# Two-photon exchange corrections in elastic lepton-proton scattering 

Dissertation<br>zur Erlangung des Grades<br>, Doktor der Naturwissenschaften "<br>am Fachbereich Physik, Mathematik und Informatik der Johannes Gutenberg-Universität Mainz

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Mainz, 2016

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#### Abstract

Elastic electron-proton scattering has been a time-honored tool to provide basic information on general properties of the proton, such as its charge distribution. At leading order, this process is described by the exchange of one photon. In recent years, two experimental approaches, with and without polarized protons, gave strikingly different results for the ratio of the electric to magnetic proton form factors. Even more recently, a mysterious discrepancy ("the proton radius puzzle") has been observed in the extraction of the proton charge radius from the muonic hydrogen versus hydrogen spectroscopy and elastic electron-proton scattering. In these experiments, two-photon exchange (TPE) contributions are the largest source of the hadronic uncertainty.

In the present work, the forward virtual Compton scattering is calculated within a dispersive formalism to determine TPE corrections. One of the amplitudes requires a subtraction function, which is estimated based on experimental data. Exploiting these results, the TPE correction to the Lamb shift for the 2 S level in muonic hydrogen is evaluated. Within a dispersion relation approach for the lepton-proton amplitudes, the hadronic TPE correction to the hyperfine splitting of the $S$ energy levels is also determined.

The TPE correction in the elastic lepton-proton scattering is given by a sum of diagrams with proton and with inelastic intermediate states. At low energies, the former yields the main TPE correction. Comparing a box graph model with the dispersion relations at fixed momentum transfer, we find agreement when performing one subtraction. Fixing the subtraction point to the TPE fit of data performed by the MAMI/A1 Collaboration, the contribution from the inelastic intermediate states in the electron-proton scattering is estimated. Additionally, a new method of analytical continuation of the elastic contribution to TPE amplitudes is developed.

At low momentum transfer, the inelastic intermediate states are included approximating the hadronic part of the TPE box graph by the near-forward unpolarized virtual Compton scattering which has the proton structure functions as input. The resulting TPE are compared with the empirical fit. Subsequently, the study is extended to larger momentum transfer. For this purpose, the pion-nucleon intermediate state in the dispersion relation approach is studied.

A further part of this work is devoted to the muon-proton scattering experiment (MUSE), which was proposed to compare the elastic scattering of electrons and muons on the proton target and to measure the proton charge radius in the muon-proton scattering. The sub-percent level of the experimental accuracy requires an account of TPE corrections. In this work, the proton TPE box graph for the muon-proton process is evaluated for the kinematics of the proposed experiment. Approximating the doubly virtual Compton tensor by the near-forward form, the inelastic TPE correction is quantified. Additionally, the contribution of the subtraction function, relevant because of the muon mass as compared to the beam energy, is studied in detail. The evaluated TPE correction provides the necessary input for the forthcoming MUSE experiment.


## Zusammenfassung

Die elastische Elektron-Proton-Streuung ist eine bewährte Methode, um Basisinformation der allgemeinen Eigenschaften des Protons zu ermitteln, wie zum Beispiel seine Ladungsverteilung. Dieser Prozess wird in niedrigster Ordnung durch einen Einphotonenaustausch beschrieben. In den letzten Jahren haben zwei experimentelle Ansätze, mit und ohne polarisierte Protonen, auffallend unterschiedliche Ergebnisse für das Verhältnis aus dem elektrischen und dem magnetische Formfaktors des Protons gegeben. Jüngst wurde in der Messung des Protonenladungsradius im muonischen Wasserstoff im Vergleich zur Wasserstoffspektroskopie und zur elastischen Elektron-Proton-Streuung eine mysteriöse Diskrepanz ("das Rätsel des Protonradius") beobachtet. In diesen Experimenten ist der Beitrag des Zweiphotonenaustausches (TPE, engl. two-photon exchange) die größte hadronische Unsicherheitsquelle.

In der vorliegenden Arbeit wird ein Dispersionsformalismus entwickelt um die TPE Korrekturen zu beschreiben aus der vorwärts virtuellen Compton Streuung. Eine der Amplituden in diesem Formalismus erfordert die Berechnung einer Subtraktionsfunktion anhand von Experimentellen Daten. Auf diesem Beitrag gestützt wird die entsprechende TPE-Korrektur zur Lamb-Verschiebung für das 2S-Niveau sowie zum hyperfeinen Aufspalten des S-Energieniveaus in muonischem Wasserstoff ausgerechnet.

Die TPE Korrektur in der elastischen Leptonen-Protonen Streuung ist gegeben als die Summe aus den Diagrammen mit dem Proton und inelastischen Zwischenzustände. Bei niedrigen Energien liefert ersteres die größten TPE-Korrekturen. Beim Vergleich eines hadronischen Modells mit nichtsubtrahierten Dispersionsrelationen bei festgehaltener Übertragung des Impulses wird vorgeschlagen, die subtrahierte Dispersionsrelation in der Elektron-Proton-Streuung zu verwenden. Ausgehend vom Subtraktionspunkt einer empirischen TPE-Anpassung der Daten der MAMI/A1-Kollaboration wird der Beitrag der inelastischen Zwischenzustände abgeschätzt. Zusätzlich wird eine neue Methode zur analytischen Fortführung des elastischen Beitrags zu den TPE-Amplituden entwickelt. Bei niedriger Impulsübertragung werden die inelastischen Zwischenzustände eingeschloßen und der hadronische Beitrag zur TPE-Boxdiagramms durch den vorwärts unpolarisierten virtuellen Compton-Streuungsprozess angeglichen und mit der empirischen Angleichung verglichen. Aufbauend darauf wird die Theorie auch auf größere Impulsübertragungen erweitert. Zu diesem Zweck wird der Pion-Nukleon Zwischenzustand im Dispersions Relations-Ansatz untersucht.

Ein weiterer Teil dieser Arbeit wird dem Myon-Proton-Streuungsexperiment (MUSE) gewidmet, das vorgeschlagen wurde, um die elastische Streuung der Elektronen und Myonen auf dem Protonentarget zu vergleichen und, um den Protonenladungsradius in der Myon-ProtonStreuung zu messen. Das Subprozent-Niveau der experimentellen Genauigkeit verlangt eine Berücksichtigung des TPE-Beitrages. In dieser Arbeit wird das TPE-Boxdiagramm für den Myon-Proton-Prozess für die Kinematik des vorgeschlagenen Experiments evaluiert. Durch eine Angleichung des doppelt virtuellen Compton-Tensors mit der Form in Vorwörtrichtung wird die inelastische TPE-Korrektur quantifiziert. Darüberhinaus wird der Beitrag der Subtraktionsfunktion im Detail untersucht, die wegen der beträchtlichen Masse des Myons relevant ist. Die evaluierte TPE-Korrektur wird wesentlich sein bei der Auswertung des bevorstehenden MUSE-Experiments.

## Chapter 1

## Introduction

### 1.1 Standard model of particle physics

The development of science in the 20th century allowed us to understand and test nature at very small length scales up to $10^{-18} \mathrm{~m}$. At first, the Quantum Mechanics became the standard framework in atomic physics accounting correctly for the non-relativistic effects on the scale of $10^{-10} \mathrm{~m}$ by a solution of the Schrödinger equation. Later, the unification of the principles of Quantum Mechanics and Special Relativity, resulting in the celebrated Dirac equation [1], allowed us to describe the fine structure of atomic spectra and to predict the existence of antiparticles. Paul Dirac (1902-1984) and Erwin Schrödinger (1887-1961) received the Nobel prize in physics for their discoveries in 1933. The existence of antiparticles was confirmed starting from the observation of positron [2] in cosmic rays in 1932 by Carl Anderson (1905 - 1991), who shared the Nobel prize in 1936 for this discovery. However, the attempts to apply in Quantum Mechanics the widely used perturbation theory to the quantized fields lead to divergences at higher orders. Additionally, from the experimental side the more precise measurements of the shift of energy levels in hydrogen [3], known as the Lamb shift nowadays, and of the electron anomalous magnetic moment [4] revealed unexplained discrepancies. For their measurements, Willis Lamb (1913-2008) and Polykarp Kusch (1911-1993) received the Nobel prize in 1955. The theoretical and experimental problems triggered the theoretical development of Quantum Electrodynamics, the theory describing the interaction of light and matter, starting with computations of Hans Bethe (1906-2005) [5]. As a consequence, in the following theoretical works [6-13] the covariant and gauge invariant formulation of Quantum Electrodynamics (QED) based on the abelian symmetry group $U(1)$ was developed. For their fundamental work with significant impact for the physics of elementary particles, Sin-Itiro Tomonaga (1906-1976), Julian Schwinger (1918-1994) and Richard Feynman (1918-1988) received the Nobel prize in 1965. QED was the first experimentally tested renormalizable Quantum Field Theory (QFT), and became a paradigm for the theoretical description of other fundamental interactions in nature.

The description in terms of QFT became very useful after the discovery of numerous particles from the atmospheric cosmic rays and later on from particle accelerators. The increasing number of discovered particles and measurements of their properties in the 1950th called for a classification of them and a description of their interactions. This led to the development of the Standard Model of elementary particles that describes electromagnetic, weak and strong forces in a closed and self-consistent way. We show all particles in the Standard Model and the tree-level interactions among them in Fig. 1.1 [14]. The strong interaction is symmetric under the non-abelian $S U(3)_{\mathrm{C}}$ gauge symmetry group and takes place between quarks and gluons. The latter ones are the self-interacting particles of the Yang-Mills $S U(3)_{\mathrm{C}}$ theory. This fact causes the vanishing of the strong interaction coupling constant at high energies (asymptotic freedom). For the discovery of this feature of Quantum Chromodynamics (QCD) Gross, Politzer and Wilczek received the Nobel prize in 2004. Due to an increase of the coupling constant at low energies, the perturbation theory in QCD is not applicable in this region. Only a numerical solution on a space-time lattice and phenomenological effective field
theory descriptions can give quantitative predictions at low energies. All quarks participate in $U(1)_{\mathrm{Y}}$ hypercharge interaction. Left-handed quarks participate also in the weak interaction entering it as $S U(2)_{\mathrm{W}}$ doublets. The weak interaction is mediated by Yang-Mills $W$ gauge bosons while the hypercharge interaction is mediated by the $B$ boson. Left-handed leptons with corresponding left-handed neutrinos and the Higgs field also participate in the $U(1)_{\mathrm{Y}}$ hypercharge interaction and weak interaction as $S U(2)_{\mathrm{w}}$ doublets. For the unified description of the weak and electromagnetic interactions Weinberg, Glashow and Salam received the Nobel prize in 1979. The Higgs field is a self-interacting complex scalar field with an $S U(2)_{\mathrm{W}} \times U(1)_{\mathrm{Y}}$ symmetric potential. At very high energies all fields of the Standard Model, except for the Higgs field, are massless, and the theory is symmetric under $S U(3)_{\mathrm{C}} \times S U(2)_{\mathrm{W}} \times U(1)_{\mathrm{Y}}$ gauge symmetry. However, when the Higgs field selects the minimum of its potential, the symmetry breaks to $S U(3)_{\mathrm{C}} \times U(1)_{\mathrm{EM}}$. $W^{0}$ and $B$ bosons can be combined in the photon, which is $U(1)_{\text {EM }}$ gauge boson, and massive $U(1)_{\text {EM }}$ neutral $Z^{0}$ field, while $W^{ \pm}$bosons became massive and charged under $U(1)_{\mathrm{EM}}$. All Standard Model fermions, except for neutrinos, receive masses through the interaction with the Higgs field.


Figure 1.1: Standard model of elementary particles and its tree-level interactions. W and Z bosons couple to all SM fermions at tree level, photon $(\gamma)$ couples only to charged particles, i.e., all fermions except for neutrinos, and gluons couple to quarks only. Gluons, $\mathrm{W} / \mathrm{Z}$ and Higgs bosons are self-interacting particles, the photon couples to $W^{ \pm}$bosons. The Higgs boson has non-zero couplings with all massive particles. The values of masses are approximately indicated. For most recent values see Review of Particle Physics (Particle Data Group Collaboration) [15].

The theoretical foundations of the Standard Model were established in 1960th and 1970th in complementary works of experimental and theoretical groups. The discovery of the charm quark in 1974 was expected [16] and predicted from the GIM mechanism [17] that explained the suppression of the Flavor Changing Neutral Currents in a consistent way. The GIM mechanism allowed to put limits on the mass of the charm quark before its discovery. Observing that CP-violation in decays of neutral kaons could not be explained in a model with four quarks, Kobayashi and Maskawa [18] generalized the Cabbibo matrix with two quark families, which relates free and weakly interacting quark eigenstates, into the Cabibbo-Kobayashi-Maskawa (CKM) matrix with three families of quarks. They have predicted bottom and top quarks by this extension. For these studies Kobayashi and Maskawa obtained the Nobel prize in 2008 after the discovery of the bottom quark in 1977 at Fermilab [19] and top quark in 1994-1995 by the D0 and CDF experiments at Fermilab [20,21]. Prior to the top quark observation, its mass was bounded from the precision measurements of the $\mathrm{W} / \mathrm{Z}$ masses through electroweak radiative corrections. The account of radiative corrections became possible after works of t'Hooft and Veltmann [22], who had shown the renormalizability of the gauge theories. They were awarded the Nobel prize in 1999 for this work.

The most complicated and powerful machine in the history of particle physics, Large Hadron Collider (LHC), was built with the aim to discover the Higgs boson, the last unobserved particle from the Standard Model. The first beam was steered around the 27 km path in 2008. It is remarkable that the predictive power of the Standard Model and results of searches from LEP at CERN and Tevatron at Fermilab allowed tuning LHC to search for a particle in the particular mass range exploiting the values of experimental parameters in the electroweak sector within the machinery of radiative corrections. After test runs and data taking, the discovery of the Higgs boson $[23,24]$ in the decay channel into two photons, which provided an experimentally clean signal, was announced on July 4, 2012 validating the Standard Model. For their theoretical proposal of a mechanism explaining the origin of mass of subatomic particles, which have been confirmed 50 years after the original ideas, François Englert and Peter Higgs received the Nobel prize in 2013.

Despite a self-consistent description of the overwhelming amount of the existing experimental data in particle physics, the Standard Model cannot explain all observed phenomena. All three neutrino flavors are assumed to be massless in the Standard Model. However, in 1998 the Super-Kamiokande experiment had discovered the neutrino flavor oscillations in atmospheric neutrinos [25] and in 2001 the oscillations were confirmed measuring the flux of solar neutrinos at Sudbury Neutrino Observatory [26]. These results indicated that neutrinos are massive particles. A new broad field of experimental and theoretical studies was opened. The leaders of these two research groups, Arthur B. McDonald and Takaaki Kajita, were awarded the Nobel prize for the discovery of neutrino oscillations in 2015.

The evidence of particle physics beyond the Standard Model makes it fascinating and challenging to find and study any experimental deviations from the theoretical predictions. Therefore, such deviations attract a lot of theoretical and experimental attention. For example, the recent unexpected 750 GeV diphoton excess at LHC at $3.6 \sigma$ [27] (2.6 $\sigma[28])$ level, seen by ATLAS (CMS), lead to more than hundred theoretical papers within the first month after the official announcement of this preliminary result. Another active field of theoretical and experimental activity arose after the measurement of the anomalous magnetic moment of the muon with sub-ppm precision with a result that differs from the theoretical prediction by $(2.2-2.7) \sigma$ standard deviations [29,30]. The improved theoretical analysis [31,32] increased this difference to $(3.6-3.7) \sigma$ and new experiments at Fermilab [33] and J-PARC [34] are planned with the aim to improve on the precision and to indicate the possible sign of new physics. The most significant discrepancy so far is related to measurements of the proton charge radius from the Lamb shift in muonic hydrogen and elastic electron-proton scattering experiments. The largest un-
certainty in the theoretical input for these two problems comes from the hadronic physics. The latter describes QCD at low energies in the non-perturbative regime and investigates hadrons as particle states. Such effective description often depends on a particular model and requires experimental information as an input. This thesis aims to decrease the model dependence and uncertainties in the typical analysis of experimental data on the proton structure at the low energy precision frontier.

### 1.2 Proton and its electromagnetic structure

The first transmutation of one nucleus into another one was performed in 1917 at the Victoria University of Manchester, UK, by the group of Ernest Rutherford (1871-1937). Using alpha radiation and pure nitrogen they had converted the nitrogen nuclei into the oxygen nuclei through the reaction ${ }^{14} \mathrm{~N}+\alpha \rightarrow{ }^{17} \mathrm{O}+p$. Rutherford had identified the hydrogen atoms in the products of the reaction by similarity with the products of the scattering of $\alpha$ particles on hydrogen atoms. This result had shown that the nitrogen nucleus contains the hydrogen nuclei as constituents. Following the Prout's hypothesis (1815) Rutherford had assumed that all other nuclei also contain the hydrogen nuclei as a building block since the latter was known as the lightest nucleus and masses of the heavier nuclei were approximately given by the integers of the hydrogen mass. A new particle was named by Rutherford as a proton in 1920 and was considered as a fundamental particle until the experimental evidence of the anomalously large magnetic moment of the proton. The group of Otto Stern (1888-1969) at the University of Hamburg in 1933 had measured the value of the magnetic moment that was in contradiction with the Dirac theory. This result indicated the composite nature of the proton. Otto Stern received the Nobel prize in 1943 partly for this discovery.

In view of the Stanford linear electron accelerator program, which expected to have a highintensity beam of relativistic electrons, Marshall Rosenbluth (1927-2003) had derived the cross section expression for the elastic scattering of relativistic electrons on the spherically symmetric proton in the assumption of the exchange of one virtual photon [35]. In 1953, the increase of electron beam energy at the Stanford University and the University of Michigan to hundreds MeV and consequently the decrease of the De Broglie wavelength of electrons allowed to probe the nucleus structure and to measure the charge densities and radii of nuclei. In 1954, the similar experiments on hydrogen were initiated by the group of Robert Hofstadter (1915-1990). These experiments demonstrated that the proton has a finite size and allowed to extract the Dirac $F_{D}$ and Pauli $F_{P}$ proton form factors (FFs) from the Rosenbluth cross section expression at different electron scattering angles [36]. ${ }^{1}$ For his pioneering studies of electron scattering in atomic nuclei and discovery concerning the structure of nucleons Robert Hofstadter received the Nobel prize in 1961 [37].

The other way to access the electromagnetic proton FFs was proposed by Akhiezer and Rekalo $[38,39]$, and separately by Dombey [40,41]. They studied the polarization observables instead of unpolarized scattering in order to precisely determine the nucleon FFs. The development of the recoil polarization technique as well as the implementation of polarized targets at electron scattering facilities led to the possibility of a second method of proton FFs extraction. Such experiments access the ratio $\mu_{p} G_{E} / G_{M}{ }^{2}$ directly from the ratio of the asymmetries with the transverse and longitudinal nucleon polarization in the elastic electron-nucleon scattering. The longitudinally polarized electron beam is used in these measurements. For squared momentum transfers $Q^{2}$ up to $8.5 \mathrm{GeV}^{2}$, over the past 15 years this ratio has been measured in a series

[^0]

Figure 1.2: Ratio $\mu_{p} G_{E} / G_{M}$ extracted by Rosenbluth method (green rhombi and hollow circles) and by recoil polarization technique (star and solid triangles, circles, squares).
of experiments at Jefferson Laboratory (JLab), Newport News [45-48], with plans to extend these measurements in the near future at the JLab 12 GeV facility to even larger $Q^{2}$ values [49]. The measurements in the polarization transfer experiments gave strikingly different results in comparison with the available data from the Rosenbluth separation method for the FF ratio at $Q^{2} \gtrsim 1 \mathrm{GeV}^{2}$, see Fig. 1.2 [50]. Two-photon exchange (TPE) processes have been proposed as a plausible solution to resolve this puzzle [51,52], see Refs. [53,54] for a review. Estimates for TPE processes were studied in a variety of different model calculations, see e.g. Refs. [51,55-66], and first phenomenological extractions of TPE observables based on available data were given in Refs. [67-72]. Furthermore, dedicated experiments to directly measure TPE observables at large momentum transfer $Q^{2} \gtrsim(1-2) \mathrm{GeV}^{2}$ have been performed in recent years measuring the polarization transfer [73]. The ratio of the cross sections of the electron-proton to positronproton elastic scattering, which deviates from unity due to the TPE correction entering the electron and positron scattering with different signs, can be used for an extraction of the TPE correction. With improved precision in comparison with first experiments [74] this ratio was measured at the VEPP-3 storage ring in Novosibirsk [75,76] and by the CLAS Collaboration at JLab [77-79], while the analysis of the OLYMPUS experiment at DESY is ongoing [80, 81].

The leading deviations of a proton from the spherical shape are described in terms of scalar electric $\alpha_{E}$ and magnetic $\beta_{M}$ polarizabilities [82]. In analogy to the classical electrodynamics, these are just coefficients between the applied electric (magnetic) field and the induced electric (magnetic) dipole moment. Polarizabilities provide a leading order correction to the Compton scattering on a proton. They are extracted experimentally from the precise measurements of the angular distribution of the unpolarized Compton scattering at low energy experiments [83] subtracting the contribution where the proton moves as a whole due to its
charge and magnetization (Born contribution) [84]. The sum $\alpha_{E}+\beta_{M}$ can be determined from the total photoabsorption cross section with the help of Baldin sum rule [85] that has its foundation in dispersion theory. With the development of the chiral perturbation theory, the scalar electromagnetic polarizabilities of nucleons are also predicted theoretically [86, 87]. The current values of the proton magnetic dipole polarizability, which are used in this thesis, are shown in Fig. 1.3 [88].


Figure 1.3: The magnetic dipole polarizability of the proton. The dispersion relation result [89] is compared with the experimental value [15], the prediction of $\mathrm{B} \chi \mathrm{PT}$ [90], fits of experimental data using the heavy baryon chiral perturbation theory $(\mathrm{HB} \chi \mathrm{PT})$ and the baryon chiral perturbation theory $(\mathrm{B} \chi \mathrm{PT})$.

At larger energies, other particles than the proton can be created resulting in the so-called inelastic final state in both $e p$ and $\gamma p$ scattering. Such process is phenomenologically described in terms of the inelastic proton structure functions. The typical cross section shape in the region of first inelastic states reflects a resonance structure. The data in this region had shown that the resonance excitations were quite large with a small non-resonant background. Up to now, the most precise data on the proton structure functions in this region comes from the series of experiments at JLab [91,92]. The first inelastic state contains the nucleon and the pion as the lightest meson and is studied in detail by the measurement of polarization observables. The data on the photo- and electroproduction of pions is fitted by the partial-wave analysis under the assumption that all relevant resonances have a Breit-Wigner form. These fits are collected into such solutions as MAID [93,94] and SAID [95]. The most precise experimental data on the pion electroproduction in the resonance region is coming from JLab at Newport News, MAMI at Mainz, ELSA at Bonn, MIT-Bates at Cambridge (US), and on the pion photoproduction from LEGS at Brookhaven and GRAAL at Grenoble.

Above the resonance region the $e p$ scattering with significant energy transfer to a proton is called the deep inelastic scattering (DIS). Most likely other particles than the proton are created in the final state. Already the first results from the electron accelerator (1967-1972), which allowed to operate the electron beam with an energy up to 21 GeV , at Stanford University had shown that deep inelastic structure functions exhibit the scaling behavior, suggested by Bjorken [96]. In the limit of high energy and high momentum transfer the proton structure functions depend only on one variable (momentum transfer) rather than the two variables
allowed by kinematics. These features were explained in the naive parton model developed by Feynman [97]. Afterward, Callan and Gross have derived the relation between the proton structure functions in the quark model and thereby have shown that partons were particles of spin-1/2 [98]. The new constituents were identified with the quarks from the Gell-Mann or Zweig model of hadrons (1964) [99, 100]. For the pioneering experimental investigations concerning deep inelastic scattering of electrons on nucleons that lead to the development of the quark model, J. Friedman, H. Kendall and R. Taylor received the Nobel prize in 1990 [101]. Later, the precise measurements of the proton structure functions in DIS region were performed with the HERA (Hadron-Electron Ring Accelerator) accelerator at DESY, Hamburg (19922007).

### 1.3 Proton charge radius puzzle

The most elementary quantity that characterizes the proton electromagnetic structure is its root mean square (rms) charge radius $R_{E}$. The conventional definition of the charge radius is given by

$$
\begin{equation*}
R_{E}^{2}=-\left.6 \frac{\mathrm{~d} G_{E}\left(Q^{2}\right)}{\mathrm{d} Q^{2}}\right|_{Q^{2}=0} \tag{1.1}
\end{equation*}
$$

Already the scattering experiments of Rutherford had estimated the order of the proton size $R_{E} \sim 1 \mathrm{fm}$. The first measurements of the proton radius as a FF slope were performed by Hofstadter's group [102]. However, they assumed the same behavior of the Dirac and Pauli proton FFs and the extracted value $R_{E}=0.74(24) \mathrm{fm}$ has quite large uncertainty. The most precise measurement of the proton charge radius from the scattering experiments was performed by the A1 Collaboration at MAMI $[103,104]$. The resulting value is given by $R_{E}=0.879(8) \mathrm{fm}[104] .{ }^{3}$

The proton charge radius can be extracted by a second independent method. The leading finite size correction to the Lamb shift in the hydrogen $S$ states is proportional to $R_{E}^{2}$ and can be evaluated using the scattering result for the charge radius. The problem can be inverted, and one can extract $R_{E}$ from the spectroscopy measurements. The increasing experimental accuracy by an order of magnitude of the atomic transitions measurements in electronic hydrogen in the 1990th [111] required to account for the mentioned proton structure corrections and allowed to perform the radius extraction. The hydrogen spectroscopy results are in a fair agreement with the results from the electron-proton scattering. The proton size measurements by the second method were realized also in experiments with muons. The Bohr radius in muonic hydrogen is 186 times smaller than in electronic hydrogen leading to a strongly enhanced (by the third power of the lepton mass) sensitivity to the proton structure. The spectroscopy experiments at PSI with muonic hydrogen were planned with the aim to increase the precision of the proton charge radius measurement by an order of magnitude [112]. It came as a big surprise that extractions of the radius from muonic hydrogen Lamb shift measurements performed by the CREMA Collaboration $[113,114]\left(R_{E}=0.84087(39) \mathrm{fm}\right)$ are in strong contradiction, by around 7 standard deviations, with values obtained from the shifts of energy levels in electronic hydrogen and deuterium [115] $\left(R_{E}=0.8758(77) \mathrm{fm}\right)$ and with elastic electron-proton scattering experiments, see Fig. 1.4 as an illustration of these results. The so-called "proton radius puzzle" has triggered a large activity and is the subject of intense debate, see e.g. Refs. [116-118] for recent reviews.

[^1]

Figure 1.4: The proton charge radius results from different experimental techniques. The upper value corresponds with the analysis of the electron-proton scattering data performed by the A1 Collaboration at MAMI [104]. The revised extraction with account of the JLab form factor ratio $\mu_{P} G_{M} / G_{E}$ measurements at low $Q^{2}$ [119] is indicated as "JLab scatt". The CODATA value [115] is given by the combined result from the electronic hydrogen and deuterium spectroscopy. The lowest value is obtained from the muonic hydrogen Lamb shift measurement performed by the CREMA Collaboration at PSI [114].

In order to clarify whether the discrepancy is coming from the proton itself, from muons or from the theoretical treatment of the bound states in QED, experiments with light nuclei are required. The high precision results of the $1 \mathrm{~S}-2 \mathrm{~S}$ splitting from the isotope shift measurement in ordinary electronic hydrogen and deuterium [120] allow to extract the difference between the deuterium $\left(R_{d}\right)$ and proton charge radii very accurately. In combination with the CODATA2010 proton charge radius value [115] ( $\left.R_{E}=0.8775(51) \mathrm{fm}\right)$ the deuterium radius is given by $R_{d}=2.1424(21) \mathrm{fm}$ that is 5 times more precise than the value coming from the scattering data $R_{d}=2.130(10) \mathrm{fm}$. The precision of the electron-deuteron scattering experiments is planned to be increased by ongoing experiments at Mainz [118, 121]. The muonic deuterium Lamb shift measurements has already been performed at PSI [122]. The obtained deuterium radius combined with the electronic isotope shift yields a smaller value of the proton charge radius, similar to the one from muonic hydrogen. The muonic deuterium radius is $3.5 \sigma$ away from the electronic spectroscopy result. Also new Lamb shift experiments with $\mu^{4} \mathrm{He}^{+}$and $\mu^{3} \mathrm{He}^{+}$ have been recently performed by the CREMA Collaboration at PSI $[118,123]$ and seem to be in agreement with corresponding electron-proton scattering results.

In the field of elastic electron-proton scattering, new high-precision unpolarized scattering experiment at very low $Q^{2}$ (PRad) is being performed at JLab [124]. They plan to measure the electron-proton scattering at very small $Q^{2}$ values down to $10^{-4} \mathrm{GeV}^{2}$ and hope to achieve a sub-percent model-independent extraction of the proton charge radius. The extraction of proton FFs at very low $Q^{2}$ at MAMI is also being pursued using the initial state radiation (ISR) technique [125]. The energy of the incoming electron can be decreased due to ISR causing a quite small value of $Q^{2}$. The first results of this experiment [126] show a good agreement with the form factors extraction from the elastic electron-proton scattering [104].

To shed further light on this puzzle, several new scattering experiments involving muons are also being planned. Among them few experiments aim to test the lepton universality in the interaction of lepton with a proton. One can compare the elastic scattering of electrons and muons on the proton target and measure the proton charge radius in the elastic muon-proton scattering in a similar way as it was done in the elastic electron-proton scattering [35, 104]. Such an elastic scattering experiment is presently being planned by the MUSE Collaboration at PSI [127]. However, the information from a quite large range of momentum transfers can be required to test the consistency of FF fit [128, 129]. Complementary, one can also compare the electron and muon pair photoproduction on the proton target as proposed in Ref. [130]. By measuring the $\mu^{-} \mu^{+}$over $e^{-} e^{+}$production with a $10^{-3}$ accuracy on a ratio of cross sections, such experiment has the potential to distinguish between both proton radius values.

### 1.4 Motivation and Outline

The limiting accuracy in extracting the proton charge radius from the Lamb shift measurements in muonic atoms is due to the TPE hadronic correction. ${ }^{4}$ To evaluate this correction, one considers the TPE graph in the forward kinematics neglecting the external particles momenta that are suppressed by the fine structure constant $\alpha=e^{2} /(4 \pi) \approx 1 / 137$. The TPE correction is expressed in terms of the forward doubly virtual Compton scattering (VVCS) amplitudes, which has been estimated phenomenologically in Refs. [131-137]. The input was used in terms of the unpolarized proton structure functions and a model for the subtraction function of the forward VVCS which causes the main theoretical uncertainty. The attempt to explain the proton radius puzzle with an enhanced subtraction function at large photon virtuality was made in Ref. [138]. The estimates from the non-relativistic QED effective field theory [139], as well as from the chiral effective field theory [140-143] include evaluation of the subtraction function at low photon virtuality. The total TPE corrections to the Lamb shift were found to be in the 10-15 \% range of the total discrepancy for the proton charge radius extractions between electron data and muonic hydrogen spectroscopy, see Ref. [88] for a detailed summary. The intriguing subtraction function, in principle, can be determined with an account of the high-energy proton structure functions data [136, 144, 145].

In Chapter 2, we extract the subtraction function at low photon virtuality from the highenergy DIS HERA data $[146,147]$ assuming the absence of a constant term at infinity and estimate its contribution to the Lamb shift in muonic hydrogen [144].

The other recent experimental achievement in the field of atomic spectroscopy is the measurement of the 2 S hyperfine splitting (HFS) in muonic hydrogen by the CREMA Collaboration at PSI [114]. Also the new measurements of the 1S hyperfine splitting (HFS) in muonic hydrogen and $\mu^{3} \mathrm{He}^{+}$with 1 ppm precision were proposed by the CREMA Collaboration [148]. Such an impressive precision requires an additional theoretical studies of the TPE correction, which has the relative size of $(6-7) \times 10^{-3}$ in $\mu \mathrm{H}$.

We introduce the forward lepton-proton scattering amplitudes and derive dispersion relations with the same steps as the derivation of sum rules for Compton or light-by-light scattering [149, 150]. Furthermore, we express the TPE corrections to the Lamb shift and HFS of S energy levels through the forward TPE amplitudes at threshold. Due to a pure convergence, we are not able to express the correction to the Lamb shift through the experimental information within dispersion relations for the lepton-proton amplitudes, but the correction to the HFS is entirely expressed through the proton spin structure. The resulting HFS correction agrees with the standard approach of Iddings, Drell and Sullivan et al. [53, 137, 151-160], subject to validity of the Burkhardt-Cottingham sum rule [161]. However, the contribution of each channel to

[^2]the TPE correction in the present approach, which is based on the on-shell information only, differs from the literature result. We reevaluate the elastic, phenomenological $\Delta(1232)$ [65] and polarizability TPE correction to S-level HFS in H and $\mu \mathrm{H}$.

The sub-percent level of precision in the scattering experiments also calls for studies of the higher order effects, as the corrections to the scattering cross sections are suppressed by one power in $\alpha$ and also in the few \% range. Modern measurements partially account for the hadronic correction through the radiative corrections which include the graph with the exchange of two photons $[162,163]$ where one of these photons is soft. Although the corrections to the Coulomb distortion in the elastic electron-proton scattering were found to be small in the low- $Q^{2}$ region [164], a high precision extraction of the proton radii, especially its magnetic radius, calls for an assessment of the model dependence of the TPE corrections. ${ }^{5}$

In Chapter 3 we introduce the general formalism of the elastic lepton-proton scattering following Refs. [52, 167-169] and relate TPE amplitudes with measured observables. Afterward, in Chapter 4 we pay an additional attention to TPE corrections from the intermediate proton state [170, 171]. We describe the evaluation of the TPE diagram in a box graph model with the on-shell form of proton-photon vertices and present the results for the TPE correction in the kinematics of the MUSE experiment for the case of electron- and muon-proton scattering. We write down dispersion relations for TPE amplitudes in the elastic electron-proton scattering [59], which do not require assumptions about the off-shell vertex, and generalize them to the case of the massive lepton scattering. We also propose the method of an analytical continuation to the unphysical region for the imaginary parts of TPE amplitudes with the proton intermediate state, which has the advantage to be valid for any parametrization of the elastic proton FFs. Furthermore, we provide a detailed comparison of the proton intermediate state TPE contribution in the dispersion relation method with the box graph model. In order to minimize the model dependence due to unknown or poorly constrained contributions from other intermediate states, we propose and apply the subtracted dispersion relation formalism in the electron-proton scattering. The subtraction constant, which encodes the less well-constrained physics at high energies, is fitted to the available data [104]. We apply the developed formalism in the kinematical region of the CLAS data [77,79] and make a prediction for the OLYMPUS experiment [81].

In the region of very low momentum transfer, the total TPE correction (sum of the proton and inelastic TPE) can be determined model-independently. In Chapter 5, we study the low momentum transfer limit [144, 172]. The leading term in the momentum transfer expansion of the TPE correction to the unpolarized electron-proton scattering cross section arises from the scattering of the relativistic massless electron on a point charged target [173]. This result in the electron-proton scattering was reproduced by R. W. Brown as the leading term in the expansion of the proton intermediate state TPE in Ref. [174]. He has also found that the subleading $Q^{2} \ln ^{2} Q^{2}$ term entirely arises from this expansion, while the subleading $Q^{2} \ln Q^{2}$ term contains also the contribution from the inelastic TPE. The inelastic $Q^{2} \ln Q^{2}$ term can be expressed in terms of an energy integral over the total photoabsorption cross section on a proton target $[166,174]$. In this work, we approximate the proton line of the TPE diagram in form of the near-forward VVCS tensor. Such approximation allows to reproduce the leading terms of the low- $Q^{2}$ expansion in the elastic electron-proton scattering [174] and generalize it to the case of massive lepton. We extend the low- $Q^{2}$ limit of the inelastic TPE beyond the leading $Q^{2} \ln Q^{2}$ term. We show that the subtraction function in the forward Compton scattering is negligible in the elastic electron-proton scattering regardless of the near-forward approximation and evaluate its contribution to the unpolarized muon-proton scattering. The

[^3]residual TPE correction is presented as a double integral of the unpolarized proton structure functions over the virtual photon energy and virtuality. We compare our results for the total TPE correction in the electron-proton scattering with the TPE fit of Ref. [104], the recent data from CLAS [79], VEPP-3 [76] and extrapolate our calculation to the low- $Q^{2}$ region of the OLYMPUS experiment [81]. Subsequently, we provide estimates for the upcoming MUSE experiment.

We next turn to the study of TPE corrections with inelastic intermediate states at larger momentum transfers. The inclusion of inelastic intermediate states in the model dependent way was performed in Refs. [65, 175, 176]. Dispersion relations estimates in the region of large momentum transfer were performed in Refs. [64,66]. In Chapter 6 we study the pionnucleon intermediate states TPE contributions to the elastic electron-proton scattering in the dispersion relation framework exploiting the invariant amplitudes of the pion electroproduction from MAID. We firstly provide a detailed comparison of the $P_{33}$ channel TPE with the nearforward calculation in the region of low momentum transfer. We present the framework how such calculations can be extended to include all $\pi N$ intermediate states.

We finish with conclusions and outlook in Chapter 7.

## Chapter 2

## Forward lepton-proton scattering and two-photon exchange (TPE) corrections to atomic energy levels

We start this thesis with studies of the elastic lepton-proton scattering in forward kinematics, see Fig. 2.1 for notations. We introduce the forward amplitudes and relate the imaginary parts of these amplitudes to experimental cross sections. For the TPE contribution, we write down dispersion relations (DRs) and express the spin-dependent amplitudes through the proton spin structure functions. We describe the forward VVCS tensor and provide an empirical estimate of the subtraction function in the spin-independent Compton amplitude $\mathrm{T}_{1}$. Afterward, we estimate the subtraction function contribution to the Lamb shift of the 2 S energy level in muonic hydrogen. We also express the TPE correction to the HFS in hydrogen-like atoms in terms of the proton spin structure functions in the DR approach and provide its evaluation for the 1 S energy level correction in the electronic and muonic hydrogen.


Figure 2.1: Forward elastic lepton-proton scattering.

### 2.1 Forward lepton-proton scattering

Elastic lepton-proton scattering in forward kinematics $l(k, h)+p(p, \lambda) \rightarrow l\left(k, h^{\prime}\right)+p\left(p, \lambda^{\prime}\right)$, where $h\left(h^{\prime}\right)$ denote the incoming (outgoing) lepton helicities and $\lambda\left(\lambda^{\prime}\right)$ the corresponding proton helicities respectively, is described by just one kinematical variable, e.g., the lepton energy in the proton rest frame (laboratory frame) $\omega$. In this frame the particle momenta are given by $p=(M, 0), k=(\omega, \boldsymbol{k})$, with the proton (lepton) mass $M(m)$. The squared energy in the lepton-proton center-of-mass (c.m.) reference frame $s$ is expressed in terms of the lepton energy in the lab frame $\omega$ as

$$
\begin{equation*}
s=(p+k)^{2}=M^{2}+2 M \omega+m^{2} . \tag{2.1}
\end{equation*}
$$

The forward elastic $l p$ scattering is described by three non-vanishing independent helicity amplitudes $T_{h^{\prime} \lambda^{\prime}, h \lambda}$, see Fig. 2.1 for the notations of kinematics and helicities,

$$
\begin{equation*}
T_{++,++}, \quad T_{+-,+-}, \quad T_{--,++} \tag{2.2}
\end{equation*}
$$

with the positive helicity denoted by + and the negative helicity by - . Two amplitudes describe processes without flip of helicities, while the amplitude $T_{--,++}$corresponds to the simultaneous flip of the lepton and proton helicities conserving the total angular momentum.

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The contribution with exchange of a fixed number of photons, which are connected to the lepton (proton) line, to the following three amplitudes has definite even-odd property with respect to the crossing $\omega \rightarrow-\omega$ :

$$
\begin{align*}
f_{ \pm}(\omega) & =\frac{1}{2}\left(T_{++++} \pm T_{+-+-}\right),  \tag{2.3}\\
g(\omega) & =\frac{1}{2} T_{--++} . \tag{2.4}
\end{align*}
$$

The Lorentz structure of the forward amplitude is then given by

$$
\begin{align*}
T_{h^{\prime} \lambda^{\prime}, h \lambda}(\omega) & =\frac{f_{+}(\omega)}{4 M m} \bar{u}\left(k, h^{\prime}\right) u(k, h) \bar{N}\left(p, \lambda^{\prime}\right) N(p, \lambda) \\
& -\frac{m f_{-}(\omega)+\omega g(\omega)}{8 M \boldsymbol{k}^{2}} \bar{u}\left(k, h^{\prime}\right) \gamma^{\mu \nu} u(k, h) \bar{N}\left(p, \lambda^{\prime}\right) \gamma_{\mu \nu} N(p, \lambda) \\
& +\frac{\omega f_{-}(\omega)+m g(\omega)}{4 M \boldsymbol{k}^{2}} \bar{u}\left(k, h^{\prime}\right) \gamma_{\mu} \gamma_{5} u(k, h) \bar{N}\left(p, \lambda^{\prime}\right) \gamma^{\mu} \gamma_{5} N(p, \lambda), \tag{2.5}
\end{align*}
$$

with $u(\bar{u})$ the lepton spinor, $\boldsymbol{k}$ the lepton momentum in the lab frame, $N(\bar{N})$ the proton spinor, $\gamma^{\mu \nu}=\frac{1}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]$, the spinors are normalized as

$$
\begin{equation*}
\bar{u}(k, h) u(k, h)=2 m, \quad \bar{N}(p, \lambda) N(p, \lambda)=2 M . \tag{2.6}
\end{equation*}
$$

We provide expressions for the helicity spinors used in this thesis in Appendix A and the relation between the invariant amplitudes and possible observables in the forward lepton-proton scattering in Appendix B.

In order to establish the even-odd properties for the invariant amplitudes under the crossing $\omega \rightarrow-\omega$, we first perform the crossing on the lepton line and relate amplitudes of the leptonproton scattering $f^{l^{-} p}(\omega)$ in the physical region $(\omega>0)$ to amplitudes of the antilepton-proton scattering $f^{l^{+} p}(-\omega)$ in the unphysical region $(\omega<0)$. Writing the general form of the amplitude as ${ }^{1}$

$$
\begin{equation*}
T(\omega)=\sum_{i=1}^{3} A_{i}(\omega) \bar{u}\left(k^{\prime}, h^{\prime}\right) O_{i} u(k, h) \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) O_{i} N(p, \lambda), \tag{2.7}
\end{equation*}
$$

with $O=\left(1, \gamma^{\mu \nu}, \gamma^{\mu} \gamma_{5}\right)$, we observe that after the replacement in the lepton line $k \rightarrow-k$, the amplitude transforms to

$$
\begin{equation*}
T^{c}(\omega)=\sum_{i=1}^{3} A_{i}(-\omega) \bar{u}\left(-k^{\prime},-h^{\prime}\right) O_{i} u(-k,-h) \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) O_{i} N(p, \lambda) . \tag{2.8}
\end{equation*}
$$

We can rewrite the lepton spinor $u$ in terms of the antilepton spinor $v$ as $u(-k,-h)=$ $-\gamma^{2} v^{*}(k, h)$, where we exploit the same form $u$ for the antilepton spinor as only particles or antiparticles participate in scattering. The expression for the helicity amplitude is given by

$$
\begin{equation*}
T^{c}(\omega)=\sum_{i=1}^{3} A_{i}(-\omega) v^{T}\left(k^{\prime}, h^{\prime}\right) \gamma_{2}^{+} \gamma_{0} O_{i} \gamma_{2} v^{*}(k, h) \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) O_{i} N(p, \lambda) . \tag{2.9}
\end{equation*}
$$

Transposing the lepton line, we obtain:

$$
\begin{equation*}
T^{c}(\omega)=\sum_{i=1}^{3} A_{i}(-\omega) \bar{v}(k, h) \gamma_{0}\left(\gamma_{2}^{+} \gamma_{0} O_{i} \gamma_{2}\right)^{T} v\left(k^{\prime}, h^{\prime}\right) \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) O_{i} N(p, \lambda) . \tag{2.10}
\end{equation*}
$$

[^4]The tensor structure of Eq. (2.5) transforms to

$$
\begin{align*}
T^{c}(\omega) & =-\frac{f_{+}(-\omega)}{4 M m} \bar{v}(k, h) v\left(k^{\prime}, h^{\prime}\right) \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) N(p, \lambda) \\
& -\frac{m f_{-}(-\omega)-\omega g(-\omega)}{8 M \boldsymbol{k}^{2}} \bar{v}(k, h) \gamma^{\mu \nu} v\left(k^{\prime}, h^{\prime}\right) \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) \gamma_{\mu \nu} N(p, \lambda) \\
& -\frac{-\omega f_{-}(-\omega)+m g(-\omega)}{4 M \boldsymbol{k}^{2}} \bar{v}(k, h) \gamma_{\mu} \gamma_{5} v\left(k^{\prime}, h^{\prime}\right) \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) \gamma^{\mu} \gamma_{5} N(p, \lambda), \tag{2.11}
\end{align*}
$$

which corresponds to the scattering of the antilepton off the proton.
According to crossing properties we can write amplitudes for the scattering of antilepton $f^{l^{+} p}$ in terms of the lepton scattering amplitudes $f^{l^{-} p}$ as

$$
\begin{align*}
f_{+}^{l^{l} p}(\omega) & =f_{+}^{l^{-p}}(-\omega),  \tag{2.12}\\
f_{-}^{l^{+} p}(\omega) & =-f_{-}^{l^{-} p}(-\omega),  \tag{2.13}\\
g^{l^{+} p}(\omega) & =g^{l^{-} p}(-\omega), \tag{2.14}
\end{align*}
$$

where $\omega$ is treated as a complex variable. The perturbative contributions with odd number of photons connected to the lepton (antilepton) line have different sign in the amplitudes of the lepton-proton and antilepton-proton scattering as compared with the contributions with an even number of photons, which have the same sign. We express the scattering amplitudes in terms of the contributions with even $f^{(2 n) \gamma}(\omega)$ and odd $f^{(2 n-1) \gamma}(\omega)$ number of photons connected to the lepton line, e.g.:

$$
\begin{equation*}
f_{-}^{l^{ \pm}}(\omega)=\sum_{n=1}^{\infty}\left(f_{-}^{(2 n) \gamma}(\omega) \pm f_{-}^{(2 n-1) \gamma}(\omega)\right) \tag{2.15}
\end{equation*}
$$

and obtain the following crossing relations for the contributions of graphs with $n$ exchanged photons on the real $\omega$ axis:

$$
\begin{align*}
f_{+}^{n \gamma}(\omega) & =(-1)^{n}\left(f_{+}^{n \gamma}(-\omega)\right)^{*},  \tag{2.16}\\
f_{-}^{n \gamma}(\omega) & =-(-1)^{n}\left(f_{-}^{n \gamma}(-\omega)\right)^{*},  \tag{2.17}\\
g^{n \gamma}(\omega) & =(-1)^{n}\left(g^{n \gamma}(-\omega)\right)^{*} . \tag{2.18}
\end{align*}
$$

The optical theorem establishes the relation between the imaginary part of the forward amplitudes and the total inclusive cross sections of $l p$ collisions:

$$
\begin{align*}
\Im f_{ \pm}(\omega) & =M|\boldsymbol{k}|\left(\sigma_{++}(\omega) \pm \sigma_{+-}(\omega)\right)  \tag{2.19}\\
\Im g(\omega) & =2 M|\boldsymbol{k}|\left(\sigma_{\|}(\omega)-\sigma_{\perp}(\omega)\right) \tag{2.20}
\end{align*}
$$

where $\sigma_{h \lambda}$ is the inclusive cross section with the incoming lepton helicity $h$ and the incoming proton helicity $\lambda ; \sigma_{\perp}\left(\sigma_{\|}\right)$is the inclusive cross section with lepton and proton polarized transversely and perpendicular (parallel) to each other. We express the latter cross sections in terms of the proton structure functions (SFs) up to the order $\alpha^{2}$ in Section 2.3 and obtain the imaginary parts of TPE amplitudes in the leading $\alpha$ order with Eqs. (2.19-2.20).

The elastic (proton) contribution to the inclusive cross section is infrared divergent. This divergence should be subtracted in a proper way in all three amplitudes. We realize this subtraction for the case of amplitudes at threshold in Section 2.6.

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### 2.2 Dispersion relations for forward TPE amplitudes

Assuming analyticity of the forward amplitudes in the entire complex $\omega$-plane, except for the branch cuts along the real axis extending from threshold to infinity, see Fig. 2.2, we can write down the standard DRs:

$$
\Re\left\{\begin{array}{l}
f_{ \pm}(\omega)  \tag{2.21}\\
g(\omega)
\end{array}\right\}=\frac{1}{\pi}\left(f_{-\infty}^{-m}+f_{m}^{\infty}\right) \frac{\mathrm{d} \omega^{\prime}}{\omega^{\prime}-\omega} \Im\left\{\begin{array}{l}
f_{ \pm}\left(\omega^{\prime}\right) \\
g\left(\omega^{\prime}\right)
\end{array}\right\}
$$

where $f$ stands for the principal-value integration.


Figure 2.2: Complex plane of the lepton energy $\omega$.
Usually, the crossing relates the amplitude of the particle scattering with the amplitude of the antiparticle scattering [177,178]. Nevertheless, for the TPE amplitudes, we exploit the crossing in one channel due to the charge independence of the TPE contributions. Using properties of the TPE amplitudes under the crossing $\omega \rightarrow-\omega$, see Eqs. (2.16-2.18), we write down DRs valid in the leading $\alpha$ order:

$$
\begin{align*}
& \Re f_{+}^{2 \gamma}(\omega)=\frac{4 M}{\pi} f_{m}^{\infty} \frac{\omega^{\prime}\left|\boldsymbol{k}^{\prime}\right| \sigma\left(\omega^{\prime}\right)}{\omega^{\prime 2}-\omega^{2}} \mathrm{~d} \omega^{\prime},  \tag{2.22}\\
& \Re f_{-}^{2 \gamma}(\omega)=\frac{2 M \omega}{\pi} f_{m}^{\infty} \frac{\left|\boldsymbol{k}^{\prime}\right|\left(\sigma_{++}\left(\omega^{\prime}\right)-\sigma_{+-}\left(\omega^{\prime}\right)\right)}{\omega^{\prime 2}-\omega^{2}} \mathrm{~d} \omega^{\prime},  \tag{2.23}\\
& \Re g^{2 \gamma}(\omega)=\frac{4 M}{\pi} f_{m}^{\infty} \frac{\omega^{\prime}\left|\boldsymbol{k}^{\prime}\right|\left(\sigma_{\|}\left(\omega^{\prime}\right)-\sigma_{\perp}\left(\omega^{\prime}\right)\right)}{\omega^{\prime 2}-\omega^{2}} \mathrm{~d} \omega^{\prime} . \tag{2.24}
\end{align*}
$$

These DRs are written for amplitudes in one channel contrary to the DRs for the forward proton-proton scattering [177-179].

The high-energy behavior of the total unpolarized inclusive cross section requires a subtraction in the DR for the amplitude $f_{+}^{2 \gamma}$, e.g., at point $\omega_{s}$ :

$$
\begin{equation*}
\Re f_{+}^{2 \gamma}(\omega)-\Re f_{+}^{2 \gamma}\left(\omega_{s}\right)=\frac{4 M\left(\omega^{2}-\omega_{s}^{2}\right)}{\pi} f_{m}^{\infty} \frac{\omega^{\prime}\left|\boldsymbol{k}^{\prime}\right| \sigma\left(\omega^{\prime}\right)}{\left(\omega^{\prime 2}-\omega^{2}\right)\left(\omega^{\prime 2}-\omega_{s}^{2}\right)} \mathrm{d} \omega^{\prime} . \tag{2.25}
\end{equation*}
$$

The lepton-proton DRs were checked in the leading QED order, see Appendix D for details. However, the Regge behavior of the proton SF $F_{1}$ makes the inclusive cross-section $\sigma$ divergent due to the virtual photons with high energy in the lab frame. Consequently the DR for leptonproton amplitudes can not be exploited for the inelastic TPE contribution to the unpolarized amplitude $f_{+}$.

The DRs of Eqs. (2.23-2.25) have the same form as DRs for the light-by-light scattering [150].

### 2.3 Relation of the forward TPE amplitudes to the proton structure functions

In this Section, we first express the total inclusive cross sections in terms of the experimentally measured proton SFs. Exploiting these relations, we express the real parts of the forward TPE amplitudes (see Fig. 2.3 for kinematics of forward TPE) as integrals over the photon energy $\nu_{\gamma}$ in the lab frame and photon virtuality $Q^{2}$.


Figure 2.3: Two-photon exchange graph in forward kinematics.
A common way to express the differential inelastic $e^{-} p$ scattering cross section assumes the exchange of one photon. The differential cross section is given by the contraction of the leptonic tensor $L^{\mu \nu}$ and the hadronic tensor $W^{\mu \nu}[15]$. It is proportional to the phase space of the final lepton (with 4-momentum $k^{\prime}=\left(\omega^{\prime}, \boldsymbol{k}^{\prime}\right)$ in the lab frame) and given by

$$
\begin{equation*}
d \sigma=\frac{e^{4}}{4 M \sqrt{\omega^{2}-m^{2}}} \frac{\mathrm{~d}^{3} \boldsymbol{k}^{\prime}}{(2 \pi)^{3} 2 \omega^{\prime}}(4 \pi) L^{\mu \nu} W_{\mu \nu}, \tag{2.26}
\end{equation*}
$$

with the unit of electric charge $e$. The kinematics are traditionally described by the kinematical Bjorken variable $x_{\mathrm{Bj}}$, the variable $y$ related to the energy transferred by the virtual photon relative to the beam energy and the momentum transfer $Q^{2}$ :

$$
\begin{equation*}
x_{\mathrm{Bj}}=\frac{Q^{2}}{2(p \cdot q)}, \quad y=\frac{(p \cdot q)}{(p \cdot k)}=\frac{Q^{2}}{2 x_{\mathrm{Bj}} M \omega}, \quad Q^{2}=-q^{2}=-\left(k-k^{\prime}\right)^{2} . \tag{2.27}
\end{equation*}
$$

The leptonic tensor is evaluated in QED. It is given by

$$
\begin{equation*}
L^{\mu \nu}=2\left(k^{\mu} k^{\prime \nu}+k^{\prime \mu} k^{\nu}+\left(m^{2}-\left(k \cdot k^{\prime}\right)\right) g^{\mu \nu}-i m \varepsilon^{\mu \nu \rho \sigma} q_{\rho} s_{\sigma}\right), \tag{2.28}
\end{equation*}
$$

where $s^{\mu}$ is the lepton spin vector: $s^{\mu} s_{\mu}=-1,(s \cdot k)=0$. The general Lorentz and gauge invariant structure of the hadronic tensor $W^{\mu \nu}$, which preserves parity and charge conjugation

Chapter 2 Forward lepton-proton scattering and two-photon exchange (TPE) corrections to atomic energy levels
invariance, is given by

$$
\begin{align*}
W_{\mu \nu} & =\left(-g_{\mu \nu}+\frac{q_{\mu} q_{\nu}}{q^{2}}\right) F_{1}\left(\nu_{\gamma}, Q^{2}\right)+\frac{\hat{p}_{\mu} \hat{p}_{\nu}}{(p \cdot q)} F_{2}\left(\nu_{\gamma}, Q^{2}\right) \\
& +i \varepsilon_{\mu \nu \alpha \beta} \frac{M q^{\alpha}}{(p \cdot q)}\left[S^{\beta} g_{1}\left(\nu_{\gamma}, Q^{2}\right)+\left(S^{\beta}-\frac{(S \cdot q)}{(p \cdot q)} p^{\beta}\right) g_{2}\left(\nu_{\gamma}, Q^{2}\right)\right], \tag{2.29}
\end{align*}
$$

with $\hat{p}_{\mu}=p_{\mu}-\frac{(p \cdot q)}{q^{2}} q_{\mu}$, the virtual photon energy in the laboratory frame $\nu_{\gamma}=(p \cdot q) / M$ and the proton SFs $F_{1}\left(\nu_{\gamma}, Q^{2}\right), F_{2}\left(\nu_{\gamma}, Q^{2}\right), g_{1}\left(\nu_{\gamma}, Q^{2}\right), g_{2}\left(\nu_{\gamma}, Q^{2}\right)$, which are extracted from the experimental data. The proton spin 4 -vector satisfies: $S^{2}=-1,(S \cdot p)=0$.

The total unpolarized cross section is given by

$$
\begin{equation*}
\frac{\mathrm{d}^{2} \sigma}{\mathrm{~d} \nu_{\gamma} d Q^{2}}=\frac{\pi \alpha^{2}}{\left(Q^{2}\right)^{2}} \frac{2}{\omega^{2}-m^{2}}\left(\frac{Q^{2}-2 m^{2}}{M} F_{1}\left(\nu_{\gamma}, Q^{2}\right)+\left(\frac{2 \omega^{2}}{\nu_{\gamma}}-2 \omega-\frac{Q^{2}}{2 \nu_{\gamma}}\right) F_{2}\left(\nu_{\gamma}, Q^{2}\right)\right) . \tag{2.30}
\end{equation*}
$$

This expression reduces to the known expression [15] in the massless limit.
Consider a scattering of longitudinally polarized leptons on the proton polarized in the lepton momentum direction $\sigma_{h \lambda}=\sigma_{+-}$and the scattering on the proton polarized in the opposite direction $\sigma_{h \lambda}=\sigma_{++}$with the proton (lepton) spin vector in the laboratory frame $S^{\mu}=(0,-\lambda \hat{\mathbf{k}})$ $\left(s^{\mu}=(|\mathbf{k}|, \omega \hat{\mathbf{k}}) / m\right)$ and $\hat{\mathbf{k}}=\mathbf{k} /|\mathbf{k}|$. For the cross sections difference we obtain:

$$
\begin{align*}
\frac{\mathrm{d}^{2} \sigma_{++}-\mathrm{d}^{2} \sigma_{+-}}{\mathrm{d} \nu_{\gamma} \mathrm{d} Q^{2}}= & \frac{4 \pi \alpha^{2}}{\nu_{\gamma} M Q^{2}} \frac{\omega}{\omega^{2}-m^{2}}\left\{-\frac{Q^{2}}{\nu_{\gamma} \omega} g_{2}\left(\nu_{\gamma}, Q^{2}\right)\right. \\
& \left.+\left(2-\frac{Q^{2}}{2\left(\omega^{2}-m^{2}\right)}\left(1+\frac{2 \nu_{\gamma} m^{2}}{Q^{2} \omega}\right)\left(1+\frac{2 \nu_{\gamma} \omega}{Q^{2}}\right)\right) g_{1}\left(\nu_{\gamma}, Q^{2}\right)\right\} \tag{2.31}
\end{align*}
$$

This expression reduces to the known expression [15], [180] in the massless limit.
Consider the scattering of transversely polarized leptons on transversely polarized protons. Denoting the averaged over the azimuthal angle cross section $\sigma_{\perp}\left(\sigma_{\|}\right)$for scattering with perpendicular (parallel) spin vectors of lepton ( $s^{\mu}=\left(0, \cos \phi_{l}, \sin \phi_{l}, 0\right)$ ) and proton ( $S^{\mu}=$ $\left.\left(0, \cos \phi_{p}, \sin \phi_{p}, 0\right)\right),{ }^{2}$ i.e., $\phi_{l}-\phi_{p}= \pm \pi / 2\left(\phi_{l}-\phi_{p}=0\right)$ for the perpendicular (parallel) configuration, we obtain:

$$
\begin{align*}
\frac{\mathrm{d}^{2} \sigma_{\perp}-\mathrm{d}^{2} \sigma_{\|}}{\mathrm{d} \nu_{\gamma} \mathrm{d} Q^{2}}= & \frac{2 \pi m \alpha^{2}}{\nu_{\gamma} M Q^{2}} \frac{1}{\omega^{2}-m^{2}}\left\{2 g_{2}\left(\nu_{\gamma}, Q^{2}\right)\right. \\
& \left.+\left(1+\frac{\omega \nu_{\gamma}}{\omega^{2}-m^{2}}\left(1+\frac{m^{2} \nu_{\gamma}}{\omega Q^{2}}+\frac{Q^{2}}{4 \nu_{\gamma} \omega}\right)\right) g_{1}\left(\nu_{\gamma}, Q^{2}\right)\right\} . \tag{2.32}
\end{align*}
$$

Consequently, a measurement of the inclusive $e^{-} p$ cross sections accesses the proton spin SFs $g_{1}$ and $g_{2}$.

The elastic scattering cross sections $l^{-} p \rightarrow l^{-} p$ are obtained by substitution of the inelastic SFs by the elastic contribution to them:

[^5]\[

$$
\begin{align*}
F_{1}^{\mathrm{el}}\left(x_{\mathrm{Bj}}, Q^{2}\right) & =\frac{1}{2} G_{M}^{2}\left(Q^{2}\right) \delta\left(1-x_{\mathrm{Bj}}\right)  \tag{2.33}\\
F_{2}^{\mathrm{el}}\left(x_{\mathrm{Bj}}, Q^{2}\right) & =\frac{G_{E}^{2}\left(Q^{2}\right)+\tau_{P} G_{M}^{2}\left(Q^{2}\right)}{1+\tau_{P}} \delta\left(1-x_{\mathrm{Bj}}\right),  \tag{2.34}\\
g_{1}^{\mathrm{el}}\left(x_{\mathrm{Bj}}, Q^{2}\right) & =\frac{1}{2} F_{D}\left(Q^{2}\right) G_{M}\left(Q^{2}\right) \delta\left(1-x_{\mathrm{Bj}}\right),  \tag{2.35}\\
g_{2}^{\mathrm{el}}\left(x_{\mathrm{Bj}}, Q^{2}\right) & =-\frac{1}{2} \tau_{P} F_{P}\left(Q^{2}\right) G_{M}\left(Q^{2}\right) \delta\left(1-x_{\mathrm{Bj}}\right), \tag{2.36}
\end{align*}
$$
\]

where $F_{D}\left(Q^{2}\right), F_{P}\left(Q^{2}\right), G_{E}\left(Q^{2}\right), G_{M}\left(Q^{2}\right)$ are the Dirac, Pauli, Sachs electric and magnetic proton form factors and $\tau_{P}=Q^{2} /\left(4 M^{2}\right)$.

As a test, we also calculated the narrow $\Delta$ production cross sections, which are obtained by substitution of the inelastic structure functions by the $\Delta$ contribution to them:

$$
\begin{align*}
F_{1}^{\Delta}\left(x_{\mathrm{Bj}}, Q^{2}\right)= & \frac{1}{2} \frac{Q_{-}^{2}\left(M+M_{\Delta}\right)^{2} \tau_{P}}{\left(M_{\Delta}^{2}-M^{2}+Q^{2}\right)^{2}}\left(3\left(G_{E}^{*}\right)^{2}+\left(G_{M}^{*}\right)^{2}\right) \delta\left(x_{\Delta}-x_{\mathrm{Bj}}\right),  \tag{2.37}\\
F_{2}^{\Delta}\left(x_{\mathrm{Bj}}, Q^{2}\right)= & \frac{Q^{2}\left(M+M_{\Delta}\right)^{2} \tau_{P}}{Q_{+}^{2}\left(M_{\Delta}^{2}-M^{2}+Q^{2}\right)}\left(3\left(G_{E}^{*}\right)^{2}+\left(G_{M}^{*}\right)^{2}+\frac{Q^{2}}{M_{\Delta}^{2}}\left(G_{C}^{*}\right)^{2}\right) \delta\left(x_{\Delta}-x_{\mathrm{Bj}}\right), \\
g_{1}^{\Delta}\left(x_{\mathrm{Bj}}, Q^{2}\right)= & -g_{2}^{\Delta}\left(x_{\mathrm{Bj}}, Q^{2}\right)+\frac{1}{8} \frac{Q_{-}^{2}\left(M+M_{\Delta}\right)^{2} \tau_{P}}{M_{\Delta} M\left(M_{\Delta}^{2}-M^{2}+Q^{2}\right)} G_{C}^{*}\left(3 G_{E}^{*}-G_{M}^{*}\right) \delta\left(x_{\Delta}-x_{\mathrm{Bj}}\right),  \tag{2.38}\\
g_{2}^{\Delta}\left(x_{\mathrm{Bj}}, Q^{2}\right)= & \frac{1}{8} \frac{Q_{+}^{-2}\left(M+M_{\Delta}\right)^{2} \tau_{P}}{M_{\Delta} M\left(M_{\Delta}^{2}-M^{2}+Q^{2}\right)^{-1} G_{C}^{*}\left(3 G_{E}^{*}-G_{M}^{*}\right) \delta\left(x_{\Delta}-x_{\mathrm{Bj}}\right)}  \tag{2.39}\\
& +\frac{\left(M+M_{\Delta}^{2} \tau_{P}\right.}{4 Q_{+}^{2}}\left(\left(G_{M}^{*}\right)^{2}-3\left(G_{E}^{*}\right)^{2}+6 G_{E}^{*} G_{M}^{*}\right) \delta\left(x_{\Delta}-x_{\mathrm{Bj}}\right), \tag{2.40}
\end{align*}
$$

where $G_{E}^{\star}\left(Q^{2}\right), G_{M}^{\star}\left(Q^{2}\right), G_{C}^{\star}\left(Q^{2}\right)$ are the electric, magnetic and Coulomb form factors of Jones and Scadron [181], which are functions of $Q^{2}$ only, $M_{\Delta}$ denotes the $\Delta$-particle mass, with the following convenient notations:

$$
\begin{equation*}
x_{\Delta}=\frac{Q^{2}}{Q^{2}+M_{\Delta}^{2}-M^{2}}, \quad Q_{ \pm}^{2}=Q^{2}+\left(M_{\Delta} \pm M\right)^{2} \tag{2.41}
\end{equation*}
$$

Substituting the expressions for the inclusive cross sections of Eqs. (2.30-2.32) into the DRs, see Eqs. (2.23-2.24), changing the integration order, as detailed in Appendix C, and expressing the spin-dependent forward TPE amplitudes in terms of the proton SFs, we obtain:

$$
\begin{align*}
\Re g^{2 \gamma}(\omega)= & \frac{4 m \alpha^{2}}{\boldsymbol{k}^{2}} \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}} \int_{\nu_{\text {thr }}}^{\infty} \frac{\mathrm{d} \nu_{\gamma}}{\nu_{\gamma}} \\
& \left\{2 \frac{\left(\omega_{0}-\left|\boldsymbol{k}_{0}\right|\right) \nu_{\gamma}+m^{2}\left(\tau_{l}+\tilde{\tau}\right)}{\left|\boldsymbol{k}_{0}\right|} g_{1}\left(\nu_{\gamma}, Q^{2}\right)\right. \\
& +\frac{m^{2}\left(\tau_{l}+\tilde{\tau}\right)+\boldsymbol{k}^{2}}{|\boldsymbol{k}|} g_{1}\left(\nu_{\gamma}, Q^{2}\right) \ln \frac{|\boldsymbol{k}|-\left|\boldsymbol{k}_{0}\right|}{|\boldsymbol{k}|+\left|\boldsymbol{k}_{0}\right|}+2 g_{2}\left(\nu_{\gamma}, Q^{2}\right) \ln \frac{|\boldsymbol{k}|-\left|\boldsymbol{k}_{0}\right|}{|\boldsymbol{k}|+\left|\boldsymbol{k}_{0}\right|} \\
& \left.+\frac{\omega \nu_{\gamma}}{|\boldsymbol{k}|} g_{1}\left(\nu_{\gamma}, Q^{2}\right) \ln \frac{(\omega+|\boldsymbol{k}|)^{2}\left(\omega_{0}^{2}-\omega^{2}\right)}{\left(\omega\left|\boldsymbol{k}_{0}\right|+|\boldsymbol{k}| \omega_{0}\right)^{2}}\right\} \tag{2.42}
\end{align*}
$$

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$$
\begin{align*}
\Re f_{-}^{2 \gamma}(\omega)= & \frac{8 \omega \alpha^{2}}{\boldsymbol{k}^{2}} \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}} \int_{\nu_{\mathrm{thr}}}^{\infty} \frac{\mathrm{d} \nu_{\gamma}}{\nu_{\gamma}} \\
& \left\{2 \frac{\left(\omega_{0}-\left|\boldsymbol{k}_{0}\right|\right) \nu_{\gamma}+m^{2}\left(\tau_{l}+\tilde{\tau}\right)}{\left|\boldsymbol{k}_{0}\right|} g_{1}\left(\nu_{\gamma}, Q^{2}\right)\right. \\
& +\frac{m^{2}\left(\tau_{l}+\tilde{\tau}\right)-\boldsymbol{k}^{2}}{|\boldsymbol{k}|} g_{1}\left(\nu_{\gamma}, Q^{2}\right) \ln \frac{|\boldsymbol{k}|-\left|\boldsymbol{k}_{0}\right|}{|\boldsymbol{k}|+\left|\boldsymbol{k}_{0}\right|} \\
& +\frac{Q^{2}|\boldsymbol{k}|}{2 \omega \nu_{\gamma}} g_{2}\left(\nu_{\gamma}, Q^{2}\right) \ln \frac{(\omega+|\boldsymbol{k}|)^{2}\left(\omega_{0}^{2}-\omega^{2}\right)}{\left(\omega\left|\boldsymbol{k}_{0}\right|+|\boldsymbol{k}| \omega_{0}\right)^{2}} \\
& \left.+\frac{\left(\omega^{2}+m^{2}\right) \nu_{\gamma}}{2 \omega|\boldsymbol{k}|} g_{1}\left(\nu_{\gamma}, Q^{2}\right) \ln \frac{(\omega+|\boldsymbol{k}|)^{2}\left(\omega_{0}^{2}-\omega^{2}\right)}{\left(\omega\left|\boldsymbol{k}_{0}\right|+|\boldsymbol{k}| \omega_{0}\right)^{2}}\right\}, \tag{2.43}
\end{align*}
$$

where we have introduced the notations:

$$
\begin{align*}
|\boldsymbol{k}| & =\sqrt{\omega^{2}-m^{2}}, \quad\left|\boldsymbol{k}_{0}\right|=\sqrt{\omega_{0}^{2}-m^{2}},  \tag{2.44}\\
\omega_{0} & =m\left(\sqrt{\tau_{l} \tilde{\tau}}+\sqrt{1+\tau_{l}} \sqrt{1+\tilde{\tau}}\right),  \tag{2.45}\\
\tau_{l} & =\frac{Q^{2}}{4 m^{2}}, \quad \tau_{P}=\frac{Q^{2}}{4 M^{2}}, \quad \tilde{\tau}=\frac{\nu_{\gamma}^{2}}{Q^{2}} . \tag{2.46}
\end{align*}
$$

Furthermore, in Eqs. $(2.42,2.43)$ the elastic threshold $\nu_{\text {thr }}$ is given by $\nu_{\text {thr }}=0$.
The leading TPE correction to the atomic energy levels is given by the values of the amplitudes at threshold $\omega=m$. The TPE amplitudes $f_{-}^{2 \gamma}, g^{2 \gamma}$ at threshold can then be expressed in terms of the proton spin SFs as

$$
\begin{align*}
& f_{-}^{2 \gamma}(m)=\frac{16 \alpha^{2}}{3} \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}} \int_{\nu_{\text {thr }}}^{\infty} \frac{\mathrm{d} \nu_{\gamma}}{\nu_{\gamma}} \frac{\left[2+\rho\left(\tau_{l}\right) \rho(\tilde{\tau})\right] g_{1}\left(\nu_{\gamma}, Q^{2}\right)-3 \rho\left(\tau_{l}\right) \rho(\tilde{\tau}) g_{2}\left(\nu_{\gamma}, Q^{2}\right) / \tilde{\tau}}{\sqrt{\tilde{\tau}} \sqrt{1+\tau_{l}}+\sqrt{\tau_{l}} \sqrt{1+\tilde{\tau}}} \\
& g^{2 \gamma}(m)=-\frac{16 \alpha^{2}}{3} \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}} \int_{\nu_{\text {thr }}}^{\infty} \frac{\mathrm{d} \nu_{\gamma}}{\nu_{\gamma}} \frac{\left[2+\rho\left(\tau_{l}\right) \rho(\tilde{\tau})\right] g_{1}\left(\nu_{\gamma}, Q^{2}\right)+3 g_{2}\left(\nu_{\gamma}, Q^{2}\right)}{\sqrt{\tilde{\tau}} \sqrt{1+\tau_{l}}+\sqrt{\tau_{l}} \sqrt{1+\tilde{\tau}}} \tag{2.47}
\end{align*}
$$

with

$$
\begin{equation*}
\rho(\tau)=\tau-\sqrt{\tau(1+\tau)} . \tag{2.49}
\end{equation*}
$$

### 2.4 Forward doubly virtual Compton scattering (VVCS) tensor

On the other hand the forward TPE amplitude can be evaluated considering the lower blob of the TPE graph in Fig. 2.3 as a forward doubly virtual Compton scattering (VVCS) process on a proton (see Fig. 2.4): $\gamma^{*}\left(q, \lambda_{1}\right)+N(p, \lambda) \rightarrow \gamma^{*}\left(q, \lambda_{2}\right)+N\left(p, \lambda^{\prime}\right)$. The forward VVCS amplitude $T_{\lambda_{2} \lambda^{\prime}, \lambda_{1} \lambda}$ can be written in terms of the forward VVCS tensor $M^{\mu \nu}$ as

$$
\begin{equation*}
T_{\lambda_{2} \lambda^{\prime}, \lambda_{1} \lambda}=\varepsilon_{\nu}\left(q, \lambda_{1}\right) \varepsilon_{\mu}^{*}\left(q, \lambda_{2}\right) \cdot \bar{N}\left(p, \lambda^{\prime}\right)\left(4 \pi M^{\mu \nu}\right) N(p, \lambda), \tag{2.50}
\end{equation*}
$$

where $\varepsilon_{\nu}, \varepsilon_{\mu}^{*}$ denote the virtual photon polarization vectors, $N, \bar{N}$ the proton spinors, and $\lambda_{1}, \lambda_{2}\left(\lambda, \lambda^{\prime}\right)$ the photon (proton) helicities.

The forward VVCS tensor $M^{\mu \nu}$ can be expressed as the sum of a symmetric (spin-independent) $M_{S}^{\mu \nu}$ and an antisymmetric (spin-dependent) $M_{A}^{\mu \nu}$ parts:

$$
\begin{equation*}
M^{\mu \nu}=M_{S}^{\mu \nu}+M_{A}^{\mu \nu}, \tag{2.51}
\end{equation*}
$$

$$
\begin{align*}
& M_{S}^{\mu \nu}=\left(-g^{\mu \nu}+\frac{q^{\mu} q^{\nu}}{q^{2}}\right) \mathrm{T}_{1}\left(\nu_{\gamma}, Q^{2}\right)+\frac{1}{M^{2}}\left(p^{\mu}-\frac{(p \cdot q)}{q^{2}} q^{\mu}\right)\left(p^{\nu}-\frac{(p \cdot q)}{q^{2}} q^{\nu}\right) \mathrm{T}_{2}\left(\nu_{\gamma}, Q^{2}\right),  \tag{2.52}\\
& M_{A}^{\mu \nu}=\frac{1}{2 M^{2}}\left[M\left\{\gamma^{\mu \nu}, \hat{q}\right\} \mathrm{S}_{1}\left(\nu_{\gamma}, Q^{2}\right)+\left(\left[\gamma^{\mu}, \gamma^{\nu}\right] q^{2}+q^{\mu}\left[\gamma^{\nu}, \hat{q}\right]+q^{\nu}\left[\hat{q}, \gamma^{\mu}\right]\right) \mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right)\right], \tag{2.53}
\end{align*}
$$

with the forward Compton amplitudes $\mathrm{T}_{1}, \mathrm{~T}_{2}, \mathrm{~S}_{1}, \mathrm{~S}_{2}$, which enter Eq. (2.51) in a gaugeinvariant way, e.g., $q_{\mu} M^{\mu \nu}=q_{\nu} M^{\mu \nu}=0$, and $\hat{a}=\gamma^{\mu} a_{\mu}$. The antisymmetric tensor can be written in a scalar form when the initial and final protons have the same spin 4 -vector $S^{\mu}$ :

$$
\begin{equation*}
M_{A}^{\mu \nu}=\frac{i}{M} \epsilon^{\mu \nu \alpha \beta} q_{\alpha} S_{\beta} \mathrm{S}_{1}\left(\nu_{\gamma}, Q^{2}\right)+\frac{i}{M^{3}} \epsilon^{\mu \nu \alpha \beta} q_{\alpha}\left((p \cdot q) S_{\beta}-(S \cdot q) p_{\beta}\right) \mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right) \tag{2.54}
\end{equation*}
$$



Figure 2.4: Forward VVCS process.

Performing the crossing of the photon lines, we obtain the symmetry properties of the forward Compton amplitudes:

$$
\begin{array}{lc}
\mathrm{T}_{1}\left(\nu_{\gamma}, Q^{2}\right)=\mathrm{T}_{1}\left(-\nu_{\gamma}, Q^{2}\right), & \mathrm{T}_{2}\left(\nu_{\gamma}, Q^{2}\right)=\mathrm{T}_{2}\left(-\nu_{\gamma}, Q^{2}\right) \\
\mathrm{S}_{1}\left(\nu_{\gamma}, Q^{2}\right)=\mathrm{S}_{1}\left(-\nu_{\gamma}, Q^{2}\right), & \mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right)=-\mathrm{S}_{2}\left(-\nu_{\gamma}, Q^{2}\right) \tag{2.56}
\end{array}
$$

Applying the optical theorem for the scalar form of the forward VVCS tensor, we relate its imaginary part to the hadronic tensor $W^{\mu \nu}$ by

$$
\begin{equation*}
\Im M^{\mu \nu}=\frac{e^{2}}{4 M} W^{\mu \nu} \tag{2.57}
\end{equation*}
$$

The absorptive parts of the forward VVCS amplitudes $\mathrm{T}_{1}, \mathrm{~T}_{2}, \mathrm{~S}_{1}, \mathrm{~S}_{2}$ are related to the proton structure functions $F_{1}, F_{2}, g_{1}, g_{2}$ by

$$
\begin{array}{ll}
\Im \mathrm{T}_{1}\left(\nu_{\gamma}, Q^{2}\right)=\frac{e^{2}}{4 M} F_{1}\left(\nu_{\gamma}, Q^{2}\right), & \Im \mathrm{T}_{2}\left(\nu_{\gamma}, Q^{2}\right)=\frac{e^{2}}{4 \nu_{\gamma}} F_{2}\left(\nu_{\gamma}, Q^{2}\right) \\
\Im \mathrm{S}_{1}\left(\nu_{\gamma}, Q^{2}\right)=\frac{e^{2}}{4 \nu_{\gamma}} g_{1}\left(\nu_{\gamma}, Q^{2}\right), & \Im \mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right)=\frac{e^{2} M}{4 \nu_{\gamma}^{2}} g_{2}\left(\nu_{\gamma}, Q^{2}\right) \tag{2.59}
\end{array}
$$

The real part of the even amplitude $\mathrm{T}_{1}$ can be expressed through a subtracted dispersion relation as

$$
\begin{equation*}
\Re \mathrm{T}_{1}\left(\nu_{\gamma}, Q^{2}\right)=\Re \mathrm{T}_{1}^{\text {Born }}\left(\nu_{\gamma}, Q^{2}\right)+\mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)+\frac{e^{2} \nu_{\gamma}^{2}}{2 \pi} f_{\nu_{\mathrm{thr}}^{\text {incl }}}^{\infty} \frac{F_{1}\left(\nu^{\prime}, Q^{2}\right) \mathrm{d} \nu^{\prime}}{M \nu^{\prime}\left(\nu^{\prime 2}-\nu_{\gamma}^{2}\right)} \tag{2.60}
\end{equation*}
$$

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with the pion-proton inelastic threshold: $\nu_{\mathrm{thr}}^{\mathrm{inel}}=m_{\pi}+\left(m_{\pi}^{2}+Q^{2}\right) /(2 M)$, where $m_{\pi}$ denotes the pion mass, and $\mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)$ is the subtraction function at zero photon energy $\nu_{\gamma}=0$, which we have conveniently defined relative to the Born contribution as specified below. The real part of the spin-independent even amplitude $\mathrm{T}_{2}$ can be obtained from an unsubtracted DR:

$$
\begin{equation*}
\Re \mathrm{T}_{2}\left(\nu_{\gamma}, Q^{2}\right)=\Re \mathrm{T}_{2}^{\text {pole }}\left(\nu_{\gamma}, Q^{2}\right)+\frac{e^{2}}{2 \pi} \int_{\substack{\text { innel } \\ \nu_{\text {thr }}}}^{\infty} \frac{F_{2}\left(\nu^{\prime}, Q^{2}\right) \mathrm{d} \nu^{\prime}}{\nu^{\prime 2}-\nu_{\gamma}^{2}} . \tag{2.61}
\end{equation*}
$$

The real part of the spin-dependent amplitudes $\mathrm{S}_{1}$ (even amplitude) and $\mathrm{S}_{2}$ (odd amplitude) also can be reconstructed within unsubtracted DRs:

$$
\begin{align*}
\Re \mathrm{S}_{1}\left(\nu_{\gamma}, Q^{2}\right) & =\Re \mathrm{S}_{1}^{\text {pole }}\left(\nu_{\gamma}, Q^{2}\right)+\frac{e^{2}}{2 \pi} f_{\substack{\nu_{\mathrm{in}}^{\text {inel }} \\
\nu_{\text {thr }}}}^{\infty} \frac{g_{1}\left(\nu^{\prime}, Q^{2}\right)}{\nu^{\prime 2}-\nu_{\gamma}^{2}} \mathrm{~d} \nu_{\gamma}^{\prime},  \tag{2.62}\\
\Re \mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right) & =\Re \mathrm{S}_{2}^{\text {pole }}\left(\nu_{\gamma}, Q^{2}\right)+\frac{e^{2} M \nu_{\gamma}}{2 \pi} f_{\substack{\nu_{\text {inl }}}}^{\infty} \frac{g_{2}\left(\nu^{\prime}, Q^{2}\right)}{\nu_{\text {thr }}^{\prime 2}\left(\nu^{\prime 2}-\nu_{\gamma}^{2}\right)} \mathrm{d} \nu^{\prime} . \tag{2.63}
\end{align*}
$$

Assuming the vanishing high-energy behavior of the $\mathrm{S}_{2}$ Compton amplitude:

$$
\begin{equation*}
\mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right) \underset{\nu_{\gamma} \rightarrow \infty}{<} \frac{1}{\nu_{\gamma}}, \tag{2.64}
\end{equation*}
$$

we can also write down the unsubtracted DR for the even amplitude $\nu_{\gamma} \mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right)$ [182]:

$$
\begin{equation*}
\Re\left(\nu_{\gamma} \mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right)\right)=\Re\left(\nu_{\gamma} \mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right)\right)^{\text {pole }}+\frac{e^{2}}{2 \pi} \int_{\substack{i \text { nel } \\ \nu_{\text {thr }}}}^{\infty} \frac{M g_{2}\left(\nu^{\prime}, Q^{2}\right)}{\nu^{\prime 2}-\nu_{\gamma}^{2}} \mathrm{~d} \nu^{\prime} \tag{2.65}
\end{equation*}
$$

Polarizabilities in Compton scattering are conventionally defined by separating the Compton amplitudes into Born and non-Born parts. The Born contributions to the forward VVCS amplitudes $\mathrm{T}_{1}^{\text {Born }}, \mathrm{T}_{2}^{\text {Born }}, \mathrm{S}_{1}^{\text {Born }}, \mathrm{S}_{2}^{\text {Born }},\left(\nu_{\gamma} \mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right)\right)^{\text {Born }}$ are given by

$$
\begin{align*}
\mathrm{T}_{1}^{\text {Born }}\left(\nu_{\gamma}, Q^{2}\right) & =\frac{\alpha}{M}\left(\frac{Q^{4} G_{M}^{2}\left(Q^{2}\right)}{Q^{4}-4 M^{2} \nu_{\gamma}^{2}-i \varepsilon}-F_{D}^{2}\left(Q^{2}\right)\right),  \tag{2.66}\\
\mathrm{T}_{2}^{\text {Born }}\left(\nu_{\gamma}, Q^{2}\right) & =4 M Q^{2} \alpha \frac{F_{D}^{2}\left(Q^{2}\right)+\frac{Q^{2}}{4 M^{2}} F_{P}^{2}\left(Q^{2}\right)}{Q^{4}-4 M^{2} \nu_{\gamma}^{2}-i \varepsilon},  \tag{2.67}\\
\mathrm{~S}_{1}^{\text {Born }}\left(\nu_{\gamma}, Q^{2}\right) & =\frac{\alpha}{2 M}\left(\frac{4 M^{2} Q^{2} F_{D}\left(Q^{2}\right) G_{M}\left(Q^{2}\right)}{Q^{4}-4 M^{2} \nu_{\gamma}^{2}-i \varepsilon}-F_{P}^{2}\left(Q^{2}\right)\right),  \tag{2.68}\\
\mathrm{S}_{2}^{\text {Born }}\left(\nu_{\gamma}, Q^{2}\right) & =-\alpha \frac{2 M^{2} \nu_{\gamma} F_{P}\left(Q^{2}\right) G_{M}\left(Q^{2}\right)}{Q^{4}-4 M^{2} \nu_{\gamma}^{2}-i \varepsilon},  \tag{2.69}\\
\left(\nu_{\gamma} \mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right)\right)^{\text {Born }} & =-\frac{\alpha}{2}\left(\frac{Q^{4} F_{P}\left(Q^{2}\right) G_{M}\left(Q^{2}\right)}{Q^{4}-4 M^{2} \nu_{\gamma}^{2}-i \varepsilon}-F_{P}\left(Q^{2}\right) G_{M}\left(Q^{2}\right)\right) . \tag{2.70}
\end{align*}
$$

Note that in the derivation of DRs as given in Eqs. (2.60-2.65) the elastic (on-shell proton pole) term contribution is given by only the first term of Eqs. (2.66-2.70). This pole
contribution differs from the Born term by

$$
\begin{align*}
\mathrm{T}_{1}^{\text {Born }}\left(\nu_{\gamma}, Q^{2}\right)-\mathrm{T}_{1}^{\text {pole }}\left(\nu_{\gamma}, Q^{2}\right) & =-\frac{\alpha}{M} F_{D}^{2}\left(Q^{2}\right)  \tag{2.71}\\
\mathrm{S}_{1}^{\text {Born }}\left(\nu_{\gamma}, Q^{2}\right)-\mathrm{S}_{1}^{\text {pole }}\left(\nu_{\gamma}, Q^{2}\right) & =-\frac{\alpha}{2 M} F_{P}^{2}\left(Q^{2}\right),  \tag{2.72}\\
\left(\nu_{\gamma} \mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right)\right)^{\text {Born }}-\left(\nu_{\gamma} \mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right)\right)^{\text {pole }} & =\frac{\alpha}{2} F_{P}\left(Q^{2}\right) G_{M}\left(Q^{2}\right) . \tag{2.73}
\end{align*}
$$

For the amplitudes $\mathrm{T}_{2}$ and $\mathrm{S}_{2}$ the pole and Born terms are identical.
Note that for the amplitude $\mathrm{T}_{1}$, which has to be subtracted, the difference between the Born and pole terms is an energy $\left(\nu_{\gamma}\right)$ independent function. We therefore have absorbed it in the definition of $\mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)$, which in Eq. (2.60) is defined as

$$
\begin{equation*}
\mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right) \equiv \mathrm{T}_{1}\left(0, Q^{2}\right)-\mathrm{T}_{1}^{\text {Born }}\left(0, Q^{2}\right) \equiv Q^{2} \beta\left(Q^{2}\right) \tag{2.74}
\end{equation*}
$$

The advantage of expressing the amplitude $\mathrm{T}_{1}$ w.r.t. to its Born contribution, results from the fact that the non-Born amplitude in Eq. (2.74) starts at $Q^{2}$, and is usually parametrized in terms of polarizabilities, i.e., the function $\beta\left(Q^{2}\right)$ at $Q^{2}=0$ is given by the magnetic polarizability $\beta_{M}: \beta(0)=\beta_{M}[135,141]$.

Note, furthermore, that when subtracting from DR for the amplitude $\nu_{\gamma} \mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right)$ of Eq. (2.65) the DR for the amplitude $\mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right)$ of Eq. (2.63) multiplied by $\nu_{\gamma}$, we obtain the "superconvergence relation". It is known as the Burkhardt-Cottingham sum rule:

$$
\begin{equation*}
\int_{\substack{\mathrm{inel} \\ \nu_{\mathrm{thr}}}}^{\infty} g_{2}\left(\nu^{\prime}, Q^{2}\right) \frac{M \mathrm{~d} \nu^{\prime}}{\nu^{\prime 2}}=\frac{1}{4} F_{P}\left(Q^{2}\right) G_{M}\left(Q^{2}\right) \tag{2.75}
\end{equation*}
$$

It has a simple form for the proton structure function $g_{2}$ expressed as a function of the Bjorken variable $x_{\mathrm{Bj}}$ :

$$
\begin{equation*}
\int_{0}^{1} \mathrm{~d} x_{\mathrm{Bj}} g_{2}\left(x_{\mathrm{Bj}}, Q^{2}\right)=0 \tag{2.76}
\end{equation*}
$$

### 2.4.1 Empirical estimate of $\mathrm{T}_{1}$ subtraction function

We start this Section by discussing an empirical estimate of the subtraction function $\mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)$, or equivalently the function $\beta\left(Q^{2}\right)$ defined through Eq. (2.74), at non-zero $Q^{2}$ from the experimental information about the inelastic electron-proton scattering.

Following the idea of Refs. [183, 184], the subtraction function can be obtained from an unsubtracted dispersion relation for the amplitude $\mathrm{T}_{1}\left(\nu_{\gamma}, Q^{2}\right)-\mathrm{T}_{1}^{\mathrm{R}}\left(\nu_{\gamma}, Q^{2}\right)$, where $\mathrm{T}_{1}^{\mathrm{R}}$ denotes a Regge amplitude which is chosen such as to match the high-energy behavior of the amplitude $\mathrm{T}_{1}$, i.e., $\mathrm{T}_{1}-\mathrm{T}_{1}^{\mathrm{R}} \rightarrow 0$ for $\nu_{\gamma} \rightarrow \infty .^{3}$ The function $\mathrm{T}_{1}^{\mathrm{R}}$ is chosen as a sum over the leading Regge trajectories:

$$
\begin{align*}
\mathrm{T}_{1}^{\mathrm{R}}\left(\nu_{\gamma}, Q^{2}\right) & \equiv-\frac{\pi \alpha}{M} \sum_{\alpha_{0}>0} \frac{\gamma_{\alpha_{0}}\left(Q^{2}\right)}{\sin \pi \alpha_{0}}\left\{\left(\nu_{0}-\nu_{\gamma}-i \varepsilon\right)^{\alpha_{0}}+\left(\nu_{0}+\nu_{\gamma}-i \varepsilon\right)^{\alpha_{0}}\right\} \\
& -\frac{\pi \alpha}{M} \sum_{\alpha_{0}>1} \frac{\alpha_{0} \nu_{0} \gamma_{\alpha_{0}}\left(Q^{2}\right)}{\sin \pi\left(\alpha_{0}-1\right)}\left\{\left(\nu_{0}-\nu_{\gamma}-i \varepsilon\right)^{\alpha_{0}-1}+\left(\nu_{0}+\nu_{\gamma}-i \varepsilon\right)^{\alpha_{0}-1}\right\}, \tag{2.77}
\end{align*}
$$

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with the intercept $\alpha_{0}>0, \nu_{0}$ is an arbitrary hadronic scale, and $\gamma_{\alpha_{0}}\left(Q^{2}\right)$ are the Regge residues. Using Eqs. (2.58), the imaginary part of $\mathrm{T}_{1}^{\mathrm{R}}$ yields the corresponding Regge structure:

$$
\begin{align*}
F_{1}^{\mathrm{R}}\left(\nu_{\gamma}, Q^{2}\right) \equiv \frac{M}{\pi \alpha} \Im \mathrm{~T}_{1}^{R}\left(\nu_{\gamma}, Q^{2}\right) & =\sum_{\alpha_{0}>0} \gamma_{\alpha_{0}}\left(Q^{2}\right)\left(\nu_{\gamma}-\nu_{0}\right)^{\alpha_{0}} \Theta\left(\nu_{\gamma}-\nu_{0}\right) \\
& +\sum_{\alpha_{0}>1} \gamma_{\alpha_{0}}\left(Q^{2}\right) \alpha_{0} \nu_{0}\left(\nu_{\gamma}-\nu_{0}\right)^{\alpha_{0}-1} \Theta\left(\nu_{\gamma}-\nu_{0}\right) \tag{2.78}
\end{align*}
$$

The Regge residues $\gamma_{\alpha_{0}}\left(Q^{2}\right)$ can be obtained by performing a fit to inclusive electroproduction data on a proton. In our work we use the Donnachie-Landshoff (DL) high-energy fit [146] to obtain the proton structure function $F_{1}$ as

$$
\begin{equation*}
F_{1}\left(\nu_{\gamma}, Q^{2}\right) \underset{\nu_{\gamma} \ggg}{\longrightarrow} \sum_{\alpha_{0}>0} \gamma_{\alpha_{0}}\left(Q^{2}\right) \nu_{\gamma}^{\alpha_{0}}, \tag{2.79}
\end{equation*}
$$

where the values of the Regge intercepts $\alpha_{0}$ and the residue functions $\gamma_{\alpha_{0}}\left(Q^{2}\right)$ are detailed in Appendix E.

By comparing Eq. (2.78) and (2.79) we notice that the second term in Eq. (2.78) is chosen such that for the Regge trajectory with $1<\alpha_{0}<2$ ("Pomeron"):

$$
\begin{equation*}
F_{1}\left(\nu_{\gamma}, Q^{2}\right)-F_{1}^{\mathrm{R}}\left(\nu_{\gamma}, Q^{2}\right) \underset{\nu_{\gamma} \gg}{\sim} \nu_{\gamma}^{\alpha_{0}-2}, \tag{2.80}
\end{equation*}
$$

whereas for the Regge trajectory with $0<\alpha_{0}<1$ (Reggeon):

$$
\begin{equation*}
F_{1}\left(\nu_{\gamma}, Q^{2}\right)-F_{1}^{\mathrm{R}}\left(\nu_{\gamma}, Q^{2}\right) \underset{\nu_{\gamma} \gg}{\sim} \nu_{\gamma}^{\alpha_{0}-1} . \tag{2.81}
\end{equation*}
$$

This ensures that in all cases the quantity $\left[F_{1}\left(\nu_{\gamma}, Q^{2}\right)-F_{1}^{\mathrm{R}}\left(\nu_{\gamma}, Q^{2}\right)\right] \rightarrow 0$ when $\nu_{\gamma} \rightarrow \infty$.
Consequently, one can write down an unsubtracted dispersion relation for $\mathrm{T}_{1}-\mathrm{T}_{1}^{\mathrm{R}}$ at fixed $Q^{2}$ as

$$
\begin{equation*}
\mathrm{T}_{1}\left(\nu_{\gamma}, Q^{2}\right)-\mathrm{T}_{1}^{\mathrm{R}}\left(\nu_{\gamma}, Q^{2}\right)=\mathrm{T}_{1}^{\text {pole }}\left(\nu_{\gamma}, Q^{2}\right)+\frac{2}{\pi} f \mathrm{~d} \nu^{\prime} \frac{\nu^{\prime} \Im\left[\mathrm{T}_{1}\left(\nu^{\prime}, Q^{2}\right)-\mathrm{T}_{1}^{\mathrm{R}}\left(\nu^{\prime}, Q^{2}\right)\right]}{\nu^{\prime 2}-\nu_{\gamma}^{2}} . \tag{2.82}
\end{equation*}
$$

Using Eqs. ( $2.58,2.71,2.74$ ), this yields an expression for $\mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)$ which expressed in terms of the squared invariant mass variable $W^{2} \equiv(p+q)^{2}=2 M \nu^{\prime}+M^{2}-Q^{2}$ as

$$
\begin{align*}
& \mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)=\mathrm{T}_{1}^{\mathrm{R}}\left(0, Q^{2}\right)+\frac{\alpha}{M} F_{D}^{2}\left(Q^{2}\right) \\
& +\frac{2 \alpha}{M} \int_{s_{\mathrm{thr}}}^{\infty} \frac{F_{1}\left(\left(W^{2}-M^{2}+Q^{2}\right) /(2 M), Q^{2}\right)-F_{1}^{\mathrm{R}}\left(\left(W^{2}-M^{2}+Q^{2}\right) /(2 M), Q^{2}\right)}{W^{2}-M^{2}+Q^{2}} \mathrm{~d} W^{2}, \tag{2.83}
\end{align*}
$$

where the lower integration limit in Eq. (2.83) $s_{\text {thr }}$ is given by

$$
\begin{equation*}
s_{\mathrm{thr}}=\min \left(s_{0} \equiv 2 M \nu_{0}+M^{2}-Q^{2}, W_{\mathrm{thr}}^{2}=\left(M+m_{\pi}\right)^{2}\right), \tag{2.84}
\end{equation*}
$$

corresponding with a branch cut of $F_{1}$ starting at $W_{\mathrm{thr}}^{2}$ and a branch cut of $F_{1}^{\mathrm{R}}$ starting at $s_{0}$. Eq. (2.83) allows to quantitatively estimate the subtraction function given the structure function $F_{1}$, the Regge fit determining $F_{1}^{\mathrm{R}}$ of the form of Eq. (2.78), as well as the corresponding value of $\mathrm{T}_{1}^{\mathrm{R}}\left(0, Q^{2}\right)$ which follows from Eq. (2.77) as

$$
\begin{equation*}
\mathrm{T}_{1}^{\mathrm{R}}\left(0, Q^{2}\right)=-\frac{2 \pi \alpha}{M} \sum_{\alpha_{0}>0} \frac{\gamma_{\alpha_{0}}\left(Q^{2}\right)}{\sin \pi \alpha_{0}}\left(\nu_{0}\right)^{\alpha_{0}}-\frac{2 \pi \alpha}{M} \sum_{\alpha_{0}>1} \frac{\alpha_{0} \nu_{0} \gamma_{\alpha_{0}}\left(Q^{2}\right)}{\sin \pi\left(\alpha_{0}-1\right)}\left(\nu_{0}\right)^{\alpha_{0}-1}, \tag{2.85}
\end{equation*}
$$

and is also fully determined by the Regge fit.
In our numerical evaluation of Eq. (2.83), we describe the proton structure function $F_{1}$ in the resonance region by the fit performed by Christy and Bosted (BC) [91]. This fit is valid in the following region of kinematical variables: $0<Q^{2}<8 \mathrm{GeV}^{2}$, and $W^{2}<9.61 \mathrm{GeV}^{2} \approx 10 \mathrm{GeV}^{2}$. For the dispersion integral in Eq. (2.83) we connect the BC fit with the DL high-energy fit starting from $W^{2}=10 \mathrm{GeV}^{2}$. The latter fit is described in Appendix E. The resulting proton structure function $F_{1}$ is shown in Fig. 2.5 as it enters the integral of Eq. (2.83). We add a $3 \%$ error band to the BC fit [91] and use the same error estimate for all Regge pole residues. We notice that at low values of $Q^{2}$, both fits either overlap or are very close to the matching point $W^{2} \approx 10 \mathrm{GeV}^{2}$. With increasing values of $Q^{2}$ there is an increasing mismatch in both fits around $W^{2}=10 \mathrm{GeV}^{2}$. This is to be expected because the BC fit has not accounted for the HERA high-energy data, and the DL fit has not accounted for the lower $W$ data. Even though a combined fit of all data would be very worthwhile, or a smooth interpolating procedure between the BC and DL fits could easily be performed, for our purpose we will only need data at a lower value of $Q^{2}$ up to about $1 \mathrm{GeV}^{2}$. For this purpose, we can just split the $W^{2}$ integral entering Eq. (2.83) in a region $W^{2}<10 \mathrm{GeV}^{2}$, where we will use the BC fit, and a region $W^{2}>10 \mathrm{GeV}^{2}$, where we will use the DL fit.


Figure 2.5: Fits of the proton structure function $F_{1}$ used in our estimates entering the dispersion integral in Eq. (2.83).

In Fig. 2.6, we demonstrate explicitly the vanishing high-energy behavior of the quantity $F_{1}-F_{1}^{\mathrm{R}}$, which is the necessary condition for the unsubtracted DR of Eq. (2.83) to hold.

We furthermore provide another consistency check of our numerical implementation. As the Regge function $\mathrm{T}_{1}^{\mathrm{R}}$ of Eq. (2.77) has an arbitrary scale $\nu_{0}$ (or equivalently $s_{0}$ ), the total result should not depend on the specific choice of this parameter. We demonstrate this in Fig. 2.7, where we illustrate how the $s_{0}$ dependence of the individual contributions in Eq. (2.83) adds up to yield the total result which is independent of $s_{0}$.

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Figure 2.6: High-energy behavior of the function $F_{1}-F_{1}^{\mathrm{R}}$ for the fixed value $s_{0}=1 \mathrm{GeV}^{2}$.


Figure 2.7: The contribution of the individual terms in Eq. (2.83) to $\mathrm{T}_{1}^{\text {subt }}(0,0)$ as function of $s_{0}$. Dashed curve: the dispersion integral contribution from the BC fit $\sim \int_{W_{\text {thr }}^{2}}^{10 \mathrm{GeV}^{2}}\left(F_{1}^{\mathrm{BC}}-F_{1}^{\mathrm{R}}\right)$. Dashed-dotted curve: the dispersion integral contribution from the DL fit $\sim \int_{10}^{\infty} \mathrm{GeV}^{2}\left(F_{1}^{\mathrm{DL}}-F_{1}^{\mathrm{R}}\right)$. Dotted curve: the dispersion integral contribution $\sim-\int_{s_{0}}^{W_{\mathrm{thr}}^{2}} F_{1}^{\mathrm{R}}$ due to $F_{1}^{\mathrm{R}}$. Dashed double-dotted curve: the contribution from the real part $\mathrm{T}_{1}^{R}(0,0)$ according to Eq. (2.85). Solid curve: sum of all terms in Eq. (2.83), yielding the $s_{0}$-independent value of $\mathrm{T}_{1}^{\text {subt }}(0,0)$.

In Fig. 2.8 we present the empirically extracted subtraction function $\mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)$ of Eq. (2.83). The subtraction function $\mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)$ should vanish linearly when $Q^{2} \rightarrow 0$ according to Eq. (2.74). This general property, therefore, provides a quality check on the accuracy of an empirical determination as described above. One notices from Fig. 2.8 that the value of $\mathrm{T}_{1}^{\text {subt }}$ at $Q^{2}=0$ is compatible with zero within $1-1.5 \sigma$. We like to notice however that at present such empirical determination can unfortunately only give the correct order of magnitude of $\mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)$. This is partly due to the non-perfect match between the proton $F_{1}$ fits for the resonance region and the large $W$ region, as we have shown in Fig. 2.5. Despite this caveat, it seems however that with increasing $Q^{2}, \mathrm{~T}_{1}^{\text {subt }}\left(0, Q^{2}\right)$ changes sign in the range somewhere between $0.1-0.4 \mathrm{GeV}^{2}$, which may be an indication of the range up to which the ChPT based results can be used. To provide a more accurate determination of the functional dependence of $\mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)$, a combined fit of all proton $F_{1}$ structure function data over the whole range of $W$, incorporating the Regge behavior at large $W$, would be desirable. At intermediate values
of $Q^{2}$, below and around $1 \mathrm{GeV}^{2}$, this will also require more accurate data in the intermediate $W$ range between $3-10 \mathrm{GeV}$. In the lower end of this range, such data can be provided by the JLab 12 GeV facility.


Figure 2.8: The empirical subtraction function of Eq. (2.83).


Figure 2.9: The empirical estimate for the magnetic polarizability $\beta\left(Q^{2}\right)$ based on Eqs. (2.74, 2.83).

Using our empirical determination of $\mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)$, we can extract $\beta\left(Q^{2}\right)$ dividing $\mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)$ by $Q^{2}$ according to Eq. (2.74). For the purpose of combining our empirical estimate of $\mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)$ with the empirical value of $\beta(0)$ as determined from the real Compton scattering, we use the central curve in the empirically determined error band of $\mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)$ (green band in Fig. 2.8) to extract $\beta\left(Q^{2}\right)$ in the range $Q^{2}>0.15 \mathrm{GeV}^{2}$, and extrapolate it by a linear function to the PDG value of $\beta_{M}$ at $Q^{2}=0$. The smooth error bands are obtained connecting the empirical dispersion result, denoted by $\beta_{\mathrm{d}}\left(Q^{2}\right)$, at $Q_{\mathrm{d}}^{2}=0.2 \mathrm{GeV}^{2}$ to the linear extrapolation of the magnetic polarizability value, denoted by $\beta_{1}\left(Q^{2}\right)$, at $Q_{1}^{2}=0.1 \mathrm{GeV}^{2}$,

$$
\begin{align*}
\beta\left(Q^{2}\right)= & \beta_{1}\left(Q^{2}\right) \Theta\left(Q_{1}^{2}-Q^{2}\right)+\beta_{\mathrm{d}}\left(Q^{2}\right) \Theta\left(Q^{2}-Q_{\mathrm{d}}^{2}\right)+ \\
& \frac{c_{1} \beta_{1}\left(Q^{2}\right)+c_{2} f\left(Q^{2}\right) \beta_{\mathrm{d}}\left(Q^{2}\right)}{1+f\left(Q^{2}\right)} \Theta\left(Q_{\mathrm{d}}^{2}-Q^{2}\right) \Theta\left(Q^{2}-Q_{1}^{2}\right), \tag{2.86}
\end{align*}
$$

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where $f\left(Q^{2}\right)$ is a function as entering the Fermi-Dirac distribution, defined as

$$
\begin{equation*}
f\left(Q^{2}\right)=e^{\frac{2 Q^{2}-Q_{\mathrm{d}}^{2}-Q_{1}^{2}}{2 a_{0}}}, \tag{2.87}
\end{equation*}
$$

and the constants $c_{1}, c_{2}, a_{0}$ were chosen as those that preserve the regularity and smoothness of error bands. The resulting function $\beta\left(Q^{2}\right)$ is displayed in Fig. 2.9. As this function encodes both the empirical information on the magnetic polarizability at $Q^{2}=0$, as well as the structure function information at larger $Q^{2}$, we will use this function in the following to provide an empirical estimate for the subtraction function contribution to the TPE correction in the Lamb shift of the muonic hydrogen 2 S energy level in Section 2.6.1 and in the elastic muon-proton scattering at small momentum transfer in Section 5.5.

### 2.5 Forward TPE amplitudes through forward VVCS amplitudes

In this Section, we express the forward lepton-proton TPE amplitude in terms of the forward VVCS amplitudes and compare these expressions with the results of DR analysis of Sections 2.2, 2.3.

The forward lepton-proton scattering TPE amplitude can be expressed in terms of the forward VVCS amplitude $M^{\mu \nu}$ as

$$
\begin{equation*}
T^{2 \gamma}(\omega)=e^{2} \int \frac{i \mathrm{~d}^{4} q}{(2 \pi)^{3}} \frac{\tilde{L}^{\mu \nu} \bar{N}\left(p, \lambda^{\prime}\right) M_{\mu \nu} N(p, \lambda)}{\left(q^{2}\right)^{2}}, \tag{2.88}
\end{equation*}
$$

with the forward leptonic tensor $\tilde{L}^{\mu \nu}$ :

$$
\begin{equation*}
\tilde{L}^{\mu \nu}=\bar{u}\left(k, h^{\prime}\right)\left(\gamma^{\mu} \frac{\hat{k}-\hat{q}+m}{(k-q)^{2}-m^{2}} \gamma^{\nu}+\gamma^{\nu} \frac{\hat{k}+\hat{q}+m}{(k+q)^{2}-m^{2}} \gamma^{\mu}\right) u(k, h) . \tag{2.89}
\end{equation*}
$$

Substituting the expression for the forward VVCS amplitude of Eqs. (2.51-2.53), we obtain:

$$
\begin{align*}
T^{2 \gamma}(\omega)= & -\frac{\bar{u}\left(k, h^{\prime}\right) u(k, h) \bar{N}\left(p, \lambda^{\prime}\right) N(p, \lambda)}{4 M m} \frac{8 \alpha}{M} \int \frac{i \mathrm{~d}^{4} q}{\pi^{2}} \\
& \frac{M^{2}(k \cdot q)^{2}\left(2 \mathrm{~T}_{1}-\mathrm{T}_{2}\right)+q^{2}\left(M^{2} m^{2} \mathrm{~T}_{1}-(k \cdot p)^{2} \mathrm{~T}_{2}\right)+2(k \cdot p)(k \cdot q)(p \cdot q) \mathrm{T}_{2}}{\left(q^{4}-4(k \cdot q)^{2}\right)\left(q^{2}\right)^{2}} \\
- & \alpha \int \frac{i \mathrm{~d}^{4} q}{\pi^{2}} \frac{\bar{u}\left(k, h^{\prime}\right)\left\{\gamma_{\mu \nu}, \hat{q}\right\} q^{2} u(k, h) \bar{N}\left(p, \lambda^{\prime}\right)\left(M\left\{\gamma^{\mu \nu}, \hat{q}\right\} \mathrm{S}_{1}+2 \gamma^{\mu \nu} q^{2} \mathrm{~S}_{2}\right) N(p, \lambda)}{4 M^{2}\left(q^{4}-4(k \cdot q)^{2}\right)\left(q^{2}\right)^{2}} . \tag{2.90}
\end{align*}
$$

Performing now the tensor decomposition in the arbitrary frame, we obtain for the TPE amplitudes:

$$
\begin{align*}
f_{+}^{2 \gamma}(\omega)= & -\frac{8 \alpha}{M} \int \frac{i \mathrm{~d}^{4} q}{\pi^{2}} \\
& \frac{M^{2}(k \cdot q)^{2}\left(2 \mathrm{~T}_{1}-\mathrm{T}_{2}\right)+q^{2}\left(M^{2} m^{2} \mathrm{~T}_{1}-(k \cdot p)^{2} \mathrm{~T}_{2}\right)+2(k \cdot p)(k \cdot q)(p \cdot q) \mathrm{T}_{2}}{\left(q^{4}-4(k \cdot q)^{2}\right)\left(q^{2}\right)^{2}}, \tag{2.91}
\end{align*}
$$

$g^{2 \gamma}(\omega)=\frac{4 m \alpha}{M^{2}} \int \frac{i \mathrm{~d}^{4} q}{\pi^{2}} \frac{\left(M^{2} q^{2}-\frac{M^{2}(k \cdot q)^{2}+m^{2}(p \cdot q)^{2}-2(k \cdot p)(k \cdot q)(p \cdot q)}{\omega^{2}-m^{2}}\right) \mathrm{S}_{1}+2 q^{2}(p \cdot q) \mathrm{S}_{2}}{\left(q^{4}-4(k \cdot q)^{2}\right) q^{2}}$,

$$
\begin{align*}
f_{-}^{2 \gamma}(\omega)= & -\frac{8 \alpha}{M^{3}} \int \frac{i \mathrm{~d}^{4} q}{\pi^{2}} \\
& \frac{\left(M^{2} q^{2}+\frac{M^{2}(k \cdot q)^{2}+m^{2}(p \cdot q)^{2}-2(k \cdot p)(k \cdot q)(p \cdot q)}{\omega^{2}-m^{2}}\right)(k \cdot p) \mathrm{S}_{1}+M^{2}(k \cdot q)\left((p \cdot q) \mathrm{S}_{1}+q^{2} \mathrm{~S}_{2}\right)}{\left(q^{4}-4(k \cdot q)^{2}\right) q^{2}} . \tag{2.93}
\end{align*}
$$

These expressions in the proton rest frame at threshold are given by

$$
\begin{align*}
f_{+}^{2 \gamma}(m) & =8 \alpha M m^{2} \int \frac{i \mathrm{~d}^{4} q}{\pi^{2}} \frac{\left(Q^{2}-2 \nu_{\gamma}^{2}\right) \mathrm{T}_{1}\left(\nu_{\gamma}, Q^{2}\right)-\left(\nu_{\gamma}^{2}+Q^{2}\right) \mathrm{T}_{2}\left(\nu_{\gamma}, Q^{2}\right)}{\left(Q^{4}-4 m^{2} \nu_{\gamma}^{2}\right) Q^{4}}  \tag{2.94}\\
f_{-}^{2 \gamma}(m) & =-g^{2 \gamma}(m)=\frac{8 m \alpha}{3 M} \int \frac{i \mathrm{~d}^{4} q}{\pi^{2}} \frac{\left(\nu_{\gamma}^{2}-2 Q^{2}\right) M \mathrm{~S}_{1}\left(\nu_{\gamma}, Q^{2}\right)-3 Q^{2} \nu_{\gamma} \mathrm{S}_{2}\left(\nu_{\gamma}, Q^{2}\right)}{\left(Q^{4}-4 m^{2} \nu_{\gamma}^{2}\right) Q^{2}} \tag{2.95}
\end{align*}
$$

The result for the spin-independent amplitude $f_{+}^{2 \gamma}(m)$ is in agreement with Ref. [135]. While the result for the spin-dependent amplitude $f_{-}^{2 \gamma}(m)$, multiplied by a factor $3 / 2$, coincide with the HFS correction of Refs. [53, 137, 151-160].

It is instructive to compare the HFS evaluation cutting only the lower blob of the TPE graph with the DR evaluation of Sections 2.2 and 2.3, see Fig. 2.10.


Figure 2.10: Forward elastic $l p$ scattering with cut of both fermions lines (left panel) and with cut of the proton line only (right panel).

Exploiting the DRs for the spin-dependent Compton amplitudes $\mathrm{S}_{1}$ and $\mathrm{S}_{2}$ of Eqs. (2.62, 2.63) and performing the Wick rotation we obtain the same expression for the amplitude $f_{-}^{2 \gamma}(m)$ at threshold, see Eq. (2.47), as with DRs for the forward $l p$ amplitudes. However, if one uses the DR for the amplitude $\nu_{\gamma} \mathrm{S}_{2}$ [88], the result coincide with the amplitude expression $-g^{2 \gamma}(m)$ of Eq. (2.48).

Evaluating the sum of the spin-dependent lepton-proton TPE amplitudes in the DR approach, see Eqs. (2.47, 2.48), we obtain:

$$
\begin{equation*}
g^{2 \gamma}(m)+f_{-}^{2 \gamma}(m)=64 \alpha^{2} M m \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{4}} \rho\left(\tau_{l}\right) \int_{0}^{1} \mathrm{~d} x_{\mathrm{Bj}} g_{2}\left(x_{\mathrm{Bj}}, Q^{2}\right)=0, \tag{2.96}
\end{equation*}
$$

that is a trivial relation due to the Burkhardt-Cottingham (BC) sum rule of Eq. (2.76).
We proved that the forward TPE amplitudes evaluated within the DR approach for the $l p$ scattering coincide with the amplitudes evaluated with a help of DRs for the forward VVCS process when accounting for the BC sum rule. However, the contribution of individual TPE intermediate states differs in these two approaches, which we will study in detail in Section 2.6.2.

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### 2.6 TPE corrections to hydrogen energy levels

In the $l p$ center-of-mass reference frame the TPE forward scattering amplitude $T^{2 \gamma}$ is expressed in terms of the invariant amplitudes $f_{+}^{2 \gamma}, f_{-}^{2 \gamma}, g^{2 \gamma}$ as

$$
\begin{equation*}
T^{2 \gamma}(\omega)=f_{+}^{2 \gamma}(\omega)+4 g^{2 \gamma}(\omega) s \cdot \boldsymbol{S}+4\left(f_{-}^{2 \gamma}(\omega)+g^{2 \gamma}(\omega)\right) s \cdot \hat{\boldsymbol{k}} \boldsymbol{S} \cdot \hat{\boldsymbol{p}}, \tag{2.97}
\end{equation*}
$$

with $\boldsymbol{s}(\boldsymbol{S})$ and $\hat{\boldsymbol{k}}(\hat{\boldsymbol{p}})$ the lepton (proton) spin and momentum direction vectors. This decomposition often arises in the analysis of the non-relativistic forward neutron-proton scattering, see e.g. Ref. [179].

### 2.6.1 TPE correction to Lamb shift in hydrogen-like atoms

It is then easy to see that, considering $T^{2 \gamma}$ as correction to the Coulomb potential, its effect on the nS-state energy level is given by

$$
\begin{equation*}
\Delta E_{\mathrm{nS}}=-\frac{\left|\psi_{\mathrm{nS}}(0)\right|^{2}}{4 M m} f_{+}^{2 \gamma}(m), \tag{2.98}
\end{equation*}
$$

with $\left|\psi_{\mathrm{nS}}(0)\right|^{2}=\alpha^{3} m_{r}^{3} /\left(\pi n^{3}\right)$ - the non-relativistic squared wave function of the hydrogen atom, where $m_{r}=M m /(M+m)$ is the reduced mass of the lepton and proton bound state. Using the DR for $f_{+}^{2 \gamma}$ with only the elastic part of the unpolarized cross section and subtracting the accounted TPE contribution in hydrogen wave functions, as well as the OPE finite size correction, we reproduce the non-relativistic limit of the TPE contribution:

$$
\begin{equation*}
\Delta E_{\mathrm{nS}}=-\frac{8 m_{r}^{4} \alpha^{5}}{\pi n^{3}} \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{5}}\left(G_{E}^{2}\left(Q^{2}\right)-2 G_{E}^{\prime}(0) Q^{2}-1\right) \tag{2.99}
\end{equation*}
$$

This correction yields the third Zemach moment term [88].
The TPE contribution due to the subtraction function provides a correction to the $2 \mathrm{~S}-$ 2P muonic hydrogen Lamb shift, which is the largest hadronic uncertainty in this precise quantity $[113,114]$. Using the ChPT based results for $\beta\left(Q^{2}\right)$ as input, this TPE correction was estimated in Refs. $[141,142]$ and found to be too small to resolve the proton radius puzzle. According to Eqs. $(2.94,2.98)$ this correction is given by

$$
\begin{equation*}
\Delta E_{2 \mathrm{~S}}^{\text {subt }}=-2 \alpha\left|\psi_{\mathrm{nS}}(0)\right|^{2} m \int \frac{i \mathrm{~d}^{4} q}{\pi^{2}} \frac{\left(Q^{2}-2 \nu_{\gamma}^{2}\right) \mathrm{T}_{1}^{\text {subt }}\left(0, Q^{2}\right)}{\left(Q^{4}-4 m^{2} \nu_{\gamma}^{2}\right) Q^{4}} \tag{2.100}
\end{equation*}
$$

Performing the Wick rotation and hyper-angular integration it can be expressed as [135]

$$
\begin{equation*}
\Delta E_{2 \mathrm{~S}}^{\mathrm{subt}}=4 \alpha\left|\psi_{\mathrm{nS}}(0)\right|^{2} \int_{0}^{\infty} \frac{\mathrm{d} Q}{Q^{2}}\left(2 \rho\left(\tau_{l}\right) \sqrt{\tau_{l}}+\sqrt{1+\tau_{l}}\right) \mathrm{T}_{1}^{\mathrm{subt}}\left(0, Q^{2}\right) . \tag{2.101}
\end{equation*}
$$

The correction from the empirically determined subtraction function in Section 2.4.1 to the 2 S energy level in the muonic hydrogen, after integration up to $Q^{2}=1 \mathrm{GeV}^{2}$, yields: ${ }^{4}$

$$
\begin{equation*}
\Delta E_{2 \mathrm{~S}}^{\text {subt }} \approx 2.3 \pm 1.3 \mu \mathrm{eV} \tag{2.102}
\end{equation*}
$$

which is in fair agreement, though slightly smaller than the estimate of Birse et al. [141]: $\Delta E_{2 \mathrm{~S}}^{\text {subt }} \approx 4.2 \pm 1.0 \mu \mathrm{eV}$. Our result of Eq. (2.102) is also within errors of the analogous evaluation of Ref. [136], where authors assumed the existence of a $J=0$ fixed pole.

[^7]
### 2.6.2 TPE correction to hyperfine splitting (HFS) in hydrogenlike atoms

The TPE contribution to the S-level HFS $\delta E_{\text {ns }}^{\mathrm{HFS}}$ is expressed in terms of the relative correction $\Delta_{\mathrm{HFS}}$ and the leading order S-level HFS $E_{\mathrm{nS}}^{\mathrm{HFS}, 0}$ (Fermi energy) as

$$
\begin{align*}
\delta E_{\mathrm{nS}}^{\mathrm{HFS}} & =\Delta_{\mathrm{HFS}} E_{\mathrm{nS}}^{\mathrm{HFS}, 0}  \tag{2.103}\\
E_{\mathrm{nS}}^{\mathrm{HFS}, 0} & =\frac{8}{3} \frac{m_{r}^{3} \alpha^{4}}{M m} \frac{\mu_{P}}{n^{3}}, \tag{2.104}
\end{align*}
$$

with the proton magnetic moment $\mu_{P} \approx 2.793$.
Considering the spin part of $T^{2 \gamma}$ as correction to the Hamiltonian of the lepton-proton spinspin interaction, we express the leading TPE proton structure correction to the S-level HFS in terms of the amplitudes $f_{-}^{2 \gamma}, g^{2 \gamma}$ at threshold $(\omega=m)$ :

$$
\begin{equation*}
\mu_{P} e^{2} \Delta_{\mathrm{HFS}}=-g^{2 \gamma}(m)+\frac{1}{2} f_{-}^{2 \gamma}(m) \tag{2.105}
\end{equation*}
$$

The TPE correction to the S-level HFS $\Delta_{0}$ of Refs. [53,137,152-160] can be obtained adding $g^{2 \gamma}(m)+f_{-}^{2 \gamma}(m)=0$ to Eq. (2.105):

$$
\begin{equation*}
\mu_{P} e^{2} \Delta_{0}=\frac{3}{2} f_{-}^{2 \gamma}(m) \tag{2.106}
\end{equation*}
$$

Consequently, we have verified the TPE correction to HFS of S energy levels. In the following, we study the difference in the individual channel contribution to HFS correction between the traditional HFS expressions and the DR approach based on the forward lepton-proton amplitudes.
Traditionally the proton intermediate state TPE correction to HFS $\Delta_{0}^{\text {el }}$ is expressed as a sum of the Zemach correction $\Delta_{\mathrm{Z}}$, with subtraction of the TPE contribution which is already accounted for in the hydrogen wave functions, and the recoil correction $\Delta_{\mathrm{R}}^{\mathrm{p}}$ :

$$
\begin{align*}
\Delta_{0}^{\mathrm{el}}= & \Delta_{\mathrm{z}}+\Delta_{\mathrm{R}}^{\mathrm{p}},  \tag{2.107}\\
\Delta_{\mathrm{Z}}= & \frac{8 \alpha m_{r}}{\pi \mu_{P}} \int_{0}^{\infty} \frac{\mathrm{d} Q}{Q^{2}}\left(G_{M}\left(Q^{2}\right) G_{E}\left(Q^{2}\right)-\mu_{P}\right),  \tag{2.108}\\
\Delta_{\mathrm{R}}^{\mathrm{p}}= & \frac{\alpha}{\pi \mu_{P}} \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}}\left\{\frac{\left[2+\rho\left(\tau_{l}\right) \rho\left(\tau_{P}\right)\right] F_{D}\left(Q^{2}\right)+3 \rho\left(\tau_{l}\right) \rho\left(\tau_{P}\right) F_{P}\left(Q^{2}\right)}{\sqrt{\tau_{P}} \sqrt{1+\tau_{l}}+\sqrt{\tau_{l}} \sqrt{1+\tau_{P}}}-\frac{4 m_{r}}{Q} G_{E}\left(Q^{2}\right)\right\} \\
& \quad \times G_{M}\left(Q^{2}\right)-\frac{\alpha}{\pi \mu_{P}} \frac{m}{M} \int_{0}^{\infty} \frac{\mathrm{d} Q}{Q} \beta_{1}\left(\tau_{l}\right) F_{P}^{2}\left(Q^{2}\right), \tag{2.109}
\end{align*}
$$

with $\beta_{1}(\tau)=-3 \tau+2 \tau^{2}+2(2-\tau) \sqrt{\tau(1+\tau)}$.
We express the elastic TPE contribution to the S-level HFS $\Delta_{\text {HFS }}^{\mathrm{el}}$ in the lepton-proton amplitudes DR framework, with subtraction of the TPE contribution which is already accounted for in the hydrogen wave functions, in terms of the proton electric and magnetic form factors as

$$
\begin{equation*}
\Delta_{\mathrm{HFS}}^{\mathrm{el}}=\frac{\alpha}{\pi \mu_{P}} \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}}\left\{\frac{2 G_{E}\left(Q^{2}\right)+\rho\left(\tau_{l}\right) \rho\left(\tau_{P}\right) G_{M}\left(Q^{2}\right)}{\sqrt{\tau_{P}} \sqrt{1+\tau_{l}}+\sqrt{\tau_{l}} \sqrt{1+\tau_{P}}} G_{M}\left(Q^{2}\right)-\frac{4 \mu_{P} m_{r}}{Q}\right\} \tag{2.110}
\end{equation*}
$$

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We reproduce the Zemach correction [151] as the non-relativistic limit of the elastic HFS correction $\Delta_{\mathrm{HFS}}^{\mathrm{el}}$. The proton non-pole term $\Delta_{\mathrm{HFS}}^{\mathrm{F}_{\mathrm{F}}^{2}}$ arising from the non-pole part of the Born amplitude $\mathrm{S}_{1}^{\text {Born }}$ of Eq. (2.68) is given by

$$
\begin{equation*}
\Delta_{\mathrm{HFS}}^{\mathrm{F}_{\mathrm{F}}^{2}}=\frac{\alpha}{\pi \mu_{P}} \frac{m}{M} \int_{0}^{\infty} \frac{\mathrm{d} Q}{Q} \beta_{1}\left(\tau_{l}\right) F_{P}^{2}\left(Q^{2}\right), \tag{2.111}
\end{equation*}
$$

was eliminated from the TPE contribution to HFS in Refs. [159, 160], see last term in Eq. (2.107). In our approach the non-pole term does not appear, and therefore does not need to be subtracted by hand. The remaining difference between expressions of Eq. (2.107) and Eq. (2.110) is given by the elastic contribution to the amplitude $g^{2 \gamma}(m)+f_{-}^{2 \gamma}(m)$.

Traditionally the polarizability correction $\Delta_{0}^{\mathrm{pol}}$ is given by [159, 160]

$$
\begin{align*}
\Delta_{0}^{\mathrm{pol}}= & \frac{2 \alpha}{\pi \mu_{P}} \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}} \int_{\nu}^{\infty} \frac{\mathrm{d} \nu_{\gamma}}{\nu_{\gamma}} \frac{\left[2+\rho\left(\tau_{l}\right) \rho(\tilde{\tau})\right] g_{1}\left(\nu_{\gamma}, Q^{2}\right)-3 \rho\left(\tau_{l}\right) \rho(\tilde{\tau}) g_{2}\left(\nu_{\gamma}, Q^{2}\right) / \tilde{\tau}}{\sqrt{\tilde{\tau}} \sqrt{1+\tau_{l}}+\sqrt{\tau_{l}} \sqrt{1+\tilde{\tau}}} \\
& +\frac{\alpha}{\pi \mu_{P}} \frac{m}{M} \int_{0}^{\infty} \frac{\mathrm{d} Q}{Q} \beta_{1}\left(\tau_{l}\right) F_{P}^{2}\left(Q^{2}\right) . \tag{2.112}
\end{align*}
$$

We express the inelastic $\alpha^{5}$-correction to the S-level HFS in the lepton-proton amplitudes DR approach in terms of the proton inelastic spin SFs $g_{1}$ and $g_{2}$ as

$$
\begin{equation*}
\Delta_{\mathrm{HFS}}^{\mathrm{inel}}=\frac{2 \alpha}{\pi \mu_{P}} \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}} \int_{\substack{\nu_{\mathrm{inel}}}}^{\infty} \frac{\mathrm{d} \nu_{\gamma}}{\nu_{\gamma}} \frac{\left[2+\rho\left(\tau_{l}\right) \rho(\tilde{\tau})\right] g_{1}\left(\nu_{\gamma}, Q^{2}\right)+\left[2-\rho\left(\tau_{l}\right) \rho(\tilde{\tau}) / \tilde{\tau}\right] g_{2}\left(\nu_{\gamma}, Q^{2}\right)}{\sqrt{\tilde{\tau}} \sqrt{1+\tau_{l}}+\sqrt{\tau_{l}} \sqrt{1+\tilde{\tau}}}, \tag{2.113}
\end{equation*}
$$

It differs from $\Delta_{0}^{\mathrm{pol}}$ by the absence of the $\Delta_{\mathrm{HFS}}^{\mathrm{F}_{\mathrm{F}}^{2}}$ contribution, which allows to expand the HFS integrand near $Q^{2}=0$ in terms of polarizabilities, and by the contribution from the spin SF $g_{2}$ to the amplitude $g^{2 \gamma}(m)+f_{-}^{2 \gamma}(m)$.

### 2.6.3 Polarizability correction evaluation

For the numerical evaluation of the polarizability correction we subtract the leading moment of the spin structure function $g_{1}$ and separate contributions from the $g_{1}$ and $g_{2}$ structure functions as [88]

$$
\begin{align*}
\Delta_{0}^{\mathrm{pol}}= & \Delta_{1}^{\mathrm{pol}}+\Delta_{2}^{\mathrm{pol}}  \tag{2.114}\\
\Delta_{1}^{\mathrm{pol}}= & \frac{\alpha}{\pi \mu_{P}} \frac{m}{M} \int_{0}^{\infty} \frac{\mathrm{d} Q}{Q} \beta_{1}\left(\tau_{l}\right)\left\{4 I_{1}\left(Q^{2}\right)+F_{P}^{2}\left(Q^{2}\right)\right\} \\
& +\frac{2 \alpha}{\pi \mu_{P}} \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}} \int_{\nu_{\text {inhl }}}^{\infty} \frac{\mathrm{d} \nu_{\gamma}}{\nu_{\gamma}}\left\{\frac{2+\rho\left(\tau_{l}\right) \rho(\tilde{\tau})}{\sqrt{\tilde{\tau}} \sqrt{1+\tau_{l}}+\sqrt{\tau_{l}} \sqrt{1+\tilde{\tau}}}-\frac{m \beta_{1}\left(\tau_{l}\right)}{\nu_{\gamma}}\right\} g_{1}\left(\nu_{\gamma}, Q^{2}\right),  \tag{2.115}\\
\Delta_{2}^{\mathrm{pol}}= & -\frac{6 \alpha}{\pi \mu_{P}} \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}} \int_{\nu_{\text {inh }}^{\text {ind }}}^{\infty} \frac{\mathrm{d} \nu_{\gamma}}{\nu_{\gamma} \tilde{\tau}} \frac{\rho\left(\tau_{l}\right) \rho(\tilde{\tau}) g_{2}\left(\nu_{\gamma}, Q^{2}\right)}{\sqrt{\tilde{\tau}} \sqrt{1+\tau_{l}}+\sqrt{\tau_{l}} \sqrt{1+\tilde{\tau}}} \tag{2.116}
\end{align*}
$$

with the first moment of the $g_{1}$ structure function given by

$$
\begin{equation*}
I_{1}\left(Q^{2}\right)=\frac{2 M^{2}}{Q^{2}} \int_{0}^{x_{0}} g_{1}\left(x, Q^{2}\right) \mathrm{d} x, \quad I_{1}(0)=-\frac{\left(\mu_{P}-1\right)^{2}}{4} \tag{2.117}
\end{equation*}
$$

In order to evaluate the contribution from $4 I_{1}+F_{P}^{2}$ we approximate $I_{1}\left(Q^{2}\right)=I_{1}(0)+$ $I^{\prime}{ }_{1}(0) Q^{2}$ up to $Q_{\mathrm{I}_{1}}=0.25 \mathrm{GeV}$, with the low energy constant $I^{\prime}{ }_{1}(0)=(7.6 \pm 2.5) \mathrm{GeV}^{-2}$ [185], and afterward we exploit the spin SFs data parametrization from Refs. [92, 186] (JLab parametrization). We show the correspondent $Q^{2}$ dependence of $4 I_{1}+F_{P}^{2}$ in Fig. 2.11.


Figure 2.11: JLab HFS integrand from $4 I_{1}+F_{P}^{2}$, corresponding with the first integral in Eq. (2.115), connected to the low- $Q^{2}$ behavior. Left panel: electronic hydrogen, right panel: muonic hydrogen.

For the remaining polarizability correction $\Delta_{1}^{\mathrm{pol}}$ and $\Delta_{2}^{\mathrm{pol}}$ coming from $g_{1}$ and $g_{2}$ we use the JLab parametrization only, which is in a fair agreement with the MAID model [94] in the region of small $Q^{2}$, see Fig. 2.12 for details.


Figure 2.12: MAID and JLab integrand in the $g_{1}$ integral of Eq. (2.115) and the $g_{2}$ integral of Eq. (2.116). Left panel: electronic hydrogen, right panel: muonic hydrogen.

We add the uncertainties coming from the Pauli form factor $F_{P}$ [104], spin structure functions $g_{1}, g_{2}$ and the parameter $I^{\prime}{ }_{1}(0)$ in quadrature under the HFS integrand and treat the uncertainties coming from the two $Q$ integration regions in the $\Delta_{1}$ and $4 I_{1}+F_{P}^{2}$ evaluations as uncorrelated uncertainties.

We present the results for the HFS correction to the 1 S energy level in $e \mathrm{H}(2 \mathrm{~S}$ in $\mu \mathrm{H})$ in Table 2.1 (2.2). Though the contributions from the $g_{1}$ and $g_{2}$ structure functions slightly differ with a previous evaluation of Ref. [160], the resulting polarizability correction is in a fair agreement

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with results of Ref. [160]: $\Delta_{0}^{\mathrm{pol}}=11.0 \pm 3.8 \mathrm{peV}$ in $e \mathrm{H}$ and $\Delta_{0}^{\mathrm{pol}}=8.0 \pm 2.6 \mu \mathrm{eV}$ in $\mu \mathrm{H}$. The source of small difference in the central value can be in the overestimated polarizability correction due to the large phenomenological value of $I^{\prime}{ }_{1}(0)$, which does not match the data in Fig. 2.11.

| $e \mathrm{H}$ | $4 I_{1}+F_{P}^{2}$ | $g_{1}$ | $\Delta_{1}^{\text {pol }}$ | $\Delta_{2}^{\text {pol }}$ | $\Delta_{0}^{\text {pol }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\Delta E_{1 S}, \mathrm{peV}$ | $11.6 \pm 2.8$ | $1.17 \pm 0.65$ | $13.8 \pm 2.8$ | $-2.61 \pm 0.8$ | $11.2 \pm 3.0$ |

Table 2.1: Contributions to the 1S HFS correction in $e \mathrm{H}$.

| $\mu \mathrm{H}$ | $4 I_{1}+F_{P}^{2}$ | $g_{1}$ | $\Delta_{1}^{\text {pol }}$ | $\Delta_{2}^{\mathrm{pol}}$ | $\Delta_{0}^{\mathrm{pol}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\Delta E_{2 S}, \mu \mathrm{eV}$ | $9.17 \pm 2.09$ | $0.61 \pm 0.35$ | $9.78 \pm 1.91$ | $-1.48 \pm 0.44$ | $8.30 \pm 1.96$ |

Table 2.2: Contributions to the 2 S HFS correction in $\mu \mathrm{H}$.

Within the dispersion relation approach, see Eqs. (2.23, 2.106), we express the polarizability correction $\Delta_{0}^{\text {pol }}$ directly in terms of the measurable inclusive inelastic $l p$ cross sections difference as

$$
\begin{equation*}
\Delta_{0}^{\mathrm{pol}}=\frac{3 M m}{\pi e^{2} \mu_{P}} \int_{\omega_{\mathrm{thr}}}^{\infty} \frac{\sigma_{++}^{\mathrm{inel}}\left(\omega^{\prime}\right)-\sigma_{+-}^{\mathrm{inel}}\left(\omega^{\prime}\right)}{\sqrt{\omega^{\prime 2}-m^{2}}} \mathrm{~d} \omega^{\prime}+\frac{\alpha}{\pi \mu_{P}} \frac{m}{M} \int_{0}^{\infty} \frac{\mathrm{d} Q}{Q} \beta_{1}\left(\tau_{l}\right) F_{P}^{2}\left(Q^{2}\right) \tag{2.118}
\end{equation*}
$$

with the pion production threshold $\omega_{\text {thr }}=m+m_{\pi}\left(2 M+2 m+m_{\pi}\right) /(2 M)$.

### 2.6.4 Resulting HFS correction evaluation

For the numerical evaluation of the HFS corrections from the proton intermediate state and the $\Delta_{\mathrm{HFS}}^{\mathrm{F}_{\mathrm{F}}^{2}}$ part of the polarizability correction $\Delta_{0}^{\text {pol }}$ we exploit the elastic form factor parametrizations from Refs. [67,104]. We make two evaluations for the $1-\sigma$ band curves coming from the elastic proton form factor uncertainties of Ref. [104], where a global analysis of form factor data with account of TPE corrections for $Q^{2}<10 \mathrm{GeV}^{2}$ was performed. We estimate the uncertainty as a half of a difference between these two curves. For the numerical evaluation of the inelastic HFS correction we exploit the spin SFs data parametrization from Refs. [92, 186] in the region of large $Q^{2}$. In the region of low- $Q^{2}$ we expand the $Q^{2}$-integrand from the proton spin SFs in terms of small $x$ and account for the leading non-vanishing moments:

$$
\begin{align*}
\Delta_{0}^{\mathrm{pol}} & \rightarrow \frac{\alpha}{2 \pi} \int_{0} \mathrm{~d} Q^{2} \frac{\rho\left(\tau_{l}\right)\left(\rho\left(\tau_{l}\right)-4\right)}{\mu_{P} M m \tau_{l}} I_{1}\left(Q^{2}\right)+\frac{\alpha}{8 \pi} \int_{0} \mathrm{~d} Q^{2} \frac{\left(9-2 \rho\left(\tau_{l}\right)\right) \rho\left(\tau_{l}\right)^{2}}{\mu_{P} M m \tau_{l}} I_{1}^{(3)}\left(Q^{2}\right) \\
& -\frac{3 \alpha}{2 \pi} \int_{0} \mathrm{~d} Q^{2} \frac{1+2 \rho\left(\tau_{l}\right)}{\mu_{P} M m} I_{2}^{(3)}\left(Q^{2}\right)  \tag{2.119}\\
\Delta_{\mathrm{HFS}}^{\mathrm{innl}} & \rightarrow \Delta_{0}^{\mathrm{pol}}-\frac{2 \alpha}{\pi} \int_{0} \mathrm{~d} Q^{2} \frac{\rho\left(\tau_{l}\right)}{\mu_{P} M m \tau_{l}} I_{2}\left(Q^{2}\right), \tag{2.120}
\end{align*}
$$

with the moments of the proton spin SFs:

$$
\begin{equation*}
I_{2}\left(Q^{2}\right)=\frac{2 M^{2}}{Q^{2}} \int_{0}^{x_{0}} g_{2}\left(x, Q^{2}\right) \mathrm{d} x=\frac{1}{4} F_{P}\left(Q^{2}\right) G_{M}\left(Q^{2}\right), \tag{2.121}
\end{equation*}
$$

$$
\begin{align*}
& I_{1}^{(3)}\left(Q^{2}\right)=\frac{8 M^{4}}{Q^{4}} \int_{0}^{x_{0}} x^{2} g_{1}\left(x, Q^{2}\right) \mathrm{d} x \underset{Q^{2} \rightarrow 0}{\longrightarrow} \frac{Q^{2} M^{2}}{2 \alpha} \gamma_{0},  \tag{2.122}\\
& I_{2}^{(3)}\left(Q^{2}\right)=\frac{8 M^{4}}{Q^{4}} \int_{0}^{x_{0}} x^{2} g_{2}\left(x, Q^{2}\right) \mathrm{d} x \underset{Q^{2} \rightarrow 0}{\longrightarrow} \frac{Q^{2} M^{2}}{2 \alpha}\left(\delta_{\mathrm{LT}}-\gamma_{0}\right), \tag{2.123}
\end{align*}
$$

with $x_{0}=Q^{2} /\left(2 M \nu_{\mathrm{thr}}^{\mathrm{inel}}\right)$ and the low energy constants values [185]:

$$
\begin{align*}
\delta_{\mathrm{LT}} & =(1.34 \pm 0.17) \times 10^{-4} \mathrm{fm}^{4}  \tag{2.124}\\
\gamma_{0} & =(-1.01 \pm 0.13) \times 10^{-4} \mathrm{fm}^{4}  \tag{2.125}\\
{I^{\prime}}_{1}(0) & =(7.6 \pm 2.5) \mathrm{GeV}^{-2} \tag{2.126}
\end{align*}
$$

In Fig. 2.13 we show the integrand $I_{\mathrm{HFS}}(Q)$ entering the HFS correction:

$$
\begin{equation*}
\Delta_{\mathrm{HFS}}=\int_{0}^{\infty} I_{\mathrm{HFS}}(Q) \mathrm{d} Q \tag{2.127}
\end{equation*}
$$



Figure 2.13: Q-dependence of the integrand $I_{\mathrm{HFS}}(Q)$ entering the HFS. A comparison is given of the integrand based on DR for the $l p$ amplitudes and based on Compton amplitudes (SF). Left panel: electronic hydrogen, right panel: muonic hydrogen.
in the case of $e \mathrm{H}$ and $\mu \mathrm{H}$. The low- $Q$ behavior based on the moments of the proton spin SFs of Eqs. (2.117, 2.121-2.123) $I_{\mathrm{HFS}}^{\mathrm{sr}}$ and the high- $Q$ behavior based on the data $I_{\mathrm{HFS}}^{\mathrm{d}}$ are almost independent of the way to evaluate the HFS. While in the region $0.2 \mathrm{GeV} \lesssim Q \lesssim 0.5 \mathrm{GeV}$ the HFS evaluation with DRs for the lepton-proton amplitudes ( $\left.\Delta_{\mathrm{HFS}}=\Delta_{\mathrm{HFS}}^{\mathrm{el}}+\Delta_{\mathrm{HFS}}^{\mathrm{inel}}\right)$ and the traditional HFS evaluation ( $\Delta_{\mathrm{HFS}}=\Delta_{0}=\Delta_{0}^{\mathrm{el}}+\Delta_{0}^{\mathrm{pol}}$ ) slightly differ (for $Q>0.5 \mathrm{GeV}$ both methods agree within $2.5 \%$ ). The available parametrization of the proton spin SFs [186] (JLab) satisfy the Burkhardt-Cottingham sum rule sufficiently enough. New data in this kinematical region will be also very useful for the HFS evaluation. In order to avoid any model dependence, we connect the two model-independent regions by the function of the Fermi-Dirac distribution type:

$$
\begin{align*}
I_{\mathrm{HFS}}(Q)= & I_{\mathrm{HFS}}^{\mathrm{sr}}(Q) \Theta\left(Q_{\mathrm{sr}}-Q\right)+I_{\mathrm{HFS}}^{\mathrm{d}}(Q) \Theta\left(Q-Q_{\mathrm{d}}\right)+ \\
& \frac{c_{1} I_{\mathrm{HFS}}(Q)+c_{2} f(Q) I_{\mathrm{HFS}}^{\mathrm{d}}(Q)}{1+f(Q)} \Theta\left(Q-Q_{\mathrm{sr}}\right) \Theta\left(Q_{\mathrm{d}}-Q\right), \tag{2.128}
\end{align*}
$$

with $f(Q)$ given by

$$
\begin{equation*}
f(Q)=e^{\frac{2 Q-Q_{\mathrm{st}}-Q_{\mathrm{d}}}{2 a_{0}}} \tag{2.129}
\end{equation*}
$$

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Furthermore, $Q_{\mathrm{sr}}=0.2 \mathrm{GeV}, Q_{\mathrm{d}}=0.5 \mathrm{GeV}, a_{0}=0.1 \mathrm{GeV}$, and the constants $c_{1}, c_{2}, Q_{\mathrm{sr}}, a_{0}$ were chosen as those that preserve the regularity and smoothness of the integrand $I_{\mathrm{HFS}}(Q)$.

In the low- $Q$ region, we make two evaluations for the $1-\sigma$ bands of the elastic proton form factors from Ref. [104]. We add the combined uncertainty from $\gamma_{0}, \delta_{L T}, I^{\prime}{ }_{1}(0)$ linearly. For the larger $Q>(0.013-0.017) \mathrm{GeV}$ region, we make evaluation for the central values of the proton elastic form factors and add the uncertainty of the proton elastic form factors to the uncertainties from $\gamma_{0}, \delta_{L T}, I^{\prime}{ }_{1}(0)$ in quadrature. The boundary $Q$ value is chosen as the value that leads to the same uncertainties in the proton intermediate state HFS contribution in both ways of the error estimate described in this paragraph. For the larger $Q>0.5 \mathrm{GeV}$ region, we add the uncertainty of the proton spin structure function parametrization in quadrature to the uncertainties coming from the proton elastic form factors. We connect the high- $Q$ integrands $I^{\mathrm{d}}$ by two curves to the $1-\sigma$ boundaries in the low- $Q$ region $I^{\text {sr }}$. We estimate the uncertainty from the difference between the integral of Eq. (2.127) for these two curves, which are shown in Fig. 2.14 for $e \mathrm{H}(\mu \mathrm{H})$, and take the averaged central value. In the region $Q^{2}>10 \mathrm{GeV}^{2}$, the sizable contribution comes only from the $\mu_{P}$ term in Eqs. (2.108-2.110) and doesn't introduce any sizable additional uncertainty.


Figure 2.14: HFS integrand $I_{\mathrm{HFS}}(Q)$ in the evaluation by DRs for $l p$ amplitudes with error bands. Left panel: electronic hydrogen, right panel: muonic hydrogen.

We evaluate the proton TPE correction to the HFS in the DR approach either for Compton amplitudes or for lepton-proton amplitudes and present results for the Zemach correction, recoil correction and the $\Delta_{\mathrm{HFS}}^{\mathrm{F}}$ (contribution in Table 2.3. We also present the result of the polarizability correction $\Delta_{0}^{\text {pol }}$ evaluation described in Section 2.6.3.

|  | $10^{6} \Delta, e \mathrm{H}$ | $10^{3} \Delta, \mu \mathrm{H}$ |
| :---: | :---: | :---: |
| Zemach, $\Delta_{\mathrm{Z}}$ | $-39.59(75)$ | $-7.36(14)$ |
| Recoil, $\Delta_{\mathrm{R}}^{\mathrm{p}}$ | $5.31(12)$ | $0.8476(8)$ |
| Total elastic, $\Delta_{0}^{\text {el }}$ | $-34.29(75)$ | $-6.51(14)$ |
| Total elastic, $\Delta_{\mathrm{HFS}}^{\mathrm{el}}$ | $-43.40(74)$ | $-7.03(14)$ |
| Non-pole $\Delta_{\mathrm{H}}^{\mathrm{F}} \mathrm{HFS}$ | $22.53(7)$ | $1.11(1)$ |
| Polarizability, $\Delta_{0}^{\text {pol }}$ | $1.91(51)$ | $0.363(86)$ |
| Total $\Delta_{\text {HFS }}$ within $l p$ DRs | $-32.80(1.56)$ | $-6.22(29)$ |
| Total, $\Delta_{\text {HFS }}=\Delta_{0}^{\text {el }}+\Delta_{0}^{\text {pol }}[159]$ | $-32.38(91)$ | $-6.15(16)$ |

Table 2.3: Finite-size TPE correction to hyperfine splitting of S energy levels in $e \mathrm{H}$ and $\mu \mathrm{H}$. The Fermi energy HFS is $5.86785 \mu \mathrm{eV}$ for the 1 S level in $e \mathrm{H}$ and 182.4432 meV for the 1S level in $\mu \mathrm{H}$.

The evaluation of the resulting HFS correction is performed for the sum of elastic and inelastic contributions using the traditional expressions and the expressions based on the DRs for $l p$ amplitudes. The latter lead to twice smaller uncertainties. However, both evaluations agree within errors, which is a good test of the proton spin structure function $g_{2}$ parametrization for this calculation. The leading Zemach correction is a bit smaller than the evaluation based on the typical proton form factors parametrization in Ref. [159] due to the suppressed low $Q^{2}$ behavior of the magnetic proton form factor measured by the A1 Collaboration at MAMI [104]. Both ways of the TPE correction evaluation as a sum $\Delta_{0}^{\mathrm{el}}+\Delta_{0}^{\mathrm{pol}}$ and as a sum $\Delta_{\mathrm{HFS}}^{\mathrm{el}}+\Delta_{\mathrm{HFS}}^{\mathrm{inel}}$ are in agreement.

The evaluation of the elastic $\Delta_{0}^{\mathrm{el}}$ and polarizability $\Delta_{0}^{\mathrm{pol}}$ correction in sum [159] has smaller uncertainty than the evaluation of the total TPE correction described above. However, we do not account for the third moments of the proton spin structure functions in this case and also add uncertainties coming from two integration regions in quadrature, while in the first method the integration of the resulting uncertainty is performed.


Figure 2.15: Contribution of the $g_{2}$ structure function to the HFS integrand $I_{\mathrm{HFS}}(Q)$ in muonic hydrogen when the first moment or the third moment are replaced by the lowenergy constants, 0 and $\delta_{\mathrm{LT}}-\gamma_{0}$ respectively.


Figure 2.16: Left panel: Relative contribution of $4 I_{1}+F_{P}^{2}$ to the HFS integrand in muonic hydrogen. Right panel: Relative contribution of $4 I_{1}+F_{P}^{2}$ and third moments of the proton spin structure functions to the HFS integrand in muonic hydrogen.

Chapter 2 Forward lepton-proton scattering and two-photon exchange (TPE) corrections to atomic energy levels

Now we study the polarazability correction to HFS in $\mu \mathrm{H}$ in detail. The account of the third moments of the spin structure function in the low $Q^{2}$ integration region increases the polarizability correction. We obtain the result within uncertainties of our previous evaluation $\Delta_{0}^{\mathrm{pol}}=395 \pm 103 \mathrm{ppm}$. The larger central value of the polarizability correction is explained by the change of sign in the low- $Q^{2}$ behavior of the $g_{2}$ contribution, see Fig. 2.15, as well as by the increased contribution from the $g_{1}$ structure function. In the following Figs. 2.16 we compare the contribution of the $4 I_{1}+F_{P}^{2}$ as well as the contribution of the next term in the moments expansion with the total polarizability integrand. The main uncertainty comes from the pure knowledge of $I_{1}(0)^{\prime}$. The contribution from the low-energy constants $\gamma_{0}$ and $\delta_{\mathrm{LT}}$ is negligible and can not be distinguished in Figs. 2.16.

### 2.6.5 Overview of 1S HFS measurement in $\mu \mathrm{H}$

In view of the forthcoming high-precision measurement of the 1S HFS in muonic hydrogen at PSI with 1 ppm precision [148], we also provide the corresponding estimates of the HFS correction from the TPE graph in Table 2.4. The uncertainty of our estimate is 164 times larger than the expected experimental accuracy. The uncertainty of the polarizability contribution is 1.5 times smaller than the uncertainty of the Zemach term and dominated by pure knowledge of $I^{\prime}{ }_{1}(0)$. The forthcoming data from EG4, SANE and g2p experiments at JLab on the proton spin structure functions $g_{1}, g_{2}$ [187-189] will improve the knowledge of the polarizability correction. The precise measurements of the proton electric and magnetic form factors in the low- $Q^{2}$ region [190] will allow to decrease the uncertainty of the Zemach term.

|  | $\Delta(\mathrm{ppm}), \mu \mathrm{H}$ | uncertainty $(\mathrm{ppm})$ |
| :---: | :---: | :---: |
| Zemach, $\Delta_{\mathrm{Z}}$ | -7360 | 140 |
| Recoil, $\Delta_{\mathrm{R}}^{\mathrm{p}}$ | 847.6 | 0.8 |
| Polarizability, $\Delta_{0}^{\text {pol }}$ | 363 | 86 |
| Total, $\Delta_{\text {HFS }}=\Delta_{0}^{\text {el }}+\Delta_{0}^{\text {pol }}$ | -6149 | 164 |

Table 2.4: Finite-size TPE correction to hyperfine splitting of the $S$ energy levels in $\mu \mathrm{H}$.
The Zemach correction can be evaluated accounting for the measured values of the proton charge and magnetic radii. We split the $Q$-integration in the Zemach contribution at a small enough scale $Q_{0}$ and exploit the radii expansion at low $Q^{2}$ as

$$
\begin{equation*}
\Delta_{\mathrm{Z}}=\frac{8 \alpha m_{r}}{\pi}\left(\int_{Q_{0}}^{\infty} \frac{\mathrm{d} Q}{Q^{2}}\left(\frac{G_{M}\left(Q^{2}\right) G_{E}\left(Q^{2}\right)}{\mu_{P}}-1\right)-\frac{R_{E}^{2}+R_{M}^{2}}{6} Q_{0}\right), \tag{2.130}
\end{equation*}
$$

with the approximate value $Q_{0} \lesssim(0.2-0.4) \mathrm{GeV}$ and the definition of the proton radii:

$$
\begin{equation*}
R_{E(M)}^{2}=-\left.\frac{6}{G_{E(M)}(0)} \frac{\mathrm{d} G_{E(M)}\left(Q^{2}\right)}{\mathrm{d} Q^{2}}\right|_{Q^{2}=0} \tag{2.131}
\end{equation*}
$$

Consequently, after accounting for all contributions at the $1-10 \mathrm{ppm}$ level, the forthcoming measurement can constrain the low- $Q^{2}$ TPE contribution to HFS $\Delta_{\text {structure }}$ with the following combination of proton radii and $I^{\prime}{ }_{1}(0)$ :

$$
\begin{equation*}
\Delta_{\text {structure }}=-\frac{4 \alpha}{3 \pi}\left(m_{r} Q_{0}\left(R_{E}^{2}+R_{M}^{2}\right)+\frac{m}{M} \frac{h\left(\tau_{l}\right)}{\mu_{P}} I^{\prime}{ }_{1}(0) m^{2}\right), \tag{2.132}
\end{equation*}
$$

where $\tau_{l}$ is taken at the point $Q=Q_{\mathrm{I}_{1}} \sim(0.1-0.3) \mathrm{GeV}$ up to which we use the low energy expansion of $I_{1}\left(Q^{2}\right)$ and

$$
\begin{equation*}
h(\tau)=(9-4 \tau) \tau^{2}+\frac{15}{2} \ln (\sqrt{\tau}+\sqrt{1+\tau})-\frac{1}{2}\left(15+22 \tau-8 \tau^{2}\right) \sqrt{\tau(1+\tau)} \tag{2.133}
\end{equation*}
$$

## Chapter 3

## Elastic lepton-proton scattering and TPE corrections

In this Chapter, we describe the elastic lepton-proton scattering in the non-forward kinematics. We first discuss the kinematical variables. For the region of quite low energies, we describe the formalism of helicity amplitudes based on the discrete symmetries in QFT. We relate the forward and non-forward amplitudes and provide the crossing properties for the leptonproton scattering amplitudes. We write down the unpolarized cross section expressions in the case of the massive lepton-proton scattering in the OPE approximation and account for the leading TPE effects. Subsequently, we obtain the corresponding relations in the limit of small electron mass relevant for electron-proton scattering experiments. We give a description of the experimental knowledge of the elastic proton FFs at low momentum transfer. We also provide well-known expressions for some experimentally accessed polarization observables in the elastic electron-proton scattering with an account of TPE amplitudes and generalize them to the case of massive lepton.

### 3.1 Kinematics of massive lepton scattering

Elastic lepton-proton scattering $l(k, h)+p(p, \lambda) \rightarrow l\left(k^{\prime}, h^{\prime}\right)+p\left(p^{\prime}, \lambda^{\prime}\right)$, where $h\left(h^{\prime}\right)$ denote the incoming (outgoing) lepton helicities and $\lambda\left(\lambda^{\prime}\right)$ the corresponding proton helicities respectively, (see Fig. 3.1) is completely described by 2 Mandelstam variables, e.g., $Q^{2}=-\left(k-k^{\prime}\right)^{2}$ - the squared momentum transfer, and $s=(p+k)^{2}$ - the squared energy in the lepton-proton center-of-mass (c.m.) reference frame.


Figure 3.1: Elastic lepton-proton scattering.
The squared momentum transfer is expressed in terms of the lepton scattering angle $\theta_{\mathrm{cm}}$ in the c.m. reference frame by

$$
\begin{equation*}
Q^{2}=-\left(k-k^{\prime}\right)^{2}=\frac{\Sigma\left(s, M^{2}, m^{2}\right)}{2 s}\left(1-\cos \theta_{\mathrm{cm}}\right), \tag{3.1}
\end{equation*}
$$

with the kinematical triangle function $\Sigma\left(s, M^{2}, m^{2}\right) \equiv\left(s-(M+m)^{2}\right)\left(s-(M-m)^{2}\right)=\Sigma_{s}$.
In terms of the laboratory frame momenta $p=(M, 0), k=(\omega, \boldsymbol{k}), k^{\prime}=\left(\omega^{\prime}, \boldsymbol{k}^{\prime}\right), p^{\prime}=$ ( $E_{p}^{\prime}, \boldsymbol{k}-\boldsymbol{k}^{\prime}$ ) the invariant variables are expressed as

$$
\begin{equation*}
Q^{2}=2 M\left(\omega-\omega^{\prime}\right), \tag{3.2}
\end{equation*}
$$

$$
\begin{align*}
s & =M^{2}+2 M \omega+m^{2}  \tag{3.3}\\
\Sigma_{s} & =4 M^{2} \mathbf{k}^{2} \tag{3.4}
\end{align*}
$$

The laboratory frame scattering angle $\theta_{\text {lab }}$ and the momentum transfer are also given by

$$
\begin{align*}
\cos \theta_{\mathrm{lab}} & =\frac{\omega \omega^{\prime}-m^{2}-M\left(\omega-\omega^{\prime}\right)}{|\mathbf{k}|\left|\mathbf{k}^{\prime}\right|}  \tag{3.5}\\
Q^{2} & =2 M \frac{\mathbf{k}^{2}\left(M+\omega \sin ^{2} \theta_{\mathrm{lab}}-\sqrt{M^{2}-m^{2} \sin ^{2} \theta_{\mathrm{lab}}} \cos \theta_{\mathrm{lab}}\right)}{(\omega+M)^{2}-\mathbf{k}^{2} \cos ^{2} \theta_{\mathrm{lab}}} \tag{3.6}
\end{align*}
$$

For the MUSE muon beam momenta $|\mathbf{k}|=0.115 \mathrm{GeV},|\mathbf{k}|=0.153 \mathrm{GeV}$, and $|\mathbf{k}|=$ 0.210 GeV [127] the kinematically allowed momentum transfer is $0<Q^{2}<\Sigma_{s} / s$, or $0<$ $Q^{2}<0.039 \mathrm{GeV}^{2}, 0<Q^{2}<0.066 \mathrm{GeV}^{2}$ and $0<Q^{2}<0.116 \mathrm{GeV}^{2}$ respectively. For the scattering angles of the experiment $20^{\circ}<\theta_{\text {lab }}<100^{\circ}$ the momentum transfer varies in the region $0.0016-0.026 \mathrm{GeV}^{2}, 0.0028-0.045 \mathrm{GeV}^{2}$ and $0.0052-0.080 \mathrm{GeV}^{2}$ respectively. In the case of electron scattering with the same momenta and experimental scattering angles the momentum transfer varies in the region $0.0016-0.027 \mathrm{GeV}^{2}, 0.0028-0.046 \mathrm{GeV}^{2}$ and $0.0052-0.082 \mathrm{GeV}^{2}$.

It is convenient to introduce the averaged momentum variables $P=\left(p+p^{\prime}\right) / 2, \quad K=$ $\left(k+k^{\prime}\right) / 2$, the $u$-channel squared energy $u=\left(k-p^{\prime}\right)^{2}$ and the crossing symmetric variable $\nu=(s-u) / 4=(K \cdot P)$ which changes sign with $s \leftrightarrow u$ channel crossing. The crossing symmetric variable can be expressed in terms of the laboratory frame variables as $\nu=M\left(\omega+\omega^{\prime}\right) / 2$. Instead of the Mandelstam invariant $s$ or the crossing symmetric variable $\nu$, one can use the virtual photon polarization parameter $\varepsilon$. In terms of $Q^{2}$ and $\nu$ the photon polarization parameter is expressed as

$$
\begin{equation*}
\varepsilon=\frac{16 \nu^{2}-Q^{2}\left(Q^{2}+4 M^{2}\right)}{16 \nu^{2}-Q^{2}\left(Q^{2}+4 M^{2}\right)+2\left(Q^{2}+4 M^{2}\right)\left(Q^{2}-2 m^{2}\right)} \tag{3.7}
\end{equation*}
$$

It varies between $\varepsilon_{0}=2 m^{2} / Q^{2}$ and 1 for the fixed momentum transfer $Q^{2}>2 m^{2}$ and between 1 and $\varepsilon_{0}$ for the fixed momentum transfer $Q^{2}<2 m^{2}$. The high energy limit corresponds to $\varepsilon=1$. The value of the critical momentum transfer $Q^{2}=2 m^{2}$, corresponding to $\varepsilon=1$ for all possible beam momenta, is given by $Q^{2} \simeq 0.022 \mathrm{GeV}^{2}$ for muon beams. This value is inside the MUSE kinematical region for all three nominal beam momenta. $\varepsilon$ has the physical interpretation of the degree of the virtual photon longitudinal polarization in the case of the one-photon exchange.

### 3.2 Invariant amplitudes and formalism of helicity amplitudes

To describe the lepton-proton scattering, there are 16 helicity amplitudes $T_{h^{\prime} \lambda^{\prime}, h \lambda}$ with arbitrary positive or negative helicities $h, h^{\prime}, \lambda, \lambda^{\prime}= \pm$ in Fig. 3.1. It is convenient to work with helicity amplitudes in the c.m. reference frame. The discrete symmetries of QCD and QED (parity and time-reversal invariance) leave just six independent amplitudes:

$$
\begin{array}{lll}
T_{1} \equiv T_{++,++}, & T_{2} \equiv T_{+-,++}, & T_{3} \equiv T_{+-,+-} \\
T_{4} \equiv T_{-+,++}, & T_{5} \equiv T_{--,++}, & T_{6} \equiv T_{-+,+-} \tag{3.8}
\end{array}
$$

The helicity amplitudes for the $l^{-} p$ elastic scattering can be expressed by the sum of six different tensor structures and generalized FFs (Lorentz invariant amplitudes) that are complex functions of two independent kinematical variables. It is common to divide the helicity amplitudes into a part without lepton helicity flip which survives in the lepton massless limit
$T^{\text {non-flip }}$, and the part with lepton helicity flip $T^{\text {flip }}$ which is proportional to the mass of the lepton $[167,169]$ (where the $T$ matrix is defined as $S=1+i T$ ):

$$
\begin{align*}
T_{h^{\prime} \lambda^{\prime}, h \lambda}^{\text {non-flip }} & =\frac{e^{2}}{Q^{2}} \bar{u}\left(k^{\prime}, h^{\prime}\right) \gamma_{\mu} u(k, h) \cdot \bar{N}\left(p^{\prime}, \lambda^{\prime}\right)\left(\gamma^{\mu} \mathcal{G}_{M}\left(\nu, Q^{2}\right)-\frac{P^{\mu}}{M} \mathcal{F}_{2}\left(\nu, Q^{2}\right)\right) N(p, \lambda) \\
& +\frac{e^{2}}{Q^{2}} \mathcal{F}_{3}\left(\nu, Q^{2}\right) \bar{u}\left(k^{\prime}, h^{\prime}\right) \gamma_{\mu} u(k, h) \cdot \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) \frac{\gamma \cdot K P^{\mu}}{M^{2}} N(p, \lambda)  \tag{3.9}\\
T_{h^{\prime} \lambda^{\prime}, h \lambda}^{\text {flip }} & =\frac{e^{2}}{Q^{2}} \frac{m}{M} \bar{u}\left(k^{\prime}, h^{\prime}\right) u(k, h) \cdot \bar{N}\left(p^{\prime}, \lambda^{\prime}\right)\left(\mathcal{F}_{4}\left(\nu, Q^{2}\right)+\frac{\gamma \cdot K}{M} \mathcal{F}_{5}\left(\nu, Q^{2}\right)\right) N(p, \lambda) \\
& +\frac{e^{2}}{Q^{2}} \frac{m}{M} \mathcal{F}_{6}\left(\nu, Q^{2}\right) \bar{u}\left(k^{\prime}, h^{\prime}\right) \gamma_{5} u(k, h) \cdot \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) \gamma_{5} N(p, \lambda) \tag{3.10}
\end{align*}
$$

The helicity amplitudes can be expressed in terms of the generalized FFs and vice versa. Using the Jacob and Wick [191] phase convention for the spinors (see explicit expressions in Appendix A), the helicity amplitudes $T_{h^{\prime} \lambda^{\prime}, h \lambda}$ for elastic lepton-proton scattering in the c. m . reference frame are expressed in terms of the generalized FFs by

$$
\begin{align*}
\Sigma_{s} \xi^{2} \frac{T_{1}}{e^{2}}= & 2\left(\frac{\Sigma_{s} Q^{2}}{\Sigma_{s}-s Q^{2}}+s-M^{2}-m^{2}\right) \mathcal{G}_{M}-2\left(s-M^{2}-m^{2}\right) \mathcal{F}_{2}+\frac{\left(s-M^{2}-m^{2}\right)^{2}}{M^{2}} \mathcal{F}_{3} \\
& +4 m^{2} \mathcal{F}_{4}+2 m^{2} \frac{s-M^{2}-m^{2}}{M^{2}} \mathcal{F}_{5}, \\
M \Sigma_{s} \xi \frac{T_{2}}{e^{2}}= & 2 M^{2}\left(s-M^{2}+m^{2}\right) \mathcal{G}_{M}-\left(\left(s-m^{2}\right)^{2}-M^{4}\right) \mathcal{F}_{2}+\left(\left(s-M^{2}\right)^{2}-m^{4}\right) \mathcal{F}_{3} \\
& +2\left(s+M^{2}-m^{2}\right) m^{2} \mathcal{F}_{4}+2\left(s-M^{2}+m^{2}\right) m^{2} \mathcal{F}_{5}, \\
\Sigma_{s} \xi^{2} \frac{T_{3}}{e^{2}}= & 2\left(s-M^{2}-m^{2}\right)\left(\mathcal{G}_{M}-\mathcal{F}_{2}\right)+\frac{\left(s-M^{2}-m^{2}\right)^{2}}{M^{2}} \mathcal{F}_{3}+4 m^{2} \mathcal{F}_{4} \\
& +2 \frac{m^{2}\left(s-M^{2}-m^{2}\right)}{M^{2}} \mathcal{F}_{5}, \\
\frac{\Sigma_{s}}{m} \xi \frac{T_{4}}{e^{2}}= & -2\left(s+M^{2}-m^{2}\right)\left(\mathcal{G}_{M}-\mathcal{F}_{2}\right)-\frac{\left(\left(s-m^{2}\right)^{2}-M^{4}\right)}{M^{2}} \mathcal{F}_{3}-2\left(s-M^{2}+m^{2}\right) \mathcal{F}_{4} \\
& -\frac{\left(\left(s-M^{2}\right)^{2}-m^{4}\right)}{M^{2}} \mathcal{F}_{5}, \\
\frac{M \Sigma_{s}}{m} \frac{T_{5}}{e^{2}=} & -4 M^{2} s \mathcal{G}_{M}+\left(s+M^{2}-m^{2}\right)^{2} \mathcal{F}_{2}-\left(s^{2}-\left(M^{2}-m^{2}\right)^{2}\right)\left(\mathcal{F}_{3}+\mathcal{F}_{4}\right)-\Sigma_{s} \mathcal{F}_{6} \\
& -\left(s-M^{2}+m^{2}\right)^{2} \mathcal{F}_{5}, \\
\frac{M \Sigma_{s}}{m} \frac{T_{6}}{e^{2}}= & 4 M^{2} s \mathcal{G}_{M}-\left(s+M^{2}-m^{2}\right)^{2} \mathcal{F}_{2}+\left(s^{2}-\left(M^{2}-m^{2}\right)^{2}\right)\left(\mathcal{F}_{3}+\mathcal{F}_{4}\right)-\Sigma_{s} \mathcal{F}_{6} \\
& +\left(s-M^{2}+m^{2}\right)^{2} \mathcal{F}_{5}, \tag{3.11}
\end{align*}
$$

with

$$
\begin{equation*}
\xi=\sqrt{\frac{Q^{2}}{\Sigma_{s}-s Q^{2}}} \tag{3.12}
\end{equation*}
$$

We consider the azimuthal angle of the scattered lepton to be $\phi=0$. Notice that following the Jacob-Wick phase convention [191], the azimuthal angular dependence of the helicity amplitudes is in general given by $T_{h^{\prime} \lambda^{\prime}, h \lambda}(\theta, \phi)=e^{i\left(\Lambda-\Lambda^{\prime}\right) \phi} T_{h^{\prime} \lambda^{\prime}, h \lambda}(\theta, 0)$, with $\Lambda=h-\lambda$ and $\Lambda^{\prime}=h^{\prime}-\lambda^{\prime}$.

The relations of Eqs. (3.11) can be inverted to yield the generalized FFs in terms of the helicity amplitudes as

$$
\begin{align*}
e^{2} \mathcal{G}_{M}= & \frac{1}{2}\left(T_{1}-T_{3}\right) \\
\Sigma_{s} e^{2} \mathcal{F}_{2}= & -2 m^{2} M^{2} T_{1}-M\left(\left(s-M^{2}\right)^{2}-m^{4}\right) \xi T_{2}-M^{2} \eta(m) T_{3} \\
& +2 m M^{2}\left(s-M^{2}+m^{2}\right) \xi T_{4}-m M\left(s-m^{2}-M^{2}\right)\left(T_{5}-T_{6}\right) \\
\frac{\Sigma_{s}}{M^{2}} e^{2} \mathcal{F}_{3}= & -\left(s-M^{2}-m^{2}\right) T_{1}-2 M\left(s-M^{2}+m^{2}\right) \xi T_{2}+\rho_{3} T_{3} \\
& +2 m\left(s+M^{2}-m^{2}\right) \xi T_{4}-2 m M\left(T_{5}-T_{6}\right) \\
\frac{\Sigma_{s}}{M} e^{2} \mathcal{F}_{4}= & -M\left(s-M^{2}-m^{2}\right) T_{1}-\left(\left(s-m^{2}\right)^{2}-M^{4}\right) \xi T_{2}+M \rho_{3} T_{3} \\
& +\frac{M\left(\left(s-M^{2}\right)^{2}-m^{4}\right)}{m} \xi T_{4}-\frac{\left(s-M^{2}-m^{2}\right)^{2}}{2 m}\left(T_{5}-T_{6}\right), \\
\frac{\Sigma_{s}}{M^{2}} e^{2} \mathcal{F}_{5}= & 2 M^{2} T_{1}+2 M\left(s+M^{2}-m^{2}\right) \xi T_{2}+\eta(M) T_{3}-\frac{\left(s-m^{2}\right)^{2}-M^{4}}{m} \xi T_{4} \\
& +\frac{M\left(s-M^{2}-m^{2}\right)}{m}\left(T_{5}-T_{6}\right), \\
e^{2} \mathcal{F}_{6}= & -\frac{M}{2 m}\left(T_{5}+T_{6}\right), \tag{3.13}
\end{align*}
$$

with

$$
\begin{align*}
\rho_{3} & =\frac{s^{3}-2 s^{2}\left(M^{2}+m^{2}\right)+\left(M^{2}-m^{2}\right)^{2}\left(s-Q^{2}\right)}{\Sigma_{s}-s Q^{2}}-M^{2}-m^{2} \\
\eta(m) & =\frac{2 m^{2}\left(\Sigma_{s}+s Q^{2}\right)+\Sigma_{s} Q^{2}}{s Q^{2}-\Sigma_{s}} \tag{3.14}
\end{align*}
$$

Exploiting the relations of Eqs. (2.3, 3.13), we can express the forward spin dependent amplitudes $f_{-}, g$ in terms of the generalized FFs in the forward limit $Q^{2} \rightarrow 0$ as $^{1}$

$$
\begin{align*}
f_{-}(\omega) & =e^{2} \mathcal{G}_{M}\left(M \omega, Q^{2}=0\right)  \tag{3.15}\\
g(\omega) & =-e^{2} \frac{m}{M} \mathcal{F}_{6}\left(M \omega, Q^{2}=0\right) \tag{3.16}
\end{align*}
$$

The contributions beyond the exchange of one photon (and other graphs with $Q^{2}=0$ pole), e.g., TPE amplitudes, to all six non-forward amplitudes satisfy the following model-independent relations in the forward limit:

$$
\begin{align*}
& \mathcal{G}_{M}\left(\nu, Q^{2}=0\right)+\frac{\nu}{M^{2}} \mathcal{F}_{3}\left(\nu, Q^{2}=0\right)+\frac{m^{2}}{M^{2}} \mathcal{F}_{5}\left(\nu, Q^{2}=0\right)=0  \tag{3.17}\\
& \mathcal{G}_{M}\left(\nu, Q^{2}=0\right)-\mathcal{F}_{2}\left(\nu, Q^{2}=0\right)+\frac{\nu}{M^{2}} \mathcal{F}_{3}\left(\nu, Q^{2}=0\right)=0  \tag{3.18}\\
& \mathcal{F}_{4}\left(\nu, Q^{2}=0\right)+\frac{\nu}{M^{2}} \mathcal{F}_{5}\left(\nu, Q^{2}=0\right)=0  \tag{3.19}\\
& \mathcal{F}_{3}\left(\nu, Q^{2}=0\right)-\mathcal{F}_{4}\left(\nu, Q^{2}=0\right)+\mathcal{F}_{6}\left(\nu, Q^{2}=0\right)=0 \tag{3.20}
\end{align*}
$$

We obtain these relations in Appendix F analyzing the forward limit of the expressions for the helicity amplitudes in terms of invariant amplitudes, see Eqs. (3.11). Consequently among six non-forward TPE amplitudes only two amplitudes are independent in the forward limit.

[^8]From the other hand, the unitarity provides constraints on the high-energy behavior of the invariant amplitudes:

$$
\begin{align*}
\mathcal{F}_{2}\left(\nu \rightarrow \infty, Q^{2}\right), \mathcal{F}_{3}\left(\nu \rightarrow \infty, Q^{2}\right), \mathcal{F}_{5}\left(\nu \rightarrow \infty, Q^{2}\right) & \lesssim \frac{1}{\nu},  \tag{3.21}\\
\mathcal{G}_{M}\left(\nu \rightarrow \infty, Q^{2}\right)+\frac{\nu}{M^{2}} \mathcal{F}_{3}\left(\nu \rightarrow \infty, Q^{2}\right) & \lesssim \frac{1}{\nu}  \tag{3.22}\\
\mathcal{G}_{M}\left(\nu \rightarrow \infty, Q^{2}\right), \mathcal{F}_{4}\left(\nu \rightarrow \infty, Q^{2}\right), \mathcal{F}_{6}\left(\nu \rightarrow \infty, Q^{2}\right) & \lesssim \text { const }, \tag{3.23}
\end{align*}
$$

which are obtained in Appendix $G$ analyzing the high-energy limit of the invariant amplitudes expressions of Eqs. (3.13).

Performing the crossing $\nu \rightarrow-\nu$ in the lepton (proton) line and rewriting the lepton (proton) spinors in term of the anti-lepton (anti-proton) spinors with the same steps as in Eqs. (2.72.10), we obtain the symmetry properties of the invariant amplitudes on the real $\nu$ axis with the number $n$ of exchanged photons:

$$
\begin{align*}
\mathcal{G}_{M}^{n \gamma}\left(\nu, Q^{2}\right) & =-(-1)^{n}\left(\mathcal{G}_{M}^{n \gamma}\left(-\nu, Q^{2}\right)\right)^{*},  \tag{3.24}\\
\mathcal{F}_{2}^{n \gamma}\left(\nu, Q^{2}\right) & =-(-1)^{n}\left(\mathcal{F}_{2}^{n \gamma}\left(-\nu, Q^{2}\right)\right)^{*},  \tag{3.25}\\
\mathcal{F}_{3}^{n \gamma}\left(\nu, Q^{2}\right) & =(-1)^{n}\left(\mathcal{F}_{3}^{n \gamma}\left(-\nu, Q^{2}\right)\right)^{*},  \tag{3.26}\\
\mathcal{F}_{4}^{n \gamma}\left(\nu, Q^{2}\right) & =(-1)^{n}\left(\mathcal{F}_{4}^{n \gamma}\left(-\nu, Q^{2}\right)\right)^{*},  \tag{3.27}\\
\mathcal{F}_{5}^{n \gamma}\left(\nu, Q^{2}\right) & =-(-1)^{n}\left(\mathcal{F}_{5}^{n \gamma}\left(-\nu, Q^{2}\right)\right)^{*},  \tag{3.28}\\
\mathcal{F}_{6}^{n \gamma}\left(\nu, Q^{2}\right) & =(-1)^{n}\left(\mathcal{F}_{6}^{n \gamma}\left(-\nu, Q^{2}\right)\right)^{*} . \tag{3.29}
\end{align*}
$$

### 3.3 One-photon exchange approximation

In the OPE approximation, the two non-zero generalized FFs in $l^{-} p$ elastic scattering $\mathcal{G}_{M}$ and $\mathcal{F}_{2}$ can be expressed in terms of the Dirac $F_{D}$ and Pauli $F_{P}$ FFs with the following expression for the helicity amplitude [192]:

$$
\begin{equation*}
T^{1 \gamma}=\frac{e^{2}}{Q^{2}} \bar{u}\left(k^{\prime}, h^{\prime}\right) \gamma_{\mu} u(k, h) \cdot \bar{N}\left(p^{\prime}, \lambda^{\prime}\right)\left(\gamma^{\mu} F_{D}\left(Q^{2}\right)+\frac{i \sigma^{\mu \nu} q_{\nu}}{2 M} F_{P}\left(Q^{2}\right)\right) N(p, \lambda), \tag{3.30}
\end{equation*}
$$

that is just a product of lepton and proton currents shown in Fig. 3.2. It is customary in


Figure 3.2: Elastic lepton-proton scattering in the OPE approximation.
experimental analysis to work with Sachs magnetic and electric FFs:

$$
\begin{equation*}
G_{M}=F_{D}+F_{P}, \quad G_{E}=F_{D}-\tau_{P} F_{P} \tag{3.31}
\end{equation*}
$$

where $\tau_{P}$ is defined as in Eqs. (2.46). For non-relativistic systems, such as atomic nuclei, the Sachs electromagnetic proton FFs have the physical interpretation as Fourier transforms of the density of the electric charge and magnetization [193]. For relativistic systems, an analogous interpretation is only possible in the infinite momentum frame [193]. In the OPE approximation, the invariant amplitudes defined in Eqs. (3.9), (3.10), can be expressed in terms of the proton FFs $\mathcal{G}_{M}^{1 \gamma}=G_{M}\left(Q^{2}\right), \mathcal{F}_{2}^{1 \gamma}=F_{P}\left(Q^{2}\right), \mathcal{F}_{3}^{1 \gamma}=\mathcal{F}_{4}^{1 \gamma}=\mathcal{F}_{5}^{1 \gamma}=\mathcal{F}_{6}^{1 \gamma}=0$. The exchange of more than one photon gives corrections of order $O(\alpha)$, with $\alpha=e^{2} /(4 \pi) \simeq 1 / 137$, to all these amplitudes.

The unpolarized differential cross section in the OPE approximation in the laboratory frame is given by

$$
\begin{equation*}
\left(\frac{\mathrm{d} \sigma_{1 \gamma}}{\mathrm{~d} \Omega}\right)_{\mathrm{lab}}=\frac{1}{64 \pi^{2} M^{2}} \frac{\left|\mathbf{k}^{\prime}\right|}{|\mathbf{k}|} \frac{M}{M+\omega-\omega^{\prime}|\mathbf{k}|} \operatorname{|\mathbf {k}^{\prime }|} \cos \theta_{\mathrm{lab}} \frac{1}{4} \sum_{\text {spin }}\left|T^{1 \gamma}\right|^{2}, \tag{3.32}
\end{equation*}
$$

with the lepton solid angle $\Omega$. Averaging over the spin states of incoming and outgoing particles we obtain in the laboratory frame:

$$
\begin{equation*}
\left(\frac{\mathrm{d} \sigma_{1 \gamma}}{\mathrm{~d} \Omega}\right)_{\mathrm{lab}}=\frac{\alpha^{2}}{2 M} \frac{\left|\mathbf{k}^{\prime}\right|}{|\mathbf{k}|} \frac{1}{M+\omega-\omega^{\prime} \frac{|\mathbf{k}|}{\left|\mathbf{k}^{\prime}\right|} \cos \theta_{\mathrm{lab}}} \frac{1-\varepsilon_{0}}{1-\varepsilon}\left(G_{M}^{2}+\frac{\varepsilon}{\tau_{P}} G_{E}^{2}\right) \tag{3.33}
\end{equation*}
$$

an analogue of the Rosenbluth expression [170, 194, 195] in agreement with Ref. [196]. The unpolarized differential cross section can equivalently be written in the compact form:

$$
\begin{equation*}
\frac{\mathrm{d} \sigma_{1 \gamma}}{\mathrm{~d} Q^{2}}=\frac{\pi \alpha^{2}}{2 M^{2} \mathbf{k}^{2}} \frac{1-\varepsilon_{0}}{1-\varepsilon}\left(G_{M}^{2}+\frac{\varepsilon}{\tau_{P}} G_{E}^{2}\right) . \tag{3.34}
\end{equation*}
$$

### 3.4 Two-photon exchange contributions

The TPE correction to the unpolarized elastic lepton-proton scattering cross section is given by the interference between the OPE amplitude and the sum of box and crossed-box graphs with two photons. The correction in the leading $\alpha$ order $\delta_{2 \gamma}$ can be defined through the difference between the cross section with account of the exchange of two photons and the cross section in the $1 \gamma$-exchange approximation $\sigma_{1 \gamma}$ by

$$
\begin{equation*}
\sigma=\sigma_{1 \gamma}\left(1+\delta_{2 \gamma}\right) . \tag{3.35}
\end{equation*}
$$

The leading TPE correction to the elastic $l^{-} p$ scattering can be expressed in terms of the TPE invariant amplitudes as

$$
\begin{equation*}
\delta_{2 \gamma}=\frac{2}{G_{M}^{2}+\frac{\varepsilon}{\tau_{P}} G_{E}^{2}}\left\{G_{M} \Re \mathcal{G}_{1}^{2 \gamma}+\frac{\varepsilon}{\tau_{P}} G_{E} \Re \mathcal{G}_{2}^{2 \gamma}+\frac{1-\varepsilon}{1-\varepsilon_{0}}\left(\frac{\varepsilon_{0}}{\tau_{P}} \frac{\nu}{M^{2}} G_{E} \Re \mathcal{G}_{4}^{2 \gamma}-G_{M} \Re \mathcal{G}_{3}^{2 \gamma}\right)\right\}, \tag{3.36}
\end{equation*}
$$

where we defined for convenience the following amplitudes:

$$
\begin{align*}
\mathcal{G}_{1}^{2 \gamma} & =\mathcal{G}_{M}^{2 \gamma}+\frac{\nu}{M^{2}} \mathcal{F}_{3}^{2 \gamma}+\frac{m^{2}}{M^{2}} \mathcal{F}_{5}^{2 \gamma},  \tag{3.37}\\
\mathcal{G}_{2}^{2 \gamma} & =\mathcal{G}_{M}^{2 \gamma}-\left(1+\tau_{P}\right) \mathcal{F}_{2}^{2 \gamma}+\frac{\nu}{M^{2}} \mathcal{F}_{3}^{2 \gamma},  \tag{3.38}\\
\mathcal{G}_{3}^{2 \gamma} & =\frac{m^{2}}{M^{2}} \mathcal{F}_{5}^{2 \gamma}+\frac{\nu}{M^{2}} \mathcal{F}_{3}^{2 \gamma},  \tag{3.39}\\
\mathcal{G}_{4}^{2 \gamma} & =\mathcal{F}_{4}^{2 \gamma}+\frac{\nu}{M^{2}\left(1+\tau_{P}\right)} \mathcal{F}_{5}^{2 \gamma} . \tag{3.40}
\end{align*}
$$

According to Eqs. (3.17-3.19) in the forward limit these amplitudes satisfy:

$$
\begin{equation*}
\mathcal{G}_{1}\left(\nu, Q^{2} \rightarrow 0\right), \mathcal{G}_{2}\left(\nu, Q^{2} \rightarrow 0\right), \mathcal{G}_{4}\left(\nu, Q^{2} \rightarrow 0\right) \rightarrow 0 \tag{3.41}
\end{equation*}
$$

Consequently, the TPE correction to the unpolarized cross section vanishes in the forward limit.

In the high-energy limit, corresponding with

$$
\begin{equation*}
\nu \rightarrow \infty, \quad \frac{1-\varepsilon}{1-\varepsilon_{0}} \rightarrow\left(1+\tau_{P}\right) \frac{Q^{2} M^{2}}{2 \nu^{2}}+\mathrm{O}\left(\frac{1}{\nu^{4}}\right), \tag{3.42}
\end{equation*}
$$

the invariant amplitudes behavior is constrained by the unitarity, see Appendix G, as

$$
\begin{align*}
\mathcal{G}_{1}\left(\nu \rightarrow \infty, Q^{2}\right), \mathcal{G}_{2}\left(\nu \rightarrow \infty, Q^{2}\right) & \lesssim \frac{1}{\nu},  \tag{3.43}\\
\mathcal{G}_{3}\left(\nu \rightarrow \infty, Q^{2}\right), \mathcal{G}_{4}\left(\nu \rightarrow \infty, Q^{2}\right) & \lesssim \text { const. } \tag{3.44}
\end{align*}
$$

Consequently, the TPE correction to the unpolarized cross section vanishes also in the highenergy limit.

Amplitudes $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ have the same symmetry properties on the real $\nu$ axis as the amplitude $\mathcal{G}_{M}$ w.r.t. $\nu \rightarrow-\nu$ :

$$
\begin{align*}
& \mathcal{G}_{1}^{n \gamma}\left(\nu, Q^{2}\right)=-(-1)^{n}\left(\mathcal{G}_{1}^{n \gamma}\left(-\nu, Q^{2}\right)\right)^{*}  \tag{3.45}\\
& \mathcal{G}_{2}^{n \gamma}\left(\nu, Q^{2}\right)=-(-1)^{n}\left(\mathcal{G}_{2}^{n \gamma}\left(-\nu, Q^{2}\right)\right)^{*} \tag{3.46}
\end{align*}
$$

### 3.5 Scattering in the limit of massless electrons and proton form factors data

The momentum transfer accessed by current experiments down to $Q^{2} \gtrsim 0.001 \mathrm{GeV}^{2}[103,104]$ is still much larger than the squared electron mass, so that to very good approximation electrons can be treated as massless particles. All traditional expressions for the elastic electron-proton scattering can be obtained taking the limit $m \rightarrow 0$ from the expressions above.

The virtual photon polarization parameter $\varepsilon$ varies between 0 and 1 , indicating the degree of the longitudinal polarization of virtual photon in case of one-photon exchange. It is customary to use this parameter in experiment. In terms of $Q^{2}$ and $\nu, \varepsilon$ is then defined as

$$
\begin{equation*}
\varepsilon=\frac{16 \nu^{2}-Q^{2}\left(Q^{2}+4 M^{2}\right)}{16 \nu^{2}+Q^{2}\left(Q^{2}+4 M^{2}\right)}=\left(1+2\left(1+\tau_{P}\right) \tan ^{2} \frac{\theta_{\mathrm{lab}}}{2}\right)^{-1} \tag{3.47}
\end{equation*}
$$

The expression for the momentum transfer in case of the massless lepton is simplified to

$$
\begin{equation*}
Q^{2}=\frac{\left(s-M^{2}\right)^{2}}{s} \sin ^{2} \frac{\theta_{\mathrm{cm}}}{2}=2 \omega \omega^{\prime}\left(1-\cos \theta_{\mathrm{lab}}\right) . \tag{3.48}
\end{equation*}
$$

All amplitudes with electron helicity flip are suppressed by the electron mass. Only three independent helicity amplitudes (in the c.m. reference frame) of Eq. (3.11) survive in the limit of massless electrons. Following the Jacob-Wick [191] phase convention, these amplitudes, $T_{1}, T_{2}, T_{3}$, can be expressed through the generalized FFs as

$$
\begin{align*}
& T_{1}=\frac{2 e^{2}}{Q^{2}}\left\{\frac{s u-M^{4}}{s-M^{2}}\left(\mathcal{F}_{2}-\mathcal{G}_{M}-\frac{s-M^{2}}{2 M^{2}} \mathcal{F}_{3}\right)+Q^{2} \mathcal{G}_{M}\right\}, \\
& T_{2}=-e^{2} \frac{\sqrt{M^{4}-s u}}{M Q}\left\{\mathcal{F}_{2}+2 \frac{M^{2}}{s-M^{2}}\left(\mathcal{F}_{2}-\mathcal{G}_{M}\right)-\mathcal{F}_{3}\right\}, \\
& T_{3}=\frac{2 e^{2}}{Q^{2}} \frac{s u-M^{4}}{s-M^{2}}\left\{\mathcal{F}_{2}-\mathcal{G}_{M}-\frac{s-M^{2}}{2 M^{2}} \mathcal{F}_{3}\right\} . \tag{3.49}
\end{align*}
$$

The invariant amplitudes can in turn be expressed through the helicity amplitudes as [197]

$$
\begin{align*}
e^{2} \mathcal{G}_{M} & =\frac{1}{2}\left\{T_{1}-T_{3}\right\} \\
e^{2} \mathcal{F}_{2} & =\frac{M Q}{\sqrt{M^{4}-s u}}\left\{-T_{2}+T_{3} \frac{M Q}{\sqrt{M^{4}-s u}}\right\} \\
e^{2} \mathcal{F}_{3} & =\frac{M^{2}}{s-M^{2}}\left\{-T_{1}-T_{2} \frac{2 M Q}{\sqrt{M^{4}-s u}}+T_{3}\left(1+Q^{2} \frac{s+M^{2}}{M^{4}-s u}\right)\right\} \tag{3.50}
\end{align*}
$$

In the OPE approximation the unpolarized cross section is given by the Rosenbluth expression [35]:

$$
\begin{equation*}
\frac{\mathrm{d} \sigma_{1 \gamma}}{\mathrm{~d} Q^{2}}=\frac{\pi \alpha^{2}}{2 M^{2} \omega^{2}} \frac{1}{1-\varepsilon}\left(G_{M}^{2}+\frac{\varepsilon}{\tau_{P}} G_{E}^{2}\right) \tag{3.51}
\end{equation*}
$$

This expression allows to extract the proton FFs by the Rosenbluth separation method. One can use the data for a cross section measurement at fixed $Q^{2}$ and vary the incoming electron energy. The charge FF $G_{E}$ is customary determined from the radiatively corrected $\varepsilon$-slope, and the magnetic $\mathrm{FF} G_{M}$ is extracted from the radiatively corrected offset at $\varepsilon=0$ of the cross section when plotted as function of $\varepsilon$. The first data [36] were described by the simple dipole form:

$$
\begin{equation*}
G_{E}\left(Q^{2}\right)=\frac{1}{\left(1+\frac{Q^{2}}{\Lambda^{2}}\right)^{2}}, \quad G_{M}\left(Q^{2}\right)=\frac{\mu_{P}}{\left(1+\frac{Q^{2}}{\Lambda^{2}}\right)^{2}} \tag{3.52}
\end{equation*}
$$

with $\Lambda^{2}=0.71 \mathrm{GeV}^{2}$. The approximation of Eqs. (3.52) is a reasonable approximation, at least, in the region of small momentum transfers $Q^{2}<1 \mathrm{GeV}^{2}$. We use it as a leading order estimate for the evaluation of the TPE corrections with the proton intermediate state.

However, the modern FFs extractions performed by A1 Collaboration at MAMI, Mainz, use directly the expressions of Eq. (3.51) and functional parametrizations of FFs in order to avoid the unnecessary kinematical limitations due to the Rosenbluth separation [104]. The overall normalization of FFs is fixed to the well-known proton charge, $G_{E}(0)=1$, and magnetic moment, $G_{M}(0)=\mu_{P}$. It was controlled experimentally [104] by the explicit luminosity measurement using an extra spectrometer. It is convenient to present the FFs normalized to the standard dipole form of Eqs. (3.52). The best fit to the recent Mainz data in comparison with fits to the older data and experimental results is shown in Figs. 3.3, 3.4 [104].

The extraction of the proton FFs, which enter the cross section expression in the leading $\alpha$ order, requires an account of the radiative corrections. The standard framework of the radiative corrections in unpolarized elastic lepton-proton scattering is described in Refs. [162, 163, 195]. The vacuum polarization correction, the lepton vertex correction and the lepton Bremsstrahlung correction are evaluated model-independently in QED, while corrections with the off-shell photon-proton vertex: the proton vertex correction and the proton Bremsstrahlung correction, introduce a model dependence due to the assumption of the on-shell form of the photon-proton vertex [198]. Additionally, the proton vertex correction does not account for the inelastic hadronic states at large photon loop momenta. The ultraviolet (UV) divergence of the lepton vertex correction and the lepton self-energy is absorbed into the charge, mass and field renormalization constants, while the proton vertex correction is UV finite due to the vanishing high- $Q^{2}$ behavior of the proton elastic FFs. The finite part of the external fermions self-energy correction is believed to be zero. The TPE graph is accounted in the approximation of one soft photon by the IR divergent expression that is distinct in Ref. [162] and Ref. [163]. In this work, we follow the Maximon and Tjon (MaTj) [163] prescription approximating the photon momenta in the numerator of the TPE loop integral only.


Figure 3.3: The form factors $G_{E}$ and $G_{M}$, normalized to the standard dipole form, as a function of $Q^{2}$. Black line: Best fit to the new Mainz data; blue area: statistical $68 \%$ confidence band; light blue area: experimental systematic error; green outer band: variation of the Coulomb correction by $\pm 50 \%$.


Figure 3.4: Same as Fig. 3.3, but for the FFs ratio $\mu_{P} G_{E} / G_{M}$.

All IR divergences in the elastic scattering amplitudes are canceled at the cross section level when summing with the inclusive soft photon radiation that should be taken into account due to the finite resolution of a real experiment. This statement for the scattering cross sections and decay rates is known as Bloch-Nordsieck theorem [199] in QED and as Kinoshita-Lee-Nauenberg theorem $[200,201]$ in the Standard Model. The IR divergence in the lepton (proton) vertex correction is canceled by the lepton (proton) Bremsstrahlung correction, the IR part of the TPE correction is canceled in sum with the interference of the lepton and proton Bremsstrahlung corrections.
In Ref. [104] the IR-finite part of the TPE correction was approximated by the first TPE result obtained by McKinley and Feshbach [173]. They evaluated the TPE contribution, corresponding with Coulomb photon couplings to the static proton (i.e., two $\gamma^{0}$ vertices). This so-called Feshbach term contribution to $\delta_{2 \gamma}$, denoted by $\delta_{F}$, can be expressed through the scattering angle in the laboratory frame $\theta_{\text {lab }}$ and the lepton velocity $v$ as

$$
\begin{equation*}
\delta_{F}=\pi \alpha v \frac{\sin \theta_{\mathrm{lab}} / 2\left(1-\sin \theta_{\mathrm{lab}} / 2\right)}{1-v^{2} \sin ^{2} \theta_{\mathrm{lab}} / 2} \tag{3.53}
\end{equation*}
$$

The residual cross section contribution was parametrized in Ref. [104] by the two-parameters linear function in $\varepsilon$ :

$$
\begin{equation*}
\delta_{2 \gamma}-\delta_{F}=-a(1-\varepsilon) \ln \left(1+b Q^{2}\right), \tag{3.54}
\end{equation*}
$$

that vanishes in the forward limit $(\varepsilon \rightarrow 1)$ and reproduces the logarithmic $Q^{2}$ behavior [67]. With account of the phenomenological TPE of Eq. (3.54) FFs fits were performed over a wide $Q^{2}$ region [104].
In the following, we consider the correction to observables due to TPE which are corrections of order $e^{2}$. The correction to the unpolarized elastic $e^{-} p$ cross section of Eq. (3.36) is expressed in terms of TPE amplitudes by
$\delta_{2 \gamma}=\frac{2}{G_{M}^{2}+\frac{\varepsilon}{\tau_{P}} G_{E}^{2}}\left\{\left(G_{M}+\frac{\varepsilon}{\tau_{P}} G_{E}\right) \Re \mathcal{G}_{M}^{2 \gamma}-\frac{\varepsilon\left(1+\tau_{P}\right)}{\tau_{P}} G_{E} \Re \mathcal{F}_{2}^{2 \gamma}+\left(G_{M}+\frac{G_{E}}{\tau_{P}}\right) \frac{\nu \varepsilon}{M^{2}} \Re \mathcal{F}_{3}^{2 \gamma}\right\}$.
Other accessible observables, which are influenced by real parts of TPE amplitudes, are double polarization observables with a polarization transfer from the longitudinally polarized electron to the recoil proton. The longitudinal polarization transfer asymmetry is defined as

$$
\begin{equation*}
P_{l}=\frac{\mathrm{d} \sigma\left(h=+, \lambda^{\prime}=+\right)-\mathrm{d} \sigma\left(h=+, \lambda^{\prime}=-\right)}{\mathrm{d} \sigma\left(h=+, \lambda^{\prime}=+\right)+\mathrm{d} \sigma\left(h=+, \lambda^{\prime}=-\right)}, \tag{3.56}
\end{equation*}
$$

and the transverse polarization transfer asymmetry is given by

$$
\begin{equation*}
P_{t}=\frac{\mathrm{d} \sigma\left(h=+, S^{\prime}=S_{\perp}\right)-\mathrm{d} \sigma\left(h=+, S^{\prime}=-S_{\perp}\right)}{\mathrm{d} \sigma\left(h=+, S^{\prime}=S_{\perp}\right)+\mathrm{d} \sigma\left(h=+, S^{\prime}=-S_{\perp}\right)}, \tag{3.57}
\end{equation*}
$$

with the spin direction of the recoil proton $S^{\prime}= \pm S_{\perp}$ in the scattering plane transverse to its momentum direction.

The following ratio is measured experimentally [73]:

$$
\begin{equation*}
-\sqrt{\frac{\tau_{P}(1+\varepsilon)}{2 \varepsilon}} \frac{P_{t}}{P_{l}}=\frac{G_{E}}{G_{M}}+\left(1+\tau_{P}\right) \frac{F_{2} \Re \mathcal{G}_{M}^{2 \gamma}-G_{M} \Re \mathcal{F}_{2}^{2 \gamma}}{G_{M}^{2}}+\left(1-\frac{2 \varepsilon}{1+\varepsilon} \frac{G_{E}}{G_{M}}\right) \frac{\nu}{M^{2}} \frac{\Re \mathcal{F}_{3}^{2 \gamma}}{G_{M}} . \tag{3.58}
\end{equation*}
$$

Experimental data on the longitudinal polarization transfer allows to reconstruct [73] its value relative to the OPE (Born) result $P_{l}^{\text {Born }}$ :

$$
\begin{align*}
\frac{P_{l}}{P_{l}^{\text {Born }}} & =1-\frac{2 \varepsilon}{1+\frac{\varepsilon}{\tau_{P}} \frac{G_{E}^{2}}{G_{M}^{2}}} \frac{1+\tau_{P}}{\tau_{P}} \frac{G_{E}}{G_{M}^{3}}\left(F_{2} \Re \mathcal{G}_{M}^{2 \gamma}-G_{M} \Re \mathcal{F}_{2}^{2 \gamma}\right) \\
& -\frac{2 \varepsilon}{1+\frac{\varepsilon}{\tau_{P}} \frac{G_{E}^{2}}{G_{M}^{2}}}\left(\frac{\varepsilon}{1+\varepsilon}\left(1-\frac{G_{E}^{2}}{\tau_{P} G_{M}^{2}}\right)+\frac{G_{E}}{\tau_{P} G_{M}}\right) \frac{\nu}{M^{2}} \frac{\Re \mathcal{F}_{3}^{2 \gamma}}{G_{M}} . \tag{3.59}
\end{align*}
$$

For further use, it will be convenient to work with amplitudes $\mathcal{G}_{1}^{2 \gamma}, \mathcal{G}_{2}^{2 \gamma}$ defined in the lepton massless limit as

$$
\begin{align*}
\mathcal{G}_{1}^{2 \gamma} & =\mathcal{G}_{M}^{2 \gamma}+\frac{\nu}{M^{2}} \mathcal{F}_{3}^{2 \gamma},  \tag{3.60}\\
\mathcal{G}_{2}^{2 \gamma} & =\mathcal{G}_{M}^{2 \gamma}-\left(1+\tau_{P}\right) \mathcal{F}_{2}^{2 \gamma}+\frac{\nu}{M^{2}} \mathcal{F}_{3}^{2 \gamma} . \tag{3.61}
\end{align*}
$$

In terms of these amplitudes, the TPE correction to the unpolarized $e^{-} p$ cross section is given by

$$
\begin{equation*}
\delta_{2 \gamma}=\frac{2}{G_{M}^{2}+\frac{\varepsilon}{\tau_{P}} G_{E}^{2}}\left\{G_{M} \Re \mathcal{G}_{1}^{2 \gamma}+\frac{\varepsilon}{\tau_{P}} G_{E} \Re \mathcal{G}_{2}^{2 \gamma}+G_{M}(\varepsilon-1) \frac{\nu}{M^{2}} \Re \mathcal{F}_{3}^{2 \gamma}\right\}, \tag{3.62}
\end{equation*}
$$

and the polarization transfer observables can be written as

$$
\begin{align*}
\frac{P_{t}}{P_{l}}=-\sqrt{\frac{2 \varepsilon}{\tau_{P}(1+\varepsilon)}}\left(\frac{G_{E}}{G_{M}}+\frac{\Re \mathcal{G}_{2}^{2 \gamma}}{G_{M}}-\frac{G_{E}}{G_{M}} \frac{\Re \mathcal{G}_{1}^{2 \gamma}}{G_{M}}+\frac{1-\varepsilon}{1+\varepsilon} \frac{G_{E}}{G_{M}} \frac{\nu}{M^{2}} \frac{\Re \mathcal{F}_{3}^{2 \gamma}}{G_{M}}\right),  \tag{3.63}\\
\frac{P_{l}}{P_{l}^{\text {Born }}=1-\frac{2 \varepsilon}{1+\frac{\varepsilon}{\tau_{P}} \frac{G_{E}^{2}}{G_{M}^{2}}}}\left\{\begin{array}{l}
\frac{G_{E}}{\tau_{P} G_{M}} \frac{\Re \mathcal{G}_{2}^{2 \gamma}}{G_{M}}-\frac{G_{E}^{2}}{\tau_{P} G_{M}^{2}} \frac{\Re \mathcal{G}_{1}^{2 \gamma}}{G_{M}} \\
\\
\\
\left.+\left(\frac{\varepsilon}{1+\varepsilon}+\frac{1}{1+\varepsilon} \frac{G_{E}^{2}}{\tau_{P} G_{M}^{2}}\right) \frac{\nu}{M^{2}} \frac{\Re \mathcal{F}_{3}^{2 \gamma}}{G_{M}}\right\} .
\end{array} .\right.
\end{align*}
$$

Accounting for the relations between TPE amplitudes in the forward and high-energy limits of Eqs. (3.21, 3.41, 3.43), the TPE correction to the measured form factors ratio of Eq. (3.63) vanishes in both limits.

The measurement of the vanishing in OPE approximation single spin asymmetry allows to cross-check theoretical TPE calculations. The asymmetry in the scattering of the unpolarized electrons on the polarized normal to the scattering plane protons ( $S= \pm S_{n}$ ) is called the target normal single spin asymmetry $A_{n}[197,202]$ :

$$
\begin{equation*}
A_{n}=\frac{\mathrm{d} \sigma\left(S=S_{n}\right)-\mathrm{d} \sigma\left(S=-S_{n}\right)}{\mathrm{d} \sigma\left(S=S_{n}\right)+\mathrm{d} \sigma\left(S=-S_{n}\right)}, \tag{3.65}
\end{equation*}
$$

and the asymmetry in the interaction of the polarized normal to the scattering plane electrons (with the spin direction of the initial electron: $s= \pm s_{n}$ ) on the unpolarized target is called the beam normal single spin asymmetry $B_{n}[169,202]$ :

$$
\begin{equation*}
B_{n}=\frac{\mathrm{d} \sigma\left(s=s_{n}\right)-\mathrm{d} \sigma\left(s=-s_{n}\right)}{\mathrm{d} \sigma\left(s=s_{n}\right)+\mathrm{d} \sigma\left(s=-s_{n}\right)} . \tag{3.66}
\end{equation*}
$$

The asymmetries of Eqs. $(3.65,3.66)$ are given by the imaginary parts of the TPE amplitudes at the leading $\alpha$ order as

$$
\begin{align*}
A_{n} & =\sqrt{\frac{2 \varepsilon(1+\varepsilon)}{\tau_{P}}} \frac{1}{G_{M}^{2}+\frac{\varepsilon}{\tau_{P}} G_{E}^{2}}\left\{-G_{M} \Im \mathcal{G}_{2}^{2 \gamma}+G_{E} \Im\left(\mathcal{G}_{1}^{2 \gamma}-\frac{\tau_{P}\left(1+\tau_{P}\right) M^{2}}{\nu} \mathcal{F}_{3}^{2 \gamma}\right)\right\} \\
B_{n} & =-\frac{m}{M} \frac{\sqrt{2 \varepsilon(1-\varepsilon)}}{G_{M}^{2}+\frac{\varepsilon}{\tau_{P}} G_{E}^{2}} \frac{\sqrt{1+\tau_{P}}}{\tau_{P}}\left\{\tau_{P} G_{M} \Im\left(\mathcal{F}_{3}^{2 \gamma}+\frac{\nu}{M^{2}} \frac{\mathcal{F}_{5}^{2 \gamma}}{1+\tau_{P}}\right)+G_{E} \Im \mathcal{G}_{4}^{2 \gamma}\right\} \tag{3.67}
\end{align*}
$$

Accounting for amplitudes with lepton helicity flip and lepton mass dependence in the kinematical factor, we provide also expressions for the single spin asymmetries of Eqs. (3.65, 3.66) in the case of massive lepton scattering:

$$
\begin{align*}
A_{n}= & \sqrt{\frac{2\left(\varepsilon-\varepsilon_{0}\right)\left(1+\varepsilon-2 \varepsilon \varepsilon_{0}\right)}{\tau_{P}}} \frac{1}{G_{M}^{2}+\frac{\varepsilon}{\tau_{P}} G_{E}^{2}} \frac{1}{\left|1-\varepsilon_{0}\right|} \\
& \left\{-G_{M}\left(\Im \mathcal{G}_{2}^{2 \gamma}+\frac{\left(1+\tau_{P}\right) m^{2}}{\nu} \Im \mathcal{G}_{4}^{2 \gamma}\right)+G_{E} \Im\left(\mathcal{G}_{1}^{2 \gamma}-\frac{\tau_{P}\left(1+\tau_{P}\right) M^{2}}{\nu} \mathcal{F}_{3}^{2 \gamma}\right)\right\},  \tag{3.69}\\
B_{n}= & -\frac{m}{M} \frac{\sqrt{2(1-\varepsilon)\left(\varepsilon-\varepsilon_{0}\right)}}{\left(G_{M}^{2}+\frac{\varepsilon}{\tau_{P}} G_{E}^{2}\right)\left|1-\varepsilon_{0}\right|} \frac{\sqrt{1+\tau_{P}}}{\tau_{P}}\left\{\tau_{P} G_{M} \Im\left(\mathcal{F}_{3}^{2 \gamma}+\frac{\nu}{M^{2}} \frac{\mathcal{F}_{5}^{2 \gamma}}{1+\tau_{P}}\right)+G_{E} \Im \mathcal{G}_{4}^{2 \gamma}\right\} \tag{3.70}
\end{align*}
$$

Note that the amplitude $\mathcal{G}_{4}^{2 \gamma}$ introduced in Eq. (3.40) appears also in the expression for the unpolarized cross section [170] for finite lepton mass, see Eq. (3.36) for details. The contribution to $A_{n}, B_{n}$ and $\delta_{2 \gamma}$ which is linear in the amplitude $\mathcal{F}_{6}^{2 \gamma}$ vanishes [170, 203]. The amplitude $\mathcal{F}_{6}^{2 \gamma}$ only shows up in double polarization observables, e.g., the transverse polarization transfer observable value relative to the Born result $P_{t}^{\mathrm{Born}}$ is given by

$$
\begin{equation*}
\frac{P_{t}}{P_{t}^{\text {Born }}}=1-\delta_{2 \gamma}+\frac{\Re \mathcal{G}_{M}^{2 \gamma}}{G_{M}}+\frac{\Re \mathcal{G}_{2}^{2 \gamma}}{G_{E}}+\frac{m^{2}}{M \omega} \frac{\left(1+\tau_{P}\right) \Re \mathcal{G}_{4}^{2 \gamma}+\tau_{P} \Re \mathcal{F}_{6}^{2 \gamma}}{G_{E}} \tag{3.71}
\end{equation*}
$$

## Chapter 4

## TPE corrections from proton intermediate state

At very low energies, where the probability to excite the proton is very small, the TPE process can be described to good approximation as photons interacting with the distributed charge and magnetic moment. The contribution with the proton intermediate state is expected to be dominant at low energies and should be studied separately as it contains the explicit pole from the proton propagator in the VVCS amplitudes. The correction to the unpolarized cross section $\delta_{2 \gamma}$ is proportional to the real parts of TPE amplitudes. In this Chapter, we describe two ways to access these real parts theoretically. One can model the photon-proton vertex with the off-shell proton and evaluate the loop integral, see Section 4.1 for details of such calculation. The model-independent way is possible within DR approach described in Section 4.2 , where we also propose the novel method of the analytical continuation of the elastic TPE amplitudes. In the following Section 4.3 we compare two calculations for the proton intermediate state contribution. Afterward, we provide predictions for the upcoming MUSE experiment in Section 4.4. We finish with the model-independent description of the electronproton scattering data within the subtracted DR formalism in Section 4.5 that allows estimating the TPE contribution from the unaccounted inelastic states. We provide the subtracted DR predictions for the kinematics of the CLAS and OLYMPUS experiments.

### 4.1 Box graph model

In this section, we use a box graph model to estimate the TPE correction to elastic leptonproton scattering at low momentum transfer. For such kinematics, we expect the dominant contribution to be given by the TPE direct box and crossed box diagrams with proton intermediate state, as shown in Fig. 4.1.


Figure 4.1: Direct and crossed TPE diagrams in elastic $l p$ scattering.
The helicity amplitudes corresponding with the TPE direct and crossed graphs are independent of the lepton charge and can be expressed as

$$
\begin{align*}
T_{\text {direct }}^{2 \gamma}= & e^{4} \int \frac{\mathrm{~d}^{4} k_{1}}{(2 \pi)^{4} i} \bar{u}\left(k^{\prime}, h^{\prime}\right) \gamma^{\mu}\left(\hat{k_{1}}+m\right) \gamma^{\nu} u(k, h) \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) \Gamma_{\mu}\left(\hat{P}+\hat{K}-\hat{k}_{1}+M\right) \Gamma_{\nu} N(p, \lambda) \\
& \frac{1}{\left(k_{1}-P-K\right)^{2}-M^{2}} \frac{1}{k_{1}^{2}-m^{2}} \frac{1}{\left(k_{1}-K-\frac{q}{2}\right)^{2}-\mu^{2}} \frac{1}{\left(k_{1}-K+\frac{q}{2}\right)^{2}-\mu^{2}}, \\
T_{\text {crossed }}^{2 \gamma}= & e^{4} \int \frac{\mathrm{~d}^{4} k_{1}}{(2 \pi)^{4} i} \bar{u}\left(k^{\prime}, h^{\prime}\right) \gamma^{\mu}\left(\hat{k_{1}}+m\right) \gamma^{\nu} u(k, h) \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) \Gamma_{\nu}\left(\hat{P}-\hat{K}+\hat{k}_{1}+M\right) \Gamma_{\mu} N(p, \lambda) \\
& \frac{1}{\left(k_{1}+P-K\right)^{2}-M^{2}} \frac{1}{k_{1}^{2}-m^{2}} \frac{1}{\left(k_{1}-K-\frac{q}{2}\right)^{2}-\mu^{2}} \frac{1}{\left(k_{1}-K+\frac{q}{2}\right)^{2}-\mu^{2}}, \tag{4.1}
\end{align*}
$$

where $P$ and $K$ are defined as in Section 3.1, $\Gamma^{\mu}$ denotes the virtual photon-proton-proton vertex, and the notation $\hat{a} \equiv \gamma^{\mu} a_{\mu}$ was used. We introduce a small photon mass $\mu$ to regulate the IR singularities. The invariant amplitudes entering Eqs. $(3.36,3.62)$ can be expressed as combination of helicity amplitudes with the help of Eqs. (3.13, 3.50).
We perform the box diagram calculation with the assumption of an on-shell form of the virtual photon-proton-proton vertex,

$$
\begin{equation*}
\Gamma^{\mu}\left(Q^{2}\right)=\gamma^{\mu} F_{D}\left(Q^{2}\right)+\frac{i \sigma^{\mu \nu} q_{\nu}}{2 M} F_{P}\left(Q^{2}\right), \tag{4.2}
\end{equation*}
$$

for two models. In the first model the proton is treated as a point particle with charge and anomalous magnetic moment, i.e., the Dirac and Pauli FFs in Eq. (4.2) have the following form:

$$
\begin{equation*}
F_{D}\left(Q^{2}\right)=1, \quad F_{P}\left(Q^{2}\right)=\mu_{P}-1 . \tag{4.3}
\end{equation*}
$$

The second model is more realistic and is based on the dipole form for the proton electromagnetic FFs of Eqs. (3.52). We call the TPE correction in the box graph model the Born TPE correction, as it is given by the Born contribution to the VVCS tensor in the proton line of the TPE graphs.

Due to the photon momentum in the numerator of the term proportional to the FF $F_{P}$ in Eq. (4.2), the high-energy (HE) behavior of the amplitudes can be different depending on whether $F_{D}$ or $F_{P}$ enters the vertex. We denote the contribution with two vector couplings by $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$, two tensor couplings by $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$, and the contributions from the mixed case by $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$, see Fig. 4.2. In Section 4.2.2 we discuss the HE behavior of the invariant amplitudes in the case of point-like couplings and in the model with dipole form of electromagnetic FFs. The inclusion of FFs leads to ultraviolet (UV) finite results for the invariant amplitudes.

We use LoopTools [204, 205] to evaluate the four-point integrals and derivatives of them, as well as to provide a numerical evaluation of the invariant amplitudes. Some technical details for the calculation in massless electron scattering on proton are described in Appendix H. The calculation is performed with the subtraction of the IR divergent terms according to the Maximon and Tjon prescription [163]. The TPE amplitude $\mathcal{G}_{M}^{2 \gamma}$ in the case of scattering of two point charges (i.e., $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ contribution with $F_{D}=1$ ) has the IR divergent term:

$$
\begin{align*}
\mathcal{G}_{M}^{\mathrm{IR}, \text { point }} & =\frac{\alpha}{\pi} \frac{s-M^{2}-m^{2}}{\sqrt{\Sigma_{s}}}\left(\ln \left(\frac{\sqrt{\Sigma_{s}}-s+(m+M)^{2}}{\sqrt{\Sigma_{s}}+s-(m+M)^{2}}\right)+i \pi\right) \ln \left(\frac{Q^{2}}{\mu^{2}}\right) \\
& -\frac{\alpha}{\pi} \frac{u-M^{2}-m^{2}}{\sqrt{\Sigma_{u}}} \ln \left(\frac{\sqrt{\Sigma_{u}}-u+(m+M)^{2}}{-\sqrt{\Sigma_{u}}-u+(m+M)^{2}}\right) \ln \left(\frac{Q^{2}}{\mu^{2}}\right), \tag{4.4}
\end{align*}
$$

with $\Sigma_{u} \equiv \Sigma\left(u, M^{2}, m^{2}\right)=\left(u-(m+M)^{2}\right)\left(u-(m-M)^{2}\right)$. In electron-proton scattering the IR divergent term simplifies to

$$
\begin{equation*}
\mathcal{G}_{M}^{\mathrm{IR}, \text { point }}=\frac{\alpha}{\pi} \ln \left(\frac{Q^{2}}{\mu^{2}}\right)\left\{\ln \left(\frac{\left|u-M^{2}\right|}{s-M^{2}}\right)+i \pi\right\} . \tag{4.5}
\end{equation*}
$$



Figure 4.2: The different contributions to the proton box diagram, depending on the different virtual photon-proton-proton vertices. The vertex with (without) the cross denotes the contribution proportional to the $F_{P}\left(F_{D}\right) \mathrm{FF}$. The different diagrams show the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ (upper left panel), $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ (upper right panel) and $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ (lower panels) vertex structures.

When including FFs, the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure gives rise to an IR divergence in the amplitudes $\mathcal{G}_{M}^{2 \gamma}, \mathcal{G}_{1}^{2 \gamma}, \mathcal{G}_{2}^{2 \gamma}$ which is given by

$$
\begin{equation*}
\mathcal{G}_{1}^{\mathrm{IR}, \mathrm{~F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}}=\mathcal{G}_{2}^{\mathrm{IR}, \mathrm{~F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}}=\mathcal{G}_{M}^{\mathrm{IR}, \mathrm{~F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}}=F_{D}\left(Q^{2}\right) \mathcal{G}_{M}^{\mathrm{IR}, \text { point }} \tag{4.6}
\end{equation*}
$$

whereas the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ contribution to the amplitudes $\mathcal{F}_{2}^{2 \gamma}, \mathcal{F}_{3}^{2 \gamma}, \mathcal{F}_{4}^{2 \gamma}, \mathcal{F}_{5}^{2 \gamma}$ and $\mathcal{F}_{6}^{2 \gamma}$ is IR finite. The $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structure gives rise to IR divergences in the amplitudes $\mathcal{G}_{M}^{2 \gamma}, \mathcal{G}_{1}^{2 \gamma}$ as well as $\mathcal{F}_{2}^{2 \gamma}$ which are given by

$$
\begin{equation*}
\mathcal{G}_{1}^{\mathrm{IR}, \mathrm{~F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{P}}}=\mathcal{G}_{M}^{\mathrm{IR}, \mathrm{~F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{P}}}=\mathcal{F}_{2}^{\mathrm{IR}, \mathrm{~F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{P}}}=F_{P}\left(Q^{2}\right) \mathcal{G}_{M}^{\mathrm{IR}, \text { point }} \tag{4.7}
\end{equation*}
$$

The correspondent IR divergence in the amplitude $\mathcal{G}_{2}^{2 \gamma}$ is given by

$$
\begin{equation*}
\mathcal{G}_{2}^{\mathrm{IR}, \mathrm{~F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{P}}}=-\tau_{P} F_{P}\left(Q^{2}\right) \mathcal{G}_{M}^{\mathrm{IR}, \text { point }} \tag{4.8}
\end{equation*}
$$

whereas the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ contribution to the amplitudes $\mathcal{F}_{3}^{2 \gamma}, \mathcal{F}_{4}^{2 \gamma}, \mathcal{F}_{5}^{2 \gamma}, \mathcal{F}_{6}^{2 \gamma}$ is IR finite. Finally, the $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure contribution to all TPE amplitudes is IR finite. All plots with TPE amplitudes are made with the subtraction of the IR divergent piece of Eqs. (4.6-4.8).

When combining all IR divergent pieces, Eq. (3.36) yields the IR divergent TPE correction to the unpolarized $l^{-} p$ cross section:

$$
\begin{align*}
\delta_{2 \gamma}^{\mathrm{IR}}= & \frac{2 \alpha}{\pi} \ln \left(\frac{Q^{2}}{\mu^{2}}\right)\left\{\frac{s-M^{2}-m^{2}}{\sqrt{\Sigma_{s}}}\left(\ln \left(\frac{\sqrt{\Sigma_{s}}-s+(m+M)^{2}}{\sqrt{\Sigma_{s}}+s-(m+M)^{2}}\right)\right)\right. \\
& \left.-\frac{u-M^{2}-m^{2}}{\sqrt{\Sigma_{u}}} \ln \left(\frac{\sqrt{\Sigma_{u}}-u+(m+M)^{2}}{-\sqrt{\Sigma_{u}}-u+(m+M)^{2}}\right)\right\} . \tag{4.9}
\end{align*}
$$

In case of the electron-proton $\left(e^{-} p\right)$ scattering the IR divergent TPE correction to the unpolarized cross section of Eq. (3.62) simplifies to

$$
\begin{equation*}
\delta_{2 \gamma}^{\mathrm{IR}}=\frac{2 \alpha}{\pi} \ln \left(\frac{Q^{2}}{\mu^{2}}\right) \ln \left(\frac{\left|u-M^{2}\right|}{s-M^{2}}\right) \tag{4.10}
\end{equation*}
$$

In order to compare with data, which are radiatively corrected, we subtract Eq. (4.10) (Eq. (4.9)) in the cross section formula of Eq. (3.62) (Eq. (3.36)). This is in agreement with the Maximon and $\mathrm{Tjon}(\mathrm{MaTj})$ prescription for the soft photon TPE contribution, i.e., $\delta_{2 \gamma \text {, soft }}^{\mathrm{MaTj}}=\delta_{2 \gamma}^{\mathrm{IR}}$, see Eq. (3.39) of Ref. [163]. Note that the $P_{t} / P_{l}, P_{l} / P_{l}^{\mathrm{Born}}, A_{n}$ and $B_{n}$ polarization observables, Eqs. (3.63, 3.64, 3.69, 3.70), are free of IR divergencies.

We have checked the model-independent relations of Eqs. (3.17-3.20) between the TPE amplitudes in the forward limit separately for the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}, \mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ and $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structures in the box graph model.

### 4.2 Dispersion relation (DR) framework

In this Section, we describe the details of the DR framework for the evaluation of the TPE amplitudes paying additional attention to the proton intermediate state contribution. The DR framework allows evaluating the TPE contribution using the on-shell one-photon exchange (OPE) information as an input. As the intermediate state is on its mass shell in the DR approach, it avoids additional model dependent assumptions about the off-shell interaction vertex. However, there are few standard complications in this approach. First of all, it requires the convergence of the DR integrals. Secondly, even for the convergent DR integrals the real part of amplitude can not be fully reconstructed as it was shown for the proton intermediate state contribution to the forward TPE amplitudes in Section 2.6.2. Additionally, in the nonforward scattering the allowed kinematical region of experiments does not always cover the full range of the DR integral, and the analytical continuation to the unphysical region is required. We describe the way to avoid these complications for the case of the proton intermediate state in this Section.

### 4.2.1 Unitarity relations

The imaginary parts of the invariant amplitudes can be obtained with the help of the unitarity equation for the scattering matrix $S$ (with $S=1+i T$ ):

$$
\begin{equation*}
S^{+} S=1, \quad T^{+} T=i\left(T^{+}-T\right) . \tag{4.11}
\end{equation*}
$$

In the c.m. reference frame we can reconstruct the imaginary part of the TPE helicity amplitude $\Im T_{h^{\prime} \lambda^{\prime}, h \lambda}^{2 \gamma}$ in the leading order in $\alpha$ by the phase space integration of the product of the OPE amplitudes from the initial to intermediate state $T_{\text {hel }, h \lambda}^{1 \gamma}$ and from the intermediate state to final state $T_{h^{\prime} \lambda^{\prime}, \text {,hel }}^{1 \gamma}$ :

$$
\begin{equation*}
\Im T_{h^{\prime} \lambda^{\prime}, h \lambda}^{2 \gamma}=\frac{1}{2} \sum_{n, \text { hel }} \prod_{i=1}^{n} \int \frac{\mathrm{~d}^{3} \mathbf{q}_{i}}{(2 \pi)^{3}} \frac{1}{2 E_{i}}\left(T_{\text {hel }, h^{\prime} \lambda^{\prime}}^{1 \gamma}\right)^{*} T_{\text {hel }, h \lambda}^{1 \gamma}(2 \pi)^{4} \delta^{4}\left(k+p-\sum_{i} q_{i}\right), \tag{4.12}
\end{equation*}
$$

with the momentum of the intermediate particle $q_{i}=\left(E_{i}, \mathbf{q}_{i}\right)$, the sum goes over all possible number $n$ of intermediate particles and all possible helicity states (denoted as "hel"). Unitarity relations allow to relate the imaginary part of the TPE amplitude to the experimental OPE input in a model-independent way.

The forward limit can be easily obtained from Eq. (4.12). Following the Jacob-Wick phase convention [191], we can factorize the dependence on the azimuthal angle of the intermediate
lepton $\phi_{1}$ in the integrands of unitarity relations as

$$
\begin{align*}
& \Im T_{1}^{2 \gamma} \sim \sum_{\text {hel }} \int \mathrm{d} \phi_{1}\left(T_{\mathrm{hel},++}^{1 \gamma}\right)^{*} T_{\mathrm{hel},++}^{1 \gamma},  \tag{4.13}\\
& \Im T_{2}^{2 \gamma} \sim \sum_{\text {hel }} \int \mathrm{d} \phi_{1}\left(T_{\mathrm{hel},+-}^{1 \gamma}\right)^{*} T_{\mathrm{hel},++}^{1 \gamma} e^{-i \phi_{1}},  \tag{4.14}\\
& \Im T_{3}^{2 \gamma} \sim \sum_{\text {hel }} \int \mathrm{d} \phi_{1}\left(T_{\mathrm{hel},+-}^{1 \gamma}\right)^{*} T_{\mathrm{hel},+-}^{1 \gamma},  \tag{4.15}\\
& \Im T_{4}^{2 \gamma} \sim \sum_{\text {hel }} \int \mathrm{d} \phi_{1}\left(T_{\mathrm{hel},-+}^{1 \gamma}\right)^{*} T_{\mathrm{hel},++}^{1 \gamma} e^{i \phi_{1}},  \tag{4.16}\\
& \Im T_{5}^{2 \gamma} \sim \sum_{\text {hel }} \int \mathrm{d} \phi_{1}\left(T_{\mathrm{hel},---}^{1 \gamma}\right)^{*} T_{\mathrm{hel},++}^{1 \gamma},  \tag{4.17}\\
& \Im T_{6}^{2 \gamma} \sim \sum_{\text {hel }} \int \mathrm{d} \phi_{1}\left(T_{\mathrm{hel},-+}^{1 \gamma}\right)^{*} T_{\mathrm{hel},+-}^{1 \gamma} e^{2 i \phi_{1}} . \tag{4.18}
\end{align*}
$$

The helicity ( $\Lambda$ ) phase factor $e^{-i \Lambda \phi_{1}}$ coming from the intermediate state cancels between the two OPE amplitudes, and we obtain the $\phi_{1}$ dependence only from the external states. Performing the lepton azimuthal angle integration first, we obtain that the amplitudes $T_{2}, T_{4}, T_{6}$ vanish in the forward limit reflecting the conservation of the angular momentum. Other relations for the amplitudes $T_{1}, T_{3}, T_{5}$ give precisely the optical theorem in the forward limit, see Eqs. (2.19, 2.20) of Chapter 2.

We first consider the unitarity relations of Eq. (4.12) for the proton intermediate state contribution, which by definition only involves on-shell amplitudes in the $1 \gamma$-exchange approximation. The unitarity relations are represented in Fig. 4.3.


Figure 4.3: Unitarity relations for the case of the elastic intermediate state contribution.
In the c.m. reference frame, the lepton energy $\omega_{\mathrm{cm}}$ and the momentum $\left|\mathbf{k}_{\mathrm{cm}}\right|$ are given by

$$
\begin{equation*}
\omega_{\mathrm{cm}}=\frac{s-M^{2}+m^{2}}{2 \sqrt{s}}, \quad\left|\mathbf{k}_{\mathrm{cm}}\right|=\frac{\sqrt{\Sigma_{s}}}{2 \sqrt{s}} \tag{4.19}
\end{equation*}
$$

The lepton initial $(k)$, intermediate $\left(k_{1}\right)$ and final $\left(k^{\prime}\right)$ momenta are given by: ${ }^{1}$

$$
\begin{align*}
k & =\left(\omega_{\mathrm{cm}}, 0,0,\left|\mathbf{k}_{\mathrm{cm}}\right|\right),  \tag{4.20}\\
k_{1} & =\left(\omega_{\mathrm{cm}},\left|\mathbf{k}_{\mathrm{cm}}\right| \sin \theta_{1} \cos \phi_{1},\left|\mathbf{k}_{\mathrm{cm}}\right| \sin \theta_{1} \sin \phi_{1},\left|\mathbf{k}_{\mathrm{cm}}\right| \cos \theta_{1}\right),  \tag{4.21}\\
k^{\prime} & =\left(\omega_{\mathrm{cm}},\left|\mathbf{k}_{\mathrm{cm}}\right| \sin \theta_{\mathrm{cm}}, 0,\left|\mathbf{k}_{\mathrm{cm}}\right| \cos \theta_{\mathrm{cm}}\right), \tag{4.22}
\end{align*}
$$

with the intermediate lepton angles $\theta_{1}$ and $\phi_{1}$.

[^9]We also introduce the relative angle between the 3-momenta of intermediate and final leptons as $\mathbf{k}_{1} \cdot \mathbf{k}^{\prime} \equiv \cos \theta_{2}$, with $\cos \theta_{2}=\cos \theta_{\mathrm{cm}} \cos \theta_{1}+\sin \theta_{\mathrm{cm}} \sin \theta_{1} \cos \phi_{1}$.

The imaginary parts of the elastic contribution to the TPE helicity amplitudes are given as an integral over the intermediate lepton angle by

$$
\begin{equation*}
\Im T_{h^{\prime} \lambda^{\prime}, h \lambda}^{2 \gamma}=\frac{1}{64 \pi^{2}} \frac{\sqrt{\Sigma_{s}}}{s} \sum_{\tilde{h} \tilde{\lambda}} \int \mathrm{~d} \Omega_{1}\left(T_{\tilde{h} \tilde{\lambda}, h^{\prime} \lambda^{\prime}}^{1 \gamma}\right)^{*} T_{\tilde{h} \tilde{\lambda}, h \lambda}^{1 \gamma} . \tag{4.23}
\end{equation*}
$$

The imaginary parts of the invariant amplitudes are given by relations of Eqs. (3.13) (Eqs. (3.50) in the massless lepton limit). The integrand in the unitarity relations can be expressed as a product of the OPE amplitudes with kinematical phases. Afterward, we exploit Eq. (4.23) for the numerical evaluations.
In case of the massless electron-proton scattering we obtain the following quite simple expressions:

$$
\begin{align*}
& \Im T_{1}^{2 \gamma}=\frac{1}{64 \pi^{2}} \frac{\sqrt{\Sigma_{s}}}{s} \int\left\{T_{1}^{1 \gamma}\left(Q_{1}^{2}\right) T_{1}^{1 \gamma}\left(Q_{2}^{2}\right)+T_{2}^{1 \gamma}\left(Q_{1}^{2}\right) T_{2}^{1 \gamma}\left(Q_{2}^{2}\right) \cos \left(\tilde{\phi}^{\prime}\right)\right\} \mathrm{d} \Omega_{1}, \\
& \Im T_{3}^{2 \gamma}=\frac{1}{64 \pi^{2}} \frac{\sqrt{\Sigma_{s}}}{s} \int\left\{T_{3}^{1 \gamma}\left(Q_{1}^{2}\right) T_{3}^{1 \gamma}\left(Q_{2}^{2}\right) \cos \left(\phi-\phi^{\prime}\right)-T_{2}^{1 \gamma}\left(Q_{1}^{2}\right) T_{2}^{1 \gamma}\left(Q_{2}^{2}\right) \cos (\phi+\tilde{\phi})\right\} \mathrm{d} \Omega_{1}, \\
& \Im T_{2}^{2 \gamma}=\frac{1}{64 \pi^{2}} \frac{\sqrt{\Sigma_{s}}}{s} \int\left\{T_{2}^{1 \gamma}\left(Q_{1}^{2}\right) T_{3}^{1 \gamma}\left(Q_{2}^{2}\right) \cos \left(\phi^{\prime}\right)+T_{1}^{1 \gamma}\left(Q_{1}^{2}\right) T_{2}^{1 \gamma}\left(Q_{2}^{2}\right) \cos (\tilde{\phi})\right\} \mathrm{d} \Omega_{1}, \tag{4.24}
\end{align*}
$$

where the phases $\phi, \phi^{\prime}, \tilde{\phi}, \tilde{\phi}^{\prime}$ are defined in Eq. (I.1) of Appendix I. The momentum transfers $Q_{1}^{2}$ and $Q_{2}^{2}$ correspond with the scattering from initial to intermediate state and with the scattering from intermediate to final state respectively. The OPE amplitudes, which were defined in Eq. (4.24) by explicitly taking out all kinematical phases, can be obtained from Eq. (3.49) after substitution of the invariant amplitudes by the corresponding FFs: $\mathcal{G}_{M} \rightarrow$ $G_{M}, \mathcal{F}_{2} \rightarrow F_{P}, \mathcal{F}_{3} \rightarrow 0$ and are given by

$$
\begin{align*}
T_{1}^{1 \gamma} & =2 e^{2} G_{M}+2 \frac{e^{2}}{Q^{2}} \frac{s u-M^{4}}{s-M^{2}}\left(F_{P}-G_{M}\right), \\
T_{2}^{1 \gamma} & =-\frac{e^{2}}{Q^{2}} \frac{\sqrt{Q^{2}\left(M^{4}-s u\right)}}{M}\left\{F_{P}+2 \frac{M^{2}}{s-M^{2}}\left(F_{P}-G_{M}\right)\right\}, \\
T_{3}^{1 \gamma} & =2 \frac{e^{2}}{Q^{2}} \frac{s u-M^{4}}{s-M^{2}}\left(F_{P}-G_{M}\right) . \tag{4.25}
\end{align*}
$$

We perform calculations separately for $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}, \mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ and $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structures (see Fig. 4.2) for both FF models described in Eqs. (3.52, 4.3). For the point-like model, we obtain analytical expressions for the imaginary part of TPE amplitudes. The imaginary parts of the invariant amplitudes due to the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure in the direct box graph are given by

$$
\begin{align*}
\Im \mathcal{G}_{M}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}} & =\alpha\left\{\ln \left(\frac{Q^{2}}{\mu^{2}}\right)-\frac{s+M^{2}}{2 s}-\frac{2\left(s-M^{2}\right)^{2}-s Q^{2}}{2\left(\left(s-M^{2}\right)^{2}-s Q^{2}\right)} \ln \frac{s Q^{2}}{\left(s-M^{2}\right)^{2}}\right\}, \\
\Im \mathcal{F}_{2}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}} & =\frac{\alpha M^{2} Q^{2}}{\left(s-M^{2}\right)^{2}-s Q^{2}}\left\{1+\frac{\left(s-M^{2}\right)^{2}}{\left(s-M^{2}\right)^{2}-s Q^{2}} \ln \frac{s Q^{2}}{\left(s-M^{2}\right)^{2}}\right\}, \\
\Im \mathcal{F}_{3}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}} & =\frac{\alpha M^{2}\left(s-M^{2}\right)}{\left(s-M^{2}\right)^{2}-s Q^{2}}\left\{\frac{s+M^{2}}{s}+\frac{\left(s-M^{2}\right)\left(2\left(s-M^{2}\right)-Q^{2}\right)}{\left(s-M^{2}\right)^{2}-s Q^{2}} \ln \frac{s Q^{2}}{\left(s-M^{2}\right)^{2}}\right\} . \tag{4.26}
\end{align*}
$$

The imaginary parts of the invariant amplitudes due to the mixed $F_{D} F_{P}$ vertex structure are given by

$$
\begin{align*}
\Im \mathcal{G}_{M}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{P}}}= & \alpha\left(\mu_{P}-1\right)\left\{\ln \left(\frac{Q^{2}}{\mu^{2}}\right)-\frac{M^{2}}{s}+\frac{2\left(s-M^{2}\right)^{2}-s Q^{2}}{2\left(\left(s-M^{2}\right)^{2}-s Q^{2}\right)} \ln \left(\frac{s Q^{2}}{\left(s-M^{2}\right)^{2}}\right)\right\}, \\
\Im \mathcal{F}_{2}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{P}}}= & \frac{\alpha\left(\mu_{P}-1\right) M^{2} Q^{2}}{\left(s-M^{2}\right)^{2}-s Q^{2}}\left\{1+\frac{\left(s-M^{2}\right)^{2}\left(s+2 M^{2}\right)-s^{2} Q^{2}}{2 M^{2}\left(\left(s-M^{2}\right)^{2}-s Q^{2}\right)} \ln \left(\frac{s Q^{2}}{\left(s-M^{2}\right)^{2}}\right)\right\} \\
& +\alpha\left(\mu_{P}-1\right) \ln \left(\frac{Q^{2}}{\mu^{2}}\right), \\
\Im \mathcal{F}_{3}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{P}}}= & \frac{\alpha\left(\mu_{P}-1\right) M^{2}}{s\left(\left(s-M^{2}\right)^{2}-s Q^{2}\right)}\left\{2 M^{2}\left(s-M^{2}\right)+s Q^{2}\right. \\
& \left.+\frac{s\left(s-M^{2}\right)^{2}\left(2\left(s-M^{2}\right)-Q^{2}\right)}{\left(s-M^{2}\right)^{2}-s Q^{2}} \ln \left(\frac{s Q^{2}}{\left(s-M^{2}\right)^{2}}\right)\right\} . \tag{4.27}
\end{align*}
$$

The imaginary parts of the invariant amplitudes due to the $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure are given by

$$
\begin{align*}
\Im \mathcal{G}_{M}^{\mathrm{F}_{\mathrm{P}} \mathrm{~F}_{\mathrm{P}}}= & \alpha\left(\mu_{P}-1\right)^{2} \frac{s-M^{2}}{2 s}\left\{1+\frac{s^{2} Q^{2}}{4 M^{2}\left(\left(s-M^{2}\right)^{2}-s Q^{2}\right)} \ln \left(\frac{s Q^{2}}{\left(s-M^{2}\right)^{2}}\right)\right\} \\
\Im \mathcal{F}_{2}^{\mathrm{F}_{\mathrm{P}} \mathrm{~F}_{\mathrm{P}}}= & -\frac{\alpha\left(\mu_{P}-1\right)^{2}}{4} \frac{\left(s-M^{2}\right) Q^{2}}{\left(s-M^{2}\right)^{2}-s Q^{2}}\left\{1+\frac{\left(s-M^{2}\right)^{2}}{\left(s-M^{2}\right)^{2}-s Q^{2}} \ln \left(\frac{s Q^{2}}{\left(s-M^{2}\right)^{2}}\right)\right\} \\
\Im \mathcal{F}_{3}^{\mathrm{F}_{\mathrm{P}} \mathrm{~F}_{\mathrm{P}}}= & -\frac{\alpha\left(\mu_{P}-1\right)^{2}}{4 s\left(\left(s-M^{2}\right)^{2}-s Q^{2}\right)}\left\{4 M^{2}\left(s-M^{2}\right)^{2}+s Q^{2}\left(s-3 M^{2}\right)\right. \\
& \left.+\frac{\left(M^{6}-3 M^{2} s^{2}+2 s^{3}-s^{2} Q^{2}\right) s Q^{2}}{\left(s-M^{2}\right)^{2}-s Q^{2}} \ln \left(\frac{s Q^{2}}{\left(s-M^{2}\right)^{2}}\right)\right\} \tag{4.28}
\end{align*}
$$

We checked that the numerical calculations of the imaginary part of the invariant amplitudes are in agreement with predictions for the target normal spin asymmetry $A_{n}$ [197], given by Eq. (3.67), for the model with dipole form of electromagnetic FFs [202].

### 4.2.2 Dispersion relations and high-energy behavior

Assuming the analyticity of TPE amplitudes in the complex $\nu$ variable one can write down the Cauchy's theorem for the fixed $Q^{2}$ value and relate the real and imaginary part through DRs. The TPE amplitudes have branching points in complex $\nu$-plane on the real axis at the particle production thresholds. As usual in DR analyses, we relate the imaginary part of TPE amplitudes (discontinuity across the cuts extending from the branching points up to infinity) to the experimental information. The dispersive integral starts from the threshold $\nu_{\text {thr }}$ corresponding with the branching point. The threshold corresponding with the elastic cut due to the proton intermediate state is located at $s=(M+m)^{2}$ or $\nu_{\mathrm{thr}}=M m-Q^{2} / 4$, so there is an integration region with intersection of $s$ - and $u$-channel cuts when $Q^{2}>4 M m$, in case of the electron-proton scattering experiments the cut intersection always happens. The threshold corresponding with the first inelastic cut due to the pion-nucleon intermediate states is given by $s=\left(M+m+m_{\pi}\right)^{2}$ or $\nu_{\mathrm{thr}}=M m+(M+m) m_{\pi}+m_{\pi}^{2} / 2-Q^{2} / 4$. The intersection of the
cuts for $e p$ scattering ( $m \approx 0$ ) happens for $Q^{2}>0.55 \mathrm{GeV}^{2}$. The threshold positions in case of the muon-proton (electron-proton) scattering are shown in Fig. 4.4 (Fig. 4.5).

The TPE amplitudes $\mathcal{G}_{M}^{2 \gamma}\left(\nu, Q^{2}\right), \mathcal{F}_{2}^{2 \gamma}\left(\nu, Q^{2}\right), \mathcal{F}_{5}^{2 \gamma}\left(\nu, Q^{2}\right), \mathcal{G}_{1}^{2 \gamma}\left(\nu, Q^{2}\right), \mathcal{G}_{2}^{2 \gamma}\left(\nu, Q^{2}\right)$ are odd functions under $\nu \rightarrow-\nu$, and the amplitudes $\mathcal{F}_{3}^{2 \gamma}\left(\nu, Q^{2}\right), \mathcal{F}_{4}^{2 \gamma}\left(\nu, Q^{2}\right), \mathcal{F}_{6}^{2 \gamma}\left(\nu, Q^{2}\right)$ are even. The amplitudes which are odd in $\nu, \mathcal{G}^{\text {odd }}$, satisfy the DR:

$$
\begin{equation*}
\Re \mathcal{G}^{\text {odd }}\left(\nu, Q^{2}\right)=\frac{2 \nu}{\pi} \int_{\nu_{\text {thr }}}^{\infty} \frac{\Im \mathcal{G}^{\text {odd }}\left(\nu^{\prime}, Q^{2}\right)}{\nu^{\prime 2}-\nu^{2}} \mathrm{~d} \nu^{\prime} . \tag{4.29}
\end{equation*}
$$

The amplitudes which are even in $\nu, \mathcal{G}^{\text {even }}$, satisfy the DR:

$$
\begin{equation*}
\Re \mathcal{G}^{\text {even }}\left(\nu, Q^{2}\right)=\frac{2}{\pi} \int_{\nu_{\text {thr }}}^{\infty} \frac{\Im \mathcal{G}^{\text {even }}\left(\nu^{\prime}, Q^{2}\right)}{\nu^{\prime 2}-\nu^{2}} \mathrm{~d} \nu^{\prime} . \tag{4.30}
\end{equation*}
$$

The imaginary part in Eqs. $(4.29,4.30)$ is taken from the $s$-channel discontinuity only. The Cauchy's theorem can be applied only when the function on the contour at infinity vanishes. Consequently, the unsubtracted DRs as given by Eqs. $(4.29,4.30)$ can only be written down for functions with appropriate high-energy (HE) behavior.

Traditionally, the Regge theory helps to estimate the HE behavior of the invariant amplitudes. However, Pomeron and mesons do not directly couple to leptons. Consequently, the HE behavior in the Regge analysis is approximate. In the high energy limit $\nu \rightarrow \infty$ with a fixed $Q^{2}$, the helicity amplitudes behavior is given by the Pomeron and meson Regge trajectories $T_{1} \simeq T_{3} \sim \nu^{x_{\mathbb{P}}} \sim \nu^{1.08}, T_{2} \simeq T_{4} \simeq T_{5} \simeq T_{6} \sim \nu^{x_{M}}$ with $0<x_{M}<0.5$. The HE behavior of the invariant amplitudes is given by the leading exponent in Eqs. (3.13). The leading contribution from the Pomeron trajectory drops out from the amplitudes $\mathcal{G}_{M}, \mathcal{F}_{3}$, resulting at $\mathcal{G}_{M} \sim \nu^{x_{A}}$ with $x_{A}<1.08$. The main contribution to $\mathcal{G}_{M}$ amplitude is given by the axial vector exchange, so we conclude $x_{A}<0$. The HE Regge behavior of the amplitudes $\mathcal{F}_{2}, \mathcal{F}_{3}, \mathcal{F}_{4}, \mathcal{F}_{5}$ is given by $\mathcal{F}_{2} \sim \nu^{x_{M}-1}, \mathcal{F}_{3} \sim \nu^{x_{\mathbb{P}}-2}, \mathcal{F}_{4} \sim \nu^{x_{M}}, \mathcal{F}_{5} \sim \nu^{x_{M}-1}$ with $x_{M}-1<-0.5$. Equivalently, the amplitudes $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ introduced in Eqs. $(3.37,3.38)$ have the HE behavior: $\mathcal{G}_{1}, \mathcal{G}_{2} \sim \nu^{x_{\mathbb{P}}-1}$. The main contribution to $\mathcal{F}_{6}$ comes from the axial vector exchange, therefore $\mathcal{F}_{6} \sim \nu^{x_{A}}$ with $x_{A}<0$, the vector exchange contribution to $\mathcal{F}_{6}$ is 0 for all energies. The expected HE exponents for the invariant amplitudes are collected in Table 4.1. Based on such Regge arguments,

| $\mathcal{G}_{M}$ | $\mathcal{F}_{2}, \mathcal{F}_{5}$ | $\mathcal{F}_{3}$ | $\mathcal{F}_{4}$ | $\mathcal{F}_{6}$ | $\mathcal{G}_{1}, \mathcal{G}_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $x_{A}<0$ | $x_{M}-1<-0.5$ | $x_{\mathbb{P}}-2<-0.9$ | $x_{M}>0$ | $x_{A}<0$ | $x_{\mathbb{P}}-1$ |

Table 4.1: The value of the HE exponent $\nu^{x}$ based on the Regge phenomenology.
we can expect applicability of the unsubtracted DR for all amplitudes except for $\mathcal{F}_{4}, \mathcal{G}_{1}$ and $\mathcal{G}_{2}$. Note that the Regge arguments do not constrain the HE behavior of the TPE correction to the measured observables due to the non-vanishing behavior of $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$.

We will next discuss the HE behavior of the invariant amplitudes for the case of the box diagram calculation with the proton intermediate state, which is explained in detail in Section 4.1. We consider first the virtual photon-proton-proton vertices as point couplings. Furthermore we consider three contributions, whether both vertices correspond with vector couplings (referred to as $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ structure), both vertices correspond with tensor couplings ( $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ structure), or whether one vertex corresponds with a vector and the second vertex with a tensor coupling ( $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ structure), as it is shown in Fig. 4.2.

In general, the HE behavior ( $\nu \gg Q^{2}, M^{2}, m^{2}$ ) of the TPE amplitudes can be parametrized as $\mathcal{G}^{2 \gamma}(\nu) \simeq \nu^{\beta}\left(c_{1}+c_{2} \nu^{\rho} \ln \nu\right)$ with $\rho \leq 0$, where the parameters can be extracted with a help of the fit to the calculation. In Tables 4.2-4.4, we show the extracted value of the leading power $\beta$ for the different invariant amplitudes and for the different cases of virtual photon-proton-proton vertices.

The amplitudes $\mathcal{G}_{M}^{2 \gamma}, \mathcal{F}_{3}^{2 \gamma}, \mathcal{F}_{6}^{2 \gamma}$ are UV divergent in the case of the point-like proton with two magnetic vertices $\left(\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}\right)$. The divergent piece is given by

$$
\begin{equation*}
\mathcal{G}_{M}^{\mathrm{UV}}=-\frac{\nu}{M^{2}} \mathcal{F}_{3}^{\mathrm{UV}}=\frac{\nu}{M^{2}} \mathcal{F}_{6}^{\mathrm{UV}} \tag{4.31}
\end{equation*}
$$

|  | $\mathcal{G}_{M}$ | $\mathcal{F}_{2}$ | $\mathcal{F}_{3}$ | $\mathcal{F}_{4}$ | $\mathcal{F}_{5}$ | $\mathcal{F}_{6}$ | $\mathcal{G}_{1}$ | $\mathcal{G}_{2}$ | $\mathcal{F}_{4}-\mathcal{F}_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\beta$ for $\Im \mathcal{G}^{2 \gamma}$ | 0 | -2 | -1 | -1 | -2 | -1 | -1 | -1 | -1 |
| $\beta$ for $\Re \mathcal{G}^{2 \gamma}$ | 0 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |

Table 4.2: The value of the leading power $\beta$ in the HE fit of the different invariant amplitudes (upper row: imaginary parts, lower row: real parts) according to the form $\mathcal{G}^{2 \gamma}(\nu) \simeq$ $\nu^{\beta}\left(c_{1}+c_{2} \nu^{\rho} \ln \nu\right)$ with $\rho \leq 0$, for the box diagram model with point-like $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure.

|  | $\mathcal{G}_{M}$ | $\mathcal{F}_{2}$ | $\mathcal{F}_{3}$ | $\mathcal{F}_{4}$ | $\mathcal{F}_{5}$ | $\mathcal{F}_{6}$ | $\mathcal{G}_{1}$ | $\mathcal{G}_{2}$ | $\mathcal{F}_{4}-\mathcal{F}_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\beta$ for $\Im \mathcal{G}^{2 \gamma}$ | 0 | -1 | -1 | 0 | -1 | 0 | -1 | -1 | -1 |
| $\beta$ for $\Re \mathcal{G}^{2 \gamma}$ | 0 | -1 | -1 | 0 | -1 | 0 | -1 | -1 | 0 |

Table 4.3: Same as Table 4.2, but for the box diagram model with point-like $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structure.

|  | $\mathcal{G}_{M}$ | $\mathcal{F}_{2}$ | $\mathcal{F}_{3}$ | $\mathcal{F}_{4}$ | $\mathcal{F}_{5}$ | $\mathcal{F}_{6}$ | $\mathcal{G}_{1}$ | $\mathcal{G}_{2}$ | $\mathcal{F}_{4}-\mathcal{F}_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\beta$ for $\Im \mathcal{G}^{2 \gamma}$ | 0 | -1 | -1 | 0 | -1 | 0 | 0 | 0 | -1 |
| $\beta$ for $\Re \mathcal{G}^{2 \gamma}$ | 1 | -1 | 0 | 0 | -1 | 0 | 0 | 0 | 0 |

Table 4.4: Same as Table 4.2, but for the box diagram model with point-like $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure.

For the case of $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ (and $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ in elastic electron-proton scattering) vertex structure, one notices that the behavior of all amplitudes is sufficient to ensure unsubtracted DRs. However, the dispersive integral for the amplitudes $\mathcal{F}_{4}^{2 \gamma}, \mathcal{F}_{6}^{2 \gamma}$ does not converge for the case of $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ and $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structures. For the case of two magnetic vertices ( $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ structure), we notice that the TPE amplitudes $\mathcal{F}_{2}^{2 \gamma}, \mathcal{F}_{5}^{2 \gamma}, \mathcal{F}_{3}^{2 \gamma}+\mathcal{F}_{6}^{2 \gamma}, \mathcal{G}_{1}^{2 \gamma}, \mathcal{G}_{2}^{2 \gamma}$ are sufficiently convergent to satisfy an unsubtracted DR. Also after the UV regularization the amplitude $\mathcal{G}_{M}^{2 \gamma}\left(\mathcal{F}_{3}^{2 \gamma}, \mathcal{F}_{6}^{2 \gamma}, \mathcal{F}_{4}^{2 \gamma}\right)$ has a real part which is behaving as $\nu\left(\nu^{0}\right)$ respectively, which in both cases leads to a constant contribution due to the contour at infinity in the Cauchy's integral formula. This constant term cannot be reconstructed from the imaginary part of the amplitude. The same constant term appears in the amplitude $\mathcal{F}_{4}^{2 \gamma}-\mathcal{F}_{6}^{2 \gamma}$, which has a convergent DR integral for the case of $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ and $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structures. To partially avoid such unknown contributions, we use in our following calculations instead of the amplitudes $\mathcal{G}_{M}^{2 \gamma}$ and $\mathcal{F}_{2}^{2 \gamma}$ the amplitudes $\mathcal{G}_{1}^{2 \gamma}$ and $\mathcal{G}_{2}^{2 \gamma}$,
defined in Eqs. $(3.37,3.38)$. As is clear from Tables 4.2-4.4, the amplitudes $\mathcal{G}_{1}^{2 \gamma}$ and $\mathcal{G}_{2}^{2 \gamma}$ are expected to satisfy the unsubtracted DR.

The inclusion of the proton FFs of Eqs. (3.52) gives a more realistic description of the proton electromagnetic properties and, usually, improves the HE behavior of TPE amplitudes. We study the HE behavior of TPE amplitudes in the model with dipole FFs and present the extracted value of the leading power $\beta$ from the HE fit in the following Tables 4.5-4.7

|  | $\mathcal{G}_{M}$ | $\mathcal{F}_{2}$ | $\mathcal{F}_{3}$ | $\mathcal{F}_{4}$ | $\mathcal{F}_{5}$ | $\mathcal{F}_{6}$ | $\mathcal{G}_{1}$ | $\mathcal{G}_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\beta$ for $\Im \mathcal{G}^{2 \gamma}$ | 0 | -2 | -1 | -1 | -2 | -1 | 0 | 0 |
| $\beta$ for $\Re \mathcal{G}^{2 \gamma}$ | -1 | -1 | -2 | -2 | -1 | -2 | -1 | -1 |

Table 4.5: The value of the leading power $\beta$ in the HE fit of the different invariant amplitudes (upper row: imaginary parts, lower row: real parts) according to the form $\mathcal{G}^{2 \gamma}(\nu) \simeq$ $\nu^{\beta}\left(c_{1}+c_{2} \nu^{\rho} \ln \nu\right)$ with $\rho \leq 0$, for the box diagram model with $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure and dipole FFs.

|  | $\mathcal{G}_{M}$ | $\mathcal{F}_{2}$ | $\mathcal{F}_{3}$ | $\mathcal{F}_{4}$ | $\mathcal{F}_{5}$ | $\mathcal{F}_{6}$ | $\mathcal{G}_{1}$ | $\mathcal{G}_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\beta$ for $\Im \mathcal{G}^{2 \gamma}$ | 0 | 0 | -1 | -1 | -2 | -1 | 0 | 0 |
| $\beta$ for $\Re \mathcal{G}^{2 \gamma}$ | -1 | -1 | -2 | 0 | -1 | 0 | -1 | -1 |

Table 4.6: Same as Table 4.5, but for the box diagram model with $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structure.

|  | $\mathcal{G}_{M}$ | $\mathcal{F}_{2}$ | $\mathcal{F}_{3}$ | $\mathcal{F}_{4}$ | $\mathcal{F}_{5}$ | $\mathcal{F}_{6}$ | $\mathcal{G}_{1}$ | $\mathcal{G}_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\beta$ for $\Im \mathcal{G}^{2 \gamma}$ | 0 | -1 | -1 | -1 | -2 | -1 | 0 | 0 |
| $\beta$ for $\Re \mathcal{G}^{2 \gamma}$ | 1 | -1 | 0 | 0 | -1 | 0 | -1 | -1 |

Table 4.7: Same as Table 4.5, but for the box diagram model with $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure.
All TPE amplitudes are UV finite in the box graph model with dipole proton FFs. Now the unsubtracted dispersive integral for the amplitudes $\mathcal{F}_{4}^{2 \gamma}, \mathcal{F}_{6}^{2 \gamma}$ converges for all vertex structures. However the amplitudes $\mathcal{G}_{M}^{2 \gamma}$ and $\mathcal{F}_{3}^{2 \gamma}, \mathcal{F}_{6}^{2 \gamma}, \mathcal{F}_{4}^{2 \gamma}$ have a real part which is behaving as $\nu$ and $\nu^{0}$ respectively in the case of $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure, consequently one can not use the unsubtracted DR for these amplitudes. Additionally, the real part of the amplitudes $\mathcal{F}_{4}^{2 \gamma}$ and $\mathcal{F}_{6}^{2 \gamma}$, in the case of $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structure, behaves as $\nu^{0}$. This constant is not reconstructed within unsubtracted DRs.

All TPE amplitudes discussed above are expected to satisfy the once-subtracted DR that can be formally obtained by subtraction at a low-energy point $\nu_{0}$ in Eqs. (4.29, 4.30):

$$
\begin{align*}
& \Re \mathcal{G}^{\text {odd }}\left(\nu, Q^{2}\right)-\Re \mathcal{G}^{\text {odd }}\left(\nu_{0}, Q^{2}\right)=\frac{2 \nu\left(\nu^{2}-\nu_{0}^{2}\right)}{\pi} f_{\nu_{\mathrm{thr}}}^{\infty} \frac{\Im \mathcal{G}^{\text {odd }}\left(\nu^{\prime}, Q^{2}\right)}{\left(\nu^{\prime 2}-\nu^{2}\right)\left(\nu^{\prime 2}-\nu_{0}^{2}\right)} \mathrm{d} \nu^{\prime},  \tag{4.32}\\
& \Re \mathcal{G}^{\text {even }}\left(\nu, Q^{2}\right)-\Re \mathcal{G}^{\text {even }}\left(\nu_{0}, Q^{2}\right)=\frac{2\left(\nu^{2}-\nu_{0}^{2}\right)}{\pi} f_{\nu_{\mathrm{thr}}}^{\infty} \frac{\nu^{\prime} \Im \mathcal{G}^{\text {even }}\left(\nu^{\prime}, Q^{2}\right)}{\left(\nu^{\prime 2}-\nu^{2}\right)\left(\nu^{\prime 2}-\nu_{0}^{2}\right)} \mathrm{d} \nu^{\prime} \tag{4.33}
\end{align*}
$$

For the amplitudes $\mathcal{G}_{M}^{2 \gamma}, \mathcal{F}_{3}^{2 \gamma}, \mathcal{F}_{4}^{2 \gamma}, \mathcal{F}_{6}^{2 \gamma}$, which are even in $\nu$, in the case of the lepton-proton scattering and for which an UV regularization has to be performed in the box diagram model
when using point-like couplings, we will in the following compare the unsubtracted DR with a once-subtracted DR. The unknown subtraction constant in the $\mathcal{F}_{3}^{2 \gamma}$ amplitude in the case of the electron-proton scattering can be fixed to the electron-proton scattering data [104].

It is also instructive to study the possible HE behavior of the real part of the amplitude $\mathcal{G}$ reconstructed within the unsubtracted DR . We start with the case of the odd amplitude $\mathcal{G}^{\text {odd }}$ and assume in the following the HE behavior, sufficient for the convergence of the DR integral, of the imaginary part $\Im \mathcal{G}^{\text {odd }}\left(\nu, Q^{2}\right) \simeq \nu^{\beta}\left(c_{1}+c_{2} \ln \nu\right)$ with the integer $\beta \leq 0$. It is convenient to represent the unsubtracted DR integral of Eq. (4.29) as a sum of two integrals with the large separation scale $\nu_{0}: \nu \gg \nu_{0} \gg Q^{2}, M^{2}, m^{2}$. Above this scale, we expect the imaginary part to behave as $\Im \mathcal{G}^{\text {odd }}\left(\nu, Q^{2}\right)=\nu^{\beta}\left(c_{1}+c_{2} \ln \nu\right)$. The corresponding real part is then given by

$$
\begin{equation*}
\Re \mathcal{G}^{\text {odd }}\left(\nu, Q^{2}\right) \simeq-\frac{2}{\pi \nu} \int_{\nu_{\mathrm{thr}}}^{\nu_{0}} \Im \mathcal{G}^{\text {odd }}\left(\nu^{\prime}, Q^{2}\right) \mathrm{d} \nu^{\prime}+\frac{2 \nu^{\beta}}{\pi} \int_{x_{0} \equiv \frac{\nu_{0}}{\nu}}^{\infty} \frac{x^{\beta}\left(c_{1}+c_{2} \ln \frac{x}{x_{0}}+c_{2} \ln \nu_{0}\right)}{x^{2}-1} \mathrm{~d} x \tag{4.34}
\end{equation*}
$$

The leading HE behavior of the first integral is given by $1 / \nu$. The logarithmic term $x^{\beta} \ln \left(x / x_{0}\right)$ from the second integral leads to the HE behavior of the real part: $\nu^{0}$ for $\beta=0,\left(\ln ^{2} \nu\right) / \nu$ for $\beta=-1$ and $\nu^{-1}$ for all other integer $\beta<-1$. Finally, the term with $x^{\beta}$ from the second integral leads to the leading HE behavior of the real part given by $(\ln \nu) / \nu$ for $\beta=-1$ and $1 / \nu$ for all other values of $\beta$. Consequently, the correspondent exponent $\tilde{\beta}$ in the HE behavior of the odd amplitude $\Re \mathcal{G}^{\text {odd }}\left(\nu, Q^{2}\right) \simeq \nu^{\tilde{\beta}}\left(\tilde{c}_{1}+\tilde{c}_{2} \ln \nu+\tilde{c}_{3} \ln ^{2} \nu\right)$, reconstructed within the unsubtracted DR , in general have the upper bound $\tilde{\beta} \leq-1$ and can be constant only in the case of the logarithmic leading behavior of the imaginary part $\Im \mathcal{G}^{\text {odd }}\left(\nu, Q^{2}\right) \sim \ln \nu$. These properties are represented in Tables 4.2-4.7 for all odd TPE amplitudes, when we expect the unsubtracted DR to be valid in the box graph model.

We next turn to the even amplitudes. We assume the HE behavior, sufficient for the convergence of the DR integral, of the imaginary part $\Im \mathcal{G}^{\text {even }}\left(\nu, Q^{2}\right) \simeq \nu^{\beta}\left(c_{1}+c_{2} \ln \nu\right)$ with the integer $\beta \leq-1$. The unsubtracted DR integral of Eq. (4.30) can then be written as

$$
\begin{equation*}
\Re \mathcal{G}^{\mathrm{even}}\left(\nu, Q^{2}\right) \simeq \frac{2}{\pi \nu^{2}} \int_{\nu_{0}}^{\nu_{\mathrm{thr}}} \nu^{\prime} \Im \mathcal{G}^{\mathrm{even}}\left(\nu^{\prime}, Q^{2}\right) \mathrm{d} \nu^{\prime}+\frac{2 \nu^{\beta}}{\pi} \int_{x_{0} \equiv \frac{\nu_{0}}{\nu}}^{\infty} \frac{x^{\beta+1}\left(c_{1}+c_{2} \ln \frac{x}{x_{0}}+c_{2} \ln \nu_{0}\right)}{x^{2}-1} \mathrm{~d} x \tag{4.35}
\end{equation*}
$$

The leading HE behavior of the real part of the first integral is given by a power $\nu^{-2}$. The logarithmic term $x^{\beta+1} \ln \left(x / x_{0}\right)$ from the second integral reflects in the HE behavior of the real part: $\nu^{-1}$ for $\beta=-1,\left(\ln ^{2} \nu\right) / \nu^{2}$ for $\beta=-2$ and $\nu^{-2}$ for all other integer $\beta<-2$. The term $x^{\beta+1}$ from the second integral leads to the leading HE behavior given by $(\ln \nu) / \nu^{2}$ for $\beta=-2$ and $1 / \nu^{2}$ for all other values of $\beta$. Consequently, the HE behavior of the real part of the even amplitude in the unsubtracted DR analysis is expected to be vanishing. The above mentioned properties are represented in Tables 4.2-4.7 for the even amplitudes that are expected to satisfy the unsubtracted DR in the box graph model.

According to studies above, the real part of the TPE amplitude reconstructed within unsubtracted dispersion relations vanishes or behaves as a constant at high energies. The latter is possible only for the odd amplitude with the logarithmic behavior of the imaginary part. Consequently, the unsubtracted dispersion relation evaluation does not contradict the unitarity constraints of Eqs. (3.21-3.23, 3.43, 3.44) when exploiting the unitarity consistent imaginary parts as an input.

The HE behavior of the invariant amplitudes in Tables 4.2-4.7 provides the vanishing behavior of the corresponding TPE correction to the unpolarized lepton-proton scattering cross section of Eqs. $(3.36,3.55)$ and to the polarization transfer observable $P_{t} / P_{l}$ of Eq. (3.63) except for the case of $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure in the model with point-like proton. Accounting for the constant HE behavior of the amplitudes $\mathcal{G}_{1}, \mathcal{G}_{2}$ imaginary parts in the case of arbitrary vertex structure except for the point-like $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ structure, the dispersion relation analysis also gives the vanishing HE behavior of the discussed above observables.

### 4.2.3 Analytical continuation into the unphysical region for proton intermediate state

To evaluate the dispersive integral at a fixed value of momentum transfer $Q^{2}$ we have to know the imaginary part of the invariant amplitude from the threshold in energy upwards. The imaginary part evaluated from the unitarity relations by performing a phase space integration over physical angles covers only the "physical" region of integration. The invariant amplitudes also have an imaginary part outside the physical region as long as one is above the threshold energy. Accounting for only the contribution of the physical region to the invariant amplitudes is in contradiction with the results obtained from the direct box graph evaluation for the electron-muon scattering [206]. Starting from the imaginary part of the invariant amplitude in the physical region, we will now discuss how to continue it analytically into the unphysical region. To illustrate the physical and unphysical regions, we show in Fig. 4.4 the Mandelstam plot for the elastic muon-proton scattering.


Figure 4.4: Physical and unphysical regions of the kinematical variables $\nu$ and $Q^{2}$ (Mandelstam plot) for the elastic muon-proton scattering. The hatched blue region corresponds to the physical region, the long-dashed green lines give the elastic threshold positions, the short-dashed brown lines give the inelastic threshold positions. The horizontal red curve indicates the line at fixed $Q^{2}$ along which the dispersive integrals are evaluated. For $Q^{2} \gtrsim 0.4 \mathrm{GeV}^{2}\left(Q^{2} \gtrsim 1 \mathrm{GeV}^{2}\right)$ the s- and u-channel elastic (pionnucleon) cuts overlap.

The boundary of the physical region is defined by the hyperbola:

$$
\begin{equation*}
\nu=\nu_{\mathrm{ph}} \equiv M m \sqrt{1+\tau_{P}} \sqrt{1+\tau_{l}} \tag{4.36}
\end{equation*}
$$

with $\tau_{l}$ and $\tau_{P}$ defined as in Eqs. (2.46). Therefore, the evaluation of the dispersive integral for the elastic intermediate state contribution to the TPE amplitudes for $Q^{2}>0$ always requires the information from the unphysical region. Note that the intersection between the backward angle branch of the hyperbola of Eq. (4.36) and the line $s=\left(M+m+m_{\pi}\right)^{2}$ describing the first inelastic threshold corresponds with $Q_{\mathrm{th}}^{2}=m_{\pi}\left(2 m+m_{\pi}\right)\left(2 M+m_{\pi}\right)(2(M+m)+$ $\left.m_{\pi}\right) /\left(M+m+m_{\pi}\right)^{2}$ (indicated by the red horizontal line in Fig. 4.4). For $Q^{2}<Q_{\mathrm{th}}^{2} \simeq$ $0.150 \mathrm{GeV}^{2}$, an analytical continuation into the unphysical region is only required for the evaluation of the cut in the box diagram due to the proton intermediate state. For $Q^{2}$ larger than this value, also the evaluation of the cut due to the $\pi N$ inelastic intermediate states requires an analytical continuation into the unphysical region. Fortunately, the kinematically allowed momentum transfer region of the MUSE experiment $Q^{2}<0.116 \mathrm{GeV}^{2}$ does not require an analytical continuation for inelastic contributions to TPE amplitudes and the method of analytical continuation discussed in this Section allows to account for all contributions from the unphysical region.

The Mandelstam plot with physical and unphysical regions for the elastic electron-proton scattering is shown in Fig. 4.5.


Figure 4.5: Same as Fig. 4.4, for the elastic electron-proton scattering.
In the massless lepton limit the physical region is bounded by the hyperbola:

$$
\begin{equation*}
\nu=\nu_{\mathrm{ph}} \equiv M^{2} \sqrt{\tau_{P}\left(1+\tau_{P}\right)}=\frac{\sqrt{Q^{2}\left(Q^{2}+4 M^{2}\right)}}{4} \tag{4.37}
\end{equation*}
$$

An analytical continuation of TPE amplitudes into the unphysical region is only required for the evaluation of the cut in the box diagram due to the proton intermediate state for $Q^{2}<m_{\pi}^{2}\left(2 M+m_{\pi}\right)^{2} /\left(M+m_{\pi}\right)^{2} \simeq 0.064 \mathrm{GeV}^{2}$ (indicated by the red horizontal line in Fig. 4.5). For larger $Q^{2}$ values, also the evaluation of the $\pi N$ inelastic intermediate states contribution requires an analytical continuation into the unphysical region.

We next discuss the integration region entering the unitarity relations for the case of the proton intermediate state contribution. The momentum transfers for the OPE processes entering the r.h.s. of the unitarity relations of Eqs. $(4.23,4.24)$ are given by

$$
\begin{equation*}
Q_{1}^{2}=\frac{\Sigma_{s}}{2 s}\left(1-\cos \theta_{1}\right), \quad Q_{2}^{2}=\frac{\Sigma_{s}}{2 s}\left(1-\cos \theta_{2}\right) . \tag{4.38}
\end{equation*}
$$

The indices 1,2 correspond to scattering from initial to intermediate state and from intermediate to final state. The momentum transfer $Q^{2}$ obtains its maximal value for backward scattering $\theta=180^{\circ}$. If $Q_{1}^{2}$ is maximal (i.e., $\theta_{1}=180^{\circ}$ ), then $Q_{2}^{2}$ can be evaluated as

$$
\begin{equation*}
Q_{1}^{2}=Q_{\max }^{2}=\frac{\Sigma_{s}}{s}, \quad Q_{2}^{2}=\frac{\Sigma_{s}}{s}-Q^{2} . \tag{4.39}
\end{equation*}
$$

The phase space integration in Eq. (4.24) maps out an ellipse in the $Q_{1}^{2}, Q_{2}^{2}$ plane, where the position of the major axis depends on the elastic scattering angle (or $Q^{2}$ ). The centre of the ellipse is located at $Q_{1}^{2}=Q_{2}^{2}=Q_{\max }^{2} / 2 \equiv Q_{c}^{2}$. For forward and backward scattering, the ellipse reduces to a line: $Q_{1}^{2}=Q_{2}^{2}$ for $\theta_{\mathrm{cm}}=0^{0}$, and $Q_{2}^{2}=Q_{\max }^{2}-Q_{1}^{2}$ for $\theta_{\mathrm{cm}}=180^{0}$. In Fig. 4.6, we show the physical integration regions for different kinematics in elastic electron-proton scattering. Note that the ellipses are enclosed in rectangular boxes of sizes $Q_{1}^{2}, Q_{2}^{2} \leq Q_{\text {max }}^{2}$, see Fig. 4.6.

We will now demonstrate the procedure of analytical continuation on the example of the integral which corresponds with one denominator (originating from one of photon propagators) on the r.h.s. of the unitarity relations in Eq. (4.24). The phase space integration entering the unitarity relations can be expressed in terms of the elliptic coordinates $\alpha$ and $\phi$ (see Appendix J) as

$$
\begin{equation*}
\int \frac{g\left(Q_{1}^{2}, Q_{2}^{2}\right) \mathrm{d} \Omega_{1}}{Q_{1,2}^{2}+\mu^{2}} \sim \int_{0}^{1} \mathrm{~d} \tilde{\alpha} \int_{0}^{2 \pi} \mathrm{~d} \phi \frac{g\left(Q_{c}^{2}(a+b \cos \phi-c \sin \phi), Q_{c}^{2}(a+b \cos \phi+c \sin \phi)\right)}{a+b \cos \phi \mp c \sin \phi}, \tag{4.40}
\end{equation*}
$$

with

$$
a=1+\frac{2 s \mu^{2}}{\Sigma_{s}}, \quad b=\sqrt{1-\tilde{\alpha}^{2}} \sqrt{1-\frac{s Q^{2}}{\Sigma_{s}}}, \quad c=\sqrt{1-\tilde{\alpha}^{2}} \sqrt{\frac{s Q^{2}}{\Sigma_{s}}}
$$

The angular integration can be performed on a unit circle in a complex plane with $z=e^{i \phi}$ :

$$
\begin{align*}
& \int_{0}^{2 \pi} \mathrm{~d} \phi \frac{g\left(Q_{1}^{2}, Q_{2}^{2}\right)}{a+b \cos \phi-c \sin \phi}=-i \oint \frac{g\left(Q_{1}^{2}, Q_{2}^{2}\right)}{b+i c} \frac{2 \mathrm{~d} z}{\left(z-z_{1}\right)\left(z-z_{2}\right)}, \\
& \int_{0}^{2 \pi} \mathrm{~d} \phi \frac{g\left(Q_{1}^{2}, Q_{2}^{2}\right)}{a+b \cos \phi+c \sin \phi}=-i \oint \frac{g\left(Q_{1}^{2}, Q_{2}^{2}\right)}{b-i c} \frac{2 \mathrm{~d} z}{\left(z-z_{3}\right)\left(z-z_{4}\right)}, \tag{4.41}
\end{align*}
$$

with poles position given by

$$
\begin{align*}
& z_{1,2}=\frac{1}{b+i c}\left(-a \pm \sqrt{a^{2}-\left(1-\tilde{\alpha}^{2}\right)}\right),  \tag{4.42}\\
& z_{3,4}=\frac{1}{b-i c}\left(-a \pm \sqrt{a^{2}-\left(1-\tilde{\alpha}^{2}\right)}\right) . \tag{4.43}
\end{align*}
$$



Figure 4.6: The phase space integration regions entering the unitarity relations for the case of a proton intermediate state in TPE graph of the elastic electron-proton scattering.

In the physical region $\Sigma_{s}>s Q^{2}$, the integral is given by the residues of the poles $z_{1}, z_{3}("+"$ sign in Eqs. (4.42, 4.43)), see Fig. 4.7.


Figure 4.7: The moduli of the pole positions in the physical region entering the angular integral in Eqs. (4.41) for the laboratory electron energy $\omega=0.3 \mathrm{GeV}$ and $\mu=10^{-6} \mathrm{GeV}$. Note that these moduli do not depend on the momentum transfer $Q^{2}$. The poles $z_{1}$ and $z_{3}$ are inside the unit circle of integration $(|z|=1)$ for all values of $\tilde{\alpha}$.


Figure 4.8: Imaginary part of the poles in the unphysical region entering the angular integral in Eqs. (4.41) for the laboratory electron energy $\omega=0.3 \mathrm{GeV}, \mu=10^{-6} \mathrm{GeV}$ and $Q^{2}=0.35 \mathrm{GeV}^{2}$ (for which $b_{0}=0.78$ and $c_{0}=1.27$ ). The poles lie on the imaginary axis in the unphysical region. The pole $z_{3}$ is outside the unit circle for the values $\tilde{\alpha}<\tilde{\alpha}_{0}=0.61$. The intersections of the new contour of integration with the imaginary axis are shown by the horizontal solid lines, corresponding with values $c_{0}-b_{0} \simeq 0.49$ (upper line) and $-c_{0}-b_{0} \simeq 2.05$ (lower line) respectively.

In the unphysical region $\Sigma_{s}<s Q^{2}$, the positions of the poles change with respect to the unit circle (Fig. 4.8), so the integral has a discontinuity at the transition point. To avoid the discontinuities, we define an analytical continuation by deforming the integration contour so as to include the poles $z_{1}$ and $z_{3}$. The integration can be done on the circle of the radius $c_{0}$ and the centre $-i b_{0}$ as

$$
\begin{equation*}
\int_{0}^{2 \pi} f\left(e^{i \phi}\right) \mathrm{d} \phi=\oint_{|z|=1}-i f(z) \frac{\mathrm{d} z}{z} \rightarrow \oint_{z=c_{0} e^{i \phi}-i b_{0}}-i f(z) \frac{\mathrm{d} z}{z} \tag{4.44}
\end{equation*}
$$

with

$$
c_{0}=\sqrt{s Q^{2} / \Sigma_{s}}, \quad b_{0}=\sqrt{-1+s Q^{2} / \Sigma_{s}}
$$

For the value $\tilde{\alpha}=0$, when the expression in brackets of Eqs. (4.42, 4.43) approaches its minimum, the positions of the poles of interest (for small photon mass parameter $\mu \rightarrow 0$ ) are given by

$$
\begin{align*}
z_{1} & =\frac{i}{b_{0}+c_{0}}\left(1-2 \mu \sqrt{\frac{s}{\Sigma_{s}}}\right)=i\left(c_{0}-b_{0}\right)\left(1-2 \mu \sqrt{\frac{s}{\Sigma_{s}}}\right) \\
z_{3} & =\frac{i}{b_{0}-c_{0}}\left(1-2 \mu \sqrt{\frac{s}{\Sigma_{s}}}\right)=-i\left(c_{0}+b_{0}\right)\left(1-2 \mu \sqrt{\frac{s}{\Sigma_{s}}}\right) . \tag{4.45}
\end{align*}
$$

These poles lie inside the deformed contour of integration which intersects the imaginary axis at $\Im z=c_{0}-b_{0}$ and $\Im z=-c_{0}-b_{0}$ respectively. We show in Fig. 4.9 that with the growth of photon mass parameter $\mu$ the poles move further away from the boundary of the integration region and therefore lie inside the new contour of integration.


Figure 4.9: Same as Fig. 4.8 for $\tilde{\alpha}=0$ as function of $\mu$.
The deformed contour includes poles from both photon propagators, consequently the procedure of analytical continuation works also for the second photon propagator in the unitarity relations of Eqs. (4.24). Therefore, through analytical continuation, the unitarity relations are able to reproduce the imaginary part of the invariant amplitudes in the unphysical region also. As a cross-check of our procedure, we show the imaginary part of the amplitude $\mathcal{G}_{M}^{2 \gamma}$ for the case
of electron-muon scattering in Fig. 4.10, as calculated using the analytically continued phase space integral, and compare it with the direct loop graph evaluation as explained in Section 4.1 [206]. We find a perfect agreement between both calculations, justifying our analytical continuation procedure for the calculation based on unitarity relations.


Figure 4.10: Comparison between two evaluations of the imaginary part of the invariant amplitude $\mathcal{G}_{M}^{2 \gamma}$ for $e^{-} \mu^{-}$scattering for $Q^{2}=0.1 \mathrm{GeV}^{2}$ corresponding with $\nu_{\mathrm{ph}}=$ $0.03 \mathrm{GeV}^{2}$. Dashed-dotted curve: box graph evaluation; solid curve (coinciding): evaluation based on the unitarity relations. The region $\nu>\nu_{\mathrm{ph}}\left(\nu<\nu_{\mathrm{ph}}\right)$ corresponds with the physical (unphysical) region respectively.

A more realistic description of the proton is obtained by including electromagnetic FFs of the dipole form. This induces additional poles for the time-like region $Q_{i}^{2}<0$ in the unitarity relations Eq. (4.24):

$$
\begin{equation*}
G_{M} \sim \frac{1}{\left(Q_{i}^{2}+\Lambda^{2}\right)^{2}}, \quad F_{P} \sim \frac{1}{\left(Q_{i}^{2}+4 M^{2}\right)\left(Q_{i}^{2}+\Lambda^{2}\right)^{2}} \tag{4.46}
\end{equation*}
$$

These poles arise from the dipole mass parameter $\Lambda\left(Q_{i}^{2}+\Lambda^{2}=0\right)$ and from the "kinematical" pole $\left(Q_{i}^{2}+4 M^{2}=0\right)$. These poles can be treated in a similar way as the poles in Eqs. (4.42, 4.43) through the replacement $\mu \rightarrow \Lambda$ or $\mu \rightarrow 2 M$. These poles lie on the same line in the complex $z$ plane as the $z_{1}, z_{2}, z_{3}, z_{4}$ poles. As soon as $\Lambda>\mu, 2 M>\mu$, the new poles satisfy $\left|z_{1}^{\prime}\right|<\left|z_{1}\right|,\left|z_{3}^{\prime}\right|<\left|z_{3}\right|,\left|z_{2}^{\prime}\right|>\left|z_{2}\right|,\left|z_{4}^{\prime}\right|>\left|z_{4}\right|$. From Fig. 4.9, where the $\mu$ dependence of the pole positions in the unphysical region is shown, we see that our procedure of analytical continuation does not change the position of the new poles with respect to the deformed integration contour after the transition to the unphysical region. We can therefore conclude that the outlined procedure of analytical continuation is also valid for the calculation with proton FFs that have poles in the time-like region.

### 4.3 Comparison of DR approach and box graph model for proton intermediate state

In this Section, we compare the model calculation of the proton intermediate state contribution to TPE amplitudes of Section 4.1 with the evaluation within the DR formalism of Section 4.2.

The results for the real and imaginary parts of the amplitudes for the case of the $F_{D} F_{D}$ ( $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ and $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ ) vertex structure in the proton model with dipole FFs are compared in Figs. 4.11-4.16 (Figs. 4.17-4.22 and Figs. 4.23-4.28). We show the unitarity relations calculation of the imaginary parts of the invariant amplitudes both in physical and unphysical regions. For the latter, we use the analytical continuation as outlined in Section 4.2.3. For the imaginary parts, we see a perfect agreement between the unitarity relations calculations and the box graph evaluation both in physical and unphysical regions. This is to be expected as the imaginary parts of the invariant amplitudes correspond with an intermediate state in the box diagram which is on its mass shell. Therefore, only on-shell information enters the imaginary parts.

For the real parts, we use the unsubtracted DRs at fixed $Q^{2}$ of Eqs. (4.29, 4.30). By comparing the DR results with the loop diagram evaluation for $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure of the real parts (for the sum of direct and crossed box diagrams), we see from Figs. 4.11-4.16 that they nicely agree over the whole physical region of $\nu$, as it was expected form the HE behavior studies of Section 4.2.2. Also the real parts of the amplitudes $\mathcal{G}_{M}^{2 \gamma}, \mathcal{F}_{2}^{2 \gamma}, \mathcal{F}_{3}^{2 \gamma}, \mathcal{F}_{5}^{2 \gamma}$ evaluated in the box graph model in the case of $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structure are in agreement with the unsubtracted DR results, see Figs. 4.17-4.19, 4.21 for details. However, the result for the real parts of the amplitudes $\mathcal{F}_{4}^{2 \gamma}, \mathcal{F}_{6}^{2 \gamma}$ in the box graph model is shifted by a constant from the result of unsubtracted DR approach as it can be seen from Figs. 4.20, 4.22. Consequently, the real parts of the amplitudes $\mathcal{F}_{4}^{2 \gamma}, \mathcal{F}_{6}^{2 \gamma}$ in the case of $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structure ${ }^{2}$ are in agreement between two types of evaluation, if one uses the once-subtracted $D R$. In the case of $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure, ${ }^{3}$ the unsubtracted DRs reproduce the box diagram model results for the amplitudes $\mathcal{G}_{1}^{2 \gamma}, \mathcal{G}_{2}^{2 \gamma}, \mathcal{F}_{5}^{2 \gamma}$, as it is shown in Figs. 4.23, 4.24, 4.27. While the results for the real parts of the amplitudes $\mathcal{F}_{4}^{2 \gamma}, \mathcal{F}_{6}^{2 \gamma}$ and $\mathcal{F}_{3}^{2 \gamma}$, shown in Figs. 4.25, 4.26, 4.28, are shifted by a constant.

Consequently, all real parts of the TPE amplitudes in the box graph model are reconstructed using once-subtracted DRs. The three amplitudes $\mathcal{G}_{1}^{2 \gamma}, \mathcal{G}_{2}^{2 \gamma}, \mathcal{F}_{5}^{2 \gamma}$ among the five TPE amplitudes, which are required for the evaluation of the cross section correction by Eq. (3.36), are reconstructed in the box graph model within the unsubtracted DR. These results were expected from the HE behavior studies in Section 4.2.2. The DR analysis for the even amplitudes $\mathcal{F}_{3}^{2 \gamma}, \mathcal{F}_{4}^{2 \gamma}, \mathcal{F}_{6}^{2 \gamma}$ requires one subtraction. One of these amplitudes, the helicity-flip amplitude $\mathcal{F}_{4}^{2 \gamma}$, contributes to the unpolarized lepton-proton scattering with massive leptons only. There is no available experimental data to fix this constant. However, the imaginary part of the amplitude $\mathcal{F}_{4}^{2 \gamma}$ is directly related to the beam normal spin asymmetry by Eqs. (3.68, 3.70).

We also compare the TPE amplitudes for the case of the electron-proton scattering in detail. As in the general case of the massive lepton-proton scattering the imaginary parts of all TPE amplitudes are the same in the box graph model and in the unitarity relations based evaluation both in physical and unphysical regions. The real parts of all three independent amplitudes (e.g., $\mathcal{G}_{M}^{2 \gamma}, \mathcal{F}_{2}^{2 \gamma}, \mathcal{F}_{3}^{2 \gamma}$ ) in the box graph model are reconstructed within the unsubtracted DR in the case of $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ and $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structures. We present this comparison in Appendix K. In the case of $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure we show the results for the amplitudes $\mathcal{G}_{1}^{2 \gamma}$ and $\mathcal{G}_{2}^{2 \gamma}$ in Figs. 4.29, 4.30. These amplitudes are UV finite in the model calculation with a point-like proton. While the real part of the $\mathcal{F}_{3}^{2 \gamma}$ amplitude requires an UV regularization in the point-like model.

[^10]

Figure 4.11: Imaginary part (left panel) and real part (right panel) of the invariant amplitude $\mathcal{G}_{M}^{2 \gamma}$ for the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure with dipole FFs in muon-proton scattering for $Q^{2}=0.03 \mathrm{GeV}^{2}$. The vertical line in the left panel corresponds with the boundary between physical and unphysical regions, i.e., $\nu_{\mathrm{ph}} \approx 0.128 \mathrm{GeV}^{2}$.


Figure 4.12: Same as Fig. 4.11, but for the invariant amplitude $\mathcal{F}_{2}^{2 \gamma}$.


Figure 4.13: Same as Fig. 4.11, but for the invariant amplitude $\mathcal{F}_{3}^{2 \gamma}$.


Figure 4.14: Same as Fig. 4.11, but for the invariant amplitude $\mathcal{F}_{4}^{2 \gamma}$.


Figure 4.15: Same as Fig. 4.11, but for the invariant amplitude $\mathcal{F}_{5}^{2 \gamma}$.


Figure 4.16: Same as Fig. 4.11, but for the invariant amplitude $\mathcal{F}_{6}^{2 \gamma}$.


Figure 4.17: Imaginary part (left panel) and real part (right panel) of the invariant amplitude $\mathcal{G}_{M}^{2 \gamma}$ for the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structure with dipole FFs in muon-proton scattering for $Q^{2}=0.03 \mathrm{GeV}^{2}$. The vertical line in the left panel corresponds with the boundary between physical and unphysical regions, i.e., $\nu_{\mathrm{ph}} \approx 0.128 \mathrm{GeV}^{2}$.


Figure 4.18: Same as Fig. 4.17, but for the invariant amplitude $\mathcal{F}_{2}^{2 \gamma}$.


Figure 4.19: Same as Fig. 4.17, but for the invariant amplitude $\mathcal{F}_{3}^{2 \gamma}$.


Figure 4.20: Same as Fig. 4.17, but for the invariant amplitude $\mathcal{F}_{4}^{2 \gamma}$.


Figure 4.21: Same as Fig. 4.17, but for the invariant amplitude $\mathcal{F}_{5}^{2 \gamma}$.


Figure 4.22: Same as Fig. 4.17, but for the invariant amplitude $\mathcal{F}_{6}^{2 \gamma}$.


Figure 4.23: Imaginary part (left panel) and real part (right panel) of the invariant amplitude $\mathcal{G}_{1}^{2 \gamma}$ for the $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure with dipole FFs in muon-proton scattering for $Q^{2}=0.03 \mathrm{GeV}^{2}$. The vertical line in the left panel corresponds with the boundary between physical and unphysical regions, i.e., $\nu_{\mathrm{ph}} \approx 0.128 \mathrm{GeV}^{2}$.


Figure 4.24: Same as Fig. 4.23, but for the invariant amplitude $\mathcal{G}_{2}^{2 \gamma}$.


Figure 4.25: Same as Fig. 4.23, but for the invariant amplitude $\mathcal{F}_{3}^{2 \gamma}$.


Figure 4.26: Same as Fig. 4.23, but for the invariant amplitude $\mathcal{F}_{4}^{2 \gamma}$.


Figure 4.27: Same as Fig. 4.23, but for the invariant amplitude $\mathcal{F}_{5}^{2 \gamma}$.


Figure 4.28: Same as Fig. 4.23, but for the invariant amplitude $\mathcal{F}_{6}^{2 \gamma}$.


Figure 4.29: Imaginary part (left panel) and real part (right panel) of the invariant amplitude $\mathcal{G}_{1}^{2 \gamma}$ for the $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure with dipole FFs in electron-proton scattering for $Q^{2}=0.1 \mathrm{GeV}^{2}$. The vertical line in the left panel corresponds with the boundary between physical and unphysical regions, i.e., $\nu_{\mathrm{ph}}=0.15 \mathrm{GeV}^{2}$.


Figure 4.30: Same as Fig. 4.29, but for the invariant amplitude $\mathcal{G}_{2}^{2 \gamma}$.


Figure 4.31: $\varepsilon$-dependence of the real part of the invariant amplitude $\mathcal{F}_{3}$ in the case of $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure with dipole FFs in electron-proton scattering for $Q^{2}=0.1 \mathrm{GeV}^{2}$. Left panel: comparison of the box diagram evaluation with unsubtracted DR. Right panel: comparison between the box diagram and DR evaluations when performing one subtraction. The calculations are shown for two different subtraction points: $\nu_{0}=1 \mathrm{GeV}^{2}$, and $\nu_{0}=2 \mathrm{GeV}^{2}$.

Consequently, the DR for the amplitude $\mathcal{F}_{3}^{2 \gamma}$ requires one subtraction. The resulting subtraction term cannot be reconstructed from the imaginary part of the amplitude $\mathcal{F}_{3}^{2 \gamma}$. This term describes the contribution of physics at high energies to low-energy processes. ${ }^{4}$ The results for the real part of the invariant amplitude $\mathcal{F}_{3}^{2 \gamma}$ in the case of $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure with dipole FFs are shown in Fig. 4.31 as a function of the photon polarization parameter $\varepsilon$, which is related to $\nu$ as

$$
\begin{equation*}
\nu=\sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \nu_{\mathrm{ph}}, \tag{4.47}
\end{equation*}
$$

with $\nu_{\text {ph }}$ defined in Eq. (4.37). One firstly notices from Fig. 4.31 (left panel) that the calculated real part of $\mathcal{F}_{3}^{2 \gamma}$ in the box graph model does not agree with the amplitude reconstructed using unsubtracted DRs. Although the box diagram calculation for $\mathcal{F}_{3}^{2 \gamma}$ is convergent for the $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure when using on-shell vertices with dipole FFs , we like to stress that this result is model dependent. We notice however that after performing one subtraction, we find an agreement between the DR calculation and the box diagram model evaluation, see right panel of Fig. 4.31.


Figure 4.32: Real parts of $\mathcal{G}_{1}^{2 \gamma}, \mathcal{G}_{2}^{2 \gamma}$, and $\mathcal{F}_{3}^{2 \gamma}$ in the elastic electron-proton scattering evaluated through unsubtracted DRs, as function of the upper integration limit $\nu_{\max }$. The plot shows the relative deviation of each amplitude from its value at $\nu_{\max }=\infty$, denoted by $\mathcal{F}(\infty)$, where $\mathcal{F}$ stands for $\mathcal{G}_{1}^{2 \gamma}, \mathcal{G}_{2}^{2 \gamma}, \mathcal{F}_{3}^{2 \gamma}$. All results are for the $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure with dipole FFs.

[^11]To test the numerical convergence for different kinematical situations, we show in Fig. 4.32 the contributions to the real parts of $\mathcal{G}_{1}^{2 \gamma}, \mathcal{G}_{2}^{2 \gamma}$, and $\mathcal{F}_{3}^{2 \gamma}$ evaluated through unsubtracted DRs, as function of the upper integration limit in the DR. We see from Fig. 4.32 that for the case of $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure with dipole FFs, the convergence of unsubtracted DRs is slowest at large (small) values of $\varepsilon$ for $\mathcal{G}_{2}^{2 \gamma}\left(\mathcal{G}_{1}^{2 \gamma}\right)$ respectively, while at intermediate values of $\varepsilon$ the slowest convergence occurs for $\mathcal{F}_{3}^{2 \gamma}$.

### 4.4 Muon-proton scattering experiment (MUSE) predictions

Using the TPE box graph evaluation from Sections 4.1, 4.2, we are able to make predictions of the leading proton intermediate state contribution in the elastic muon-proton scattering. We present in Fig. 4.33 the predictions for $\delta_{2 \gamma}$ in the box graph model of Section 4.1 (Born TPE) in terms of the different vertex structures for the kinematical region of the MUSE experiment described in Section 3.1. This experiment aims to measure the scattering of electrons and positrons, muons and antimuons on a proton target in order to test the lepton universality, to access the TPE correction directly and to extract the proton charge radius from the muon scattering data.


Figure 4.33: TPE correction to the unpolarized elastic $\mu^{-} p$ cross section evaluated within the box graph model (Born TPE) for three nominal muon beam momenta. The total correction is shown by the black solid curves, the contribution from the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ structure of photon-proton-proton vertices is shown by the red dashed curves, the contribution from the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ structure by the green dashed-dotted curves, and the contribution from the $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ structure by the blue dotted curves.

One notices from Fig. 4.33 that the $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure does not contribute significantly
to the cross section, while the main contribution comes from the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure. The contribution from the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structure rises when increasing the momentum transfer. This contribution is significant only for the largest values of the momentum transfer of the MUSE experiment. In magnitude, the TPE correction varies between $0.25 \%$ and $0.5 \%$.

As the TPE correction depends on the proton FFs, we also investigated the change when using dipole FFs versus proton electric FF of the form: $G_{E}\left(Q^{2}\right)=\left(1-0.14265 \times Q^{2}\right) G_{M}\left(Q^{2}\right) / \mu_{P}$, obtained accounting for the polarization transfer data [207] by the linear in $Q^{2}$ term. We obtain that $1>\delta_{2 \gamma}\left(G_{E}\right.$ from $\left.[207]\right) / \delta_{2 \gamma}\left(\right.$ dipole $\left.G_{E}\right)>0.95,0.96,0.965$ in $\mu^{-} p$ scattering for the beam momenta $k=115 \mathrm{MeV}, 153 \mathrm{MeV}, 210 \mathrm{MeV}$ respectively and $1>\delta_{2 \gamma}\left(G_{E}\right.$ from $[207]) / \delta_{2 \gamma}\left(\right.$ dipole $\left.G_{E}\right)>0.992,0.988,0.981$ in $e^{-} p$ scattering for the same beam momenta.


Figure 4.34: Born TPE correction to the unpolarized cross section for three different muon beam momenta. The TPE correction to elastic $\mu^{-} p$ scattering is shown by the blue solid curves. The black dashed-dotted curves show the elastic $e^{-} p$ scattering correction. The elastic $\mu^{-} p$ scattering correction without account of muon helicity flip is shown by the red dashed curves.

We show a comparison between the TPE corrections to elastic electron-proton and elastic muon-proton scattering in Fig. 4.34. One sees that the TPE correction in the case of the muon-proton scattering is up to a factor three smaller than the correction in the case of the electron-proton scattering with the same lepton beam momenta. The contribution of the helicity-flip amplitudes $\mathcal{F}_{4}^{2 \gamma}, \mathcal{F}_{5}^{2 \gamma}$ plays a significant role for $\mu^{-} p$ scattering in the kinematical region of the proposed experiment. It contributes with a sign, opposite from the contribution of the amplitudes without helicity flip, and significantly reduces the correction. We found that for the higher momentum transfer $Q^{2} \sim 1-2 \mathrm{GeV}^{2}$ the contribution of helicity flip amplitudes does not play a significant role and the predictions for elastic $\mu^{-} p$ scattering only slightly deviate
from the predictions for elastic $e^{-} p$ scattering, in agreement with the findings of Ref. [194].


Figure 4.35: TPE correction to the unpolarized elastic $\mu^{-} p$ cross section evaluated for three nominal muon beam momenta within the unsubtracted DR framework. The total correction is shown by the black solid curves, the contribution from the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ structure of photon-proton-proton vertices is shown by the red dashed curves, the contribution from the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ structure by the green dashed-dotted curves, and the contribution from the $F_{P} F_{P}$ structure by the blue dotted curves. For comparison, the evaluation in the box diagram model (Born TPE) is shown by the doubledotted black curve.

On the following plots in Fig. 4.35 we present the prediction for $\delta_{2 \gamma}$ within unsubtracted DRs in terms of the different vertex structures and compare it with the box graph model results. The contribution from the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure in the unsubtracted DR formalism is the same as in the box graph model. The negative contribution from the $F_{P} F_{P}$ vertex structure can not be neglected in the unsubtracted DR formalism due to the sizable difference in $\mathcal{F}_{4}^{\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}}$ amplitude comparing to the box graph model. The contribution from the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structure in the unsubtracted DR evaluation is negative as opposed to the box graph model contribution. The difference between two evaluations is given mainly by the amplitude $\mathcal{F}_{4}^{2 \gamma}$. Moreover, the unsubtracted DR evaluation of only the elastic intermediate state TPE contribution in the forward limit yields: $\mathcal{G}_{4}^{2 \gamma}\left(\nu, Q^{2} \rightarrow 0\right) \neq 0$. Consequently, such calculation does not satisfy the expected vanishing low- $Q^{2}$ behavior of the cross section, see Section 3.4 for details. The correct evaluation of this amplitude within DRs requires the $Q^{2}$-dependent subtraction function. In absence of data to fix this constant, the box graph model estimate of the TPE correction from the proton intermediate state may be used as a first guidance on
the size of TPE. However, the MUSE experiment will be able to provide measurements for all three beam momenta in the region $0.0052 \mathrm{GeV}^{2}<Q^{2}<0.027 \mathrm{GeV}^{2}$ allowing, in principle, to fix the subtraction points to data.

### 4.5 Subtracted DR formalism: comparison with $e p$ data

We next discuss predictions for unpolarized cross section and polarization transfer observables in the elastic electron-proton scattering and compare them with existing data.

For a phenomenological evaluation of the TPE contribution to elastic electron-proton scattering, we like to minimize any model dependence due to higher energy contributions. In a full calculation, such contributions arise from the inelastic states which always will require some approximate treatment to which we will turn in Chapters 5 and 6 . To minimize any such uncertainties and to provide a more flexible formalism when applied to data, we propose to consider a DR formalism with one subtraction for the amplitude $\mathcal{F}_{3}^{2 \gamma}$. The subtraction constant will be obtained by a fit to the elastic electron-proton scattering observables, in the region where precise data are available.

We next discuss the implementation of such a subtracted DR formalism for the TPE contribution and provide a detailed comparison to different observables. The TPE correction to the unpolarized elastic electron-proton scattering cross section in Eq. (3.62) can be expressed as the sum of a term evaluated using an unsubtracted DR and a term arising from the $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ contribution to $\mathcal{F}_{3}^{2 \gamma}$, which we will evaluate by performing a subtraction:

$$
\begin{equation*}
\delta_{2 \gamma}=\delta_{2 \gamma}^{0}+f\left(\nu, Q^{2}\right) \Re \mathcal{F}_{3}^{\mathrm{FPF}_{\mathrm{P}}}, \tag{4.48}
\end{equation*}
$$

with

$$
\begin{equation*}
\delta_{2 \gamma}^{0}=\frac{1}{G_{M}^{2}+\frac{\varepsilon}{\tau_{P}} G_{E}^{2}}\left(2 G_{M} \Re \mathcal{G}_{1}^{2 \gamma}+2 \frac{\varepsilon}{\tau_{P}} G_{E} \Re \mathcal{G}_{2}^{2 \gamma}+2 G_{M}(\varepsilon-1) \frac{\nu}{M^{2}} \Re \mathcal{F}_{3}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}+\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{P}}}\right), \tag{4.49}
\end{equation*}
$$

and

$$
\begin{equation*}
f\left(\nu, Q^{2}\right)=\frac{2 G_{M}(\varepsilon-1)}{G_{M}^{2}+\frac{\varepsilon}{\tau_{P}} G_{E}^{2}}\left(\frac{\nu}{M^{2}}\right) \tag{4.50}
\end{equation*}
$$

The polarization transfer observables of Eqs. $(3.63,3.64)$ can also be expressed as the model independent terms $\left(\frac{P_{t}}{P_{l}}\right)^{0},\left(\frac{P_{l}}{P_{l}^{\text {Born }}}\right)^{0}$ and the contribution due to $\mathcal{F}_{3}^{\mathrm{F}}{ }^{\mathrm{FFP}}$ as

$$
\begin{align*}
\frac{P_{t}}{P_{l}} & =\left(\frac{P_{t}}{P_{l}}\right)^{0}+g\left(\nu, Q^{2}\right) \Re \mathcal{F}_{3}^{\mathrm{FP}_{\mathrm{P}}},  \tag{4.51}\\
\frac{P_{l}}{P_{l}^{\text {Born }}} & =\left(\frac{P_{l}}{P_{l}^{\text {Born }}}\right)^{0}+h\left(\nu, Q^{2}\right) \Re \mathcal{F}_{3}^{\mathrm{FP}_{\mathrm{P}}}, \tag{4.52}
\end{align*}
$$

with

$$
\begin{align*}
g\left(\nu, Q^{2}\right) & =-\sqrt{\frac{2 \varepsilon}{\tau_{P}(1+\varepsilon)}} \frac{1-\varepsilon}{1+\varepsilon} \frac{G_{E}}{G_{M}^{2}}\left(\frac{\nu}{M^{2}}\right),  \tag{4.53}\\
h\left(\nu, Q^{2}\right) & =-\frac{2 \varepsilon}{\tau_{P} G_{M}^{2}+\varepsilon G_{E}^{2}} \frac{1}{G_{M}} \frac{\varepsilon \tau_{P} G_{M}^{2}+G_{E}^{2}}{1+\varepsilon}\left(\frac{\nu}{M^{2}}\right) . \tag{4.54}
\end{align*}
$$

The predictions for the elastic electron-proton scattering observables can be made with one subtraction point at $\nu=\nu_{0}$, which we express as

$$
\begin{align*}
\delta_{2 \gamma}\left(\nu, Q^{2}\right) & =\delta_{2 \gamma}^{0}\left(\nu, Q^{2}\right)+f\left(\nu, Q^{2}\right)\left[\Re \mathcal{F}_{3}^{\mathrm{FP}_{\mathrm{P}}}\left(\nu, Q^{2}\right)-\Re \mathcal{F}_{3}^{\mathrm{F}_{\mathrm{P}} \mathrm{~F}_{\mathrm{P}}}\left(\nu_{0}, Q^{2}\right)\right] \\
& +f\left(\nu, Q^{2}\right) \Re \mathcal{F}_{3}^{\mathrm{FP}_{\mathrm{P}} \mathrm{~F}_{\mathrm{P}}}\left(\nu_{0}, Q^{2}\right), \tag{4.55}
\end{align*}
$$

where $\Re \mathcal{F}_{3}^{\mathrm{FPFP}_{\mathrm{P}}}\left(\nu, Q^{2}\right)-\Re \mathcal{F}_{3}^{\mathrm{FPFP}}\left(\nu_{0}, Q^{2}\right)$ is calculated using the subtracted DR , and where we determine the subtraction function $\Re \mathcal{F}_{3}^{\mathrm{FP}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}}\left(\nu_{0}, Q^{2}\right)$ through $\delta_{2 \gamma}\left(\nu_{0}, Q^{2}\right)$, which has to be obtained from the experiment, as

$$
\begin{equation*}
\Re \mathcal{F}_{3}^{\mathrm{FP}_{\mathrm{P}} \mathrm{~F}_{\mathrm{P}}}\left(\nu_{0}, Q^{2}\right)=\frac{\delta_{2 \gamma}\left(\nu_{0}, Q^{2}\right)-\delta_{2 \gamma}^{0}\left(\nu_{0}, Q^{2}\right)}{f\left(\nu_{0}, Q^{2}\right)} . \tag{4.56}
\end{equation*}
$$

We can then insert this subtraction term (for every fixed value of $Q^{2}$ ) into Eqs. (4.51, 4.52) and make predictions for the $\nu$ or $\epsilon$ dependence of these observables as

$$
\begin{align*}
\left(\frac{P_{t}}{P_{l}}\right)\left(\nu, Q^{2}\right) & =\left(\frac{P_{t}}{P_{l}}\right)^{0}\left(\nu, Q^{2}\right)+g\left(\nu, Q^{2}\right)\left[\Re \mathcal{F}_{3}^{\mathrm{FP}}{ }^{\mathrm{FP}}\left(\nu, Q^{2}\right)-\Re \mathcal{F}_{3}^{\mathrm{FP}_{\mathrm{P}}}\left(\nu_{0}, Q^{2}\right)\right] \\
& +g\left(\nu, Q^{2}\right) \frac{\delta_{2 \gamma}\left(\nu_{0}, Q^{2}\right)-\delta_{2 \gamma}^{0}\left(\nu_{0}, Q^{2}\right)}{f\left(\nu_{0}, Q^{2}\right)},  \tag{4.57}\\
\left(\frac{P_{l}}{P_{l}^{\text {Born }}}\right)\left(\nu, Q^{2}\right) & =\left(\frac{P_{l}}{P_{l}^{\text {Born }}}\right)^{0}\left(\nu, Q^{2}\right)+h\left(\nu, Q^{2}\right)\left[\Re \mathcal{F}_{3}^{\mathrm{FPF}_{\mathrm{P}}}\left(\nu, Q^{2}\right)-\Re \mathcal{F}_{3}^{\mathrm{FP}_{\mathrm{P}}}\left(\nu_{0}, Q^{2}\right)\right] \\
& +h\left(\nu, Q^{2}\right) \frac{\delta_{2 \gamma}\left(\nu_{0}, Q^{2}\right)-\delta_{2 \gamma}^{0}\left(\nu_{0}, Q^{2}\right)}{f\left(\nu_{0}, Q^{2}\right)} . \tag{4.58}
\end{align*}
$$

In the following, we determine the subtraction term from the unpolarized cross section measurements [104], and show our predictions for the different observables. The TPE correction to the unpolarized elastic electron-proton scattering evaluated in the model calculation of Section 4.1, with the Feshbach term subtracted, is shown in Fig. 4.36 for a small value of $\varepsilon$. It is seen from Fig. 4.36 that the departure of the TPE correction from the Feshbach term strongly increases with increasing $Q^{2}$. This is mainly due to the contribution from the $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure.


Figure 4.36: Model prediction for the TPE correction $\delta_{2 \gamma}-\delta_{F}$, with $\delta_{F}$ is the Feshbach term of Eq. (5.1), for $\varepsilon=0.01$.

To compare our DR results for the proton intermediate state contribution with the data, we perform, for every fixed value of $Q^{2}$, one subtraction for the amplitude $\mathcal{F}_{3}^{2 \gamma}$ with the subtraction point fixed by one cross section result, which we take from Ref. [104]. For comparison, we also show the result for the box diagram model in Fig. 4.37. The difference between the results for different choices of the subtraction point corresponds to the uncertainty of our procedure. We would like to notice that for $Q^{2}$ larger than around $1 \mathrm{GeV}^{2}$ the account of inelastic intermediate states becomes increasingly important. Additionally, a description in terms of intermediate hadronic states ceases to be valid for large momentum transfer: due to the scattering off individual quarks, one will go over into a partonic picture [55,56,61,62,69].


Figure 4.37: Subtracted DR based prediction for the TPE corrections $\delta_{2 \gamma}-\delta_{F}$, in comparison with the Born TPE, unsubtracted DR prediction, for $\varepsilon=0.01$, and with the parametrization of experimental data [104], for $\varepsilon=0$ (blue band). The subtracted DR predictions are shown for three choices of the subtraction point: $\varepsilon_{0}=0.2,0.5,0.8$.


Figure 4.38: $\varepsilon$ dependence of the Born TPE correction to the unpolarized cross section for the elastic electron-proton scattering for different momentum transfers.

We next discuss in more detail the TPE evaluations using proton intermediate state only to test the validity of this approximation. At low momentum transfer, the model calculation approaches the Feshbach limit, and is in agreement with the experimental results and previous evaluations performed by Blunden, Melnitchouk and Tjon [51], see Fig. 4.38 for our results. The TPE correction to the unpolarized elastic electron-proton scattering evaluated in the box diagram model of Section 4.1 is shown in Fig. 4.39 as a function of $\varepsilon$ for the momentum transfers $Q^{2}=0.05 \mathrm{GeV}^{2}$ and $Q^{2}=1 \mathrm{GeV}^{2}$.


Figure 4.39: Model prediction for the TPE correction for $Q^{2}=0.05 \mathrm{GeV}^{2}$ (upper panel) and $Q^{2}=1 \mathrm{GeV}^{2}$ (lower panel). Solid curve: full box diagram model (Born TPE) result; dashed-dotted curve: $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex contribution only. The experimental results from the MAMI/A1 Collaboration [104] are shown by the blue bands.

We also investigate the change in the input of TPE calculation exploiting the proton electric FF of the form: $G_{E}\left(Q^{2}\right)=\left(1-0.14265 \times Q^{2}\right) G_{M}\left(Q^{2}\right) / \mu_{P}[207]$. We obtain at $Q^{2}=0.05 \mathrm{GeV}^{2}$ that $1<\delta_{2 \gamma}\left(G_{E}\right.$ from [207] $) / \delta_{2 \gamma}\left(\right.$ dipole $\left.G_{E}\right)<1.045$ and at $Q^{2}=0.5 \mathrm{GeV}^{2}$ that $0.94<\delta_{2 \gamma}\left(G_{E}\right.$ from $[207]) / \delta_{2 \gamma}\left(\right.$ dipole $\left.G_{E}\right)<1$ respectively with the largest deviation observed at forward angles.


Figure 4.40: Subtracted DR based predictions for the TPE corrections for $Q^{2}=0.05 \mathrm{GeV}^{2}$ (upper panel) and $Q^{2}=1 \mathrm{GeV}^{2}$ (lower panel), in comparison with the unsubtracted DR prediction as well as with the Born TPE. The subtracted DR curves correspond with three choices for the subtraction points: $\varepsilon_{0}=0.2,0.5,0.8$. The blue bands correspond with the experimental result from the fit of Ref. [104].

We next show our predictions at low momentum transfer based on the subtracted DR framework. As seen from Fig. 4.40, the subtracted DR result describes the data better in the region of intermediate $\varepsilon$. For higher $\varepsilon$ values, i.e., higher energies, the contribution of inelastic intermediate states becomes important and the agreement between theory and experiment becomes worse. One also notices clear deviations at lower values of $\varepsilon$. This may arise due to the assumption in the experimental TPE analysis of a linear $\varepsilon$-behavior for the difference $\delta_{2 \gamma}-\delta_{F}$. The theoretical calculations show non-linear behavior in $\varepsilon$ for this region.

For $Q^{2} \approx 0.206 \mathrm{GeV}^{2}$, the CLAS Collaboration has recently performed measurements of the ratio of $e^{+} p$ to $e^{-} p$ elastic scattering cross section [77]. Its deviation from unity is directly related to the TPE corrections. Furthermore, the ratio $P_{t} / P_{l}$ was measured for momentum transfer values $Q^{2}=0.298 \mathrm{GeV}^{2}$ [119] and $Q^{2}=0.308 \mathrm{GeV}^{2}[208]$ in Hall A at JLab. In Figs. 4.41, 4.42 we show the theoretical estimates for physical observables based on the subtracted DR prediction. We fix the subtracted amplitude $\mathcal{F}_{3}^{2 \gamma}$ according to Eq. (4.56), by using the unpolarized cross section analysis of Ref. [104] at one point in $\varepsilon$ as input. We choose the subtraction point $\varepsilon_{0}=0.83$, which is in the $\varepsilon$-range of both experiments. For both observables we use the FFs from the $P_{t} / P_{l}$ measurement of Ref. [119]. We extract the TPE correction $\delta_{2 \gamma}$ from the CLAS data of the cross section ratio $R_{2 \gamma}=\sigma\left(e^{+} p\right) / \sigma\left(e^{-} p\right)$ by $\delta_{2 \gamma} \approx\left(1+\delta_{\text {even }}\right) \times\left(1-R_{2 \gamma}\right) / 2$, where $\delta_{\text {even }} \approx-0.2$ is the total charge-even radiative correction factor according to Ref. [77]. Note that for the CLAS data, which have been radiatively corrected according to the Mo and Tsai (MT) procedure [162] in Ref. [77], we applied the correction $\delta_{2 \gamma \text {, soft }}^{\mathrm{MT}}-\delta_{2 \gamma, \text { soft }}^{\mathrm{MaTj}}$ to the data in order to compare them relative to the Maximon and $\mathrm{Tjon}(\mathrm{MaTj})$ procedure which we follow in this paper. The uncertainty band shown for the subtracted DR analysis arises from the experimental uncertainty entering through the subtraction. We conclude from Figs. 4.41, 4.42 that all measurements are in agreement for low momentum transfer and the TPE corrections in the CLAS data and the $P_{t} / P_{l}$ measurements of Refs. [77], [119] are described by the elastic contribution within the errors of the experiments.


Figure 4.41: Comparison of the subtracted DR prediction for the TPE correction for $Q^{2}=$ $0.206 \mathrm{GeV}^{2}$ with the data [77], with the unsubtracted DR prediction and with the Born TPE. The subtraction point used in the DR analysis is $\varepsilon_{0}=0.83$.


Figure 4.42: Comparison of the subtracted DR prediction for the ratio $R=-\mu_{p} \sqrt{\frac{1+\varepsilon}{\varepsilon} \tau_{P}} \frac{P_{t}}{P_{l}}$ for $Q^{2}=0.298 \mathrm{GeV}^{2}$ with the data [119, 208], with the unsubtracted DR prediction and with the Born TPE. The subtraction point used in the DR analysis is $\varepsilon_{0}=$ 0.83 .


Figure 4.43: Comparison of the subtracted DR prediction for the TPE correction for $Q^{2}=$ $0.85 \mathrm{GeV}^{2}$ (left panel) and $Q^{2}=1.45 \mathrm{GeV}^{2}$ (right panel) with the data [79], with the unsubtracted DR prediction and with the Born TPE. The value of $\varepsilon_{0}$ in the subtracted DR analysis indicates the subtraction point which was used.

We show the similar comparison with the recent CLAS data from the measurement of $R_{2 \gamma}$ at larger momentum transfer $Q^{2} \approx 0.85 \mathrm{GeV}^{2}$ and $Q^{2} \approx 1.45 \mathrm{GeV}^{2}$ [79] in Fig. 4.43. We take the subtraction point from the fit of Ref. [104]. With account of the proton intermediate state only the measured TPE correction is in agreement with an empirical fit [104]. The comparison with the VEPP-3 data renormalized [76] according to the empirical fit of Ref. [104] is shown in Fig. 4.44. The experimental points at $Q^{2}=0.298 \mathrm{GeV}^{2}$ and $Q^{2}=1.51 \mathrm{GeV}^{2}$ are in agreement with the Born TPE. The measured TPE correction is in agreement with an empirical fit [104] for $Q^{2}>0.8 \mathrm{GeV}^{2}$. The subtracted DR is able to describe all points. The
subtracted DR prediction for the OLYMPUS experiment [81], which measures the ratio of $e^{+} p$ to $e^{-} p$ cross section $R_{2 \gamma}$ exploiting the 2 GeV lepton beam, is shown in Fig. 4.45. According to it, the TPE correction vanishes at $Q^{2}=0$ and changes the sign somewhere in the region $0.3 \mathrm{GeV}^{2} \lesssim Q^{2} \lesssim 1.2 \mathrm{GeV}^{2}$. The largest TPE correction corresponds to larger momentum transfer values of the experiment. The experiment has finished the data taking period in the kinematical region $0.4 \mathrm{GeV}^{2}<Q^{2}<2.2 \mathrm{GeV}^{2}$ and the analysis is going to be finished soon.


Figure 4.44: TPE correction measurements of Ref. [76] in comparison with the subtracted DR prediction, the Born TPE and empirical fit of Ref. [104] evaluated for the experimental $\left(Q^{2}, \varepsilon\right)$ values. The VEPP-3 [76] data points correspond to the following kinematics: $Q^{2}=0.298 \mathrm{GeV}^{2}, \quad \varepsilon=0.932 ; \quad Q^{2}=0.83 \mathrm{GeV}^{2}, \varepsilon=0.404$; $Q^{2}=0.976 \mathrm{GeV}^{2}, \varepsilon=0.272 ; Q^{2}=1.51 \mathrm{GeV}^{2}, \varepsilon=0.449$.


Figure 4.45: The subtracted DR prediction for the TPE correction in the scattering of 2 GeV electron beam on a proton target (OLYMPUS [81]) in comparison with the Born TPE and the unsubtracted DR result.

We next discuss the polarization transfer observables for momentum transfer $Q^{2} \approx 2.5 \mathrm{GeV}^{2}$, where data have been taken both for $P_{t}$ and $P_{l}$ separately [72,73]. In our theoretical predictions, we use the proton FFs taken from the $P_{t} / P_{l}$ ratio measurement. To evaluate the TPE invariant amplitudes, we use the dipole FFs as an input. The comparison with the data for the ratio $P_{t} / P_{l}$ is shown in Fig. 4.46. As one sees, the present data for $P_{t} / P_{l}[73]$ does not allow to extract a TPE effect, indicating a cancellation between the three TPE amplitudes for this specific observable. The comparison with the data [73] for the absolute polarization transfer observable $P_{l} / P_{l}^{\text {Born }}[73]$ is also shown in Fig. 4.46. It shows that the point at $\varepsilon=0.635$ with $P_{l} / P_{l}^{\text {Born }}=1.007 \pm 0.005$ is consistent with the proton contribution only, but the point at $\varepsilon=0.785$ with $P_{l} / P_{l}^{\text {Born }}=1.023 \pm 0.006$ requires further theoretical investigations, e.g., account of inelastic intermediate states which are relevant at these larger momentum transfers. The specific property of the subtracted DR analysis for the ratio $P_{l} / P_{l}^{\mathrm{Born}}$ is the divergence of the errors for $\varepsilon \rightarrow 1$ as $1 / \sqrt{1-\varepsilon}$. The divergent behavior is inherited from the function $h\left(\nu, Q^{2}\right) \sim \nu \sim 1 / \sqrt{1-\varepsilon}$ entering the Eq. (4.58) in the product with the subtraction point uncertainty.


Figure 4.46: Comparison of the subtracted DR predictions for the ratio $R=-\mu_{p} \sqrt{\frac{1+\varepsilon}{\varepsilon} \tau_{P} \frac{P_{t}}{P_{l}}}$ (left panel) and $P_{l} / P_{l}^{\text {Born }}$ (right panel) for $Q^{2}=2.5 \mathrm{GeV}^{2}$ with the data of Ref. [73], with the unsubtracted DR prediction and with the Born TPE. The subtraction point used in the DR analysis is $\varepsilon_{0}=0.785$.

For a phenomenological extraction of TPE amplitudes, it is useful to define the following TPE amplitudes:

$$
\begin{align*}
& \Upsilon_{M}=\Re\left(\frac{\mathcal{G}_{M}^{2 \gamma}}{G_{M}}\right), \Upsilon_{E}=\Re\left(\frac{\mathcal{G}_{M}^{2 \gamma}-\left(1+\tau_{P}\right) \mathcal{F}_{2}^{2 \gamma}}{G_{M}}\right), \\
& \Upsilon_{1}=\Re\left(\frac{\mathcal{G}_{1}^{2 \gamma}}{G_{M}}\right),  \tag{4.59}\\
& \Upsilon_{2}=\Re\left(\frac{\mathcal{G}_{2}^{2 \gamma}}{G_{M}}\right), \quad \Upsilon_{3}=\frac{\nu}{M^{2}} \Re\left(\frac{\mathcal{F}_{3}^{2 \gamma}}{G_{M}}\right) .
\end{align*}
$$

These amplitudes have been extracted [72] from the experimental data at $Q^{2}=2.5 \mathrm{GeV}^{2}$.
In Fig. 4.47, we show the results for the elastic contribution to the amplitudes $\Upsilon_{1}$ and $\Upsilon_{2}$, which coincide with the Born TPE in case of these amplitudes. The amplitude $\Upsilon_{2}$ calculated within the box graph model is in agreement with the amplitude extracted from the experimental data in the covered by experiment region $(\varepsilon \sim 0.6-0.9)$. The amplitude $\Upsilon_{2}$ is at the per mille level. While the amplitude $\Upsilon_{1}$ is at the percent level and deviates from the experimental result for smaller $\varepsilon$ values.



Figure 4.47: $\varepsilon$-dependence of the amplitude $\Upsilon_{1}$ (left panel) and $\Upsilon_{2}$ (right panel) for $Q^{2}=$ $2.5 \mathrm{GeV}^{2}$. The light band corresponds with the phenomenological extraction of Ref. [72]. Solid curve: Born TPE.

In Fig. 4.48, we show the subtracted DR results for the amplitudes $\Upsilon_{3}, \Upsilon_{M}$ and $\Upsilon_{E}$. We take the subtraction point from the unpolarized cross section measurements [104]. The subtracted DR analysis of these amplitudes leads to the $1 / \sqrt{1-\varepsilon}$ divergence of the errors for $\varepsilon \rightarrow 1$ as in the case of the ratio $P_{l} / P_{l}^{\text {Born }}$ due to the factor $\nu$ coming with the subtracted amplitude $\mathcal{F}_{3}$. The amplitudes evaluated within the box graph model are different with the amplitudes extracted from experimental data in the region of available data $(\varepsilon \sim 0.6-0.9)$. We expect that inelastic intermediate states can lead to a large contribution for momentum transfer $Q^{2} \sim 2.5 \mathrm{GeV}^{2}$ and explain at least some part of this discrepancy.


Figure 4.48: $\varepsilon$-dependence of the amplitudes $\Upsilon_{3}$ (upper panel), $\Upsilon_{E}$ (central panel) and $\Upsilon_{M}$ (lower panel) for $Q^{2}=2.5 \mathrm{GeV}^{2}$. The light band corresponds with the phenomenological extraction of Ref. [72]. The dark (blue) band is the subtracted DR result for the proton intermediate state, with subtraction point $\varepsilon_{0}=0.785$. The size of the band reflects the uncertainty of the data at the subtraction point, according to Eqs. (4.56, 4.59).

## Chapter 5

## Low- $Q^{2}$ limit of TPE correction including inelastic intermediate states

In this Chapter, we account for all inelastic intermediate states in the region of low momentum transfer approximating the proton side of the TPE graph by the near-forward unpolarized doubly virtual Compton scattering. We first reproduce the low momentum transfer expansion of the Born TPE in the elastic electron-proton scattering and generalize it to the case of massive lepton in Section 5.1. Deriving this expansion in the near-forward approximation in Sections 5.2 and 5.3 , we confirm the validity of the approximation for the proton TPE contribution. Inelastic states contribution is given by the subtraction function in the unpolarized forward Compton amplitude $T_{1}$ and unpolarized proton structure functions $F_{1}, F_{2}$. In the elastic electron-proton scattering, the subtraction function contribution is suppressed by the square of the electron mass. We study its contribution in the muon-proton scattering in Section 5.5 and evaluate the resulting TPE correction from the heavy-baryon, the baryon chiral perturbation theory and empirically determined subtraction functions. Furthermore, we reproduce the leading $Q^{2} \ln Q^{2}$ term of the inelastic TPE in the electron-proton scattering and evaluate the proton structure functions contribution numerically in Section 5.6. We compare the total TPE correction, the sum of the Born TPE and inelastic TPE in the near-forward approximation, with the TPE fit performed by the MAMI/A1 Collaboration and the experimental data points from CLAS and VEPP-3. We also make predictions for the MUSE kinematics and low- $Q^{2}$ region of the OLYMPUS experiment.

### 5.1 Low- $Q^{2}$ expansion of elastic TPE correction

### 5.1.1 Elastic $e^{-} p$ scattering

The limit of low momentum transfer is relevant to extract the proton charge radius from elastic scattering data. In the forward limit, corresponding with $Q^{2} \rightarrow 0$ and $\varepsilon \rightarrow 1$, the TPE correction to the cross section is given by Coulomb photons from the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ structure of virtual photon-proton-proton vertices. This result was first obtained for the electron-proton scattering in the Dirac theory as the first order cross section correction by McKinley and Feshbach [173], see Eq. (3.53) for the whole analytical expression. In case of the elastic electron-proton scattering the Feshbach correction simplifies to

$$
\begin{equation*}
\delta_{F}=\frac{\alpha \pi \sin \frac{\theta_{\mathrm{lab}}}{2}}{1+\sin \frac{\theta_{\mathrm{lab}}}{2}} \approx \frac{\alpha \pi \sqrt{1-\varepsilon}}{\sqrt{1-\varepsilon}+\sqrt{1+\varepsilon}} \tag{5.1}
\end{equation*}
$$

It is instructive to provide some analytical expressions for $\delta_{2 \gamma}$ in the forward limit resulting from the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex contribution to the full box diagram calculation.

For the case of electron scattering off massless quarks (taken with unit charge) the TPE correction is given by [56]

$$
\begin{align*}
\delta_{2 \gamma}=\frac{\alpha}{\pi}\left\{2 \ln \left(\frac{Q^{2}}{\mu^{2}}\right) \ln \left(\frac{1-y}{1+y}\right)\right. & +\frac{y}{1+y^{2}}\left[\ln ^{2}\left(\frac{1+y}{2 y}\right)+\ln ^{2}\left(\frac{1-y}{2 y}\right)+\pi^{2}\right] \\
& \left.-\frac{y}{1+y^{2}}\left[\ln \left(\frac{1-y^{2}}{4 y^{2}}\right)-y \ln \left(\frac{1+y}{1-y}\right)\right]\right\} \tag{5.2}
\end{align*}
$$

with $y=\sqrt{1-\varepsilon} / \sqrt{1+\varepsilon}$ and $Q^{2}=4 y \nu$. In the forward limit $\left(Q^{2} \rightarrow 0\right.$ and $\varepsilon \rightarrow 1$, at finite $\left.\nu\right)$ we recover the Feshbach term and find large logarithmic correction terms in $(1-\varepsilon)$ :

$$
\begin{equation*}
\delta_{2 \gamma}-\delta_{2 \gamma}^{I R} \longrightarrow \delta_{F}+\frac{\alpha}{\pi} \sqrt{\frac{1-\varepsilon}{2}} \ln (2(1-\varepsilon))\left[\frac{1}{2} \ln (2(1-\varepsilon))+1\right], \tag{5.3}
\end{equation*}
$$

where the IR divergent TPE is given by the massless limit of Eq. (4.10) and the limit of the Feshbach correction is given by

$$
\begin{equation*}
\delta_{F} \rightarrow \frac{\alpha \pi Q}{2 \omega} . \tag{5.4}
\end{equation*}
$$

For the case of forward scattering off a massive point particle we also give the analytical form of the momentum transfer expansion of the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex contribution for the model with point particles. In the forward direction, only the contribution from the amplitude $\mathcal{G}_{2}^{2 \gamma}$ defined in Eq. (3.38) survives since $\delta_{2 \gamma} \rightarrow 2 \Re \mathcal{G}_{2}^{2 \gamma}$ in the forward limit. The $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ point vertex contribution to the imaginary part of $\mathcal{G}_{2}^{2 \gamma}$ is obtained from Eqs. (4.26) as

$$
\begin{equation*}
\Im \mathcal{G}_{2}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}}=\alpha\left\{\ln \left(\frac{Q^{2}}{\mu^{2}}\right)+\frac{Q^{2}}{4 s}+\frac{Q^{2}}{8} \frac{s}{\nu^{2}-\nu_{\mathrm{ph}}^{2}} \ln \left(\frac{s Q^{2}}{\left(s-M^{2}\right)^{2}}\right)\right\}, \tag{5.5}
\end{equation*}
$$

with $\nu_{\mathrm{ph}}$ as defined in Eq. (4.37). Using the dispersion relation of Eq. (4.29), we can express the real part of $\mathcal{G}_{2}^{\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}}$ in the forward limit (for $Q^{2} \ll M^{2}, M^{2} \omega^{2} / s$ ) in terms of $Q^{2}$ and the electron beam energy $\omega$ in the laboratory frame as

$$
\begin{align*}
\Re \mathcal{G}_{2}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}} \longrightarrow \frac{\alpha}{\pi}\{ & -\frac{Q^{2}}{M \omega} \ln \left(\frac{Q^{2}}{\mu^{2}}\right)+\pi^{2} \frac{Q}{4 \omega} \\
& \left.+\frac{Q^{2}}{2 M \omega} \ln \left(\frac{Q}{2 \omega}\right)\left[\ln \left(\frac{Q}{2 \omega}\right)+1\right]+\mathrm{O}\left(\frac{Q^{2}}{M^{2}}, \frac{s Q^{2}}{M^{2} \omega^{2}}\right)\right\} \tag{5.6}
\end{align*}
$$

Note that we can equivalently express Eq. (5.6) through the variable $\varepsilon$ using the kinematical relation $Q / \omega \simeq \sqrt{2(1-\varepsilon)}$, which holds in the forward direction. Eq. (5.6) then allows to directly express $\delta_{2 \gamma}$ in the forward direction as

$$
\begin{equation*}
\delta_{2 \gamma}-\delta_{2 \gamma}^{I R} \longrightarrow \delta_{F}+\frac{\alpha}{\pi} \frac{Q^{2}}{M \omega} \ln \left(\frac{Q}{2 \omega}\right)\left[\ln \left(\frac{Q}{2 \omega}\right)+1\right]+\mathrm{O}\left(\frac{Q^{2}}{M^{2}}, \frac{s Q^{2}}{M^{2} \omega^{2}}\right) \tag{5.7}
\end{equation*}
$$

where the leading finite term (proportional to $Q / \omega$ ) is obtained as the limit of the Feshbach correction term $\delta_{F}$, and where subleading logarithmic correction terms are also shown. We found that our forward limit result of Eq. (5.7) agrees with an expression obtained some time ago [174] by the direct evaluation of the loop integrals. We can similarly study the contributions of the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ and $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structures to the amplitude $\mathcal{G}_{2}^{2 \gamma}$ in the case of a point-like proton comparing their imaginary parts $\Im \mathcal{G}_{2}^{\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}}$ and $\Im \mathcal{G}_{2}^{\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}}$ with the expression for the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex
structure $\Im \mathcal{G}_{2}^{\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}}$ of Eq. (5.5). The corresponding imaginary parts can be obtained from Eqs. $(4.27,4.28)$ as

$$
\begin{align*}
& \Im \mathcal{G}_{2}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{P}}}=\frac{\alpha \mu_{P} Q^{2}}{4 M^{2}}\left\{-\ln \left(\frac{Q^{2}}{\mu^{2}}\right)+\frac{2 M^{2}}{s}-\frac{Q^{2}}{8} \frac{s}{\nu^{2}-\nu_{\mathrm{ph}}^{2}} \ln \left(\frac{s Q^{2}}{\left(s-M^{2}\right)^{2}}\right)\right\}  \tag{5.8}\\
& \Im \mathcal{G}_{2}^{\mathrm{F}_{\mathrm{P}} \mathrm{~F}_{\mathrm{P}}}=\frac{\alpha\left(\mu_{P}-1\right)^{2} Q^{2}}{8 M^{2}}\left\{\frac{2 M^{2}-s}{s}-\frac{Q^{2}}{8} \frac{s}{\nu^{2}-\nu_{\mathrm{ph}}^{2}} \ln \left(\frac{s Q^{2}}{\left(s-M^{2}\right)^{2}}\right)-\ln \left(\frac{s Q^{2}}{\left(s-M^{2}\right)^{2}}\right)\right\} \tag{5.9}
\end{align*}
$$

The Feshbach correction and the subleading logarithmic terms in the real part of the amplitude $\mathcal{G}_{2}^{\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}}$, Eq. (5.6), arise from the logarithmic term in Eq. (5.5). Analogous terms are suppressed by the pre-factor $Q^{2} / M^{2}$ in the imaginary parts for the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ and $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structures in comparison with the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure. The additional logarithmic term in Eq. (5.9) also leads to corrections of higher order in $Q$ in comparison with Eq. (5.6).

In Fig. 5.1, we compare the limit of the Feshbach correction with the full box diagram calculation of $\delta_{2 \gamma}$ for point-like proton at low momentum transfers and beam energies corresponding with experiments at MAMI and JLab. One sees that at low $Q^{2}$, the leading TPE contribution is given by the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure, and approaches the Feshbach term in the forward direction. We furthermore see that at low $Q^{2}$, the leading corrections to the Feshbach result are given by the logarithmic terms given in Eq. (5.7).

In Fig. 5.2, we compare the analogous results using the dipole model for the proton FFs.
It is instructive also to provide expressions for the real parts of all other TPE amplitudes in the case of $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure in the forward limit. Subtracting the IR divergent terms in the amplitudes $\mathcal{G}_{1}^{\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}}, \mathcal{G}_{M}^{\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}}$, see Eqs. (4.6, 5.6), and the finite constant terms in the amplitudes $\mathcal{G}_{M}^{\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}}$ and $\mathcal{F}_{3}^{\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}}$, we obtain the following leading $Q^{2}$-dependent terms: ${ }^{1}$

$$
\begin{align*}
& \Re \mathcal{G}_{1}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}} \longrightarrow \frac{\alpha}{\pi}\left\{\pi^{2} \frac{Q}{2 \omega}+\frac{Q^{2}}{2 M \omega} \ln \left(\frac{Q}{2 \omega}\right)\left[\ln \left(\frac{Q}{2 \omega}\right)+3\right]+\mathrm{O}\left(\frac{Q^{2}}{M^{2}}, \frac{s Q^{2}}{M^{2} \omega^{2}}\right)\right\},(5  \tag{5.10}\\
& \Re \mathcal{G}_{M}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}} \longrightarrow \frac{\alpha}{\pi}\left\{-\pi^{2} \frac{Q}{4 \omega}-\frac{Q^{2}}{2 M \omega} \ln \left(\frac{Q}{2 \omega}\right)\left[\ln \left(\frac{Q}{2 \omega}\right)-1\right]+\mathrm{O}\left(\frac{Q^{2}}{M^{2}}, \frac{s Q^{2}}{M^{2} \omega^{2}}\right)\right\},  \tag{5.11}\\
& \Re \mathcal{F}_{2}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}} \longrightarrow \frac{\alpha}{\pi}\left\{\pi^{2} \frac{Q}{4 \omega}+\frac{Q^{2}}{M \omega} \ln \left(\frac{Q}{2 \omega}\right)+\mathrm{O}\left(\frac{Q^{2}}{M^{2}}, \frac{s Q^{2}}{M^{2} \omega^{2}}\right)\right\},  \tag{5.12}\\
& \Re \mathcal{F}_{3}^{\mathrm{FD}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}} \longrightarrow \frac{\alpha}{\pi}\left\{\pi^{2} \frac{2 Q M}{4 \omega^{2}}+\frac{Q^{2}}{\omega^{2}} \ln \left(\frac{Q}{2 \omega}\right)\left[\ln \left(\frac{Q}{2 \omega}\right)+1\right]+\mathrm{O}\left(\frac{Q^{2}}{M^{2}}, \frac{s Q^{2}}{M^{2} \omega^{2}}\right)\right\} .(5 \tag{5.13}
\end{align*}
$$

These expressions are in agreement with the low- $Q^{2}$ expansion of the amplitudes obtained by the direct box graph evaluation in case of the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure. Consequently, the $Q, Q^{2} \ln Q^{2}, Q^{2} \ln ^{2} Q^{2}$ terms are model independent results which should be respected by any hadronic TPE calculation.

### 5.1.2 Elastic $\mu^{-} p$ scattering

In this Section, we start by studying the quality of some approximate expressions for the proton contribution in the low- $Q^{2}$ limit in case of the massive lepton-proton scattering, for which an analytical form can be provided.

[^12]

Figure 5.1: The forward limit of the TPE correction to the unpolarized electron-proton scattering cross-section in the model with a point-like proton for $\omega=0.18 \mathrm{GeV}$ (upper panels) and $\omega=1.1 \mathrm{GeV}$ (lower panels). For clarity, the contribution relative to the Feshbach term is shown on the right panels for the logarithmic correction term of Eq. (5.7), for the total $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex contribution to the box diagram (Dirac particle), and for the full box diagram calculation, also including the $F_{D} F_{P}$ and $F_{P} F_{P}$ contributions (point-like proton).


Figure 5.2: The forward limit of the TPE correction in the model with dipole proton FFs for $\omega=0.18 \mathrm{GeV}$ (left panel) and $\omega=1.1 \mathrm{GeV}$ (right panel).

The leading term of the momentum transfer expansion of the TPE correction $\delta_{2 \gamma}$ of Eq. (5.4) is modified by the recoil correction due to the final lepton mass:

$$
\begin{equation*}
\delta_{F} \rightarrow \frac{\alpha \pi Q}{2 \omega}\left(1+\frac{m}{M}\right) \tag{5.14}
\end{equation*}
$$

We present the derivation of this leading term in the momentum transfer expansion of the TPE correction in Appendix L.

As a next step, one may consider the TPE correction in the scattering of two point-like Dirac particles (corresponding with two $\gamma^{\mu}$ couplings). When studying the low- $Q^{2}$ expansion $Q^{2} \ll m^{2}, M^{2}, M^{2}|\boldsymbol{k}|^{2} / s$, of the TPE correction in the elastic muon-proton scattering with a point-like proton $\delta_{2 \gamma}^{\mathrm{QED}}$ neglecting its magnetic moment, we find the analogue of the Feshbach term of Eq. (5.14), the IR divergent piece $\delta_{2 \gamma}^{\mathrm{IR}}$, and a logarithmic correction:

$$
\begin{align*}
\delta_{2 \gamma}^{\mathrm{QED}} & \rightarrow \delta_{2 \gamma}^{\mathrm{IR}}+\delta_{F}+\frac{\alpha Q^{2}}{2 \pi M \omega} \ln \frac{Q^{2}}{M \omega}\left(1+\frac{2 \omega}{|\mathbf{k}|} \ln \frac{|\mathbf{k}|-\omega+m}{|\mathbf{k}|+\omega-m}\right)+\mathrm{O}\left(\frac{Q^{2}}{M^{2}}, \frac{Q^{2}}{m^{2}}, \frac{s Q^{2}}{M^{2}|\boldsymbol{k}|^{2}}\right) \\
\delta_{2 \gamma}^{\mathrm{IR}} & \rightarrow \frac{\alpha \omega Q^{2}}{\pi M|\mathbf{k}|^{2}} \ln \frac{\mu^{2}}{Q^{2}}\left(1+\frac{m^{2}}{\omega|\mathbf{k}|} \ln \frac{|\mathbf{k}|-\omega+m}{|\mathbf{k}|+\omega-m}\right) \tag{5.15}
\end{align*}
$$

where $|\mathbf{k}|$ is the muon momentum in the lab frame.
We also provide the more general expansion of $\delta_{2 \gamma}^{\mathrm{QED}}$ in the low- $Q^{2} \operatorname{limit} Q^{2} \ll M^{2}, M^{2}|\boldsymbol{k}|^{2} / s$, where $Q^{2}$ needs not be very small relative to the squared lepton mass. For such expansion, the leading $Q^{2}$ terms are given by

$$
\begin{align*}
\delta_{2 \gamma}^{\mathrm{QED}} \rightarrow & \delta_{2 \gamma}^{\mathrm{IR}}+\frac{\alpha \pi Q}{2 \omega}+\frac{\alpha \omega Q^{2}}{2 \pi M|\mathbf{k}|^{2}}\left(\frac{\frac{Q^{2}}{4}}{m^{2}+\frac{Q^{2}}{4}}+\frac{2|\mathbf{k}|}{\omega} \ln \frac{|\mathbf{k}|-\omega+m}{|\mathbf{k}|+\omega-m}\right) \ln \frac{Q^{2}}{M \omega} \\
& +\frac{16 \alpha \pi Q^{2}}{M \omega|\mathbf{k}|^{2}} \frac{\omega^{2} \frac{Q^{4}}{8}+m^{2} \omega^{2} Q^{2}+m^{4}|\mathbf{k}|^{2}}{m^{2}+\frac{Q^{2}}{4}} C\left(m, Q^{2}\right)+\mathrm{O}\left(\frac{Q^{2}}{M^{2}}, \frac{s Q^{2}}{M^{2}|\boldsymbol{k}|^{2}}\right) \tag{5.17}
\end{align*}
$$

with

$$
\begin{align*}
C\left(m, Q^{2}\right)= & \frac{1}{16 \pi^{2}} \frac{1}{Q^{2}} \frac{1}{\sqrt{1+\frac{4 m^{2}}{Q^{2}}}}\left\{\ln \frac{Q^{2}}{m^{2}} \ln \left(\frac{1+\sqrt{1+\frac{4 m^{2}}{Q^{2}}}}{-1+\sqrt{1+\frac{4 m^{2}}{Q^{2}}}}\right)-\operatorname{Li}_{2}\left(\frac{2}{1+\sqrt{1+\frac{4 m^{2}}{Q^{2}}}}\right)\right. \\
& \left.-\operatorname{Li}_{2}\left(\frac{1-\sqrt{1+\frac{4 m^{2}}{Q^{2}}}}{2}\right)-\frac{1}{2} \ln ^{2}\left(\frac{2}{-1+\sqrt{1+\frac{4 m^{2}}{Q^{2}}}}\right)+\frac{5}{6} \pi^{2}\right\} \tag{5.18}
\end{align*}
$$

where $\operatorname{Li}_{2}(x)$ denotes the dilogarithm function. The leading IR divergent piece is given by Eq. (5.16). The two limits of the function $C\left(m, Q^{2}\right)$, when $m \rightarrow 0$ and when $Q^{2} \rightarrow 0$, are of a special interest:

$$
\begin{align*}
C\left(m \rightarrow 0, Q^{2}\right) & =\frac{\frac{1}{2} \ln ^{2} \frac{m^{2}}{Q^{2}}+\frac{2 \pi^{2}}{3}}{16 \pi^{2} Q^{2}} \\
C\left(m, Q^{2} \rightarrow 0\right) & =\frac{1}{32 m Q}+\frac{-2+\ln \frac{Q^{2}}{m^{2}}}{32 \pi^{2} m^{2}} \tag{5.19}
\end{align*}
$$

In the limit $Q^{2} \ll m^{2}$, the result of Eq. (5.17) reduces to the expression of Eq. (5.15). When taking the massless limit $m^{2} \ll Q^{2}$ of Eq. (5.17), we also recover the expression of Eq. (5.7) [172, 174].

We show in Fig. 5.3 (left panel) the comparison between the Feshbach term, the TPE contribution for point-like Dirac particles, and the TPE for a point-like proton, with an inclusion of the magnetic moment contribution. It is seen that the Feshbach correction of Eq. (3.53) with an account of the recoil correction factor $(1+m / M)$ describes the result for point-like Dirac particles quite well in the kinematics of the MUSE experiment.

We also show in Fig. 5.3 (right panel) the effect of the proton FFs, according to the full numerical calculation of Ref. [170]. In the low $Q^{2}$ kinematics of the MUSE experiment, the inclusion of the FFs provides a reduction of the TPE by around $40 \%$ at $Q^{2} \approx 0.025 \mathrm{GeV}^{2}$, consequently one should use the full numerical calculation of Section 4.1 [170] (corresponding with the Born TPE result in Fig. 5.3) in MUSE kinematics.


Figure 5.3: Left panel: TPE correction in elastic muon-proton scattering for the case of a point-like proton neglecting the contribution from $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure, compared with the case where one neglects the magnetic moment (Dirac particle), as well as the Feshbach result (corresponding with Coulomb photon exchange). Right panel: TPE correction for the case of the proton with electric and magnetic form factors of the dipole form. We compare the box graph calculation with the Feshbach term corrected by the recoil correction $1+m / M$.

In the forward direction, only the amplitudes $\mathcal{G}_{2}^{2 \gamma}, \mathcal{G}_{4}^{2 \gamma}$ defined in Eqs. (3.38, 3.40) contribute to the leading powers of the TPE correction of Eq. (3.36). The resulting correction $\delta_{2 \gamma}$ is approximated in the forward limit as

$$
\begin{equation*}
\delta_{2 \gamma} \rightarrow 2 \Re\left(\mathcal{G}_{2}^{2 \gamma}+\frac{m^{2}}{M \omega} \mathcal{G}_{4}^{2 \gamma}\right) . \tag{5.20}
\end{equation*}
$$

We provide the leading terms in the low- $Q^{2}$ expansion, $Q^{2} \ll M^{2}, M^{2}|\boldsymbol{k}|^{2} / s$, for the real parts of TPE amplitudes in the case of $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure after subtraction of the IR divergent terms from Eqs. (4.6):

$$
\begin{align*}
\Re \mathcal{G}_{2}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}}= & \frac{\alpha \pi \omega Q}{4|\mathbf{k}|^{2}}+\frac{\alpha \omega Q^{2}}{4 \pi M|\mathbf{k}|^{2}}\left(1+\frac{2 \omega^{2}+m^{2}}{\omega|\mathbf{k}|} \ln \frac{\omega-|\mathbf{k}|-m}{\omega+|\mathbf{k}|-m}\right) \ln \frac{Q^{2}}{M \omega} \\
& +\frac{16 \alpha \pi \omega Q^{2}\left(m^{2}+\frac{Q^{2}}{4}\right)}{M|\mathbf{k}|^{2}} C\left(m, Q^{2}\right), \tag{5.21}
\end{align*}
$$

$$
\begin{align*}
\Re \mathcal{G}_{4}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}}= & -\frac{\alpha \pi M Q}{4|\mathbf{k}|^{2}}-\frac{\alpha Q^{2}}{4 \pi|\mathbf{k}|^{2}}\left(\frac{\omega^{2}}{m^{2}+\frac{Q^{2}}{4}}+3 \frac{\omega}{|\mathbf{k}|} \ln \frac{\omega-|\mathbf{k}|-m}{\omega+|\mathbf{k}|-m}\right) \ln \frac{Q^{2}}{M \omega} \\
& -\frac{8 \alpha \pi m^{2} Q^{2}\left(\omega^{2}+m^{2}+\frac{Q^{4}}{8 m^{2}}\right)}{\left(m^{2}+\frac{Q^{2}}{4}\right)|\mathbf{k}|^{2}} C\left(m, Q^{2}\right) . \tag{5.22}
\end{align*}
$$

According to Eqs. $(3.18,3.19)$ the amplitudes $\mathcal{G}_{2}^{\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}}$ and $\mathcal{G}_{4}^{\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}}$ have no constant term in the low- $Q^{2}$ expansion.

Subtracting the IR divergent and constant terms that are related by Eqs. (3.17-3.20), we obtain the leading $Q^{2}$-dependent terms as

$$
\begin{align*}
& \Re \mathcal{G}_{M}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}}= \frac{\alpha \omega Q^{2}}{4 \pi M|\mathbf{k}|^{2}}\left(1-\frac{2|\mathbf{k}|}{\omega} \ln \frac{\omega-|\mathbf{k}|-m}{\omega+|\mathbf{k}|-m}\right) \ln \frac{Q^{2}}{M \omega} \\
&-\frac{8 \alpha \pi \omega Q^{2}\left(m^{2}+\frac{Q^{2}}{2}\right)}{M|\mathbf{k}|^{2}} C\left(m, Q^{2}\right)-\frac{\alpha \pi \omega Q}{4|\mathbf{k}|^{2}},  \tag{5.23}\\
& \Re \mathcal{F}_{2}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}}= \frac{\alpha \omega Q^{2}}{4 \pi M|\mathbf{k}|^{2}}\left(1+\frac{\omega^{2}}{|\mathbf{k}|^{2}}+\frac{m^{2}}{|\mathbf{k}|^{2}} \frac{4 \omega^{2}-m^{2}}{\omega|\mathbf{k}|} \ln \frac{\omega-|\mathbf{k}|-m}{\omega+|\mathbf{k}|-m}\right) \ln \frac{Q^{2}}{M \omega} \\
&+\frac{16 \alpha \pi \omega Q^{2}\left(m^{2}+\frac{Q^{2}}{4}\right)\left(m^{2}+\frac{Q^{2}}{2}\right)}{M|\mathbf{k}|^{4}} C\left(m, Q^{2}\right)+\frac{\alpha \pi \omega\left(\omega^{2}+m^{2}\right) Q}{4|\mathbf{k}|^{4}},  \tag{5.24}\\
& \Re \mathcal{F}_{3}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}}= \frac{\alpha Q^{2}}{4 \pi|\mathbf{k}|^{2}}\left(1+\frac{\omega^{2}}{|\mathbf{k}|^{2}}+\frac{4 \omega^{2}-m^{2}}{|\mathbf{k}|^{2}} \frac{\omega}{|\mathbf{k}|} \ln \frac{\omega-|\mathbf{k}|-m}{\omega+|\mathbf{k}|-m}\right) \ln \frac{Q^{2}}{M \omega} \\
&+\frac{8 \alpha \pi m^{2} Q^{2}\left(3 \omega^{2}-m^{2}+\frac{\omega^{2} Q^{2}}{m^{2}}\right)}{|\mathbf{k}|^{4}} C\left(m, Q^{2}\right)+\frac{\alpha \pi M\left(3 \omega^{2}-m^{2}\right) Q}{4|\mathbf{k}|^{4}},  \tag{5.25}\\
& \Re \mathcal{F}_{4}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}}= \frac{\alpha \pi\left(\omega^{2}+m^{2}\right) M Q}{4|\mathbf{k}|^{4}}+\frac{\alpha \omega^{2} Q^{2}}{4 \pi|\mathbf{k}|^{4}}\left(1+\frac{\omega^{2}+2 m^{2}}{\omega|\mathbf{k}|} \ln \frac{\omega-|\mathbf{k}|-m}{\omega+|\mathbf{k}|-m}\right) \ln \frac{Q^{2}}{M \omega} \\
& \Re \mathcal{F}_{5}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}}=-\frac{8 \alpha \pi Q^{2}\left(m^{2}+\frac{Q^{2}}{2}\right)\left(\omega^{2}+m^{2}\right)}{|\mathbf{k}|^{4}} C\left(m, Q^{2}\right),  \tag{5.26}\\
& 2|\mathbf{k}|^{4}-\frac{\alpha M \omega M^{2}}{4 \pi|\mathbf{k}|^{4}}\left(1+\frac{|\mathbf{k}|^{2}}{m^{2}+\frac{Q^{2}}{4}}+\frac{4 \omega^{2}-m^{2}}{\omega|\mathbf{k}|} \ln \frac{\omega-|\mathbf{k}|-m}{\omega+|\mathbf{k}|-m}\right) \ln \frac{Q^{2}}{M \omega} \\
&-\frac{8 \alpha \pi \omega Q^{2} m^{2}\left(\omega^{2}+m^{2}+\frac{Q^{4}}{4 m^{2}}\right) M}{\left(m^{2}+\frac{Q^{2}}{4}\right)|\mathbf{k}|^{4}} C\left(m, Q^{2}\right),  \tag{5.27}\\
& \Re \mathcal{F}_{6}^{\mathrm{F}_{\mathrm{D}} \mathrm{~F}_{\mathrm{D}}}= \frac{\alpha Q^{2}}{4 \pi|\mathbf{k}|^{2}}\left(1+\frac{\omega}{|\mathbf{k}|} \ln \frac{\omega-|\mathbf{k}|-m}{\omega+|\mathbf{k}|-m}\right) \ln \frac{Q^{2}}{M \omega}, \tag{5.28}
\end{align*}
$$

These results were obtained as the low- $Q^{2}$ limit of the TPE box and crossed box graphs in the elastic scattering of a massive lepton on a point charge. Taking the limit $Q^{2} \gg m^{2}$, we obtain Eqs. (5.11-5.13).

### 5.2 TPE correction in terms of VVCS amplitudes

In this Section, we express the TPE correction $\delta_{2 \gamma}$ in terms of the VVCS tensor and study the TPE correction in detail for the case of the proton intermediate state contribution. The exchange of two photons contributes to the elastic lepton-proton scattering through the TPE diagram shown in Fig. 5.4.


Figure 5.4: Two-photon exchange (TPE) diagram.

The lower blob in Fig. 5.4 is given by the doubly virtual Compton scattering (VVCS) process on a proton (see Fig. 5.5): $\gamma^{*}\left(q_{1}, \lambda_{1}\right)+N(p, \lambda) \rightarrow \gamma^{*}\left(q_{2}, \lambda_{2}\right)+N\left(p^{\prime}, \lambda^{\prime}\right)$. The VVCS amplitude $T_{\lambda_{2} \lambda^{\prime}, \lambda_{1} \lambda}$ can be written in terms of the VVCS tensor $M^{\mu \nu}$ as

$$
\begin{equation*}
T_{\lambda_{2} \lambda^{\prime}, \lambda_{1} \lambda}=\varepsilon_{\nu}\left(q_{1}, \lambda_{1}\right) \varepsilon_{\mu}^{*}\left(q_{2}, \lambda_{2}\right) \cdot \bar{N}\left(p^{\prime}, \lambda^{\prime}\right)\left(4 \pi M^{\mu \nu}\right) N(p, \lambda), \tag{5.29}
\end{equation*}
$$

where $\varepsilon_{\nu}, \varepsilon_{\mu}^{*}$ denote the virtual photon polarization vectors, $N, \bar{N}$ the proton spinors, and $\lambda_{1}, \lambda_{2}\left(\lambda, \lambda^{\prime}\right)$ the photon (proton) helicities. The initial and final virtual photon momenta $q_{1}$ and $q_{2}$ are related to the momentum transfer $q$ and the loop variable $\tilde{q}$ in Fig. 5.4 by

$$
\begin{equation*}
q_{1}=\tilde{q}+\frac{q}{2}, \quad q_{2}=\tilde{q}-\frac{q}{2} . \tag{5.30}
\end{equation*}
$$



Figure 5.5: Non-forward doubly virtual Compton scattering (VVCS) process.
The TPE correction, $\delta_{2 \gamma}$ of Eq. (3.35), for the lepton-proton $\left(l^{-} p\right)$ scattering is in general given by the interference of the one-photon exchange $T^{1 \gamma}$ and the two-photon exchange $T^{2 \gamma}$ amplitudes,

$$
\begin{equation*}
\delta_{2 \gamma}=\frac{2 \Re\left(\sum_{\text {spin }} T^{2 \gamma}\left(T^{1 \gamma}\right)^{*}\right)}{\sum_{\text {spin }}\left|T^{1 \gamma}\right|^{2}} . \tag{5.31}
\end{equation*}
$$

The OPE expression in the denominator of Eq. (5.31) is given by

$$
\begin{equation*}
\sum_{\text {spin }}\left|T^{1 \gamma}\right|^{2}=\frac{8 e^{4}}{\tau_{P}} \frac{1-\varepsilon_{0}}{1-\varepsilon}\left(\varepsilon G_{E}^{2}\left(Q^{2}\right)+\tau_{P} G_{M}^{2}\left(Q^{2}\right)\right) \tag{5.32}
\end{equation*}
$$

where $G_{E}\left(G_{M}\right)$ are the proton electric (magnetic) form factors respectively, and with kinematical quantities $\tau_{P}$ and $\varepsilon_{0}$ defined as in Sections 2.3, 3.1. The interference between the OPE and TPE amplitudes in the numerator of Eq. (5.31) can be expressed as

$$
\begin{gather*}
2 \Re\left(\sum_{\text {spin }} T^{2 \gamma}\left(T^{1 \gamma}\right)^{*}\right)=\Re \frac{4 \pi e^{4}}{Q^{2}} \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \frac{L^{\mu \nu \alpha} H_{\mu \nu \alpha}}{\left(\left(\tilde{q}-\frac{q}{2}\right)^{2}-\mu^{2}\right)\left(\left(\tilde{q}+\frac{q}{2}\right)^{2}-\mu^{2}\right)}, \\
L^{\mu \nu \alpha}=\operatorname{Tr}\left\{\left(\gamma^{\mu} \frac{\hat{K}-\hat{\tilde{q}}+m}{(K-\tilde{q})^{2}-m^{2}} \gamma^{\nu}+\gamma^{\nu} \frac{\hat{K}+\hat{\tilde{q}}+m}{(K+\tilde{q})^{2}-m^{2}} \gamma^{\mu}\right)(\hat{k}+m) \gamma^{\alpha}\left(\hat{k}^{\prime}+m\right)\right\}, \\
H^{\mu \nu \alpha}=\operatorname{Tr}\left\{M^{\mu \nu}(\hat{p}+M) \Gamma^{\alpha}\left(Q^{2}\right)\left(\hat{p^{\prime}}+M\right)\right\} . \tag{5.33}
\end{gather*}
$$

Furthermore, the VVCS tensor $M^{\mu \nu}$ reads, in the notation of Refs. [209, 210], ${ }^{2}$ as

$$
\begin{gather*}
M^{\mu \nu}=\alpha \sum_{i \in J} B_{i}\left(q_{1}^{2}, q_{2}^{2}, q_{1} \cdot q_{2}, \tilde{q} \cdot P\right) T_{i}^{\mu \nu}, \\
J=\{1, \ldots, 21\} \backslash\{5,15,16\}, \tag{5.34}
\end{gather*}
$$

where the 18 independent tensors $T_{i}^{\mu \nu}$ were constructed to be gauge invariant, and the non-Born parts are free of kinematical singularities and constraints, following the procedure outlined in Ref. [211]. The invariant amplitudes $B_{i}$ satisfy definite transformation properties with respect to photon crossing as well as charge conjugation combined with proton crossing as detailed in Refs. [209, 210]. The lack of knowledge of the amplitudes $B_{i}$ for the doubly-virtual case does not allow one to evaluate the TPE correction in the general case. However, in the limit of low momentum transfer one can approximate the lower VVCS process by the forward virtual Compton scattering, which is studied experimentally relatively well.

In the following, we firstly study the TPE correction to the unpolarized elastic lepton-proton scattering cross section coming from the proton intermediate state in the low- $Q^{2}$ limit. This will set the stage to apply the formalism subsequently for inelastic intermediate states. The correction of Eqs. (5.7, 5.15, 5.17) can be obtained by the graph with two Dirac structures $\gamma^{\mu}$ in the photon-proton-proton vertices. The VVCS tensor $M_{\text {QED }}^{\mu \nu}$ in this case is given by the sum of three nonvanishing terms [209-211]:

$$
\begin{equation*}
M_{\mathrm{QED}}^{\mu \nu}=\alpha\left(B_{2}^{\mathrm{QED}} T_{2}^{\mu \nu}+B_{10}^{\mathrm{QED}} T_{10}^{\mu \nu}+B_{17}^{\mathrm{QED}} T_{17}^{\mu \nu}\right) \tag{5.35}
\end{equation*}
$$

The amplitudes $B_{2}^{\mathrm{QED}}, B_{10}^{\mathrm{QED}}, B_{17}^{\mathrm{QED}}$ and the expressions for the corresponding tensors are given in Appendix M.

The leading contribution to the TPE correction is obtained by the contribution from the unpolarized VVCS amplitude $B_{2}^{\text {QED }}$ only. Neglecting the subleading terms in the momentum transfer expansion in case of the proton intermediate state (i.e., taking the limit $Q^{2} \ll$ $\left.M^{2}, s Q^{2} / \Sigma_{s}\right)$, the leading terms in the $Q^{2}$ expansion for a Dirac point particle can then be obtained from

[^13]\[

$$
\begin{align*}
\delta_{2 \gamma}^{Q E D} \rightarrow \frac{16 Q^{2} e^{2}}{M \omega} & \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \Pi_{P}^{+} \Pi_{P}^{-} \Pi_{K}^{+} \Pi_{K}^{-} \Pi_{Q}^{+} \Pi_{Q}^{-}\left\{\left(2 \tilde{q}^{2}+\frac{3}{2} Q^{2}\right)(K \cdot P)(K \cdot \tilde{q})(P \cdot \tilde{q})\right. \\
& -2(K \cdot \tilde{q})^{2}(P \cdot \tilde{q})^{2}-\left(\tilde{q}^{2}+\frac{Q^{2}}{4}\right)\left(M^{2}(K \cdot \tilde{q})^{2}+m^{2}(P \cdot \tilde{q})^{2}\right) \\
& \left.-\left(\tilde{q}^{2}+\frac{Q^{2}}{4}\right)^{2}(K \cdot P)^{2}+\frac{Q^{2}}{2 M \omega}(K \cdot \tilde{q})(P \cdot \tilde{q})\left((P \cdot \tilde{q})^{2}+(K \cdot \tilde{q})^{2}\right)\right\}, \tag{5.36}
\end{align*}
$$
\]

where we have introduced the propagator notations:

$$
\begin{equation*}
\Pi_{P}^{ \pm}=\frac{1}{(P \pm \tilde{q})-M^{2}}, \quad \Pi_{K}^{ \pm}=\frac{1}{(K \pm \tilde{q})-m^{2}}, \quad \Pi_{Q}^{ \pm}=\frac{1}{(\tilde{q} \pm q / 2)-\mu^{2}} \tag{5.37}
\end{equation*}
$$

The expression in Eq. (5.36) reproduces all terms in the low-Q ${ }^{2}$ expansion of Eq. (5.17). The spin-dependent structures $B_{10}^{\mathrm{QED}} T_{10}^{\mu \nu}$ and $B_{17}^{\mathrm{QED}} T_{17}^{\mu \nu}$ give rise to higher powers in $Q^{2}$ in the expansion of the TPE correction in comparison with the contribution of the unpolarized structure $B_{2}^{\mathrm{QED}} T_{2}^{\mu \nu}$. Consequently, the leading $Q^{2}$ terms in the expansion coming from the graph with two vertices $\gamma_{\mu}$ is reproduced in a correct way with Eq. (5.36). As an example, we describe the derivation of the Feshbach correction limit, see Eqs. (5.4, 5.14), in Appendix L.
The VVCS tensor for a Dirac point particle corresponding with all three structures of Eq. (5.35) results in the TPE correction $\delta_{2 \gamma}^{\mathrm{QED}}$ given by

$$
\begin{align*}
\delta_{2 \gamma}^{\mathrm{QED}}= & \frac{32(K \cdot P) e^{2}}{M^{2}\left(\varepsilon+\tau_{P}\right)} \frac{1-\varepsilon}{1-\varepsilon_{0}} \Re \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \Pi_{P}^{+} \Pi_{P}^{-} \Pi_{K}^{+} \Pi_{K}^{-} \Pi_{Q}^{+} \Pi_{Q}^{-} \\
& \times\left\{\left(2 \tilde{q}^{2}+\frac{3}{2} Q^{2}\right)(K \cdot P)(K \cdot \tilde{q})(P \cdot \tilde{q})\right. \\
& -\left(\tilde{q}^{2}+\frac{Q^{2}}{4}\right)\left(\left(P^{2}+\frac{Q^{2}}{4}\right)(K \cdot \tilde{q})^{2}+\left(K^{2}+\frac{Q^{2}}{4}\right)(P \cdot \tilde{q})^{2}\right) \\
& -\left(\tilde{q}^{2}+\frac{Q^{2}}{4}\right)^{2}\left((K \cdot P)^{2}-\frac{Q^{2}}{4}\left(M^{2}+m^{2}\right)+\frac{(q \cdot \tilde{q})^{2}-Q^{2} \tilde{q}^{2}}{8}\right) \\
& +\frac{Q^{2}}{2(P \cdot K)}(K \cdot \tilde{q})(P \cdot \tilde{q})\left((P \cdot \tilde{q})^{2}+(K \cdot \tilde{q})^{2}\right)-2(K \cdot \tilde{q})^{2}(P \cdot \tilde{q})^{2} \\
& +\frac{\left(P^{2}+K^{2}\right) Q^{2}}{4(K \cdot P)}(K \cdot \tilde{q})(P \cdot \tilde{q})\left(\tilde{q}^{2}-\frac{3}{4} Q^{2}\right) \\
& \left.-\frac{(K \cdot \tilde{q})(P \cdot \tilde{q})}{2(K \cdot P)}\left((q \cdot \tilde{q})^{2} \frac{Q^{2}}{4}+Q^{2}\left(\frac{5 \tilde{q}^{2} Q^{2}}{8}-\frac{\tilde{q}^{4}}{4}-\frac{17}{64} Q^{4}\right)\right)\right\} . \tag{5.38}
\end{align*}
$$

### 5.3 Near-forward approximation of VVCS

The VVCS amplitudes $B_{1}, B_{2}, B_{3}, B_{4}, B_{19}$ contribute to the unpolarized forward virtual Compton scattering tensor.

The full proton intermediate state contribution from the unpolarized VVCS amplitudes is described by two nonvanishing amplitudes,

$$
\begin{align*}
B_{1}^{\text {Born }}= & \frac{1}{M} \frac{\tilde{\nu}^{2}}{\left(q_{1} \cdot q_{2}\right)^{2}-(2 M \tilde{\nu})^{2}}\left\{F_{P}\left(Q_{1}^{2}\right) F_{P}\left(Q_{2}^{2}\right)\right. \\
& \left.+\frac{\left(q_{1} \cdot q_{2}\right)}{\tilde{\nu}^{2}}\left(F_{D}\left(Q_{1}^{2}\right) F_{P}\left(Q_{2}^{2}\right)+F_{P}\left(Q_{1}^{2}\right) F_{D}\left(Q_{2}^{2}\right)+F_{P}\left(Q_{1}^{2}\right) F_{P}\left(Q_{2}^{2}\right)\right)\right\}, \\
B_{2}^{\text {Born }=} & \frac{1}{M} \frac{1}{\left(q_{1} \cdot q_{2}\right)^{2}-(2 M \tilde{\nu})^{2}}\left\{F_{D}\left(Q_{1}^{2}\right) F_{D}\left(Q_{2}^{2}\right)-\frac{\left(q_{1} \cdot q_{2}\right)}{4 M^{2}} F_{P}\left(Q_{1}^{2}\right) F_{P}\left(Q_{2}^{2}\right)\right\}, \tag{5.39}
\end{align*}
$$

with $Q_{1}^{2} \equiv-q_{1}^{2}, Q_{2}^{2} \equiv-q_{2}^{2}$, the kinematical relation $\left(q_{1} \cdot q_{2}\right)=-\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)$, and the usual notations:

$$
\begin{equation*}
M \tilde{\nu}=(P \cdot \tilde{q}), \quad \tilde{Q}^{2}=-\tilde{q}^{2} \tag{5.41}
\end{equation*}
$$

These amplitudes contribute to the VVCS tensor of Eq. (5.34) with the following tensor structures:

$$
\begin{align*}
& T_{1}^{\mu \nu}=-\left(q_{1} \cdot q_{2}\right) g^{\mu \nu}+q_{1}^{\mu} q_{2}^{\nu},  \tag{5.42}\\
& T_{2}^{\mu \nu}=-4(P \cdot \tilde{q})^{2} g^{\mu \nu}-4\left(q_{1} \cdot q_{2}\right) P^{\mu} P^{\nu}+4(P \cdot \tilde{q})\left(P^{\nu} q_{1}^{\mu}+P^{\mu} q_{2}^{\nu}\right) . \tag{5.43}
\end{align*}
$$

The other unpolarized amplitudes vanish in the Born approximation, i.e.:

$$
\begin{equation*}
B_{3}^{\text {Born }}=0, \quad B_{4}^{\text {Born }}=0, \quad B_{19}^{\text {Born }}=0 . \tag{5.44}
\end{equation*}
$$

When evaluating the TPE correction, one can simplify the calculation by using explicitly gauge invariance, i.e., $q_{2}^{\mu} L_{\mu \nu \alpha}=0$ and $q_{1}^{\nu} L_{\mu \nu \alpha}=0$ for the virtual photon momenta $q_{1}, q_{2}$. Consequently, we can use the identities:

$$
\begin{equation*}
q_{1}^{\mu} L_{\mu \nu \alpha}=q^{\mu} L_{\mu \nu \alpha}, \quad q_{2}^{\nu} L_{\mu \nu \alpha}=-q^{\nu} L_{\mu \nu \alpha} . \tag{5.45}
\end{equation*}
$$

Approximating the arguments of proton form factors as: $Q_{1}^{2} \approx Q_{2}^{2} \approx \tilde{Q}^{2}-Q^{2} / 4$, the $B_{1}$ and $B_{2}$ Born contributions can be obtained from the effective near-forward VVCS tensor,

$$
\begin{align*}
M^{\mu \nu}\left(\tilde{\nu}, \tilde{Q}^{2}\right)= & \left(-g^{\mu \nu}+\frac{q^{\mu} q^{\nu}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}\right) \mathrm{T}_{1}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \\
& +\frac{1}{M^{2}}\left(P^{\mu}+\frac{M \tilde{\nu}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}} q^{\mu}\right)\left(P^{\nu}-\frac{M \tilde{\nu}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}} q^{\nu}\right) \mathrm{T}_{2}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \tag{5.46}
\end{align*}
$$

The expression of Eq. (5.46) gives the correct forward limit, when $q=0$. For a Dirac point particle, we checked that Eq. (5.46) corresponds with the exact result for the unpolarized VVCS tensor when replacing $F_{D}\left(Q^{2}\right) \rightarrow 1$ and $F_{P}\left(Q^{2}\right) \rightarrow 0$ in Eqs. (2.66) and (2.67). At low $Q^{2}$, it leads to Eq. (5.36) and gives the leading terms in the low- $Q^{2}$ expansion of the proton contribution to $\delta_{2 \gamma}$ of Eq. (5.17). We call the approximation of Eq. (5.46) the near-forward approximation.

We now turn to the inelastic contributions to the unpolarized forward Compton amplitudes $\mathrm{T}_{1}, \mathrm{~T}_{2}$. They are expressed in terms of the VVCS invariant amplitudes $B_{i}$ in the forward kinematics by

$$
\begin{align*}
& \mathrm{T}_{1}\left(\tilde{\nu}, \tilde{Q}^{2}\right)=\alpha\left(-\tilde{Q}^{2} B_{1}+4 M^{2} \tilde{\nu}^{2} B_{2}-\tilde{Q}^{4} B_{3}+4 M \tilde{\nu} \tilde{Q}^{2} B_{4}\right)  \tag{5.47}\\
& \mathrm{T}_{2}\left(\tilde{\nu}, \tilde{Q}^{2}\right)=4 M^{2} \tilde{Q}^{2} \alpha\left(B_{2}+\tilde{Q}^{2} B_{19}\right) \tag{5.48}
\end{align*}
$$

In this work, we choose the tensor form of Eq. (5.46) to describe the inelastic TPE correction and only keep the amplitudes $B_{1}$ and $B_{2}$ in the region of small momentum transfer.

We describe the VVCS invariant amplitudes $B_{i}\left(q_{1}^{2}, q_{2}^{2}, q_{1} \cdot q_{2}, \tilde{q} \cdot P\right)$ as

$$
\begin{equation*}
B_{i}\left(q_{1}^{2}, q_{2}^{2}, q_{1} \cdot q_{2}, \tilde{q} \cdot P\right) \simeq B_{i}\left(-\tilde{Q}^{2}+\frac{Q^{2}}{4},-\tilde{Q}^{2}+\frac{Q^{2}}{4},-\tilde{Q}^{2}+\frac{Q^{2}}{4}, M \tilde{\nu}\right) \tag{5.49}
\end{equation*}
$$

where we use the approximation $q_{1}^{2} \approx q_{2}^{2} \approx-\tilde{Q}^{2}+Q^{2} / 4$.
The near-forward approximation allows one to exactly obtain the first two terms of the inelastic TPE expansion in the elastic electron-proton scattering coming from the proton structure functions $F_{1}$ and $F_{2}$ :

$$
\begin{equation*}
\delta_{2 \gamma}^{\text {inel }} \approx a(\omega) Q^{2} \ln Q^{2}+b(\omega) Q^{2} \tag{5.50}
\end{equation*}
$$

In our calculation, we keep the $Q^{2}$ dependence in all kinematical factors, but we do not pretend on the exact validity of our approximations beyond the expansion of Eq. (5.50) due to the contribution of doubly virtual Compton amplitudes besides $B_{1}$ and $B_{2}$. The near-forward approximation of Eq. (5.46) is valid only in the region of small momentum transfer $Q^{2}$. In Section 5.6.2, we explicitly study the range in $Q^{2}$ over which such expansion is expected to provide a good approximation.

We obtain the TPE correction substituting the near-forward approximation of the VVCS tensor of Eq. (5.46) into the general expression for the cross section correction $\delta_{2 \gamma}$ of Eq. (5.31):

$$
\begin{align*}
\delta_{2 \gamma}^{\text {inel }}= & \frac{4 \pi G_{E}}{\varepsilon G_{E}^{2}+\tau_{P} G_{M}^{2}} \frac{2}{M} \frac{1-\varepsilon}{1-\varepsilon_{0}} \Re \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \Pi_{Q}^{+} \Pi_{Q}^{-} \\
& \times\left\{-2 m^{2}(K \cdot P)\left(\Pi_{K}^{-}+\Pi_{K}^{+}\right) \mathrm{T}_{1}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right)\right. \\
& -2 \frac{\tilde{Q}^{2}+\frac{Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}\left((K \cdot P)(K \cdot \tilde{q})-\frac{Q^{2}}{4}(P \cdot \tilde{q})\right)\left(\Pi_{K}^{-}-\Pi_{K}^{+}\right) \mathrm{T}_{1}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \\
& +2(K \cdot P)\left((K \cdot P)^{2}-\frac{Q^{2}}{4} P^{2}\right) \frac{\Pi_{K}^{-}+\Pi_{K}^{+}}{M^{2}} \mathrm{~T}_{2}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \\
& +\frac{Q^{2}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}\left((P \cdot \tilde{q})(K \cdot \tilde{q}) P^{2}-(P \cdot \tilde{q})^{2}(K \cdot P)\right) \frac{\Pi_{K}^{-}+\Pi_{K}^{+}}{M^{2}} \mathrm{~T}_{2}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \\
& +\left(P^{2}+\frac{(P \cdot \tilde{q})^{2} Q^{2}}{\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}}\right)\left((K \cdot P)(K \cdot \tilde{q})-\frac{Q^{2}}{4}(P \cdot \tilde{q})\right) \frac{\Pi_{K}^{-}-\Pi_{K}^{+}}{M^{2}} \mathrm{~T}_{2}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \\
& \left.-2(P \cdot \tilde{q}) \frac{\tilde{Q}^{2}-\frac{3 Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}\left((K \cdot P)^{2}-\frac{Q^{2}}{4} P^{2}\right) \frac{\Pi_{K}^{-}-\Pi_{K}^{+}}{M^{2}} \mathrm{~T}_{2}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right)\right\} . \tag{5.51}
\end{align*}
$$

The forward VVCS amplitudes $T_{1}, T_{2}$ are related to the experimentally measured unpolarized proton structure functions $F_{1}, F_{2}$ by dispersion relations of Eqs. (2.60, 2.61), see Section 2.4 for details. It is convenient to rewrite these DRs in terms of the squared invariant mass variable $W^{2} \equiv(P+\tilde{q})^{2}:$

$$
\begin{align*}
\Re \mathrm{T}_{1}\left(\tilde{\nu}, \tilde{Q}^{2}\right) & =\Re \mathrm{T}_{1}^{\text {Born }}\left(\tilde{\nu}, \tilde{Q}^{2}\right)+\mathrm{T}_{1}^{\text {subt }}\left(0, \tilde{Q}^{2}\right) \\
& +\frac{2 \tilde{\nu}^{2}}{\pi} \int_{W_{\text {thr }}^{2}}^{\infty} \frac{e^{2} M F_{1}\left(\left(W^{2}-P^{2}+\tilde{Q}^{2}\right) /(2 M), \tilde{Q}^{2}\right) \mathrm{d} W^{2}}{\left(W^{2}-P^{2}+\tilde{Q}^{2}\right)\left((P+\tilde{q})^{2}-W^{2}+i \varepsilon\right)\left((P-\tilde{q})^{2}-W^{2}+i \varepsilon\right)}, \\
\Re \mathrm{T}_{2}\left(\tilde{\nu}, \tilde{Q}^{2}\right) & =\Re \mathrm{T}_{2}^{\text {Born }}\left(\tilde{\nu}, \tilde{Q}^{2}\right)+\frac{1}{\pi} \int_{W_{\mathrm{thr}}^{2}}^{\infty} \frac{e^{2} M F_{2}\left(\left(W^{2}-P^{2}+\tilde{Q}^{2}\right) /(2 M), \tilde{Q}^{2}\right) \mathrm{d} W^{2}}{\left((P+\tilde{q})^{2}-W^{2}+i \varepsilon\right)\left((P-\tilde{q})^{2}-W^{2}+i \varepsilon\right)}, \tag{5.52}
\end{align*}
$$

with the pion-proton inelastic threshold: $W_{\mathrm{thr}}^{2}=\left(M+m_{\pi}\right)^{2} \approx 1.15 \mathrm{GeV}^{2}$.
In order to evaluate the inelastic TPE contribution using the forward non-Born VVCS amplitudes, we need the information on the proton structure functions $F_{1}$ and $F_{2}$ as well as to specify the subtraction function $\mathrm{T}_{1}^{\text {subt }}\left(0, \tilde{Q}^{2}\right)$. We evaluate the subtraction function contribution in Sections 5.4, 5.5 and the dispersive contribution due to the unpolarized structure functions $F_{1}$ and $F_{2}$ in Section 5.6.2.

### 5.4 Subtraction function contribution in $e p$ scattering

Before discussing the proton structure functions contribution to $\delta_{2 \gamma}$, we first discuss the subtraction term in the $\mathrm{T}_{1}$ amplitude in the elastic electron-proton scattering regardless of the near-forward approximation of Eq. (5.46). The subtraction function $\mathrm{T}_{1}^{\text {subt }}\left(0, \tilde{Q}^{2}\right)$ is expressed in terms of only two VVCS amplitudes $B_{1}$ and $B_{3}$ as

$$
\begin{equation*}
\mathrm{T}_{1}^{\text {subt }}\left(0, \tilde{Q}^{2}\right)=-\alpha\left(B_{1}\left(Q^{2}\right) Q^{2}+B_{3}\left(Q^{2}\right) Q^{4}\right), \tag{5.54}
\end{equation*}
$$

with the relevant VVCS tensor structures given by

$$
\begin{align*}
& T_{1}^{\mu \nu}=-\left(q_{1} \cdot q_{2}\right) g^{\mu \nu}+q_{1}^{\mu} q_{2}^{\nu},  \tag{5.55}\\
& T_{3}^{\mu \nu}=q_{1}^{2} q_{2}^{2} g^{\mu \nu}+\left(q_{1} \cdot q_{2}\right) q_{1}^{\nu} q_{2}^{\mu}-\frac{q_{1}^{2}+q_{2}^{2}}{2}\left(q_{1}^{\nu} q_{1}^{\mu}+q_{2}^{\nu} q_{2}^{\mu}\right)+\frac{q_{1}^{2}-q_{2}^{2}}{2}\left(q_{1}^{\nu} q_{1}^{\mu}-q_{2}^{\nu} q_{2}^{\mu}\right) . \tag{5.56}
\end{align*}
$$

The TPE correction due to the subtraction term, $\delta_{2 \gamma}^{\text {subt }}$, arising from the amplitudes $B_{1}, B_{3}$, is obtained from Eq. (5.31) as

$$
\begin{align*}
\delta_{2 \gamma}^{\text {subt }} \sim & \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}}\left(B_{1} \Pi_{Q}^{+} \Pi_{Q}^{-}\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}\right)+B_{3}\right) \Pi_{K}^{+} \Pi_{K}^{-} \\
& \times\left\{\frac{Q^{2}}{4}(P \cdot \tilde{q})(K \cdot \tilde{q})-(K \cdot P)(K \cdot \tilde{q})^{2}\right\} \equiv I_{1}+I_{2} . \tag{5.57}
\end{align*}
$$

We express the loop integral of the first term of Eq. (5.57), denoted by $I_{1}$, as

$$
\begin{equation*}
I_{1}=\frac{Q^{2}}{4} \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}}\left(B_{1} \Pi_{Q}^{+} \Pi_{Q}^{-}\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}\right)+B_{3}\right)(P \cdot \tilde{q})(K \cdot \tilde{q}) \Pi_{K}^{+} \Pi_{K}^{-}=a_{K}(K \cdot P), \tag{5.58}
\end{equation*}
$$

where we have defined $a_{K}$ as

$$
\begin{equation*}
a_{K} K^{\mu}=\frac{Q^{2}}{16} \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}}\left(B_{1} \Pi_{Q}^{+} \Pi_{Q}^{-}\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}\right)+B_{3}\right) \tilde{q}^{\mu}\left(\Pi_{K}^{-}-\Pi_{K}^{+}\right) . \tag{5.59}
\end{equation*}
$$

On the other hand, since $K^{2}=Q^{2} / 4$ in the electron massless limit, we can rewrite the integral $I_{1}$ as

$$
\begin{equation*}
I_{1}=(K \cdot P) \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}}\left(B_{1} \Pi_{Q}^{+} \Pi_{Q}^{-}\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}\right)+B_{3}\right)(K \cdot \tilde{q})^{2} \Pi_{K}^{+} \Pi_{K}^{-}, \tag{5.60}
\end{equation*}
$$

which exactly cancels the integral $I_{2}$ from the second term in Eq. (5.57). Consequently, the subtraction function contribution to unpolarized elastic electron-proton scattering vanishes in the limit of massless electrons.
It is instructive to study the subtraction function contribution to the TPE amplitude $T^{2 \gamma}$ of elastic lepton-proton scattering, accounting for a finite lepton mass. The amplitude is given by

$$
\begin{equation*}
T^{2 \gamma}=e^{2} \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{3}} \frac{\tilde{L}^{\mu \nu} \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) M_{\mu \nu} N(p, \lambda)}{\left(\left(\tilde{q}-\frac{q}{2}\right)^{2}-\mu^{2}\right)\left(\left(\tilde{q}+\frac{q}{2}\right)^{2}-\mu^{2}\right)} \tag{5.61}
\end{equation*}
$$

with the leptonic tensor:

$$
\begin{equation*}
\tilde{L}^{\mu \nu}=\bar{u}\left(k^{\prime}, h^{\prime}\right)\left(\gamma^{\mu} \frac{\hat{K}-\hat{\tilde{q}}+m}{(K-\tilde{q})^{2}-m^{2}} \gamma^{\nu}+\gamma^{\nu} \frac{\hat{K}+\hat{\tilde{q}}+m}{(K+\tilde{q})^{2}-m^{2}} \gamma^{\mu}\right) u(k, h) . \tag{5.62}
\end{equation*}
$$

We study the TPE contribution due to the VVCS amplitude $B_{1}$ first. The $-\left(q_{1} \cdot q_{2}\right) g^{\mu \nu}$ term contribution is given by

$$
\begin{equation*}
-e^{4} \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} B_{1}\left(2 m\left(\tilde{q}^{2}+\frac{Q^{2}}{4}\right) \bar{u} u+4(K \cdot \tilde{q}) \bar{u} \hat{\tilde{q}} u\right) \cdot(\bar{N} N)\left(\tilde{q}^{2}+\frac{Q^{2}}{4}\right) \Pi_{K}^{+} \Pi_{K}^{-} \Pi_{Q}^{+} \Pi_{Q}^{-} . \tag{5.63}
\end{equation*}
$$

Using the gauge symmetry in the tensor $T_{1}^{\mu \nu}$, the $q_{1}^{\mu} q_{2}^{\nu}$ term contribution is expressed as

$$
\begin{equation*}
e^{4} Q^{2} \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} B_{1}(2(K \cdot \tilde{q}) \bar{u} \hat{\tilde{q} u}) \cdot(\bar{N} N) \Pi_{K}^{+} \Pi_{K}^{-} \Pi_{Q}^{+} \Pi_{Q}^{-} . \tag{5.64}
\end{equation*}
$$

Denoting

$$
\begin{equation*}
e^{4} Q^{2} \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} B_{1} \cdot\left(2(K \cdot \tilde{q}) \tilde{q}^{\mu}\right) \Pi_{K}^{+} \Pi_{K}^{-} \Pi_{Q}^{+} \Pi_{Q}^{-}=a_{K}^{(1)} K^{\mu}, \tag{5.65}
\end{equation*}
$$

the contribution of the $q_{1}^{\mu} q_{2}^{\nu}$ term is given by

$$
\begin{equation*}
m a_{K}^{(1)}(\bar{u} u) \cdot(\bar{N} N) . \tag{5.66}
\end{equation*}
$$

Note that, according to the symmetry properties of the VVCS amplitudes [209], the expansion of Eq. (5.65) also contains no $q^{\mu}$ term for the case of VVCS amplitude $B_{1}\left(q_{1}^{2}, q_{2}^{2},\left(q_{1} \cdot q_{2}\right),(P \cdot \tilde{q})\right)$ subtracted at an arbitrary point $\nu_{0}:(P \cdot \tilde{q})=M \nu_{0}$. The amplitude $B_{3}$ contributes through only one tensor structure $q_{1}^{2} q_{2}^{2} g^{\mu \nu}$ in the similar way as the amplitude $B_{1}$ contributes through the structure $-\left(q_{1} \cdot q_{2}\right) g^{\mu \nu}$.

We conclude that the $\mathrm{T}_{1}$ subtraction function contributes to the $(\bar{u} u) \cdot(\bar{N} N)$ term in the elastic lepton-proton scattering amplitude. ${ }^{3}$ This contribution is suppressed by the lepton mass. In the language of effective field theories, the chiral (or axial) symmetry on the lepton side forbids the $(\bar{u} u) \cdot(\bar{N} N)$ type structure for massless electrons, and therefore the contribution of the subtraction function vanishes in the massless lepton limit.

[^14]
### 5.5 Subtraction function contribution in $\mu p$ scattering

In the present Section we discuss the TPE correction due to the subtraction function $T_{1}^{\text {subt }}\left(0, \tilde{Q}^{2}\right)$ in the elastic muon-proton scattering. For this purpose, we compare three different estimates for $\beta\left(\tilde{Q}^{2}\right)$ defined through Eq. (2.74). At low $Q^{2}$ we use existing estimates from heavy-baryon and baryon chiral perturbation theory and compare them with an empirical determination of $\beta\left(\tilde{Q}^{2}\right)$ based on the high-energy behavior of the Compton amplitude, which is described in Section 2.4.1.

### 5.5.1 Heavy-baryon Chiral Perturbation Theory subtraction function

Firstly, we show the fit of Ref. [141] obtained by matching the heavy-baryon chiral perturbation theory (HBChPT) result to a dipole behavior:

$$
\begin{equation*}
\beta\left(\tilde{Q}^{2}\right)=\frac{\beta_{M}}{\left(1+\tilde{Q}^{2} / \Lambda^{2}\right)^{2}}, \quad \Lambda=530-842 \mathrm{MeV} \tag{5.67}
\end{equation*}
$$

with the value of the magnetic polarizability $\beta_{M}=(2.5 \pm 0.4) \times 10^{-4} \mathrm{fm}^{3}$ taken from PDG [15]. For the purpose of showing error bands in our numerical estimates, we choose the lower and upper edges of such bands to correspond with the values: $\Lambda=530 \mathrm{MeV}, \beta_{M}=2.1 \times 10^{-4} \mathrm{fm}^{3}$ and $\Lambda=842 \mathrm{MeV}, \beta_{M}=2.9 \times 10^{-4} \mathrm{fm}^{3}$ respectively. The resulting bands for $\mathrm{T}_{1}^{\text {subt }}\left(0, \tilde{Q}^{2}\right)$ are shown in Fig. 5.6, and correspondingly for $\beta\left(\tilde{Q}^{2}\right)$ in Fig. 5.7 (blue bands).


Figure 5.6: The empirical subtraction function of Eq. (2.83) in comparison with the subtraction functions from HBChPT of Birse et al. [141], and from BChPT [142].

### 5.5.2 Baryon Chiral Perturbation Theory subtraction function

Secondly, we also show the prediction for $\beta\left(\tilde{Q}^{2}\right)$ resulting from the covariant baryon chiral perturbation theory (BChPT) [142], with $\beta$ decomposed as

$$
\begin{equation*}
\beta\left(\tilde{Q}^{2}\right)=\beta_{\pi N}\left(\tilde{Q}^{2}\right)+\beta_{\Delta}\left(\tilde{Q}^{2}\right)+\beta_{\pi \Delta}\left(\tilde{Q}^{2}\right) \tag{5.68}
\end{equation*}
$$

with $\beta_{\pi N}\left(\tilde{Q}^{2}\right)$ the $\mathcal{O}\left(p^{3}\right)$ diamagnetic polarizability contribution from $\pi N$ loops given by Eq. (22) of Ref. [142], $\beta_{\Delta}\left(\tilde{Q}^{2}\right)$ the paramagnetic contribution of the $\Delta$-resonance to the magnetic
polarizability [90] and $\beta_{\pi \Delta}\left(\tilde{Q}^{2}\right)$ the $\mathcal{O}\left(p^{7 / 2}\right)$ at $p \simeq m_{\pi}$ diamagnetic polarizability contribution from $\pi \Delta$ loops [90].

In Fig. 5.6 (Fig. 5.7), we compare the heavy-baryon and baryon ChPT predictions for $\mathrm{T}_{1}^{\text {subt }}\left(0, \tilde{Q}^{2}\right)\left(\beta\left(\tilde{Q}^{2}\right)\right)$ with the empirical determinations from Section 2.4.1. Notice that the HBChPT value of $\beta(0)$ is taken from a fit to data (PDG 2014) whereas the baryon ChPT value of $\beta(0)$ results from the sum of the positive paramagnetic part due to the $s$-channel $\Delta$ excitation $\beta_{\Delta}(0) \simeq 7 \times 10^{-4} \mathrm{fm}^{3}$, and the negative diamagnetic part due to $\pi N$ and $\pi \Delta$ loops, i.e., $\beta_{\pi N}(0)=-2 \times 10^{-4} \mathrm{fm}^{3}$ and $\beta_{\pi \Delta}(0)=-1.2 \times 10^{-4} \mathrm{fm}^{3}$.


Figure 5.7: The empirical estimate for the magnetic polarizability $\beta\left(\tilde{Q}^{2}\right)$ based on Eqs. (2.74, 2.83 ) and including the constraint from the experimental value $\beta_{M}=\beta(0)$ compared with the HBChPT result of Birse et al. [141] normalized to the PDG value $\beta(0)=$ $(2.5 \pm 0.4) \times 10^{-4} \mathrm{fm}^{3}$ [15], and with the BChPT result [142].

### 5.5.3 TPE correction from the subtraction function

Using Eqs. (5.31-5.33), we can now estimate the TPE correction due to the $\mathrm{T}_{1}^{\text {subt }}\left(0, \tilde{Q}^{2}\right)$ contribution to the first term in the near-forward VVCS tensor, see Eq. (5.46) for the form of decomposition of this tensor. Performing the traces in Eq. (5.33) explicitly, the subtraction function results in the following TPE correction in the region of small momentum transfers:

$$
\begin{align*}
\delta_{2 \gamma}^{\text {subt }} & =\frac{32 \pi G_{E}}{\varepsilon G_{E}^{2}+\tau_{P} G_{M}^{2}} \frac{1-\varepsilon}{1-\varepsilon_{0}} \frac{1}{M} \Re \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \beta\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \Pi_{K}^{+} \Pi_{K}^{-} \Pi_{Q}^{+} \Pi_{Q}^{-} \\
& \times\left\{(K \cdot P) m^{2}\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}+\frac{1}{2}\left(Q^{2}(P \cdot \tilde{q})-4(K \cdot P)(K \cdot \tilde{q})\right)(K \cdot \tilde{q})\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}\right)\right\}, \tag{5.69}
\end{align*}
$$

where the lepton (photon) propagators $\Pi_{K}^{ \pm}\left(\Pi_{Q}^{ \pm}\right)$are defined as in Eqs. (5.37). The second term within the curly brackets of Eq. (5.69) can be simplified to yield the expression:

$$
\begin{align*}
\delta_{2 \gamma}^{\text {subt }} & =\frac{32 \pi m^{2} G_{E}}{\varepsilon G_{E}^{2}+\tau_{P} G_{M}^{2}} \frac{1-\varepsilon}{1-\varepsilon_{0}} \frac{(K \cdot P)}{M} \\
& \times \Re \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \beta\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \Pi_{K}^{+} \Pi_{K}^{-} \Pi_{Q}^{+} \Pi_{Q}^{-}\left\{\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}-\frac{2(K \cdot \tilde{q})^{2}}{m^{2}+\frac{Q^{2}}{4}}\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}\right)\right\}, \tag{5.70}
\end{align*}
$$

making explicit the overall proportionality of $\delta_{2 \gamma}^{\text {subt }}$ to the squared lepton mass $m^{2}$.
The integration in Eq. (5.70) is performed through a Wick rotation, as detailed in Appendix N , and the resulting TPE correction is given by

$$
\begin{equation*}
\delta_{2 \gamma}^{\text {subt }}=\frac{m^{2} G_{E}\left(Q^{2}\right)}{\varepsilon G_{E}^{2}\left(Q^{2}\right)+\tau_{P} G_{M}^{2}\left(Q^{2}\right)} \frac{1-\varepsilon}{1-\varepsilon_{0}} \frac{(K \cdot P)}{M} \frac{Q}{K} \int_{x_{\min }}^{\infty} f(x, a) \beta\left(\frac{Q^{2}(x-1)}{4}\right) \frac{\mathrm{d} x}{4 \pi}, \tag{5.71}
\end{equation*}
$$

in terms of the dimensionless variable $x=4 \tilde{Q}^{2} / Q^{2}$ with the weighting function $f(x, a)$ :

$$
\begin{align*}
f(x, a)= & -\frac{2 x^{\Theta(1-x)}}{\sqrt{1+a}} \Theta(x)+\frac{1+2 a-x}{1+a} \frac{|1-x|}{1+x}\left\{\ln \left|\frac{x-z}{x+z}\right| \Theta(x) \Theta(1-x)\right. \\
& \left.+\ln \left|\frac{z+1}{z-1}\right| \Theta(x-1)+\ln \left|\frac{x-z}{x+z} \frac{z-1}{z+1}\right| \Theta\left(x-x_{\min }\right) \Theta(-x)\right\} \tag{5.72}
\end{align*}
$$

with

$$
\begin{equation*}
z=\frac{1-x-\sqrt{(1+x)^{2}+4 a x}}{2 \sqrt{1+a}}, \quad x_{\min }=-(\sqrt{1+a}-\sqrt{a})^{2}, \quad a=\frac{4 m^{2}}{Q^{2}} . \tag{5.73}
\end{equation*}
$$

At low momentum transfers the result of Eq. (5.71) starts from a term proportional to $Q^{2}$. We show the $x$ (or $\tilde{Q}^{2}$ ) dependence of the weighting function of Eq. (5.72) in Fig. 5.8.


Figure 5.8: The weighting function $f$ of Eq. (5.72) for the range of $Q^{2}$ values of the MUSE experiment.

It was speculated in Ref. [138] that in order to explain the proton radius puzzle it would require a huge enhancement of $\beta\left(\tilde{Q}^{2}\right)$ at large $\tilde{Q}^{2}$. In order to account for the experimentally observed discrepancy on $\Delta E_{2 S}$ of around $310 \mu \mathrm{eV}$ [118], it would require an around two orders of magnitude larger TPE correction than the naturally expected result from the ChPT estimates. For this purpose, an ad-hoc subtraction function, proposed to be added as an extra contribution on top of the ChPT based subtraction functions discussed above, was conjectured in Ref. [138] with the following functional form:

$$
\begin{equation*}
\beta_{\mathrm{extra}}\left(\tilde{Q}^{2}\right)=\left(\frac{\tilde{Q}^{2}}{M_{0}^{2}}\right)^{2} \frac{\beta_{M}}{\left(1+\tilde{Q}^{2} / \Lambda_{0}^{2}\right)^{5}}, \quad M_{0}=0.5 \mathrm{GeV}, \quad \Lambda_{0}=3.92 \mathrm{GeV} \tag{5.74}
\end{equation*}
$$

In such scenario, the large $\tilde{Q}^{2}$ region would also dominate the TPE correction to the elastic muon-proton scattering, and the integral of Eq. (5.71) would be approximated by

$$
\begin{equation*}
\delta_{2 \gamma, 0}^{\text {subt }} \approx-\frac{3 m^{2} G_{E}}{\varepsilon G_{E}^{2}+\tau_{P} G_{M}^{2}} \frac{1-\varepsilon}{1-\varepsilon_{0}} \frac{(K \cdot P)}{\pi M} \int_{0}^{\infty} \beta\left(\tilde{Q}^{2}\right) \frac{\mathrm{d} \tilde{Q}^{2}}{\tilde{Q}^{2}} \approx-\frac{3 Q^{2} m^{2}}{2 \pi \omega} \int_{0}^{\infty} \beta\left(\tilde{Q}^{2}\right) \frac{\mathrm{d} \tilde{Q}^{2}}{\tilde{Q}^{2}}, \tag{5.75}
\end{equation*}
$$

where the last step gives the approximate expression in the limit $Q^{2} \ll M^{2}, M \omega, \omega^{2}$. This approximation corresponds in magnitude with the result of Ref. [138] for $\mu^{-} p$ scattering, however it differs by an overall sign.

In Fig. 5.9 we compare the TPE correction to elastic muon-proton scattering (for MUSE kinematics) due to the above discussed ChPT as well as empirically determined subtraction functions. To estimate the size of uncertainties of the BChPT result [142], we plot a band corresponding with a variation of the upper integration limit in Eq. (5.71) between $\tilde{Q}^{2}=$ $0.9-5 \mathrm{GeV}^{2}$. We notice that the HBChPT and BChPT results are in agreement within their uncertainties. The TPE correction due to the empirically extracted subtraction function is also


Figure 5.9: Subtraction function contribution to the TPE correction in elastic muon-proton scattering for the muon lab momentum $\mathrm{k}=153 \mathrm{MeV}$. Blue band: result for the HBChPT based subtraction function [141]. Pink band: result for the BChPT based subtraction function [142]. Green band: result based on the empirical subtraction function, corresponding with the result shown in Fig. 5.7. Solid curve: result based on the conjectured subtraction function of Ref. [138]. The (black) dashed-dotted curve is the Feshbach term of Eq. (3.53) for a point-like Dirac particle corrected by the recoil factor $(1+m / M)$. The sign labels on the curve show the sign of the corresponding expressions for $\mu^{-} p$ scattering.
shown in Fig. 5.9, giving a similar though slightly smaller result. This can be understood as the empirically determined $\beta\left(\tilde{Q}^{2}\right)$ changes sign as function of $\tilde{Q}^{2}$. The region of $\tilde{Q}^{2}$ contributing to the above result is shown in Fig. 5.10. One sees that the TPE integral has largely converged for an upper integration limit value of around $\tilde{Q}_{\max }^{2} \sim 1 \mathrm{GeV}^{2}$.

In Fig. 5.9, we furthermore also show the TPE correction to elastic muon-proton scattering resulting from the subtraction function conjectured in Ref. [138] to explain the proton radius puzzle through enhancing the TPE corrections by nearly two orders of magnitude. Even though


Figure 5.10: The dependence of the integral of Eq. (5.71) on the upper integration limit $\tilde{Q}_{\max }^{2}$ for three different estimates of the subtraction function $\beta\left(\tilde{Q}^{2}\right)$ as described in the text.
the weighting functions entering the TPE corrections in the muonic hydrogen Lamb shift and the elastic muon-proton scattering are different, one notices from Fig. 5.9 that the subtraction function of Ref. [138] also yields a nearly two order of magnitude larger TPE correction for the elastic muon-proton scattering. To put this in perspective, we also display in Fig. 5.9 the model independent estimate of the elastic TPE contribution, which has to be added on top of the inelastic TPE contribution, and which is due to the Feshbach term of Eq. (3.53) corrected by the recoil factor $(1+m / M)$. One notices that the use of such large subtraction function would yield an inelastic TPE correction to elastic muon-proton scattering which in magnitude already would exceed the elastic Feshbach contribution around $Q^{2}=0.02 \mathrm{GeV}^{2}$, and would increase further with increasing $Q^{2}$.

### 5.6 TPE correction from unpolarized proton structure functions

In this Section, we evaluate the inelastic TPE correction coming from the unpolarized proton structure functions $\delta_{2 \gamma}^{\mathrm{F}_{1}, \mathrm{~F}_{2}}$. Using Eqs. (5.31, 5.32) and working out the traces in Eq. (5.33), the corresponding contributions from the unpolarized proton structure functions $F_{1}$ and $F_{2}$ to $\delta_{2 \gamma}^{\text {inel }}$ are given by

$$
\begin{align*}
& \delta_{2 \gamma}^{F_{1}}=F \Re \int_{W_{\mathrm{thr}}^{2}}^{\infty} \mathrm{d} W^{2} \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \Pi_{Q}^{+} \Pi_{Q}^{-} F_{1}\left(W^{2}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \frac{(P \cdot \tilde{q})^{2}}{W^{2}-P^{2}+\tilde{Q}^{2}} \\
& \times \frac{\left\{A\left(\Pi_{K}^{-}+\Pi_{K}^{+}\right)+B\left(\Pi_{K}^{-}-\Pi_{K}^{+}\right)\right\}}{\left((P+\tilde{q})^{2}-W^{2}\right)\left((P-\tilde{q})^{2}-W^{2}\right)},  \tag{5.76}\\
& \delta_{2 \gamma}^{F_{2}}=F \Re \int_{W_{\mathrm{thr}}^{2}}^{\infty} \mathrm{d} W^{2} \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \Pi_{Q}^{+} \Pi_{Q}^{-} F_{2}\left(W^{2}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \\
& \times \frac{\left\{C\left(\Pi_{K}^{-}+\Pi_{K}^{+}\right)+D\left(\Pi_{K}^{-}-\Pi_{K}^{+}\right)\right\}}{\left((P+\tilde{q})^{2}-W^{2}\right)\left((P-\tilde{q})^{2}-W^{2}\right)}, \tag{5.77}
\end{align*}
$$

with the following notations:

$$
\begin{align*}
F & =\frac{8 e^{2}}{M^{2}} \frac{G_{E}}{\varepsilon G_{E}^{2}+\tau_{P} G_{M}^{2}} \frac{1-\varepsilon}{1-\varepsilon_{0}}, \\
A & =-4 m^{2}(K \cdot P), \\
B & =\frac{\tilde{Q}^{2}+\frac{Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}\left(Q^{2}(P \cdot \tilde{q})-4(K \cdot P)(K \cdot \tilde{q})\right), \\
C & =\frac{1}{2}(K \cdot P)\left(4(K \cdot P)^{2}-Q^{2} P^{2}\right)+\frac{Q^{2}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}(P \cdot \tilde{q})\left((K \cdot \tilde{q}) P^{2}-(P \cdot \tilde{q})(K \cdot P)\right), \\
D & =-\frac{1}{4}\left(P^{2}+\frac{(P \cdot \tilde{q})^{2} Q^{2}}{\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}}\right)\left(Q^{2}(P \cdot \tilde{q})-4(K \cdot P)(K \cdot \tilde{q})\right) \\
& -\frac{1}{2}(P \cdot \tilde{q}) \frac{\tilde{Q}^{2}-\frac{3 Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}\left(4(K \cdot P)^{2}-Q^{2} P^{2}\right) . \tag{5.78}
\end{align*}
$$

We express the resulting correction $\delta_{2 \gamma}^{\mathrm{F}_{1}, \mathrm{~F}_{2}}$ as a two-dimensional integral over the unpolarized proton structure functions $F_{1}$ and $F_{2}$ :

$$
\begin{align*}
\delta_{2 \gamma}^{\mathrm{F}_{1}, \mathrm{~F}_{2}}=\delta_{2 \gamma}^{\mathrm{F}_{1}}+\delta_{2 \gamma}^{\mathrm{F}_{2}}=\int \mathrm{d} W^{2} \mathrm{~d} \tilde{Q}^{2} & \left\{w_{1}\left(W^{2}, \tilde{Q}^{2}\right) F_{1}\left(W^{2}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right)\right. \\
& \left.+w_{2}\left(W^{2}, \tilde{Q}^{2}\right) F_{2}\left(W^{2}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right)\right\} \tag{5.79}
\end{align*}
$$

To obtain the result of Eq. (5.79), starting from Eqs. (5.76) and (5.77), we perform the integration, using two choices of Breit frames with the aim to cross check the method and its application. In a first choice, which we denote as $P$-frame, the kinematics of the external particles is defined by

$$
\begin{equation*}
K=\left(K_{0}, 0,0,|\vec{K}|\right), \quad P=P(1,0,0,0), \quad q=Q(0,1,0,0) . \tag{5.80}
\end{equation*}
$$

In a second choice, which we denote as $K$-frame, the kinematics of the external particles is defined by

$$
\begin{equation*}
K=K(1,0,0,0), \quad P=\left(P_{0}, 0,0,|\vec{P}|\right), \quad q=Q(0,1,0,0) . \tag{5.81}
\end{equation*}
$$

In either of these frames, we evaluate the weighting functions $w_{1}, w_{2}$ by performing a Wick rotation in Eqs. (5.76) and (5.77) in the complex $\tilde{q}_{0}$ variable plane. The resulting TPE correction is given by the sum of the integral along the imaginary $\tilde{q}_{0}$ axis and the contributions from the poles which are crossed by the integration contour. ${ }^{4}$ In addition to the photon and lepton propagator poles, there exist the hadronic poles coming from the dispersion relation propagators:

$$
\begin{equation*}
\Pi_{H}^{ \pm}=\frac{1}{(\tilde{q} \pm P)^{2}-W^{2}} . \tag{5.82}
\end{equation*}
$$

The pole positions are shown in Fig. 5.11.
The integration contour does not cross the photon poles $\Pi_{q}$. The expressions of Eqs. (5.76) and (5.77) are symmetric with respect to the change of integration variable $\tilde{q} \rightarrow-\tilde{q}$. Exploiting

[^15]

Figure 5.11: The position of the $\tilde{q}_{0}$ poles for different propagators. The lepton poles $\left(\Pi_{K}\right)$ and hadronic poles $\left(\Pi_{H}\right)$ contribute only in a limited region of integration variables.
this symmetry, we only need to calculate the residues of the upper half plane poles and double the result. The leptonic pole $\Pi_{K}^{-}$and the hadronic pole $\Pi_{H}^{-}$are moving poles; they contribute in the limited region of $W^{2}, \tilde{Q}^{2}$ variables. We show the corresponding regions in Fig. 5.12 (Fig. 5.13 ) in case of the electron-proton (muon-proton) scattering for the limit of low- $Q^{2}$ scattering. We provide the expressions for the regions of these poles contributions and weighting functions $w_{1}, w_{2}$ in Appendix P.


Figure 5.12: Integration ranges for the different contributions to $\delta_{2 \gamma}^{\mathrm{F}_{1}, \mathrm{~F}_{2}}$ in electron-proton scattering.

$$
\mathrm{P}(1,0,0,0) \text { frame }
$$



Figure 5.13: Integration ranges for the different contributions to $\delta_{2 \gamma}^{\mathrm{F}_{1}, \mathrm{~F}_{2}}$ in muon-proton scattering.

### 5.6.1 The leading $Q^{2} \ln Q^{2}$ inelastic TPE correction in $e p$ scattering

In this Section, we describe the way to obtain the leading $Q^{2} \ln Q^{2}$ inelastic TPE contribution to the unpolarized cross section in elastic electron-proton scattering in terms of the total photoabsorption cross section $\sigma_{T}[174]$ in our approach. For the region of small momentum transfer, we exploit the approximation [182]:

$$
\begin{align*}
& F_{1}\left(W^{2}, \tilde{Q}^{2}\right) \approx \frac{(P \cdot \tilde{q})}{\pi e^{2}} \sigma_{T}\left(W^{2}\right),  \tag{5.83}\\
& F_{2}\left(W^{2}, \tilde{Q}^{2}\right) \approx \frac{\tilde{Q}^{2}}{\pi e^{2}} \sigma_{T}\left(W^{2}\right) \tag{5.84}
\end{align*}
$$

In the $P$-frame, the leading correction is coming from the leptonic pole, whereas in the $K$-frame, it originates from the hadronic pole. We present its derivation in the $K$-frame. The pole $\tilde{q}_{0}=P_{0}-\sqrt{(\vec{P}-\overrightarrow{\tilde{q}})^{2}+W^{2}}$ contributes in the invariant mass region $W^{2}<P_{0}^{2} \approx$ $4(K \cdot P)^{2} / Q^{2}$. For the leading contribution, we consider the $W^{2}$ integration up to infinity. For the case of small momentum transfer values, the integration region in the $\tilde{Q}^{2}$ variable is given by

$$
\begin{equation*}
\alpha_{0} Q^{2} \leq \tilde{Q}^{2} \leq \frac{16(K \cdot P)^{2}}{Q^{2}}, \tag{5.85}
\end{equation*}
$$

with

$$
\begin{equation*}
\alpha_{0}=\left(\frac{W^{2}-M^{2}}{4(K \cdot P)}\right)^{2} \tag{5.86}
\end{equation*}
$$

We can replace the upper integration limit by infinity, but the approximation of Eqs. (5.83) and (5.84) in terms of the photoabsorption cross section is valid only up to some hadronic scale
$\Lambda$, which reproduces the $\tilde{Q}^{2}$ behavior of unpolarized structure functions. ${ }^{5}$ In the present work, we go beyond such approximation and directly use the unpolarized proton structure functions $F_{1}, F_{2}$ with their $\tilde{Q}^{2}$ dependence as an input. The integration region $\tilde{Q}^{2}>\Lambda^{2}$ does not contribute to the term proportional to $Q^{2} \ln Q^{2}$, and therefore we integrate up to the squared hadronic scale $\Lambda^{2}$.

The logarithmic term comes from the region $\tilde{Q}^{2} \gg Q^{2}$. Accounting also for the pole condition $(P \cdot \tilde{q})=\left(P^{2}-\tilde{Q}^{2}-W^{2}\right) / 2 \gg Q^{2}$, we get:

$$
\begin{align*}
\delta_{2 \gamma}^{\mathrm{F} 1} & =\frac{2 Q^{2} e^{2}}{(K \cdot P)} \Re \int_{W_{\mathrm{thr}}^{2}}^{\infty} \mathrm{d} W^{2} \int \frac{\mathrm{~d}^{3} \overrightarrow{\tilde{q}}}{(2 \pi)^{3}} \frac{K \tilde{q}_{0}\left(\Pi_{K}^{-}-\Pi_{K}^{+}\right)}{\tilde{Q}^{4}} \frac{F_{1}\left(W^{2}, \tilde{Q}^{2}\right)}{P_{0}-\tilde{q}_{0}},  \tag{5.87}\\
\delta_{2 \gamma}^{\mathrm{F}_{2}} & =4 Q^{2} e^{2} \Re \int_{W_{\mathrm{thr}}^{2}}^{\infty} \mathrm{d} W^{2} \int \frac{\mathrm{~d}^{3} \overrightarrow{\tilde{q}}}{(2 \pi)^{3}} \frac{(P \cdot \tilde{q})\left(\Pi_{K}^{-}-\Pi_{K}^{+}\right)-(K \cdot P)\left(\Pi_{K}^{-}+\Pi_{K}^{+}\right)}{\tilde{Q}^{4}\left(W^{2}-P^{2}+\tilde{Q}^{2}\right)} \frac{F_{2}\left(W^{2}, \tilde{Q}^{2}\right)}{P_{0}-\tilde{q}_{0}} . \tag{5.88}
\end{align*}
$$

With an account of the proton structure functions approximation of Eqs. (5.83) and (5.84) the leading term of the TPE correction expansion is expressed as

$$
\begin{align*}
\delta_{2 \gamma}^{\mathrm{F} 1} & =-\frac{Q^{4}}{(K \cdot P)^{2}} \Re \int_{W_{\mathrm{thr}}^{2}}^{\infty} \frac{\mathrm{d} W^{2}}{16 \pi^{3}} \sigma_{T}\left(W^{2}\right) \int_{\alpha_{0} Q^{2}}^{\Lambda^{2}} \mathrm{~d} \tilde{Q}^{2} \frac{(P \cdot \tilde{q})}{\tilde{Q}^{4}}\left(L_{1}^{-}-L_{1}^{+}\right)  \tag{5.89}\\
\delta_{2 \gamma}^{\mathrm{F}_{2}} & =\frac{Q^{2} K}{(K \cdot P)} \Re \int_{W_{\mathrm{thr}}^{2}}^{\infty} \frac{\mathrm{d} W^{2}}{4 \pi^{3}} \sigma_{T}\left(W^{2}\right) \int_{\alpha_{0} Q^{2}}^{\Lambda^{2}} \mathrm{~d} \tilde{Q}^{2} \frac{(K \cdot P)\left(L_{0}^{-}+L_{0}^{+}\right)-(P \cdot \tilde{q})\left(L_{0}^{-}-L_{0}^{+}\right)}{\tilde{Q}^{2}(P \cdot \tilde{q})} \tag{5.90}
\end{align*}
$$

with the following integrals:

$$
\begin{align*}
L_{0}^{ \pm} & =\int_{0}^{\tilde{q}_{0}^{M}} \mathrm{~d} \tilde{q}_{0} \Pi_{K}^{ \pm}=-\int_{0}^{\tilde{q}_{0}^{M}} \frac{\mathrm{~d} \tilde{q}_{0}}{\tilde{Q}^{2} \mp Q \tilde{q}_{0}}= \pm \frac{1}{Q} \ln \frac{W^{2}-P^{2} \mp 2(K \cdot P)}{W^{2}-P^{2}}  \tag{5.91}\\
L_{1}^{ \pm} & =\int_{0}^{\tilde{q}_{0}^{M}} \tilde{q}_{0} \mathrm{~d} \tilde{q}_{0} \Pi_{K}^{ \pm}=-\int_{0}^{\tilde{q}_{0}^{M}} \frac{\tilde{q}_{0} \mathrm{~d} \tilde{q}_{0}}{\tilde{Q}^{2} \mp Q \tilde{q}_{0}} \approx \frac{\tilde{Q}^{2}}{Q^{2}}\left(\ln \frac{W^{2}-P^{2} \mp 2(K \cdot P)}{W^{2}-P^{2}} \pm \frac{2(K \cdot P)}{W^{2}-P^{2}}\right) \\
\tilde{q}_{0}^{M} & =\frac{2(K \cdot P)}{W^{2}-P^{2}} \frac{\tilde{Q}^{2}}{Q} . \tag{5.92}
\end{align*}
$$

Performing the $\tilde{Q}^{2}$ integration, we obtain the leading logarithmic contributions,

$$
\begin{equation*}
\delta_{2 \gamma}^{\mathrm{F} 1}=\frac{Q^{2}}{8 \pi^{3}} \ln \left(\frac{Q^{2}}{\Lambda^{2}}\right) \int_{W_{\mathrm{thr}}^{2}}^{\infty} \frac{\mathrm{d} W^{2}}{M \omega}\left(\frac{W^{2}-P^{2}}{4 M \omega} \ln \frac{2(K \cdot P)+P^{2}-W^{2}}{2(K \cdot P)-P^{2}+W^{2}}+1\right) \sigma_{T}\left(W^{2}\right), \tag{5.93}
\end{equation*}
$$

[^16]\[

$$
\begin{align*}
\delta_{2 \gamma}^{\mathrm{F}_{2}} & =\frac{Q^{2}}{8 \pi^{3}} \ln \left(\frac{Q^{2}}{\Lambda^{2}}\right) \int_{W_{\mathrm{thr}}^{2}}^{\infty} \frac{\mathrm{d} W^{2}}{M \omega} \ln \frac{\left(W^{2}-P^{2}\right)^{2}}{\left(2(K \cdot P)+P^{2}-W^{2}\right)\left(2(K \cdot P)-P^{2}+W^{2}\right)} \sigma_{T}\left(W^{2}\right) \\
& +\frac{Q^{2}}{4 \pi^{3}} \ln \left(\frac{Q^{2}}{\Lambda^{2}}\right) \int_{W_{\mathrm{thr}}^{2}}^{\infty} \frac{\mathrm{d} W^{2}}{W^{2}-P^{2}} \ln \frac{2(K \cdot P)+P^{2}-W^{2}}{2(K \cdot P)-P^{2}+W^{2}} \sigma_{T}\left(W^{2}\right) \tag{5.94}
\end{align*}
$$
\]

which sum up to the known expression of Ref. [174].
When also accounting for the $Q^{2}$ terms in the expansion of Eq. (5.79) for the TPE correction, the hadronic scale $\Lambda$ dependence in Eqs. (5.93) and (5.94) drops out.

Our numerical studies of the inelastic TPE contribution in elastic muon-proton scattering indicate that, in the limit $Q^{2} \ll m^{2}, M^{2}, M \omega$, the momentum transfer expansion starts with a $Q^{2}$ term and contains no $Q^{2} \ln Q^{2}$ type of non-analyticity. This is unlike the above described elastic electron-proton scattering case, where in the limit $m^{2} \ll Q^{2} \ll M^{2}, M \omega$ a non-analytic behavior of the type $Q^{2} \ln Q^{2}$ is present [172,174].

### 5.6.2 Numerical evaluation of structure functions contribution

Having verified the leading $Q^{2} \ln Q^{2}$ contribution for the inelastic TPE in the electron-proton scattering, we next discuss the numerical evaluation of Eq. (5.79) including the $\tilde{Q}^{2}$ dependence of the structure functions.

As a numerical check, we evaluate the weighting functions $w_{1}$ and $w_{2}$, see Eq. (5.79), both in the $K$-frame and $P$-frame for the correction to electron-proton scattering in Figs. 5.14 (Figs. 5.15 in case of the muon-proton scattering). When adding the integral along the imaginary axis with the pole contributions, we checked that we obtain the same result in both frames. Despite the finite $\tilde{Q}^{2}$ ranges of the poles contribution, which are distinct in the $K$ - and $P$-frames, the resulting weighting functions are continuous. The boundaries of the $K$-pole region and the lower boundary of the $P$-pole region, shown in Figs. 5.12, 5.13, are covered by Figs. 5.14, 5.15. The weighting functions $w_{1}$ and $w_{2}$ have a singularity at $\tilde{Q}^{2}=Q^{2} / 4$, when the photons are on their mass shell in our approximation, i.e., $q_{1}^{2}=q_{2}^{2}=0$. The weighting function $w_{2}$ has a discontinuity in the first derivative at $\tilde{Q}^{2}=0$, where the integral along the imaginary axis starts to contribute.

We perform the $\tilde{Q}^{2}$ integration first and express the resulting TPE correction from the unpolarized proton structure functions $\delta_{2 \gamma}^{\mathrm{F}_{1} \mathrm{~F}_{2}}$ in terms of the $W^{2}$ integral as

$$
\begin{equation*}
\delta_{2 \gamma}^{\mathrm{F}_{1} \mathrm{~F}_{2}}=\int_{W_{\mathrm{thr}}^{2}}^{\infty} f(W) \mathrm{d} W^{2} . \tag{5.95}
\end{equation*}
$$

In Fig. 5.16, we present the result for the $W^{2}$ integrand in the elastic electron-proton ( $e^{-} p$ ) scattering for different inputs of proton structure functions and compare our full calculation with the approximation of Eqs. (5.93) and (5.94) with $\Lambda \approx 0.6 \mathrm{GeV}$. To describe the $\tilde{Q}^{2}$ dependence of the unpolarized structure functions, we use the empirical fit performed by Christy and Bosted (BC) [91], while for the logarithmic approximation in Fig. 5.16, we also use the SAID Partial-Wave Analysis Facility [95] for the photoabsorption cross section. The BC fit is valid in the region $0<\tilde{Q}^{2}<8 \mathrm{GeV}^{2}, M+m_{\pi}<W<3.1 \mathrm{GeV}$. This fit is not very accurate in the threshold region near $\tilde{Q}^{2}=0$, although it is still compatible with the error bars of the photoproduction cross sections. We checked that difference between using the SAID and BC fit in the region $W<1.15 \mathrm{GeV}$ and $0.9 \times Q^{2} / 4<\tilde{Q}^{2}<1.1 \times Q^{2} / 4$ only amounts to the relative change in $\delta_{2 \gamma}^{\mathcal{F}_{1} \mathrm{~F}_{2}}$ at the $3 \%-4 \%$ level for the kinematics shown in Fig. 5.16. For the result of


Figure 5.14: The weighting functions of Eq. (5.79), with the absolute value plotted and where the sign labels show the corresponding signs of $w_{1}$ and $w_{2}$ for $e^{-} p$ scattering. As a check, the weighting functions are evaluated in two frames, yielding exactly the same result. The kinematics is chosen as shown in the figure. In the upper panels, the weighting functions $w_{1}$ and $w_{2}$ are shown in the region $-\frac{Q^{2}}{4} \leq \tilde{Q}^{2} \leq \frac{Q^{2}}{2}$ and in the lower panels for $\frac{Q^{2}}{2} \leq \tilde{Q}^{2} \leq 8 \omega^{2}$.


Figure 5.15: Same as Fig. 5.14, but in case of the muon-proton scattering in the upper panel for $0 \leq \tilde{Q}^{2} \leq \frac{Q^{2}}{2}$ and in the lower panels for $\frac{Q^{2}}{2} \leq \tilde{Q}^{2} \leq 8 \mathbf{k}^{2}$.


Figure 5.16: $W^{2}$ integrand $f$ of Eq. (5.95) for two different external kinematics in elastic electron-proton scattering. The integrand with the unpolarized proton structure functions from BC [91] is shown by the blue solid curve. The leading logarithmic correction in the approximation of Eqs. (5.93) and (5.94) with $\Lambda=0.6 \mathrm{GeV}$ and the photoabsorption cross section from the fit of BC (SAID [95]) is shown by the blue dashed (the red dash-dotted) curves. The dominant $\pi N$-channel contribution is shown for the SAID fit by the red dotted curve, which only differs in a visible way from the red dashed-dotted curve for $W>1.3 \mathrm{GeV}$.
$\delta_{2 \gamma}^{\mathrm{F}_{1} \mathrm{~F}_{2}}$ beyond the $Q^{2} \ln Q^{2}$ approximation, we use the BC fit for the region $0<\tilde{Q}^{2}<12 \mathrm{GeV}^{2}$, where the integrand behaves in a smooth way. The relative contribution to $\delta_{2 \gamma}^{\mathrm{F}_{1} \mathrm{~F}_{2}}$ from the uncovered by data region $8<\tilde{Q}^{2}<12 \mathrm{GeV}^{2}$ is expected to be smaller than $0.1 \%$. We perform the $W^{2}$ integration up to $W=4 \mathrm{GeV}$. The extrapolation from $W=3.1 \mathrm{GeV}$ (upper bound of the BC fit) to $W=4 \mathrm{GeV}$ leads to an additional relative contribution to $\delta_{2 \gamma}^{\mathrm{F}_{1} \mathrm{~F}_{2}}$ of less than $2 \%$. We have also checked on the SAID parametrization that the region $W>4 \mathrm{GeV}$ has a relative
contribution to $\delta_{2 \gamma}^{\mathrm{F}_{1} \mathrm{~F}_{2}}$ of less than $2 \%-3 \%$, when interpolating the SAID parametrization to the Regge behavior. The main inelastic TPE contribution is given by the $\pi N$-channel. The singular peak at $W^{2}=M^{2}+2 M \omega$ corresponds to the quasireal photon singularity (when both photons in the two-photon box are quasireal and collinear with either lepton). This singularity appears only in the electron-proton scattering for the beam energies above the pion production threshold.

In order to clarify the validity of the inelastic TPE estimates in electron-proton scattering, we study numerically the low- $Q^{2}$ expansion of the $\delta_{2 \gamma}^{\mathrm{F}_{1} \mathrm{~F}_{2}}$ coming from the $F_{1}$ and $F_{2}$ structure functions. In Fig. 5.17, we present the ratio between the TPE correction $\delta_{2 \gamma}^{\mathrm{F}_{1} \mathrm{~F}_{2}}\left(Q^{2}\right)$ and the low- $Q^{2}$ fit $\delta_{2 \gamma}^{\mathrm{F}_{1} \mathrm{~F}_{2}, \text { fit }}\left(Q^{2}\right)$ in the form of Eq. (5.50) for the energies of available data. We compare the $Q^{2} / \omega^{2}$ and $Q^{2} /(M \omega)$ expansions. We perform the fit in either the variable $Q^{2} / \omega^{2}$ or $Q^{2} /(M \omega)$ in a range which is 100 times smaller than the range displayed in Fig. 5.17. The comparison of our full calculation with such a fit over an extended range provides us then with a quantitative argument on the $Q^{2}$ range where such an expansion holds. If we use as a criterion that the full calculation stays within $10 \%$ of the fit, we can see from Fig. 5.17 that for energies corresponding with available data an expansion of the type of Eq. (5.50) holds for $Q^{2} \lesssim \omega^{2}$ and requires $Q^{2} \lesssim(M \omega) / 5$. We expect the same type of expansion for the TPE contributions from other amplitudes and stress on the validity of such criterion only in the limited region of beam energies.


Figure 5.17: $Q^{2}$ dependence of ratio of the inelastic TPE correction $\delta_{2 \gamma}^{\mathrm{F}_{1} \mathrm{~F}_{2}}$ to the low- $Q^{2}$ fit of the form $\delta_{2 \gamma}^{\mathrm{F}_{1} \mathrm{~F}_{2}, \text { fit }}=a(\omega) Q^{2} \ln Q^{2}+b(\omega) Q^{2}$. The ratio is shown as function of $Q^{2} /(M \omega)$ on the left panel and $Q^{2} / \omega^{2}$ on the right panel.

In Figs. $5.18,5.19$ we compare the $W$ dependence of the integrand $f(W)$ in Eq. (5.95) for the $\mu^{-} p$ and $e^{-} p$ elastic scattering processes. We find that the $\tilde{Q}^{2}$ integrations in the elastic muon-proton scattering are well saturated when performed up to $\tilde{Q}^{2}=8 \mathrm{GeV}^{2}$, which is the largest value covered by the BC fit. As a test, we extended the BC fit beyond its fit region and found that the relative contribution from the region $8 \mathrm{GeV}^{2}<\tilde{Q}^{2}<12 \mathrm{GeV}^{2}$ is smaller than $0.015 \%$. Figs. 5.18, 5.19 show results in different kinematical regions corresponding with the MUSE experiment. The TPE corrections to $e^{-} p$ are sizably larger than for the $\mu^{-} p$ case at low $Q^{2}$. With increasing $Q^{2}$, the $\mu^{-} p$ TPE corrections increase, as is evident from the result at lower beam momentum in Fig. 5.19, where at $Q^{2}=0.03 \mathrm{GeV}^{2}$ both corrections reach similar sizes. We furthermore notice the absence of the quasi-real photon singularity for the $\mu^{-} p$ scattering. To estimate the inelastic TPE correction to elastic muon-proton scattering, we find that the $W$ integration in Eq. (5.95) is well saturated when performed up to 3.1 GeV ,


Figure 5.18: $W$ dependence of the integrand $f(W)$ which determines the inelastic TPE correction, as given by Eq. (5.95). The integrand is shown for the case of elastic $e^{-} p$ and $\mu^{-} p$ scattering. The external kinematics (indicated on the plots) correspond with the MUSE experiment.


Figure 5.19: Same as Fig. 5.18, but for the lepton momentum $\mathrm{k}=115 \mathrm{MeV}$.
which is the largest value covered by the BC fit. When again extending the BC fit beyond its fit range, for the purpose of a test, we checked that the relative contribution from the region $3.1 \mathrm{GeV}<W<4 \mathrm{GeV}$ to $\delta_{2 \gamma}^{\mathrm{F}_{1} \mathrm{~F}_{2}}$ is smaller than $1.5 \%$. We estimate the uncertainties of the numerical integration coming from the integration regions outside the BC fit and from the inaccuracies in the BC fit at $5-6 \%$ level.

### 5.7 TPE correction at low- $Q^{2}$

### 5.7.1 Correction to $e^{-} p$ cross section

The resulting inelastic TPE correction as a function of $Q^{2}$ for the beam energy $\omega=0.18 \mathrm{GeV}$ corresponding to the lower energy range at MAMI is shown in Fig. 5.20. We compare the Feshbach term for point-like particles, the Born TPE correction of Section 4.1, the approximation of Eqs. (5.93) and (5.94) with $\Lambda \approx 0.6 \mathrm{GeV}$ and the result of Ref. [166]. The difference between both $Q^{2} \ln Q^{2}$ curves comes from the term of order $Q^{2}$ and appears due to the different choices of $\Lambda$. The approximation of Eqs. (5.93) and (5.94) implies the same hadronic scale for all intermediate states, while the hadronic scale in Ref. [166] depends on the intermediate state. We present our results for the total TPE correction including the $\tilde{Q}^{2}$ dependence in the structure functions and extrapolate them to the region $Q^{2} \gtrsim M \omega$. Increasing the momentum transfer, the inelastic TPE correction shows a clear departure from the $Q^{2} \ln Q^{2}$ term reducing the latter value. With an account of the inelastic intermediate states, the TPE correction in the low- $Q^{2}$ region comes closer to the Feshbach correction in comparison with the TPE correction in a box graph model only (Born TPE). The inelastic TPE correction has the same order of magnitude and the opposite sign in comparison with the proton form factor effects in the Born TPE.

We compare the TPE corrections as a function of the $\varepsilon$ variable for $Q^{2}=0.05 \mathrm{GeV}^{2}$ $\left(Q^{2}=0.25 \mathrm{GeV}^{2}\right)$ in Fig. 5.21 (5.22). The inelastic excitations compensate the proton form factor effects, and the resulting TPE correction comes closer to the Feshbach term. For the small momentum transfer $Q^{2}=0.05 \mathrm{GeV}^{2}$, where we expect the validity of the near-forward approximation for $\varepsilon \gtrsim 0.7$ or $Q^{2} /(M \omega) \lesssim 1 / 5$, our calculation is in good agreement with the empirical TPE fit of Ref. [104] in the region $\varepsilon>0.35-0.4$ as one notices from Fig. 5.21. One should definitely account for contributions beyond the near-forward approximation and unpolarized proton structure functions for smaller $\varepsilon$ values. We show the region of small $\varepsilon$ with the aim to illustrate the characteristic features of our calculation. In the limit $\varepsilon \rightarrow 0$ the inelastic TPE correction vanishes. ${ }^{6}$ Increasing the momentum transfer, as shown in Fig. 5.22, the predicted TPE correction is found to be in reasonable agreement with the empirical fit of Ref. [104], confirming the proton charge radius values extracted with this TPE correction.

We compare the experimental TPE correction results from measurements of the ratio of elastic $e^{+} p$ to $e^{-} p$ scattering cross section of Refs. [76,79] with the empirical fit of Ref. [104], the Born TPE and total TPE in the near-forward approximation in Fig. 5.23. The CLAS data is in agreement with the total TPE correction in the near-forward approximation. However, the VEPP-3 data point of Ref. [76] is in better agreement with the Born TPE result and agrees with the total TPE only after the renormalization [76] according to the empirical fit of Ref. [104]. We also extrapolate our calculation in the near-forward approximation to the low- $Q^{2}$ region of the OLYMPUS experiment [81] kinematics in Fig. 5.23. The sign change of the TPE correction at $Q^{2} \simeq 0.59 \mathrm{GeV}^{2}$ is in agreement with the subtracted DR prediction of Fig. 4.45.

[^17]

Figure 5.20: $Q^{2}$ dependence of the TPE correction $\delta_{2 \gamma}$ to $e^{-} p \rightarrow e^{-} p$ for electron lab energy $\omega=0.180 \mathrm{GeV}$ (for which the kinematically allowed region is $Q^{2}<0.094 \mathrm{GeV}^{2}$ ). The Feshbach term for point-like particles, the Born TPE of Section 4.1 based on the box graph evaluation with dipole form factors, and the total TPE correction as the sum of Born TPE and inelastic TPE are presented (upper panel). The inelastic contribution is compared with the leading logarithmic approximation of Eqs. (5.93) and (5.94) with $\Lambda=0.6 \mathrm{GeV}$ and the result of Ref. [166] (lower panel). The experimental input for the proton structure functions is taken from the Christy-Bosted fit [91]. The vertical dashed lines restrict the region of validity of the expansion for the inelastic term $Q^{2} \lesssim(M \omega) / 5$ as follows from Fig. 5.17.


Figure 5.21: Same as Fig. 5.20, but for the fixed value $Q^{2}=0.05 \mathrm{GeV}^{2}$ as function of $\varepsilon$ in comparison with the empirical TPE fit using the data of Ref. [104] (A1 Collaboration, blue bands).



Figure 5.22: Same as Fig. 5.21, but for the fixed value $Q^{2}=0.25 \mathrm{GeV}^{2}$.


Figure 5.23: TPE correction measurements of Refs. [76, 79] in comparison with the total TPE given as a sum of the Born TPE and inelastic contributions in the near-forward approximation (shown by stars), the Born TPE (shown by squares) and empirical fit of Ref. [104] evaluated for the experimental $\left(Q^{2}, \varepsilon\right)$ values. The Born TPE (dashed curve) and total TPE result (solid curve) are also shown for the kinematics of the OLYMPUS experiment [81] with 2 GeV lepton beam as a function of the momentum transfer. The CLAS [79] data points correspond to the following kinematics: $Q^{2}=0.23 \mathrm{GeV}^{2}, \varepsilon=0.92 ; \quad Q^{2}=0.34 \mathrm{GeV}^{2}, \varepsilon=0.89$; $Q^{2}=0.45 \mathrm{GeV}^{2}, \varepsilon=0.89$. The VEPP-3 [76] data point corresponds to $Q^{2}=0.298 \mathrm{GeV}^{2}, \varepsilon=0.93$. This point was renormalized [76] according to the empirical fit of Ref. [104].

### 5.7.2 Correction to $\mu^{-} p$ cross section

The resulting inelastic TPE corrections to the elastic $\mu^{-} p$ scattering cross section are shown in Figs. 5.24 as a function of $Q^{2}$ for three values of muon beam momentas, corresponding with the MUSE kinematics. We notice that for the low momentum transfer corresponding with the MUSE kinematics, the inelastic TPE corrections to elastic $\mu^{-} p$ scattering are very small, at these low $Q^{2}$ values in the range of $\delta_{2 \gamma} \sim 0.05 \%$. This is well below the anticipated cross section precision of around $1 \%$ of the MUSE experiment. Furthermore, we notice that the TPE corrections due to the subtraction function and the dispersive $F_{1}, F_{2}$ structure functions integrals come with opposite signs, leading to a partial cancellation.

We present in Figs. 5.25 the total TPE correction as a sum of the Born TPE correction of Section 4.1, corresponding with a proton intermediate state, and the inelastic TPE of this work using the empirically estimated subtraction function of Section 2.4.1. We compare our result with the Feshbach term of Eq. (3.53) for a point-like Dirac particle corrected by the recoil factor


Figure 5.24: TPE correction for elastic $\mu^{-} p$ scattering for three different muon lab momenta as planned in the MUSE experiment. The TPE correction due to the subtraction function is shown for three subtraction function inputs: Birse et al. [141] (blue bands), BChPT [142] (solid curves), and the empirical determination as described in Section 2.4.1 (green bands). The inelastic TPE correction due to the dispersion integrals over the proton structure functions $F_{1}$ and $F_{2}$ is shown by the dashed-dotted curves. The resulting total inelastic TPE correction (sum of both) is shown by the pink bands using the empirically extracted subtraction function of Section 2.4.1 and accounting for a $6 \%$ uncertainty in $\delta_{2 \gamma}^{\mathrm{F}_{1} \mathrm{~F}_{2}}$.


Figure 5.25: The total TPE correction for elastic $\mu^{-} p$ scattering is shown as sum of the Born TPE, the TPE correction from the $F_{1}$ and $F_{2}$ proton structure functions and the TPE correction from the empirically estimated subtraction function of Section 2.4.1. It is compared with the Feshbach term for point-like particles, see Eq. (3.53), corrected by the recoil factor $(1+m / M)$, the Born TPE correction from Section 4.1 and the $e^{-} p$ total TPE correction.
$(1+m / M)$, with the Born TPE correction, and with the corresponding TPE correction for elastic $e^{-} p$ scattering of Section 5.7.1. In contrast to the electron-proton scattering case, where the subtraction function contribution is negligible [172] as it is proportional to the lepton mass squared, in the muon-proton scattering the inelastic proton structure function contribution is partially cancelled by the $\mathrm{T}_{1}$ subtraction function resulting in a very small inelastic TPE correction for the MUSE kinematics. Only with increased lepton beam energy or when going to larger $Q^{2}$ values one needs to start accounting for the inelastic TPE correction, which shifts the total correction a little closer to the Feshbach result.

## Chapter 6

## $\pi N$ intermediate state TPE contribution

In order to provide theoretical predictions for the TPE correction to elastic $l p$ scattering at arbitrary scattering angles, we now turn to the study of the TPE correction due to the $\pi N$ intermediate state within dispersion relations. We firstly discuss the production of pions in the lepton-proton scattering in Sections 6.1, 6.2. Subsequently we use the unitarity relations in Section 6.3 to construct the imaginary part of the TPE amplitude arising from the $\pi N$ intermediate state. In Section 6.4, a DR analysis is performed analogous as in Chapter 4 to reconstruct the real part. We present first results for the $\pi N$ intermediate state TPE contribution to electron-proton scattering coming from the $P_{33}$ partial wave, corresponding with the $\Delta(1232)$ resonance contribution.

### 6.1 Kinematics of pion production in lepton-proton scattering

The production of pions in the scattering of leptons off a proton target $l(k, h)+p(p, \lambda) \rightarrow$ $l\left(k_{1}, h_{1}\right)+p\left(p_{1}, \lambda_{1}\right)+\pi^{0}\left(p_{\pi}\right)$ or $l(k, h)+p(p, \lambda) \rightarrow l\left(k_{1}, h_{1}\right)+n\left(p_{1}, \lambda_{1}\right)+\pi^{+}\left(p_{\pi}\right)$, see Fig. 6.1, ${ }^{1}$ is completely described by five kinematical variables. We conventionally use the squared energy $s=(p+k)^{2}$ in the lepton-proton c.m. frame, the squared momentum transfer on the lepton line $Q^{2}=-q^{2}=-\left(k-k_{1}\right)^{2}$, which is just the photon virtuality in the one-photon exchange approximation, the momentum transfer variable on the nucleon line $t=\left(p_{1}-p\right)^{2}$, the squared invariant mass of the pion-nucleon system $W^{2}=\left(p_{1}+p_{\pi}\right)^{2}$ and one relative angle between the electron and hadron production planes.


Figure 6.1: Pion production in the lepton-proton scattering.

We study the kinematics of the pion production in the lepton-proton c.m. reference frame, as we exploit this frame relating the lepton-proton helicity amplitudes to invariant amplitudes in Sections 3.2, 3.5. The initial (final) lepton energy $\omega_{\mathrm{cm}}\left(\omega_{1}\right)$ and the momentum $\left|\mathbf{k}_{\mathrm{cm}}\right|\left(\left|\mathbf{k}_{1}\right|\right)$ are given by

$$
\begin{equation*}
\omega_{\mathrm{cm}}=\frac{s-M^{2}+m^{2}}{2 \sqrt{s}}, \quad\left|\mathbf{k}_{\mathrm{cm}}\right|=\frac{\sqrt{\Sigma_{s}}}{2 \sqrt{s}} \tag{6.1}
\end{equation*}
$$

[^18]\[

$$
\begin{equation*}
\omega_{1}=\frac{s-W^{2}+m^{2}}{2 \sqrt{s}}, \quad\left|\mathbf{k}_{1}\right|=\frac{\sqrt{\Sigma\left(s, W^{2}, m^{2}\right)}}{2 \sqrt{s}} \tag{6.2}
\end{equation*}
$$

\]

with $\Sigma_{s}$ defined as in Section 3.1.
The initial $(k)$ and final $\left(k_{1}\right)$ lepton 4 -momenta are given by ${ }^{2}$

$$
\begin{align*}
k & =\left(\omega_{\mathrm{cm}}, 0,0,\left|\mathbf{k}_{\mathrm{cm}}\right|\right)  \tag{6.3}\\
k_{1} & =\left(\omega_{1},\left|\mathbf{k}_{1}\right| \sin \theta_{1} \cos \phi_{1},\left|\mathbf{k}_{1}\right| \sin \theta_{1} \sin \phi_{1},\left|\mathbf{k}_{1}\right| \cos \theta_{1}\right) \tag{6.4}
\end{align*}
$$

with the lepton scattering angles $\theta_{1}$ and $\phi_{1}$.
The pion 4-momentum $p_{\pi}=\left(E_{\pi}, \vec{p}_{\pi}\right)$ can be expressed in terms of the pion angles $\theta_{\pi}, \phi_{\pi}$ as

$$
\begin{align*}
\vec{p}_{\pi} & =p_{\pi}\left(\sin \theta_{\pi} \cos \phi_{\pi}, \sin \theta_{\pi} \sin \phi_{\pi}, \cos \theta_{\pi}\right), \\
E_{\pi} & =\sqrt{\vec{p}_{\pi}^{2}+m_{\pi}^{2}} . \tag{6.5}
\end{align*}
$$

The relative angle between the final lepton and pion $\Theta_{\pi}=\left(\hat{k}_{1} \cdot \hat{p}_{\pi}\right)$ is given by

$$
\begin{equation*}
\cos \Theta_{\pi}=\sin \theta_{1} \sin \theta_{\pi} \cos \left(\phi_{1}-\phi_{\pi}\right)+\cos \theta_{1} \cos \theta_{\pi} \tag{6.6}
\end{equation*}
$$

The initial $(p)$ and final $\left(p_{1}\right)$ nucleon 4 -momenta are given by

$$
\begin{align*}
p & =\left(\frac{s+M^{2}-m^{2}}{2 \sqrt{s}}, 0,0,-\frac{\sqrt{\sum\left(s, M^{2}, m^{2}\right)}}{2 \sqrt{s}}\right)  \tag{6.7}\\
p_{1} & =\left(E_{1}, \vec{p}_{1}\right), \quad \vec{p}_{1}=-\vec{k}_{1}-\vec{p}_{\pi}, \quad E_{1}=\sqrt{\vec{p}_{1}^{2}+M^{2}} \tag{6.8}
\end{align*}
$$

We firstly analyze the special cases in the phase space of the final particles. The minimum of the final lepton momentum $\left|\mathbf{k}_{1}\right|=0$ corresponds to the geometric configuration, when the pion and nucleon are moving in opposite directions, i.e.:

$$
\begin{align*}
E_{\pi}+E_{1} & =\sqrt{s}-m \equiv \sqrt{\tilde{s}}  \tag{6.9}\\
p_{\pi} & =p_{1}=\frac{\sqrt{\sum\left(\tilde{s}, M^{2}, m_{\pi}^{2}\right)}}{2 \sqrt{\tilde{s}}} \tag{6.10}
\end{align*}
$$

The maximum of $\left|\mathbf{k}_{1}\right|$ corresponds to the minimum of $W^{2}=\left(M+m_{\pi}\right)^{2}$. It is reached for the case of the same pion and nucleon momentum directions, which is opposite to the lepton momentum direction, i.e., $\cos \Theta_{\pi}=-1$. The corresponding particle momenta are given by

$$
\begin{align*}
\left|\mathbf{k}_{1}\right| & =k_{1}^{\max }=\frac{\sqrt{\Sigma\left(s,\left(M+m_{\pi}\right)^{2}, m^{2}\right)}}{2 \sqrt{s}}  \tag{6.11}\\
p_{1} & =\frac{M}{M+m_{\pi}} k_{1}^{\max }  \tag{6.12}\\
p_{\pi} & =\frac{m_{\pi}}{M+m_{\pi}} k_{1}^{\max } \tag{6.13}
\end{align*}
$$

Accounting for the energy conservation, we obtain the following expression for the pion momentum:

$$
\begin{equation*}
p_{\pi}^{ \pm}=-\left(\frac{W^{2}-M^{2}+m_{\pi}^{2}}{W^{2}+k_{1}^{2} \sin ^{2} \Theta_{\pi}}\right) \frac{k_{1} \cos \Theta_{\pi}}{2} \pm \frac{\sqrt{\Sigma\left(W^{2}, M^{2}, m_{\pi}^{2}\right)-4 m_{\pi}^{2} k_{1}^{2} \sin ^{2} \Theta_{\pi}}}{W^{2}+k_{1}^{2} \sin ^{2} \Theta_{\pi}} \frac{\sqrt{s}-\omega_{1}}{2} \tag{6.14}
\end{equation*}
$$

[^19]In the kinematical region $s \Sigma\left(W^{2}, M^{2}, m_{\pi}^{2}\right) \geq m_{\pi}^{2} \Sigma\left(s, W^{2}, m^{2}\right)$, only the solution $p_{\pi}^{+}$is positive. In the region $s \Sigma\left(W^{2}, M^{2}, m_{\pi}^{2}\right) \leq m_{\pi}^{2} \Sigma\left(s, W^{2}, m^{2}\right)$, both solutions are positive and can be realized. In this case, the pion can be scattered only into the backward cone w. r. t. the lepton momentum direction, $\Theta_{\pi}>\Theta_{\pi}^{0}>\frac{\pi}{2}$, with the cone boundary $\Theta_{\pi}^{0}$ given by

$$
\begin{equation*}
\sin ^{2} \Theta_{\pi}^{0}=\frac{s \Sigma\left(W^{2}, M^{2}, m_{\pi}^{2}\right)}{m_{\pi}^{2} \Sigma\left(s, W^{2}, m^{2}\right)}, \tag{6.15}
\end{equation*}
$$

and two distinct geometric configurations are possible. For $\Theta_{\pi}^{0}>\pi / 2$, this corresponds with $W_{\min }$ value, above which a physical solution exists, of (in the lepton massless limit):

$$
\begin{equation*}
W_{\min }^{2}=\frac{M^{2}+m_{\pi}^{2} \cos ^{2} \Theta_{\pi}^{0}}{s-m_{\pi}^{2} \sin ^{2} \Theta_{\pi}^{0}} s+\frac{2 M m_{\pi} s}{s-m_{\pi}^{2} \sin ^{2} \Theta_{\pi}^{0}} \sqrt{1+\frac{\Sigma\left(s, M^{2}, m_{\pi}^{2}\right)}{4 M^{2} s} \sin ^{2} \Theta_{\pi}^{0}} . \tag{6.16}
\end{equation*}
$$

The bifurcation value of the pion-nucleon invariant mass $W_{0}^{2}$, corresponding to $\sin ^{2} \Theta_{\pi}^{0}=1$, is given by

$$
\begin{equation*}
W_{0}^{2}=\frac{s+M^{2}-m_{\pi}^{2}-m^{2}}{s-m_{\pi}^{2}} m_{\pi} \sqrt{s}+\frac{s M^{2}-m_{\pi}^{2} m^{2}}{s-m_{\pi}^{2}} \underset{\sqrt{s} \geq M+m+m_{\pi}}{\geq}\left(M+m_{\pi}\right)^{2} . \tag{6.17}
\end{equation*}
$$

This value is given by the pion-nucleon threshold $W_{0}^{2}=\left(M+m_{\pi}\right)^{2}$ for the beam energy at the pion production threshold, i.e., $s=\left(M+m+m_{\pi}\right)^{2}$, and grows with increasing lepton beam energy.


Figure 6.2: Pion momentum as a function of the invariant mass $W$ of the $\pi^{0} p$ state in the pion electroproduction process with the electron beam energy $\omega=0.18 \mathrm{GeV}$ (left panel) and $\omega=1.1 \mathrm{GeV}$ (right panel). The vertical dashed line of $W_{0}$ indicates the invariant mass value below which there exist two distinct solutions for the pion momentum.

We show the dependence of $p_{\pi}$ on the invariant mass $W$ of the pion-nucleon state ( $\pi^{0} p$ ) for the typical kinematics of experiments in Fig. 6.2. At the lowest MAMI energy ( $\omega=0.18 \mathrm{GeV}$ ), corresponding with $W_{0} \approx 1.07331 \mathrm{GeV}$ and $W_{\mathrm{thr}} \approx 1.07325 \mathrm{GeV}$, the contribution of the region with two solutions is negligible. While at larger beam energies, both solutions should be taken into account.

### 6.2 Invariant amplitudes for pion electroproduction

We are going to evaluate the $\pi N$ channel TPE contribution to the real parts of the lepton-proton invariant amplitudes, see Fig. 6.3. The dispersive evaluation of these contributions requires the pion production amplitudes in the OPE approximation as an input. Moreover, the experimental extraction of the pion production amplitudes is performed in the OPE approximation. For the corresponding input, we consider the production of pions by the virtual photon [212-214] as it is implemented in the MAID model [93, 94].


Figure 6.3: TPE graph with $\pi N$ intermediate state.

Accounting for gauge invariance as well as a parity conservation in strong and electromagnetic interactions, the pion production by the virtual photon can be completely described by six independent helicity amplitudes. The same amplitudes describe the interaction of the lepton electromagnetic current with the pion production current. The conventional Lorentz invariant expression for the pion production amplitude $T_{\pi N}^{1 \gamma}$ is given by

$$
\begin{equation*}
T_{\pi N}^{1 \gamma}=-\frac{e}{Q^{2}} \bar{u}\left(k_{1}, h_{1}\right) \gamma_{\mu} u(k, h) \cdot \bar{N}\left(p_{1}, \lambda_{1}\right) J_{\pi N}^{\mu}\left(p_{\pi} ; p_{1}, \lambda_{1} ; p, \lambda\right) N(p, \lambda) \tag{6.18}
\end{equation*}
$$

with the pion production current $J_{\pi N}^{\mu}$ expressed as a sum of six covariants $M_{i}^{\mu}$ :

$$
\begin{align*}
J_{\pi N}^{\mu}\left(p_{\pi} ; p_{1}, \lambda_{1} ; p, \lambda\right) & =\sum_{i=1}^{6} A_{i}\left(W^{2}, t, Q^{2}\right) M_{i}^{\mu}  \tag{6.19}\\
M_{1}^{\mu} & =-\frac{1}{2} i \gamma_{5}\left(\gamma^{\mu} \hat{q}-\hat{q} \gamma^{\mu}\right)  \tag{6.20}\\
M_{2}^{\mu} & =2 i \gamma_{5}\left(P^{\mu} q \cdot\left(p_{\pi}-\frac{1}{2} q\right)-\left(p_{\pi}-\frac{1}{2} q\right)^{\mu} q \cdot P\right)  \tag{6.21}\\
M_{3}^{\mu} & =-i \gamma_{5}\left(\gamma^{\mu} q \cdot p_{\pi}-\hat{q} p_{\pi}^{\mu}\right)  \tag{6.22}\\
M_{4}^{\mu} & =-2 i \gamma_{5}\left(\gamma^{\mu} q \cdot P-\hat{q} P^{\mu}\right)-2 M M_{1}^{\mu}  \tag{6.23}\\
M_{5}^{\mu} & =i \gamma_{5}\left(q^{\mu} q \cdot p_{\pi}+Q^{2} p_{\pi}^{\mu}\right)  \tag{6.24}\\
M_{6}^{\mu} & =-i \gamma_{5}\left(\hat{q} q^{\mu}+Q^{2} \gamma^{\mu}\right) \tag{6.25}
\end{align*}
$$

with $P=\left(p+p_{1}\right) / 2$ and invariant amplitudes $A_{i}$ which are completely described by three Mandelstam variables, e.g., $W^{2}, Q^{2}$ and $t$. This form of the amplitude is manifestly gauge invariant, i.e., each covariant $M_{i}^{\mu}$ satisfies $q_{\mu} M_{i}^{\mu}=0$. For the numerical implementation we exploit the invariant amplitudes $A_{i}$ from the MAID model (version 2007) [93, 94]. ${ }^{3}$

[^20]We exploit also the conjugated amplitude for the second photon in Fig. 6.3. The nucleon current enters it in the complex conjugated form:

$$
\begin{equation*}
\left(\bar{N}\left(p_{1}, \lambda_{1}\right) J_{\pi N}^{\mu}\left(p_{\pi} ; p_{1}, \lambda_{1} ; p, \lambda\right) N(p, \lambda)\right)^{*}=\bar{N}(p, \lambda) \tilde{J}_{\pi N}^{\mu}\left(p_{\pi} ; p_{1}, \lambda_{1} ; p, \lambda\right) N\left(p_{1}, \lambda_{1}\right) \tag{6.26}
\end{equation*}
$$

with the conjugated pion production vector:

$$
\begin{equation*}
\tilde{J}_{\pi N}^{\mu}\left(p_{\pi} ; p_{1}, \lambda_{1} ; p, \lambda\right)=\sum_{i=1}^{6} A_{i}^{*}\left(W^{2}, t, Q^{2}\right) \tilde{M}_{i}^{\mu} \tag{6.27}
\end{equation*}
$$

and covariants:

$$
\begin{array}{lll}
\tilde{M}_{1}^{\mu}=-M_{1}^{\mu}, & \tilde{M}_{2}^{\mu}=M_{2}^{\mu}, & \tilde{M}_{3}^{\mu}=-M_{3}^{\mu} \\
\tilde{M}_{4}^{\mu}=-M_{4}^{\mu}, & \tilde{M}_{5}^{\mu}=M_{5}^{\mu}, & \tilde{M}_{6}^{\mu}=-M_{6}^{\mu} \tag{6.28}
\end{array}
$$

### 6.3 Unitarity and DRs

To obtain the imaginary parts of the TPE amplitudes we exploit the unitarity relations from Section 4.2.1. In case of the $\pi N$ intermediate state the unitarity relations of Eqs. (4.12), see Fig. 6.4, are simplified to [197]

$$
\begin{equation*}
\Im T_{h^{\prime} \lambda^{\prime}, h \lambda}^{2 \gamma}=\int \frac{e^{2} \mathrm{~d}^{3} \vec{k}_{1}}{(2 \pi)^{3} 4 \omega_{1}} \frac{\bar{u}\left(k^{\prime}, h^{\prime}\right) \gamma_{\mu}\left(\hat{k_{1}}+m\right) \gamma_{\nu} u(k, h) \cdot \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) W^{\mu \nu}\left(p^{\prime}, \lambda^{\prime} ; k_{1} ; p, \lambda\right) N(p, \lambda)}{Q_{1}^{2} Q_{2}^{2}}, \tag{6.29}
\end{equation*}
$$

with the expression for the non-forward hadronic tensor:

$$
\begin{align*}
W^{\mu \nu}\left(p^{\prime}, \lambda^{\prime} ; k_{1} ; p, \lambda\right)= & \int \frac{\mathrm{d}^{3} \vec{p}_{1}}{(2 \pi)^{3} 2 E_{1}} \int \frac{d^{3} \vec{p}_{\pi}}{(2 \pi)^{3} 2 E_{\pi}}(2 \pi)^{4} \delta^{4}\left(k+p-k_{1}-p_{1}-p_{\pi}\right) \\
& \times \sum_{\lambda_{1}} \tilde{J}_{\pi N}^{\mu}\left(p_{\pi} ; p_{1}, \lambda_{1} ; p^{\prime}, \lambda^{\prime}\right) J_{\pi N}^{\nu}\left(p_{\pi} ; p_{1}, \lambda_{1} ; p, \lambda\right) \tag{6.30}
\end{align*}
$$



Figure 6.4: Unitarity relations for the case of the $\pi N$ intermediate state contribution.

The momentum transfers of the OPE processes entering the r.h.s. of the unitarity relations of Eq. (6.29) are given by

$$
\begin{align*}
Q_{1}^{2} & =\frac{\left(s-M^{2}+m^{2}\right)\left(s-W^{2}+m^{2}\right)-4 m^{2} s}{2 s}-\frac{\sqrt{\Sigma_{s} \Sigma\left(s, W^{2}, m^{2}\right)}}{2 s} \cos \theta_{1},  \tag{6.31}\\
Q_{2}^{2} & =\frac{\left(s-M^{2}+m^{2}\right)\left(s-W^{2}+m^{2}\right)-4 m^{2} s}{2 s}-\frac{\sqrt{\Sigma_{s} \Sigma\left(s, W^{2}, m^{2}\right)}}{2 s} \cos \theta_{2} \tag{6.32}
\end{align*}
$$

which are simplified in the case of the massless lepton scattering to

$$
\begin{equation*}
Q_{1}^{2}=\frac{\left(s-W^{2}\right)\left(s-M^{2}\right)}{2 s}\left(1-\cos \theta_{1}\right), \quad Q_{2}^{2}=\frac{\left(s-W^{2}\right)\left(s-M^{2}\right)}{2 s}\left(1-\cos \theta_{2}\right) . \tag{6.33}
\end{equation*}
$$

The momentum transfers region is bounded by an ellipse in the $Q_{1}^{2}, Q_{2}^{2}$ plane, similar to the ellipse in Section 4.2.3 in case of the elastic intermediate state. The maximal value of the momentum transfers $Q_{1}^{2}, Q_{2}^{2}$ is given by $Q_{\max }^{2}=\left(s-W^{2}\right)\left(s-M^{2}\right) / s$. Consequently, the size of the ellipse depends now on the $\pi N$ invariant mass. We show this dependence in Fig. 6.5.


Figure 6.5: The relation between the invariant mass $W$ and the size of the momentum transfers integration region $Q_{\max }^{2}$ entering the unitarity relations for the case of a $\pi N$ intermediate state in the TPE graph of the elastic electron-proton scattering. The kinematics is chosen as shown in figure.

The elastic TPE contribution of Section 4.2.1 is obtained by replacing in the hadronic tensor of Eq. (6.30) the pion production current by the proton current and changing the phase space integration as

$$
\begin{align*}
J_{\pi N}^{\mu}\left(p_{\pi} ; p_{1}, \lambda_{1} ; p, \lambda\right) & \rightarrow G_{M} \gamma^{\mu}-F_{P} \frac{P^{\mu}}{M},  \tag{6.34}\\
\int \frac{\mathrm{~d}^{3} \vec{p}_{1}}{(2 \pi)^{3} 2 E_{1}} \int \frac{d^{3} \vec{p}_{\pi}}{(2 \pi)^{3} 2 E_{\pi}}(2 \pi)^{4} \delta^{4}\left(k+p-k_{1}-p_{1}-p_{\pi}\right) & \rightarrow 2 \pi \delta\left(W^{2}-M^{2}\right) . \tag{6.35}
\end{align*}
$$

For the $\pi N$ intermediate state we perform the numerical integration in the following variables:

$$
\begin{equation*}
W^{2}, \theta_{1}, \phi_{1}, \theta_{\pi}, \phi_{\pi} \tag{6.36}
\end{equation*}
$$

The expression for the imaginary part simplifies to

$$
\begin{align*}
\Im T_{h^{\prime} \lambda^{\prime}, h \lambda}^{2 \gamma} & =\frac{e^{2}}{64 \pi^{3}} \int \mathrm{~d} W^{2} \mathrm{~d} \Omega_{1} \frac{\left|\vec{k}_{1}\right|}{\sqrt{s}} \frac{1}{Q_{1}^{2} Q_{2}^{2}} \\
& \times \bar{u}\left(k^{\prime}, h^{\prime}\right) \gamma_{\mu}\left(\hat{k_{1}}+m\right) \gamma_{\nu} u(k, h) \cdot \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) W^{\mu \nu}\left(p^{\prime}, \lambda^{\prime} ; k_{1} ; p, \lambda\right) N(p, \lambda) \tag{6.37}
\end{align*}
$$

with the hadronic tensor:

$$
\begin{equation*}
W^{\mu \nu}\left(p^{\prime}, \lambda^{\prime} ; k_{1} ; p, \lambda\right)=\sum_{\lambda_{1}} \int \frac{p_{\pi}^{2} \mathrm{~d} \Omega_{\pi}}{(4 \pi)^{2}} \frac{\tilde{J}_{\pi N}^{\mu}\left(p_{\pi} ; p_{1}, \lambda_{1} ; p^{\prime}, \lambda^{\prime}\right) J_{\pi N}^{\nu}\left(p_{\pi} ; p_{1}, \lambda_{1} ; p, \lambda\right)}{p_{\pi}\left(\sqrt{s}-\omega_{1}\right)+E_{\pi} k_{1} \cos \left(\hat{k}_{1}, \hat{p}_{\pi}\right)} \tag{6.38}
\end{equation*}
$$

In the kinematical region $s \Sigma\left(W^{2}, M^{2}, m_{\pi}^{2}\right) \leq m_{\pi}^{2} \Sigma\left(s, W^{2}, m^{2}\right)$ we have to sum over both solutions for $p_{\pi}$, see Eq. (6.14). Performing the numerical integration, we obtain the imaginary parts of the lepton-proton TPE amplitudes in the leading $\alpha$ order within Eqs. (3.13) and (3.50).

The pion electroproduction amplitudes in MAID are available in the restricted kinematical region $W<W_{\max }=2.5 \mathrm{GeV}$. Consequently we integrate over the whole phase space in the unitarity relation of Eq. (6.37) in the region of the crossing symmetric variable:

$$
\begin{equation*}
\nu<\nu_{0}=\frac{W_{\max }^{2}+2 m W_{\max }-M^{2}}{2}-\frac{Q^{2}}{4} \tag{6.39}
\end{equation*}
$$

with $\nu_{0} \approx 2.67 \mathrm{GeV}^{2}$ at $Q^{2}=0.05 \mathrm{GeV}^{2}$. For larger $\nu>\nu_{0}$, we truncate the $W$ integration at $W=W_{\max }$ accounting only for the available kinematical region in MAID. Consequently, the imaginary parts of invariant amplitudes will have a discontinuity at the point $\nu=\nu_{0}$.

We first limit ourselves to the $\Delta$-resonance region, which corresponds to the $P_{33}{ }^{4}$ partial wave in MAID. We present the resulting $P_{33}$ partial wave contribution to imaginary parts of the electron-proton TPE amplitudes $\mathcal{G}_{1}^{P_{33}}, \mathcal{G}_{2}^{P_{33}}, \mathcal{G}_{M}^{P_{33}}, \mathcal{F}_{3}^{P_{33}}$ for $Q^{2}=0.05 \mathrm{GeV}^{2}$ in Fig. 6.6.


Figure 6.6: Imaginary part of $P_{33}$ partial wave contribution to the electron-proton TPE amplitudes $\mathcal{G}_{1}^{P_{33}}, \mathcal{G}_{2}^{P_{33}}$ (left panel) and $\mathcal{G}_{M}^{P_{33}}, \mathcal{F}_{3}^{P_{33}}$ (right panel) for momentum transfer $Q^{2}=0.05 \mathrm{GeV}^{2}$.

We checked that the numerical calculations of the imaginary part of the invariant amplitudes are in reasonable agreement with the analogous evaluation of the $\pi N$ channel contribution to the target normal spin asymmetry $A_{n}[197]$ of Eq. (3.67). Our result is shown in Fig. 6.7. We also checked numerically that the amplitudes $\mathcal{G}_{1}, \mathcal{G}_{2}, \mathcal{F}_{2}$ vanish in the limit $Q^{2} \rightarrow 0$ and the amplitudes $\mathcal{G}_{M}, \mathcal{F}_{3}$ remain finite in agreement with Eqs. (3.17, 3.18, 3.41).

[^21]

Figure 6.7: $P_{33}$ partial wave contribution to the target normal single spin asymmetry for momentum transfer $Q^{2}=0.05 \mathrm{GeV}^{2}$.

Having specified the imaginary parts, we next evaluate the DRs of Eqs. (4.29) and (4.30) to obtain the real parts of the electron-proton TPE amplitudes. The results are shown in Figs. 6.8-6.11 for different values of the upper integration limit in the $\mathrm{DR} \nu_{\max }$.


Figure 6.8: Real part of the $P_{33}$ partial wave contribution to the electron-proton TPE amplitude $\mathcal{G}_{1}^{P_{33}}$ for momentum transfer $Q^{2}=0.05 \mathrm{GeV}^{2}$ for different values of the upper integration limit in the $\mathrm{DR} \nu_{\max }=2.67,5,6 \mathrm{GeV}^{2}$. On the left (right) panel we show amplitudes as function of the $\nu(\varepsilon)$ variable.


Figure 6.9: Same as Fig. 6.8, but for electron-proton TPE amplitude $\mathcal{G}_{2}^{P_{33}}$.


Figure 6.10: Same as Fig. 6.8, but for electron-proton TPE amplitude $\mathcal{G}_{M}^{P_{33}}$.


Figure 6.11: Same as Fig. 6.8, but for electron-proton TPE amplitude $\mathcal{F}_{3}^{P_{33}}$.

## $6.4 \pi N P_{33}$ partial wave contribution to TPE in $e^{-} p$ scattering

Substituting the real parts of TPE amplitudes into Eq. (3.62), we obtain the unsubtracted DR result for the electron-proton scattering cross section correction. We estimate the uncertainties of the DR calculation by varying the upper integration limit $\nu_{\text {max }}$ in the range $\nu_{0}<\nu_{\max }<$ $6 \mathrm{GeV}^{2}$, and show the corresponding error band in Fig. 6.12. We furthermore compare in

Fig. 6.12 the $P_{33}$ partial wave DR result with the $P_{33}$ partial wave contribution in the nearforward approximation of Chapter 5 (the upper integration region is restricted by MAID to $W=2 \mathrm{GeV})$.


Figure 6.12: $\pi N P_{33}$ partial wave TPE correction within the unsubtracted DRs is compared with the near-forward approximation for $Q^{2}=0.05 \mathrm{GeV}^{2}$ and the subtracted DR prediction in the near-forward approximation with a subtraction point $\left(\varepsilon_{0}=0.8\right)$ fixed to the unsubtracted $\mathrm{DR}\left(\nu_{\max }=6 \mathrm{GeV}\right)$ calculation. The region to the right of the vertical dashed line denotes the validity region of the near-forward approximation.

In Fig. 6.13, we compare the near-forward approximation corresponding to the $\mathcal{F}_{2}$ tensor structure with the unsubtracted DRs contribution of the amplitude $\mathcal{F}_{2}$ (the W integration is restricted to $W=2 \mathrm{GeV}$ ). Two methods are in a reasonable agreement over the whole $\varepsilon$ region. Consequently, the difference in Fig. 6.12 is given by $\mathcal{G}_{M}$ and $\mathcal{F}_{3}$ contributions.


Figure 6.13: Same as Fig. 6.12, but for the contribution of the invariant amplitude $\mathcal{F}_{2}$ within unsubtracted DRs.

We next perform the subtracted DR analysis in the assumption of the dominant contribution coming from the TPE amplitude $\mathcal{F}_{2}$, which is the only non-vanishing amplitude in the nearforward approximation. In this case, the TPE correction to the unpolarized elastic electronproton scattering cross section of Eq. (3.55) can be expressed as

$$
\begin{equation*}
\delta_{2 \gamma}\left(\nu, Q^{2}\right) \approx \zeta\left(\nu, Q^{2}\right) \Re \mathcal{F}_{2}^{2 \gamma}\left(\nu, Q^{2}\right), \tag{6.40}
\end{equation*}
$$

with

$$
\begin{equation*}
\zeta\left(\nu, Q^{2}\right)=-\frac{2\left(1+\tau_{P}\right) G_{E}}{G_{E}^{2}+\frac{\tau_{P}}{\varepsilon} G_{M}^{2}} . \tag{6.41}
\end{equation*}
$$

Fixing the subtraction constant in the model evaluation $\delta_{2 \gamma}^{0}\left(\nu, Q^{2}\right)$ to the unsubtracted DR result $\delta_{2 \gamma}^{\mathrm{DR}}\left(\nu_{0}, Q^{2}\right)$, we can predict the TPE correction for other values of $\nu$ by

$$
\begin{equation*}
\delta_{2 \gamma}\left(\nu, Q^{2}\right)=\delta_{2 \gamma}^{0}\left(\nu, Q^{2}\right)+\frac{\zeta\left(\nu, Q^{2}\right)}{\zeta\left(\nu_{0}, Q^{2}\right)}\left(\delta_{2 \gamma}^{\mathrm{DR}}\left(\nu_{0}, Q^{2}\right)-\delta_{2 \gamma}^{0}\left(\nu_{0}, Q^{2}\right)\right) . \tag{6.42}
\end{equation*}
$$

We present our subtracted DR prediction in Fig. 6.12. Such analysis is valid only in the region to the right from the vertical dashed line, where we observe a reasonable agreement between both methods of the $P_{33}$ channel TPE evaluation. This indicates on the necessity of the subtraction also in the amplitude $\mathcal{F}_{2}^{2 \gamma}$ accounting for the $\pi N$ intermediate state.

In Fig. 6.14 we compare the inelastic TPE evaluated in the near-forward approximation of Chapter 5 with the BC fit (valid for $W<3.1 \mathrm{GeV}$ ), $P_{33}$ partial wave and $\pi N$ contributions from the MAID (valid for $W<2 \mathrm{GeV}$ ) as well as a combination of the MAID and BC fit $(2 \mathrm{GeV}<W<3.1 \mathrm{GeV})$ as an input, denoted in Fig. 6.14 by $\pi \mathrm{N}$ MAID $(\mathrm{W}<2 \mathrm{GeV})+\mathrm{SF}$. One notices that the main inelastic TPE contribution is given by the $\pi N$ channel and that the relative contribution of the $\pi N$ channel to the total inelastic result decreases for larger $\varepsilon$.


Figure 6.14: Evaluating in the near-forward approximation of Eqs. (5.76, 5.77), the $P_{33}$ partial wave and full $\pi N$ TPE corrections with input from MAID are compared with the inelastic TPE based on the BC fit and with the calculation based on the combination of the MAID and BC fit ( $2 \mathrm{GeV}<W<3.1 \mathrm{GeV}$ ), denoted by $\pi$ N MAID $(\mathrm{W}<2 \mathrm{GeV})+$ SF, for $Q^{2}=0.05 \mathrm{GeV}^{2}$. The region to the right of the vertical dashed line denotes the validity region of the near-forward approximation.

In Fig. 6.15 we compare the sum of the proton and $P_{33}$ partial wave contributions (with an upper integration limit $\nu_{\max }=6 \mathrm{GeV}^{2}$ ) within the unsubtracted DR framework with the total TPE, elastic TPE and empirical TPE fit [104]. In the forward (large $\varepsilon$ ) region, the pion-nucleon $P_{33}$ partial wave accounts up to half of the inelastic TPE correction.


Figure 6.15: Unsubtracted DR evaluation of $\pi N$ TPE correction from the $P_{33}$ partial wave in comparison with the elastic TPE and sum of Born and total inelastic TPE in the near-forward approximation for $Q^{2}=0.05 \mathrm{GeV}^{2}$.

The account of the $P_{33}$ partial wave in the subtracted DR formalism of Section 4.5 allows to reduce the model-dependent uncertainty as it is shown in Fig. 6.16. The remaining disagreement between the prediction and the empirical fit calls for an account of other $\pi N$ partial waves and higher lying channels ( $\pi \pi N, \ldots$ ).


Figure 6.16: Subtracted DR based prediction for elastic and $\pi N$ TPE correction from the $P_{33}$ partial wave for $Q^{2}=0.05 \mathrm{GeV}^{2}$ in comparison with the elastic TPE correction within the subtracted DR formalism of Section 4.5. The subtracted DR curves correspond with two choices for the subtraction points: $\varepsilon_{0}=0.2,0.8$.

## Chapter 7

## Conclusion and outlook

The precision of modern experiments in atomic spectroscopy and elastic lepton-proton scattering is well below the size of the TPE correction, which contributes the largest theoretical uncertainty. This correction also plays an important role in the proton form factor puzzle at large momentum transfer, partially resolving it. To improve on the precision of the proton charge radius extractions, we studied and evaluated the correction from the diagram with two exchanged photons to various physical observables at zero and low momentum transfers.

The hadronic correction from the TPE box graph to the atomic $S$ energy levels at leading $\alpha$ order is determined by the box diagram in the forward kinematics. One evaluates this correction considering DRs for the forward doubly virtual Compton scattering (VVCS) tensor. Firstly, we classified the lepton-proton scattering amplitudes in the forward kinematics and expressed the TPE correction in terms of the forward virtual Compton amplitudes. This calculation requires the input of one subtraction function which we estimated. We furthermore calculated its TPE effect on the Lamb shift of the 2 S energy level in muonic hydrogen from the inelastic electronproton scattering data in the resonance and deep inelastic scattering regions combined with the experimental value of the proton magnetic polarizability. It was found that the extracted subtraction function is compatible in magnitude with the chiral perturbation theory calculation. Our evaluation of the Lamb shift correction, $\Delta E_{2 \mathrm{~S}}^{\text {subt }} \approx 2.3 \pm 1.3 \mu \mathrm{eV}$, is in a fair agreement, though slightly smaller than the estimate of Birse and McGovern: $\Delta E_{2 \mathrm{~S}}^{\text {subt }} \approx 4.2 \pm 1.0 \mu \mathrm{eV}$, due to the sign change in the empirically estimated subtraction function. The upcoming JLab 12 GeV data will furthermore improve on the precision of our evaluation. With dispersion relations for the forward lepton-proton scattering amplitudes, we have expressed the $O\left(\alpha^{5}\right)$ TPE proton structure correction to the S-level HFS in hydrogen-like atoms in terms of the experimentally accessible inclusive cross sections of $l p$ scattering. Dispersion relations provide a new way to determine the HFS correction explicitly accounting for the double spin-flip leptonproton amplitude. The result for the individual channel HFS contribution is distinct with the standard approach. However, when accounting for contributions from all possible channels through the Burkhardt-Cottingham sum rule both methods agree. We have reevaluated the TPE correction to HFS in the electronic and muonic hydrogen connecting the region with small photons virtualities, which is expressed through moments of the proton spin structure functions (SFs), to the region with larger photons virtualities, where the precise data (parametrizations) exist. The resulting HFS correction value is similar to the previous evaluations with slightly smaller uncertainties. The precise experimental (theoretical) knowledge of the proton spin SFs, the proton magnetic form factor and moments of the proton spin structure functions at low momentum transfer is required to reduce the model dependence and uncertainties in the HFS evaluation further, which be crucial in the analysis of the 1S HFS measurement at PSI. The developed formalism can be generalized for the evaluation of the TPE correction to levels with other orbital angular momentum and for the studies of the $\gamma Z$ box diagram contribution to the atomic energy levels. The evaluation of the forward TPE amplitudes at threshold can be used as a model independent way to obtain the parameters of the effective interaction between the lepton and the proton at low energies.

Next, we described the helicity formalism of the elastic lepton-proton scattering. We ex-
pressed the TPE correction $\delta_{2 \gamma}$ to the unpolarized elastic lepton-proton scattering in terms of the invariant amplitudes. We expressed all six non-forward TPE amplitudes at $Q^{2}=0$ in terms of two forward lepton-proton amplitudes and proved the vanishing behavior of the TPE correction $\delta_{2 \gamma}$. Subsequently, we described the lepton massless limit and provided the known expressions for the TPE correction to the cross section and double polarization observables. We also obtained the expressions for the single spin asymmetries for the massive lepton scattering.

In the small momentum transfer region, the main contribution to TPE correction is coming from the proton intermediate state. We have evaluated this contribution in a dispersion relation approach and compared it with the box graph model (Born TPE), based on the on-shell electromagnetic vertices. The imaginary parts are the same in both approaches. However, the real parts differ in two helicity-flip amplitudes and one helicity conserving amplitude $\mathcal{F}_{3}^{2 \gamma}$. With the aim to minimize the model dependence and decrease the contribution from the high-energy region in analysis of the electron-proton scattering data, we applied the subtracted dispersion relation for the amplitude $\mathcal{F}_{3}^{2 \gamma}$. We took the subtraction point from an empirical fit to data. Choosing different subtraction points, we quantitatively estimated the contribution from the inelastic intermediate states. However, the data fit is not trustable in the region of low momentum transfer and small $\varepsilon$, where the data is not available. Going to large $Q^{2}$, the slope of the TPE correction in the region $\varepsilon \rightarrow 1$ is predicted to change sign. We provided predictions for kinematics of the OLYMPUS experiment based on the subtracted DR formalism with the proton intermediate state. The existent CLAS data [77,79] were found to be in agreement with the empirical TPE fit of the A1 Collaboration at MAMI [104] accounting only for the proton intermediate state. Additionally, we have developed a new method of analytical continuation of the elastic TPE amplitudes. The advantage of this method is that it performs the analytical continuation within a broad range of the proton form factors parametrizations as an input.

Besides the Born TPE, we accounted for the inelastic TPE by using the VVCS tensor in a near-forward approximation and expressed the corresponding TPE correction as an integral over the unpolarized proton structure functions. We checked that this approximation reproduces the existing results for the low momentum transfer expansion of the TPE correction. It allows to go beyond the known expansion accounting for the momentum transfer dependence of the proton structure functions. In the near-forward approximation the TPE correction was found to be in good agreement with the empirical TPE fit of the A1 Collaboration at MAMI [104], CLAS measurements [79] and VEPP-3 data point at low $Q^{2}$ [76] normalized to the empirical TPE fit. Increasing the momentum transfer, the TPE correction starts to deviate from the data fit, still following its shape. We also extrapolated the resulting TPE to low- $Q^{2}$ region of the OLYMPUS experiment [81].

In order to make theoretical estimates of the inelastic TPE correction at larger momentum transfer and arbitrary scattering angles, we next exploited a DR framework. We evaluated the contribution from the $P_{33}$ partial-wave of pion electroproduction, corresponding with the $\Delta(1232)$ resonance, within DRs. We took the invariant amplitudes of the pion electroproduction from the MAID partial-wave analysis. The dispersive evaluation gave results of the same order as the near-forward approximation in the region of its applicability. The latter approximation is in a reasonable agreement with the contribution of the amplitude $\mathcal{F}_{2}^{2 \gamma}$ (or $\mathcal{G}_{2}^{2 \gamma}$ in the $\mathcal{G}_{1}^{2 \gamma}, \mathcal{G}_{2}^{2 \gamma}, \mathcal{F}_{3}^{2 \gamma}$ decomposition) to the cross section correction evaluated within unsubtracted DRs over the whole $\varepsilon$ region at low momentum transfers. The $P_{33}$ partial-wave accounts for up to half of the inelastic TPE correction. The next step will be to evaluate the contribution from other $\pi N$ partial waves numerically and to extend the DR formalism to the region of larger momentum transfers.

With the aim to evaluate the correction in the proposed muon-proton scattering experiment (MUSE), we extended the general formalism of TPE corrections in the elastic unpolarized
scattering to the case of the finite mass lepton. As a first step, we have evaluated the Born TPE correction. The estimates for the TPE correction to the muon-proton scattering cross section vary between $0.25 \%$ and $0.5 \%$. These estimates are up to a factor three smaller, as compared with TPE corrections for the case of elastic electron-proton scattering for the same beam momenta. This is due to the contribution of lepton helicity-flip amplitudes, which has an opposite sign as compared with the contribution of non-flip amplitudes and significantly reduces the correction. Our calculations reproduced the expected low momentum transfer limit of TPE in muon-proton scattering. The inelastic intermediate states were studied within the near-forward approximation. Contrary to the case of massless lepton-proton scattering, the subtraction function of the unpolarized forward Compton scattering contributes to the TPE correction due to the sizable muon mass scale. We have evaluated this contribution from the chiral perturbation theory and from the empirically estimated subtraction function. This contribution is comparable in magnitude to the unpolarized proton structure functions contribution and has an opposite sign. In MUSE kinematics, the elastic TPE contribution largely dominates, and the size of the inelastic TPE is within the anticipated error of the forthcoming data.

All performed calculations will be relevant as radiative corrections in elastic lepton-proton scattering experiments improving on the precision of the form factor measurements at low momentum transfer and extractions of proton radii by the MUSE Collaboration, PRad experiment at JLab and other experiments. TPE corrections will also be relevant in the precise determination of the proton magnetic radius by the MAGIX experiment at MESA (Mainz) as well as for the measurement of the weak mixing angle by the P2 experiment at MESA.

## Appendix A

## Dirac spinors in the Jacob-Wick phase convention

The solution of the free Dirac equation in the momentum space $(\hat{k}+m) u(k, h)=0$ describes a lepton with the four-momentum $k\left(k_{0}, \vec{k}\right)$ and the helicity $h$. In the Dirac representation, the solution is given by

$$
\begin{equation*}
u(k, h)=\binom{\sqrt{k_{0}+m} \times \chi_{h}(\theta, \phi)}{h \sqrt{k_{0}-m} \times \chi_{h}(\theta, \phi)} \tag{A.1}
\end{equation*}
$$

with the particle momentum $\vec{k}=|\vec{k}|(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ and the helicity eigenstates $\chi_{h}$ :

$$
\begin{equation*}
\chi_{+}=\binom{\cos \frac{\theta}{2}}{e^{i \phi} \sin \frac{\theta}{2}}, \quad \chi_{-}=\binom{-e^{-i \phi} \sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \tag{A.2}
\end{equation*}
$$

The solution is normalized as $u^{+}\left(k, h^{\prime}\right) u(k, h)=2 k_{0} \delta_{h, h^{\prime}}$. The corresponding solution for the antiparticle is given by

$$
\begin{equation*}
v(k, h)=\binom{-\sqrt{k_{0}-m} \times \chi_{-h}(\theta, \phi)}{h \sqrt{k_{0}+m} \times \chi_{-h}(\theta, \phi)} . \tag{A.3}
\end{equation*}
$$

According to the Jacob and Wick [191] phase convention, the helicity eigenstates of a proton $\tilde{\chi}_{\lambda}$ at rest are defined reverse to the helicity eigenstates of a lepton:

$$
\begin{equation*}
\tilde{\chi}_{+}=\binom{-e^{-i \phi} \sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}, \quad \tilde{\chi}_{-}=\binom{\cos \frac{\theta}{2}}{e^{i \phi} \sin \frac{\theta}{2}} . \tag{A.4}
\end{equation*}
$$

The proton spinor is then given by

$$
\begin{equation*}
N(p, \lambda)=\binom{\sqrt{p_{0}+M} \times \tilde{\chi}_{\lambda}(\theta, \phi)}{\lambda \sqrt{p_{0}-M} \times \tilde{\chi}_{\lambda}(\theta, \phi)} \tag{A.5}
\end{equation*}
$$

with the following normalization: $N^{+}\left(p, \lambda^{\prime}\right) N(p, \lambda)=2 p_{0} \delta_{\lambda, \lambda^{\prime}}$.
The relations of Eqs. $(3.11,3.49)$ are derived in the center of mass frame. In this frame the kinematical factors in spinors are expressed in terms of the Mandelstam variable $s$ as

$$
\begin{align*}
k_{0}+m=\frac{(\sqrt{s}+m)^{2}-M^{2}}{2 \sqrt{s}}, & k_{0}-m=\frac{(\sqrt{s}-m)^{2}-M^{2}}{2 \sqrt{s}}  \tag{A.6}\\
p_{0}+M=\frac{(\sqrt{s}+M)^{2}-m^{2}}{2 \sqrt{s}}, & p_{0}-M=\frac{(\sqrt{s}-M)^{2}-m^{2}}{2 \sqrt{s}} \tag{A.7}
\end{align*}
$$

## Appendix B

## Forward lepton-proton scattering observables

The forward elastic scattering cross section in the c.m. reference frame is given by

$$
\begin{equation*}
\frac{\mathrm{d} \sigma}{\mathrm{~d} \Omega}(\theta=0)=\frac{\left|f_{+}(\omega)\right|^{2}+\left|f_{-}(\omega)\right|^{2}+2|g(\omega)|^{2}}{64 \pi^{2}\left(M^{2}+2 M \omega+m^{2}\right)} \tag{B.1}
\end{equation*}
$$

All possible single-spin asymmetries are zero in the forward scattering. We denote the lepton spin asymmetry in scattering on the polarized proton as $A$ and in scattering on the unpolarized proton with the polarization transfer to the final proton as $P$. The asymmetries for the longitudinally polarized lepton and the longitudinally polarized proton in the forward scattering are expressed as

$$
\begin{align*}
& A_{l}=\frac{\mathrm{d} \sigma_{+-}-\mathrm{d} \sigma_{--}}{\mathrm{d} \sigma_{+-}+\mathrm{d} \sigma_{--}}=-2 \frac{\Re\left(f_{+} f_{-}^{*}\right)+|g|^{2}}{\left|f_{+}\right|^{2}+\left|f_{-}\right|^{2}+2|g|^{2}}, \\
& P_{l}=\frac{\mathrm{d} \sigma_{+-}-\mathrm{d} \sigma_{--}}{\mathrm{d} \sigma_{+-}+\mathrm{d} \sigma_{--}}=-2 \frac{\Re\left(f_{+} f_{-}^{*}\right)-|g|^{2}}{\left|f_{+}\right|^{2}+\left|f_{-}\right|^{2}+2|g|^{2}} . \tag{B.2}
\end{align*}
$$

The asymmetries for the transversely polarized lepton and the transversely polarized proton in the forward scattering are expressed as

$$
\begin{align*}
A_{t} & =\frac{\mathrm{d} \sigma_{\uparrow \uparrow}-\mathrm{d} \sigma_{\uparrow \downarrow}}{\mathrm{d} \sigma_{\uparrow \uparrow}+\mathrm{d} \sigma_{\uparrow \downarrow}}=2 \frac{\Re\left(\left(f_{+}+f_{-}\right) g^{*}\right)}{\left|f_{+}\right|^{2}+\left|f_{-}\right|^{2}+2|g|^{2}}, \\
P_{t} & =\frac{\mathrm{d} \sigma_{\uparrow \uparrow}-\mathrm{d} \sigma_{\uparrow \downarrow}}{\mathrm{d} \sigma_{\uparrow \uparrow}+\mathrm{d} \sigma_{\uparrow \downarrow}}=2 \frac{\Re\left(\left(f_{+}-f_{-}\right) g^{*}\right)}{\left|f_{+}\right|^{2}+\left|f_{-}\right|^{2}+2|g|^{2}} \tag{B.3}
\end{align*}
$$

The asymmetries in the case of one transverse and one longitudinal polarizations vanish.
The elastic cross section at threshold is given by the $S$ wave only, therefore it is an angular independent quantity. From the other hand, the total elastic cross section at threshold is expressed in terms of the scattering length $a$ as $\sigma=4 \pi a^{2}$ that is related to the forward amplitudes by

$$
\begin{equation*}
a^{2}=\frac{\left|f_{+}(m)\right|^{2}+\left|f_{-}(m)\right|^{2}+2|g(m)|^{2}}{64 \pi^{2}(M+m)^{2}} \tag{B.4}
\end{equation*}
$$

We also define the triplet scattering lengths for the elastic scattering of the state with parallel (anti-parallel) fermions spin directions along the beam line $a_{t, 1}\left(a_{t, 0}\right)$ and the singlet (averaged triplet) scattering length $a_{s}\left(a_{t}\right)$. The scattering length $a$ is expressed in terms of the singlet and triplet scattering lengths as

$$
\begin{equation*}
a^{2}=\frac{1}{4} a_{s}^{2}+\frac{3}{4} a_{t}^{2}=\frac{1}{4}\left(a_{s}^{2}+a_{t, 0}^{2}+2 a_{t, 1}^{2}\right) \tag{B.5}
\end{equation*}
$$

We express the triplet $\left(a_{t}, a_{t, 0}, a_{t, 1}\right)$ and singlet $\left(a_{s}\right)$ scattering lengths in terms of the double spin asymmetries as

$$
\begin{align*}
a_{s}^{2} & =\left(1-A_{l}-2 A_{t}\right) a^{2},  \tag{B.6}\\
a_{t}^{2} & =\left(1+\frac{1}{3} A_{l}+\frac{2}{3} A_{t}\right) a^{2},  \tag{B.7}\\
a_{t, 1}^{2} & =\left(1+\frac{4}{3} A_{l}-\frac{1}{3} A_{t}\right) a^{2},  \tag{B.8}\\
a_{t, 0}^{2} & =\left(1-\frac{2}{3} A_{l}+\frac{5}{3} A_{t}\right) a^{2} . \tag{B.9}
\end{align*}
$$

The above expressions of this Appendix are valid in scattering of two fermions of arbitrary nature. For example, in the case of neutron-proton scattering.

In elastic lepton-proton scattering, it is convenient to decompose the forward scattering amplitudes into a sum of OPE and TPE contributions. The TPE amplitudes, except for $\Re f_{+}^{2 \gamma}$, are obtained with DRs and unitarity relations described in Sections 2.1, 2.2. The OPE amplitudes are real and given by

$$
\begin{align*}
f_{-}^{1 \gamma} & =e^{2} \mu_{P},  \tag{B.10}\\
g^{1 \gamma} & =0 . \tag{B.11}
\end{align*}
$$

The vacuum polarization correction is zero in the forward scattering. The lepton vertex correction does not change the amplitude $g$, modifies the amplitude $f_{-}$by the lepton anomalous magnetic moment $a_{l}: \delta f_{-}=a_{l} e^{2}$ and contributes to the amplitude $f_{+}$. The proton vertex correction with the proton intermediate state contributes to $f_{-}$and $f_{+}$amplitudes [163]. The contribution of inelastic intermediate states is expected to be small, however, this correction requires an additional theoretical investigation. We combine all orders contributions to the amplitude $\Re f_{+}$. Therefore, the forward amplitudes are completely expressed in terms of the total inclusive cross sections, see Eqs. (2.19, 2.20, 2.23, 2.24), at $O\left(\alpha^{2}\right)$ up to the one unknown spin-independent amplitude $\Re f_{+}$.

## Appendix C

## Forward invariant amplitudes in terms of the proton structure functions

Substituting expressions for the inclusive cross sections of Eqs. (2.30-2.32) into DRs, see Eqs. (2.22-2.24), we change the integration order and express the forward TPE amplitudes in terms of the proton SFs:

$$
\begin{align*}
\Re f_{+}^{2 \gamma}(\omega)=4 \alpha^{2} M \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}} \int_{\nu_{\text {thr }}}^{\infty} \frac{\mathrm{d} \nu_{\gamma}}{\nu_{\gamma}} & \left\{\frac{4 m^{2} \nu_{\gamma}}{M Q^{2}}\left(1-2 \tau_{l}\right) I_{1}^{0} F_{1}\left(\nu_{\gamma}, Q^{2}\right)\right. \\
& \left.+\left(I_{1}^{0}+\frac{4 m \nu_{\gamma}}{Q^{2}} I_{2}^{0}-\frac{I_{3}^{0}}{\tau_{l}}\right) F_{2}\left(\nu_{\gamma}, Q^{2}\right)\right\},  \tag{C.1}\\
\Re f_{-}^{2 \gamma}(\omega)=16 \alpha^{2} \omega \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}} \int_{\nu_{\text {thr }}}^{\infty} \frac{\mathrm{d} \nu_{\gamma}}{\nu_{\gamma}} & \left\{\left(-I_{1}^{0}+\left(\tau_{l}+\tilde{\tau}\right) I_{1}^{1}+\frac{\nu_{\gamma}}{m}\left(\frac{I_{0}^{0}}{2}+I_{0}^{1}\right)\right) g_{1}\left(\nu_{\gamma}, Q^{2}\right)\right. \\
& \left.+\frac{Q^{2}}{2 m \nu_{\gamma}} I_{0}^{0} g_{2}\left(\nu_{\gamma}, Q^{2}\right)\right\},  \tag{C.2}\\
\Re g^{2 \gamma}(\omega)=8 \alpha^{2} m \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}} \int_{\nu_{\text {thr }}}^{\infty} \frac{\mathrm{d} \nu_{\gamma}}{\nu_{\gamma}} & \left\{\left(I_{1}^{0}+\left(\tau_{l}+\tilde{\tau}\right) I_{1}^{1}+\frac{\nu_{\gamma}}{m}\left(I_{0}^{0}+I_{0}^{1}\right)\right) g_{1}\left(\nu_{\gamma}, Q^{2}\right)\right. \\
& \left.+2 I_{1}^{0} g_{2}\left(\nu_{\gamma}, Q^{2}\right)\right\}, \tag{C.3}
\end{align*}
$$

with the DR master integrals (we introduce the cut-off $\Lambda$ in the divergent integrals):

$$
\begin{align*}
& I_{0}^{0}=\int_{\omega_{0}}^{\infty} \frac{-m}{\omega^{\prime 2}-\omega^{2}} \frac{\mathrm{~d} \omega^{\prime}}{\sqrt{\omega^{\prime 2}-m^{2}}}=\frac{m}{2|\boldsymbol{k}| \omega} \ln \frac{\omega+|\boldsymbol{k}|}{\omega-|\boldsymbol{k}|} \frac{\omega\left|\boldsymbol{k}_{0}\right|-|\boldsymbol{k}| \omega_{0}}{\omega\left|\boldsymbol{k}_{0}\right|+|\boldsymbol{k}| \omega_{0}},  \tag{C.4}\\
& I_{1}^{0}=\int_{\omega_{0}}^{\infty} \frac{-\omega^{\prime}}{\omega^{\prime 2}-\omega^{2}} \frac{\mathrm{~d} \omega^{\prime}}{\sqrt{\omega^{\prime 2}-m^{2}}}=\frac{1}{2|\boldsymbol{k}|} \ln \frac{|\boldsymbol{k}|-\left|\boldsymbol{k}_{0}\right|}{|\boldsymbol{k}|+\left|\boldsymbol{k}_{0}\right|},  \tag{C.5}\\
& I_{2}^{0}=\frac{1}{m} \int_{\omega_{0}}^{\Lambda} \frac{-\omega^{\prime 2}}{\omega^{\prime 2}-\omega^{2}} \frac{\mathrm{~d} \omega^{\prime}}{\sqrt{\omega^{\prime 2}-m^{2}}}=\frac{1}{m} \ln \frac{2 \Lambda}{\omega_{0}+\left|\boldsymbol{k}_{0}\right|}+\frac{\omega}{2 m|\boldsymbol{k}|} \ln \frac{\omega+|\boldsymbol{k}| \boldsymbol{\omega}\left|\boldsymbol{k}_{0}\right|-|\boldsymbol{k}| \omega_{0}}{\omega-|\boldsymbol{k}|} \frac{\omega\left|\boldsymbol{k}_{0}\right|+|\boldsymbol{k}| \omega_{0}}{},  \tag{C.6}\\
& I_{3}^{0}=\frac{1}{m^{2}} \int_{\omega_{0}}^{\Lambda} \frac{-\omega^{\prime 3}}{\omega^{\prime 2}-\omega^{2}} \frac{\mathrm{~d} \omega^{\prime}}{\sqrt{\omega^{\prime 2}-m^{2}}}=\frac{\left|\boldsymbol{k}_{0}\right|-\Lambda}{m^{2}}+\frac{\omega^{2}}{2|\boldsymbol{k}| m^{2}} \ln \frac{|\boldsymbol{k}|-\left|\boldsymbol{k}_{0}\right|}{|\boldsymbol{k}|+\left|\boldsymbol{k}_{0}\right|},  \tag{C.7}\\
& I_{0}^{1}=\int_{\omega_{0}}^{\infty} \frac{-m^{3}}{\omega^{\prime 2}-\omega^{2}} \frac{\mathrm{~d} \omega^{\prime}}{\left(\omega^{\prime 2}-m^{2}\right)^{3 / 2}}=\frac{\omega_{0} m}{\left|\boldsymbol{k}_{0}\right| \boldsymbol{k}^{2}}-\frac{m}{\boldsymbol{k}^{2}}+\frac{m^{3}}{2|\boldsymbol{k}|^{3} \omega} \ln \frac{\omega+|\boldsymbol{k}| \omega\left|\boldsymbol{k}_{0}\right|-|\boldsymbol{k}| \omega_{0}}{\omega-|\boldsymbol{k}|} \frac{\omega\left|\boldsymbol{k}_{0}\right|+|\boldsymbol{k}| \omega_{0}}{\omega} \tag{C.8}
\end{align*}
$$

$$
\begin{equation*}
I_{1}^{1}=\int_{\omega_{0}}^{\infty} \frac{-m^{2} \omega^{\prime}}{\omega^{\prime 2}-\omega^{2}} \frac{\mathrm{~d} \omega^{\prime}}{\left(\omega^{\prime 2}-m^{2}\right)^{3 / 2}}=\frac{m^{2}}{\boldsymbol{k}^{2}\left|\boldsymbol{k}_{0}\right|}+\frac{m^{2}}{2|\boldsymbol{k}|^{3}} \ln \frac{|\boldsymbol{k}|-\left|\boldsymbol{k}_{0}\right|}{|\boldsymbol{k}|+\left|\boldsymbol{k}_{0}\right|} . \tag{C.9}
\end{equation*}
$$

The reasonable result for the proton or narrow $\Delta$ contribution, when we are allowed to change the $\omega$ and $Q^{2}, \nu_{\gamma}$ integration order, is given by the once-subtracted dispersion relation of Eq. (2.25). It can be obtained from Eq. (C.1) by $\Re f_{+}^{2 \gamma}(\omega)-\Re f_{+}^{2 \gamma}\left(\omega_{s}\right)$. The $\nu_{\gamma}$ integrals for the spin-dependent amplitudes of Eqs. $(2.42,2.43)$ are convergent. In this Appendix we express also the TPE amplitude $\Re f_{+}^{2 \gamma}$ directly in terms of the proton SFs:

$$
\begin{align*}
\Re f_{+}^{2 \gamma}(\omega)- & \Re f_{+}^{2 \gamma}(m)=\frac{4 m \alpha^{2}}{|\boldsymbol{k}|} \\
& \times \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}} \int_{\nu_{\text {thr }}}^{\infty} \frac{\mathrm{d} \nu_{\gamma}}{\nu_{\gamma}}\left(\frac{\nu_{\gamma}\left(1-2 \tau_{l}\right)}{2 m \tau_{l}}\left(\ln \frac{|\boldsymbol{k}|-\left|\boldsymbol{k}_{0}\right|}{|\boldsymbol{k}|+\left|\boldsymbol{k}_{0}\right|}+\frac{2|\boldsymbol{k}|}{\left|\boldsymbol{k}_{0}\right|}\right) F_{1}\left(\nu_{\gamma}, Q^{2}\right)\right. \\
& +\frac{2 M \nu_{\gamma}}{m Q^{2}}\left(\omega \ln \frac{(\omega+|\boldsymbol{k}|)^{2}\left(\omega_{0}^{2}-\omega^{2}\right)}{\left(\omega\left|\boldsymbol{k}_{0}\right|+|\boldsymbol{k}| \omega_{0}\right)^{2}}+2\left(\omega_{0}-\left|\boldsymbol{k}_{0}\right|\right) \frac{|\boldsymbol{k}|}{\left|\boldsymbol{k}_{0}\right|}\right) F_{2}\left(\nu_{\gamma}, Q^{2}\right) \\
& \left.-\frac{M}{m}\left(\frac{\omega^{2}-m^{2} \tau_{l}}{2 m^{2} \tau_{l}} \ln \frac{|\boldsymbol{k}|-\left|\boldsymbol{k}_{0}\right|}{|\boldsymbol{k}|+\left|\boldsymbol{k}_{0}\right|}+\frac{1-\tau_{l} \mid}{\tau_{l}} \frac{|\boldsymbol{k}|}{\left|\boldsymbol{k}_{0}\right|}\right) F_{2}\left(\nu_{\gamma}, Q^{2}\right)\right),  \tag{C.10}\\
\Re f_{+}^{2 \gamma}(\omega)= & \frac{4 m \alpha^{2}}{|\boldsymbol{k}|} \int_{0}^{\infty} \frac{\mathrm{d} Q^{2}}{Q^{2}} \int_{\nu_{\text {thr }}}^{\infty} \frac{\mathrm{d} \nu_{\gamma}}{\nu_{\gamma}}\left(\frac{\nu_{\gamma}\left(1-2 \tau_{l}\right)}{2 m \tau_{l}} F_{1}\left(\nu_{\gamma}, Q^{2}\right) \ln \frac{|\boldsymbol{k}|-\left|\boldsymbol{k}_{0}\right|}{|\boldsymbol{k}|+\left|\boldsymbol{k}_{0}\right|}\right. \\
& -\left(\frac{M\left(\omega^{2}-m^{2} \tau_{l}\right)}{2 m^{3} \tau_{l}} \ln \frac{|\boldsymbol{k}|-\left|\boldsymbol{k}_{0}\right|}{|\boldsymbol{k}|+\left|\boldsymbol{k}_{0}\right|}+\frac{|\boldsymbol{k}| M\left(\left|\boldsymbol{k}_{0}\right|+\nu_{\gamma} \ln \frac{\omega_{0}+\left|\boldsymbol{k}_{0}\right|}{2}\right)}{m^{3} \tau_{l}}\right) F_{2}\left(\nu_{\gamma}, Q^{2}\right) \\
& \left.+\frac{2 M \omega \nu_{\gamma}}{m Q^{2}} F_{2}\left(\nu_{\gamma}, Q^{2}\right) \ln \frac{(\omega+|\boldsymbol{k}|)^{2}\left(\omega_{0}^{2}-\omega^{2}\right)}{\left(\omega\left|\boldsymbol{k}_{0}\right|+|\boldsymbol{k}| \omega_{0}\right)^{2}}\right) . \tag{C.11}
\end{align*}
$$

For the amplitude $f_{+}^{2 \gamma}(\omega)$ we write only the regular part of the unsubtracted DR, i.e., we drop the $\omega$-independent terms with $\ln \Lambda, \Lambda$ in the divergent integrals $I_{2}^{0}, I_{3}^{0}$, see Eqs. (C.6, C.7).
Due to the Regge behavior of the $F_{1}$ proton structure function given by the Pomeron exchange, the $\nu_{\gamma}$ integrals are divergent, and the DR for the amplitude $f_{+}^{2 \gamma}$ is not applicable for the inelastic intermediate states TPE.

## Appendix D

## Forward DRs verification in Quantum Electrodynamics

In this Appendix, we verify the lepton-proton forward dispersion relations in QED. We reconstruct the real parts of TPE amplitudes with the relations of Eqs. (2.23-2.25) and compare them to the sum of the direct and crossed box graphs. The OPE helicity amplitude $T^{1 \gamma}$ for the lepton scattering off the charged point proton, see Fig. 3.2, is given by

$$
\begin{equation*}
T^{1 \gamma}=\frac{e^{2}}{Q^{2}+\mu^{2}} \bar{u}\left(k^{\prime}, h^{\prime}\right) \gamma^{\mu} u(k, h) \bar{N}\left(p^{\prime}, \lambda^{\prime}\right) \gamma_{\mu} N(p, \lambda) . \tag{D.1}
\end{equation*}
$$

We introduce the finite mass $\mu$ with the aim to have no deal with IR divergences.
The relevant OPE cross sections are given by

$$
\begin{align*}
\sigma^{1 \gamma}(\omega)= & \frac{4 \pi M^{2} \alpha^{2}}{\Sigma_{s}}\left\{\frac{\Sigma_{s}}{2 M^{2} s}+\left(\frac{s}{M^{2}}+\frac{4 \omega^{2}}{\mu^{2}}+\frac{\mu^{2}}{2 M^{2}}\right) \frac{\Sigma_{s}}{\Sigma_{s}+s \mu^{2}}\right. \\
& \left.-\frac{s+\mu^{2}}{M^{2}} \ln \frac{\Sigma_{s}+s \mu^{2}}{s \mu^{2}}\right\},  \tag{D.2}\\
\sigma_{++}^{1 \gamma}(\omega)-\sigma_{+-}^{1 \gamma}(\omega)= & \frac{16 \pi M \alpha^{2}}{\Sigma_{s}^{2}}\left\{\left(\omega \Sigma_{s}+2 \mu^{2}\left(m^{2}+M \omega\right)(\omega+M)\right) \ln \frac{\Sigma_{s}+s \mu^{2}}{s \mu^{2}}\right. \\
& \left.-\frac{\Sigma_{s}\left(\Sigma_{s}\left(2 \omega s-M\left(\omega^{2}-m^{2}\right)\right)+2 s\left(m^{2}+M \omega\right)(\omega+M) \mu^{2}\right)}{s\left(\Sigma_{s}+s \mu^{2}\right)}\right\}, \\
\sigma_{\perp}^{1 \gamma}(\omega)-\sigma_{\|}^{1 \gamma}(\omega)= & \frac{4 \pi \alpha^{2} M m}{\Sigma_{s}^{2}}\left\{\frac{2 s \mu^{2} \Sigma_{s}}{\Sigma_{s}+s \mu^{2}}+\left(\Sigma_{s}-2 s \mu^{2}\right) \ln \frac{\Sigma_{s}+s \mu^{2}}{s \mu^{2}}\right\} . \tag{D.3}
\end{align*}
$$

The high energy behavior of the relevant cross sections in the OPE approximation is following:

$$
\begin{equation*}
\sigma^{1 \gamma}(\omega) \underset{\omega \gg}{\sim} \omega^{0}, \quad \sigma_{++}^{1 \gamma}(\omega)-\sigma_{+-}^{1 \gamma}(\omega) \underset{\omega \gg}{\sim} \omega^{-1} \ln \omega, \quad \sigma_{\perp}^{1 \gamma}(\omega)-\sigma_{\|}^{1 \gamma}(\omega) \underset{\omega \gg}{\sim} \omega^{-2} \ln \omega . \tag{D.4}
\end{equation*}
$$

The unsubtracted DR for the amplitude $f_{+}^{2 \gamma}$ of Eq. (2.22) is divergent as $\omega$. Therefore, we use the subtracted DR of Eq. (2.25) for this amplitude.

The helicity amplitude corresponding with the TPE direct box graph $T^{2 \gamma}$ is given by

$$
\begin{equation*}
T^{2 \gamma}=-e^{4} \int \frac{i \mathrm{~d}^{4} q}{(2 \pi)^{4}} \frac{\bar{u}\left(k, h^{\prime}\right) \gamma^{\mu}(\hat{k}-\hat{q}+m) \gamma^{\nu} u(k, h) \bar{N}\left(p, \lambda^{\prime}\right) \gamma_{\mu}(\hat{p}+\hat{q}+M) \gamma_{\nu} N(p, \lambda)}{\left((p+q)^{2}-M^{2}\right)\left((k-q)^{2}-m^{2}\right)\left(q^{2}-\mu^{2}\right)\left(q^{2}-\mu^{2}\right)} \tag{D.5}
\end{equation*}
$$

We find the contribution of the direct box graph to the forward amplitudes $f_{+}^{2 \gamma, \text { dir }}, f_{-}^{2 \gamma, \text { dir }}$ with exchange of two photons (see Fig. 2.3) multiplying the fermion spinors by the spin projection operators. Then we sum over all possible polarizations evaluating the traces of the Dirac matrices. The direct amplitudes $f_{+}^{2 \gamma, \text { dir }}, f_{-}^{2 \gamma, \text { dir }}$ are given by

$$
\begin{align*}
f_{+}^{2 \gamma, \text { dir }}= & -8 e^{4} \frac{\partial}{\partial \mu^{2}} \int \frac{i \mathrm{~d}^{4} q}{(2 \pi)^{4}} \frac{1}{(p+q)^{2}-M^{2}} \frac{1}{(k-q)^{2}-m^{2}} \frac{1}{q^{2}-\mu^{2}} \\
& \times\left(2 M^{2} \nu_{\gamma}^{2}-q^{2} M \nu_{\gamma}-(p \cdot q)\left(2 M \nu_{\gamma}+m^{2}+(k \cdot q)\right)+(k \cdot q)\left(2 M \nu_{\gamma}+M^{2}\right)\right), \\
f_{-}^{2 \gamma, \text { dir }=} & 8 e^{4} \frac{\partial}{\partial \mu^{2}} \int \frac{i \mathrm{~d}^{4} q}{(2 \pi)^{4}} \frac{M}{(p+q)^{2}-M^{2}} \frac{m}{(k-q)^{2}-m^{2}} \frac{1}{q^{2}-\mu^{2}}\left(q^{2}(s \cdot S)-(q \cdot s)(q \cdot S)\right), \tag{D.6}
\end{align*}
$$

with the lepton and proton spin vectors in the laboratory frame $s=(|\boldsymbol{k}|, 0,0, \omega) / m$ and $S=(0,0,0,-1)$. For the double spin-flip amplitude $g^{2 \gamma, \text { dir }}$ we use the decomposition in terms of the scalar integrals, since the evaluation of traces can not be exploited here due to the different spin directions of initial and final fermions. Furthermore, we repeat the same steps for the crossed box graph contribution.
The optical theorem of Eqs. $(2.19,2.20)$, the once-subtracted DR for the amplitude $f_{+}^{2 \gamma}$ of Eq. (2.25), the unsubtracted DRs for the amplitudes $f_{-}^{2 \gamma}, g^{2 \gamma}$ of Eqs. (2.23, 2.24) and the amplitudes properties of Eqs. (2.16-2.18) under the crossing $\omega \rightarrow-\omega$ were checked comparing to the sum of the direct and crossed box graphs.

## Appendix E

## Regge poles residues of the proton structure function $F_{1}$ from high-energy data

The high-energy limit ( $\nu_{\gamma}$ very large at fixed $Q^{2}$ ) of the proton structure function $F_{1}$ is often parameterized through a Regge pole fit as

$$
\begin{equation*}
F_{1}\left(\nu_{\gamma}, Q^{2}\right) \underset{\nu_{\gamma} \gg}{\longrightarrow} \sum_{\alpha_{0} \geq 0} \gamma_{\alpha_{0}}\left(Q^{2}\right) \nu_{\gamma}^{\alpha_{0}}, \tag{E.1}
\end{equation*}
$$

where $\gamma_{\alpha_{0}}\left(Q^{2}\right)$ are the leading Regge poles residues, which can be extracted from the highenergy inclusive electron-proton scattering data. We will determine these residues from the Donnachie-Landshoff (DL) fit [146] to data for the proton structure function $F_{2}$ in the region of very small Bjorken variable $x_{\mathrm{Bj}} \equiv Q^{2} /\left(2 M \nu_{\gamma}\right)$ :

$$
\begin{equation*}
F_{2}\left(\nu_{\gamma}=\frac{Q^{2}}{2 M x_{\mathrm{Bj}}}, Q^{2}\right) \underset{x_{\mathrm{Bj}} \ll 1}{\longrightarrow} f_{0}\left(Q^{2}\right) x_{\mathrm{Bj}}^{-\varepsilon_{0}}+f_{1}\left(Q^{2}\right) x_{\mathrm{Bj}}^{-\varepsilon_{1}}+f_{2}\left(Q^{2}\right) x_{\mathrm{Bj}}^{-\varepsilon_{2}}, \tag{E.2}
\end{equation*}
$$

where

$$
\begin{align*}
& f_{0}\left(Q^{2}\right)=A_{0}\left(\frac{Q^{2}}{1+Q^{2} / Q_{0}^{2}}\right)^{1+\varepsilon_{0}}\left(1+Q^{2} / Q_{0}^{2}\right)^{\varepsilon_{0} / 2}  \tag{E.3}\\
& f_{1}\left(Q^{2}\right)=A_{1}\left(\frac{Q^{2}}{1+Q^{2} / Q_{1}^{2}}\right)^{1+\varepsilon_{1}}  \tag{E.4}\\
& f_{2}\left(Q^{2}\right)=A_{2}\left(\frac{Q^{2}}{1+Q^{2} / Q_{2}^{2}}\right)^{1+\varepsilon_{2}} \tag{E.5}
\end{align*}
$$

with parameters values (using GeV units for all mass scales) [146]:

$$
\begin{align*}
& A_{0}=0.00151, \quad A_{1}=0.658, \quad A_{2}=1.01, \\
& Q_{0}^{2}=7.85, \quad Q_{1}^{2}=0.6, \quad Q_{2}^{2}=0.214, \\
& \varepsilon_{0}=0.452, \quad \varepsilon_{1}=0.0667, \quad \varepsilon_{2}=-0.476 . \tag{E.6}
\end{align*}
$$

The $F_{1}$ structure function in the high-energy region is then obtained as

$$
\begin{equation*}
F_{1}\left(\nu_{\gamma}, Q^{2}\right) \underset{\nu_{\gamma} \gg}{\longrightarrow} \frac{M \nu_{\gamma}}{Q^{2}} \frac{F_{2}\left(\nu_{\gamma}, Q^{2}\right)}{1+R}, \tag{E.7}
\end{equation*}
$$

where $R \equiv \sigma_{L}^{\gamma p} / \sigma_{T}^{\gamma p}$ is the ratio of longitudinal to transverse virtual photon absorption cross sections on a proton. We will use the experimental result $R_{0}=0.23 \pm 0.04$ at $Q^{2}>1.5 \mathrm{GeV}^{2}$ from the H1 and ZEUS Collaborations [147], and approximate $R$ in our numerical estimates by the following expression, independent of $W^{2} \equiv 2 M \nu_{\gamma}+M^{2}-Q^{2}$ :

$$
\begin{equation*}
R=R\left(Q^{2}\right)=R_{0} \Theta\left(Q^{2}-1.5 \mathrm{GeV}^{2}\right)+R_{\mathrm{BC}}\left(Q^{2}\right) \Theta\left(-Q^{2}+1.5 \mathrm{GeV}^{2}\right), \tag{E.8}
\end{equation*}
$$

where $R_{\mathrm{BC}}\left(Q^{2}\right)$ is value obtained in the Christy and Bosted fit [91] evaluated at $W^{2} \approx$ $2.63 \mathrm{GeV}^{2} .{ }^{1}$ The latter corresponds with the $W^{2}$ value for which the ratio $R$ from the BC fit $R_{\mathrm{BC}}\left(Q^{2}=1.5 \mathrm{GeV}^{2}\right) \approx 0.23$, and thus goes over into the H1/ZEUS value at $Q^{2}>1.5 \mathrm{GeV}^{2}$. We use the relative uncertainties from the data of Ref. [147] in the whole $Q^{2}$ region. We show the resulting functional form of $R\left(Q^{2}\right)$ in Fig. E.1, and compare its value with the data from Refs. [215-217] in the range $Q^{2}<1.5 \mathrm{GeV}^{2}$. We notice that our parameterization of $R$ yields a good agreement with the data.


Figure E.1: $Q^{2}$ dependence of the ratio $R=\sigma_{L}^{\gamma p} / \sigma_{T}^{\gamma p}$. The experimental result $R_{0}=0.23 \pm 0.04$ in the region $Q^{2}>1.5 \mathrm{GeV}^{2}$ from H1 and ZEUS [147] is connected with the ratio taken from the BC fit [91] (central curve). The error band reflects the experimental uncertainty in the value of $R$ from the fit to the H1 and ZEUS data. The data points are taken from Refs. [215-217].

Adopting the above Regge parameterization for $F_{2}$, with the ratio $R$ from Eq. (E.8), we obtain from Eq. (E.7) for $F_{1}$ the following Regge pole residues entering Eq. (E.1):

$$
\begin{equation*}
\gamma_{1+\varepsilon_{i}}\left(Q^{2}\right)=\frac{1}{2} \frac{f_{i}\left(Q^{2}\right)}{1+R\left(Q^{2}\right)}\left(\frac{2 M}{Q^{2}}\right)^{1+\varepsilon_{i}} . \tag{E.9}
\end{equation*}
$$

[^22]
## Appendix F

## Relations between TPE amplitudes in the forward limit

In this Appendix, we study the forward limit of the invariant amplitudes beyond the OPE approximation, and contributions with $Q^{2}=0$ poles, exploiting the helicity amplitudes expressions through the invariant amplitudes of Eqs. (3.11). The $Q^{2}$-expansion of coefficients allows to obtain the following expressions for the helicity amplitudes:

$$
\begin{align*}
\frac{T_{1}}{e^{2}}= & 2 \mathcal{G}_{M}+\frac{4 M^{2}}{Q^{2}}\left(\frac{\omega}{M}\left(\mathcal{G}_{M}-\mathcal{F}_{2}+\frac{\omega}{M} \mathcal{F}_{3}\right)+\frac{m^{2}}{M^{2}}\left(\mathcal{F}_{4}+\frac{\omega}{M} \mathcal{F}_{5}\right)\right) \\
& -\frac{4 s}{\Sigma_{s}}\left(M \omega\left(\mathcal{G}_{M}-\mathcal{F}_{2}+\frac{\omega}{M} \mathcal{F}_{3}\right)+m^{2}\left(\mathcal{F}_{4}+\frac{\omega}{M} \mathcal{F}_{5}\right)\right)+\mathrm{O}\left(Q^{2}\right),  \tag{F.1}\\
\frac{T_{2}}{e^{2}}= & \frac{4 M}{Q} \frac{\left(m^{2}+M \omega\right)\left(\mathcal{G}_{M}+\frac{\omega}{M} \mathcal{F}_{3}+\frac{m^{2}}{M^{2}} \mathcal{F}_{5}\right)+\left(M^{2}+M \omega\right)\left(\frac{m^{2}}{M^{2}} \mathcal{F}_{4}-\frac{\omega}{M} \mathcal{F}_{2}\right)}{\sqrt{\Sigma_{s}}}+\mathrm{O}(Q), \\
T_{3}= & T_{1}-2 e^{2} \mathcal{G}_{M},  \tag{F.2}\\
\frac{T_{4}}{e^{2}}= & -\frac{4 m}{Q} \frac{\left(M^{2}+M \omega\right)\left(\mathcal{G}_{M}-\mathcal{F}_{2}+\frac{\omega}{M} \mathcal{F}_{3}\right)+\left(m^{2}+M \omega\right)\left(\mathcal{F}_{4}+\frac{\omega}{M} \mathcal{F}_{5}\right)}{\sqrt{\Sigma_{s}}}+\mathrm{O}(Q), \quad \text { (F.1 }  \tag{F.3}\\
\frac{T_{5}}{e^{2}}= & 4 \frac{M m(M+\omega)^{2} \mathcal{F}_{2}-m(M+\omega)\left(m^{2}+M \omega\right)\left(\mathcal{F}_{3}+\mathcal{F}_{4}\right)-\frac{m}{M}\left(m^{2}+M \omega\right)^{2} \mathcal{F}_{5}}{\Sigma_{s}}  \tag{F.4}\\
& -\frac{4 M m s \mathcal{G}_{M}}{\Sigma_{s}}-\frac{m}{M} \mathcal{F}_{6},  \tag{F.5}\\
T_{6}= & -T_{5}-\frac{2 m}{M} e^{2} \mathcal{F}_{6} . \tag{F.6}
\end{align*}
$$

The unitarity, $\left|T_{i}\right| \lesssim 1$, and the amplitudes expressions of Eqs. (3.13) provide the convergent low- $Q^{2}$ behavior of all invariant amplitudes entering Eqs. (F.1-F.6). The absence of the divergent in $Q$ terms in all helicity amplitudes leads to

$$
\begin{align*}
& \frac{\omega}{M}\left(\mathcal{G}_{M}-\mathcal{F}_{2}+\frac{\omega}{M} \mathcal{F}_{3}\right)+\frac{m^{2}}{M^{2}}\left(\mathcal{F}_{4}+\frac{\omega}{M} \mathcal{F}_{5}\right)=0  \tag{F.7}\\
& \left(m^{2}+M \omega\right)\left(\mathcal{G}_{M}+\frac{\omega}{M} \mathcal{F}_{3}+\frac{m^{2}}{M^{2}} \mathcal{F}_{5}\right)+M(M+\omega)\left(\frac{m^{2}}{M^{2}} \mathcal{F}_{4}-\frac{\omega}{M} \mathcal{F}_{2}\right)=0  \tag{F.8}\\
& \left(M^{2}+M \omega\right)\left(\mathcal{G}_{M}-\mathcal{F}_{2}+\frac{\omega}{M} \mathcal{F}_{3}\right)+\left(m^{2}+M \omega\right)\left(\mathcal{F}_{4}+\frac{\omega}{M} \mathcal{F}_{5}\right)=0 \tag{F.9}
\end{align*}
$$

These equations and the conservation of the total angular momentum, i.e., $T_{6}\left(Q^{2} \rightarrow 0\right)=0$, allow to write down four model-independent relations for the lepton-proton scattering amplitudes
in the forward limit beyond the contributions with $Q^{2}=0$ pole:

$$
\begin{align*}
\mathcal{G}_{1}\left(\nu, Q^{2}=0\right)= & \mathcal{G}_{M}\left(\nu, Q^{2}=0\right)+\frac{\nu}{M^{2}} \mathcal{F}_{3}\left(\nu, Q^{2}=0\right)+\frac{m^{2}}{M^{2}} \mathcal{F}_{5}\left(\nu, Q^{2}=0\right)=0, \\
\mathcal{G}_{2}\left(\nu, Q^{2}=0\right)= & \mathcal{G}_{M}\left(\nu, Q^{2}=0\right)-\mathcal{F}_{2}\left(\nu, Q^{2}=0\right)+\frac{\nu}{M^{2}} \mathcal{F}_{3}\left(\nu, Q^{2}=0\right)=0,  \tag{F.10}\\
\mathcal{G}_{4}\left(\nu, Q^{2}=0\right)= & \mathcal{F}_{4}\left(\nu, Q^{2}=0\right)+\frac{\nu}{M^{2}} \mathcal{F}_{5}\left(\nu, Q^{2}=0\right)=0,  \tag{F.12}\\
& \mathcal{F}_{3}\left(\nu, Q^{2}=0\right)-\mathcal{F}_{4}\left(\nu, Q^{2}=0\right)+\mathcal{F}_{6}\left(\nu, Q^{2}=0\right)=0 . \tag{F.13}
\end{align*}
$$

The following relations between two amplitudes are valid:

$$
\begin{array}{r}
\nu \mathcal{F}_{2}\left(\nu, Q^{2}=0\right)-m^{2} \mathcal{F}_{4}\left(\nu, Q^{2}=0\right)=0, \\
M^{2} \mathcal{F}_{2}\left(\nu, Q^{2}=0\right)+m^{2} \mathcal{F}_{5}\left(\nu, Q^{2}=0\right)=0 . \tag{F.15}
\end{array}
$$

The TPE amplitudes $\mathcal{G}_{M}^{2 \gamma}\left(\nu, Q^{2}=0\right), \mathcal{F}_{6}^{2 \gamma}\left(\nu, Q^{2}=0\right)$ in the forward limit are directly related to the forward amplitudes $f_{-}^{2 \gamma}, g^{2 \gamma}$ by means of Eqs. (3.15, 3.16). Consequently, all TPE amplitudes in the forward limit can be reconstructed from the experimental data on the proton spin structure functions.

In order to obtain the spin-independent forward amplitude $f_{+}^{2 \gamma}$ one should subtract the divergent in $Q$ behavior in a proper way that requires additional studies in general case. The knowledge of the $Q^{2}$-slope of the invariant amplitudes $\mathcal{G}_{M}^{2 \gamma}, \mathcal{F}_{2}^{2 \gamma}, \mathcal{F}_{3}^{2 \gamma}, \mathcal{F}_{4}^{2 \gamma}, \mathcal{F}_{5}^{2 \gamma}$ is also required:

$$
\begin{equation*}
f_{+}^{2 \gamma}(\omega)=f_{-}^{2 \gamma}(\omega)+4 M \omega\left(\mathcal{G}_{2}^{2 \gamma}+\frac{m^{2}}{M \omega} \mathcal{G}_{4}^{2 \gamma}\right)^{\prime}=f_{-}^{2 \gamma}(\omega)+\left.2 M \omega \frac{\mathrm{~d} \delta_{2 \gamma}\left(\omega, Q^{2}\right)}{\mathrm{d} Q^{2}}\right|_{Q^{2}=0} . \tag{F.16}
\end{equation*}
$$

The amplitudes $Q^{2}$-slope term is just a slope of the TPE correction $\delta_{2 \gamma}$ of Eqs. (3.36, 5.20). The evaluation of the TPE correction to the Lamb shift within Eq. (2.98) requires to take the second limit $\omega \rightarrow m$ and can be performed only when limits $Q^{2} \rightarrow 0$ and $\omega \rightarrow m$ commute.

## Appendix G

## Relations between TPE amplitudes in the high-energy limit

We study the high-energy limit, corresponding to $\nu \rightarrow \infty$, of the invariant amplitudes beyond the OPE approximation, exploiting the invariant amplitudes expressions of Eqs. (3.13) and the amplitudes definitions of Eqs. (3.37-3.40). The leading terms in the $\nu \rightarrow \infty$ expansion are given by

$$
\begin{align*}
e^{2} \mathcal{G}_{M} & =\frac{1}{2}\left(T_{1}-T_{3}\right)  \tag{G.1}\\
e^{2} \mathcal{F}_{2} & =-\frac{M}{2 \nu}\left(Q T_{2}+m\left(T_{5}-T_{6}\right)\right)+\mathrm{O}\left(\frac{1}{\nu^{2}}\right)  \tag{G.2}\\
e^{2} \mathcal{F}_{3} & =-\frac{M^{2}}{2 \nu}\left(T_{1}-T_{3}\right)+\mathrm{O}\left(\frac{1}{\nu^{2}}\right)  \tag{G.3}\\
e^{2} \mathcal{F}_{4} & =-\frac{M}{2 m}\left(T_{5}-T_{6}\right)+\mathrm{O}\left(\frac{1}{\nu}\right)  \tag{G.4}\\
e^{2} \mathcal{F}_{5} & =-\frac{M^{2}}{2 m \nu}\left(Q T_{4}-M\left(T_{5}-T_{6}\right)\right)+\mathrm{O}\left(\frac{1}{\nu^{2}}\right)  \tag{G.5}\\
e^{2} \mathcal{F}_{6} & =-\frac{M}{2 m}\left(T_{5}+T_{6}\right)  \tag{G.6}\\
e^{2} \mathcal{G}_{1} & =-\frac{Q}{8 \nu}\left(4 M T_{2}-Q\left(T_{1}+T_{3}\right)\right)+\mathrm{O}\left(\frac{1}{\nu^{2}}\right)  \tag{G.7}\\
e^{2} \mathcal{G}_{2} & =\frac{Q}{8 M \nu}\left(M Q\left(T_{1}+T_{3}\right)+Q^{2} T_{2}+4 M m T_{4}+m Q\left(T_{5}-T_{6}\right)\right)+\mathrm{O}\left(\frac{1}{\nu^{2}}\right)  \tag{G.8}\\
e^{2} \mathcal{G}_{3} & =-e^{2} \mathcal{G}_{M}+\mathrm{O}\left(\frac{1}{\nu}\right)  \tag{G.9}\\
e^{2} \mathcal{G}_{4} & =-\frac{Q}{8 M m\left(1+\tau_{P}\right)}\left(4 M T_{4}+Q\left(T_{5}-T_{6}\right)\right)+\mathrm{O}\left(\frac{1}{\nu}\right) \tag{G.10}
\end{align*}
$$

Accounting for the unitarity condition $\left|T_{i}\right| \lesssim 1$, we obtain the constraints of Eqs. (3.21-3.23, $3.43,3.44)$ on the HE behavior of the invariant amplitudes:

$$
\begin{align*}
\mathcal{F}_{2}\left(\nu \rightarrow \infty, Q^{2}\right), \mathcal{F}_{3}\left(\nu \rightarrow \infty, Q^{2}\right), \mathcal{F}_{5}\left(\nu \rightarrow \infty, Q^{2}\right) & \lesssim \frac{1}{\nu},  \tag{G.11}\\
\mathcal{G}_{M}\left(\nu \rightarrow \infty, Q^{2}\right)+\frac{\nu}{M^{2}} \mathcal{F}_{3}\left(\nu \rightarrow \infty, Q^{2}\right) & \lesssim \frac{1}{\nu}  \tag{G.12}\\
\mathcal{G}_{M}\left(\nu \rightarrow \infty, Q^{2}\right), \mathcal{F}_{4}\left(\nu \rightarrow \infty, Q^{2}\right), \mathcal{F}_{6}\left(\nu \rightarrow \infty, Q^{2}\right) & \lesssim \text { const, }  \tag{G.13}\\
\mathcal{G}_{1}\left(\nu \rightarrow \infty, Q^{2}\right), \mathcal{G}_{2}\left(\nu \rightarrow \infty, Q^{2}\right) & \lesssim \frac{1}{\nu},  \tag{G.14}\\
\mathcal{G}_{3}\left(\nu \rightarrow \infty, Q^{2}\right), \mathcal{G}_{4}\left(\nu \rightarrow \infty, Q^{2}\right) & \lesssim \text { const. } \tag{G.15}
\end{align*}
$$

In the model with point-like proton described in Eqs. (4.3), these constraints are valid for $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ and $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structures. However, the amplitudes $\mathcal{G}_{M}, \mathcal{F}_{3}, \mathcal{G}_{1}, \mathcal{G}_{2}$ violate unitarity
in the case of $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure. In the proton model with dipole electromagnetic form factors described in Eqs. (3.52), these relations are always violated by the imaginary part of the amplitudes $\mathcal{G}_{1}, \mathcal{G}_{2}$, by the imaginary part of the amplitude $\mathcal{F}_{2}$ in the case of $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structure ${ }^{1}$ and by the real part of the amplitudes $\mathcal{G}_{M}, \mathcal{F}_{3}$ in the case of $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure. In case of the inelastic intermediate states, these relations are expected to be valid.

[^23]
## Appendix H

## Box diagram results in terms of the LoopTools integrals: massless lepton scattering

We describe the details of the box diagram calculation for the point-like model below. The helicity amplitude from the direct TPE diagram $T_{\text {dir }}$ is given by Eqs. (4.1) and has the following structure:

$$
\begin{gather*}
T_{\mathrm{dir}}=N^{0} A_{\mathrm{dir}}+N_{\alpha} A_{\mathrm{dir}}^{\alpha}+N_{\alpha \beta} A_{\mathrm{dir}}^{\alpha \beta}+N_{\alpha \beta \gamma} A_{\mathrm{dir}}^{\alpha \beta \gamma}+N_{\alpha \beta \gamma \delta} A_{\mathrm{dir}}^{\alpha \beta \gamma \delta},  \tag{H.1}\\
\left(A_{\mathrm{dir}}, A_{\mathrm{dir}}^{\alpha}, A_{\mathrm{dir}}^{\alpha \beta}, A_{\mathrm{dir}}^{\alpha \beta \gamma}, A_{\mathrm{dir}}^{\alpha \beta \gamma \delta}\right)=i \int \frac{\mathrm{~d}^{4} k_{1}}{(2 \pi)^{4}} \\
\frac{\left(1, k_{1}^{\alpha}, k_{1}^{\alpha} k_{1}^{\beta}, k_{1}^{\alpha} k_{1}^{\beta} k_{1}^{\gamma}, k_{1}^{\alpha} k_{1}^{\beta} k_{1}^{\gamma} k_{1}^{\delta}\right)}{\left(\left(k_{1}-P-K\right)^{2}-M^{2}\right)\left(k_{1}^{2}-m^{2}\right)\left(\left(k_{1}-K-\frac{q}{2}\right)^{2}-\mu^{2}\right)\left(\left(k_{1}-K+\frac{q}{2}\right)^{2}-\mu^{2}\right)}, \tag{H.2}
\end{gather*}
$$

with $N$ - spinor expression with free indices. The contraction is done with momenta from the expansion of the integrals $A_{\mathrm{dir}}, A_{\mathrm{dir}}^{\alpha}, \ldots$ in terms of the on-shell momenta. These integrals are invariant under the replacement $q \rightarrow-q$ and can be expressed as

$$
\begin{align*}
A_{\mathrm{dir}}^{\alpha} & =a_{s}(P+K)^{\alpha}+a_{P} P^{\alpha} \\
A_{\mathrm{dir}}^{\alpha \beta} & =a_{s s}(P+K)^{\alpha}(P+K)^{\beta}+a_{s P} P^{[\alpha,}(P+K)^{\beta]}+a_{P P} P^{\alpha} P^{\beta}+a_{q q} q^{\alpha} q^{\beta}+a_{00} g^{\alpha \beta} \\
A_{\mathrm{dir}}^{\alpha \beta \gamma} & =a_{s s s}(P+K)^{\alpha}(P+K)^{\beta}(P+K)^{\gamma}+a_{P P P} P^{\alpha} P^{\beta} P^{\gamma}+a_{s 00} g^{[\alpha, \beta,}(P+K)^{\gamma]} \\
& +a_{s s P} P^{[\alpha,}(P+K)^{\beta,}(P+K)^{\gamma]}+a_{s P P} P^{[\alpha,} P^{\beta,}(P+K)^{\gamma]}+a_{P q q} q^{[\alpha,} q^{\beta,} P^{\gamma]} \\
& +a_{P 00} g^{[\alpha, \beta,} P^{\gamma]}+a_{s q q} q^{[\alpha,} q^{\beta,}(P+K)^{\gamma]}, \\
A_{\mathrm{dir}}^{\alpha \beta \gamma \delta} & =a_{s s s s}(P+K)^{\alpha}(P+K)^{\beta}(P+K)^{\gamma}(P+K)^{\delta}+a_{P P P P} P^{\alpha} P^{\beta} P^{\gamma} P^{\delta}+a_{q q q q} q^{\alpha} q^{\beta} q^{\gamma} q^{\delta} \\
& +a_{s s s P} P^{[\alpha,}(P+K)^{\beta,}(P+K)^{\gamma,}(P+K)^{\delta]}+a_{s P P P} P^{[\alpha,} P^{\beta,} P^{\gamma,}(P+K)^{\delta]} \\
& +a_{s s P P} P^{[\alpha,} P^{\beta,}(P+K)^{\gamma,}(P+K)^{\delta]}+a_{0000} g^{[\alpha, \beta,} g^{\gamma, \delta]}+a_{s P 00} g^{[\alpha, \beta,}(P+K)^{\gamma,} P^{\delta]} \\
& +a_{P P 00} g^{[\alpha, \beta,} P^{\gamma,} P^{\delta]}+a_{q q 00} g^{[\alpha, \beta,} q^{\gamma,} q^{\delta]}+a_{s s 00} g^{[\alpha, \beta,}(P+K)^{\gamma,}(P+K)^{\delta]} \\
& +a_{s P q q}(P+K)^{[\alpha,} P^{\beta,} q^{\gamma,} q^{\delta]}+a_{P P q q} q^{[\alpha,} q^{\beta,} P^{\gamma,} P^{\delta]}+a_{s s q q} q^{[\alpha,} q^{\beta,}(P+K)^{\gamma,}(P+K)^{\delta]}, \tag{H.3}
\end{align*}
$$

where all non-equivalent permutations are accounted only once. The integrals $a_{s}, a_{p}, \ldots$ can be expressed through the LoopTools four-point functions:

$$
\begin{equation*}
D\left(p_{1}^{2}, p_{2}^{2}, p_{3}^{2}, p_{4}^{2},\left(p_{1}+p_{2}\right)^{2},\left(p_{2}+p_{3}\right)^{2}, m_{1}^{2}, m_{2}^{2}, m_{3}^{2}, m_{4}^{2}\right) \tag{H.4}
\end{equation*}
$$

with the following kinematics:

$$
\begin{gather*}
m_{1}=m, \quad m_{2}=\mu, \quad m_{3}=M, \quad m_{4}=\mu \\
p_{1}=-k^{\prime}, \quad p_{2}=-p^{\prime}, \quad p_{3}=p, \quad p_{4}=k \\
p_{12} \equiv\left(p_{1}+p_{2}\right)^{2}=s, \quad p_{23} \equiv\left(p_{2}+p_{3}\right)^{2}=t \\
k_{1}=-k^{\prime}, \quad k_{2}=-(P+K), \quad k_{3}=-(P+K)+p=-k \tag{H.5}
\end{gather*}
$$

Appendix H Box diagram results in terms of the LoopTools integrals: massless lepton scattering

Exploiting the relations between the LoopTools four-point functions:

$$
\begin{array}{cl}
D_{1}=D_{3}, & D_{11}=D_{33}, \quad D_{12}=D_{23}, \quad D_{001}=D_{003} \\
D_{111}=D_{333}, & D_{113}=D_{133}, \quad D_{112}=D_{233}, \quad D_{223}=D_{122} \tag{H.6}
\end{array}
$$

we express the integrals in the tensor decomposition of Eqs. (H.3) as

$$
\begin{gather*}
a_{P}=2 D_{1}, \quad a_{S}=-2 D_{1}-D_{2}, \quad a_{q q}=\frac{1}{2}\left(D_{11}-D_{13}\right), \\
a_{P P}=2\left(D_{11}+D_{13}\right), \quad a_{S S}=2\left(D_{11}+D_{12}+D_{13}+D_{23}\right)+D_{22}, \\
a_{s P}=-2\left(D_{11}+D_{12}+D_{13}\right), \quad a_{00}=D_{00}, \\
a_{P 00}=2 D_{001}, \quad a_{S 00}=-2 D_{001}-D_{002}, \\
a_{P P P}=2\left(D_{111}+3 D_{113}\right), \quad a_{s s s}=-2 D_{111}-6 D_{112}-6 D_{113}-6 D_{122}-6 D_{123}-D_{222}, \\
a_{s s P}=2\left(D_{111}+2 D_{112}+3 D_{113}+D_{122}+2 D_{123}\right), \\
a_{s P P}=-2\left(D_{111}+D_{112}+3 D_{113}+D_{123}\right), \\
a_{q q s}=\frac{1}{2}\left(-D_{111}-D_{112}+D_{113}+D_{123}\right), a_{q q P}=\frac{1}{2}\left(D_{111}-D_{113}\right), \\
a_{q q 00}=\frac{1}{2}\left(D_{0011}-D_{0013}\right), \\
a_{P P 00}=2\left(D_{0011}+D_{0013}\right), \quad a_{S S 00}=2\left(D_{0011}+D_{0012}+D_{0013}\right)+D_{0022}, \\
a_{s P 00}=-2\left(D_{0011}+D_{0012}+D_{0013}\right), \quad a_{0000}=D_{0000}, \\
a_{s s s s}=D_{1111}+8 D_{1112}+8 D_{1113}+12 D_{1122}+24 D_{1123}, \\
+6 D_{1133}+8 D_{1222}+12 D_{1223}+D_{2222}+D_{3333}, \\
a_{P P P P}=D_{1111}+8 D_{1113}+6 D_{1133}+D_{3333}, \\
a_{q q q q}=\frac{1}{16}\left(D_{1111}-8 D_{1113}+6 D_{1133}+D_{3333}\right), \\
a_{s s s P}=-D_{1111}-6 D_{1112}-8 D_{1113}-6 D_{1122}-18 D_{1123} \\
-6 D_{1133}-2 D_{1222}-6 D_{1223}-D_{3333}, \\
a_{s P P P}=-D_{1111}-2 D_{1112}-8 D_{1113}-6 D_{1123}-6 D_{1133}-D_{3333}, \\
a_{s P q q}=\frac{1}{4}\left(-D_{1111}-2 D_{1112}+2 D_{1123}+2 D_{1133}-D_{3333}\right), \\
a_{P P q q}=\frac{1}{4}\left(D_{1111}-2 D_{1133}+D_{3333}\right), \\
a_{s s P P}=D_{1111}+4 D_{1112}+8 D_{1113}+2 D_{1122}+12 D_{1123}+6 D_{1133}+2 D_{1223}+D_{3333},(\mathrm{H}
\end{gather*}
$$

up to an overall factor $-\pi^{2} /(2 \pi)^{4}$.
Among the LoopTools four-point functions the following are IR divergent:

$$
\begin{equation*}
D_{1}, D_{11}, D_{111}, D_{1111} \tag{H.8}
\end{equation*}
$$

Others are IR free. The only UV divergent four-point function is $D_{0000}$.
The result for the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure of virtual photon-proton-proton vertices for the direct diagram is given by

$$
\begin{align*}
\mathcal{G}_{M}^{2 \gamma} & =e^{2}\left(-2\left(\left(s-M^{2}\right)^{2}+Q^{2} M^{2}\right) a_{p}-2\left(\left(s-M^{2}\right)^{2}+Q^{2} s\right) a_{s}\right. \\
& +12\left(s-M^{2}+\frac{Q^{2}}{3}\right) a_{00}+2\left(\left(s-M^{2}\right)\left(s+3 M^{2}\right)+Q^{2} s\right) a_{s p}-4\left(s-M^{2}\right) Q^{2} a_{q q} \\
& \left.+2\left(s^{2}-M^{4}+Q^{2} s\right) a_{s s}+4\left(s-M^{2}\right)\left(M^{2}+\frac{Q^{2}}{4}\right) a_{p p}\right)  \tag{H.9}\\
\mathcal{F}_{2}^{2 \gamma} & =-2 M^{2} e^{2} Q^{2}\left(a_{p p}+a_{s p}\right) \tag{H.10}
\end{align*}
$$

$$
\begin{align*}
\mathcal{F}_{3}^{2 \gamma} & =e^{2}\left(4\left(s-M^{2}\right) M^{2} a_{p}+4\left(s-M^{2}\right) M^{2} a_{s}-8 M^{2}\left(M^{2}+\frac{Q^{2}}{4}\right) a_{p p}\right. \\
& \left.-4\left(s+M^{2}\right) M^{2} a_{s s}-24 M^{2} a_{00}-4\left(s+3 M^{2}\right) M^{2} a_{s p}+8 Q^{2} M^{2} a_{q q}\right) \tag{H.11}
\end{align*}
$$

The result for the $F_{D} F_{P}$ vertex structure for the direct diagram is given by

$$
\begin{align*}
\mathcal{G}_{M}^{2 \gamma} & =e^{2}\left(4\left(M^{2}+\frac{Q^{2}}{4}\right)\left(s-M^{2}-\frac{Q^{2}}{2}\right) a_{P P P}+4 s\left(s-M^{2}\right) a_{s s s}-4 Q^{2}\left(s-M^{2}\right) a_{s q q}\right. \\
& +\left(\left(s+M^{2}\right)\left(4 s-Q^{2}\right)-8 M^{2}\left(M^{2}+\frac{Q^{2}}{4}\right)\right) a_{s P P}+12\left(2\left(s-M^{2}\right)-Q^{2}\right) a_{P 0} \\
& \left.-2 Q^{2}\left(2\left(s-M^{2}\right)-Q^{2}\right) a_{P q q}+\left(2 s\left(4 s-Q^{2}\right)-4 M^{2}\left(s+M^{2}\right)\right) a_{s s P}\right) \\
& -e^{2}\left(Q^{2}\left(s+M^{2}\right) a_{p}+2 Q^{2} s a_{s}+\left(4\left(\left(s-M^{2}\right)^{2}-Q^{2} s\right)+Q^{2}\left(M^{2}+\frac{Q^{2}}{4}\right)\right) a_{p p}\right. \\
& \left.+\left(4\left(s-M^{2}\right)^{2}-3 Q^{2} s\right) a_{s s}-\left(8\left(s-M^{2}\right)+4 Q^{2}\right) a_{00}\right) \\
& -e^{2}\left(\left(8\left(s-M^{2}\right)^{2}+7 Q^{2} s-Q^{2} M^{2}\right) a_{s p}+48 Q^{2}\left(s-M^{2}\right) a_{s 0}\right),  \tag{H.12}\\
\mathcal{F}_{2}^{2 \gamma} & =e^{2}\left(-Q^{2}\left(s+M^{2}\right) a_{p}-2 Q^{2} s a_{s}+Q^{2}\left(2 s+M^{2}+\frac{Q^{2}}{4}\right) a_{p p}+\right. \\
& \left.+3 Q^{2} s a_{s s}+8 Q^{2} a_{00}+6 Q^{2} s a_{s p}-Q^{4} a_{q q}\right) \\
& -e^{2}\left(3 Q^{2}\left(M^{2}+\frac{Q^{2}}{4}\right) a_{P P P}+Q^{2} s a_{s s s}+Q^{2}\left(4 s+M^{2}\right) a_{s s P}\right. \\
& \left.+Q^{2}\left(3 s+4 M^{2}+\frac{Q^{2}}{4}\right) a_{s P P}+6 Q^{2} a_{s 0}+18 Q^{2} a_{P 0}-3 Q^{4} a_{P q q}-Q^{4} a_{s q q}\right)  \tag{H.13}\\
\mathcal{F}_{3}^{2 \gamma} & =2 M^{2} e^{2}\left(2\left(2\left(s-M^{2}\right)-Q^{2}\right) a_{p p}+4\left(s-M^{2}\right) a_{s s}-8 a_{00}+\left(8\left(s-M^{2}\right)-Q^{2}\right) a_{s p}\right) \\
& -8 M^{2} e^{2}\left(\left(M^{2}+\frac{Q^{2}}{4}\right) a_{P P P}+s a_{s s s}+\left(2 s+M^{2}\right) a_{s s P}+\left(s+2 M^{2}+\frac{Q^{2}}{4}\right) a_{s P P}\right. \\
& \left.+6 a_{s 0}+6 a_{P 0}-Q^{2} a_{P q q}-Q^{2} a_{s q q}-Q^{2} a_{q q}\right) . \tag{H.14}
\end{align*}
$$

The result for the $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure for the direct diagram is given by

$$
\begin{aligned}
\mathcal{G}_{M}^{2 \gamma} & =e^{2}\left(\frac{18\left(s-M^{2}-\frac{Q^{2}}{4}\right)}{M^{2}} a_{0000}-11 \frac{Q^{2}\left(s-M^{2}-\frac{19 Q^{2}}{44}\right)}{M^{2}} a_{q q 00}\right. \\
& +11 \frac{\left(M^{2}+\frac{Q^{2}}{4}\right)\left(s-M^{2}-\frac{19 Q^{2}}{44}\right)}{M^{2}} a_{P P 00} \\
& +\frac{1}{2}\left(\frac{2 s\left(s+M^{2}\right)^{2}}{M^{2}}-\left(s-M^{2}\right)^{2}-8 M^{2} s-Q^{2} s\right) a_{s s s P} \\
& +\left(\frac{7 s^{2}-\frac{3 Q^{2} s}{4}}{M^{2}}-3 s-4 M^{2}\right) a_{S S 00}+\left(\frac{7 s^{2}-\frac{3 Q^{2} s}{4}}{M^{2}}+8 s-15 M^{2}-\frac{19 Q^{2}}{4}\right) a_{s P 00} \\
& +\frac{\left(M^{2}+\frac{Q^{2}}{4}\right)^{2}\left(2\left(s-M^{2}\right)-Q^{2}\right)}{2 M^{2}} a_{P P P P}+\frac{Q^{4}\left(2 s-2 M^{2}-Q^{2}\right)}{2 M^{2}} a_{q q q q} \\
& +\frac{\left(M^{2}+\frac{Q^{2}}{4}\right)\left(3 s^{2}+2 M^{2} s-5 M^{4}-Q^{2}\left(s+2 M^{2}\right)\right)}{2 M^{2}} a_{s P P P}+\frac{s\left(s^{2}-M^{4}\right)}{2 M^{2}} a_{s s s s}
\end{aligned}
$$

Appendix H Box diagram results in terms of the LoopTools integrals: massless lepton scattering

$$
\begin{align*}
& -\frac{Q^{2}\left(M^{2}+\frac{Q^{2}}{4}\right)\left(2 s-2 M^{2}-Q^{2}\right)}{M^{2}} a_{P P q q}-\frac{Q^{2}\left(3 s^{2}-2 M^{2} s-M^{4}-Q^{2} s\right)}{2 M^{2}} a_{s s q q} \\
& -\frac{Q^{2}\left(3 s^{2}+2 M^{2} s-5 M^{4}-Q^{2}\left(s+2 M^{2}\right)\right)}{2 M^{2}} a_{s P q q} \\
& \left.+\frac{s\left(s+Q^{2}\right)\left(4 s-Q^{2}\right)+2 M^{2} s\left(12 s-5 Q^{2}\right)-M^{4}\left(12 s+5 Q^{2}\right)-16 M^{6}}{8 M^{2}} a_{s s P P}\right) \\
& +e^{2}\left(-\frac{s\left(8 s^{2}+s Q^{2}-3 Q^{4}\right)+2 M^{2} s\left(6 s-11 Q^{2}\right)-M^{4}\left(48 s-9 Q^{2}\right)+28 M^{6}}{8 M^{2}} a_{s P P}\right. \\
& -\frac{\left(M^{2}+\frac{Q^{2}}{4}\right)\left(3\left(s-M^{2}\right)^{2}+Q^{2} M^{2}-2 Q^{2} s\right)}{2 M^{2}} a_{P P P}-\frac{s\left(3\left(s-M^{2}\right)^{2}-Q^{2} s\right)}{2 M^{2}} a_{s s s} \\
& -\frac{s^{2}\left(5 s-2 Q^{2}\right)-M^{2} s\left(6 s+Q^{2}\right)-3 M^{4} s+4 M^{6}}{2 M^{2}} a_{s s P} \\
& -\frac{\left(11 s^{2}-24 M^{2} s+13 M^{4}-5 Q^{2} s\right)}{M^{2}} a_{s 00}-\frac{\left(s-M^{2}\right)\left(7 s-9 M^{2}-4 Q^{2}\right)}{M^{2}} a_{P 00} \\
& \left.+\frac{Q^{2}\left(3\left(s-M^{2}\right)^{2}-Q^{2}\left(2 s-M^{2}\right)\right)}{2 M^{2}} a_{P q q}+\frac{Q^{2}\left(5\left(s-M^{2}\right)^{2}-3 Q^{2} s\right)}{2 M^{2}} a_{s q q}\right) \\
& +e^{2}\left(-\frac{\left(\left(s-M^{2}\right)^{3}-s Q^{2}\left(s-M^{2}\right)\right)}{2 M^{2}} a_{p}+\frac{\left(s-M^{2}\right)\left(3\left(s-M^{2}\right)^{2}-2 Q^{2} s\right)}{2 M^{2}} a_{s s}\right. \\
& +\frac{4\left(s-M^{2}\right)^{3}-2 Q^{2}\left(s-M^{2}\right)^{2}-Q^{4}\left(s+M^{2}\right)}{8 M^{2}} a_{p p}-\frac{\left(s-M^{2}\right)^{3}-Q^{2} s\left(s-M^{2}\right)}{2 M^{2}} a_{s} \\
& \left.+\frac{\left(s-M^{2}\right)\left(4\left(s-M^{2}\right)^{2}-Q^{2}\left(3 s-M^{2}\right)\right)}{2 M^{2}} a_{s p}+\frac{4\left(s-M^{2}\right)^{2}-Q^{2}\left(3 s-M^{2}\right)}{M^{2}} a_{00}\right), \\
& \mathcal{F}_{2}^{2 \gamma}=-e^{2}\left(6 Q^{2} a_{P P 00}+2 Q^{2} a_{S S 00}+8 Q^{2} a_{s P 00}+\frac{Q^{2} s}{4} a_{s s s s}+\frac{3 Q^{2}}{4}\left(M^{2}+\frac{Q^{2}}{4}\right) a_{P P P P}\right.  \tag{H.15}\\
& +\frac{Q^{2}}{4}\left(5 s+M^{2}\right) a_{s s s P}+\frac{Q^{2}}{4}\left(3 s+7 M^{2}+Q^{2}\right) a_{s P P P}-Q^{4} a_{s P q q}-\frac{3 Q^{4}}{4} a_{P P q q} \\
& \left.+\frac{Q^{2}}{16}\left(28 s+20 M^{2}+Q^{2}\right) a_{s s P P}-\frac{Q^{4}}{4} a_{s s q q}\right) \\
& +e^{2}\left(\frac{Q^{2}}{8}\left(4 s-Q^{2}\right) a_{P P P}+\frac{Q^{2}}{2}\left(2 s+M^{2}\right) a_{s s P}+\frac{Q^{2}}{2}\left(2 s+M^{2}+\frac{Q^{2}}{4}\right) a_{s P P}\right. \\
& \left.+3 Q^{2} a_{s 00}-2 Q^{2} a_{P 00}+\frac{Q^{4}}{2} a_{P q q}-\frac{Q^{4}}{2} a_{s q q}+\frac{Q^{2} s}{2} a_{s s s}\right) \\
& +e^{2}\left(\frac{Q^{2}}{4}\left(2\left(s-M^{2}\right)-Q^{2}\right) a_{p p}+Q^{4} a_{q q}-2 Q^{2} a_{00}+\frac{Q^{2}}{2}\left(s-M^{2}\right) a_{s p}\right),  \tag{H.16}\\
& \mathcal{F}_{3}^{2 \gamma}=e^{2}\left(-36 a_{0000}+22 Q^{2} a_{q q 00}-22\left(M^{2}+\frac{Q^{2}}{4}\right) a_{P P 00}-2\left(7 s+4 M^{2}-Q^{2}\right) a_{S S 00}\right. \\
& -2\left(7 s+15 M^{2}-Q^{2}\right) a_{s P 00}-s\left(s+M^{2}-\frac{Q^{2}}{4}\right) a_{s s s s}-2\left(M^{2}+\frac{Q^{2}}{4}\right)^{2} a_{P P P P} \\
& -\left(\frac{s}{2}\left(4 s-Q^{2}\right)+\frac{M^{2}}{4}\left(20 s-Q^{2}\right)+M^{4}\right) a_{s s s P}-2 Q^{4} a_{q q q q}
\end{align*}
$$

$$
\begin{align*}
& +\frac{Q^{2}}{4}\left(12 s+20 M^{2}-Q^{2}\right) a_{s P q q}+4 Q^{2}\left(M^{2}+\frac{Q^{2}}{4}\right) a_{P P q q} \\
& \left.-\frac{1}{16}\left(16 s^{2}+8 Q^{2} s+4 M^{2}\left(28 s-Q^{2}\right)+64 M^{4}-Q^{4}\right) a_{s s P P}\right) \\
& -\frac{e^{2}}{4}\left(\left(M^{2}+\frac{Q^{2}}{4}\right)\left(12 s+20 M^{2}-Q^{2}\right) a_{s P P P}-Q^{2}\left(12 s+4 M^{2}-Q^{2}\right) a_{s s q q}\right) \\
& +e^{2}\left(\frac{1}{4}\left(M^{2}+\frac{Q^{2}}{4}\right)\left(12 s-12 M^{2}+5 Q^{2}\right) a_{P P P}+\left(3 s\left(s-M^{2}\right)-\frac{5 Q^{2} s}{4}\right) a_{s s s}\right. \\
& +\left(\frac{5 s}{2}\left(2 s-Q^{2}\right)-M^{2}\left(s+\frac{5 Q^{2}}{4}+4 M^{4}\right)\right) a_{s s P}-Q^{2}\left(3\left(s-M^{2}\right)-\frac{5 Q^{2}}{4}\right) a_{P q q} \\
& +\left(22 s-26 M^{2}-\frac{15 Q^{2}}{2}\right) a_{s 00}+\left(14 s-18 M^{2}-\frac{15 Q^{2}}{2}\right) a_{P 00} \\
& \left.+\left(2 s^{2}+\frac{5}{4} M^{2}\left(4 s-3 Q^{2}\right)-7 M^{4}-\frac{5 Q^{4}}{16}\right) a_{s P P}-\frac{5}{4} Q^{2}\left(4\left(s-M^{2}\right)-Q^{2}\right) a_{s q q}\right) \\
& +e^{2}\left(\left(\left(s-M^{2}\right)^{2}-Q^{2} s\right) a_{p}+\left(\left(s-M^{2}\right)^{2}-Q^{2} s\right) a_{s}\right. \\
& -\left(\left(s-M^{2}+\frac{Q^{2}}{4}\right)^{2}-Q^{2} s-\frac{5 Q^{4}}{16}\right) a_{p p}-\left(4 s\left(s-M^{2}\right)^{2}-3 Q^{2} s\right) a_{s p} \\
& \left.-\left(3\left(s-M^{2}\right)^{2}-2 Q^{2} s\right) a_{s s}-\left(8\left(s-M^{2}\right)-6 Q^{2}\right) a_{00}+Q^{2}\left(2\left(s-M^{2}\right)-Q^{2}\right) a_{q q}\right) . \tag{H.17}
\end{align*}
$$

The crossed diagram contribution to the invariant amplitudes can be obtained from the direct diagram by the replacement $s \rightarrow u$, and with a sign according to the crossing relations of Eqs. (3.24-3.29).

We now describe the details of the calculation for the dipole form of electric and magnetic FFs, see Eqs. (3.52). The Pauli and Dirac FFs have the following expressions:

$$
\begin{align*}
& F_{P}=-\frac{\left(\mu_{P}-1\right) \Lambda^{4} 4 M^{2}}{\left(q^{2}-\Lambda^{2}\right)^{2}\left(q^{2}-4 M^{2}\right)}, \\
& F_{D}=\frac{\mu_{P} \Lambda^{4}}{\left(q^{2}-\Lambda^{2}\right)^{2}}+\frac{\left(\mu_{P}-1\right) \Lambda^{4} 4 M^{2}}{\left(q^{2}-\Lambda^{2}\right)^{2}\left(q^{2}-4 M^{2}\right)} . \tag{H.18}
\end{align*}
$$

The amplitudes for the case of dipole form of electromagnetic FFs can be obtained from the point-like model expressions after differentiation of the photon propagators with respect to the IR parameter $\mu^{2}$ and replacement of this parameter either by $\Lambda^{2}$ or by $4 M^{2}$. The two terms with different order of vector and tensor couplings for the case of the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structure are not the same in this calculation and should be considered separately. Also the tensor expressions for the integrals $A$ are not symmetric under the replacement $q \rightarrow-q$ in case of both terms for the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structure. We perform the Passarino-Veltman decomposition [218] in terms of the LoopTools momenta in this case.

## Appendix I

## Phases entering the unitarity relations

The unitarity relation phases entering Eq. (4.24) can be expressed in terms of the Mandelstam variables as

$$
\begin{align*}
\cos \phi^{\prime} & =\frac{1}{\sqrt{4 Q^{2} Q_{1}^{2} x x_{1} x_{2}}}\left(-Q_{2}^{2}+Q^{2} x+Q_{1}^{2} x_{1}+\frac{s Q_{2}^{2}}{\Sigma_{s}} Q_{2}^{2}\left(x Q_{1}^{2}+x_{1} Q^{2}\right)\right), \\
\cos \tilde{\phi} & =\frac{1}{\sqrt{4 Q^{2} Q_{2}^{2} x x_{2}}}\left(x Q_{2}^{2}+x_{2} Q^{2}-Q_{1}^{2}\right), \\
\cos \tilde{\phi}^{\prime} & =\frac{1}{\sqrt{4 Q_{1}^{2} Q_{2}^{2} x_{1} x_{2}}}\left(-Q^{2}+x_{2} Q_{1}^{2}+x_{1} Q_{2}^{2}\right), \\
\cos \left(\phi-\phi^{\prime}\right) & =\frac{1}{2 x x_{1} x_{2}}\left(x^{2}+x_{1}^{2}+x_{2}^{2}-1+2 \frac{s^{3}}{\Sigma_{s}^{3}} Q^{2} Q_{1}^{2} Q_{2}^{2}\right), \\
\cos (\phi+\tilde{\phi}) & =\frac{1}{\sqrt{4 x_{1} x_{2} Q_{1}^{2} Q_{2}^{2}}} \frac{1}{x}\left(-Q_{1}^{2} x_{1}-Q_{2}^{2} x_{2}+Q^{2}-\frac{s Q^{2}}{\Sigma_{s}}\left(Q_{2}^{2}+Q_{1}^{2}\right) x\right), \tag{I.1}
\end{align*}
$$

with

$$
\begin{align*}
x & \equiv \frac{1}{2}\left(1+\cos \theta_{\mathrm{cm}}\right)=1-\frac{s Q^{2}}{\Sigma_{s}}, \\
x_{1} & \equiv \frac{1}{2}\left(1+\cos \theta_{1}\right)=1-\frac{s Q_{1}^{2}}{\Sigma_{s}} \\
x_{2} & \equiv \frac{1}{2}\left(1+\cos \theta_{2}\right)=1-\frac{s Q_{2}^{2}}{\Sigma_{s}} . \tag{I.2}
\end{align*}
$$

## Appendix J

## Different integration coordinates in the unitarity relations

The boundaries of the ellipse mentioned in Section 4.2.3 correspond to $\cos ^{2} \phi_{1}=0$. Defining $z_{1} \equiv \cos \theta_{1}, z_{2} \equiv \cos \theta_{2}, z \equiv \cos \theta_{\mathrm{cm}}$, the ellipse equation is given by

$$
\begin{equation*}
1-z^{2}-z_{1}^{2}-z_{2}^{2}=-2 z z_{1} z_{2} \tag{J.1}
\end{equation*}
$$

The coordinates $z_{1}, z_{2}$ can be rotated by $45^{0}$, so that the new coordinate system coincides with the axes of the ellipse:

$$
\begin{equation*}
\tilde{z}_{1}=-\frac{1}{\sqrt{2}}\left(z_{1}+z_{2}\right), \quad \tilde{z}_{2}=\frac{1}{\sqrt{2}}\left(z_{1}-z_{2}\right) \tag{J.2}
\end{equation*}
$$

The $\tilde{z}_{2}$-axis corresponds to the line $Q_{1}^{2}=Q_{2}^{2}$, whereas the $\tilde{z}_{1}$-axis corresponds to the line $Q_{2}^{2}=Q_{\max }^{2}-Q_{1}^{2}$. The phase space integration in terms of new coordinates is expressed as

$$
\begin{equation*}
\int \mathrm{d} \Omega=2 \int_{-1}^{1} \mathrm{~d} \cos \theta_{1} \int_{0}^{\pi} \mathrm{d} \phi_{1}=\frac{2}{\sqrt{1-z^{2}}} \int \mathrm{~d} \tilde{z}_{1} \mathrm{~d} \tilde{z}_{2} \frac{1}{|\tilde{\alpha}|} \tag{J.3}
\end{equation*}
$$

with $\tilde{\alpha} \equiv \sin \theta_{1} \sin \phi_{1}$. The ellipse equation is then given by

$$
\begin{equation*}
\frac{\tilde{z}_{1}^{2}}{1+z}+\frac{\tilde{z}_{2}^{2}}{1-z}=1 \tag{J.4}
\end{equation*}
$$

The integration of Eq. J. 3 maps out the whole surface of the ellipse. It is therefore convenient to introduce the elliptic coordinates $\tilde{\alpha}, \phi$ as

$$
\begin{align*}
& \tilde{z}_{1}=\sqrt{1-\tilde{\alpha}^{2}} \sqrt{1+z} \cos (\phi), \\
& \tilde{z}_{2}=\sqrt{1-\tilde{\alpha}^{2}} \sqrt{1-z} \sin (\phi), \tag{J.5}
\end{align*}
$$

which satisfy

$$
\begin{equation*}
\frac{\tilde{z}_{1}^{2}}{1+z}+\frac{\tilde{z}_{2}^{2}}{1-z}=1-\tilde{\alpha}^{2} \tag{J.6}
\end{equation*}
$$

The photons virtualities $Q_{1}^{2}, Q_{2}^{2}$ are symmetric in terms of the elliptic coordinates $\tilde{\alpha}, \phi$. The phase space integration in terms of these elliptic coordinates can then be expressed as

$$
\begin{equation*}
\int \mathrm{d} \Omega=\frac{2}{\sqrt{1-z^{2}}} \int \mathrm{~d} \tilde{z}_{1} \mathrm{~d} \tilde{z}_{2} \frac{1}{|\tilde{\alpha}|}=2 \int_{0}^{1} \mathrm{~d} \tilde{\alpha} \int_{0}^{2 \pi} \mathrm{~d} \phi \tag{J.7}
\end{equation*}
$$

## Appendix K

## Comparison of DR approach and hadronic model for the proton intermediate state in the electron-proton scattering.

We compare the real and imaginary parts of the TPE amplitudes for the typical kinematics of the elastic electron-proton scattering experiments. As it was discussed in Section 4.3 all imaginary parts of the TPE amplitudes in the box graph model coincide with the imaginary parts evaluated with a help of the unitarity relations. The real parts of all TPE amplitudes in the box graph model are the same as the real parts in the unsubtracted DR approach in the case of $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ and $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structures, as it is shown in Figs. K.1-K.6.

We can see from Fig. K. 7 that the real part of the amplitude $\mathcal{G}_{M}^{2 \gamma}$ in the case of $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure differs between the hadronic model and unsubtracted DR evaluation. It also shows the convergence of the DR result at HE limit, when $\varepsilon \rightarrow 1$, and the divergence of $\Re \mathcal{G}_{M}^{F_{P} F_{P}}$ in the hadronic model.


Figure K.1: Imaginary part (left panel) and real part (right panel) of the invariant amplitude $\mathcal{G}_{M}^{2 \gamma}$ for the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}$ vertex structure with dipole FFs in electron-proton scattering for $Q^{2}=0.1 \mathrm{GeV}^{2}$. The vertical line in the left panel corresponds with the boundary between physical and unphysical regions, i.e., $\nu_{\mathrm{ph}}=0.15 \mathrm{GeV}^{2}$.

Appendix $K$ Comparison of $D R$ approach and hadronic model for the proton intermediate state in the electron-proton scattering.


Figure K.2: Same as Fig. K.1, but for the invariant amplitude $\mathcal{F}_{2}^{2 \gamma}$.


Figure K.3: Same as Fig. K.1, but for the invariant amplitude $\mathcal{F}_{3}^{2 \gamma}$.


Figure K.4: Imaginary part (left panel) and real part (right panel) of the invariant amplitude $\mathcal{G}_{M}^{2 \gamma}$ for the $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structure with dipole FFs in electron-proton scattering for $Q^{2}=0.1 \mathrm{GeV}^{2}$. The vertical line in the left panel corresponds with the boundary between physical and unphysical regions, i.e., $\nu_{\mathrm{ph}}=0.15 \mathrm{GeV}^{2}$.


Figure K.5: Same as Fig. K.4, but for the invariant amplitude $\mathcal{F}_{2}^{2 \gamma}$.


Figure K.6: Same as Fig. K.4, but for the invariant amplitude $\mathcal{F}_{3}^{2 \gamma}$.


Figure K.7: Same as Fig. 4.29, but for the invariant amplitude $\mathcal{G}_{M}^{2 \gamma}$.

## Appendix L

## Evaluation of the forward limit of the Feshbach correction

The leading term in the low momentum transfer $(Q)$ expansion of $\delta_{2 \gamma}$ comes from terms of the following type in Eq. (5.36):

$$
\begin{equation*}
\delta_{2 \gamma}^{\mathrm{QED}} \rightarrow-\frac{16 M Q^{2} e^{2}}{\omega} \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \Pi_{P}^{+} \Pi_{P}^{-} \Pi_{K}^{+} \Pi_{K}^{-} \Pi_{Q}^{+} \Pi_{Q}^{-}\left(\tilde{q}^{2}+\frac{Q^{2}}{4}\right)(K \cdot \tilde{q})^{2} . \tag{L.1}
\end{equation*}
$$

We perform the integration in Euclidean space in this Appendix. The Euclidean coordinates $\tilde{q}_{E}$ can be expressed through the Minkowski coordinates $(\tilde{\nu}, \overrightarrow{\tilde{q}})$ by

$$
\begin{equation*}
\tilde{q}_{E}\left(\tilde{q}_{E}^{0}, \overrightarrow{\tilde{q}}_{E}\right)=\tilde{q}_{E}(-i \tilde{\nu}, \overrightarrow{\tilde{q}}), \quad\left(a_{E} \cdot b_{E}\right)=-(a \cdot b), \tag{L.2}
\end{equation*}
$$

where we use the index " E " for the notation of vectors in Euclidean space.
The leading term of the Feshbach correction can then be cast into the form:

$$
\begin{equation*}
\delta_{2 \gamma}^{0}=\frac{M Q^{2} e^{2}}{\omega} \int \frac{\mathrm{~d}^{4} \tilde{q}_{E}}{(2 \pi)^{4}}\left(\Pi_{P}^{-}+\Pi_{P}^{+}\right)\left(\Pi_{Q}^{+}+\Pi_{Q}^{-}\right) \Pi_{Q}, \tag{L.3}
\end{equation*}
$$

with $\Pi_{Q}=1 /\left(\tilde{q}_{E}^{2}+\frac{Q^{2}}{4}+\mu^{2}\right)$. We can expand $\Pi_{P, Q}^{ \pm}$as Gegenbauer polynomials $\left(C_{n}(z)\right)$ generating functions:

$$
\begin{align*}
& \frac{1}{\left(\tilde{q}_{E}+\frac{q_{E}}{2}\right)^{2}+\mu^{2}}=\frac{2 z_{\mu}}{\tilde{q}_{E} Q} \sum_{n=0}^{\infty}(-1)^{n} z_{\mu}^{n} C_{n}\left(\hat{\tilde{\vec{q}}}_{E} \hat{\vec{q}}_{E}\right), \\
& \frac{1}{\left(\tilde{q}_{E}-\frac{q_{E}}{2}\right)^{2}+\mu^{2}}=\frac{2 z_{\mu}}{\tilde{q}_{E} Q} \sum_{n=0}^{\infty} z_{\mu}^{n} C_{n}\left(\hat{\vec{q}}_{E} \hat{\vec{q}}_{E}\right), \\
& \frac{1}{\left(\tilde{q}_{E}+P_{E}\right)^{2}+M^{2}}=\frac{i z_{M}}{\tilde{q}_{E} P_{E}} \sum_{n=0}^{\infty}(-1)^{n}\left(i z_{M}\right)^{n} C_{n}\left(\hat{\tilde{\vec{q}}_{E}} \hat{\vec{P}}_{E}\right), \\
& \frac{1}{\left(\tilde{q}_{E}-P_{E}\right)^{2}+M^{2}}=\frac{i z_{M}}{\tilde{q}_{E} P_{E}} \sum_{n=0}^{\infty}\left(i z_{M}\right)^{n} C_{n}\left(\hat{\vec{q}}_{E} \hat{\vec{P}}_{E}\right), \tag{L.4}
\end{align*}
$$

with

$$
\begin{align*}
z_{M} & =\frac{Q}{4 M x}\left(1-x^{2}+\sqrt{\left(1-x^{2}\right)^{2}+\frac{16 M^{2}}{Q^{2}} x^{2}}\right) \\
z_{\mu} & =\frac{1+x^{2}+\tilde{\mu}^{2}-\sqrt{\left(1+x^{2}+\tilde{\mu}^{2}\right)^{2}-4 x^{2}}}{2 x} \\
x & =\frac{2 \tilde{q}_{E}}{Q}, \quad \tilde{\mu}^{2}=4 \mu^{2} / Q^{2} . \tag{L.5}
\end{align*}
$$

Using the Gegenbauer polynomials value $C_{n}(0)=(-1)^{n / 2} \cdot\left(1+(-1)^{n}\right) / 2$ and the orthogonality relation for vectors $q, x, y$ in Euclidean space:

$$
\begin{equation*}
\int \mathrm{d} \Omega(\hat{q}) C_{n}(\hat{q} \hat{x}) C_{m}(\hat{q} \hat{y})=\frac{2 \pi^{2}}{n+1} \delta_{m, n} C_{n}(\hat{x} \hat{y}) \tag{L.6}
\end{equation*}
$$

the integral of Eq. (L.3) simplifies to the following expression:

$$
\begin{equation*}
\delta_{2 \gamma}^{0}=\frac{4 \alpha Q}{\pi \omega} \int_{0}^{\infty} \frac{\tilde{q}_{E} \mathrm{~d} \tilde{q}_{E}}{\tilde{q}_{E}^{2}+\frac{Q^{2}}{4}} \sum_{n=0}^{\infty} \frac{\left(z_{M} z_{\mu}\right)^{2 n+1}}{2 n+1}=\frac{2 \alpha Q}{\pi \omega} \int_{0}^{\infty} \frac{x \mathrm{~d} x}{1+x^{2}} \ln \left(\frac{1+z_{M} z_{\mu}}{1-z_{M} z_{\mu}}\right) \tag{L.7}
\end{equation*}
$$

For $x \rightarrow 0$ the product $z_{M} z_{\mu} \rightarrow \frac{Q}{2 M}\left(1-\frac{\tilde{\mu}^{2}}{1+\tilde{\mu}^{2}}\right)$ is IR finite. Numerically the result of the integral does not depend on small $\frac{Q}{M}$ and $\tilde{\mu}^{2}$ values, so we can neglect terms of order $\tilde{\mu}^{2}, \frac{Q}{M}$. The resulting TPE correction is given by

$$
\begin{equation*}
\delta_{2 \gamma}^{0}=\frac{2 \alpha Q}{\pi \omega} \int_{0}^{\infty} \frac{x \mathrm{~d} x}{1+x^{2}} \ln \left(\frac{1+x}{|1-x|}\right)+O\left(Q^{2} \ln Q^{2}\right)=\frac{\alpha \pi Q}{2 \omega}+\mathrm{O}\left(Q^{2} \ln Q^{2}\right) \tag{L.8}
\end{equation*}
$$

reproducing the low- $Q^{2}$ behavior of the Feshbach correction, see Eq. (5.4).
Accounting also for the suppressed by the lepton mass term in Eq. (5.36),

$$
\begin{equation*}
\delta_{2 \gamma}^{\mathrm{QED}} \rightarrow-\frac{m^{2}}{M^{2}} \frac{16 M Q^{2} e^{2}}{\omega} \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \Pi_{P}^{+} \Pi_{P}^{-} \Pi_{K}^{+} \Pi_{K}^{-} \Pi_{Q}^{+} \Pi_{Q}^{-}\left(\tilde{q}^{2}+\frac{Q^{2}}{4}\right)(P \cdot \tilde{q})^{2} \tag{L.9}
\end{equation*}
$$

we reproduce the low- $Q^{2}$ behavior in case of the massive lepton scattering of Eq. (5.14):

$$
\begin{equation*}
\delta_{2 \gamma}^{0}=\frac{\alpha \pi Q}{2 \omega} \frac{M+m}{M}+\mathrm{O}\left(Q^{2} \ln Q^{2}\right) \tag{L.10}
\end{equation*}
$$

## Appendix M

## Some VVCS amplitudes and tensor structures

The three non-vanishing VVCS amplitudes for the case of two Dirac couplings $\gamma^{\mu}$ in the photon-proton-proton vertices in Fig. (5.5) are given by

$$
\begin{align*}
B_{2}^{\mathrm{QED}} & =\frac{1}{M} \Pi_{P}^{+} \Pi_{P}^{-} \\
B_{10}^{\mathrm{QED}} & =-\frac{1}{2 M} \Pi_{P}^{+} \Pi_{P}^{-} \\
B_{17}^{\mathrm{QED}} & =-\frac{(P \cdot \tilde{q})}{M} \Pi_{P}^{+} \Pi_{P}^{-} \tag{M.1}
\end{align*}
$$

with the propagator notations of Eqs. (5.37), and the relevant VVCS tensor structures have the following form:

$$
\begin{align*}
T_{2}^{\mu \nu}= & -4(P \cdot \tilde{q})^{2} g^{\mu \nu}-4\left(q_{1} \cdot q_{2}\right) P^{\mu} P^{\nu}+4(P \cdot \tilde{q})\left(P^{\nu} q_{1}^{\mu}+P^{\mu} q_{2}^{\nu}\right) \\
T_{10}^{\mu \nu}= & -8\left(q_{1} \cdot q_{2}\right) P^{\mu} P^{\nu}+4(P \cdot \tilde{q})\left(P^{\nu} q_{1}^{\mu}+P^{\mu} q_{2}^{\nu}\right)+4 M\left(q_{1} \cdot q_{2}\right)\left(P^{\mu} \gamma^{\nu}+P^{\nu} \gamma^{\mu}\right) \\
& -4 M(P \cdot \tilde{q})\left(q_{1}^{\mu} \gamma^{\nu}+q_{2}^{\nu} \gamma^{\mu}\right)-2(P \cdot \tilde{q})\left\{\left(q_{2}^{\nu} \gamma^{\mu}-q_{1}^{\mu} \gamma^{\nu}\right) \gamma \cdot \tilde{q}-\gamma \cdot \tilde{q}\left(q_{2}^{\nu} \gamma^{\mu}-q_{1}^{\mu} \gamma^{\nu}\right)\right\} \\
& -2\left(q_{1} \cdot q_{2}\right)(P \cdot \tilde{q})\left(\gamma^{\nu} \gamma^{\mu}-\gamma^{\mu} \gamma^{\nu}\right) \\
& +M\left(q_{1} \cdot q_{2}\right)\left\{\left(\gamma^{\nu} \gamma^{\mu}-\gamma^{\mu} \gamma^{\nu}\right) \gamma \cdot \tilde{q}+\gamma \cdot \tilde{q}\left(\gamma^{\nu} \gamma^{\mu}-\gamma^{\mu} \gamma^{\nu}\right)\right\}, \\
T_{17}^{\mu \nu}= & -4(P \cdot \tilde{q}) g^{\mu \nu}+2\left(P^{\nu} q_{1}^{\mu}+P^{\mu} q_{2}^{\nu}\right)+4 M g^{\mu \nu} \gamma \cdot \tilde{q}-2 M\left(q_{1}^{\mu} \gamma^{\nu}+q_{2}^{\nu} \gamma^{\mu}\right) \\
& +\left\{\left(q_{2}^{\nu} \gamma^{\mu}-q_{1}^{\mu} \gamma^{\nu}\right) \gamma \cdot \tilde{q}-\gamma \cdot \tilde{q}\left(q_{2}^{\nu} \gamma^{\mu}-q_{1}^{\mu} \gamma^{\nu}\right)\right\}+\left(q_{1} \cdot q_{2}\right)\left(\gamma^{\nu} \gamma^{\mu}-\gamma^{\mu} \gamma^{\nu}\right) \tag{M.2}
\end{align*}
$$

## Appendix N

## Subtraction function TPE correction: evaluation of integrals

The first integral in the subtraction function TPE correction of Eq. (5.70) is of the type:

$$
\begin{equation*}
I_{1}=\int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} g\left(\tilde{Q}^{2}\right) \Pi_{Q}^{+} \Pi_{Q}^{-} \Pi_{K}^{+} \Pi_{K}^{-}, \tag{N.1}
\end{equation*}
$$

where $g\left(\tilde{Q}^{2}\right)$ is given by

$$
\begin{equation*}
g\left(\tilde{Q}^{2}\right)=\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2} \beta\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \tag{N.2}
\end{equation*}
$$

Accounting for the symmetry $\tilde{q} \rightarrow-\tilde{q}$, the integral $I_{1}$ can be expressed as

$$
\begin{equation*}
I_{1}=\frac{1}{2} \int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \frac{g\left(\tilde{Q}^{2}\right) \Pi_{K}^{+}\left(\Pi_{Q}^{+}+\Pi_{Q}^{-}\right)}{\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}\right)} . \tag{N.3}
\end{equation*}
$$

The azimuthal angle integration is trivial, the polar angle integration gives the same result for both terms in Eq. (N.3), such that the integral $I_{1}$ can be written as

$$
\begin{equation*}
I_{1}=\int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \frac{g\left(\tilde{Q}^{2}\right) \Pi_{K}^{+} \Pi_{Q}^{+}}{\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}\right)} \tag{N.4}
\end{equation*}
$$

We evaluate the integral conveniently in the lepton Breit frame defined by

$$
\begin{equation*}
K=K(1,0,0,0), \quad q=Q(0,0,0,1), \tag{N.5}
\end{equation*}
$$

and perform the integral through a Wick rotation. The integration contour crosses the lepton propagator poles in this frame during the Wick rotation, as detailed in Section 5.6 [172]. The integral of Eq. (N.4) is given by the sum of the integral along the imaginary axis, which we denote by $I_{1}^{W}$, and the lepton pole contribution, which we denote by $I_{1}^{p}$.

The integral along the imaginary axis $I_{1}^{W}$ can be evaluated by the Gegenbauer polynomial technique, see Appendix L for some technical details [172]. It results in an integral over the dimensionless variable $x \equiv 4 \tilde{Q}^{2} / Q^{2}$ :

$$
\begin{equation*}
I_{1}^{W}=\frac{1}{Q^{4} \sqrt{1+a}} \int_{0}^{\infty} \frac{\mathrm{d} x}{2 \pi^{2}} \frac{g\left(\frac{x Q^{2}}{4}\right)}{\left|1-x^{2}\right|} \ln \left|\frac{z+1}{z-1}\right|, \tag{N.6}
\end{equation*}
$$

with the notations of Eq. (5.73).
The contribution of the pole $\tilde{q}_{0}=K-\sqrt{\tilde{q}^{2}+m^{2}}+i \varepsilon$ which is enclosed by the Wick rotation contour to the integral $I_{1}$ is given by

$$
\begin{equation*}
I_{1}^{p}=\frac{1}{Q^{4} \sqrt{1+a}} \int_{x_{\min }}^{1} \frac{\mathrm{~d} x}{2 \pi^{2}} \frac{g\left(\frac{x Q^{2}}{4}\right)}{1-x^{2}} \ln \left|\frac{(z-1)(x-z)}{(z+1)(x+z)}\right| \Theta\left(x+(\sqrt{1+a}-\sqrt{a})^{2}\right) \tag{N.7}
\end{equation*}
$$

with $a$ defined in Eq. (5.73). Note that the lower integration limit in Eq. (N.7) is given by

$$
\begin{equation*}
x_{\min }=-(\sqrt{1+a}-\sqrt{a})^{2}, \tag{N.8}
\end{equation*}
$$

which has limits $x_{\text {min }} \rightarrow-1$ for $m^{2} \ll Q^{2}$, and $x_{\text {min }} \rightarrow-\frac{Q^{2}}{16 m^{2}}$ for $Q^{2} \ll m^{2}$.
Subsequently, we evaluate the second integral in Eq. (5.70):

$$
\begin{equation*}
I_{2}=-\int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \frac{2(K \cdot \tilde{q})^{2}}{K^{2}}\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}\right) \beta\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \Pi_{K}^{+} \Pi_{K}^{-} \Pi_{Q}^{+} \Pi_{Q}^{-} \tag{N.9}
\end{equation*}
$$

Performing similar steps as for the $I_{1}$ integral we obtain:

$$
\begin{equation*}
I_{2}=\int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \frac{\beta\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)}{2 K^{2}}\left\{-\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \Pi_{K}^{+} \Pi_{Q}^{+}+\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}\right) \Pi_{Q}^{+} \Pi_{Q}^{-}\right\} . \tag{N.10}
\end{equation*}
$$

The integral from the first term can be obtained from the $I_{1}$ integral of Eqs. (N.6, N.7) with

$$
\begin{equation*}
g\left(Q^{2}\right)=-\frac{1}{2 K^{2}}\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}\right) \beta\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right) . \tag{N.11}
\end{equation*}
$$

The integral from the second term of Eq. (N.10) can be performed by the Gegenbauer polynomial technique for the angular integration. The result is given by

$$
\begin{equation*}
\int \frac{i \mathrm{~d}^{4} \tilde{q}}{(2 \pi)^{4}} \frac{\Pi_{Q}^{+} \Pi_{Q}^{-}}{2 K^{2}}\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}\right) \beta\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)=-\frac{1}{32 \pi^{2}(1+a)} \int_{0}^{\infty} x^{n} \beta\left(\frac{(x-1) Q^{2}}{4}\right) \mathrm{d} x \tag{N.12}
\end{equation*}
$$

with $n=1$ for $x<1$ and $n=0$ for $x>1$.
Summing up all contributions we obtain the result of Eq. (5.71).

## Appendix O

## Contribution from the Wick rotation pole to the Lamb shift of S energy levels

The leading contribution from TPE graphs to the Lamb shift of atomic S energy levels starts at order $O\left(\alpha^{5}\right)$. In this Appendix, we prove the vanishing contribution of Wick rotation pole in this order.

Substituting the near-forward approximation of the VVCS tensor, see Eq. (5.46) and Section 5.3, into the general expression for the TPE amplitude of Eq. (5.61), we obtain:

$$
\begin{align*}
T^{2 \gamma}= & -\alpha \int \frac{i \mathrm{~d}^{4} \tilde{q}}{\pi^{2}} \Pi_{Q}^{+} \Pi_{Q}^{-}(\bar{N} N) \times\left\{m(\bar{u} u)\left(\Pi_{K}^{-}+\Pi_{K}^{+}\right) \mathrm{T}_{1}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right)\right. \\
& +(\bar{u} \hat{\tilde{q}} u) \frac{\tilde{Q}^{2}+\frac{Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}\left(\Pi_{K}^{-}-\Pi_{K}^{+}\right) \mathrm{T}_{1}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \\
& -(\bar{u} \hat{P} u)\left((P \cdot K) \frac{\Pi_{K}^{-}+\Pi_{K}^{+}}{M^{2}}-(P \cdot \tilde{q}) \frac{\tilde{Q}^{2}-\frac{3 Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}} \frac{\Pi_{K}^{-}-\Pi_{K}^{+}}{M^{2}}\right) \mathrm{T}_{2}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \\
& -(\bar{u} \hat{\tilde{q} u} u)\left(P^{2}+\frac{(P \cdot \tilde{q})^{2} Q^{2}}{\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}}\right) \frac{\Pi_{K}^{-}-\Pi_{K}^{+}}{2 M^{2}} \mathrm{~T}_{2}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \\
& \left.+(\bar{u}(\hat{q} \hat{\tilde{q}}-\hat{\tilde{q}} \hat{q}) \hat{P} u) \frac{(P \cdot \tilde{q})}{\tilde{Q}^{2}-\frac{Q^{2}}{4}} \frac{\Pi_{K}^{-}+\Pi_{K}^{+}}{M^{2}} \mathrm{~T}_{2}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right)\right\} . \tag{O.1}
\end{align*}
$$

The spin-independent amplitude is evaluated exploiting the following spinor relations:

$$
\begin{array}{r}
\frac{1}{2} \sum_{\text {spin }} \bar{N} N=\frac{1}{2} \operatorname{Tr}(\hat{P}+M)=2 M, \\
\frac{1}{2} \sum_{\text {spin }} \bar{u} u=\frac{1}{2} \operatorname{Tr}(\hat{K}+m)=2 m, \\
\frac{1}{2} \sum_{\text {spin }} \bar{u} \hat{P} u=\frac{1}{2} \operatorname{Tr}(\hat{P}(\hat{K}+M))=2(P \cdot K), \\
\frac{1}{2} \sum_{\text {spin }} \bar{u} \hat{\tilde{q}} u=\frac{1}{2} \operatorname{Tr}(\hat{\tilde{q}}(\hat{K}+M))=2(K \cdot \tilde{q}), \\
\frac{1}{2} \sum_{\text {spin }} \bar{u} \hat{\tilde{q}} u=\frac{1}{2} \operatorname{Tr}((\hat{q} \hat{\tilde{q}}-\hat{\tilde{q}} \hat{q}) \hat{P}(\hat{K}+M))=0 . \tag{O.2}
\end{array}
$$

Therefore, the forward unpolarized lepton-proton scattering amplitude, which reproduces $f_{+}^{2 \gamma}(\omega)$ of Eq. (2.91) in the forward limit $Q^{2} \rightarrow 0$, can be expressed as

$$
\begin{align*}
T^{2 \gamma}\left(\omega, Q^{2}\right) \rightarrow & 4 M \alpha \int \frac{i \mathrm{~d}^{4} \tilde{q}^{2}}{\pi^{2}} \Pi_{Q}^{+} \Pi_{Q}^{-} \times\left\{-m^{2}\left(\Pi_{K}^{-}+\Pi_{K}^{+}\right) \mathrm{T}_{1}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right)\right. \\
& -(K \cdot \tilde{q}) \frac{\tilde{Q}^{2}+\frac{Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}\left(\Pi_{K}^{-}-\Pi_{K}^{+}\right) \mathrm{T}_{1}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \\
& +(P \cdot K)^{2} \frac{\Pi_{K}^{-}+\Pi_{K}^{+}}{M^{2}} \mathrm{~T}_{2}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \\
& -(P \cdot \tilde{q})(P \cdot K) \frac{\tilde{Q}^{2}-\frac{3 Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}} \frac{\Pi_{K}^{-}-\Pi_{K}^{+}}{M^{2}} \mathrm{~T}_{2}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \\
& \left.+\left(P^{2}+\frac{(P \cdot \tilde{q})^{2} Q^{2}}{\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}}\right) \frac{\Pi_{K}^{-}-\Pi_{K}^{+}}{2 M^{2}}(K \cdot \tilde{q}) \mathrm{T}_{2}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right)\right\} . \tag{0.3}
\end{align*}
$$

Studying the pole $\tilde{q}_{0}=K-\sqrt{\tilde{q}^{2}+m^{2}}$ contribution in the $K$-frame, where it contributes for $\tilde{q}<Q / 2$, we can use the low-energy expansion of the forward VVCS amplitudes:

$$
\begin{align*}
& \mathrm{T}_{1}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \approx\left(\alpha_{E}+\beta_{M}\right) \tilde{\nu}^{2}+\beta_{M}\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \approx-2 m \beta_{M} \tilde{q}_{0}  \tag{0.4}\\
& \mathrm{~T}_{2}\left(\tilde{\nu}, \tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \approx\left(\alpha_{E}+\beta_{M}\right)\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \approx-2 m\left(\alpha_{E}+\beta_{M}\right) \tilde{q}_{0} \tag{0.5}
\end{align*}
$$

Accounting for the symmetry $q \rightarrow-q$ property of the integral and performing the azimuthal and polar angle integrations, we obtain for the resulting $K$-pole contribution to the unpolarized TPE amplitude in the leading $Q^{2} \ll M^{2}, m^{2}$ order:

$$
\begin{equation*}
T^{2 \gamma}\left(m, Q^{2}\right) \rightarrow 8 \alpha \alpha_{E} M m Q \int_{0}^{1} x \mathrm{~d} x \frac{1-x^{2}}{1+x^{2}} \ln \frac{1+x}{1-x} . \tag{0.6}
\end{equation*}
$$

Due to the momentum transfer in the numerator, the pole contribution is suppressed by $\alpha$.

## Appendix P

## TPE correction in terms of the unpolarized proton structure functions

In this Appendix, we present the expressions for the pole contributions as well as the contribution from the integral along the imaginary axis to the weighting functions $w_{1}, w_{2}$, which appear in the TPE correction of Eq. (5.79).

We first present the results in the $P$-frame, defined by Eq. (5.80). The contribution to the weighting functions $w_{1}, w_{2}$ arising from the leptonic pole $\tilde{q}_{0}=K_{0}-\sqrt{(\vec{K}-\overrightarrow{\tilde{q}})^{2}+m^{2}}$ in the $P$-frame is given by

$$
\begin{align*}
w_{1}\left(W^{2}, \tilde{Q}^{2}\right) & =\frac{2 \alpha}{\pi} \frac{G_{E}}{\varepsilon G_{E}^{2}+\tau G_{M}^{2}} \frac{P^{2}}{M^{2}} \frac{Q^{2}}{(K \cdot P)^{2}+M^{4} \tau(1+\tau)} \frac{1}{|\vec{K}|} \frac{P^{2} \tilde{Q}_{0}^{2}}{\tilde{Q}^{2}+\frac{Q^{2}}{4}} \frac{\tilde{A}+\tilde{B}}{\left(W^{2}-P^{2}+\tilde{Q}^{2}\right)^{2}}, \\
w_{2}\left(W^{2}, \tilde{Q}^{2}\right) & =\frac{2 \alpha}{\pi} \frac{G_{E}}{\varepsilon G_{E}^{2}+\tau G_{M}^{2}} \frac{P^{2}}{M^{2}} \frac{Q^{2}}{(K \cdot P)^{2}+M^{4} \tau(1+\tau)} \frac{1}{|\vec{K}|} \frac{1}{\tilde{Q}^{2}+\frac{Q^{2}}{4}} \frac{\tilde{C}+\tilde{D}}{W^{2}-P^{2}+\tilde{Q}^{2}}, \tag{P.1}
\end{align*}
$$

with coefficients $\tilde{A}+\tilde{B}$ and $\tilde{C}+\tilde{D}$ following from Eqs. (5.78) as

$$
\begin{align*}
\tilde{A}+\tilde{B}= & -\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}-2 m^{2}\right)(K \cdot P)-2 \frac{\tilde{Q}^{2}+\frac{Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}} \frac{Q^{2}}{4} P \tilde{q}_{0}, \\
\tilde{C}+\tilde{D}= & -(K \cdot P)\left((K \cdot P)^{2}-\frac{3 Q^{2}}{16} P^{2}-\frac{\tilde{Q}^{2}}{4} P^{2}\right)+\frac{Q^{2}}{8} P^{3} \tilde{q}_{0}+\frac{Q^{2}}{8} \frac{Q^{2} P^{3} \tilde{q}_{0}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}} \\
& +P \tilde{q}_{0} \frac{\tilde{Q}^{2}-\frac{3 Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}(K \cdot P)^{2}+\frac{3}{4} \frac{P^{2} \tilde{q}_{0}^{2} Q^{2}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}(K \cdot P)+\frac{Q^{2}}{8} \frac{P^{3} \tilde{q}_{0}^{3} Q^{2}}{\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}} . \tag{P.2}
\end{align*}
$$

In Eq. (P.2), $\tilde{q}_{0}^{n}$ stands for the sum of two integrals with either $\pm$ signs:

$$
\begin{equation*}
\tilde{q}_{0}^{n} \rightarrow \int_{0 \text { or } \tilde{q}_{0}^{-}}^{\tilde{q}_{0}^{+}} \frac{\mathrm{d} \tilde{q}_{0}}{\sqrt{\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}-Q^{2} \tilde{q}_{0}^{2}+\frac{Q^{2}}{4|\tilde{K}|^{2}}\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}+2 \tilde{q}_{0} K_{0}\right)^{2}}} \frac{\tilde{q}_{0}^{n}}{P^{2} \pm 2 P \tilde{q}_{0}-\tilde{Q}^{2}-W^{2}} . \tag{P.3}
\end{equation*}
$$

The integration regions are given by

$$
\begin{gather*}
0 \leq \tilde{q}_{0} \leq \tilde{q}_{0}^{+} \quad \text { for } \quad\left(-|\vec{K}|+\sqrt{K_{0}^{2}-m^{2}}\right)^{2} \leq \tilde{Q}^{2} \leq\left(|\vec{K}|+\sqrt{K_{0}^{2}-m^{2}}\right)^{2}, \\
\tilde{q}_{0}^{-} \leq \tilde{q}_{0} \leq \tilde{q}_{0}^{+} \quad \text { for } \quad-(K-m)^{2} \leq \tilde{Q}^{2} \leq\left(-|\vec{K}|+\sqrt{K_{0}^{2}-m^{2}}\right)^{2}, \tag{P.4}
\end{gather*}
$$

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with

$$
\begin{equation*}
\tilde{q}_{0}^{ \pm}=\frac{K_{0}}{2}\left(1-\frac{m^{2}}{K^{2}}\right)-\frac{K_{0} \tilde{Q}^{2}}{2 K^{2}} \pm \frac{|\vec{K}|}{2 K^{2}} \sqrt{\left((K+m)^{2}+\tilde{Q}^{2}\right)\left((K-m)^{2}+\tilde{Q}^{2}\right)} . \tag{P.5}
\end{equation*}
$$

We do not consider the hadronic pole in the $P$-frame. It contributes only in the region of large momentum transfer:

$$
\begin{equation*}
Q^{2} \geq 8 M m_{\pi}\left(1+\frac{m_{\pi}}{2 M}\right) \approx 1.09 \mathrm{GeV}^{2} \tag{P.6}
\end{equation*}
$$

The contribution to the weighting functions arising from the integral along the imaginary axis in the $P$-frame is given by

$$
\begin{align*}
w_{1}\left(W^{2}, \tilde{Q}^{2}\right)= & \frac{2 \alpha}{\pi} \frac{G_{E}}{\varepsilon G_{E}^{2}+\tau G_{M}^{2}} \frac{P^{2}}{M^{2}} \frac{Q^{2}}{(K \cdot P)^{2}+M^{4} \tau(1+\tau)} \frac{\tilde{Q}^{2}}{\tilde{Q}^{2}+\frac{Q^{2}}{4}} \\
& \times \Re \int_{0}^{\pi} \frac{(P \cdot \tilde{q})^{2}}{W^{2}-P^{2}+\tilde{Q}^{2}} \frac{(\tilde{A}+\tilde{B}) \sin ^{2} \psi \mathrm{~d} \psi}{\left((P+\tilde{q})^{2}-W^{2}\right)\left((P-\tilde{q})^{2}-W^{2}\right)}, \\
w_{2}\left(W^{2}, \tilde{Q}^{2}\right)= & \frac{2 \alpha}{\pi} \frac{G_{E}}{\varepsilon G_{E}^{2}+\tau G_{M}^{2}} \frac{P^{2}}{M^{2}} \frac{Q^{2}}{(K \cdot P)^{2}+M^{4} \tau(1+\tau)} \frac{\tilde{Q}^{2}}{\tilde{Q}^{2}+\frac{Q^{2}}{4}} \\
& \times \Re \int_{0}^{\pi} \frac{(\tilde{C}+\tilde{D}) \sin ^{2} \psi \mathrm{~d} \psi}{\left((P+\tilde{q})^{2}-W^{2}\right)\left((P-\tilde{q})^{2}-W^{2}\right)}, \tag{P.7}
\end{align*}
$$

with the notations:

$$
\begin{align*}
\tilde{A}+\tilde{B}= & -2 \frac{\tilde{Q}^{2}+\frac{Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}(K \cdot P) J_{0}+2\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}-2 m^{2}\right)(K \cdot P) I_{0}^{c}+\frac{\tilde{Q}^{2}+\frac{Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}} Q^{2}(P \cdot \tilde{q}) I_{0}^{c}, \\
\tilde{C}+\tilde{D}= & \left(\left(P^{2}+\frac{(P \cdot \tilde{q})^{2} Q^{2}}{\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}}\right)(K \cdot P)+\frac{Q^{2}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}(P \cdot \tilde{q}) P^{2}\right) \frac{J_{0}}{2} \\
& +2(K \cdot P)\left((K \cdot P)^{2}-\frac{Q^{2}}{4} P^{2}\right) I_{0}^{c}-\frac{Q^{2}}{2}(P \cdot \tilde{q}) P^{2} I_{0}^{c}-\frac{Q^{2}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}(P \cdot \tilde{q})^{2}(K \cdot P) I_{0}^{c} \\
& -\left(P^{2}+\frac{(P \cdot \tilde{q})^{2} Q^{2}}{\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}}\right) \frac{Q^{2}}{4}(P \cdot \tilde{q}) I_{0}^{c}-2(P \cdot \tilde{q}) \frac{\tilde{Q}^{2}-\frac{3 Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}\left((K \cdot P)^{2}-\frac{Q^{2}}{4} P^{2}\right) I_{0}^{c} \\
& -\left(P^{2}+\frac{(P \cdot \tilde{q})^{2} Q^{2}}{\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}}\right)(K \cdot P)\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right) \frac{I_{0}^{c}}{2} \tag{P.8}
\end{align*}
$$

and master integrals:

$$
\begin{align*}
& J_{0}=\int_{-1}^{1} \frac{\mathrm{~d} x}{\sqrt{a^{2}+b^{2} x^{2}}}=\frac{1}{b} \ln \frac{\sqrt{a^{2}+b^{2}}+b}{\sqrt{a^{2}+b^{2}}-b}, \\
& I_{0}^{c}=\int_{-1}^{1} \frac{\mathrm{~d} x}{\sqrt{a^{2}+b^{2} x^{2}}} \frac{1}{c+g x}=\frac{1}{\sqrt{a^{2} g^{2}+c^{2} b^{2}}} \ln \frac{\left(\sqrt{a^{2}+b^{2}} c+\sqrt{a^{2} g^{2}+c^{2} b^{2}}\right)^{2}}{a^{2}\left(c^{2}-g^{2}\right)}, \\
& a^{2}=\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}-Q^{2} \tilde{q}_{0}^{2}=\left(\tilde{Q}^{2}+\frac{Q^{2}}{4}\right)^{2}-b^{2}, \quad b^{2}=Q^{2} \tilde{q}^{2}, \quad \tilde{q}=|\overrightarrow{\tilde{q}}|, \tag{P.9}
\end{align*}
$$

with

$$
\begin{equation*}
c=\tilde{Q}^{2}-\frac{Q^{2}}{4}+2 K_{0} \tilde{q}_{0}, \quad g^{2}=4 \vec{K}^{2} \tilde{Q}^{2} \sin ^{2} \psi \tag{P.10}
\end{equation*}
$$

The integral along the imaginary axis contributes in the range $0 \leq \tilde{Q}^{2} \leq \infty$.
As a check of our calculations, we also provide the expressions evaluated in the $K$-frame, defined by Eq. (5.81). The leptonic pole $\tilde{q}_{0}=K-\sqrt{\tilde{q}^{2}+m^{2}}$ contribution to $w_{1}$, $w_{2}$ evaluated in the $K$-frame is given by

$$
\begin{align*}
w_{1}\left(W^{2}, \tilde{Q}^{2}\right) & =\frac{\alpha}{4 \pi} \frac{G_{E}}{\varepsilon G_{E}^{2}+\tau G_{M}^{2}} \frac{P^{2}}{M^{2}} \frac{\tilde{q} Q^{3}}{(K \cdot P)^{2}+M^{4} \tau(1+\tau)} \frac{\tilde{A}+\tilde{B}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}} \\
w_{2}\left(W^{2}, \tilde{Q}^{2}\right) & =\frac{\alpha}{4 \pi} \frac{G_{E}}{\varepsilon G_{E}^{2}+\tau G_{M}^{2}} \frac{P^{2}}{M^{2}} \frac{\tilde{q} Q^{3}}{(K \cdot P)^{2}+M^{4} \tau(1+\tau)} \frac{\tilde{C}+\tilde{D}}{\tilde{Q}^{2}+\frac{Q^{2}}{4}} \tag{P.11}
\end{align*}
$$

with the notations:

$$
\begin{align*}
\tilde{A}+\tilde{B}= & c_{+} I_{0}^{c_{+}}-c_{-} I_{0}^{c_{-}}-\frac{8 m^{2}}{Q^{2}}\left(P_{0} \tilde{q}_{0}+(K \cdot P) \frac{\tilde{Q}^{2}-\frac{Q^{2}}{4}}{\tilde{Q}^{2}+\frac{Q^{2}}{4}}\right)\left(I_{0}^{c_{-}}+I_{0}^{c_{+}}-\frac{2 J_{0}}{c_{0}}\right), \\
\tilde{C}+\tilde{D}= & \frac{Q^{2}}{4\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}} c_{0}\left(c_{-} I_{0}^{c_{-}}-c_{+} I_{0}^{c_{+}}\right)+\frac{16(K \cdot P)^{3}}{Q^{2} c_{0}}\left(I_{0}^{c_{-}}+I_{0}^{c_{+}}\right) \\
& +\frac{2(K \cdot P)}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}\left(\left(\frac{\tilde{Q}^{2}-\frac{3 Q^{2}}{4}}{Q^{2}} 4(K \cdot P)+2 P_{0} \tilde{q}_{0}\right)\left(I_{0}^{c_{-}}-I_{0}^{c_{+}}\right)+2 J_{0}-c_{+} I_{0}^{c_{+}}-c_{-} I_{0}^{c_{-}}\right) \\
& +\frac{4 K \tilde{q}_{0}+\tilde{Q}^{2}-\frac{5 Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}} P^{2}\left(I_{0}^{c_{+}}-I_{0}^{c_{-}}\right)+2\left(\tilde{q}_{0}-2 K\right) \frac{P_{0} P^{2}}{c_{0}}\left(I_{0}^{c_{+}}+I_{0}^{c_{-}}\right) \\
& +\frac{2 m^{2}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}} P_{0} \tilde{q}_{0} c_{0}\left(I_{0}^{c_{-}}+I_{0}^{c_{+}}-\frac{2 J_{0}}{c_{0}}\right)+\frac{8 m^{2}}{Q^{2}} \tilde{q}_{0} \frac{P_{0} P^{2}}{c_{0}}\left(I_{0}^{c_{-}}+I_{0}^{c_{+}}\right), \tag{P.12}
\end{align*}
$$

in terms of the integrals of Eqs. (P.9), where
$c_{ \pm}=W^{2}-P^{2}+\tilde{Q}^{2} \mp 2 P_{0} \tilde{q}_{0}, \quad c_{0}=W^{2}-P^{2}+\tilde{Q}^{2}, \quad g^{2}=4 \frac{(K \cdot P)^{2}-P^{2} K^{2}}{K^{2}} \tilde{q}^{2}$.
The $\tilde{Q}^{2}$ integration region is given by

$$
\begin{equation*}
-(K-m)^{2} \leq \tilde{Q}^{2} \leq \frac{Q^{2}}{4} \tag{P.14}
\end{equation*}
$$

The contribution to $w_{1}, w_{2}$ arising from the hadronic pole $\tilde{q}_{0}=P_{0}-\sqrt{(\vec{P}-\overrightarrow{\tilde{q}})^{2}+W^{2}}$ in the $K$-frame is given by

$$
\begin{align*}
w_{1}\left(W^{2}, \tilde{Q}^{2}\right) & =\frac{\alpha}{2 \pi} \frac{G_{E}}{\varepsilon G_{E}^{2}+\tau G_{M}^{2}} \frac{P^{2}}{M^{2}} \frac{Q^{2}}{(K \cdot P)^{2}+M^{4} \tau(1+\tau)} \frac{1}{|\vec{P}|} \frac{\tilde{A}+\tilde{B}}{\tilde{Q}^{2}+\frac{Q^{2}}{4}} \\
w_{2}\left(W^{2}, \tilde{Q}^{2}\right) & =\frac{\alpha}{2 \pi} \frac{G_{E}}{\varepsilon G_{E}^{2}+\tau G_{M}^{2}} \frac{P^{2}}{M^{2}} \frac{Q^{2}}{(K \cdot P)^{2}+M^{4} \tau(1+\tau)} \frac{1}{|\vec{P}|} \frac{\tilde{C}+\tilde{D}}{\tilde{Q}^{2}+\frac{Q^{2}}{4}} \frac{1}{(P \cdot \tilde{q})}, \tag{P.15}
\end{align*}
$$

with the notations:

$$
\begin{align*}
\tilde{A}+\tilde{B}= & 2 \frac{\tilde{Q}^{2}+\frac{Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}\left((K \cdot P) K\left(L_{1}^{-}-L_{1}^{+}\right)-\frac{Q^{2}}{4}(P \cdot \tilde{q})\left(L_{0}^{-}-L_{0}^{+}\right)\right) \\
& +2 m^{2}(K \cdot P)\left(L_{0}^{-}+L_{0}^{+}\right), \\
\tilde{C}= & (K \cdot P)\left(2(K \cdot P)^{2}-\frac{Q^{2} P^{2}}{2}-\frac{Q^{2}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}(P \cdot \tilde{q})^{2}\right)\left(L_{0}^{-}+L_{0}^{+}\right) \\
& +\frac{Q^{2} K}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}(P \cdot \tilde{q}) P^{2}\left(L_{1}^{-}+L_{1}^{+}\right), \\
\tilde{D}= & \left(P^{2}+\frac{(P \cdot \tilde{q})^{2} Q^{2}}{\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}}\right)\left((K \cdot P) K\left(L_{1}^{-}-L_{1}^{+}\right)-\frac{Q^{2}}{4}(P \cdot \tilde{q})\left(L_{0}^{-}-L_{0}^{+}\right)\right) \\
& -2(P \cdot \tilde{q}) \frac{\tilde{Q}^{2}-\frac{3 Q^{2}}{4}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}}\left((K \cdot P)^{2}-\frac{Q^{2} P^{2}}{4}\right)\left(L_{0}^{-}-L_{0}^{+}\right), \tag{P.16}
\end{align*}
$$

and master integrals:

$$
\begin{align*}
L_{0}^{ \pm} & =\int_{0}^{\tilde{q}_{0}^{M}} \mathrm{~d} \tilde{q}_{0} \frac{\Pi_{K}^{ \pm}}{D_{\gamma}}, \quad L_{1}^{ \pm}=\int_{0}^{\tilde{q}_{0}^{M}} \tilde{q}_{0} \mathrm{~d} \tilde{q}_{0} \frac{\Pi_{K}^{ \pm}}{D_{\gamma}} \\
D_{\gamma} & =\sqrt{\left(\tilde{Q}^{2}-\frac{Q^{2}}{4}\right)^{2}-Q^{2} \tilde{q}_{0}^{2}+\frac{Q^{2}}{|\vec{P}|^{2}}\left((P \cdot \tilde{q})-P_{0} \tilde{q}_{0}\right)^{2}} \\
\tilde{q}_{0}^{M} & =\frac{(P \cdot \tilde{q}) P_{0}+|\vec{P}| \sqrt{(P \cdot \tilde{q})^{2}+P^{2} \tilde{Q}^{2}}}{P^{2}} \tag{P.17}
\end{align*}
$$

The integration region is given by

$$
\begin{align*}
\left(|\vec{P}|-\sqrt{P_{0}^{2}-W^{2}}\right)^{2} & \leq \tilde{Q}^{2} \leq\left(|\vec{P}|+\sqrt{P_{0}^{2}-W^{2}}\right)^{2} \\
W_{\mathrm{thr}}^{2} & \leq W^{2} \leq P_{0}^{2} \tag{P.18}
\end{align*}
$$

The contribution to the weighting functions arising from the integral along the imaginary axis in the $K$-frame is given by

$$
\begin{align*}
w_{1}\left(W^{2}, \tilde{Q}^{2}\right)= & \frac{\alpha}{8 \pi^{2}} \frac{G_{E}}{\varepsilon G_{E}^{2}+\tau G_{M}^{2}} \frac{P^{2}}{M^{2}} \frac{Q^{4}}{(K \cdot P)^{2}+M^{4} \tau(1+\tau)} \frac{\tilde{Q}^{2}}{\tilde{Q}^{2}-\frac{Q^{2}}{4}} \\
& \times \Re \int_{0}^{\pi} \frac{(\tilde{A}+\tilde{B}) \sin ^{2} \psi \mathrm{~d} \psi}{\tilde{Q}^{2}-\frac{Q^{2}}{4}+2 i K \tilde{Q} \cos \psi}, \\
w_{2}\left(W^{2}, \tilde{Q}^{2}\right)= & \frac{\alpha}{8 \pi^{2}} \frac{G_{E}}{\varepsilon G_{E}^{2}+\tau G_{M}^{2}} \frac{P^{2}}{M^{2}} \frac{Q^{4}}{(K \cdot P)^{2}+M^{4} \tau(1+\tau)} \frac{\tilde{Q}^{2}}{\tilde{Q}^{2}+\frac{Q^{2}}{4}} \\
& \times \Re \int_{0}^{\pi} \frac{(\tilde{C}+\tilde{D}) \sin ^{2} \psi \mathrm{~d} \psi}{\tilde{Q}^{2}-\frac{Q^{2}}{4}+2 i K \tilde{Q} \cos \psi}, \tag{P.19}
\end{align*}
$$

with the notations of Eqs. (P.12, P.13) and integrals of Eqs. (P.9). The integral along the imaginary axis contributes in the range $0 \leq \tilde{Q}^{2} \leq \infty$.

In numerical evaluations of the TPE correction, we use the analytical expressions for the master integrals of the poles contributions. The integration over the hyperangle $\psi$ for the integral along the imaginary axis is performed numerically.
The experimentally inaccessible region $-Q^{2} / 4<\tilde{Q}^{2}<0$, corresponding with negative $\tilde{Q}^{2}$ values, is present in the $\tilde{Q}^{2}$ integration. For this relatively small region, we approximate the unpolarized proton structure functions by the following relations: $F_{1}\left(W^{2},-\tilde{Q}^{2}\right) \approx$ $F_{1}\left(W^{2}, \tilde{Q}^{2}\right), F_{2}\left(W^{2},-\tilde{Q}^{2}\right) \approx-F_{2}\left(W^{2}, \tilde{Q}^{2}\right)$, according to the approximation of Eqs. (5.83) and (5.84).

## Oleksandr Tomalak



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## Education

11/2012-10/2016 Ph. D. in Theoretical Physics, Johannes Gutenberg University, Mainz, Germany, Summa Cum Laude, top 10\%.

Thesis Two-photon exchange corrections in elastic lepton-proton scattering
Supervisor Prof. Dr. Marc Vanderhaeghen
09/2010-06/2012 Master of Science in Theoretical Physics, Taras Shevchenko National University, Kyiv, Ukraine, Red Diploma, top 10\%.
Specialized in Nuclear and Particle Physics, Quantum Field Theory
Diploma Generation of gravitational waves during magnetic instability in early universe
Supervisor Dr. Yurii Shtanov
09/2006-06/2010 Bachelor of Science in Physics, Taras Shevchenko National University, Kyiv, Ukraine, Red Diploma, top 10\%.
Specialized in Nuclear and Particle Physics, Quantum Field Theory
Diploma Muons registration efficiency determination for $\mathrm{F}_{2}$ proton structure function measurement in ZEUS experiment
Supervisor Dr. Vladimir Aushev

## Awards and Honors

11/2015 Winner of the poster competition, EINN 2015, Paphos, Cyprus
01/2015 Best poster prize, 53rd International Winter Meeting on Nuclear Physics, Bormio, Italy
11/2012-11/2014 Fellowship in the Graduate School "Symmetry Breaking in Fundamental Interactions", Johannes Gutenberg University, Mainz, Germany (top 10 \% admitted)
06/2012 Alumnus of the year, Taras Shevchenko National University, Kyiv, Ukraine
07/2006 Silver medal, 37th International Physics Olympiad, Singapore

## Soft skills

11/2015 Frontiers and Careers in Photonuclear Physics, Annabelle Hotel, Paphos, Cyprus
08/2014 Frontiers and Careers in Photonuclear Physics, MIT, Cambridge, MA, US
11/2013 Time and Self Management, Johannes Gutenberg University, Mainz, Germany

## Computer skills

Mathematica, Latex, DataGraph; C, C++, Fortran, Python; Root. Windows, Linux, Mac OS, office and dictionary software.

## Work Experience

11/2012-10/2016 PhD student, Johannes Gutenberg University of Mainz, Mainz, Germany. Research on two-photon exchange corrections in elastic lepton-proton scattering.
06/2015 Guest scientist, Department of Physics, University of Pavia, Pavia, Italy. Comparison of the results for the pion-nucleon contribution to the single spin asymmetry in elastic electron-proton scattering with Prof. Dr. Barbara Pasquini.
2010-2011 Member of D0 collaboration, D0 Experiment, Fermilab, Batavia, IL, USA.
Calorimeter and muon system monitoring, DAQ system monitoring. Estimation of purity for Jet Energy Scale group. Measurement of $\mathrm{W}+\mathrm{c}+\mathrm{jets}$ to $\mathrm{W}+$ jets cross sections ratio in QCD group. Contact person: Dr. Dmitry Bandurin, Florida State University, FL, US.
2009-2010 Bachelor student, ZEUS Experiment, DESY, Hamburg, Germany.
Efficiencies of muon reconstruction algorithms at ZEUS experiment for 2006, 2007 data. Contact person: Dr. Massimo Corradi, INFN, Bologna, Italy.

## Teaching Experience

04/2014-08/2014 Assistant, Johannes Gutenberg University of Mainz, Mainz, Germany.
Classical Electrodynamics. Combine exercise questions, correct students solutions, give consultations. Lecture when professor was out of town.
10/2013-02/2014 Assistant, Johannes Gutenberg University of Mainz, Mainz, Germany.
Advanced Quantum Mechanics and Quantum Field Theory. Combine exercise questions, correct students solutions, give consultations. Lecture when professor was out of town.
04/2013-08/2013 Assistant, Johannes Gutenberg University of Mainz, Mainz, Germany. Advanced Quantum Mechanics and Quantum Field Theory. Correct students solutions, give consultations. Lecture when professor was out of town.
09/2008-05/2009 Assistant, Scientific and Educational Centre of Bogolyubov Institute for Theoretical Physics, Kyiv, Ukraine.
Classical Mechanics seminars for first-year undergraduate students.
10/2007-12/2009 Assistant, Ukrainian Lyceum of Physics and Mathematics, Kyiv, Ukraine. Physics Olympiad preparation for high school students.

## Peer-reviewed scientific papers

## M. Sydorenko, O. Tomalak and Yu. Shtanov

Magnetic fields and chiral asymmetry in the early hot universe
JCAP 10, 018 (2016), arXiv:1607.04845 [astro-ph.CO]

## O. Tomalak and M. Vanderhaeghen

Two-photon exchange correction in elastic unpolarized muon-proton scattering at low momentum transfer
Eur. Phys. J. C 76, no. 3, 125 (2016), arXiv:1512.09113 [hep-ph]

## O. Tomalak and M. Vanderhaeghen

Two-photon exchange correction in elastic unpolarized electron-proton scattering at small momentum transfer
Phys. Rev. D 93, 013023 (2016), arXiv:1508.03759 [hep-ph]

## O. Tomalak and M. Vanderhaeghen

Subtracted dispersion relation estimate of two-photon exchange
EPJ A Highlight and Europhysics News 46/3

## O. Tomalak and M. Vanderhaeghen

Subtracted dispersion relation formalism for the two-photon exchange correction in elastic electron-proton scattering: comparison with data
Eur. Phys. J. A 51, no. 2, 24 (2015), arXiv:1408.5330 [hep-ph]

## O. Tomalak and M. Vanderhaeghen

Two-photon exchange corrections in elastic muon-proton scattering Phys. Rev. D 90, 013006 (2014), arXiv:1405.1600 [hep-ph]

## International Conferences, Workshops and Schools

04/2016 Two-photon exchange corrections in elastic lepton-proton scattering APS DNP 2016 Fall Meeting, Vancouver, BC, Canada (talk)
04/2016 Two-photon exchange corrections in elastic lepton-proton scattering APS April Meeting 2016, Salt Lake City, UT, US (talk)
11/2015 Two-photon exchange corrections in elastic lepton-proton scattering EINN 2015, Annabelle Hotel, Paphos, Cyprus (talk as a winner of the poster session)
09/2015 Two-photon exchange corrections in elastic lp scattering at low momentum transfer Probing Hadron Structure with Lepton and Hadron Beams, Erice, Sicily, Italy (talk)
06/2015 Two-photon exchange corrections in elastic lepton-proton scattering Department of Physics, University of Pavia, Pavia, Italy (seminar)
03/2015 Two-photon exchange corrections in elastic lepton-proton scattering German Physical Society Spring Meeting, Heidelberg, Germany (talk)
01/2015 Two-photon exchange corrections in elastic lepton-proton scattering 53rd International Winter Meeting on Nuclear Physics, Bormio, Italy (poster)
08/2014 Two-photon exchange corrections in elastic electron-proton and muon-proton scattering. Elastic contribution. Dispersive framework Frontiers and Careers in Photonuclear Physics, MIT, Cambridge, MA, US (talk)
06/2014 Two-photon exchange corrections in elastic electron-proton and muon-proton scattering. Elastic contribution. Dispersive framework Mainz Institute for Theoretical Physics (MITP) Workshop on "Proton Radius Puzzle", Schloss Waldthausen, Mainz, Germany (talk)
03/2014 Two-photon exchange corrections in elastic ep scattering. Dispersive framework German Physical Society Spring Meeting, Frankfurt, Germany (talk)
$01 / 2014 \mathrm{GGI}$ lectures on the theory of fundamental interactions Galileo Galilei Institute for Theoretical Physics, Firenze, Italy (participant)
11/2013 Two-photon exchange corrections in elastic ep scattering. Dispersive framework EINN 2013, Annabelle Hotel, Paphos, Cyprus (talk)
06/2013 Non-perturbative QCD: Hadron Structure and Hadronic Matter ICTP SFAIR, Sao Paulo, Brazil (participant)
08/2011 Wolfram Mathematica School on Theoretical Physics "Scattering Amplitudes and AdS/CFT"
Perimeter Institute, Canada (participant)

## Languages

Russian Native
English, Ukrainian Fluent
German Basics

## Personal interests

- Mountaineering - Skiing
- Running - Cycling


## Versicherung

## für das Gesuch um Zulassung zur Promotion am Fachbereich 08

Hiermit versichere ich gemäß § 12 Abs. 3e der Promotionsordnung des Fachbereichs 08, Physik, Mathematik und Informatik der Johannes Gutenberg-Universität Mainz vom 02.12.2013:
a) Ich habe die jetzt als Dissertation vorgelegte Arbeit selbständig verfasst. Es wurden ausschließlich die angegebenen Quellen und Hilfsmittel verwendet. Von der Ordnung zur Sicherung guter wissenschaftlicher Praxis in Forschung und Lehre und vom Verfahren zum Umgang mit wissenschaftlichem Fehlverhalten habe ich Kenntnis genommen.
b) Ich habe oder hatte die jetzt als Dissertation vorgelegte Arbeit nicht schon als Prüfungsarbeit für eine andere Prüfung eingereicht *)
teh hatte-die jezt als-Dissertation vorgelegte Arbeit-als-Prüfungsarbeit für folgende-Prüfung
eingereicht:*)
(Bezeichnung der Prüfung)
(Bezeichnung und Ort der Prüfungsstelle)
c) Ich hatte weder die jetzt als Dissertation vorgelegte Arbeit noch Teile davon an einer anderen Stelle als Dissertation eingereicht *)
teh hatte die folgende Abhandlung mit anstehendem Ergebnis als Disseftation eingereicht *)


## Bibliography

[1] Paul A. M. Dirac. The quantum theory of the electron. Proc. Roy. Soc. Lond., A117:610624, 1928.
[2] C. D. Anderson. The Positive Electron. Phys. Rev., 43:491-494, 1933.
[3] Willis E. Lamb and Robert C. Retherford. Fine Structure of the Hydrogen Atom by a Microwave Method. Phys. Rev., 72:241-243, 1947.
[4] H. M. Foley and P. Kusch. On the Intrinsic Moment of the Electron. Phys. Rev., 73:412412, 1948.
[5] H. A. Bethe. The Electromagnetic shift of energy levels. Phys. Rev., 72:339-341, 1947.
[6] S. Tomonaga. On a relativistically invariant formulation of the quantum theory of wave fields. Prog. Theor. Phys., 1:27-42, 1946.
[7] Julian S. Schwinger. On Quantum electrodynamics and the magnetic moment of the electron. Phys. Rev., 73:416-417, 1948.
[8] Julian S. Schwinger. Quantum electrodynamics. I A covariant formulation. Phys. Rev., 74:1439, 1948.
[9] R. P. Feynman. Space-time approach to quantum electrodynamics. Phys. Rev., 76:769789, 1949.
[10] R. P. Feynman. The Theory of positrons. Phys. Rev., 76:749-759, 1949.
[11] R. P. Feynman. Mathematical formulation of the quantum theory of electromagnetic interaction. Phys. Rev., 80:440-457, 1950.
[12] F. J. Dyson. The Radiation theories of Tomonaga, Schwinger, and Feynman. Phys. Rev., 75:486-502, 1949.
[13] F. J. Dyson. The S matrix in quantum electrodynamics. Phys. Rev., 75:1736-1755, 1949.
[14] Wikipedia. Standard model — wikipedia, the free encyclopedia, 2016. [Online; accessed 08-June-2016].
[15] K. A. Olive et al. Review of Particle Physics. Chin. Phys., C38:090001, 2014.
[16] J. D. Bjorken and S. L. Glashow. Elementary Particles and SU(4). Phys. Lett., 11:255257, 1964.
[17] S. L. Glashow, J. Iliopoulos, and L. Maiani. Weak Interactions with Lepton-Hadron Symmetry. Phys. Rev., D2:1285-1292, 1970.
[18] Makoto Kobayashi and Toshihide Maskawa. CP Violation in the Renormalizable Theory of Weak Interaction. Prog. Theor. Phys., 49:652-657, 1973.
[19] S. W. Herb et al. Observation of a Dimuon Resonance at 9.5 GeV in $400-\mathrm{GeV}$ ProtonNucleus Collisions. Phys. Rev. Lett., 39:252-255, 1977.
[20] S. Abachi et al. Search for the top quark in $p \bar{p}$ collisions at $\sqrt{s}=1.8 \mathrm{TeV}$. Phys. Rev. Lett., 72:2138-2142, 1994.
[21] F. Abe et al. Observation of top quark production in $\bar{p} p$ collisions. Phys. Rev. Lett., 74:2626-2631, 1995.
[22] Gerard 't Hooft and M. J. G. Veltman. Regularization and Renormalization of Gauge Fields. Nucl. Phys., B44:189-213, 1972.
[23] Serguei Chatrchyan et al. Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC. Phys. Lett., B716:30-61, 2012.
[24] Georges Aad et al. Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC. Phys. Lett., B716:1-29, 2012.
[25] Y. Fukuda et al. Evidence for oscillation of atmospheric neutrinos. Phys. Rev. Lett., 81:1562-1567, 1998.
[26] Q. R. Ahmad et al. Measurement of the rate of $\nu_{e}+d \rightarrow p+p+e^{-}$interactions produced by ${ }^{8} B$ solar neutrinos at the Sudbury Neutrino Observatory. Phys. Rev. Lett., 87:071301, 2001.
[27] ATLAS Collaboration. Search for resonances decaying to photon pairs in $3.2 \mathrm{fb}^{-1}$ of $p p$ collisions at $\sqrt{s}=13 \mathrm{TeV}$ with the ATLAS detector. 2015.
[28] CMS Collaboration. Search for new physics in high mass diphoton events in proton-proton collisions at 13 TeV .2015.
[29] G. W. Bennett et al. Measurement of the negative muon anomalous magnetic moment to 0.7 ppm. Phys. Rev. Lett., 92:161802, 2004.
[30] G. W. Bennett et al. Final Report of the Muon E821 Anomalous Magnetic Moment Measurement at BNL. Phys. Rev., D73:072003, 2006.
[31] Michel Davier, Andreas Hoecker, Bogdan Malaescu, and Zhiqing Zhang. Reevaluation of the Hadronic Contributions to the Muon g-2 and to $\alpha\left(M_{Z}^{2}\right)$. Eur. Phys. J., C71:1515, 2011. [Erratum: Eur. Phys. J.C72,1874(2012)].
[32] Fred Jegerlehner. Application of Chiral Resonance Lagrangian Theories to the Muon $g-2$. Acta Phys. Polon., B44(11):2257-2266, 2013.
[33] B. Lee Roberts. The Fermilab muon (g-2) project. Nucl. Phys. Proc. Suppl., 218:237-241, 2011.
[34] Hiromi Iinuma. New approach to the muon g-2 and EDM experiment at J-PARC. J. Phys. Conf. Ser., 295:012032, 2011.
[35] M. N. Rosenbluth. High Energy Elastic Scattering of Electrons on Protons. Phys. Rev., 79:615-619, 1950.
[36] Robert Hofstadter. Electron scattering and nuclear structure. Rev. Mod. Phys., 28:214254, 1956.
[37] Robert Hofstadter. The electron-scattering method and its application to the structure of nuclei and nucleons. Nobel Lecture, December 11, 1961.
[38] A. I. Akhiezer and Mikhail P. Rekalo. Polarization phenomena in electron scattering by protons in the high energy region. Sov. Phys. Dokl., 13:572, 1968. [Dokl. Akad. Nauk Ser. Fiz.180,1081(1968)].
[39] A. I. Akhiezer and Mikhail P. Rekalo. Polarization effects in the scattering of leptons by hadrons. Sov. J. Part. Nucl., 4:277, 1974. [Fiz. Elem. Chast. Atom. Yadra4,662(1973)].
[40] Norman Dombey. Scattering of polarized leptons at high energy. Rev. Mod. Phys., 41:236-246, 1969.
[41] Norman Dombey. An experiment to measure the electron form factor of the proton. Phys. Lett., B29:588-589, 1969.
[42] D.R. Yennie, M. M. Levy, and D.G. Ravenhall. Electromagnetic Structure of Nucleons. Rev. Mod. Phys., 29:144-157, 1957.
[43] F. J. Ernst, R. G. Sachs, and K. C. Wali. Electromagnetic form factors of the nucleon. Phys. Rev., 119:1105-1114, 1960.
[44] L. N. Hand, D. G. Miller, and Richard Wilson. Electric and Magnetic Form Factor of the Nucleon. Rev. Mod. Phys., 35:335, 1963.
[45] M. K. Jones et al. $G_{E_{p}} / G_{M_{p}}$ ratio by polarization transfer in polarized $\overrightarrow{e p} \rightarrow e \vec{p}$. Phys. Rev. Lett., 84:1398-1402, 2000.
[46] O. Gayou et al. Measurement of $G_{E_{p}} / G_{M_{p}}$ in polarized $\vec{e} p \rightarrow e \vec{p}$ to $Q^{2}=5.6 \mathrm{GeV}^{2}$. Phys. Rev. Lett., 88:092301, 2002.
[47] V. Punjabi et al. Proton elastic form factor ratios to $Q^{2}=3.5 \mathrm{GeV}^{2}$ by polarization transfer. Phys. Rev., C71:055202, 2005. [Erratum: Phys. Rev.C71,069902(2005)].
[48] A. J. R. Puckett et al. Recoil Polarization Measurements of the Proton Electromagnetic Form Factor Ratio to $Q^{2}=8.5 \mathrm{GeV}^{2}$. Phys. Rev. Lett., 104:242301, 2010.
[49] Vina Punjabi. The proton form factor measurements at Jefferson Lab, past and future. Phys. Part. Nucl., 45:163-166, 2014.
[50] V. Punjabi, C. F. Perdrisat, M. K. Jones, E. J. Brash, and C. E. Carlson. The Structure of the Nucleon: Elastic Electromagnetic Form Factors. Eur. Phys. J., A51:79, 2015.
[51] P. G. Blunden, W. Melnitchouk, and J. A. Tjon. Two-photon exchange and elastic electron-proton scattering. Phys. Rev. Lett., 91:142304, 2003.
[52] Pierre A. M. Guichon and M. Vanderhaeghen. How to reconcile the Rosenbluth and the polarization transfer method in the measurement of the proton form factors. Phys. Rev. Lett., 91:142303, 2003.
[53] Carl E. Carlson and Marc Vanderhaeghen. Two-Photon Physics in Hadronic Processes. Ann. Rev. Nucl. Part. Sci, 57:171-204, 2007.
[54] J. Arrington, P. G. Blunden, and W. Melnitchouk. Review of two-photon exchange in electron scattering. Prog. Part. Nucl. Phys., 66:782-833, 2011.
[55] Y. C. Chen, A. Afanasev, S. J. Brodsky, C. E. Carlson, and M. Vanderhaeghen. Partonic calculation of the two-photon exchange contribution to elastic electron-proton scattering at large momentum transfer. Phys. Rev. Lett., 93:122301, 2004.
[56] Andrei V. Afanasev, Stanley J. Brodsky, Carl E. Carlson, Yu-Chun Chen, and Marc Vanderhaeghen. The two-photon exchange contribution to elastic electron-nucleon scattering at large momentum transfer. Phys. Rev., D72:013008, 2005.
[57] P. G. Blunden, W. Melnitchouk, and J. A. Tjon. Two-photon exchange in elastic electronnucleon scattering. Phys. Rev., C72:034612, 2005.
[58] Mikhail Gorchtein. Dispersive contributions to $e^{+} p / e^{-} p$ cross section ratio in forward regime. Phys. Lett., B644:322-330, 2007.
[59] Dmitry Borisyuk and Alexander Kobushkin. Two-photon exchange in dispersion approach. Phys. Rev., C78:025208, 2008.
[60] Dmitry Borisyuk and Alexander Kobushkin. Perturbative QCD predictions for twophoton exchange. Phys. Rev., D79:034001, 2009.
[61] Nikolai Kivel and Marc Vanderhaeghen. Two-photon exchange in elastic electron-proton scattering: QCD factorization approach. Phys. Rev. Lett., 103:092004, 2009.
[62] N. Kivel and M. Vanderhaeghen. Two-photon exchange corrections to elastic electronproton scattering at large momentum transfer within the SCET approach. JHEP, 04:029, 2013.
[63] Krzysztof M. Graczyk. Comparison of neural network and hadronic model predictions of the two-photon exchange effect. Phys. Rev., C88:065205, 2013.
[64] Dmitry Borisyuk and Alexander Kobushkin. Two-photon exchange amplitude with $\pi \mathrm{N}$ intermediate states: $\mathrm{P}_{33}$ channel. Phys. Rev., C89(2):025204, 2014.
[65] Hai-Qing Zhou and Shin Nan Yang. $\Delta(1232)$ resonance contribution to two-photon exchange in electron-proton scattering revisited. Eur. Phys. J., A51(8):105, 2015.
[66] Dmitry Borisyuk and Alexander Kobushkin. Two-photon exchange amplitude with $\pi N$ intermediate states: Spin-1/2 and spin-3/2 channels. Phys. Rev., C92(3):035204, 2015.
[67] J. Arrington, W. Melnitchouk, and J. A. Tjon. Global analysis of proton elastic form factor data with two-photon exchange corrections. Phys. Rev., C76:035205, 2007.
[68] Yu-Chun Chen, Chung-Wen Kao, and Shin-Nan Yang. Is there model-independent evidence of the two-photon exchange effect in the electron-proton elastic scattering crosssection? Phys. Lett., B652:269-274, 2007.
[69] Dmitry Borisyuk and Alexander Kobushkin. Phenomenological analysis of two-photon exchange effects in proton form factor measurements. Phys. Rev., C76:022201, 2007.
[70] M. A. Belushkin, H. W. Hammer, and U. G. Meißner. Model-independent extraction of two-photon effects in elastic electron-proton scattering. Phys. Lett., B658:138-142, 2008.
[71] I. A. Qattan, A. Alsaad, and J. Arrington. Reexamination of phenomenological twophoton exchange corrections to the proton form factors and $e^{ \pm} p$ scattering. Phys. Rev., C84:054317, 2011.
[72] Julia Guttmann, Nikolai Kivel, Mehdi Meziane, and Marc Vanderhaeghen. Determination of two-photon exchange amplitudes from elastic electron-proton scattering data. Eur. Phys. J., A47:77, 2011.
[73] M. Meziane et al. Search for effects beyond the Born approximation in polarization transfer observables in $\vec{e} p$ elastic scattering. Phys. Rev. Lett., 106:132501, 2011.
[74] J. Mar, Barry C. Barish, Jerome Pine, D. H. Coward, H. C. DeStaebler, J. Litt, A. Minten, Richard E. Taylor, and Martin Breidenbach. A comparison of electron-proton and positron-proton elastic scattering at four momentum transfers up to $5(\mathrm{GeV} / \mathrm{c})^{2}$. Phys. Rev. Lett., 21:482-484, 1968.
[75] A. V. Gramolin et al. Measurement of the two-photon exchange contribution in elastic ep scattering at VEPP-3. Nucl. Phys. Proc. Suppl., 225-227:216-220, 2012.
[76] I. A. Rachek et al. Measurement of the two-photon exchange contribution to the elastic $e^{ \pm} p$ scattering cross sections at the VEPP-3 storage ring. Phys. Rev. Lett., 114(6):062005, 2015.
[77] M. Moteabbed et al. Demonstration of a novel technique to measure two-photon exchange effects in elastic $e^{ \pm} p$ scattering. Phys. Rev., C88:025210, 2013.
[78] D. Adikaram et al. Towards a resolution of the proton form factor problem: new electron and positron scattering data. Phys. Rev. Lett., 114:062003, 2015.
[79] D. Rimal et al. Measurement of two-photon exchange effect by comparing elastic $e^{ \pm} p$ cross sections. 2016.
[80] M. Kohl. The OLYMPUS experiment at DESY. AIP Conf. Proc., 1160:19-23, 2009.
[81] R. Milner et al. The OLYMPUS Experiment. Nucl. Instrum. Meth., A741:1-17, 2014.
[82] Abraham Klein. Low-Energy Theorems for Renormalizable Field Theories. Phys. Rev., 99:998-1008, 1955.
[83] V. Goldansky, O. Karpukhin, A. Kutsenko, and V. Pavlovskaya. Elastic $\gamma p$ scattering at 40 to 70 MeV and polarizability of the proton. Nucl. Phys., 18:473-491, 1960.
[84] John Powell. Note on the Bremsstrahlung Produced by Protons. Phys. Rev., 75:32-34, 1949.
[85] A. Baldin. Polarizability of nucleons. Nucl. Phys., 18:310-317, 1960.
[86] Veronique Bernard, Norbert Kaiser, and Ulf G. Meißner. Chiral expansion of the nucleon's electromagnetic polarizabilities. Phys. Rev. Lett., 67:1515-1518, 1991.
[87] Veronique Bernard, Norbert Kaiser, and Ulf G. Meißner. Nucleons with chiral loops: Electromagnetic polarizabilities. Nucl. Phys., B373:346-370, 1992.
[88] Franziska Hagelstein, Rory Miskimen, and Vladimir Pascalutsa. Nucleon Polarizabilities: from Compton Scattering to Hydrogen Atom. Prog. Part. Nucl. Phys., 88:29-97, 2016.
[89] Martin Schumacher. Polarizability of the nucleon and Compton scattering. Prog. Part. Nucl. Phys., 55:567-646, 2005.
[90] Vadim Lensky, Judith McGovern, and Vladimir Pascalutsa. Predictions of covariant chiral perturbation theory for nucleon polarisabilities and polarised Compton scattering. Eur. Phys. J., C75(12):604, 2015.
[91] M. E. Christy and Peter E. Bosted. Empirical fit to precision inclusive electron-proton cross sections in the resonance region. Phys. Rev., C81:055213, 2010.
[92] S. E. Kuhn, J. P. Chen, and E. Leader. Spin Structure of the Nucleon - Status and Recent Results. Prog. Part. Nucl. Phys., 63:1-50, 2009.
[93] D. Drechsel, O. Hanstein, S. S. Kamalov, and L. Tiator. A Unitary isobar model for pion photoproduction and electroproduction on the proton up to 1 GeV . Nucl. Phys., A645:145-174, 1999.
[94] D. Drechsel, S. S. Kamalov, and L. Tiator. Unitary Isobar Model - MAID2007. Eur. Phys. J., A34:69-97, 2007.
[95] R. A. Arndt, W. J. Briscoe, I. I. Strakovsky, and R. L. Workman. Analysis of pion photoproduction data. Phys. Rev., C66:055213, 2002.
[96] J. D. Bjorken. Asymptotic Sum Rules at Infinite Momentum. Phys. Rev., 179:1547-1553, 1969.
[97] Richard P. Feynman. Very high-energy collisions of hadrons. Phys. Rev. Lett., 23:14151417, 1969.
[98] C. G. Callan and David J. Gross. Crucial Test of a Theory of Currents. Phys. Rev. Lett., 21:311-313, 1968.
[99] Murray Gell-Mann. A Schematic Model of Baryons and Mesons. Phys. Lett., 8:214-215, 1964.
[100] G. Zweig. An SU(3) model for strong interaction symmetry and its breaking. Version 1. 1964.
[101] Henry W. Kendall. Deep inelastic scattering: Experiments on the proton and the observation of scaling. Rev. Mod. Phys., 63:597-614, 1991.
[102] R. W. Mcallister and R. Hofstadter. Elastic Scattering of $188-\mathrm{MeV}$ Electrons From the Proton and the $\alpha$ Particle. Phys. Rev., 102:851-856, 1956.
[103] J. C. Bernauer et al. High-precision determination of the electric and magnetic form factors of the proton. Phys. Rev. Lett., 105:242001, 2010.
[104] J. C. Bernauer et al. Electric and magnetic form factors of the proton. Phys. Rev., C90(1):015206, 2014.
[105] I. T. Lorenz, Ulf-G. Meißner, H. W. Hammer, and Y. B. Dong. Theoretical Constraints and Systematic Effects in the Determination of the Proton Form Factors. Phys. Rev., D91(1):014023, 2015.
[106] Keith Griffioen, Carl Carlson, and Sarah Maddox. Consistency of electron scattering data with a small proton radius. Phys. Rev., C93(6):065207, 2016.
[107] Douglas W. Higinbotham, Al Amin Kabir, Vincent Lin, David Meekins, Blaine Norum, and Brad Sawatzky. Proton radius from electron scattering data. Phys. Rev., C93(5):055207, 2016.
[108] Gabriel Lee, John R. Arrington, and Richard J. Hill. Extraction of the proton radius from electron-proton scattering data. Phys. Rev., D92(1):013013, 2015.
[109] John Arrington and Ingo Sick. Evaluation of the proton charge radius from ep scattering. J. Phys. Chem. Ref. Data, 44:031204, 2015.
[110] John Arrington. An examination of proton charge radius extractions from ep scattering data. J. Phys. Chem. Ref. Data, 44:031203, 2015.
[111] F. Schmidt-Kaler, D. Leibfried, M. Weitz, and T. W. Hansch. Precision measurement of the isotope shift of the 1S- 2 S transition of atomic hydrogen and deuterium. Phys. Rev. Lett., 70:2261-2264, 1993.
[112] Randolf Pohl et al. The muonic hydrogen Lamb shift experiment. Can. J. Phys., 83(4):339-349, 2005.
[113] Randolf Pohl et al. The size of the proton. Nature, 466:213-216, 2010.
[114] Aldo Antognini et al. Proton Structure from the Measurement of 2S-2P Transition Frequencies of Muonic Hydrogen. Science, 339:417-420, 2013.
[115] Peter J. Mohr, Barry N. Taylor, and David B. Newell. CODATA Recommended Values of the Fundamental Physical Constants: 2010. Rev. Mod. Phys., 84:1527-1605, 2012.
[116] Randolf Pohl, Ronald Gilman, Gerald A. Miller, and Krzysztof Pachucki. Muonic hydrogen and the proton radius puzzle. Ann. Rev. Nucl. Part. Sci., 63:175-204, 2013.
[117] Jan C. Bernauer and Randolf Pohl. The proton radius problem. Sci. Am., 310(2):18-25, 2014.
[118] Carl E. Carlson. The Proton Radius Puzzle. Prog. Part. Nucl. Phys., 82:59-77, 2015.
[119] X. Zhan et al. High-Precision Measurement of the Proton Elastic Form Factor Ratio $\mu_{p} G_{E} / G_{M}$ at low $Q^{2}$. Phys. Lett., B705:59-64, 2011.
[120] Christian G. Parthey, Arthur Matveev, Janis Alnis, Randolf Pohl, Thomas Udem, Ulrich D. Jentschura, Nikolai Kolachevsky, and Theodor W. Hänsch. Precision Measurement of the Hydrogen-Deuterium 1S-2S Isotope Shift. Phys. Rev. Lett., 104:233001, 2010.
[121] Michael O. Distler. Electron-deuteron scattering at Mainz. MITP Proton Radius Puzzle Workshop, Mainz, Germany, 2014.
[122] Randolf Pohl et al. Laser spectroscopy of muonic deuterium. Science, 353(6300):669-673, 2016.
[123] A. Antognini. Muonic atoms: from atomic to nuclear and particle physics. Electromagnetic Interactions with Nucleons and Nuclei, Paphos, Cyprus, 2015.
[124] Ashot Gasparian. The PRad experiment and the proton radius puzzle. EPJ Web Conf., 73:07006, 2014.
[125] M. Mihoviloviç and H. Merkel. Initial state radiation experiment at MAMI. AIP Conf. Proc., 1563:187-190, 2013.
[126] M. Mihoviloviç. ISR experiment at MAMI. New Vistas in Low-Energy Precision Physics, Mainz, Germany, 2016.

## Bibliography

[127] R. Gilman et al. Studying the Proton "Radius" Puzzle with $\mu p$ Elastic Scattering. 2013.
[128] E. Kraus, K. E. Mesick, A. White, R. Gilman, and S. Strauch. Polynomial fits and the proton radius puzzle. Phys. Rev., C90(4):045206, 2014.
[129] Ingo Sick and Dirk Trautmann. Proton root-mean-square radii and electron scattering. Phys. Rev., C89(1):012201, 2014.
[130] Vladyslav Pauk and Marc Vanderhaeghen. Lepton universality test in the photoproduction of $e^{-} e^{+}$versus $\mu^{-} \mu^{+}$pairs on a proton target. Phys. Rev. Lett., 115(22):221804, 2015.
[131] Krzysztof Pachucki. Theory of the Lamb shift in muonic hydrogen. Phys. Rev., A53:20922100, 1996.
[132] Krzysztof Pachucki. Proton structure effects in muonic hydrogen. Phys. Rev., A60:35933598, 1999.
[133] R. N. Faustov and A. P. Martynenko. Proton polarizability and Lamb shift in muonic hydrogen. Phys. Atom. Nucl., 63:845-849, 2000. [Yad. Fiz.63,915(2000)].
[134] A. P. Martynenko. Proton polarizability effect in the Lamb shift of the hydrogen atom. Phys. Atom. Nucl., 69:1309-1316, 2006.
[135] Carl E. Carlson and Marc Vanderhaeghen. Higher-order proton structure corrections to the Lamb shift in muonic hydrogen. Phys. Rev., A84:020102, 2011.
[136] Mikhail Gorchtein, Felipe J. Llanes-Estrada, and Adam P. Szczepaniak. Muonic-hydrogen Lamb shift: Dispersing the nucleon-excitation uncertainty with a finite-energy sum rule. Phys. Rev., A87(5):052501, 2013.
[137] Michael I. Eides, Howard Grotch, and Valery A. Shelyuto. Theory of light hydrogen like atoms. Phys. Rept., 342:63-261, 2001.
[138] Gerald A. Miller. Proton Polarizability Contribution: Muonic Hydrogen Lamb Shift and Elastic Scattering. Phys. Lett., B718:1078-1082, 2013.
[139] Richard J. Hill and Gil Paz. Model independent analysis of proton structure for hydrogenic bound states. Phys. Rev. Lett., 107:160402, 2011.
[140] David Nevado and Antonio Pineda. Forward virtual Compton scattering and the Lamb shift in chiral perturbation theory. Phys. Rev., C77:035202, 2008.
[141] Michael C. Birse and Judith A. McGovern. Proton polarisability contribution to the Lamb shift in muonic hydrogen at fourth order in chiral perturbation theory. Eur. Phys. J., A48:120, 2012.
[142] Jose Manuel Alarcon, Vadim Lensky, and Vladimir Pascalutsa. Chiral perturbation theory of muonic hydrogen Lamb shift: polarizability contribution. Eur. Phys. J., C74(4):2852, 2014.
[143] Clara Peset and Antonio Pineda. The two-photon exchange contribution to muonic hydrogen from chiral perturbation theory. Nucl. Phys., B887:69-111, 2014.
[144] Oleksandr Tomalak and Marc Vanderhaeghen. Two-photon exchange correction to muonproton elastic scattering at low momentum transfer. Eur. Phys. J., C76(3):125, 2016.
[145] Irinel Caprini. Constraints on the virtual nucleon Compton scattering in a new dispersive formalism. Phys. Rev., D93(7):076002, 2016.
[146] A. Donnachie and P. V. Landshoff. Does the hard pomeron obey Regge factorization? Phys. Lett., B595:393-399, 2004.
[147] V. Andreev et al. Measurement of inclusive ep cross sections at high $Q^{2}$ at $\sqrt{s}=225$ and 252 GeV and of the longitudinal proton structure function $F_{L}$ at HERA. Eur. Phys. J., C74(4):2814, 2014.
[148] A. Antognini. Hyperfine splittings in muonic hydrogen and $\mathrm{He}^{3}$. Open Users Meeting BV47, PSI, Switzerland, 2016.
[149] Murray Gell-Mann, M. L. Goldberger, and Walter E. Thirring. Use of causality conditions in quantum theory. Phys. Rev., 95:1612-1627, 1954.
[150] Vladimir Pascalutsa and Marc Vanderhaeghen. Sum rules for light-by-light scattering. Phys. Rev. Lett., 105:201603, 2010.
[151] A. C. Zemach. Proton Structure and the Hyperfine Shift in Hydrogen. Phys. Rev., 104:1771-1781, 1956.
[152] C. K. Iddings and P. M. Platzman. Nuclear Structure Correction to the Hyperfine Structure in Hydrogen. Phys. Rev., 113:192-197, 1959.
[153] C. K. Iddings. Structure of the Proton and the Hyperfine Shift in Hydrogen. Phys. Rev., 138:B446-B458, 1965.
[154] S. D. Drell and Jeremiah D. Sullivan. Polarizability contribution to the hydrogen hyperfine structure. Phys. Rev., 154:1477-1498, 1967.
[155] R. N. Faustov. The proton structure and hyperfine splitting of hydrogen energy levels. Nucl. Phys., 75:669, 1966.
[156] G. M. Zinovjev, B. V. Struminski, R. N. Faustov, and V. L. Chernyak. Proton structure and the hyperfine splitting in the hydrogen atom. Yad. Fiz., 11:1284, 1970.
[157] Geoffrey T. Bodwin and D. R. Yennie. Some Recoil Corrections to the Hydrogen Hyperfine Splitting. Phys. Rev., D37:498, 1988.
[158] R. N. Faustov, E. V. Cherednikova, and A. P. Martynenko. Proton polarizability contribution to the hyperfine splitting in muonic hydrogen. Nucl. Phys., A703:365-377, 2002.
[159] Carl E. Carlson, Vahagn Nazaryan, and Keith Griffioen. Proton structure corrections to electronic and muonic hydrogen hyperfine splitting. Phys. Rev., A78:022517, 2008.
[160] Carl E. Carlson, Vahagn Nazaryan, and Keith Griffioen. Proton structure corrections to hyperfine splitting in muonic hydrogen. Phys. Rev., A83:042509, 2011.
[161] Hugh Burkhardt and W. N. Cottingham. Sum rules for forward virtual Compton scattering. Annals Phys., 56:453-463, 1970.
[162] Luke W. Mo and Yung-Su Tsai. Radiative Corrections to Elastic and Inelastic ep and $\mu p$ Scattering. Rev. Mod. Phys., 41:205-235, 1969.
[163] L. C. Maximon and J. A. Tjon. Radiative corrections to electron-proton scattering. Phys. Rev., C62:054320, 2000.
[164] Peter G. Blunden and Ingo Sick. Proton radii and two-photon exchange. Phys. Rev., C72:057601, 2005.
[165] R. N. Lee and A. I. Milstein. Coulomb corrections to electron scattering on the extended source and the proton charge radius. 2014.
[166] Mikhail Gorchtein. Forward sum rule for the $2 \gamma$-exchange correction to the charge-radius extraction from elastic electron scattering. Phys. Rev., C90(5):052201, 2014.
[167] M. Goldberger, Y. Nambu, and R. Oehme. Dispersion relations for nucleon-nucleon scattering. Ann. of Phys., 2:226-282, 1957.
[168] P. Van Nieuwenhuizen and J. Oberholzer. Invariant amplitudes and double dispersion relations for electron-muon scattering. Nucl. Phys., B41:626-642, 1972.
[169] M. Gorchtein, Pierre A. M. Guichon, and M. Vanderhaeghen. Beam normal spin asymmetry in elastic lepton-nucleon scattering. Nucl. Phys., A741:234-248, 2004.
[170] O. Tomalak and M. Vanderhaeghen. Two-photon exchange corrections in elastic muonproton scattering. Phys. Rev., D90(1):013006, 2014.
[171] O. Tomalak and M. Vanderhaeghen. Subtracted dispersion relation formalism for the two-photon exchange correction to elastic electron-proton scattering: comparison with data. Eur. Phys. J., A51(2):24, 2015.
[172] O. Tomalak and M. Vanderhaeghen. Two-photon exchange correction in elastic unpolarized electron-proton scattering at small momentum transfer. Phys. Rev., D93(1):013023, 2016.
[173] William A. McKinley and Herman Feshbach. The Coulomb Scattering of Relativistic Electrons by Nuclei. Phys. Rev., 74:1759-1763, 1948.
[174] R. W. Brown. Comparison of the Scattering of Electrons and Positrons from Protons at Small Angles. Phys. Rev., D1:1432-1444, 1970.
[175] S. Kondratyuk, P. G. Blunden, W. Melnitchouk, and J. A. Tjon. $\Delta$ Resonance Contribution to Two-Photon Exchange in Electron-Proton Scattering. Phys. Rev. Lett., 95:172503, 2005.
[176] S. Kondratyuk and P. G. Blunden. Contribution of spin $1 / 2$ and $3 / 2$ resonances to twophoton exchange effects in elastic electron-proton scattering. Phys. Rev., C75:038201, 2007.
[177] W. Grein and P. Kroll. Dispersion theoretic analysis of the proton-proton helicity amplitudes at $t=0$. Nucl. Phys., B137:173, 1978.
[178] P. Kroll. Rising cross-section and the real part of the forward amplitude in proton-proton scattering. Lett. Nuovo Cim., 7S2:745-749, 1973. [Lett. Nuovo Cim.7,745(1973)].
[179] Pierre A. M. Guichon. Forward (p, n) reactions and nucleon-nucleon scattering. Nucl. Phys., A402:541, 1983.
[180] Johannes Blumlein and Nikolai Kochelev. On the twist-2 and twist-3 contributions to the spin-dependent electroweak structure functions. Nucl. Phys., B498:285-309, 1997.
[181] H. F. Jones and M. D. Scadron. Multipole $\gamma N-\Delta$ Form Factors and Resonant Photoand Electroproduction. Annals Phys., 81:1-14, 1973.
[182] D. Drechsel, B. Pasquini, and M. Vanderhaeghen. Dispersion relations in real and virtual Compton scattering. Phys. Rept., 378:99-205, 2003.
[183] J. Gasser and H. Leutwyler. Implications of Scaling for the Proton-Neutron Mass Difference. Nucl. Phys., B94:269, 1975.
[184] J. Gasser, M. Hoferichter, H. Leutwyler, and A. Rusetsky. Cottingham formula and nucleon polarisabilities. Eur. Phys. J., C75(8):375, 2015.
[185] Vladimir Pascalutsa and Marc Vanderhaeghen. Polarizability relations across real and virtual Compton scattering processes. Phys. Rev., D91:051503, 2015.
[186] K. A. Griffioen, S. E. Kuhn, and N. Guler. Personal communication. 2015-2016.
[187] X. Zheng. The EG4 experiment at Jefferson Lab. AIP Conf. Proc., 1155:135-144, 2009.
[188] Seonho Choi. Spin asymmetries on nucleon experiment at Jefferson Lab. AIP Conf. Proc., 1388:480-483, 2011.
[189] A. Camsonne, J. P. Chen, D. Crabb, and K. Slifer. Jlab E08-027 (g2p) experiment.
[190] A. Denig and H. Merkel. Precision hadron and particle physics at MAGIX/MESA. New Vistas in Low-Energy Precision Physics, Mainz, Germany, 2016.
[191] M. Jacob and G. C. Wick. On the general theory of collisions for particles with spin. Annals Phys., 7:404-428, 1959. [Annals Phys.281,774(2000)].
[192] L. L. Foldy. The Electromagnetic Properties of Dirac Particles. Phys. Rev., 87:688-693, 1952.
[193] D. E. Soper. Infinite-momentum helicity states. Phys. Rev., D5:1956-1962, 1972.
[194] Dian-Yong Chen and Yu-Bing Dong. Two-photon exchange in the $\ell+p \rightarrow \ell+p$ process with a massive lepton. Phys. Rev., C87(4):045209, 2013.
[195] A. V. Gramolin, V. S. Fadin, A. L. Feldman, R. E. Gerasimov, D. M. Nikolenko, I. A. Rachek, and D. K. Toporkov. A new event generator for the elastic scattering of charged leptons on protons. J. Phys., G41(11):115001, 2014.
[196] B. M. Preedom and R. Tegen. Nucleon electromagnetic form factors from scattering of polarized muons or electrons. Phys. Rev., C36:2466-2472, 1987.
[197] B. Pasquini and M. Vanderhaeghen. Resonance estimates for single spin asymmetries in elastic electron-nucleon scattering. Phys. Rev., C70:045206, 2004.
[198] M. Vanderhaeghen, J. M. Friedrich, D. Lhuillier, D. Marchand, L. Van Hoorebeke, and J. Van de Wiele. QED radiative corrections to virtual Compton scattering. Phys. Rev., C62:025501, 2000.
[199] F. Bloch and A. Nordsieck. Note on the Radiation Field of the electron. Phys. Rev., 52:54-59, 1937.
[200] T. Kinoshita. Mass singularities of Feynman amplitudes. J. Math. Phys., 3:650-677, 1962.
[201] T. D. Lee and M. Nauenberg. Degenerate Systems and Mass Singularities. Phys. Rev., 133:B1549-B1562, 1964.
[202] A. De Rujula, J. M. Kaplan, and E. De Rafael. Elastic scattering of electrons from polarized protons and inelastic electron scattering experiments. Nucl. Phys., B35:365389, 1971.
[203] G. I. Gakh, M. Konchatnyi, A. Dbeyssi, and E. Tomasi-Gustafsson. Model independent study of massive lepton elastic scattering on the proton, beyond the Born approximation. Nucl. Phys., A934:52-72, 2014.
[204] Thomas Hahn. Automatic loop calculations with FeynArts, FormCalc, and LoopTools. Nucl. Phys. Proc. Suppl., 89:231-236, 2000.
[205] G. J. van Oldenborgh and J. A. M. Vermaseren. New Algorithms for One Loop Integrals. Z. Phys., C46:425-438, 1990.
[206] P. Van Nieuwenhuizen. Muon-electron scattering cross section to order $\alpha^{3}$. Nucl. Phys., B28:429-454, 1971.
[207] C. F. Perdrisat, V. Punjabi, and M. Vanderhaeghen. Nucleon Electromagnetic Form Factors. Prog. Part. Nucl. Phys., 59:694-764, 2007.
[208] G. Ron et al. Low- $Q^{2}$ measurements of the proton form factor ratio $\mu_{p} G_{E} / G_{M}$. Phys. Rev., C84:055204, 2011.
[209] D. Drechsel, G. Knochlein, A. Yu. Korchin, A. Metz, and S. Scherer. Structure analysis of the virtual Compton scattering amplitude at low energies. Phys. Rev., C57:941-952, 1998.
[210] D. Drechsel, G. Knochlein, A. Yu. Korchin, A. Metz, and S. Scherer. Low-energy and low-momentum representation of the virtual Compton scattering amplitude. Phys. Rev., C58:1751-1757, 1998.
[211] R. Tarrach. Invariant Amplitudes for Virtual Compton Scattering Off Polarized Nucleons Free from Kinematical Singularities, Zeros and Constraints. Nuovo Cim., A28:409, 1975.
[212] Ph. Dennery. Theory of the Electro- and Photoproduction of $\pi$ Mesons. Phys. Rev., 124:2000, 1961.
[213] Frits A. Berends, A. Donnachie, and D. L. Weaver. Photoproduction and electroproduction of pions. 1. Dispersion relation theory. Nucl. Phys., B4:1-53, 1967.
[214] B. Pasquini, D. Drechsel, and L. Tiator. Invariant amplitudes for pion electroproduction. Eur. Phys. J., A34:387-403, 2007.
[215] L. W. Whitlow, Stephen Rock, A. Bodek, E. M. Riordan, and S. Dasu. A Precise extraction of $R=\sigma_{L} / \sigma_{T}$ from a global analysis of the SLAC deep inelastic ep and ed scattering cross-sections. Phys. Lett., B250:193-198, 1990.
[216] S. Dasu et al. Measurement of kinematic and nuclear dependence of $R=\sigma_{L} / \sigma_{T}$ in deep inelastic electron scattering. Phys. Rev., D49:5641-5670, 1994.
[217] M. Arneodo et al. Measurement of the proton and deuteron structure functions, $F_{2}^{p}$ and $F_{2}^{d}$, and of the ratio $\sigma_{L} / \sigma_{T}$. Nucl. Phys., B483:3-43, 1997.
[218] G. Passarino and M. J. G. Veltman. One Loop Corrections for $e^{+} e^{-}$Annihilation Into $\mu^{+} \mu^{-}$in the Weinberg Model. Nucl. Phys., B160:151, 1979.

## List of Acronyms

| TPE | Two-Photon Exchange |
| :--- | :--- |
| VVCS | doubly Virtual Compton Scattering |
| HFS | HyperFine Splitting |
| FF | Form Factor |
| DR | Dispersion Relation |
| MUSE | MUon-proton Scattering Experiment |
| ChPT | Chiral Perturbation Theory |
| SF | Structure Function |
| QED | Quantum ElectroDynamics |
| QFT | Quantum ChromoDynamics |
| QCD | Charge and Parity |
| GIM mechanism | Glashow-Iliopoulos-Maiani mechanism |
| CP | Cabibbo-Kobayashi-Maskawa matrix |
| CKM matrix | Fermi National Accelerator Laboratory |
| Fermilab | Daropean Organization for Nuclear Research |
| D0 | Collider Detector at Fermilab, experiment at Fermilab |
| CDF | Large Hadron Collider |
| LHC | Large Electron-Positron Collider |
| LEP | CERN |

## List of Acronyms

| DIS | Deep Inelastic Scattering |
| :---: | :---: |
| HERA | Hadron-Electron Ring Accelerator |
| DESY | Deutsches Elektronen-Synchrotron |
| JLab | Thomas Jefferson National Accelerator Facility |
| SAID | Scattering Analysis Interactive Dialin |
| MAID | Mainz Analysis Interactive Dialin |
| MAMI | Mainz Microtron |
| ELSA | Electron Stretcher and Accelerator |
| MIT-Bates | Massachusetts Institute of Technology Bates Linear Accelerator |
| LEGS | Laser Electron Gamma Source |
| GRAAL | GRenoble Anneau Accelerateur Laser |
| VEPP | Colliding electron-positron beams (in Russian) |
| CEBAF | Continuous Electron Beam Accelerator Facility |
| CLAS | CEBAF Large Acceptance Spectrometer |
| OLYMPUS | pOsitron-proton and eLectron-proton elastic scattering to test the hYpothesis of Multi-Photon exchange Using doriS |
| PDG | Particle Data Group |
| $\mathbf{H B} \chi \mathbf{P T}$ | Heavy Baryon ChPT |
| $\mathbf{B} \chi \mathbf{P T}$ | Baryon ChPT |
| NLO | Next-to-Leading Order |
| PSI | Paul Scherrer Institute |
| CREMA | Charge Radius Experiment with Muonic Atoms |
| CODATA | Committee on Data for Science and Technology |
| PRad | Proton Radius experiment at JLab |

## List of Acronyms

ISR Initial State Radiation
C. m. Center of mass

OPE One-Photon Exchange
DL Donnachie and Landshoff
BC Christy and Bosted
IR InfraRed
UV UltraViolet

SANE Spin Asymmetries of the Nucleon Experiment
MaTj Maximon and Tjon
HE High Energy
MT Mo and Tsai
$P_{33} \quad$ Partial wave with the pion angular momentum $\mathrm{L}=1$, spin $\mathrm{J}=3 / 2$, isospin $\mathrm{I}=3 / 2$

## List of Notations

$$
\begin{aligned}
& \alpha \\
& e \\
& \mu_{P} \\
& M, m, m_{\pi} \\
& Q^{2}, \nu_{\gamma}, \tilde{\tau}=\frac{\nu_{\gamma}^{2}}{Q^{2}} \\
& \tau_{P}=\frac{Q^{2}}{4 M^{2}}, \tau_{l}=\frac{Q^{2}}{4 m^{2}} \\
& \rho(\tau)=\tau-\sqrt{\tau(1+\tau)} \\
& \nu_{\mathrm{thr}}^{\mathrm{inel}}=m_{\pi}+\frac{m_{\pi}^{2}+Q^{2}}{2 M} \\
& x_{\mathrm{Bj}}=\frac{Q^{2}}{2 M \nu_{\gamma}} \\
& \omega \\
& k, k_{1}, k^{\prime} \\
& p, p_{1}, p^{\prime} \\
& K(P) \\
& \theta_{1}, \phi_{1} \\
& \theta_{\mathrm{cm}} \\
& q_{1}, q_{2} \\
& \tilde{q}=\frac{q_{1}+q_{2}}{2} \\
& s, u \\
& \Sigma_{s}=\Sigma\left(s, M^{2}, m^{2}\right) \\
& \nu=\frac{s-u}{4} \\
& \nu_{\mathrm{ph}}
\end{aligned}
$$

Electromagnetic coupling constant

## Electric charge

Proton magnetic moment

Proton, lepton and pion masses

Photon virtuality and energy in lab frame

Kinematical combinations

Combination in correction to energy levels

Pion production threshold

Bjorken kinematical variable

$$
W^{2} \quad \text { Invariant mass of the recoil proton system }
$$

Initial lepton energy in the lab frame

Initial, intermediate and final lepton momenta

Initial, intermediate and final proton momenta

Initial and final lepton (proton) momenta average
Lepton (intermediate) scattering angles

Lepton scattering angle in the c. m. frame

Photons in the TPE box graph momenta

Average of the photons momenta
s- and u-channel c. m. frame squared energies

## $\Sigma_{s}=\left(s-(M+m)^{2}\right)\left(s-(M-m)^{2}\right)$

Crossing symmetric variable

Lepton-proton physical region boundary

Photon polarization parameter

Boundary value of $\varepsilon$

Sec. (2.4)

Sec. (2.6.2)
Secs. (2.1, 2.4)

Sec. (2.3), Eqs. (2.46)

Sec. (2.3), Eqs. (2.46)

Eq. (2.49)

Sec. (2.4), Eq. (2.60)

Eqs. (2.27)

Secs. (2.4.1, 5.3)

Secs. (2.1, 3.1)

Secs. (3.1, 4.2.1, 6.1)

Secs. (3.1, 6.1)

Secs. (3.1, 6.2)
Secs. (4.2.1, 6.1)
Sec. (3.1)

Eqs. (5.30)

Eqs. (5.30)

Eq. (2.1), Sec. (3.1)

Sec. (3.1)

Sec. (3.1)

Eqs. (4.36, 4.37)

Eqs. (3.7, 3.47)
Sec. (3.1)

## List of Notations

| $f_{+}, f_{-}, g$ | Forward lepton-proton scattering amplitudes | Eqs. (2.3, 2.4, 2.5) |
| :---: | :---: | :---: |
| $\mathcal{G}_{M}, \mathcal{F}_{2}-\mathcal{F}_{6}, \mathcal{G}_{1}, \mathcal{G}_{2}$ | Non-forward lepton-proton scattering amplitudes | Eqs. (3.9, 3.37, 3.38) |
| $\mathrm{T}_{1}, \mathrm{~T}_{2}, \mathrm{~S}_{1}, \mathrm{~S}_{2}$ | Forward VVCS amplitudes | Eq. (2.51) |
| $F_{1}, F_{2}, g_{1}, g_{2}$ | Proton structure functions | Eq. (2.29) |
| $G_{M}, G_{E}, F_{D}, F_{P}$ | Proton elastic form factors | Eqs. (3.30, 3.31) |
| $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}, \mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}, \mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ | Dirac, Dirac and Pauli, Pauli proton vertices | Fig. (4.2) |
| $\mu$ | IR regulator | Eq. (4.1) |
| $M^{\mu \nu}$ | VVCS tensor | Eqs. (2.51, 5.34) |
| $B_{i}$ | VVCS amplitudes | Eq. (5.34) |
| $T_{i}^{\mu \nu}$ | VVCS tensor structures | Eq. (5.34) |
| $\delta_{2 \gamma}$ | TPE correction to unpolarized cross section | Eq. (3.35) |
| $\beta\left(Q^{2}\right)$ | Magnetic polarizability: $T_{1}^{\text {subt }}\left(0, Q^{2}\right)=\beta\left(Q^{2}\right) Q^{2}$ | Eq. (2.74) |
| $T, T^{1 \gamma}, T^{2 \gamma}$ | Helicity amplitudes | Eqs. (2.5, 3.9, 3.30) |
| $K$-pole | Pole from the lepton propagator | Eqs. (5.37) |
| $P$-pole | Pole from the proton propagator | Eqs. (5.37) |
| $P$-frame | Breit frame with $P=P(1,0,0,0)$ | Eqs. (5.80) |
| $K$-frame | Breit frame with $K=K(1,0,0,0)$ | Eqs. (5.81) |

Submission date: June 13, 2016
Oral exam date: October 11, 2016


[^0]:    ${ }^{1}$ The unit fermi $\left(1 \mathrm{fm}=10^{-15} \mathrm{~m}\right)$ was first introduced in Ref. [36].
    ${ }^{2}$ Nowadays it is more convenient to work in terms of the Sachs electric $G_{E}$ and magnetic $G_{M}$ FFs [42-44] as they enter the Rosenbluth expression without the interference term $G_{E} G_{M}$. The Sachs form factors are normalized to the proton electric charge and magnetic moment $\mu_{P}$ as $G_{E}(0)=1, G_{M}(0)=\mu_{P}$.

[^1]:    ${ }^{3}$ Note that the revised analyses of the electron-proton scattering data in Refs. [105-107] gave a result consistent with the muonic hydrogen value $R_{E} \approx 0.84 \mathrm{fm}$, while the result of analyses in Refs. [108-110] is in agreement with the value of Ref. [104].

[^2]:    ${ }^{4}$ The TPE hadronic correction to the Lamb shift in electronic hydrogen is few orders of magnitude lower than the current experimental accuracy due to the suppression by the electron mass.

[^3]:    ${ }^{5}$ Note that the TPE correction coming from the Coulomb photons [165] and inelastic intermediate states [166] can also be important in the extraction of the electric charge radius according to Refs. [165, 166].

[^4]:    ${ }^{1}$ We distinguish the initial and final particles momenta in order to exploit the same steps of derivation for the case of non-forward scattering in Section 3.2.

[^5]:    ${ }^{2}$ The nontrivial relation $\sigma_{\perp}=\sigma$ holds for the lepton-proton scattering. We have obtained this relation in one-photon exchange (OPE) approximation of Eq. (2.26) and have proved it by the direct cross sections evaluation for the case of elastic scattering and by exploiting the symmetry properties of the helicity amplitudes for the scattering to arbitrary channel $l p \rightarrow l X$.

[^6]:    ${ }^{3}$ This assumes that $T_{1}\left(\nu_{\gamma}, Q^{2}\right)$ does not have a fixed $J=0$ pole behavior when $\nu_{\gamma} \rightarrow \infty$.

[^7]:    ${ }^{4}$ Note that for the total TPE correction to the muonic hydrogen 2 S level one needs to add to the subtraction function contribution also the dispersive contribution, which was evaluated based on data in Ref. [135].

[^8]:    ${ }^{1}$ The TPE contribution to the unpolarized helicity amplitude $\left(T_{1}+T_{3}\right)^{2 \gamma} \sim Q^{-1}$ at low $Q$ due to the leading contribution from the scattering of two point-like charges. Subtracting the divergent contributions one can determine the residual amplitude $f_{+}$consistently as described in Appendix F.

[^9]:    ${ }^{1}$ In the lepton massless limit $\omega_{\mathrm{cm}}=\left|\mathbf{k}_{\mathrm{cm}}\right|=\left(s-M^{2}\right) /(2 \sqrt{s})$.

[^10]:    ${ }^{2}$ The amplitudes $\mathcal{F}_{4}^{2 \gamma}, \mathcal{F}_{6}^{2 \gamma}$ are UV finite in the case of the point-like proton with $\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{P}}$ vertex structure.
    ${ }^{3}$ The amplitudes $\mathcal{G}_{M}^{2 \gamma}, \mathcal{F}_{3}^{2 \gamma}, \mathcal{F}_{6}^{2 \gamma}$ are UV divergent in the case of the point-like proton with $\mathrm{F}_{\mathrm{P}} \mathrm{F}_{\mathrm{P}}$ vertex structure, see Eq. (4.31).

[^11]:    ${ }^{4}$ The overall pre-factor $1 / Q^{2}$ in the amplitudes decomposition of Eqs. (3.9), (3.10) provides the vanishing contribution from all possible lepton-proton counter-terms at $Q^{2}=0$. That is the reason of the agreement between two types of evaluation of the all channels TPE correction to the hydrogen HFS in Chapter 2.

[^12]:    ${ }^{1}$ The constant terms in the electron-proton scattering amplitudes at vanishing $Q^{2}$ are related by $\mathcal{F}_{2}^{\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}}\left(\nu, Q^{2}=0\right)=0$ and $\mathcal{G}_{M}^{\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}}\left(\nu, Q^{2}=0\right)+\nu / M^{2} \mathcal{F}_{3}^{\mathrm{F}_{\mathrm{D}} \mathrm{F}_{\mathrm{D}}}\left(\nu, Q^{2}=0\right)=0$.

[^13]:    ${ }^{2}$ Note, however, that our convention for the photon field four-momenta indices $\mu$ and $\nu$ (see Fig. 5.5) is opposite to Refs. [209], [210], which changes the sign of the tensors that are antisymmetric under $\mu \leftrightarrow \nu$. Conveniently, we use the amplitudes $B_{i}$ with an overall " - " sign relative to the notation of Refs. [209, 210].

[^14]:    ${ }^{3}$ That is equivalent to a subtraction in the dispersion relation for the $\mathcal{F}_{4}$ invariant amplitude.

[^15]:    ${ }^{4}$ The contribution of these poles to the Lamb shift of atomic energy levels is discussed in Appendix O.

[^16]:    ${ }^{5}$ Note that the numerical evaluation of Ref. [166] corresponds with a specific choice of the hadronic scale, $\Lambda^{2}=\left(W^{2}-M^{2}\right)^{2} /\left(M^{2}+2 M \omega\right)$.

[^17]:    ${ }^{6}$ The inelastic TPE amplitude of Eq. (5.61) in the near-forward approximation contributes only to $\mathcal{F}_{2}$ and $\mathcal{F}_{4}$ invariant amplitudes. In the elastic ep scattering the $\mathcal{F}_{2}^{2 \gamma}$ contribution to the unpolarized cross section of Eq. (3.55) enters with a factor $\varepsilon$ and, consequently, vanishes in the backward limit $\varepsilon \rightarrow 0$.

[^18]:    ${ }^{1}$ We define the final lepton and nucleon momenta and helicities with the index " 1 " in order to use them as the intermediate state in the unitarity relation.

[^19]:    ${ }^{2}$ In the lepton massless limit $\omega_{\mathrm{cm}}=\left|\mathbf{k}_{\mathrm{cm}}\right|=\left(s-M^{2}\right) /(2 \sqrt{s}), \omega_{1}=\left|\mathbf{k}_{1}\right|=\left(s-W^{2}\right) /(2 \sqrt{s})$.

[^20]:    ${ }^{3}$ The covariants $M_{i}^{\mu}$ depend only on kinematics of the pion photoproduction process $\gamma^{\star} N \rightarrow \pi N$.

[^21]:    ${ }^{4} L_{2 \mathrm{I} 2 \mathrm{~J}}$ is the partial wave with the pion angular momentum $L=1$, isospin $\mathrm{I}=3 / 2$ and spin $\mathrm{J}=3 / 2$.

[^22]:    ${ }^{1}$ The $Q^{2}$ expansion of the ratio R in the Christy and Bosted fit starts from $Q^{4}$ term instead of $Q^{2}$. The ratio enters our evaluation in the combination $1+R$ and does not influence our results significantly due to the small absolute value of $R$. The evaluation of the correction to physical observables in Chapters 2 and 5 at low- $Q^{2}$ region relies on the magnetic polarizability value but not on the ratio $R$.

[^23]:    ${ }^{1}$ Note that the proper subtraction of the IR divergent piece, which has a constant HE limit of the imaginary part and vanishing limit $(1 / \nu)$ of the real part, can restore the unitarity in the case of $F_{D} F_{D}$ and $F_{D} F_{P}$ vertex structures.

