilizing the following lemma in [DM82].

Lemma 4.21 ([DM82, Proposition 2.21])
Let $f\colon G \to G'$ be a homomorphism of affine group schemes over k and let ω^f be the corresponding functor $\mathrm{Rep}_\mathbb{Q}(G') \to \mathrm{Rep}_\mathbb{Q}(G)$.

1. *f is faithfully flat if and only if ω^f is fully faithful and every subobject of $\omega^f(X')$, for $X' \in \mathrm{Rep}_\mathbb{Q}(G')$, is isomorphic to the image of a subobject of X'.*

2. *f is a closed immersion if and only if every object of $\mathrm{Rep}_\mathbb{Q}(G)$ is isomorphic to a subquotient of an object of the form $\omega^f(X')$, where $X' \in \mathrm{Rep}_\mathbb{Q}(G')$.*

Theorem 4.22
Let L be a finite Galois extension of k with Galois group $G = \mathrm{Gal}(L|k)$. Then there exists a split exact sequence

$$1 \to \mathrm{G}(\mathrm{MT}(L)) \to \mathrm{G}(\mathrm{MAT}(L|k)) \rightleftarrows \mathrm{Gal}(L|k) \to 1.$$

Before we prove the theorem, we give a lemma used in the proof.

Lemma 4.23
Let L be a finite Galois extension of k. Let $\phi\colon\ \mathrm{Spec}\,L \to \mathrm{Spec}\,k$.

1. *Every* $M \in \mathrm{MT}(L)$ *is a direct summand of* $\phi^*\phi_* M$.
2. *Every* $M \in \mathrm{MAT}(L|k)$*, such that* $\phi^* M$ *is in the additive subcategory of* $\mathrm{MT}(L)$ *that is generated by* $\mathbb{Q}_L(0)$*, is in* $\mathrm{MA}(L|k)$.

Proof. 1. We prove the more general claim that every $M \in \mathrm{DMT}(L)$ is a direct summand of $\phi^*\phi_* M$. For $M = \mathbb{Q}_L(q)$, $q \in \mathbb{Z}$, we have $\phi^*\phi_* \mathbb{Q}_L(q) \simeq \mathbb{Q}_L(q)^{\oplus n}$, where $n = [L : k]$. Now, the claim follows by induction. Let A, B be objects in $\mathrm{DMT}(L)$ such that A and B are direct summands of $\phi^*\phi_* A$ respectively $\phi^*\phi_* B$ and let $f \colon A \to B$. Then it is easy to see that $\mathrm{Cone}(f)$ is a direct summand of $\mathrm{Cone}(\phi^*\phi_* f)$ and that $\mathrm{Cone}(\phi^*\phi_* f) \simeq \phi^*\phi_* \mathrm{Cone}(f)$ since the functors ϕ^* and ϕ_* are triangulated. Therefore, every motive $M \in \mathrm{DMT}(L)$ and is a direct summand of $\phi^*\phi_* M$.

2. Let $M \in \mathrm{MAT}(L|k)$ such that $\phi^* M \in \langle \mathbb{Q}_L(0) \rangle$, i.e. in the additive subcategory of $\mathrm{MT}(L)$ that is generated by $\mathbb{Q}_L(0)$. We claim that $M \in \mathrm{MAT}(L|k)$ is a direct summand of $\phi_*\phi^* M$. By the same arguments as in the proof of the first claim, it is enough to check this for the objects $\phi_* \mathbb{Q}_L(q)$, $q \in \mathbb{Z}$. Now, $\phi_*\phi^*\phi_* \mathbb{Q}_L(q) \simeq \phi_* \mathbb{Q}_L(q)^{\oplus n}$. Therefore, M is a direct summand of $\phi_*\phi^* M$. On the other hand, $\phi_*\phi^* M$ is by assumption in the subcategory generated by $\phi_* \mathbb{Q}_L(0)$ which equals $\mathrm{MA}(L|k)$. Since $\mathrm{MA}(L|k)$ is closed under direct summands, M is in $\mathrm{MA}(L|k)$.

$\square$

Proof of Theorem 4.22. We use the same arguments as in the proof of [HMS17, Theorem 9.1.16].
Let $\phi \colon \operatorname{Spec} L \to \operatorname{Spec} k$. The map $G(\mathrm{MT}(L)) \to G(\mathrm{MAT}(L|k))$ is induced by the functor $\phi^* \colon \mathrm{DM}(L) \to \mathrm{DM}(k)$. This functor restricts to a functor $\phi^* \colon \mathrm{DMAT}(L|k) \to \mathrm{DMT}(L)$. Note that the functor ϕ^* is triangulated and $\phi^*\phi_* \mathbb{Q}_L(q) \simeq \mathbb{Q}_L(q)^{\oplus n}$ for $q \in \mathbb{Z}$ to see that the image of ϕ^* is actually in $\mathrm{DMT}(L)$. Since the functor ϕ^* is compatible with the weight filtrations and hence the t-structures on $\mathrm{DMAT}(L|k)$ and $\mathrm{DMT}(L)$, it restricts to a functor $\phi^* \colon \mathrm{MAT}(L|k) \to \mathrm{MT}(L)$ which is the dual of the map $G(\mathrm{MT}(L)) \to G(\mathrm{MAT}(L|k))$. By Lemma 4.21, this map is a closed immersion if every motive $M \in \mathrm{MT}(L)$ is a subquotient of an object in the image of ϕ^* which follows by Lemma 4.23.

$G(\mathrm{MAT}(L|k)) \to \mathrm{Gal}(L|k) = G(\mathrm{MA}(L|k))$ is the homomorphism dual to the inclusion functor $i \colon \mathrm{MA}(L|k) \to \mathrm{MAT}(L|k)$. By Lemma 4.21, this map is surjective if the functor i is fully faithful and its image is closed under subquotients. Clearly, i is fully faithful and $\mathrm{MA}(L|k)$ is semi-simple by Remark 4.5, hence the second condition is satisfied.

Furthermore, the functor $\oplus_q \mathrm{gr}_q^W\colon \mathrm{MAT}(L|k) \to \mathrm{MA}(L|k)$ defines a right inverse to the inclusion $\mathrm{MA}(L|k) \to \mathrm{MAT}(L|k)$ and therefore, induces a splitting to the homomorphism $G(\mathrm{MAT}(L|k)) \to \mathrm{Gal}(L|k)$.

Lastly, we have to show that the sequence is exact at $\mathrm{G}(\mathrm{MAT}(L|k))$. We claim that $\mathrm{Gal}(L|k)$ is the cokernel of the the map $\mathrm{G}(\mathrm{MT}(L)) \to \mathrm{G}(\mathrm{MAT}(L|k))$. We consider the dual map $\phi^*\colon \mathrm{MAT}(L|k) \to \mathrm{MT}(L)$. Let $\mathcal{A}$ be the biggest full Tannakian subcategory of $\mathrm{MT}(L)$ containing the objects M such that $\eta_M = \mathrm{id}_M$ for all $\eta \in \mathrm{G}(\mathrm{MT}(L))$. Furthermore, we denote by $\mathcal{B}$ the biggest full Tannakian subcategory of $\mathrm{MAT}(L|k)$ such that the image of $\mathcal{B}$ under ϕ^* is in $\mathcal{A}$. By Tannaka duality, $\mathrm{G}(\mathcal{B})$ is the cokernel of the map $\mathrm{G}(\mathrm{MT}(L)) \to \mathrm{G}(\mathrm{MAT}(L|k))$. Hence, we need to show that $\mathcal{D} \sim \mathrm{MA}(L|k)$. In Lemma 2.30 we have computed the Tannaka group $\mathrm{G}(\mathrm{MT}(L))$. The proof of Lemma 4.21 shows that $\mathcal{C} = \mathrm{gr}_0 \mathrm{MT}(L) = \langle \mathbb{Q}_L(0)\rangle$. Therefore, $\mathcal{D} = \{M \in \mathrm{MAT}(L|k)\colon \phi^*M \in \langle \mathbb{Q}_L(0)\rangle\}$ and we claim that $\mathcal{D} = \mathrm{MA}(L|k)$. The inclusion $\mathrm{MA}(L|k) \hookrightarrow \mathcal{D}$ is obvious since $\phi^*\phi_*\,\mathbb{Q}_L(0) \simeq \mathbb{Q}_L(0)^{\oplus n} \in \langle \mathbb{Q}_L(0)\rangle$. For the converse, we need to show that every motive $M \in \mathrm{MAT}(L|k)$ such that $\phi^*M \in \langle \mathbb{Q}_L(0)\rangle$ is in $\mathrm{MA}(L|k)$. This follows by Lemma 4.23.

This completes the proof. $\square$

Theorem 4.24
Let k be a number field and let $\bar{k}$ denote its algebraic closure. Then there exists a split exact sequence

$$1 \to \mathrm{G}(\mathrm{MT}(\bar{k})) \to \mathrm{G}(\mathrm{MAT}(k)) \to \mathrm{Gal}(\bar{k}|k) \to 1.$$

Proof. The map $\mathrm{G}(\mathrm{MT}(\bar{k})) \to G(\mathrm{MAT}(k))$ is again induced by the functor $\phi^*\colon \mathrm{MAT}(k) \to \mathrm{MT}(\bar{k})$, where $\phi\colon \operatorname{Spec}\bar{k} \to \operatorname{Spec} k$. Note that while ϕ itself is not finite, the functor ϕ^* factors for any motive $\psi_*\,\mathbb{Q}_L(n)$ through ψ^* and therefore, $\phi^*\psi_*\,\mathbb{Q}_L(q) \simeq \mathbb{Q}_{\bar{k}}(q)^{\oplus n}$ for some finite number n. Then the statement follows by the same arguments as in the proof of Theorem 4.22. $\square$

4.2 Cell modules with Galois action

As we have seen in the previous chapter, the derived category $\mathcal{D}^f_{\mathcal{N}(k)}$ of Adams graded modules over the cycle algebra $\mathcal{N}(k)$ is equivalent to the triangulated category of Tate motives $\mathrm{DMT}(k)$ over k. The goal of this section is to construct a triangulated category $\mathcal{D}(k)$ of cell modules containing $\mathcal{D}^f_{\mathcal{N}(k)}$ that is equivalent to the triangulated category of Artin-Tate motives $\mathrm{DMAT}(k)$ over k and that restricts to the equivalence $\mathcal{D}^f_{\mathcal{N}(k)} \to \mathrm{DMT}(k)$ given in Spitzweck's representation theorem (Theorem 3.31).

Furthermore, $\mathcal{D}(k)$ should carry a non-degenerate t-structure whose heart is a Tannakian category that we denote by $\mathcal{A}(k)$. The t-structure should be compatible with the equivalence $\mathcal{D}(k) \to \mathrm{DMAT}(k)$ and therefore, induce an equivalence of Tannakian categories $\mathcal{A}(k) \to \mathrm{MAT}(k)$.

To achieve this goal we construct a triangulated tensor category $\mathcal{D}(L|k)$ of $\mathcal{N}(L)$-cell modules with $\mathrm{Gal}(L|k)$-action for every finite Galois extension L over k that is equivalent to $\mathrm{DMAT}(L|k)$ with the desired properties and such that for a tower of Galois extensions $L|K|k$ there exists an embedding $\mathcal{D}(K|k) \to \mathcal{D}(L|k)$. The union of these categories is our desired category $\mathcal{D}(k)$.

Let L be a finite Galois extension of k of degree n with Galois group $G = \mathrm{Gal}(L|k)$. Let $\phi\colon \operatorname{Spec} L \to \operatorname{Spec} k$.

4.2.1 Idea

Recall that $\mathcal{M}_{\mathcal{N}(L)}$ is the category of Adams graded dg modules over the cycle algebra $\mathcal{N}(L)$ that we defined in section 3.1 and $\mathcal{CM}^f_{\mathcal{N}(L)}$ is the category of finite cell modules over $\mathcal{N}(L)$. Their homotopy categories are denoted by $\mathcal{KM}_{\mathcal{N}(L)}$ and $\mathcal{KCM}^f_{\mathcal{N}(L)}$ respectively.

Recall further that $\mathbb{Q}_{\mathcal{N}(L)}(q)[p]$ is the Tate object in $\mathcal{M}_{\mathcal{N}(L)}$ which is the free rank one $\mathcal{N}(L)$-module with generator b_q having Adams degree $-q$, cohomological degree $-p$ and $db_q = 0$, where d is its differential.

The algebra map $\phi^{\#}\colon \mathcal{N}(k) \to \mathcal{N}(L)$ induces a pair of adjoint functors $\phi^*\colon \mathcal{M}_{\mathcal{N}(k)} \to \mathcal{M}_{\mathcal{N}(L)}$ (extension of scalars) and $\phi_*\colon \mathcal{M}_{\mathcal{N}(L)} \to \mathcal{M}_{\mathcal{N}(k)}$ (restriction of scalars). The functor ϕ_* allows us to consider $\mathcal{N}(L)$-modules as $\mathcal{N}(k)$-modules, i.e. $\phi_* \mathbb{Q}_{\mathcal{N}(L)}(0)$ is just $\mathcal{N}(L)$ considered as an $\mathcal{N}(k)$-module and similarly for $\phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p]$, where $q, p \in \mathbb{Z}$. In the same way as in the case of Artin-Tate motives we can consider the full triangulated subcategory of $\mathcal{KM}_{\mathcal{N}(k)}$ generated by the objects $\phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p]$.

In DMAT$(L|k)$ we have

$$\mathrm{Hom}_{\mathrm{DMAT}(L|k)}(\phi_* \mathbb{Q}_L(0), \phi_* \mathbb{Q}_L(q)[p]) \simeq \mathrm{Hom}_{\mathrm{DMT}(L)}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p])[G].$$

Thus, if we define the full thick triangulated subcategory of $\mathcal{KM}_{\mathcal{N}(k)}$ generated by the objects $\phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p]$, we have to check whether

$$\mathrm{Hom}_{\mathcal{KM}_{\mathcal{N}(k)}}(\phi_* \mathbb{Q}_{\mathcal{N}(L)}(0), \phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p])$$

and

$$\mathrm{Hom}_{\mathcal{KM}_{\mathcal{N}(L)}}(\mathbb{Q}_{\mathcal{N}(L)}(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p])[G] \simeq \mathrm{Hom}_{\mathrm{DMT}(L)}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p])[G]$$

agree. Otherwise, these categories cannot be equivalent.

$G = \mathrm{Gal}(L|k)$ acts canonically on $\mathcal{N}(L)$ by sending a cycle W to its conjugate cycle W^σ for $\sigma \in G$. Therefore, we have a G-action on every free $\mathcal{N}(L)$-module. Note that if M is a dg $\mathcal{N}(L)$-module (or an $\mathcal{N}(L)$-cell module), the action of G does not necessarily commute with the differential on M. However, this is the case for the modules $\mathbb{Q}_{\mathcal{N}(L)}(q)[p]$, q, $p \in \mathbb{Z}$. Thus, every $\sigma \in G$ induces an ($\mathcal{N}(k)$-linear) endomorphism on $\mathbb{Q}_{\mathcal{N}(L)}(q)[p]$, i.e. an endomorphisms on $\phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p]$ in $\mathcal{M}_{\mathcal{N}(k)}$.

Clearly, every $\mathcal{N}(L)$-module morphism $f\colon \mathbb{Q}_{\mathcal{N}(L)}(0) \to \mathbb{Q}_{\mathcal{N}(L)}(q)[p]$ gives an $\mathcal{N}(k)$-module morphism $\phi_* f\colon \phi_* \mathbb{Q}_{\mathcal{N}(L)}(0) \to \phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p]$ that we also denote by f. Composing the endomorphism σ on $\mathbb{Q}_{\mathcal{N}(L)}(q)[p]$ with f defines another $\mathcal{N}(k)$-module morphism

$$f \circ \sigma\colon \phi_* \mathbb{Q}_{\mathcal{N}(L)}(0) \to \phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p].$$

As in the case of motives, this gives an injective map:

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{KM}_{\mathcal{N}(L)}}(\mathbb{Q}_{\mathcal{N}(L)}(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p])[G] \\ &\qquad\qquad \hookrightarrow \mathrm{Hom}_{\mathcal{KM}_{\mathcal{N}(k)}}(\phi_* \mathbb{Q}_{\mathcal{N}(L)}(0), \phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p]).\end{aligned}$$

But the other inclusion is not evident, namely if every $\mathcal{N}(k)$-linear morphism $\phi_* \mathbb{Q}_{\mathcal{N}(L)}(0) \to \phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p]$ is necessarily a $\mathbb{Q}$-linear combination of morphisms of the form

$$f \circ \sigma, \text{ where } \sigma \in G \text{ and } f \in \mathrm{Hom}_{\mathcal{KM}_{\mathcal{N}(L)}}(\phi_* \mathbb{Q}_{\mathcal{N}(L)}(0), \phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p]).$$

Thus, the full triangulated subcategory of $\mathcal{M}_{\mathcal{N}(k)}$ generated by the objects $\phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p]$ might not be equivalent to the respective category $\mathrm{DMAT}(L|k)$ of motives.

An intuitive approach to solve this issue is to consider not the full triangulated subcategory of $\mathcal{KM}_{\mathcal{N}(k)}$ generated by the objects $\phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p]$, $p, q \in \mathbb{Z}$, but the triangulated subcategory generated by the $\mathcal{N}(k)$-modules $\phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p]$ and putting

$$\begin{aligned}
&\mathrm{Hom}(\phi_* \mathbb{Q}_{\mathcal{N}(L)}(0), \phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p]) \\
&\qquad := \mathrm{Hom}_{\mathcal{KM}_{\mathcal{N}(L)}}(\mathbb{Q}_{\mathcal{N}(L)}(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p])[G] \\
&\qquad \subset \mathrm{Hom}_{\mathcal{KM}_{\mathcal{N}(k))}}(\phi_* \mathbb{Q}_{\mathcal{N}(L)}, \phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)[p]).
\end{aligned}$$

However, there is no nice machinery for defining a non-full triangulated subcategory generated by certain objects and morphisms.

To circumvent this we construct a triangulated category of modules containing the category $\mathcal{D}^f_{\mathcal{N}(k)}$ and "base objects" $M_L(q)[p]$, q, $p \in \mathbb{Z}$, such that

$$\mathrm{Hom}(M_L(0), M_L(q)[p]) \simeq \mathrm{Hom}_{\mathcal{KM}_{\mathcal{N}(L)}}(\mathbb{Q}_{\mathcal{N}(L)}(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p])[G]$$

and then consider inside this category the full triangulated subcategory generated by the objects $M_L(q)[p]$.

A candidate for such a category containing $\mathcal{D}^f_{\mathcal{N}(k)}$ arises in the following way. The functor $\phi^* \colon \mathcal{KCM}^f_{\mathcal{N}(k)} \to \mathcal{KCM}^f_{\mathcal{N}(L)}$ identifies $\mathcal{D}^f_{\mathcal{N}(k)} \sim \mathcal{KCM}^f_{\mathcal{N}(k)}$ with a triangulated subcategory of $\mathcal{KCM}^f_{\mathcal{N}(L)}$. Note that this subcategory is not full, but rather by Lemma 4.20 we have

$$\mathrm{Hom}_{\mathcal{KM}_{\mathcal{N}(k)}}(\mathbb{Q}_{\mathcal{N}(k)}(0), \mathbb{Q}_{\mathcal{N}(k)}(q)[p]) \simeq \mathrm{Hom}_{\mathcal{KM}_{\mathcal{N}(L)}}(\mathbb{Q}_{\mathcal{N}(L)}(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p])^G,$$

where the G-action is given by conjugation. On the other hand, $\mathrm{Hom}_{\mathcal{KM}_{\mathcal{N}(L)}}(\mathbb{Q}_{\mathcal{N}(L)}(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p])^G$ is exactly the $\mathbb{Q}$-vector space of G-equivariant morphisms $\mathbb{Q}_{\mathcal{N}(L)}(0) \to \mathbb{Q}_{\mathcal{N}(L)}(q)[p]$ with respect to the natural action of G on $\mathcal{N}(L)$. Therefore, we consider in the following the category of $\mathcal{N}(L)$-cell modules with G-action and G-equivariant morphisms.

4.2.2 Definition

We denote by $G\text{-}\mathcal{CM}^f_{\mathcal{N}(L)}$ the category of finite $\mathcal{N}(L)$-cell modules with G-action. Its objects are finite $\mathcal{N}(L)$-cell modules M together with a bi-degree preserving G-action on M that is compatible with the module structure and the differential on M in the following sense. For all $a \in \mathcal{N}(L)$, $m \in M$ and $\sigma \in G$:

$$\sigma(dm) = d(\sigma m) \quad \text{and} \quad \sigma(a \cdot m) = \sigma(a) \cdot \sigma(m).$$

The morphisms $f\colon M \to N$ in $G\text{-}\mathcal{CM}^f_{\mathcal{N}(L)}$ are exactly the morphisms $f\colon M \to N$ in $\mathcal{CM}^f_{\mathcal{N}(L)}$ that are G-equivariant with respect to the G-action on M and N.

Recall that for M, $N \in \mathcal{M}_{\mathcal{N}(L)}$ we have the Adams graded $\mathcal{N}(L)$-module $\mathcal{H}om_{\mathcal{N}(L)}(M, N)$, where $\mathcal{H}om_{\mathcal{N}(L)}(M, N)^p(q)$ consists of $\mathcal{N}(L)$-linear maps $f\colon M \to N$ (they do not need to be compatible with the differentials d_M and d_N on M and N respectively) such that $f(M^a(b)) \subset N^{a+p}(s+q)$. The differential of $f \in \mathcal{H}om_{\mathcal{N}(L)}(M, N)^p(q)$ is given by

$$df(m) := d_N(f(m)) + (-1)^{p+1} f(d_M(m))$$

for $m \in M$. Note that for p even (especially for $p = 0$) $df = 0$ is equivalent to f commuting with the differentials d_M and d_N.

Now, if M and N are in $G\text{-}\mathcal{CM}^f_{\mathcal{N}(L)}$, then $\mathcal{H}om_{\mathcal{N}(L)}(M, N)$ is also endowed with a G-action that is given by conjugation: $f^\sigma := \sigma^{-1} \circ f \circ \sigma$. It is easy to see that f^σ is $\mathcal{N}(L)$-linear if f is $\mathcal{N}(L)$-linear. The G-equivariant maps are exactly the maps that are invariant under this G-action. Furthermore, the G-action is compatible with the differential on $\mathcal{H}om_{\mathcal{N}(L)}(M, N)$:

$$\begin{aligned} df^\sigma(m) &= d_N(\sigma^{-1} \circ f \circ \sigma(m)) + (-1)^{p+1}\sigma^{-1} \circ f \circ \sigma(d_M(m)) \\ &= \sigma^{-1} \circ d_N(f(\sigma(m))) + (-1)^{p+1}\sigma^{-1} \circ f(d_M(\sigma(m))) \\ &= (\sigma^{-1} \circ df \circ \sigma)(m) = (df)^\sigma(m). \end{aligned}$$

In other words, the differential of a G-equivariant map is again G-equivariant and the G-equivariant maps form a subcomplex of $\mathcal{H}om_{\mathcal{N}(L)}(M, N)$. We denote this subcomplex by $\mathcal{H}om_{\mathcal{N}(L)}(M, N)^G$.

We denote by $G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$ the homotopy category of $G\text{-}\mathcal{CM}^f_{\mathcal{N}(L)}$. Thus, the objects of $G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$ are the objects of $G\text{-}\mathcal{CM}^f_{\mathcal{N}(L)}$ and for M, $N \in G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$ the morphisms are given by

$$\mathrm{Hom}_{G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}}(M, N) := \mathrm{H}^0(\mathcal{H}om_{\mathcal{N}(L)}(M, N)(0)^G).$$

Since the G-action commutes with the differential of $\mathcal{H}om_{\mathcal{N}(L)}(M, N)$, we have for all M, $N \in G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$ the equality

$$\begin{aligned} \mathrm{H}^0(\mathcal{H}om_{\mathcal{N}(L)}(M, N)(0)^G) &\simeq \mathrm{H}^0(\mathcal{H}om_{\mathcal{N}(L)}(M, N)(0))^G \\ &\simeq \mathrm{Hom}_{\mathcal{KM}_{\mathcal{N}(L)}}(M, N)^G. \end{aligned}$$

Let $f\colon M \to N$ be a map in $G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$. Then, the G-actions on M and N induce a natural G-action on $\mathrm{Cone}(f)$ that is defined by $\sigma(m, n) := (\sigma(m), \sigma(n))$ for $m \in M$, $n \in N$. Clearly, this action is compatible with the $\mathcal{N}(L)$-module structure of $\mathrm{Cone}(f)$ since $\mathrm{Cone}(f)$ is just the direct sum $M[1] \oplus N$ as a $\mathcal{N}(L)$-module. Furthermore, the differential of $\mathrm{Cone}(f)$ is compatible with the G-action:

$$\begin{aligned} d(\sigma(m, n)) &= (-d_M(\sigma(m)), f(\sigma(m) + d_N(\sigma(n)))) \\ &= (-\sigma(d_M(m)), \sigma(f(m) + \sigma(d_N(n))) \\ &= \sigma(d(m, n)) \end{aligned}$$

since d_M, d_N and f commute with all $\sigma \in G$. Therefore, $\mathrm{Cone}(f)$ is in $G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$ if $f\colon M \to N$ is G-equivariant. Note that the canonical projection $\mathrm{Cone}(f) \to M[1]$ and the canonical inclusion $N \to \mathrm{Cone}(f)$ are G-equivariant.

Similarly, the tensor product over $\mathcal{N}(L)$ of two $\mathcal{N}(L)$-modules M and N in $G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$ is in $G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$. The G-action on $M \otimes_{\mathcal{N}(L)} N$ is given by $\sigma(m \otimes n) := \sigma(m) \otimes \sigma(n)$ and is again compatible with the differential $d(m \otimes n) := d_M(m) \otimes n + (-1)^{\deg m} m \otimes d_N(n)$ on $M \otimes_{\mathcal{N}(L)} N$:

$$\begin{aligned} d(\sigma(m, n)) &= d_M(\sigma(m)) \otimes \sigma(n) + (-1)^{\deg m} \sigma(m) \otimes d_N(\sigma(n)) \\ &= \sigma(d_M m) \otimes \sigma(n) + (-1)^{\deg m} \sigma(m) \otimes \sigma(d_N n) \\ &= \sigma(d(m \otimes n)). \end{aligned}$$

Proposition 4.25
$G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$ is triangulated category, where the distinguished triangles are those triangles that are isomorphic to a cone sequence.

Proof. We have already seen that $\mathcal{KCM}_{\mathcal{N}(L)}$ is a triangulated category. Therefore, the maps needed to satisfy the axioms (TR1)-(TR2) exist in $\mathcal{KCM}_{\mathcal{N}(L)}$. It remains to show that these maps are G-equivariant if the maps in the assumptions are.

Clearly, (TR1) is satisfied for $G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$ since the maps id and 0 are G-equivariant.

For (TR2) we have to show that if $A \xrightarrow{f} B \to C$ is a distinguished triangle in G-$\mathcal{KCM}^f_{\mathcal{N}(L)}$, then so is $B \to C \to A[1]$ and vice versa. This follows by the fact that $A[1]$ is isomorphic to $\mathrm{Cone}(B \to C)$. The isomorphism is given by $\phi^k = (-f^{k+1}, \mathrm{id}_A^{k+1}, 0)\colon A[1] \to \mathrm{Cone}(B \to C)$ with inverse $\psi^k = (0, \mathrm{id}_A^{k+1}, 0)$ which are G-equivariant maps since the maps f, id_A and id_B are G-equivariant. See [Sos12, Lemma 2.6] for details.

Similarly, one proves (TR3) and (TR4). Again the maps required in (TR3) and (TR4) exist in $\mathcal{KCM}^f_{\mathcal{N}(L)}$ and are G-equivariant if the maps in the assumption are G-equivariant. See the proof of Proposition 1.5 and [Sos12, Theorem 2.7] for details. □

As mentioned before, we can identify $\mathcal{D}^f_{\mathcal{N}(k)}$ with a full subcategory of G-$\mathcal{KCM}^f_{\mathcal{N}(L)}$ via the functor $\phi^*\colon \mathcal{KCM}^f_{\mathcal{N}(k)} \to \mathcal{KCM}^f_{\mathcal{N}(L)}$. To see this, the image of every finite $\mathcal{N}(k)$-cell module M under ϕ^* must be endowed with a natural G-action. For $M = \mathbb{Q}_{\mathcal{N}(k)}(q)$, $q \in \mathbb{Z}$, we have $\phi^* M \simeq \mathbb{Q}_{\mathcal{N}(L)}(q)$ which carries the natural action of G on $\mathcal{N}(L)$. If $f\colon \mathbb{Q}_{\mathcal{N}(k)}(0) \to \mathbb{Q}_{\mathcal{N}(k)}(q)$ is a morphism in $\mathcal{KCM}^f_{\mathcal{N}(k)}$, then $\phi^* f\colon \mathbb{Q}_{\mathcal{N}(L)}(0) \to \mathbb{Q}_{\mathcal{N}(L)}(q)[p]$ is G-equivariant by Lemma 4.20 and the fact that the equivalence $\mathcal{KCM}^f_{\mathcal{N}(L)} \to \mathrm{DMAT}(L)$ respects the G-action. We have already seen that $\mathrm{Cone}(\phi^* f)$ carries a natural G-action induced by the G-actions on $\mathbb{Q}_{\mathcal{N}(L)}(0)$ and $\mathbb{Q}_{\mathcal{N}(L)}(q)[p]$. Since the objects $\mathbb{Q}_{\mathcal{N}(k)}(q)[p]$, q, $p \in \mathbb{Z}$, generate $\mathcal{KCM}^f_{\mathcal{N}(k)}$ as a triangulated category, we have endowed $\phi^* M$ with a G-action induced by the G-action on $\mathcal{N}(L)$ for every object $M \in \mathcal{KCM}^f_{\mathcal{N}(k)}$ giving the following theorem:

Theorem 4.26
The image of every $M \in \mathcal{KCM}^f_{\mathcal{N}(k)}$ under the functor

$$\phi^*\colon \mathcal{KCM}^f_{\mathcal{N}(k)} \to \mathcal{KCM}^f_{\mathcal{N}(L)}$$

can be endowed with a natural G-action yielding a fully faithful tensor triangulated functor

$$\phi^*\colon \mathcal{KCM}^f_{\mathcal{N}(k)} \to G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$$

identifying $\mathcal{KCM}^f_{\mathcal{N}(k)}$ with a full triangulated subcategory of G-$\mathcal{KCM}^f_{\mathcal{N}(L)}$.

Proof. We have already described the G-action on $\phi^* M$ for every $M \in \mathcal{KCM}^f_{\mathcal{N}(k)}$. To obtain a functor $\phi^*\colon \mathcal{KCM}^f_{\mathcal{N}(k)} \to G$-$\mathcal{KCM}^f_{\mathcal{N}(L)}$, we need to check that $\phi^* f\colon \phi^* M \to \phi^* N$ is G-equivariant for every morphism

$f\colon M \to N$ in $\mathcal{KCM}^f_{\mathcal{N}(k)}$. For $M = \mathbb{Q}_{\mathcal{N}(k)}(0)$ and $N = \mathbb{Q}_{\mathcal{N}(k)}(q)[p]$, this is true by Lemma 4.20. For arbitrary M and N, the $\mathcal{N}(L)$-modules with Galois action $\phi^* M$ and $\phi^* N$, where we ignore the differential, are given as direct sums of the base objects $\mathbb{Q}_{\mathcal{N}(L)}(q)[p] \simeq \phi^* \mathbb{Q}_{\mathcal{N}(k)}(q)[p]$ and morphisms $\phi^* f\colon \phi^* M \to \phi^* N$ are given as sums of G-equivariant maps $\mathbb{Q}_{\mathcal{N}(L)}(0) \to \mathbb{Q}_{\mathcal{N}(L)}(q)[p]$ and hence also G-equivariant.

Therefore, ϕ^* defines indeed a functor

$$\phi^*\colon \mathcal{KCM}^f_{\mathcal{N}(k)} \to G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$$

that induces a bijection

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(k)}}(\mathbb{Q}_{\mathcal{N}(k)}(0), \mathbb{Q}_{\mathcal{N}(k)}(q)[p]) \\ &\qquad\simeq \mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(L)}}(\mathbb{Q}_{\mathcal{N}(L)}(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p])^G. \end{aligned}$$

Since the objects $\mathbb{Q}_{\mathcal{N}(k)}(q)[p]$, $q, p \in \mathbb{Z}$ generate $\mathcal{KCM}^f_{\mathcal{N}(k)}$ as a triangulated category and the functor ϕ^* is triangulated, ϕ^* gives an equivalence of $\mathcal{KCM}^f_{\mathcal{N}(k)}$ with the full triangulated subcategory of $G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$ generated by the objects $\mathbb{Q}_{\mathcal{N}(L)}(q)[p]$, $q, p \in \mathbb{Z}$. $\square$

Our goal is to define a category that is equivalent to $\mathrm{DMAT}(L|k)$. Inside $G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$ we already identified a full triangulated subcategory that is equivalent to $\mathrm{DMT}(k)$, where the objects $\mathbb{Q}_{\mathcal{N}(L)}(q)$ correspond to the Tate motives $\mathbb{Q}_k(q)$. We still need to find $\mathcal{N}(L)$-modules that correspond to the Artin-Tate motives $\phi_* \mathbb{Q}_L(q)$, i.e. objects $M_L(q) \in G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$, $q \in \mathbb{Z}$, such that

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(L)}}(M_L(0), M_L(q)[p])^G \\ &\qquad\simeq \mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_* \mathbb{Q}_L(0), \phi_* \mathbb{Q}_L(q)[p]) \\ &\qquad\simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p])[G] \\ &\qquad\simeq \mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(L)}}(\mathbb{Q}_{\mathcal{N}(L)}(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p])[G]. \end{aligned}$$

Furthermore, $\mathbb{Q}_{\mathcal{N}(L)}(q)$ should be a direct summand of $M_L(q)$ in $G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$ such that

$$\mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(L)}}(M_L(0), \mathbb{Q}_{\mathcal{N}(L)}(0))^G \simeq \mathbb{Q}$$

and

$$\mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(L)}}(\mathbb{Q}_{\mathcal{N}(L)}(0), M_L(0))^G \simeq \mathbb{Q}$$

mirroring the results in Example 4.9 for motives.

Therefore, $M_L(q)$ should be the direct sum of copies of $\mathbb{Q}_{\mathcal{N}(L)}(q)$ as a $\mathcal{N}(L)$-module, but the G-action on $M(q)$ cannot be the canonical G-action on every direct summands (since in this case there are too many projections $M_L(q) \to \mathbb{Q}_{\mathcal{N}(L)}(q)$).

The $\mathcal{N}(L)$-cell modules $\mathcal{N}(L)[G](q)$, $q \in \mathbb{Z}$, have the desired properties. These are the free rank $n = [L\colon k]$ modules with generators $\tau \in G$ having Adams degree $-q$, cohomological degree 0 and $d(\tau) = 0$. The G-action is defined by

$$\sigma(\sum_{\tau \in G} a_\tau \tau) := \sum_{\tau \in G} \sigma(a_\tau)\sigma \circ \tau.$$

Lemma 4.27
Let k be a number field. Let L be a finite Galois extension of k with Galois group $G = \mathrm{Gal}(L|k)$. Let $M_L(q) := \mathcal{N}(L)[G](q)$. Then:

$$\mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(L)}}(M_L(0), M_L(q)[p])^G \simeq \mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(L)}}(\mathbb{Q}_{\mathcal{N}(L)}(0)\, \mathbb{Q}_{\mathcal{N}(L)}(q)[p])[G].$$

Furthermore, we have:

$$\mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(\mathcal{L})}}(\mathbb{Q}_{\mathcal{N}(L)}(0), M_L(q)[p])^G \simeq \mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(\mathcal{L})}}(\mathbb{Q}_{\mathcal{N}(L)}(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p]),$$
$$\mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(\mathcal{L})}}(M_L(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p])^G \simeq \mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(\mathcal{L})}}(\mathbb{Q}_{\mathcal{N}(L)}(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p]).$$

Proof. For the first claim let $f\colon M_L(0) \to M_L(q)[p]$ be a G-equivariant map. Since f is $\mathcal{N}(L)$-linear, f is determined by its values on $\nu \in G$. Let $f(\nu) =: \sum_{\tau \in G} f_{\tau,\nu}\tau$, where $f_{\tau,\nu} \in \mathcal{N}(L)^p(q)$. Then:

$$f(\sigma\nu) = \sum_\tau f_{\tau,\sigma\nu}\tau.$$

On the other hand,

$$\sigma(f(\nu)) = \sum_\tau \sigma(f_{\tau,\nu})\sigma \circ \tau = \sum_\tau \sigma(f_{\sigma^{-1}\tau,\nu})\tau.$$

Equating coefficients and putting $\nu = \mathrm{id}$ yields the equality

$$f_{\tau,\sigma} = \sigma(f_{\sigma^{-1}\tau,\mathrm{id}})$$

for all $\sigma, \tau \in G$. In other words, f is already determined by $f(\mathrm{id})$, i.e. by giving $f_{\tau,\mathrm{id}} \in \mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(L)}}(\mathbb{Q}_{\mathcal{N}(L)}(0)\, \mathbb{Q}_{\mathcal{N}(L)}(q)[p])$ for all $\tau \in G$.

For the second isomorphism consider a map $f\colon \mathbb{Q}_{\mathcal{N}(L)}(0) \to M_L(q)[p]$. It is determined by $f(1) =: \sum_\tau f_\tau \tau$, where $f_\tau \in \mathcal{N}(L)^p(q)$. If f is G-equivariant, we have

$$\sum_\tau f_\tau \tau = f(1) = \sigma(f(1)) = \sum_\tau \sigma(f_\tau)\sigma \circ \tau = \sum_\tau \sigma(f_{\sigma^{-1}\tau})\tau$$

for all $\sigma \in G$. By equating coefficients, we see that for $\tau = \mathrm{id}$: $\sigma(f_{\sigma^{-1}}) = f_{\mathrm{id}}$ or, equivalently, $f_\sigma = \sigma(f_{\mathrm{id}})$ for all $\sigma \in G$. Therefore, f is determined by $f_{\mathrm{id}} \in \mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(\mathcal{L})}}(\mathbb{Q}_{\mathcal{N}(L)}(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p])$.

For the last isomorphism we write $f_\tau := f(\tau)$ for a G-equivariant map $f\colon M_L(0) \to \mathbb{Q}_{\mathcal{N}(L)}(q)[p]$. Then for all $\sigma, \tau \in G$:

$$\sigma(f_\tau) = \sigma(f(\tau)) = f(\sigma \circ \tau) = f_{\sigma\tau}.$$

For $\tau = \mathrm{id}$ we get $f_\sigma = \sigma(f_{\mathrm{id}})$. Again, f is determined by f_{id} and the claim follows. □

Corollary 4.28
$\mathbb{Q}_{\mathcal{N}(L)}(q)$ is a direct summand of $M_L(q)$ in G-$\mathcal{KCM}^f_{\mathcal{N}(L)}$.

Proof. Without loss of generality we assume $q = 0$. Recall that $n = |G|$. By the proof of the previous lemma, the maps

$$\begin{aligned} s\colon \mathbb{Q}_{\mathcal{N}(L)}(0) \to M_L(q), &\quad a \mapsto \frac{1}{n}\sum_\tau a\tau, \\ p\colon M_L(q) \to \mathbb{Q}_{\mathcal{N}(L)}(q), &\quad \sum_\tau a_\tau \tau \mapsto \sum_\tau a_\tau \end{aligned}$$

are G-equivariant and satisfy $p \circ s = \mathrm{id}_{\mathbb{Q}_{\mathcal{N}(L)}(q)}$. □

We see that the modules $M_L(q) = \mathcal{N}(L)[G](q)$ behave like the Artin-Tate motives $\phi_* \mathbb{Q}_L(q)$ in $\mathrm{DMAT}(L|k)$. Therefore, we consider the full thick triangulated subcategory of G-$\mathcal{KCM}^f_{\mathcal{N}(L)}$ generated by the objects $M_L(q)$, $q \in \mathbb{Z}$, to obtain a category that is equivalent to $\mathrm{DMAT}(L|k)$.

Definition 4.29
Let $\mathcal{D}(L|k)$ be the full thick triangulated subcategory of G-$\mathcal{KCM}^f_{\mathcal{N}(L)}$ generated by the objects $M_L(q) := \mathcal{N}(L)[G](q)$, $q \in \mathbb{Z}$.

If we denote by $\mathcal{D}(k)_L$ the full triangulated subcategory (without taking direct summands) of G-$\mathcal{KCM}^f_{\mathcal{N}(L)}$ generated by the objects $M_L(q)$, $q \in \mathbb{Z}$, then $\mathcal{D}(L|k)$ is given as the pseudo-abelian hull of $\mathcal{D}(k)_L$.

Lemma 4.30
$\mathcal{D}^f_{\mathcal{N}(k)}$ is a full triangulated subcategory of $\mathcal{D}(L|k)$.

Proof. We identify $\mathcal{D}^f_{\mathcal{N}(k)}$ with $\mathcal{KCM}^f_{\mathcal{N}(k)}$. The essential image of the functor

$$\phi^*\colon \mathcal{KCM}^f_{\mathcal{N}(k)} \to G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$$

in Theorem 4.26 is in $\mathcal{D}(L|k)$ since $\phi^*\,\mathbb{Q}_{\mathcal{N}(k)}(q) \simeq \mathbb{Q}_{\mathcal{N}(L)}(q)$, $q \in \mathbb{Z}$, is a direct summand of $M_L(q)$ by Corollary 4.28. □

Now we are able to prove that the categories $\mathcal{D}(L|k)$ and $\mathrm{DMAT}(L|k)$ are indeed equivalent.

Theorem 4.31
Let k be a number field. Let L be a finite Galois extension of k of degree n with Galois group G. Then there is a tensor functor

$$\Phi_{L|k}\colon \mathcal{D}(L|k) \to \mathrm{DM_{gm}}(k)_{\mathbb{Q}}$$

that induces an equivalence of triangulated tensor categories

$$\Phi_{L|k}\colon \mathcal{D}(L|k) \to \mathrm{DMAT}(L|k).$$

Restricted to the full subcategory $\mathcal{D}^f_{\mathcal{N}(k)}$ of $\mathcal{D}(L|k)$, $\Phi_{L|k}$ agrees with the equivalence

$$\Phi_k\colon \mathcal{D}^f_{\mathcal{N}(k)} \to \mathrm{DMT}(k)$$

in Spitzweck's representation theorem 3.31.

Proof. We prove the theorem by giving an equivalence of $\mathcal{D}(k)_L$ and $\mathrm{DMAT}(k)_L$ which then in turn induces an equivalence of the respective pseudo-abelian hulls $\mathcal{D}(L|k) \to \mathrm{DMAT}(L|k)$.

As in the proof of Spitzweck's representation theorem 3.31, we do not consider the category $\mathcal{D}(k)_L$ but instead the equivalent category of $\mathcal{N}(L)$-cell modules in $\mathcal{D}(k)_L$ with the choice of a basis over $\mathcal{N}(L)$. Again, we construct a functor from the full triangulated subcategory $\mathcal{D}(k)_{L,\geq 0}$ of $\mathcal{D}(k)_L$ that is generated by the objects $M_L(q)$, $q \geq 0$, to $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$. $\mathcal{D}(k)_{L,\geq 0}$ consists of the cell modules with Adams-degree concentrated in non-positive degrees. The same arguments as in the proof of Lemma 3.32 show that $\mathcal{D}(k)_L$ is equivalent to the category obtained by inverting the functor $-\otimes M_L(1)$ on the triangulated subcategory $\mathcal{D}(k)_{L,\geq 0}$ of $\mathcal{D}(k)_L$ and a tensor functor $\mathcal{D}(k)_{L,\geq 0} \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ extends canonically to the desired tensor functor $\mathcal{D}(k)_L \to \mathrm{DM_{gm}}(k)$.

Any $M \in \mathcal{D}(k)_L$ is a finite direct sum of copies of $\mathcal{N}(L)[G]$ as a $\mathcal{N}(L)$-module with G-action. Therefore, we can choose elements $m_j \in M$, $j \in J$, such that

$$M \simeq \bigoplus_{j \in J} \mathcal{N}(L)[G] m_j \simeq \bigoplus_{j,\tau} \mathcal{N}(L)(\tau m_j),$$

i.e. the basis of M is given as $\{\tau m_j\}_{j,\tau}$ with differential

$$d(\tau m_j) = \sum_{i,\nu} c_{i,j,\nu,\tau} \cdot \nu m_i$$

and G-action

$$\sigma(\tau m_j) = (\sigma \circ \tau) m_j.$$

The condition $d \circ d = 0$ implies

$$d(c_{k,j,\sigma,\mathrm{id}}) = \sum_{i,\nu} (-1)^{\deg c_{i,j,\nu,\mathrm{id}}} c_{i,j,\nu,\mathrm{id}} c_{k,i,\sigma,\nu}$$

for all k, $j \in J$ and all $\sigma \in G$. The compatibility of the differential and G-action on M yields the equation $\sigma(c_{i,j,\mathrm{id},\tau}) = c_{i,j,\sigma,\sigma\tau}$ for all i, $j \in J$ and all $\sigma, \nu \in G$. Therefore, $\{c_{i,j,\mathrm{id},\tau}\}_{\tau \in G}$ already determines $c_{i,j,\nu,\sigma}$ for all σ, $\nu \in G$.

We put

$$\Phi_{L|k}(M) := \phi_* \left(\bigoplus_{j \in J} \mathcal{N}_L(r_j)[n_j] \mu_j \right),$$

where $\phi\colon \operatorname{Spec} L \to \operatorname{Spec} k$, $\{\mu_j\}$ is a formal basis over $\mathcal{N}(L)$, $-r_j$ is the Adams degree of m_j and $-n_j$ is the cohomological degree of m_j. The differential δ on $\Phi_{L|k}(M)$ is defined by

$$\delta(c \cdot \mu_j) := d_{\mathcal{N}(L)}(c) \cdot \mu_j + (-1)^{\deg c} \sum_{i,\tau} \tau(c) c_{i,j,\mathrm{id},\tau} \cdot \mu_i,$$

where $c \in \mathcal{N}(L)$. Note that δ does not satisfy the Leibniz rule for all $c \in \mathcal{N}(L)$, but for all $c \in \mathcal{N}(k)$ since $\tau(c) = c$ for all $\tau \in G$. By $d \circ d = 0$ and using the compatibility of d and the G-action on M, it follows $\delta \circ \delta = 0$, giving a well-defined object in $\mathrm{DM}^{\mathrm{eff}}(k)_{\mathbb{Q}}$.

If $f\colon M \to N$ is a G-equivariant morphism of $\mathcal{N}(L)$-cell modules with G-action, we choose bases $\{\tau m_j\}$ and $\{\tau n_j\}$ of M respectively N with corresponding bases $\{\mu_j\}$ and $\{\nu_j\}$ of $\Phi_{L|k}(M)$ and $\Phi_{L|k}(N)$ respectively. Let

$$f(\tau m_j) = \sum_{i,\sigma} f_{i,j,\sigma,\tau}(\sigma n_i).$$

The condition to be G-equivariant is equivalent to the equation $f_{i,j,\sigma,\sigma\tau} = \sigma(f_{i,j,\mathrm{id},\tau})$ for all $i, j \in J$ and all $\sigma, \tau \in G$. We put

$$\Phi_{L|k}(f)(c \cdot \mu_j) := \sum_{i,\tau} \tau(c) f_{i,j,\mathrm{id},\tau} \cdot \nu_i$$

for $c \in \mathcal{N}(L)$. Again, note that $\Phi_{L|k}(f)$ is not $\mathcal{N}(L)$-linear, but $\mathcal{N}(k)$-linear, since $\tau(c) = c$ for all $c \in \mathcal{N}(k)$ and all $\tau \in G$.

We omit the (easy but tedious) computation that $\Phi_{L|k}(g \circ f) = \Phi_{L|k}(g) \circ \Phi_{L|k}(f)$ showing that $\Phi_{L|k}$ is indeed a functor.

Since $\Phi_{L|k}(M_L(q)) \simeq \phi_* \mathbb{Q}_L(q)$ for $q \geq 0$, the image of $\Phi_{L|k}$ is in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)_{\mathbb{Q}}$.

From the definitions above it is easy (but again tedious) to deduce that the functor $\Phi_{L|k}$ respects cone sequences and hence distinguished triangles. Furthermore, $\Phi_{L|k}$ commutes with the respective translation functors.

To see that $\Phi_{L|k}$ is a tensor functor, it is enough to show

$$M_L(0) \otimes_{\mathcal{N}(L)} M_L(q) \simeq \oplus_{i=1}^n M_L(q).$$

Clearly, as $\mathcal{N}(L)$-modules both sides are free modules of rank n^2. The set of generators of the left hand side is given by $\{\sigma \otimes \tau\}_{\sigma,\tau}$ and on the right hand side by $\{\tau m_i\}_{\tau,i}$. Writing $G = \{\sigma_1, \ldots, \sigma_n\}$ and sending $(\sigma \otimes \sigma\sigma_i) = \sigma(\mathrm{id} \otimes \sigma_i)$ to $\sigma m_i = \sigma(\mathrm{id}\, m_i)$ defines an $\mathcal{N}(L)$-module isomorphism that is compatible with the respective G-actions. The differentials of all generators is 0, hence it is an isomorphism of $\mathcal{N}(L)$-cell modules with G-action.

Therefore, $\Phi_{L|k}$ induces a triangulated tensor functor

$$\Phi_{L|k} \colon \mathcal{D}(k)_L \to \mathrm{DM}_{\mathrm{gm}}(k).$$

Furthermore, we have

$$\mathrm{Hom}_{\mathcal{KCM}}(M_L(0), M_L(q)[p])^G \simeq \mathrm{Hom}_{\mathrm{DM}(k)_{\mathbb{Q}}}(\phi_* \mathbb{Q}_L(0), \phi_* \mathbb{Q}_L(q)[p])$$

since by Lemma 4.27 and Example 4.9 both agree with

$$\mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(L)}}(\mathbb{Q}_{\mathcal{N}(L)}(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p])[G] \simeq \mathrm{Hom}_{\mathrm{DM}(L)_{\mathbb{Q}}}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p])[G]$$

and $\Phi_{L|k}$ gives a bijection between these $\mathbb{Q}$-vector spaces.

Since the objects $M_L(q)$, $q \in \mathbb{Z}$, generate $\mathcal{D}(k)_L$ as a triangulated category, $\Phi_{L|k}$ is fully faithful. On the other hand, $\mathrm{DMAT}(k)_L$ is generated by the objects $\phi_* \mathbb{Q}_L(q) \simeq \Phi_{L|k}(M_L(q))$ as a full triangulated subcategory of $\mathrm{DM_{gm}}(k)_{\mathbb{Q}}$. Hence, the essential image of the functor is $\mathrm{DMAT}(k)_L$.

Therefore, $\Phi_{L|k}$ is an equivalence $\mathcal{D}(k)_L \to \mathrm{DMAT}(k)_L$ that induces an equivalence of triangulated categories between the pseudo-abelian hulls:

$$\Phi_{L|k} \colon \mathcal{D}(L|k) \to \mathrm{DMAT}(L|k).$$

For the last claim, recall that $\phi^* \mathbb{Q}_{\mathcal{N}(k)}(0) \in \mathcal{D}(L|k)$ is given as the image of the idempotent endomorphism

$$\alpha \colon \sum_\tau a_\tau \tau \mapsto \frac{1}{n} \sum_\sigma (\sum_\tau a_\tau)\sigma$$

on $M_L(q)$. Now,

$$\Phi_{L|k}(\alpha) = \frac{1}{n} \sum_\sigma \sigma$$

on $\phi_* \mathbb{Q}_L(0)$. Therefore, $\Phi_{L|k}(\phi^* \mathbb{Q}_{\mathcal{N}(k)}(0)) \simeq \mathbb{Q}_k(0)$ and since $\Phi_{L|k}$ is triangulated, it identifies $\mathcal{D}^f_{\mathcal{N}(k)}$ with $\mathrm{DMT}(k)$.

Every finite $\mathcal{N}(k)$-cell module $M \simeq \oplus_i \mathcal{N}(L) m_i$ is isomorphic to the direct summand generated by $\{\sum_i \sum_\tau a_i \tau m_i\}$ of $\oplus_i \mathcal{N}(L)[G] m_i$, showing $\Phi_{L|k}(M) \simeq \Phi_k(M)$ for all $M \in \mathcal{D}^f_{\mathcal{N}(k)}$. Identifying $M \in \mathcal{D}^f_{\mathcal{N}(k)}$ with a direct summand makes it easy to see that $\Phi_{L|k}(f) \simeq \Phi_k(f)$ for all morphisms f in $\mathcal{D}^f_{\mathcal{N}(k)}$ and likewise for the differentials. This completes the proof. □

4.2.3 Embeddings

If we are given an intermediate Galois extension $k \subset K \subset L$ with Galois group $H = \mathrm{Gal}(K|k)$, we have seen in section 4.1 that $\mathrm{DMAT}(K|k)$ is a full subcategory of $\mathrm{DMAT}(L|k)$. Now, we prove the same fact for the categories $\mathcal{D}(K|k)$ and $\mathcal{D}(L|k)$. For $K = k$ we have already seen this in Lemma 4.30 and the same arguments show:

Lemma 4.32
Let K be an intermediate Galois extension $k \subset K \subset L$ over k. Then $\mathcal{D}(K|k)$ is a full triangulated tensor subcategory of $\mathcal{D}(L|k)$.

Proof. Let $\varphi \colon \operatorname{Spec} L \to \operatorname{Spec} K$ and $\psi \colon \operatorname{Spec} K \to \operatorname{Spec} k$. We identify $N = \mathrm{Gal}(L|K)$ with a subgroup of G such that $G = N \cdot H$. Let $m := [L \colon K]$.

We apply Theorem 4.26 for $k = K$ and obtain a functor

$$\varphi^*\colon \mathcal{KCM}^f_{\mathcal{N}(K)} \to N\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}.$$

Applying the functor φ^* to $H\text{-}\mathcal{KCM}^f_{\mathcal{N}(K)}$ yields a functor

$$\varphi^*\colon H\text{-}\mathcal{KCM}^f_{\mathcal{N}(K)} \to G\text{-}\mathcal{KCM}^f_{\mathcal{N}(L)}$$

with $\varphi^*(\mathcal{N}(K)[H](q)) \simeq \mathcal{N}(L)[H](q)$, $q \in \mathbb{Z}$. $\mathcal{N}(L)[H](q)$ is a direct summand of $M_L(q) = \mathcal{N}(L)[G](q)$, namely the image of the idempotent endomorphism

$$\alpha\colon \sum_\tau a_\tau \tau \mapsto \frac{1}{m} \sum_{\substack{\nu\in N,\\ \sigma\in H}} \left(\sum_{\tau\in N} a_{\tau\sigma}\right) \nu\sigma$$

on $M_L(q)$. Therefore, we get a triangulated tensor functor

$$\varphi^*\colon \mathcal{D}(K|k) \to \mathcal{D}(L|k).$$

It is fully faithful since

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(K)}}(M_K(0), M_K(q)[p])^H \\
&\qquad\simeq \mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(K)}}(\mathbb{Q}_{\mathcal{N}(K)}(0), \mathbb{Q}_{\mathcal{N}(K)}(q)[p])[H] \\
&\qquad\simeq (\mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(K)}}(\mathbb{Q}_{\mathcal{N}(L)}(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p])[H])^N \\
&\qquad\simeq (\mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(K)}}(\mathcal{N}(L)[H](0), \mathcal{N}(L)[H](q)[p])^H)^N \\
&\qquad\simeq \mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(K)}}(\mathcal{N}(L)[H](0), \mathcal{N}(L)[H](q)[p])^G.
\end{aligned}$$

□

The embeddings $\mathcal{D}(K|k) \to \mathcal{D}(L|k)$ are compatible with the equivalences $\mathcal{D}(L|k) \to \mathrm{DMAT}(L|k)$ as the following lemma shows.

Lemma 4.33
Let $L|K|k$ be a tower of finite Galois extensions over a number field k. Then the equivalences $\Phi_{L|k}\colon \mathcal{D}(L|k) \to \mathrm{DMAT}(L|k)$ and $\Phi_{K|k}\colon \mathcal{D}(K|k) \to \mathrm{DMAT}(K|k)$ given in Theorem 4.31 respect the embeddings $\mathrm{DMAT}(K|k) \to \mathrm{DMAT}(L|k)$ and $\mathcal{D}(K|k) \to \mathcal{D}(L|k)$, i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}(K|k) \arrow[r, "\sim"] \arrow[d] & \mathrm{DMAT}(K|k) \arrow[d] \\
\mathcal{D}(L|k) \arrow[r, "\sim"] & \mathrm{DMAT}(L|k).
\end{tikzcd}$$

Proof. Let $m = [L\colon K]$ and $N = \mathrm{Gal}(L|K) \subset G$. Recall that $\varphi^* M_K(q)$ is given as the image of the idempotent endomorphism

$$\alpha\colon \sum_\tau a_\tau \tau \mapsto \frac{1}{m} \sum_{\substack{\nu \in N,\\ \sigma \in H}} \left(\sum_{\tau \in N} a_{\tau\sigma} \right) \nu\sigma$$

on $M_L(q)$. $\varphi_* \mathbb{Q}_K(q)$ is exactly the image of the idempotent endomorphism $\Phi_{L|k}(\alpha) = \frac{1}{m} \sum_{\sigma \in N} \sigma$ on $\phi_* \mathbb{Q}_L(q)$ in $\mathrm{DMAT}(L|k)$. □

4.2.4 t-structure

We constructed a triangulated tensor category $\mathcal{D}(L|k)$ that is equivalent to $\mathrm{DMAT}(L|k)$. Our next goal is to define a t-structure on $\mathcal{D}(L|k)$ that is preserved under the equivalence yielding an equivalence of the hearts of the respective t-structures.

The equality

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{D}(L|k)}(M_L(0), M_L(q)[p]) \\ &\qquad \simeq \mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(L)}}(M_L(0), M_L(q)[p])^G \\ &\qquad \simeq \mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(L)}}(\mathbb{Q}_{\mathcal{N}(L)}(0), \mathbb{Q}_{\mathcal{N}(L)}(q)[p])[G] \end{aligned}$$

ensures that $\mathrm{Hom}_{\mathcal{D}(L|k)}(M_L(0), M_L(q)[p])$ vanishes if and only if $\mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(L)}}(\mathbb{Q}_L(0), \mathbb{Q}_L(q)[p])$ does. This makes it easy to imitate the construction of the t-structure on $\mathcal{KCM}^f_{\mathcal{N}(L)}$ to define a t-structure on $\mathcal{D}(L|k)$.

For any $m \in \mathbb{Z}$ we define the category $W_{\leq m}\mathcal{D}(L|k)$ as the full thick triangulated subcategory of $\mathcal{D}(L|k)$ generated by the objects $M_L(-q)$, $q \leq m$. Dually, we define $W_{>m}\mathcal{D}(L|k)$ to be the full thick triangulated subcategory generated by $M_L(-q)$, $q > m$. $(W_{\leq m}\mathcal{D}(L|k), W_{>m}\mathcal{D}(L|k))$ defines a t-structure on $\mathcal{D}(L|k)$ for every $m \in \mathbb{Z}$, since

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{D}(L|k)}(M_L(a)[i], M_L(b)[j]) \\ &\qquad \simeq \mathrm{Hom}_{\mathcal{D}(L|k)}(M_L(0)[i], M_L(b-a)[j]) \\ &\qquad \simeq \mathrm{Hom}_{\mathcal{KCM}_{\mathcal{N}(L)}}(\mathbb{Q}_{\mathcal{N}(L)}(0)[i], \mathbb{Q}_{\mathcal{N}(L)}(b-a)[j])[G] \\ &\qquad \simeq 0 \end{aligned}$$

for $b < a$.

We denote the corresponding truncation functors by

$$W_{\leq m} \colon \mathcal{D}(L|k) \to W_{\leq m}\mathcal{D}(L|k)$$

and

$$W_{>m} \colon \mathcal{D}(L|k) \to W_{>m}\mathcal{D}(L|k).$$

For $a \leq b$ we denote by $W_{[a,b]}\mathcal{D}(L|k)$ the full thick triangulated subcategory generated by $M_L(-q)$, $a \leq q \leq b$. We write gr_a^W for the functor $W_{[a,a]}$ and $\mathrm{gr}_a^W \mathcal{D}(L|k)$ for the subcategory $W_{[a,a]}\mathcal{D}(L|k)$.

Since the weight structures W on $\mathrm{DMAT}(L|k)$ and $\mathcal{D}(L|k)$ are defined in the same way using the "base objects" $M_L(q)$ respectively $\phi_* \mathbb{Q}_L(q)$ and since $\Phi_{L|k}(M_L(q)) \simeq \phi_* \mathbb{Q}_L(q)$, the following statement is obvious:

Proposition 4.34
The equivalence $\Phi_{L|k} \colon \mathcal{D}(L|k) \to \mathrm{DMAT}(L|k)$ is compatible with the weight filtrations on $\mathcal{D}(L|k)$ and $\mathrm{DMAT}(L|k)$.

Let $\mathcal{D}(L|k)_q^{\leq 0}$ be the smallest full thick additive subcategory of $\mathcal{D}(L|k)$ containing the objects $M_L(-q)[p]$, where $p \leq 0$, that is closed under cones. Dually, let $\mathcal{D}(L|k)_q^{\geq 0}$ be the smallest full thick additive subcategory of $\mathcal{D}(L|k)$ containing the objects $M_L(-q)[p]$, where $p \geq 0$, and that is closed under fibres.

We define $\mathcal{D}(L|k)^{\leq 0}$ as the full thick subcategory of $\mathcal{D}(L|k)$ consisting of the objects $M \in \mathcal{D}(L|k)$ with $\mathrm{gr}_q^W M \in \mathcal{D}(L|k)_q^{\leq 0}$ and dually the full thick subcategory $\mathcal{D}(L|k)^{\geq 0}$. Let $\mathcal{A}(L|k) := \mathcal{D}(L|k)^{\leq 0} \cap \mathcal{D}(L|k)^{\geq 0}$.

Theorem 4.35
The equivalence $\Phi_{L|k} \colon \mathcal{D}(L|k) \to \mathrm{DMAT}(L|k)$ identifies the full subcategories $\mathcal{D}(L|k)^{\leq 0}$ and $\mathcal{D}(L|k)^{\geq 0}$ of $\mathcal{D}(L|k)$ with the full subcategories $\mathrm{DMAT}(k)^{\leq 0}$ and $\mathrm{DMAT}(k)^{\geq 0}$ respectively of $\mathrm{DMAT}(L|k)$. In particular, the functor $\Phi_{L|k}$ induces an equivalence of categories

$$\mathcal{A}(L|k) \to \mathrm{MAT}(L|k).$$

Proof. This follows by the fact that $\Phi_{L|k}(M_L(q)[p]) = \phi_* \mathbb{Q}_L(q)[p]$ and that the equivalence $\Phi_{L|k} \colon \mathcal{D}(L|k) \to \mathrm{DMAT}(L|k)$ preserves the weight structures by Proposition 4.34. $\square$

As in the case of Artin-Tate motives, $(\mathcal{D}(L|k)^{\leq 0}, \mathcal{D}(L|k)^{\geq 0})$ is a non-degenerate t-structure on $\mathcal{D}(L|k)$. This can be seen by imitating the proof of Theorem 4.13 or using the equivalence $\mathcal{D}(L|k) \to \mathrm{DMAT}(L|k)$. Furthermore, the category $\mathrm{gr}_0^W \mathcal{D}(L|k)$ is equal to the full thick subcategory of $\mathcal{D}(L|k)$ generated by the object $M_L(0)$ which in turn is equivalent to $\mathrm{gr}_0^W \mathrm{DMAT}(L|k)$ by Theorem 4.31 and Theorem 4.35. Therefore, Remark 4.5 gives an equivalence of $\mathrm{gr}_0^W \mathcal{D}(L|k)$ and $\mathrm{D}^b(\mathrm{MA}(L|k))$, the bounded derived category of the abelian category $\mathrm{MA}(L|k)$ of representations of the Galois group $G = \mathrm{Gal}(L|k)$ of L over k in finitely generated $\mathbb{Q}$-vector spaces. The object in $\mathrm{MA}(L|k)$ corresponding to $M_L(0)$ is just the $\mathbb{Q}$-vector space $\mathbb{Q}[G]$ endowed with its natural G-action.

Similarly, tensoring with $\mathbb{Q}_{\mathcal{N}(L)}(q)$, $q \in \mathbb{Z}$, gives an equivalence $\mathrm{gr}_q^W \mathcal{D}(L|k) \sim \mathrm{gr}_0^W \mathcal{D}(L|k) \sim D^b(\mathrm{MA}(L|k))$. This restricts to an equivalence of $\mathrm{gr}_q^W \mathcal{A}(L|k)$ and $\mathrm{MA}(L|k)$.

We summarise our results about the t-structure on $\mathcal{D}(L|k)$ in the following two theorems:

Theorem 4.36
Let k be a number field. Let L be a finite Galois extension of k of degree n with Galois group $G = \mathrm{Gal}(L|k)$. Then:

1. *$(\mathcal{D}(L|k)^{\leq 0}, \mathcal{D}(L|k)^{\geq 0})$ is a non-degenerate t-structure on $\mathcal{D}(L|k)$ with heart $\mathcal{A}(L|k)$ containing all direct summands of the objects $M_L(q)$, $q \in \mathbb{Z}$.*
2. *$\mathcal{A}(L|k)$ is equal to the smallest abelian subcategory of $\mathcal{A}(L|k)$ which contains (the direct summands of) the objects $M_L(q)$, $q \in \mathbb{Z}$, and is closed under extensions in $\mathcal{A}(L|k)$.*
3. *The tensor operation in $\mathcal{D}(L|k)$ makes $\mathcal{A}(L|k)$ a rigid $\mathbb{Q}$-linear abelian tensor category.*
4. *The functor $\mathrm{gr}_*^W : \mathcal{A}(L|k) \to \mathbb{Q}\text{-}\mathrm{Vec}_{\mathbb{Q}}$ which is defined by the composition of $\oplus_q \mathrm{gr}_q^W : \mathcal{A}(L|k) \to \mathrm{D}^b(\mathrm{MA}(L|k))$ and the forgetful functor to the category of (graded) $\mathbb{Q}$-vector spaces is an exact fibre functor making $\mathcal{A}(L|k)$ a Tannakian category.*
5. *Each object M in $\mathcal{A}(L|k)$ has a canonical weight filtration by subobjects*
$$0 \subset \ldots \subset W_{m-1}M \subset W_m M \subset \ldots \subset M.$$
This filtration is functorial and exact in M. It is uniquely characterized by the properties of being finite (i.e. $W_m M = 0$ for m

small and $W_m M = M$ *for* m *large), and of admitting subquotients* $\mathrm{gr}_m^W M = W_m M / W_{m-1} M \in \mathrm{gr}_m^W \mathcal{A}(L|k)$, $m \in \mathbb{Z}$.

6. *The natural maps*

$$\mathrm{Ext}^p_{\mathcal{A}(L|k)}(M, N) \to \mathrm{Hom}^p_{\mathcal{D}(L|k)}(M, N)$$

are isomorphisms, for all p*, and all* M, $N \in \mathcal{A}(L|k)$*. Both sides are zero for* $p \geq 2$.

Proof. This follows from the equivalence $\mathcal{D}(L|k) \sim \mathrm{DMAT}(L|k)$ that is compatible with the respective weight filtrations by Proposition 4.34 and Theorem 4.13 stating the corresponding results for $\mathrm{DMAT}(L|k)$. □

Now, we can rephrase Theorem 4.35 as follows:

Theorem 4.37
The equivalence $\Phi_{L|k}\colon \mathcal{D}(L|k) \to \mathrm{DMAT}(L|k)$ *is compatible with the t-structures* $(\mathcal{D}(L|k)^{\leq 0}, \mathcal{D}(L|k)^{\geq 0})$ *and* $(\mathrm{DMAT}(k)^{\leq 0}, \mathrm{DMAT}(k)^{\geq 0})$ *on* $\mathcal{D}(L|k)$ *and* $\mathrm{DMAT}(L|k)$ *respectively.*
In particular, the functor $\Phi_{L|k}$ *induces an equivalence of Tannakian categories*

$$\mathcal{A}(L|k) \to \mathrm{MAT}(L|k)$$

transforming the fibre functor gr_*^W *on* $\mathcal{A}(L|k)$ *into the fibre functor* gr_*^W *on* $\mathrm{MAT}(L|k)$.

Corollary 4.38
For K *an intermediate Galois extension* $k \subset K \subset L$*, the embedding* $\mathcal{D}(K|k) \to \mathcal{D}(L|k)$ *also preserves the t-structure.*

Proof. This can be seen directly from the definitions or by applying the commutative diagram in Lemma 4.33 and using Theorem 4.37. □

The category $\mathrm{DMAT}(k)$ of Artin-Tate motives over k is given as the full subcategory of $\mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ generated by the categories $\mathrm{DMAT}(L|k)$, L a finite Galois extension of k. On the modules side there is no obvious counterpart for the category $\mathrm{DM}_{\mathrm{gm}}(k)$, i.e. a category that contains the categories $\mathcal{D}(L|k)$ as full triangulated subcategories.

But it is still possible to define a union $\mathcal{D}(k)$ of the categories $\mathcal{D}(L|k)$, $L|k$ a finite Galois extension. The class of objects of $\mathcal{D}(k)$ is the union of the objects of $\mathcal{D}(L|k)$, $L|k$ Galois. Recall that for a tower of Galois extensions

$L|K|k$, $\mathcal{D}(K|k)$ is a full subcategory of $\mathcal{D}(L|k)$ and therefore, this union is not disjoint.

If M and N are two objects in $\mathcal{D}(k)$, we can find two finite Galois extensions L and K of k such that $M \in \mathcal{D}(L|k)$ and $N \in \mathcal{D}(K|k)$. Then there exists another finite Galois extension E of k containing L and K. Therefore, M, $N \in \mathcal{D}(E|k)$. Now, we define

$$\mathrm{Hom}_{\mathcal{D}(k)}(M,N) := \mathrm{Hom}_{\mathcal{D}(E|k)}(M,N).$$

This is independent of the choice of the field E. Indeed, if E' is another finite Galois extension containing L and K, then again there exists a finite Galois extension F containing E and E' and $\mathcal{D}(E|k)$ and $\mathcal{D}(E'|k)$ are full subcategories of $\mathcal{D}(F|k)$ by Lemma 4.32.

Clearly, $\mathcal{D}(L|k)$ is a full subcategory of $\mathcal{D}(k)$ for all Galois extensions L of k. Furthermore, $\mathcal{D}(k)$ is exactly the subcategory of $\mathcal{D}(k)$ generated by the categories $\mathcal{D}(L|k)$, $L|k$ Galois. Since the categories $\mathcal{D}(L|k)$ are triangulated, $\mathcal{D}(k)$ has a natural structure of a triangulated category.

Furthermore, the tensor structure on $\mathcal{D}(L|k)$ induces a tensor product on $\mathcal{D}(k)$ since the tensor structures are compatible with the embeddings $\mathcal{D}(K|k) \to \mathcal{D}(L|k)$, $L|K|k$ a tower of Galois extensions.

By the previous discussion, we have proven the following theorem:

Theorem 4.39
Let k be a number field.
$\mathcal{D}(k)$ is a triangulated tensor category. If L is a finite Galois extension of k, then $\mathcal{D}(L|k)$ is a full triangulated tensor subcategory of $\mathcal{D}(k)$.

The t-structures on the subcategories $\mathcal{D}(L|k)$ induce a t-structure on $\mathcal{D}(k)$:

Let $\mathcal{D}(k)^{\leq 0}$ be the union of the full subcategories $\mathcal{D}(L|k)^{\leq 0}$, $L|k$ finite Galois, of $\mathcal{D}(k)$. Dually, let $\mathcal{D}(k)^{\geq 0}$ be the union of the full subcategories $\mathcal{D}(L|k)^{\geq 0}$ of $\mathcal{D}(k)$. Define $\mathcal{A}(k) := \mathcal{D}(k)^{\leq 0} \cap \mathcal{D}(k)^{\geq 0}$ which equals the union of the Tannakian categories $\mathcal{A}(L|k)$.

Since the embeddings $\mathcal{D}(K|k) \to \mathcal{D}(L|k)$ for $L|K|k$ respect the t-structures by Corollary 4.38, $(\mathcal{D}(k)^{\leq 0}, \mathcal{D}(k)^{\geq 0})$ defines a t-structure on $\mathcal{D}(k)$.

Theorem 4.40
Let k be a number field.

1. *$(\mathcal{D}(k)^{\leq 0}, \mathcal{D}(k)^{\geq 0})$ is a non-degenerate t-structure on $\mathcal{D}(k)$ with heart $\mathcal{A}(k)$ containing all direct summands of the objects $\phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)$, where L is a finite Galois extension of k, $\phi\colon \operatorname{Spec} L \to \operatorname{Spec} k$ and $q \in \mathbb{Z}$.*

2. *$\mathcal{A}(k)$ is equal to the smallest abelian subcategory of $\mathcal{A}(k)$ which contains (the direct summands of) the objects $\phi_* \mathbb{Q}_{\mathcal{N}(L)}(q)$, L a finite Galois extension of k and $q \in \mathbb{Z}$, and is closed under extensions in $\mathcal{A}(k)$.*

3. *The tensor operation in $\mathcal{D}(k)$ makes $\mathcal{A}(k)$ a rigid $\mathbb{Q}$-linear abelian tensor category.*

4. *The functor $\operatorname{gr}^W_*\colon \mathcal{A}(k) \to \mathbb{Q}\text{-}\mathrm{Vec}_{\mathbb{Q}}$ which is defined by the composition of $\oplus_q \operatorname{gr}^W_q\colon \mathcal{A}(k) \to \mathrm{D}^b(\mathrm{MA}(k))$ and the forgetful functor to the category of (graded) $\mathbb{Q}$-vector spaces is an exact fibre functor making $\mathcal{A}(k)$ a Tannakian category.*

5. *Each object M in $\mathcal{A}(k)$ has a canonical weight filtration by subobjects*

$$0 \subset \ldots \subset W_{m-1}M \subset W_m M \subset \ldots \subset M.$$

This filtration is functorial and exact in M. It is uniquely characterized by the properties of being finite (i.e. $W_m M = 0$ for n small and $W_m M = M$ for m large), and of admitting subquotients $\operatorname{gr}^W_m M = W_m M / W_{m-1} M \in \operatorname{gr}^W_m \mathcal{A}(k)$, $m \in \mathbb{Z}$.

6. *The natural maps*

$$\operatorname{Ext}^p_{\mathcal{A}(k)}(M, N) \to \operatorname{Hom}^p_{\mathcal{D}(k)}(M, N)$$

are isomorphisms, for all p, and all M, $N \in \mathcal{A}(k)$. Both sides are zero for $p \geq 2$.

Proof. This follows by Theorem 4.36 that states the corresponding properties for the full triangulated subcategories $\mathcal{D}(L|k)$ of $\mathcal{D}(k)$, where L is a finite Galois extension of k. □

Furthermore, the functors $\Phi_{L|k}\colon \mathcal{D}(L|k) \to \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ in Theorem 4.35 yield a functor $\Phi\colon \mathcal{D}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ that induces the desired equivalence $\mathcal{D}(k) \to \mathrm{DMAT}(k)$:

Theorem 4.41
Let k be a number field. There is a natural exact functor

$$\Phi\colon \mathcal{D}(k) \to \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$$

that induces an equivalence of triangulated tensor categories

$$\Phi\colon \mathcal{D}(k) \to \mathrm{DMAT}(k).$$

Furthermore, the functor Φ *is compatible with the weight filtrations in* $\mathcal{D}(k)$ *and* $\mathrm{DMAT}(k)$ *and yields an equivalence of Tannakian categories*

$$\Phi\colon \mathcal{A}(k) \to \mathrm{MAT}(k).$$

Restricted to the the full subcategory $\mathcal{D}^f_{\mathcal{N}(k)}$ *of* $\mathcal{D}(k)$*, the functor* Φ *agrees with the functor*

$$\Phi_k\colon \mathcal{D}^f_{\mathcal{N}(k)} \to \mathrm{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$$

in Spitzweck's representation theorem 3.31.

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