The F-pure threshold of quasi-homogeneous polynomials

Dissertation
zur Erlangung des Grades
„Doktor der Naturwissenschaften“
am Fachbereich 08 – Physik, Mathematik und Informatik
der Johannes Gutenberg-Universität
in Mainz

vorgelegt von
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Mainz, im August 2017

1. Gutachter:
2. Gutachter:

Datum der mündlichen Prüfung: 27.10.2017

Abstract

To any polynomial $f \in K[x_0, \dots, x_n]$, where K is a field of characteristic $p > 0$, one can attach an invariant called the F-pure threshold, first defined by Takagi and Watanabe. This invariant is the characteristic p analogue of the log-canonical threshold in characteristic zero. The F-pure threshold, which is a rational number, is a quantitative measure of the severity of the singularity of f. Smaller values of the F-pure threshold correspond to a "worse" singularity.
Inspired by the work of Bhatt and Singh we compute the F-pure threshold of quasi-homogeneous polynomials, i.e. polynomials $f \in K[x_0, \dots, x_n]$ which are homogeneous with respect to some $\mathbb{N}$-grading of $K[x_0, \dots, x_n]$. In particular, we consider the case of a Calabi-Yau hypersurface, i.e. a hypersurface given by a quasi-homogeneous polynomial f in $n+1$ variables $x_0, \dots, x_n$ of degree equal to the degree of $x_0 \cdots x_n$. Moreover, we relate the F-pure threshold of $f \in R = K[x_0, \dots, x_n]$ to a numerical invariant of $X = \mathrm{Proj}(R/fR)$, namely the order of vanishing of the so-called Hasse invariant on a certain deformation space of X.
In the second part of this thesis we turn our attention away from the Hasse invariant towards an important invariant in the theory of formal groups. Namely, we give a connection between the F-pure threshold of a polynomial and the height of the corresponding Artin-Mazur formal group. For this, we consider a quasi-homogeneous polynomial $f \in \mathbb{Z}[x_0, \dots, x_n]$ of degree w equal to the degree of $x_0 \cdots x_n$ and denote by X the hypersurface given by $f = 0$. We show that the F-pure threshold of the reduction $f_p \in \mathbb{F}_p[x_0, \dots, x_n]$ is equal to the log-canonical threshold of f if and only if the height of the Artin-Mazur formal group associated to $H^{n-1}(X, \mathbb{G}_{m,X})$ is equal to 1. We also prove that a similar result holds for Fermat hypersurfaces of degree greater than $n+1$. Furthermore, we give examples of weighted Delsarte surfaces which show that other values of the F-pure threshold of a quasi-homogeneous polynomial of degree w cannot be characterized by the height.

Zusammenfassung

Man kann jedem Polynom $f \in K[x_0, \ldots, x_n]$, wobei K ein Körper der Charakteristik $p > 0$ ist, eine Invariante zuordnen, die als F-reine Schwelle bezeichnet wird und die zuerst von Takagi und Watanabe definiert wurde. Diese Invariante ist das Charakteristik p-Analogon der logkanonischen Schwelle in Charakteristik null. Die F-reine Schwelle, die eine rationale Zahl ist, ist ein quantitatives Maß dafür, wie „schlimm" die Singularität von f ist. Kleinere Werte der F-reinen Schwelle entsprechen einer „schlimmeren" Singularität.

Inspiriert durch die Arbeit von Bhatt und Singh berechnen wir die F-reine Schwelle von quasihomogenen Polynomen, das heißt Polynomen $f \in K[x_0, \ldots, x_n]$, die homogen sind bezüglich einer $\mathbb{N}$-Graduierung von $K[x_0, \ldots, x_n]$. Insbesondere betrachten wir den Fall einer Calabi-Yau-Hyperfläche, das heißt einer Hyperfläche, die durch ein quasihomogenes Polynom f in $n+1$ Variablen $x_0, \ldots, x_n$ gegeben ist und deren Grad gleich dem Grad von $x_0 \cdots x_n$ ist. Außerdem stellen wir einen Zusammenhang zwischen der F-reinen Schwelle von $f \in R = K[x_0, \ldots, x_n]$ und einer numerischen Invariante von $X = \mathrm{Proj}(R/fR)$ her, der Verschwindungsordnung der sogenannten Hasseinvariante auf einem bestimmten Deformationsraum von X.

Im zweiten Teil dieser Arbeit lenken wir unsere Aufmerksamkeit von der Hasseinvariante auf eine wichtige Invariante in der Theorie der formalen Gruppen. Wir geben einen Zusammenhang zwischen der F-reinen Schwelle eines Polynoms und der Höhe der entsprechenden Artin-Mazur formalen Gruppe an. Dazu betrachten wir ein quasihomogenes Polynom $f \in \mathbb{Z}[x_0, \ldots, x_n]$ vom Grad w, der gleich dem Grad von $x_0 \cdots x_n$ ist, und bezeichnen mit X die durch $f = 0$ definierte Hyperfläche. Wir zeigen, dass die F-reine Schwelle der Reduktion $f_p \in \mathbb{F}_p[x_0, \ldots, x_n]$ genau dann gleich der logkanonischen Schwelle von f ist, wenn die Höhe der Artin-Mazur formalen Gruppe, die zu $H^{n-1}(X, \mathbb{G}_{m,X})$ assoziiert ist, gleich 1 ist. Wir beweisen auch, dass ein ähnliches Ergebnis für Fermathyperflächen vom Grad größer als $n+1$ gilt. Darüber hinaus geben wir Beispiele für gewichtete Delsartehyperflächen an, die zeigen, dass andere Werte der F-reinen Schwelle eines quasihomogenen Polynoms von Grad w nicht durch die Höhe charakterisiert werden können.

Contents

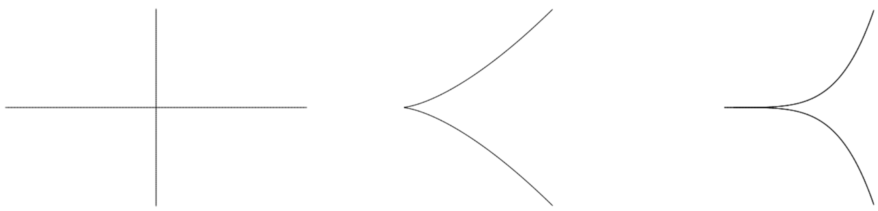
Introduction

Throughout this thesis, we consider a polynomial $f \in K[x_0, \dots, x_n]$ over a field K, vanishing at a point $x \in K^{n+1}$. The polynomial f is called *singular* at x if and only if $\frac{\partial f}{\partial x_i} = 0$ for all $i = 0, \dots, n$. But how "bad" is the singularity of f at x? Can we measure the severity of the singularity of f at x?

The most basic measurement of singularities is probably given by the multiplicity. Since f is called singular at x if all the first order partials vanish at x, it is natural to say that f is "more singular" at x if also all the second order partials vanish and so on. This leads to the definition of the *multiplicity* of f at x given by

$$\mathrm{mult}_x(f) \coloneqq \max\left\{ d \;\middle|\; \text{all partial derivatives of order} < d \text{ vanish at } x \right\}.$$

If one chooses coordinates in a way such that x is the origin, then the multiplicity is given by the degree of the lowest degree term of f.
However, it turns out that the multiplicity is too crude to give a good measurement of singularities. For example, if we consider the following three polynomials



$xy = 0$ $\qquad$ $y^2 - x^3 = 0$ $\qquad$ $y^2 - x^9 = 0$

then one computes that $\mathrm{mult}_0(f) = 2$ in all three cases (if $\mathrm{char}(K) \neq 2$), but the singularity gets "worse" from left to right. Thus, in order to distinguish these singularities, we need another measurement.

In characteristic zero one such measurement is given by the *log-canonical threshold* $\mathrm{lct}(f)$ of a polynomial f. This invariant was first defined analytically via integration. In this context, the log-canonical threshold can be interpreted as one of the numbers in the so-called spectrum of the singularity (see [Ste89]). In the context of birational geometry (where it was first introduced by Shokurov in [Sho93]) the log-canonical threshold can be defined using resolution of singularities. Roughly speaking, the idea is to say that f is "more singular" if the number of blowings-up required to resolve f (and their complicatedness) is big. Moreover, the log-canonical threshold can be interpreted as a critical number for the behavior of certain associated ideals, called the multiplier ideals (see section 1.3), which have emerged as an important tool in algebraic geometry.

In general, it is difficult to compute the log-canonical threshold but since we are mainly working with quasi-homogeneous polynomials (i.e. polynomials which are homogeneous with respect to an $\mathbb{N}$-grading of the polynomial ring), for our purposes only the following example will be interesting:

Example ([HNBWZ16, Theorem 6.2])**.** Let $f \in \mathbb{Q}[x_0, \ldots, x_n]$ be a quasi-homogeneous polynomial of degree d with an isolated singularity (for the precise definitions see section 1.1). Let w be the degree of $x_0 \cdots x_n$. Then,

$$\operatorname{lct}(f) = \begin{cases} \frac{w}{d}, & \text{if } d \geq w \\ 1, & \text{otherwise.} \end{cases}$$

Using this, we can compute the log-canonical thresholds of the three polynomials from above:

(1) Let $f = xy$ with grading $\deg(x) := 1$ and $\deg(y) := 1$, then, $\operatorname{lct}(f) = 1$.

(2) Let $f = y^2 - x^3$ with grading $\deg(x) := 2$ and $\deg(y) := 3$, then, $\operatorname{lct}(f) = \frac{5}{6}$.

(3) Let $f = y^2 - x^9$ with grading $\deg(x) := 2$ and $\deg(y) := 9$, then, $\operatorname{lct}(f) = \frac{11}{18}$.

Some properties of the log-canonical threshold can already be discovered by looking at the above examples; as, for instance, that the log-canonical threshold is a positive rational number, which is bounded above by 1 (see [Laz04b, Example 9.3.16]). Moreover, smaller values of the log-canonical threshold correspond to "worse" singularities.

The analogue of the log-canonical threshold in characteristic $p > 0$ is the *F-pure threshold* $\operatorname{fpt}(f)$ of a polynomial f, first defined by Mustaţă, Takagi and Watanabe in [TW04] and [MTW05]. The main ingredient in the definition of the F-pure threshold is the Frobenius map. Let $R := K[x_0, \ldots, x_n]$ be the polynomial ring over a field K of characteristic $p > 0$ with maximal ideal $\mathfrak{m} := (x_0, \ldots, x_n)$. Then, by

$$\begin{aligned} F : R &\to R, \\ r &\mapsto r^p \end{aligned}$$

we denote the Frobenius map on R. Let $q = p^e$ and denote by $\mathfrak{a}^{[q]} := \big(a^q \bigm| a \in \mathfrak{a}\big)$ the Frobenius power of an ideal $\mathfrak{a} \subset R$. For an element $f \in \mathfrak{m}$ one defines

$$\mu_f(q) \coloneqq \min \left\{ k \in \mathbb{N} \bigm| f^k \in \mathfrak{m}^{[q]} \right\}$$

and observes that $\mu_f(1) = 1$ and $1 \leq \mu_f(q) \leq q$. Furthermore, $\left\{ \frac{\mu_f(p^e)}{p^e} \right\}_{e \geq 0}$ is a non-increasing sequence of positive rational numbers. Hence, one defines:

Definition. *The* F-pure threshold *of f is defined as*

$$\operatorname{fpt}(f) \coloneqq \lim_{e \to \infty} \frac{\mu_f(p^e)}{p^e}.$$

Although, birational geometry in positive characteristic has not yet been developed so far, the F-pure threshold is nevertheless an important invariant. Again, the F-pure threshold is a rational number in $(0,1]$ (see [BMS08, Theorem 3.1]) and similarly as in characteristic zero, the F-pure threshold can be interpreted as a jumping number of the so-called test ideal introduced by Hara and Yoshida in [HY03] (see section 1.2).

Comparing the F-pure threshold of f with the multiplicity of f it turns out that

$$\frac{1}{\operatorname{mult}_0(f)} \leq \operatorname{fpt}(f) \leq \frac{n+1}{\operatorname{mult}_0(f)},$$

i.e. $\operatorname{fpt}(f)$ behaves like $\frac{1}{\operatorname{mult}_0(f)}$. However, we will see that it is a more subtle invariant. For example, if one computes the F-pure threshold of the polynomials we considered at the beginning, one gets that $\operatorname{fpt}(xy) = 1$ for all p but

$$\operatorname{fpt}(y^2 - x^3) = \begin{cases} \frac{1}{2}, & \text{if } p = 2 \\ \frac{2}{3}, & \text{if } p = 3 \\ \frac{5}{6}, & \text{if } p \equiv 1 \bmod 6 \\ \frac{5}{6} - \frac{1}{6p}, & \text{if } p \equiv 5 \bmod 6 \end{cases}$$

i.e. the F-pure threshold of $y^2 - x^3$ is smaller than the one of xy. If one further computes the F-pure threshold of $y^2 - x^9$ then it turns out that

$$\operatorname{fpt}(y^2 - x^9) = \begin{cases} \frac{1}{2}, & \text{if } p = 2 \\ \frac{5}{9}, & \text{if } p = 3 \\ \frac{11}{18}, & \text{if } p \equiv 1 \bmod 18 \\ \frac{11}{18} - \frac{1}{18p}, & \text{if } p \equiv 5 \bmod 18 \\ \frac{11}{18} - \frac{5}{18p}, & \text{if } p \equiv 7 \bmod 18 \\ \frac{11}{18} - \frac{7}{18p^3}, & \text{if } p \equiv 11 \bmod 18 \\ \frac{11}{18} - \frac{5}{18p^2}, & \text{if } p \equiv 13 \bmod 18 \\ \frac{11}{18} - \frac{7}{18p}, & \text{if } p \equiv 17 \bmod 18 \end{cases}$$

and again, the values of the F-pure threshold get smaller. Therefore, the F-pure threshold distinguishes between these types of singularities in the sense that (roughly speaking) smaller values of the F-pure threshold correspond to “worse” singularities.

By looking at the above examples, one can already observe the following: Comparing the log-canonical threshold of a polynomial $f \in \mathbb{Z}[x_0, \dots, x_n]$ with the F-pure threshold of its reduction $f_p \in \mathbb{F}_p[x_0, \dots, x_n]$ it turns out that

$$\mathrm{fpt}(f_p) \leq \mathrm{lct}(f)$$

for $p \gg 0$ and

$$\lim_{p \to \infty} \mathrm{fpt}(f_p) = \mathrm{lct}(f)$$

(see [MTW05, Theorem 3.3 & Theorem 3.4]). Furthermore, it is conjectured that for infinitely many primes p one has $\mathrm{fpt}(f_p) = \mathrm{lct}(f)$ (see [MTW05, Conjecture 3.6]). But this conjecture is wide open and was formulated in many contexts, for example by using log-canonical pairs and F-pure pairs (see section 1.3). Moreover, this conjecture is connected to other open problems in arithmetic geometry, such as the weak ordinarity conjecture of Serre (see [MS11] and [Mus12] or section 1.3).

Of course, there are many other interesting invariants of singularities such as the Milnor number (see [Mil68]), which plays an important role in the topological study of singularities. However, in this thesis we will only deal with the log-canonical threshold and the F-pure threshold.

The F-pure threshold and the Hasse invariant

In general, the F-pure threshold has been computed in many cases. For example, in his dissertation, Hernández computes the F-pure threshold of diagonal and binomial hypersurfaces using base p expansions (see [Her] or section 1.4 for an overview).

More generally, in [BS15] the authors compute the F-pure threshold of a homogeneous polynomial. They first give an upper and a lower bound for the numbers $\mu_f(p)$. Using these bounds, they compute a list of possible values for $\mu_f(p)$. Then they explain how to “lift” this to compute a list of possibe values for $\mu_f(p^e)$. Finally, as a limit they get a list of possible F-pure thresholds. We generalize this procedure to the case of a quasi-homogeneous polynomial. In particular, we consider the case of a Calabi-Yau hypersurface and obtain the following result which generalizes Theorem 4.1 of [BS15]:

Theorem (see Theorem 2.9). *Let $K[x_0, \dots, x_n]$ be the graded polynomial ring with $\alpha_i := \deg(x_i)$. Let $f \in K[x_0, \dots, x_n]$ be a quasi-homogeneous polynomial of degree $w := \sum_{i=0}^{n} \alpha_i$ with an isolated singularity. Then:*

(1) $\mu_f(p) = p - h$, where $0 \leq h \leq n-1$ is an integer.

(2) $\mu_f(pq) = p\mu_f(q)$ for all q with $q \geq n-1$.

(3) If $p \geq n-1$, then $\operatorname{fpt}(f) = 1 - \frac{h}{p}$, where $0 \leq h \leq n-1$.

In particular, in the case of a curve we get:

Theorem (see Theorem 2.12). *Let $C := \operatorname{Proj}(R/fR)$ be the curve given by a quasi-homogeneous polynomial $f \in R := K[x,y,z]$ of degree equal to the degree of xyz with an isolated singularity. Then*

$$\operatorname{fpt}(f) = \begin{cases} 1, & \text{if } C \text{ is ordinary} \\ 1 - \frac{1}{p}, & \text{otherwise.} \end{cases}$$

Here, a curve C is (by definition) *ordinary* if and only if the map on $H^1(C, \mathcal{O}_C)$ induced by Frobenius is bijective. This theorem is a generalization of the two-dimensional case of the main theorem of Bhatt and Singh ([BS15]), which says that the F-pure threshold of an elliptic curve E given by a homogeneous polynomial $f \in K[x,y,z]$ of degree three is 1 if E is ordinary and $1 - \frac{1}{p}$ otherwise. In contrast to the paper of Bhatt and Singh our proof does not rely on deformation-theoretic arguments and hence gives a more elementary approach to this result.

More generally, Bhatt and Singh relate the F-pure threshold of a homogeneous polynomial $f \in R = K[x_0, \dots, x_n]$ to a numerical invariant of $X := \operatorname{Proj}(R/fR)$, namely the order of vanishing of the so-called Hasse invariant on a certain deformation space of X. Again, we generalize this result to the quasi-homogeneous case. For this, we consider the family $\pi : \mathfrak{X} \to \mathrm{Hyp}_w$ of hypersurfaces of degree $w := \deg(x_0 \cdots x_n)$ in the weighted projective space $\mathbb{P}^n(\alpha_0, \dots, \alpha_n)$. Our chosen hypersurface $X = \operatorname{Proj}(R/fR)$ gives a point $[X]$ in Hyp_w.
The *Hasse invariant* H of a suitable family $\pi : \mathfrak{X} \to S$ of varieties in characteristic p is the element in

$$\begin{aligned} \operatorname{Hom}\left(R^N \pi_*^{(1)} \mathcal{O}_{\mathfrak{X}^{(1)}}, R^N \pi_* \mathcal{O}_{\mathfrak{X}}\right) &\cong \operatorname{Hom}\left(\left(R^N \pi_* \mathcal{O}_{\mathfrak{X}}\right)^p, R^N \pi_* \mathcal{O}_{\mathfrak{X}}\right) \\ &\cong \operatorname{Hom}\left(\mathcal{O}_S, \left(R^N \pi_* \mathcal{O}_{\mathfrak{X}}\right)^{1-p}\right) \\ &\cong H^0\left(S, \left(R^N \pi_* \mathcal{O}_{\mathfrak{X}}\right)^{1-p}\right) \end{aligned}$$

induced by the relative Frobenius $\mathrm{Frob}_{\mathfrak{X}/S}$. Here the relative Frobenius is given by the following diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[dr, dashed, "\mathrm{Frob}_{\mathfrak{X}/S}"] \arrow[drr, bend left, "\mathrm{Frob}_{\mathfrak{X}}"] \arrow[ddr, bend right, "\pi"'] & & \\
& \mathfrak{X}^{(1)} \arrow[r] \arrow[d, "\pi^{(1)}"] & \mathfrak{X} \arrow[d, "\pi"] \\
& S \arrow[r, "\mathrm{Frob}_S"] & S.
\end{tikzcd}$$

Now, fix $s \in S$ and an integer $t > 0$ and let $t[s]$ be the order t neighbourhood of s. Then, the order of vanishing of the Hasse invariant at the point $s \in S$ is given by $\mathrm{ord}_s(H) \coloneqq \max\left\{t \mid i^*H = 0 \text{ where } i : t[s] \hookrightarrow S\right\}$. The second main result of this paper, generalizing [BS15] to the quasi-homogeneous case, is the following:

Theorem (see Theorem 2.16)**.** *If $p \geq w(n-2)+1$, then $\mathrm{fpt}(f) = 1 - \frac{h}{p}$, where h is the order of vanishing of the Hasse invariant at $[X] \in \mathrm{Hyp}_w$ on the deformation space $\mathfrak{X}$ of $X \subset \mathbb{P}^n(\alpha_0, \ldots, \alpha_n)$.*

In the homogeneous case this result was proven by Bhatt and Singh in [BS15] by pointing out a connection between the order of vanishing of the Hasse invariant and the injectivity of the map $H^{n-1}(X, \mathcal{O}_X) \xrightarrow{a_t} H^{n-1}(tX, \mathcal{O}_{tX})$ induced by Frob_R, where tX is the order t neighbourhood of X in $\mathbb{P}^n(\alpha_0, \ldots, \alpha_n)$. We generalize this statement to the quasi-homogeneous case using local cohomology instead of sheaf cohomology, i.e. we consider the map a_t as a map $H^n_{\mathfrak{m}}(R/f)_0 \xrightarrow{a_t} H^n_{\mathfrak{m}}\left(R/f^t\right)_0$, which makes our approach rather explicit.

We should also mention the paper [HNBWZ16] of Hernández, Núñez-Betancourt, Witt and Zhang, where the authors compute the possible values of the F-pure threshold of a quasi-homogeneous polynomial of arbitrary degree using base p expansions. In particular, as a corollary they get the same list of possible F-pure thresholds as we obtain here in the case of a Calabi-Yau hypersurface.

The F-pure threshold and the height of quasi-homogeneous polynomials

In the third chapter of this thesis we turn our attention away from the Hasse invariant towards another important invariant. Namely, we give a connection between the F-pure threshold of a quasi-homogeneous polynomial $f \in \mathbb{Z}[x_0, \ldots, x_n]$ and the height of the Artin-Mazur formal group associated to $H^{n-1}(X, \mathbb{G}_{m,X})$, where $X \subset \mathbb{P}^n_{\mathbb{Z}}(\alpha_0, \ldots, \alpha_n)$ is the hypersurface given by f.

In this thesis, a *one-dimensional formal group law* over a commutative ring R with identity is a power series $F(x,y) \in R[\![x,y]\!]$, such that

$$F(x, F(y,z)) = F(F(x,y), z) \text{ and}$$

$$F(x,y) \equiv x + y \bmod \deg \geq 2.$$

It is called commutative, if one has in addition that $F(x,y) = F(y,x)$.
If R is a ring of characteristic zero, then every one-dimensional commutative formal group law $F(x,y)$ over R determines a unique power series $l(\tau)$ with coefficients in $R \otimes \mathbb{Q}$ such that

$$l(\tau) \equiv \tau \bmod \deg \geq 2 \text{ and}$$

$$F(x,y) = l^{-1}\left(l(x) + l(y)\right).$$

The power series $l(\tau)$ is called the *logarithm* of the formal group law $F(x,y)$. One can write

$$l(\tau) = \tau + \sum_{m=2}^{\infty} \frac{b_{m-1}}{m} \tau^m$$

with $b_{m-1} \in R$.
The *height* of a formal group law, which is either infinite or an integer greater or equal to 1, uniquely characterizes one-dimensional formal group laws over an algebraically closed field of positive characteristic by Lazard [Laz55] (see Theorem 3.9).
Let $F(x,y)$ be a one-dimensional formal group law over a field K of characteristic $p > 0$, and let $[p]_F(x)$ be the multiplication by p, i.e.

$$[p]_F(x) = \underbrace{x +_F x +_F \ldots +_F x}_{p \text{ times}},$$

where $x +_F y := F(x,y)$. Then one can show (see [Haz78, section 18.3.1]) that either $[p]_F(x) = 0$ or there is a power $q = p^r$ of p such that $[p]_F(x) = \beta(x^q)$ for some power series β with $\beta(x) \not\equiv 0 \bmod \deg \geq 2$. The height $\mathrm{ht}(F)$ of F is infinite if and only if $[p]_F(x) = 0$ and $\mathrm{ht}(F) = r$ if $q = p^r$ is the highest power of p such that $[p]_F(x) = \beta(x^q)$. If R is a local ring of characteristic zero with residue field K of characteristic $p > 0$ and $F(x,y)$ is a one-dimensional formal group law over R, then we define the height of $F(x,y)$ as the height of the reduction $\overline{F}(x,y)$ of $F(x,y)$ over K.
In the theory of formal groups one can also choose the point of view of functors. For this, let $\mathfrak{Nilalg}_R$ denote the category of nil-R-algebras, i.e. of R-algebras in which every element is nilpotent. The *formal affine* 1*-space* over R is defined as the forgetful functor

$$\mathbb{A}^1_R : \mathfrak{Nilalg}_R \to \mathfrak{Sets},$$

which sends a nil-R-algebra N to the set N and which sends a morphism f to the underlying map f. A *one-dimensional formal group* over R is a functor

$$F : \mathfrak{Nilalg}_R \to \mathfrak{Abelian\ Groups},$$

such that $V \circ F \cong \mathbb{A}^1_R$, where $V : \mathfrak{Abelian\ Groups} \to \mathfrak{Sets}$ is the forgetful functor. We will see that one can associate to a commutative formal group law $F(x,y) \in R[\![x,y]\!]$ a functor $F : \mathfrak{Nilalg}_R \to \mathfrak{Abelian\ Groups}$, where the group structure is given by the power series F. Conversely, given a functor $F : \mathfrak{Nilalg}_R \to \mathfrak{Abelian\ Groups}$, then F is defined by a formal group law (see section 3.1.2).
Now, consider the one-dimensional multiplicative formal group law $\mathbb{G}_m$, which is given by $\mathbb{G}_m(x,y) := x + y + xy$ and where the logarithm is

$$l(\tau) = \log(1+\tau) = \sum_{n\geq 1} (-1)^{n+1} \frac{1}{n} \tau^n.$$

As a functor, the one-dimensional multiplicative formal group law is the following:

$$\mathbb{G}_m(N) = (1+N)^{\times},$$

where $(1+N)^{\times}$ is the set of all formal sums $1+u$, $u \in N$, with the multiplication given by $(1+u)(1+v) = 1+u+v+uv$.
If X is a scheme over R and $i \in \mathbb{N}_0$, then one can construct the following diagram:

$$\begin{tikzcd}
\mathfrak{Nilalg}_R \arrow[r, "\mathcal{O}_X \otimes_R -"] \arrow[dr, "\mathbb{G}_{m,\mathcal{O}_X}"] \arrow[ddr, "H^i(X{,}\mathbb{G}_{m,\mathcal{O}_X})"'] & \mathfrak{Sheaves\ of\ nil\text{-}}\mathrm{R}\mathfrak{\text{-}algebras\ on}\ X \arrow[d, "\mathbb{G}_m"] \\
& \mathfrak{Sheaves\ of\ abelian\ groups\ on}\ X \arrow[d, "H^i"] \\
& \mathfrak{Abelian\ Groups}
\end{tikzcd}$$

Here $\mathcal{O}_X \otimes_R -$ assigns to a nil-R-algebra A the sheaf $\mathcal{O}_X \otimes_R A$ associated with the pre-sheaf $U \mapsto \Gamma(U, \mathcal{O}_X) \otimes_R A$ for U open. The functor $\mathbb{G}_m$ assigns to a sheaf $\mathfrak{a}$ of nil-R-algebras on X the sheaf of abelian groups $\mathbb{G}_m(\mathfrak{a})$ defined by $\Gamma(U, \mathbb{G}_m(\mathfrak{a})) = \mathbb{G}_m(\Gamma(U, \mathfrak{a}))$ for $U \subset X$ open. The functor H^i is taking i-th cohomology and the functors $\mathbb{G}_{m,\mathcal{O}_X}$ and $H^i(X, \mathbb{G}_{m,\mathcal{O}_X})$ are defined by the commutativity of the above diagram. Writing $\mathbb{G}_{m,X}$ instead of $\mathbb{G}_{m,\mathcal{O}_X}$, the functors

$$H^i(X, \mathbb{G}_{m,X}) : \mathfrak{Nilalg}_R \to \mathfrak{Abelian\ Groups}$$

are called *Artin-Mazur functors.* These functors are not necessarily formal groups, but we will use a criterion of Stienstra (see Theorem 3.5) to show that it is a formal group in all cases that will be considered in this thesis. The main theorem of the first part of chapter 3 is the following:

Theorem (see Theorem 3.11). *Let $\mathbb{Z}[x_0, \ldots, x_n]$ be the graded polynomial ring with $\alpha_i \coloneqq \deg(x_i)$ and set $w \coloneqq \alpha_0 + \ldots + \alpha_n$. Let $f \in \mathbb{Z}[x_0, \ldots, x_n]$ be a quasi-homogeneous polynomial of degree w and type $\alpha \coloneqq (\alpha_0, \ldots, \alpha_n)$ with an isolated singularity such that the greatest common divisor of all coefficients of f is 1. Furthermore, let X be the hypersurface in $\mathbb{P}^n_{\mathbb{Z}}(\alpha)$ defined by f and let $f_p \in \mathbb{F}_p[x_0, \ldots, x_n]$ be the reduction of f modulo p. Assume that $p \geq w(n-2)+1$. Then $\operatorname{fpt}(f_p) = 1 = \operatorname{lct}(f)$ if and only if $\operatorname{ht}\left(H^{n-1}\left(X, \mathbb{G}_{m,X}\right)\right) = 1$.*

Furthermore, we show that a similar result holds for Fermat hypersurfaces of degree $> n+1$:

Proposition (see Example 3.14 & Corollary 3.17). *Let $f = x_0^d + \ldots + x_n^d \in \mathbb{Z}[x_0, \ldots, x_n]$ with $d = n+k$ for $k \geq 2$ and such that $n \geq 2(k-1)$. Furthermore, let $d \not\equiv 0 \bmod p$. Then $H^{n-1}\left(X, \mathbb{G}_{m,X}\right)$ is a direct sum of formal groups of dimension 1 and these formal groups are all of height 1 if and only if $\operatorname{fpt}(f_p) = \operatorname{lct}(f) = \frac{n+1}{d}$.*

We will see that the above statements imply that the F-pure threshold is equal to the log-canonical threshold if and only if the height of the corresponding Artin-Mazur formal group is equal to its dimension. Since $\operatorname{fpt}(f_p) \leq \operatorname{lct}(f)$ for all $p \gg 0$, this means that the F-pure threshold is equal to its greatest possible value if and only if the height is equal to its smallest possible value. We suspect that this could hold more generally for quasi-homogeneous polynomials. All computations of the height and the F-pure threshold in concrete examples support this.

The second half of chapter 3 is dedicated to the following. Let $R \coloneqq K[x_0, \ldots, x_n]$ be the graded polynomial ring with $\alpha_i \coloneqq \deg(x_i)$ over an algebraically closed field K of characteristic $p > 0$. Let $f \in R$ be a quasi-homogeneous polynomial of degree $w \coloneqq \alpha_0 + \ldots + \alpha_n$ and type $\alpha \coloneqq (\alpha_0, \ldots, \alpha_n)$ with an isolated singularity. Theorem 2.9 together with Theorem 2.16 yield that

$$\operatorname{fpt}(f) = 1 - \frac{h}{p}$$

with $0 \leq h \leq n-1$ for $p \geq w(n-2)+1$, where h is the order of vanishing of the Hasse invariant on a certain deformation space of $X \coloneqq \operatorname{Proj}(R/fR) \subset \mathbb{P}^n(\alpha)$. Theorem 3.11 shows that $h = 0$ if and only if $\operatorname{ht}\left(H^{n-1}\left(X, \mathbb{G}_{m,X}\right)\right) = 1$.

Therefore, it is natural to ask whether the other possible values of the F-pure threshold (i.e. $1 \leq h \leq n-1$) can also be characterized by $\operatorname{ht}\left(H^{n-1}\left(X, \mathbb{G}_{m,X}\right)\right)$. For this, in the second part of chapter 3 we will introduce weighted Delsarte $K3$ surfaces, which are convenient for explicit calculations of the height. First, we explain a method of Goto ([Got04]) to compute their height and then we will give two examples of weighted Delsarte $K3$ surfaces which show that the answer to the above question is negative. The first example will have the same height but different F-pure threshold and the second one will have the same F-pure threshold but the height will differ for two different primes p.

Acknowledgements

The acknowledgements are left out in this version due to data privacy reasons.

Chapter 1

The F-pure threshold of a polynomial

In this chapter we give an introduction to the theory of the F-pure threshold of a polynomial. In the first section we set up the notation for the rest of this thesis. In the second part we give the definition of the F-pure threshold and briefly explain that the F-pure threshold can be interpreted as a critical number for the behavior of the so-called test ideal. In the third section, we introduce the log-canonical threshold and compare it with the F-pure threshold, which is its characteristic $p > 0$ analogue. Furthermore, we give a short overview about some results concerning the computation of the F-pure threshold. These are the results that mainly inspired us to work on the F-pure threshold.

1.1 Quasi-homogeneous polynomials with an isolated singularity

In this section we fix some notation for the rest of this thesis. For further information we refer the reader to [Kun97], for example.
Throughout this thesis, K will denote a field of characteristic $p > 0$ and

$$R \coloneqq K[x_0, \dots, x_n]$$

will be the polynomial ring over K in $n+1$ variables. By

$$\mathfrak{m} \coloneqq (x_0, \dots, x_n)$$

we will denote the maximal ideal of R generated by the variables of R.
An $(n+1)$-tuple $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}_{>0}^{n+1}$ defines a grading on $R = K[x_0, \dots, x_n]$ by setting

$$\deg(x_i) \coloneqq \alpha_i \text{ and } \deg\left(x^k\right) \coloneqq \sum_{i=0}^{n} \alpha_i k_i,$$

where $k = (k_0, \ldots, k_n)$ is a multi-index and $x^k = x_0^{k_0} \cdots x_n^{k_n}$. It follows that $R = \bigoplus_d R_d$, where

$$R_d := \left\{ \sum_{k=(k_0,\ldots,k_n)} p_k x^k \;\middle|\; \sum_{i=0}^{n} \alpha_i k_i = d, p_k \in K \right\}$$

and the elements of R_d are called *quasi-homogeneous polynomials of degree d and type α*. For an element $f \in R_d$ we have

$$f\left(\lambda^{\alpha_0} x_0, \ldots, \lambda^{\alpha_n} x_n\right) = \lambda^d f(x_0, \ldots, x_n) \text{ for all } \lambda \in K.$$

We set

$$w := \sum_{i=0}^{n} \alpha_i = \deg\left(x_0 \cdots x_n\right).$$

A sequence $f_1, \ldots, f_m \in R$ is called a *regular sequence* if the image of f_i in $R/(f_1, \ldots, f_{i-1})$ is a non zero-divisor $(1 \leq i \leq m)$ and if $(f_1, \ldots, f_m) \neq R$. We say that an element $f \in R_d$ has an *isolated singularity* if

$$\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}$$

is a regular sequence. Furthermore, the *Jacobian ideal* of $f \in R$ is

$$J(f) := \left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

and the *Milnor number* of f is defined by

$$\mu(f) := \dim_K R/J(f).$$

It is well-known that f has an isolated singularity if and only if $\mu(f) < \infty$. This can be shown with explicit bounds for $\mu(f)$ by using the Poincaré series (see section 2.1).
The projective variety $\mathbb{P}_K^n(\alpha) := \operatorname{Proj} K[x_0, \ldots, x_n]$ is called the *weighted projective space over K of type α*.

1.2 Definition of the F-pure threshold

As already mentioned, the F-pure threshold was first introduced by Takagi and Watanabe in [TW04]. However, we mainly follow the approach of Blickle, Mustaţă and Smith in [BMS08] since their definition of the F-pure threshold is more suitable for the computations in this thesis. Both definitions agree by section 2.3 of [BMS08].

Let $R = K[x_0, \ldots, x_n]$ be the polynomial ring over a field K of characteristic $p > 0$ with maximal ideal $\mathfrak{m} = (x_0, \ldots, x_n)$. The main ingredient in the definition of the F-pure threshold is the Frobenius map or p-th power map on R given by

$$\begin{aligned} F : R &\to R, \\ r &\mapsto r^p. \end{aligned}$$

Let $q = p^e$ be a power of p and denote by

$$\mathfrak{a}^{[q]} := \left(a^q \mid a \in \mathfrak{a}\right)$$

the Frobenius power of an ideal $\mathfrak{a} \subset R$. For $f \in \mathfrak{m}$ one defines

$$\mu_f(q) := \min\left\{k \in \mathbb{N} \mid f^k \in \mathfrak{m}^{[q]}\right\}$$

and observes that $\mu_f(1) = 1$ and that

$$1 \leq \mu_f(q) \leq q.$$

Furthermore,

$$\mu_f(pq) \leq p\mu_f(q), \tag{1.1}$$

since $f^{\mu_f(q)} \in \mathfrak{m}^{[q]}$ implies that $f^{p\mu_f(q)} \in \mathfrak{m}^{[pq]}$. Hence, $\left\{\frac{\mu_f(p^e)}{p^e}\right\}_{e\geq 0}$ is a non-increasing sequence of positive rational numbers and one defines:

Definition 1.1. *The* F-pure threshold *of f is*

$$\operatorname{fpt}(f) := \lim_{e\to\infty} \frac{\mu_f(p^e)}{p^e}.$$

It was shown in [BMS08, Theorem 3.1] that the F-pure threshold is a rational number in $(0, 1]$. Moreover, the F-pure threshold can also be interpreted as a jumping number of the so-called test ideal introduced by Hara and Yoshida in [HY03]. Again, we follow [BMS08], where it is also shown that their definition of the test ideal coincides with the one in [HY03] (see Proposition 2.22 of [BMS08]). Let K be an F-*finite* field, i.e. a field K such that the Frobenius morphism $F : K \to K$, $F(u) = u^p$, is finite, and denote by $R = K[x_0, \ldots, x_n]$ the polynomial ring over K. For an element $f \in R$, denote by $f^{[1/p^m]}$ the (unique) minimal ideal J such that $f \in J^{[p^m]}$. Let $c \in \mathbb{R}_+$. Then, for every $m \geq 1$ we have

$$\left(f^{\lceil cp^m \rceil}\right)^{[1/p^m]} \subseteq \left(f^{\lceil cp^{m+1} \rceil}\right)^{[1/p^{m+1}]},$$

where $\lceil u \rceil$ denotes the smallest integer $\geq u$ (see Lemma 2.8 of [BMS08]). Since R is noetherian, this sequence of ideals stabilizes.

Definition 1.2. *The* test ideal *of f with exponent c is defined as*

$$\tau\left(f^c\right) := \left(f^{\lceil cp^m \rceil}\right)^{[1/p^m]}$$

for $m \gg 0$.

The test ideal has the following properties (see [BMS08]):

(1) For $c \in \mathbb{R}_+$ sufficiently small, $\tau\left(f^c\right)$ is the unit ideal.

(2) If $c > d$, then $\tau\left(f^c\right) \subseteq \tau\left(f^d\right)$.

(3) $\operatorname{fpt}(f) = \sup\left\{c \mid \tau\left(f^c\right) = (1)\right\}$.

(4) For $\epsilon > 0$ small enough, one has $\tau\left(f^c\right) = \tau\left(f^{c+\epsilon}\right)$.

(5) There exist $c \in \mathbb{R}_+$, such that $\tau\left(f^{c-\epsilon}\right) \supsetneq \tau\left(f^c\right)$ for all positive ϵ.

The numbers $c_i \in \mathbb{R}_+$ with $\tau\left(f^{c_i-\epsilon}\right) \neq \tau\left(f^{c_i}\right)$ for every $\epsilon > 0$ are called *F-jumping numbers* of f. They are discrete and rational (see [BMS08, Theorem 3.1]). In particular, the F-pure threshold is the smallest F-jumping number.

1.3 Comparison with the situation in characteristic zero

Originally, the motivation of Takagi and Watanabe [TW04] to study the F-pure threshold was to investigate the so-called log-canonical threshold, which is an important invariant in characteristic zero. It turned out that the F-pure threshold itself is an interesting invariant and behaves in a similar way as the log-canonical threshold, i.e. one can say that the F-pure threshold is the characteristic $p > 0$ analogue of the log-canonical threshold.

The log-canonical threshold of a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ was first defined analytically, via integration. In [Sho93] Shokurov introduced the log-canonical threshold in the context of birational geometry, where it is defined using resolution of singularities. Similarly as in the characteristic $p > 0$ case, the log-canonical threshold can be defined as a jumping number for a certain associated ideal, namely of the so-called multiplier ideal.

Hironaka proved in [Hir64], that for a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ there always exists a so-called *log resolution* of f. This is a proper, birational morphism $\pi : X \to \mathbb{C}^{n+1}$ from a smooth variety X, such that the support of the divisor $F_\pi := \operatorname{div}(f \circ \pi)$ has simple normal crossings and such that $F_\pi + E$ has simple normal crossings. Here, E is the exceptional divisor, i.e. the locus of points on X at which π is not an isomorphism.

A divisor $D = \sum_i D_i$ on a nonsingular variety X of dimension $n+1$ has *simple normal crossings* if D is reduced, each D_i is smooth and D is defined in a neighborhood of any point by an equation in local analytic coordinates of the type $z_1 \cdots z_k$ for some $k \le n+1$. A divisor $\sum_i a_i D_i$ has *simple normal crossing support* if the underlying reduced divisor $\sum_i D_i$ has simple normal crossings (see Definition 4.1.1 of [Laz04a]).

The *multiplier ideal* of f with exponent $c \in \mathbb{R}_+$ is defined via the log resolution π of f, namely

$$\mathcal{I}(f^c) := \pi_* \mathcal{O}_X \left(\lceil K_\pi - cF_\pi \rceil \right),$$

where $K_\pi := \operatorname{div}\left(\operatorname{Jac}_{\mathbb{C}}(\pi)\right)$. If $(U, y_0, \ldots, y_n)$ are local coordinates on X and if we write $\pi(y_0, \ldots, y_n) = (f_0, \ldots, f_n)$ on U then $\operatorname{Jac}_{\mathbb{C}}(\pi)$ is defined on U as $\det\left(\frac{\partial f_i}{\partial y_j}\right)$. Remark that the definition of the multiplier ideal is independent of the choice of the log resolution π (see [Laz04b, Theorem 9.2.18]). Furthermore, it has the following properties (see [Laz04b]):

(1) For $c \in \mathbb{R}_+$ sufficiently small, $\mathcal{I}(f^c)$ is the unit ideal.

(2) If $c > d$, then $\mathcal{I}(f^c) \subseteq \mathcal{I}\left(f^d\right)$.

(3) For $\epsilon > 0$ small enough, one has $\mathcal{I}(f^c) = \mathcal{I}(f^{c+\epsilon})$.

(4) There exist $c \in \mathbb{R}_+$, such that $\mathcal{I}(f^{c-\epsilon}) \supsetneq \mathcal{I}(f^c)$ for all positive ϵ.

The numbers $c_i \in \mathbb{R}_+$ with $\mathcal{I}(f^{c_i-\epsilon}) \neq \mathcal{I}(f^{c_i})$ for every $\epsilon > 0$ are called *jumping numbers*. They are discrete and rational (see [ELSV04, Lemma 1.3]). In particular, the *log-canonical threshold* is defined as the smallest jumping number, i.e.

$$\operatorname{lct}(f) := \sup\left\{c \,\middle|\, \mathcal{I}(f^c) = (1)\right\}.$$

The log-canonical threshold is a positive rational number bounded above by one (see [Laz04b, Example 9.3.16]). In general it is hard to compute the log-canonical threshold, but in special cases there are algorithms to compute it, for example in the monomial case (see [How01]), in the toric case (see [Bli04]) and in the case of two variables (see [Tuc10]). The reason that the log-canonical threshold can be computed in these cases, is that a resolution of singularities can be understood. Moreover, we want to mention the following result which will be very important in chapter 3:

Example 1.3 ([HNBWZ16, Theorem 6.2])**.** Let $f \in \mathbb{Q}[x_0, \ldots, x_n]$ be a quasi-homogeneous polynomial of degree d and type α with an isolated singularity. Let w be the degree of $x_0 \cdots x_n$. Then,

$$\operatorname{lct}(f) = \begin{cases} \frac{w}{d}, & \text{if } d \ge w \\ 1, & \text{otherwise.} \end{cases}$$

Using this formula it is easy to compute the log-canonical threshold in the following cases:

(1) Let $f \in \mathbb{Q}[x_0, \dots, x_n]$ be a quasi-homogeneous polynomial of degree w with an isolated singularity. Then by the above $\operatorname{lct}(f) = 1$.

(2) Another example that will be considered in chapter 3 is the following. Let $f := x_0^d + \dots + x_n^d$ be the Fermat hypersurface of degree d with $d \geq n + 1$. Then $\operatorname{lct}(f) = \frac{n+1}{d}$.

We conclude this chapter with a comparison between the situation in characteristic zero and the situation in characteristic $p > 0$. Let $f \in \mathbb{Q}[x_0, \dots, x_n]$ or equivalently $f \in \mathbb{Z}[x_0, \dots, x_n]$ and denote by $f_p \in \mathbb{F}_p[x_0, \dots, x_n]$ the reduction of f modulo p. Fix a log resolution π of f. The corresponding multiplier ideal $\mathcal{I}(f^c) \subset \mathbb{Z}[x_0, \dots, x_n]$ is generated over $\mathbb{Z}$, thus we can reduce modulo p and denote the result by $\mathcal{I}(f^c)_p$.

Theorem 1.4 ([HY03, Theorem 6.8]). *For all $p \gg 0$ and for all c*

$$\tau(f_p^c) \subseteq \mathcal{I}(f^c)_p.$$

Fix c, then for all $p \gg 0$ (depending on c)

$$\tau(f_p^c) = \mathcal{I}(f^c)_p.$$

We reformulate the above results in terms of the corresponding thresholds.

Theorem 1.5 ([MTW05, Theorem 3.3 & Theorem 3.4]). *Let $f \in \mathbb{Z}[x_0, \dots, x_n]$ with reduction $f_p \in \mathbb{F}_p[x_0, \dots, x_n]$. Then*

$$\operatorname{fpt}(f_p) \leq \operatorname{lct}(f)$$

for $p \gg 0$ and

$$\lim_{p \to \infty} \operatorname{fpt}(f_p) = \operatorname{lct}(f).$$

A further connection between the log-canonical threshold and the F-pure threshold of the reduction is conjectured:

Conjecture 1.6 ([MTW05, Conjecture 3.6]). *There are infinitely many primes p such that* $\operatorname{fpt}(f_p) = \operatorname{lct}(f)$.

This conjecture is wide open and is related to other open problems in birational geometry which we will briefly mention now.
First, the above comparison of the F-pure threshold with the log-canonical threshold can also be described by using some class of F-singularities. We say that a commutative ring R of characteristic $p > 0$ is *F-pure* if the inclusion $R^{p^e} \subseteq R$ splits as a map of R^{p^e}-modules for some $e \geq 1$ (or equivalently for all $e \geq 1$), where $R^{p^e} := \left\{ r^{p^e} \,\middle|\, r \in R \right\}$ (see [HR76]). The notion of F-purity can be extended to pairs of the form (R, f^c), where f is a non-zero, non-unit element of R and c is a non-negative real parameter (see [HW02]). The pair (R, f^c) is

called F-pure if the inclusion $R^{p^e} \cdot f^{\lfloor c(p^e-1)\rfloor} \subseteq R$ splits as a map of R^{p^e}-modules for all $e \gg 0$. Clearly, R is F-pure if and only if (R, f^0) is F-pure and one can show that

$$\operatorname{fpt}(f) = \sup\left\{c \geq 0 \;\middle|\; (R, f^c) \text{ is } F\text{-pure}\right\}$$

(see [MTW05]). In characteristic zero there is a similar notion (see [Kol97]). Namely, for a variety X over $\mathbb{C}$ and a hypersurface $Y \subseteq X$ one defines the notion of a log-canonical pair (X, Y^c) via resolution of singularities. Similarly as in characteristic $p > 0$ the log-canonical threshold is equal to the supremum over all c, such that the pair (X, Y^c) is log-canonical. The above results then correspond to the following theorem:

Theorem 1.7 ([HW02]). *The pair $(\mathbb{Q}[x_0, \ldots, x_n], f^c)$ is log-canonical if the pairs $(\mathbb{F}_p[x_0, \ldots, x_n], f_p^c)$ are F-pure for infinitely many $p \gg 0$.*

The opposite direction of this theorem is equivalent to Conjecture 1.6.

In [MS11] and [Mus12] Mustaţă and Srinivas relate the open question on the connection between multiplier ideals and test ideals after reduction to characteristic $p > 0$ to the weak ordinarity conjecture of Serre. For this, let $X \subseteq \mathbb{P}^n_{\mathbb{Q}}$ be a projective variety and choose homogeneous polynomials $f_1, \ldots, f_r \in \mathbb{Z}[x_0, \ldots, x_n]$ whose images in $\mathbb{Q}[x_0, \ldots, x_n]$ generate the ideal of X. For a prime p consider the projective variety $X_p \subseteq \mathbb{P}^n_{\mathbb{F}_p}$ defined by the ideal generated by the reductions of $f_1, \ldots, f_r$ in $\mathbb{F}_p[x_0, \ldots, x_n]$ (remark that for $p \gg 0$, X_p does not depend on the choice of $f_1, \ldots, f_r$).

Conjecture 1.8 ([MS11, Conjecture 1.1]). *Let X be a smooth, geometrically connected (i.e. $X \times_{\operatorname{Spec}(\mathbb{Q})} \operatorname{Spec}(\overline{\mathbb{Q}})$ is connected), n-dimensional projective variety over $\mathbb{Q}$. Then there are infinitely many primes p such that the endomorphism induced by the Frobenius on $H^n\left(X_p, \mathcal{O}_{X_p}\right)$ is bijective.*

This conjecture is also wide open. Moreover, in [MS11] and [Mus12] Mustaţă and Srinivas show, that this conjecture is equivalent to the expected relation between F-pure threshold and log-canonical threshold mentioned in Conjecture 1.6.

1.4 Some results on the computation of the F-pure threshold

In general it is difficult to compute the F-pure threshold. However, in the following, we give a short overview of some cases, where the computation of the F-pure threshold is possible and which are of particular interest for the rest of this thesis.
As already mentioned at the end of the last section, it was conjectured in [MTW05] that $\operatorname{fpt}(f_p) = \operatorname{lct}(f)$ for infinitely many primes p. As a hint into this direction, they consider the following example.

Example 1.9 ([MTW05, Example 4.2]). Let $f \in \mathbb{Z}[x_0, \dots, x_n]$. Write $f = \sum_{i=1}^{r} c_i x^{a_i}$, where $a_i = (a_{i,0}, \dots, a_{i,n}) \in \mathbb{N}^{n+1}$ and all c_i are non-zero. Assume that $a_1, \dots, a_r$ are affinely independent, i.e. if $\sum_i \lambda_i a_i = 0$ and $\sum_i \lambda_i = 0$ for $\lambda = (\lambda_i) \in \mathbb{Q}^r$, then $\lambda_i = 0$ for all i. Moreover, assume that for every $j \leq n$ there is an $i \leq r$ with $a_{i,j} > 0$. Define

$$Q := \left\{ b = (b_1, \dots, b_r) \in \mathbb{R}_+^r \;\middle|\; \sum_{i=1}^{r} b_i a_{ij} \leq 1 \text{ for all } j \right\},$$

then

$$\operatorname{lct}(f) = \max_{b \in Q} \sum_i b_i.$$

If $\operatorname{lct}(f) = 1$, then choose any $v \in Q \cap \mathbb{Q}^r$ such that $\sum_i v_i = 1$. If $\operatorname{lct}(f) < 1$, let v be one of the vertices of Q, such that $\sum_i v_i = \max_{b \in Q} \sum_i b_i$. Take N such that $N v_i$ is an integer for all i. Then $\operatorname{fpt}(f_p) = \operatorname{lct}(f)$ for all p with $p \equiv 1 \bmod N$.

In chapter 3 we will need the following special case of the above example. Let $f := x_0^d + \ldots + x_n^d$. Then

$$Q = \left\{ (b_0, \dots, b_n) \in \mathbb{R}_+^{n+1} \;\middle|\; b_j d \leq 1 \text{ for all } j \right\}$$

and $\operatorname{lct}(f) = \frac{n+1}{d}$. Choose $v = (\frac{1}{d}, \dots, \frac{1}{d})$ and let $N = d$. Then the above example shows that $\operatorname{fpt}(f_p) = \operatorname{lct}(f) = \frac{n+1}{d}$ if $p \equiv 1 \bmod d$.

More generally, Hernández computes in his thesis (see [Her]) the F-pure threshold of a diagonal hypersurface using base p expansions.
For this, let $R = K[x_0, \dots, x_n]$ be the polynomial ring over an F-finite field K of characteristic $p > 0$.

Definition 1.10 ([Her, Definition 4.1]). *Let $\alpha \in (0, 1]$. The* non-terminating base p expansion *of α is an expression*

$$\alpha = \sum_{i \geq 1} \frac{a_i}{p^i}$$

with $0 \leq a_i \leq p - 1$, such that for all $N > 0$ there exists an $e \geq N$ with $a_e \neq 0$.

Note that for every $\alpha \in (0, 1]$ the non-terminating base p expansion is unique. For example,

$$\frac{1}{p} = \frac{0}{p} + \sum_{i \geq 2} \frac{p-1}{p^i}.$$

Definition 1.11 ([Her, Definition 4.5]). *Let $\alpha \in (0,1]$ and fix $e \geq 1$.*

(1) The number $c_e(\alpha) := a_e$ denotes the e-th digit *in the non-terminating base p expansion of α. By convention, $c_e(0) = 0$.*

(2) The e-th truncation *of the non-terminating base p expansion of α is defined by*
$$\langle\alpha\rangle_e := \frac{c_1(\alpha)}{p} + \ldots + \frac{c_e(\alpha)}{p^e}.$$
By convention, $\langle 0\rangle_e = 0$.

With this notation we can state the theorem of Hernández about the F-pure threshold of a diagonal hypersurface. Remember, that $R = K[x_0, \ldots, x_n]$ denotes the polynomial ring over an F-finite field K of characteristic $p > 0$.

Theorem 1.12 ([Her, Theorem 8.1]). *Let $(d_0, \ldots, d_n) \in \mathbb{N}_{>0}^{n+1}$ and let $f := u_0 x_0^{d_0} + \ldots + u_n x_n^{d_n} \in R$. If*
$$L := \sup\left\{ N \;\middle|\; c_e\left(\frac{1}{d_0}\right) + \ldots + c_e\left(\frac{1}{d_n}\right) \leq p-1 \textit{ for all } 0 \leq e \leq N \right\},$$
then
$$\operatorname{fpt}(f) = \begin{cases} \frac{1}{d_0} + \ldots + \frac{1}{d_n}, & \textit{if } L = \infty \\ \left\langle \frac{1}{d_0} \right\rangle_L + \ldots + \left\langle \frac{1}{d_n} \right\rangle_L + \frac{1}{p^L}, & \textit{if } L < \infty. \end{cases}$$

Example 1.13. Using this result, Hernández is able to recover a result of [MTW05, Example 4.3], namely he computes the F-pure threshold of $f := x^2 + y^3$. For example, if $p = 3$ then
$$\frac{1}{2} = \sum_{i \geq 1} \frac{1}{3^i} \quad \text{and} \quad \frac{1}{3} = \sum_{i \geq 2} \frac{2}{3^i}$$
which means
$$c_e\left(\frac{1}{2}\right) = 1 \text{ for all } e \text{ and}$$
$$c_1\left(\frac{1}{3}\right) = 0 \text{ and } c_e\left(\frac{1}{3}\right) = 2 \text{ for } e \geq 2.$$
Therefore, $L = 1$ and
$$\operatorname{fpt}(f) = \left\langle \frac{1}{2} \right\rangle_1 + \left\langle \frac{1}{3} \right\rangle_1 + \frac{1}{3} = \frac{1}{3} + 0 + \frac{1}{3} = \frac{2}{3}.$$
Similarly one computes the remaining cases and gets
$$\operatorname{fpt}(x^2 + y^3) = \begin{cases} \frac{1}{2}, & \text{if } p = 2 \\ \frac{2}{3}, & \text{if } p = 3 \\ \frac{5}{6}, & \text{if } p \equiv 1 \bmod 6 \\ \frac{5}{6} - \frac{1}{6p}, & \text{if } p \equiv 5 \bmod 6. \end{cases}$$

By using methods similar to those in the above example, Hernández computes the F-pure threshold of a Fermat hypersurface of degree d in terms of the residue of p modulo d:

Corollary 1.14 ([Her, Corollary 8.3])**.** *Let $f := u_1x_1^d + \ldots + u_dx_d^d \in K[x_1, \ldots, x_d]$. Then*

$$\operatorname{fpt}(f) = \begin{cases} \frac{1}{p^l}, & \text{if } p^l \leq d < p^{l+1} \text{ for some } l \geq 1 \\ 1 - \frac{a-1}{p}, & \text{if } 0 < d < p \text{ and } p \equiv a \bmod d \text{ with } 1 \leq a < d. \end{cases}$$

Furthermore, Hernández computes the F-pure threshold of a binomial hypersurface, i.e. a K-linear combination of two distinct monomials. For this he uses a polytope associated to the binomial hypersurface, which is the same as the one defined in Example 1.9, and gives a formula for the F-pure threshold in terms of this polytope. See [Her, Theorem 9.14] for a precise statement.

The article that has provided the main inspiration for the second chapter of this thesis (and that generalizes the results of Hernández) is the article [BS15] of Bhatt and Singh.
Let K be a field of characteristic $p > 0$. In the first part of their paper, Bhatt and Singh explain how to compute a list of possible F-pure thresholds of a homogeneous polynomial $f \in K[x_0, \ldots, x_n]$. For this, they give a lower and an upper bound for the numbers $\mu_f(q)$, $q = p^e$, that were defined in section 1.2. Using these bounds one can compute a list of possible values for $\mu_f(p)$. Then they explain how to compute $\mu_f(pq)$ if $\mu_f(p)$ is given. In this way, they get a list of possible values for $\mu_f(p^e)$ and can finally compute a list of possible F-pure thresholds. As our first main result, we will generalize this to the quasi-homogeneous case in section 2.1.
In [BS15], the authors moreover consider Calabi-Yau hypersurfaces, i.e. hypersurfaces given by a polynomial $f \in K[x_0, \ldots, x_n]$ of degree $n+1$ and obtain the following:

Theorem 1.15 ([BS15, Theorem 4.1])**.** *Let $R = K[x_0, \ldots, x_n]$ be the polynomial ring over a field K of characteristic $p > 0$. Furthermore, let $f \in R$ be a homogeneous polynomial of degree $n+1$ with an isolated singularity and let $X := \operatorname{Proj}(R/fR)$ be the Calabi-Yau hypersurface defined by f. Then*

(1) $\mu_f(p) = p - h$, where h is an integer with $0 \leq h \leq n-1$,

(2) $\mu_f(pq) = p\mu_f(q)$ for all $q = p^e$ with $q \geq n-1$.

(3) If $p \geq n-1$, then $\operatorname{fpt}(f) = 1 - \frac{h}{p}$, where $0 \leq h \leq n-1$.

(4) Let $p \geq n^2 - n - 1$. Then the integer $0 \leq h \leq n-1$ equals the order of vanishing of the Hasse invariant on a certain deformation space of $X \subset \mathbb{P}^n$.

For the definition of the Hasse invariant see the introduction or section 2.3. The proof of Theorem 1.15 can be found in [BS15], we will generalize it to the case of a quasi-homogenous polynomial in chapter 2. While the proof of the fourth part of Theorem 1.15 uses deformation-theoretic arguments in [BS15], our proof gives a more elementary approach to this result.
The two-dimensional case of Theorem 1.15 says the following:

Theorem 1.16. *Let E be an elliptic curve given by a homogeneous polynomial $f \in K[x, y, z]$ of degree 3. Then*

$$\mathrm{fpt}(f) = \begin{cases} 1, & \textit{if } E \textit{ is ordinary} \\ 1 - \frac{1}{p}, & \textit{otherwise.} \end{cases}$$

Here, a curve E is called *ordinary*, if and only if the endomorphism on $H^1(E, \mathcal{O}_E)$ induced by Frobenius is bijective, otherwise it is called *supersingular* (i.e. the map on $H^1(E, \mathcal{O}_E)$ induced by Frobenius is zero). In chapter 3 we will come back to the notion of supersingularity and its connection to the F-pure threshold.

In the paper [HNBWZ16] of Hernández, Núñez-Betancourt, Witt and Zhang the authors use base p expansions to compute the possible values of the F-pure threshold of a quasi-homogeneous polynomial of arbitrary degree. In particular, their list of possible F-pure thresholds agrees with the one we will obtain in chapter 2 in the case of a Calabi-Yau hypersurface.
Let $R = K[x_0, \dots, x_n]$ be the polynomial ring over a field K of characteristic $p > 0$. For $b \in \mathbb{N}_{>0}$ and $m \in \mathbb{Z}$ denote by $[\![m \ \% \ b]\!]$ the least positive residue of m modulo b, i.e. $1 \leq [\![m \ \% \ b]\!] \leq b$ for all $m \in \mathbb{Z}$. Furthermore, if p and b are relatively prime, denote by

$$\mathrm{ord}(p, b) := \min \left\{ k \geq 1 \;\middle|\; [\![p^k \ \% \ b]\!] = 1 \right\}$$

the order of p modulo b. Note that $\mathrm{ord}(p, 1) = 1$. Then:

Theorem 1.17 ([HNBWZ16, Theorem 3.5])**.** *Let $R = K[x_0, \dots, x_n]$ be the polynomial ring over a field K of characteristic $p > 0$ graded by $\alpha_i = \deg(x_i)$. Let f be a quasi-homogeneous polynomial with an isolated singularity and write*

$$\lambda := \min \left\{ \frac{\sum_{i=0}^n \alpha_i}{\deg(f)}, 1 \right\} = \frac{a}{b}$$

in lowest terms (i.e. $\lambda = \mathrm{lct}(f)$).

(1) If $\mathrm{fpt}(f) \neq \lambda$, then

$$\mathrm{fpt}(f) = \lambda - \frac{[\![ap^L \ \% \ b]\!] + bE}{bp^L} = \langle \lambda \rangle_L - \frac{E}{p^L},$$

for some pair $(L, E) \in \mathbb{N}^2$ with $L \geq 1$ and $0 \leq E \leq n - \left\lceil \frac{[\![ap^L \ \% \ b]\!] + a}{b} \right\rceil$.

(2) If $p > (n-1)b$ and $p \nmid b$, then $1 \leq L \leq \operatorname{ord}(p, b)$.

(3) If $p > (n-1)b$ and $p > b$, then $a < [\![ap^e \ \% \ b]\!]$ for all $1 \leq e \leq L-1$.

(4) If $p > nb$, then there exists a unique pair (L, E) satisfying the conditions in (1).

For example, if $f \in R = K[x_0, \dots, x_n]$ is a quasi-homogeneous polynomial of degree $\deg(f) = \sum_{i=0}^{n} \deg(x_i)$ with an isolated singularity and if $p > n-1$ and $\operatorname{fpt}(f) \neq 1$, then

$$\operatorname{fpt}(f) = 1 - \frac{a}{p}$$

for some integer $1 \leq a \leq n-1$, which generalizes the result of [BS15].

Example 1.18 ([HNBWZ16, Example 4.7])**.** Let $f \in K[x, y]$ be a quasi-homogeneous polynomial with isolated singularity and suppose that $\frac{\deg(xy)}{\deg(f)} = \frac{2}{7}$. Let $p \geq 11$. We claim:

$$\begin{aligned}
&\text{If } p \equiv 1 \bmod 7, \text{ then } \operatorname{fpt}(f) = \frac{2}{7} \text{ or } \operatorname{fpt}(f) = \frac{2}{7} - \frac{2}{7p}.\\
&\text{If } p \equiv 2 \bmod 7, \text{ then } \operatorname{fpt}(f) = \frac{2}{7} - \frac{4}{7p} \text{ or } \operatorname{fpt}(f) = \frac{2}{7} - \frac{1}{7p^2}.\\
&\text{If } p \equiv 3 \bmod 7, \text{ then } \operatorname{fpt}(f) = \frac{2}{7} - \frac{4}{7p^2} \text{ or } \operatorname{fpt}(f) = \frac{2}{7} - \frac{5}{7p^3}\\
&\qquad\qquad\qquad\qquad\quad \text{or } \operatorname{fpt}(f) = \frac{2}{7} - \frac{1}{7p^4}.\\
&\text{If } p \equiv 4 \bmod 7, \text{ then } \operatorname{fpt}(f) = \frac{2}{7} - \frac{1}{7p}.\\
&\text{If } p \equiv 5 \bmod 7, \text{ then } \operatorname{fpt}(f) = \frac{2}{7} - \frac{3}{7p} \text{ or } \operatorname{fpt}(f) = \frac{2}{7} - \frac{1}{7p^2}.\\
&\text{If } p \equiv 6 \bmod 7, \text{ then } \operatorname{fpt}(f) = \frac{2}{7} \text{ or } \operatorname{fpt}(f) = \frac{2}{7} - \frac{5}{7p} \text{ or } \operatorname{fpt}(f) = \frac{2}{7} - \frac{2}{7p^2}.
\end{aligned}$$

We will only prove the case $p \equiv 1 \bmod 7$. Suppose $\operatorname{fpt}(f) \neq \frac{2}{7}$, then

$$\operatorname{fpt}(f) = \frac{2}{7} - \frac{[\![2p^L \ \% \ 7]\!] + 7E}{7p^L}.$$

By the second statement of Theorem 1.17 we have $1 \leq L \leq \operatorname{ord}(p, 7) = 1$, since $p \equiv 1 \bmod 7$, i.e. $L = 1$. Therefore, $[\![2p^L \ \% \ 7]\!] = [\![2p \ \% \ 7]\!] = 2$ and by the first statement of Theorem 1.17 this yields $E = 0$. Altogether this means that

$$\operatorname{fpt}(f) = \frac{2}{7} - \frac{[\![2p \ \% \ 7]\!]}{7p} = \frac{2}{7} - \frac{2}{7p}.$$

In many cases the authors of [HNBWZ16] were able to verify, that the list obtained by the above procedure (and some additional results about the base p expansion of the F-pure threshold) is in fact minimal, i.e. each value in the list is the F-pure threshold of some polynomial (see [HNBWZ16, Remark 4.8]). There are also examples, where the F-pure threshold is precisely determined:

Example 1.19 ([HNBWZ16, Example 4.9]). Let $f \in K[x,y]$ be quasi-homogeneous with an isolated singularity and with $\frac{\deg(xy)}{\deg(f)} = \frac{3}{5}$. For example, we can take $f \coloneqq x^5 + x^3y + xy^2$ with $\deg(x) \coloneqq 1$ and $\deg(y) \coloneqq 2$. Furthermore, let $p \geq 7$.

$$\begin{aligned}
&\text{If } p \equiv 1 \bmod 5, \text{ then } \operatorname{fpt}(f) = \frac{3}{5}.\\
&\text{If } p \equiv 2 \bmod 5, \text{ then } \operatorname{fpt}(f) = \frac{3}{5} - \frac{1}{5p}.\\
&\text{If } p \equiv 3 \bmod 5, \text{ then } \operatorname{fpt}(f) = \frac{3}{5} - \frac{2}{5p^2}.\\
&\text{If } p \equiv 4 \bmod 5, \text{ then } \operatorname{fpt}(f) = \frac{3}{5} - \frac{2}{5p}.
\end{aligned}$$

In the second chapter, where we generalize the results of Bhatt and Singh to the quasi-homogeneous case, we will use similar methods as the authors of [HNBWZ16] used to prove Theorem 1.17. In particular, we get the same list of possible F-pure thresholds as they obtain in the case of a Calabi-Yau hypersurface.

Chapter 2

The F-pure threshold and the Hasse invariant

In this chapter we extend the results of [BS15] mentioned in chapter 1 to the case of a quasi-homogeneous polynomial. In the first section, we explain how to compute the F-pure threshold of a quasi-homogeneous polynomial. For this, we first give a lower and an upper bound for $\mu_f(p)$ and then explain how to compute $\mu_f(p^e)$ if $\mu_f(p)$ is given. Using these results, we give a list of possible F-pure thresholds of a Calabi-Yau hypersurface. In the second part we consider the case of a curve given by a quasi-homogeneous polynomial f in three variables x, y, z of degree equal to the degree of xyz. In the third part of this chapter, we proceed with the general case of a Calabi-Yau hypersurface and relate its F-pure threshold to the Hasse invariant.
To a large extent the proofs are analogous to the ones in [BS15]; the results of this chapter have appeared in [Mül18].

2.1 The F-pure threshold of a quasi-homogeneous polynomial

Let $R = K[x_0, \ldots, x_n]$ be the graded polynomial ring with $\alpha_i = \deg(x_i)$ over a field K of characteristic $p > 0$ and let $\mathfrak{m} = (x_0, \ldots, x_n)$. Furthermore, for $f \in \mathfrak{m}$ we defined in chapter 1

$$\mu_f(q) = \min\left\{k \in \mathbb{N} \mid f^k \in \mathfrak{m}^{[q]}\right\}$$

for $q = p^e$ and

$$\operatorname{fpt}(f) = \lim_{e\to\infty} \frac{\mu_f(p^e)}{p^e}.$$

The definition of $\mu_f(q)$ yields $f^{\mu_f(q)-1} \notin \mathfrak{m}^{[q]}$ and therefore we get that $f^{p\mu_f(q)-p} \notin \mathfrak{m}^{[pq]}$. Together with the inequality $\mu_f(pq) \leq p\mu_f(q)$ (see (1.1)) we deduce $p\mu_f(q) - p + 1 \leq \mu_f(pq) \leq p\mu_f(q)$, which implies

$$\mu_f(q) = \left\lceil \frac{\mu_f(pq)}{p} \right\rceil . \tag{2.1}$$

Now, let $f \in R = K[x_0, \dots, x_n]$ be a quasi-homogeneous polynomial of degree d and type $\alpha := (\alpha_0, \dots, \alpha_n)$ and let $t \leq q$ be an integer. Then the Frobenius iterate $F^e : R/fR \to R/fR$ lifts to a map $R/fR \to R/f^qR$. We compose this map with the canonical surjection $R/f^qR \to R/f^tR$ and obtain a map $\widetilde{F^e_t} : R/fR \to R/f^tR$.

Lemma 2.1. *Let $f \in R$ be a quasi-homogeneous polynomial of degree d and type α and let $t \leq q$ be an integer. Then $\mu_f(q) > q - t$ if and only if the map $\widetilde{F^e_t} : H^n_{\mathfrak{m}}(R/fR) \to H^n_{\mathfrak{m}}(R/f^tR)$ is injective.*

For the proof of this lemma, we will need the so-called socle of a module over a ring. Let us briefly recall its definition and some important property of the socle.
Let M be a graded R-module. The *socle* of M is defined by

$$\operatorname{Soc}(M) := \left\{ x \in M \;\middle|\; \mathfrak{m} \cdot x = 0 \right\}.$$

Lemma 2.2. *Let M be an R-module of finite length. Then for every submodule $U \subset M$ with $U \neq 0$ we also have $U \cap \operatorname{Soc}(M) \neq 0$.*

Proof. Since M is of finite length and U is a submodule of M, we know that U is also of finite length. Therefore, there exists an $l \in \mathbb{N}$ such that

$$\mathfrak{m}^l U = 0 \text{ but } \mathfrak{m}^{l-1} U \neq 0.$$

Obviously $\mathfrak{m}^{l-1}U \subset U$ and since $\mathfrak{m} \cdot \mathfrak{m}^{l-1}U = 0$ it follows that $\mathfrak{m}^{l-1}U \subset \operatorname{Soc}(M)$. Hence $U \cap \operatorname{Soc}(M) \neq 0$. □

Proof of Lemma 2.1. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & R(-d) \arrow[r, "f"] \arrow[d, "f^{q-t}F^e"] & R \arrow[r] \arrow[d, "F^e"] & R/fR \arrow[r] \arrow[d, "\widetilde{F^e_t}"] & 0 \\
0 \arrow[r] & R(-dt) \arrow[r, "f^t"] & R \arrow[r] & R/f^tR \arrow[r] & 0,
\end{tikzcd}$$

which gives an induced diagram of local cohomology modules

$$\begin{tikzcd}
0 \arrow[r] & H^n_{\mathfrak{m}}(R/fR) \arrow[r] \arrow[d, "\widetilde{F^e_t}"] & H^{n+1}_{\mathfrak{m}}(R)(-d) \arrow[r, "f"] \arrow[d, "f^{q-t}F^e"] & H^{n+1}_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F^e"] & 0 \\
0 \arrow[r] & H^n_{\mathfrak{m}}(R/f^tR) \arrow[r] & H^{n+1}_{\mathfrak{m}}(R)(-dt) \arrow[r, "f^t"] & H^{n+1}_{\mathfrak{m}}(R) \arrow[r] & 0.
\end{tikzcd}$$

We first show that the map F^e is injective. For this, it suffices to show that F^e acts injectively on the socle

$$\operatorname{Soc}\left(H^{n+1}_{\mathfrak{m}}(R)\right) = \left\langle \left[\frac{1}{x_0 \ldots x_n}\right] \right\rangle,$$

which follows from $F^e\left(\left[\frac{1}{x_0 \ldots x_n}\right]\right) = \left[\frac{1}{x_0^q \cdot \ldots \cdot x_n^q}\right] \neq 0$. Now by the five lemma $\widetilde{F^e_t}$ is injective if and only if $f^{q-t}F^e$ is injective. Again by looking at the socle one shows that $f^{q-t}F^e$ is injective if and only if

$$f^{q-t}F^e\left(\left[\frac{1}{x_0 \ldots x_n}\right]\right) = \left[\frac{f^{q-t}}{x_0^q \ldots x_n^q}\right] \neq 0,$$

which is equivalent to $f^{q-t} \notin \mathfrak{m}^{[q]}$. By the definition of $\mu_f(q)$ this is equivalent to $\mu_f(q) > q - t$. □

This cohomological description of $\mu_f(q)$ will be very useful in the following sections. Now we want to compute the F-pure threshold of a quasi-homogeneous polynomial. As a first step we give a lower and an upper bound for $\mu_f(p)$ (see Proposition 2.5 and Proposition 2.6) from which one obtains bounds for $\mu_f(p^e)$. We start with some lemmata which are similar to results shown in [HNBWZ16]. For the proof of these, let us recollect some facts about the *Poincaré series* $H_M(T)$ of a finitely generated graded R-module M, which is defined by

$$H_M(T) \coloneqq \sum_j \dim_K (M_j)\, T^j,$$

where M_j is the homogeneous part of M of degree j.
First, we want to compute $H_R(T)$. For this, we use the fact that if $f \in R$ is a homogeneous element of degree d, which is a non zero-divisor of M, then

$$H_{M/fM}(T) = (1 - T^d)H_M(T).$$

Hence, $H_{R/x_n}(T) = (1 - T^{\alpha_n})\, H_R(T)$ and inductively we get

$$1 = H_{R/(x_0,\ldots,x_n)}(T) = \prod_{i=0}^{n} (1 - T^{\alpha_i})\, H_R(T).$$

Thus,

$$H_R(T) = \prod_{i=0}^{n} \frac{1}{(1 - T^{\alpha_i})}.$$

Now, let $f_0, \ldots, f_m \in R = K[x_0, \ldots, x_n]$ be quasi-homogeneous polynomials of type α with $\deg(f_i) = d_i$, $0 \leq i \leq m$, such that $f_0, \ldots, f_m$ is a regular sequence. Then

$$H_{R/(f_0,\ldots,f_m)}(T) = \prod_{j=0}^{m} \left(1 - T^{d_j}\right) H_R(T) = \frac{\prod\limits_{j=0}^{m} \left(1 - T^{d_j}\right)}{\prod\limits_{i=0}^{n} (1 - T^{\alpha_i})}.$$

In particular, it is shown in [Kun97, p. 213] that for $m = n$ one has

$$\dim_K \left(R/(f_0, \ldots, f_n)\right) = \lim_{T \to 1} \prod_{i=0}^{n} \frac{\left(1 - T^{d_i}\right)}{\left(1 - T^{\alpha_i}\right)} = \prod_{i=0}^{n} \frac{d_i}{\alpha_i},$$

i.e. $R/(f_0, \ldots, f_n)$ is a finite dimensional K-algebra.
If we have $f \in R_d$ with an isolated singularity, then $\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}$ form a regular sequence and we know that $\deg\left(\frac{\partial f}{\partial x_i}\right) = d - \alpha_i$. Therefore the Milnor number is given by

$$\mu(f) = \dim_K \left(R/J(f)\right) = \prod_{i=0}^{n} \frac{d - \alpha_i}{\alpha_i}.$$

Using the Poincaré series we can now prove the following lemma:

Lemma 2.3. *Let $f \in R = K[x_0, \ldots, x_n]$ be a quasi-homogeneous polynomial of degree d and type α with an isolated singularity. Then*

$$R_{\geq (n+1)d - 2w + 1} \subset J(f),$$

where $w = \sum_{i=0}^{n} \alpha_i$.

Proof. Since f has an isolated singularity, there exists a $k \in \mathbb{N}$ such that $\mathfrak{m}^k \subset J(f)$. This means that $\left(R/J(f)\right)_i = 0$ for all i greater than some $N \in \mathbb{N}$. Thus, the Poincaré series

$$H_{R/J(f)}(T) = \frac{\prod_{j=0}^{n} \left(1 - T^{d - \alpha_j}\right)}{\prod_{i=0}^{n} \left(1 - T^{\alpha_i}\right)}$$

must be a polynomial, hence $\prod_{j=0}^{n} \left(1 - T^{d - \alpha_j}\right)$ is divisible by $\prod_{i=0}^{n} \left(1 - T^{\alpha_i}\right)$. Therefore,

$$\deg\left(H_{R/J(f)}\right) = \left((n+1)d - \sum_{j=0}^{n} \alpha_j\right) - \sum_{i=0}^{n} \alpha_i = (n+1)d - 2w.$$

This means that $\left(R/J(f)\right)_i = 0$ for all $i \geq (n+1)d - 2w + 1$, thus, $R_{\geq (n+1)d - 2w + 1}$ must be a subset of $J(f)$. $\square$

Let R be a commutative ring, let I be an ideal of R and let S be a subset of R. Then the *ideal quotient*

$$(I :_R S) := \left\{ r \in R \;\middle|\; rS \subset I \right\}$$

is an ideal of R.

Lemma 2.4. *We have*

$$\left(\mathfrak{m}^{[q]} :_R R_{\geq (n+1)d-2w+1}\right) \setminus \mathfrak{m}^{[q]} \subset R_{\geq (q+1)w-(n+1)d}.$$

Proof. Suppose the statement is false, then there exists a monomial

$$\lambda := x_0^{q-1-b_0} \dots x_n^{q-1-b_n} \subset \left(\mathfrak{m}^{[q]} :_R R_{\geq (n+1)d-2w+1}\right) \setminus \mathfrak{m}^{[q]}$$

of degree

$$\deg(\lambda) = (q-1)w - \sum_{i=0}^{n} b_i \alpha_i < (q+1)w - (n+1)d.$$

Equivalently,

$$\sum_{i=0}^{n} b_i \alpha_i > (q-1)w - (q+1)w + (n+1)d = (n+1)d - 2w,$$

thus the monomial $\eta := x_0^{b_0} \dots x_n^{b_n}$ is an element of $R_{\geq (n+1)d-2w+1}$. Therefore, $x_0^{q-1} \dots x_n^{q-1} = \lambda \cdot \eta \in \mathfrak{m}^{[q]}$, which is a contradiction. □

This yields a lower and an upper bound for $\mu_f(q)$.

Proposition 2.5. *Let $f \in R$ be a quasi-homogeneous polynomial of degree d and type α with an isolated singularity. If $p \nmid \mu_f(q)$, then*

$$\mu_f(q) \geq \frac{w(q+1) - nd}{d}.$$

Proof. With $k := \mu_f(q)$ we have $f^k \in \mathfrak{m}^{[q]}$. The partial derivatives $\frac{\partial}{\partial x_i}$ map $\mathfrak{m}^{[q]}$ to $\mathfrak{m}^{[q]}$ and therefore

$$k f^{k-1} \frac{\partial f}{\partial x_i} \in \mathfrak{m}^{[q]}$$

for all i. Since k is non-zero in K, it follows

$$f^{k-1} J(f) \subset \mathfrak{m}^{[q]}.$$

By the definition of k we know that $f^{k-1} \notin \mathfrak{m}^{[q]}$ so that

$$f^{k-1} \in \left(\mathfrak{m}^{[q]} :_R J(f)\right) \setminus \mathfrak{m}^{[q]}.$$

By Lemma 2.3 and Lemma 2.4 it follows that

$$\left(\mathfrak{m}^{[q]} :_R J(f)\right) \setminus \mathfrak{m}^{[q]} \subset \left(\mathfrak{m}^{[q]} :_R R_{\geq (n+1)d-2w+1}\right) \setminus \mathfrak{m}^{[q]} \subset R_{\geq (q+1)w-(n+1)d}.$$

This means that $f^{k-1} \in R_{\geq (q+1)w-(n+1)d}$. Hence,

$$d(k-1) = \deg(f^{k-1}) \geq w(q+1) - (n+1)d$$

and therefore $k \geq \frac{w(q+1)-nd}{d}$. □

Proposition 2.6. *Let $f \in R$ be a quasi-homogeneous polynomial of degree d and type α. Then*

$$\mu_f(q) \leq \left\lceil \frac{wq - w + 1}{d} \right\rceil$$

for all $q = p^e$.

Proof. For $k \in \mathbb{N}$ one has $f^k \in \mathfrak{m}^{[q]}$ if $dk \geq w(q-1)+1$. Thus,

$$\mu_f(q) = \min\left\{k \;\middle|\; f^k \in \mathfrak{m}^{[q]}\right\} \leq \left\lceil \frac{wq - w + 1}{d} \right\rceil.$$

□

The following proposition explains how to compute $\mu_f(pq)$ if $\mu_f(q)$ is given, when certain conditions are fulfilled.

Proposition 2.7. *Let $f \in R$ be a quasi-homogeneous polynomial of degree d and type α with an isolated singularity.*

(1) If $\frac{\mu_f(q)-1}{q-1} = \frac{w}{d}$ for some $q = p^s$, then $\frac{\mu_f(pq)-1}{pq-1} = \frac{w}{d}$.

(2) Suppose $p \geq nd - d - w + 1$. If $\frac{\mu_f(q)}{q} < \frac{w}{d}$ for some $q = p^s$, then $\mu_f(pq) = p\mu_f(q)$. In particular,

$$\frac{\mu_f(pq)}{pq} = \frac{\mu_f(q)}{q},$$

thus the sequence $\left\{\frac{\mu_f(p^e)}{p^e}\right\}_{p^e \geq q}$ is constant.

In particular, the second statement of this proposition shows the following: Let $f \in \mathbb{Z}[x_0, \ldots, x_n]$ be a quasi-homogeneous polynomial such that the reduction $f_p \in \mathbb{F}_p[x_0, \ldots, x_n]$ satisfies the conditions of Proposition 2.7. If the F-pure threshold of f_p does not coincide with the log-canonical threshold of f (see Example 1.3) then the proposition shows that the denominator of the F-pure threshold of f_p is a power of p.

Proof of Proposition 2.7. To prove the first statement, assume that $\frac{\mu_f(q)-1}{q-1} = \frac{w}{d}$ for some $q = p^e$. Then $f^{\mu_f(q)-1}$ has degree

$$d(\mu_f(q) - 1) = d \cdot \frac{w}{d} \cdot (q-1) = w(q-1).$$

Since we also know that $f^{\mu_f(q)-1} \notin \mathfrak{m}^{[q]}$, it follows that $f^{\mu_f(q)-1}$ generates $\operatorname{Soc}\left(R/\mathfrak{m}^{[q]}\right)$. Therefore, $\left(f^{\mu_f(q)-1}\right)^{\frac{pq-1}{q-1}}$ generates $\operatorname{Soc}\left(R/\mathfrak{m}^{[pq]}\right)$, so

$$(\mu_f(q) - 1)\,\frac{pq-1}{q-1} = \mu_f(pq) - 1.$$

Rearranging the terms one obtains $\frac{\mu_f(pq)-1}{pq-1} = \frac{\mu_f(q)-1}{q-1} = \frac{w}{d}$.
In order to prove the second statement, note that by equation (1.1) we only need to show that $\mu_f(pq) < p\mu_f(q)$ cannot occur. Therefore, suppose that $\mu_f(pq) < p\mu_f(q)$. First we will show that $\mu_f(pq)$ cannot be a multiple of p. Suppose $\mu_f(pq) = pl$ for some l. Then, by equation (2.1) it follows that $p\mu_f(q) = pl = \mu_f(pq)$, which is a contradiction. We can now use Proposition 2.5 and get

$$\mu_f(pq) \geq \frac{w(pq+1) - nd}{d}.$$

Using $\mu_f(pq) < p\mu_f(q)$ and the second assumption, namely $d\mu_f(q) < wq$, we get

$$w(pq+1) - nd \leq d\mu_f(pq) \leq dp\mu_f(q) - d \leq wpq - p - d.$$

This gives $p \leq nd - d - w$, which contradicts our assumption on p. □

To see how one can calculate the F-pure threshold of a quasi-homogeneous polynomial using Proposition 2.5, Proposition 2.6 and Proposition 2.7, let us consider an example.

Example 2.8. Let $f := xy^2 + x^4$, then f is quasi-homogeneous of degree 8 and type $\alpha := (2, 3)$. Let $p \neq 2$. We claim that

$$\operatorname{fpt}(f) = \begin{cases} \frac{5}{8}, & \text{if } p \equiv 1, 3 \bmod 8 \\ \frac{5}{8} - \frac{1}{8p}, & \text{if } p \equiv 5 \bmod 8 \\ \frac{5}{8} - \frac{3}{8p}, & \text{if } p \equiv 7 \bmod 8. \end{cases}$$

By Proposition 2.6 we get

$$\mu_f(p) \leq \left\lceil \frac{5p-4}{8} \right\rceil = \left\lceil p - \frac{3p+4}{8} \right\rceil \leq p - 1.$$

With Proposition 2.5 it follows

$$\mu_f(p) \geq \left\lceil \frac{5p-3}{8} \right\rceil.$$

Since $\left\lceil \frac{5p-4}{8} \right\rceil \leq \left\lceil \frac{5p-3}{8} \right\rceil$, we conclude that $\mu_f(p) = \left\lceil \frac{5p-3}{8} \right\rceil$.
First, let $p \equiv 1 \bmod 8$, then $\mu_f(p) = \frac{5}{8}p + \frac{3}{8}$. With the first part of Proposition 2.7 it follows that $\mu_f(q) = \frac{5}{8}q + \frac{3}{8}$. Hence, $\operatorname{fpt}(f) = \lim_{e\to\infty} \frac{\mu_f(p^e)}{p^e} = \frac{5}{8}$.
Secondly, let $p \equiv 3 \bmod 8$, then $\mu_f(p) = \frac{5}{8}p + \frac{1}{8}$. Since we cannot use Proposition 2.7 for $\mu_f(p)$, we compute $\mu_f(p^2)$ in the same way we computed $\mu_f(p)$. We get $\mu_f(p^2) = \frac{5}{8}p^2 + \frac{3}{8}$. Now, with the first part of Proposition 2.7 it follows $\mu_f(q) = \frac{5}{8}q + \frac{3}{8}$ for all $q = p^e \geq p^2$. Hence, $\operatorname{fpt}(f) = \lim_{e\to\infty} \frac{\mu_f(p^e)}{p^e} = \frac{5}{8}$.
Now let $p \equiv 5 \bmod 8$, then $\mu_f(p) = \frac{5}{8}p - \frac{1}{8}$. With the second part of Proposition 2.7 it follows that $\left\{ \frac{\mu_f(q)}{q} \right\}_q$ is a constant sequence. Hence, $\operatorname{fpt}(f) = \frac{5}{8} - \frac{1}{8p}$.

The last case is $p \equiv 7 \bmod 8$. Here $\mu_f(p) = \frac{5}{8}p - \frac{3}{8}$. With the second part of Proposition 2.7 it follows that $\left\{\frac{\mu_f(q)}{q}\right\}_q$ is a constant sequence. Hence, $\mathrm{fpt}(f) = \frac{5}{8} - \frac{3}{8p}$.

As a consequence of the above results we obtain the following theorem, which is very similar to the one given in [BS15] for homogeneous polynomials.

Theorem 2.9. *Let $f \in K[x_0, \ldots, x_n]$ be a quasi-homogeneous polynomial of degree $w = \sum_{i=0}^{n} \alpha_i$ with an isolated singularity. Then:*

(1) $\mu_f(p) = p - h$, where $0 \leq h \leq n-1$ is an integer.

(2) $\mu_f(pq) = p\mu_f(q)$ for all q with $q \geq n-1$.

(3) If $p \geq n-1$, then $\mathrm{fpt}(f) = 1 - \frac{h}{p}$, where $0 \leq h \leq n-1$.

Proof. First, suppose $\mu_f(p) = p$. Then by Proposition 2.7 (1) we get $\mu_f(q) = q$ for all q, so the first two assertions follow.

Now, suppose $\mu_f(p) < p$. Then by Proposition 2.5 it follows that

$$\mu_f(p) \geq \frac{w(p+1) - nw}{w} = p + 1 - n,$$

which gives $\mu_f(p) = p - h$ with $0 < h \leq n-1$. To prove the second assertion, suppose $\mu_f(pq) < p\mu_f(q)$ (see equation (1.1)). Then $p \nmid \mu_f(pq)$, since otherwise $\mu_f(pq) = p\mu_f(q)$ by equation (2.1). Thus Proposition 2.5 yields

$$\mu_f(pq) \geq pq + 1 - n.$$

Since $\mu_f(p) \leq p - 1$, it follows with equation (1.1) that $\mu_f(q) \leq q - \frac{q}{p}$ and therefore

$$\mu_f(pq) \leq p\mu_f(q) - 1 \leq pq - q - 1.$$

Combining these two results we get

$$pq + 1 - n \leq \mu_f(pq) \leq pq - q - 1,$$

which is equivalent to $q \leq n - 2$. The third assertion easily follows from (1) and (2). $\square$

In the homogeneous case, Bhatt and Singh [BS15] relate the integer h that appears in Theorem 2.9 to the so-called Hasse invariant. The aim of the next two sections is to answer the following question: What is h in the quasi-homogeneous case? For this, we first consider the case of a curve and then we pass on to the case $n > 2$.

2.2 The case of a curve

In this section let $R \coloneqq K[x,y,z]$, where K is a perfect field of characteristic $p > 0$ and let $\mathfrak{m} \coloneqq (x,y,z)$ be the maximal ideal of R. Let

$$\deg(x) \coloneqq \alpha_x, \deg(y) \coloneqq \alpha_y \text{ and } \deg(z) \coloneqq \alpha_z$$

and let $f \in R$ be a quasi-homogeneous polynomial of degree $w \coloneqq \alpha_x + \alpha_y + \alpha_z$ and type $\alpha \coloneqq (\alpha_x, \alpha_y, \alpha_z)$ with an isolated singularity. By

$$C \coloneqq \operatorname{Proj}(R/fR)$$

we denote the elliptic curve given by f. Then by [Rei, Proposition 3.3 and 3.4] the curve C is in fact projective, thus by [CL98, Exposé III, 4.4] C is ordinary if and only if the map

$$F : H^1(C, \mathcal{O}_C) \to H^1(C, \mathcal{O}_C)$$

induced by the Frobenius is bijective. Since K is perfect, F is bijective if and only if F is injective.
In order to prove the main theorem of this section, we need the following two lemmata.

Lemma 2.10. *If the coefficient of* $(xyz)^{p-1}$ *in* f^{p-1} *is non-zero, then* $\operatorname{fpt}(f) = 1$. *Otherwise* $\operatorname{fpt}(f) = 1 - \frac{1}{p}$.

Proof. First, suppose that the coefficient of $(xyz)^{p-1}$ in f^{p-1} is non-zero. This means that the coefficient of $\frac{1}{xyz}$ in $\frac{f^{p-1}}{(xyz)^p}$ is non-zero. By Lemma 2.1 this is equivalent to $\mu_f(p) > p-1$. Since $\mu_f(p) \leq p$, we get $\mu_f(p) = p$. Therefore,

$$\mu_f(q) = q$$

by Proposition 2.7 (1) and it follows $\operatorname{fpt}(f) = 1$.
Now, let the coefficient of $(xyz)^{p-1}$ in f^{p-1} be zero, which implies that the coefficient of $\frac{1}{xyz}$ in $\frac{f^{p-1}}{(xyz)^p}$ is zero. Again, by Lemma 2.1 this is equivalent to $\mu_f(p) \leq p-1$, thus

$$\frac{\mu_f(p)}{p} \leq 1 - \frac{1}{p} < 1.$$

By Proposition 2.7 (2) it follows that the sequence $\left\{\frac{\mu_f(q)}{q}\right\}_q$ is constant. Since $\operatorname{fpt}(f) \in \left\{1, 1-\frac{1}{p}\right\}$ by Theorem 2.9 (3), this gives $\operatorname{fpt}(f) = \frac{\mu_f(p)}{p} = 1 - \frac{1}{p}$. ☐

If $f \in K[x_0, \ldots, x_n]$ is quasi-homogeneous of degree $w \coloneqq \sum_{i=0}^n \alpha_i$ (where $\alpha_i \coloneqq \deg(x_i)$), then a similar argument as above shows: If $p \geq w(n-2)+1$, then $\operatorname{fpt}(f) = 1$ if and only if the coefficient of $(x_0 \cdots x_n)^{p-1}$ in f^{p-1} is non-zero.

Lemma 2.11. *Let $C = \operatorname{Proj}(R/fR)$ be the curve given by the quasi-homogeneous polynomial $f \in K[x,y,z]$ of degree w and type α with an isolated singularity. Then $H^1(C, \mathcal{O}_C) = \operatorname{Soc}\left(H^2_{\mathfrak{m}}(R/fR)\right)$.*

Proof. Using local cohomology [ILL$^+$07, Theorem 13.21], we get

$$H^1(C, \mathcal{O}_C) = H^2_{\mathfrak{m}}(R/fR)_0.$$

Thus, it is enough to show that $H^2_{\mathfrak{m}}(R/fR)_0 = \operatorname{Soc}\left(H^2_{\mathfrak{m}}(R/fR)\right)$. By Lemma 2.1 we know that $H^2_{\mathfrak{m}}(R/fR)$ is a submodule of $H^3_{\mathfrak{m}}(R)(-w)$ and we know that

$$\operatorname{Soc}\left(H^3_{\mathfrak{m}}(R)(-w)\right) = \left\langle \left[\frac{1}{xyz}\right] \right\rangle,$$

which is the degree zero part of $H^3_{\mathfrak{m}}(R)(-w)$. Since

$$\mathfrak{m} \cdot H^2_{\mathfrak{m}}(R/fR)_0 \subset H^2_{\mathfrak{m}}(R/fR)_{>0} = 0,$$

it follows that $H^2_{\mathfrak{m}}(R/fR)_0 \subseteq \operatorname{Soc}\left(H^2_{\mathfrak{m}}(R/fR)\right)$. Furthermore, $H^2_{\mathfrak{m}}(R/fR) \neq 0$, hence

$$\operatorname{Soc}\left(H^2_{\mathfrak{m}}(R/fR)\right) = H^2_{\mathfrak{m}}(R/fR) \cap \operatorname{Soc}\left(H^3_{\mathfrak{m}}(R)(-w)\right) \neq 0$$

(see Lemma 2.2). Therefore, $H^2_{\mathfrak{m}}(R/fR)_0 = \operatorname{Soc}\left(H^2_{\mathfrak{m}}(R/fR)\right)$. □

Using these two lemmata, we deduce the main theorem of this section, which generalizes the two-dimensional case of the main theorem of [BS15] to the quasi-homogeneous case.

Theorem 2.12. *Let $C = \operatorname{Proj}(R/fR)$ be the curve given by the quasi-homogeneous polynomial $f \in K[x,y,z]$ of degree w and type α with an isolated singularity. Then*

$$\operatorname{fpt}(f) = \begin{cases} 1, & \text{if } C \text{ is ordinary} \\ 1 - \frac{1}{p}, & \text{otherwise.} \end{cases}$$

Proof. The curve C is ordinary if and only if $F : H^1(C, \mathcal{O}_C) \to H^1(C, \mathcal{O}_C)$ is injective. Using Lemma 2.11, we have shown that C is ordinary if and only if

$$F : \operatorname{Soc}\left(H^2_{\mathfrak{m}}(R/fR)\right) \to \operatorname{Soc}\left(H^2_{\mathfrak{m}}(R/fR)\right)$$

is injective. But this is equivalent to the fact that

$$\widetilde{F^1_1} : H^2_{\mathfrak{m}}(R/fR) \to H^2_{\mathfrak{m}}(R/fR)$$

is injective (see Lemma 2.1). Equivalently, the coefficient of $\frac{1}{xyz}$ in $\frac{f^{p-1}}{(xyz)^p}$ (which is the coefficient of $(xyz)^{p-1}$ in f^{p-1}) is non-zero. By Lemma 2.10 the result follows. □

We conclude this section with some examples.

Example 2.13. Up to permutation there are three solutions $(a, b, c) \in \mathbb{N}^3$ of the equation $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, namely

$$(3,3,3), (2,4,4) \text{ and } (2,3,6).$$

We will consider the corresponding elliptic singularities that occur in the classification of Arnold directly after the ADE-singularities (see for example [AGZV85]):

$$\begin{aligned} \tilde{E_6} = P_8 : \quad & x^3 + y^3 + z^3 + \lambda xyz = 0, \\ \tilde{E_7} = X_9 : \quad & x^2 + y^4 + z^4 + \lambda xyz = 0, \\ \tilde{E_8} = J_{10} : \quad & x^2 + y^3 + z^6 + \lambda xyz = 0. \end{aligned}$$

The aim is to compute the F-pure threshold of these three polynomials. In order to do this by Lemma 2.10 it is enough to compute the coefficient of $(xyz)^{p-1}$ in the $(p-1)$-th power of the respective polynomial.
Let us start with $f_\lambda \coloneqq x^3 + y^3 + z^3 + \lambda xyz$, which is (quasi)-homogeneous of degree 3 and type $\alpha \coloneqq (1,1,1)$. First, we compute f_λ^{p-1}:

$$\begin{aligned} & \left(x^3 + y^3 + z^3 + \lambda xyz\right)^{p-1} \\ & = \sum_{n=0}^{p-1} \binom{p-1}{n} \lambda^{p-1-n} (xyz)^{p-1-n} (x^3 + y^3 + z^3)^n \\ & = \sum_{n=0}^{p-1} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{p-1}{n} \binom{n}{k} \binom{n-k}{l} \lambda^{p-1-n} x^{3k+p-1-n} y^{3l+p-1-n} z^{3(n-k-l)+p-1-n}. \end{aligned}$$

Thus, in order to compute the coefficient of $(xyz)^{p-1}$, we need to solve the following three equations:

$$3k = n,\ 3l = n \ \text{ and } \ 3(n-k-l) = n.$$

Using this, the coefficient of $(xyz)^{p-1}$ in f_λ^{p-1} is

$$\varphi(\lambda) \coloneqq \sum_{s=0}^{\lfloor \frac{p-1}{3} \rfloor} \binom{p-1}{3s} \binom{3s}{s} \binom{2s}{s} \lambda^{p-1-3s} = \sum_{s=0}^{\lfloor \frac{p-1}{3} \rfloor} \frac{(3s)!}{(s!)^3} (-1)^{3s} \lambda^{p-1-3s}$$

since $\binom{p-1}{3s} \equiv (-1)^{3s} \bmod p$. Therefore,

$$\operatorname{fpt}(f_\lambda) = \begin{cases} 1, & \text{if } \varphi(\lambda) \neq 0 \\ 1 - \frac{1}{p}, & \text{if } \varphi(\lambda) = 0. \end{cases}$$

Now, let us consider $f_\lambda \coloneqq x^2 + y^4 + z^4 + \lambda xyz$, which is quasi-homogeneous of degree 4 and type $\alpha \coloneqq (2,1,1)$. Similar to the above we compute that the coefficient of $(xyz)^{p-1}$ in f_λ^{p-1} is given by

$$\varphi(\lambda) \coloneqq \sum_{s=0}^{\lfloor \frac{p-1}{4} \rfloor} \frac{(4s)!}{(2s)!(s!)^2} \lambda^{p-1-4s}.$$

Therefore,

$$\operatorname{fpt}(f_\lambda) = \begin{cases} 1, & \text{if } \varphi(\lambda) \neq 0 \\ 1 - \frac{1}{p}, & \text{if } \varphi(\lambda) = 0. \end{cases}$$

Lastly, we consider $f_\lambda \coloneqq x^2 + y^3 + z^6 + \lambda xyz$, which is quasi-homogeneous of degree 6 and type $\alpha \coloneqq (3, 2, 1)$. The coefficient of $(xyz)^{p-1}$ in f_λ^{p-1} is given by

$$\varphi(\lambda) \coloneqq \sum_{s=0}^{\lfloor \frac{p-1}{6} \rfloor} \frac{(6s)!}{(3s)!(2s)!(s!)} \lambda^{p-1-6s}.$$

Therefore,

$$\operatorname{fpt}(f_\lambda) = \begin{cases} 1, & \text{if } \varphi(\lambda) \neq 0 \\ 1 - \frac{1}{p}, & \text{if } \varphi(\lambda) = 0. \end{cases}$$

Remark 2.14. Consider the period

$$\psi(\lambda) \coloneqq \frac{1}{(2\pi i)^3} \oint \frac{\lambda xyz}{f_\lambda} \frac{dx}{x} \frac{dy}{y} \frac{dz}{z}$$

of $f_\lambda = x^a + y^b + z^c + \lambda xyz$, where (a, b, c) is one of the three triples of Example 2.13 and $\lambda \neq 0$. One can compute that

$$\psi(\lambda) = \sum_{n=0}^{\infty} \left(\frac{-1}{\lambda}\right)^n \left[\left(\frac{x^a + y^b + z^c}{xyz}\right)^n\right]_0,$$

where $[-]_0$ denotes the degree zero part. By computing the degree zero part of

$$\left(\frac{x^a + y^b + z^c}{xyz}\right)^n = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} x^{ak-n} y^{bl-n} z^{c(n-k-l)-n},$$

which is given by the relations $ak = n$, $bl = n$ and $c(n - k - l) = n$, it is easy to see that the corresponding polynomial

$$\psi(\lambda) = \sum_{n=0}^{\lfloor \frac{p-1}{w} \rfloor} \left(\frac{-1}{\lambda}\right)^n \left[\left(\frac{x^a + y^b + z^c}{xyz}\right)^n\right]_0$$

is equal to the polynomial $\varphi(\lambda)$ computed in Example 2.13.

Example 2.15. Next, we want to consider the $T_{a,b,c}$–singularities given by

$$f_\lambda = x^a + y^b + z^c + \lambda xyz \quad \text{with} \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1 \quad \text{and} \quad \lambda \neq 0.$$

Since f_λ is not quasi-homogeneous, we cannot use Lemma 2.10. Instead, we remember that the F-pure threshold of f_λ was defined by $\operatorname{fpt}(f_\lambda) = \lim_{e\to\infty} \frac{\mu_{f_\lambda}(p^e)}{p^e}$ with $\mu_{f_\lambda}(p^e) = \min\left\{k \in \mathbb{N} \mid f_\lambda^k \in \mathfrak{m}^{[p^e]}\right\}$. We will show that the coefficient of $(xyz)^{q-1}$ in f_λ^{q-1} is 1, where $q = p^e$. This means that $f_\lambda^{q-1} \notin \mathfrak{m}^{[q]}$, but obviously $f_\lambda^q \in \mathfrak{m}^{[q]}$. Thus, $\mu_{f_\lambda}(p^e) = p^e$ and therefore $\operatorname{fpt}(f_\lambda) = 1$ for all λ.
Now, it remains to compute the coefficient of $(xyz)^{q-1}$ in f_λ^{q-1}:

$$\begin{aligned}
&\left(x^a + y^b + z^c + \lambda xyz\right)^{q-1} \\
&= \sum_{n=0}^{q-1} \binom{q-1}{n} \lambda^{q-1-n} (xyz)^{q-1-n} (x^a + y^b + z^c)^n \\
&= \sum_{n=0}^{q-1} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{q-1}{n} \binom{n}{k} \binom{n-k}{l} \lambda^{q-1-n} x^{ak+q-1-n} y^{bl+q-1-n} z^{c(n-k-l)+q-1-n}.
\end{aligned}$$

We have to solve the equations

$$ak = n, \; bl = n \quad \text{and} \quad c(n-k-l) = n$$

but since $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$, the third equation is never satisfied except for $n = k = l = 0$. Therefore, the coefficient of $(xyz)^{q-1}$ in f_λ^{q-1} is $\varphi(\lambda) = \lambda^{q-1} \equiv 1$.

2.3 The general case

Now, let us come back to the situation of Theorem 2.9. Let K be a field of characteristic $p > 0$. Recall that we consider $R = K[x_0, \ldots, x_n]$ with maximal ideal $\mathfrak{m} = (x_0, \ldots, x_n)$ and let $f \in R$ be a quasi-homogeneous polynomial of degree $w = \sum_{i=0}^n \alpha_i$ with an isolated singularity, where $\alpha_i = \deg(x_i)$.
Similar to the homogeneous case ([BS15]) we want to relate the integer h that appears in Theorem 2.9 to the order of vanishing of the Hasse invariant on some deformation space of $X = \operatorname{Proj}(R/fR)$. For this, let us first fix some more notation.
We consider the family $\pi : \mathfrak{X} \to \operatorname{Hyp}_w$ of hypersurfaces of degree w in the weighted projective space $\mathbb{P}^n(\alpha_0, \ldots, \alpha_n)$. Our chosen hypersurface $X = \operatorname{Proj}(R/fR)$ gives a point $[X]$ in Hyp_w. Set

$$G := \sum_{i=1}^{m} \tilde{s}_i g_i \in K[x_0, \ldots, x_n, \tilde{s}_1, \ldots, \tilde{s}_m],$$

where $\{g_1, \ldots, g_m\} \subset K[x_0, \ldots, x_n]$ is the set of monomials of degree w and we set $\deg(\tilde{s}_i) \coloneqq 0$ for $1 \leq i \leq m$ (such that G is quasi-homogeneous of degree w). The family π of hypersurfaces of degree w in $\mathbb{P}^n(\alpha_0, \ldots, \alpha_n)$ is given by

$$\begin{tikzcd}
\mathfrak{X} = \operatorname{Proj}_{\mathbb{P}^{m-1}}\left(\mathcal{O}_{\mathbb{P}^{m-1}}[x_0, \ldots, x_n]/G\right) \arrow[r, "i"] \arrow[dr, "\pi"] & \operatorname{Proj}_{\mathbb{P}^{m-1}}\left(\mathcal{O}_{\mathbb{P}^{m-1}}[x_0, \ldots, x_n]\right) \arrow[d] \\
& \mathbb{P}^{m-1},
\end{tikzcd} \tag{2.2}$$

where i is a closed immersion. If $f = \sum_{i=1}^m f_i g_i$, $f_i \in K$, is the defining equation of X in the weighted projective space $\mathbb{P}^n(\alpha_0, \ldots, \alpha_n)$, then X is the fiber over $(f_1, \ldots, f_m)$, i.e. $[X] \in \mathrm{Hyp}_w$ is equal to $V\left(\tilde{s}_i - f_i \mid i \in \{1, \ldots, m\}\right)$. The aim of this section is to prove the following theorem:

Theorem 2.16. *If $p \geq w(n-2)+1$, then the integer h in Theorem 2.9 is the order of vanishing of the Hasse invariant at $[X] \in \mathrm{Hyp}_w$ on the deformation space $\mathfrak{X}$ of $X \subset \mathbb{P}^n(\alpha_0, \ldots, \alpha_n)$ described above.*

First, let us recall the definition of the *Hasse invariant* of a family of varieties in characteristic p (see for example [BS15]). Fix a proper flat morphism $\pi : \mathfrak{X} \to S$ of relative dimension N between noetherian $\mathbb{F}_p$-schemes and assume that the formation of $R^N\pi_*\mathcal{O}_{\mathfrak{X}}$ is compatible with base change, i.e. if we have the following pullback diagram

$$\begin{tikzcd}
\mathfrak{X} \times_S T \arrow[r, "i^{(1)}"] \arrow[d, "\pi^{(1)}"] & \mathfrak{X} \arrow[d, "\pi"] \\
T \arrow[r, "i"] & S,
\end{tikzcd}$$

then $i^* R^N\pi_*\mathcal{O}_{\mathfrak{X}} \cong R^N\pi^{(1)}_* {i^{(1)}}^* \mathcal{O}_{\mathfrak{X}}$. Furthermore, assume that $R^N\pi_*\mathcal{O}_{\mathfrak{X}}$ is a line bundle.
Consider the Frobenius twist $\mathfrak{X}^{(1)} = \mathfrak{X} \times_{\mathrm{Frob}_S} S$ of $\mathfrak{X}$, which gives the following diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[dr, dashed, "\mathrm{Frob}_{\mathfrak{X}/S}"] \arrow[drr, bend left, "\mathrm{Frob}_{\mathfrak{X}}"] \arrow[ddr, bend right, "\pi"'] & & \\
& \mathfrak{X}^{(1)} \arrow[r] \arrow[d, "\pi^{(1)}"] & \mathfrak{X} \arrow[d, "\pi"] \\
& S \arrow[r, "\mathrm{Frob}_S"] & S.
\end{tikzcd}$$

By the base change assumption we have

$$R^N\pi^{(1)}_*\mathcal{O}_{\mathfrak{X}^{(1)}} \cong \mathrm{Frob}_S^* R^N\pi_*\mathcal{O}_{\mathfrak{X}} \cong \left(R^N\pi_*\mathcal{O}_{\mathfrak{X}}\right)^p. \tag{2.3}$$

The relative Frobenius $\mathrm{Frob}_{\mathfrak{X}/S}$ induces a map $\mathcal{O}_{\mathfrak{X}^{(1)}} \to \left(\mathrm{Frob}_{\mathfrak{X}/S}\right)_* \mathcal{O}_{\mathfrak{X}}$ and this induces a map

$$H : R^N\pi_*^{(1)}\mathcal{O}_{\mathfrak{X}^{(1)}} \to R^N\pi_*^{(1)}\left(\mathrm{Frob}_{\mathfrak{X}/S}\right)_* \mathcal{O}_{\mathfrak{X}},$$

which is called the Hasse invariant of the family π. Since $\pi = \pi^{(1)} \circ \mathrm{Frob}_{\mathfrak{X}/S}$ the Hasse invariant is an element of $\mathrm{Hom}\left(R^N\pi_*^{(1)}\mathcal{O}_{\mathfrak{X}^{(1)}}, R^N\pi_*\mathcal{O}_{\mathfrak{X}}\right)$. Using (2.3) it follows that

$$\begin{aligned}\mathrm{Hom}\left(R^N\pi_*^{(1)}\mathcal{O}_{\mathfrak{X}^{(1)}}, R^N\pi_*\mathcal{O}_{\mathfrak{X}}\right) &\cong \mathrm{Hom}\left(\left(R^N\pi_*\mathcal{O}_{\mathfrak{X}}\right)^p, R^N\pi_*\mathcal{O}_{\mathfrak{X}}\right)\\ &\cong \mathrm{Hom}\left(\mathcal{O}_S, \left(R^N\pi_*\mathcal{O}_{\mathfrak{X}}\right)^{1-p}\right)\\ &\cong H^0\left(S, \left(R^N\pi_*\mathcal{O}_{\mathfrak{X}}\right)^{1-p}\right),\end{aligned}$$

hence, the Hasse invariant H is a section of a line bundle.
Next, we want to give a description of the *order of vanishing of the Hasse invariant.* For this, fix $s \in S$ and an integer $t \geq 0$. Let $t[s]$ be the order t neighbourhood of s, i.e. it is defined by the t-th power of the ideal defining s, and let $t\mathfrak{X}_s \subset \mathfrak{X}$ respectively $t\mathfrak{X}_s^{(1)} \subset \mathfrak{X}^{(1)}$ be the corresponding neighbourhoods of the fibers of π respectively of $\pi^{(1)}$, i.e. we have the following cartesian diagrams

$$\begin{tikzcd} t\mathfrak{X}_s \arrow[r, "\widetilde{i}"] \arrow[d, "\widetilde{\pi}"] & \mathfrak{X} \arrow[d, "\pi"] \\ t[s] \arrow[r, hook, "i"] & S \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} t\mathfrak{X}_s^{(1)} \arrow[r, "\widetilde{i^{(1)}}"] \arrow[d, "\widetilde{\pi^{(1)}}"] & \mathfrak{X}^{(1)} \arrow[d, "\pi^{(1)}"] \\ t[s] \arrow[r, hook, "i"] & S. \end{tikzcd}$$

The map $\mathrm{Frob}_{\mathfrak{X}/S}$ induces maps $t\mathfrak{X}_s \to t\mathfrak{X}_s^{(1)}$ for all t and hence maps

$$\phi_t : H^N\left(t\mathfrak{X}_s^{(1)}, \mathcal{O}_{t\mathfrak{X}_s^{(1)}}\right) \to H^N\left(t\mathfrak{X}_s, \mathcal{O}_{t\mathfrak{X}_s}\right).$$

The two diagrams above demonstrate that the maps ϕ_t are given by i^*H, where $i : t[s] \hookrightarrow S$. For this, consider

$$i^*H : i^*R^N\pi_*^{(1)}\mathcal{O}_{\mathfrak{X}^{(1)}} \to i^*R^N\pi_*^{(1)}\left(\mathrm{Frob}_{\mathfrak{X}/S}\right)_* \mathcal{O}_{\mathfrak{X}}.$$

Here, by the base change assumption, we have

$$i^*R^N\pi_*^{(1)}\mathcal{O}_{\mathfrak{X}^{(1)}} = R^N\widetilde{\pi^{(1)}}_*\left(\widetilde{i^{(1)}}\right)^*\mathcal{O}_{\mathfrak{X}^{(1)}} = R^N\widetilde{\pi^{(1)}}_*\mathcal{O}_{t\mathfrak{X}_s^{(1)}} = H^N\left(t\mathfrak{X}_s^{(1)}, \mathcal{O}_{t\mathfrak{X}_s^{(1)}}\right)$$

and similarly

$$\begin{aligned}i^*R^N\pi_*^{(1)}\left(\mathrm{Frob}_{\mathfrak{X}/S}\right)_*\mathcal{O}_{\mathfrak{X}} = i^*R^N\pi_*\mathcal{O}_{\mathfrak{X}} &= R^N\widetilde{\pi}_*\widetilde{i}^*\mathcal{O}_{\mathfrak{X}} = R^N\widetilde{\pi}_*\mathcal{O}_{t\mathfrak{X}_s}\\ &= H^N\left(t\mathfrak{X}_s, \mathcal{O}_{t\mathfrak{X}_s}\right),\end{aligned}$$

since $\pi = \pi^{(1)} \circ \mathrm{Frob}_{\mathfrak{X}/S}$. Using this identification of ϕ_t with i^*H, one obtains the following lemma:

Lemma 2.17. *The order of vanishing of the Hasse invariant H at the point $s \in S$ is* $\operatorname{ord}_s(H) = \max\{t \mid \phi_t = 0\}$.

Later, we will need the following reformulation of the Hasse invariant (for a proof see [BS15]):

Lemma 2.18 ([BS15, Lemma 4.5])**.** *If $\psi_t : H^N(\mathfrak{X}_s, \mathcal{O}_{\mathfrak{X}_s}) \to H^N(t\mathfrak{X}_s, \mathcal{O}_{t\mathfrak{X}_s})$ induced by* $\mathrm{Frob}_{\mathfrak{X}}$ *is non-zero for some $t \leq p$, then* $\operatorname{ord}_s(H)+1 = \min\{t \mid \psi_t \neq 0\}$.

Now, let us come back to our family π of hypersurfaces of degree w in $\mathbb{P}^n(\alpha_0, \ldots, \alpha_n)$. Using diagram (2.2), one can check that π is a proper morphism of relative dimension $n-1$ between noetherian $\mathbb{F}_p$-schemes and Lemma 9.3.4 of [FGI+05] shows that π is also flat. Furthermore, one can prove that $R^{n-1}\pi_*\mathcal{O}_{\mathfrak{X}}$ is a line bundle (see for example [Ogu01, p. 35]). The compatibility of $R^{n-1}\pi_*\mathcal{O}_{\mathfrak{X}}$ with base change follows from [Mum66, p. 51] or EGA III ([Gro63, 7.7]) since $H^n(\mathfrak{X}_s, \mathcal{O}_{\mathfrak{X}_s}) = 0$. Thus, the order of vanishing of the Hasse invariant at $[X] \in \mathrm{Hyp}_w$ on the space of hypersurfaces $\mathfrak{X}$ is defined.
In order to start with the proof of Theorem 2.16, let us first remark that it suffices to consider the affine situation, i.e. we work on the left side of the following diagram

$$\begin{tikzcd}
\mathrm{Proj}_{\mathbb{A}^m\setminus 0}\left(\mathcal{O}_{\mathbb{A}^m\setminus 0}[x_0, \ldots, x_n]/G\right) \arrow[r] \arrow[d] & \mathrm{Proj}_{\mathbb{P}^{m-1}}\left(\mathcal{O}_{\mathbb{P}^{m-1}}[x_0, \ldots, x_n]/G\right) \arrow[d] \\
(\mathbb{A}^m \setminus 0) \times \mathbb{P}^n(\alpha_0, \ldots, \alpha_n) \arrow[r] \arrow[d] & \mathrm{Proj}_{\mathbb{P}^{m-1}}\left(\mathcal{O}_{\mathbb{P}^{m-1}}[x_0, \ldots, x_n]\right) \arrow[d] \\
\mathbb{A}^m \setminus 0 \arrow[r] & \mathbb{P}^{m-1}.
\end{tikzcd}$$

Remember that $f = \sum_{i=1}^m f_i g_i$ is the defining equation of X and $G = \sum_{i=1}^m \tilde{s}_i g_i$. Changing coordinates via $s_i = \tilde{s}_i - f_i$, one obtains

$$X' := \pi^{-1}([X]) = \mathrm{Proj}\left(\tilde{R}/(F,s)\tilde{R}\right),$$

where $\tilde{R} := K[x_0, \ldots, x_n, s_1, \ldots, s_m]$, $F := f + \sum_{i=0}^m s_i g_i$ and $s = (s_1, \ldots, s_m)$. Furthermore, let tX respectively tX' be the order t neighbourhood of X in $\mathbb{P}^n(\alpha_0, \ldots, \alpha_n)$ respectively X' in $\mathfrak{X}$, i.e.

$$tX = \mathrm{Proj}\left(R/f^t R\right) \quad \text{and} \quad tX' = \mathrm{Proj}_{K[s]/s^t}\left(\tilde{R}/(F, s^t)\tilde{R}\right).$$

Proof of Theorem 2.16. For $1 \leq t \leq p$ we consider the following commutative diagram

$$\begin{tikzcd}
\tilde{R}/(F,s) \arrow[r, "\mathrm{Frob}_{\tilde{R}}"] \arrow[d, equal] & \tilde{R}/\left(F^p, s^{[p]}\right) \arrow[r, "a"] & \tilde{R}/(F, s^p) \arrow[r, "\tilde{pr}_1"] & \tilde{R}/(F, s^t) \arrow[r, "\tilde{pr}_2"] & \tilde{R}/(F,s) \arrow[d, equal] \\
R/f \arrow[r, "\mathrm{Frob}_R"] & R/f^{[p]} \arrow[u, "h_1"] \arrow[r, equal] & R/f^p \arrow[u, "h_2"] \arrow[r, "pr_1"] & R/f^t \arrow[u, "h_3"] \arrow[r, "pr_2"] & R/f,
\end{tikzcd}$$

where the maps $\tilde{pr}_1, \tilde{pr}_2, pr_1$ and pr_2 are the evident projections and the map a is given by the inclusion $\left(F^p, s^{[p]}\right) \subset (F, s^p)$, since $(s_1^p, \ldots, s_m^p) \subset (s_1, \ldots, s_m)^p$. The maps h_1, h_2 and h_3 are defined as follows: Obviously, we have a map $\varphi_1 : R \hookrightarrow \tilde{R} \twoheadrightarrow \tilde{R}/F^p \twoheadrightarrow \tilde{R}/\left(F^p, s^{[p]}\right)$, which induces the map h_1 if and only if $f^p \in \operatorname{Ker}(\varphi_1) = \left(F^p, s^{[p]}\right)$, which follows from

$$f^p = \left(F - \sum_{i=1}^m s_i g_i\right)^p = F^p + \left(-\sum_{i=1}^m s_i g_i\right)^p \in \left(F^p, s^{[p]}\right).$$

Similarly, we have a map $\varphi_2 : R \hookrightarrow \tilde{R} \twoheadrightarrow \tilde{R}/F \twoheadrightarrow \tilde{R}/\left(F, s^t\right)$, which induces the map h_3 if and only if $f^t \in \operatorname{Ker}(\varphi_2) = \left(F, s^t\right)$. But this is true, since

$$f^t = \left(F - \sum_{i=1}^m s_i g_i\right)^t = h \cdot F + g \cdot \left(\sum_{i=1}^m s_i g_i\right)^t \in \left(F, s^t\right)$$

for some $h, g \in \tilde{R}$. The same argument for $t = p$ gives h_2.
Passing to cohomology and taking the degree zero parts yields the following commutative diagram

$$\begin{array}{ccc} H^n_{\mathfrak{m}}\left(\tilde{R}/(F,s)\right)_0 & \xrightarrow{b_t} & H^n_{\mathfrak{m}}\left(\tilde{R}/\left(F,s^t\right)\right)_0 \\ \| & & \uparrow{\scriptstyle c_t} \\ H^n_{\mathfrak{m}}(R/f)_0 & \xrightarrow{a_t} & H^n_{\mathfrak{m}}\left(R/f^t\right)_0 . \end{array} \tag{2.4}$$

In order to prove Theorem 2.16, we now show the following four equalities

$$\begin{aligned} \operatorname{ord}_s(H) + 1 &= \min\left\{t \,\middle|\, H^{n-1}\left(X', \mathcal{O}_{X'}\right) \xrightarrow{b_t} H^{n-1}\left(tX', \mathcal{O}_{tX'}\right) \text{ injective }\right\} \\ &= \min\left\{t \,\middle|\, H^{n-1}\left(X, \mathcal{O}_{X}\right) \xrightarrow{a_t} H^{n-1}\left(tX, \mathcal{O}_{tX}\right) \text{ injective }\right\} \\ &= \min\left\{t \,\middle|\, H^n_{\mathfrak{m}}(R/fR) \xrightarrow{\widetilde{F^1_t}} H^n_{\mathfrak{m}}\left(R/f^tR\right) \text{ injective }\right\} \\ &= h + 1 \end{aligned} \tag{2.5}$$

and then clearly we get $h = \operatorname{ord}_s(H)$ (here we used [ILL+07, Theorem 13.21] to replace local cohomology by sheaf cohomology).
The first equality follows by Lemma 2.18. For the proof of the third equation we need the following lemma about the injectivity of Frobenius on the negatively graded part of local cohomology modules:

Lemma 2.19. *Let $f \in R$ be a quasi-homogeneous polynomial of degree d and type α with an isolated singularity. Let $t \leq p$. Then for $p \geq nd - w - td + 1$ the Frobenius action*

$$\widetilde{F^1_t} : \left[H^n_{\mathfrak{m}}(R/fR)\right]_{<0} \to \left[H^n_{\mathfrak{m}}(R/f^tR)\right]_{<0}$$

is injective.

Proof. As in the proof of Lemma 2.1 for $e = 1$ we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & [H^n_{\mathfrak{m}}(R/fR)]_{\leq -1} & \longrightarrow & \left[H^{n+1}_{\mathfrak{m}}(R)\right]_{\leq -d-1} & \longrightarrow & \cdots \\ & & \big\downarrow \widetilde{F^1_t} & & \big\downarrow f^{p-t}F & & \\ 0 & \longrightarrow & \left[H^n_{\mathfrak{m}}(R/f^tR)\right]_{\leq -p} & \longrightarrow & \left[H^{n+1}_{\mathfrak{m}}(R)\right]_{\leq -dt-p} & \longrightarrow & \cdots . \end{array}$$

and again it suffices to prove the injectivity of $f^{p-t}F$.
For this, let $\left[\frac{g}{(x_0\cdots x_n)^{q/p}}\right]$ be an element of $\left[H^{n+1}_{\mathfrak{m}}(R)\right]_{\leq -d-1}$, where g is quasi-homogeneous of degree $\deg(g) \leq w\frac{q}{p} - d - 1$ for some q. Suppose that $\left[\frac{g}{(x_0\cdots x_n)^{q/p}}\right] \in \operatorname{Ker}\left(f^{p-t}F\right)$, which means that

$$0 = f^{p-t}F\left(\left[\frac{g}{(x_0\cdots x_n)^{q/p}}\right]\right) = \left[\frac{f^{p-t}g^p}{(x_0\cdots x_n)^q}\right].$$

Therefore, $f^{p-t}g^p \in \mathfrak{m}^{[q]}$. Let

$$k := \min\left\{l \mid f^l g^p \in \mathfrak{m}^{[q]}\right\},$$

then $0 \leq k \leq p - t$. We want to show that $k = 0$, so suppose $k \neq 0$. Arguing as in the proof of Proposition 2.5, we get $f^{k-1}g^p J(f) \subset \mathfrak{m}^{[q]}$. By Lemma 2.3 and Lemma 2.4 it follows that

$$\begin{aligned} f^{k-1}g^p \in \left(\mathfrak{m}^{[q]} :_R J(f)\right) \setminus \mathfrak{m}^{[q]} &\subset \left(\mathfrak{m}^{[q]} :_R R_{\geq (n+1)d-2w+1}\right) \setminus \mathfrak{m}^{[q]} \\ &\subset R_{\geq (q+1)w-(n+1)d}, \end{aligned}$$

which implies that $(k-1)d + p\deg(g) = \deg(f^{k-1}g^p) \geq (q+1)w - (n+1)d$. Since $k \leq p - t$ and $\deg(g) \leq w\frac{q}{p} - d - 1$, a short computation yields $p \leq nd - w - td$, which is a contradiction. Therefore, $k = 0$, which means that $g^p \in \mathfrak{m}^{[q]}$. Thus, $\left[\frac{g}{(x_0\cdots x_n)^{q/p}}\right] = 0$ and therefore, $f^{p-t}F$ is injective. □

Using this, we can now prove the third equality of (2.5). Since $p \geq w(n-2)+1 \geq w(n-1-t)+1$ for $1 \leq t \leq p$ and using Lemma 2.19, we know that $\widetilde{F^1_t}$ is injective in negative degrees. Then, since $a_t = \left(\widetilde{F^1_t}\right)_0$, the asserted equality follows.
Next, we prove the fourth equation. By the first part of Theorem 2.9 we know that $\mu_f(p) = p-h$ and this gives the two inequalities $\mu_f(p) > p-h-1 = p-(h+1)$ and $\mu_f(p) \leq p - h$. Thus, by Lemma 2.1 it follows that $\widetilde{F^1_{h+1}}$ is injective, but $\widetilde{F^1_h}$ is not injective. Therefore, $h + 1 = \min\left\{t \mid \widetilde{F^1_t} \text{ injective}\right\}$.
Hence, to prove Theorem 2.16 it only remains to show the second equality of (2.5), which is the most difficult part of the proof. First, to simplify the notation

we will write in the following f^t instead of $f^t \cdot R$, f^{t-1}/f^t instead of $f^{t-1} \cdot R/f^t \cdot R$ and so forth.
Now, diagram (2.4) shows that it is sufficient to show that c_t is injective for all t. For this, consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & f^{t-1}/f^t \arrow[r] \arrow[d, "h_4"] & R/f^t \arrow[r] \arrow[d, "h_3"] & R/f^{t-1} \arrow[r] \arrow[d, "h_3"] & 0 \\
0 \arrow[r] & \left(F, s^{t-1}\right)/\left(F, s^t\right) \arrow[r] & \tilde{R}/\left(F, s^t\right) \arrow[r] & \tilde{R}/\left(F, s^{t-1}\right) \arrow[r] & 0.
\end{tikzcd}$$

Since $f^{t-1} \in \left(F, s^{t-1}\right)$, the map $\varphi_3 : f^{t-1} \hookrightarrow \left(F, s^{t-1}\right) \twoheadrightarrow \left(F, s^{t-1}\right)/\left(F, s^t\right)$ induces h_4. Passing to local cohomology we obtain the following diagram (notice that the diagram has been splitted in two lines due to space reasons)

$$\begin{tikzcd}
H^{n-1}_{\mathfrak{m}}\left(R/f^{t-1}\right)_0 \arrow[r] \arrow[d] & H^n_{\mathfrak{m}}\left(f^{t-1}/f^t\right)_0 \arrow[r, "a"] \arrow[d, "\phi_t"] & {} \\
H^{n-1}_{\mathfrak{m}}\left(\tilde{R}/\left(F, s^{t-1}\right)\right)_0 \arrow[r] & H^n_{\mathfrak{m}}\left(\left(F, s^{t-1}\right)/\left(F, s^t\right)\right)_0 \arrow[r, "b"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "a"] & H^n_{\mathfrak{m}}\left(R/f^t\right)_0 \arrow[r, "d"] \arrow[d, "c_t"] & H^n_{\mathfrak{m}}\left(R/f^{t-1}\right)_0 \arrow[d, "c_{t-1}"] \\
{} \arrow[r, "b"] & H^n_{\mathfrak{m}}\left(\tilde{R}/\left(F, s^t\right)\right)_0 \arrow[r, "e"] & H^n_{\mathfrak{m}}\left(\tilde{R}/\left(F, s^{t-1}\right)\right)_0 .
\end{tikzcd}$$

The rest of the proof mainly consists of the following three steps: We first show that b is injective and then conclude that it suffices to prove the injectivity of ϕ_t for all t in order to show that c_t is injective for all t. As a last step, we prove the injectivity of ϕ_t for all t.
Step 1: The map b is injective:
To prove this, we show that $H^{n-1}_{\mathfrak{m}}\left(\tilde{R}/\left(F, s^{t-1}\right)\right) = 0$. For $t = 1$ this is clear. For $t = 2$ we get

$$H^{n-1}_{\mathfrak{m}}\left(\tilde{R}/\left(F, s^{t-1}\right)\right) = H^{n-1}_{\mathfrak{m}}\left(\tilde{R}/(F, s)\right) = H^{n-1}_{\mathfrak{m}}(R/f) = 0.$$

Now let $t \geq 3$ and consider the exact sequence

$$0 \longrightarrow \left(F, s^{t-1}\right)/\left(F, s^t\right) \longrightarrow \tilde{R}/\left(F, s^t\right) \longrightarrow \tilde{R}/\left(F, s^{t-1}\right) \longrightarrow 0.$$

This gives the long exact sequence

$$H^{n-1}_{\mathfrak{m}}\left(\left(F, s^{t-1}\right)/\left(F, s^t\right)\right) \longrightarrow H^{n-1}_{\mathfrak{m}}\left(\tilde{R}/\left(F, s^t\right)\right) \longrightarrow H^{n-1}_{\mathfrak{m}}\left(\tilde{R}/\left(F, s^{t-1}\right)\right).$$

By induction, we know that $H^{n-1}_{\mathfrak{m}}\left(\tilde{R}/\left(F,s^{t-1}\right)\right)=0$ and in the following we will show that $H^{n-1}_{\mathfrak{m}}\left(\left(F,s^{t-1}\right)/\left(F,s^{t}\right)\right)=0$. Using this and the exact sequence above, it follows that $H^{n-1}_{\mathfrak{m}}\left(\tilde{R}/\left(F,s^{t}\right)\right)=0$. Therefore, it remains to show that $H^{n-1}_{\mathfrak{m}}\left(\left(F,s^{t-1}\right)/\left(F,s^{t}\right)\right)=0$. For this, we compute

$$\begin{aligned}\left(F,s^{t-1}\right)/\left(F,s^{t}\right)&=\left(F+s^{t}+s^{t-1}\right)/\left(F+s^{t}\right)\\&\cong\left(s^{t-1}\right)/\left(s^{t-1}\cap\left(F+s^{t}\right)\right)\\&\cong\left(s^{t-1}/s^{t}\right)/F(s^{t-1}/s^{t})\\&\cong\left(s^{t-1}/s^{t}\right)/f\left(s^{t-1}/s^{t}\right)\\&\cong R/f\otimes_{K}\left(s^{t-1}/s^{t}\right).\end{aligned}\tag{2.6}$$

Hence, using [ILL+07, Proposition 7.15], we get

$$\begin{aligned}H^{n-1}_{\mathfrak{m}}\left(\left(F,s^{t-1}\right)/\left(F,s^{t}\right)\right)&\cong H^{n-1}_{\mathfrak{m}}\left(R/f\otimes_{K}\left(s^{t-1}/s^{t}\right)\right)\\&\cong H^{n-1}_{\mathfrak{m}}\left(R/f\right)\otimes_{K}\left(s^{t-1}/s^{t}\right)\\&=0.\end{aligned}$$

Step 2: The second step of the proof is to show that it suffices to prove the injectivity of ϕ_t for all t, in order to prove the injectivity of c_t for all t. Again, we do this by induction on t. It is easy to see that c_1 is injective if and only if ϕ_1 is injective. Now, suppose $c_1,\ldots,c_{t-1}$ are injective. Then by a diagram chase and using that b and ϕ_t are injective, one can prove that c_t is also injective.
Step 3: Now, as a last step, we prove the injectivity of

$$\phi_t : H^{n}_{\mathfrak{m}}\left(f^{t-1}/f^{t}\right)_0\to H^{n}_{\mathfrak{m}}\left(\left(F,s^{t-1}\right)/\left(F,s^{t}\right)\right)_0.$$

For this, consider the projective resolution

$$0\longrightarrow f^{t}\longrightarrow f^{t-1}\longrightarrow f^{t-1}/f^{t}\longrightarrow 0$$

of f^{t-1}/f^{t} (remark that (f^t) can be identified with $R(-tw)$). Tensoring the sequence

$$0\longrightarrow R(-w)\xrightarrow{f}R\longrightarrow R/f\longrightarrow 0$$

with s^{t-1}/s^{t} over K yields the projective resolution

$$0\to R(-w)\otimes_K\left(s^{t-1}/s^{t}\right)\xrightarrow{f}R\otimes_K\left(s^{t-1}/s^{t}\right)\to R/f\otimes_K\left(s^{t-1}/s^{t}\right)\to 0$$

of $R/f\otimes_K\left(s^{t-1}/s^{t}\right)\cong\left(F,s^{t-1}\right)/\left(F,s^{t}\right)$ (see (2.6)). Altogether, we have the following situation

$$\begin{tikzcd}0 \arrow[r] & (f^{t-1})\cdot f \arrow[r] \arrow[d, dashed, "\theta"] & f^{t-1} \arrow[r] \arrow[d, dashed, "\theta"] & f^{t-1}/f^{t} \arrow[r] \arrow[d, dashed, "\theta"] & 0\\ 0 \arrow[r] & R(-w)\otimes_K\left(s^{t-1}/s^{t}\right) \arrow[r, "f"] & R\otimes_K\left(s^{t-1}/s^{t}\right) \arrow[r] & R/f\otimes_K\left(s^{t-1}/s^{t}\right) \arrow[r] & 0,\end{tikzcd}$$

where the map θ is induced by h_4 and sends f^{t-1} to $\left(\sum_{i=1}^{m} s_i g_i\right)^{t-1}$ considered as an element of $R/f \otimes_K \left(s^{t-1}/s^t\right) \cong \left(F, s^{t-1}\right) / \left(F, s^t\right)$, i.e. if we define $\{f_j\}$ to be a basis of the polynomials of degree $t-1$ in the variables $y_1, \dots y_m$ then

$$\theta\left(f^{t-1}\right) = \sum_j c_j f_j\left(g_1, \dots, g_m\right) \otimes f_j\left(s_1, \dots, s_m\right)$$

for some coefficients $c_j \in K$. If we apply the functor $\operatorname{Hom}_R\left(-, R(-w)\right)_0$ to the above diagram, we get a map

$$\psi_t : \operatorname{Hom}_R\left(R(-w) \otimes_K \left(s^{t-1}/s^t\right), R(-w)\right)_0 \to \operatorname{Hom}\left(f^t, R(-w)\right)_0,$$

which sends a map φ to $\varphi \circ \theta$ and induces a map

$$\psi_t : \operatorname{Ext}^1_R\left(\left(F, s^{t-1}\right) / \left(F, s^t\right), R(-w)\right)_0 \to \operatorname{Ext}^1_R\left(f^{t-1}/f^t, R(-w)\right)_0.$$

Here,

$$\operatorname{Ext}^1_R\left(f^{t-1}/f^t, R(-w)\right) = \operatorname{Hom}_R\left(f^t, R(-w)\right) \Big/ f \cdot \operatorname{Hom}_R\left(f^{t-1}, R(-w)\right)$$

and

$$\begin{aligned}&\operatorname{Ext}^1_R\left(\left(F, s^{t-1}\right) / \left(F, s^t\right), R(-w)\right)\\ &= \operatorname{Hom}_R\left(R(-w) \otimes_K \left(s^{t-1}/s^t\right), R(-w)\right) \Big/ f \cdot \operatorname{Hom}_R\left(R \otimes_K \left(s^{t-1}/s^t\right), R(-w)\right).\end{aligned}$$

By the functoriality of the local duality (see [BH93, Theorem 3.6.19]) the map ψ_t is equal to the map $\phi_t^\vee$ since ϕ_t is induced by the map θ by passing to local cohomology and then taking the degree zero parts.
Now, the idea is to prove the surjectivity of $\phi_t^\vee = \psi_t$ instead of proving the injectivity of ϕ_t by using the above description as Ext-modules. Therefore, we want to examine ψ_t more closely:

$$\begin{array}{ccc}\operatorname{Hom}_R\left(R(-w) \otimes_K \left(s^{t-1}/s^t\right), R(-w)\right)_0 & \xrightarrow{\cong} & \left(R(-w) \otimes_K \left(s^{t-1}/s^t\right)^\vee\right)_0 \\ \Big\downarrow{\scriptstyle \psi_t} & & \\ \operatorname{Hom}\left(f^t, R(-w)\right)_0 & & \\ \Big\downarrow{\scriptstyle \cong} & & \\ \operatorname{Hom}\left(R(-tw), R(-w)\right)_0 & & \\ \Big\downarrow{\scriptstyle \cong} & & \\ R((t-1)w)_0 = R_{(t-1)w}, & & \end{array}$$

where the first vertical isomorphism is given by the identification of f^t with $R(-tw)$ and the second vertical isomorphism is given by $\varphi \mapsto \varphi(1)$.

Using the above, we get

$$\theta\left(rf^t\right) = \theta\left(r \cdot f \cdot f^{t-1}\right) = rf\theta\left(f^{t-1}\right) = rf\sum_j c_j f_j\left(g_1, \ldots, g_m\right) \otimes f_j\left(s_1, \ldots, s_m\right)$$

for some coefficients $c_j \in K$. For an element $1 \otimes \delta_i \in \left(R(-w) \otimes_K \left(s^{t-1}/s^t\right)^\vee\right)_0$, where $\{\delta_i\}$ is a dual basis of s^{t-1}/s^t, we have

$$\begin{aligned}
\psi_t\left(1 \otimes \delta_i\right) &= \left[rf^t \mapsto \left(1 \otimes \delta_i\right)\left(\theta\left(rf^t\right)\right)\right] \\
&= \left[rf^t \mapsto \left(1 \otimes \delta_i\right)\left(rf\sum_j c_j f_j\left(g_1, \ldots, g_m\right) \otimes f_j\left(s_1, \ldots, s_m\right)\right)\right] \\
&= \left[rf^t \mapsto rfc_i f_i\left(g_1, \ldots, g_m\right)\right].
\end{aligned}$$

As an element of $\operatorname{Hom}\left(R(-tw), R(-w)\right)_0$ this map is given by

$$r \mapsto rc_i f_i\left(g_1, \ldots, g_m\right)$$

and via the last vertical isomorphism this map is sent to

$$c_i f_i\left(g_1, \ldots, g_m\right) \in R_{(t-1)w}.$$

With these observations the surjectivity of ψ_t becomes clear, since the set $\{f_j\}$ forms a basis of the space of polynomials of degree $t-1$. $\square$

Chapter 3

The F-pure threshold and the height of quasi-homogeneous polynomials

In the first part of this chapter we give a short introduction to formal groups and we define the Artin-Mazur functors, which give examples of formal groups. Furthermore, we give the definition of the height of a formal group. This is an important invariant, uniquely characterizing one-dimensional formal groups over an algebraically closed field of positive characteristic by Lazard ([Laz55]).
In the second part, we consider a quasi-homogeneous polynomial $f \in \mathbb{Z}[x_0, \ldots, x_n]$ of degree w equal to the degree of $x_0 \cdots x_n$ and denote by X the hypersurface given by $f = 0$. We show that the F-pure threshold of the reduction $f_p \in \mathbb{F}_p[x_0, \ldots, x_n]$ is equal to the log-canonical threshold of f if and only if the height of the Artin-Mazur formal group associated to $H^{n-1}(X, \mathbb{G}_{m,X})$ is equal to 1. We also prove that a similar result holds for Fermat hypersurfaces of degree $> n + 1$.
In the third part of this chapter, we show that other values of the F-pure threshold cannot be characterized by the height. For this, we introduce weighted Delsarte $K3$ surfaces and explain a method of Goto ([Got04]) to compute their height. Using these results, we give two examples of weighted Delsarte $K3$ surfaces, where the first one has the same height but different F-pure threshold for varying p and the second one has the same F-pure threshold but the height will differ for two different primes p.
The results of this chapter are contained in the preprint [Mül17].

3.1 Formal groups

In the theory of formal groups one can choose the point of view of formal power series or the point of view of functors – we will sketch both in what follows.

For further information about the point of view of formal power series we refer the reader to [Frö68], [Haz78], [Hon70] and [Vla16]. In [Sti87] and [Zin84] the authors also treat the point of view of functors.

3.1.1 Formal group laws

During this section let R be a commutative ring with identity element and let $m \in \mathbb{N}$. Let $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ be two sets of m variables.

Definition 3.1. *An* m-dimensional formal group law *over* R *is an* m*-tuple of power series* $F(x, y) = (F_1(x, y), \dots, F_m(x, y))$ *with* $F_i(x, y) \in R[\![x, y]\!]$*, such that*

$$F(x, F(y, z)) = F(F(x, y), z) \text{ and}$$

$$F(x, y) \equiv x + y \bmod \deg \geq 2.$$

A formal group law is called commutative*, if one has in addition that*

$$F_i(x, y) = F_i(y, x)$$

for all i*.*

One can check that there is a unique m-tuple of power series $i_F(x)$ such that $F(x, i_F(x)) = F(i_F(x), x) = 0$ (see [Frö68, Chapter I, §3, Proposition 1]). The first condition above then shows that $F(x, 0) = x$ and $F(0, y) = y$.

Let F and G be two formal group laws over R of dimension m_F and m_G respectively. A *homomorphism* $F(x, y) \to G(x, y)$ over R is an m_G-tuple of power series φ in m_F variables such that $\varphi(x) \equiv 0 \bmod \deg \geq 1$ and

$$\varphi\left(F(x, y)\right) = G\left(\varphi(x), \varphi(y)\right).$$

The homomorphism $\varphi(x)$ is an *isomorphism* if there exists a homomorphism $\psi(x) : G(x, y) \to F(x, y)$ such that $\varphi(\psi(x)) = x$ and $\psi(\varphi(x)) = x$. One can show that $\varphi(x)$ is an isomorphism if and only if the Jacobian matrix $J(\varphi)$ of $\varphi(x)$ is invertible (see [Haz78, section A.4.6]). Here, $J(\varphi)$ is the matrix

$$J(\varphi) := \begin{pmatrix} a_{11} & \dots & a_{1m_F} \\ \vdots & & \vdots \\ a_{m_G 1} & \dots & a_{m_G m_F} \end{pmatrix}$$

if $\varphi(x) = (\varphi_1(x), \dots, \varphi_{m_G}(x))$ with $\varphi_j(x) = a_{j1}x_1 + \ldots + a_{jm_F}x_{m_F} \bmod \deg \geq 2$. The morphism $\varphi(x)$ is said to be a *strict isomorphism* if $J(\varphi) = \mathbb{1}$, i.e. if $\varphi(x) \equiv x \bmod \deg \geq 2$.

If R is a ring of characteristic zero, then every m-dimensional commutative formal group law $F(x, y)$ over R determines a unique m-tuple

$$l(\tau) = (l_1(\tau), \ldots, l_m(\tau))$$

of power series in an m-tuple of variables $\tau = (\tau_1, \ldots, \tau_m)$ with coefficients in $R \otimes \mathbb{Q}$ such that

$$l(\tau) \equiv \tau \bmod \deg \geq 2 \text{ and}$$
$$F(x, y) = l^{-1}\left(l(x) + l(y)\right)$$

(see [Hon70, Theorem 1]). This m-tuple $l(\tau)$ is called the *logarithm* of the formal group law $F(x, y)$. In the one-dimensional case one can write

$$l(\tau) = \tau + \sum_{n=2}^{\infty} \frac{b_{n-1}}{n} \tau^n$$

with $b_{n-1} \in R$. The name of the logarithm comes from the following example:

Example 3.2. We consider the *one-dimensional additive formal group law* $\mathbb{G}_a$ and the *one-dimensional multiplicative formal group law* $\mathbb{G}_m$, which are both defined over $\mathbb{Z}$. The additive formal group law is given by

$$\mathbb{G}_a(x, y) \coloneqq x + y$$

with logarithm $l(\tau) = \tau$. The multiplicative formal group law is given by

$$\mathbb{G}_m(x, y) \coloneqq x + y + xy$$

and the logarithm is $l(\tau) = \log(1 + \tau) = \sum_{n \geq 1} (-1)^{n+1} \frac{1}{n} \tau^n$.

3.1.2 Formal groups as functors

Now, let us come to the point of view of functors. In this section a formal group law will always be assumed to be commutative.

Let $\mathfrak{Nilalg}_R$ denote the category of nil-R-algebras, i.e. of R-algebras in which every element is nilpotent. The *formal affine m-space* over R is defined as the functor

$$\mathbb{A}^m_R : \mathfrak{Nilalg}_R \to \mathfrak{Sets},$$

which sends a nil-R-algebra N to the set $N^{(m)} := N \oplus \cdots \oplus N$ with m summands and which sends a morphism f to the map $f \times \cdots \times f$.

Definition 3.3. *An* m-dimensional formal group *over* R *is a functor*

$$F : \mathfrak{Nilalg}_R \to \mathfrak{Abelian\ Groups},$$

such that $V \circ F \cong \mathbb{A}^m_R$*, where* $V : \mathfrak{Abelian\ Groups} \to \mathfrak{Sets}$ *is the forgetful functor.*

Remark that we write "formal group law" when we consider F as a power series whereas we write "formal group" when F is considered as a functor. However, we will show now that given a formal group law $F(x,y) \in R[\![x,y]\!]$ one can associate to F a formal group (i.e. a functor $F : \mathfrak{Nilalg}_R \to \mathfrak{Abelian\ Groups}$) and vice versa.
First, let $F(x,y) = (F_1(x,y), \ldots, F_m(x,y))$ with $F_i(x,y) \in R[\![x,y]\!]$ be an m-dimensional formal group law over R. We can associate to F the functor

$$\begin{aligned} F : \mathfrak{Nilalg}_R &\to \mathfrak{Abelian\ Groups} \\ N &\mapsto \left(N^{(m)}, +_F\right), \end{aligned}$$

where $+_F$ is defined as follows. Given two elements $s = (s_1, \ldots, s_m)$, $t = (t_1, \ldots, t_m) \in N^{(m)}$, then $F_i(s,t)$ is a finite sum (since the power series is applied to nilpotent elements) and gives a well-defined element of N. Hence, we define

$$\begin{aligned} +_F : N^{(m)} \times N^{(m)} &\to N^{(m)} \\ (s,t) &\mapsto (F_1(s,t), \ldots, F_m(s,t)). \end{aligned}$$

Definition 3.1 together with the existence of the tuple of power series i_F imply that $+_F$ defines a group structure on $N^{(m)}$. Furthermore, it is clear, that the composition of the resulting functor with the forgetful functor V is $\mathbb{A}^m_R$.
For the opposite direction, let $F : \mathfrak{Nilalg}_R \to \mathfrak{Abelian\ Groups}$ be a formal group with $V \circ F \cong \mathbb{A}^m_R$. Let $\mathfrak{m} \subset R[\![x,y]\!]$ be the ideal, which is generated by the indeterminates, i.e. $\mathfrak{m} = (x_1, \ldots, x_m, y_1, \ldots, y_m)$. Clearly, $\mathfrak{m}/\mathfrak{m}^l$ is an object of $\mathfrak{Nilalg}_R$ for all $l \in \mathbb{N}$ and hence $F\left(\mathfrak{m}/\mathfrak{m}^l\right)$ is an abelian group with underlying set $\left(\mathfrak{m}/\mathfrak{m}^l\right)^{(m)}$. In this group we compute the sum

$$(x_1, \ldots, x_m) +_F (y_1, \ldots, y_m) = \left(F_1^{(l)}, \ldots, F_m^{(l)}\right),$$

with $F_i^{(l)} \in \mathfrak{m}/\mathfrak{m}^l$. By assumption,

$$F\left(\mathfrak{m}/\mathfrak{m}^{l+1}\right) \to F\left(\mathfrak{m}/\mathfrak{m}^l\right)$$

is a group homomorphism and therefore, $F_i^{(l+1)} \equiv F_i^{(l)} \bmod \mathfrak{m}^l$ for all i. Thus, we find power series F_i such that $F_i \equiv F_i^{(l)}$ mod deg $\geq l$. It remains to show, that the tuple $(F_i)_i$ defines an m-dimensional formal group law. Associativity and commutativity follow from the fact that the $F\left(\mathfrak{m}/\mathfrak{m}^l\right)$ are abelian groups. Furthermore, we have to show that

$$F_i(x,y) \equiv F_i^{(2)}(x,y) \equiv x_i + y_i \text{ mod deg} \geq 2.$$

We know that $F_i^{(2)} \in \mathfrak{m}/\mathfrak{m}^2$, hence we can write

$$F_i^{(2)}(x,y) = \sum_{j=1}^{m} a_{ij}x_j + \sum_{j=1}^{m} b_{ij}y_j.$$

But since F is commutative, we get that $a_{ij} = b_{ij}$ for all i and j. Using that $(x_1, \ldots, x_m) +_F (0, \ldots, 0)$ should be equal to $(x_1, \ldots, x_m)$, we get

$$a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

and hence $F_i(x,y) \equiv x_i + y_i \bmod \deg \geq 2$.

Example 3.4. The one-dimensional additive and the one-dimensional multiplicative formal group laws are seen as functors in the following way:

$$\begin{aligned} \mathbb{G}_a(N) &= (N, +), \\ \mathbb{G}_m(N) &= (1+N)^{\times}, \end{aligned}$$

where $(N,+)$ is the group N with the usual addition and $(1+N)^{\times}$ is the set of all formal sums $1+u$, $u \in N$, with the multiplication given by $(1+u)(1+v) = 1 + u + v + uv$.

3.1.3 Artin-Mazur functors

Let F be a formal group over R, X a scheme over R, J a sheaf of R-algebras on X and $i \in \mathbb{N}_0$. Then one can construct the following diagram:

$$\begin{tikzcd}
\mathfrak{Nilalg}_R \arrow[r, "J\otimes_R -"] \arrow[dr, "F_J"] \arrow[ddr, "H^i(X,F_J)"'] & \mathfrak{Sheaves\ of\ nil\text{-}}\mathrm{R}\mathfrak{\text{-}algebras\ on\ } X \arrow[d, "F"] \\
 & \mathfrak{Sheaves\ of\ abelian\ groups\ on\ } X \arrow[d, "H^i"] \\
 & \mathfrak{Abelian\ Groups}
\end{tikzcd}$$

Here $J \otimes_R -$ assigns to a nil-R-algebra N the sheaf $J \otimes_R N$ associated with the pre-sheaf $U \mapsto \Gamma(U,J) \otimes_R N$ for U open. The functor F assigns to a sheaf $\mathfrak{a}$ of nil-R-algebras the sheaf of abelian groups $F(\mathfrak{a})$ defined by $\Gamma(U, F(\mathfrak{a})) = F(\Gamma(U, \mathfrak{a}))$ for $U \subset X$ open. The functor H^i is taking i-th cohomology and the functors F_J and $H^i(X, F_J)$ are defined by the commutativity of the above diagram.
In the following, we will use the diagram above with $F = \mathbb{G}_m$ and $J = \mathcal{O}_X$. Writing $\mathbb{G}_{m,X}$ instead of $\mathbb{G}_{m,\mathcal{O}_X}$, the functors

$$H^i(X, \mathbb{G}_{m,X}) : \mathfrak{Nilalg}_R \to \mathfrak{Abelian\ Groups}$$

introduced in [AM77] are called *Artin-Mazur functors.* These functors are not necessarily formal groups, but in the following, we will often use a criterion of Stienstra for when $H^{\bullet}(X, \mathbb{G}_{m,X})$ is a formal group:

Theorem 3.5 ([Sti87, Theorem 1]). *Let R be a noetherian ring and let X be a subscheme of $\mathbb{P}_R^n$ defined by the ideal $(F_1, \ldots, F_r)$, where $F_1, \ldots, F_r$ is a regular sequence of homogeneous polynomials in $R[x_0, \ldots, x_n]$. Let $d_i := \deg(F_i)$ and $d := \sum_{i=1}^r d_i$. If X is flat over R and $d_i \geq d - n \geq 1$ for all i, then $H^{n-r}(X, \mathbb{G}_{m,X})$ is a formal group over R of dimension $\binom{d-1}{n}$.*
Furthermore, assume that R is flat over $\mathbb{Z}$ and set

$$J := \left\{ i = (i_0, \ldots, i_n) \in \mathbb{Z}^{n+1} \;\middle|\; i_0, \ldots, i_n \geq 1, i_0 + \ldots + i_n = d \right\}.$$

Then there is a formal group law for $H^{n-r}(X, \mathbb{G}_{m,X})$ whose logarithm is the tuple $(l_i(\tau))_{i \in J}$ of power series in $\tau = (\tau_i)_{i \in J}$ with

$$l_i(\tau) = \sum_{m \geq 1} \sum_{j \in J} \frac{b_{m-1,i,j}}{m} \tau_j^m,$$

where

$$b_{m-1,i,j} := \textit{coefficient of } x_0^{mj_0 - i_0} \cdots x_n^{mj_n - i_n} \textit{ in } (F_1 \cdots F_r)^{m-1}.$$

One can show that the above criterion also holds in the quasi-homogeneous case. In particular, we will consider the quasi-homogeneous case with $r = 1$, i.e. let X be the subscheme of $\mathbb{P}_R^n(\alpha)$ defined by a quasi-homogeneous polynomial $F \in R[x_0, \ldots, x_n]$ of type $\alpha = (\alpha_0, \ldots, \alpha_n)$ and degree $d \geq w$. If X is flat over R, then $H^{n-1}(X, \mathbb{G}_{m,X})$ is a formal group over R. If in addition R is flat over $\mathbb{Z}$, then the set J is given by

$$J := \left\{ i = (i_0, \ldots, i_n) \in \mathbb{Z}^{n+1} \;\middle|\; i_0, \ldots, i_n \geq 1, i_0\alpha_0 + \ldots + i_n\alpha_n = d \right\}$$

and the dimension of $H^{n-1}(X, \mathbb{G}_{m,X})$ is given by $\#J$. The formula for the logarithm is similar to the one in Theorem 3.5.

Definition 3.6. *The functor $H^2(X, \mathbb{G}_{m,X})$ is called the* formal Brauer group *of X (at least if it is a formal group).*

3.1.4 The height of a formal group law

Let $F(x, y)$ be an m-dimensional formal group law over a perfect field K of characteristic $p > 0$. An important invariant of the formal group law is the *height* $\mathrm{ht} = \mathrm{ht}(F)$, which is either infinite or an integer greater or equal to 1. Consider the multiplication by p endomorphism, which is given by

$$[p]_F(x) := \underbrace{x +_F x +_F \ldots +_F x}_{p \text{ times}},$$

where $x +_F y := F(x, y)$ and write $[p]_F(x) = (H_1(x), \ldots, H_m(x))$. We say that $F(x, y)$ is of finite height, if the ring $K[\![x_1, \ldots, x_m]\!]$ is a finitely generated

module over the subring $K[\![H_1(x),\dots,H_m(x)]\!]$. In this case, $K[\![x_1,\dots,x_m]\!]$ is free of rank p^r, $r\in\mathbb{N}$, over $K[\![H_1(x),\dots,H_m(x)]\!]$ and $\mathrm{ht}(F):=r$ is called the height of $F(x,y)$ (see [Haz78, Definition 18.3.8]).
If R is a local ring of characteristic zero with residue field K of characteristic $p>0$ and $F(x,y)$ is an m-dimensional formal group law over R, then we define the height of $F(x,y)$ as the height of the reduction $\overline{F}(x,y)$ of $F(x,y)$ over K.
If $F(x,y)$ is a one-dimensional formal group law over a field K of characteristic $p>0$, then this definition says the following: Let $[p]_F(x)$ be the multiplication by p as above. Then one can show (see [Haz78, section 18.3.1]) that either $[p]_F(x)=0$ or there is a power $q=p^r$ of p such that $[p]_F(x)=\beta(x^q)$ for some power series β with $\beta(x)\not\equiv 0 \bmod \deg\geq 2$. Then $\mathrm{ht}(F)=\infty$ if and only if $[p]_F(x)=0$ and $\mathrm{ht}(F)=r$ if $q=p^r$ is the highest power of p such that $[p]_F(x)=\beta(x^q)$.

Lemma 3.7. *Let $F=G\times H$ be a formal group law, which is the product of two formal group laws G and H over a perfect field K of characteristic $p>0$ of height ht_G respectively ht_H. Then F has height $\mathrm{ht}_G+\mathrm{ht}_H$.*

Proof. Write

$$[p]_G(x)=(G_1(x),\dots,G_{m_G}(x)) \quad\text{and}\quad [p]_H(x)=(H_1(x),\dots,H_{m_H}(x))\,.$$

Since G has height ht_G, we know that $K[\![x_1,\dots,x_{m_G}]\!]$ is a finitely generated module over the subring $K[\![G_1(x),\dots,G_{m_G}(x)]\!]$ of rank p^{ht_G} and since H has height ht_H, we know that $K[\![y_1,\dots,y_{m_H}]\!]$ is a finitely generated module over the subring $K[\![H_1(y),\dots,H_{m_H}(y)]\!]$ of rank p^{ht_H}. Therefore, it follows that $K[\![x_1,\dots,x_{m_G},y_1,\dots,y_{m_H}]\!]$ is a finitely generated module over the subring $K[\![G_1(x),\dots,G_{m_G}(x),H_1(y),\dots,H_{m_H}(y)]\!]$ of rank $p^{\mathrm{ht}_G}p^{\mathrm{ht}_H}=p^{\mathrm{ht}_G+\mathrm{ht}_H}$. (Also see Proposition 28.2.6 and Corollary 28.2.9 of [Haz78]). $\square$

Example 3.8. Consider the one-dimensional additive formal group law $\mathbb{G}_a$ and the multiplicative formal group law $\mathbb{G}_m$ as above. It is easy to see that

$$\mathrm{ht}\,(\mathbb{G}_a)=\infty \quad\text{and}\quad \mathrm{ht}\,(\mathbb{G}_m)=1.$$

Furthermore, one can show that $\mathrm{ht}\,(\mathbb{G}_a\times\mathbb{G}_m)=\infty$ but $\mathrm{ht}\,(\mathbb{G}_m\times\mathbb{G}_m)=2$.

The height is an important tool since it completely determines one-dimensional formal group laws:

Theorem 3.9 ([Laz55, Corollaire 1 & Théorème IV])**.** *Let K be an algebraically closed field of positive characteristic.*

(1) For every integer $\mathrm{ht}\geq 1$ and for $\mathrm{ht}=\infty$ there exists a one-dimensional formal group law of height ht over K.

(2) Two one-dimensional formal group laws over K are isomorphic if and only if they have the same height.

3.2 Connection between the F-pure threshold and the height of quasi-homogeneous polynomials

The aim of this section is to give a connection between the F-pure threshold of the reduction $f_p \in \mathbb{F}_p[x_0, \dots, x_n]$ of a quasi-homogeneous polynomial $f \in \mathbb{Z}[x_0, \dots, x_n]$ and the height of the Artin-Mazur formal group associated to $H^{n-1}(X, \mathbb{G}_{m,X})$, which was defined in section 3.1.3, where X denotes the hypersurface given by f. For this, we first need the following result, which is an immediate consequence of Theorem 2 of [Vla16]:

Lemma 3.10. *Let R be the ring of integers of a complete absolutely unramified discrete valuation field of characteristic zero and residue characteristic $p > 0$, equipped with a lift of the p-th power Frobenius on the residue field R/pR. Let $F(x, y) \in R[\![x, y]\!]$ be a formal group law of dimension 1 with logarithm*

$$l(\tau) = \sum_{m=1}^{\infty} \frac{b_{m-1}}{m} \tau^m,$$

where $\{b_m\}_{m \geq 0}$ is a sequence of elements of R with $b_0 = 1$. Then $\mathrm{ht}(F) = 1$ if and only if $\mathrm{ord}_p(b_{p-1}) = 0$.

Proof. First, let $\mathrm{ht}(F) = 1$. Then, by Theorem 2 (i) of [Vla16], we get $\mathrm{ord}_p(b_{p-1}) = 1 - \left\lfloor \frac{1}{1} \right\rfloor = 0$.
For the opposite direction, let $\mathrm{ht}(F) \neq 1$. Then we have two cases. The first case is $\mathrm{ht}(F) = \infty$, which yields $\mathrm{ord}_p(b_{p-1}) \geq 1$ by Theorem 2 (i) of [Vla16]. The second case is $\mathrm{ht}(F) < \infty$ and $\mathrm{ht}(F) \neq 1$. Then, again by Theorem 2 (i) of [Vla16], we conclude that $\mathrm{ord}_p(b_{p-1}) \geq 1 - \left\lfloor \frac{1}{\mathrm{ht}(F)} \right\rfloor = 1$, since $\mathrm{ht}(F) > 1$. $\square$

Using this, we are able to prove the main theorem of this section.

Theorem 3.11. *Let $\mathbb{Z}[x_0, \dots, x_n]$ be the graded polynomial ring with $\alpha_i := \deg(x_i)$ and let $w := \alpha_0 + \ldots + \alpha_n$. Let $f \in \mathbb{Z}[x_0, \dots, x_n]$ be a quasi-homogeneous polynomial of degree w and type $\alpha := (\alpha_0, \dots, \alpha_n)$ with an isolated singularity such that the greatest common divisor of all coefficients of f is 1. Furthermore, let X be the hypersurface in $\mathbb{P}^n_{\mathbb{Z}}(\alpha)$ defined by f and let $f_p \in \mathbb{F}_p[x_0, \dots, x_n]$ be the reduction of f modulo p. Assume that $p \geq w(n-2)+1$. Then $\mathrm{fpt}(f_p) = 1 = \mathrm{lct}(f)$ if and only if $\mathrm{ht}\left(H^{n-1}(X, \mathbb{G}_{m,X})\right) = 1$.*

Proof. In order to use Theorem 1 of [Sti87] (which also holds for quasi-homogeneous polynomials), we first show that X is flat over $\mathbb{Z}$. Since $\mathbb{Z}$ is a Dedekind domain it is enough to show that R/f is torsionfree over $\mathbb{Z}$, where $R = \mathbb{Z}[x_0, \dots, x_n]$ (see [GW10, Proposition B.86]). In order to show this, it is enough to show that multiplication by a prime q is injective on R/f for all primes q. Using the fact that the greatest common divisor of all coefficients of f is 1 one gets the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R \arrow[r, "\cdot q"] \arrow[d, "\cdot f"] & R \arrow[r] \arrow[d, "\cdot f"] & R/q \arrow[r] \arrow[d, "\cdot f"] & 0 \\
0 \arrow[r] & R \arrow[r, "\cdot q"] \arrow[d] & R \arrow[r] \arrow[d] & R/q \arrow[r] \arrow[d] & 0 \\
 & R/f \arrow[r, "\cdot q"] \arrow[d] & R/f \arrow[r] \arrow[d] & R/(f,q) & \\
 & 0 & 0 & &
\end{tikzcd}$$

and the snake lemma then shows that q is injective on R/f for all primes q. Hence, Theorem 1 of [Sti87] yields that $H^{n-1}(X, \mathbb{G}_{m,X})$ is a formal group of dimension 1. Using the notation of Theorem 3.5 we have $J = \{(1, \ldots, 1)\}$, since $d = w$. Therefore, the logarithm of the formal group law is given by

$$l(\tau) = \sum_{m \geq 1} \frac{b_{m-1}}{m} \tau^m,$$

where b_{m-1} is the coefficient of $(x_0 \cdots x_n)^{m-1}$ in f^{m-1}.
Clearly, by Example 1.3 we have that $\operatorname{lct}(f) = 1$. Using the remark after Lemma 2.10 we have that $\operatorname{fpt}(f_p) = 1$ if and only if $b_{p-1} \not\equiv 0 \bmod p$. Furthermore, $b_{p-1} \not\equiv 0 \bmod p$ if and only if $\operatorname{ord}_p(b_{p-1}) = 0$. Finally, by Lemma 3.10 it follows that $\operatorname{ord}_p(b_{p-1}) = 0$ if and only if $\operatorname{ht}\left(H^{n-1}(X, \mathbb{G}_{m,X})\right) = 1$. $\square$

Example 3.12. Let $f := x_0^d + \ldots + x_n^d \in \mathbb{Z}[x_0, \ldots, x_n]$ with $d = n+1$ and let X be the Fermat hypersurface in $\mathbb{P}^n_{\mathbb{Z}}$ given by f. Let $p \geq (n+1)(n-2)+1$. Then by Theorem 3.11 we have that $\operatorname{fpt}(f_p) = 1 = \operatorname{lct}(f)$ if and only if $\operatorname{ht}\left(H^{n-1}(X, \mathbb{G}_{m,X})\right) = 1$.

The aim of the rest of this section is to show that a similar result as the above also holds for

$$f = x_0^d + \ldots + x_n^d \in \mathbb{Z}[x_0, \ldots, x_n] \text{ with } d = n+k \text{ and } k \geq 2.$$

Here, again by Example 1.3 we know that $\operatorname{lct}(f) = \frac{n+1}{d}$. Before we consider the case $k \geq 3$, we start with $k = 2$. For this, we need the following lemma, which holds in a more general setting.

Lemma 3.13. *Let $f \in K[x_0, \ldots, x_n]$ be a quasi-homogeneous polynomial of degree d and type $\alpha = (\alpha_0, \ldots, \alpha_n)$, where K is a field of characteristic $p > 0$. Let $w \coloneqq \alpha_0 + \ldots + \alpha_n$. If $wq \equiv x \bmod d$ with $1 \leq x \leq w-1$ for some $q = p^e$, then $\operatorname{fpt}(f) < \frac{w}{d}$.*

Proof. By Proposition 2.6 we have that $\mu_f(q) \leq \left\lceil \frac{wq-w+1}{d} \right\rceil$. By the assumption we get

$$\mu_f(q) \leq \left\lceil \frac{wq-w+1}{d} \right\rceil \leq \frac{wq-w+1+(w-2)}{d} = \frac{wq-1}{d}.$$

Therefore, $\mu_f(q) < \frac{wq}{d}$ and $\operatorname{fpt}(f) \leq \frac{\mu_f(q)}{q} < \frac{w}{d}$. □

Example 3.14. Let $f = x_0^d + \ldots + x_n^d \in \mathbb{Z}[x_0, \ldots, x_n]$ with $d = n+2$ and let X be the Fermat hypersurface in $\mathbb{P}^n_{\mathbb{Z}}$ given by f. We claim that the formal group $H^{n-1}(X, \mathbb{G}_{m,X})$ is the direct sum of $n+1$ copies of a one-dimensional formal group law F and that $\operatorname{fpt}(f_p) = \operatorname{lct}(f) = \frac{n+1}{n+2}$ if and only if $\operatorname{ht}(F) = 1$.
For the proof of this, we use the notation of Theorem 3.5 and compute

$$J = \{(2,1,\ldots,1), (1,2,1,\ldots,1), \ldots, (1,\ldots,1,2)\}.$$

For $i, j \in J$ we denote by $b_{m-1,i,j}$ the coefficient of $x_0^{mj_0-i_0} \ldots x_n^{mj_n-i_n}$ in f^{m-1}. It is an easy computation to see that $b_{m-1,i,j} = 0$ if $i \neq j$. Therefore, by using Theorem 3.5 the logarithm of $H^{n-1}(X, \mathbb{G}_{m,X})$ is given by $(l_i(\tau_i))_{i \in J}$, where

$$l_i(\tau_i) = \sum_{m \geq 1} \frac{b_{m-1,i,i}}{m} \tau_i^m.$$

Hence, the formal group $H^{n-1}(X, \mathbb{G}_{m,X})$ is the direct sum of $n+1$ copies of the one-dimensional formal group law $F(\tau, \eta) = l^{-1}(l(\tau) + l(\eta))$ with logarithm

$$l(\tau) = \sum_{k \geq 0} \frac{1}{kd+1} \frac{(kd)!}{k!^n (2k)!} \tau^{kd+1} = \sum_{m \geq 1} \frac{b_{m-1}}{m} \tau^m,$$

where

$$b_{m-1} = \begin{cases} \frac{(kd)!}{k!^n (2k)!}, & \text{if } m = 1 + kd \text{ for some } k \in \mathbb{Z} \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Now, we show that $\operatorname{fpt}(f_p) = \operatorname{lct}(f) = \frac{n+1}{n+2}$ if and only if $\operatorname{ht}(F) = 1$.
For this, remark that by Lemma 3.10 it follows that $\operatorname{ht}(F) = 1$ if and only if $\operatorname{ord}_p(b_{p-1}) = 0$, which is equivalent to $b_{p-1} \not\equiv 0 \bmod p$. For $p \equiv 1 \bmod d$ we have that $kd = p-1$ and $2k = \frac{2(p-1)}{d}$ for some $k \in \mathbb{Z}$ and both expressions are smaller than p. Since $\operatorname{ord}_p(s!) = \sum_{i \geq 1} \left\lfloor \frac{s}{p^i} \right\rfloor$ for all $s \in \mathbb{N}$, this means that the

p-adic valuation of $\frac{(kd)!}{k!^n(2k)!}$ is zero. Using this together with (3.1) we conclude that $b_{p-1} \not\equiv 0 \bmod p$ if and only if $p \equiv 1 \bmod d$.
Hence, it remains to prove that $p \equiv 1 \bmod d$ is equivalent to $\operatorname{fpt}(f_p) = \operatorname{lct}(f)$. Using Example 4.2 of [MTW05] (see Example 1.9) one gets that $\operatorname{fpt}(f_p) = \operatorname{lct}(f)$ if $p \equiv 1 \bmod d$. Now let $\operatorname{fpt}(f_p) = \operatorname{lct}(f)$. By using the contraposition of Lemma 3.13 we conclude that $(n+1)q \equiv x \bmod n+2$ with $x \in \{0, n+1\}$ for all q. If $x = 0$ for $q = p$ we get $(n+1)p \equiv 0 \bmod n+2$. Therefore $p \equiv 0 \bmod n+2$, which is a contradiction. If $x = n+1$ one gets $(n+1)p \equiv n+1 \bmod n+2$, hence $p \equiv 1 \bmod n+2$, i.e. $p \equiv 1 \bmod d$.

Next, we consider the general case $d = n+k$ with $k \geq 3$ and $n \geq 2(k-1)$.

Proposition 3.15. *Let $f = x_0^d + \ldots + x_n^d \in \mathbb{Z}[x_0, \ldots, x_n]$ with $d = n+k$ for $k \geq 3$ and such that $n \geq 2(k-1)$ and let X be the hypersurface in $\mathbb{P}^n_{\mathbb{Z}}$ given by f. Then the formal group $H^{n-1}(X, \mathbb{G}_{m,X})$ is the direct sum of $\binom{n+k-1}{n}$ one-dimensional formal groups, which are all of height 1 if and only if $p \equiv 1 \bmod d$.*

Proof. As in Theorem 3.5 let

$$J = \left\{ i = (i_0, \ldots, i_n) \in \mathbb{Z}^{n+1} \;\middle|\; i_0, \ldots, i_n \geq 1 \text{ and } i_0 + \ldots + i_n = d \right\}$$

and for $i, j \in J$ let $b_{m-1,i,j}$ be the coefficient of $x_0^{mj_0 - i_0} \cdots x_n^{mj_n - i_n}$ in f^{m-1}. We prove the proposition via the following steps:
Step 1: We show, that for $i \neq j$ one has $b_{m-1,i,j} = 0$: Since $k \geq 2$ and $n \geq 2(k-1)$ it follows that $n \geq k$. The elements of the set J are tuples $i = (i_0, \ldots, i_n)$ with $i_0, \ldots, i_n \geq 1$ and

$$i_0 + \ldots + i_n = d = n + k = (n+1) + (k-1) \leq (n+1) + (n-1),$$

i.e. each entry i_s is at least one and further $k-1 \leq n-1$ has to be distributed to the entries of i.
Since $n \geq 2(k-1)$, it follows that $\frac{n+1}{2} > k-1$, i.e. for all $i \in J$ we know that more than half of the entries of a tuple $i \in J$ are equal to 1. This means that if $j = (j_0, \ldots, j_n) \in J$ is a second tuple, then there exists at least one position s with $i_s = 1 = j_s$.
Now write

$$f^{m-1} = \sum_{\beta_0 + \ldots + \beta_n = m-1} \binom{m-1}{\beta_0, \ldots, \beta_n} x_0^{d\beta_0} \cdots x_n^{d\beta_n}.$$

Then we have $d\beta_t = mj_t - i_t$ for all t and in particular $d\beta_s = m-1$. The last equality shows that $m \equiv 1 \bmod d$ and the first equality then yields

$$0 \equiv mj_t - i_t \bmod d \equiv j_t - i_t \bmod d,$$

i.e. $j_t \equiv i_t \bmod d$ for all t. But since $i_t, j_t \leq n$ and $d = n+k > n$ it follows that $i_t = j_t$ for all t and therefore $i = j$.

Step 2: The first step of this proof means, that the logarithm $l(\tau)$ of the formal group $H^{n-1}(X, \mathbb{G}_{m,X})$ of dimension $\#J = \binom{n+k-1}{n}$ is given by $(l_i(\tau_i))_{i\in J}$, where

$$l_i(\tau_i) = \sum_{m\geq 1} \frac{b_{m-1,i,i}}{m} \tau_i^m$$

and one can compute that

$$b_{m-1,i,i} = \begin{cases} \frac{(kd)!}{(ki_0)!\cdots(ki_n)!}, & \text{if } m = 1 + kd \text{ for some } k \in \mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}$$

Step 3: By step 1 and 2 we know that $H^{n-1}(X, \mathbb{G}_{m,X})$ is the direct sum of $\binom{n+k-1}{n}$ formal groups $(F_i)_{i\in J}$, where $F_i(\tau_i, \eta_i) = l_i^{-1}(l_i(\tau_i) + l_i(\eta_i))$. We prove that $\mathrm{ht}(F_i) = 1$ for all $i \in J$ if and only if $p \equiv 1 \bmod d$. For this, Lemma 3.10 shows that $\mathrm{ht}(F_i) = 1$ if and only if $b_{p-1,i,i} \not\equiv 0 \bmod p$. For $p \equiv 1 \bmod d$, we have that $kd = p-1$ and $ki_r = \frac{p-1}{d} i_r < p-1$ for all $0 \leq r \leq n$ and hence the p-adic valuation of $\frac{(kd)!}{(ki_0)!\cdots(ki_n)!}$ is zero. Therefore, it follows that $b_{p-1,i,i} \not\equiv 0 \bmod p$ if and only if $p \equiv 1 \bmod d$. □

The following proposition computes the F-pure threshold of Fermat hypersurfaces.

Proposition 3.16. *Let $f = x_0^d + \ldots + x_n^d \in \mathbb{Z}[x_0, \ldots, x_n]$ with $d = n + k$ for $k \geq 2$ and such that $n > k-2$. Furthermore, let $d \not\equiv 0 \bmod p$. Then $p \equiv 1 \bmod d$ if and only if $\mathrm{fpt}(f_p) = \mathrm{lct}(f) = \frac{n+1}{d}$.*

Proof. First, let $p \equiv 1 \bmod d$. Then by Example 4.2 of [MTW05] it follows that $\mathrm{fpt}(f_p) = \mathrm{lct}(f)$.
Now, we show that if $p \not\equiv 1 \bmod d$, then $\mathrm{fpt}(f_p) < \mathrm{lct}(f)$. For this, remember that $\mathrm{fpt}(f_p) = \lim_{e\to\infty} \frac{\mu_{f_p}(p^e)}{p^e}$, where $\mu_{f_p}(p^e) = \min\left\{k \in \mathbb{N} \mid f_p^k \in \mathfrak{m}^{[p^e]}\right\}$ and

$$f_p^k = \sum_{\beta_0+\ldots+\beta_n=k} \binom{k}{\beta_0, \ldots, \beta_n} x_0^{d\beta_0} \cdots x_n^{d\beta_n}.$$

We claim that $\frac{\mu_{f_p}(p)}{p} < \frac{n+1}{d}$. Once we have shown this, it follows that $\mathrm{fpt}(f_p) \leq \frac{\mu_{f_p}(p)}{p} < \frac{n+1}{d}$. In order to show $\frac{\mu_{f_p}(p)}{p} < \frac{n+1}{d}$ or equivalently $\mu_{f_p}(p) < \frac{p(n+1)}{d}$ it is enough to show that $f_p^{\left\lfloor \frac{p(n+1)}{d} \right\rfloor} \in \mathfrak{m}^{[p]}$, since $d \not\equiv 0 \bmod p$. For this, it is enough to show that

$$d \left\lceil \frac{\left\lfloor \frac{p(n+1)}{d} \right\rfloor}{n+1} \right\rceil \geq p,$$

We now consider the following two cases:

Case 1: $n+1 \not| \left\lfloor \frac{p(n+1)}{d} \right\rfloor$
Clearly one has $\left\lfloor \frac{p(n+1)}{d} \right\rfloor + 1 \geq \frac{p(n+1)}{d}$, hence $\frac{\left\lfloor \frac{p(n+1)}{d} \right\rfloor}{n+1} + \frac{1}{n+1} \geq \frac{p}{d}$. Since $n+1$ does not divide $\left\lfloor \frac{p(n+1)}{d} \right\rfloor$ by assumption, this last inequality yields $\left\lceil \frac{\left\lfloor \frac{p(n+1)}{d} \right\rfloor}{n+1} \right\rceil \geq \frac{p}{d}$.

Case 2: $n+1 \mid \left\lfloor \frac{p(n+1)}{d} \right\rfloor$
Write $p = \lambda d + r$ with $0 \leq r < d$ and $r \neq 1$ since $p \not\equiv 1 \bmod d$. Thus $n+1$ divides

$$\left\lfloor \frac{p(n+1)}{d} \right\rfloor = \left\lfloor \frac{(\lambda d + r)(n+1)}{d} \right\rfloor = \left\lfloor \lambda(n+1) + \frac{r(n+1)}{d} \right\rfloor$$
$$= \lambda(n+1) + \left\lfloor \frac{r(n+1)}{d} \right\rfloor.$$

Since $\frac{r}{d} < 1$, it follows that $\frac{r(n+1)}{d} < n+1$ and by the above $\left\lfloor \frac{r(n+1)}{d} \right\rfloor$ must be divisible by $n+1$, hence we conclude that $\left\lfloor \frac{r(n+1)}{d} \right\rfloor = 0$. This means that $\frac{r(n+1)}{d} < 1$ or equivalently $r < \frac{d}{n+1} = \frac{n+k}{n+1}$. By assumption $k < n+2$, therefore we have $r < \frac{n+k}{n+1} < \frac{2n+2}{n+1} = 2$. Since $r \neq 1$ this means that $r = 0$ and $p = d$ since p is a prime. But this is a contradiction to our assumptions. $\square$

In [Her] Hernández already computed the F-pure threshold of diagonal hypersurfaces using base p expansions (see section 1.4). In particular, he obtained the F-pure threshold of the hypersurface $f = x_0^{n+1} + \ldots + x_n^{n+1}$ and showed that $\mathrm{fpt}(f) = 1$ if and only if $p \equiv 1 \bmod n+1$ (see [Her, Corollary 8.3]). We obtain this special case as an immediate consequence of the preceding proposition for $d = n+1$. It seems conceivable that his more general result about diagonal hypersurfaces (see [Her, Theorem 8.1]) can also be used to show the above proposition.
Combining Proposition 3.15 and Proposition 3.16 we obtain:

Corollary 3.17. *Let $f = x_0^d + \ldots + x_n^d \in \mathbb{Z}[x_0, \ldots, x_n]$ with $d = n+k$ for $k \geq 3$ and such that $n \geq 2(k-1)$ and let X be the hypersurface in $\mathbb{P}^n_{\mathbb{Z}}$ given by f. Furthermore, let $d \not\equiv 0 \bmod p$. Then $H^{n-1}(X, \mathbb{G}_{m,X})$ is a direct sum of formal groups of dimension 1, which are all of height 1 if and only if* $\mathrm{fpt}(f_p) = \mathrm{lct}(f) = \frac{n+1}{d}$.

In [Kob75, section V. 2.] it was shown that under the assumptions of Corollary 3.17 the action of the Frobenius on $H^{n-1}(X, \mathcal{O}_X)$ is bijective if and only if $p \equiv 1 \bmod d$. Therefore, the two conditions of Corollary 3.17 are also equivalent to the Frobenius action on $H^{n-1}(X, \mathcal{O}_X)$ being bijective.
In the proofs of Corollary 3.17 and Theorem 3.11 we have seen that the F-pure threshold is equal to the log-canonical threshold if and only if the height of the corresponding Artin-Mazur formal group is equal to its dimension. Or,

equivalently, since $\operatorname{fpt}(f_p) \leq \operatorname{lct}(f)$ for $p \gg 0$, the F-pure threshold is equal to its greatest possible value if and only if the height is equal to its smallest possible value (see Lemma 3.7). Hence, it is a natural question to ask if this is the case for all quasi-homogeneous polynomials:

Question 3.18. *Let $f \in R[x_0, \ldots, x_n]$ be a quasi-homogeneous polynomial such that $H^{n-1}(X, \mathbb{G}_{m,X})$ is a formal group and let the notation be as above. Is it true that $\operatorname{fpt}(f_p) = \operatorname{lct}(f)$ if and only if the height of $H^{n-1}(X, \mathbb{G}_{m,X})$ is equal to its dimension?*

So far we did not find any counterexample for this, which leads us to suspect that the answer to the above question is positive.

3.3 The height of the formal Brauer group of a weighted Delsarte $K3$ surface

Let $R = \mathbb{Z}[x_0, \ldots, x_n]$ be the graded polynomial ring with $\alpha_i = \deg(x_i)$. Let $f \in R$ be a quasi-homogeneous polynomial of degree $w = \alpha_0 + \ldots + \alpha_n$ and type $\alpha = (\alpha_0, \ldots, \alpha_n)$ with an isolated singularity such that the greatest common divisor of all coefficients of f is 1. Let $f_p \in \mathbb{F}_p[x_0, \ldots, x_n]$ be the reduction of f modulo p and assume $p \geq w(n-2)+1$. Theorem 2.9 and Theorem 2.16 yield that

$$\operatorname{fpt}(f_p) = 1 - \frac{h}{p}$$

with $0 \leq h \leq n-1$, where h is the order of vanishing of the Hasse invariant on a certain deformation space of $\operatorname{Proj}(R/f_pR) \subset \mathbb{P}^n(\alpha)$. Theorem 3.11 shows that $h = 0$ if and only if $\operatorname{ht}\left(H^{n-1}(X, \mathbb{G}_{m,X})\right) = 1$, where X is the hypersurface in $\mathbb{P}^n_{\mathbb{Z}}(\alpha)$ defined by f.
Therefore, one may ask whether the other possible values of the F-pure threshold (i.e. $1 \leq h \leq n-1$) can also be characterized by $\operatorname{ht}\left(H^{n-1}(X, \mathbb{G}_{m,X})\right)$. However, in section 3.4 we will give two examples of weighted Delsarte $K3$ surfaces which show that the answer is negative.
For this, we give a short introduction to $K3$ surfaces in the first part of this section and introduce the notion of supersingular $K3$ surfaces, i.e. $K3$ surfaces where the height of the formal Brauer group is infinite. In the second part of this section we introduce crystalline cohomology and give a criterion for a $K3$ surface to be supersingular in terms of crystalline cohomology. In the third section we define weighted Delsarte $K3$ surfaces following [Got04]. One of the reasons for introducing weighted Delsarte $K3$ surfaces is that they are convenient for explicit calculations of the height. This is due to the fact, that a weighted Delsarte $K3$ surface X is birational to the quotient of a Fermat surface. By examining the associated quotient map, one can extract informations about the supersingularity of X from the supersingularity of the corresponding Fermat surface. This will be done in section 3.3.4 and section 3.3.5. In section 3.3.6 we finally compute

the height of the formal Brauer group of a weighted Delsarte $K3$ surface if it is finite.

3.3.1 Supersingular $K3$ surfaces

In this section let K be an algebraically closed field. The aim of this section is to give a short introduction to $K3$ surfaces and to define supersingular $K3$ surfaces. For further information we refer the reader to [Lie16], [Lie15], [Huy16] and [VA16].

Definition 3.19. *A* $K3$ surface *is a smooth and projective surface X over a field K such that*

$$\omega_X \cong \mathcal{O}_X \text{ and } H^1(X, \mathcal{O}_X) = 0.$$

Example 3.20. Let K be an algebraically closed field of characteristic $p > 0$ and let X be a smooth quartic surface in $\mathbb{P}^3_K$. Then, taking cohomology in the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_K}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3_K} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

together with $H^1\left(\mathbb{P}^3_K, \mathcal{O}_{\mathbb{P}^3_K}\right) = 0$ and $H^2\left(\mathbb{P}^3_K, \mathcal{O}_{\mathbb{P}^3_K}(-4)\right) = 0$ gives $H^1(X, \mathcal{O}_X) = 0$. Furthermore, by [Har77, Exercise II. 8.4]

$$\omega_X \cong \mathcal{O}_X(4 - 3 - 1) = \mathcal{O}_X.$$

Thus, X is a $K3$ surface. A particularly interesting special case is given by the Fermat quartic in $\mathbb{P}^3_K$ defined by $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ over a field K of characteristic $\mathrm{char}(K) \neq 2$.

Before we can introduce the notion of a supersingular $K3$ surface, we first need to recall some basic facts about the Néron-Severi group of a smooth surface. For this, let X be a smooth surface over K. By $\mathrm{Div}(X)$ we denote the group of divisors on X, i.e. the free group generated by the prime divisors on X. There are three equivalence relations one can put on $\mathrm{Div}(X)$:

(1) *Linear equivalence*: Two divisors $C, D \in \mathrm{Div}(X)$ are called linearly equivalent, if their difference $C - D$ is principal.

(2) *Algebraic equivalence*: Two divisors $C, D \in \mathrm{Div}(X)$ are called algebraically equivalent, if there exists a connected curve T, two closed points $t_0, t_1 \in T$ and a divisor E on $X \times T$, flat over T, such that $E|_{X \times \{t_0\}} - E|_{X \times \{t_1\}} = C - D$.

(3) *Numerical equivalence*: Two divisors $C, D \in \mathrm{Div}(X)$ are called numerically equivalent, if $(C, E)_X = (D, E)_X$ for all $E \in \mathrm{Div}(X)$. Here,

$$(-, -)_X : \mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$$

is the intersection pairing on X introduced in [Har77, V. 1].

One can show (see [VA16, section 1.3]) that

Linear equivalence $\Rightarrow$ Algebraic equivalence $\Rightarrow$ Numerical equivalence.

Definition 3.21. *The* Picard group $\operatorname{Pic}(X)$ *is defined to be the quotient of* $\operatorname{Div}(X)$ *by the linear equivalence relation. Let* $\operatorname{Pic}^0(X) \subset \operatorname{Pic}^\tau(X) \subset \operatorname{Pic}(X)$, *where*

$$\operatorname{Pic}^\tau(X) := \left\{ L \in \operatorname{Pic}(X) \;\middle|\; (L, L')_X = 0 \textit{ for all } L' \in \operatorname{Pic}(X) \right\}$$

is the set of numerically trivial classes and $\operatorname{Pic}^0(X)$ *is the set of classes algebraically equivalent to zero. The* Néron-Severi group *of X is defined to be*

$$\operatorname{NS}(X) := \operatorname{Pic}(X)/\operatorname{Pic}^0(X).$$

Furthermore, let $\operatorname{Num}(X) := \operatorname{Pic}(X)/\operatorname{Pic}^\tau(X)$.

Proposition 3.22 ([Nér52, Théorème 2], [LN59])**.** *The Néron-Severi group is finitely generated. Its rank is called the* Picard number $\rho(X) = \operatorname{rank}\operatorname{NS}(X)$.

Proposition 3.23 ([Huy16, Chapter 1, Proposition 2.4])**.** *Let X be a $K3$ surface. Then the natural surjections*

$$\operatorname{Pic}(X) \to \operatorname{NS}(X) \to \operatorname{Num}(X)$$

are isomorphisms.

Now let X be a $K3$ surface over an algebraically closed field K of positive characteristic. In [AM77, section II, part 4] Artin and Mazur proved that the Artin-Mazur functor $H^2(X, \mathbb{G}_{m,X})$ of X is in fact a formal group of dimension 1. Moreover, in [Art74, Theorem (0.1)] Artin showed that if the formal Brauer group $H^2(X, \mathbb{G}_{m,X})$ of a $K3$ surface X has finite height ht then $\rho(X) \leq 22 - 2\,\mathrm{ht}$. Since $\rho(X) \geq 1$ for a $K3$ surface X, this gives that the height ht of the formal Brauer group of a $K3$ surface satisfies $1 \leq \mathrm{ht} \leq 10$ or $\mathrm{ht} = \infty$.

Definition 3.24. *Let X be a $K3$ surface over a field K of positive characteristic and let* ht *be the height of its formal Brauer group. Then X is called* ordinary *if* $\mathrm{ht} = 1$ *and X is called* Artin-supersingular *if* $\mathrm{ht} = \infty$.

Example 3.25. Let $X \subseteq \mathbb{P}^3_K$ be the Fermat surface given by $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ over an algebraically closed field K of characteristic $p > 2$ and let ht be the height of its formal Brauer group. Then

$$\mathrm{ht} = \begin{cases} \infty, & \text{if } p \equiv 3 \bmod 4 \\ 1, & \text{if } p \equiv 1 \bmod 4, \end{cases}$$

i.e. X is Artin-supersingular if and only if $p \equiv 3 \bmod 4$.

For surfaces, Shioda introduced another notion of supersingularity, characterized by the Picard rank of the variety (see [Shi74, p. 235]). In positive characteristic one has that $\rho(X) \leq 22$ for a $K3$ surface X (see [Igu60]). This bound is sharp since Tate ([Tat65]) and Shioda ([Shi77]) showed that there exist $K3$ surfaces with Picard rank 22 in positive characteristic. For example, the Fermat surface $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ has Picard number 22 if and only if $p \equiv 3 \bmod 4$.

Definition 3.26. *Let X be a $K3$ surface over an algebraically closed field of positive characteristic. Then X is called* Shioda-supersingular *if $\rho(X) = 22$.*

The example of the Fermat quartic already indicates that there is a connection between Artin-supersingularity and Shioda-supersingularity. Indeed, Theorem (0.1) of [Art74] shows that Shioda-supersingular $K3$ surfaces are Artin-supersingular. But the converse is also true:

Theorem 3.27 ([Lie15, Theorem 2.3])**.** *Let X be a $K3$ surface in odd characteristic. Then X is Shioda-supersingular if and only if it is Artin-supersingular.*

Thus, in the following, we will not distinguish between the two notions of supersingularity.

3.3.2 Supersingularity and crystalline cohomology

In the following sections we will need a criterion for a $K3$ surface to be supersingular using crystalline cohomology. Since the crystalline cohomology groups are modules over the ring of Witt vectors, let us first give a short introduction to this ring. After this, we will give a brief overview of crystalline cohomology. For a more detailed introduction to crystalline cohomology, we refer the reader to [BO78], [CL98], [Ill79], [Ill94] and [Lie16].

Witt ring

Let K be a perfect field of characteristic $p > 0$. Associated to K there exists the so-called *Witt ring* $W(K)$ of K, such that

(1) $W(K)$ is a discrete valuation ring of characteristic zero,

(2) the unique maximal ideal $\mathfrak{m}$ of $W(K)$ is generated by p and the residue field $W(K)/\mathfrak{m}$ is isomorphic to K,

(3) $W(K)$ is complete with respect to the $\mathfrak{m}$-adic topology,

(4) every $\mathfrak{m}$-adically complete discrete valuation ring of characteristic zero with residue field K contains $W(K)$ as a subring,

(5) $W(K)$ is functorial in K

(see [Ser79, Chapter II, §5 & §6]). Remark that the fourth property shows that $W(K)$ is unique up to isomorphism.
Let us give a short explanation of an explicit construction of $W(K)$. For a prime p the *Witt polynomials* are defined as follows:

$$W_n(x_0, \ldots, x_n) := \sum_{i=0}^{n} p^i x_i^{p^{n-i}} \in \mathbb{Z}[x_0, \ldots, x_n].$$

It is shown in [Ser79, Chapter II, §6, Theorem 6], that there exist unique polynomials S_n and P_n in $2(n+1)$ variables with coefficients in $\mathbb{Z}$, such that

$$W_n(x_0, \ldots, x_n) + W_n(y_0, \ldots, y_n) = W_n(S_n(x_0, \ldots, x_n, y_0, \ldots, y_n)),$$
$$W_n(x_0, \ldots, x_n) \cdot W_n(y_0, \ldots, y_n) = W_n(P_n(x_0, \ldots, x_n, y_0, \ldots, y_n)).$$

For example (see [Ser79, Chapter II, §6])

$$S_0(x_0, y_0) = x_0 + y_0, \quad S_1(x_0, x_1, y_0, y_1) = x_1 + y_1 + \frac{x_0^p + y_0^p - (x_0 + y_0)^p}{p},$$
$$P_0(x_0, y_0) = x_0 \cdot y_0, \quad P_1(x_0, x_1, y_0, y_1) = y_0^p x_1 + y_1 x_0^p + p x_1 y_1.$$

The *truncated Witt ring* $W_n(R)$ is as a set R^n and the ring structure is given by

$$\begin{aligned}
&(x_0, \ldots, x_{n-1}) + (y_0, \ldots, y_{n-1}) \\
&\qquad := (S_0(x_0, y_0), \ldots, S_{n-1}(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1})), \\
&(x_0, \ldots, x_{n-1}) \cdot (y_0, \ldots, y_{n-1}) \\
&\qquad := (P_0(x_0, y_0), \ldots, P_{n-1}(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1})).
\end{aligned}$$

It turns out that $W_n(R)$ is a ring with zero $0 = (0, \ldots, 0)$ and unit $1 = (1, 0, \ldots, 0)$ (see [Ser79, Chapter II, §6, Theorem 7]). For example, $W_1(R)$ is just the ring R with its usual addition and multiplication.
If R is a ring of characteristic $p > 0$, we define

$$\begin{aligned}
V : W_n(R) &\to W_{n+1}(R) \\
(x_0, \ldots, x_{n-1}) &\mapsto (0, x_0, \ldots, x_{n-1}), \\
\sigma : W_n(R) &\to W_{n-1}(R) \\
(x_0, \ldots, x_{n-1}) &\mapsto (x_0^p, \ldots, x_{n-2}^p).
\end{aligned}$$

The additive map V is called *Verschiebung* and the ring homomorphism σ is called *Frobenius.* These two maps have the following property

$$V \circ \sigma = \sigma \circ V = p \cdot \mathrm{id}_{W_n(R)},$$

where multiplication by p means adding p times any vector with itself. Moreover, for all $n \geq 2$ we have a projection $W_n(R) \to W_{n-1}(R)$ onto the first $n-1$ components.

Definition 3.28. *The* ring of Witt vectors $W(R)$ *is defined as*

$$W(R) := \varprojlim_n W_n(R).$$

Equivalently, $W(R)$ can be defined using the above construction for $R^{\mathbb{N}}$, then $W_n(R) = W(R)/V^n W(R)$, where $V(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$.

Example 3.29 ([Ser79, Chapter II, §6])**.** Witt vectors can be seen as a generalization of p-adic numbers, since $W_n(\mathbb{F}_p) \cong \mathbb{Z}/p^n\mathbb{Z}$ and $W(\mathbb{F}_p) \cong \mathbb{Z}_p$.

If X is a scheme, we can sheafify the above construction to obtain sheaves of rings $W_n(\mathcal{O}_X)$ and $W(\mathcal{O}_X)$, i.e. if $(X, \mathcal{O}_X)$ is a ringed space then $U \mapsto W_n(\mathcal{O}_X(U))$ respectively $U \mapsto W(\mathcal{O}_X(U))$ define a sheaf denoted by $W_n(\mathcal{O}_X)$ respectively $W(\mathcal{O}_X)$.

Crystalline cohomology

Now, let us give a short survey of crystalline cohomology. For this, let K be a perfect field of characteristic $p > 0$, $W := W(K)$ the ring of Witt vectors and $W_n := W_n(K)$ the truncated Witt ring.
Crystalline cohomology of a scheme X over K is defined as

$$H^i_{\text{cris}}(X/W) := \varprojlim_n H^i_{\text{cris}}(X/W_n),$$

where $H^i_{\text{cris}}(X/W_n) = H^i\left(\text{Cris}(X/W_n), \mathcal{O}_{X/W_n}\right)$ is the cohomology of the crystalline site of X over W_n with values in the sheaf of rings $\mathcal{O}_{X/W_n}$ (for a precise definition of this see the references at the beginning of this section).
Crystalline cohomology is a Weil cohomology theory. In particular, this means that $H^i_{\text{cris}}(X/W)$ is a contravariant functor in X. Furthermore, the cohomology groups $H^i_{\text{cris}}(X/W)$ respectively $H^i_{\text{cris}}(X/W_n)$ are finitely generated W-modules respectively W_n-modules and $H^i_{\text{cris}}(X/W) = 0$ if $i < 0$ or $i > 2\dim(X)$. Moreover, we have Poincaré duality, there exists a Lefschetz fixed point formula and so on.
Finally, crystalline cohomology refines the notion of de Rham cohomology for schemes in the following sense. First, for $n = 1$ one has

$$H^i_{\text{dR}}(X/K) \cong H^i_{\text{cris}}(X/W_1) = H^i_{\text{cris}}(X/K)$$

for all $i \geq 0$. Furthermore, if X is smooth and projective over K and if there exists a projective lift of X to W, i.e. a smooth projective scheme $\mathcal{X} \to \text{Spec}(W)$ such that $\mathcal{X} \times_{\text{Spec}(W)} \text{Spec}(K) \cong X$, then

$$H^i_{dR}(\mathcal{X}/W) \cong H^i_{\text{cris}}(X/W).$$

In the following we will be interested in the crystalline cohomology of a $K3$ surface, since it will help us to decide whether the $K3$ surface is supersingular or not.

Proposition 3.30 ([Lie16, Proposition 2.3]). *Let X be a $K3$ surface. Then*

i	0	1	2	3	4
$\operatorname{rank}_W H^i_{cris}(X/W)$	1	0	22	0	1

The above proposition shows that $H^2_{\text{cris}}(X/W)$ is the only interesting cohomology group of a $K3$ surface X. The aim for the rest of this section is to explain how to use this second crystalline cohomology of a $K3$ surface to determine supersingularity.

Connection to supersingularity

Remember that $W = W(K)$ for a perfect field K of positive characteristic p and that $\sigma : W \to W$ denotes the Frobenius morphism. An *F-crystal* (M, φ) over K is a free W-module M of finite rank together with an injective and σ-linear map $\varphi : M \to M$, i.e. φ is injective, additive and

$$\varphi(r \cdot m) = \sigma(r) \cdot \varphi(m)$$

for all $r \in W$ and $m \in M$. An *F-isocrystal* (V, φ) is a finite dimensional K-vector space V together with an injective and σ-linear map $\varphi : V \to V$. A *morphism* $u : (M, \varphi) \to (N, \psi)$ of F-crystals (respectively F-isocrystals) is a W-linear (respectively K-linear) map $M \to N$ such that $\varphi \circ u = u \circ \psi$. An *isogeny* of F-crystals is a morphism $u : (M, \varphi) \to (N, \psi)$ of F-crystals, such that the induced map $M \otimes_W K \to N \otimes_W K$ is an isomorphism of F-isocrystals.
For the rest of this section, we will consider the following two examples of F-crystals.

Example 3.31 ([CL98, Exposé II, 1.1]).

(1) Let X be a smooth, proper variety over K, then for all $i \geq 0$

$$H^i_{\text{cris}}(X/W)/\text{torsion}$$

is a free W-module of finite rank. By functoriality, the absolute Frobenius morphism $F : X \to X$ induces a σ-linear map

$$\varphi : H^i_{\text{cris}}(X/W)/\text{torsion} \to H^i_{\text{cris}}(X/W)/\text{torsion}.$$

One can show that φ is injective, hence $\left(H^i_{\text{cris}}(X/W)/\text{torsion}, \varphi\right)$ is an F-crystal.

(2) Let $W_\sigma \langle T \rangle$ be the non-commutative polynomial ring in T over W with the relations

$$T \cdot r = \sigma(r) \cdot T$$

for all $r \in W$. Let $a = \frac{r}{s} \in \mathbb{Q}_{\geq 0}$ with non-negative and coprime integers r and s. Then

$$M_a \coloneqq W_\sigma \langle T \rangle / (T^s - p^r)$$

together with

$$\begin{aligned} M_a &\to M_a \\ m &\mapsto T \cdot m \end{aligned}$$

defines an F-crystal (M_a, φ). The rational number $a = \frac{r}{s}$ is called the *slope* of M_a.

These last examples are crucial for the following isogeny classification.

Theorem 3.32 ([Man63]). *If (M, φ) is an F-crystal over an algebraically closed field of positive characteristic, then (M, φ) is isogeneous to a direct sum of F-crystals of the form*

$$(M, \varphi) \sim \bigoplus_{a \in \mathbb{Q}_{\geq 0}} M_a^{n_a}.$$

The elements in the set

$$\left\{ a \in \mathbb{Q}_{\geq 0} \;\middle|\; n_a \neq 0 \right\}$$

are called the *slopes* of (M, φ). For every slope a of (M, φ), the integer

$$\lambda_a \coloneqq n_a \cdot \operatorname{rank}_W M_a$$

is called the *multiplicity* of the slope a. If (M, φ) is an F-crystal over a perfect field K, its slopes and multiplicities are defined to be the ones of $(M, \varphi) \otimes_{W(K)} W\left(\overline{K}\right)$, where $\overline{K}$ is an algebraic closure of K.
Let (M, φ) be an F-crystal and let

$$0 \leq a_1 < a_2 < \ldots < a_m$$

be its slopes with respective multiplicities $\lambda_1, \ldots, \lambda_m$. The *Newton polygon* of (M, φ) is defined to be the graph of the piecewise linear function Nwt_H from the interval $[0, \operatorname{rank} M]$ to $\mathbb{R}$, such that $\mathrm{Nwt}_M(0) = 0$ and with

$$\begin{aligned} &\text{slope } a_1 \text{ if } 0 \leq t < \lambda_1, \\ &\text{slope } a_2 \text{ if } \lambda_1 \leq t < \lambda_1 + \lambda_2, \\ &\qquad \vdots \end{aligned}$$

Remark that the Newton polygon is convex and determines the F-crystal up to isogeny.
Let X be a $K3$ surface. Then it is shown in [Lie16, Proposition 2.5] that $H^i_{\text{cris}}(X/W)$ is a free W-module for all $i \geq 0$. In particular, we have seen that the only interesting cohomology is $H^2_{\text{cris}}(X/W)$, which is free of rank 22 (see Proposition 3.30).

Proposition 3.33 ([Lie16, Proposition 6.17]). *Let X be a $K3$ surface. Then, there are only 12 possibilities for the Newton polygon of the F-crystal $H^2_{cris}(X/W)$. These possibilities are determined by the height* ht *of the formal Brauer group of X in the following sense:*

(1) If $\mathrm{ht} < \infty$, then the slopes and multiplicites of the Newton polygon of $H^2_{cris}(X/W)$ are as follows:

slope	$1 - \frac{1}{\mathrm{ht}}$	1	$1 + \frac{1}{\mathrm{ht}}$
multiplicity	ht	$22 - 2\,\mathrm{ht}$	ht

(2) If $\mathrm{ht} = \infty$, then the Newton polygon of $H^2_{cris}(X/W)$ is of slope 1 with multiplicity 22.

Summarizing the results of this section we have the following criterion for a $K3$ surface to be supersingular:

Theorem 3.34. *Let X be a $K3$ surface in odd characteristic. Then the following are equivalent:*

(1) X is Shioda-supersingular, i.e. $\rho(X) = 22$.

(2) X is Artin-supersingular, i.e. the height of the formal Brauer group of X is infinite.

(3) The Newton polygon of $H^2_{cris}(X/W)$ has slope 1.

More generally, the height of a Calabi-Yau variety X of dimension n over an algebraically closed field K (i.e. a smooth projective variety X over K such that $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim(X)$ and $\omega_X \cong \mathcal{O}_X$) can be computed using crystalline cohomology, namely

$$\mathrm{ht} = \dim_L H^n(X, W\mathcal{O}_X) \otimes_W L = \dim_L \left(H^n_{\mathrm{cris}}(X/W) \otimes_W L\right)_{[0,1)}$$

if $H^n(X, W\mathcal{O}_X) \otimes_W L \neq 0$ and $\mathrm{ht} = \infty$ otherwise (see section 6.3 of [Lie16]). Here, L denotes the quotient field of $W = W(K)$ and $(\cdot)_{[0,1)}$ denotes the subspace of slopes strictly less than 1.

3.3.3 Weighted Delsarte $K3$ surfaces

In this section, we introduce weighted Delsarte $K3$ surfaces. For further references for this section see [Shi86], [Got04], [Got00], [Got96b], [Got03] and [Got96a].

Let K be an algebraically closed field of characteristic $p > 2$ and let $\alpha := (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ be a tuple of positive integers, such that

$$p \nmid \alpha_i \text{ for } 0 \leq i \leq 3 \text{ and}$$

$$\gcd(\alpha_i, \alpha_j, \alpha_k) = 1 \text{ for all } \{i, j, k\} \subset \{0, 1, 2, 3\}.$$

Let $\mathbb{P}^3(\alpha) := \operatorname{Proj} K[x_0, x_1, x_2, x_3]$ with $\deg(x_i) := \alpha_i$ for $0 \leq i \leq 3$ be the weighted projective 3-space over K of type α.
Let m be a positive integer such that $p \nmid m$ and let $A = (a_{ij}) \in \mathbb{Z}^{4\times 4}$ be a matrix such that

(1) $a_{ij} \geq 0$ and $p \nmid a_{ij}$ for all (i, j),

(2) given j there is some i, such that $a_{ij} = 0$,

(3) $p \nmid \det(A)$,

(4) $\sum_{j=0}^{3} \alpha_j a_{ij} = m$ for all $0 \leq i \leq 3$, i.e. $A\alpha^T = \big(m, m, m, m\big)^T$.

Definition 3.35. *A* weighted Delsarte surface *in $\mathbb{P}^3(\alpha)$ of degree m with matrix A is defined to be the surface $X_A \subset \mathbb{P}^3(\alpha)$ given by*

$$F_A := \sum_{i=0}^{3} x_0^{a_{i0}} x_1^{a_{i1}} x_2^{a_{i2}} x_3^{a_{i3}} = 0.$$

This type of surfaces were considered first by Delsarte ([Del51]) in the homogeneous case, i.e. the case where $\alpha = (1, 1, 1, 1)$. Later Shioda ([Shi86]) investigated Delsarte surfaces in $\mathbb{P}^3$ to demonstrate the effectiveness of his algorithm for computing Picard numbers of surfaces. Finally, the weighted version of Delsarte surfaces was introduced in [Got96b].
Weighted Delsarte surfaces are in general singular surfaces. We denote by $\widetilde{X_A}$ the minimal resolution (of singularities) of X_A. Here, a resolution of singularities $f : Y \to X$ is called *minimal* if for every resolution $g : Z \to X$, there is a morphism $h : Z \to Y$ such that $f \circ h = g$, i.e.

$$\begin{tikzcd} Y \arrow[r, "f"] & X \\ Z \arrow[u, dashed, "h"] \arrow[ur, "g"] & \end{tikzcd}$$

commutes. The fact that $\widetilde{X_A}$ exists and is unique up to isomorphism is shown in [Abh56] and [Lip69, Corollary 27.3] respectively.

Definition 3.36 ([Dol82, section 3.1])**.** *Let X be a closed subscheme of a weighted projective space $\mathbb{P}^3(\alpha)$ and let $p : \mathbb{A}^4 \setminus \{0\} \to \mathbb{P}^3(\alpha)$ be the canonical projection.*

(1) The scheme closure of $p^{-1}(X)$ in $\mathbb{A}^4$ is called the affine quasicone *over X.*

(2) A closed subscheme $X \subset \mathbb{P}^3(\alpha)$ is called quasi-smooth, *if its affine quasicone is smooth outside the origin.*

In particular, if $X = X_A$ is a weighted Delsarte surface, then X_A is quasi-smooth if and only if $(0,0,0,0)$ is the only point, where all four partial derivatives of F_A vanish. In other words, X_A is quasi-smooth if F_A has an isolated singularity at the origin.

Definition 3.37 ([Dim86, Definition 1]). *A hypersurface X in $\mathbb{P}^3(\alpha)$ is* in general position relative to $\mathbb{P}^3(\alpha)_{\text{sing}}$ *if* $\operatorname{codim}_X(X \cap \mathbb{P}^3(\alpha)_{\text{sing}}) \geq 2$*, where* $\mathbb{P}^3(\alpha)_{\text{sing}}$ *denotes the singular locus of* $\mathbb{P}^3(\alpha)$*.*

In order to show that a hypersurface is in general position relative to $\mathbb{P}^3(\alpha)_{\text{sing}}$, we will use the following proposition:

Proposition 3.38.

(1) For a point $P = (x_0 : x_1 : x_2 : x_3)$ in $\mathbb{P}^3(\alpha)$, let

$$I_P \coloneqq \left\{ i \;\middle|\; 0 \leq i \leq 3, x_i \neq 0 \right\}$$

be the subset of indices corresponding to non-zero x_i's. Then the singular locus of $\mathbb{P}^3(\alpha)$ is given by

$$\mathbb{P}^3(\alpha)_{\text{sing}} = \left\{ P \in \mathbb{P}^3(\alpha) \;\middle|\; \gcd(\alpha_i \mid i \in I_P) \geq 2 \right\}.$$

(2) If X is quasi-smooth and in general position relative to $\mathbb{P}^3(\alpha)_{\text{sing}}$, then $X \cap \mathbb{P}^3(\alpha)_{\text{sing}}$ is equal to the singular locus of X.

For the proof of the first part of this proposition see [DD85, Proposition 7], for the proof of the second part see [Dim86, Proposition 8].

Example 3.39 ([Got03, Remark 1.3]). Let us give an example of a quasi-smooth hypersurface that is not in general position relative to $\mathbb{P}^3(\alpha)_{\text{sing}}$. Let $\alpha \coloneqq (1,2,2,3)$, $m \coloneqq 9$ and $p \neq 2,3$. Consider the hypersurface X_A of degree 9 in $\mathbb{P}^3(\alpha)$ given by

$$F_A \coloneqq x_0^9 + x_0 x_1^4 + x_2^3 x_3 + x_3^3 = 0.$$

The Jacobian ideal of F_A is given by $J(F_A) = \left(9x_0^8 + x_1^4, 4x_0x_1^3, 3x_2^2x_3, x_2^3 + 3x_3^2\right)$ and it is easy to see that $(0,0,0,0)$ is the only point, where all four partial derivatives vanish. Therefore, X_A is quasi-smooth. By Proposition 3.38 we see that $\mathbb{P}^3(\alpha)_{\text{sing}}$ consists of $(0 : x_1 : x_2 : 0)$ and $(0 : 0 : 0 : 1)$. Therefore, $X_A \cap \mathbb{P}^3(\alpha)_{\text{sing}} = \{(0 : x_1 : x_2 : 0)\}$, which has codimension 1 in X_A, hence, X_A is not in general position relative to $\mathbb{P}^3(\alpha)_{\text{sing}}$.

Example 3.40. Now, let $\alpha \coloneqq (1,1,1,2)$, $m \coloneqq 5$ and $p \neq 2,5$. Consider the hypersurface X_A of degree 5 in $\mathbb{P}^3(\alpha)$ given by

$$F_A \coloneqq x_0^5 + x_1^5 + x_2^5 + x_0x_3^2 = 0.$$

Then the Jacobian ideal of F_A is given by $J(F_A) = \left(5x_0^4 + x_3^2, 5x_1^4, 5x_2^4, 2x_0x_3\right)$ and again it is easy to see that $(0,0,0,0)$ is the only point, where all four partial derivatives vanish, which means that X_A is quasi-smooth. By Proposition 3.38 we see that $\mathbb{P}^3(\alpha)_{\text{sing}}$ consists of $(0:0:0:1)$. Therefore, $X_A \cap \mathbb{P}^3(\alpha)_{\text{sing}} = \{(0:0:0:1)\}$, which has codimension ≥ 2 in X_A. Hence, X_A is in general position relative to $\mathbb{P}^3(\alpha)_{\text{sing}}$.

Lemma 3.41 ([Got03, section 1]). *Let X_A be a weighted Delsarte surface in $\mathbb{P}^3(\alpha)$ of degree m with matrix A. Assume that X_A is quasi-smooth and in general position relative to $\mathbb{P}^3(\alpha)_{\text{sing}}$. Then the minimal resolution $\widetilde{X_A}$ of X_A is a $K3$ surface if and only if $m = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$.*

Definition 3.42. *Let X_A be a weighted Delsarte surface in $\mathbb{P}^3(\alpha)$ of degree m with matrix A. Let X_A be quasi-smooth and in general position relative to $\mathbb{P}^3(\alpha)_{\text{sing}}$ and let $m = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$. Then we call X_A a weighted Delsarte $K3$ surface in $\mathbb{P}^3(\alpha)$ of degree m with matrix A.*

There are exactly 95 pairs of m and α which give $K3$ surfaces in $\mathbb{P}^3(\alpha)$ (see [Rei, section 4.5]). For a complete list of these 95 pairs see [IF00, section 13.3].

3.3.4 Relation of weighted Delsarte surfaces to Fermat surfaces

As already mentioned, a weighted Delsarte $K3$ surface X_A is birational to the quotient of a Fermat surface. This will be proven in this section following the ideas of [Shi86, section 3].

Let X_A be a weighted Delsarte $K3$ surface in $\mathbb{P}^3(\alpha)$ of degree m with matrix A given by

$$F_A := \sum_{i=0}^{3} x_0^{a_{i0}} x_1^{a_{i1}} x_2^{a_{i2}} x_3^{a_{i3}}$$

and let $d := |\det(A)|$. By X_d we denote the Fermat surface of degree d in $\mathbb{P}^3$ given by the vanishing of

$$F_d := y_0^d + y_1^d + y_2^d + y_3^d,$$

by X_1 we denote the subspace of $\mathbb{P}^3$ given by the vanishing of

$$F_1 := z_0 + z_1 + z_2 + z_3.$$

Denote by $B = (b_{ij})$ the adjugate matrix of the matrix A, i.e. $AB = \det(A) \cdot \mathbb{1}_4$. Then we have rational maps

$$X_d \overset{\varphi}{\dashrightarrow} X_A \overset{\psi}{\dashrightarrow} X_1$$

given by

$$\varphi(y_0 : y_1 : y_2 : y_3) = (x_0 : x_1 : x_2 : x_3), \quad x_i = \prod_{j=0}^{3} y_j^{b_{ij}}$$

and

$$\psi\left(x_0 : x_1 : x_2 : x_3\right) = \left(z_0 : z_1 : z_2 : z_3\right), \quad z_i = \prod_{j=0}^{3} x_j^{a_{ij}}.$$

Since $AB = \det(A) \cdot \mathbb{1}_4$, it is easy to check that φ and ψ are well-defined and that the composition is given by the morphism

$$(y_0 : y_1 : y_2 : y_3) \mapsto \left(y_0^d : y_1^d : y_2^d : y_3^d\right).$$

Now, we claim that the extensions of the function fields $K(X_d)/K(X_1)$ is a Galois extension with Galois group

$$\Gamma \coloneqq \left\{(\zeta_0 : \zeta_1 : \zeta_2 : \zeta_3) \mid \zeta_i^d = 1, \zeta_i \in K\right\} \subset \operatorname{Aut}\left(X_d\right),$$

where Γ acts on X_d by the rule

$$(\zeta_0 : \zeta_1 : \zeta_2 : \zeta_3)(y_0 : y_1 : y_2 : y_3) \coloneqq (\zeta_0 y_0 : \zeta_1 y_1 : \zeta_2 y_2 : \zeta_3 y_3).$$

For this, let us first briefly recall the definition of the function field of a projective variety $Y \subseteq \mathbb{P}^n = \operatorname{Proj} K[x_0, \ldots, x_n]$ (see [Har77, Chapter I, section 3] for more details). Let U_i be the open set $x_i \neq 0$ and let

$$\begin{aligned} \phi_i : U_i &\to \mathbb{A}^n \\ (a_0, \ldots, a_n) &\mapsto \left(\frac{a_0}{a_i}, \ldots, \frac{a_n}{a_i}\right), \end{aligned}$$

where $\frac{a_i}{a_i}$ is omitted. Define Y_i to be the image of $Y \cap U_i$ under ϕ_i. Then $K(Y) \coloneqq K(Y_i)$ (remark that this definition does not depend on the choice of the affine patch, hence we assume $i = 0$ in the following). If $Y \subseteq \mathbb{A}^n$ is an affine variety, then $K(Y)$ is given by the quotient field of $A(Y) \coloneqq K[x_1, \ldots, x_n]/I(Y)$. By the above description we have

$$K(X_d) = \operatorname{Quot}\left(\frac{K\left[\frac{y_1}{y_0}, \frac{y_2}{y_0}, \frac{y_3}{y_0}\right]}{\left(1 + \left(\frac{y_1}{y_0}\right)^d + \left(\frac{y_2}{y_0}\right)^d + \left(\frac{y_3}{y_0}\right)^d = 0\right)}\right)$$

and

$$K(X_1) = \operatorname{Quot}\left(\frac{K\left[\frac{z_1}{z_0}, \frac{z_2}{z_0}, \frac{z_3}{z_0}\right]}{\left(1 + \frac{z_1}{z_0} + \frac{z_2}{z_0} + \frac{z_3}{z_0} = 0\right)}\right).$$

Now, the above claim that the Galois group of the extension $K(X_d)/K(X_1)$ is equal to Γ follows. Indeed, $K(X_d)$ is obtained by adjoining three times a d-th root of unity to $K(X_1)$, hence the Galois group of this extension is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^3$. Passing from affine coordinates to projective coordinates we see $\Gamma \cong (\mathbb{Z}/d\mathbb{Z})^3$.

Proposition 3.43. *Setting*

$$\Gamma_A := \left\{ (\zeta_0 : \zeta_1 : \zeta_2 : \zeta_3) \in \Gamma \;\middle|\; \prod_{j=0}^{3} \zeta_j^{b_{ij}} \text{ is independent of } i \right\},$$

we have $K(X_A) = K(X_d)^{\Gamma_A}$.

Proof. By the Galois correspondence it is enough to show that the Galois group of the extension $K(X_d)/K(X_A)$, which is a subgroup of Γ, is equal to Γ_A. First, let $\sigma = (\zeta_0 : \zeta_1 : \zeta_2 : \zeta_3) \in \Gamma_A$. Then,

$$\sigma\left(\frac{x_i}{x_0}\right) = \sigma\left(\frac{\prod_{j=0}^{3} y_j^{b_{ij}}}{\prod_{j=0}^{3} y_j^{b_{0j}}}\right) = \frac{\prod_{j=0}^{3} \zeta_j^{b_{ij}} y_j^{b_{ij}}}{\prod_{j=0}^{3} \zeta_j^{b_{0j}} y_j^{b_{0j}}} = \frac{\left(\prod_{j=0}^{3} \zeta_j^{b_{ij}}\right) x_i}{\left(\prod_{j=0}^{3} \zeta_j^{b_{0j}}\right) x_0} = \frac{x_i}{x_0}$$

for all i, since $\prod_{j=0}^{3} \zeta_j^{b_{ij}}$ is independent of i. This means that σ acts as the identity on $K(X_A)$.
For the opposite direction, let

$$\frac{x_i}{x_0} = \sigma\left(\frac{x_i}{x_0}\right) = \frac{\left(\prod_{j=0}^{3} \zeta_j^{b_{ij}}\right) x_i}{\left(\prod_{j=0}^{3} \zeta_j^{b_{0j}}\right) x_0}$$

for every $\sigma \in \Gamma_A \subset \Gamma$ and for all i. But this means, that $\prod\limits_{j=0}^{3} \zeta_j^{b_{ij}}$ has to be independent of i. ☐

Altogether, $K(X_A) = K(X_d)^{\Gamma_A} = K\left(X_d/\Gamma_A\right)$ and this means that X_A is birational to the quotient X_d/Γ_A.

3.3.5 Supersingular weighted Delsarte $K3$ surfaces

The aim of this section is to prove a criterion of [Got04] for the minimal resolution of a weighted Delsarte $K3$ surface to be supersingular. As already mentioned in section 3.3.2 we can use crystalline cohomology to check supersingularity of $K3$ surfaces. Hence, let us first compute the second crystalline cohomology of the Fermat surface X_d from section 3.3.4 and of the weighted Delsarte $K3$ surface X_A.

Proposition 3.44 ([Shi79b, section 1], [Shi79a]). *Let X_d be the Fermat surface in $\mathbb{P}^3$ of degree d. Define*

$$\mathfrak{A}(X_d) := \left\{ q = (q_0, q_1, q_2, q_3) \middle| \, q_i \in \mathbb{Z}/d\mathbb{Z}, q_i \neq 0 (0 \leq i \leq 3), \sum_{i=0}^{3} q_i = 0 \bmod d \right\}.$$

Then

$$H^2_{cris}(X_d/W) \cong V(0) \oplus \bigoplus_{q \in \mathfrak{A}(X_d)} V(q),$$

where the $V(q)$'s are W-modules of rank 1 defined in [Got96b].

Using that X_A is birational to the quotient X_d/Γ_A the above decomposition of the second crystalline cohomology group of X_d induces a decomposition of $H^2_{\mathrm{cris}}(X_A/W)$ as follows.

Proposition 3.45 ([Got96b, Proposition 2.1], [Got00, Proposition 2.3]). *Let X_A be a weighted Delsarte $K3$ surface in $\mathbb{P}^3(\alpha)$ of degree m with matrix A and let $\widetilde{X_A}$ be its minimal resolution. Define*

$$\mathfrak{A}(X_A) := \left\{ q = (q_0, q_1, q_2, q_3) \in \mathfrak{A}(X_d) \;\middle|\; qA = (0,0,0,0) \bmod d \right\}.$$

Then

$$H^2_{cris}((X_d/\Gamma_A)/W) \cong H^2_{cris}(X_d/W)^{\Gamma_A} \cong V(0) \oplus \bigoplus_{q \in \mathfrak{A}(X_A)} V(q)$$

and

$$H^2_{cris}(\widetilde{X_A}/W) \approx H^2_{cris}(X_d/W)^{\Gamma_A},$$

where $\approx$ means that $H^2_{cris}(\widetilde{X_A}/W)$ and $H^2_{cris}(X_d/W)^{\Gamma_A}$ only differ by classes of exceptional cycles.

Definition 3.46. *For each element $q = (q_0, q_1, q_2, q_3) \in \mathfrak{A}(X_d)$ define the* length *of q to be*

$$\|q\| := \sum_{i=0}^{3} \left\langle \frac{q_i}{d} \right\rangle - 1,$$

where $\left\langle \frac{q_i}{d} \right\rangle$ denotes the fractional part of $\frac{q_i}{d}$.

We claim that $\|q\| \in \mathbb{Z}$ for all $q \in \mathfrak{A}(X_d)$. For the proof of this, let $q = (q_0, q_1, q_2, q_3) \in \mathfrak{A}(X_d)$. Then $\left\langle \frac{q_i}{d} \right\rangle = \frac{\overline{q_i}}{d}$, where $\overline{a}$ denotes the representative of a modulo d between 0 and $d-1$. Furthermore, we know that $\sum_{i=0}^{3} q_i \equiv 0 \bmod d$. Altogether, this gives

$$\|q\| = \sum_{i=0}^{3} \left\langle \frac{q_i}{d} \right\rangle - 1 = \sum_{i=0}^{3} \frac{\overline{q_i}}{d} - 1 = \frac{1}{d} \sum_{i=0}^{3} \overline{q_i} - 1 \in \mathbb{Z}.$$

Lemma 3.47. *For all $q \in \mathfrak{A}(X_d)$ one has $\|q\| \in \{0, 1, 2\}$.*

Proof. Let $q = (q_0, q_1, q_2, q_3)$, then clearly $\|q\| \geq 0$, since $q_i \neq 0 \bmod d$ and $\|q\| \in \mathbb{Z}$. Moreover

$$\|q\| = \sum_{i=0}^{3} \left\langle \frac{q_i}{d} \right\rangle - 1 = \frac{1}{d} \sum_{i=0}^{3} \overline{q_i} - 1 < \frac{1}{d} 4d - 1 = 3.$$

□

Proposition 3.48. *Let X_A be a weighted Delsarte $K3$ surface with matrix A. Then, there exists a unique element $q_{ss} \in \mathfrak{A}(X_A)$ such that $\|q_{ss}\| = 0$ and a unique element $\tilde{q} \in \mathfrak{A}(X_A)$ such that $\|\tilde{q}\| = 2$.*

We will show in the proof that these elements are given by

$$q_{ss} = (1, 1, 1, 1)\,\mathrm{adj}(A),$$

where $\mathrm{adj}(A)$ is the adjugate matrix of A and

$$\tilde{q} = (d, d, d, d) - q_{ss}.$$

Proof of Proposition 3.48. We first show the uniqueness of the element $q_{ss} = (q_0, q_1, q_2, q_3)$. If we assume $1 \leq q_i < d$ for $0 \leq i \leq 3$, then q_{ss} is the unique element satisfying

$$q_{ss}A \equiv (0, 0, 0, 0) \bmod d \quad \text{and} \quad q_0 + q_1 + q_2 + q_3 = d.$$

Since $\widetilde{X_A}$ is $K3$ we know that

$$A\alpha^T = \begin{pmatrix} m \\ m \\ m \\ m \end{pmatrix} = \sum_{i=0}^{3} \alpha_i \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

The condition $q_{ss}A \equiv (0, 0, 0, 0) \bmod d$ means that there exists some $\beta = (\beta_0, \beta_1, \beta_2, \beta_3) \in \mathbb{Z}^4$ with $q_{ss}A = d\beta$. Since $q_i > 0$, $a_{ij} \geq 0$ and $d \neq 0$, we get that $\beta_i > 0$ for $0 \leq i \leq 3$. We have

$$d\left(\sum_{i=0}^{3} \beta_i \alpha_i\right) = d\beta\alpha^T = q_{ss}A\alpha^T = q_{ss} \begin{pmatrix} m \\ m \\ m \\ m \end{pmatrix} = m \sum_{i=0}^{3} q_i = md.$$

Therefore, $\sum_{i=0}^{3} \beta_i \alpha_i = m = \sum_{i=0}^{3} \alpha_i$, which means that $\beta_i = 1$ for $0 \leq i \leq 3$ since all $\beta_i > 0$. Thus, q_{ss} is given by the solution of $q_{ss}A = (d, d, d, d)$, i.e. $q_{ss} = (1, 1, 1, 1)\,\mathrm{adj}(A)$, where $\mathrm{adj}(A)$ is the adjugate matrix of A.

Now the uniqueness of $\tilde{q} = (p_0, p_1, p_2, p_3)$ follows from the uniqueness of q_{ss} as follows. If we assume $1 \le p_i < d$ for $0 \le i \le 3$, then $\tilde{q}$ is the unique element satisfying

$$\tilde{q}A \equiv (0,0,0,0) \bmod d \quad \text{and} \quad p_0 + p_1 + p_2 + p_3 = 3d.$$

Clearly, the element $(d,d,d,d) - q_{ss}$ satisfies these conditions. Now, assume that there exist two elements with these three conditions, say $\tilde{q} = (p_0, p_1, p_2, p_3)$ and $\tilde{q}'$. Then the element $(d,d,d,d) - \tilde{q} = (d - p_0, d - p_1, d - p_2, d - p_3)$ satisfies $1 \le d - p_i < d$ since $1 \le p_i < d$ and

$$\sum_{i=0}^{3}(d - p_i) = 4d - \sum_{i=0}^{3} p_i = 4d - 3d = d.$$

Moreover, $(d-p_0, d-p_1, d-p_2, d-p_3)A = (0,0,0,0) \bmod d$ since we know that $\tilde{q}A \equiv (0,0,0,0) \bmod d$. But the same holds true for the element $(d,d,d,d) - \tilde{q}'$. Hence, by the uniqueness of q_{ss} it follows that $(d,d,d,d) - \tilde{q} = (d,d,d,d) - \tilde{q}'$ and therefore $\tilde{q} = \tilde{q}'$. $\square$

Definition 3.49. *Given the element $q_{ss} = (q_0, q_1, q_2, q_3)$, we define*

$$e_A := \frac{d}{\gcd(q_0, q_1, q_2, q_3, d)}.$$

Now, we are able to formulate the main theorem of this section:

Theorem 3.50 ([Got04, Proposition 2.2 & Remark 2.1])**.** *Let X_A be a weighted Delsarte $K3$ surface with matrix A. Then the height of the formal Brauer group of the minimal resolution $\widetilde{X_A}$ of X_A is infinite (i.e. $\widetilde{X_A}$ is supersingular) if and only if $p^\mu \equiv -1 \bmod e_A$ for some integer $\mu \ge 1$.*

The aim of the rest of this section is to prove Theorem 3.50. For this, we need some more lemmata.

Lemma 3.51. *For all t with $\gcd(t,d) = 1$ one has*

$$\|tq_{ss}\| = \begin{cases} 0, & \text{if } t \equiv 1 \bmod e_A \\ 1, & \text{if } t \not\equiv \pm 1 \bmod e_A \\ 2, & \text{if } t \equiv -1 \bmod e_A. \end{cases}$$

Proof. We first show that $tq_{ss} = (tq_0, tq_1, tq_2, tq_3) \in \mathfrak{A}(X_d)$. For this, suppose there exists a t with $\gcd(t,d) = 1$ and such that $tq_i \equiv 0 \bmod d$ for some i. Since $\gcd(t,d) = 1$ this would give $q_i \equiv 0 \bmod d$ which is a contradiction to $1 \le q_i$. Hence $tq_{ss} \in \mathfrak{A}(X_d)$ and by Lemma 3.47 we have $\|tq_{ss}\| \in \{0,1,2\}$. We can now prove the assertion by showing

$$\textbf{(I)} : \|tq_{ss}\| = 0 \text{ if and only if } t \equiv 1 \bmod e_A.$$
$$\textbf{(II)} : \|tq_{ss}\| = 2 \text{ if and only if } t \equiv -1 \bmod e_A.$$

Let us start with the proof of **(I)**:
First, let $t \equiv 1 \bmod e_A$, i.e. $t = 1 + e_A s$ for some s and let

$$g := \gcd(q_0, q_1, q_2, q_3, d),$$

i.e. $ge_A = d$. Then, $gt = g + (ge_A)s = g + ds$. Using this, one gets

$$\left\langle \frac{tq_i}{d} \right\rangle = \left\langle \frac{1}{g}\frac{gtq_i}{d} \right\rangle = \left\langle \frac{1}{g}\frac{(g+ds)q_i}{d} \right\rangle = \left\langle \frac{1}{g}\left(\frac{gq_i}{d} + sq_i\right) \right\rangle = \left\langle \frac{q_i}{d} + \frac{sq_i}{g} \right\rangle$$
$$= \left\langle \frac{q_i}{d} \right\rangle$$

since g divides q_i for all i. Therefore, $\|tq_{ss}\| = \|q_{ss}\| = 0$.
For the opposite direction let $\|tq_{ss}\| = 0$, i.e. $\frac{1}{d}\sum_{i=0}^{3} \overline{tq_i} = 1$ or equivalently $\sum_{i=0}^{3} \overline{tq_i} = d$. Moreover, $1 \leq \overline{tq_i} < d$, hence, the element $\left(\overline{tq_0}, \overline{tq_1}, \overline{tq_2}, \overline{tq_3}\right)$ satisfies the three properties

- $1 \leq \overline{tq_i} < d$,
- $\sum\limits_{i=0}^{3} \overline{tq_i} = d$ and
- $\left(\overline{tq_0}, \overline{tq_1}, \overline{tq_2}, \overline{tq_3}\right) A = (0,0,0,0) \bmod d$.

But q_{ss} is unique with these three conditions, therefore, we get that

$$tq_i \equiv q_i \bmod d$$

for all i. Now, let $n_i, m \in \mathbb{Z}$ such that $\sum_{i=0}^{3} n_i q_i + md = g$, hence $\sum_{i=0}^{3} n_i q_i \equiv g \bmod d$. Since $(t-1)q_i \equiv 0 \bmod d$ by the above, it follows $(t-1)q_i n_i \equiv 0 \bmod d$ and moreover,

$$(t-1)g \equiv \sum_{i=0}^{3} (t-1)q_i n_i \equiv 0 \bmod d.$$

Thus, $(t-1)g = ds$ for some s, which means that $t - 1 = \frac{d}{g}s = e_A s$. This means that $t \equiv 1 \bmod e_A$.
The proof of **(II)** is quite similar:
Let $t \equiv -1 \bmod e_A$, i.e. $t = -1 + e_A s$ for some s and again we get $gt = -g + (ge_A)s = -g + ds$. Similarly as above, this gives

$$\left\langle \frac{tq_i}{d} \right\rangle = \left\langle \frac{1}{g}\frac{gtq_i}{d} \right\rangle = \left\langle \frac{1}{g}\frac{(-g+ds)q_i}{d} \right\rangle = \left\langle \frac{1}{g}\left(\frac{-gq_i}{d} + sq_i\right) \right\rangle = \left\langle \frac{-q_i}{d} + \frac{sq_i}{g} \right\rangle$$
$$= \left\langle \frac{d - q_i}{d} \right\rangle.$$

This yields

$$\|tq_{ss}\| = \sum_{i=0}^{3} \frac{d-q_i}{d} - 1 = 3 - \sum_{i=0}^{3} \frac{q_i}{d} = 2.$$

For the opposite direction, let $\|tq_{ss}\| = 2$ i.e. $\frac{1}{d}\sum_{i=0}^{3} \overline{tq_i} = 3$ or equivalently $\sum_{i=0}^{3} \overline{tq_i} = 3d$. Again,

- $1 \leq \overline{tq_i} < d$,
- $\sum\limits_{i=0}^{3} \overline{tq_i} = 3d$ and
- $\left(\overline{tq_0}, \overline{tq_1}, \overline{tq_2}, \overline{tq_3}\right) A = (0,0,0,0) \bmod d$.

By Proposition 3.48 we know that $(d-q_0, d-q_1, d-q_2, d-q_3)$ is unique with these three conditions, hence, we get that

$$tq_i \equiv d - q_i \equiv -q_i \bmod d$$

for all i. Similarly as above, one deduces that $(t+1)g \equiv 0 \bmod d$ and then $t \equiv -1 \bmod e_A$. $\square$

Definition 3.52. *Let f_d denote the order of p modulo d and let*

$$H_d := \left\{ p^i \bmod d \,\middle|\, 0 \leq i < f_d \right\}.$$

For $q \in \mathfrak{A}(X_d)$ we define as in [GY95]

$$A_{H_d}(q) := \sum_{t \in H_d} \|tq\|.$$

Lemma 3.53. *Let f_{e_A} denote the order of p modulo e_A. Then:*

$$\sum_{i=0}^{f_{e_A}-1} \left\|p^i q_{ss}\right\| = \begin{cases} f_{e_A}, & \text{if } p^\mu \equiv -1 \bmod e_A \text{ for some } \mu \\ f_{e_A} - 1, & \text{otherwise.} \end{cases}$$

Proof. If there exists no μ such that $p^\mu \equiv -1 \bmod e_A$, then

$$\begin{aligned} \left\|p^0 q_{ss}\right\| &= \|q_{ss}\| = 0 \text{ and} \\ \left\|p^i q_{ss}\right\| &= 1 \text{ for all } 1 \leq i < f_{e_A}. \end{aligned}$$

This gives

$$A_{H_{e_A}}(q_{ss}) = \sum_{i=0}^{f_{e_A}-1} \left\|p^i q_{ss}\right\| = f_{e_A} - 1.$$

Now, suppose that there exists some μ such that $p^\mu \equiv -1 \bmod e_A$. Without loss of generality, we can assume that $\mu < f_{e_A}$. Then

$$\begin{aligned}\left\|p^0 q_{ss}\right\| &= \|q_{ss}\| = 0,\\ \left\|p^\mu q_{ss}\right\| &= 2 \text{ and}\\ \left\|p^i q_{ss}\right\| &= 1 \text{ for all } i \neq 0, \mu\end{aligned}$$

Altogether, this gives

$$\sum_{i=0}^{f_{e_A}-1} \left\|p^i q_{ss}\right\| = 2 + (f_{e_A} - 2) = f_{e_A}.$$

□

Lemma 3.54. *With the notation from above we have:*

$$A_{H_d}(q_{ss}) = \begin{cases} f_d, & \textit{if } p^\mu \equiv -1 \bmod e_A \textit{ for some } \mu \\ \frac{f_d}{f_{e_A}}(f_{e_A} - 1), & \textit{otherwise.}\end{cases}$$

Proof. First, note that

$$\sum_{i=0}^{f_d-1} \left\|p^i q_{ss}\right\| = \frac{f_d}{f_{e_A}} \sum_{i=0}^{f_{e_A}-1} \left\|p^i q_{ss}\right\|.$$

Therefore

$$A_{H_d}(q_{ss}) = \begin{cases} \frac{f_d}{f_{e_A}} f_{e_A}, & \text{if } p^\mu \equiv -1 \bmod e_A \text{ for some } \mu \\ \frac{f_d}{f_{e_A}}(f_{e_A} - 1), & \text{otherwise.}\end{cases}$$

□

Finally, we are able to prove Theorem 3.50, which claims that the height of the formal Brauer group of $\widetilde{X_A}$ is infinite (i.e. $\widetilde{X_A}$ is supersingular) if and only if $p^\mu \equiv -1 \bmod e_A$ for some integer $\mu \geq 1$.

Proof of Theorem 3.50. By Theorem 3.34 we know that $\widetilde{X_A}$ is supersingular if and only if the Newton polygon of $H^2_{\text{cris}}(\widetilde{X_A}/W)$ is of slope 1. We already stated in Proposition 3.44 and Proposition 3.45 that we have the two decompositions

$$H^2_{\text{cris}}(X_d/W) \cong V(0) \oplus \bigoplus_{q \in \mathfrak{A}(X_d)} V(q) \tag{3.2}$$

and

$$H^2_{\text{cris}}(\widetilde{X_A}/W) \approx H^2_{\text{cris}}(X_d/W)^{\Gamma_A} \cong V(0) \oplus \bigoplus_{q \in \mathfrak{A}(X_A)} V(q), \tag{3.3}$$

where $H^2_{\text{cris}}(\widetilde{X_A}/W)$ and $H^2_{\text{cris}}(X_d/W)^{\Gamma_A}$ only differ by exceptional cycles. Now, the crucial part in this proof is to use [SY89, section 5]. In this paper, the authors use this motivic decomposition (3.2) and show that $H^2_{\text{cris}}(X_d/W)$ is of pure slope 1 if and only if

$$A_{H_d}(q) = f_d \text{ for all } q \in \mathfrak{A}(X_d) \text{ with } \|q\| = 0$$

(see [SY89, Proposition 5.11]). Since we do not want to go into the details of this motivic point of view we refer the reader to [SY89] and [GY95] for more information.
By equation (3.3) the crystalline cohomology of $\widetilde{X_A}$ is (up to exceptional cycles) a subspace of the crystalline cohomology of X_d. Moreover, the exceptional cycles in $H^2_{\text{cris}}(\widetilde{X_A}/W)$ are given by elements of length 1 (see [Got95]). Hence, $\widetilde{X_A}$ is supersingular if and only if

$$A_{H_d}(q) = f_d \text{ for all } q \in \mathfrak{A}(X_A) \text{ with } \|q\| = 0.$$

But by Proposition 3.48 we know that there is only one $q \in \mathfrak{A}(X_A)$ with the property that $\|q\| = 0$, namely the element q_{ss}. Therefore, $\widetilde{X_A}$ is supersingular if and only if $A_{H_d}(q_{ss}) = f_d$ and by Lemma 3.54 this is the case if and only if there exists an integer μ such that $p^\mu \equiv -1 \bmod e_A$. □

3.3.6 Computing the height of the formal Brauer group of a weighted Delsarte $K3$ surface

The aim of this section is to compute the height of the formal Brauer group of a weighted Delsarte $K3$ surface if it is finite:

Theorem 3.55 ([Got04, Theorem 3.2]). *Let X_A be a weighted Delsarte $K3$ surface with matrix A. Assume that there is no integer $\mu \geq 1$ such that $p^\mu \equiv -1 \bmod e_A$. Then the height of the formal Brauer group of the minimal resolution $\widetilde{X_A}$ of X_A is equal to the order of p modulo e_A.*

Proof. Since there is no integer $\mu \geq 1$ such that $p^\mu \equiv -1 \bmod e_A$, by Theorem 3.50 we know that the height of the formal Brauer group of the minimal resolution $\widetilde{X_A}$ of X_A is finite. Hence, combining Proposition 3.33 and the proof of Proposition 4.6.1 of [Yui99], the height ht can be computed as

$$1 - \frac{1}{\text{ht}} = \frac{A_{H_d}(q_{ss})}{f_d}$$

or equivalently

$$\text{ht} = \frac{f_d}{f_d - A_{H_d}(q_{ss})}$$

(also see [GY95, chapter 3]). By Lemma 3.54 we have

$$A_{H_d}(q_{ss}) = \frac{f_d}{f_{e_A}}(f_{e_A} - 1),$$

therefore $\text{ht} = f_{e_A}$. □

3.4 Counterexamples

The motivation to study weighted Delsarte $K3$ surfaces was to give two examples of weighted Delsarte $K3$ surfaces, where the first one has the same height but different F-pure threshold for varying p and the second one has the same F-pure threshold but the height will differ for two different primes p. Remember that K is an algebraically closed field of characteristic $p > 0$.

Example 3.56. Assume that $p \neq 2, 5$. Consider

$$F_A := x_0^5 + x_1^5 + x_2^5 + x_0 x_3^2 \in K[x_0, x_1, x_2, x_3],$$

which is quasi-homogeneous of degree 5 and weight $\alpha := (1, 1, 1, 2)$. Let X_A be the weighted Delsarte surface in $\mathbb{P}^3(1, 1, 1, 2)$ defined by F_A, i.e. defined by the matrix

$$A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}.$$

Using the methods of [Got04] we computed the height of the formal Brauer group of the minimal resolution $\widetilde{X_A}$ of X_A. First, remark that we have already shown in Example 3.40 that X_A is quasi-smooth and in general position relative to $\mathbb{P}^3(\alpha)_{\text{sing}}$. Furthermore, $m = 5 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$ and therefore the minimal resolution $\widetilde{X_A}$ of X_A is a $K3$ surface. One has $\det(A) = 250$ and $q_{ss} = (25, 125, 50, 50)$, hence $e_A = 10$. Thus, Theorem 3.50 shows that the height of the formal Brauer group of $\widetilde{X_A}$ is infinite if and only if there exists some $\mu \geq 1$ such that $p^\mu \equiv -1 \bmod 10$, i.e. $p \equiv 3, 7, 9 \bmod 10$.
Using the results of chapter 2 we also computed the F-pure threshold of F_A. First we computed $\mu_{F_A}(p)$ and obtained:

p	11	13	17	19
$\mu_{F_A}(p)$	11	12	16	17

By the first claim of Proposition 2.7 this means that $\mu_{F_A}(11^e) = 11^e$ and therefore, $\operatorname{fpt}(F_A) = 1$ for $p = 11$. For $p = 13, 17, 19$ by the second statement of Proposition 2.7 we get that the sequence $\left\{\frac{\mu_{F_A}(p^e)}{p^e}\right\}_e$ is constant, hence $\operatorname{fpt}(F_A) = 1 - \frac{1}{p}$ for $p = 13, 17$ and $\operatorname{fpt}(F_A) = 1 - \frac{2}{p}$ for $p = 19$. These results were also verified by the PosChar-package of Macaulay 2 ([BBH$^+$]). This package uses methods similar to the ones described in chapter 1 and chapter 2. For example, if the given polynomial is diagonal or binomial, then the F-pure threshold is computed by the results of Hernández (for more details see [BBH$^+$]). Altogether, we obtained the following:

p	11	13	17	19
ht	1	∞	∞	∞
$\mathrm{fpt}(F_A)$	1	$1-\frac{1}{p}$	$1-\frac{1}{p}$	$1-\frac{2}{p}$

In particular, one can see that for $p = 17$ and $p = 19$ the height is the same but the F-pure threshold is different.

Example 3.57. Assume that $p \neq 2, 3$. Consider

$$F_A \coloneqq x_0^8 x_1 + x_1^6 x_2 + x_2^3 + x_0 x_3^2 \in K[x_0, x_1, x_2, x_3],$$

which is quasi-homogeneous of degree 9 and weight $\alpha \coloneqq (1, 1, 3, 4)$. Let X_A be the weighted Delsarte surface in $\mathbb{P}^3(1, 1, 3, 4)$ defined by F_A, i.e. defined by the matrix

$$A = \begin{pmatrix} 8 & 1 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}.$$

Here, $J(F_A) = (8x_0^7 x_1 + x_3^2, x_0^8 + 6x_1^5 x_2, x_1^6 + 3x_2^2, 2x_0 x_3)$ and it is easy to see that $(0, 0, 0, 0)$ is the only point, where all four partial derivatives vanish. This means that X_A is quasi-smooth. By Proposition 3.38 we know that $\mathbb{P}^3(\alpha)_{\text{sing}}$ consists of $(0 : 0 : 1 : 0)$ and $(0 : 0 : 0 : 1)$, therefore $X_A \cap \mathbb{P}^3(\alpha)_{\text{sing}} = \{(0 : 0 : 0 : 1)\}$ which has codimension ≥ 2 in X_A, so X_A is in general position relative to $\mathbb{P}^3(\alpha)_{\text{sing}}$. Furthermore, $m = 9 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$ and therefore the minimal resolution $\widetilde{X_A}$ of X_A is a $K3$ surface. We compute that $\det(A) = 288$, $q_{ss} = (18, 144, 45, 81)$ and $e_A = 32$. Using Theorem 3.55 we get that the height of the formal Brauer group of $\widetilde{X_A}$ is given by

$$\mathrm{ht} = \begin{cases} 1, & \text{if } p \equiv 1 \bmod 32 \\ 2, & \text{if } p \equiv \pm 15 \bmod 32 \\ 4, & \text{if } p \equiv \pm 7, \pm 9 \bmod 32 \\ 8, & \text{if } p \equiv \pm 3, \pm 5, \pm 11, \pm 13 \bmod 32. \end{cases}$$

Again, we computed the integers $\mu_{F_A}(p)$ and obtained $\mu_{F_A}(p) = p - 1$ for $p = 11, 13, 17, 19$. By the second part of Proposition 2.7 this means that $\mathrm{fpt}(F_A) = 1 - \frac{1}{p}$ for $p = 11, 13, 17, 19$. Altogether we have:

p	11	13	17	19
ht	8	8	2	8
$\mathrm{fpt}(F_A)$	$1-\frac{1}{p}$	$1-\frac{1}{p}$	$1-\frac{1}{p}$	$1-\frac{1}{p}$

In particular, in this case the F-pure threshold is $1 - \frac{1}{p}$ for all p, but the height differs.

Altogether, the two examples above show that if a weighted Delsarte $K3$ surface has the same height for varying p, then this does not imply that it also has the same F-pure threshold for varying p and vice versa.

Appendix A

MuPAD implementation

In this section we present an implementation in MuPAD, which uses Theorem 3.50 and Theorem 3.55 to compute the height of the formal Brauer group of the minimal resolution of a weighted Delsarte $K3$ surface.

The only input of the program is the polynomial $F_A \in K[x_0, x_1, x_2, x_3]$ which defines the weighted Delsarte $K3$ surface X_A (see section 3.3.3 for details) and a set of primes p, where p is equal to the characteristic of the field K (see lines 1 and 2 on the next page).

For each p, the program translates the polynomial F_A into the corresponding matrix A and computes its determinant d (lines 4 - 9). Using d and A, in line 10 the program calculates the unique element $q_{ss} = (q_0, q_1, q_2, q_3)$, which is given by

$$q_{ss} = (1, 1, 1, 1)\,\mathrm{adj}(A)$$

by Proposition 3.48. Finally, the number

$$e_A = \frac{d}{\gcd(q_0, q_1, q_2, q_3, d)}$$

is computed (line 11).

Now, there are two possibilities. If there exists an integer i such that $p^i \equiv -1 \bmod e_A$, then by Theorem 3.50 the height is infinite (lines 14 - 17). Otherwise, the height is finite and is equal to the order of p modulo e_A by Theorem 3.55 (lines 18 - 21).

Since $\gcd(p, e_A) = 1$ (otherwise, $p \mid e_A \mid d$ which is a contradiction to the assumptions of section 3.3.3), we know that $p^{\varphi(e_A)} \equiv 1 \bmod e_A$, where φ denotes Euler‘s totient function. Therefore, in order to check the two possibilities above we only need to check all powers p^i where $1 \leq i \leq \varphi(e_A)$ (see line 13).

For example, let us compute the height in the case of Example 3.56:

```
F_A := poly(x_0^5+x_1^5+x_2^5+x_0*x_3^2,[x_0,x_1,x_2,x_3]):
for p in {3,7,11,13,17,19,23} do

  f:=poly2list(F_A):
  A:=matrix([[f[1][2][1],f[1][2][2],f[1][2][3],f[1][2][4]],
             [f[2][2][1],f[2][2][2],f[2][2][3],f[2][2][4]],
             [f[3][2][1],f[3][2][2],f[3][2][3],f[3][2][4]],
           [f[4][2][1],f[4][2][2],f[4][2][3],f[4][2][4]]]):
  d:=abs(linalg::det(A)):
  q_ss:=matrix([[1,1,1,1]])*linalg::adjoint(A):
  e_A:=d/gcd(q_ss[1],q_ss[2],q_ss[3],q_ss[4],d):

  for i from 1 to numlib::phi(e_A) do
    if (p^i mod e_A)=e_A-1 then
      print(Unquoted, "For p = ".expr2text(p).
            " the height is infinite."):
      break:
    elif (p^i mod e_A)=1 then
      print(Unquoted, "For p = ".expr2text(p).
              " the height is ".expr2text(i). "."):
      break:
    end_if:
  end_for:
end_for:
```

The output is the following:

```
For p = 3 the height is infinite.
For p = 7 the height is infinite.
For p = 11 the height is 1.
For p = 13 the height is infinite.
For p = 17 the height is infinite.
For p = 19 the height is infinite.
For p = 23 the height is infinite.
```

Next, let us compute the height in the case of Example 3.57, i.e.

$$F_A := x_0^8 x_1 + x_1^6 x_2 + x_2^3 + x_0 x_3^2$$

for $p \in \{5, 7, 11, 13, 17, 19, 23\}$. Then, the output is the following:

```
For p = 5 the height is 8.
For p = 7 the height is 4.
For p = 11 the height is 8.
For p = 13 the height is 8.
For p = 17 the height is 2.
For p = 19 the height is 8.
For p = 23 the height is 4.
```

In section 3.3.1, we have seen that the height of a $K3$ surface is an integer between 1 and 10 if it is finite. In [Yui99] Yui gave examples of $K3$ surfaces with height $1, 2, 3, 4, 6$ or 10 using so-called weighted diagonal or quasi-diagonal $K3$ surfaces. In his paper [Got04], Goto generalized these results and was able to give examples of $K3$ surfaces with height $5, 8$ and 9. The height 7, which is still missing cannot be realized by the methods of Goto (see [Got04, Remark 4.1]). We already computed examples with height $1, 2, 4, 8$ and now we want to give examples of height $3, 5, 6, 9, 10$. First, let

$$F_A := x_0^4 + x_0 x_1^3 + x_1 x_2^3 + x_2 x_3^3,$$

which is quasi-homogeneous of degree 4 and type $\alpha := (1, 1, 1, 1)$. For $p \neq 2, 3$ it is an easy computation to show that $(0, 0, 0, 0)$ is the only point where all four partial derivatives $(4x_0^3 + x_1^3, 3x_0x_1^2 + x_2^3, 3x_1x_2^2 + x_3^3, 3x_2x_3^2)$ vanish, which means that the corresponding hypersurface X_A is quasi-smooth. Moreover, $X_A \cap \mathbb{P}^3(\alpha)_{\text{sing}} = \emptyset$, hence X_A is in general position relative to $\mathbb{P}^3(1, 1, 1, 1)_{\text{sing}}$. Altogether, X_A is a weighted Delsarte $K3$ surface and the height is given by:

```
For p = 5 the height is infinite.
For p = 7 the height is 9.
For p = 11 the height is infinite.
For p = 13 the height is 9.
For p = 17 the height is infinite.
For p = 19 the height is 3.
For p = 23 the height is infinite.
```

Now, let

$$F_A := x_0^5 x_1 + x_1^5 x_2 + x_2^3 x_3 + x_3^2,$$

which is quasi-homogeneous of degree 6 and type $\alpha := (1, 1, 1, 3)$. Here, $J(F_A) = (5x_0^4 x_1, x_0^5 + 5x_1^4 x_2, x_1^5 + 3x_2^2 x_3, x_2^3 + 2x_3)$. Again, for $p \neq 2, 3, 5$ it is an easy computation to see that $(0, 0, 0, 0)$ is the only point where all four partial derivatives vanish, hence the corresponding hypersurface X_A is quasi-smooth. Moreover, $\mathbb{P}^3(\alpha)_{\text{sing}}$ consists of the point $(0 : 0 : 0 : 1)$, therefore $X_A \cap \mathbb{P}^3_{\text{sing}} = \emptyset$, hence X_A is in general position relative to $\mathbb{P}^3(\alpha)_{\text{sing}}$. Therefore, X_A is a weighted Delsarte $K3$ surface and the corresponding height for $p \in \{7, 11, 13, 17, 19, 23\}$ is:

```
For p = 7 the height is infinite.
For p = 11 the height is 5.
For p = 13 the height is infinite.
For p = 17 the height is infinite.
For p = 19 the height is infinite.
For p = 23 the height is infinite.
```

Let
$$F_A := x_0^{21} + x_1^7 + x_2^3 + x_0x_3^2,$$
which is quasi-homogeneous of degree 21 and type $\alpha := (1, 3, 7, 10)$. Let $p \neq 2, 3, 7$. Since $J(F_A) = (21x_0^{20} + x_3^2, 7x_1^6, 3x_2^2, 2x_0x_3)$ it is easy to see that the corresponding hypersurface X_A is quasi-smooth. Moreover, $\mathbb{P}^3(\alpha)_{\text{sing}}$ consists of $(0:1:0:0)$, $(0:0:1:0)$ and $(0:0:0:1)$, hence it follows that $X_A \cap \mathbb{P}^3(\alpha)_{\text{sing}} = \{(0:0:0:1)\}$, which means that X_A is also in general position relative to $\mathbb{P}^3(\alpha)_{\text{sing}}$. Altogether, X_A is a weighted Delsarte $K3$ surface and the height is given by:

```
For p = 5 the height is infinite.
For p = 11 the height is 6.
For p = 13 the height is 2.
For p = 17 the height is infinite.
For p = 19 the height is 6.
For p = 23 the height is 6.
```

Let
$$F_A := x_0^{12} + x_1^3 + x_2^2 + x_0x_3^{11},$$
which is quasi-homogeneous of degree 12 and type $\alpha := (1, 4, 6, 1)$. Let $p \neq 2, 3, 11$. Since $J(F_A) = (12x_0^{11} + x_3^{11}, 3x_1^2, 2x_2, 11x_0x_3^{10})$ it is again an easy computation to see that $(0, 0, 0, 0)$ is the only point, where all four partial derivatives vanish. Therefore, the corresponding hypersurface X_A is quasi-smooth. Moreover, $\mathbb{P}^3(\alpha)_{\text{sing}}$ consists of $(0 : x_1 : x_2 : 0)$, hence we have $\operatorname{codim}_{X_A}\left(X_A \cap \mathbb{P}^3(\alpha)_{\text{sing}}\right) \geq 2$ and X_A is also in general position relative to $\mathbb{P}^3(\alpha)_{\text{sing}}$. Altogether, X_A is a weighted Delsarte $K3$ surface and the height is given by:

```
For p = 5 the height is 10.
For p = 7 the height is 10.
For p = 13 the height is 10.
For p = 17 the height is infinite.
For p = 19 the height is 10.
For p = 23 the height is 2.
```

We want to conclude this section with a short remark about the runtime of the above program. First, even for very large p (for example for p in the order of 10^{1000}) the program returns the result immediately since we are working modulo e_A (in the above examples, the result was returned in less than 0.016s). So the only case, where the computation possibly becomes slower, is the case, where e_A is big. But in the above examples, the biggest e_A that appeared was 66 and here the height was computed immediately.

Lebenslauf

Aus datenschutzrechtlichen Gründen wurde der Lebenslauf aus der Online-Version dieser Dissertation entfernt.

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