Dissertation

Effective Bounds for the Negativity of Shimura Curves on Hilbert Modular Surfaces

zur Erlangung des Grades

Doktor der Naturwissenschaften

gemäß der Promotionsordnung vom 02. Dezember 2013
am Fachbereich 08 – Physik, Mathematik und Informatik
Johannes Gutenberg Universität Mainz

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Mainz, im Juli 2016

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

Abstract

Fachbereich 08 – Physik, Mathematik und Informatik

Mathematik

Doktor der Naturwissenschaften

**Effective Bounds
for the Negativity
of Shimura Curves
on Hilbert Modular Surfaces**

by Sonia SAMOL

The Bounded Negativity Conjecture states that for each smooth projective surface X defined over a field of characteristic zero there exists a number $b(X) \geq 0$ such that the self-intersection number C^2 for every reduced, irreducible curve $C \subset X$ is bounded below by $b(X)$, i.e.

$$C^2 \geq -b(X).$$

In this thesis, we consider Hirzebruch-Zagier curves on Hilbert modular surfaces and give explicit bounds for the self-intersection numbers in these cases. More general, we give a bound for the self-intersection number of reduced, irreducible Shimura curves C on Hilbert modular surfaces X, generalising a result from the literature from compact Hilbert modular surfaces to non-compact Hilbert modular surfaces.
We compare the resulting bounds with the actual self-intersection numbers of Hirzebruch-Zagier curves calculated with Pari/GP.

Contents

List of Abbreviations

$A = N(\mathfrak{a})$	the norm of an ideal $\mathfrak{a} \subset \mathcal{O}_K$
$\mathfrak{a}$	fractional ideal in $\mathcal{O}_K$
$\mathfrak{b}$	fractional ideal in $\mathcal{O}_K$
c_1^2	Chern number
c_2	Chern number
$\mathfrak{Cl}$	class group
$\mathfrak{Cl}^+$	narrow class group
D	Discriminant of K
e	Euler number
F_B	skew-hermitian curve
F_N	modular curve
$f(\mathfrak{a})$	function on ideals
h, h_K	class number of K
h'	modified class number
$\mathbb{H}$	complex upper half plane
$H_p(N)$	sum of class numbers
Im	imaginary part
$I_p(n)$	function on n
K	totally real number field
k	complex number field
m	content of quadratic form
N	norm
$\mathcal{O}_K$	ring of integers of K
$\mathfrak{o}$	ring of integers of k
p	prime number
$\mathbb{P}$	projective space
PSL	projective special linear group
$\mathrm{PSL}_2(\mathcal{O}_K, \mathfrak{a})$	Hilbert modular group
S_k	resolution curve
Sq, Sq^+	squaring map
T_N	Hirzebruch-Zagier curve
Tr	trace
U_K	units of $\mathcal{O}_K$
u_k	local coordinate in cusp resolution
v_k	local coordinate in cusp resolution

w_k	reduced quadratic irrationality
X	Hilbert modular surface
$X^{\mathfrak{a}}$	Hilbert modular surface ass. to $\mathfrak{a}$
$\bar{X}, \bar{X}^{\mathfrak{a}}$	Baily-Borel compactification of $X, X^{\mathfrak{a}}$
γ	Euler-Mascheroni constant
$\Gamma, \Gamma(\mathcal{O}_K, \mathfrak{a})$	Hilbert modular group
Δ	discriminant
ζ_K	Dedekind zeta function on K
$\nu_p(N)$	exponent of p in N
σ	cusp
φ	quadratic form of special point
χ_p	Legendre symbol $\left(\frac{\cdot}{p}\right)$
ω	volume form
$\gg 0$	totally positive
(a, b)	greatest common divisor of a and b
$[a, b, c]$	quadratic form $ax^2 + bxy + cy^2$

Introduction

This thesis was motivated by the Bounded Negativity Conjecture (BNC) which can be stated as follows.

Conjecture (Bounded Negativity Conjecture)**:** *For each smooth projective surface X defined over a field of characteristic zero there exists a number $b(X) \geq 0$ such that*

$$C^2 \geq -b(X)$$

for every reduced, irreducible curve $C \subset X$.

It is unclear where the origins of the conjecture lay, but it probably dates back to the beginning of the 20th century. It is believed to have been mentioned by Federigo Enriques to his last student, Alfredo Franchetta.
Algebraic surfaces with infinitely many curves that have negative self-intersection are well-known. For example Nagata ([22]) showed that for X the projective plane blown up in the base locus of a general elliptic pencil there are infinitely many curves $C \subset X$ with negative self-intersection, but all negative curves C on X have self-intersection equal to -1.
That it is necessary to restrict the conjecture to the characteristic zero case has been known for a long time. A counterexample in characteristic $p > 0$ was for example given in Hartshorne's book „Algebraic Geometry“:

Example ([8, Exercise V.1.10])**:** *Let C be a smooth curve of genus $g \geq 2$ defined over a field of characteristic $p > 0$. Let X be the product surface $X = C \times C$. The graph Γ_q of the Frobenius morphism defined by taking the $q = p^r$−th power is a smooth curve of genus g and self-intersection number $\Gamma_q^2 = q(2-2g)$. With r going to infinity we obtain a sequence of smooth curves of genus g whose self-intersection numbers tend to minus infinity.*

In recent years, there has been made a lot of progress on verifying the conjecture in characteristic zero in special cases. For this thesis, the most important result was proven in 2011: In [2] it was shown that for example for Shimura curves on compact Hilbert modular surfaces the conjecture is true.

Theorem 0.1 ([2, Prop. 3.5] Bauer, Harbourne, Knutsen, Küronya, Müller-Stach, Roulleau, Szemberg)**:** *For a Shimura curve C, not necessarily smooth, on a compact Hilbert modular surface X the inequality*

$$C^2 \geq -6c_2(X)$$

holds, where $c_2(X)$ *is the second Chern class of X. Moreover, there is only a finite number of Shimura curves with $C^2 < 0$.*

This theorem gave rise to the problems we consider in this thesis: The first problem is whether we can give a sharper bound for the negative self-intersection numbers in the special case of Hirzebruch-Zagier curves on Hilbert modular surfaces. The second is how the bound given in the theorem changes when we consider non-compact Hilbert modular surfaces.

Structure of the thesis

In the first three chapters of this thesis we will look at Hirzebruch-Zagier curves on Hilbert modular surfaces and give an effective lower bound for their self-intersection numbers.
We start by giving basic definitions in Chapter 1.
Let $p \equiv 1 \bmod 4$ be a prime, $\mathcal{O}_K$ the ring of integers of $K = \mathbb{Q}(\sqrt{p})$ and $\mathfrak{a}$ an ideal in $\mathcal{O}_K$ with $\text{norm}(\mathfrak{a}) = A$. The group $\text{SL}_2(\mathcal{O}_K, \mathfrak{a})$ consisting of matrices

$$\begin{pmatrix} \mathcal{O}_K & \mathfrak{a}^{-1} \\ \mathfrak{a} & \mathcal{O}_K \end{pmatrix}$$

with determinant equal to 1 acts proper discontinuously on $\mathbb{H}^2$, the product of the upper half plane with itself, by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (z_1, z_2) \mapsto \left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'} \right).$$

We define the non-compact quotient $X^{\mathfrak{a}} := \mathbb{H}^2/\text{SL}_2(\mathcal{O}_K, \mathfrak{a})$. It can be compactified by adding finitely many points and resolving the singularities by rational curves S_k. This yields the Hirzebruch compactification $\bar{X}^{\mathfrak{a}} = X^{\mathfrak{a}} \cup \bigcup_k S_k$.
For a given integer N, the Hirzebruch-Zagier curves T_N are defined as

the image in $X^{\mathfrak{a}}$ of all points $(z_1, z_2) \in \mathbb{H}^2$ satisfying the equation

$$a\sqrt{p}z_1z_2 + \lambda z_2 - \lambda' z_1 + \frac{b}{A}\sqrt{p} = 0$$

with $a, b \in \mathbb{Z}, \lambda \in \mathcal{O}_K, \lambda\lambda' + \frac{abp}{A} = N$ in $X^{\mathfrak{a}}$ and $\mathrm{norm}(\mathfrak{a}) = A$.
The curves T_N can be extended to curves on $\bar{X}^{\mathfrak{a}}$, these curves will be called T_N^c and can be written as

$$T_N^c = T_N + \sum_k \alpha(N,k) S_k$$

where the coefficients $\alpha(N,k)$ are uniquely defined rational numbers determined by $T_N \cdot S_k = 0$ for all curves S_k.

In 1976, Friedrich Hirzebruch and Don Zagier proved in [14] that the generating series for the intersection numbers of two Hirzebruch-Zagier cycles T_M^c and T_N^c is a classical modular form of weight 2 by giving an explicit formula for their intersection numbers. The precise formulation is as follows (see also Theorem 2.28 in this thesis):

Theorem (F. Hirzebruch, D. Zagier)**:** *Let M, N be positive integers with $\nu_p(N) \leq \nu_p(M)$, where ν_p denotes the exponent of p. Then the intersection number of the homology classes T_M^c and T_N^c on the compact surface $\bar{X}^{\mathfrak{a}}$ is given by*

$$T_M^c T_N^c = \frac{1}{2} \sum_{d|(M,N)} d\left(\chi_p(d) + \chi_p(AN/d)\right)\left(H_p(MN/d^2) + I_p(MN/d^2)\right),$$

with

$$H_p(n) = \sum_{\substack{x \in \mathbb{Z} \\ x^2 \leq 4n \\ x^2 \equiv 4n \bmod p}} H\left(\frac{4n - x^2}{p}\right),$$

$$H(n) = \begin{cases} -\frac{1}{12} & \text{if } n = 0 \\ \sum_{d^2|n} h'\left(-\frac{n}{d^2}\right) & \text{else,} \end{cases}$$

$$h'(\Delta) = \begin{cases} \frac{1}{3} & \text{if } \Delta = -3 \\ \frac{1}{2} & \text{if } \Delta = -4 \\ h(\Delta) & \text{if } \Delta \equiv 0 \text{ or } 1 \bmod 4, \Delta \leq -4, \end{cases}$$

where $h(\Delta)$ is the class number of positive definite primitive binary integral quadratic forms with discriminant Δ and

$$I_p(n) = \frac{1}{\sqrt{p}} \sum_{\substack{\lambda \in \mathcal{O} \\ \lambda > 0, \lambda' > 0 \\ \lambda\lambda' = n}} \min(\lambda, \lambda').$$

Chapter 2 deals with the proof of this formula. It is separated into four sections, where Section 2.1 treats the part of the self-intersection number coming from the volumes of the curves. Section 2.2 attends to what Hirzebruch and Zagier call the transversal intersection numbers of the curves T_M and T_N in so-called special points, which contribute the main part to the intersection numbers. In Section 2.3 we look at the contribution of the cusps to the intersection-number, until we can finally prove the theorem above by Hirzebruch and Zagier in Theorem 2.28 in the last section of Chapter 2.
We then use this formula to find an explicit bound for the negativity of the Hirzebruch-Zagier curves in Chapter 3.
We first show in 3.1 that the contribution from the cusps, which can be written as

$$C_p(N) = \frac{1}{2}\sum_{n|N} nI_p\left(\frac{N^2}{n^2}\right)\left(\chi_p(n) + \chi_p\left(\frac{NA}{n}\right)\right),$$

is always non-negative, so in the following sections it is enough to consider just a part of the self-intersection number, namely

$$T_N^2 = \frac{1}{2}\sum_{n|N} nH_p\left(\frac{N^2}{n^2}\right)\left(\chi_p(n) + \chi_p\left(\frac{NA}{n}\right)\right).$$

In Section 3.2 we consider the case $\chi_p(A) = -1$, where $A = \text{norm}(\mathfrak{a})$ in $\bar{X}^{\mathfrak{a}}$. But in this case we will see that the self-intersection number T_N^2 is always positive.
The more interesting case concerning the Bounded Negativity Conjecture will be dealt with in Section 3.3. We can again use the formula given above to find an explicit bound for the negativity of the Hirzebruch-Zagier curves. To that purpose we give estimates for the appearing terms with the aim of ending up with a polynomial of degree 2 in N, where N is the discriminant of the Hirzebruch-Zagier curves. We deal with this in lemmas 3.8, 3.9 and 3.10 of this thesis. Then, the determination of the minimum is an easy polynomial extremum calculation.
First, we restrict to the case $N = \prod_{i=1}^{k} p_i$ with $p_1, .., p_k$ pairwise different prime numbers with $\chi_p(p_i) = 1$, because otherwise the self-intersection numbers will be less negative and hence not interesting for the Bounded Negativity Conjecture. But then the sum simplifies to

$$T_N^2 = \sum_{n|N} nH_p\left(\frac{N^2}{n^2}\right).$$

This sum can be split into a negative and a positive part where the negative part is just $-\frac{1}{6}\sigma_1(N)$ and can be measured by Guy Robin's estimate for the divisor sum.
For the positive part we get a sum over some class numbers. The sum

is defined over solutions in $\mathbb{Z}$ of the equation

$$x^2 \leq 4n,\ x^2 \equiv 4n \bmod p.$$

We approximate the solutions and insert Paley's inequality for the class numbers in Lemma 3.10 to get an estimate for the self-intersection numbers.
The only remaining obstacle to detecting the minimum of the self-intersection numbers lies now in the logarithmic terms which occur both in Paley's estimate for the class numbers and in Robin's estimate for the divisor sum. But we solve this issue by replacing them with their biggest possible value in the interval we are interested in.

Therefore, we get an effective version of Theorem 0.1 for the special case of Hirzebruch-Zagier curves T_N on Hilbert modular surfaces, namely Theorem 3.12 in this thesis:

Theorem: *Let $N = \prod_{i=1}^{k} p_i$ with $p_1, .., p_k$ pairwise different prime numbers with $\chi_p(p_i) = 1$, then the self-intersection number T_N^2 of Hirzebruch-Zagier curves on the Hilbert modular surface $\bar{X}^{\mathfrak{a}} = \overline{\mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})}$, where $\mathcal{O}_K$ is the ring of integers of $K = \mathbb{Q}(\sqrt{p})$, satisfies*

$$T_N^2 \geq -O(p^{\frac{3}{2}}).$$

More precisely,

$$T_N^2 \geq \frac{-1}{192}\frac{c^2}{\delta}p^{\frac{3}{2}}p^{4k\varepsilon} + p^{\frac{1}{2}}\frac{\delta}{24p^{2k\varepsilon}}$$

for all N, where $\delta = \frac{\pi}{12\mathrm{e}^{\gamma}}$, $c = \mathrm{e}^{\gamma} + 0.6482$, ε depends on p and for $p > 17$ can be chosen as $\frac{\log(\log(\log(p)))}{\log(p)}$, $k = \frac{3}{2(1-\varepsilon)}$, so the bound only depends on p.

In Chapter 4 we follow closely the approach in [2], using results by Miyaoka to generalize Theorem 0.1 for non-quaternionic Hilbert modular surfaces. Here, we look at Shimura curves that are a generalisation of Hirzebruch-Zagier curves to arbitrary Hilbert modular surfaces. In literature, Hirzebruch-Zagier curves and Shimura curves are often used synonymously and recently, these curves are mainly referred to as *special curves*. In Chapter 4 of this thesis, we will use the term Shimura curve to stay in the terminology of [2] and to distinguish from the bound in Chapter 3 that was calculated for Hilbert modular surfaces X over totally real number fields K with prime discriminant p, where $p \equiv 1 \bmod 4$. This yields Theorem 4.6 in this thesis:

Theorem: *For a Shimura curve C on a Hilbert modular surface X we have*

$$C^2 \geq -9d_2(\bar{X}),$$

where $d_2(\bar{X}) := 3c_2(\bar{X}) - K_{\bar{X}}^2$ and $\bar{X}$ is the Hirzebruch compactification of X.

For the special case of a Hilbert modular surface over K a real quadratic field with discriminant D that we considered in Chapter 3 for the special case of D a prime number congruent to 1 mod 4, a bound for the second Chern class can be calculated using [27].

By this approach we then get Theorem 4.14 in this thesis:

Theorem: *For a reduced irreducible Shimura curve C on a Hilbert modular surface $\bar{X} = \overline{\mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K)}$, where $\mathcal{O}_K$ the ring of integers of K a real quadratic field with discriminant D, the self-intersection number is bounded by*

$$\begin{aligned} C^2 \geq& -\left(\frac{3}{10}\sum_{x\in\mathbb{Z}}\sigma_1\left(\frac{D-x^2}{4}\right) + 18\sum_{x<\sqrt{D}}\sigma_0\left(\frac{D-x^2}{4}\right) + \frac{3}{2}h(-3D)\right. \\ &\left. + \frac{27\pi}{8\mathrm{e}^\gamma}\frac{D^{\frac{1}{2}}}{\log\log(4D)} + \frac{3\cdot 5\sqrt{3}\pi}{2\mathrm{e}^\gamma}\frac{D^{\frac{1}{2}}}{\log\log(3d)} + \frac{3\cdot\sqrt{3}\pi}{2\mathrm{e}^\gamma}\frac{D^{\frac{1}{2}}}{\log\log(3D)}\right) \\ \geq& -\left(\left(\frac{3}{10}D^{\frac{3}{2}} + 18D^{\frac{1}{2}}\right)\left(\frac{3}{2\pi^2}\log^2(D) + 1.05\log D\right) + \frac{3}{2}\frac{\sqrt{3D}}{\pi}\log(3D)\right. \\ &\left. + \frac{27\pi}{8\mathrm{e}^\gamma}\frac{D^{\frac{1}{2}}}{\log\log(4D)} + \frac{3\cdot 5\sqrt{3}\pi}{2\mathrm{e}^\gamma}\frac{D^{\frac{1}{2}}}{\log\log(3D)} + \frac{3\cdot\sqrt{3}\pi}{2\mathrm{e}^\gamma}\frac{D^{\frac{1}{2}}}{\log\log(3D)}\right). \end{aligned}$$

1 Basics

1.1 Hilbert modular group and Hilbert modular surfaces

We summarise some basic definitions. We start with binary quadratic forms, which we need to introduce the class numbers $h(\cdot)$ of binary quadratic forms. These class numbers we will use to define the number-theoretic functions $H(\cdot)$ (see equation 1.1) and $H_p(\cdot)$ (see equation 1.2) that will play an important role later on in this thesis. Then we give some basic definitions concerning Hilbert modular surfaces until we can finally define Hirzebruch-Zagier curves on Hilbert modular surfaces. We will mainly follow van der Geer's definitions in [27].

1.1.1 Binary quadratic forms

Let M be a free oriented $\mathbb{Z}$-module of rank 2. Regarding a $\mathbb{Z}$-basis of M, all elements of M can be written as pairs (x, y) of integers. A quadratic form is given by a homogeneous polynomial of degree 2. It is called *binary* if it is given in two variables (x, y), i.e. it has the form $Q(x, y) = ax^2 + bxy + cy^2$. For $a, b, c \in \mathbb{Z}$ the *discriminant* of Q is given by $\Delta = b^2 - 4ac$. The discriminant of a quadratic form is either divisible by 4 or congruent to 1 mod 4.
On the other hand, every integer $D \equiv 0$ or 1 mod 4 is the discriminant of some integral binary quadratic form and every discriminant can be written as

$$D = D_0 f^2$$

with D_0 a discriminant and f a positive integer. A discriminant D is called *fundamental discriminant* if $f = 1$ in every decomposition. It can be shown that D is a fundamental discriminant if and only if

1. $D \equiv 1 \bmod 4$ and D is squarefree or

2. $D = 4m$, $m \equiv 2$ or $3 \bmod 4$ and m is squarefree.

The *content* of the form Q is the greatest common divisor (a, b, c). The form Q is *primitive* if and only if $(a, b, c) = 1$, it is *positive definite* if and only if $\Delta < 0$ and $a > 0$.
For $\Delta < 0$ we denote by $h(\Delta)$ the number of isomorphism classes of primitive positive definite quadratic forms with discrimant Δ. $h(\Delta)$ is zero if Δ is not a discriminant.

Two modules M and M' with associated forms Q and Q' are called isomorphic if there exists an orientation-preserving isomorphism $M \to M'$ that carries Q to Q'.
For $\Delta < 0$ we define (see Hurwitz, [15])

$$H(-\Delta) = \sum_{d^2|\Delta} h'\left(-\frac{\Delta}{d^2}\right), \tag{1.1}$$

where $h'(-3) = \frac{1}{3}, h'(-4) = \frac{1}{2}$ and else $h'(\Delta) = h(\Delta)$.
For p a prime number with $p \equiv 1 \bmod 4$ we also consider

$$H_p(n) = \sum_{\substack{x\in\mathbb{Z} \\ x^2\leq 4n \\ x^2\equiv 4n \bmod p}} H\left(\frac{4n - x^2}{p}\right). \tag{1.2}$$

1.1.2 The action of the Hilbert modular group

Let D be the discriminant of the real quadratic number field K, $\mathcal{O}_K$ the ring of algebraic integers of K.
For $\lambda \in K$ we denote the action of the non-trivial Galois automorphism of K on λ by λ', the conjugate of λ in K. Then an element $\lambda \in K$ is called *totally positive* if $\lambda > 0$ and $\lambda' > 0$, we notate this by $\lambda \gg 0$. The norm $\mathrm{N}(\lambda)$ for an element $\lambda \in K$ is then defined as $N(\lambda) = \lambda\lambda'$, the trace $\mathrm{Tr}(\lambda)$ as $\mathrm{Tr}(\lambda) = \lambda + \lambda'$.
The group

$$\mathrm{GL}_2(K) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in K, ad - bc \gg 0\right\}$$

can be embedded into $\mathrm{GL}_2(\mathbb{R})^2$ by using the two embeddings $K \hookrightarrow \mathbb{R}$.
Furthermore, we have

$$\mathrm{PGL}_2^+(K) = \mathrm{GL}_2(K) / \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in K^+\right\}.$$

Definition 1.1: *For K a real quadratic number field and $\mathcal{O}_K$ its ring of integers the Hilbert modular group is defined as*

$$\mathrm{PSL}_2(\mathcal{O}_K) = \mathrm{SL}_2(\mathcal{O}_K)/\left\{\pm 1\right\} \tag{1.3}$$

and denoted by Γ_K.

Denote by $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the upper half plane of $\mathbb{C}$, then

$$\mathrm{GL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc > 0 \right\}$$

acts on $\mathbb{H}$ by fractional linear transformations $z \mapsto \frac{az+b}{cz+d}$ and $\mathrm{SL}_2(\mathcal{O}_K)$ acts proper discontinuously on $\mathbb{H}^2$, the product of the upper half plane $\mathbb{H}$ with itself, by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (z_1, z_2) \mapsto \left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'} \right).$$

The domain $\mathbb{H}^2$ is contained in $\mathbb{P}^1(\mathbb{C}) \otimes \mathbb{P}^1(\mathbb{C})$ and the two embeddings of K into the real numbers induce an embedding

$$\mathbb{P}^1(K) \;\rightarrow\; \mathbb{P}^1(\mathbb{R})^2 \subset \; \mathbb{P}^1(\mathbb{C})^2.$$

The orbit of $\mathbb{P}^1(K)$ under any group $\Gamma \subset \mathrm{PGL}_2(K)$ is called the *cusp* of Γ. We will also call a representative of an orbit *cusp*.
Let $\sigma = (\alpha : \beta) \in \mathbb{P}^1(K)$, and assume α and β to be integral. Then we associate to σ the well-defined class of the ideal (α, β) in $\mathcal{O}_K$ generated by α and β in the ideal class group $\mathfrak{Cl}(K)$ of K.
We say that two non-zero fractional ideals $\mathfrak{a}, \mathfrak{b}$ in $\mathfrak{Cl}(K)$ are equivalent in the *narrow sense* if there exists a $\lambda \in K$ with $\mathrm{N}(\lambda) > 0$ and $\lambda\mathfrak{a} = \mathfrak{b}$. The equivalence classes form a group $\mathfrak{Cl}^+(K)$.
Let $Sq^+ : \mathfrak{Cl}^+ \rightarrow \mathfrak{Cl}^+$ be the homomorphism defined by squaring. Then $\mathfrak{Cl}^+/Sq^+(\mathfrak{Cl})$ is called the *genus group.* A coset of $Sq^+(\mathfrak{Cl}^+)$ is called a *genus* and $Sq^+(\mathfrak{Cl}^+)$ itself is called a *principal genus.* Two non-zero fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ belong to the same genus if there exists a $\lambda \in K$ with $\mathrm{N}(\lambda) > 0$ and $\mathrm{N}(\lambda)\mathrm{N}(\mathfrak{a}) = \mathrm{N}(\mathfrak{b})$.

Proposition 1.2 ([27, Chapter 1, Proposition 1.1])**:** *The association $(\alpha : \beta) \mapsto [(\alpha, \beta)]$, where $[(\alpha, \beta)]$ is the class of (α, β) in $\mathfrak{Cl}(K)$, sets up a bijective correspondence between the set of cusps of Γ_K and the ideal class group $\mathfrak{Cl}(K)$ of K. The number of cusps of Γ_K equals the class number h of K.*

The cusp given by $(1 : 0) \in \mathbb{P}^1(K)$ is called the cusp at ∞. An element $\sigma = (\alpha : \beta)$ of $\mathbb{P}^1(K)$, where $\alpha, \beta \in \mathcal{O}_K$, can be transformed to ∞ by the inverse of an unimodular matrix $M_\sigma = \begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix}$ with α^* , $\beta^* \in \mathfrak{a}^{-1}$,

$\mathfrak{a} = (\alpha, \beta)$.
Let U_K be the group of units of $\mathcal{O}_K$. Then the isotropy group of the cusp ∞ in $\mathrm{PSL}_2(\mathcal{O}_K \oplus \mathfrak{a}^2)$ equals

$$\left\{ \begin{pmatrix} \varepsilon & \mu \\ 0 & \varepsilon^{-1} \end{pmatrix} : \varepsilon \in U_K, \mu \in \mathfrak{a}^{-2} \right\} / \{ \pm 1\}$$

or equivalently

$$\left\{ \begin{pmatrix} \varepsilon^2 & \mu \\ 0 & 1 \end{pmatrix} : \varepsilon \in U_K, \mu \in \mathfrak{a}^{-2} \right\}$$

as a transformation group. The pair $(\mathfrak{a}^2, U_K^2)$ is called *the type of the cusp* σ.
Let $\sigma = (\alpha : \beta) \in \mathbb{P}^1(K)$. For $z \in \mathbb{H}$ we define

$$\mu(\sigma, z) = \frac{N(\mathfrak{a})N(y)}{|N(-\beta z + \alpha)|^2}, \mathfrak{a} = (\alpha, \beta), y_i = \mathrm{Im}(z_i).$$

Then μ is invariant under the choice of α and $\beta \in \mathcal{O}_K$ and for any $\gamma \in \Gamma_K$ there is the invariance property

$$\mu(\gamma\sigma, \gamma z) = \mu(\sigma, z).$$

The *sphere of influence* of σ is then defined as

$$F_\sigma = \left\{ z \in \mathbb{H}^2 : \mu(\sigma, z) \geq \mu(\tau, z) \text{ for all } \tau \in \mathbb{P}^1(K) \right\}. \tag{1.4}$$

Proposition 1.3 ([27, Chapter 1, Proposition 2.3])**:** *The action of* Γ_K *in the interior of* F_σ *reduces to that of the isotropy group* Γ_σ *of* σ.

A generalisation of the Hilbert modular group is obtained by taking the automorphism group of a projective $\mathcal{O}_K$-module of rank 2 modulo its centre. Any projective $\mathcal{O}_K$-module of rank 2 is isomorphic to a module of the form $\mathfrak{a} \oplus \mathfrak{b}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are fractional ideals of K. Since $\mathfrak{a} \oplus \mathfrak{b} \cong \mathcal{O}_K \oplus \mathfrak{ab}$ the isomorphism class of the module depends only on the ideal class of $\mathfrak{ab}$ in $\mathfrak{Cl}(K)$.

Definition 1.4: *Let*

$$\begin{pmatrix} \mathcal{O}_K & \mathfrak{a}^{-1} \\ \mathfrak{a} & \mathcal{O}_K \end{pmatrix}$$

be the maximal order in the matrix algebra $M_2(K)$*. Then* $SL_2(\mathcal{O}_K, \mathfrak{a})$ *is defined as the group consisting of matrices of this order with determinant* 1*, its image in* $PGL_2(K)$ *is denoted by* $\Gamma(\mathcal{O}_K, \mathfrak{a})$ *and called generalised Hilbert modular group.*

Two maximal orders in $M_2(K)$ are conjugate if and only if the corresponding non-zero ideals $\mathfrak{a}$ and $\mathfrak{a}_1$ belong to the same genus, which means that there exists a totally positive $\lambda \in K$ and an ideal $\mathfrak{b}$ such that $\mathfrak{a}_1 = \lambda\mathfrak{b}^2\mathfrak{a}$. Furthermore, there is a matrix $A \in \begin{pmatrix} \mathfrak{a} & \mathfrak{a}^{-1}\mathfrak{a}_1^{-1} \\ \mathfrak{a}\mathfrak{a}_1 & \mathfrak{a}^{-1} \end{pmatrix}$ with $\det(A) = 1$ such that

$$\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} A^{-1}\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a}_1)A \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a}). \tag{1.5}$$

This gives the equivalence of $\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a}_1)$ and $\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$ as transformation groups of $\mathbb{H}^2$.

1.1.3 Hurwitz-Maaß extension and Hilbert modular surfaces

We now describe the Hurwitz-Maaß extension $\Gamma_m(\mathcal{O}_K, \mathfrak{a})$ of $\Gamma(\mathcal{O}_K, \mathfrak{a})$ for a fixed ideal $\mathfrak{a}$ in $\mathcal{O}_K$ to be able to give the definition of Hilbert modular surfaces.
Let $\mathfrak{g}_1, ..., \mathfrak{g}_\mathfrak{r}$ be integral ideals in $\mathcal{O}_K$ representing the r elements of $\ker Sq$ (the kernel of Sq), where

$$Sq : \mathfrak{Cl}(K) \to \mathfrak{Cl}^+(K)$$

is again the homomorphism given by squaring. We then consider the subset

$$\bigcup_{i=1}^{r} \left\{ \alpha \in \mathfrak{g}_\mathfrak{i} \begin{pmatrix} \mathcal{O}_K & \mathfrak{a}^{-1} \\ \mathfrak{a} & \mathcal{O}_K \end{pmatrix} : \det\alpha \gg 0, (\det\alpha) = \mathfrak{g}_i^2 \right\} \tag{1.6}$$

of $M_2(K)$, where $(\det\alpha)$ is the ideal generated by $\det(\alpha)$.

Definition 1.5: *The Hurwitz-Maaß extension of* $\Gamma(\mathcal{O}_K, \mathfrak{a})$*, notated by* $\Gamma_m(\mathcal{O}_K, \mathfrak{a})$*, is defined as the image of the set of matrices given in (1.6) in* $PGL_2(K)$*. It is a discrete subgroup of* $PGL_2^+(\mathbb{R})^2$ *and it contains the generalised Hilbert modular group* $\Gamma(\mathcal{O}_K, \mathfrak{a})$ *as a normal subgroup.*

For $\Gamma \subset \Gamma_m(\mathcal{O}_K, \mathfrak{a})$ of finite index, the *Hilbert modular surface* is defined as the quotient $\mathbb{H}^2/\Gamma$. We are only interested in the special case $\Gamma = \Gamma(\mathcal{O}_K, \mathfrak{a})$, where for $\mathfrak{a} = \mathcal{O}_K$ we get $\Gamma(\mathcal{O}_K, \mathfrak{a}) = \Gamma_K$.
For convenience, we denote the quotient by $X^\mathfrak{a} = \mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$ instead of $\mathbb{H}^2/\Gamma(\mathcal{O}_K, \mathfrak{a})$, where $\Gamma(\mathcal{O}_K, \mathfrak{a})$ was defined as the image of $\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$ in $\mathrm{PGL}_2^+(K)$. It is a non-compact complex surface with finitely many quotient singularities. The quotient singularities are in $1:1$ correspondence to the conjugacy classes of maximal finite cyclic subgroups of $\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$, which are represented by the points $z \in \mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$ with non-trivial isotropy group and are called *elliptic fixed points* (the exact definition is given in Section 1.1.4 and Section 2.2).

$X^{\mathfrak{a}}$ can be compactified by adding finitely many points (namely $h(p)$ many, where $h(p) = \#\mathfrak{Cl}(K)$ for $K = \mathbb{Q}(p)$ is the class number introduced in Section 1.1.1) and resolving the singularities created. This yields the Hirzebruch compactification $\bar{X}^{\mathfrak{a}} = X^{\mathfrak{a}} \cup \bigcup_k S_k$ where the S_k are rational curves (for a good description of these curves see for example [27, Chapter 2, 11, Chapter 2] or Section 2.3 in this thesis).

We can restrict ourselves to groups $\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$ where $\mathfrak{a}$ runs through a set of representatives of all genera of K, because for an ideal $\mathfrak{a}_1$ lying in the same genus as $\mathfrak{a}$ the map in (1.5) induces an isomorphism

$$\rho_{\lambda,\mathfrak{a}} : X^{\mathfrak{a}_1} \xrightarrow{\sim} X^{\mathfrak{a}} \tag{1.7}$$

(with λ as in the definition of (1.5)). Since the surface $X^{\mathfrak{a}}$ only depends on the genus of $\mathfrak{a}$, we choose a prime ideal $\mathfrak{a}$ for each genus and denote the norm of $\mathfrak{a}$ by A.

1.1.4 Elliptic fixed points

The Hilbert modular groups Γ_K and $\Gamma(\mathcal{O}_K, \mathfrak{a})$ for a real field K do in general not act freely on $\mathbb{H}^2$. The action is indeed free on the interior of the sphere of influence F_σ (1.4) for a cusp σ, so the points $z \in \mathbb{H}^2$ with non-trivial isotropy group lie on the boundary of F_σ.
An element $(\alpha_1, \alpha_2) \in \mathrm{GL}_2(\mathbb{R})^2$ is called *elliptic* if

$$\mathrm{tr}(\alpha_i)^2 - 4\det(\alpha_i) < 0,\ i = 1, 2.$$

A point $z \in \mathbb{H}^2$ is called an *elliptic point* for a discrete subgroup G of $\mathrm{GL}_2(\mathbb{R})^2$ if it occurs as a fixed point of an elliptic element of G. For $z \in \mathbb{H}^2$ let

$$G_z = \{g \in G : G(z) = z\}$$

be the isotropy group of z.
Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})^2$ be an elliptic element with fixed point $z = (z_1, z_2)$,

$$z_i = \frac{a_i - d_i}{2c_i} + \frac{1}{2|ci|}\sqrt{\mathrm{tr}(\alpha_i)^2 - 4\det(\alpha_i)}. \tag{1.8}$$

By the transformation

$$\zeta_i \mapsto \frac{\zeta_i - z_i}{\zeta_i - \bar{z}_i}$$

the isotropy group $\bar{G}_z$ becomes a group of rotations around the origin of an 2-fold product of unit discs. Each element is determined by its rotation factor $r = (r_1, r_2)$ with

$$r_i = \exp(2i\theta_j), \quad \cos(\theta_j) = \frac{\mathrm{tr}(\alpha_j)}{2\sqrt{\det(\alpha_j)}}, \quad c_j \sin(\theta_j) > 0.$$

If K is a real quadratic field and $G = \mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$ then α is elliptic only if $\mathrm{tr}(\alpha) \gg 4 \det \alpha = 4$, so the trace of α can only be one of the following

$$0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}, \pm\frac{1 \pm \sqrt{5}}{2}$$

and the order of such an element equals $2, 3, 4, 6$ or 5 respectively. If the discriminant D of K is different from $5, 8$ and 12, then only order 2 and 3 occur. Those of order 2 have rotation factor $(-1, -1)$, those of order 3 are either generated by an element α with rotation factor (ζ_3, ζ_3) or (ζ_3, ζ_3^{-1}) with ζ_3 a primitive third root of 1. For $D > 12$ and $\Gamma = \Gamma_K$ Prestel (see [24]) showed a_2, a_3^+, a_3^-, the numbers of quotient singularities of type $(2; 1, 1), (3; 1, 1)$ and $(3; 1, -1)$, are given as

$$a_2(\Gamma) = \begin{cases} h(-4D) & \text{if } D \equiv 1 \bmod 4 \\ 3h(-D) & \text{if } D/4 \equiv 2 \bmod 4 \\ 10h(-D) & \text{if } D/4 \equiv 3 \bmod 8 \\ 4h(-D/4) & \text{if } D/4 \equiv 7 \bmod 8, \end{cases}$$

$$a_3^+(\Gamma) = \begin{cases} \frac{1}{2}h(-3D) & \text{if } D \not\equiv 0 \bmod 3 \\ 4h(-D/3) & \text{if } D \equiv 3 \bmod 9 \\ 3h(-D/3) & \text{if } D \equiv 6 \bmod 9, \end{cases}$$

$$a_3^-(\Gamma) = \begin{cases} \frac{1}{2}h(-3D) & \text{if } D \not\equiv 0 \bmod 3 \\ h(-D/3) & \text{if } D \equiv 3 \bmod 9 \\ 0 & \text{if } D \equiv 6 \bmod 9. \end{cases}$$

1.2 Hirzebruch-Zagier curves F_N and T_N

From now on, let $p \equiv 1 \bmod 4$ be a prime number, $K = \mathbb{Q}(\sqrt{p})$ and $\mathcal{O}_K$ the ring of integers of K. Then we can define now the Hirzebruch-Zagier curves F_N and T_N on $X^{\mathfrak{a}}$ for any ideal $\mathfrak{a} \subset \mathcal{O}_K$ with $\mathrm{norm}(\mathfrak{a}) = A$.

Definition 1.6: *A skew-hermitian matrix*

$$B = \begin{pmatrix} a\sqrt{p} & \lambda \\ -\lambda' & \frac{b}{A}\sqrt{p} \end{pmatrix}$$

is called $\mathfrak{a}-$*integral if* a *and* b *are integers and* $\lambda \in \mathfrak{a}^{-1}$*, where* λ' *is the conjugate of* λ*.*
If there is no integer $n > 1$ *with* $(\frac{a}{n}, \frac{b}{n}, \frac{\lambda}{n}) \in \mathbb{Z}^2 \times \mathfrak{a}^{-1}$*, then* B *is called primitive.*

Definition 1.7: *For a primitive,* $\mathfrak{a}-$*integral, skew-hermitian matrix* B *the curve* F_B *is defined as the image of the set*

$$\left\{(z_1, z_2) \in \mathbb{H}^2 \cup \mathbb{P}^1(K) : (z_2, 1)B\begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0\right\}$$

in $X^{\mathfrak{a}}$*. With* $A =$*Norm*$(\mathfrak{a})$ *the curve* F_N *is defined as*

$$F_N := \bigcup_{\substack{B \text{ as above} \\ \det(B)=\frac{N}{A}}} F_B. \tag{1.9}$$

Remark 1.8: *With* $\chi_p(n) = \left(\frac{n}{p}\right)$ *the Legendre symbol, the curve* F_N *is non-empty if* $\chi_p(NA) \neq -1$ *and compact if* N *is not the norm of an ideal in the genus of* $\mathfrak{a}$*. Franke ([7, Corollary 2.11]) showed that over a real quadratic field* K *with prime discriminant* p *the curve* F_N *consists of only one component if* $p^2 \not| N$*.*

Definition 1.9: *With the conditions of the definition above, the Hirzebruch-Zagier curve* T_N *is defined as*

$$T_N = \bigcup_{\substack{t \geq 1 \\ t^2 | N}} F_{\frac{N}{t^2}}. \tag{1.10}$$

Remark 1.10: *For* N *squarefree we have* $T_N = F_N$ *and thus the curve* T_N *is irreducible for* N *squarefree by Franke.*

The isomorphisms $\rho_{\lambda,\mathfrak{a}} : X^{\mathfrak{a}_1} \to X^{\mathfrak{a}}$ for two ideals $\mathfrak{a}_1, \mathfrak{a}$ lying in the same genus defined in (1.7) map the curves F_N to F_N and T_N to T_N, so we can assume

$$(N, A) = 1.$$

The curves T_N can also be extended to curves on the compactification $\bar{X}^{\mathfrak{a}}$, then they can be written as

$$T_N^c = T_N + \sum_k \alpha(N, k) S_k$$

where the coefficients $\alpha(N, k)$ are uniquely defined rational numbers determined by $T_N \cdot S_k = 0$ for all curves S_k.

2 Intersection numbers of the curves T_M and T_N

In this chapter we describe the intersection number of two Hirzebruch-Zagier curves T_M and T_N for $M, N \in \mathbb{Z}$ in the Hilbert modular surface $X^{\mathfrak{a}} = \mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$. For convenience, we write X instead of $X^{\mathfrak{a}}$ in this chapter.
The curves T_N are given as

$$T_N = \bigcup_{n^2|N} F_N,$$

where $F_N \subset X$ was defined in equation (1.9).
For MN a square the curves T_M and T_N have a common component. To calculate the contribution of this component to the intersection number, we need to calculate the volume of the curve $T_{(M,N)}$, where (M, N) is the greatest common divisor of M and N. The volume of the curves T_N is calculated in Section 2.1.
If MN is not a square then all the intersections of F_M and F_N are called transversal and occur at so called *special points*. We associate to each such special point $\mathfrak{z} \in X$ a positive binary quadratic form $\varphi_{\mathfrak{z}}$, and the curves T_M and T_N meet in $\mathfrak{z}$ if and only if the form $\varphi_{\mathfrak{z}}$ represents both M and N. We study those special points and the associated quadratic forms in Section 2.2 and determine for how many special points $\mathfrak{z} \in X$ the associated form $\varphi_{\mathfrak{z}}$ represents a given positive definite form φ. We use this to evaluate the transversal intersection numbers of the curves T_M and T_N on X in Theorem 2.24.
In Section 2.3 we study how the curves T_N meet the curves S_k of the cusp resolution and evaluate the (self-)intersection of the curves T_N on $\bar{X} = X \bigcup S_k$ and therefore get the intersection number in all cases for the homology classes $[T_M]$ and $[T_N^c]$ on the compact surface $\bar{X}$.

2.1 Volume of the curve T_N

We calculate the volume of the curve T_N with the help of a result by Eichler in [6].
For a skew-hermitian matrix

$$B = \begin{pmatrix} a\sqrt{p} & \lambda \\ -\lambda' & \frac{b}{A}\sqrt{p} \end{pmatrix}$$

we define Γ_B to be the group consisting of elements $T \in \mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$ with $B = \pm T^* B T$, where T^* is the conjugate transpose of T. Then $\Gamma_B/\{\pm 1\}$ is a discrete subgroup of $\mathrm{PGL}_2^+(\mathbb{R})$ and therefore acts properly discontinuously and effectively on the upper half plane $\mathbb{H}$, so we can consider the curve $\mathbb{H}/\Gamma_B$. It is a singularity-free model of the component of F_N that is represented by B and the canonical map

$$\begin{aligned} \mathbb{H}/\Gamma_B &\to X \\ z &\mapsto \left(z, \frac{\lambda' z - b\sqrt{p}/A}{a\sqrt{p}z + \lambda}\right) \end{aligned}$$

sends $\mathbb{H}/\Gamma_B$ with degree 1 to this component. We consider the invariant volume form $\omega := -\frac{1}{2\pi}\frac{dx \wedge dy}{y^2}$ on $\mathbb{H}$, $z = x + iy \in \mathbb{H}$. Then the volume of $\mathbb{H}/\Gamma$ for a discrete subgroup Γ of $\mathrm{PGL}_2^+(\mathbb{R})$ is given by the well-defined integral

$$\mathrm{vol}(\mathbb{H}/\Gamma) := \int_{\mathbb{H}/\Gamma} \omega.$$

The volume $\mathrm{vol}(\mathbb{H}/\Gamma_B)$ only depends on the equivalence class of B in $\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$ (since two different representatives give rise to two groups that are conjugate to each other in $\mathrm{SL}_2(\mathbb{R})$ (see [9, Kapitel 3])) so it can be considered as the volume of the corresponding component of the curve F_N.
Furthermore, we define the subgroup $\varepsilon_B := \{T \in \Gamma_B : B = T^* B T\}$. Then the index of ε_B in Γ_B is 1 or 2. ε_B can be described as the group of units with norm 1 of an order O in an indefinite rational quaternion algebra, which is why the first part of this section will deal with orders in quaternion algebras. The order O lies in a maximal order O^{max} and the group $\varepsilon_B^{max} \subset \mathrm{SL}_2(\mathbb{R})$ of units with norm 1 of the order O^{max} is an extension of the group ε_B of finite index.
Eichler calculated the volume of $\mathbb{H}/\varepsilon_B^{max}$, so the aim of this section is to calculate the volume of the curves T_N by using Eichler's calculation of $\mathrm{vol}(\mathbb{H}/\varepsilon_B^{max})$.

2.1.1 Quaternion algebra

An algebra $\mathfrak{A}$ of rank 4 over a field k is called *quaternion algebra* if there exists a basis $(1, i, j, ij)$ of $\mathfrak{A}$ and numbers $a, b \in k^\times$ such that

$x = 1x = x1$ for all $x \in \mathfrak{A}$ and $i^2 = a, j^2 = b, ij = -ji$.

Example 2.1: *For $a = b = -1$ this yields the common example of a quaternion algebra over $\mathbb{C}$, the Hamilton quaternions.*

We denote this by writing

$$\mathfrak{A} \cong \left(\frac{a,b}{k}\right).$$

For $x = x_1 1 + x_2 i + x_3 j + x_4 ij, (x_1, ..., x_4 \in k)$ the conjugate of x is defined as $\bar{x} := x_1 1 - x_2 i - x_3 j - x_4 ij$, the norm of x is given by $N(x) := x\bar{x}$ and the trace of x as $Tr(x) := x + \bar{x}$. The norm gives rise to a quadratic form $N : \mathfrak{A} \to k$ with associated bilinear form $\beta(x, y)$ given by $\beta(x, y) = N(x + y) - N(x) - N(y) = Tr(x\bar{y})$.

If a is not a square in k we have the following description of the quaternion algebra $\left(\frac{a,b}{k}\right)$:
Via the embedding $k(\sqrt{a}) \hookrightarrow \left(\frac{a,b}{k}\right)$ given by $(1 \mapsto 1, \sqrt{a} \mapsto i)$, $\left(\frac{a,b}{k}\right)$ is both a $k(\sqrt{a})$ left- and right-module with $(1, j)$ as a basis. The vector space structures are related by $\alpha j = j\alpha'$ for all $\alpha \in k(\sqrt{a})$. If we write u instead of j we get $\left(\frac{a,b}{k}\right) \cong k(\sqrt{a}) \oplus k(\sqrt{a})u$, where the ring structure is defined by $u^2 = b$ and $u\alpha = \alpha' u$.
From now on, we consider rational quaternion algebras over $k = \mathbb{Q}$. We distinguish between *definite* and *indefinite* rational quaternion algebras. The algebra $\left(\frac{a,b}{\mathbb{Q}}\right)$ is (in-)definite if the norm considered as a quadratic form is (in-)definite, i.e. $\left(\frac{a,b}{\mathbb{Q}}\right)$ is definite if and only if a and b are both negative.
An element of a rational quaternion algebra is called *integral* if its norm and trace are in $\mathbb{Z}$. An *order* O in $\left(\frac{a,b}{\mathbb{Q}}\right)$ is a subring with 1 that only contains integral quaternions and that is free as a $\mathbb{Z}$-modul and of rank 4. The norm restricted to an order O can be interpreted as an integral quaternion quadratic form which is called the *norm form* of O. The discriminant of this form is called the *discriminant* of O. It is always a square in $\mathbb{Z}$. We call an order *maximal* if it is maximal with respect to inclusion, that means if it is not properly contained in any other order. All maximal orders of a quaternion algebra have the same discriminant Δ, because their quadratic forms can be transferred into each other via rational linear transformations with determinant ± 1.

We define the *reduced discriminant* of $\left(\frac{a,b}{\mathbb{Q}}\right)$ as

$$d := \begin{cases} \sqrt{\Delta} & \text{if } \left(\frac{a,b}{\mathbb{Q}}\right) \text{ is indefinite} \\ -\sqrt{\Delta} & \text{if } \left(\frac{a,b}{\mathbb{Q}}\right) \text{ is definite.} \end{cases}$$

The absolute value of the reduced discriminant d is equal to the product of all primes p for which $\left(\frac{a,b}{\mathbb{Q}_p}\right)$ is a division ring. Thus, it is easy to calculate. For our purposes, we only need a special case:

Theorem 2.2 ([9, Satz 3.3])**:** *Let N be a positive number such that $\left(\frac{p}{AN}\right) \neq -1$. Then the reduced discriminant d of $\left(\frac{p,-n/Ap}{\mathbb{Q}}\right)$ is the product of the prime divisors q of N satisfying*

$$\left(\frac{p}{q}\right) = -1 \text{ and } \nu_q(N) \equiv 1 \bmod 2 \text{ or}$$
$$\left(\frac{p}{q}\right) = 0 \text{ and } \left(\frac{NA,p}{q}\right) = -1,$$

where $\nu_q(N)$ denotes the exponent of q in N.

Proof. A prime number q divides d if and only if there is no non-trivial solution of

$$\alpha\alpha' + \gamma\gamma'\frac{N}{Ap} = 0$$

with $\alpha, \gamma \in \mathbb{Q}(\sqrt{p})_q$, where $\mathbb{Q}(\sqrt{p})_q := \mathbb{Q}(\sqrt{p}) \otimes \mathbb{Q}_q$, $\mathbb{Q}_q$ the q-adic numbers.
If $\left(\frac{p}{q}\right) = 1$, then there exist non-trivial solutions of $\alpha\alpha' = 0$. Otherwise the existence of a non-trivial solution of the above equation is equivalent to the existence of a non-trivial solution of the equation

$$\alpha\alpha' + ApN = 0$$

with $\alpha \in \mathbb{Q}(\sqrt{p})_q$, so we get

$$q|d \Leftrightarrow \left(\frac{p}{q}\right) \neq 1 \text{ and } \left(\frac{-ApN,p}{q}\right) = \left(\frac{NA,p}{q}\right) = -1.$$

□

We need the following result by Eichler:

Theorem 2.3 ([6, Satz 11])**:** *Let $\left(\frac{a,b}{\mathbb{Q}}\right)$ an indefinite rational quaternion algebra with reduced discriminant d, O^{max} a maximal order in $\left(\frac{a,b}{\mathbb{Q}}\right)$ and ε^{max} the group of units of norm 1 in O^{max}, then ε^{max} acts properly discontinuously on $\mathbb{H}$ and we have*

$$\mathrm{vol}(\mathbb{H}/\varepsilon^{max}) = \frac{1}{6}\varphi(d)$$

with φ the Euler-function.

Proof. See [6]. □

Now let N be a positive integer and B a skew-hermitian matrix

$$B = \begin{pmatrix} a\sqrt{p} & \lambda \\ -\lambda' & \frac{b}{A}\sqrt{p} \end{pmatrix}$$

of discriminant N satisfying the following conditions:

1. $(a, ApN) = 1$
2. $a^2 | b$
3. $\lambda \in q^{\nu_q(N)+1}\mathfrak{A}^{-1}$ for all $q|N$ **or** $\lambda \in p^{\frac{\nu_p(N)}{2}}\mathfrak{A}^{-1}$ and $\lambda\lambda' A - N \equiv 0 \bmod p^{\nu_p(N)+6}$ if p divides N and $\lambda\lambda' A - N \equiv 0 \bmod p^{\nu_p(N)+2}$ has a solution $\lambda \in \mathfrak{A}^{-1}$.

It can be shown that such a skew-hermitian matrix exists in every equivalence class. For the rest of this chapter we assume that B is a matrix satisfying the three conditions above.

Definition 2.4: *For N and B as above we define*

1. $\mathfrak{A}_B := \left\{ T \in M_2(\mathbb{Q}(\sqrt{p})) : T^*B = B\tilde{T} \right\}$ *with* $\tilde{T} = det(T)T^{-1}$ *if* $det(T) \neq 0$ *and* T^* *the transpose conjugate of* T,
2. $O_B := \mathfrak{A}_B \cap \begin{pmatrix} \mathcal{O}_K & \mathfrak{a}^{-1} \\ \mathfrak{a} & \mathcal{O}_K \end{pmatrix}$.

Then the group ε_B that we defined at the beginning of Section 2.1 is equal to the group of units of O_B with determinant 1 (see [9, Kapitel 3]).

Lemma 2.5 ([9, Lemma 3.6])**:** *Let $N, B, \mathfrak{A}_B$ and O_B as above. Then:*

1. $\mathfrak{A}_B \cong \left(\frac{p, -N/Ap}{\mathbb{Q}}\right)$ *is an indefinite quaternion algebra.*
2. $O_B \subset \mathfrak{A}_B$ *is an order of discriminant N^2, and*

$$\begin{aligned} O_B &\cong \left\{ \begin{pmatrix} \delta + \frac{\lambda}{\sqrt{p}}\gamma & -\frac{N}{Ap}\gamma' \\ \gamma & \delta' - \frac{\lambda'}{\sqrt{p}}\gamma' \end{pmatrix} : \delta \in o_b, \gamma \in \mathfrak{m}_b \right\} \\ &= o_b + \left(\frac{\lambda}{\sqrt{p}} + u \right) \mathfrak{m}_b := O, \end{aligned}$$

where o_B is the order of the conductor b in $\mathbb{Q}(\sqrt{p})$ (see [10] or Section 2.2.2 for the definition of the conductor), $\mathfrak{m}_b$ is the o_b-module that consists of the elements γ of $\frac{1}{b}\mathfrak{a}$ for which $\lambda\gamma - \lambda'\gamma'$ is integral and u was defined before.

Proof. See [9, Lemma 3.6]. □

Now we can construct a maximal order O^{max} that contains O.
If d is the reduced norm of $\mathfrak{A}_B$, then $\frac{N}{d}$ is the norm of an ideal $\mathfrak{n} \in \mathcal{O}_K$

which is prime to the ideal b. Then $\mathfrak{n}_b := \mathfrak{n} \cap o_b$ is an o_b-module of the same norm. Thus, we get:

Lemma 2.6 ([9, Lemma 3.7])**:** *$O^{max} := o_b + \left(\frac{\lambda}{\sqrt{p}} + u\right) \mathfrak{n}_b^{-1}\mathfrak{m}_b$ is a maximal order in $\left(\frac{p,-N/pA}{\mathbb{Q}}\right)$.*

Let ε^{max} (resp. ε) be the group of units with norm 1 in O^{max} (resp. O). Then the index $[\varepsilon^{max} : \varepsilon]$ allows us with the help of Theorem 2.3 to determine the volume of the curves F_N.
For a prime number q dividing $\frac{N}{d}$ we consider the $\mathbb{Z}/q^{r(q)}\mathbb{Z}$-module

$$O^{max} \otimes_{\mathbb{Z}} \mathbb{Z}/q^{r(q)}\mathbb{Z} =: L_q^{max}$$

with $r(q) := \nu_q\left(\frac{N}{d}\right)$ if $q \neq 2$ and $r(2) = \nu_2\left(\frac{N}{d}\right) + 2$.
The canonical map $\varphi_q : O^{max} \to O^{max} \otimes \mathbb{Z} \to L_q^{max}$ is a ring homomorphism and the norm $N : O^{max} \to \mathbb{Z}$ induces via φ_q a map

$$N_q : L_q^{max} \to \mathbb{Z}/q^{r(q)}\mathbb{Z}.$$

For $x, y \in L_q^{max}$ we have $N_q(xy) = N_q(x)N_q(y)$ and an element in L_q^{max} is a unit in L_q^{max} if and only if $N_q(x) \in (\mathbb{Z}/q^{r(q)}\mathbb{Z})^\times$. We denote the multiplicative group of units with norm 1 in L_q^{max} by U_q^{max}. Furthermore, we denote by L_q the image of O under φ_q and define U_q to be $U_q^{max} \cap L_q$.
U_q^{max}(resp. U_q) is the kernel of the group homomorphism

$$N_q : (L_q^{max})^\times \ \to \ (\mathbb{Z}/q^{r(q)}\mathbb{Z})^\times$$

(resp. the restriction to $L_q^\times$), so

$$[U_q^{max} : U_q] = \frac{|(L_q^{max})^\times||N_q(L_q^\times)|}{|N_q((L_q^{max})^\times)|\ |L_q^\times|}.$$

Lemma 2.7 ([9, Lemma 3.9])**:** *Let q be a prime number that divides $\frac{N}{d}$. Then*

$$|(L_q^{max})^\times| = \begin{cases} (q^{r(q)-1})^4(q^4 - q^3 - q^2 + q) & \text{if } q \not| d \\ (q^{r(q)-1})^4(q^4 - q^2) & \text{if } q|d \end{cases}$$

and

$$|(L_q)^\times| = \frac{(q^{r(q)-1})^4}{q^{\nu_q(N/d)}} \cdot \begin{cases} (q^4 - 2q^3 + q^2) & \text{if } \left(\frac{p}{q}\right) = 1 \text{ or } q = p \text{ and } q^2 \not| N \\ (q^4 - q^2) & \text{if } \left(\frac{p}{q}\right) = -1 \\ (q^4 - q^3) & \text{else.} \end{cases}$$

Every element of $(\mathbb{Z}/q^{r(q)}\mathbb{Z})^\times$ is the norm of an element in $L_q^\times$ if $q \neq p$

or $q = p$ and $\nu_q(N) = 1$. Otherwise $N_q(L_q^\times)$ is a subgroup of index 2 in $(\mathbb{Z}/q^{r(q)}\mathbb{Z})^\times$.

2.1.2 Volume of the curves F_N and T_N

With these lemmas we are now able to compute the volume of T_N. With the above Theorem 2.3 by Eichler and Lemmas 2.5 and 2.7 we get

Theorem 2.8: *[9, Satz 3.10] Let N be a positive integer, then*

$$\mathrm{vol}(F_N) = -\frac{1}{12} N(1 + \chi_p(NA)) \prod_{q|N} \alpha_q,$$

where

$$\alpha_q = \begin{cases} \left(1 + \left(\frac{p}{q}\right)/q\right) & \text{if } q \neq p \\ \left(1 + \left(\frac{NA}{q}\right)/q\right) & \text{if } q = p, \nu_q(N) = 1 \\ (1 - 1/q^2) & \text{if } q = p, \nu_q(N) \geq 2. \end{cases}$$

Because all irreducible curves have finite volume, the volume of T_N is well-defined. Since

$$T_N = \bigcup_{n^2|N} F_{N/n^2},$$

we have

$$\mathrm{vol}(T_N) = \sum_{n^2|N} \mathrm{vol}(F_N)$$

yielding the following corollary:

Corollary 2.9 ([14, p. 82, 7, Korollar 2.4.8, 9, Korollar 3.11])**:** *The volume of the curve T_N is given by*

$$\mathrm{vol}(T_N) = -\frac{1}{12} \sum_{d|N} \left(\chi_p(d) + \chi_p\left(\frac{AN}{d}\right)\right) d.$$

We will not give the proof here. In [9, Korollar 3.11] it is proven by calculating the left-hand side and the right-hand side of the formula directly and then showing that they are equal in every possible case. In [7, Korollar 2.4.8] it is proven inductively.

2.2 Special points and transversal intersection

In this section we take a look at so called special points in which the curves T_M and T_N intersect. With special points we will mean both the intersection points in $X = \mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$ and their preimages in $\mathbb{H}^2$.
In the first Subsection 2.2.1 we will see that to a special point $z \in \mathbb{H}^2$ we can associate a binary positive definite quadratic form φ_z that is defined on an oriented lattice $\mathfrak{M}_z$. The $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of φ_z only depends on the image $\mathfrak{z}$ of z in X.
The main result of Subsection 2.2.1 is given in Theorem 2.13 stating how often the equivalence class of a given form φ is represented as a form of a special point in X. To prove this theorem, we first show that every special point in X can be represented by a point on the diagonal via a convenient isomorphism, so it is enough to count the number of $\Gamma_0(N)$-equivalence classes of certain quadratic irrational numbers in $\mathbb{H}$ (see 2.16).
We will then learn about modules of imaginary quadratic fields in subsection 2.2.2, before we use this knowledge in Subsection 2.2.3 to look at representations of numbers by positive definite binary quadratic forms to be able to prove the main theorem of this subsection, namely Theorem 2.18.
This allows us to finally prove the main theorem of this section, Theorem 2.24.

2.2.1 Binary quadratic forms and special points

Before we look at special points and the transversal intersection, we need some terminology concerning binary quadratic forms.
Let L be an oriented $\mathbb{Z}$-lattice of rank 2 and $\{e_1, e_2\}$ a basis of L that is compatible with the orientation. A quadratic form $\varphi : L \to \mathbb{Z}$ can be written as

$$\varphi(ue_1 + ve_2) = \alpha u^2 + \beta uv + \gamma v^2,$$

where $\alpha = \varphi(e_1), \beta = \varphi(e_1 + e_2) - \varphi(e_1) - \varphi(e_2), \gamma = \varphi(e_2)$. We will denote such a quadratic form by $[\alpha, \beta, \gamma]$. As in Section 1.1.1 the discriminant of the quadratic form φ is given by $\beta^2 - 4\alpha\gamma$, the content of φ by the greatest common divisor (α, β, γ). If the content is 1, then the quadratic form is called primitive. Two forms $[\alpha_1, \beta_1, \gamma_2]$ and $[\alpha_2, \beta_2, \gamma_2]$ are equivalent if they can be transformed into one another by an element of $\mathrm{SL}_2(\mathbb{Z})$ (equivalences between quadratic forms should always be orientation preserving).

Let $\mathfrak{M}$ be the quaternionic lattice of matrices

$$\begin{pmatrix} a\sqrt{p} & \lambda \\ -\lambda' & \frac{b}{A}\sqrt{p} \end{pmatrix}$$

with $\lambda \in \mathfrak{a}^{-1}$, $a, b \in \mathbb{Z}$ and $A = \mathrm{norm}(\mathfrak{a})$. We get an indefinite quadratic form of discriminant p^3 by

$$\mathfrak{M} \ni B \mapsto A \cdot \det B \in \mathbb{Z}.$$

Then for $z = (z_1, z_2) \in \mathbb{H}^2$ the lattice

$$\mathfrak{M}_z := \left\{ B \in \mathfrak{M} : (z_2, 1) B \begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0 \right\}$$

is of rank $0, 1$ or 2. A point $z \in \mathbb{H}^2$ is called *special* if the rank of $\mathfrak{M}_z$ is 2. The restriction of the form

$$\begin{aligned} \mathbb{R}^3 &\longrightarrow \mathbb{C} \cong \mathbb{R}^2 \\ (x_1, x_2, x_3) &\mapsto x_1 z_1 z_2 + x_2 z_2 + x_3 z_1 \end{aligned}$$

to $\mathfrak{M}_z$ is then a binary quadratic form denoted by φ_z.

The form φ_z is positive definite because for $0 \neq B \in \mathfrak{M}_z$

$$\mathrm{Im}(z_2) = \frac{\det B}{|a\sqrt{p} z_1 + \lambda|^2} \mathrm{Im}(z_1) > 0.$$

Considering the 4-dimensional vector space $\mathfrak{M} \otimes \mathbb{R}$,

$$(\mathfrak{M} \otimes \mathbb{R})_z := \left\{ B \in \mathfrak{M} \otimes \mathbb{R} : (z_2, 1) B \begin{pmatrix} z_1 \\ 1 \end{pmatrix} = 0 \right\}$$

is a 2-dimensional $\mathbb{R}$-vector space for all $z \in \mathbb{H}^2$. Thus, we get an embedding of $\mathbb{H}^2$ into the Grassmannian $\mathfrak{G}$ of 2-dimensional subspaces of $\mathfrak{M} \otimes \mathbb{R}$. Looking also at the Grassmannian $\hat{\mathfrak{G}}$ of 2-dimensional oriented subspaces of $\mathfrak{M} \otimes \mathbb{R}$ we get a canonical 2-fold cover $\hat{\mathfrak{G}} \to \mathfrak{G}$. Since $\mathbb{H}^2$ is simply connected, there are two possibilities of lifting the embedding $\mathbb{H} \to \mathfrak{G}$ to an embedding $\mathbb{H} \to \hat{\mathfrak{G}}$ of which we choose one for the rest of this chapter.

For $T \in \mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$ the automorphism of $\mathfrak{M}$ given by $B \mapsto T^* B T$ sends the lattice $\mathfrak{M}_{Tz}$ orientation preserving to $\mathfrak{M}_z$. This makes the quadratic forms φ_{Tz} and φ_z $\mathrm{SL}_2(\mathbb{Z})$-equivalent. Therefore we can consider a quadratic form $\varphi_{\mathfrak{z}}$ for a special point $\mathfrak{z} \in X$, where we call a point $\mathfrak{z} \in X$ *special* if the representatives of $\mathfrak{z} \in \mathbb{H}^2$ are special. The form $\varphi_{\mathfrak{z}}$ is well-defined up to $\mathrm{SL}_2(\mathbb{Z})$-equivalence.

Definition 2.10: *For φ a positive definite binary quadratic form we define the number $s(\varphi)$ of points $\mathfrak{z} \in X$ with $\varphi_{\mathfrak{z}} \cong \varphi$ by:*

$$s(\varphi) := w_\varphi \sum_{\substack{\mathfrak{z} \in X: \\ \varphi_{\mathfrak{z}} \cong \varphi}} \frac{1}{v_{\mathfrak{z}}},$$

where w_φ is the order of $Aut(\varphi)$, the group of orientation preserving automorphisms of φ, and $v_{\mathfrak{z}}$ is the order of the isotropy group of $SL_2(\mathcal{O}_K, \mathfrak{a})$ at a point $z \in \mathbb{H}^2$ that represents $\mathfrak{z}$.

Lemma 2.11 ([14, Lemma 1, 9, Lemma 4.3.])**:** *Let z be a special point. Then the discriminant of $\varphi_z : \mathfrak{M}_z \to \mathbb{Z}$ is divisible by p and φ_z represents only quadratic residues modulo p. The content m of φ_z is not divisible by p^2 or by any prime q with $\left(\frac{q}{p}\right) = -1$. If m is divisible by p and the discriminant of $\frac{1}{p}\varphi_z$ is also divisible by p, then $\frac{1}{p}\varphi_z$ represents only quadratic residues mod p.*

From this lemma we know that a primitive form $\frac{1}{m}\varphi_z$ of a special point z represents numbers x with

$$\left(\frac{p}{xA}\right) = \left(\frac{m}{p}\right).$$

There exist infinitely many prime numbers x solving this equation. We choose one of them that does not divide mpA and call it q_0. q_0 is the norm of an ideal because both m and A are norms of ideals and

$$\left(\frac{p}{q_0}\right) = \left(\frac{p}{q_0 A}\right) = \left(\frac{m}{p}\right) = 1,$$

so $q_0 m$ is the norm of an ideal $\mathfrak{b}$ which lies in the same genus as $\mathfrak{a}$ since

$$\left(\frac{q_0 m A}{p}\right) = \left(\frac{q_0 A}{p}\right)\left(\frac{m}{p}\right) = \left(\frac{p}{q_0 A}\right)\left(\frac{m}{p}\right) = 1.$$

Lemma 2.12 ([9, Lemma 4.4.])**:** *Let m be the content of the quadratic form φ which satisfies the conditions in the lemma before. Then there exists a prime number q_0 that is prime to mpA such that φ represents the number $N = q_0 m$ with N being a norm of an ideal $\mathfrak{b}$ that lies in the same genus as $\mathfrak{a}$.*

For such a quadratic form φ all the special points $\mathfrak{z} \in X$ with $\varphi_{\mathfrak{z}} \cong \varphi$ have to lie on F_N because φ represents N primitively.
We will now prove:

Theorem 2.13 ([9, Satz 4.2.])**:** *Let φ be a positive definite binary quadratic form representing N with content m of discriminant $\Delta < 0$*

and let $\varphi_0 := \frac{1}{m}\varphi$ *be the associated primitive form. Then:*

$$s(\varphi) = \frac{1}{2}\left(1 + \left(\frac{mA}{p}\right)\right)\chi_p(\varphi_0)\beta_p(m)h'\left(\frac{-\Delta}{p}\right),$$

where

$$\chi_p(\varphi_0) = \begin{cases} 0 & \text{if } p \not| \Delta_0 \\ 1 & \text{if } p|\Delta \text{ and } \varphi_0 \text{ represents only quadratic residues mod } p \\ -1 & \text{otherwise} \end{cases}$$

and

$$\beta_p(m) = \begin{cases} \prod_{q|m}(1 + \chi_p(q)) & \text{if } p^2 \not| m \\ 0 & \text{if } p^2|m. \end{cases} \tag{2.1}$$

We start the proof with the case where N is the norm of a primitive ideal $\mathfrak{b}$ in the genus of $\mathfrak{a}$, so the form φ with discriminant Δ represents $N = q_0 m$. As we have seen in (1.7), every equation $\mathfrak{b} = \eta\mathfrak{c}^2\mathfrak{a}$ ($\eta \gg 0$, $\mathfrak{c}$ an ideal) induces an isomorphism

$$\rho_{\eta,\mathfrak{c}} : \mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{b}) \xrightarrow{\cong} \mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a}).$$

Furthermore, for every component of F_N there exist suitable η and $\mathfrak{c}$ such that this component is the image under $\rho_{\eta,\mathfrak{c}}$ of the curve in $\mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{b})$ that is given by the diagonal $\mathbb{H}$ in $\mathbb{H}^2$ ($t \mapsto (t,t)$). We choose for every component $F_{N,i}$ of F_N such an isomorphism and call it ρ_i.
If p does not divide N, then the subgroup Γ of $\mathrm{SL}_2(N)$ that maps the diagonal of $\mathbb{H}^2$ to itself is

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), c \equiv 0 \bmod N\right\},$$

otherwise $\Gamma_0(N)$ is a subgroup of Γ of index 2.
The map

$$\mathbb{H}/\Gamma \to \mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{b})$$

induced by the diagonal maps the curve $\mathbb{H}/\Gamma$ with degree 1 to its image. Under the isomorphisms ρ_i that we chose above the lattice $\mathfrak{M}$ corresponds to the lattice

$$\mathfrak{N} = \left\{B = \begin{pmatrix} a\sqrt{p} & \lambda \\ -\lambda' & \frac{c}{N}\sqrt{p} \end{pmatrix} : a, b \in \mathbb{Z}, \lambda \in \mathfrak{a}^{-1}\right\}$$

and the quadratic form $A \cdot \det$ on $\mathfrak{M}$ corresponds to the form

$$N \cdot \det : \mathfrak{N} \longrightarrow \mathbb{Z}$$
$$B \mapsto N \cdot \det B.$$

A point t in $\mathbb{H}$ (or $\rho_i([t,t])$ for all i) is special if and only if there exists an equation

$$a_0Nt^2 + b_0t + c_0 = 0 \tag{2.2}$$

with $a_0, b_0, c_0 \in \mathbb{Z}, a_0 > 0, (a_0, b_0, c_0) = 1$.
Let $\mathfrak{N}_t$ be the lattice consisting of matrices $B \in \mathfrak{N}$ with

$$(t,1)B\begin{pmatrix} t \\ 1 \end{pmatrix} = 0.$$

Then it can be shown that

$$\mathfrak{N}_t = \mathbb{Z}B_t + \mathbb{Z}C_0, \quad \text{with } A_t = \begin{pmatrix} a_0\sqrt{p} & \lambda_0 \\ -\lambda_0' & \frac{c_0}{N}\sqrt{p} \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $\lambda_0 = b_0\frac{M+\sqrt{p}}{2N}$ with $M \in \mathbb{Z}$ such that $\mathfrak{b}^{-1}$ is equal to $\mathbb{Z}\frac{M+\sqrt{p}}{2N} + \mathbb{Z}1$ (see [14, p. 66]).
For fixed ρ_i all the bases (A_t, C_0) have the same orientation. Without loss of generality we assume that it is the same as the one we have chosen before. Then the quadratic form φ_t for a special point t is given by

$$[a_0c_0p + b_0^2\frac{M^2-p}{4N}, b_0M, N]. \tag{2.3}$$

The discriminant of this form is $p(b_0^2 - 4a_0c_0N) = p\Delta_1$, where Δ_1 is the discriminant of (2.2). The content of φ_t is divisible by p if and only if p divides b_0 and p divides the discriminant of $\frac{1}{p}\varphi_p$ if and only if p divides a_0c_0. Thus, we get that the content of this form is equal to m if and only if $m = (a_0c_0, b_0, N)p$ and p divides m or if $m = (a_0c_0, b_0, N)$ and p does not divide m. A prime number q that is not equal to p and that does not divide m cannot divide M, so if it divides the content m of φ with exponent α, it also has to divide b_0 at least with that exponent. So we get:

Lemma 2.14 ([9, Lemma 4.5.])**:** *Let φ be a binary quadratic form with discriminant $\Delta < 0$ and satisfying the conditions of Lemma 2.11. Assume that the content m of φ is not divisible by 4 and $N = q_0m = N(\mathfrak{b})$ for some ideal $\mathfrak{b} \subset \mathcal{O}_K$ with $\mathfrak{b} = \eta\mathfrak{c}^2\mathfrak{a}$, where $\eta \gg 0$ and $\mathfrak{c}$ an ideal. Then a special point $\mathfrak{z} \in \mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$ with $\varphi_\mathfrak{z} \cong \varphi$ corresponds under the right isomorphism*

$$\rho_{\eta,\mathfrak{c}} : \mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{b}) \overset{\cong}{\to} \mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$$

to a point t on the diagonal of $\mathbb{H}^2/SL_2(\mathcal{O}_K, \mathfrak{b})$. This point satisfies the equation

$$a_0Nt^2 + b_0t + c_0 = 0$$

with $a_0, b_0, c_0 \in \mathbb{Z}, a_0 > 0, (a_0, b_0, c_0) = 1$ and the corresponding form has discriminant

$$\Delta_1 = b_0^2 - 4a_0c_0N = \frac{\Delta}{p}.$$

On the other hand, each point $t \in \mathbb{H}$ that satisfies the conditions above represents a special point whose quadratic form represents N with content m and discriminant Δ.

Definition 2.15: *We denote by $h'(\Delta)$ the modified class number of primitive positive definite binary quadratic forms of discriminant $\Delta < 0$, where the class of the form φ is counted with multiplicity $\frac{2}{w_\varphi}$.*

$$h'(\Delta) = \begin{cases} \frac{1}{3} & \text{if } \Delta = -3 \\ \frac{1}{2} & \text{if } \Delta = -4 \\ h(\Delta) & \text{if } \Delta \equiv 0 \text{ or } 1 \bmod 4, \Delta \leq -4, \end{cases}$$

where $h(\Delta)$ is the usual class number that counts every class with multiplicity 1.

We have:

Lemma 2.16 ([14, Lemma 2, 9, Lemma 4.6.])**:**

1. *Let $N > 0, \Delta < 0, m > 0$ be given with $\Delta \equiv 0$ or 1 mod 4, $m|N, m^2|\Delta, \frac{N}{m}$ square-free and prime to m.*
   Furthermore, let $\frac{\Delta}{m^2} \equiv 0, 1$ mod 4 and every odd prime number q that divides Δ and N, but not m, fulfills $q^2 \not|\Delta$. If 2 divides $(\frac{N}{m}, \Delta)$ then $\frac{\Delta}{4}$ has to be congruent to 2 or 3 mod 4. Then the number of $\Gamma_0(N)$-equivalence classes of positive definite quadratic forms $[aN, b, c]$ with

   $$(a, b, c) = 1, \quad b^2 - 4acN = \Delta, \quad (N, b, ac) = m,$$

   each form φ being counted with multiplicity $\frac{2}{|Aut(\varphi)\cap\Gamma_0(N)|}$ is given by

   $$h'(\Delta)2^\nu \prod_q \left(1 + \left(\frac{\Delta}{q}\right)\right),$$

   where ν is the number of distinct prime factors of m and the product is taken over all primes q diving $\frac{N}{m}$.

2. *Let $N > 0, \Delta < 0, m > 0$, $\Delta \equiv 2^{k-1}$ mod $2^k, N \equiv 2^{k-3}$ mod 2^{k-2} $(k \in \mathbb{N}, k \geq 4)$, m odd and $m|N, m^2|\Delta, \frac{N}{2^{k-3}m}$ square-free and prime to m. Every odd prime number q that divides Δ and N, but not m, fulfills $q^2 \not|\Delta$. Then the number of $\Gamma_0(N)$-equivalence classes of positive definite quadratic forms $[aN, b, c]$ with*

   $$(a, b, c) = 1, b^2 - 4acN = \Delta, (N, b, ac) = m,$$

   each form φ being counted with multiplicity $\frac{2}{|Aut(\varphi)\cap\Gamma_0(N)|}$, is given by

   $$h'(\Delta)2^\nu \prod_q \left(1 + \left(\frac{\Delta}{q}\right)\right).$$

Proof. By the requirements on the positive definite quadratic forms $[aN,b,c]$ we know that every prime number q dividing m divides either a or c but not both. So (N,b,a) and (N,b,c) are relatively prime, have product m and are $\Gamma_0(N)$-invariant. The value of b is also $\Gamma_0(N)$-invariant and $b \equiv mx \bmod 2N$ with x satisfying $x^2 \equiv \frac{\Delta}{m^2} \bmod \frac{4N}{m}$. But there are exactly

$$\prod_q \left(1 + \left(\frac{\Delta}{q}\right)\right)$$

such residue classes and 2^ν decompositions $m = m_1 m_2$, so the only thing left to prove is that for each decomposition $m = m_1 m_2$ and each value of x the number of $\Gamma_0(N)$-equivalence classes of forms $[aN,b,c]$ satisfying the conditions of the lemma and

$$(N,b,a) = m_1, \ (N,b,c) = m_2, \ b \equiv mx \bmod 2N$$

is $h'(\Delta)$.

This is done by showing that the map

$$\varphi = [aN,b,c] \mapsto \tilde{\varphi} = [aN_1,b,cN_2]$$

with $N = N_1 N_2, (N_1,N_2) = 1$ and N_2 containing only those prime divisors that divide m_2, is a $1:1$-correspondence between $\Gamma_0(N)$ equivalence classes of forms $[aN,b,c]$ satisfying the conditions and $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of primitive forms of discriminant Δ. A detailed proof of this can be found in [14, Lemma 2]. □

Now we finish the proof of Theorem 2.13. By the above lemma we can assume that m is not divisible by p^2 or any prime q with $\left(\frac{q}{p}\right) \neq 1$. Furthermore, we can choose $\chi_p(\varphi_0)$ to be 0 or 1. Now, we first assume φ to have no non-trivial automorphisms and not to be equal to $\varphi_{\mathfrak{z}}$ for one of the finitely many quotient singularities $\mathfrak{z} \in X$, so $v_{\mathfrak{z}} = w_{\mathfrak{z}} = 2$.

First, we consider the case $p \nmid m$, then we have $p \nmid N$ and $(N,M) = 1$. The lemma above shows that there are

$$h'(\Delta/p) 2^\nu \prod_{q|N/m_1} \left(1 + \left(\frac{\Delta_1}{q}\right)\right)$$

Γ-equivalence classes of points $t \in \mathbb{H}$, where ν the number of different primes dividing m_1 and $p \nmid N$.

Now we calculate for how many of these points we obtain that $\varphi_t \cong \varphi$. Because

$$\left(\frac{\Delta}{q_0}\right) = \left(\frac{p}{q_0}\right) = 1,$$

we have $\left(\frac{\Delta_1}{q_0}\right) = 1$, so there are two $\mathrm{SL}_2(\mathbb{Z})$-equivalent representations of q_0 by primitive forms of discriminant Δ/m^2. There are again two cases to consider:

1. φ is – up to $\mathrm{SL}_2(\mathbb{Z})$-equivalence – the only form with discriminant Δ with content m that represents $N = q_0 m$ and φ represents N twice.
2. Both forms $\varphi = [\alpha, \beta, \gamma]$ and $[\gamma, \beta, \alpha](\cong [\alpha, -\beta, \gamma])$ are $\mathrm{GL}_2(\mathbb{Z})$-equivalent, but not $\mathrm{SL}_2(\mathbb{Z})$-equivalent and each one represents N exactly once.

In the first case the $2h'(\Delta_1)2^\nu$ points $[t] \in \mathbb{H}/\Gamma$ with $\varphi_t \cong \varphi$ are mapped in pairs to $h'(\Delta_1)2^\nu$ special points $\mathfrak{z}$ of $\mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$ with $\varphi_\mathfrak{z} \cong \varphi$. Then F_N has one or two components and since φ represents N twice, this means that these points are double points in F_N, so each point has two preimages under

$$\bigcup_i \mathbb{H}/\Gamma \times \{i\} \to \mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$$
$$([t], i) \mapsto \rho_i([t, t]).$$

In the second case every special point is only considered once, but then also only in half of the cases $\varphi_t \cong \varphi$ is true, so we get $2^\nu h'(\Delta_1) = \beta_p(m) h'(\Delta/p)$ in both cases.
Now we consider the case $p|m$. Then $(N, M) = p$, $p \nmid \frac{M^2-p}{4N}$ and also $p^2|\Delta$, so $p|\Delta_1$. The content of the form (2.3) is either (N, b_0, a_0c_0) or $p(N, b_0, a0c_0)$ (see discussion before Lemma 2.14) depending on whether p does divide $\Delta_0 = \Delta/m^2$ or not.
By Lemma 2.16 there exist

$$h'(\Delta_1)2^\nu \left(1 + \left(\frac{\Delta_1}{q_0}\right)\right)$$

different $\Gamma_0(N)$-equivalence classes of forms $[a_0N, b_0, c_0]$ with value (N, b_0, a_0c_0), ν being the number of prime factors of (N, b_0, a_0c_0). We have $2^\nu = (1+\chi_p(\varphi_0))\beta_p(m)$. The rest can be done as in the case $p \nmid m$, the final result only has to be divided by $|\Gamma : \Gamma_0(N)| = 2$.
Now the only thing missing to finish the proof of Theorem 2.13 are the isotropy and automorphism groups.
To calculate $s(\varphi)$ we count each point $\mathfrak{z}$ with multiplicity $\frac{w_\varphi}{v_\mathfrak{z}}$, where $w_\varphi = |\mathrm{Aut}(\varphi)|$ and $v_\mathfrak{z} = |\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})_\mathfrak{z}|$.
For the map $\pi : \mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})_\mathfrak{z} \to \mathrm{Aut}(\varphi)$ we have

$$\frac{w_\varphi}{v_\mathfrak{z}} = \frac{|\mathrm{Aut}(\varphi)|}{|\mathrm{Im}(\pi)|} \cdot \frac{1}{|\mathrm{Ker}(\pi)|}.$$

The first factor represents the number of *distinct* $t \in \mathbb{H}/\Gamma$ mapping onto $\mathfrak{z}$, the second factor is equal to $2/|\Gamma_t|$, where Γ_t is the isotropy group of t in Γ. In Lemma 2.16 each $\Gamma_0(N)$-equivalence class of forms was counted in this way, so the proof counts each point $\mathfrak{z}$ in the way required in the theorem. Thus, we have proven Theorem 2.13. □

2.2.2 Modules in imaginary quadratic fields

Let k be an imaginary quadratic field with discriminant D and M a free $\mathbb{Z}$-module of rank 2 contained in k. Two modules M_1 and M_2 are called equivalent if there exists an element $\alpha \in k$ such that $\alpha\ M_1 = \ M_2$. An *order* of k is a module that is also a subring of the ring of integers $\mathfrak{o}$ of k containing the element 1. The *conductor* f of an order $\mathfrak{o}'$ is defined as the number $|\mathfrak{o} : \mathfrak{o}'| = f$. For every $f \in \mathbb{N}$ there exists a unique order $\mathfrak{o}_f$ with this property. The set of all $\alpha \in k$ with $\alpha M \subset M$ is an order $\mathfrak{o}(M)$ for every module M, the conductor f of this order is also called the conductor of M and $\mathfrak{o}(M) = \mathfrak{o}_f$. The *norm* $N(M)$ of a module M is characterized as the unique rational number such that the quadratic form

$$x \mapsto \frac{x\bar{x}}{N(M)}\ (x \in M, \bar{x} = \text{ conjugate of } x)$$

is integral and primitive. That quadratic form has discriminant Df^2. Every module is oriented by the basis z, w with $z/w \in \mathbb{H}$.
There is a bijection between the set of equivalence classes of modules of discriminant Δ and the set of isomorphism classes of primitive positive definite integral quadratic forms of discriminant Δ. We have $\Delta = Df^2$, where D is the discriminant of a field k, a so-called fundamental discriminant. The number of equivalence classes of modules with discriminant Δ is the class number $h(\Delta)$. Every module is equivalent to a module with basis $z, 1$ with $z \in \mathbb{H}$, so we get a bijection between the set of equivalence classes of all modules in all imaginary quadratic fields and the set of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of points in $\mathbb{H}$ satisfying a quadratic equation over $\mathbb{Q}$.
If φ is a primitive form and M the corresponding module with $\mathbb{Z}$-basis $z, 1$ ($z \in \mathbb{H}$) then the group $\mathrm{Aut}(\varphi)$, the group $\mathrm{Aut}(M)$ of units of $\mathfrak{o}(M)$ and the isotropy group of z in $\mathrm{SL}_2(\mathbb{Z})$ are isomorphic.

The function $H(n)$ which is defined for $n > 0$ as the number of equivalence classes of all positive definite forms of discriminant $-n$ (counted with multiplicity), can instead also be defined as the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of points $z \in \mathbb{H}$ satisfying

$$\alpha z^2 + \beta z + \gamma = 0,\ \beta^2 - 4\alpha\gamma = -n,$$

where $(\alpha, \beta, \gamma) \in \mathbb{Z}^3$ is arbitrary and where a point equivalent to i or $\exp(\pi i/3)$ is counted with multiplicity $\frac{1}{2}$ or $\frac{1}{3}$ respectively. Then we have

$$H(n) = \sum_{d^2|n} h'\left(-\frac{n}{d^2}\right),$$

where d runs through the natural numbers such that $-n/d^2$ is a discriminant.

Two modules M_1 and M_2 in k can be multiplied. For f_1 the conductor of M_1 and f_2 the conductor of M_2 we have that $f = (f_1, f_2)$ is the conductor of M_1M_2 and $N(M_1M_2) = N(M_1)N(M_2)$. The equivalence classes of modules with a fixed conductor f form a group $G(Df^2)$ of order $h(Df^2)$ and for every divisor d of f the map

$$M \mapsto \mathfrak{o}_d M$$

induces a homomorphism of $G(Df^2)$ onto $G(Dd^2)$, which can be used to prove:

Proposition 2.17 ([14, Proposition 1])**:** *Let D be a fundamental discriminant and f a positive integer. Then*

$$h'(Df^2) = h'(D)\gamma_D(f),$$

where

$$\gamma_D(f) = f \prod_{q|f} \left(1 - \left(\frac{D}{q}\right) \Big/ q\right)$$

and q runs through all primes dividing f.

2.2.3 The representation of forms by binary quadratic forms

Now we want to prove the following theorem:

Theorem 2.18 ([14, Theorem 2])**:** *For φ a positive definite binary quadratic form with content m and discriminant $\Delta < 0$ let $\varphi_0 := \frac{1}{m}\varphi$ be the associated primitive form. Then:*

$$s_0(\varphi) = \sum_{d|m} \frac{1}{2} \left(\chi_p(d) + \chi_p\left(\frac{mA}{d}\right)\chi_p(\varphi_0)\right) dH\left(\frac{-\Delta}{pd^2}\right),$$

where

$$H(n) = \begin{cases} -\frac{1}{12} & \text{if } n = 0 \\ \sum_{d^2|n} h'\left(-\frac{n}{d^2}\right) & \text{else.} \end{cases}$$

Before we can prove the theorem, we first need some results about representations of numbers by primitive binary quadratic forms.

Proposition 2.19 ([14, Proposition 2])**:** *Let $D_1 f^2 < 0$ be a discriminant, where D_1 is the discriminant of an imaginary quadratic field. Let $\mathcal{D}$ be a fundamental discriminant that divides $D_1 f^2$ as a discriminant (that means $D_1 f^2/\mathcal{D} \equiv 0, 1 \mod 4$). Then we can rewrite this as $D_1 f^2/\mathcal{D} = \mathcal{D}^\circ g^2$, where $\mathcal{D}^\circ$ is a fundamental discriminant and g an* *integer. For $n \in \mathbb{N}$ we define*

$$r_{\mathcal{D}^\circ}^{\mathcal{D}}(g,n)$$

as the number of $SL_2(\mathbb{Z})$-inequivalent representations of n by primitive binary quadratic forms φ of discriminant $D_1 f^2 = \mathcal{D}\mathcal{D}^\circ g^2$, where every quadratic form φ is counted with multiplicity $\chi_{\mathcal{D}}(\varphi)$. Then we have:

1. *$r_{\mathcal{D}^\circ}^{\mathcal{D}}(g,n)$ is simultaneously multiplicative in g and n, i.e. for $g_1, g_2, n_1, n_2 \in \mathbb{N}$ with $(g_1 n_1, g_2 n_2) = 1$ we have*

$$r_{\mathcal{D}^\circ}^{\mathcal{D}}(g_1 g_2, n_1 n_2) = r_{\mathcal{D}^\circ}^{\mathcal{D}}(g_1, n_1) r_{\mathcal{D}^\circ}^{\mathcal{D}}(g_2, n_2).$$

2. *For q a prime number define*

$$R_{\mathcal{D}^\circ,q}^{\mathcal{D}}(t,u) = \sum_{\alpha,\beta=0}^{\infty} r_{\mathcal{D}^\circ}^{\mathcal{D}}(q^\alpha, q^\beta) t^\alpha u^\beta$$

as a generating series. Then

$$R_{\mathcal{D}^\circ,q}^{\mathcal{D}}(t,u) = \frac{(1-\chi_{\mathcal{D}}(q)tu)(1-\chi_{\mathcal{D}^\circ}(q)tu)}{(1-qtu^2)(1-\chi_{\mathcal{D}^\circ}(q)u)(1-t)(1-\chi_{\mathcal{D}}(q)u)}.$$

The proof of this proposition is given in [14, p. 69] for $\mathcal{D} = 1$ and can be given analogously for general $\mathcal{D}$, which can be found in [9, Kapitel 5]. Using the proposition we can now look at the representations of binary forms.
For modules M_1, M_2 in k with conductors f_1, f_2 we consider the module

$$\mathrm{Hom}(M_1, M_2) = \{\alpha \in k \mid \alpha M_1 \subset M_2\}$$

on which there is the quadratic form

$$\varphi_{M_1,M_2}(\alpha) = |M_2 : \alpha M_1|.$$

Lemma 2.20 ([14, Lemma 3])**:** *The content of the quadratic form φ_{M_1,M_2} equals $m = f_1 f_2 / f^2$, where $f = (f_1, f_2)$. The primitive quadratic form $\frac{1}{m}\varphi_{M_1,M_2}$ belongs to the module $\overline{M}_1 M_2$.*

Let ψ be a quadratic form with discriminant $D_1 f_1^2 m_1^2 < 0$. For a fundamental discriminant $\mathcal{D}$ we define

$$r_{D_1}^{(\mathcal{D})}(\psi; m_2, f_2)$$

as the number of $\mathrm{SL}_2(\mathbb{Z})$-inequivalent representations of ψ by quadratic forms φ of discriminant $D_1 f_2^2 m_2^2$ with content m_2, where every representation by such a φ is counted with multiplicity $\chi_{\mathcal{D}}(\psi/m_1)\chi_{\mathcal{D}}(\varphi/m_2)$.

If ψ is the quadratic form

$$x \mapsto \psi(x) = m_1 \frac{x\bar{x}}{N(M_1)}, \ x \in M_1,$$

then such a representation is given by a module $M_2 \subset k = \mathbb{Q}(\sqrt{D})$ with conductor f_2 and an element $\alpha \in \mathrm{Hom}(M_1, M_2)$ with

$$|M_2 : \alpha M_1| = \varphi_{M_1,M_2}(\alpha) = \frac{m_1 f_1}{m_2 f_2}. \tag{2.4}$$

Therefore $r^{(\mathcal{D})}_{D_1}(\psi; m_2, f_2)$ is the number of equivalence classes (M_2, α) with α a homomorphism from M_1 to M_2 satisfying (2.4), where (M_2, α) is considered equivalent to $(\tilde{M_2}, \tilde{\alpha})$ if there exists an element $\gamma \in k$ such that $\gamma M_2 = \tilde{M_2}$ and $\tilde{\alpha} = \gamma\alpha$.
Furthermore, we define for a fundamental discriminant $D_1 < 0$ and a natural number f

$$\gamma_{D_1}(f) := f \prod_{q|f} \left(1 - \left(\frac{D_1}{q}\right) \Big/ q\right) = \left(\frac{h'(D_1 f^2)}{h'(D_1)}\right).$$

(See [3, p. 170] for more on $\gamma_D(f)$.)

Proposition 2.21 ([14, Prop 3,3']): *With the conditions of Proposition 2.19 we have:*

1. $r^{(\mathcal{D})}_{D_1}(\psi; m_2, f_2)$ *depends only on* m_1, m_2, f_1, f_2.
2. $r^{(\mathcal{D})}_{D_1}(\psi; m_2, f_2) = 0$ *if* $\mathcal{D}$ *does not divide* $D_1 f_1^2$ *and* $D_1 f_2^2$ *as a discriminant. If* $\mathcal{D}$ *divides* $D_1 f_1^2$ *and* $D_1 f_2^2$ *as a discriminant, then there is a fundamental discriminant* $\mathcal{D}^\circ$ *and natural numbers* g_1, g_2 *with* $D_1 f_1^2 = \mathcal{D}\mathcal{D}^\circ g_1^2$ *and* $D_1 f_2^2 = \mathcal{D}\mathcal{D}^\circ g_2^2$. *Writing* $r^{(\mathcal{D})}_{\mathcal{D}^\circ}(m_1, g_1; m_2, g_2)$ *instead of* $r^{(\mathcal{D})}_{D_1}(\psi; m_2, f_2)$ *we get*

$$r^{(\mathcal{D})}_{\mathcal{D}^\circ}(m_1, g_1; m_2, g_2) = r^{(\mathcal{D})}_{\mathcal{D}^\circ}(g, \frac{m_1 g_1^2}{m_2, g_2^2}) \frac{\gamma_{D_1}(f_2)}{\gamma_{D_1}(f)},$$

   for $g = (g_1, g_2), f = (f_1, f_2), r^{\mathcal{D}}_{\mathcal{D}^\circ}(g, n) = 0$ *if* $n \notin \mathbb{Z}$.
   Furthermore, $r^{\mathcal{D}}_{\mathcal{D}^\circ}(m_1, g_1; m_2, g_2)$ *is simultaneously multiplicative in* m_1, g_1, m_2, g_2.

Proof. For a fixed module M_1 sending the module M_2 to the product $\bar{M}_1 M_2$ yields a map $G(D_1 f_2^2) \to G(D_1 f^2)$ from the group of equivalence classes of modules with conductor f_2 to the group of equivalence classes of modules with conductor f. Since $\bar{M}_1 M_2 = \mathfrak{o}_f \bar{M}_1 \mathfrak{o}_f M_2$ we know that the preimage of an element of $G(D_1 f^2)$ has

$$\frac{|G(D_1 f_2^2)|}{|G(D_1 f^2)|}$$

elements. We get

$$\frac{|G(D_1 f_2^2)|}{|G(D_1 f^2)|} = \frac{\gamma_D(f_2)}{\gamma_D(f)} \frac{|\mathrm{Aut}(M_2)|}{|\mathrm{Aut}(\bar{M}_1 M_2)|}. \tag{2.5}$$

Now we have to count the pairs (M_2, α) with $\alpha \in \mathrm{Hom}(M_1, M_2)$ such that (2.4) holds. But by Lemma 2.20 we know that this is the number of pairs (M_2, α) with $\alpha \in \bar{M}_1 M_2$ such that

$$\frac{\alpha\bar{\alpha}}{N(M_1)N(M_2)} = \frac{m_1}{m_2}\frac{f^2}{f_2^2}.$$

The number of possible α is given by $r_{D_1}^{(\mathcal{D})}(\psi; m_2, f_2) \cdot |\mathrm{Aut}(\bar{M}_1 M_2)|$. So to get the number of pairs (M_2, α) we have to multiply it with the expression in (2.5) and divide by $|\mathrm{Aut}(M_2)|$. □

If M is a module in an imaginary quadratic field k and φ_0 the corresponding primitive quadratic form, we put $\chi_p(M) = \chi_p(\varphi_0)$. So $\chi_p(M) = 0$ if the discriminant Δ of M is not divisible by p and $\chi_p(M) = \pm 1$ if $p|\Delta$.
Let ψ be again a form of discriminant $D_1 f_1^2 m_1^2 < 0$ with D_1 a fundamental discriminant and m_1 content of ψ, then we define

$$\rho_{D_1}(\psi; m_2, f_2) \tag{2.6}$$

as the number of $\mathrm{SL}_2(\mathbb{Z})$-inequivalent representations of ψ by quadratic forms of discriminant $D_1 f_2^2 m_2^2$ with content m_2, where every representation of φ is counted with multiplicity

$$\frac{1}{2}\left(1 + \left(\frac{m_2 A}{p}\right)\right)\chi_p(\varphi_0),$$

where $\varphi_0 = \frac{1}{m_2}\varphi$.

Proposition 2.22: *For ρ_{D_1} defined as above*

$$\rho_{D_1}(\psi; m_2, f_2) = \frac{1}{2} r_{D_1}^{(p)}(\psi; m_2, f_2)\chi_p(\psi/m)\chi_p(m_2 A).$$

Proof. See Prop 3 in [14], also p.73 in [14]. □

Now for a positive definite binary quadratic form ψ of discriminant $\Delta = D_1 f_1^2 m^2$ let $s_0(\psi)$ be the number of $\mathrm{SL}_2(\mathbb{Z})$-inequivalent representations of ψ by forms of special points $z \in \mathbb{H}^2$, where every representation is counted with multiplicity $1/|\mathrm{Ker}(\pi)|$, where π is given by

$$\pi : \mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})_z \to \mathrm{Aut}(\varphi_z),$$

the operation of the group homomorphism induced by $\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$. Then $s_0(\psi)$ is 0 if Δ/p is not a discriminant.
Now we can give the proof of Theorem 2.18:

Proof. We write again m_1 instead of m and $\Delta = D_1 f_1^2 m_1^2$, where D_1 is a fundamental discriminant and $\mathcal{D}^\circ$ is the discriminant of $\mathbb{Q}(\sqrt{D_1 p})$. Theorem 2.13 implies

$$s_0(\psi) = \sum_{m_2, f_2} \rho_{D_1}(\psi, m_2, f_2)\beta_p(m_2, D_1 f_2^2 m_2^2) h'(D_1 f_2^2 m_2^2/p),$$

where ρ and β_p are given as in (2.6) and (2.1) and $h'(n) = 0$ if n is not integral. With Proposition 2.22 it suffices to prove

$$\begin{aligned} &\sum_{n|m_1} \chi_p(n)\chi_p(m_1/n)\chi_p(\psi_0) H(-D_1 f_1^2/pn^2) \\ &= \sum_{m_2, f_2} \chi_p(m_2)\chi_p(\psi_0) r^p_{D_1}(\psi; m_2, f_2)\beta_p(m_2, D_1 f_2^2 m_2^2) h'(D_1 m_2^2/p). \end{aligned} \tag{2.7}$$

We write $H(-\Delta_1) = \sum_{c^2|\Delta_1} h'(\Delta_1/c^2)$ and $h'(\mathcal{D}^\circ f^2) = h'(\mathcal{D}^\circ)\gamma_{\mathcal{D}^\circ}(f)$. Then we can divide both sides of (2.7) by $h'(\mathcal{D}^\circ)$ and define u_1 by $D_1 f_1^2 m_1^2 = \mathcal{D}^\circ u_1^2 p$ to get

$$\begin{aligned} &\sum_{n|m_1} \sum_{b|\frac{u_1}{n}} n\chi_p(n)\chi_p(m_1/n)\gamma_{\mathcal{D}^\circ}(b) \\ &= \sum_{m_2, g_2} \chi_p(m_2)\beta_p(m_2, \mathcal{D}^\circ p u_2^2)\gamma_{\mathcal{D}^\circ}(u_2) r^p_{\mathcal{D}^\circ}(m_1, g_1; m_2, g_2). \end{aligned}$$

On the right-hand-side we replace g_i by $u_i\alpha_{\mathcal{D}}/m_i$, where $\alpha_{\mathcal{D}}$ is defined by $D^0 = D_1\alpha_{\mathcal{D}}^2$, where D^0 is the discriminant of $\mathbb{Q}(\sqrt{D_1})$, then we get

$$\sum_{m_2, u_2} \chi_p(m_2)\beta_p(m_2 \mathcal{D}^\circ p u_2^2)\gamma_{\mathcal{D}^\circ} u_2 r^p_{\mathcal{D}^\circ}(m_1, \frac{u_1\alpha_{\mathcal{D}}}{m_1}; m_2, \frac{u_2\alpha_{\mathcal{D}}}{m_2}),$$

where $r^p_{\mathcal{D}^\circ}(x, y; v, w) = 0$ if one of the numbers x, y, v, w is not integral. Both sides of (2.7) are simultaneously multiplicative in m_1, u_1, so it is enough to prove (2.7) in the case where m_1 and u_1 can be written as powers of some prime number q. Now we write $a(r, s)$ for the left-hand-side of (2.7) with $m_1 = q^r$ and $u_1 = q^{r+s}, q^s = g_1/\alpha_{\mathcal{D}}$, where q is a prime number that does not divide $\alpha_{\mathcal{D}}$. Then we define

$$A_0(x, y) = \sum_{r, s \geq 0} a(r, s) x^r y^s$$

and get

$$\begin{aligned} A_0(x, y) &= \sum_{r, s \geq 0} a(r, s) x^r y^s \\ &= \frac{1 - \chi_{\mathcal{D}^\circ}(q)y - \chi_p(q)qxy + \chi_{\mathcal{D}^\circ}(q)\chi_p(q)(-x + xy - qxy)}{(1 - \chi_p(q)qx)(1 - \chi_p(q)x)(1 - y)(1 - \chi_p(q)qx)(1 - qy)} \end{aligned}$$

$$= \begin{cases} \frac{1-\chi_{\mathcal{D}^\circ}(q)y-\chi_{\mathcal{D}}(q)qxy+\chi_{\mathcal{D}^\circ}(q)\chi_{\mathcal{D}}(q)(-x+xy-qxy)}{(1-\chi_p(q)qx)(1-\chi_p(q)x)(1-y)(1-\chi_p(q)qx)(1-qy)} & \text{if } q \neq p \\ \frac{1-\chi_{\mathcal{D}^\circ}(q)y}{(1-y)(1-qy)} & \text{if } q = p. \end{cases}$$

Now let q be a prime number dividing $\alpha_{\mathcal{D}}$ with exponent i. Then let $a_i(r,s)$ for $i = 1,2,3$ be the left-hand-side of (2.7) for $m_1 = q^r$, $u_1 = q^{r+s-i}$, with $i \geq 2$ only possible for $q = 2$. Then with

$$A_i(x,y) = \sum_{r,s\geq 0} a_i(r,s)x^r y^s$$

we get

$$A_1(x,y) = \frac{y + \chi_{\mathcal{D}^\circ}(q)x - \chi_{\mathcal{D}}(q)xy - \chi_{\mathcal{D}^\circ}(q)qxy}{(1-\chi_{\mathcal{D}}(q)x)(1-\chi_{\mathcal{D}}(q)xq)(1-y)(1-qy)},$$

and with $x' = \chi_{\mathcal{D}}(q)x$ we get

$$qA_2(x,y) = A_1(x,y) - \sum_{r,s;r+s\geq 1} x'^r y^s = A_1(x,y) - \frac{x'+y-x'y}{(1-x')(1-y)}$$

and

$$\begin{aligned} qA_3(x,y) &= A_2(x,y) - \sum_{r,s;r+s\geq 2} x'^r y^s \\ &= A_2(x,y) - \frac{x'^2+y^2+x'y-x'y-x'y^2-x'^2y}{(1-x')(1-y)}. \end{aligned}$$

For the right-hand-side of (2.7) we consider for the coefficients

$$\chi_p(m_2)\beta_p(q^k, \mathcal{D}^\circ p u_2^2)\gamma_{\mathcal{D}^\circ} u_2$$

and q a prime number that does not divide $\alpha_{\mathcal{D}}$ the following formal series:

$$B_0(x,y) = \sum_{k,l} \chi_p(q^k)\beta_p(q^k, \mathcal{D}^\circ p q^{2(k+l)})\gamma_{\mathcal{D}^\circ}(q^{k+l})x^k y^l.$$

Then we have

$$B_0(x,y) = \begin{cases} \frac{\chi_{\mathcal{D}^\circ}(q)}{q} + (1-\chi_{\mathcal{D}^\circ}(q)q)\frac{1+qx'}{(1-\chi_p(q)qx')(1-qy)} & \text{if } q \neq p \\ \frac{\chi_{\mathcal{D}^\circ}(q)}{q} + (1-\chi_{\mathcal{D}^\circ}(q)q)\frac{1}{(1-qy)} & \text{if } q = p, \end{cases}$$

where for $q = 2$ and $q = p$ the summands for $k \geq 2$ are zero.
For $q|\alpha_{\mathcal{D}}$ we first consider the case $\nu_q(\alpha_{\mathcal{D}}) = 1$, then we look at

$$\begin{aligned} B_1(x,y) &= \sum_{k,l;k+l>0} \chi_p(q^k)\beta_p(q^k, p\mathcal{D}^\circ q^{2(k+l-1)})\gamma_{\mathcal{D}^\circ}(q^{k+l-1})x^k y^l \\ &= \frac{1}{q}\left(\frac{1+x'q}{1-qy} - 1\right). \end{aligned}$$

For $\nu_q(\alpha_{\mathcal{D}}) = 2$ only $q = 2$ is possible, and the terms $k + l - 1$ become $k + l - 2$ and we get

$$B_2(x, y) = \frac{1}{q}(B_1(x, y) - y - x') + x'^2.$$

For the last case, we get

$$B_3(x, y) = \frac{1}{q}(B_2(x, y) - x'^2 - y^2 - x'y) + x'^3 + x'^2 y.$$

For every prime number q the functions $r^p_{\mathcal{D}^\circ}(q^m, q^n; q^k, q^l)$ now define a $\mathbb{Q}[[x]]$-linear endomorphism G_q on the $\mathbb{Q}$-vector space $\mathbb{Q}[[x, y]]$:

$$\begin{aligned} G_q : \sum_{m,n} a_{m,n} x^m y^n &\mapsto \sum_{m,n} a'_{m,n} x^m y^n \\ \text{with } a'_{m,n} &= \sum_{k,l} r^p_{\mathcal{D}^\circ}(q^m, q^n; q^k, q^l) a_{k,l}. \end{aligned}$$

Since

$$r^p_{\mathcal{D}^\circ}(q^m, q^n \alpha_{\mathcal{D}}; q^k, q^l \alpha_{\mathcal{D}}) = r^p_{\mathcal{D}^\circ}(q^m, q^{n+\nu_q \alpha_{\mathcal{D}}}; q^k, q^{l+\alpha_{\mathcal{D}}})$$

it is enough to show for every q and every $i \in \{0, 1, 2, 3\}$ that

$$G_q(B_i) = A_i.$$

This can be done exactly as in the proof of Theorem 2 in [14] by using the above equations. □

2.2.4 Transversal intersections of the curves T_N

Now we are finally able to calculate the transversal intersection number of the curves T_M and T_N on $X = \mathbb{H}/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$. If MN is not a square, then T_M and T_N do not have a common component, and all the intersections of T_M and T_N are what Hirzebruch and Zagier call transversal.

Remark 2.23: *The term transversal might be confusing, but Hirzebruch and Zagier use it to describe the intersection of two curves which have no common component.*

If MN is a square and they do have a common component, then this common component is exactly the curve $T_{(M,N)}$. Consider the projection

$$\mathrm{pr} : \mathbb{H}^2 \to \mathbb{H}/\ \mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a}).$$

We call the intersection in a special point $\mathfrak{z}$ of two curves T and $\tilde{T}$ *transversal* if the lifted curves $\mathrm{pr}^{-1}(T)$ and $\mathrm{pr}^{-1}(\tilde{T})$ intersect transversally in one (and therefore in every) preimage z of $\mathfrak{z}$.
Now for $\mathfrak{z} \in X$ represented by $z \in \mathbb{H}^2$ the *local transversal intersection number* of T_M and T_N is defined as follows:
If k_1 components of $\mathrm{pr}^{-1}(T_M)$, k_2 components of $\mathrm{pr}^{-1}(T_N)$ and k_3 components of the one-dimensional part of $\mathrm{pr}^{-1}(T_M \cap T_N)$ meet in z, then the local transversal intersection number is given by

$$(T_M T_N)_{\mathfrak{z}}^{tr} := \frac{k_1 k_2 - k_3}{v_{\mathfrak{z}}/2},$$

where $v_{\mathfrak{z}}$ is the order of the isotropy group $\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})_z$, so $v_{\mathfrak{z}}/2$ is the order of the isotropy group of $\mathrm{SL}_2(\mathcal{O}_K)/\{\pm 1\}$ at z.
There are only finitely many transversal intersection points of T_M and T_N, so

$$(T_M T_N)_X^{tr} := \sum_{\mathfrak{z} \in X} (T_M T_N)_{\mathfrak{z}}^{tr}$$

is well-defined.

A transversal intersection of T_M and T_N in a point $\mathfrak{z} \in X$ represented by $z \in \mathbb{H}^2$ is given by an ordered pair (A, B) of linearly independent elements of $\mathfrak{M}_z$ with $\det A = M, \det B = N$, so the transversal intersections exist only in special points (see Section (1.1.1)). All ordered pairs (A, B) determine an orientation of $\mathfrak{M}_z$, but an orientation was already chosen before. Let $(T_M T_N)_{\mathfrak{z}}$ be the transversal intersection number of T_N and T_M in $\mathfrak{z}$ in the sense of rational homology manifolds. Then the intersection $(T_M T_N)_{\mathfrak{z}}$ is $\frac{1}{v_{\mathfrak{z}}}$ times the number of ordered pairs (A, B) with A, B linearly independent elements of $\mathfrak{M}_{\mathfrak{z}}$ with $\varphi_z(A) = M$ and $\varphi_z(B) = N$.
The convention on the orientation allows only the sign change from A, B to $-A, -B$ and each $v_{\mathfrak{z}}$ contains the factor 2, with $v_{\mathfrak{z}}/2$ the order of the isotropy group of $\mathrm{SL}_2(\mathcal{O}_K)/\{\pm 1\}$ at z. Since $\mathrm{SL}_2(\mathcal{O}_K)/\{\pm 1\}$ acts effectively on $\mathbb{H}^2$, intersection theory on rational homology requires division by $v_{\mathfrak{z}}/2$.
For each pair (A, B) we get a quadratic form by restricting φ_z to $\mathbb{Z}A \oplus \mathbb{Z}B$, this quadratic form is given by $[M, b, N]$ with

$$b = \det(A + B) - \det(A) - \det(B).$$

The form is positive definite and has discriminant divisible by p (see Lemma 2.11), so the integer b has to satisfy $4MN - b^2 < 0$ and $4MN - b^2 \equiv 0 \bmod p$.
On the other hand, we can consider the quadratic forms $\psi = [M, b, N]$ over $\mathbb{Z} \oplus \mathbb{Z}$ satisfying those conditions. The number of $\mathrm{SL}_2(\mathcal{O}_K)_z$-inequivalent orientation-preserving embeddings $j : \mathbb{Z} \otimes \mathbb{Z} \to \mathfrak{M}_z$ with $\varphi_z \times j = \psi$ is equal to $(T_M T_N)_{\mathfrak{z}}$. Therefore, the transversal intersection

number is given by

$$(T_M T_N)_X^{tr} = \sum_{\substack{\mathfrak{z} \in X \\ \mathfrak{z} \text{ special}}} (T_M T_N)_{\mathfrak{z}}$$

and thus

$$(T_M T_N)_X^{tr} = \sum_{\substack{b \in \mathbb{Z} \\ b^2 < 4MN \\ b^2 \equiv 4MN \bmod p}} s_0([M, b, N]). \tag{2.8}$$

Theorem 2.24 ([14, Theorem 3])**:** *Let M and N be positive integers, $\nu_p(N) \leq \nu_p(M)$. Then the number of transversal intersections of curves T_M and T_N on the Hilbert modular surface X is given by*

$$T_M T_N{}_X^{tr} = \frac{1}{2} \sum_{d|(M,N)} d \left(H_p^0 \left(\frac{MN}{d^2} \right) \right) \left(\chi_p(n) + \chi_p \left(\frac{NA}{n} \right) \right),$$

with

$$H_p(n) = \sum_{\substack{x \in \mathbb{Z} \\ x^2 \leq 4n \\ x^2 \equiv 4n \bmod p}} H \left(\frac{4n - x^2}{p} \right),$$

$$H(n) = \begin{cases} -\frac{1}{12} & \text{if } n = 0 \\ \sum_{d^2|n} h' \left(-\frac{n}{d^2} \right) & \text{else,} \end{cases}$$

$$h'(\Delta) = \begin{cases} \frac{1}{3} & \text{if } \Delta = -3 \\ \frac{1}{2} & \text{if } \Delta = -4 \\ h(\Delta) & \text{if } \Delta \equiv 0 \text{ or } 1 \bmod 4, \Delta \leq -4, \end{cases}$$

Proof. By (2.8) and Theorem 2.24 we have

$$2(T_M T_N)_X^{tr} = \sum_{\substack{b \in \mathbb{Z} \\ b^2 < 4MN \\ b^2 \equiv 4MN \bmod p}} \sum_{d|(M,b,N)} \chi_p(d) d H \left(\frac{4MN - b^2}{pd^2} \right)$$

$$+ \sum_{\substack{b \in \mathbb{Z} \\ b^2 < 4MN \\ b^2 \equiv 4MN \bmod p}} \chi_p \left(\frac{[M, b, N]}{(M, b, N)} \right) \sum_{d|(M,b,N)} \chi_p \left(\frac{(M, b, N)A}{d} \right) d H \left(\frac{4MN - b^2}{pd^2} \right).$$

In the first sum we rewrite b as $b = dx$ with $x^2 \equiv 4MN/d^2 \bmod p$, because the terms contributing are those with $p \nmid d$. Then the first sum equals

$$\sum_{d|(M,N)} \chi_p(d) \sum_{\substack{x\in\mathbb{Z} \\ x^2<4MN/d^2 \\ x^2\equiv 4MN/d^2 \bmod p}} H((4MN/d^2-x^2)/p)$$
$$= \sum_{d|(M,N)} \chi_p(d) d H_p^0(MN/d^2).$$

In the second sum we only have to sum over those b which satisfy $(4MN-b^2)/(M,b,N)^2 \equiv 0 \bmod p$, hence $\frac{N}{(M,b,N)} \not\equiv 0 \bmod p$, so

$$\chi_p\left(\frac{[M,b,N]}{(M,b,N)}\right) = \chi_p\left(\frac{N}{(M,b,N)}\right).$$

This yields the equation

$$\chi_p\left(\frac{[M,b,N]}{(M,b,N)}\right)\chi_p\left(\frac{(M,b,N)A}{d}\right) = \chi_p\left(\frac{NA}{d}\right)$$

and we see as before that the second sum equals

$$\sum_{d|(M,N)} \chi_p\left(\frac{NA}{d}\right) dH_p^0(MN/d^2).$$

□

2.3 Intersection with cusp resolutions

As we have seen in Section 1.1.3, $X^{\mathfrak{a}}$ can be compactified by adding $h(p)$ points, the cusps. These cusps are in 1 : 1 correspondence with parabolic orbits of $\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})$ under the operation of this group on $\mathbb{P}^1(\mathbb{R})$ (see Section 1.1.2).
By identifying $(m:n) \in \mathbb{P}^1(K)$ with the ideal $\mathcal{O}_K m + \mathfrak{a}^{-1}n$ we get a bijection between the set of parabolical orbits and the ideal class group of K in the normal sense.
The cusp corresponding to the ideal class of $\mathfrak{b} = \mathcal{O}_K m + \mathfrak{a}^{-1}n$ is of type $(\mathfrak{a}^{-1}\mathfrak{b}^{-2}, U^2)$ in the sense of [11], where U^2 is the group of squares of units of $\mathcal{O}_K$. That means that every small enough neighbourhood of a cusp in $X^{\mathfrak{a}}$ looks like a neighbourhood of the cusp ∞ of the quotient space of $\mathbb{H}^2$ under the operation of the group of matrices of the form

$$\begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix}, \quad \text{with} \varepsilon \in U^2, \quad \mu \in \mathfrak{a}^{-1}\mathfrak{b}^{-2}.$$

The cusps can be resolved by rational curves S_k and for a fitting index set I we can describe the compactification of $X^{\mathfrak{a}}$ by

$$\bar{X}^{\mathfrak{a}} = X^{\mathfrak{a}} \cup \bigcup_{k \in I} S_k.$$

For every $k \in I$ there is a reduced quadratic irrationality

$$w_k = \frac{M_k + \sqrt{p}}{2N_k}, \quad 0 < w_k' < 1 < w_k,$$

and in this way we obtain all possible reduced quadratic irrationalities w of discriminant p with the added property that the corresponding ideal $\mathfrak{b}$ with

$$\mathfrak{b}^{-1} = \mathbb{Z}w + \mathbb{Z}1$$

is lying in the genus of $\mathfrak{a}$.
The coefficients of the section matrix $S_k \circ S_l$ $(k, l \in I)$ depend on the continued fraction expansion of w_k.
For an ideal $\mathfrak{b}$ in $\mathcal{O}_K$ we define

$$f(\mathfrak{b}) = \frac{1}{\sqrt{p}} \sum_{\substack{(\lambda)=\mathfrak{b} \\ \lambda>0, \lambda'>0}} \min(\lambda, \lambda').$$

Then the main result of this section is Proposition 2.27 stating

$$(T_M T_N)_\infty = \sum_{\substack{N(\mathfrak{b})=M \\ N(\mathfrak{c})=N}} f(\mathfrak{b}\mathfrak{c}),$$

where $(T_M T_N)_\infty$ is the intersection number of T_M and T_N coming from the cusps, the exact definition of $(T_M T_N)_\infty$ is given in (2.10).

2.3.1 The rational curves S_k of the cusp resolution

The curves T_N can be extended to curves on the compactification of $X^{\mathfrak{a}}$, which is given by $\bar{X}^{\mathfrak{a}} = X^{\mathfrak{a}} \cup \bigcup_{k \in I} S_k$, where the S_k are rational curves, I a finite index set. Those extended curves will also be denoted by T_N. The rational curves S_k of the cusp resolutions are arranged in finitely many cycles corresponding to the $h(p)$ cusps and generate a direct summand of $H_2(\bar{X}^{\mathfrak{a}}, \mathbb{Q})$. To each k corresponds a reduced quadratic irrationality of discriminant p, namely

$$w_k = \frac{M_k + \sqrt{p}}{2N_k}, 0 < w_k' < 1 < w_k,$$

where M_k and N_k are natural numbers. Each of these w_k determines a primitive ideal $\mathfrak{b}_k$ with $\mathfrak{b}_k^{-1} = \mathbb{Z}w_k + \mathbb{Z}$, $N(\mathfrak{b}_k) = N_k$ and

$$\mathfrak{b}_k = \left(N_k, \frac{M_k + \sqrt{p}}{2}\right) = \mathbb{Z}N_k + \mathbb{Z}\frac{M_k + \sqrt{p}}{2}.$$

The integers operate on the index set, the operation of $n \in \mathbb{Z}$ on $k \in I$ will be denoted by $k + n$. We have

$$w_k = b_k - \frac{1}{w_{k+1}}$$

with $b_k \in \mathbb{Z}$ and $b_k = [w_k] + 1 \geq 2$.

The section matrix $S_k \circ S_l, k, l \in I$, has the form

$$S_k \circ S_l = \begin{cases} 1 & \text{if } k = l \pm 1, \\ -b_k & \text{if } k = l, \\ 0 & \text{otherwise.} \end{cases}$$

The extended curve T_N meets each curve S_k of the resolution with some multiplicity $T_N \circ S_k$. Since $\det(S_k \circ S_l) \neq 0$ we can look at the divisor

$$T_N^c = T_N + \sum_k \alpha(N, k) S_k$$

with the property $T_N^c \circ S_j = 0$ for all j. To calculate the rational coefficients $\alpha(N, k)$ we need to invert the intersection matrix $(S_k \circ S_l)$. For an ideal $\mathfrak{b}$ in $\mathcal{O}_K$ we define

$$f(\mathfrak{b}) = \frac{1}{\sqrt{p}} \sum_{\substack{(\lambda)=\mathfrak{b} \\ \lambda>0,\lambda'>0}} \min(\lambda, \lambda'),$$

so $f(\mathfrak{b}) = 0$ if $\mathfrak{b}$ is not a principal ideal. For $\mathfrak{b}$ principal, we have $\mathfrak{b} = (\mu)$ for some μ and the sum is infinite but converges. For ε the generator of the infinite cyclic group U^+ of totally positive units of $\mathcal{O}_K$ which satisfy $\varepsilon < 1, \varepsilon' > 1$ with ε' the conjugate of ε and μ such that

$$\varepsilon^2 \leq \frac{\mu}{\mu'} \leq 1$$

we have by Hirzebruch and Zagier (see [14, p. 79])

$$f((\mu)) = \frac{1}{\sqrt{p}} \left(\sum_{j=0}^{\infty} \mu\varepsilon^j + \sum_{j=1}^{\infty} \mu'\varepsilon^j \right) = \frac{1}{\sqrt{p}} \left(\frac{\mu}{1-\varepsilon} + \frac{\mu'\varepsilon}{1-\varepsilon} \right). \tag{2.9}$$

Proposition 2.25 ([14, Prop 4])**:** *The inverse matrix of the intersection matrix $(S_k \circ S_l)$ is the matrix $(-f(\mathfrak{b}_k\mathfrak{b}_l))$, where $\mathfrak{b}_k$ resp. $\mathfrak{b}_l$ are the ideals associated to the curves S_k resp. S_l.*

Proof. See the proof in [14]. □

The coefficient $\alpha(N,k)$ of S_k in the formula of the divisor T_N^c is now given in terms of the matrix $f(\mathfrak{b}_k\mathfrak{b}_l)$:

$$\alpha(N,k) = \sum_l f(\mathfrak{b}_k\mathfrak{b}_l')(S_l T_N).$$

From now on we write X instead of $X^{\mathfrak{a}}$ and $\bar{X}$ instead of $\bar{X}^{\mathfrak{a}}$ in the rest of this section. For the intersection numbers in the sense of rational homology we then get

$$\begin{aligned}(T_M T_N^c)_{\bar{X}} &= (T_M^c T_N)_{\bar{X}} = (T_M^c T_N^c)_{\bar{X}} \\ &= (T_M T_N)_{\bar{X}} + \sum_{k,l} f(\mathfrak{b}_k\mathfrak{b}_l{}')(S_k T_M)(S_l T_N).\end{aligned}$$

If MN is not a square, then the curves T_M and T_N have no common components. The two curves meet in finitely many points of the compact surface $\bar{X}$. $(T_M T_N)_{\bar{X}}$ breaks up into the part coming from $(T_M T_N)_X$ and the sum of the intersection multiplicities of T_M and T_N in points of $\bigcup S_k = \bar{X}\backslash X$. If this sum is denoted by $(T_M T_N)_{\bar{X}\backslash X}$ we put

$$(T_M T_N)_\infty = (T_M T_N)_{\bar{X}\backslash X} + \sum_{k,l} f(\mathfrak{b}_k\mathfrak{b}_l)(S_k T_M)(S_l T_N) \tag{2.10}$$

and get

$$T_M T_N^c = (T_M T_N)_X + (T_M T_N)_\infty.$$

Proposition 2.26 ([14, Prop. 5])**:** *Assume that MN is not a square. Then*

$$(T_M T_N)_\infty = \sum_{\substack{N(\mathfrak{b})=M \\ N(\mathfrak{c})=N}} f(\mathfrak{b}\mathfrak{c}),$$

where the sum is taken over all ideals $\mathfrak{b}$, $\mathfrak{c}$ in $\mathcal{O}_K$ satisfying $N(\mathfrak{b}) = M$, $N(\mathfrak{c}) = N$.

Proof. The intersection of T_M with a small neighbourhood of $\bigcup S_k$ is the union of all curves

$$u_k^q = v_k^p, \quad p \geq 0, \quad q \geq 0, \quad p^2 N_{k-1} + pqM_k + q^2 N_k = M,$$

for q and p relatively prime and where (u_k, v_k) is the local coordinate system in which S_k is given by $v_k = 0$ and S_{k-1} by $u_k = 0$.
The characteristic $(k|q,p)$ (see [11, p. 247] for more on the characteristic) determines an ideal $\mathfrak{b}$ with $N(\mathfrak{b}) = M$ so that the curves correspond bijectively to ideals $\mathfrak{b}$ in $\mathcal{O}_K$ with $N(\mathfrak{b}) = M$. Note that $(k|0,1)$ has to be identified with $(k+1|1,0)$.
For T_N we get similar equations near $\bigcup S_k$, it is the union of the curves

$$u_l^t = v_l^s, \quad s \geq 0, \quad t \geq 0, \quad s^2 N_{l-1} + stM_l + t^2 N_l = N,$$

and those correspond bijectively to ideals $\mathfrak{c}$ in $\mathcal{O}_K$ with $N(\mathfrak{c}) = N$. The curve $u_k^q = v_k^p$ intersects S_k with multiplicity q and S_{k-1} with multiplicity p, the curve $u_l^t = u_l^s$ intersects S_l with multiplicity t and S_{l-1} with multiplicity s, no other intersections with curves S_j occur. In a sufficiently small neighbourhood these curves only intersect if $k = l$. Their intersection point is the origin of the k-th coordinate system u_k, v_k. The intersection numbers of the affine curves $u^q - v^p = 0$ and $u^t - v^s = 0$ at their origin is for $pt - qs \neq 0$ equal to $\min(pt, qs)$, so the intersection numbers of the curves near $\bigcup S_j$ is $\delta_{kl} \min(pt, qs)$.
Each pair of such curves contributes the following to $(T_M T_N)_\infty$:

$$\delta_{kl} \min(pt, qs) + f(\mathfrak{b}_{k-1}\mathfrak{b}'_{l-1})ps + f(\mathfrak{b}_{k-1}\mathfrak{b}'_l)pt + f(\mathfrak{b}_k\mathfrak{b}'_{l-1})qs + f(\mathfrak{b}_k\mathfrak{b}'_l)qt.$$

Now we have to show that this is equal to $f(\mathfrak{a}\mathfrak{b}')$, but that is an easy calculation that was done in [14, p. 82].

□

2.3.2 Self-intersection and the adjunction formula

We now show that Proposition 2.26 is also true for MN a square.
For positive integers M, N we define

$$(T_M T_N)_X = (T_M T_N)_X^{tr} + \operatorname{vol}((T_M \cap T_N)^1)$$

with $(T_M \cap T_N)^1$ the 1-dimensional part of $T_M \cap T_N$, which is empty if MN is not a square and equals $T_{(M,N)}$ if MN is a square, where (M, N) is the greatest common divisor of M and N. For the definition of $(T_M T_N)^{tr}$ see Section 2.2.4.
Assume that MN is a square and that $\nu_p(N) \leq \nu_p(M)$ holds, then we have $\nu_p((M, N)) = \nu_p(N)$. $N/(M, N)$ is also a square and prime to p. Hence, we have $\chi_p(N/d) = \chi_p((M, N)/d)$ and by Section 2.1

$$\operatorname{vol}(T_{(M,N)}) = \sum_{d|(M,N)} d(\chi_p(d) + \chi_p(AN/d))H(0),$$

where $H(0)$ is defined as $-1/12$. We define the function H_p by

$$H_p(n) = \sum_{\substack{x\in\mathbb{Z} \\ x^2\leq 4n \\ x^2\equiv 4n \bmod p}} H\left(\frac{4n - x^2}{p}\right).$$

(For the definition of $H(\cdot)$ see (1.1)). If n is not a square, we have $H_p(n) = H_p^0(n)$, where $H_p^0(\cdot)$ was defined in Theorem 2.24. If $n > 0$ is a square, we have $H_p(n) - H_p^0(n) = 2H(0)$, so we get

$$(T_M T_N)_X = \frac{1}{2} \sum_{d|(M,N)} d(\chi_p(d) + \chi_p(NA))H_p(MN/d^2)$$

if $\nu_p(N) \leq \nu_p(M)$.

If MN is not a square we have

$$(T_M T_N^c)_{\bar{X}} = (T_M T_N)_X + (T_M T_N)_\infty.$$

We now want to show that Proposition 2.26 is also true for MN a square.

Proposition 2.27 ([14, Proposition 6])**:** *For the curves T_M, T_N on the Hilbert modular surface $\bar{X}$ we have*

$$T_M T_N^c = T_M^c T_N = (T_M T_N)_X + (T_M T_N)_\infty$$

with

$$(T_M T_N)_\infty = \sum f(\mathfrak{a}\mathfrak{b}').$$

Before we can prove this proposition, we first need to learn something about the adjunction formula.

Self-intersection numbers can be calculated with the adjunction formula (see [11, Section 0.6]): For a compact curve T on a singularity free complex surface Y the Euler number of the singularity free model $\tilde{T}$ of T is given by

$$e(\tilde{T}) = c_1(T) - TT + \sum_{x \in Y} \mu_x(T),$$

where c_1 is the first Chern class of Y and $\mu_x(T)$ is the Plücker number of T in x, a non-negative integer that only depends on the germ of T in x which is positive if and only if x is a singular point of T. If for example r non-singular branches of T intersect pairwise transversally in x, then $\mu_x(T) = r(r-1)$.
The adjunction formula can be generalized to surfaces Y with isolated fixed points, but then the Euler number $e(\tilde{T})$ has to be replaced by some modified Euler number $e'(\tilde{T})$ if there are quotient singularities of Y on T.
For this we choose for each $x \in Y$ a sufficiently small neighbourhood U_x of x in Y, an open ball V_x around the origin of $\mathbb{C}^2$ with a linear action of a finite group G_x on V_x which is free on $V_x - \{0\}$ and a map $\pi : V_x \to U_x$ with $\pi^{-1}(x) = 0$. This map induces an isomorphism $V_x/G_x \to U_x$. For T a compact curve in Y passing through x we consider $\pi^{-1}(T \cap U_x)$ and define

$$\mu_x(T) := \frac{1}{|G_x|}\mu_0(\pi^{-1}(T \cap U_x)).$$

For every irreducible branch $T_{x,j}$ of T in x the inverse image under π consists of $\nu_{x,j}$ irreducible branches, each covering $T_{x,j}$ with multiplicity

$r_{x,j} = |G_x|/\nu_{x,j}$. Then the modified Euler number $e'(\tilde{T})$ is defined as

$$e'(\tilde{T}) = e(\tilde{T}) - \sum_{x,j} \frac{r_{x,j} - 1}{r_{x,j}},$$

where T' is the singularity free model of T and the sum is over all irreducible branches of T in quotient singularities of Y. Let Y' be the non-singular surface obtained from Y by removing all of the quotient singularities. Then $H^2(Y', \mathbb{Q}) \cong H^2(Y, \mathbb{Q})$ by Mayer-Vietoris, so the first Chern class of Y' defines a class $c_1 \in H^2(Y; \mathbb{Q})$. Hence, we get the equation

$$e'(\tilde{T}) = c_1(T) - TT + \sum_{x \in Y} \mu_x(T)$$

for the modified Euler number.

Now let T_1 and T_2 be compact curves on Y. Let $T = T_1 \cap T_2$ be the one-dimensional part with $T_1' = T_1 - T, T_2' = T_2 - T$. Then the equation above becomes

$$T_1 T_2 = T(T_1' + T_2') + T_1' T_2' + \sum_{x \in Y} \mu_x(T) - c_1[T] - e'(\tilde{T}),$$

which can be simplified to

$$T_1 T_2 = \sum_{x \in Y} \mu_x(T_1, T_2) + c_1[T] - e'(\tilde{T})$$

with

$$\mu_x(T_1, T_2) = (T(T_1' + T_2'))_x + (T_1' T_2')_x + \mu_x(T),$$

where $(T(T_1' + T_2'))_x$ and $(T_1' T_2')_x$ are the local intersection numbers.

This allows us to prove Proposition 2.26 also for MN a square. For $\mathfrak{z} \in X$ with $|SL_2(\mathcal{O}_K, \mathfrak{a})| = v_{\mathfrak{z}}$ such that r branches of the inverse image of $T_{(M,N)}$ in $\mathbb{H}^2$ pass through a representative z of $\mathfrak{z}$ in $\mathbb{H}^2$ we have

$$\mu_{\mathfrak{z}}(T_{(M,N)}) = \frac{2r(r-1)}{v_{\mathfrak{z}}} = (T_{(M,N)} T_{(M,N)})_{\mathfrak{z}}^{tr}$$

and

$$\mu_{\mathfrak{z}}(T_M, T_N) = (T_M T_N)_{\mathfrak{z}}^{tr}.$$

The curve $T_{(M,N)}$ near $\bigcup S_j$ is given by local equations $u_k^b - v_k^a = 0$ corresponding to the ideals $\mathfrak{c}$ in $\mathcal{O}_K$ with $N(\mathfrak{c}) = (M, N)$, the characteristic of $\mathfrak{c}$ being $(k|a, b)$. Locally such a curve has (a, b) branches and hence the number of cusps of $T_{(M,N)}$ is $\sum_{\mathfrak{c}} (a, b)$ where the sum is over all ideals $\mathfrak{c}$ with norm (M, N). For the non-singular model $\tilde{T}_{(M,N)}$ of $T_{(M,N)}$ we have

$$e'(\tilde{T}_{(M,N)}) = \mathrm{vol}(T_{(M,N)}) + \sum_{\mathfrak{c}} (a, b).$$

From [11] we get

$$\tilde{c}_1[T_{(M,N)}] = 2\ \mathrm{vol}(T_{(M,N)}) + T_{(M,N)} \cdot (\sum_k S_k) = 2\ \mathrm{vol}(T_{(M,N)}) + \sum_{\mathfrak{c}} (a+b),$$

where $\tilde{c}_1 \in H^2(\tilde{X}, \mathbb{Q})$ is the first Chern class of $\tilde{X}$. Together with the above formulas this becomes

$$(T_M T_N)_{\tilde{X}} = (T_M T_N)_X + \sum_c (a + b - (a,b)) + \sum_x \mu_x(T_M, T_N),$$

where the second sum runs over $x \in T_M \cap T_N \cap \bigcup S_j$. To calculate $\mu_x(T_M, T_N)$ we have to calculate $\mu_x(A, B)$ for the curve A given by $u_k^q = v_k^p$, B the curve $u_k^t = v_k^s$ and x the origin in the coordinate system (u_k, v_k). If $pt - qs \neq 0$, then A and B have no common branch in x and so $\mu_x(A,B)$ is equal to the local intersection number $\min(pt, qs)$ of A and B in x. If $ps - qt = 0$, then for α and β such that $\frac{p}{s} = \frac{q}{t} = \frac{\alpha}{\beta}$ with $(\alpha, \beta) = 1$ and with $\frac{p}{\alpha} = \frac{s}{\beta} = a$ and $\frac{q}{\alpha} = \frac{t}{\beta} = b$, we have

$$\mu_x(A, B) = \min(pt, qs) - (a+b) + (a,b),$$

as was calculated in [14, p. 85].
The ideal $\mathfrak{b}$ with characteristic $(k|p,q)$ has norm M, the ideal $\mathfrak{c}$ with characteristic $(k|s,t)$ has norm N. But we can write $\frac{\mathfrak{b}}{\alpha} = \frac{\mathfrak{c}}{b} = \mathfrak{d}$ with norm (M,N) and characteristic $(k|a,b)$ and every ideal $\mathfrak{d}$ is obtained in this way. Therefore, we get

$$(T_M T_N)_{\tilde{X}} = (T_M T_N)_X + \sum_{\substack{N(\mathfrak{b})=M \\ N(\mathfrak{c})=N}} \delta_{kl} \min(pt, qs),$$

where the sum is over all the ideals with fitting norm and characteristic. Since

$$(T_M T_N^c) = (T_M T_N)_{\bar{X}} + \sum_{k,l} f(\mathfrak{b}_k \mathfrak{b}_l{}') (S_k T_M)(S_l T_N),$$

the proof is now identical to the one for MN not a square.

Now we define $I_p(N)$ by

$$I_p(N) = \frac{1}{\sqrt{p}} \sum_{\substack{\lambda \in \mathcal{O} \\ \lambda > 0, \lambda' > 0 \\ \lambda\lambda' = N}} \min(\lambda, \lambda'). \tag{2.11}$$

Then we get

$$(T_M T_N)_\infty = \sum_{\substack{N(\mathfrak{a})=M \\ N(\mathfrak{b})=N}} f(\mathfrak{a}\mathfrak{b}) = \sum_{d|(M,N)} d\chi_p(d) I_p(MN/d^2).$$

(See [14, Lemma 5] for a proof.) We can rewrite this as

$$(T_M T_N)_\infty = \frac{1}{2} \sum_{d|(M,N)} (d\chi_p(d) + d\chi_p(NA/d)) I_p(MN/d^2)$$

if $\nu_p(N) \leq \nu_p(M)$.

2.4 Formula for the intersection numbers of the curves T_M and T_N

Combining the results of the preceding sections, the intersection numbers for the curves T_M and T_N are given by:

Theorem 2.28 ([14, Theorem 4])**:** *Let M, N be positive integers with $\nu_p(N) \leq \nu_p(M)$. Then the intersection number of the homology classes T_M^c and T_N^c on the compact surface $\bar{X}^{\mathfrak{a}}$ is given by*

$$T_M^c T_N^c = \frac{1}{2} \sum_{d|(M,N)} d\left(\chi_p(d) + \chi_p(AN/d)\right)\left(H_p(MN/d^2) + I_p(MN/d^2)\right),$$

with

$$H_p(n) = \sum_{\substack{x\in\mathbb{Z} \\ x^2 \leq 4n \\ x^2 \equiv 4n \bmod p}} H\left(\frac{4n - x^2}{p}\right),$$

$$H(n) = \begin{cases} -\frac{1}{12} & \text{if } n = 0 \\ \sum_{d^2|n} h'\left(-\frac{n}{d^2}\right) & \text{else,} \end{cases}$$

$$h'(\Delta) = \begin{cases} \frac{1}{3} & \text{if } \Delta = -3 \\ \frac{1}{2} & \text{if } \Delta = -4 \\ h(\Delta) & \text{if } \Delta \equiv 0 \text{ or } 1 \bmod 4, \Delta \leq -4, \end{cases}$$

where $h(\Delta)$ is the class number of positive definite primitive binary integral quadratic forms with discriminant Δ and

$$I_p(n) = \frac{1}{\sqrt{p}} \sum_{\substack{\lambda\in\mathcal{O} \\ \lambda>0, \lambda'>0 \\ \lambda\lambda'=n}} \min(\lambda, \lambda').$$

For $M = N$ this becomes

$$T_N^2 = \frac{1}{2} \sum_{n|N} n \left(H_p\left(\frac{N^2}{n^2}\right) + I_p\left(\frac{N^2}{n^2}\right)\right)\left(\chi_p(n) + \chi_p\left(\frac{NA}{n}\right)\right). \quad (2.12)$$

Remark 2.29: *Theorem 2.28 also holds for K a real quadratic number field of discriminant D, where D is not a prime number with slight adjustments (see [9, Satz 5.7]). Then the formula becomes*

$$T_M^c T_N^c = \sum_{d|(M,N)} d(H_D(MN/d^2)+I_D(MN/d^2)) \prod_{p|D} \left(\frac{\chi_{D(p)}(d) + \chi_{D(p)}(\frac{AP_p}{d})}{2} \right),$$

where for every prime number p dividing D the number P_p is defined as

$$P_p = \begin{cases} M & \text{if } \nu_p(M) \leq \nu_p(N) \\ N & \text{else} \end{cases}$$

and

$$D = \prod_{p|D} D(p)$$

with $D(p) = p^r$ a prime discriminant for some $r \in \mathbb{N}$. $H_D(\cdot)$ and $I_D(\cdot)$ are defined by replacing p with D in the definitions of $H_p(\cdot)$ and $I_p(\cdot)$. The Hirzebruch-Zagier curves T_N are then empty if for one $p|D$ we have $\chi_{D(p)}(NB) = -1$. The proofs are very similar to the prime case. Additional work is needed to prove everything in case $p = 2$ is a divisor of D.

3 Self-intersection of Hirzebruch-Zagier curves on $\bar{X}^{\mathfrak{a}}$

In this chapter we will give an effective bound for the self-intersection number of Hirzebruch-Zagier curves by using the explicit formula given in Theorem 2.28. First, we will look at the contribution of the cusps that is given by $I_p(N)$. We will see that this is always non-negative, so it will be enough to consider only the self-intersection of the curves T_N and not the compactified curves T_N^c, because $T_N^2 \geq {T_N^c}^2$.
In the second section we consider $\bar{X}^{\mathfrak{a}} = \overline{\mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})}$ in the case $\chi_p(A) = -1$ where $A = \mathrm{norm}(\mathfrak{a})$. Then we will see that the self-intersection numbers of the Hirzebruch-Zagier curves are always non-negative, so the bound for the self-intersection number of these special curves is $b_{X^{\mathfrak{a}}}^{HZ} = 0$.
In the last section we consider the case $\chi_p(A) = 1$. In this case, the self-intersection numbers are indeed negative, and we will show that with $\delta := \frac{\pi}{12\mathrm{e}^{\gamma}}$, $c := \mathrm{e}^{\gamma} + 0.6482$, $\varepsilon := \frac{\log(\log(\log(p)))}{\log(p)}$ for $p > 17$, $k := \frac{3}{2(1-\varepsilon)}$

$$T_N^2 \geq \left(\frac{-1}{192}\frac{c^2}{\delta}\right) p^{\frac{3}{2}} p^{4k\varepsilon} + p^{\frac{1}{2}} \frac{\delta}{24p^{2k\varepsilon}}$$

for all N, so the self-intersection number of the Hirzebruch-Zagier curves is bounded from below and the bound only depends on p.

3.1 The contribution of the cusps

In Section 2.3, (see equation (2.9) and (2.11)) we have seen that the contribution of the cusps, that we will now call $C_p(N)$, to the self-intersection number ${T_N^c}^2$ can be calculated as

$$C_p(N) := \frac{1}{2}\sum_{n|N} nI_p\left(\frac{N^2}{n^2}\right)\left(\chi_p(n) + \chi_p\left(\frac{NA}{n}\right)\right), \tag{3.1}$$

where $I_p(N)$ is given by

$$I_p(N) = \frac{1}{\sqrt{p}} \sum_{\substack{\lambda\in\mathcal{O} \\ \lambda>0,\lambda'>0 \\ \lambda\lambda'=n}} \min(\lambda,\lambda') = \frac{1}{\sqrt{p}} \sum_{\substack{\mu\in\mathcal{O} \\ \mu>0,\mu'>0 \\ \mu\mu'=N}} \frac{\mu}{1-\varepsilon} - \frac{\mu'\varepsilon}{1-\varepsilon'}.$$

First, we need to recall some elementary number theory.

Theorem 3.1 (Quadratic reciprocity, see for example [20, Satz 8.7])**:** *Let $p \neq q$ be two odd prime numbers. Then*

$$\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{p}{q}\right).$$

Since we have $p \equiv 1 \bmod 4$, we get for every odd prime number q that $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)$, so if $\left(\frac{q}{p}\right) = -1$ then $\left(\frac{p}{q}\right) = -1$. Furthermore, we have:

Theorem 3.2 ([20, Satz 17.14])**:** *Let $\mathcal{O}_K$ be the ring of integers of a quadratic number field $K = \mathbb{Q}(\sqrt{d})$ and q a prime number in $\mathbb{Z}$. If $q \nmid 2p$, then q is prime in $\mathcal{O}_K$ if and only if $\chi_q(d) = -1$.*

So if we have a prime number q with $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right) = -1$ and $q \nmid 2p$, then q is also a prime number in $\mathcal{O}_K$.

Lemma 3.3: *Let T_M^c and T_N^c be two Hirzebruch-Zagier curves such that $N = qM$, where q is a prime number with $\chi_p(q) = -1$, then*

$$I_p((qM)^2) = qI_p(M^2).$$

Proof. By quadratic reciprocity and Theorem 3.2 q is a prime number in $\mathcal{O}_K$. If we have an element $\mu \in \mathcal{O}_K$ with norm $\mu = (qM)^2$, we know that μ can be written as $\mu = q\eta$ because q is a prime number in $\mathcal{O}_K$ and therefore the only element (up to sign) that has norm q^2. So we get

$$\begin{aligned} I_p((qM)^2) &= \frac{1}{\sqrt{p}} \sum_{\substack{\mu \in \mathcal{O} \\ \mu > 0, \mu' > 0 \\ \mu\mu' = (qM)^2}} \frac{\mu}{1-\varepsilon} - \frac{\mu'\varepsilon}{1-\varepsilon'} \\ &= \frac{1}{\sqrt{p}} \sum_{\substack{\eta \in \mathcal{O} \\ \eta > 0, \eta' > 0 \\ \eta\eta' = M^2}} \frac{q\eta}{1-\varepsilon} - \frac{q\eta'\varepsilon}{1-\varepsilon'} \\ &= q\frac{1}{\sqrt{p}} \sum_{\substack{\eta \in \mathcal{O} \\ \eta > 0, \mu' > 0 \\ \eta\eta' = M^2}} \frac{\eta}{1-\varepsilon} - \frac{\eta'\varepsilon}{1-\varepsilon'} = qI_p(M^2). \end{aligned}$$

□

But now it is easy to see that the contribution of the cusps is non-negative and we get:

Proposition 3.4: *Let T_N^c be a Hirzebruch-Zagier curve on the Hilbert modular surface $\bar{X}^{\mathfrak{a}} = \overline{\mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})}$ defined over $K = \mathbb{Q}(\sqrt{p})$ and $\mathcal{O}_K$ the ring of integers of K. Then the contribution of the cusps $C_p(N)$ to the self-intersection number ${T_N^c}^2$ is non-negative:*

$$C_p(N) \geq 0.$$

Proof. If $N = \prod_{i \in I} p_i$ with $\chi_p(p_i) = 1$ for all p_i, then for $\chi_p(A) = -1$ the self-intersection number ${T_N^c}^2$ is zero for all N, because then $\chi_p(NA) = -1$.
For $\chi_p(A) = 1$ we have

$$\begin{aligned} C_p(N) &= \frac{1}{2} \sum_{n|N} n I_p \left(\frac{N^2}{n^2}\right) \left(\chi_p(n) + \chi_p\left(\frac{N}{n}\right)\right) \\ &= \sum_{n|N} n I_p \left(\frac{N^2}{n^2}\right) \geq 0 \end{aligned}$$

by the definition of $I_p(N)$.
If N has a prime divisor q with $\chi_p(q) = -1$ and $N = Mq$ for some M, we can rewrite $C_p(N)$ as follows:

$$\begin{aligned} C_p(N) &= \frac{1}{2} \sum_{n|N} n I_p \left(\frac{N^2}{n^2}\right) \left(\chi_p(n) + \chi_p\left(\frac{AN}{n}\right)\right) \\ &= \frac{1}{2} \sum_{\substack{q|n \\ n|Mq}} n I_p \left(\frac{N^2}{n^2}\right) \left(\chi_p(n) + \chi_p\left(\frac{AN}{n}\right)\right) \\ &+ \frac{1}{2} \sum_{\substack{q \nmid n \\ n|M}} n I_p \left(\frac{N^2}{n^2}\right) \left(\chi_p(n) + \chi_p\left(\frac{AN}{n}\right)\right) \\ &= \frac{1}{2} \sum_{d|M} dq I_p \left(\frac{M^2}{d^2}\right) \left(\chi_p(dq) + \chi_p\left(\frac{AN}{dq}\right)\right) \\ &+ \frac{1}{2} \sum_{\substack{q \nmid n \\ n|M}} n I_p \left(\frac{N^2}{n^2}\right) \left(\chi_p(n) + \chi_p\left(\frac{AN}{n}\right)\right) \\ &= -\frac{1}{2} q \sum_{d|M} d I_p \left(\frac{M^2}{d^2}\right) \left(\chi_p(d) + \chi_p\left(\frac{AN}{d}\right)\right) \\ &+ \frac{1}{2} \sum_{\substack{q \nmid n \\ n|M}} n I_p \left(\frac{(Mq)^2}{n^2}\right) \left(\chi_p(n) + \chi_p\left(\frac{AN}{n}\right)\right) \end{aligned}$$

$$\begin{aligned}
&\overset{\text{Lemma 3.3}}{=} -\frac{1}{2}q\sum_{d|M} dI_p\left(\frac{M^2}{d^2}\right)\left(\chi_p(d)+\chi_p\left(\frac{AN}{d}\right)\right)\\
&+\frac{1}{2}\sum_{\substack{q\nmid n\\ n|M}} nqI_p\left(\frac{M^2}{n^2}\right)\left(\chi_p(n)+\chi_p\left(\frac{AN}{n}\right)\right)\\
&=-\frac{1}{2}q\sum_{d|M} dI_p\left(\frac{M^2}{d^2}\right)\left(\chi_p(d)+\chi_p\left(\frac{AN}{d}\right)\right)\\
&+\frac{1}{2}q\sum_{n|M} nI_p\left(\frac{M^2}{n^2}\right)\left(\chi_p(n)+\chi_p\left(\frac{AN}{n}\right)\right)\\
&=0.
\end{aligned}$$

So we have shown that $C_p(N)\geq 0$ in all cases. □

3.2 Effective bound for Hirzebruch-Zagier curves T_N in $\bar{X}^{\mathfrak{a}}$ in the case $\chi_p(A)=-1$

In this section we consider the Hirzebruch-Zagier curves in the modular surface $\bar{X}^{\mathfrak{a}}$, where $\chi_p(A)=-1$. We will see that the self-intersection number is always non-negative in this case.
We have seen in Section 3.1 that the contribution of the cusps is always zero in this case, so the self-intersection number of the curve T_N[1] is given by:

$$T_N^2=\frac{1}{2}\sum_{n|N} nH_p\left(\frac{N^2}{n^2}\right)\left(\chi_p(n)+\chi_p\left(\frac{NA}{n}\right)\right),$$

with

$$H_p(n)=\sum_{\substack{x\in\mathbb{Z}\\ x^2\leq 4n\\ x^2\equiv 4n \bmod p}} H\left(\frac{4n-x^2}{p}\right),$$

$$H(n)=\begin{cases}-\frac{1}{12} & \text{if } n=0\\ \sum_{d^2|n} h'\left(-\frac{n}{d^2}\right) & \text{else,}\end{cases}$$

$$h'(\Delta)=\begin{cases}\frac{1}{3} & \text{if } \Delta=-3\\ \frac{1}{2} & \text{if } \Delta=-4\\ h(\Delta) & \text{if } \Delta\equiv 0 \text{ or } 1 \bmod 4, \Delta\leq -4,\end{cases}$$

[1]Normally in literature self-intersection numbers are considered for compact curves, but in the rest of this Chapter we write T_N instead of $T_N^{\mathfrak{c}}$, because by Section 3.1 the difference in the self-intersection numbers is non-negative. Therefore all the estimates are also true for the curves $T_N^{\mathfrak{c}}$.

where $h(\Delta)$ is the class number of positive definite primitive binary integral quadratic forms with discriminant Δ. Furthermore, we define $H_p^0(N) := H_p(N) - 2H(0)$.

First, we look at the case where $N = q$ is a prime number, then $\chi_p(q)$ has to be -1, otherwise the self-intersection number is 0. But then we get:

$$\begin{aligned} T_q^2 &= \frac{1}{2}\sum_{n|q} n\left(H_p\left(\frac{q^2}{n^2}\right)\right)\left(\chi_p(n) + \chi_p\left(\frac{qA}{n}\right)\right) \\ &= H_p(q^2) - qH_p(1) \\ &= H_p^0(q^2) + \frac{q-1}{6} \\ &> 0, \end{aligned}$$

so the self-intersection number is obviously positive in this case.
Now we consider the general case.

Remember the following well-known fact about class numbers (see for example [3, p. 170])

Proposition 3.5: *Let D be a fundamental discriminant. Then*

$$h'(Df^2) = h'(D)\gamma_D(f),$$

where

$$\gamma_D(f) = f\prod_{q|f}\left(1 - \left(\frac{D}{q}\right)\Big/q\right)$$

and q runs through all primes dividing f.

Let $N = \prod_{i\in I} q_i$ for some index set I such that $\chi_p(q_i) = -1$ for an odd number of pairwise different prime numbers q_i, so that we get $\chi_p(AN) = 1$ and the self-intersection number T_N^2 is not zero. Then we have

$$\begin{aligned}
T_N^2 =& \frac{1}{2}\sum_{n|N} n \left(H_p\left(\frac{N^2}{n^2}\right)\right)\left(\chi_p(n) + \chi_p\left(\frac{NA}{n}\right)\right) \\
=& H_p(N^2) - \sum_i q_i H_p\left(\left(\frac{N}{q_i}\right)^2\right) + \sum_{\substack{i,j\in I \\ i\neq j}} q_i q_j H_p\left(\left(\frac{N}{q_i q_j}\right)^2\right) \\
&- \dots + \sum_i \frac{N}{q_i} H_p\left(q_i^2\right) - N H_p(1) \\
=& H_p^0(N^2) - \sum_i q_i H_p^0\left(\left(\frac{N}{q_i}\right)^2\right) + \sum_{\substack{i,j\in I \\ i\neq j}} q_i q_j H_p^0\left(\left(\frac{N}{q_i q_j}\right)^2\right) - \dots \\
&+ \sum_i \frac{N}{q_i} H_p^0\left(q_i^2\right) + \frac{1}{6}\underbrace{\left(N - \sum_i N/q_i + \sum_{i\neq j} N/q_i q_j - \dots + \sum_i q_i - 1\right)}_{:=\beta(N)}.
\end{aligned}$$

First, we look at the term $\beta(N)$. It is an alternating divisor sum, so it can be written as

$$\beta(N) = \sum_{n|N} n\lambda\left(\frac{N}{n}\right),$$

where $\lambda(n) = (-1)^{\Omega(n)}$ is the Liouville function and $\Omega(n)$ denotes the number of prime powers dividing n. It is known (see [26]) that $\beta(N) \geq \sqrt{N}$, so $\beta(N)$ is positive.

Now we have to look at the remaining terms of the sum, the ones that involve H_p^0. $H_p^0(n)$ is by definition positive for all $n \in \mathbb{N}$.
Consider $n = q_1 q_2 q_3$ for $n|N$ with $\chi_p(q_i) = -1$ for $i = 1, 2, 3$, q_i pairwise coprime, but not necessarily prime numbers. The summands appear in pairs of one positive summand and one negative summand. Such a pair consists of the summands $H_p^0(n^2)$ and $-\sum_i q_i H_p^0((n/q_i)^2)$. $H_p^0(\cdot)$ is defined as a sum over $H(\cdot)$, so it is enough to show that for every summand $H(\cdot)$ with negative sign, there is a summand with positive sign equalising it. The sum of $H_p^0(\cdot)$ runs over all $x \in \mathbb{Z}$ satisfying both $x^2 < 4n^2$ and $x^2 \equiv 4n^2 \bmod p$.

But for every solution of the equation

$$x^2 < 4\left(\frac{n}{q_i}\right)^2, x^2 \equiv 4\left(\frac{n}{q_i}\right)^2 \mod p \tag{3.2}$$

there are „lifts“ to a solution of

$$x^2 < 4n^2, x^2 \equiv 4n^2 \bmod p, \tag{3.3}$$

namely for every solution y of the equation (3.2), $q_i y$ is a solution to

the equation (3.3).
So we know that if the term

$$s = \frac{4\left(\frac{n}{q_i}\right)^2 - y^2}{p}$$

appears in the sum $-\sum_i q_i H_p^0((n/q_i)^2)$, which means that one of the summands in $-\sum_i q_i H_p^0((n/q_i)^2)$ is equal to

$$H\left(\frac{4\left(\frac{n}{q_i}\right)^2 - y^2}{p}\right),$$

then

$$\tilde{s} = \frac{4n^2 - (yq_i)^2}{p} = q_i^2 \frac{4\left(\frac{n}{q_i}\right)^2 - y^2}{p},$$

appears in $H_p^0(n^2)$, so one of the summands is equal to

$$H\left(\frac{4n^2 - (yq_i)^2}{p}\right) = H\left(q_i^2 \frac{4\left(\frac{n}{q_i}\right)^2 - y^2}{p}\right).$$

$H(\cdot)$ is defined as a sum over class numbers, but we know by Proposition 3.5 that if $E := \frac{4M^2 - x^2}{p}$ is a fundamental discriminant, then we have

$$h'(q^2 E) = \begin{cases} (q-1)h'(E) & \text{if} \quad \left(\frac{E}{q}\right) = 1 \\ qh'(E) & \text{if} \quad \left(\frac{E}{q}\right) = 0 \\ (q+1)h'(E) & \text{if} \quad \left(\frac{E}{q}\right) = -1. \end{cases}$$

If E is not a fundamental discriminant it can be uniquely written as $E = f^2 E_0$ with E_0 a fundamental discriminant. If $f = \prod_{i \in I} p_i$ for some prime numbers p_i, then

$$h'(q^2 E) = (q \pm 1) \prod_i (p_i \pm 1) h'(E_0).$$

By the definition of $H(\cdot)$ we have

$$H(\tilde{s}) = \sum_{d^2 | \tilde{s}} h'\left(-\frac{\tilde{s}}{d^2}\right).$$

But by the discussion above we also know that

$$H(\tilde{s}) = \sum_{d^2 | \tilde{s}} h'\left(-\frac{\tilde{s}}{d^2}\right) = q_i \sum_{d^2 | s} h'\left(-\frac{s}{d^2}\right) + \sum_{\substack{d^2 | \tilde{s} \\ d \nmid q_i}} h'\left(-\frac{\tilde{s}}{d^2}\right),$$

so there is a summand in $H_p^0(N^2)$ with positive sign equalising the

negative part coming from $-q_i H_p^0((n/q_i)^2)$.
If $n = q_1 q_2 m$ then the solutions of

$$x \in \mathbb{Z}, x^2 < 4\left(\frac{n}{q_1}\right)^2, x^2 \equiv 4\left(\frac{n}{q_1}\right)^2 \mod p$$

and

$$x \in \mathbb{Z}, x^2 < 4\left(\frac{n}{q_2}\right)^2, x^2 \equiv 4\left(\frac{n}{q_2}\right)^2 \mod p$$

could give lift to the same solution of

$$x \in \mathbb{Z}, x^2 < 4n^2, x^2 \equiv 4n^2 \mod p,$$

but then we know that

$$q_1^2 \frac{4\left(\frac{n}{q_1}\right)^2 - y^2}{p} = q_2^2 \frac{4\left(\frac{n}{q_2}\right)^2 - z^2}{p},$$

so y^2 must be divisible by q_2^2 and z^2 must be divisible by q_1^2 and it can be written as

$$q_1^2 q_2^2 \frac{4\left(m^2\right) - w^2}{p}.$$

So if two solutions in different sums give lifts to the same solution, then we get two different summands in which those occur, so the intersection number is still positive.
Thus, we get that $T_N^2 \geq 0$ for all N.

Therefore, we have proven

Theorem 3.6: *For the Hirzebruch-Zagier curves T_N on the Hilbert modular surface $\bar{X}^{\mathfrak{a}} = \overline{\mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})}$, where $\mathrm{norm}(\mathfrak{a}) = A$ and $\chi_p(A) = -1$, the self-intersection number T_N^2 is non-negative:*

$$T_N^2 \geq 0.$$

3.3 Effective bound for Hirzebruch-Zagier curves T_N^c in $\bar{X}^{\mathfrak{a}}$ in the case $\chi_p(A) = 1$

The aim of this section is to calculate a bound $b(\bar{X}^{\mathfrak{a}})$ for the self-intersection numbers ${T_N^c}^2$ of Hirzebruch-Zagier curves on the Hilbert modular surface $\bar{X}^{\mathfrak{a}} = \overline{\mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})}$ in the case $\chi_p(A) = 1$. Then the formula for ${T_N^c}^2$ becomes

$${T_N^c}^2 = \frac{1}{2} \sum_{n|N} n \left(H_p\left(\frac{N^2}{n^2}\right) + I_p\left(\frac{N^2}{n^2}\right)\right)\left(\chi_p(n) + \chi_p\left(\frac{N}{n}\right)\right).$$

We have seen in Section 3.1 that the contribution of the cusps is zero if one of the divisors p_i of $N = \prod_{i\in I} p_i$ satisfies $\chi_p(p_i) = -1$, and strictly positive if $\chi_p(p_i) = 1$ for all p_i. Therefore, it is enough to consider the following formula for the self-intersection number of the curve T_N:

$$T_N^2 = \frac{1}{2}\sum_{n|N} n\left(H_p\left(\frac{N^2}{n^2}\right)\right)\left(\chi_p(n) + \chi_p\left(\frac{N}{n}\right)\right),$$

where $H_p(\cdot), H(\cdot)$ and $h(\cdot)$ are defined as before.

First, we use this formula to show that there exist only finitely many curves T_N with negative self-intersection. Then, we use this to find a bound $b(\bar{X}^{\mathfrak{a}})$ such that $T_N^2 \geq -b(\bar{X}^{\mathfrak{a}})$.

Let $N = \prod_{i=1}^{k} p_i$ with $p_1, .., p_k$ prime numbers with $\chi_p(p_i) = 1$. If the curve T_N is irreducible, then the prime numbers p_i have to be pairwise different. We will assume this throughout this chapter (even though it would not change the proofs if we did not). As in [14] we write $H_p(n) = -\frac{1}{6}n + H_p^0(n)$, where

$$H_p^0(n) = \sum_{\substack{x\in\mathbb{Z} \\ x^2<4n \\ x^2\equiv 4n \bmod p}} H\left(\frac{4n - x^2}{p}\right)$$

has only summands bigger or equal to zero.
Since we look at a sum over class numbers, we need an estimate for those numbers:

Proposition 3.7 (Paley, Littlewood)**:** *Let $d > 0$ be a squarefree positive integer and K a quadratic field with discriminant $-d$. Then*

$$h(-d) \geq \frac{\pi}{24e^{\gamma}}\frac{\sqrt{d}}{\log\log(d)} =: \tilde{h}(-d), \tag{3.4}$$

where $h(-d)$ is the class number and $\gamma := \lim_{n\to\infty}\sum_{k=1}^{n}(\frac{1}{k} - \log n)$ is the Euler-Mascheroni constant.

Proof. It is well known (see for example [5, p. 49]) that the class number for a quadratic field with negative discriminant is given by

$$h(-d) = \frac{w\sqrt{|d|}}{2\pi}L(1,\chi),$$

where

$$w = \begin{cases} 6 & \text{if } d = -3 \\ 4 & \text{if } d = -4 \\ 2 & \text{if } d \leq -4, \end{cases}$$

and $L(1,\chi)$ is the Dirichlet L-series based on a character χ at $s = 1$. Paley [23] and Littlewood [17] gave a lower bound for $L(1,\chi)$ that proves the above estimate. □

Lemma 3.8: *Let $N = \prod_{i=1}^{k} p_i$ with $p_1,..,p_k$ pairwise different prime numbers with $\chi_p(p_i) = 1$, n a divisor of N, then*

$$H_p^0\left(n^2\right) \geq \frac{\pi n}{12\mathrm{e}^{\gamma}\sqrt{p}\log\log(4n^2)}\left(\frac{4n}{3p} + \frac{p}{24n}\right).$$

Proof. We will prove this Lemma in three steps:

$$\begin{aligned} H_p^0\left(n^2\right) &\overset{1)}{\geq} \sum_{k=0}^{\lfloor\frac{2n}{p}\rfloor-1} H'\left(\frac{4n^2 - \left(\frac{p-1}{2} + kp\right)^2}{p}\right) \\ &\overset{2)}{\geq} \frac{\pi n}{12\mathrm{e}^{\gamma}\sqrt{p}\log\log(4n^2)} \sum_{k=0}^{\lfloor\frac{2n}{p}\rfloor-1}\left(1 - \left(\frac{\left(\frac{p-1}{2}\right)^2 + kp(p-1) + k^2p^2}{2n^2}\right)\right) \\ &\overset{3)}{\geq} \frac{\pi n}{12\mathrm{e}^{\gamma}\sqrt{p}\log\log(4n^2)}\left(\frac{4n}{3p} + \frac{p}{24n}\right). \end{aligned}$$

At first we want to write

$$H_p^0(n) = \sum_{\substack{x\in\mathbb{Z} \\ x^2<4n \\ x^2\equiv 4n \bmod p}} H\left(\frac{4n - x^2}{p}\right)$$

in a more attainable way, namely we want to get rid of the x in the formula and formulate it instead with the help of the known number p. But we know that

$$\frac{4n^2 - x^2}{p} \geq \frac{4n^2 - (\frac{p-1}{2})^2}{p},$$

because there exists a solution x smaller than $\frac{p-1}{2}$ for the equation $x \in \mathbb{Z}, x^2 < 4n^2, x^2 \equiv 4n \bmod p$. Furthermore, one gets at least $\lfloor\frac{2n}{p}\rfloor$ solutions for the above equation, namely if x is a solution, then the numbers $x + kp$ for $1 \leq k \leq \lfloor\frac{2n}{p}\rfloor - 1$ are also solutions and so we get the wanted inequality by inserting them into the original formula.
Now we use the inequality 1) and the formula of Proposition 3.4 for the

resulting $d = \frac{4n^2 - \left(\frac{p-1}{2} + kp\right)^2}{p}$ to get

$$\begin{aligned} h\left(-\frac{4n^2 - \left(\frac{p-1}{2} + kp\right)^2}{p}\right) &\geq \frac{\pi}{24e^\gamma} \frac{\sqrt{\frac{4n^2 - ((p-1)/2)^2 + kp(p-1) + k^2p^2}{p}}}{\log(\log(4n^2))} \\ &\geq \frac{\pi}{24e^\gamma} \frac{\sqrt{4n^2(1 - \frac{((p-1)/2)^2 + kp(p-1) + k^2p^2}{4n^2})}}{\sqrt{p}\log(\log(4n^2))} \\ &= \frac{\pi n}{12e^\gamma} \frac{\sqrt{(1 - \frac{((p-1)/2)^2 + kp(p-1) + k^2p^2}{4n^2})}}{\sqrt{p}\log(\log(4n^2))}. \end{aligned}$$

To get rid of the square root we use the inequality $\sqrt{1-x} \geq 1 - x$ for $0 \leq x \leq 1$. Then the term above is bigger or equal to

$$\frac{\pi n}{12e^\gamma \sqrt{p} \log\log(4n^2)} \left(1 - \left(\frac{\left(\frac{p-1}{2}\right)^2 + kp(p-1) + k^2p^2}{4n^2}\right)\right),$$

and so the inequality 2) is proven.
Now part 3) is just an easy calculation of sums. Recall that the Gaussian sum is calculated as

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

and the sum over squares as

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1).$$

Considering that $\frac{2n}{p}$ is not an integer we get an inequality

$$\begin{aligned} &\sum_{k=0}^{\lfloor \frac{2n}{p} \rfloor - 1} \left(1 - \left(\frac{\left(\frac{p-1}{2}\right)^2 + kp(p-1) + k^2p^2}{4n^2}\right)\right) \\ &\geq \frac{2n}{p} - \frac{2n}{p}\left(\frac{(p-1)^2}{16n^2}\right) - \frac{p-1}{4n}\left(\frac{2n}{p} - 1\right) - \frac{p}{12} \frac{\left(\frac{2n}{p} - 1\right)\left(\frac{4n}{p} - 1\right)}{n} \\ &= \frac{4n}{3p} + \frac{p}{24n} + \frac{1}{p} - \frac{1}{8pn} \\ &\geq \frac{4n}{3p} + \frac{p}{24n}, \end{aligned}$$

since we have that $\frac{1}{p} > \frac{1}{8pn}$. This completes the proof. □

Now we insert this estimate for $H_p^0(n^2)$ into the formula for T_N^2 to get:

Lemma 3.9: *Let T_N be a Hirzebruch-Zagier curve on the Hilbert modular surface $\bar{X}^{\mathfrak{a}}$ with $N = \prod_{i=1}^{k} p_i$, $p_1, .., p_k$ pairwise different prime numbers with $\chi_p(p_i) = 1$, then*

$$T_N^2 \geq -\frac{1}{6} cN \log\log(N) + \frac{N\delta}{\sqrt{p}\log\log(4N^2)}\left(\frac{4N}{3p} + \frac{p}{24N}\right),$$

with $\delta := \frac{\pi}{12\mathrm{e}^{\gamma}}$, $c := \mathrm{e}^{\gamma} + 0.6482$.

Proof.

$$\begin{aligned}
T_N^2 &= \frac{1}{2}\sum_{n|N} n\left(H_p\left(\frac{N^2}{n^2}\right)\right)\\
&= \sum_{n|N} nH_p^0\left(\left(\frac{N}{n}\right)^2\right) - \frac{1}{6}\sum_{n|N} n\\
&= -\frac{1}{6}\sigma_1(N) + \sum_{n|N} n \sum_{\substack{x\in\mathbb{Z}\\ x^2<4(N/n)^2\\ x^2\equiv 4(N/n)^2 \bmod p}} H\left(\frac{4(N/n)^2 - x^2}{p}\right)\\
&\overset{\text{Lemma 3.8}}{\geq} -\frac{1}{6}\sigma_1(N) + \frac{\pi N}{12\mathrm{e}^{\gamma}\sqrt{p}\log\log(4N^2)} \cdot \sum_{n|N, n\neq 1}\left(\frac{4n}{3p} + \frac{p}{24n}\right)\\
&= -\frac{1}{6}\sigma_1(N) + \frac{\delta N}{6\sqrt{p}\log\log(4N^2)}\left(\frac{4(\sigma_1(N)-1)}{3p} + \frac{p(\sigma_1(N)-N)}{24N}\right)\\
&\geq -\frac{1}{6}\sigma_1(N) + \frac{\delta N}{6\sqrt{p}\log\log(4N^2)}\left(\frac{4N}{3p} + \frac{p}{24N}\right),
\end{aligned}$$

since $\sigma_1(N) \geq N + 1$.
Furthermore, we need an estimate for $\sigma_1(N)$, but Robin showed in [25] that for $N \geq 3$ we have

$$\sigma_1(N) < \mathrm{e}^{\gamma} N \log\log(N) + 0.6482\frac{N}{\log\log(N)} \leq (\mathrm{e}^{\gamma} + 0.6482) N \log\log(N).$$

□

Now we can use this formula to show that for N big enough the self-intersection number T_N^2 will always be positive.

Lemma 3.10: *The minimum of the self-intersection number T_N^2 of Hirzebruch-Zagier curves T_N on the Hilbert modular surface $\bar{X}^{\mathfrak{a}}$ is obtained for $N \leq p^k$, where $k \geq \frac{3}{2(1-\varepsilon)}$ for some ε depending on p (ε can be chosen as $\frac{\log(\log(\log(p)))}{\log(p)}$ for $p > 17$).*

Proof. First, we know that there exists $\varepsilon > 0$ with $\frac{1}{\log\log N} \geq N^{-\varepsilon}$ and $\varepsilon \to 0$ as $N \to \infty$.

By Lemma 3.9

$$T_N^2 \geq -\frac{1}{6}cN\log\log(N) + \frac{N\delta}{\sqrt{p}\log\log(4N^2)}\left(\frac{4N}{3p} + \frac{p}{24N}\right),$$

and inserting the estimate for $\log\log(N)$ we get

$$T_N^2 \geq -\frac{1}{6}cN^{1+\varepsilon} + \frac{N^{1-2\varepsilon}\delta}{\sqrt{p}}\left(\frac{4N}{3p} + \frac{p}{24N}\right).$$

The leading term of the positive right side is

$$\frac{N^{1-2\varepsilon}\delta}{\sqrt{p}}\frac{4N}{3p},$$

so we know that the minimum has be to obtained before the exponent of p of that term is bigger than the exponent of the negative term. Putting $N = p^k$ and looking at the exponent we get

$$\begin{aligned} k(1+\varepsilon) &\leq k(2-2\varepsilon) - 3/2 \\ \Leftrightarrow \frac{3}{2} &\leq 2k - 2\varepsilon k - k - k\varepsilon \\ \Leftrightarrow \frac{3}{2} &\leq k(1-3\varepsilon) \\ \Leftrightarrow k &\geq \frac{3}{2(1-3\varepsilon)}. \end{aligned}$$

□

Remark 3.11: *For* $\varepsilon = \frac{1}{10}$, $\frac{1}{\log\log N} \geq N^{-\varepsilon}$ *is true for* $N \geq 2$, *so* $T_{p^{\frac{15}{7}}} \geq 0$ *for all* p.

Theorem 3.12: *Let* $N = \prod_{i=1}^{k} p_i$ *with* $p_1, .., p_k$ *pairwise different prime numbers with* $\chi_p(p_i) = 1$, *then the self-intersection number* T_N^2 *of Hirzebruch-Zagier curves on the Hilbert modular surface* $\bar{X}^{\mathfrak{a}} = \overline{\mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{a})}$, *where* $\mathcal{O}_K$ *is the ring of integers of* $K = \mathbb{Q}(\sqrt{p})$, *satisfies*

$$T_N^2 \geq O(p^{\frac{3}{2}}).$$

More precisely,

$$T_N^2 \geq \frac{-1}{192}\frac{c^2}{\delta}p^{\frac{3}{2}}p^{4k\varepsilon} + p^{\frac{1}{2}}\frac{\delta}{24p^{2k\varepsilon}}$$

for all N, *where* $\delta = \frac{\pi}{12e^{\gamma}}$, $c = e^{\gamma} + 0.6482$, ε *depends on* p *and can be chosen as* $\frac{\log(\log(\log(p)))}{\log(p)}$ *for* $p > 17$, $k = \frac{3}{2(1-\varepsilon)}$, *so the bound only depends on* p.

Proof. By Lemma 3.10 the minimum has to be obtained for some N

smaller than p^k, so we know that for the interval, where the minimum is obtained, $\log\log(N) \leq p^{k\varepsilon}$ for k and ε as in Lemma 3.10.
Therefore, we can replace $\log\log(N)$ by $p^{k\varepsilon}$ in the formula and get

$$T_N^2 \geq -\frac{1}{6}cNp^{k\varepsilon} + \frac{N\delta}{\sqrt{p}p^{2k\varepsilon}}\left(\frac{4N}{3p} + \frac{p}{24N}\right) =: t(N).$$

To get the minimum of the self-intersection numbers we differentiate $t(N)$ with respect to N and get

$$t'(N) = -\frac{1}{6}cp^{k\varepsilon} + \frac{\delta}{\sqrt{p}p^{2k\varepsilon}}\frac{8N}{3p}.$$

This is equal to 0 for

$$N_{\min} = \frac{1}{16}\frac{c}{\delta}p^{\frac{3}{2}}p^{3k\varepsilon}.$$

Since the second derivative of $t(N)$ is always positive, this is indeed the minimum. Inserting this into the equation, we get

$$T^2_{N_{\min}} \geq \left(\frac{-1}{192}\frac{c^2}{\delta}\right)p^{\frac{3}{2}}p^{4k\varepsilon} + p^{\frac{1}{2}}\frac{\delta}{24p^{2k\varepsilon}}.$$

Thus, as p tends to ∞, $\varepsilon \to 0$ and for the asymptomatic behaviour for $T^2_{N_{\min}}$ we get

$$T^2_{N_{\min}} \geq \frac{-1}{192}\frac{c^2}{\delta}p^{\frac{3}{2}}.$$

Inserting this into the estimate for T_N^2, we get that

$$T_N^2 \geq \left(\frac{-1}{192}\frac{c^2}{\delta}\right)p^{\frac{3}{2}}p^{4k\varepsilon} + p^{\frac{1}{2}}\frac{\delta}{24p^{2k\varepsilon}}$$

for all N.

□

Remark 3.13: *If the Riemann Hypothesis is true, we can take $\delta := \frac{\pi}{6\mathrm{e}^{\gamma}}$ for all p and $c := \mathrm{e}^{\gamma}$ for all $N \geq 5041$.*

4 BNC on Hilbert modular surfaces

In this chapter, we will generalise Proposition 3.5 in [2] stating that for reduced, irreducible Shimura curves C on a compact Hirzebruch modular surface X the self-intersection number of the curves C^2 is bounded by $-6c_2(X)$, where $c_2(X)$ is the second Chern class of X, to non-compact Hilbert modular surfaces.

Here, we use the term *Shimura curve* as a generalisation of the Hirzebruch-Zagier curves that we considered in the first chapters on Hilbert modular surfaces over a totally real quadratic number field K of discriminant $p \equiv 1 \bmod 4$. For an arbitrary Hilbert modular surface X we consider for a point $z \in \mathbb{H}$ the composition of the map $z \mapsto (z, z)$ and the projection $pr : \mathbb{H}^2 \to X$. Let σ be the generator of the Galois group of K, then for a matrix $M \in \mathrm{GL}_2^{\times}(K)$ we consider the twisted diagonal $z \mapsto (Mz, M^\sigma z)$. Then the images of these twisted diagonals under the projection pr are called Shimura curves. We will always assume Shimura curves to be compact.
In literature, Hirzebruch-Zagier curves and Shimura curves are often used synonymously and recently, these curves are mainly referred to as *special curves*. In this chapter, we will use the term Shimura curve to stay in the terminology of [2] and to distinguish from the bound in Chapter 3.

In the first part of this chapter, we will adjust the proof in [2] to show that for an irreducible, reduced curve $C \subset X$ for X a Hilbert modular surface that is not necessarily compact, the self-intersection number C^2 is bounded below by $b(X) = -9d_2(\bar{X})$ with $d_2(\bar{X}) = 3c_2(\bar{X}) - K_{\bar{X}}^2$ such that

$$C^2 \geq b(X)$$

(see Theorem 4.6), where $\bar{X}$ is the Hirzebruch compactification of the Hilbert modular surface X described in Chapter 1.

Remark: *If we consider compact Hilbert modular surfaces X like in [2], we have $K_X^2 = 2c_2(X)$ and therefore would get the bound*

$$b(X) = -9c_2(X)$$

by this approach.

In the second part of this chapter, we will first give estimates for the second Chern class $c_2(\bar{X})$ and then use this to get an effective bound for Shimura curves $C \subset \bar{X}$ for X a Hilbert modular surface over a real quadratic field K with discriminant D by this approach. We will get by Theorem 4.14 that the self-intersection number C^2 is bigger than

$$\begin{aligned}
&\geq -\left(\frac{3}{10}\sum_{x\in\mathbb{Z}}\sigma_1\left(\frac{D-x^2}{4}\right) + 18\sum_{x<\sqrt{D}}\sigma_0\left(\frac{D-x^2}{4}\right) + \frac{3}{2}h(-3D)\right.\\
&\left. + \frac{27\pi}{8e^\gamma}\frac{D^{\frac{1}{2}}}{\log\log(4D)} + \frac{3\cdot 5\sqrt{3}\pi}{2e^\gamma}\frac{D^{\frac{1}{2}}}{\log\log(3d)} + \frac{3\cdot\sqrt{3}\pi}{2e^\gamma}\frac{D^{\frac{1}{2}}}{\log\log(3D)}\right)\\
&\geq -\left(\left(\frac{3}{10}D^{\frac{3}{2}} + 18D^{\frac{1}{2}}\right)\left(\frac{3}{2\pi^2}\log^2(D) + 1.05\log D\right) + \frac{3}{2}\frac{\sqrt{3D}}{\pi}\log(3D)\right.\\
&\left. + \frac{27\pi}{8e^\gamma}\frac{D^{\frac{1}{2}}}{\log\log(4D)} + \frac{3\cdot 5\sqrt{3}\pi}{2e^\gamma}\frac{D^{\frac{1}{2}}}{\log\log(3D)} + \frac{3\cdot\sqrt{3}\pi}{2e^\gamma}\frac{D^{\frac{1}{2}}}{\log\log(3D)}\right).
\end{aligned}$$

4.1 BNC for non-quaternionic Hilbert modular surfaces

In [2] it was shown that for compact Hilbert modular surfaces X there is a bound for the self-intersection number of Shimura curves $C \subset X$, namely $C^2 \geq -6c_2(X)$, where $c_2(X)$ is the second Chern class of X. In this section we will follow the proof of [2, Theorem 1] closely to generalise the theorem to non-quaternionic Hilbert modular surfaces. First, as in the proof of [2, Theorem 1] we use the following theorem by Miyaoka ([18])

Theorem 4.1 ([18, Theorem 1.3])**:** *Let X be a surface of non-negative Kodaira dimension, let C be an irreducible curve of geometric genus g on X and K_X the canonical divisor. Then for all $\alpha \in [0,1]$*

$$\frac{\alpha^2}{2}(C^2 + 3C\cdot K_X - 6g + 6) - 2\alpha(C\cdot K_X - 3g + 3) + 3c_2 - K_X^2 \geq 0.$$

Remark 4.2: *The theorem also holds for $C = \sum_i C_i$ a reduced reducible curve with $g(C) - 1 = \sum_i(g(C_i) - 1)$, see [18, Remark G].*

Remark 4.3: *Note that* $X = \mathbb{H}^2/SL_2(\mathcal{O}_K)$ *with* $\mathcal{O}_K$ *the ring of integers of* $K = \mathbb{Q}(\sqrt{p})$, $p \equiv 1 \mod 4$, *has negative Kodaira dimension if and only if* $p \in \{5, 13, 17\}$, *so the following theorems are not valid in these three cases (see [13]).*

Let C be a curve of genus g on a Hilbert modular surface X. Then again as in [2] we define

$$\delta := p_a - g = \frac{1}{2}(K_X \cdot C + C^2 - 2g + 2), \tag{4.1}$$

where K_X is the canonical divisor of X and p_a is the arithmetic genus of C. δ is always non-negative.

Theorem 4.4 (Hirzebruch-Höfer proportionality theorem [1, 16, Chapter 17])**:** *Let* C *be a Shimura curve on a Hilbert modular surface* X. *With* $\rho(C) = 2\,(\deg S_C - S_{\bar{X}} \cdot C)$, *where* $S_{\bar{X}} = \bar{X} \setminus X$ *is a strict normal crossing divisor and* $S_C = C \cap S_{\bar{X}}$ *is the boundary divisor, we have*

$$(K_{\bar{X}} + S_{\bar{X}})C + 2C^2 + \rho(C) = 4\delta.$$

Idea of proof. In [21] it was shown that for a non-singular curve C on a Hilbert modular surface X the Hirzebruch-Höfer proportionality is given by

$$2C^2 + 2\text{deg}S_C = -K_{\bar{X}}C + S_{\bar{X}}C. \tag{4.2}$$

In the singular case, the adjunction formula is given by

$$e(C) = -2g(C) + 2 = -(K_{\bar{X}} + C)C + \sum_{x_i} \mu(C, x_i),$$

where the sum runs over all singular points x_i of C and

$$\sum_{x_i} \mu(C, x_i) = \mu = 2\delta$$

(see for example [4, Proposition 1.2.1]). δ is equal to the number of nodes of C if C is nodal. Therefore, formula (4.2) for singular curves changes to

$$2C^2 + 2\text{deg}S_C = -K_{\bar{X}}C + S_{\bar{X}}C + 4\delta$$

or equivalently

$$(K_{\bar{X}} + S_{\bar{X}})C + 2C^2 + \rho(C) = 4\delta.$$

□

Remark 4.5: *Another proof of Theorem 4.4 using Hodge theory can be found in [16, Chapter 17].*

Using Theorem 4.4 we are able to show:

Theorem 4.6: *For a Shimura curve C on a Hilbert modular surface X and its Hirzebruch compactification $\bar{X}$ we have*

$$C^2 \geq -9d_2(\bar{X}),$$

where $d_2(\bar{X}) := 3c_2(\bar{X}) - K_{\bar{X}}^2$.

Proof. As in [2, Theorem 3.5.] we use the polynomial $P(\alpha)$ which is obtained by solving for g in equation (4.1) and inserting it into the equation of Theorem 4.1. Writing d_2 instead of $d_2(\bar{X})$ we get

$$P(\alpha) = \alpha^2(3\delta - C^2) + \alpha(CK_{\bar{X}} + 3C^2 - 6\delta) + d_2 \geq 0.$$

With $\rho(C) = 2\,(\deg S_C - S_{\bar{X}} \cdot C)$, where $S_{\bar{X}} = \bar{X} \setminus X$ is a strict normal crossing divisor and $S_C = C \cap S_{\bar{X}}$ is the boundary divisor, we get by Theorem 4.4

$$(K_{\bar{X}} + S_{\bar{X}})C + 2C^2 + \rho(C) = 4\delta.$$

Therefore, the polynomial becomes

$$P(\alpha) = \alpha^2(3\delta - C^2) + \alpha(C^2 - S_{\bar{X}} \cdot C - \rho(C) - 2\delta) + d_2 \geq 0.$$

The minimum of $P(\alpha)$ is attained for

$$\alpha_0 := \frac{2\delta + S_{\bar{X}} \cdot C + \rho(C) - C^2}{2(3\delta - C^2)}.$$

Evaluating the condition $P(\alpha_0) \geq 0$ we get

$$\begin{aligned} 2d_2 + 2\sqrt{d_2^2 + \delta d_2 + d_2 S_{\bar{X}} \cdot C} &\geq 2\delta + S_{\bar{X}} \cdot C + \rho(C) - C^2 \\ &\geq 2d_2 - 2\sqrt{d_2^2 + \delta d_2 + d_2 S_{\bar{X}} \cdot C + d_2\rho(C)}. \end{aligned}$$

If $C^2 - S_{\bar{X}} \cdot C \geq 2\delta$, then $C^2 \geq 0$, but for $C^2 - S_{\bar{X}} \cdot C < 2\delta$ we get the lower bound

$$\begin{aligned} C^2 &\geq 2\delta + S_{\bar{X}} \cdot C + \rho(C) - 2d_2 - 2\sqrt{d_2^2 + \delta d_2 + d_2 S_{\bar{X}} \cdot C + d_2\rho(C)} \\ &\geq 2\delta + S_{\bar{X}} \cdot C + \rho(C) - 2d_2 - 2\sqrt{d_2^2 + \delta d_2} - 2\sqrt{d_2 S_{\bar{X}} \cdot C} - 2\sqrt{d_2\rho(C)}, \end{aligned}$$

where we used the triangle inequality in the last step.
For $f(x) = x - 2\sqrt{d_2 x}$ the minimum is obtained for $x = d_2$ because the differential is

$$f'(x) = 1 - \sqrt{\frac{d_2}{x}},$$

and $f(d_2) = -d_2$.
Therefore, $S_{\bar{X}} \cdot C - 2\sqrt{d_2 S_{\bar{X}} \cdot C} \geq -d_2$ and $\rho(C) - 2\sqrt{d_2\rho(C)} \geq -d_2$

and a lower bound of C^2 is given by

$$C^2 \geq 2\delta - 4d_2 - 2\sqrt{d_2^2 + \delta d_2}.$$

We see that this is non-negative for $\delta \geq \frac{5+\sqrt{13}}{2} d_2$.

For $\delta < \frac{5+\sqrt{13}}{2} d_2$ we have $C^2 < 0$ and

$$C^2 \geq 2\delta - 4d_2 - 2\sqrt{d_2^2 + \delta d_2}.$$

Since $-2\sqrt{d_2^2 + \delta d_2} > -2\sqrt{d_2^2 + \frac{5+\sqrt{13}}{2} d_2^2}$, we get

$$C^2 \geq 2\delta - 4d_2 - 2\sqrt{\frac{7+\sqrt{13}}{2} d_2^2},$$

and hence the desired result $C^2 \geq \left(-4 - 2\sqrt{\frac{7+\sqrt{13}}{2}}\right) d_2 \geq -9d_2$. □

4.2 Effective bound

In this section we apply the result of Theorem 4.6 to the surfaces we discussed in Chapter 3 to get a second bound for the self-intersection number of the Hirzebruch-Zagier curves T_N^c. To do this, we first give an estimate for the Chern class of Hilbert modular surfaces. We will consider a quadratic field K with discriminant D and ring of integers $\mathcal{O}_K$. For the definition of the Hilbert modular group Γ_K see 1.3.

Theorem 4.7 ([27, Chapter 4, Theorem 2.5.])**:** *Let $\Gamma \subset PGL_2(\mathbb{R})$ be commensurable with the Hilbert modular group Γ_K, let X be the Hilbert modular surface $\mathbb{H}^2/\Gamma$ and $\bar{X}$ its Hirzebruch compactification. Then*

$$c_2(\bar{X}) = \mathrm{vol}(X) + l(\bar{X}) + \sum a(\Gamma; n, a, b)\left(l(n; a, b) + \frac{n-1}{n}\right),$$

where

$$\begin{aligned} a(\Gamma; n, a, b) &= \#\text{ quotient singularities of } \mathbb{H}^2/\Gamma \text{ of type } (n; a, b) \\ l(n; a, b) &= \#\text{ curves in the resolution of a quotient singularity} \\ &\quad\ \text{of type } (n; a, b) \\ l(\bar{X}) &= \#\text{ curves in the resolution of the cusps.} \end{aligned}$$

Since there are no singularities other than those of type $(2; 1, 1)$, $(3; 1, 1)$ and $(3; 1, -1)$ for $D > 13$ (see [24]), Theorem 4.7 can then be simplified to:

Theorem 4.8 ([13, Section 2.4(30), 27, Chapter 7])**:** *For X a Hilbert modular surface with discriminant D the second Chern class is given by*

$$c_2(\bar{X}) = 2\zeta_K(-1) + \frac{3}{2}a_2 + \frac{5}{3}a_3^+ + \frac{8}{3}a_3^- + l(\bar{X}),$$

where $l(\bar{X})$ is the number of curves in the resolution of the cusps, $\zeta_K(s)$ is the Dedekind zeta-function of K and a_2, a_3^+, a_3^- are the numbers of quotient singularities of type $(2;1,1)$, $(3;1,1)$ and $(3;1,-1)$ respectively.

Since our goal is to determine an effective bound for the self-intersection numbers of the curves $C \subset \bar{X}$ we need to calculate the five appearing terms in Theorem 4.8.

The volume of X in Theorem 4.7 is given by the following theorem:

Theorem 4.9 (Siegel, [27, Chapter 4, Theorem 1.1.])**:** *Let Γ_K be the Hilbert modular group for a totally real number field K and $X = \mathbb{H}^2/\Gamma_K$. Then the volume of $\mathbb{H}^2/\Gamma_K$ is given by*

$$\int_{\mathbb{H}^2/\Gamma_K} \omega = 2\zeta_K(-1),$$

where $\omega = \frac{(dx^2)+(dy^2)}{y^2}$ for $z = x + iy$ and $\zeta_K(s)$ is the Dedekind zeta-function of K.
Let $\Gamma \subset PGL_2(\mathbb{R})$ be commensurable with the Hilbert modular group Γ_K, then:

$$\mathrm{vol}(\mathbb{H}^2/\Gamma) = [\Gamma_K : \Gamma]2\zeta_K(-1).$$

In the special case of a real quadratic number field K the Dedekind zeta-function is easy to calculate. It is given by:

Theorem 4.10 (Siegel, [27, Theorem 6.5.])**:** *Let K be a real quadratic number field with discriminant D. Then*

$$\zeta_K(-1) = \frac{1}{60}\sum_{x\in\mathbb{Z}} \sigma_1\left(\frac{D-x^2}{4}\right),$$

where $\sigma_1(x) = 0$ if $x \leq 0$ and $\sigma_1(x) = \sum_{d|x} d$ if $x \geq 1$.

We also need to compute $l(\bar{X})$. We give a short summary of some results about $l(\bar{X})$, for more details we refer to [13, Chapter 2] and [27].
A cusp is described by a pair (M, V), where M is a complete $\mathbb{Z}-$module

in the real quadratic field K and V is a subgroup of finite index in the infinite cyclic group U_M^+ of all positive units ε with $\varepsilon M = M$ (see Section 2.3). We define $l(M,V)$ to be the number of curves in the cusp resolution cycle of the cusp described by (M,V). For $l(M) := l(M,U_M^+)$ we have $l(M,V) = [U_M^+ : V] \cdot l(M)$ (see [13, Chapter 2 (24)]). For the cusps of the Hilbert modular group the modules M are always strictly equivalent to ideals in the ring $\mathcal{O}_K$ of integers of K. Those strict equivalence classes correspond to narrow ideal classes in $\mathfrak{Cl}^+$. For an ideal $\mathfrak{a} \subset \mathcal{O}_K$ the map $\mathfrak{a} \mapsto \mathfrak{a}^{-2}$ induces a homomorphism $Sq : \mathfrak{Cl}^+ \to \mathfrak{Cl}^+$ (see Chapter 1). The cusps are of type $(\mathfrak{a}^{-2}, U^2)$, where U denotes the group of units of $\mathcal{O}_K$. $U^+ = U^2$ if and only if there exists a unit of negative norm, otherwise $[U^+ : U^2] = 2$. Since $l(M)$ only depends on the narrow equivalence class of M we can define

$$l(\bar{X}) = \sum_{\mathfrak{a} \in \mathfrak{Cl}^+} l(Sq(\mathfrak{a})).$$

It can be shown that $l(\bar{X})$ equals the number of all reduced quadratic rationalities w (i.e. $w > 1 > w' > 0$) of discriminant D (which is defined as the discriminant of the field K that the Hilbert modular surface X is defined over). Therefore, we have for w satisfying $aw^2 + bw + c = 0$, where $b^2 - ac = D$ and $(a,b,c) = 1$ (see also [27, Chapter 4.3(3)]),

$$l(\bar{X}) = \#\{(a,b,c) \in \mathbb{Z}^3 : b^2 - 4ac = D, \frac{b + \sqrt{D}}{2a} > 1 > \frac{b - \sqrt{D}}{2a} > 0\}.$$

Put $x = b - 2a$. Then c is determined by x and a and we get

$$l(\bar{X}) = \#\{(a,x) \in \mathbb{Z}^2 : a > 0, x^2 < D, x^2 \equiv D \bmod 4a, x + 2a > \sqrt{D}\}.$$

But this is equivalent to

$$2l(\bar{X}) = \#\{(a,x) \in \mathbb{Z}^2 : a > 0, x^2 < D, x^2 \equiv D \bmod 4a\}.$$

Hence, we have an estimate for $l(\bar{X})$ in [27, Chapter 4.3(3)]:

$$l(\bar{X}) = \frac{1}{2} \sum_{\substack{x^2 < D \\ x^2 \equiv D \bmod 4}} \sum_{\substack{a > 0 \\ a \mid (D - x^2)/4}} 1 = \frac{1}{2} \sum_{x < \sqrt{D}} \sigma_0\left(\frac{D - x^2}{4}\right),$$

where $\sigma_0(n) = \sum_{d|n} 1$ for n an integer and d a positive divisor of n, otherwise $\sigma_0(n) = 0$. We can bound the divisor sum by

Lemma 4.11 ([27, Chapter 7, Lemma 5.3])**:** *Let D be the discriminant of a real quadratic field. Then*

$$\frac{1}{2} \sum_{x < \sqrt{D}} \sigma_0\left(\frac{D - x^2}{4}\right) \leq \begin{cases} D^{1/2}\left(\frac{3}{2\pi^2} \log^2 D + 1.05 \log D\right) & \text{for all } D \\ D^{1/2}\left(\frac{3}{4\pi^2} \log^2 D + 0.8 \log D\right) & \text{if } D \equiv 0 \mod 4. \end{cases}$$

But we can also use this bound to get a bound for $\sigma_1(\cdot)$, since:

$$\begin{aligned}\sum_{x\in\mathbb{Z}}\sigma_1\left(\frac{D-x^2}{4}\right) &\leq \sum_{x\in\mathbb{Z}} D\sigma_0\left(\frac{D-x^2}{4}\right)\\ &\leq D^{\frac{3}{2}}\left(\frac{3}{2\pi^2}\log^2(D)+1.05\log D\right).\end{aligned}$$

Now we still need estimates for the part coming from the quotient singularities.
By Section 1.1.4, for $D>13$ this is equal to $\frac{3}{2}a_2+\frac{5}{3}a_3^+ + \frac{8}{3}a_3^-$, where a_2, a_3^+ and a_3^- are the numbers of quotient singularities of type $(2;1,1)$, $(3;1,1)$ and $(3;1,-1)$ respectively. By Chapter 1 in [27] we know that $a_2 = h(-4p), a_3^+ \geq 4h\left(\frac{-p}{3}\right)$ and $a_3^- \geq \frac{1}{2}h(-3p)$, where $h(\cdot)$ is the class number. With Paley's inequality (Lemma 3.4) we get

$$\begin{aligned}a_2 &\geq \frac{\pi}{12\mathrm{e}^\gamma}\frac{\sqrt{D}}{\log\log(4D)}\\ a_3^+ &\geq \frac{\sqrt{3}\pi}{6\mathrm{e}^\gamma}\frac{\sqrt{D}}{\log\log(3D)}\\ a_3^- &\geq \frac{\sqrt{3}\pi}{48\mathrm{e}^\gamma}\frac{\sqrt{D}}{\log\log(3D)}.\end{aligned}$$

The last estimate we need is for the canonical divisor $K_{\bar{X}}^2$. But there we know:

Theorem 4.12: *[12, Chapter 2] Let $K_{\bar{X}}^2$ be the canonical divisor of $\bar{X}$, the Hirzebruch compactification of the Hilbert modular surface X over a real quadratic field K of discriminant $D \geq 29$. Then*

$$K_{\bar{X}}^2 = 4\zeta_K(-1) - \frac{h(-3D)}{6} - l(\bar{X}).$$

The terms $\zeta_K(-1)$ and $l(\bar{X})$ were already discussed in this chapter and as an upper bound for the class number we know

Lemma 4.13: *[27, Chapter 7, Lemma 5.2] If $-d < -4$ is a fundamental discriminant then $h(-d) \leq \frac{\sqrt{d}}{\pi}\log(d)$.*

Combining all these computations gives:

Theorem 4.14: *For a reduced irreducible Shimura curve C on a Hilbert modular surface $\bar{X} = \overline{\mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K)}$ for $\mathcal{O}_K$ the ring of integers of a totally real number field K with discriminant $D \geq 29$ the self-intersection* *number C^2 is bounded by*

$$
\begin{aligned}
C^2 \geq &- \Bigg(\frac{3}{10} \sum_{x \in \mathbb{Z}} \sigma_1 \left(\frac{D - x^2}{4} \right) + 18 \sum_{x < \sqrt{D}} \sigma_0 \left(\frac{D - x^2}{4} \right) + \frac{3}{2} h(-3D) \\
&+ \frac{27\pi}{8e^\gamma} \frac{D^{\frac{1}{2}}}{\log\log(4D)} + \frac{3 \cdot 5\sqrt{3}\pi}{2e^\gamma} \frac{D^{\frac{1}{2}}}{\log\log(3d)} + \frac{3 \cdot \sqrt{3}\pi}{2e^\gamma} \frac{D^{\frac{1}{2}}}{\log\log(3D)} \Bigg) \\
\geq &- \Bigg(\Big(\frac{3}{10} D^{\frac{3}{2}} + 18 D^{\frac{1}{2}} \Big) \Big(\frac{3}{2\pi^2} \log^2(D) + 1.05 \log D \Big) + \frac{3}{2} \frac{\sqrt{3D}}{\pi} \log(3D) \\
&+ \frac{27\pi}{8e^\gamma} \frac{D^{\frac{1}{2}}}{\log\log(4D)} + \frac{3 \cdot 5\sqrt{3}\pi}{2e^\gamma} \frac{D^{\frac{1}{2}}}{\log\log(3D)} + \frac{3 \cdot \sqrt{3}\pi}{2e^\gamma} \frac{D^{\frac{1}{2}}}{\log\log(3D)} \Bigg).
\end{aligned}
$$

Remark 4.15: *In Theorem 3.12 we calculated a bound using the specific formula for the curves T_N^c. As expected, the bound given in Theorem 4.14 for arbitrary Shimura curves yields less good results for the curves T_N^c. This can be seen in the coefficients of $p^{\frac{3}{2}}$, which is ≈ -0.2 in the bound in Theorem 3.12 and ≈ -0.4 in Theorem 4.14 and in the different signs of the coefficients of $p^{\frac{1}{2}}$. Concrete examples of bounds for certain discriminants are given in the tables in the Appendix and illustrate the difference between the two bounds.*

Tables of bounds of Self-intersection numbers

In this appendix, there are two tables treating the self-intersection numbers of Hirzebruch-Zagier curves.
In the first column some exemplary prime numbers p are given for which we consider $K = \mathbb{Q}(\sqrt{p})$ with ring of integers $\mathcal{O}_K$ and Hilbert modular surface $\bar{X} = \overline{\mathbb{H}^2/\mathrm{SL}_2(\mathcal{O}_K)}$. The second column contains the discrimant N_{min} of the curve T_{min} for which the minimum of the self-intersection numbers is actually obtained. (In the first table only squarefree N are considered, because the Bounded Negativity Conjecture is about irreducible curves, in the second table all N are considered.) The self-intersection number T_{min}^2 of this curve is given in the third column. (The source code for a program written in Pari/GP to calculate the self-intersection numbers is given in Appendix 4.2).
In the fourth column the bound determined in Theorem 3.12 can be read off for the corresponding p and in the fifth column the bound calculated in Theorem 4.14 is given as $-9d_2(\bar{X}) = -27c_2(\bar{X}) + 9K_{\bar{X}}^2$ with $c_2(\bar{X})$ the second Chern class of $\bar{X}$.
In the last two rows the bounds are rounded down.

p	N_{min}	T^2_{min}	$b_{T^2_N}(X)$	$-9d_2(\bar{X})$
29	7	-4/3	-8.2	-108
113	26	-7	-67.4	-531
521	130	-42	-668.1	-2820.4
809	793	-206/3	-1292.9	-4592.9
1129	805	-136	-2131.6	-8490.8
1297	966	-176	-2624.8	-9100.7
1777	1326	-230	-4209.5	-12426.2
2689	2010	-486	-7836.1	-24007.6
3769	3705	-732	-13003.5	-35099.1
4201	3570	-854	-15302.2	-41949.4
5009	5005	-976	-19922.9	-42290.5
6217	4242	-1248	-27548.7	-64871.4
8329	7410	-2140	-42719.2	-102727.8
9817	9726	-2172	-54664.3	-116798.7
12049	9030	-3506	-74329.9	-179434.7

TABLE 1: Self-intersection numbers for N squarefree

p	N_{min}	T^2_{min}	$b_{T^2_N}(X)$	$-9d_2(\bar{X})$
29	7	-4/3	-8.2	-108
113	26	-7	-67.4	-531
521	260	-62	-668.1	-2820.4
809	404	-91	-1292.9	-4592.9
1129	576	-1087/6	-2131.6	-14264.4
1297	648	-461/2	-2624.8	-9100.7
1777	936	-319	-4209.5	- -12426.2
2221	1575	-886/3	-5882.1	-23591.9
2689	2016	-652	-7836.1	–24007.6
3769	2808	-942	-13003.5	-35099.1
4201	4032	-3382/3	-15302.2	-41949.4
5009	4720	-1108	-19922.9	-42290.5
6217	4536	-1774	-27548.7	-64871.4
8329	8280	-2796	-42719.2	-102727.8
9817	9792	-2951	-54664.3	-116798.7

TABLE 2: Self-intersection numbers for all N

Source Code

This is the source code for the calculation of the self-intersection numbers of Hirzebruch-Zagier curves in *Pari/GP*. The original code was written by the authors of [2] and adjusted by the author of this thesis.

```
//Calculation of the modified class number of a primitive
//positive definite form with discriminant n:
classnoHZ(n)={
local(k);
if (n==-3, k= 1/3);
if(n==-4, k= 1/2);
if((n!=-3 & n!=-4), k=qfbclassno(n));
k
}

//Calculation of H(n)
H(n)={
 local(s=0,i,d);
 if(n==0,s=-1/12,
 d=divisors(n);
 forstep(i=1,length(d),1,
     if(issquare(d[i]) & ((-n/d[i])%4==0  (-n/d[i])%4==1),
        s=s+classnoHZ(-n/d[i]))));
 s
}

//Calculation of H_p(n)
Hp(n,p)={
 local(x,hh=0,fn=4*n);
 if(fn%p==0,hh=hh+H(fn/p));
 forstep(x=1,floor(2*sqrt(n)),1,
     if((x^2-fn)%p==0,
        hh=hh+2*H((fn-x^2)/p));
     );
 hh
}

//Calculation of the self-intersection number T_N^2
//A the norm of the chosen ideal
SIB(N,p,A)={
 local(d,tn=0,m);
```

```
 forstep(d=1,N,1,
 m=N/d;
 if(N%d==0;
  tn=tn+(kronecker(d,p)+kronecker(A*m,p))*d*Hp(m^2,p));
 );
 (1/2)*tn
}

squarefree(N)={
local(V,i);
V=divisors(N);
for (i=2,length(V),
if(issquare(V[i])==1, return(1)))
}

//Calculation of the minimum of the self-intersection
//numbers T_N^2 for a ≤ N ≤ b, A the norm of the chosen ideal
mini(a,b,p,A)={
 local(N,V,MinSI,i,j);
 V=vector(b-a+1);
 forstep(N=a,b,1,
   if(gcd(N,B)==1 & squarefree(N)!=1
         & kronecker(A*N,p)!=-1,
   V[N-a+1]=SIB(N,p,A);));
MinSI=vecmin(V);
if((MinSI==0) ), print("the minimum is 0"));
forstep(i=1,b-a+1,1,
if(V[i]==MinSI & MinSI != 0,
print("minimum is given for N=", a+i-1,"
  and it is =", MinSI)))

}
```

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