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RESEARCH ARTICLE

The dynamics of Pareto distributed wealth in a small open economy

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Abstract
We study a small open economy with labor, capital accumulation, random death, taxation and a government budget balanced in the long run. We offer methods that provide ordinary differential equations for means and analytical expressions for densities. The latter is achieved by solving stochastic differential equations analytically and deriving the density from this solution. Starting from any distribution, the aggregate distribution converges, both on a transition path towards a steady state and on a transition path towards balanced growth, to a Pareto-distribution. We provide an intuitive economic interpretation for a stationary long-run density with an infinite mean in an economy on a balanced growth path. We also show how government tax policy can lead to non-monotonic links between the equilibrium growth rate of the economy and risk aversion of households.

Keywords Analytical dynamics of mean and distribution · Wealth · Government budget · Stochastic differential equation

JEL Classification C61 · D31 · E21

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1 Introduction

Distributional analyses gain in importance in times when wealth or income appear to become increasingly unequally distributed across individuals. An observed distribution that differs at two points in time can be understood in (at least) two ways. First, by comparative static analysis of a stationary distribution or by studying the transition process from the first realization of the distribution to the second. As the former is the dominating approach in the literature and the latter is often performed numerically, we focus on the latter from an analytical perspective for the case of a 'perpetual youth' model.

We model the age process of representatives of a dynasty by a stochastic differential equation. Wealth of the dynasty is derived from this age process while individuals are alive. Newborns are endowed with a constant initial wealth level (and one unit of labor supplied inelastically). This setup implies a stochastic differential equation for the wealth process as well. We study dynasties in a small open economy framework with free international capital flows. We investigate convergence properties towards a steady state and a balanced growth path.

We emphasize that our setup reflects the age process of a typical 'perpetual youth' Yaari-Blanchard model (Yaari 1965; Blanchard 1985). The wealth process in our setup is an alternative representation of the evolution of wealth in Jones (2014, 2015). An endowment that is identical at birth reminds of Kasa and Lei (2018), among others (see below for a detailed comparison with the literature). The density of the 'Steindl-model' as presented in Gabaix et al. (2016, sect. 4.2.3) is a special case of our density (as an example, we work with an arbitrary initial condition). The major difference of our paper consists in our novel approach for deriving the density. Gabaix et al. (2016) do not provide a derivation. Beare and Toda (2022) present new tools for understanding the shape of stationary distributions in a discrete time setup (see Beare et al. 2022 for a continuous time analysis). Our approach is less abstract and we focus on the shape of distributions over time.

Our main contribution lies in the analysis of transition dynamics. The crucial departure from the economic birth–death literature that enables this analysis lies in a modelling choice. We represent the age process resulting from a birth–death process by a stochastic differential equation (SDE). As a consequence, the wealth process in our economy can also be represented by an SDE.

Our main results on the dynamics of distributions are obtained by steps that differ from the more popular Fokker–Planck equations (FPEs). We rather solve the wealth SDE and derive distributions from these solutions. This approach has not been followed—to the best of our knowledge—in economics before. We show that our wealth process converges to a Pareto distribution in the limit.[1] Starting from any arbitrary initial condition, the well-known link between an exponential age and a Pareto wealth distribution is absent in transition. We characterize transitional dynamics of the wealth distribution analytically and illustrate them graphically.

Our transitional analysis allows us to provide an intuitive economic interpretation to a so far purely technical property of Pareto distributions. Pareto distributions with a shape exponent smaller than 1 have a mean of infinity, i.e. a finite mean does not exist. Nevertheless, the distribution exists and can be illustrated graphically in the standard way. On a transition towards a balanced growth path, we observe a truncated Pareto distribution with a finite mean. In the limit, the economy is on a balanced growth path (with individual wealth constantly growing) and the Pareto distribution has (and needs to have) an infinite mean.

We also obtain an interesting result concerning equilibrium growth rates. Risk aversion plays a key role in the growth rate of our economy that is new to the endogenous growth or optimal saving literature. The level of risk aversion determines the *sign* of the growth rate (in a non-monotonic way) and not just the level. The effect is present when the government balances its budget in the long run via the capital tax. This effect is absent when the labor tax is employed for ensuring a balanced budget.

The next section briefly surveys the literature to which we relate our work. Section 3 introduces the model. Section 4 provides the stochastic background of our analysis. Section 5 provides findings on the evolution of expected individual wealth and government wealth. Section 6 first derives conditions for a steady state versus a

[1] The discussion paper version also proves many properties of the age process. Mean age always converges to a constant, the age process converges to an exponential distribution in the long run, starting from an arbitrary distribution. More details are in Birkner et al. (2021) or in an appendix available upon request.

balanced growth path, illustrates the difference between time paths for expected values and realizations and offers our analytical characterization of transitional dynamics of distributions. The final section concludes.

2 Related literature

Stochastic differential equations Stochastic differential equations have been used in economics at least since Merton (1971). Subsequently and more recently, contributions include Wälde (1999, 2005), He and Krishnamurthy (2011, 2013), Brunnermeier and Sannikov (2014) or Di Tella (2017). We are not aware of analytical solutions to SDEs in the economics literature that are employed to derive analytical expressions for the evolution of densities over time.

Birth–death process We employ Poisson processes to model birth–death processes as many other papers in the wealth-distribution literature (Cao and Luo 2017; Gabaix et al. 2016; Kasa and Lei 2018; Aoki and Nirei 2017). We share with others (Blanchard 1985; Benhabib et al. 2016; Benhabib and Bisin 2006; Gabaix et al. 2016; Kaymak and Poschke 2016; Kasa and Lei 2018; Toda 2014; Benhabib et al. 2011; Aoki and Nirei 2017; Benhabib et al. 2019; Itskhoki and Moll 2019) that death and birth rates are identical. This differs from analyses that allow rates of birth and death to differ leading to population growth (Jones 2014, 2015; Cao and Luo 2017). d'Albis (2007) considers age-dependent death rates.

Identical initial endowment Newborns in our model receive a constant initial endowment. This captures the idea of equality of chances with respect to initial wealth.[2] This assumption is also made by Kasa and Lei (2018). Newborns born at the same point in time receive identical endowments also in Jones (2015). This endowment can grow over time, however. The literature employing insurance companies in finite-life models in the tradition of Yaari (1965) and Blanchard (1985) redistributes wealth intragenerationally. Both use insurance companies selling an annuity to a consumer who then receives an income flow until the point of death. After that, the insurance company dissolves the annuity and is relieved of any further payment Yaari (1965), or receives the individual's total wealth Blanchard (1985). In models in the Blanchard-tradition, all individuals also have an identical initial wealth level (of zero).[3]

Kolmogorov backward equations Analysis of the mean is facilitated by employing Kolmogorov backward equations. An introduction can be found in Stokey (2008, ch. 3.7). They are also applied in finance papers like Cox and Ross (1976), Aoki (1995), Kawai (2009) or Eberlein and Glau (2014).

Kolmogorov forward/Fokker–Planck equations Fokker–Planck equations (FPEs) became very popular recently and we share the belief in their usefulness with Benhabib et al. (2016), Achdou et al. (2020), Smith et al. (2013), Boucekkine et al. (2022), Jones

[2] We ignore other determinants of equality of chances such as cognitive and non-cognitive skills or family background. We also acknowledge a long literature studying alternative redistribution schemes. Recent contributions include Cao and Luo (2017), Benhabib and Bisin (2006) and Benhabib et al. (2011, 2016).

[3] Including an annuity in our model would lead to a different deterministic evolution of wealth over time. It would not change our main points, however. Thus, we share the ideas of Jones (2014, 2015), Toda (2014, p. 329), or Cao and Luo (2017) and omit insurance markets.

and Kim (2017), Cao and Luo (2017), Aoki and Nirei (2017), Kaplan et al. (2018), Nuño and Moll (2018) and Itskhoki and Moll (2019).[4,5]

Probability theory Given the nature of our project, we used textbooks on probability theory. They include Øksendal (1998), Kallenberg (1997) or Privault (2018). In order to understand stochastic integral equations, their solutions, and the infinitesimal generator of Markov processes, Protter (1995) is helpful. Davis (1993b) establishes a theorem on the evolution of an expected value as being entirely determined by a generator for certain assumptions which is essential when analyzing the mean.

Poisson processes Going beyond the analysis of wealth distributions, we emphasize that Poisson processes are ubiquitous in other parts of economics as well. Modelling strategies to which our method of analyzing the mean could be applied include models of R&D (Aghion et al. 2001; Grossman and Helpman 1991, 2005; Klette and Kortum 2004; Aghion and Howitt 1992). Search and matching models in the tradition of Diamond (1982), Mortensen (1982) and Pissarides (1985) also build on Poisson processes, for instance Lee and Wang (2021), as do some business cycles models (Brunnermeier and Sannikov 2014; Wälde 2005; He and Krishnamurthy 2011, 2013, or Di Tella 2017) or the trade literature in the tradition of Melitz (2003).

Pareto and double-Pareto distributions Pareto distributions have become very popular recently (Piketty and Zucman 2015). They appeared in the analysis of top income changes (Saez and Zucman 2016), income growth per person, population growth Jones (2015), financial deregulation, (corporate) taxes (Cao and Luo 2017; Zhu 2019) or bequests and saving rate inequality (Benhabib et al. 2019). Some models derive a 'double Pareto distribution' for wealth. This can be achieved by introducing a diffusion process in a model with exponentially distributed lifetimes (Reed 2001, 2003; Toda 2012, 2014).[6] Our focus is on analytical results for transitional dynamics of distributions. We believe that they can also be applied to (appropriately modified) double-Pareto structures.[7,8]

Analytical densities Some analyses in economics work with analytical densities. An early contribution is by Merton (1975) who obtains a steady-state density of per-capita wealth in a Solow growth model with geometric Brownian motion for population size.

[4] Achdou et al. (2014) provide an overview of partial differential equation models in macroeconomics. Ahn et al. (2017) describe numerical methods for continuous time models.

[5] (Bayer and Wälde 2015, p. 4) provide a short survey on the use of FPEs in economics prior to these papers (see Benhabib and Bisin 2006, for an example). Bayer and Wälde (2010, Section 5) showed how to derive FPEs for relatively general cases (using a Bewley-Huggett-Aiyagari model as example).

[6] See also the analyses by Benhabib et al. (2016), Reed (2001) or Toda and Walsh (2015).

[7] So far double-Pareto findings are built on a combination of Brownian motion and exponential age. Toda (2014) writes that "the double Pareto property is robust in the sense that it depends only on multiplicative growth and the geometric age distribution and not on the details of the stochastic process governing growth". Gabaix (1999) conjectures that the power law should hold even if the multiplicative process is time-varying. Hence, obtaining double-Pareto findings employing Poisson processes only seems possible.

[8] A paper very close to ours in spirit is Benhabib and Bisin (2006), as was kindly pointed out to us by Jess Benhabib and Alberto Bisin after having completed our study. We share the optimal saving structure, the death-birth process, the distributional nature of government activity and the intention to understand transitional wealth dynamics. We differ from their analysis in inter alia our explicit use of SDEs (from which we derive all of our findings), in our rigorous foundation in stochastics, in our more general treatment of the government's budget constraint (it is not balanced at each point in time leading to richer equilibrium conditions) and in our analytical and graphical characterization of the wealth density over time.

Gabaix et al. (2016) also present an analytical density in their study of the distribution of income. Their analysis builds on the Laplace transform (moment generating function) of the density described by the Kolmogorov Forward Equation (i.e. Fokker–Planck equation). Kasa and Lei (2018) follow this approach and derive a stationary distribution of wealth. Aoki and Nirei (2017) also provide a stationary distribution. We differ from these approaches in that we offer a method by which the density can be derived independently of partial differential equations: We propose an SDE and solve it. Based on this solution, the density is presented.

Government debt and fiscal policy Government debt has been analyzed from many different perspectives. Government deficit as opposed to tax income is central in the Ricardian equivalence literature in the tradition of Barro (1974) and Weil (1989). A stochastic analysis of government debt based on Brownian motion is in Benavie et al. (1996) or Turnovsky (1997, 2000). A new perspective on optimal taxation also taking optimal government debt into account is surveyed in Kocherlakota (2010). As government debt per se is not the focus of our analysis here, we only point out that our modelling choice of employing SDEs (and looking at long-run expected values) for the government budget constraint is natural for models with birth–death processes.

What matters for our analysis is the tax choice how to balance the budget in the long run and the effect of this choice, capital or labor tax, on the equilibrium growth rate of the economy. Equilibrium growth rates are central to growth models. As Jaimovich and Rebelo (2016) nicely put it, various growth models (Solow-type-models, human-capital based models à la Lucas 1988, or semi-endogenous growth models in the spirit of Jones 1999) predict long-run growth rates that are independent of tax policy. By contrast, the presence of knowledge externalities in R&D based growth models led to large discussion of the effect of taxes/R&D subsidies on long-run growth and welfare (see e.g. Segerstrom 2007 and the references therein). Obviously, a comparison between the effect of labour taxes and capital (income) taxes in growth is not possible in models where labour is the only factor of production. Such a comparison does take place in sunspot models where tax rates influence the emergence of continua of equilibria (Ben-Gad 2003; Park 2009).[9] Our mechanism for long-run growth or a steady-state equilibrium is (i) the difference between the after (wealth) tax interest rate and the time preference rate at the individual level and (ii) the effect government policy has on the wealth tax.

3 The model

The model presentation starts from small agents (one individual and one dynasty), passes by a large agent (the government) and ends in equilibrium (of the small open economy with two factors of production).

[9] Romer-style models are also employed to study optimal tax policies (see Gross and Klein 2022 for a recent contribution).

3.1 The individual

Each individual is endowed with a time preference rate $\hat{\rho} > 0$ and has a finite life that ends at a random point in time. This point T is exponentially distributed with parameter δ, denoted death rate. The individual maximizes expected utility $E_t \int_t^T e^{-\hat{\rho}[s-t]} u(c(s))\, ds$, where expectations are formed with respect to $T > t$ given information up to t. Instantaneous utility is $u(c(s))$ and the individual chooses the time path of consumption $c(s)$. It is well-known from Blanchard-Yaari models that this maximization problem is identical to maximizing a deterministic objective function

$$U(t) = \int_t^\infty e^{-\rho[s-t]} u(c(s))\, ds, \tag{1}$$

where discounting takes place at the rate $\rho = \hat{\rho} + \delta$, i.e. adding the death rate δ to the time preference rate. As the objective function shows, the individual cares about own consumption only. They do not value bequests or utility of offsprings. Consumption $c(s)$ is therefore perceived to be deterministic. All bequests in our model will be accidental. We consider a standard, constant relative risk aversion (henceforth CRRA), instantaneous utility function

$$u(c(t)) = \frac{c^{1-\sigma} - 1}{1-\sigma}. \tag{2}$$

The budget constraint of our individual is deterministic as well and reads

$$\dot{a}(t) = (r - \tau_a)\, a(t) + (1 - \tau_w)\, w - c(t). \tag{3}$$

Wealth increases through net labor income $(1 - \tau_w)\, w$ and net interest $(r - \tau_a)$ on individual current wealth $a(t)$. Wealth and wage taxes are levied and collected by the state.[10] The gross wage and the interest rate are constant due to our small-open-economy setup (see below). Consumption reduces wealth accumulation and the price of the consumption good is normalized to one. The time derivative of wealth is denoted by the usual $\dot{a}(t)$.

When we solve the individual's maximization problem, a guess and verify approach yields optimal consumption in closed form (see Appendix A.1). Consumption (which we require to be positive imposing a lower bound on wealth)

$$c(t) = \phi\, [a(t) + W] \tag{4}$$

is a constant share

$$\phi \equiv \frac{\rho - (1-\sigma)(r - \tau_a)}{\sigma} \tag{5}$$

[10] The wealth tax τ turns into a capital income tax τ_c when we replace τ by $r\tau_c$. The budget constraint (3) would then read $\dot{a}(t) = (1 - \tau_c)\, ra(t) - c(t)$.

out of wealth $a(t)$ and the present value of net labor income,

$$W \equiv \frac{(1-\tau_w)\,w}{r-\tau_a}. \tag{6}$$

Deriving this solution also shows that consumption grows at a rate

$$z \equiv \frac{r-\tau_a-\rho}{\sigma}. \tag{7}$$

As long as the net interest rate $r-\tau_a$ exceeds the time preference rate ρ, wealth of the individual increases over time.

Now imagine an individual is born at t_B. Age of the individual is then $t-t_B$. Using (4), (3) and endowing the individual with some initial wealth $a(t_B)$, wealth is a function of age and follows (see Appendix A.1.3)

$$a(t) = (a(t_B)+W)\,e^{z(t-t_B)} - W. \tag{8}$$

This finding is well-known from many closed-form solutions or steady-state properties: Wealth growth is also driven by the constant exponential growth rate z.

3.2 The dynasty

Turning to a dynasty i, an offspring is born once an individual dies. A dynasty is therefore characterized by a stochastic age process and a stochastic wealth process. We describe both of them by stochastic differential equations driven by Poisson processes. This is the key novelty of our paper from a methodological perspective.

3.2.1 Age

We start by specifying the age process. As emphasized above, our specification is representative of age processes in many papers employing a birth–death framework with constant population size. Our findings obtained below are possible as we model this age process by a stochastic differential equation. It reads

$$dX_i(t) = b\,dt - X_i(t_-)\,dQ_i^\delta(t). \tag{9}$$

Age of the currently alive individual of dynasty i at a point in time t is denoted by $X_i(t)$. It increases linearly and deterministically in time with slope b. When age and time are measured in the same units, b equals one. Age drops to zero at random points in time, i.e. when the increment $dQ_i^\delta(t)$ of the Poisson process $(Q(t))_{t\geq 0}$ equals one. Poisson processes $Q_i^\delta(t)$ are independent of each other.

The arrival rate of this Poisson process is the constant death rate introduced above before (1). Age dropping to zero means that an individual that dies is replaced by a newly born offspring of age zero. Population size L therefore remains constant. We denote the initial age of the currently alive individual of dynasty i by x_i.[11]

3.2.2 Wealth

We can now describe the wealth process of a dynasty i. While alive, an individual accumulates wealth according to (8). When death hits according to (9), all wealth of a dynasty goes to the state which, in turn, endows the newborn with some initial endowment $\bar{a}$.[12] This wealth accumulation process, based on (8) and (10), is captured by (see Appendix A.1.4)

$$dA_i\left(X_i\left(t\right)\right) = z\left[A_i\left(X_i\left(t\right)\right) + W\right]dt + \left[\bar{a} - A_i\left(X_i\left(t_{-}\right)\right)\right]dQ_i^{\delta}\left(t\right). \qquad (10)$$

Wealth A_i of dynasty i whose currently alive member has age $X_i\left(t\right)$ at time t changes according to a deterministic and a stochastic part. The deterministic part describes optimal wealth accumulation (8) at the individual level as long as the individual is alive. The stochastic part shows that in the case of death at t, wealth is reduced by the wealth level $A_i\left(X_i\left(t_{-}\right)\right)$ at t_{-}, i.e. an instant before death.[13] Wealth is increased by $\bar{a}$ such that the newborn starts with this initial endowment.[14]

3.3 The government

Consider a government that levies taxes on wealth and labor income, collects all wealth at the moment of death and endows all newborns with an initial constant amount of wealth. We can express the change in government wealth based on *one dynasty* by the following SDE[15]

$$dG_i\left(A_i\left(t\right)\right) = \left(\tau_a A_i\left(t\right) + \tau_w w\right)dt + \left[A_i\left(t_{-}\right) - \bar{a}\right]dQ_i^{\delta}\left(t\right). \qquad (11)$$

The deterministic sources of income are given by tax revenues $\tau_a A_i\left(t\right)$ and $\tau_w w$. A stochastic source is wealth $A_i\left(t_{-}\right)$ of individuals being transferred to the state at the moment of death, i.e. when $dQ_i^{\delta}\left(t\right) = 1$. Government spending consists in endowing the newborn with a constant amount of wealth $\bar{a}$. We do not impose a

[11] Parameters of the process (9) could differ across dynasties. Concerning the age process, we only allow for differences in initial age in this paper.

[12] As discussed above, the absence of planned bequests and identical endowments of newborns is a common assumption in the literature (Kasa and Lei 2018; Jones 2015, and models in the Blanchard 1985, tradition).

[13] Moll et al. (2021) allow for "random dissipation shocks" that imply that households are left with zero wealth after such a shock.

[14] An obvious extension reduces the inheritance tax from 100% as in (10) to some $0 < \tau_b < 1$. The wealth constraint would read $dA_i\left(X_i\left(t\right)\right) = z\left[A_i\left(X_i\left(t\right)\right) + W\right]dt + \left[\bar{a} - \tau_b A_i\left(X_i\left(t_{-}\right)\right)\right]dQ_i^{\delta}\left(t\right)$. Extending our analyses for the mean and aggregate equilibrium is straightforward. The analysis of distributional dynamics is an order of magnitude more complex. We return to this issue once we have understood distributional dynamics of wealth following (10).

[15] Even though $A_i\left(X_i\left(t\right)\right)$ more precisely describes the deterministic link between wealth and age than $A_i\left(t\right)$, we will employ the latter when appropriate.

balanced government budget at each instant (as e.g. Benhabib and Bisin 2006, or Benhabib et al. 2016). We rather allow the government to trade government bonds on the international capital market.[16]

When we denote total government wealth by $G(t)$, its evolution follows from summing over all L dynasties,

$$dG(t) = \Sigma_{i=1}^{L} \left\{ (\tau_a A_i(t) + \tau_w w)\, dt + \left[A_i(t_-) - \bar{a} \right] dQ_i^{\delta}(t) \right\}. \tag{12}$$

3.4 The small open economy

We study a small open economy. All of our distributional findings below concerning wealth can therefore be understood as findings describing the population of a small open economy.[17] In this small open economy, international capital flows fix the domestic interest rate r.

Concerning production processes, we employ a standard neoclassical technology employing capital and labor as factors of production, $Y = Y(K, L)$.[18] Profit maximization of firms joint with the fixed interest rate r fixes the domestic capital stock and thereby also the gross wage w.[19] Domestic production is constant as well. Households can nevertheless grow richer and experience exponential consumption growth as they accumulate wealth abroad.

4 Mathematical background: describing the mean of a stochastic process

This section discusses principles behind computing means in Sect. 4.1. Section 4.2 looks at a linear stochastic differential equation (SDE) that describes a stochastic process. This section also computes the time derivative of the mean of this stochastic process. We propose two approaches: a "fast and intuitive" approach and one that follows a general rigorous approach from stochastic theory. Both approaches yield the same results.[20]

4.1 Preliminaries

We are interested in a class of real-valued stochastic processes $(X(t))_{t\geq 0}$. This class can be described as solutions of an SDE driven by a Poisson process $(Q(t))_{t\geq 0}$ with

[16] This allows to study wealth of domestic dynasties and government wealth independently of each other.

[17] This idea was employed previously in Bayer et al. (2019).

[18] This extends the AK approach of e.g. Toda (2014). Our earlier version (Birkner et al. 2021) also followed the AK approach.

[19] In the presence of unemployment as in Bayer et al. (2019), the domestic capital stock would adjust accordingly.

[20] Readers interested in understanding means can go to Sect. 4.2 immediately. The analysis of the mean can be understood without the rigorous background in Sect. 4.1.

intensity $\lambda > 0$. Given suitable functions $f, g : \mathbb{R} \to \mathbb{R}$, the SDE takes the form[21]

$$dX(t) = f(X(t))dt + g(X(t_-))dQ(t), \quad t \geq 0. \tag{13}$$

Intuitively, the dynamics of the solution to (13) is the following: The path $(X(t))$ moves along solution curves of the ordinary differential equation $\dot{x} = f(x)$. Whenever the Poisson process $(Q(t))$ jumps at a certain time, say τ, the process jumps from its position $X(\tau_-)$ immediately before τ to its new position $X(\tau) = X(\tau_-) + g(X(\tau_-))$.

For completeness, let us briefly discuss the general mathematical set-up behind (13): The process is defined on a filtered probability space[22] $\left(\Omega, \mathcal{F}, P, \tilde{\mathcal{F}}\right)$ where Ω is the sample space, $\mathcal{F}$ is a $\sigma-$algebra and P is a probability measure on $\mathcal{F}$ and $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_t)_{t\geq 0}$ is a filtration of sub-σ-algebras of $\mathcal{F}$. Strictly speaking, $X : [0, \infty) \times \Omega \to \mathbb{R}$ is then a function of time and "randomness", where $X(t, \omega)$ is the (random) value of the process at time $t \geq 0$ in the sample point ω. We will follow the usual approach and suppress the dependence on ω in the notation, so $X(t)$ denotes the real-valued random variable which describes the state of the process at a fixed time $t \geq 0$. We will write $(X(t)) \equiv (X(t))_{t\geq 0}$ to denote the (random) path of the process and sometimes simply write X to denote the process when the context is clear. We will try to follow the usual notational convention to denote random variables by capital letters and possible (fixed) values by small letters. A Poisson process $(Q(t))_{t\geq 0}$ with intensity $\lambda > 0$ on $\left(\Omega, \mathcal{F}, P, \tilde{\mathcal{F}}\right)$ is a process with $Q(0) = 0$ which is constant between jumps of size $+1$ with the property that $Q(t) - Q(s)$ is independent of $\tilde{\mathcal{F}}_s$ and Poisson distributed with mean $\lambda(t - s)$ for any $0 \leq s < t$.

Under suitable assumptions,[23] it is known that (13) has a unique solution for any starting value $X(0)$ (which could itself be random) which is adapted to the filtration $\tilde{\mathcal{F}}$ and has right-continuous paths. Furthermore, if $X(0)$ has finite expectation $E[|X(0)|] < \infty$, we have then $E[|X(t)|] < \infty$ for all $t > 0$ as well. The analogous statement holds for second moments.

The solutions are semi-martingales and also (strong) Markov processes.[24] This allows to use tools both from stochastic analysis and from the theory of Markov processes in order to analyze the behavior of the process $(X(t))$.

For the Markov process viewpoint, we need a family P_x, $x \in \mathbb{R}$ of probability measures on $(\Omega, \mathcal{F})$ where for given $x \in \mathbb{R}$, P_x describes the law when starting from the fixed $x = X(0)$, in particular $P_x(X(0) = x) = 1$. We will write expectations with

21 As usual, we understand (13) to be a shorthand notation for the equation $X(t) = X(0) + \int_0^t f(X(s))\, ds + \int_0^t g(X(s_-))\, dQ(s)$ with $t \geq 0$.

22 We can and will assume that the "usual conditions" are fulfilled, i.e., $\tilde{\mathcal{F}}$ is right-continuous and complete, see e.g., Garcia and Griego (1994, p. 338).

23 We will either assume that f is Lipschitz continuous, that is there exist $c_f < \infty$ so that $|f(x) - f(y)| \leq c_f|x - y|$ holds for all $x, y \in \mathbb{R}$ and that g is either Lipschitz continuous or bounded (see for example García and Griego 1994, Theorem 6.2).

24 In fact, they belong to the class of piece-wise deterministic Markov processes: Between the jump times of $(Q(t))$, $(X(t))$ follows a differentiable curve. Such processes are discussed in much greater detail in Davis (1993a). See also Garcia and Griego (1994, p. 362) for the Markov property.

respect to P_x as E_x such that

$$E_x[h(X(t))] = E[h(X(t)) \mid X(0) = x], \tag{14}$$

where $h : \mathbb{R} \to \mathbb{R}$ is (a suitable) test function.[25] The transition semigroup[26] is $(P_t)_{t\geq 0}$ where P_t is defined via

$$P_t h(x) \equiv E_x[h(X(t))], \quad x \in \mathbb{R}. \tag{15}$$

In an economic spirit, if the test function $h : \mathbb{R} \to \mathbb{R}$, represented utility, $P_t h(x)$ is the expected value at time 0 of the random utility $h(X(t))$ at time t given that we know $X(0) = x$.

The generator $\mathcal{A}$ of a (Feller) transition semigroup $(P_t)_{t\geq 0}$ is defined as

$$\mathcal{A}h\,(x) = \lim_{t\to 0, t>0} \frac{P_t h(x) - h(x)}{t}, \quad x \in \mathbb{R} \tag{16}$$

for functions h in its domain $\mathcal{D}(\mathcal{A})$. By definition, $\mathcal{D}(\mathcal{A})$ consists of all functions $h \in C(\mathbb{R})$ for which the limit on the right-hand side of (16) exists (in the "strong" sense of the supremum norm on $C(\mathbb{R})$, i.e., uniformly in x). For a more probabilistic interpretation of (16), we re-write this as $\mathcal{A}h\,(x) = \lim_{t\to 0, t>0} \frac{E_x[h(X(t))]-h(x)}{t}$. Thus, for a very small positive time $t > 0$ and a given starting point x, we have

$$E_x[h(X(t)] \approx h(x) + t\mathcal{A}h\,(x)$$

and, hence, $\mathcal{A}h(x)$ describes approximately[27] how the mean of $h(X(t))$ changes from its initial value $h(x)$ over a very short time interval.

The generator for the solution of (13) looks as follows[28]

$$\mathcal{A}h\,(x) = f(x)h'(x) + \lambda\,[h(x + g(x)) - h(x)] \tag{17}$$

and $\mathcal{D}(\mathcal{A})$ contains all differentiable functions $h \in C(\mathbb{R})$ such that the derivative $h' \in C(\mathbb{R})$.

Let us briefly discuss why (17) holds. For an intuitive approach, consider $h \in \mathcal{D}(\mathcal{A})$, $X(0) = x$, $t > 0$ very small. Then $P_x(Q(t) = 1) = \lambda t + O(t^2)$, $P_x(Q(t) = 0) = 1 - \lambda t + O(t^2)$, $P_x(Q(t) \geq 2) = O(t^2)$. On the event $\{Q(t) = 0\}$ (no jump before

[25] Suitable means that the expectation in (14) is well-defined. This is, for example, the case when h is measurable and bounded or non-negative.

[26] The semigroup property means $P_t P_s = P_{t+s}$, compare e.g. Protter (2004, p. 35). It is known that for our examples $(P_t)_{t\geq 0}$ is a so-called Feller transition semigroup, see e.g. (Davis 1993a, Theorem (27.6)), that is $P_t : C(\mathbb{R}) \to C(\mathbb{R})$ where $C(\mathbb{R})$ denotes the set of continuous functions which vanish at $\pm\infty$. This is mathematically convenient since it allows to work on the Banach space $C(\mathbb{R})$.

[27] A precise meaning of $\approx$ is here that in fact $E_x[h(X(t)] = h(x) + t\mathcal{A}h\,(x) + o(t)$ as $t \downarrow 0$, where the "error term" $o(t)$ goes to 0 faster than linearly in t.

[28] See, e.g., Garcia and Griego (1994, pp. 361–362).

time t), we have, by linearizing the ODE, $X(t) \approx x + tf(x)$; on the event $\{Q(t) = 1\}$ we have $X(t) \approx x + g(x)$. Hence

$$\begin{aligned} E_x[h(X(t))] &\approx (1-\lambda t)h(x+tf(x)) + \lambda t h(x+g(x)) \\ &= h(x+tf(x)) + \lambda t\,[h(x+g(x)) - h(x)] - \lambda t\,[h(x+tf(x)) - h(x)]\,. \end{aligned}$$

Subtracting $h(x)$ on both sides, dividing by t and letting $t \downarrow 0$ then yields (17) (use the chain rule on $\frac{d}{dt}h(x+tf(x))$ and observe that $\lambda t\big(h(x+tf(x)) - h(x)\big) = O(t^2)$).

A rigorous argument goes as follows: Applying to $(X(t))_{t\geq 0}$ the chain rule for paths of bounded variation,[29] we find

$$\begin{aligned} h(X(t)) &= h(X(0)) + \int_0^t h'(X(s))f(X(s))\,ds \\ &\quad + \int_0^t \Big(h\big(X(s_-) + g(X(s_-))\big) - h(X(s_-))\Big)dQ(s) \\ &= h(X(0)) + \int_0^t h'(X(s))f(X(s))\,ds \\ &\quad + \lambda \int_0^t \Big(h\big(X(s_-) + g(X(s_-))\big) - h(X(s_-))\Big)ds \\ &\quad + \int_0^t \Big(h\big(X(s_-) + g(X(s_-))\big) - h(X(s_-))\Big)d[Q(s) - \lambda s]. \end{aligned}$$

By martingale properties of compensated Poisson processes (see, e.g. García and Griego 1994, Thm. 5.3), the process

$$\int_0^t \Big(h\big(X(s_-) + g(X(s_-))\big) - h(X(s_-))\Big)d[Q(s) - \lambda s], \quad t \geq 0$$

is a martingale, in particular its expectation equals 0. Thus, taking expectations with respect to P_x shows

$$\begin{aligned} E_x[h(X(t))] &= h(x) + E_x\left[\int_0^t h'(X(s))f(X(s))\,ds\right] \\ &\quad + \lambda E_x\left[\int_0^t \Big(h\big(X(s_-) + g(X(s_-))\big) - h(X(s_-))\Big)ds\right] \\ &= h(x) + \int_0^t E_x\left[h'(X(s))f(X(s))\right]ds \\ &\quad + \lambda \int_0^t E_x\left[\Big(h\big(X(s_-) + g(X(s_-))\big) - h(X(s_-))\Big)\right]ds, \end{aligned}$$

[29] See, e.g., Garcia and Griego (1994, p. 344).

where we used Fubini's theorem in the second equation. Thus

$$\frac{E_x[h(X(t))] - h(x)}{t} = \frac{1}{t}\int_0^t E_x\left[h'(X(s))f(X(s))\right]ds$$
$$+ \frac{\lambda}{t}\int_0^t E_x\left[\left(h\big(X(s_-) + g(X(s_-))\big) - h(X(s_-))\right)\right]ds.$$

Using the fact that $\lim_{s\downarrow 0} X(s) = \lim_{s\downarrow 0} X(s_-) = X(0)$ because paths are right-continuous, this shows (17) by taking $t \downarrow 0$.

This a good place to highlight the difference between Kolmogorov backward equations and Kolmogorov forward equations (aka Fokker–Planck equations). If distributional properties are to be understood, the forward equation is applied. If the interest lies in the mean, the backward equation can be used (see for instance Kallenberg 1997, p. 192). For Markov processes the general Kolmogorov backward equation reads (compare Davis 1993a, p.30, equation 14.11)

$$\frac{d}{dt}E_x\left[h\left(X\left(t\right)\right)\right] = E_x\left[\left(\mathcal{A}h\right)\left(X\left(t\right)\right)\right] \tag{18}$$

for all functions $h \in \mathcal{D}(\mathcal{A})$.[30]

We are particularly interested in computing the mean

$$\mu(x,t) \equiv E_x[X(t)] \tag{19}$$

for processes of the form (13). We will do this in the following section.

4.2 An example

We start by looking at a stochastic process $X(t)$ described by a SDE,

$$dX\left(t\right) = -aX\left(t\right)dt + bdQ\left(t\right) \tag{20}$$

with $X\left(0\right) > 0$ and $a, b > 0$. To connect (20) to (13), set $f(x) = -ax$ and $g(x) = b$. This implies that $X\left(t\right) \geq 0\ \forall\ t$ as the deterministic decay is exponential, i.e. $X\left(t\right)$ approaches zero asymptotically in the absence of jumps. The arrival rate of $Q(t)$ is given by the constant $\lambda > 0$. The support of $X\left(t\right)$ is $\mathbb{R}_+^*$, i.e. neither zero nor infinity are included, $]0, \infty[$. The support is infinitely large as in principle $Q(t)$ can jump very often relative to the speed of a. Figure 1 shows one possible realization of process (20).

[30] As further information, a brief and reader-friendly introduction is García and Griego (1994). Standard texts include Davis (1993a), Protter (1995), Privault (2018), Kallenberg (1997) and Liggett (2010). In particular, Liggett (2010, ch. 3) has a very readable introduction to Feller processes.

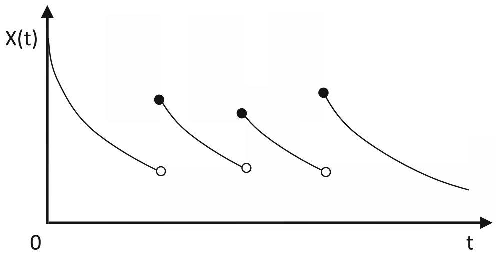

Fig. 1 One possible realization of the stochastic process $(X(t))_{t\geq 0}$ in (20)

4.2.1 The mean (simple approach)

We will now derive the expected value $E_x[X(t)]$ in a rather straightforward way. The linearity of (20) helps in this respect.

In a first step, we express the SDE in (20) in its integral version. It reads $X(t) - X(0) = -a\int_0^t X(s)\,ds + b\int_0^t dQ(s)$. When we apply the expectations operator E_x from (14), we get

$$E_x[X(t)] - E_x[X(0)] = -a\int_0^t E_x[X(s)]\,ds + bE_x\left[\int_0^t dQ(s)\right]$$
$$= -a\int_0^t E_x[X(s)]\,ds + b\lambda\int_0^t ds. \tag{21}$$

We can pull the expectations operator inside the integral as the appropriate version of a Fubini theorem holds (see Philip 2004, p. 207 or Bichteler and Lin 1995, p. 277, example 4.1 for more background). The second equality uses the martingale result of Garcia and Griego (1994, theorem 5.3).

In a second step, we rewrite (21) employing $\mu(x,t)$ from (19) and obtain $\mu(x,t) - \mu(x,0) = -a\int_0^t \mu(x,s)\,ds + b\lambda\int_0^t ds$. Computing the derivative with respect to time t gives

$$\frac{d\mu(x,t)}{dt} \equiv \dot{\mu}(x,t) = -a\mu(x,t) + b\lambda. \tag{22}$$

The Kolmogorov backward equation has thus turned into an ordinary differential equation. It describes the change over time of the expected value of $X(t)$. Expectations are formed from the perspective of the initial point of the process, here 0. The initial condition for $t = 0$ is $\mu(x,0) = x$.

It would then be straightforward to study the properties of this ODE. As one can easily verify, the mean converges to the fixpoint $\mu^* = b\lambda/a$ from above and below.

4.2.2 The mean (generic approach)

We now show how to derive the ODE for the mean in (22) in a way more closely related to Sect. 4.1. The idea consists in using the identity function $h(x) = x$ as a test function and insert it into the corresponding Kolmogorov backward equation.

With $h(x) = x$, $h'(x) = 1$, (18) becomes

$$\frac{d}{dt} E_x\left[X(t)\right] = E_x\left[(\mathcal{A}h)(X(t))\right] = E_x\left[-aX(t) + b\right] = -aE_x\left[X(t)\right] + b.$$

Replacing $E_x\left[X(t)\right]$ by $\mu(x, t)$ from (19) again, yields (22). Hence, we can either work with the integral version of an SDE and form expectations as in Sect. 4.2 or we use insights from Sect. 4.1 to obtain an ODE for the mean of our stochastic process. The second approach also shows why the first approach works so nicely: We need that for $h(x) = x$, $\mathcal{A}h(x)$ is an affine function of x.

5 Mean dynamics for dynasties, the population and the government

We now apply the results of Sect. 4 to the model from Sect. 3. There is no need to apply our findings to an individual as an individual experiences a deterministic evolution of age and wealth up to death. We therefore study expectations for dynasties, the population and the government. The main part focuses on the analysis of expected private and public wealth.[31]

5.1 Expected wealth

- A dynasty

Our main variable of interest is dynasty wealth. We define the expected level of dynasty wealth, $\eta(a_i, t)$, as

$$\eta(a_i, t) = E_a(A_i(t)) \equiv E\left[A_i(t) \,|A_i(0) = a_i\right]. \tag{23}$$

Following the intuitive description from above, we are at an initial point in time 0, consider an individual with initial age $X_i(0)$ and endow them with initial wealth $A_i(0) = a_i$. The mean $\eta(a_i, t)$ then provides the expected value of wealth for individual i with initial wealth a_i at a future point in time t.

Expected wealth can be described by a linear differential equation (see Appendix A.2),

$$\dot{\eta}(a_i, t) = zW + \delta\bar{a} + (z - \delta)\,\eta(a_i, t). \tag{24}$$

Expected wealth depends on the death rate δ, on endowment $\bar{a}$ of a newborn and on the growth rate z of individual consumption and wealth from (7). Solving this equation

[31] The analysis of expected age is very similar and is available in our discussion paper version Birkner et al. (2021) or in an online appendix available upon request.

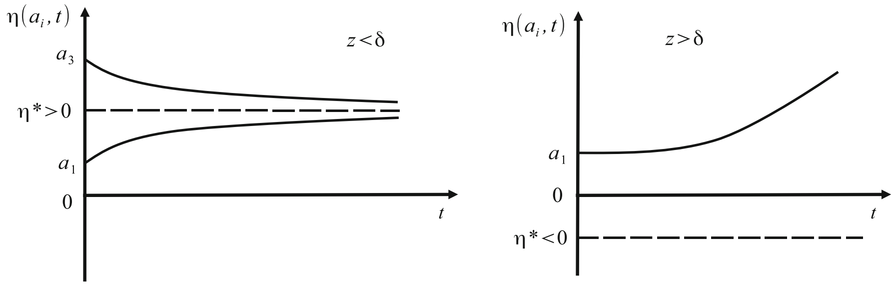

Fig. 2 Expected wealth as a function of parameters z and δ

yields

$$\eta(a_i, t) = \left(a_i - \eta^*\right) e^{(z-\delta)t} + \eta^*, \quad \text{with } \eta^* \equiv -\frac{zW + \delta\bar{a}}{z - \delta}. \tag{25}$$

The solution shows that expected wealth can rise or fall over time. The convergence or growth rate for expected wealth is $z - \delta = \frac{r - \tau_a - \rho}{\sigma} - \delta$, which can be positive or negative, depending on all those five parameters.[32] When $z < \delta$, the long-run expected wealth level η^* is positive. The initial expected value is a_i which can be of course larger or smaller than η^*. As illustrated in the left panel of Fig. 2, any initial wealth level converges to η^* at the convergence rate $z - \delta$, i.e. η^* is a globally stable fix point. This fix point is larger than the initial endowment $\bar{a}$ as long as $z < \delta$ since death rate $\delta \in (0, 1)$. Expected wealth falls when initial wealth a_i is above the long-run expected wealth level η^*, otherwise, it rises.

For the empirically more relevant case of $z > \delta$, the stationary level η^* is negative. When the initial condition $a_i = \eta^*$, which would require initial debt, the expected wealth level always stays at η^*. For $a_i < \eta^*$, expected wealth falls to minus infinity. For $a_i > \eta^*$, expected wealth rises without bound at the growth rate $z - \delta$. Hence, η^* is an unstable fix point.

- The population

Average wealth of the population at t is defined as the average over dynasty wealth levels $A_i(t)$ given constant population size L,

$$\bar{A}(t) \equiv \Sigma_{i=1}^{L} A_i(t) / L. \tag{26}$$

As dynasty wealth from (10) is stochastic, we need to form expectations for any point in time $t > 0$ in order to be able to make any model predictions. We obtain

$$E\bar{A}(t) = E\left[\Sigma_{i=1}^{L} A_i(t) / L\right] = \Sigma_{i=1}^{L} E_a\left[A_i(t)\right] / L. \tag{27}$$

[32] We do not study the case of $\delta = z$ as it is a special case which does not promise further insights. The differential equation (24) would display a constant on the right-hand side and the solution (25) would be a linear function of time.

As we can pull the expectation operator into the sum, we end up with a familiar expression, namely $E_a[A_i(t)]$. As $E_a[A_i(t)] = \eta(a_i, t)$ from (23), the expected population mean equals the mean over expected dynasty means, $E\bar{A}(t) = \Sigma_{i=1}^{L}\eta(a_i, t)/L$. Employing the solution for expected dynasty wealth (25) yields

$$E\bar{A}(t) = \left(\Sigma_{i=1}^{L} a_i/L - \eta^*\right) e^{(z-\delta)t} + \eta^* \tag{28}$$

where η^* is the same expression as defined in (25) for expected dynasty wealth.

Here we need to distinguish the two cases of $z-\delta$ being positive or negative as well. As $z > \delta$ is the empirically relevant case, we focus on this assumption. The long-run average wealth level η^* in our economy is then negative and an unstable fix point. As the initial average wealth $\Sigma_{i=1}^{L} a_i/L$ needs to be positive by empirical relevance, expected average wealth increases at the exponential rate $z - \delta > 0$.

If we assume that population size goes to infinity, i.e. $L \to \infty$, the variance of average wealth $\bar{A}(t)$ from (26), as a result of the law of large numbers,[33] tends towards 0. Hence, in any practical sense average wealth $\bar{A}(t)$ equals the expected value $E\bar{A}(t)$.

5.2 Expected government wealth

The government levies taxes and endows newborns with wealth. runs a tax scheme based on inheritances. Is this scheme feasible for government wealth? Under which conditions will the government exhibit a balanced budget in the long run? To start answering these questions, we first study the evolution of expected government wealth as an outcome of applying its tax scheme to one dynasty only. We then aggregate over all dynasties. The full answer will be obtained when we study equilibrium in Sect. 6.1.

- Expected government wealth based on one dynasty

We define expected government wealth following (11) as

$$\gamma(a_i, t) \equiv E\left[G_i(A_i(X(t)))\,|A_i(X(0)) = a_i\right]. \tag{29}$$

The initial condition a_i is the same as in (23). Following methods from above, the mean $\gamma(a_i, t)$ follows (see Appendix A.3)

$$\dot{\gamma}(a_i, t, \tau_a, \tau_w) = -\delta\bar{a} + \tau_w w + (\tau_a + \delta)\,\eta(a_i, t). \tag{30}$$

The dynamics can be easily understood when comparing this ODE with the ODE for expected wealth of a dynasty in (24). Expected wealth of a dynasty rises in δ (via a first channel) as the dynasty receives $\bar{a}$ when an offspring is born. Expected wealth of the government falls in δ as a new offspring is an expenditure for the government. Expected wealth of a dynasty falls in δ (via a second channel) as the household loses

33 The summands in $\Sigma_{i=1}^{N} A_i(X_i(t))$ are independent of each other as $A_i(X_i(t))$ is a deterministic function of random age X_i and, given independence of Poisson processes in (9), random variables X_i are independent of each other.

expected wealth $\eta(a_i, t)$. By contrast, government wealth rises in this second channel as the government receives this expected wealth. Expected wealth of the household rises at the rate of z, resulting from the optimal consumption decision of the household. Wealth of the government rises at τ_a and τ_w as these are the tax rates applied to wealth of the dynasty and constant labor income, respectively.

While ODEs between the dynasty and government level have very similar interpretations, the solution of (30) looks very different from the solution at the dynasty level. This is not surprising as the right-hand side of the government's expected wealth (for this dynasty) contains expected wealth of the dynasty. Hence, (30) is not an autonomous differential equation but needs to be solved by taking the solution of the dynasty budget constraint (25) into account. The solution to (30) reads (see Appendix A.5.2)

$$\gamma(a_i, t, \tau_a, \tau_w) = G_{i,0} + \left((\tau_a + \delta)\eta^* - \delta\bar{a} + \tau_w w\right)t + \frac{(\tau_a + \delta)(a_i - \eta^*)}{z - \delta}\left(e^{(z-\delta)t} - 1\right), \tag{31}$$

where $G_{i,0}$ describes the government wealth at the initial point in time 0 stemming from dynasty i.

- Expected total government wealth

Similar to (12), expected total wealth of the government is simply the sum over dynasty-specific means from (31), $\Gamma(t, \tau_a) = \Sigma_{i=1}^{L}\gamma(a_i, t)$. After some steps (see Appendix. A.3.3), we obtain an expression for expected government wealth per capita that reads

$$\frac{\Gamma(t, \tau_a, \tau_w)}{L} = \frac{G_0}{L} + \left((\tau_a + \delta)\eta^* - \delta\bar{a} + \tau_w w\right)t + \frac{(\tau_a + \delta)\left(\bar{A}(0) - \eta^*\right)}{z - \delta}\left(e^{(z-\delta)t} - 1\right). \tag{32}$$

Juxtaposing this equation with (31) shows that initial wealth $G_{i,0}$ is replaced by G_0/L and dynasty initial wealth a_i is replaced by average initial wealth $\bar{A}(0)$ defined as in (26).[34]

Both the solution (31) and (32) show that expected government wealth from one dynasty can rise or fall over time. Anticipating equilibrium analysis to some extent, the exponential term clearly shows that one necessary condition for a steady state is $z < \delta$. This makes sure that the final term in (A.28) approaches a constant. A steady state also requires that the term in front of t equals zero at each instant,

$$\tau_a\eta^* + \tau_w w = \delta\left[\bar{a} - \eta^*\right]. \tag{33}$$

If η^* is larger in equilibrium than $\bar{a}$ (which holds for $z > 0$ as shown in the discussion of Fig. 2), this condition suggests either a negative wealth tax and/or labor income tax:

[34] We can apply a law of large numbers for per capita government wealth in the same way as we did for average individual wealth $\bar{A}(t)$ from (26).

A positive government income per birth, $\eta^* > \bar{a}$, implies subsidies to capital and/or labor income, $\tau_a < 0$ and/or τ_w. Based on (33), the government uses the wealth and a labor income tax to ensure the budget is balanced in a long-run steady state. We will return to this condition shortly.

6 Aggregate and distributional findings

We now characterize equilibrium in our small open economy. Subsequently, we describe distributional properties of wealth. For analytical convenience, we first consider the case of an economy with capital as the only factor of production (as e.g. in Toda 2014; Benhabib et al. 2016 or Kasa and Lei 2018). We then study the neoclassical economy with capital and labor.

6.1 Steady state and balanced growth path equilibrium for the AK case

Depending on parameter values, the model ends up in a stationary equilibrium or on a growth path. We say that our economy is in a steady-state equilibrium when both individual variables (e.g. dynasty wealth) and aggregate variables (e.g. government wealth) converge to stationary values. The economy is in a growth equilibrium when individual and aggregate variables converge to a balanced growth path where (most) variables grow at identical rates. Interestingly, distributions can be stationary on a balanced growth path.

6.1.1 Convergence to a general steady-state equilibrium

As the expected wealth analysis, summarized in Fig. 2, has shown, a partial stationary equilibrium holds if $z < \delta$, i.e. when the rate of wealth growth falls short to the death rate δ. Expected wealth of a dynasty (25) as well as expected average wealth of the population (28) converge to their long-run value η^*, where the latter is described by (25). With constant wealth, consumption and utility are constant as well, where the rate z is treated as exogenous by individuals.

In order to describe the general stationary equilibrium, however, we need to consider the evolution of the government budget as well. For a steady-state equilibrium, we require that government wealth approaches a long-run constant value. We now study the corresponding conditions.

- A stationary government wealth implies an endogenous tax system

In a steady-state equilibrium, the term linear in time in (25) and (28) must vanish. This is condition (33). In the AK case, $w = 0$ and (33) simplifies to

$$\tau_a \eta^* = \delta \left(\bar{a} - \eta^* \right), \tag{34}$$

where η^* from (25) now reads

$$\eta^* = -\frac{\delta}{z - \delta} \bar{a}. \tag{35}$$

If η^* is larger in equilibrium than $\bar{a}$ (which holds for $z > 0$ as shown in the discussion of Fig. 2), this condition suggests a negative tax rate: A positive government income per birth, $\eta^* > \bar{a}$, implies subsidies to capital income, $\tau_a < 0$.

In order to determine τ_a, we start from (34) and employ η^* from (35). After some steps (see Appendix A.4), the resulting tax rate reads

$$\tau_a = \frac{r - \rho}{1 - \sigma}. \tag{36}$$

This constant tax rate balances the budget of the government in the long run. Short run average wealth of the population or current government wealth do not matter. Given this endogenous tax rate, the growth rate z from (7) adjusts. After some simple steps (see Appendix B.7), the equilibrium wealth growth rate reads

$$z = \frac{r - \rho}{\sigma - 1}. \tag{37}$$

- Conditions for convergence to a steady-state equilibrium

So far, we obtained two necessary conditions for a steady-state equilibrium. First, steady state requires $z < \delta$ as (i) expected household wealth then approaches a constant and as (ii) the second term of the wealth expression for the government (32) also approaches a constant. Second, steady state requires an endogenous tax rate, the condition for τ_a in (36). This tax rate makes sure that government wealth approaches a constant in the long run. The endogenous tax rate led to the new expression for z in (37). A steady state for both the household and the government level therefore requires that $z < \delta$ also holds for z from (37).

To understand when $z < \delta$, consider Fig. 3. It plots z from (37) as a function of σ. The left panel displays the case of $r > \rho$, the right panel of $r < \rho$. There is a pole at $\sigma = 1$. We understand when $z < \delta$ by defining a threshold level σ^* that implies $z = \delta$. This threshold level is given by

$$\sigma^* \equiv \frac{r - \rho + \delta}{\delta} \tag{38}$$

and is also shown in both panels for an example of δ.

When $r > \rho$, the threshold level is $\sigma^* > 1$ for all positive δ. There is therefore a steady state (where $z < \delta$ with endogenous τ_a from (36)) if and only if $\sigma < 1$ or $\sigma > \sigma^*$. For levels in-between, i.e. for $1 < \sigma < \sigma^*$, the economy is on a growth path. When $r < \rho$ (right panel) and $\delta < -(r - \rho)$ (as *not* drawn in the panel), the threshold level is negative, $\sigma^* < 0$. There is a steady state if and only if $\sigma > 1$. For $\sigma < 1$, the economy is on a growth path. By contrast, when $r < \rho$ and $\delta > -(r - \rho)$ (as drawn in the right panel), the threshold level is between zero and one, $0 < \sigma^* < 1$. There is a steady state if and only if $\sigma < \sigma^*$ or $\sigma > 1$. For $\sigma^* < \sigma < 1$, the economy is on a growth path. These conditions are summarized in the Table 1.

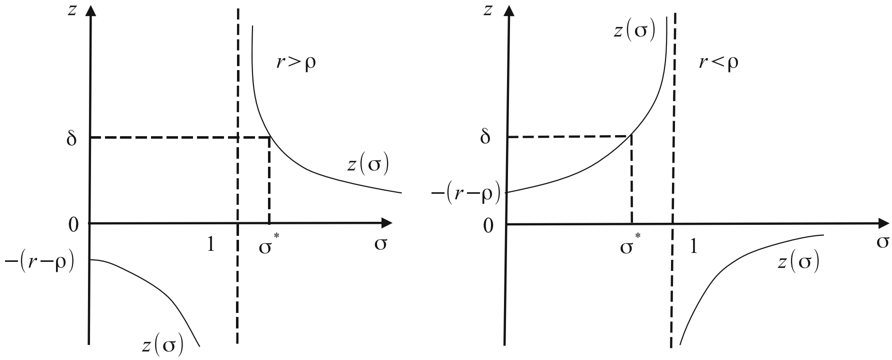

Fig. 3 The steady-state condition for σ for $r > \rho$ (left) and $r < \rho$ (right)

Table 1 Parameter conditions for steady state and growth path

$r > \rho$		$\sigma < 1$ $\sigma^* < \sigma$	steady state
		$1 < \sigma < \sigma^*$	growth path
$r < \rho$	$\delta < -(r - \rho)$	$\sigma > 1$	steady state
		$\sigma < 1$	growth path
	$\delta > -(r - \rho)$	$\sigma < \sigma^*$ $1 < \sigma$	steady state
		$\sigma^* < \sigma < 1$	growth path

- The importance of risk aversion for the equilibrium type

We would like to emphasize the importance of σ in determining the equilibrium type. To the best of our knowledge, risk aversion never played this role in any models of the optimal growth or new growth theory. Risk aversion (in optimal saving rules of the type $\dot{c}/c = (r - \rho)/\sigma$) amplifies the growth rate, but does not have an effect on the sign of the growth rate – whether the economy ends up in a steady state or on a balanced growth path.

This importance of σ is illustrated in Fig. 4. The horizontal axis plots the difference between r and ρ, the vertical axis plots risk aversion σ. Consider first the case of a positive difference $r - \rho$. With a risk aversion below 1, the economy ends up in a steady state. As $z < 0$, wealth falls over time. This holds in individual data but not for empirical aggregate averages over the lifetime. This is therefore the empirically less relevant steady state. A risk aversion equal to 1 (see Appendix B.9) or larger than 1 but still below σ^* implies a balanced growth path. When σ rises further, we return to a steady-state economy. For this region of $\sigma > \sigma^*$, z is positive such that expected dynasty wealth $\eta(t, a_i)$ rises and approaches the steady state from below (as illustrated by the trajectory starting at a_1 in the left panel of Fig. 2).

When $r - \rho$ is negative but still larger than $-\delta$, an increase in σ also moves the economy through three regimes. With risk aversion below σ^*, the economy is in a

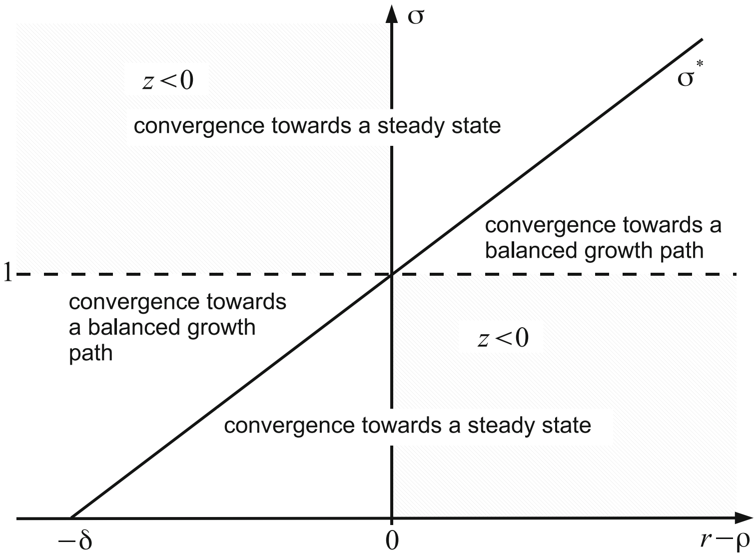

Fig. 4 Steady state and balanced growth path regions

steady state with positive z. Wealth evolves as just illustrated by $\eta\,(t, a_1)$ in Fig. 2. A higher σ brings us to a growth equilibrium and risk aversion above one leads to an empirically non-convincing steady state with $z < 0$.

For $r - \rho < -\delta$, the economy starts (at low σ) in a growth equilibrium. Risk aversion exceeding 1 yields the steady state just described.

Why does σ play this role here? The precise channel through which σ affects growth rates is not the individual growth rate z from (7) but the equilibrium growth rate z from (37). This hints to the crucial role of tax policy. In fact, when we analyze the full model and study the effect of a labor tax in Sect. 6.2.1 below, we will see that the equilibrium growth rate z falls monotonically in σ when the government employs the labor tax to balance its budget. Hence, it is the effect of σ on the capital tax τ_a in (37) through the balanced budget condition that leads to this new role of σ here.

What is the intuition behind this channel of σ? The parameter should be understood here in its interpretation (of its inverse) as intertemporal elasticity of substitution (and not in terms of risk aversion): the channel through which σ acts is through its effect on the wealth growth rate z of an individual while alive, i.e. in the absence of any risk. The higher $1/\sigma$, the higher the individual (deterministic) growth rate z from (7) (for $r - \tau_a - \rho > 0$). This implies a higher average or expected wealth level η^*, see (35) or the left panel of Fig. 2. When we write (34) as$(\tau_a + \delta)\,\eta^* = \delta\bar{a}$, we see that the government can lower the tax on wealth when η^* rises. Hence, σ affects the individual growth rate z directly and indirectly through its effect on τ_a. This leads to the sign-effect of σ on the equilibrium growth rate z in (37) beyond the standard level effect.

An interpretation in the same spirit can be given for r falling short of ρ.

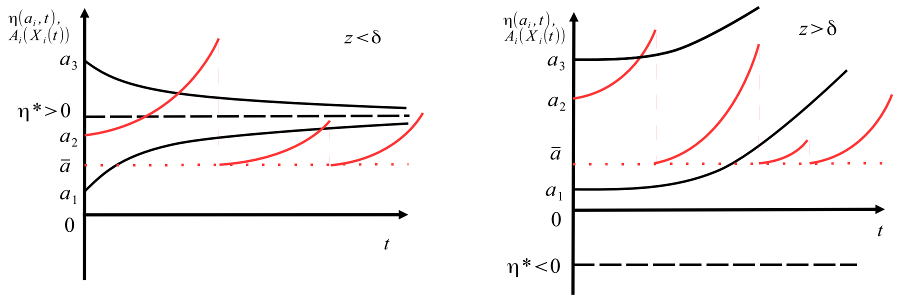

Fig. 5 Realized wealth paths (red) versus expected wealth paths (black) of a dynasty

- Equilibrium convergence to a steady state

We can now summarize equilibrium dynamics of means in our small open economy. The economy starts with initial wealth levels a_i for dynasties i. Expected dynasty wealth converges to a steady state following (35). Average wealth in our economy follows (28) altered accordingly. These paths are illustrated in the left panel of Fig. 2.

Expected government wealth under these conditions follows

$$\frac{\Gamma(t,\tau_a)}{L} = \frac{G_0}{L} + \left(\bar{A}(0) - \eta^*\right)\left(1 - e^{(z-\delta)t}\right). \tag{39}$$

It can rise or fall over time, depending on whether initial wealth $\bar{A}(0)$ of households lies above or below expected wealth η^*. Interestingly, the per-capita government wealth in the steady state, $G_0/L + \bar{A}(0) - \eta^*$, displays two initial conditions. Usually (as e.g. in dynasty wealth (25) or average wealth (28)), initial conditions vanish in the long run. Here, they persist as (initial) government wealth is not directly owned by households and therefore not subject to the death-birth process (as discussed after (11)).

Concerning realized consumption and wealth growth while alive, given the optimal consumption share ϕ from (38) and τ_a from (36), consumption reads $c(t) = ra(t)$. This illustrates that, if taxes are instantaneously chosen such that the long-term budget is balanced, the optimal, utility-maximizing share of wealth consumed is equal to the gross before-tax interest rate. From the budget constraint (3), wealth therefore falls at the rate of τ_a. Remember that τ_a can be positive or negative, depending on parameter values.

- Expected values versus realizations

Let us be explicit about the difference between an expected evolution and realizations in our model. To this end, Fig. 5 illustrates expected wealth dynamics versus realized wealth while alive.

Equilibrium dynamics for expected wealth in a steady-state economy are shown in black in the left panel of Fig. 5. In addition to Fig. 2, this panel also shows an example of a realized wealth path in red. The corresponding paths for the growing economy are in the right panel to which we turn later.

In a steady-state economy with a long-run balanced government budget, the link between expected dynasty wealth η^* and initial endowment $\bar{a}$ from (35) adjusts due to the endogenous tax rate and the implied new accumulation rate z from (37). After some steps (see Appendix B.7), expected dynasty wealth reads

$$\eta^* = \frac{(1-\sigma)\,\delta}{r-\rho+(1-\sigma)\,\delta}\bar{a}. \tag{40}$$

Obviously, η^* exceeds $\bar{a}$ if $r < \rho$ and falls short of it for $r > \rho$. The left panel shows the case of a $z < \delta$ economy converging towards a steady state with $r < \rho$. When we look at expected dynasty wealth $\eta\,(a_i, t)$ in black, it approaches η^* irrespective of initial conditions a_1 or a_3. By contrast, when we look at an example of a realized growth path $A_i\,(t)$ of a dynasty i from (10), given w and τ_w equal 0, in red, it starts at the initial level a_2 and grows at the constant rate z as long as the current representative of the dynasty stays alive. Whenever the individual is replaced by an offspring, wealth jumps to $\bar{a}$. The black curves also represent realized average wealth in the economy as a whole, i.e. $\bar{A}\,(t)$ from (28).

6.1.2 Convergence towards a balanced growth path

Figure 4 shows parameter values for which the economy finds itself on a growth path. On such a path, condition (33) making sure that the government wealth approaches a constant is not required. It would be enough to think of government wealth (or debt) as staying within a certain range of GDP or overall wealth (think of the Maastricht criteria of the EU).

- Convergence of government wealth to a balanced growth path

In this vein, we divide government wealth (32) per capita by aggregate average wealth (28) and obtain

$$\frac{\Gamma\,(t,\tau_a)\,/L}{E\bar{A}\,(t)} = \frac{G_0/L + ((\tau_a+\delta)\,\eta^* - \delta\bar{a})\,t + \frac{\tau_a+\delta}{z-\delta}\left(\bar{A}\,(0)-\eta^*\right)\left(e^{(z-\delta)t}-1\right)}{\left(\bar{A}\,(0)-\eta^*\right)e^{(z-\delta)t}+\eta^*}. \tag{41}$$

When we now consider the long run, i.e. $t \to \infty$, both the linear growth expression, $((\tau_a+\delta)\,\eta^* - \delta\bar{a})\,t$, and the exponential growth expression, $\left(\bar{A}\,(0)-\eta^*\right)e^{(z-\delta)t}$, tend towards infinity. As the exponential term grows faster than linear or constant terms, limit arguments yield

$$\begin{aligned}\lim_{t\to\infty}\frac{\Gamma\,(t,\tau_a)\,/L}{E\bar{A}\,(t)} &= \lim_{t\to\infty}\frac{\frac{G_0/L+\left((\tau_a+\delta)\eta^*-\delta\bar{a}\right)t}{e^{(z-\delta)t}} + \frac{\tau_a+\delta}{z-\delta}\left[1-\frac{1}{e^{(z-\delta)t}}\right]\left(\bar{A}\,(0)-\eta^*\right)}{\bar{A}\,(0)-\eta^*+\frac{\eta^*}{e^{(z-\delta)t}}} \\ &= \frac{\tau_a+\delta}{z-\delta}.\end{aligned} \tag{42}$$

In the long run, government wealth relative to expected average wealth is constant. The government budget is balanced asymptotically even though in absolute terms government wealth features linear and exponential growth. In the long run, government wealth follows the same growth path as dynasty wealth—independently of the tax rate τ_a.

Even though the government has one additional degree of freedom in a growing economy as compared to an economy that converges to a steady state, the tax is subject to one constraint: It must not be too large such that $z > \delta$ still holds. This is the case as long as the tax does not exceed an upper bound (see Appendix B.7)

$$z > \delta \Leftrightarrow \tau_a < \tau_a^* \equiv r - \rho - \delta\sigma. \quad (43)$$

If it did, individual returns to wealth would fall too much and wealth growth z would become smaller than the death rate. The economy would return to a steady-state equilibrium.[35]

- Equilibrium convergence to a balanced growth path

Equilibrium dynamics in our growing small open economy are as follows. Expected wealth of a dynasty starts from an initial value and grows at a rate $z - \delta$ as described in (25). Population average wealth follows (28). The debt to GDP ratio (41) in our growing economy is potentially non-monotonic over time. It starts at $t = 0$ at $\frac{G_0/L}{\bar{A}(0)}$ and converges to $\frac{\tau_a+\delta}{z-\delta}$ from (41).

Equilibrium dynamics for expected and realized wealth are shown in the right panel of Fig. 5. (Remember that (40) only holds for the left panel.) As for the steady-state economy, black curves show expected growth paths for dynasties, $\eta(a_i, t)$, given initial conditions a_1 or a_3. Both grow at the same rate and there is no convergence in expectation. Realized wealth of subsequent representatives of one dynasty starting at initial wealth a_2 are shown by the red curve. Each offspring starts at $\bar{a}$ and experiences higher wealth growth than expected wealth growth. Given that average wealth $\bar{A}(t)$ from (28) is again (as in the steady-state economy) also represented by the black curves, each individual becomes richer over life relative to the population average.

6.2 The full model

Section 6.1 characterized equilibrium for the AK case. We now return to the model with two factors of production. The government now also levies a labor tax.

6.2.1 Convergence to a steady-state equilibrium

Like in the AK case a necessary condition for a steady state is $z < \delta$. If this holds, expected wealth η^* takes on a positive, constant value in the long run as shown in (25). As in the AK case, a steady state also requires a balanced government budget in the long run. In order to determine the corresponding tax rates, we start from the

[35] If by accident τ takes on the expression (36), the linear component in (42) is removed. The debt-to-wealth ratio still converges to the same constant in the limit.

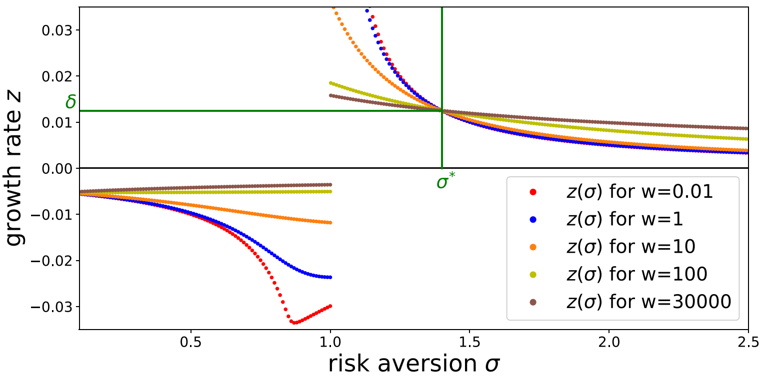

Fig. 6 Wealth growth z as a function of risk aversion σ for different wage levels

condition for a balanced budget in (33) and rewrite it as

$$f(\tau_a, \tau_w) = 0 \text{ where } f(\tau_a, \tau_w) \equiv (\tau_a + \delta)\,\eta^* + \tau_w w - \delta\bar{a}. \tag{44}$$

- The endogenous capital income tax τ_a

After quite some analysis (see Appendix A.5.1), one can show that there are two tax rates τ_a that both yield a long-run balanced budget constraint. With the lower tax, agents would experience a higher individual wealth growth rate z from (7), with the higher tax, the economy would be in a low-growth regime. We focus on the high-growth regime.

Interestingly, after some further steps, we find qualitatively very similar behaviour to our AK findings. Risk aversion σ again crucially determines whether the economy converges towards a steady state or towards a balanced growth path. Analytically, equilibrium is described by three equations in three unknows. The growth rate is $z = (r - \tau_a - \rho)/\sigma$ from (7), the tax is implicitly given by $(\tau_a + \delta)\,\eta^* - \delta\bar{a} + \tau_w w = 0$ from (33) and expected long-run wealth amounts to $\eta^* = -\left(z\frac{(1-\tau_w)w}{r-\tau_a} + \delta\bar{a}\right)/(z - \delta)$ from (25) and (6). When we solve for the growth rate z numerically and plot it as a function of σ, adjusting simultaneously for τ_a and η^*, we obtain Fig. 6.

As in the left panel of Fig. 3 for $r > \rho$, Fig. 6 shows a pole at around $\sigma \approx 1$. This pole was at $\sigma = 1$ in Fig. 3. To the left of it, the individual wealth growth rate z is smaller than the death rate—the economy converges to a steady state. Between the pole and the threshold level σ^*, the growth rate z exceeds the death rate and the economy converges to a balanced growth rate. Interestingly, the threshold level σ^* is the same as in the AK model in (38). When risk-aversion increases beyond σ^*, z falls and the economy eventually approaches a steady state again. These qualitative findings hold for various levels of the wage. We therefore conclude that the basic economic reasoning is the same as discussed after Fig. 4 for the AK case. We do not go into details here for $r < \rho$. We now rather focus on the effects of the labor tax.

- The endogenous labor income tax τ_w

If we let the government balance its budget via the labor tax τ_w, finding the appropriate tax rate is much simpler. Starting from the budget condition (33) and noting that expected wealth η^* from (25) is linear in τ_w, this condition can easily be solved for a budget-balancing τ_w. We expect this tax rate to be positive for $\tau_a = 0$ as the labor tax would then be the only source of government income to finance initial life-endowment $\bar{a}$ in addition to wealth transfers at death. When τ_a takes ever-increasing values, τ_w would turn negative at some point, just as τ_a can take negative values.

In any case, the labor tax constitutes an additional instrument giving the government a continuum of tax policies that all balance the budget in the long run. There is one strong difference between employing labor taxes and wealth taxes. When the labor tax is employed for balancing the budget, the growth rate z from (7) becomes independent of the government policy τ_w as the growth rate z only displays the tax rate τ_a. When we study the effect of a change in risk aversion on growth in the economy as we did in Figs. 3, 4 and 6, the non-monotonicity of z in σ is the effect of changes in risk aversion σ on the tax rate τ_a for capital. When the government employs the labour tax τ_w to balance the budget, a change in σ has the usual monotonic effect on z. With low risk aversion, the growth rate would be high, with high risk aversion, it would be low. Risk aversion σ would no longer influence the sign of the growth rate but—as always in standard optimal saving models—only the level. The choice of the policy instrument by the government therefore clearly has important equilibrium effects.

6.2.2 Convergence towards a balanced growth path

Our analysis of convergence to the steady state has shown that an economy with two factors of production behaves qualitatively identical to an AK economy. Our discussion of the convergence of the two-factor economy towards a balanced growth path is therefore short and follows the arguments of Sect. 6.1.2.

Government wealth per aggregate average wealth would now be given by the ratio of government wealth (32) to aggregate average wealth in (28). This is very similar to the AK expression (41) with the only difference of the addition term $\tau_w w$. The argument from Sect. 6.1.2 persists that an exponential term grows faster than a linear term. The wealth tax τ_a therefore again becomes a free policy instrument, subject to the upper bound (43) such that z exceeds δ.

6.3 Distributional dynamics

How do our aggregate equilibrium dynamics square with transitional distributional dynamics of wealth? We now present analytical findings on the dynamics of the wealth distributions on the adjustment paths towards a steady state or towards the balanced growth path.[36] We then explain how distributional dynamics and aggregate findings fit together.

Before going into details, we would like to point out that well-understood links between an exponential distribution of age and a Pareto distribution of wealth (e.g. Benhabib and Bisin, 2018, p. 1277) also exist in our framework—but only in the long run when distributions are stationary. Applying the Edgeworth translation method means computing the density of a fixed function (wealth as a function of age) of a random variable (age). When we face an arbitrary (cross-section) distribution of age (e.g. on the transition towards the stationary exponential distribution), we can still compute the corresponding wealth distribution by the Edgeworth method, given an initial cross-sectional distribution of wealth. As will become clear, we derive the wealth distribution independently of the age distribution, however.

6.3.1 The wealth distribution

We now describe the derivation of the wealth distribution in detail. We first undertake the fundamental analytical steps. Subsequently, we illustrate the dynamics of the wealth density for an initial mass point and for an initial (non-degenerate) distribution.

- Deriving distributional dynamics

Define the probability that realized dynasty wealth $A_i(t)$ from (10) for an initial wealth level of a_i and at a point in time t lies within a certain range or set $B \subset \mathbb{R}$ by

$$\pi(a_i, B, t) \equiv P(A_i(t) \in B | A_i(0) = a_i). \tag{45}$$

We introduce an indicator function $\boldsymbol{I}_A(z) = 1$ if $z \in A$ and zero otherwise.

The essential step in translating this definition into informative expressions consists in solving the SDE (10). Given the framework defined and discussed in Sect. 4.1 and given an initial condition $A_i(0) = a_i \geq 0$, the unique solution (strong and weak solutions coincide in this framework) to (10) reads (see Appendix A.6.1)

$$A_i(t) = \boldsymbol{I}_{Q_i^\delta(t)}(0)\, a_i(t) + \left(1 - \boldsymbol{I}_{Q_i^\delta(t)}(0)\right)\left((\bar{a} + W)\, e^{z[t-T]} - W\right). \tag{46}$$

where we defined

$$a_i(t) \equiv (a_i + W)\, e^{zt} - W \tag{47}$$

in the spirit of (8) as the wealth level at t that is held by an individual with initial wealth a_i. Also, T marks the most recent point in time before t where a jump of $(Q_i^\delta(s))_{s\geq 0}$ occurred, i.e. where a member of dynasty i deceased for the last time.

[36] We know from more abstract (probability-based) work by Bayer et al. (2019) that processes like our age and the related wealth process are characterized by the existence of a unique long-term distribution which is stable. The latter means that for all (meaningful) initial distributions, an initial distribution converges over time to this unique and stable long-term distribution.

The indicator function $\boldsymbol{I}_{Q_i^\delta(t)}(0)$ equals 1 for $Q_i^\delta(t) = 0$ and zero otherwise. The former represents the absence of death: the individual initially representing dynasty i lives on to accumulate wealth based on the initial value of wealth a_i, the present value of labor income W and rate z. The latter describes the opposite, namely an individual being born as a result of the previous individual's death. Wealth initially starts with $\bar{a}$, present value of labor income of the newborn W and then exponentially accumulates over the time span between birth date T and today t at rate z.

We now specify the set B from (45) as $B = [\bar{a}, x]$. We also assume, to avoid tedious case-by-case analyses, that $a_i > \bar{a}$. We can then rewrite the probability in (45) as $\pi(a_i, x, t) \equiv P(A_i(t) \leq x | A_i(0) = a_i)$. Building on the solution in (46), this probability can be expressed by (see Appendix A.6.2)

$$\pi(a_i, x, t) = e^{-\delta t} \boldsymbol{I}_B(a_i(t)) + \int_B \frac{\delta}{z} \frac{(\bar{a} + W)^{\frac{\delta}{z}}}{(v + W)^{\frac{\delta}{z}+1}} \boldsymbol{I}_{[\bar{a}, \bar{a}(t)]}(v)\, dv, \tag{48}$$

where we defined, in the same spirit as (47),

$$\bar{a}(t) \equiv (\bar{a} + W) e^{zt} - W. \tag{49}$$

To understand the expression (48), consider three ranges for x. Initially, imagine x is small, i.e. $\bar{a} \leq x < \bar{a}(t)$. Then $\boldsymbol{I}_B(a_i(t)) = 0$ and $\boldsymbol{I}_{[\bar{a}, \bar{a}(t)]}(x) = 1$. The probability (48) reads

$$\pi(a_i, x, t) = \int_{\bar{a}}^{x} \frac{\delta}{z} \frac{(\bar{a} + W)^{\frac{\delta}{z}}}{(v + W)^{\frac{\delta}{z}+1}} dv \text{ for } \bar{a} < x < \bar{a}(t). \tag{50}$$

In the second range $\bar{a}(t) < x < a_i(t)$, it still holds that $\boldsymbol{I}_B(a_i(t)) = 0$. In addition, the second indicator function is zero, $\boldsymbol{I}_{[\bar{a}, \bar{a}(t)]}(v) = 0$ for all $v > \bar{a}(t)$. Hence, we can replace the general set B by a lower bound $\bar{a}$ and an upper bound $\bar{a}(t)$ such that (48) reads

$$\pi(a_i, x, t) = \int_{\bar{a}}^{\bar{a}(t)} \frac{\delta}{z} \frac{(\bar{a} + W)^{\frac{\delta}{z}}}{(v + W)^{\frac{\delta}{z}+1}} dv \text{ for } \bar{a}(t) < x < a_i(t). \tag{51}$$

Note that the integral in (51) is not a function of x but only of time t. For the third range when $x \geq a_i(t)$, the probability that $A_i(t)$ is smaller than x is one, $\pi(a_i, x, t) = 1$.

In simple words, when we are interested in the probability that wealth x is small (the first range), this probability can only come from being reborn. Wealth of an individual that lived as of 0 would be $a_i(t)$ and would be too high. Hence, we only consider the range of wealth from endowment $\bar{a}$ at birth to the wealth level x of interest, as shown in (50). The integrand in (50) is the Pareto density with shape parameter δ/z. It follows from the Pareto density in the general expression (48) which in turn is the outcome of a simple parameter substitution (see Appendix A.6.2) starting from (46). Why do we see the Pareto density in (50) from an intuitive perspective? First, when we are

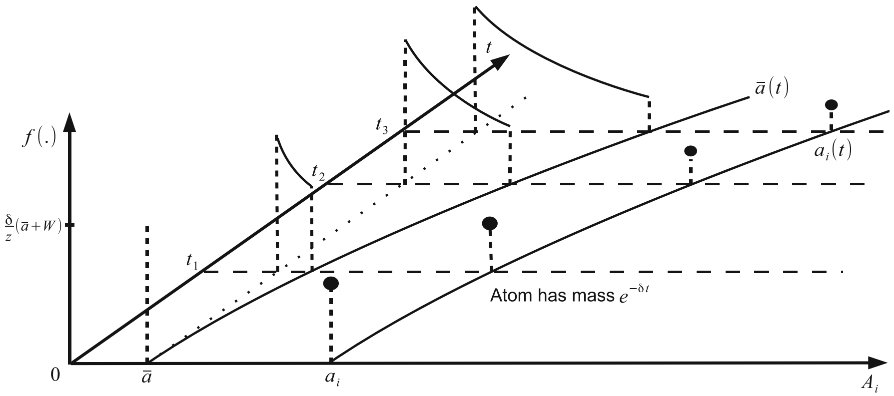

Fig. 7 Dynamics of the wealth density for an initially degenerate distribution (wealth path $\bar{a}(t)$ from (49) and $a_i(t)$ from (47))

interested in small wealth levels x, small levels would result from just being reborn. When being young, wealth cannot be much larger than initial endowment $\bar{a}$. Second, we obtain a Pareto distribution in the short run for the same reason that there are Pareto distributions in the long run: There is exponentially distributed age. We can start our analysis of the wealth distribution from any initial age or wealth distribution. As soon as an individual is reborn, however, our age process (9) makes sure that age is back to an exponential distribution. With exponential age distribution, it seems intuitive, that we obtain a (truncated) Pareto distribution for wealth in the transition.

When our wealth level x of interest is a bit larger (second range), we integrate in (51) over the entire range from $\bar{a}$ to $\bar{a}(t)$. When we think about its construction, we integrate over the entire density apart from the probability of not having died. So the integral in (51) equals $1 - e^{-\delta t}$ where $e^{-\delta t}$ is the probability of still being alive at t.

- Illustration for an initial mass point

We have described the distribution function most generally in (48). Figure 7 illustrates this expression for an initial mass point. When we start from an initial condition $A_i(0) = a_i$, the probability to hold wealth a_i in $t = 0$ equals one. At any point t, the wealth distribution has a probability mass of $e^{-\delta t}$ at $a_i(t)$ where δ is again the death rate from the age process (9). As long as the individuals do not die, they start with a_i and their present value of labor income W and then accumulate wealth at the rate of z. The probability to survive until t is given by the probability mass $e^{-\delta t}$.

Now imagine the individual is replaced by an offspring. Wealth jumps to $\bar{a}$. The maximum wealth level that can be reached by an offspring is $\bar{a}(t)$. This requires exactly one jump at $t = 0$. As offsprings can be replaced again, there is an expanding support $[\bar{a}, \bar{a}(t)]$ within which wealth is (truncated) Pareto distributed, as shown in (50). Expressing the distribution by the more reader-friendly density (with masspoint), we obtain

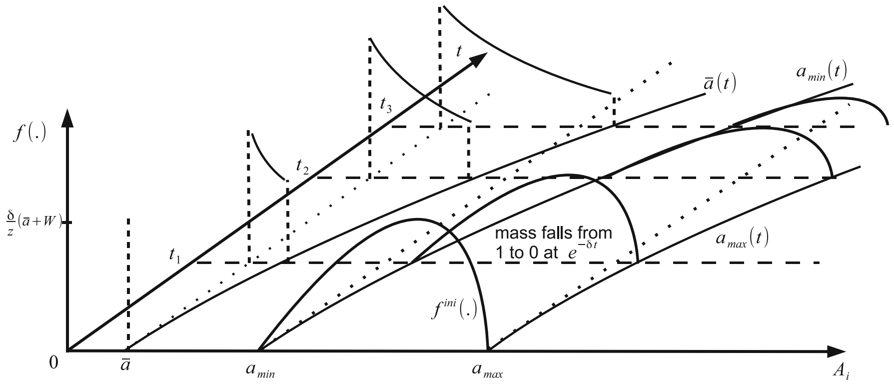

Fig. 8 Dynamics of the wealth density with an arbitrary initial wealth density (wealth path $\bar{a}(t)$ from (49)) and bounds from (54))

$$f(A_i, t) = \left\{ \begin{array}{c} e^{-\delta t} \\ \frac{\delta}{z} \frac{\bar{a}^{\frac{\delta}{z}}}{A_i^{\frac{\delta}{z}+1}} \end{array} \right\} \text{ for } A_i \left\{ \begin{array}{l} = a_i(t) \\ \in [\bar{a}, \bar{a}(t)] \end{array} \right\}. \tag{52}$$

As the mass point loses mass over time at rate δ, the truncated Pareto density gains in mass at rate δ. As we assume $a_i > \bar{a}$, the mass point at $a_i(t)$ is always to the right of the upper bound of the Pareto support. In the long run, the mass-point vanishes and the support of the Pareto density is $[\bar{a}, \infty[$.

- Illustration for an initial distribution

Now consider Fig. 8 where the initial condition is given by an initial distribution instead of a fixed parameter a_i. We could think of this initial distribution as being Pareto whose parameters follow from a certain tax policy being characterized by $\bar{a}_{\text{old}}$ and a tax rate $\tau_{a,\text{old}}$ translating into a z_{old}. We could also allow for a δ_{old}. The support would be given by $[\bar{a}_{\text{old}}, \infty[$ if this tax policy had been going on forever. This initial distribution would have a mass of 1 (see Appendix A.6.5) and reads $f(A_i(0), 0) = \kappa [\bar{a}_{\text{old}} + W]^{\kappa} / (a_i + W)^{\kappa+1}$ for $a_i \geq \bar{a}_{\text{old}}$ and with shape parameter $\kappa \equiv \delta_{\text{old}}/z_{\text{old}}$.

One could also allow for an *arbitrary* initial distribution if the economy had been subject to various shocks before changes in tax parameters. Assume the initial distribution of our random variable can be described by a density $f^{\text{ini}}(A_i)$ with support $]a_{\min}, a_{\max}[$ which could run from minus to plus infinity. Following the steps of Appendix A.6.3, the distribution of wealth can then most easily be described by its density. When $\bar{a} \leq a_{\min}$ or $\bar{a} \geq a_{\max}$, i.e. when the initial and the new (sub-) densities are non-overlapping,[37] the density reads

$$f(A_i, t) = \left\{ \begin{array}{c} e^{-(z+\delta)t} f^{\text{ini}}\left((A_i + W) e^{-zt} - W\right) \\ \frac{\delta}{z} \frac{(\bar{a}+W)^{\frac{\delta}{z}}}{(A_i+W)^{\frac{\delta}{z}+1}} \end{array} \right\} \text{ for } A_i \in \left\{ \begin{array}{c}]a_{\min}(t), a_{\max}(t)[\\ [\bar{a}, \bar{a}(t)] \end{array} \right\}, \tag{53}$$

[37] If they do overlap, the sub-densities need to be added up in the respective ranges (see also appendix A.6.3).

where we defined the lower and the upper bound as

$$a_{\min}(t) \equiv (a_{\min} + W)\, e^{zt} - W,\ a_{\max}(t) \equiv (a_{\max} + W)\, e^{zt} - W. \tag{54}$$

To understand this density most easily, assume $a_{\min} > \bar{a}$, as drawn in Fig. 8. As time goes by, the support of the initial distribution moves to the right. The bounds are $a_{\min}(t)$ and $a_{\max}(t)$ which are defined in the same spirit as (47) and (49). This points to a central property of the densities and their evolution over time. Fundamentally, wealth and bounds of wealth change over time as determined by the consumption-saving decision of households. The structure of (47), (49) and (54) are directly inherited from wealth accumulation in (8) while alive. Hence, $a_{\min}(t)$ and $a_{\max}(t)$ simply describe the evolution of wealth over time starting at $a_{\min}$ and $a_{\max}$, respectively. The initial density $f^{\text{ini}}(A_i)$ also shifts to the right at rate z which is visible by the factor e^{-zt} in its argument. The factor $e^{-(z+\delta)t} = e^{-zt}e^{-\delta t}$ in front of $f^{\text{ini}}(A_i)$ serves two purposes. First, the term e^{-zt} makes sure that the right shift with rate z does not change the overall probability mass of $f^{\text{ini}}(A_i)$ over the expanding support $]a_{\min}(t), a_{\max}(t)[$. The term $e^{-\delta t}$ makes sure that the mass of the wealth distribution is $e^{-\delta t}$, which reflects the death process.

6.3.2 The link between the distribution and the mean

The analytical analysis of distributional dynamics in our growth equilibrium provides a natural interpretation for a distribution with a non-existing mean. For our economy converging to a steady state, it is not hard to imagine that mean wealth converging to η^* (left panel in Fig. 2) goes hand in hand with a density that converges to a stable density (Figs. 7 or 8). But how does an exploding mean (right panel in Fig. 2), i.e. wealth growing at a constant rate in our growing economy, square with a density that is stationary in the long run?

The answer comes from the property that the Pareto distribution has an undefined mean, i.e. a mean of infinity, for a shape parameter below one, i.e. for $z > \delta$. Let us compute the mean over a range $[\bar{a}, a(t)]$, where in this case $a(t)$ is a short-cut for the upper bound $\bar{a}(t)$ of our (truncated) Pareto density $f^{\text{trunc}}(A_i, t) \equiv \frac{\delta}{z}(\bar{a} + W)^{\frac{\delta}{z}} / (A_i + W)^{\frac{\delta}{z}+1}$ from the lower row in (53). We then obtain (see Appendix A.6.4) $\int_{\bar{a}}^{a} A_i f^{\text{trunc}}(A_i, t)\, dA_i = \omega\bar{a}^{\omega}\left[\frac{a(t)^{1-\omega} - \bar{a}^{1-\omega}}{1-\omega}\right]$ where $\omega \equiv \delta/z$ is the shape parameter. This mean approaches infinity for $\omega < 1$ when the upper bound $a(t)$ becomes larger and larger. Hence, in the long run, our truncated Pareto density turns into an untruncated Pareto density with a non-existing (i.e. infinitely large) mean.

This can be seen in Fig. 7. For any finite t, we have a finite mean of wealth. Yet, mean wealth grows and approaches infinity as the long-run density is Pareto with $z > \delta$ and a support $[\bar{a}, \infty[$ and therefore an infinite mean. Average wealth of the economy grows without bound, yet the distribution of wealth approaches a stationary distribution with an infinite mean. A non-existing mean can therefore simply be understood as the mean of individual wealth in a growing economy.

6.3.3 A note on Fokker–Planck equations

Before concluding, we would like to point out the link to Fokker–Planck equations. Following the usual steps (see the references in the literature section or Appendix A.7), the FPE for dynasty wealth reads

$$\frac{\partial p\left(A_i,t\right)}{\partial t}=-\left(z+\delta\right)p\left(A_i,t\right)-z\left[A_i+W\right]\frac{\partial p\left(A_i,t\right)}{\partial A_i}.$$

The analytical solution and its illustration in the above figures lead to three observations. The transition from the original to the new distribution in Fig. 8 can best be understood by a transfer of probability mass from one distribution to another. The original density is characterized by a uniform loss of density at rate δ across its entire range. This simply means that individuals of each wealth level die at the same rate. It is also characterized by an exponential shift to the right driven by the growth of its lower bound.

The new density is characterized by $\frac{\partial p(A_i,t)}{\partial t}=0$ at each point in time. The new density gains probability mass by an increasing upper bound, not by rising values for any given A_i. Third, the entire density is characterized by non-differentiability (with respect to wealth) at $\bar{a}e^{zt}$ and $a_{\max}e^{zt}$.

Approximating this evolution by a numerical solution to the FPE could easily miss these points. We acknowledge that more complex models do not allow for analytical solutions and solving FPEs is the only option to understand model properties. Yet, features of the analytical structures idenepsied here are bound to be present in more complex models as well.

6.3.4 Outlook: generalizing the tax scheme and closing the economy

Let us briefly return to the issue of extending the SDE on wealth (10) as discussed in Footnote 14. The task is considerably more complex than above as can be seen from the solution of the SDE (10) in (46). Our above solution is "simple" as the initial wealth endowment after being reborn $\bar{a}$ is irrespective of the previous wealth level $A_i\left(X\left(t_{-}\right)\right)$.[38] Hence, the solution (46) displays one term only in addition to the case of no jump.

When we allow for an inheritance tax lower than 100%, initial wealth after being reborn is a function of wealth of the previous dynasty representative. The generalization of (10) leads to a generalized solution of (46) with a countable but infinite number of terms. These terms consist of multiple integrals. Understanding their property is the objective of future research.

Undertaking a general equilibrium analysis in the sense of closing the economy and endogenizing the interest rate is conceptionally straightforward. Aggregate wealth in a closed economy would be identical to the capital stock. In a sufficiently large (as measured by population size L) economy, the capital stock would follow from

[38] This differs from e.g. intentional bequest as studied, inter alia, by Citanna (2007) or Bossmann et al. (2007).

an equality between capital per capita and expected average wealth given in (28), $K(t)/L = E\bar{A}(t)$. When the latter changes, the capital stock and thereby the interest rate would change over time. While closed-form solutions for consumption in (4) can be generalized to time-varying interest rates, the linearity in (10) would get lost and thereby the analytical solutions of densities on the transitional path.[39]

Other extensions that should preserve tractability include (unanticipated) shocks to interest rates, discontinuous or branching lineages of dynasties and population growth. We need to leave this for future work.

7 Conclusion

We studied a small open economy with two factors of production, finitely lived households and a government. We describe the death-birth process of members of dynasties by a stochastic differential equation. This allows us to describe expected wealth of a dynasty and expected government wealth by ordinary differential equations.

The economy approaches either a steady state or a balanced growth path, depending on the interest rate, time preference rate, death rate and risk aversion. Especially the latter is crucial for pinning down equilibrium properties when the government employs a wealth tax to balance its budget. When the latter is achieved by a labor tax, the sign of the equilibrium growth rate does not depend on risk aversion.

Solving our SDE for dynasty wealth, we can analytically describe the transition of the wealth distribution from any initial distribution to its long-run Pareto distribution. These transitions are illustrated both for initial degenerate and for well-behaved distributions. We explain how a balanced growth path at the aggregate level is consistent with a stationary wealth distribution in the long run. The key is the mean of a Pareto distribution that approaches infinity when the shape parameter is smaller than one. This provides an economic interpretation of distributions with non-existent, i.e. infinite means.

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Declarations

Conflict of interest The authors Matthias Birkner, Niklas Scheuer and Klaus Wälde have no interests to declare.

[39] Closing the economy for a steady state analysis is of course straightforward. Expected average wealth would be constant (in a steady state) and given by η^*. The capital stock per capita would be given by η^* as well and the interest rate would be constant. The wealth distribution would be Pareto and have a finite mean.

References

Achdou, Y., Buera, F., Lasry, J., Lions, P., Moll, B.: Partial differential equation models in macroeconomics. Philos. Trans. R. Soc. A **372**(20130397), 1–19 (2014)

Achdou, Y., Han, J., Lasry, J., Lions, P., Moll, B.: Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach. Princeton University, Princeton (2020)

Aghion, P., Howitt, P.: A model of growth through creative destruction. Econometrica **60**, 323–351 (1992)

Aghion, P., Harris, C., Howitt, P., Vickers, J.: Competition, imitation and growth with step-by-step innovation. Rev. Econ. Stud. **68**(467), 492 (2001)

Ahn, S., Kaplan, G., Moll, B., Winberry, T., Wolf, C.: When Inequality Matters for Macro and Macro Matters for Inequality. CESifo Working Papers, CESifo Working Paper No. 6581 (2017)

Aoki, M.: Economic fluctuations with interactive agents: dynamic and stochastic externalities. Jpn. Econ. Rev. **46**(2), 148–165 (1995)

Aoki, S., Nirei, M.: Zipf's law, Pareto's law, and the evolution of top incomes in the United States. Am. Econ. J. Macroecon. **9**(3), 36–71 (2017)

Barro, R.J.: Are government bonds net wealth? J. Polit. Econ. **82**(6), 1095–1117 (1974)

Bayer, C., Wälde, K.: Matching and Saving in Continuous Time: Proofs. CESifo Working Paper 3026-A (2010)

Bayer, C., Wälde, K.: The Dynamics of Distributions in Continuous-Time Stochastic Models. www.waelde.com/pub (2015)

Bayer, C., Rendall, A.D., Wälde, K.: The invariant distribution of wealth and employment status in a small open economy with precautionary savings. J. Math. Econ. **85**, 17–37 (2019)

Beare, B.K., Seo, W.-K., Toda A.A.: Tail behavior of stopped Lévy processes with Markov modulation. Econom. Theory **(forthcoming)** (2022)

Beare, B.K., Toda, A.A.: Determination of Pareto exponents in economic models driven by Markov multiplicative processes. Econometrica (2022)

Benavie, A., Grinols, E., Turnovsky, S.J.: Adjustment costs and investment in a stochastic endogenous growth model. J. Monet. Econ. **38**, 77–100 (1996)

Ben-Gad, M.: Fiscal policy and indeterminacy in models of endogenous growth. J. Econ. Theory **108**, 322–344 (2003)

Benhabib, J., Bisin, A.: The Distribution of Wealth and Redistributive Policies. New York University, New York (2006)

Benhabib, J., Bisin, A., Zhu, S.: The distribution of wealth and fiscal policy in economies with finitely lived agents. Econometrica **79**(1), 123–157 (2011)

Benhabib, J., Bisin, A., Zhu, S.: The distribution of wealth in the Blanchard–Yaari model. Macroecon. Dyn. **20**(2), 466–481 (2016)

Benhabib, J., Bisin, A., Luo, M.: Wealth distribution and social mobility in the US: a quantitative approach. Am. Econ. Rev. **109**(5), 1623–47 (2019)

Bichteler, K., Lin, S.: On the stochastic Fubini theorem. Stoch. Int. J. Probab. Stoch. Process. **54**(3–4), 271–279 (1995)

Birkner, M., Scheuer, N., Wälde, K.: (2021) The Dynamics of Pareto Distributed Wealth in a Small Open Economy. CESifo Working Paper 9058

Blanchard, O.J.: Debt, deficits, and finite horizons. J. Polit. Econ. **93**, 223–47 (1985)

Bossmann, M., Kleiber, C., Wälde, K.: Bequests, taxation and the distribution of wealth in a general equilibrium model. J. Public Econ. **91**, 1247–1271 (2007)

Boucekkine, R., Prieur, F., Ruan, W., Zou, B.: Uncertainty-driven symmetry-breaking and stochastic stability in a generic differential game of lobbying. Econ. Theory (2022). https://doi.org/10.1007/s00199-021-01403-3

Brunnermeier, M.K., Sannikov, Y.: A macroeconomic model with a financial sector. Am. Econ. Rev. **104**(2), 379–421 (2014)

Cao, D., Luo, W.: Persistent heterogeneous returns and top end wealth inequality. Rev. Econ. Dyn. **26**, 301–326 (2017)
Citanna, A.: The dynamics of wealth distribution with asymmetric incentives and endogenous matching. Econ. Theory **33**(2), 243–261 (2007). https://doi.org/10.1007/s00199-006-0133-3
Cox, J.C., Ross, S.A.: The valuation of options for alternative stochastic processes. J. Financ. Econ. **3**(1–2), 145–166 (1976)
d'Albis, H.: Demographic structure and capital accumulation. J. Econ. Theory **132**(1), 411–434 (2007)
Davis, M.: Markov Models and Optimization, Volume 49 of Monographs on Statistics and Applied Probability (1993b)
Davis, M.: Markov Models and Optimization. Monographs on Statistics and Applied Probability, vol. 49. Chapman and Hill, New York (1993a)
Di Tella, S.: Uncertainty shocks and balance sheet recessions. J. Polit. Econ. **125**(6), 2038–2081 (2017)
Diamond, P.A.: Aggregate demand management in search equilibrium. J. Polit. Econ. **90**, 881–94 (1982)
Eberlein, E., Glau, K.: Variational solutions of the pricing PIDEs for European options in Lévy models. Appl. Math. Finance **21**(5), 417–450 (2014)
Gabaix, X.: Zipf's law for cities: an explanation. Q. J. Econ. **114**(3), 739–767 (1999)
Gabaix, X., Lasry, J.-M., Lions, P.-L., Moll, B., Qu, Z.: The dynamics of inequality. Econometrica **84**(6), 2071–2111 (2016)
García, M.A., Griego, R.J.: An elementary theory of stochastic differential equations driven by a Poisson process. Commun. Stat. Stoch. Models **10**(2), 335–363 (1994)
Gross, T., Klein, P.: Optimal tax policy and endogenous growth through innovation. J. Public Econ. **209**, 104645 (2022)
Grossman, G.M., Helpman, E.: Quality ladders in the theory of growth. Rev. Econ. Stud. **58**, 43–61 (1991)
Grossman, G., Helpman, E.: Outsourcing in an global economy. Rev. Econ. Stud. **72**, 135–159 (2005)
He, Z., Krishnamurthy, A.: A model of capital and crises. Rev. Econ. Stud. **79**(2), 735–777 (2011)
He, Z., Krishnamurthy, A.: Intermediary asset pricing. Am. Econ. Rev. **103**(2), 732–70 (2013)
Itskhoki, O., Moll, B.: Optimal development policies with financial frictions. Econometrica **87**(1), 139–173 (2019)
Jaimovich, N., Rebelo, S.: Nonlinear effects of taxation on growth. J. Polit. Econ. **125**(1), 265–291 (2016)
Jones, C., Kim, J.: A Schumpeterian model of top income inequality. J. Polit. Econ. **(forthcoming)** (2017)
Jones, C.I.: Simple models of pareto income and wealth inequality. Stanford, pp. 1–13 (2014)
Jones, C.I.: Growth: with or without scale effects? Am. Econ. Rev. **89**(2), 139–144 (1999)
Jones, C.I.: Pareto and Piketty: the macroeconomics of top income and wealth inequality. J. Econ. Perspect. **29**(1), 29–46 (2015)
Kallenberg, O.: Foundations of Modern Probability. Springer, New York (1997)
Kaplan, G., Moll, B., Violante, G.L.: Monetary policy according to HANK. Am. Econ. Rev. **108**, 697–743 (2018)
Kasa, K., Lei, X.: Risk, uncertainty, and the dynamics of inequality. J. Monet. Econ. **94**, 60–78 (2018)
Kawai, R.: Sensitivity analysis and density estimation for the Hobson–Rogers stochastic volatility model. Int. J. Theor. Appl. Finance **12**(03), 283–295 (2009)
Kaymak, B., Poschke, M.: The evolution of wealth inequality over half a century: the role of taxes, transfers and technology. J. Monet. Econ. **77**, 1–25 (2016)
Klette, T.J., Kortum, S.: Innovating firms and aggregate innovation. J. Polit. Econ. **112**(5), 986–1018 (2004)
Kocherlakota, N.: The New Dynamic Public Finance. Princeton University Press, Princeton (2010)
Lee, J., Wang, H.: Ranking and search effort in matching. Econ. Theory (2021). https://doi.org/10.1007/s00199-021-01391-4
Liggett, T.M.: Continuous Time Markov Processes: An Introduction, vol. 113. American Mathematical Society, Providence (2010)
Lucas, R.E., Jr.: On the mechanics of economic development. J. Monet. Econ. **22**, 3–42 (1988)
Melitz, M.J.: The impact of trade on intra-industry reallocations and aggregate industry productivity. Econometrica **71**(6), 1695–1725 (2003)
Merton, R.C.: Optimum consumption and portfolio rules in a continuous-time model. J. Econ. Theory **3**(4), 373–414 (1971)
Merton, R.C.: An asymptotic theory of growth under uncertainty. Rev. Econ. Stud. **42**(3), 375–393 (1975)
Moll, B., Rachel, L., Restrepo, P.: Uneven Growth: Automation's Impact on Income and Wealth Inequality. LSE (2021)

Mortensen, D.T.: Property rights and efficiency in mating, racing, and related games. Am. Econ. Rev. **72**, 968–79 (1982)
Nuño, G., Moll, B.: Social optima in economies with heterogeneous agents. Rev. Econ. Dyn. **28**, 150–180 (2018)
Øksendal, B.: Stochastic Differential Equations, 5th edn. Springer, Berlin (1998)
Park, H.: Ramsey fiscal policy and endogenous growth. Econ. Theory **39**, 377–398 (2009). https://doi.org/10.1007/s00199-008-0337-9
Philip, P.: Stochastic Integration and Differential Equations. Applied Mathematics, 2nd edn. Springer, New York (2004)
Piketty, T., Zucman, G.: Wealth and inheritance in the long run. In: Atkinson, A., Bourgignon, F. (eds.) Handbook of Income Distribution, vol. 2B, pp. 1304–1366. Elsevier, Amsterdam (2015)
Pissarides, C.A.: Short-run equilibrium dynamics of unemployment vacancies, and real wages. Am. Econ. Rev. **75**, 676–90 (1985)
Privault, N.: Understanding Markov Chains. Springer, New York (2018)
Protter, P.: Stochastic Integration and Differential Equations. Springer, Berlin (1995)
Reed, W.J.: The Pareto, Zipf and other power laws. Econ. Lett. **74**(1), 15–19 (2001)
Reed, W.J.: The Pareto law of incomes-an explanation and an extension. Physica A **319**, 469–486 (2003)
Saez, E., Zucman, G.: Wealth inequality in the United States since 1913: evidence from capitalized income tax data. Q. J. Econ. **131**, 519–578 (2016)
Segerstrom, P.S.: Intel economics. Int. Econ. Rev. **48**(1), 247–280 (2007)
Sennewald, K.: Controlled stochastic differential equations under Poisson uncertainty and with unbounded utility. J. Econ. Dyn. Control **31**, 1106–1131 (2007)
Sennewald, K., Wälde, K.: It's lemma and the Bellman equation for Poisson processes: an applied view. J. Econ. **89**(1), 1–36 (2006)
Smith, E., Foley, D.K., Good, B.H.: Unhedgeable shocks and statistical economic equilibrium. Econ. Theory **52**(1), 187–235 (2013). https://doi.org/10.1007/s00199-011-0663-1
Stokey, N.L.: The Economics of Inaction: Stochastic Control Models with Fixed Costs. Princeton University Press, Princeton (2008)
Toda, A.A.: The double power law in income distribution: explanations and evidence. J. Econ. Behav. Org. **84**(1), 364–381 (2012)
Toda, A.A.: Incomplete market dynamics and cross-sectional distributions. J. Econ. Theory **154**, 310–348 (2014)
Toda, A.A., Walsh, K.: The double power law in consumption and implications for testing Euler equations. J. Polit. Econ. **123**(5), 1177–1200 (2015)
Turnovsky, S.J.: International Macroeconomic Dynamics. MIT Press, Cambridge (1997)
Turnovsky, S.J.: Methods of Macroeconomic Dynamics. MIT Press, Cambridge (2000)
Wälde: Applied Intertemporal Optimization. Know Thyself - Academic Publishers (2012). www.waelde.com/KTAP
Wälde, K.: Optimal saving under Poisson uncertainty. J. Econ. Theory **87**, 194–217 (1999)
Wälde, K.: Endogenous growth cycles. Int. Econ. Rev. **46**, 867–894 (2005)
Wälde, K.: Production technologies in stochastic continuous time models. J. Econ. Dyn. Control **35**, 616–622 (2011)
Weil, P.: Overlapping families of infinitely-lived agents. J. Public Econ. **38**(2), 183–198 (1989)
Yaari, M.: Uncertain lifetime, life insurance, and the theory of the consumer. Rev. Econ. Stud. **32**(2), 137–150 (1965)
Zhu, S.: A Becker–Tomes model with investment risk. Econ. Theor. **67**(4), 951–981 (2019). https://doi.org/10.1007/s00199-018-1103-2