QUOTIENTS FOR NON-REDUCTIVE GROUP ACTIONS AND APPLICATIONS TO MODULI SPACES OF MATRIX FACTORISATIONS

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VERSICHERUNG

Ich versichere, dass ich die vorliegende Arbeit selbständig verfasst und ausschließlich die angegebenen Quellen und Hilfsmittel verwendet sowie Zitate kenntlich gemacht habe.

Mainz, den

Abstract

Classical geometric invariant theory as developed by D. Mumford in his book *Geometric Invariant Theory* [26] only applies to actions of a reductive group. G. Bérczi, B. Doran, T. Hawes, and F. Kirwan [3] construct quotients for non-reductive group actions on projective and irreducible schemes under the assumption that the unipotent radical admits a positive grading.

In the first part of this thesis, we study actions of a non-reductive group G on a separated scheme X of finite type over the base field. We formulate a definition for semi-stable and stable points with respect to a pair (K, L) of two G-linearisations and a chosen Levi-factor of G. If the line bundle K is ample, then the locus of semi-stable points admits a good quotient which contains a geometric quotient of the locus of stable points as an open subset. We give a sufficient condition such that the good quotient of the locus of semi-stable points is projective. Further, we prove a Hilbert-Mumford-style criterion to compute the set of semi-stable points. This generalises results by G. Bérczi, B. Doran, T. Hawes and F. Kirwan [3].

In the second part of this thesis, we apply the results of non-reductive geometric invariant theory to construct compactifications of moduli spaces of matrix factorisations of Shamash type. We examine two cases in particular. Let an elliptic quintic curve or a twisted quartic curve be contained in a cubic threefold which is cut out by a homogeneous form f of degree three. We obtain a matrix factorisation of f from the minimal resolution of the homogeneous coordinate ring of the curve over the homogeneous coordinate ring of the ambient projective space with a construction by J. Shamash. For both types of matrix factorisations, we give sufficient numerical conditions on the chosen linearisations (K, L) such that the good quotient of semi-stable generalised matrix factorisations by the action of the automorphism group is projective. This quotient contains a geometric quotient of the locus of stable matrix factorisations as an open subset.

ZUSAMMENFASSUNG

Die von D. Mumford in seinem Buch *Geometric Invariant Theory* [26] entwickelte geometrische Invariantentheorie ist zunächst nur für Operationen einer reduktiven Gruppe anwendbar. G. Bérczi, B. Doran, T. Hawes und F. Kirwan [3] konstruieren projektive Quotienten für Wirkungen nicht-reduktiver Gruppen auf irreduziblen und projektiven Schemata unter der Annahme, dass das unipotente Radikal der Gruppe positiv graduiert ist.

Im ersten Teil dieser Promotionsschrift untersuchen wir Operationen einer nicht-reduktiven Gruppe G auf einem separierten Schema X von endlichem Typ über dem Grundkörper. Wir formulieren einen Stabilitätsbegriff bezüglich eines Paares (K, L) aus G-Linearisierungen und der Wahl eines Levi-Faktors von G. Falls das Geradenbündel K ampel ist, so ist existiert ein guter Quotient des Ortes halbstabiler Punkte, welcher einen geometrischen Quotienten des Ortes stabiler Punkte als offene Teilmenge enthält. Wir geben eine hinreichende Bedingung, unter der der gute Quotient des Ortes halbstabiler Punkte projektiv ist. Außerdem beweisen wir ein Kriterium zur Berechnung der halbstabilen Punkte, welches als Hilbert-Mumford-Kriterium verstanden werden kann. Dies verallgemeinert Ergebnisse von G. Bérczi, B. Doran, T. Hawes und F. Kirwan [3].

Im zweiten Teil der vorliegenden Arbeit verwenden wir die erzielten Ergebnisse in der geometrischen Invariantentheorie für nicht-reduktive Gruppen, um Kompaktifizierungen von Modulräumen von Matrixfaktorisierungen des Shamash Typs zu konstruieren. Wir untersuchen insbesondere zwei Klassen von Matrixfaktorisierungen. Es liege eine elliptische Kurve fünften Grades oder eine rationale Normkurve vierten Grades auf einer kubischen Dreifaltigkeit, welche von einer homogenen Form f dritten Grades definiert wird. Aus der minimalen Auflösung des homogenen Koordinatenringes der Kurve über dem homogenen Koordinatenring des umgebenden projektiven Raumes erhalten wir mit einer Konstruktion von J. Shamash eine Matrixfaktorisierung von f. Für beide Klassen von Matrixfaktorisierungen bestimmen wir numerische Bedingungen an die jeweils gewählten Linearisierungen (K, L), unter denen ein projektiver guter Quotient verallgemeinerter Matrixfaktorisierungen für die Wirkung der Autormorphismengruppe existiert. Er enthält als offene Teilmenge einen geometrischen Quotienten des Ortes stabiler Matrixfaktorisierungen.

INTRODUCTION

The content of this thesis is twofold. In his book *Geometric Invariant Theory* [26], published in 1965, D. Mumford investigates the existence of quotients by algebraic group actions of reductive groups in the realm of algebraic geometry. Since then, his work had numerous important applications such as the construction of moduli spaces of curves of a given genus or moduli spaces of sheaves. The major part of this thesis is concerned with the action of a general algebraic group which is not necessarily reductive. Here, we build on ideas developed by Bérczi et al. in [3].

D. Eisenbud [14] introduced *matrix factorisations* in 1980. By now, they are a widely studied object in representation theory and algebraic geometry as well as in string theory. In the second part of this thesis, we apply the foregoing results about non-reductive group actions to construct compactifications of moduli spaces of matrix factorisations of Shamash type. In particular, we examine two cases: matrix factorisations associated to elliptic quintic curves and matrix factorisations associated to rational normal curves of degree four, lying on a fixed cubic three-fold respectively.

Non-reductive geometric invariant theory. Geometric invariant theory as contained in [26] does not apply if the operating group is non-reductive. There have been several attempts to deal with the action of an arbitrary affine algebraic group. But, if we want to construct moduli spaces together with possible compactifications, it is important to have an explicit description of the loci of semi-stable and stable points such that projective good respectively geometric quotients exist. J.-M. Drézet and G. Trautmann achieved this in 2003 in [12] in the case of moduli spaces of decomposable morphisms of sheaves. They reduced the problem to classical geometric invariant theory by embedding the non-reductive group into a reductive group, as well as the space acted upon into a larger variety on which the reductive group acts.

To our knowledge, the most general approach to the construction of quotients for non-reductive group actions is taken by Bérczi et al. in [3]. The reader's attention may also be drawn to [21] which rewrites most of [3].

Let G be an affine algebraic group over an algebraically closed field k of characteristic zero. Then G admits a semi-direct product decomposition

$$G = R_u(G) \rtimes R$$

where $R_u(G)$ denotes the unipotent radical of G and R is a reductive group. Let G act on a separated scheme X of finite type over k and let the action be linearised by a line bundle $L \in \operatorname{Pic} X$. Bérczi et al. identify in [3] an open set $X_0 \subseteq X$ such that X_0 is G-invariant and an $R_u(G)$-torsor with base $Y := X_0/R_u(G)$. Then they apply classical geometric invariant theory to the induced action of R on Y. If $N \in \operatorname{Pic} X$ denotes the descent of L^m to Y for some $m > 0$, then the action of R on Y is canonically linearised by N. The difficulty lies in describing the preimages

$$\varphi^{-1}(Y^{ss,R}(N)) \text{ and } \varphi^{-1}(Y^{s,R}(N))$$

which are then taken as the loci of semi-stable respectively stable points. Bérczi et al. describe those preimages after they twisted the linearisation L appropriately.

Unfortunately, we were not able to grasp every detail in [3], but we proved similar statements with different arguments. The reader may consult the notes at the end of Section 3 for a short discussion of their results. Most notably, we do not need the irreducibility assumption on X as in [3] and were able to describe the preimages above under more general assumptions than done in [21].

Let $K \in \operatorname{Pic} X$ be a second linearisation and let $L_d := K^d \otimes L$, $d > 0$. We make the following definition.

Definition. A point $x \in X$ is semi-stable with respect to the pair (K, L) if

1. there exists $f \in H^0(X, K^p)^G$ for some $p > 0$ such that $x \in X_f$, and X_f is affine and a trivial $R_u(G)$-torsor, which admits a T-equivariant section for some maximal torus T in R,
2. the orbit $R_u(G).x$ is contained in $X^{ss,R}(L_d)$ for all $d \gg 0$ sufficiently large.

A point $x \in X$ is stable with respect to (K, L) if x is semi-stable, its orbit $G.x$ is closed inside the locus of semi-stable points and the stabiliser $\operatorname{Stab}_G(x)$ is finite. The loci of semi-stable and stable points are denoted by $X^{ss,G}(K, L)$ and $X^{s,G}(K, L)$ respectively.

The notions of G-semi-stable and stable points with respect to (K, L) are well-defined by Lemma 3.10.

Theorem A. *Let K be ample. The sets of semi-stable and stable points with respect to (K, L) are open. Moreover:*

1. *There exists a good quotient*

$$\pi\colon X^{ss,G}(K, L) \to Z$$

and an ample line bundle $P \in \operatorname{Pic} Z$ such that $\pi^* P \cong (L_d)^a$ for some $d \gg 0$ and $a > 0$.

2. *There exists an open set $Z' \subseteq Z$ such that $\pi^{-1}(Z') = X^{s,G}(K, L)$ and*

$$\pi\colon X^{s,G}(K, L) \to Z'$$

is a geometric quotient.

Finally, we give a sufficient condition for the projectivity of the quotient Z and a simple description as in [3] of the locus of semi-stable points, which may be seen as a non-reductive Hilbert-Mumford-criterion.

Definition. Let $G = R_u(G) \rtimes R$ be an affine algebraic group with chosen Levi factor R. A central one-parameter subgroup $\lambda\colon \mathbb{G}_{m,k} \to Z(R)$ is called a positive grading of $R_u(G)$ if all weights of the adjoint action of λ on $\operatorname{Lie} R_u(G)$ are positive.

Definition. Let $\chi \in X(G)$. We denote by $L(\chi)$ the twisted linearisation, i.e.

$$g * y := \chi(g)g.y, \quad y \in L_x, g \in G$$

for all $x \in X$.

We prove the following generalisation of [3, Thm. 2.4, Cor. 7.10, Rmk. 7.11].

Theorem B. *Let $L \in \operatorname{Pic}^G X$ be ample and X projective. We assume that there exists a character $\chi \in X(G)$ such that $\langle \chi, \lambda \rangle$ is the maximal weight of λ on $H^0(X, L^m)$ for some $m > 0$ for which L^m is very ample and set $K := L^m(-\chi)$. If $\operatorname{Stab}_{R_u(G)}(x)$ is trivial for all $x \in X^{ss,R}(K)$, then the good quotient Z of $X^{ss,G}(K, L)$ in Theorem A is projective. Furthermore,*

$$x \in X^{ss,G}(K, L) \iff R_u(G).x \subseteq X^{ss,R}(L_d) \text{ for all } d \gg 0.$$

Moduli spaces of matrix factorisations. Matrix factorisations were originally introduced over a regular local ring in [14]. The theory is analogous for a graded ring

$$S := \bigoplus_{d \in \mathbb{N}_0} S_d$$

with $S_0 = k$ which is finitely generated over S_0. A graded matrix factorisation of $f \in S_d$ is a pair (φ, ψ) of homogeneous morphisms of degree zero

$$F \xrightarrow{\varphi} G \xrightarrow{\psi} F(d)$$

between finite free graded S-modules such that $\psi\varphi = f \cdot \mathrm{id}_F$ and $\varphi\psi = f \cdot \mathrm{id}_G$.

The group $\operatorname{Aut}(F) \times \operatorname{Aut}(G)$ acts by the rule

$$(\gamma_1, \gamma_2).(\varphi, \psi) = (\gamma_2 \varphi \gamma_1^{-1}, \gamma_1 \psi \gamma_2^{-1}) \text{ for all } (\gamma_1, \gamma_2) \in \operatorname{Aut}(F) \times \operatorname{Aut}(G)$$

on the locus of matrix factorisations of format (F, G).

To our knowledge, a construction of moduli spaces of matrix factorisations has not yet been achieved. For f of ADE type, H. Kajiura et al. construct a special Bridgeland stability condition on the homotopy category of matrix factorisations in [23]. For general cubic fourfolds containing a plane and certain $K3$ surfaces, Y. Toda constructs such a stability condition in [34] and later in [33]. The main disadvantage of these constructions is, that the authors do not work intrinsically on the homotopy category of matrix factorisations. We are not aware of a stability condition stated purely in terms of matrix factorisations in the mathematical literature. J. Walcher proposes one in [36].

Applying our results from non-reductive geometric invariant theory, we obtain invariant theoretic notions of semi-stability and stability which lead to compactifications of moduli spaces of a special class of matrix factorisations.

Let V be a finite dimensional vector space and let $S := S^{\bullet}V$ denote the symmetric algebra. We consider matrix factorisations of $f \in S^dV$ of positive degree d and assume that

$$F = S^{m_1}(-a_1) \oplus S^{m_2}(-a_2) \text{ and } G = S^{n_1}(-b_1) \oplus S^{n_2}(-b_2)$$

with $a_2 > a_1$ and $b_2 > b_1$. We write φ and ψ as block matrices

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} \text{ and } \psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}$$

with components $\varphi_{ij} \in \mathrm{Hom}(S^{m_j}(-a_j), S^{n_i}(-b_i))$ and for ψ analogously.

The following terminology draws back to a construction performed by J. Shamash in [31].

Definition. A matrix factorisation $(\varphi\colon F \to G, \psi\colon G \to F(d))$ of f is of Shamash type if $\varphi_{21} = 0$. Let $MF_f^{Sh}(F, G) \subseteq \mathrm{Hom}(F, G) \oplus \mathrm{Hom}(G, F(d))$ denote the subset of matrix factorisations of f of Shamash type.

The automorphism group of graded matrix factorisations of Shamash type can be described as follows. Let H_F denote the group

$$\left\{ \begin{pmatrix} f_1 & u \\ 0 & f_2 \end{pmatrix} \in \mathrm{Aut}(F) \;\middle|\; f_1 \in \mathrm{GL}_{m_1}, f_2 \in \mathrm{GL}_{m_2}, u\colon S^{m_2}(-a_2) \to S^{m_1}(-a_1) \right\}$$

and H_G the group

$$\left\{ \begin{pmatrix} g_1 & v \\ 0 & g_2 \end{pmatrix} \in \mathrm{Aut}(G) \;\middle|\; g_1 \in \mathrm{GL}_{n_1}, g_2 \in \mathrm{GL}_{n_2}, v\colon S^{n_2}(-b_2) \to S^{n_1}(-b_1) \right\}.$$

Then $H_F \times H_G$ acts as before on $(\varphi, \psi) \in MF_f^{Sh}(F, G)$ by the rule

$$(\gamma_1, \gamma_2).(\varphi, \psi) = (\gamma_2 \varphi \gamma_1^{-1}, \gamma_1 \psi \gamma_2^{-1}) \text{ for all } (\gamma_1, \gamma_2) \in H_F \times H_G.$$

Since the diagonal subgroup $\Delta = \{(\lambda \cdot \mathrm{id}_F, \lambda \cdot \mathrm{id}_G) \mid \lambda \in \mathbb{G}_{m,k}\}$ acts trivially, we consider the action of $\Gamma := (H_F \times H_G)/\Delta$. As a Levi-factor of Γ, we choose the reductive group

$$R := (\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \mathrm{GL}_{m_1} \times \mathrm{GL}_{m_2})/\Delta$$

embedded in Γ by setting $u = 0$ and $v = 0$.

There are canonical isomorphisms

$$\mathrm{Hom}_S(S^{m_j}(-a_j), S^{n_i}(-b_i)) \cong \mathrm{Hom}_k(k^{m_j} \otimes S^{a_j - b_i}V^*, k^{n_i})$$

and

$$\mathrm{Hom}_S(S^{n_i}(-b_i), S^{m_j}(-a_j + d)) \cong \mathrm{Hom}_k(k^{n_i} \otimes S^{b_i - a_j + d}V^*, k^{m_j})$$

induced from the restriction to the degree zero part. Therefore, the affine space

$$\{(\varphi, \psi) \in \mathrm{Hom}(F, G) \oplus \mathrm{Hom}(G, F(d)) \mid \varphi_{21} = 0\}$$

is isomorphic to the space of quiver representations $R(Q, w)$ of the weighted quiver Q

$$\begin{tikzcd}
\bullet \arrow[r, shift left, "S^{a_1-b_1}V^*"] & \bullet \arrow[l, shift left, "S^{b_1-a_1+d}V^*"] \arrow[r, shift left, "S^{b_1-a_2+d}V^*"] & \bullet \arrow[l, shift left, "S^{a_2-b_1}V^*"] \arrow[r, shift left, "S^{a_2-b_2}V^*"] & \bullet \arrow[l, shift left, "S^{b_2-a_2+d}V^*"] \arrow[lll, bend right, "S^{b_2-a_1+d}V^*"']
\end{tikzcd}$$

of dimension vector $w := (n_2, m_2, n_1, m_1)$ where we numbered the edges $0, \ldots, 3$ from right to left. We will not distinguish between (φ, ψ) as S-module homomorphisms or as quiver representations in our notation.

The next step is to projectivize in order to obtain a projective quotient by Theorem A. The boundary will contain zero factorisations, i.e. (φ, ψ) such that $\varphi\psi = 0 = \psi\varphi$.

Definition. The affine variety of generalised matrix factorisations of f of Shamash type of format (F, G) is the closed subset of $R(Q, w)$ given by

$$MF_f^{gen.Sh}(F, G) := \{(\varphi, \psi) \in R(Q, w) \mid \ \exists \lambda \in k : \varphi\psi = \lambda f \cdot \mathrm{id}_G, \psi\varphi = \lambda f \cdot \mathrm{id}_F\}$$

together with the reduced induced subscheme structure.

The quotient of $MF_f^{gen.Sh}(F, G) \setminus \{0\}$ by the action of $\mathbb{G}_{m,k}$ by scalar multiplication on $R(Q, w)$ is denoted by

$$\overline{MF}_f^{gen.Sh}(F, G) \subseteq \mathbb{P}(R(Q, w)^*).$$

The group Γ acts on $\mathbb{P}(R(Q, w)^*)$ such that $\mathcal{O}_{\mathbb{P}(R(Q,w)^*)}(1)$ is a linearisation. Let L denote the restriction of $\mathcal{O}_{\mathbb{P}(R(Q,w)^*)}(1)$ to $\overline{MF}_f^{gen.Sh}(F, G)$.

Let $\theta = (e_0, \ldots, e_3)$ be a sequence of integers such that

$$e_0 w_0 + \ldots + e_3 w_3 = e_0 n_2 + e_1 m_2 + e_2 n_1 + e_3 m_1 = 0.$$

We may associate to θ a character χ_θ of Γ by

$$\chi_\theta(f_1, f_2, g_1, g_2) = \det(f_1)^{e_3} \det(f_2)^{e_1} \det(g_1)^{e_2} \det(g_2)^{e_0}$$

for all $(f_1, f_2, g_1, g_2) \in \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \mathrm{GL}_{m_1} \times \mathrm{GL}_{m_2}$.

There is one more condition needed on θ:

$$e_0 w_0 + \ldots e_i w_i < 0 \text{ for all } i = 0, \ldots, 2 \qquad (*)$$

In particular, $3e_3 m_1 + 2e_2 n_1 + e_1 m_2$ is positive. The linearisation K is defined by

$$K := L^{3e_3 m_1 + 2e_2 n_1 + e_1 m_2}(-\chi_\theta).$$

Finally, let $L_\infty := K^b \otimes L$ for $b \gg 0$ such that the sets of R-semi-stable and stable points are independent of b if we make b larger. This is possible by Proposition 3.6.

Theorem C. *If* $\mathrm{Stab}_{R_u(\Gamma)}([\varphi,\psi])$ *is trivial for all* R*-semi-stable points*

$$[\varphi,\psi] \in \overline{MF}_f^{gen.Sh}(F,G)^{ss,R}(K),$$

then the set of factorisations $[\varphi,\psi]$ *such that the orbit* $R_u(\Gamma).[\varphi,\psi]$ *is contained in the* R*-semi-stable locus with respect to* L_∞ *is open and admits a projective good quotient* Z *for the* Γ*-action. Furthermore, the open set of* Γ*-stable factorisations* $[\varphi,\psi]$ *which are not zero-factorisations admits a geometric quotient which is open in* Z.

A. King investigates semi-stability and stability of quiver representations in [24]. Here, he introduces the notion of θ-semi-stability. We prove that a generalised matrix factorisation $[\varphi,\psi] \in \overline{MF}_f^{gen.Sh}(F,G)$ is R-semi-stable with respect to K if and only if the representation of the weighted A_4-quiver defined by

$$k^{m_1} \xrightarrow[S^{a_1-b_1}V^*]{\varphi_{11}} k^{n_1} \xrightarrow[S^{b_1-a_2+d}V^*]{\psi_{21}} k^{m_2} \xrightarrow[S^{a_2-b_2}V^*]{\varphi_{22}} k^{n_2}$$

is $(-\theta)$-semi-stable. If all e_i are assumed to be non-zero, then $(*)$ is a necessary condition for this quiver representation to have $-\theta$-semi-stable points.

We prove that Theorem C applies to matrix factorisations associated to elliptic quintic or twisted quartic curves which lie on a cubic threefold respectively. The question of how to express the invariant theoretic notions of semi-stability and stability in terms of matrix factorisations remains and the geometry of the constructed quotients is open for further investigations.

Elliptic quintics and twisted quartics on cubic threefolds. An elliptic quintic curve E in $\mathbb{P}^4$ is a smooth irreducible curve of genus one embedded by the global sections of a line bundle of degree five on E. A twisted quartic curve C is a $\mathbb{P}^1$ embedded into $\mathbb{P}^4$ by the global sections of $\mathcal{O}_{\mathbb{P}^1}(4)$. Let $X \subseteq \mathbb{P}^4$ be a cubic threefold defined by a cubic homogeneous polynomial f. For a study of the moduli space of elliptic quintics or twisted quartics lying on X see [18] for example.

Let V denote a five dimensional vector space and $S = S^\bullet V$ the homogeneous coordinate ring of $\mathbb{P}(V)$ as before.

We denote with S_E and S_C the homogeneous coordinate rings of E respectively C in $\mathbb{P}(V)$. The minimal resolution of S_E over S has the form

$$0 \longrightarrow S(-5) \longrightarrow S^5(-3) \longrightarrow S^5(-2) \longrightarrow S \longrightarrow S_E \longrightarrow 0,$$

by [15, Thm. 6.26]. If X scheme-theoretically contains E, i.e. $I_X \subseteq I_E$, then f annihilates S_E. We may therefore apply Shamash's construction 4.3 which provides us with a matrix factorisation (φ,ψ) of f of format

$$F_E = S(-8) \oplus S^5(-8) \text{ and } G_E = S^5(-6) \oplus S(-6)$$

with the property $\varphi_{21} = 0$. Let $\overline{\mathrm{MF}} := \overline{MF}_f^{gen.Sh}(F_E,G_E)^{ss,\Gamma}(K,L)$.

Theorem D. *Assume that $\theta = (e_0, e_1, e_2, e_3)$ is such that $e_1 < 0$ and $e_2 > 0$. The locus of semi-stable generalised matrix factorisations $\overline{\mathrm{MF}}^{ss,\Gamma}(K, L)$ admits a projective good quotient Z for the Γ-action which contains as an open subset a geometric quotient of the locus of Γ-stable factorisations which are not zero-factorisations. Furthermore,*

$$[\varphi, \psi] \in \overline{\mathrm{MF}}^{ss,\Gamma}(K, L) \Longleftrightarrow R_u(\Gamma).[\varphi, \psi] \subseteq \overline{\mathrm{MF}}^{ss,R}(L_\infty).$$

Similarly, starting from the minimal free resolution of S_C, we obtain matrix factorisations (φ, ψ) of f of format

$$F_C = S^3(-7) \oplus S^6(-8) \text{ and } G_C = S^8(-6) \oplus S(-6)$$

with $\varphi_{21} = 0$. Let $\overline{\mathrm{MF}} := \overline{MF}_f^{gen.Sh}(F_C, G_C)$.

Theorem E. *Assume that $\theta = (e_0, e_1, e_2, e_3)$ is such that*

$$e_1 < 0, e_2 > 0 \textit{ and } e_1 + e_2 < 0.$$

The locus of semi-stable generalised matrix factorisations $\overline{\mathrm{MF}}^{ss,\Gamma}(K, L)$ admits a projective good quotient Z for the Γ-action which contains as an open subset a geometric quotient of the locus of Γ-stable factorisations which are not zero-factorisations. Furthermore,

$$[\varphi, \psi] \in \overline{\mathrm{MF}}^{ss,\Gamma}(K, L) \Longleftrightarrow R_u(\Gamma).[\varphi, \psi] \subseteq \overline{\mathrm{MF}}^{ss,R}(L_\infty).$$

Structure of the thesis. In Section 1, we review the algebraic construction of the exponential map and the logarithm for unipotent groups. The notion of a positive grading is introduced and it is shown how its existence leads to a special composition series of the unipotent radical of a connected solvable group. In Section 2, we revisit good and geometric quotients as well as principal fibre bundles and show that every good quotient of a free action of a unipotent group is a Zariski locally trivial principal fibre bundle. We then recall the basic definitions and key results of classical geometric invariant theory. The technical heart of this thesis is Section 3. Here, we prove Theorems A and B. In Section 4, we review graded matrix factorisations and Shamash's construction. Before proving Theorems C, D and E in Section 6, we briefly remind the reader in Section 5 of the work [24] by A. King.

Additional work together with C. Böhning and H.-C. von Bothmer. I was fortunate to take part in a project by C. Böhning and H.-C. von Bothmer in the beginning of 2022. The result are three papers [5], [6] and [7]. I was not involved in the writing of [5] and [7]. My contributions are calculations for the proof of [5, Thm. 2.7] and some remark on an earlier draft of C. Böhning and H.-C. von Bothmer which lead to the final definition and study of $\widetilde{\mathcal{S}}^{ss}$ in [7]. I wrote [6] except parts of the proofs of [6, Lem. 1.1] and [6, Thm. 3.4].

CONTENTS

1. Generalities About Affine Algebraic Groups

1.1. **The exponential map and logarithm for unipotent groups.** Let G be an affine algebraic group over the base field k with multiplication $m\colon G \times G \to G$ and neutral element $e\colon \operatorname{Spec} k \to G$. If $A = k[G]$ denotes the coordinate ring, the tangent space at e is the vector space of point derivations $T_eG = \operatorname{Der}(A_{\mathfrak{m}_e}, k)$.

Example 1.1. If $G = \mathrm{GL}_{n,k}$ with coordinates X_{ij}, the tangent space at the identity is given by the span of all partial derivatives $\frac{\partial}{\partial X_{ij}}$ evaluated at the point id. The map

$$T_{\mathrm{id}} \mathrm{GL}_{n,k} \to M^n(k), \xi \mapsto (\xi(X_{ij}))_{ij}$$

is an isomorphism of vector spaces.

For general G, the Lie algebra is defined as the space of left invariant vector fields

$$\operatorname{Lie} G := \{V \in \operatorname{Der}(A, A) \mid (\mathrm{id} \otimes V) \circ m^* = m^* \circ V\}$$

together with the commutator $[V_1, V_2] = V_1 \circ V_2 - V_2 \circ V_1$. As vector spaces the tangent space is isomorphic to the Lie algebra via

$$D\colon T_eG \to \operatorname{Lie} G, \xi \mapsto D_\xi := (\mathrm{id} \otimes \xi) \circ m^*$$

with inverse $\theta(V) := e^* \circ V$. Furthermore, if $\rho\colon G \to H$ is a morphism of affine algebraic groups, the differential at e will define a Lie algebra homomorphism of the corresponding Lie algebras (see [22, Thm. III.9.1]).

Following [22], we introduce the product operation

$$\xi \cdot \eta := \xi \otimes \eta \circ m^*$$

between two k-liner maps $\xi, \eta\colon A \to k$.

Lemma 1.2. *Let $\xi_1, \ldots, \xi_r \in T_eG$.*

1. *The operation $\cdot$ is associative.*
2. $e^* \circ D_{\xi_1} \circ \ldots \circ D_{\xi_r} = \prod_{i=1}^r \xi_i$.
3. *If $G = \mathrm{GL}_{n,k}$, then $(\xi_1 \cdot \xi_2)(X_{ij}) = \sum_k \xi_1(X_{ik})\xi_2(X_{kj})$. In particular, the Lie bracket on $\operatorname{Lie} G$ is transported to the ordinary commutator of matrices under the composition of θ with the isomorphism from Example 1.1.*

Proof. The associativity uses the associativity of G as follows. Let $\xi, \eta, \nu\colon A \to k$ be linear maps, then

$$(\xi \cdot \eta) \cdot \nu = \xi \otimes \eta \otimes \nu \circ m^* \otimes \mathrm{id} \circ m^* = \xi \otimes \eta \otimes \nu \circ \mathrm{id} \otimes m^* \circ m^* = \xi \cdot (\eta \cdot \nu).$$

In the second claim the case $r = 1$ is handled by the already mentioned fact that θ is inverse to D. For $r > 1$, write $m^*(f) = \sum_i g_i \otimes h_i$ for some $f \in A$, then by induction

$$(e^* \circ D_{\xi_1} \circ \ldots \circ D_{\xi_r})(f) = \sum_j \left(\prod_{i=1}^{r-1} \xi_i\right)(g_j)\xi_r(h_j) = \left(\prod_{i=1}^{r} \xi_i \cdot \xi_r\right)(f).$$

The third claim follows readily from the identity $m^*(X_{ij}) = \sum_k X_{ik} \otimes X_{kj}$ for the coordinate functions X_{ij} on $\mathrm{GL}_{n,k}$. $\square$

We will now introduce the exponential map and logarithm for unipotent groups over a field of characteristic zero as defined in [20, Chap. VIII.1].

For any $x \in \mathrm{GL}(V)$, $\dim V < \infty$, there exist unique $x_s, x_u \in \mathrm{GL}(V)$ such that $x = x_s x_u$, x_s is semisimple, x_u is unipotent and $x_s x_u = x_u x_s$. This is the multiplicative Jordan decomposition of x. If V is infinite dimensional and a union of finite dimensional subspaces which are stable under x, one has a multiplicative decomposition $x = x_s x_u$ which restricts on every stable finite dimensional subspace to the multiplicative Jordan decomposition.

Similarly, for $x \in \mathrm{End}(V)$, $\dim V < \infty$, there exists the unique additive Jordan decomposition: $x = x_s + x_n$, i.e. x_s is semisimple, x_n is nilpotent and $x_s x_n = x_n x_s$. Of course this carries again over to an infinite dimensional vector space V which is a union of stable finite dimensional ones.

Let G act by right translation on its coordinate ring A:

$$\rho\colon G \to \mathrm{GL}(A), x \mapsto [y \mapsto (\rho_x f)(y) := f(yx)]$$

This is in fact a rational representation, i.e. A is the union of stable finite dimensional regular subrepresentations.

In the special case of $G = \mathrm{GL}_{n,k}$, one can show that $\rho_x = \rho_{x_s}\rho_{x_u}$ is the multiplicative Jordan decomposition of ρ_x.

Let $V \subseteq A$ be a finite-dimensional ρ-invariant subspace, i.e. $\rho_x(V) \subseteq V$ for all $x \in G$, and let $f_1, \dots, f_n \in V$ denote a basis. We will show that it is stable under D_ξ for all $\xi \in T_e G$. Since V is stable, we can choose $m_{ij} \in A$ such that

$$m^* f_i = \sum_j f_j \otimes m_{ji}.$$

Since $\rho_g f_i = \sum_i m_{ji}(g) f_j$, one finds that $\rho|_V$ is given with respect to the basis $\{f_i\}_i$ by

$$\rho|_V \colon G \to \mathrm{GL}_{n,k}, g \mapsto (m_{ij}(g)).$$

We conclude

$$d(\rho|_V)_e \colon T_e G \to M^n(k), \xi \mapsto (\xi(m_{ij})).$$

On the other hand, $D_\xi(f_i) = \sum_j \xi(m_{ji}) f_j$ for all $\xi \in T_e G$. Hence D_ξ stabilises V and $D_\xi|_V = d(\rho|_V)_e(\xi)$ as endomorphisms of V.

If $G = \mathrm{GL}_{n,k}$, one can prove that D_ξ has Jordan decomposition $D_{\xi_s} + D_{\xi_n}$. Together with the already mentioned fact about the multiplicative Jordan decomposition $\rho_x = \rho_{x_s}\rho_{x_u}$ one deduces the following theorem for general G.

Theorem 1.3 ([22, Thm. 15.3.]). *Let $x \in G$ and $\xi \in T_eG$.*

1. *There exist unique $x_s, x_u \in G$ such that $x = x_s x_u$, x_s and x_u commute and $\rho_x = \rho_{x_s}\rho_{x_u}$ is the multiplicative Jordan decomposition of ρ_x.*
2. *There exist unique $\xi_s, \xi_n \in T_eG$ such that $D_\xi = D_{\xi_s} + D_{\xi_n}$ is the additive Jordan decomposition of D_ξ.*
3. *If $\varphi\colon G \to G'$ is a morphism of affine algebraic groups, then $\varphi(x_s) = \varphi(x)_s$, $\varphi(x_u) = \varphi(x)_u$, $d\varphi_e(\xi_s) = d\varphi_e(\xi)_s$ and $d\varphi_e(\xi_n) = d\varphi_e(\xi)_n$.*

Definition 1.4. Let $x \in G$ and $\xi \in T_eG$.

1. The unique decomposition $x = x_s x_u$ from Theorem 1.3 is called the (mulitplicative) Jordan decomposition of x. We call x semi-simple (respectively unipotent) if $x = x_s$ (respectively $x = x_u$).
2. The unique decomposition $\xi = \xi_s + \xi_n$ from Theorem 1.3 is called the (additive) Jordan decomposition of ξ. We call ξ semisimple (respectively nilpotent) if $\xi = \xi_s$ (respectively $\xi = \xi_n$).

Definition 1.5. An affine algebraic group U is called unipotent if every element $u \in U$ is unipotent.

Lemma 1.6. *Let $x \in G$ be a unipotent element. The expression*

$$\log(\rho_x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\rho_x - \mathrm{id})^n$$

is a well-defined derivation.

Proof. We follow [20, Proof of Lem. VI.2.1]. Since x is unipotent, ρ_x is unipotent on every finite dimensional stable subspace of V. Hence the infinite sum is locally finite. Similarly $\exp(m \log \rho_x)$, $m \in \mathbb{Z}$, is a well-defined expression, and it follows from formal properties of power series, that

$$\exp(m \log \rho_x) = \rho_x^m. \tag{1}$$

Let t be a variable and extend every k-endomorphism of A to a $k[t]$-linear endomorphism of $A[t]$. Now $\exp(t \log \rho_x)$ makes sense as an endomorphism of $A[t]$. Let $f, g \in A$, then

$$\exp(t \log \rho_x)(fg) - \exp(t \log \rho_x)(f) \exp(t \log \rho_x)(g)$$

is a polynomial $p(t) \in A[t]$. But we must have $p(m) = 0$ for every $m \in \mathbb{Z}$ by Equation (1) which implies $p(t) = 0$. Considering the coefficient of t yields

$$\log(\rho_x)(fg) = \log(\rho_x)(f)g + \log(\rho_x)(g)f.$$

□

Definition 1.7. For all unipotent elements $x \in G$ let $\log(x) := e^* \circ \log(\rho_x) \in T_eG$. The logarithm is defined by

$$\log\colon \{x \in G \mid x \text{ is unipotent}\} \to T_eG, x \mapsto \log(x).$$

Lemma 1.8. *If $U \subseteq G$ is a unipotent closed subgroup, $\log(x) \in T_eU$ for all $x \in U$.*

Proof. Let A/I be the coordiante ring of U. Note that $\log(\rho_x)$ stabilises every subspace of A which is stabilised by ρ_x. Hence, if $x \in U$ we must have $\log(\rho_x)(I) \subset I$ which implies $\log(x) \in T_eU$. □

Lemma 1.9. *Let $x \in \mathrm{GL}_{n,k}$ be unipotent. We have*

$$\log(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x - \mathrm{id})^n$$

under the isomorphism $T_e \,\mathrm{GL}_{n,k} \cong M^n(k)$ from Example 1.1.

Proof. Summing up the terms

$$e^* \circ (\rho_x - \mathrm{id})^n(X_{ij}) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} X_{ij}(x^k) = (x - \mathrm{id})^n$$

with appropriate coefficients gives the claim. □

Lemma 1.10. *Let $\varphi\colon G \to H$ be a morphism of affine groups. For all unipotent elements $x \in G$ the identity $\log(\varphi(x)) = d\varphi_e(\log(x))$ holds.*

Proof. By Theorem 1.3, $\varphi(x)$ is unipotent. We note that $\varphi^* \circ \rho_{\varphi(x)} = \rho_x \circ \varphi^*$. This implies $\varphi^* \circ \log(\rho_{\varphi(x)}) = \log(\rho_x) \circ \varphi^*$. We obtain

$$e_H^* \circ \log(\rho_{\varphi(x)}) = e_G^* \circ \varphi^* \circ \log(\rho_{\varphi(x)}) = e_G^* \circ \log(\rho_x) \circ \varphi^* = d\varphi_e(e_G^* \circ \log(\rho_x))$$

which is what we wanted to show. □

Next, we define the exponential map.

Lemma 1.11. *If $\xi \in T_eG$ is a nilpotent element,*

$$\exp(D_\xi) = \sum_{n=0}^{\infty} \frac{D_\xi^n}{n!} \in \mathrm{End}_k(A)$$

is a well-defined ring homomorphism.

Proof. Since ξ is nilpotent the infinite sum is locally finite. Let $f, g \in A$, then

$$\begin{aligned}\exp(D_\xi)(fg) &= \sum_{n=0}^{\infty} \frac{1}{n!} D_\xi^n(fg) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (D_\xi^{n-k} f)(D_\xi^k g) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!} (D_\xi^{n-k} f) \frac{1}{k!} (D_\xi^k g) \\ &= \exp(D_\xi)(f) \cdot \exp(D_\xi)(g)\end{aligned}$$

which shows that $\exp(D_\xi)$ is multiplicative. Since it is obviously additive and maps 1 to 1, it is a ring homomorphism. $\square$

Definition 1.12. For any nilpotent element $\xi \in T_e G$, let $\exp(\xi)$ denote the closed point in G defined by $e^* \circ \exp(D_\xi) \colon A \to k$. The map

$$\exp\colon \{\xi \in T_e G \mid D_\xi \text{ is nilpotent}\} \to G, \xi \mapsto \exp(\xi)$$

is called the exponential map.

Lemma 1.13. *If $U \subseteq G$ is a unipotent closed subgroup, then $\exp(\xi) \in U$ for all $\xi \in T_e U$.*

Proof. If $V \subseteq A$ is a finite dimensional ρ invariant subspace, $d(\rho|_V)_e(\xi) = D_\xi|_V$ for all $\xi \in T_e U \subseteq T_e G$. Since U is unipotent, $D_\xi|_V$ is nilpotent, whence ξ is nilpotent and $\exp(\xi)$ is defined. Furthermore, if A/I denotes the coordinate ring of U, then $m^*(I) \subseteq A \otimes I + I \otimes A$. This implies $D_\xi(I) \subset I$. Therefore $\exp(D_\xi)$ descends to a ring homomorphism in $\operatorname{End}(A/I)$. Hence, $e^* \circ \exp(D_\xi)$ defines a point in U. $\square$

Lemma 1.14. *If $U \subseteq \mathrm{GL}_{n,k}$ is a unipotent closed subgroup and $\xi \in T_e U$, then*

$$\exp(\xi) = \sum_{k=0}^{\infty} \frac{1}{k!} (\xi(X_{ij}))_{ij}^k,$$

i.e. the exponential map is given by the exponential series under the identification $T_e \mathrm{GL}_{n,k} \cong M^n(k)$ from Example 1.1.

Proof. Follows from Lemma 1.2. $\square$

Lemma 1.15. *Let $\varphi\colon G \to H$ be a morphism of affine groups and $\xi \in \operatorname{Lie} G$ a nilpotent element. Then the following identity holds*

$$\varphi(\exp(\xi)) = \exp(d\varphi_e(\xi)).$$

Proof. By Theorem 1.3 $d\varphi_e(\xi)$ is nilpotent whenever $\xi \in T_eG$ is. Hence, the right hand side is defined. Let e_G respectively e_H denote the neutral elements. We have to prove

$$e_G^* \circ \exp(D_\xi) \circ \varphi^* = e_H^* \circ \exp(D_{\xi\circ\varphi^*}).$$

First we observe

$$\begin{aligned} \varphi^* \circ D_{\xi\circ\varphi^*} &= \varphi^* \circ \mathrm{id}_{k[H]} \otimes (\xi \circ \varphi^*) \circ m_H^* \\ &= \mathrm{id}_{k[G]} \otimes \xi \circ \varphi^* \otimes \varphi^* \circ m_H \\ &= \mathrm{id}_{k[G]} \otimes \xi \circ m_G^* \circ \varphi^* \\ &= D_\xi \circ \varphi^* \end{aligned}$$

which implies $\varphi^* \circ \exp(D_{\xi\circ\varphi^*}) = \exp(D_\xi) \circ \varphi^*$. The result follows by composing with e_G^* from the left. □

Theorem 1.16 ([20, Thm. VIII.1.1.]). *For any closed unipotent subgroup $U \subseteq G$, the exponential map $\exp\colon T_eU \to U$ and the logarithm map $\log\colon U \to T_eU$ are mutually inverse morphisms of varieties. They induce a bijection between the set of all closed unipotent subgroups $U \subseteq G$ and the set of all Lie subalgebras $\mathfrak{n} \subseteq T_eG$ consisting only of nilpotent elements.*

Proof. We fix a faithful representation for G. Then the first part of the theorem follows from the formal properties of the exponential and logarithm series by Lemmas 1.9 and 1.14. We are left to show, that $\exp(\mathfrak{n})$ is a unipotent closed subgroup of G for any Lie subalgebra $\mathfrak{n} \subseteq T_eG$ consisting solely of nilpotent elements.

Let $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathrm{Lie}\, G)$ denote the adjoint representation of G. Then its differential at e is the adjoint representation $\mathrm{ad}\colon \mathrm{Lie}\, G \to \mathrm{End}(\mathrm{Lie}\, G)$ of $\mathrm{Lie}\, G$. We conclude from Lemma 1.15

$$\exp(\xi)\nu \exp(\xi)^{-1} = \exp(\mathrm{ad}\,\xi)(\nu)$$

for all nilpotent elements $\xi \in T_eG$ and all $\nu \in T_eG$. If ν is nilpotent, we may apply the expontial map:

$$\exp(\xi)\exp(\nu)\exp(\xi)^{-1} = \exp(\exp(\mathrm{ad}\,\xi)(\nu)) \tag{2}$$

We will use induction on $\dim \mathfrak{n}$ to show that $\exp(\mathfrak{n})$ is a closed unipotent subgroup of G. If $\mathfrak{n} = k\xi$, $\exp\colon \mathfrak{n} \to G$ is in fact a morphism of algebraic groups, whence the image is closed. Since ξ is nilpotent, $\exp(\mathfrak{n})$ is unipotent.

Now suppose that $\dim \mathfrak{n} > 1$. Since $\mathfrak{n}$ is a nilpotent Lie algebra, there exists a decomposition $\mathfrak{n} = \mathfrak{m} + k\xi$, where $\mathfrak{m}$ is an ideal which does not contain ξ. By induction and what we already proved $\exp(\mathfrak{m})$ and $\exp(k\xi)$ are unipotent closed subgroups of G. Furthermore, $\exp(k\xi)$ normalizes $\exp(\mathfrak{m})$ by Equation (2). This implies that $\exp(\mathfrak{m})\exp(k\xi)$ is a closed subgroup of G. By [20, Prop. V.2.2] it is a unipotent group, call it N.

We must have $\mathfrak{n} = \mathfrak{m} + k\xi \subseteq T_eN$. On the other hand, since N is the image under the multiplication map $\exp(\mathfrak{m}) \times \exp(k\xi) \to N$, $\dim N \leq \dim \mathfrak{n}$. This implies first $T_eN = \mathfrak{n}$ and then $N = \exp(T_eN) = \exp(\mathfrak{n})$. $\square$

1.2. **Infinitesimal Actions.** Let $\sigma\colon G \times X \to X$ be a left action of $G = \operatorname{Spec} A$ on some affine scheme $X = \operatorname{Spec} R$ of finite type over k. We denote with $\sigma^*\colon R \to A \otimes R$ the dual action.

Definition 1.17. The map

$$\delta\colon T_eG \to \operatorname{Der}(R, R), \xi \mapsto \delta_\xi := (\xi \otimes \mathrm{id}) \circ \sigma^*$$

is called the infinitesimal action of σ.

We will denote by

$$\lambda\colon G \to \operatorname{GL}(R), x \mapsto [y \mapsto (\lambda_x f)(y) = f(x^{-1}y)]$$

the left action by left translation of G on R. This is in fact a rational representation.

Lemma 1.18. *If $V \subseteq R$ is a finite dimensional λ-stable subspace and $\xi \in T_eG$, δ_ξ stabilises V and $\delta_\xi|_V = -d(\lambda|_V)_e(\xi)$.*

Proof. The proof is parallel to the computation of the differential of the action ρ by right translation on A done in section 1.1. We skip it. $\square$

Lemma 1.19. *Let $\sigma\colon U \times X \to X$ be an action of an affine unipotent group U on X.*

1. *A function $f \in R$ is U-invariant if and only if $\delta_\xi(f) = 0$ for alle $\xi \in T_eU$.*
2. *A subspace $V \subseteq R$ is invariant if and only if $\delta_\xi(V) \subseteq V$ for all $\xi \in T_eU$.*

Proof. Recall that $f \in R$ is invariant iff it is λ-invariant iff $\sigma^*(f) = 1 \otimes f$ and that $V \subseteq R$ is an invariant subspace iff it is λ-stable iff $\sigma^*(V) \subseteq A \otimes V$. Hence, by definition of δ_ξ, $\delta_\xi(f) = 0$ and $\delta_\xi(V) \subseteq V$ for all $\xi \in T_eU$ and invariant f and V.

Note that Lemmas 1.18 and 1.15 imply

$$\lambda(\exp(\xi)) = \sum_{k=0}^{\infty} \frac{(-\delta_\xi)^k}{k!} \tag{3}$$

for all $\xi \in T_eU$. The converses follow from the surjectivity of the exponential map, see Theorem 1.16. $\square$

Definition 1.20. We call any map $\delta\colon T_eG \to \operatorname{Der}(R, R)$ such that $-\delta$ is a Lie algebra homomorphism, an infinitesimal action. An infinitesimal action δ is called integrable if there exists an action $\sigma\colon G \times X \to X$ such that δ is the infinitesimal action of σ.

Lemma 1.21. *Let U be an affine unipotent group. Mapping an action $\sigma\colon U \times X \to X$ to its infinitesimal action gives a one to one correspondence*

$$\{\textit{affine group actions } \sigma\} \leftrightarrow \{\textit{integrable infinitesimal actions } \delta\}.$$

Proof. Any action σ is determined by the induced action λ on R. But λ on the other hand is by Equation 3 determined by δ. $\square$

For later use, we finish this section with a remark about the orbits of unipotent group actions.

Lemma 1.22 ([22, Exercise 17.6.8])**.** *Let $\sigma\colon U \times X \to X$ be an action of an affine unipotent group on the affine scheme X. Then every orbit is closed.*

Proof. Let $O \subseteq X$ be an orbit and $\overline{O}$ its closure. Without loss of generality we may assume $X = \overline{O}$. Then O is open in X. If we assume that O is not closed, we can choose a non-vanishing function $f \in R$ such that $f|_{X \setminus O} = 0$. Since λ is a rational representation, the subspace V spanned by all translates $\lambda_x f$, $x \in U$, is finite dimensional and obviously λ-stable. Every function $g \in V$ vanishes on the complement of O. Every regular representation of a unipotent group has a non zero fixed point. Let $g \in V$ be one. But a fix point of λ has to be a constant function since O is dense in X. Because $g \in V$, we must have $g = 0$, which gives the contradiction. $\square$

1.3. **Solvable Groups.** Recall that the affine algebraic group G is called solvable if its derived series $(\mathscr{D}^i G)_{i \in \mathbb{N}_0}$ terminates in $\{1\}$ where $\mathscr{D}^0 G := G$ and

$$\mathscr{D}^{i+1} G := (\mathscr{D}^i G, \mathscr{D}^i G)$$

is a commutator subgroup.

Let $T(n,k) \subseteq \mathrm{GL}_n(k)$ denote the solvable subgroup of all upper triangular matrices, $U(n,k) \subseteq T(n,k)$ the unipotent subgroup of all matrices with ones on the diagonal and let $D(n,k) \subseteq T(n,k)$ be the subgroup of all diagonal matrices.

Let G be a connected solvable affine algebraic group. By Lie's and Kolchin's Theorem G can be embedded in $T(n,k)$ as a closed subgroup for some $n \in \mathbb{N}$.

By intersecting the exact sequence

$$1 \longrightarrow U(n,k) \longrightarrow T(n,k) \longrightarrow D(n,k) \longrightarrow 1$$

with G, we obtain an exact sequence

$$1 \longrightarrow R_u(G) \longrightarrow G \longrightarrow T \longrightarrow 1$$

where we denoted with $R_u(G)$ the set of all unipotent elements in G. Since it is the restriction of G to $U(n,k)$ it has to be a closed normal subgroup of G. Furthermore, T has to be a torus since it is a closed connected subgroup of $D(n,k)$.

To show that this sequence splits and that G decomposes as $R_u(G) \rtimes T$, it is enough to find a torus $T' \subseteq G$ of the same dimension as T. It follows from $R_u(G) \cap T' = \{1\}$ that the multiplication morphism $R_u(G) \rtimes T' \to G$ is bijective which in turn implies in characteristic zero that it is in fact an isomorphism of algebraic groups. Further, T' must map onto T.

Theorem 1.23 ([22, Thm. 19.3]). *Let G be a connected solvable affine algebraic group. Then all maximal tori in G are conjugate to each other. If T is any such torus, then G decomposes as a semi-direct product $G = R_u(G) \rtimes T$.*

Let $G = R_u(G) \rtimes T \subseteq \mathrm{GL}(V)$ be a connected solvable subgroup with chosen maximal torus T. Since $R_u(G)$ is normal in G, the torus T operates on $\mathrm{Lie}\, U$ via the adjoint action $\mathrm{Ad}\colon G \to \mathrm{Lie}\, G$.

Proposition 1.24. *There exists a non-zero T-weight vector $v \in V$ which is fixed by all $u \in R_u(G)$.*

Proof. By what we have already seen about the exponential map, it is enough to find a non-zero T-weight vector $v \in V$ such that $\mathrm{Lie}(R_u(G)).v = 0$. We can choose a basis $\{D_i\}_i$ of $\mathrm{Lie}\, R_u(G)$ which consists of T-weight vectors with respect to the adjoint action. Since $R_u(G)$ is a unipotent subgroup of $\mathrm{GL}(V)$, we must have

$$\bigcap_i \mathrm{Ker}\, D_i \neq 0.$$

by Lie-Kolchin. We note that for $t \in T$ acting on V

$$t.(D_i v) = \mathrm{Ad}(t)(D_i)(t.v) = \chi_i(t) D_i(t.v) \tag{4}$$

if χ_i is the T-weight of D_i. Hence, T acts on every $\mathrm{Ker}\, D_i$ and we can choose any non-zero T-weight vector from $\bigcap_i \mathrm{Ker}\, D_i$. □

Let $v \in V$ be a non zero T-weight vector which is a common eigenvector for $R_u(G)$ and let $W = V/kv$. This is again a representation of G. Let H be its image in $\mathrm{GL}(W)$. Then $R_u(H)$ will be the image of $R_u(G)$ and the image of T will be a maximal torus in H. Furthermore, we observe that one can always choose T-weight vectors as representatives of a basis of a generalized T-eigenspace in W. Hence, we conclude by induction:

Corollary 1.25. *V has a basis consisting of T-weight vectors such that $G \subseteq T(n,k)$ with respect to this basis. In particular, $T = G \cap D(n,k)$.*

One can say a little bit more if $R_u(G) = \mathbb{G}_{a,k}$.

Notation 1.26. We will denote the character group of an affine algebraic group G

$$\{\chi\colon G \to \mathbb{G}_{m,k} \mid \chi \text{ morphism of algebraic groups } \}$$

by $X(G)$. If $G = T$ is a torus of rank m, we have $X(T) \cong \mathbb{Z}^m$ and we will write $X(T)$ as an additive group. Further, if $\chi \in X(G)$ and $\lambda\colon \mathbb{G}_{m,k} \to G$ is a one-parameter subgroup, we define $\langle \chi, \lambda \rangle$ by $(\chi \circ \lambda)(t) = t^{\langle \chi, \lambda \rangle}$.

Let $G = \mathbb{G}_{a,k} \rtimes T$ be a semi-direct product of the additive group with some torus and let $\sigma\colon G \to \mathrm{GL}_{n,k}$ denote a representation. If $A := d\sigma_e(\log(1))$, $1 \in \mathbb{G}_{a,k}(k)$, then

$$\sigma(x) = \sigma(\exp(x \log(1))) = \exp(xA)$$

for all $x \in \mathbb{G}_{a,k}(k)$. Furthermore, the matrix A is nilpotent. Hence, with respect to a suitable basis, it is given as a block diagonal matrix $\mathrm{diag}(J(n_1), \ldots, J(n_r))$ with Jordan blocks $J(n_i)$ of size n_i with eigenvalue zero.

Assume that T acts with weight $\chi \in X(T)$ on $\mathrm{Lie}\,\mathbb{G}_{a,k}$. Let $V_1 = k^2$ denote the faithful representation

$$\sigma_1\colon G \to \mathrm{GL}_{2,k}, (x;t) \mapsto \begin{pmatrix} \chi(t) & x \\ 0 & 1 \end{pmatrix}$$

and $V_n := S^n V_1$. Then one checks, that the representation V_n restricted to $\mathbb{G}_{a,k}$ is isomorphic to the action of $\mathbb{G}_{a,k}$ on k^{n+1} given by $\exp(xJ(n+1))$, $x \in \mathbb{G}_{a,k}$.

The following Lemma occurs in [2] without a proof.

Lemma 1.27. *Let V be a representation of $G = \mathbb{G}_{a,k} \rtimes T$. Then there are characters $\mu_i \in X(T)$ such that*

$$V \cong \bigoplus_{i=1}^{r} k^{(\mu_i)} \otimes V_{n_i}$$

equivariantly, where $k^{(\mu_i)}$ denotes the one dimensional representation of G given by the character $\mu_i \in X(T)$.

Proof. We have to prove that there exist a basis for V which consists of T-weight vectors and brings A into Jordan Normal Form. Let $d \in \mathbb{N}$ be minimal with the property that $A^d = 0$ and let $K_l = \mathrm{Ker}\, A^l$, $l = 0, \ldots, d$. Note, that if $W \subseteq V$ is a T-invariant subspace, the image AW is invariant as well. This follows because A has to be a T-weight vector for the adjoint action of T on $\mathrm{Lie}\,\mathbb{G}_{a,k}$ and by Calculation (4). Now we choose inductively (starting with d) for every $m = 1, \ldots, d$ a T-invariant complement W_m in K_m to

$$K_{m-1} \oplus A(W_{m-1}) \oplus \ldots \oplus A^{d-m}(W_d).$$

If $\{v_j^{(m)}\}_j$ denotes a basis of T-weight vectors for every W_m, the basis vectors

$$A^{m-1}v_j^{(m)}, A^{m-2}v_j^{(m)}, \ldots, v_j^{(m)}$$

correspond to a Jordan block of size m. If μ is the T-weight of $v_j^{(m)}$, $A^l v_j^{(m)}$ has weight $\mu + l \cdot \chi$. The claim follows. $\square$

1.4. **Positive Gradings.** Let G be an affine algebraic group. Recall that the radical $R(G)$ of G is the identity component of the (unique) largest normal solvable subgroup. The closed subgroup $R_u(G) := R_u(R(G))$ is called the unipotent radical of G. Furthermore, R is called reductive, if $R_u(G) = \{e\}$.

Theorem 1.28 ([20, Thm. VIII.4.3.])**.** *There exists a reductive subgroup $R \subseteq G$ such that $G = R_u(G) \rtimes R$. If M is any reductive subgroup of G, there exists $x \in R_u(G)$ such that $xMx^{-1} \subset R$.*

Any reductive subgroup $R \subseteq G$ such that $G = R_u(G) \rtimes R$ is called a Levi-factor of G. Theorem 1.28 does in general not remain true without the assumption $\mathrm{char}(k) = 0$.

Definition 1.29 ([3])**.** Let $G = R_u(G) \rtimes R$ be an affine algebraic group with chosen Levi-factor R. A central one-parameter subgroup $\lambda\colon \mathbb{G}_{m,k} \to Z(R)$ is called a positive grading of $R_u(G)$ if all weights of the adjoint action of λ on $\mathrm{Lie}\, R_u(G)$ are positive.

Proposition 1.30. *Let $G = R_u(G) \rtimes T$ be a connected solvable group with fixed maximal torus T and a positive grading $\lambda\colon \mathbb{G}_{m,k} \to T$. There exists a composition series*

$$\{e\} = U_0 \lhd U_1 \lhd \ldots \lhd U_d = R_u(G)$$

with the following properties:

1. *The filtration is stable under the conjugation action of T on $R_u(G)$.*
2. *$U_i/U_{i-1} \cong \mathbb{G}_{a,k}$ for all $i = 1, \ldots, d$.*
3. *The projection $U_i \to U_i/U_{i-1}$ splits T-equivariantly for all $i = 1, \ldots, d$. In particular, the weight of λ on $\mathrm{Lie}\, U_i/U_{i-1}$ is positive.*

Proof. Let $\omega_1 > \omega_2 > \ldots > \omega_k > 0$ be the weights of λ on $\mathrm{Lie}\, R_u(G)$. We set

$$\mathfrak{n}'_l := \bigoplus_{i=1}^{l} (\mathrm{Lie}\, R_u(G))_{\omega_i}.$$

Since T acts on every $\mathfrak{n}'_l$, we can refine the filtration $\{\mathfrak{n}'_l\}$ to a T-stable filtration

$$0 = \mathfrak{n}_0 \subseteq \mathfrak{n}_1 \subseteq \ldots \subseteq \mathfrak{n}_d = \mathrm{Lie}\, R_u(G)$$

such that $\dim \mathfrak{n}_i = i$ for all $i = 0 \ldots, d$.

Note that we have for all weight vectors $\xi, \nu \in \mathrm{Lie}\, R_u(G)$ of weights $\chi \in X(T)$ respectively $\mu \in X(T)$, that $[\xi, \nu] \in (\mathrm{Lie}\, R_u(G))_{\chi+\mu}$. In particular, $[\mathfrak{n}_i, \mathfrak{n}_j] \subseteq \mathfrak{n}_{i-1}$ for $i \leq j$. Hence, $\mathfrak{n}_{i-1}$ is an ideal in $\mathfrak{n}_i$. The quotient $\mathfrak{n}_i/\mathfrak{n}_{i-1}$ is one-dimensional and T-stable. Therefore we have a semi-direct product decomposition of $\mathfrak{n}_i$

$$\mathfrak{n}_i = \mathfrak{n}_{i-1} \oplus \mathfrak{n}_i/\mathfrak{n}_{i-1}$$

which is T-equivariant. We apply the exponential map and obtain by Theorem 1.16 a T-stable filtration

$$\{e\} = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_d = R_u(G).$$

such that every inclusion $U_{i-1} \subseteq U_i$ is normal, $U_i \to U_i/U_{i-1}$ splits, and U_i/U_{i-1} is isomorphic to $\mathbb{G}_{a,k}$. $\square$

2. Generalities about Algebraic Quotients

2.1. **Notions of algebraic quotients.** Let us recall different notions of quotients for a given action $\sigma\colon G \times X \to X$ of an affine algebraic group G on a scheme X. The main reference is [26].

Definition 2.1. A tupel (Y, φ) consisting of a scheme Y and a morphism $\varphi\colon X \to Y$ is called a geometric quotient for the action σ if

1. φ is affine and invariant, i.e. the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d, "\varphi"] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

   commutes.
2. φ is surjective and the fibres are exactly the orbits of σ.
3. φ is submersive, i.e. $U \subseteq Y$ is open if and only if $\varphi^{-1}(U) \subseteq X$ is open.
4. $\varphi^\sharp\colon \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ induces an isomorphism $\mathcal{O}_Y \cong (\varphi_*\mathcal{O}_X)^G$.

Note that if (Y, φ) is a geometric quotient, (Y, φ) equals the quotient space in the category of locally ringed spaces.

Definition 2.2. A tupel (Y, φ) as in Definition 2.1 is called a good quotient if

1. φ is affine and invariant.
2. $\varphi^\sharp\colon \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ induces an isomorphism $\mathcal{O}_Y \cong (\varphi_*\mathcal{O}_X)^G$.
3. If $W \subseteq X$ is an invariant closed subset, $\varphi(W)$ is closed and for any family $\{W_i\}_i$ of invariant closed subsets

$$\varphi\left(\bigcap_{i\in I} W_i\right) = \bigcap_{i \in I} \varphi(W_i).$$

We observe that every geometric quotient is a good quotient, but the converse is not true in general.

Lemma 2.3. *Every good quotient (Y, φ) is submersive. In particular, if (Y, φ) is a good quotient and all orbits are closed, it is a geometric quotient.*

Proof. If (Y, φ) is a good quotient, φ is dominant by Property 2 in Definition 2.2 and hence set-theoretically surjective by Property 3 applied to $W = X$. Let $Z \subseteq Y$ be any subset such that its preimage is closed. Since $\varphi^{-1}(Z)$ is invariant, $\varphi(\varphi^{-1}(Z)) = Z$ is closed by property 4. Hence, φ is submersive. □

If a geometric quotient exists, and if X and Y are of finite type over k and connected, all orbits must have the same dimension. This holds because under those assumptions $\dim G = \dim G.x + \dim_x \varphi^{-1}(\varphi(x))$ for all $x \in X$ and both terms on the right are upper semi-continuous in x.

Definition 2.4. A tupel (Y, φ) consisting of a scheme Y and a morphism $\varphi\colon X \to Y$ is called a categorical quotient for the action σ if φ is invariant and for any other scheme Z together with an invariant morphism $\varphi'\colon X \to Z$ there exists a unique morphism $\psi\colon Y \to Z$ such that $\varphi' = \psi \circ \varphi$.

By [26, §2, Rmk. 6] every good quotient (and henceforth every geometric quotient) is a categorical quotient. In particular, geometric respectively good quotients are unique if they exist.

Lemma 2.5. *Let G be a unipotent group acting on X. If there exists a good quotient $\varphi\colon X \to Y$, and if X and Y are of finite type over k, then φ is a geometric quotient.*

Proof. Let (Y, φ) denote a good quotient for σ. Being a good quotient is local on the target. Hence, without loss of generality we may assume that X and Y are both affine. By Lemma 1.22 all orbits of G on X are closed. So (Y, φ) is a geometric quotient. □

Definition 2.6. A scheme X together with a left (respectively right) G-action and an invariant morphism $\varphi\colon X \to Y$ is called a left (respectively right) G-fibration. If $\varphi'\colon X' \to Y$ is another G-fibration, a morphism of G-fibrations is a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "\varphi'"] & X \arrow[d, "\varphi"] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

where g is G-equivariant.

Example 2.7. Let G act by left (respectively right) multiplication on the first factor of $G \times Y$ for some scheme Y. Then $\mathrm{pr}_2\colon G \times Y \to Y$ is referred to as the trivial left (respectively right) G-fibration over Y.

Example 2.8. Let $\varphi\colon X \to Y$ be a G-fibration and $f\colon Y' \to Y$ a morphism. The pull-back $f^*(\varphi)\colon Y' \times_Y X \to Y'$ carries a natural structure of a G-fibration.

Definition 2.9. A left (respectively right) G-fibration $\varphi\colon X \to Y$ is called left (respectively right) G-torsor if there exists an fpqc morphism $f\colon Y' \to Y$ and a morphism of G-fibrations

$$\begin{tikzcd} G \times Y' \arrow[r, "f'"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d, "\varphi"] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

such that the induced morphism $G \times Y' \to Y' \times_Y X$ is an isomorphism of fibrations.

Example 2.10. Let G act on a scheme F from the left, $\sigma\colon G \times F \to F$, and on itself by right multiplication. Then the induced right action $(g', f).g = (g'g, g^{-1}.f)$ makes σ into a right G-torsor. In fact, $\varphi\colon G \times F \to G \times F, (g, f) \mapsto (g, g^{-1}f)$ gives an isomorphism of right G-fibrations:

$$\begin{tikzcd} G \times F \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}_2"'] & & G \times F \arrow[dl, "\sigma"] \\ & F & \end{tikzcd}$$

Lemma 2.11. *A G-fibration $\varphi\colon X \to Y$ is a G-torsor if and only if*

1. *φ is faithfully flat and of finite type.*
2. *The morphism*

$$\Psi = \mathrm{pr}_2 \times \sigma\colon G \times X \to X \times_Y X, (x, g) \mapsto (x, g.x)$$

is an isomorphism, i.e. the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d, "\varphi"] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

is cartesian.

Proof. Let $\varphi\colon X \to Y$ be a G-torsor. Then there exists an fpqc morphism $f\colon Y' \to Y$ such that $f^*(\varphi)$ is trivial. Since G is faithfully flat and of finite type, φ is faithfully flat and of finite type by faithfully flat descend. Furthermore, the pull-back of Ψ with f will be an isomorphism, so Ψ is one too.

Conversely, if the first condition is satisfied, one can take $\varphi\colon X \to Y$ as an fpqc covering which trivializes φ because of the second condition. $\square$

Remark 2.12. Assume that $\varphi\colon X \to Y$ is a G-torsor. If $f\colon Y' \to Y$ is any morphism, the pull-back $f^*(\varphi)\colon Y' \times_Y X \to Y'$ is isomorphic to the trivial fibration if and only if there exists a section $s\colon Y' \to X$, i.e. $\varphi \circ s = f$:

$$\begin{tikzcd} & X \arrow[d, "\varphi"] \\ Y' \arrow[ur, "s"] \arrow[r, "f"] & Y \end{tikzcd}$$

Further, the section s induces an isomorphism of G-fibrations

$$\Psi\colon Y' \times G \to Y' \times_Y X, (y, g) \mapsto (y, g.s(y)).$$

2.2. **Compositions of torsors.** Let G be an affine algebraic group with a normal closed subgroup $N \triangleleft G$ and $\varphi_1 \colon X \to Y$ an N-torsor such that the action of N on X extends to an action of G. We have an induced action of G/N on Y. Let us assume that Y is a G/N-torsor $\varphi_2 \colon Y \to Z$.

$$\begin{tikzcd} X \arrow[rr, "\varphi_2 \circ \varphi_1"] \arrow[dr, "\varphi_1"'] & & Z \\ & Y \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

Proposition 2.13 ([3, Lem. 1.20]). *If there exists a closed subgroup H such that G decomposes as an inner semi-direct product $G = NH$, the composition $\varphi := \varphi_2 \circ \varphi_1$ is a G-torsor. If φ_1 and φ_2 are trivial, so is φ.*

Proof. First we reduce to the case where φ_1 is a trivial N-torsor and φ_2 a trivial H-torsor. For this we consider the following diagram where every square is cartesian:

$$\begin{tikzcd} X \times_Z X \arrow[r] \arrow[d] & Y \times_Z X \arrow[r] \arrow[d] & X \arrow[d, "\varphi_1"] \\ X \times_Z Y \arrow[r] \arrow[d] & Y \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "\varphi_2"] \\ X \arrow[r, "\varphi_1"] & Y \arrow[r, "\varphi_2"] & Z \end{tikzcd}$$

The pull-back of φ_2 with φ_2 is a trivial H-torsor $\mathrm{pr}_1 \colon Y \times_Z Y \to Y$. Hence, the H-torsor $\mathrm{pr}_1 \colon X \times_Z Y \to X$ on the lower left is trivial as well. We claim that the N-torsor $X \times_Z X \to X \times_Z Y$ on the upper left is actually isomorphic as N-fibration to the pullback $\mathrm{pr}_{13} \colon (X \times_Y X) \times_Z X \to X \times_Z Y$ of the trivial N-torsor $\mathrm{pr}_1 \colon X \times_Y X \to X$ and therefore trivial as well. For this we construct a morphism Φ of N-fibrations:

$$\begin{tikzcd} (X \times_Y X) \times_Z Y \arrow[rr, "\Phi"] \arrow[dr, "\mathrm{pr}_{13}"'] & & X \times_Z X \arrow[dl, "(\mathrm{pr}_1, \varphi_1 \circ \mathrm{pr}_2)"] \\ & X \times_Z Y & \end{tikzcd}$$

This is enough to prove the claim, because any morphism of fibrations which are N-torsors, is automatically an isomorphism. Since φ_1 and φ_2 are torsors, there exists for all $(x_1, x_2, y) \in (X \times_Y X) \times_Z Y$ a unique $n \in N$ and $h \in H$ such that $nx_1 = x_2$ and $h\varphi_1(x_1) = y$. It is easy to see that these assignments are actually morphisms of schemes. We define

$$\Phi(x_1, x_2, y) = (x_1, nhx_2)$$

which is a morphism of N-fibrations.

Henceforth, we assume, that we are in the situation

$$N \times (H \times X) \xrightarrow{\mathrm{pr}_{23}} H \times X \xrightarrow{\mathrm{pr}_2} X,$$

where N acts on $N \times (H \times X)$ and H on $H \times X$ by multiplication on itself respectively. We know further, that pr_{23} is H-equivariant. Hence, there is a morphism $\Phi\colon X \to N$ such that H acts on $(e, e, x) \in N \times H \times X$ as $h.(e, e, x) = (\Phi(h, x), h, x)$ and therefore

$$(nh).(e, e, x) = (n\Phi(h, x), h, x)$$

for alle $x \in X$, $n \in N$ and $h \in H$. We now define the following morphisms:

$$\Phi_N\colon N \times H \times X \to N, (n, h, x) \mapsto n\Phi(h, x)^{-1}$$
$$\Phi_H\colon N \times H \times X \to H, (n, h, x) \mapsto h$$

where the exponent minus one means inversion inside N. We then have by construction

$$(n, h, x) = \Phi_N(n, h, x)\Phi_H(n, h, x).(e, e, x).$$

Note that Φ_N and Φ_H are unique with this property.

One checks, that Φ_H is N-invariant and H-equivariant and that Φ_N is N-equvariant and $\Phi_N(h.(n', h', x)) = h\Phi_N(n', h', x)h^{-1}$. Therefore, the morphism

$$\Psi\colon N \times H \times X \to G \times X, (n, h, x) \mapsto (\Phi_N(n, h, x)\Phi_H(n, h, x), x)$$

is well-defined, G-equivariant and easily seen to be an isomorphism. □

2.3. **Geometric quotients and torsors.** Let G denote an affine algebraic group with no further assumptions.

Lemma 2.14. *Every G-torsor $\varphi\colon X \to Y$ is a geometric quotient.*

Proof. This is well-known, see [9] for example. By faithfully flat descend, φ is affine, surjective and submersive. Since Ψ is in particular surjective, the fibres of φ are exactly the G-orbits. We are left to show Property 4 of Definition 2.1. Because φ is faithfully flat, it is enough to show, that the pullback α of $\varphi^\sharp$ induces an isomorphism

$$\alpha\colon \varphi^*\mathcal{O}_Y \to \varphi^*(\varphi_*\mathcal{O}_X)^G.$$

First note, that by flat base change applied to the cartesian square from Lemma 2.11, we have a canonical isomorphism $\mathrm{pr}_{2*}\,\sigma^*\mathcal{O}_X \cong \varphi^*\varphi_*\mathcal{O}_X$. Furthermore

$$\mathrm{pr}_{2*}\,\sigma^*\mathcal{O}_X \cong \mathrm{pr}_{2*}\,\mathcal{O}_{G\times X} = \mathcal{O}_G(G) \otimes \mathcal{O}_X \cong \mathcal{O}_G(G) \otimes \varphi^*\mathcal{O}_Y,$$

henceforth $\mathcal{O}_G(G) \otimes \varphi^*\mathcal{O}_Y \cong \varphi^*\varphi_*\mathcal{O}_X$ canonically. We have a dual action of G on both sides. The action on the left comes from the multiplication on G. The action on X gives a dual action on the $\mathcal{O}_Y$ algebra $\varphi_*\mathcal{O}_X$. This pulls back to a dual action on $\varphi^*\varphi_*\mathcal{O}_X$. One checks that the isomorphism $\varphi^*\varphi_*\mathcal{O}_X \cong \mathcal{O}_G(G) \otimes \varphi^*\mathcal{O}_Y$ is G-equivariant. Since taking invariants commutes with flat base change, see [30, Lemma I.2.2], we obtain $\varphi^*\mathcal{O}_Y \cong \varphi^*(\varphi_*\mathcal{O}_X)^G$, which is in fact the same map as α. □

Lemma 2.15 ([32, §5, Kor. 1])**.** *Assume that G is reductive, that X is reduced, separated and of finite type, and that there exists a geometric quotient (Y, φ). Then φ is a G-torsor if and only if the stabiliser of every point $x \in X$ is trivial.*

Proof. Since φ is affine, and being a G-torsor is Zariski local on the target, we can assume that both X and Y are affine. The only if part is clear. Let $y = \varphi(x)$ be a point in Y. Since all stabilisers are zero-dimensional, every G orbit is closed. Hence, there exists a slice $S \subseteq X$ at x such that the diagram

$$\begin{tikzcd} G \times S \arrow[r, "\sigma"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d, "\varphi"] \\ S \arrow[r, "\sigma/G"] & Y \end{tikzcd}$$

is cartesian and σ/G is étale. So $\sigma/G\colon S \to Y$ is an étale neighborhood of y which trivializes φ. □

To conclude for general G, that a geometric quotient is a G-torsor, one has to assume more than that G acts set-theoretically free.

Definition 2.16. The G-action σ is called free, if

$$\Psi = \mathrm{pr}_2 \times \sigma\colon G \times X \to X \times X, (g, x) \mapsto (x, g.x)$$

is a closed immersion.

Proposition 2.17 ([26, Chap 0, §4, Prop.0.9])**.** *Assume that (Y, φ) is a geometric quotient, X and Y are separated and of finite type over k and σ is free. Then (Y, φ) is a G-torsor.*

Remark 2.18. Unfortunately, we do not know of a set-theoretically free action by some affine algebraic group G on a separated scheme X of finite type over k, such that there exists a geometric quotient (Y, φ) which is not a G-torsor.

2.4. **Special algebraic groups.** We continue the discussion of torsors. The material is taken from [29], [17] and [25] as well as from my personal notes from lecture series held by my supervisor Manfred Lehn at the University Mainz. Without loss of generality, we examine only right actions $\sigma\colon X \times G \to X$.

Definition 2.19. A G-fibration $\varphi\colon X \to Y$ is locally trivial if for every $y \in Y$ there exists an open neighborhood $U \subseteq Y$ such that the restricted fibration $\varphi^{-1}(U)$ is isomorphic to the trivial fibration $U \times G$.

If for every $y \in Y$ there exists an open neighborhood $U \subseteq Y$ and a finite étale morphism $f\colon V \to U$ such that the pulled-back fibration $V \times_Y X$ is isomorphic to the trivial fibration $V \times G$ over V, we call $\varphi\colon X \to Y$ locally isotrivial.

Remark 2.20. Let $\varphi\colon X \to Y$ be a G-fibration. If φ is locally isotrivial or trivial, φ is a G-torsor.

The aim of this section is to explain the following lemma.

Lemma 2.21. *Let U be a unipotent group which acts free on some separated scheme X of finite type over k. If there exists a good quotient (Y, φ), then φ is a locally trivial U-torsor.*

Example 2.22. Let $H \subseteq G$ be a closed subgroup of the affine algebraic group G and denote the quotient for the right multiplication of H on G with $\varphi\colon G \to G/H$. Then φ is locally isotrivial by [29, Prop. 3].

Example 2.23. Consider the additive group $\mathbb{G}_{a,k}$ as a subgroup of $\mathrm{SL}_{2,k}$ via

$$\mathbb{G}_{a,k} \to \mathrm{SL}_{2,k}, g \mapsto \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}.$$

Then $\mathbb{G}_{a,k}$ is the stabiliser of the first standard basis vector $e_1 \in \mathbb{A}^2_k$ of the standard representation of $\mathrm{SL}_{2,k}$. Hence, the quotient $\mathrm{SL}_{2,k}/\mathbb{G}_{a,k}$ is given by the orbit map

$$\pi\colon\ \mathrm{SL}_{2,k} \to \mathbb{A}^2_k \setminus \{0\}, A \mapsto Ae_1.$$

The right $\mathbb{G}_{a,k}$-torsor π admits over the open affine subsets $\{x_i \neq 0\}$, $i = 1, 2$, a section. Therefore, π is locally trivial.

Definition 2.24. An affine algebraic group G is called special, if every G-torsor is locally trivial.

Lemma 2.25 ([29, Lem. 6])**.** *Let $H \subseteq G$ be a closed normal subgroup. If H and G/H are special, then G is special.*

Proof (Sketch). Let Y be a base scheme considered with the Zariski topology. Then the first Čech cohomology set $\check{\mathrm{H}}^1(Y, G)$ computes the set of isomorphism classes of locally trivial G-torsors over Y. The lemma follows from the exact sequence

$$\dots \longrightarrow \check{\mathrm{H}}^1(Y, H) \longrightarrow \check{\mathrm{H}}^1(Y, G) \longrightarrow \check{\mathrm{H}}^1(Y, G/H).$$

We refer the reader to [25, Chap. III, §4] for details concerning the construction and properties of the first Čech cohomology set. $\square$

Let $\varphi\colon X \to Y$ be a right G-torsor and F some scheme over k on which G acts from the left. We let G act on $X \times F$ by $g.(x, f) = (x.g^{-1}, g.f)$.

Proposition 2.26 ([17, p. 295])**.** *The geometric quotient $(X \times F)/G =: X \times^G F$ exists, if F is affine. The quotient $X \times F \to X \times^G F$ is a G-torsor. Furthermore, if $G \subseteq H$ is a subgroup of some affine algebraic group H and we let G act on H via left multiplication, then there is a right H-action on $X \times^G H$ induced by right multiplication on H and $X \times^G H \to Y$ is an H-torsor.*

Proof. Let $Y' \to Y$ be an fpqc covering such that the pull-back of φ is isomorphic to the trivial G-torsor over Y'. By Remark 2.12, there exists a section $s\colon Y' \to X$ such that

$$\Psi\colon Y' \times G \to Y' \times_Y X, (y, g) \mapsto (y, s(y).g)$$

is an isomorphism of G-fibrations. We set $Y'' = Y' \times_Y Y'$ as well as $Y''' = Y' \times_Y Y''$. Let us translate via Ψ the canonical descent datum

$$\mathrm{pr}_1^*(Y' \times_Y X) \cong \mathrm{pr}_2^*(Y'' \times_Y X)$$

for the descent of $Y' \times_Y X \to Y'$ to $X \to Y$ into a descent datum for $Y' \times G$. For this we compute the isomorphism $\Phi\colon Y'' \times G \to Y'' \times G$ in the following diagram

$$\begin{tikzcd}
Y'' \times G \arrow[r, "\cong"'] \arrow[rrrr, bend left, "\Phi"] \arrow[dd, "\mathrm{pr}_{13}"] & \mathrm{pr}_1^*(Y' \times_Y X) \arrow[rr] \arrow[dd] \arrow[dr] & & \mathrm{pr}_2^*(Y' \times_Y X) \arrow[dl] \arrow[dd] & Y'' \times G \arrow[l, "\cong"] \arrow[dd, "\mathrm{pr}_{23}"'] \\
& & Y'' \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "\mathrm{pr}_2"] & & \\
Y' \times G \arrow[r, "\Psi"] & Y' \times_Y X \arrow[r] & Y' & Y' \times_Y X \arrow[l] & Y' \times G \arrow[l, "\Psi"']
\end{tikzcd}$$

where the map $\mathrm{pr}_1^*(Y' \times_Y X) \to \mathrm{pr}_2^*(Y' \times_Y X)$ is the canonical isomorphism and the outer squares and both quadrangles in the middle are cartesian. First we see that there exists a morphism of schemes $\alpha\colon Y' \times Y' \to G$ such that

$$s(y_1) = s(y_2).\alpha(y_1, y_2)$$

for all $y_i \in Y'$. This follows from the isomorphism $G \times X \to X \times_Y X$ in Lemma 2.11. This implies $\Phi(y_1, y_2, g) = (y_1, y_2, \alpha(y_1, y_2)g)$. Note that the cocycle condition $\mathrm{pr}_{13}^* \Phi = \mathrm{pr}_{23}^* \Phi \circ \mathrm{pr}_{12}^* \Phi$ translates into $\alpha(y_2, y_3)\alpha(y_1, y_2) = \alpha(y_1, y_3)$.

We construct now the quotient $(X \times G)/G$ by descent. Namely, let $\rho\colon G \times F \to F$ denote the left action of G on F.

First, $\mathrm{pr}_1\colon Y' \times G \times F \to Y'$ descends to $X \times F \to Y$ with respect to the descend datum $\Phi \times \mathrm{id}_F\colon Y'' \times G \times F \to Y'' \times G \times F$.

Second, Φ induces a descent datum $\widetilde{\Phi}(y_1, y_2, f) = (y_1, y_2, \alpha(y_1, y_2).f)$ for a descent of $\mathrm{pr}_1\colon Y' \times F \to Y'$.

$$\begin{array}{ccc} Y'' \times G \times F & \xrightarrow{\Phi \times \mathrm{id}_F} & Y'' \times G \times F \\ \downarrow{\scriptstyle \mathrm{id}_{Y''} \times \rho} & & \downarrow{\scriptstyle \mathrm{id}_{Y''} \times \rho} \\ Y'' \times F & \xrightarrow{\widetilde{\Phi}} & Y'' \times F \end{array}$$

If F is assumed to be affine, descent is effective. Hence, $Y' \times F$ descends along $Y' \to Y$ as well as the morphism $\mathrm{id}_{Y''} \times \rho$. Let's denote the descent of $Y' \times F$ by $X \times^G G$. We obtain the commutative diagram

$$\begin{array}{ccc} Y' \times G \times F & \longrightarrow & X \times F \\ \downarrow & & \downarrow \\ Y' \times F & \longrightarrow & X \times^G F \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

where the lower and outer squares are cartesian, whence the upper square is cartesian too. Since $\mathrm{id}_{Y'} \times \rho$ is isomorphic to the trivial G-torsor, $X \times F \to X \times^G F$ is a G-torsor, so in particular a geometric quotient.

Let us assume, that F is some affine algebraic group H containing G as a subgroup. Henceforth, the construction above applied to the left action of G on H gives a G-torsor $X \times H \to X \times^G H$. Since the action of H by right multiplication on H on $X \times H$ commutes with the left action of G on $X \times H$, we have an induced right action of H on $X \times^G H$. It is clear from the construction above, that after pull-back to Y', $X \times^G H$ becomes isomorphic to the trivial H torsor over Y'. The proposition follows. □

Definition 2.27. In the situation of Proposition 2.26, we call $X \times^G F \to Y$ the associated fibre bundle with fibre F.

Remark 2.28. We use the notations from Proposition 2.26. The closed immersion $Y' \times G \to Y' \times H$ coming from the inclusion of $G \subseteq H$ as a closed subgroup, descends to a closed immersion $X \to X \times^G H$. This is nothing else than the composition

$$X \xhookrightarrow{(\mathrm{id}, e)} X \times H \xrightarrow{/G} X \times^G H.$$

Proposition 2.29 ([17, Exp. XI, Prop. 5.1])**.** *The group* $\mathrm{GL}_{n,k}$ *is special.*

Proof (Sketch). Let $\varphi\colon X \to Y$ be a $\mathrm{GL}_{n,k}$-torsor. We can form the associated fibre bundle $X \times^{\mathrm{GL}_{n,k}} \mathbb{A}^n_k \to Y$ where $\mathbb{A}^n_k$ is the standard representation of $\mathrm{GL}_{n,k}$. By construction, this is a geometric vector bundle of rank n. In fact, if $Y' \to Y$ is an fpqc covering of Y such that the pull-back of X is the trivial $\mathrm{GL}_{n,k}$-fibration, then $X \times^{\mathrm{GL}_{n,k}} \mathbb{A}^n_k$ will be the relative spectrum over Y of the symmetric algebra of a descent of $\mathcal{O}^n_{Y'}$ to Y. But the descent of a locally free sheaf along an fpqc morphism is locally free.

The claim follows, since X is as a $\mathrm{GL}_{n,k}$-fibration actually isomorphic to the frame bundle of $X \times^{\mathrm{GL}_{n,k}} \mathbb{A}^n_k \to Y$. □

Theorem 2.30. *Every G-torsor $\varphi\colon X \to Y$ is locally isotrivial. Furthermore, if there exists a faithful representation $G \to H$ such that H is a special affine algebraic group and the quotient morphism $\pi\colon H \to G\backslash H$ is locally trivial, then G is special.*

Proof. Let $G \to H$ be a faithfull representation, where H is a special affine algebraic group. Such a representation always exists, since if H is not given by assumption, we can always take $H = \mathrm{GL}_{n,k}$. We form the associated fibre bundle $X \times^G H$. Because H is special, there exists a Zariski open covering $\{U_i\}_i$ of Y such that the H-torsor $X \times^G H \to Y$ is trivial over every U_i. Hence, there exists a section $s_i\colon U_i \to X \times^G H$ for all i.

Let $\pi\colon H \to G\backslash H$ denote the quotient morphism. There exists a Zariski open covering $\{V_j\}_j$ of $G\backslash H$ such that every V_j has a finite étale covering V'_j which admits a section $t_j\colon V'_j \to H$ (see Example 2.22).

Let $p\colon X \times^G HtoG\backslash H$ denote the canonical projection and U_{ij} the preimage of V_j under $p \circ s_i$. We form the fibre product $U'_{ij} := U_{ij} \times_{V_j} V'_j$ which is then finite étale over U_{ij}.

Together with the closed immersion $\iota\colon X \to X \times^G H$ from Remark 2.28 the situation looks as follows

$$\begin{tikzcd}
 & & & H \arrow[d, "\pi"'] & & & \\
X \arrow[r, hook] \arrow[dr] & X \times^G H \arrow[rr, "p"] \arrow[d] & & G\backslash H & \supseteq & V_j & V'_j \arrow[l] \arrow[lllu, "t_j"'] \\
 & Y & \supseteq & U_i \arrow[u, "p\circ s_i"] \arrow[llu, "s_i"] & \supseteq & U_{ij} \arrow[u] & U'_{ij} \arrow[l, "\mathrm{pr}_1"'] \arrow[u, "\mathrm{pr}_2"']
\end{tikzcd}$$

If we let $t_{ij} := t_j \circ \mathrm{pr}_2$ and $s'_{ij} := s_i \circ \mathrm{pr}_1$, then $\pi \circ t_{ij} = p \circ s'_{ij}$. We define

$$\sigma_{ij} := s'_{ij}.t_{ij}^{-1}\colon U'_{ij} \to X \times^G H,$$

where the exponent minus one means inversion in H. We have that $p \circ \sigma_{ij}$ is constant and maps all of U'_{ij} to the point in $G\backslash H$ representing the orbit of the neutral element. Hence, σ_{ij} maps into the image of ι. It is a section by construction. This proves the first part of the theorem.

For the second part, let us assume that π is locally trivial. Hence we can choose $V'_j = V_j$ and the claim follows. $\square$

Example 2.31. The special linear algebraic group $\mathrm{SL}_{n,k}$ is special. This follows since the determinant $\det\colon \mathrm{GL}_{n,k} \to \mathbb{G}_{m,k}$ admits a section $a \mapsto \mathrm{diag}(a, 1, \dots, 1)$ (even as groups!).

Proposition 2.32 ([29, Prop. 14])**.** *Every connected solvable algebraic group is special.*

Proof. By Theorem 1.23 we have a semidirect product decomposition $G = R_u(G) \rtimes T$ for some torus T. Since T is isomorphic to a direct product of copies of $\mathbb{G}_{m,k} = \mathrm{GL}_{1,k}$, we conclude from Proposition 2.29 and Lemma 2.25 that T is special. By the same lemma, we are left to show that $R_u(G)$ is special. But since $R_u(G)$ is a unipotent group, it admits a decomposition series where every factor is isomorphic to $\mathbb{G}_{a,k}$ which is special by Example 2.23 and Example 2.31. Hence $R_u(G)$ is special. $\square$

Proof of Lemma 2.21. This follows from Lemma 2.5 together with Propositions 2.17 and 2.32. $\square$

2.5. **Classical geometric invariant theory.** We will give a brief summary of parts of classical geometric invariant theory needed, see [26], [27] or [11]. Let X be a separated scheme of finite type over k, G an affine algebraic group acting on X and $p\colon L \to X$ a line bundle on X.

Definition 2.33. A G-linearisation on L is a G-action $G \times L \to L$ such that

1. for all $g \in G$ and $y \in L$ we have $p(g.y) = g.p(y)$ and
2. for all $x \in X$ and $g \in G$ the map
$$L_x \to L_{g.x}, y \mapsto g.y$$
is linear.

A morphism of two G-linearised line bundles L and M over X is a G-equivariant bundle map $L \to M$.

Remark 2.34. Let $\sigma\colon G \times X \to X$ denote the group action. If we consider L as an invertible sheaf on X, giving a G-linearisation on L is the same as giving an ismorphism
$$\Phi\colon \sigma^* L \to \mathrm{pr}_2^* L$$
which satisfies a certain co-cycle condition (see [26, Def. 1.6]).

If L and M are G-linearised, so are L^{-1} and $M \otimes L$ in a natural way. We denote with $\mathrm{Pic}^G X$ the group of G-linearised line bundles modulo isomorphisms of linearised line bundles. Note that, if $f\colon Y \to X$ is a G-equivariant map of G-schemes, then the linearisation of L induces a linearisation on f^*L. Hence, we get a well-defined pull-back map $f^*\colon \operatorname{Pic}^G X \to \operatorname{Pic}^G Y$.

Example 2.35. If V is a G-representation, we have an induced G-action on $\mathbb{P}(V^*)$. The tautological bundle $\mathcal{O}_{\mathbb{P}(V^*)}(-1)$ carries a natural G-linearisation coming from the action on V. Hence, $\mathcal{O}_{\mathbb{P}(V^*)}(1)$ carries the dual G-linearisation. If not mentioned otherwise, we will in this situation always use this linearisation on $\mathcal{O}_{\mathbb{P}(V^*)}(1)$.

Over any G-invariant open $U \subseteq X$ the action on L induces an action on the space of sections $H^0(U, L)$.

Proposition 2.36. *Let $\varphi\colon X \to Y$ be a G-torsor.*

1. *The pull-back $\varphi^*\colon \operatorname{Pic} Y \to \operatorname{Pic}^G X$ is an isomorphism of groups.*
2. *Let $M \in \operatorname{Pic} Y$ and $L \in \operatorname{Pic}^G X$ such that $\varphi^* M \cong L$ as linearised line bundles. The pull-back map $\varphi^*\colon H^0(Y, M) \to H^0(X, L)^G$ is an isomorphism.*
3. *Let M and L be as above. Let $G \rtimes H$ be an extension such that the G-action on X and the linearisation on L both extend to $G \rtimes H$. The H-linearisation on L descends to an H-linearisation on M of the induced H-action on Y such that the pull-back $\varphi^*\colon H^0(Y, M) \to H^0(X, L)^G$ is H-equivariant.*

Proof. This is an application of the theory of descent along torsors, see [35, Thm. 4.46] or [8, Chap. 6]. We will sketch the arguments.

First, since φ is G-invariant, the pull-back of any line bundle $M \in \operatorname{Pic} Y$ along φ will carry a canonical G-linearisation and any section over Y will pull-back to an invariant section. For the converse one might argue parallel to [8, Chap. 6.2, Example B], to show that any G-linearised line bundle $L \in \operatorname{Pic}^G X$ descends to a line bundle $M \in \operatorname{Pic} Y$. Furthermore, let $\sigma\colon G \times X \to X$ denote the G-action. We note that a section $f \in H^0(X, L)$ is G-invariant iff the corresponding morphism of line bundles $f \in \operatorname{Hom}_X(\mathcal{O}_X, L)$ is G-invariant, where $\mathcal{O}_X$ is the trivial line bundle with the trivial linearisation. This is the case iff $\sigma^* f = \operatorname{pr}_2^* f$. Hence, any invariant section $f \in H^0(X, L)^G$ descends by [8, Chap. 6.1, Prop. 1]. This proves parts one and two.

For part three, we note, that the canonical morphism of geometric vector bundles $\varphi^* M \to M$ is itself a G-torsor. Further, $\varphi^* M \cong L$ as G-linearised line bundles. Hence, we do have an induced action of H on M coming from the action of $G \rtimes H$ on L. This action is in fact an H-linearisation on M and one checks that the pull-back $\varphi^*\colon H^0(Y, M) \to H^0(X, L)^G$ is H-equivariant. $\square$

Definition 2.37. Let $L \to X$ be a G-linearised line bundle.

1. A point $x \in X$ is called semi-stable with respect to L if there exists $n > 0$ and an invariant section $s \in H^0(X, L^n)^G$ such that $s(x) \neq 0$ and X_s is affine.

2. A point $x \in X$ is called stable with respect to L if there exists a section s as in part 1 such that $s(x) \neq 0$, X_s is affine, $\mathrm{Stab}_G(x')$ is finite for all $x' \in X_s$ and all orbits in X_s are closed.

We denote the set of semi-stable (respectively stable) points with respect to the G-action on X linearised by L with $X^{ss,G}(L)$ (respectively $X^s, G(L)$). If G is clear from the context, we may drop it in the notation.

Remark 2.38. Note that $X^{ss}(L) = X^{ss}(L^n)$ and $X^s(L) = X^s(L^n)$ for all $n > 0$.

Remark 2.39. Assume that L is very ample (in the sense of [19]). Hence, there exists a finite dimensional subspace $V \subseteq H^0(X, L)$ whose sections give rise to an immersion $I\colon X \to \mathbb{P}(V)$ such that $L \cong I^*\mathcal{O}_{\mathbb{P}(V)}(1)$. In fact, if L is G-linearised one can choose V as a G-stable subspace and I will then be G-equivariant for the induced action on $\mathbb{P}(V)$. Furthermore, $L \cong I^*\mathcal{O}_{\mathbb{P}(V)}(1)$ as G-linearised line bundles, see [26, Prop. 1.7].

If X is proper over k we may take $V = H^0(X, L)$. We then have

$$X^{ss}(L) = I^{-1}(\mathbb{P}(V)^{ss}(\mathcal{O}_{\mathbb{P}(V)}(1))) \text{ and } X^s(L) = I^{-1}(\mathbb{P}(V)^s(\mathcal{O}_{\mathbb{P}(V)}(1)))$$

by [26, Thm. 1.19] and the remark following the proof of [26, Cor. 1.20]. If X is not proper, those equalities hold for a carefully chosen V after one replaces L by some high enough positive multiple L^n, see the proof of [26, Amplification 1.8] for details.

Theorem 2.40 ([26, Thm. 1.10])**.** *Assume that G is a reductive algebraic group and that L is G-linearised.*

1. *The set of semi-stable points $X^{ss}(L)$ admits a good quotient (Y, φ).*
2. *There is an ample invertible sheaf M on Y and $n > 0$ such that $\varphi^*M \cong L^n$.*
3. *There is an open subset $U \subseteq Y$ such that $\varphi^{-1}U = X^s(L)$ and $(U, \varphi|_{X^s(L)})$ is a geometric quotient of $X^s(L)$.*

Let X be projective and $L = \mathcal{O}_X(1) \in \mathrm{Pic}^G X$ be very ample. By Remark 2.39, we can think of X as a subscheme of $\mathbb{P}(H^0(X, L))$ where the G-action is given by the induced linear action on global sections of L. If $x \in X$ is a closed point, we denote by x^* a lift to the affine space $H^0(X, L)^* \setminus \{0\}$. Furthermore, we fix a maximal torus $T \subseteq G$ and an isomorphism $X(T) \cong \mathbb{Z}^n$.

Definition 2.41. Let $x^* = \sum_{\chi \in X(T)} v_\chi$ be the decomposition of x^* into T-weight vectors. We denote with

$$\mathrm{wt_T}(x^*) = \{\chi \in X(T) \mid v_\chi \neq 0\}$$

the finite set of weights of the non-zero weight components of x^*. Let $\overline{\mathrm{wt_T}}(x^*)$ denote the convex hull of $\mathrm{wt_T}(x^*)$ in $\mathbb{R} \otimes \mathbb{Z}^n$.

Theorem 2.42 ([11, Thm. 9.2])**.** *Let G be reductive, and X projective over k with very ample sheaf $L = \mathcal{O}_X(1)$. Let $\overline{\mathrm{wt}_{\mathrm{T}}}(x^*)^\circ$ denote the interior of $\overline{\mathrm{wt}_{\mathrm{T}}}(x^*)$. Then*

$$x \in X^{ss,T}(L) \Leftrightarrow 0 \in \overline{\mathrm{wt}_{\mathrm{T}}}(x^*)$$
$$x \in X^{ss,T}(L) \Leftrightarrow 0 \in \overline{\mathrm{wt}_{\mathrm{T}}}(x^*)^\circ$$

The next theorem is a version of the well-known Hilbert-Mumford criterion. It is a numerical criterion to compute the loci of semi-stable and stable points.

Theorem 2.43. *Let X be projective with ample invertible sheaf L and G reductive. For a fixed maximal torus T in G, the following equivalences hold:*

$$x \in X^{ss,G}(L) \Leftrightarrow g.x \in X^{ss,T}(L) \;\; \textit{for all} \;\; g \in G$$
$$x \in X^{s,G}(L) \Leftrightarrow g.x \in X^{s,T}(L) \;\; \textit{for all} \;\; g \in G$$

Proof. If X is proper, this is [11, Thm. 9.3] and follows from the Hilbert-Mumford criterion [26, Thm. 2.1]. If X is not proper, we may replace L by some sufficiently large tensor power and choose an embedding $X \to \mathbb{P}(V)$, $V \subseteq H^0(X, L)$, as in Remark 2.39 such that

$$X^{ss,G}(L) = I^{-1}(\mathbb{P}(V)^{ss,G}(\mathcal{O}_{\mathbb{P}(V)}(1))) \;\text{ and }\; X^{s,G}(L) = I^{-1}(\mathbb{P}(V)^{s,G}(\mathcal{O}_{\mathbb{P}(V)}(1))),$$

and the analogous equalities hold for the loci of semi-stable and stable points with respect to T. This allows us to compute the loci of stable and semi-stable points in X by applying the theorem for proper X to the proper scheme $\mathbb{P}(V)$. □

3. Constructing Quotients

From now on we assume that all schemes are separated and of finite type over k.

3.1. **The affine case.**

We will give a criteria for the existence of a quotient for a unipotent group action on an affine scheme developed in [3]. The construction stems from the following lemma.

Lemma 3.1 ([16, Lem. 3.1],[10, Chap. 4, §7, Lem. 4.7.5])**.** *Let A be a ring and $D \in \operatorname{Der}(A, A)$ a locally nilpotent derivation. Assume that there exists $a \in A$ such that $D(a) = 1$ and let $\pi\colon A \to A/(a) =: \overline{A}$ denote the canonical projection. Then*

$$\chi\colon A \to \overline{A}[Y], x \mapsto \sum_{n\geq 0} \frac{1}{n!}\pi(D^n a)Y$$

is a ring isomorphism from A to the polynomial ring in one variable Y over $\overline{A}$ with inverse given by $\chi^{-1}(Y) = a$ and

$$\chi^{-1}(\pi(x)) = \sum_{n\geq 0} \frac{(-1)^m}{m!}(D^m x)a^m \text{ for all } x \in A.$$

Furthermore, the derivation $\chi D\chi^{-1}$ is the formal derivative $\frac{\partial}{\partial Y}$.

Proof. One checks that χ is a ring homomorphism and that there exists exactly one homomorphism $\overline{A}[Y] \to A$ which fulfills the formulas for χ^{-1} given in the Lemma. Now it is a short calculation to show that they are inverse to each other and that the derivation $\chi D\chi^{-1}$ equals $\frac{\partial}{\partial Y}$. □

Corollary 3.2. *Let $\mathbb{G}_{a,k}$ act on an affine scheme $X = \operatorname{Spec} A$. Let $\xi \in \operatorname{Lie} \mathbb{G}_{a,k} \setminus \{0\}$ and denote with $\delta_\xi \in \operatorname{Der}(A, A)$ the infinitesimal action of ξ on A. If there exists $a \in A$ such that $\delta_\xi(a)$ is a unit and invariant, then X is a trivial $\mathbb{G}_{a,k}$-torsor.*

Proof. After replacing a with $a\delta_\xi(a)^{-1}$ we have $\delta_\xi(a) = 1$ and are henceforth in the situation of Lemma 3.1. Therefore $A^{\mathbb{G}_{a,k}}$ is isomorphic to $\operatorname{Ker} \Delta = \overline{A}$. Furthermore, Δ is the infinitesimal action of the translation action of $\mathbb{G}_{a,k}$ on the first factor of

$$\operatorname{Spec} \overline{A}[Y] \cong \mathbb{A}^1_k \times \operatorname{Spec} A^{\mathbb{G}_{a,k}}.$$

Since the infinitesimal action determines the action by Lemma 1.21, the claim follows. □

Let $\mathbb{G}_{a,k} \rtimes T$ be an extension of the additive group by some torus T. We assume that there exists a positive grading $\lambda\colon \mathbb{G}_{m,k} \to T$ (see Definition 1.29). Let $\mathbb{G}_{a,k} \rtimes T$ act on some affine scheme $X = \operatorname{Spec} A$ of finite type over k. We further fix some non-zero element $\xi \in \operatorname{Lie} \mathbb{G}_{a,k}$.

Lemma 3.3. *If ξ has T-weight μ for the adjoint action and $a \in A$ has T-weight ν for the action by left-translation on A, then $\delta_\xi(a)$ has T-weight $\mu + \nu$.*

Proof. Let $V \subseteq A$ be a subspace stable under left-translation $\tau\colon \mathbb{G}_{a,k} \rtimes T \to \mathrm{GL}(A)$. By Lemma 1.18, we know that $\delta_\xi|_V = -d(\tau|_V)_e(\xi)$. Hence,

$$\begin{aligned} t.\delta_\xi(a) &= -t.d(\tau|_V)_e(\xi)(a) = -\operatorname{Ad}(\tau(t))(d(\tau|_V)_e(\xi))t.a \\ &= -\nu(t)d(\tau|_V)_e(\operatorname{Ad}(t)(\xi))a = (\mu+\nu)(t)\delta_\xi(a). \end{aligned}$$

for all $t \in T$. □

Proposition 3.4 ([3, Lem. 4.2])**.** *If all weights of the action of λ by left-translation on A are non-positive and $\mathrm{Stab}_{\mathbb{G}_{a,k}}(x) = \{e\}$ for all $x \in X$, then there exists a T-weight vector $a \in A$ such that $\delta_\xi(a) = 1$. In particular, X is a trivial $\mathbb{G}_{a,k}$-torsor.*

Proof. Let $l > 0$ be the λ-weight of ξ. Then the application of δ_ξ shifts the λ-weights by l. Denote with $A = \bigoplus_{n \le 0} A_n$ the weight space decomposition with respect to λ. We claim that

$$\mathfrak{a} := \delta_\xi(A_{-l}) \oplus \bigoplus_{n<0} A_n$$

is an ideal in A. It is certainly closed under addition. Let $f \in \mathfrak{a}$ and $a \in A$. To show that $af \in \mathfrak{a}$, it is enough to consider the case where a and f have both weight zero. Choose $h \in A_{-l}$ such that $\delta_\xi(h) = f$. Since all weights on A are non-positive and $\delta_\xi(a) = 0$, we have $\delta_\xi(ah) = a\delta_\xi(h) = af$.

Next, we prove that $\mathfrak{a} = A$. If not, we can choose a maximal ideal $\mathfrak{m} \supseteq \mathfrak{a}$. Let $W \subseteq A_0$ be a complementary subspace to $\mathfrak{a}$, $A = W \oplus \mathfrak{a}$. We may write $m = w + a$ with $w \in W$, $a \in \mathfrak{a}$ for all $m \in \mathfrak{m}$. Then

$$\delta_\xi(m) = \delta_\xi(w) + \delta_\xi(a) = \delta_\xi(a)$$

which shows that $\delta_\xi(m) \in \mathfrak{m}$, whence $\mathfrak{m}$ is stable under δ_ξ. This is in contradiction to $\mathrm{Stab}_{\mathbb{G}_{a,k}}(x) = \{e\}$ for all $x \in X$.

Hence, there exists $a \in A_{-l}$ such that $\delta_\xi(a) = 1$. Let μ be the T-weight of ξ. If we decompose $a = \sum_i a_{\mu_i}$ in its T-weight components, we see that we must have $a_{-\mu} \neq 0$ and $\delta_\xi(a_{-\mu}) = 1$. This proves the proposition. □

Let $U_T := U \rtimes T$ be an extension of a unipotent group U by some torus T such that there exists a positive grading $\lambda\colon \mathbb{G}_{m,k} \to T$. We let U_T act on some affine scheme $X = \operatorname{Spec} A$ of finite type over k.

In the next theorem, the existence of a T-equivariant section is our own contribution.

Theorem 3.5 ([3, Prop. 7.4])**.** *If the weights of λ on A are non-positive and $\mathrm{Stab}_U(x)$ is trivial for all $x \in X$, then X is a trivial U-torsor. It admits a section which is equivariant for the induced T-action on the base.*

Proof. By induction on $d := \dim U$. If $d = 1$, we must have $U \cong \mathbb{G}_{a,k}$ and X is a trivial U-torsor by Proposition 3.4. Further, there exists a T-weight vector $a \in A$ by

Proposition 3.4 such that by Lemma 3.1, a is transcendental over the invariant ring A^U and $A = A^U[a]$. Since a is a T-weight vector, the natural projection $p\colon A \to A/(a)$ defines a T-equivariant section.

Let $d \geq 2$. By Proposition 1.30 there exist T-stable subgroups $U', H \subset U$ such that U' is normal, H is isomophic to $\mathbb{G}_{a,k}$ and U decomposes as a semi-direct product $U = U'H$. Since λ is a positive grading for U it will be one for $U' \rtimes T$ and $H \rtimes T$ too. Hence, we can apply the induction hypothesis to U'.

Let $\pi\colon X \to Y$ denote the (trivial) U'-torsor. Since U' is normal in $U \rtimes T$, we have an induced action of $H \rtimes T$ on Y. Note that $Y = \operatorname{Spec} A^{U'}$. Hence, all weights of λ on $A^{U'} \subseteq A$ are certainly non-positive as well. Further, let $h \in \operatorname{Stab}_H(y)$ for some $y \in Y$. We choose some preimage $x \in X$ of y. Then there exists some $u' \in U'$ such that $u'hx = x$. Since both $\operatorname{Stab}_U(x)$ and $U' \cap H$ are trivial, we must have $h = e$. Hence, we can apply the $d = 1$ case. The composition of trivial torsors is a trivial torsor by Proposition 2.13 and the composition of T-equivariant sections will be a T-equivariant section. □

3.2. **The general case.** Let G be an affine algebraic group acting on some scheme X which is separated and of finite type over k. We assume that the action is linearised by two line bundles $K, L \in \operatorname{Pic} X$ and define $L_d := K^d \otimes L$, $d > 0$. We fix a Levi-decomposition $G = R_u(G) \rtimes R$, see Theorem 1.28. Since L_d is G-linearised it is in particular an R-linearisation.

Proposition 3.6. *If X is projective and K as well as L are ample, the loci of stable and semi-stable points $X^{s,R}(L_d)$ and $X^{ss,R}(L_d)$ are independent of $d > 0$ for d sufficiently large. Furthermore, $X^{ss,R}(L_d) \subseteq X^{ss,R}(K)$ for $d \gg 0$.*

Proof. Those are results contained in [28]. If $\pi\colon V \to W$ is any R-equivariant morphism between projective varieties, the action on W is linearised by an ample line bundle N and the action on V is linearised by an ample line bundle M, the loci of stable respectively semi-stable points in V with respect to the line bundles $M_d := \pi^* N^d \otimes M$ are independent of $d > 0$ for d sufficiently large. This is corollary [28, Cor. Sect. 5].

The second part of the proposition follows from part a) of [28, Thm. 2.2]. □

Notation 3.7. For arbitrary X, whenever there exists a $d_0 \in \mathbb{N}$ such that the loci $X^{ss,R}(L_d)$ and $X^{s,R}(L_d)$ are independent of d for $d \geq d_0$, then $x \in X$ is called semi-stable respectively stable with respect to L_∞ if it is semi-stable respectively stable with respect to L_d for some (and hence all) $d \geq d_0$. Define

$$X^{ss,R}(L_\infty) := X^{ss,R}(L_{d_0}) \text{ and } X^{s,R}(L_\infty) := X^{s,R}(L_{d_0}).$$

We define the following condition on a section $f \in H^0(X, K^p)$, $p > 0$.

$(*)$ *f is G-invariant, X_f is affine and a trivial $R_u(G)$-torsor over the base $X_f/R_u(G)$, which admits a T-equivariant section for some maximal torus T in R.*

Remark 3.8. If $X_f \to X_f/R_u(G)$ is an $R_u(G)$-torsor admitting a T-equivariant section s for some maximal torus T, then there exists a T-equivariant section for any choice of a maximal torus. For if T' is another maximal torus in R, then there exists $r \in R$ such that $rTr^{-1} = T'$. Hence $\hat{s}(y) := r.s(r^{-1}.y)$ will be a T'-equivariant section.

Definition 3.9. A point $x \in X$ is semi-stable with respect to the pair (K, L) if

1. there exists $f \in H^0(X, K^p)^G$ that fulfills condition $(*)$ and $x \in X_f$, and
2. the orbit $R_u(G).x$ is contained in $X^{ss,R}(L_d)$ for all $d \gg 0$.

A point $x \in X$ is stable with respect to (K, L) if x is semi-stable, its orbit $G.x$ is closed inside the locus of semi-stable points and the stabiliser $\mathrm{Stab}_G(x)$ is finite. The loci of semi-stable and stable points are denoted by $X^{ss,G}(K, L)$ and $X^{s,G}(K, L)$ respectively.

Lemma 3.10. *Let $f \in H^0(X, K^p)^R$, $p > 0$, be an R-invariant section. The intersection*

$$X_f \cap X^{ss,R}(L_d)$$

is independent of d for $d \gg 0$. In particular, the loci of semi-stable and stable points in Definition 3.9 are well-defined.

Proof. We claim that

$$X^{ss,R}(L_d) \cap X_f \subseteq X^{ss,R}(L_{d+1}) \cap X_f$$

for all $d > 0$. This implies the lemma, since X is noetherian. Let $d > 0$. We can choose $g_1, \dots, g_s \in H^0(X, L_d^q)^G$ such that

$$X^{ss,R}(L_d) = \bigcup_j X_{g_j}.$$

For every choice of $a, b \in \mathbb{N}$, the intersection $X^{ss,R}(L_d) \cap X_f$ is therefore covered by all $X_{f^a g_j^b}$. Let a, b such that $ap = bq$. Then $f^a g_j^b$ is an R-invariant section in $K^{ap} \otimes L_d^{bq} = L_{d+1}^{bq}$. This proves the claim.

For the last statement of the lemma, we note that if f is not only R-invariant but even a G-invariant section, the whole orbit $R_u(G).x$ is contained in X_f. Hence, we proved that the loci of semi-stable and stable points defined in Definition 3.9 are well-defined. $\square$

Remark 3.11. We have $X^{ss,G}(K,L) = X^{ss,G}(K^p, L^q)$ for all $p, q > 0$, which immediately gives the analogous statement for the set of stable points. Note first that $X^{ss,G}(K,L)$ is unchanged if we replace both K and L by the same positive power. Second, replacing K with some positive power does not change the set of semi-stable points either. Hence, we get

$$X^{ss,G}(K^p, L^q) = X^{ss,G}(K^{qp}, L^q) = X^{ss,G}(K^p, L) = X^{ss,G}(K, L).$$

Remark 3.12. In the next section, we will twist a given G-linearised line bundle, say $K \in \operatorname{Pic}^G X$, by some character of G to obtain enough sections $f \in H^0(X, K^p)^G$ such that X_f is an $R_u(G)$-torsor. In many cases, this twist will cause the stable locus $X^{s,R}(K)$ to be empty. That is the reason for introducing the second line bundle L. Defining semi-stability and stability with respect to L_d, $d \gg 0$, is a generalization of the assumption that L is well-adapted in [3] or [21].

Theorem 3.13. *Let K be ample. The sets of semi-stable and stable points with respect to (K, L) are open. Moreover:*

1. *There exists a good quotient*

$$\pi \colon X^{ss,G}(K,L) \to Z$$

and a $d_0 \in \mathbb{N}$ such that for all $d \geq d_0$ there is an ample line bundle $P \in \operatorname{Pic} Z$ with the property that $\pi^* P \cong (L_d)^a$ for some $a > 0$.

2. *There exists an open set $Z' \subseteq Z$ such that $\pi^{-1}(Z') = X^{s,G}(K,L)$ and*

$$\pi \colon X^{s,G}(K,L) \to Z'$$

is a geometric quotient.

Let us give the reader some guidance on the strategy of the proof. Taking X_0 as the union of all X_f such that f satisfy condition $(*)$, we show that X_0 is a $R_u(G)$-torsor over some base Y. Let (M, N) denote the descent of (K, L) to Y. We will prove that the loci $Y^{ss,R}(N_\infty)$ and $Y^{s,R}(N_\infty)$ are well-defined and that their preimages in X_0 equal $X^{ss,G}(K,L)$ respectively $X^{s,G}(K,L)$. Here we use the existence of T-equivariant local sections and Theorem 2.43 for which we need K to be ample. Finally, if Z denotes the good quotient of $Y^{ss,R}(N_\infty)$ by the R-action, Z will be a good quotient of $X^{ss,G}(K,L)$ for the G-action which has all the claimed properties.

Proof of Theorem 3.13. As mentioned above, let

$$X_0 := \bigcup_f X_f$$

where the union runs over all f which satisfy condition $(*)$. We choose a finite number $f_1, \ldots, f_m$ such that X_{f_i} cover X_0. Without loss of generality we may assume that there exists some $p > 0$ such that $f_i \in H^0(X, K^p)^G$ and by Remark 3.11 we may replace K by K^p, so we assume that $f_i \in H^0(X,K)^G$. Every X_{f_i} is an $R_u(G)$-torsor over some base Y_i. In particular, all Y_i are categorical quotients. These categorical quotients glue to an $R_u(G)$-torsor $\varphi\colon X_0 \to Y$. Furthermore, K descends to an ample line bundle M on Y and every f_i descends to a section $s_i \in H^0(Y,M)$ such that $Y_{s_i} = Y_i$, see the proof of [26, Thm. 1.10] for the construction of Y, M and s_i.

Observe, that we clearly have the inclusions $X^{ss,G}(K,L) \subseteq X_0 \subseteq X^{ss,R}(K)$.

The Levi-factor R acts on Y such that φ is R-equivariant. By Proposition 2.36, the G-linearisation on K induces a G-linearisation on M such that s_i is in fact an R-invariant section. By the same proposition, the line bundle L descends to an R-linearised bundle $N \in \operatorname{Pic}^R Y$. We set $N_d := M^d \otimes N$ for $d > 0$, hence $\varphi^* N_d = L_d$.

Lemma 3.14. *The loci of semi-stable and stable points $Y^{ss,R}(N_d)$ and $Y^{s,R}(N_d)$ are independent of $d \gg 0$. Furthermore,*

$$\varphi^{-1}(Y^{ss,R}(N_\infty)) = X^{ss,G}(K,L) \quad \textit{and} \quad \varphi^{-1}(Y^{s,R}(N_\infty)) = X^{s,G}(K,L).$$

Proof. By Lemma 3.10, the set of semi-stable points $Y^{ss,R}(N_d)$ is independent of d for $d \gg 0$, because Y is covered by all Y_{s_i} and $s_i \in H^0(Y,M)^R$ as mentioned above. This implies that also the stable locus is independent of d for d sufficiently large.

Note that, because K is ample, L_d is ample for $d \gg 0$. Since φ is R-equivariant, it follows from Theorem 2.43, that $x \in \varphi^{-1}(Y^{ss,R}(N_d))$ if and only if $r.x$ is contained in the preimage $\varphi^{-1}(Y^{ss,T}(N_d))$ for all $r \in R$ and a chosen maximal torus T in R. By Remark 3.8 we may assume that evey $R_u(G)$-torsor $X_{f_i} \to Y_{s_i}$ admits a T-equivariant section for a fixed torus T, independent of i.

Claim. $\varphi^{-1}(Y^{ss,T}(N_d)) = X_0 \cap X^{ss,R_u(G)T}(K,L)$ for all $d \gg 0$.

Before we prove the claim, let us finish the proof of the lemma. By Theorem 2.43 and the R-equivariance of φ, we conclude that $x \in \varphi^{-1}(Y^{ss,R}(N_d))$ if and only if $r.x \in \varphi^{-1}(Y^{ss,T}(N_d))$ for all $r \in R$. This is in turn equivalent to $r.x \in X_0$ and $ur.x \in X^{ss,T}(L_d)$, $d \gg 0$, for all $r \in R$ and $u \in R_u(G)$ by the claim. But this is the case if and only if $x \in X_0$ and $u.x \in X^{ss,R}(L_d)$ for all $u \in R_u(G)$ again by Theorem 2.43. Up to proving the claim, we have proved the lemma in the case of semi-stable points. The analogous statement about the locus of stable points follows from Lemma 3.15 below, since $X^{ss,G}(K,L) \subseteq X_0$ and X_0 admits geometric quotient for the $R_u(G)$-action.

To prove the claim, let first be $t \in H^0(Y,(N_d)^p)^T$. Then $\varphi^*(t) \in H^0(X_0,(L_d)^p)^T$. Note that

$$\varphi^{-1}(Y_t) = (X_0)_{\varphi^*(t)} = \bigcup_{i=1}^{n} X_{f_i^k \varphi^*(t)}$$

for all $k > 0$. If we choose k large enough and as a multiple of p, say $k = lp$, then $f_i^k \varphi^*(t)$ extends to a T-invariant section in $H^0(X,(L_{d+l})^p)^T$ for every i. Hence, making l sufficiently large, we obtain

$$\varphi^{-1}(Y^{ss,T}(N_d)) \subseteq X_0 \cap X^{ss,R_u(G)T}(K,L).$$

For the reverse inclusion, let $x \in X_0 \cap X^{ss,R_u(G)T}(K,L)$. We choose f_{i_0} such that $x \in X_{f_{i_0}}$. Since f_{i_0} is a G-invariant, we know that the orbit $R_u(G).x$ is contained in $X_{f_{i_0}}$. Let s denote a T-equivariant section of the $R_u(G)$-torsor $X_{f_{i_0}} \to Y_{i_0}$. The $R_u(G)$-orbit of x intersects the image of s in exactly one point, say x_0. By definition of the semi-stable points $X^{ss,R_u(G)T}(K,L)$ there exists an invariant section $g \in H^0(X,(L_d)^p)^T$, $p > 0$ and $d \gg 0$, such that $g(x_0) \neq 0$. Hence the pull-back $s^*(g) \in H^0(Y_{s_{i_0}},(N_d)^p)$ is non-vanishing in $\varphi(x_0) = \varphi(x)$. Since s is T-equivariant, $s^*(g)$ is T-invariant as well. For $k = lp$ large enough, $s_{i_0}^k s^*(g)$ extends to a global invariant section of $(N_{d+l})^p$ on Y. This shows $\varphi(x) \in Y^{ss,T}(N_d)$ for $d \gg 0$, which proves the claim. ☐

We conclude from Lemma 3.14 immediately that $X^{ss,G}(K,L)$ and $X^{s,G}(K,L)$ are open. Furthermore, let $\psi\colon Y^{ss,R}(N_\infty) \to Z$ denote the good quotient for the group action by the reductive group R on Y. Then $\pi := \psi \circ \varphi\colon X^{ss,G}(K,L) \to Z$ is a good G-quotient since it is the composition of a good $R_u(G)$-quotient with a good R-quotient. Let $d_0 \in \mathbb{N}$ such that $Y^{ss,R}(N_d)$ is independent of d for $d \geq d_0$. There is some $a > 0$ such that the line bundle $(N_d)^a$ descends to an ample line bundle P on Z, see [26, Thm. 1.10]. Hence, $(L_d)^a$ descends to P which completes the proof of part one of Theorem 3.13.

Finally, if Z' denotes the geometric quotient of $Y^{s,R}(N_d)$ by R, then the preimage $\pi^{-1}(Z')$ is the locus of G-stable points $X^{s,G}(K,L)$ and Z' its good quotient by G. Since all G-orbits are closed in $X^{s,G}(K,L)$, Z' is in fact a geometric quotient by Lemma 2.3, proving all parts of Theorem 3.13. ☐

Lemma 3.15 ([4, Lem. 3.3.1])**.** *Let G be an affine algebraic group and $N \subseteq G$ a normal subgroup with factor group $H = G/N$. Let X be a separated scheme of finite type over k on which G acts. Assume that $\varphi\colon X \to Y$ is an N-torsor. The following statements about $x \in X$ and $y = \varphi(x)$ are equivalent:*

1. *The orbit $G.x$ is closed in X and $\mathrm{Stab}_G(x)$ is finite.*
2. *The orbit $H.y$ is closed in Y and $\mathrm{Stab}_H(y)$ is finite.*

Proof. The result [4, Lem. 3.3.1] is stronger. Here, we give an alternative and shorter proof of the statement needed in the proof of Theorem 3.13. Let $\pi\colon G \to H$ denote the quotient morphism. We consider the following commutative diagram

$$\begin{tikzcd}
G \arrow[r, "\psi_x"] \arrow[d, "\pi"] & G.x \arrow[r] \arrow[d, "\varphi"] & X \arrow[d, "\varphi"] \\
H \arrow[r, "\psi_y"] & H.y \arrow[r] & Y
\end{tikzcd}$$

where we denoted the orbit morphisms with ψ_x and ψ_y. Since π and φ are N-torsors, the left square is in fact cartesian.

If we assume the first assertion, then $H.y$ is closed since it is the image of the closed N-invariant subset $G.x$ under φ. Furthermore, ψ_x is quasi-finite. Since this property descends along any fpqc morphism, ψ_y is quasi-finite too. Hence, $\mathrm{Stab}_H(y)$ is finite.

Let us assume the second assertion. Then $G.x$ is closed as it is the preimage of $H.y$ under φ. Furthermore, since ψ_y is quasi-finite, so is the pull-back ψ_x. $\square$

Remark 3.16. In the situation of Theorem 3.13, any G-invariant global section g of $(L_d)^{ka}$, $d \geq d_0$, descends to a section $g' \in H^0(Y, (N_d)^{ka})^R$ since it is $R_u(G)$-invariant. Further, it is part of the package of classic geometric invariant theory that g' descends to a global section of P^k.

Remark 3.17. We can drop the assumption K ample from Theorem 3.13 if we alter Definition 3.9. Instead of assuming the existence of a T-equivariant section of the $R_u(G)$-torsor X_f, we need to impose that an R-equivariant section exists. Then we no longer appeal to Theorem 2.43 in the proof of Lemma 3.14 and follow directly the reasoning in the claim, where we replace T with R.

Remark 3.18. Assume that $G = R_u(G) \rtimes R$ acts on an affine variety X in such a way that X is a trivial $R_u(G)$-torsor over some base Y. We do not know if the assertion that there exists an R- (respectively T-) equivariant section in Condition $(*)$ is redundant or not. Let us consider the following example. Let G be a semi-direct product $G = N \rtimes R$ with R reductive and N such that there exists an injective group homorphism $f\colon R \to N$. Note that such an f does not exist, if N is unipotent and non-trivial. Let $\Lambda \subseteq R$ be a non-trivial subgroup and let $N \times R$ act on $N \times R/\Lambda$ by

$$(n, r).(n', [r']) := (nn'f(r)^{-1}, [rr']).$$

Then $\mathrm{pr}_2\colon N \times R/\Lambda \to R/\Lambda$ is a trivial N-torsor, but R acts on the base in such a way that every point has a non-trivial stabiliser whereas the R-action on $N \times R/\Lambda$ is free, since f is injective. Therefore, pr_2 admits no R-equivariant sections.

Definition 3.19. Let $\chi \in X(G)$ be a character and $L \in \mathrm{Pic}^G X$ a linearisation. We denote with $L(\chi)$ the twisted linearisation, i.e.

$$g * y := \chi(g)g.y, \quad y \in L_x, g \in G$$

for all $x \in X$.

We assume that there exists a positive grading $\lambda\colon \mathbb{G}_{m,k} \to Z(R)$ of $R_u(G)$.

Theorem 3.20 ([3, Thm. 2.4, Cor. 7.10, Rmk. 7.11])**.** *Let $L \in \mathrm{Pic}^G X$ be ample and X projective. Assume that there exists a character $\chi \in X(G)$ such that $\langle \chi, \lambda \rangle$ is the maximal weight of λ on $H^0(X, L^m)$ for some $m > 0$ for which L^m is very ample, and* *set $K := L^m(-\chi)$. If $\mathrm{Stab}_{R_u(G)}(x)$ is trivial for all $x \in X^{ss,R}(K)$, then the good quotient Z of $X^{ss,G}(K,L)$ in Theorem 3.13 is projective. Furthermore*

$$x \in X^{ss,G}(K,L) \iff R_u(G).x \subseteq X^{ss,R}(L_\infty).$$

Theorem 3.20 is more general than [3, Thm. 2.4] since we do not assume that X is irreducible. Its proof will occupy the rest of this section. As before, let Y denote the quotient of X_0 by $R_u(G)$. The aim is to construct a projective closure $\overline{Y}$ of Y together with an R-linearisation $\overline{N}$ such that $Y^{ss,R}(N_\infty) = \overline{Y}^{ss,R}(\overline{N})$. Then Z will be the quotient by the R-action on the semi-stable locus of the projective scheme $\overline{Y}$ and hence projective. The general strategy and Theorem 3.5, which is crucial in the construction, is also contained in [3]. The reader may please consult the notes at the end of the chapter for a further comparison to [3].

Lemma 3.21. *Let $\omega := \langle \chi, \lambda \rangle$ be the maximal weight of λ on $H^0(X, L^m)$. Then $dn\omega$ is the maximal weight of λ on $S^d H^0(X, L^{mn})$ for all $n, d > 0$.*

Proof. It is enough to prove the claim for $d = 1$. Let ν denote the maximal weight of λ on $H^0(X, L^{mn})$. Since L^m is very ample, the space of global sections $H^0(X, L^{mn})$ induces for all $n > 0$ a closed immersion $\iota_n \colon X \to \mathbb{P}(H^0(X, L^{mn}))$. Let $x \in \iota_n(X)$ and choose a lift $x^* \in H^0(X, L^{mn})^*$. Let $x^* = \sum_{k \in \mathbb{Z}} v_k$ be the weight space decomposition of x^* with respect to the λ-action. If $v_{-\nu} \neq 0$ then the limit point $\overline{x} := \lim_{t \to 0} \lambda(t).x$ will be the line in $H^0(X, L^{mn})^*$ spanned by $v_{-\nu}$. Note that $\iota_n(X)$ cannot be contained in any proper linear subspace. Hence $x \in \iota_n(X)$ can be chosen in such a way that $v_{-\nu} \neq 0$. This implies that ν is the weight on the geometric fibre of $\mathcal{O}_{\mathbb{P}(H^0(X,L^{mn}))}(1)$ over $\overline{x} \in \iota_n(X)$. Hence ν is the weight on the geometric fibre $(L^{mn})_{\iota_n^{-1}(\overline{x})} = (L_{\iota_n^{-1}(\overline{x})})^{mn}$. On the other hand the maximum of all weights of λ on geometric fibres of L^{mn} over fixed points is $n\omega$. This implies $\nu = n\omega$. $\square$

Lemma 3.22. *If $f \in H^0(X, K^n)^R$ is an R-invariant section, then f is an $n\chi$-weight vector for the R-action on $H^0(X, L^{mn})$. Moreover, let $\omega = \langle \chi, \lambda \rangle$. We have the following inclusions*

$$(S^d H^0(X, K^n))^R \subseteq (S^d H^0(X, L^{mn}))_{nd\omega} \subseteq (S^d H^0(X, L^{mn}))^{R_u(G)}$$

for all $n, d > 0$.

Proof. A section $f \in H^0(X, K^n)$ is R-invariant if and only if $r.f = \chi(r)^{-n} f$ for all $r \in R$. This proves the first claim. By passing to the symmetric power, this also shows the first inclusion since $\langle \chi, \lambda \rangle = \omega$.

Let $\rho \colon G \to \mathrm{GL}(S^\bullet H^0(X, L^{mn}))$ denote the action of G induced by the linearisation on L^{mn}. Then ρ defines a left-action σ of G on $\operatorname{Spec} S^\bullet H^0(X, L^{mn})$. The action σ has the property that its action by left-translation on $S^\bullet H^0(X, L^{mn})$ equals ρ. Let

$$\delta \colon T_e G \to \mathrm{Der}(S^\bullet H^0(X, L^{mn}), S^\bullet H^0(X, L^{mn}))$$

denote the infinitesimal action. Then

$$(S^d H^0(X, L^{mn}))^{R_u(G)} = \bigcap_{\xi \in T_e R_u(G)} \operatorname{Ker}(\delta_\xi)$$

by Lemma 1.19. Note that δ_ξ is homogeneous of degree zero for any $\xi \in T_e G$.

Let $f \in S^\bullet H^0(X, L^{mn})$ with λ-weight $nd\omega$ and $\xi \in T_e R_u(G)$. Since λ is a positive grading and because $nd\omega$ is the maximal λ-weight on $S^\bullet H^0(X, L^{mn})$, we must have $\delta_\xi(f) = 0$ by Lemma 3.3. This proves the second inclusion. □

Corollary 3.23. *We have* $(S^d H^0(X, K^n))^R = (S^d H^0(X, K^n))^G$ *for all* $n, d > 0$.

Proof. Since $\chi|_{R_u(G)}$ is trivial, we must have

$$(S^d H^0(X, K^n))^{R_u(G)} = (S^d H^0(X, L^{mn}))^{R_u(G)}.$$

Therefore Lemma 3.22 implies the inclusion from left into right. There is nothing to show for the reverse inclusion. □

We may replace L by L^m and henceforth assume that $K = L(-\chi)$. Then K is very ample by assumption. Let

$$S := \Gamma_*(X, \mathcal{O}_X) = \bigoplus_{p \geq 0} H^0(X, K^p)$$

be the total homogeneous coordinate ring of X with respect to K. After passing to higher powers if necessary we may further assume that the canonical map

$$S^\bullet H^0(X, K) \to S$$

is surjective and that the invariant ring S^R is generated in degree one. Let $f_1, \ldots, f_m$ denote a basis of $H^0(X, K)^R$. These functions then generate S^R. Since both canonical maps

$$S^{dr} H^0(X, K) \to H^0(X, K^{rd}) \text{ and } S^d H^0(X, K^r) \to H^0(X, K^{rd})$$

have the same image, the canonical map

$$S^\bullet H^0(X, K^r) \to S^{(r)} = \bigoplus_{p \geq 0} H^0(X, K^{rp})$$

is surjective for all $r > 0$. If we replace in the following K by some positive power, we replace $f_1, \ldots, f_m$ by their monomials of appropriate degree and rename them again by $f_1, \ldots, f_m$. In this way we may always assume that $f_1, \ldots, f_m$ generate S^R.

The set $X^{ss,R}(K)$ is covered by X_{f_i} for $i = 1, \ldots, m$ since $f_1, \ldots, f_m$ generate the invariant ring S^R. By Lemma 3.22, each X_{f_i} is $R_u(G)$-invariant. Moreover by the same lemma, every section f_i is a λ-weight vector of maximal weight in $H^0(X, L)$.

Hence, by Lemma 3.21, all weights of λ are non-positive on the homogeneous localisation $A_{(f_i)}$ for

$$A := \bigoplus_{p \geq 0} H^0(X, L^p).$$

Since $X_{f_i} = \operatorname{Spec} A_{(f_i)}$, we conclude from Theorem 3.5 that X_{f_i} is a Zariski-trivial $R_u(G)$-torsor admitting a T-equivariant section for some maximal torus T in R containing the image of λ. This shows that under the additional hypotheses of Theorem 3.20, the set X_0 in the proof of Theorem 3.13 equals $X^{ss,R}(K)$.

As before (K, L) descends to (M, N) and every section $f_i \in H^0(X, K)$ descends to a section $s_i \in H^0(Y, N)$. The affine open sets Y_{s_i} cover Y.

Next we find embeddings of Y into different projective spaces. Let $V_{n,d}$ be the image of the restriction map

$$H^0(X, (L_d)^n)^{R_u(G)} \to H^0(X_0, (L_d)^n)^{R_u(G)}$$

composed with the R-equivariant isomorphism

$$H^0(X_0, (L_d)^n)^{R_u(G)} \cong H^0(Y, (N_d)^n)$$

induced by the pull-back with $\varphi\colon X_0 \to Y$, see Proposition 2.36.

Note that by assumption $K \cong L$ and $M \cong N$ as $R_u(G)$-linearised line bundles, i.e. forgetting the R-linearisation, since every character of a unipotent group is trivial. All monomials

$$s^\alpha := s_1^{\alpha_1} \cdots s_m^{\alpha_m} \text{ for } \alpha \in \mathbb{N}_0^m \text{ with } |\alpha| := \alpha_1 + \ldots + \alpha_m = n(d+1)$$

are global sections of $(N_d)^n$ with the property that $\varphi^*(s^\alpha) = f^\alpha$. Since f^α is defined on all of X for all α, $s^\alpha \in V_{n,d}$ for all α. Therefore, every $V_{n,d}$ determines a morphism $\iota_{n,d}\colon Y \to \mathbb{P}(V_{n,d})$ because Y is covered by Y_{s^α}. The morphism $\iota_{n,d}$ is R-equivariant with respect to the R-action on $V_{n,d}$. The bundle $\mathcal{O}_{\mathbb{P}(V_{n,d})}(1)$ carries a canonical R-linearisation. It pulls back to $(N_d)^n$ as R-linearised line bundle. Note that

$$\iota_{n,d}^* \mathcal{O}_{\mathbb{P}(V_{n,d})}(1)(-n\chi) = (N_d)^n(-n\chi) = M^{n(d+1)}$$

as R-linearised line bundles.

Lemma 3.24. *There exists $n_0 \in \mathbb{N}$ such that*

$$\iota_{n,d}\colon\ Y \longrightarrow \mathbb{P}(V_{n,d})^{ss,R}(\mathcal{O}(1)(-n\chi))$$

is a closed immersion for all $n(d+1) \geq n_0$. In particular $\iota_{n,d}\colon Y \to \mathbb{P}(V_{n,d})$ is an immersion for $n(d+1) \geq n_0$.

Proof. For given $n, d \in \mathbb{N}$, let

$$\rho_{n,d}\colon S^\bullet(H^0(X, (L_d)^n)^{R_u(G)}) \to (S^{(n(d+1))})^{R_u(G)}$$

denote the canonical map which fits in the following commutative square.

$$\begin{tikzcd} S^\bullet(H^0(X,(L_d)^n)^{R_u(G)}) \arrow[r, two heads] \arrow[d, "\rho_{n,d}"] & S^\bullet V_{n,d} \arrow[d, "\iota^*_{n,d}"] \\ (S^{(n(d+1))})^{R_u(G)} \arrow[r] & \bigoplus_{p\geq 0} H^0(Y,(N_d)^{np}) \end{tikzcd}$$

We set $W_{n,d} := H^0(X, K^{n(d+1)})^{R_u(G)} = H^0(X,(L_d)^n)^{R_u(G)}$. The horizontal surjection in the top line of the square induces an R-equivariant closed immersion

$$\mathbb{P}(V_{n,d}) \hookrightarrow \mathbb{P}(W_{n,d})$$

such that the pull-back of the canonical linearisation $\mathcal{O}_{\mathbb{P}(W_{n,d})}(1)$ is $\mathcal{O}(1)(-n\chi)$.

Therefore

$$\mathbb{P}(V_{n,d})^{ss,R}(\mathcal{O}(1)(-n\chi)) = \mathbb{P}(V_{n,d}) \cap \mathbb{P}(W_{n,d})^{ss,R}(\mathcal{O}(1)).$$

Next, we construct an admissible covering of $\mathbb{P}(W_{n,d})^{ss,R}(\mathcal{O}(1))$ by principal open subsets. If $h_1,\dots,h_l \in (S^\bullet W_{n,d})^R$ are homogeneous ring generators, then

$$\mathbb{P}(W_{n,d})^{ss,R}(\mathcal{O}(1)) = \bigcup_{i=1}^{l} \mathbb{P}(W_{n,d})_{h_i}$$

Let $\overline{h_i}$ denote the image of h_i in $(S^{(n(d+1))})^R$ under the composition

$$(S^\bullet W_{n,d})^R \subseteq (S^\bullet H^0(X,K^{n(d+1)}))^R \twoheadrightarrow (S^{(n(d+1))})^R\ .$$

By our assumption from the beginning of the proof of Theorem 3.20, $(S^{(n(d+1))})^R$ is generated by the images of all f^α with $|\alpha| = n(d+1)$, which we denote in the following by $g_1,\dots,g_m$. We may therefore choose coefficients $c^i_\beta \in k$ such that

$$\overline{h_i} = \sum_{|\beta|=\deg h_i} c^i_\beta g^\beta$$

in $(S^{(n(d+1))})^R$. Note that $g_i \in (W_{n,d})^R$ for all $i = 1,\dots,m$. The collection of elements

$$g_1,\dots,g_m, h_1 - \sum_{|\beta|=\deg h_1} c^1_\beta g^\beta, \dots, h_l - \sum_{|\beta|=\deg h_l} c^l_\beta g^\beta \in S^\bullet W_{n,d}$$

is a second set of homogeneous ring generators for the invariant ring $(S^\bullet W_{n,d})^R$. Therefore,

$$\mathbb{P}(W_{n,d})^{ss,R}(\mathcal{O}(1)) = \bigcup_{i=1}^{m} \mathbb{P}(W_{n,d})_{g_i} \cup \bigcup_{j=1}^{l} \mathbb{P}(W_{n,d})_{h_i - \sum_{|\beta|=\deg h_i} c^i_\beta g^\beta}$$

Now we intersect with $\mathbb{P}(V_{n,d})$. The restriction of all sections $h_i - \sum_{|\beta|=\deg h_i} c^i_\beta g^\beta$ to Y vanishes. This follows from the fact that

$$\rho_{n,d}(h_i - \sum_{|\beta|=\deg h_i} c^i_\beta g^\beta) = 0$$

by construction of c^i_β, and the fact that the square at the beginning of this proof is commutative. To prove the lemma, it is therefore enough to show that $\iota_{n,d}(Y)$ is closed inside

$$\bigcup_{|\alpha|=n(d+1)} \mathbb{P}(V_{n,d})_{s^\alpha}$$

for (n,d) still to be specified. (The reader may please recall that we denoted the descent of f_i to Y by s_i.)

Note that

$$Y_{s^\alpha} = \operatorname{Spec} \mathcal{O}_X(X_{f^\alpha})^{R_u(G)} = \operatorname{Spec}(S_{(f^\alpha)})^{R_u(G)}.$$

since as before X_{f^α} is a trivial $R_u(G)$-torsor over the base Y_{s^α} by Theorem 3.5.

Claim. Localisation commutes with taking invariants, i.e. $(S_{f_i})^{R_u(G)} = (S^{R_u(G)})_{f_i}$.

Proof. Note that we do not assume that S is integral. Since f_i is $R_u(G)$-invariant we clearly have $(S_{f_i})^{R_u(G)} \supseteq (S^{R_u(G)})_{f_i}$. For the other inclusion let $\frac{s}{f_i^p} \in (S_{f_i})^{R_u(G)}$. Hence for every $g \in R_u(G)$ there exists a $q(g) \in \mathbb{N}$ such that

$$f_i^{q(g)}(g.s - s) = 0$$

in S. This means that $g.s - s$ is contained in the ideal $(0 : f_i^\infty)$ of S. Since S is noetherian there exists some q_0 such that $(0 : f_i^\infty) = (0 : f_i^{q_0})$. This implies $g.(f_i^{q_0} s) = f_i^{q_0} s$ for all $g \in R_u(G)$ and therefore

$$\frac{s}{f_i^p} = \frac{f_i^{q_0} s}{f_i^{p+q_0}} \in S^{R_u(G)}_{(f_i)}$$

□

Let α be a multiindex such that $|\alpha| = n(d+1)$. The diagram

$$\begin{tikzcd}
(S^\bullet V_{n,d})_{(s^\alpha)} \arrow[r, "\iota^\sharp_{n,d}"] & S^{R_u(G)}_{(f^\alpha)} \\
(S^\bullet W_{n,d})_{(f^\alpha)} \arrow[u, two heads] \arrow[ur, "\rho_{n,d}"'] &
\end{tikzcd}$$

commutes. Therefore it is enough to find n_0 such that the localisation of $\rho_{n,d}$ with respect to f^α in the diagram is surjective for any choice of α with the property that $|\alpha| = n(d+1) \geq n_0$. The invariant rings $S^{R_u(G)}_{(f_i)}$ are the coordinate rings of the base of the $R_u(G)$-torsor X_{f_i}. This torsor is trivial and admits therefore a section.

Hence, the invariant ring $S_{(f_i)}^{R_u(G)}$ is finitely generated for all i. This implies together with the above claim, that there exists n_0 with the following property. For any choice of n, d with $n(d+1) \geq n_0$, there exists $R_u(G)$-invariant global sections $g_1, \dots, g_l$ of $K^{n(d+1)} = (L_d)^n$ such that

$$\frac{g_1}{f_i^{n(d+1)}}, \dots, \frac{g_l}{f_i^{n(d+1)}} \in S_{(f_i)}^{R_u(G)}$$

are ring generators for all i.

Let $n, d \in \mathbb{N}$ have the property that $n(d+1) \geq n_0$ and let α be a multiindex for which $|\alpha| = n(d+1)$. We choose i such that $\alpha_i \geq 1$ and may write

$$S_{(f^\alpha)}^{R_u(G)} = (S_{(f_i)}^{R_u(G)})_{\frac{f^\alpha}{f_i^{|\alpha|}}}.$$

Therefore, the elements

$$\frac{g_1}{f^\alpha}, \dots, \frac{g_l}{f^\alpha}, \frac{f_i^{|\alpha|}}{f^\alpha} \text{ and } \frac{f^\alpha}{f_i^{|\alpha|}} = \frac{f^\alpha f^{|\alpha|(\alpha - e_i)}}{f^{|\alpha|\alpha}}$$

generate the ring $S_{(f^\alpha)}^{R_u(G)}$. Since all are contained in the image of the localisation of $\rho_{n,d}$ with respect to f^α, we conclude that in the above triangular diagram, $\rho_{n,d}$ is surjective. This proves the lemma. $\square$

Lemma 3.25. *Let $n_0 \in \mathbb{N}$ be as in Lemma 3.24. Let $Y^{ss,R}(N_\infty)$ denote the semi-stable locus $Y^{ss,R}(N_d)$ for $d \gg 0$. There exists n and d with $n(d+1) \geq n_0$ such that*

$$Y^{ss,R}(N_\infty) = \iota^{-1}(\mathbb{P}(V_{n,d})^{ss,R}(\mathcal{O}(1)))$$

Proof. The inclusion from right into left is clear by pull-back of sections.

For the other direction let $d \gg 0$ be large enough so that $Y^{ss,R}(N_d) = Y^{ss,R}(N_\infty)$ and choose global section $t_1, \dots, t_k \in H^0(Y, (N_d)^n)^R$ such that $Y^{ss,R}(N_\infty)$ is covered by Y_{t_j}, $j = 1, \dots, k$. Hence for every $p \geq 0$ the collection of all $Y_{t_j s_i^p}$ cover $Y^{ss,R}(N_\infty)$ as well. Let $p = nq$ for some $q \in \mathbb{N}$. Then

$$t_j s_i^p \in H^0(Y, (N_{d+q})^n)^R \cong H^0(X_0, (L_{d+q})^n)^G.$$

Let $t_j s_i^p$ be represented by $u_j f_i^p$ on the right hand side. For every p sufficiently large, $u_j f_i^p$ extends to a global G-invariant section of $(L_{d+q})^n$. Hence if we choose q large enough, $t_j s_i^p$ is an element in $(V_{n,d+q})^R$ for all i, j. This is what we had to show. $\square$

Proof of Theorem 3.20. Let n and d be as in Lemma 3.25. We set $V := V_{n,d}$ and replace Y by $\iota_{n,d}(Y) \subseteq \mathbb{P}(V)$. We may assume that $\mathcal{O}(1)$ restricts to N_d as R-linearised line bundle after we replaced (K, L) and (M, N) by (K^n, L^n) and (M^n, N^n) respectively.

The induced R-action on $\mathbb{P}(V)$ is linearised by the line bundles

$$F_e := (\mathcal{O}(1)(-\chi))^e \otimes \mathcal{O}(1) \in \mathrm{Pic}^R(\mathbb{P}(V)), e > 0.$$

Since $\mathcal{O}(1)(-\chi)|_Y = N_d(-\chi) = M^{d+1}$, we also have $F_e|_Y = N_{e(d+1)+d}$ for all positive e. By Proposition 3.6, there is an $e_0 > 0$ such that $\mathbb{P}(V)^{ss,R}(F_e) = \mathbb{P}(V)^{ss,R}(F_{e_0})$ for all $e \geq e_0$.

Claim. $Y^{ss,R}(N_\infty) = Y \cap \mathbb{P}(V)^{ss,R}(F_\infty)$.

Proof of claim. Note that every s_i^{d+1} is an element in $H^0(\mathbb{P}(V), \mathcal{O}(1)(-\chi))^R$. Let x be a point in the intersection $Y \cap \mathbb{P}(V)^{ss,R}(\mathcal{O}(1))$. This intersection equals $Y^{ss,R}(N_\infty)$ by Lemma 3.25. Then there exists some global section $g \in H^0(\mathbb{P}(V), \mathcal{O}(p))^R$ such that $g(y) \neq 0$. Since Y is covered by all Y_{s_i} there exists some s_i with $s_i(y) \neq 0$. Hence $(s_i^{d+1})^{pe} g$ is an R-invariant global section of $(F_e)^p$ on $\mathbb{P}(V)$ for all $e \in \mathbb{N}$. This proves the inclusion from left into right.

For the reverse inclusion, it is enough to observe that for every $e, p > 0$, every invariant section of $(F_e)^p$ restricts to an invariant section of $(N_{e(d+1)+d})^p$ over Y. □

Let $\overline{Y}$ denote the scheme-theoretic closure of Y in $\mathbb{P}(V)$. Since all our schemes are noetherian all morphisms are quasi-compact. Hence the underlying topological space of $\overline{Y}$ is the topological closure of Y in $\mathbb{P}(V)$. We linearize the induced R-action on $\overline{Y}$ with $\overline{N} := F_e|_{\overline{Y}}$ for some $e \gg 0$ chosen large enough so that

$$\mathbb{P}(V)^{ss,R}(F_e) = \mathbb{P}(V)^{ss,R}(F_\infty).$$

Since

$$\mathbb{P}(V)^{ss,R}(F_e) \subseteq \mathbb{P}(V)^{ss,R}(\mathcal{O}(1)(-\chi))$$

by Proposition 3.6 and since Y is closed in $\mathbb{P}(V)^{ss,R}(\mathcal{O}(1)(-\chi))$ by Lemma 3.24, Y is closed in $\mathbb{P}(V)^{ss,R}(F_e)$. We calculate

$$\overline{Y}^{ss,R}(\overline{N}) = \overline{Y} \cap \mathbb{P}(V)^{ss,R}(F_e) = Y \cap \mathbb{P}(V)^{ss,R}(F_e) = Y^{ss,R}(N_\infty).$$

This proves the theorem. □

3.3. **Further analysis of stability.** We continue with the setting from Theorem 3.20. The last statement

$$x \in X^{ss,G}(K, L) \iff R_u(G).x \subseteq X^{ss,R}(L_\infty)$$

about the locus $X^{ss,G}(K, L)$ may be understood as a non-reductive Hilbert-Mumford criterion. The analogous statement for stable points

$$x \in X^{s,G}(K, L) \iff R_u(G).x \subseteq X^{s,R}(L_\infty) \qquad (*)$$

is claimed in [3, Cor. 7.10]. We were not able to reproduce it but made some progress if G is a connected solvable group. Henceforth let us assume that $G = U_T := U \rtimes T$, where U is a unipotent group and T a torus.

Lemma 3.26. *Let $x \in X$. If $\operatorname{Stab}_U(x)$ is trivial, then $\operatorname{Stab}_{U_T}(x)$ is finite if and only if $\operatorname{Stab}_T(u.x)$ is finite for all $u \in U$.*

Proof. One direction is easy:

$$\operatorname{Stab}_T(u.x) \subseteq \operatorname{Stab}_{U_T}(u.x) = u \operatorname{Stab}_{U_T}(x) u^{-1}$$

Hence, if $\operatorname{Stab}_{U_T}(x)$ is finite, then so is $\operatorname{Stab}_T(u.x)$ for all $u \in U$.

For the reverse direction we restrict the exact sequence

$$1 \longrightarrow U \longrightarrow U_T \longrightarrow T \longrightarrow 1$$

to $\operatorname{Stab}_{U_T}(x)$. Since $\operatorname{Stab}_U(x)$ is trivial we obtain

$$1 \longrightarrow 1 \longrightarrow \operatorname{Stab}_{U_T}(x) \longrightarrow S \longrightarrow 1$$

Therefore $\operatorname{Stab}_{U_T}(x)$ is isomorphic to a closed subgroup of T since any bijective group homomorphism is an isomorphism in characteristic zero. This implies that the connected component $\operatorname{Stab}_{U_T}(x)^\circ$ of the neutral element is a torus. Therefore there exists a $u \in U$ such that

$$\operatorname{Stab}_{U_T}(u.x)^\circ = u \operatorname{Stab}_{U_T}(x)^\circ u^{-1} \subseteq T.$$

Hence, $\operatorname{Stab}_{U_T}(u.x)^\circ$ is trivial since $\operatorname{Stab}_T(u.x)$ is finite. Therefore, $\operatorname{Stab}_{U_T}(u.x)$ is finite which implies that $\operatorname{Stab}_{U_T}(x)$ is finite. $\square$

Corollary 3.27. *Assume that the hypotheses of Theorem 3.20 are satisfied with the solvable group $G = U_T$ and that in addition $X^{ss,T}(L_\infty) = X^{s,T}(L_\infty)$, then*

$$X^{ss,U_T}(K, L) = X^{s,U_T}(K, L).$$

In particular there exists a projective geometric quotient of $X^{ss,U_T}(K, L)$ by U_T.

Proof. Let $x \in X^{ss,U_T}(K, L)$. Then

$$U.x \subseteq X^{ss,T}(L_\infty) = X^{s,T}(L_\infty)$$

Therefore $\operatorname{Stab}_T(u.x)$ is finite for all $u \in U$. This implies that $\operatorname{Stab}_{U_T}(x)$ is finite by Lemma 3.26. In particular all U_T-orbits in $X^{ss,U_T}(K, L)$ have the same dimension and are therefore closed in $X^{ss,U_T}(K, L)$. $\square$

Lemma 3.28. *Let $x \in X^{ss,U_T}(K, L)$ have a finite stabiliser $\operatorname{Stab}_{U_T}(x)$. Assume that the hypotheses of Theorem 3.20 are satisfied with $G = U_T$. The following statements are equivalent.*

1. *The orbit $U_T.x$ is closed in $X^{ss,U_T}(K, L)$.*
2. *The orbit $T.(u.x)$ is closed in $X^{ss,U_T}(K, L)$ for all $u \in U$.*

Proof of Lemma 3.28. We assume the first assertion. Let $\psi_x \colon U_T \to U_T.x$ denote the orbit map. Since $\mathrm{Stab}_{U_T}(x)$ is finite, it follows that ψ_x is quasi-finite. Standard arguments in the theory of homogeneous spaces imply that ψ_x is finite and therefore in particular closed. Since Tu is certainly closed in U_T, the orbit $T.(u.x)$ is closed in $U_T.x$ and henceforth closed in $X^{ss,U_T}(K,L)$.

Let us assume the second assertion. We use the notations from the proof of Theorem 3.13. Then the open sets $X^{ss,U_T}(K,L)_{f_i}$, $i = 1,\ldots,m$, cover $X^{ss,U_T}(K,L)$ and

$$\varphi\colon X^{ss,U_T}(K,L)_{f_i} \to Y^{ss,T}(N_\infty)_{s_i}$$

is a U-torsor for all $i = 1,\ldots,m$ admitting a T-equivariant section, call it t_i. It is enough to show that $U_T.x$ is closed in $X^{ss,U_T}(K,L)_{f_i}$ for all i such that $f_i(x) \neq 0$. For such an i we may choose $u \in U$ such that $u.x$ is in the image of t_i. Since t_i is a section, it is a closed immersion. As it is also T-equivariant, it follows that $T.\varphi(x)$ is closed in $Y^{ss,T}(N_\infty)_{s_i}$ because $T.(u.x)$ is closed by assumption. But then

$$\varphi^{-1}(T.\varphi(x)) = U_T.x$$

is closed in $X^{ss,T}(K,L)_{f_i}$. □

3.4. **Notes.** We comment on the results in [3] and [21]. The theorems in the foregoing section stem from the attempt to understand the methods and proofs in [3]. Most notably we did not comprehend [3, Lem. 7.8] as well as the Remarks [3, Rem. 2.5 and Rem. 7.11] which claim a weakening of a certain stabiliser condition which in turn would imply Theorem 3.20.

V. Hoskins and J. Jackson explain in [21] parts of [3] again, but under stronger assumptions. The proofs in [3] and [21] rely heavily on the Hilbert-Mumford-criterion in the version of Theorem 2.42. In the paragraph before [21, Thm. 2.28], they write:

[...] one would like a description of $\mathrm{dom}(q)$ *in terms of* T*-weights on* Y*, but passing to the* U*-quotient involves deleting some of these weights.*

We were also unable to understand how Bérczi et al. handled this possible deleting of weights while passing to the $R_u(G)$-quotient. To circumvent this problem we did not apply the Hilbert-Mumford-criterion. Instead, we used the existence of a T-equivariant section for every $R_u(G)$-torsor $X_{f_i} \to Y_i$ in the proof of Theorem 3.13 and 3.20, see also Theorem 3.5. This also has the advantage that we do not need to assume that X is projective in Theorem 3.13, as is done throughout in [3] and [21].

Further, to prove projectivity of the good quotient in [3] the authors assume that X is irreducible and use the notion of an *enveloping quotient* which they introduced in an earlier work [4]. We prove Theorem 3.20 without the irreducible assumption on X and refer solely to classical geometric invariant theory. The irreducibility assumption on X, that is crucial in [3], is not mentioned anymore in [21].

Since [21] seems to us to contain the newest formulations of the theorems in [3], we prefer to state Theorem 2.9 from [21], so the reader may compare it to Theorem 3.20.

Let G be an affine algebraic group acting on a projective variety X/k and let L be a very ample G-linearisation. We choose a Levi-decomposition $G = U \rtimes R$, where U denotes the unipotent radical and R is a reductive subgroup of G. Further, let $\lambda\colon \mathbb{G}_{m,k} \to Z(R)$ be a positive grading and let $\omega_{min} < \omega_{min+1} < \ldots < \omega_n$ denote the weights of λ on $H^0(X,L)^*$. They introduce the following sets:

$$\begin{aligned} Z_{min} &:= X \cap \mathbb{P}(H^0(X,L)_{max}) \\ &= \{x \in X^{\lambda(\mathbb{G}_{m,k})} \mid \lambda \text{ acts on } L^*|_x \text{ with weight } \omega_{min}\} \\ X_{min} &:= \{x \in X \mid \lim_{t\to 0} \lambda(t).x \in Z_{min}\} \end{aligned}$$

Let $\overline{R} := R/\lambda(\mathbb{G}_{m,k})$ and denote by Z^{ss}_{min} respectively Z^{s}_{min} the loci of $\overline{R}$-semi-stable respectively stable sets in Z_{min} with respect to L. Further, let

$$\begin{aligned} X^{ss}_{min} &:= \{x \in X \mid \lim_{t\to 0} \lambda(t).x \in Z^{ss}_{min}\} \\ X^{s}_{min} &:= \{x \in X \mid \lim_{t\to 0} \lambda(t).x \in Z^{s}_{min}\} \end{aligned}$$

Definition 3.29. The linearisation L is adapted if $\omega_{min} < 0 < \omega_{min+1}$.

Let $\hat{U} := U \rtimes \lambda(\mathbb{G}_{m,k})$.

Definition 3.30. Let L be adapted.

1. The $\hat{U}$-stable locus is defined by
$$X^{\hat{U}-s} := X_{min} \setminus UZ_{min} = \bigcap_{u\in U} uX^{\lambda(\mathbb{G}_{m,k})-s}$$
2. The G-stable locus is defined by
$$X^{G-s} := X^{s}_{min} \setminus UZ^{s}_{min}.$$

We are now in the position to state the results in [21] which are analogous to ours.

Theorem 3.31 ([21, Thm. 2.29])**.** *Let L be adapted. If $\dim \operatorname{Stab}_U(x) = 0$ for all $x \in X^{ss}_{min}$ then the set $Y^{ss}_{min} \setminus UZ^{ss}_{min}$ admits a geometric $\hat{U}$-quotient Y*

$$q_{\hat{U}}\colon X^{ss}_{min} \setminus UZ^{ss}_{min} \to (X^{ss}_{min} \setminus UZ^{ss}_{min})/\hat{U} =: Y$$

Furthermore, let q denote the composition of this quotient with the good quotient $q_{\overline{R}}$ by the induced $\overline{R}$ action on Y linearised by the descent of L:

$$q\colon\ X^{ss}_{min} \setminus UZ^{ss}_{min} \xrightarrow{q_{\hat{U}}} Y \overset{q_{\overline{R}}}{\dashrightarrow} X/\!\!/G := Y^{ss,\overline{R}}/\!\!/\overline{R}$$

Then q defines a projective good quotient on its domain of definition $\operatorname{dom}(q)$. If in addition $\dim \operatorname{Stab}_{\overline{R}}(z) = 0$ for all $z \in Z^{ss}_{min}$, then $X^{G-s} = \operatorname{dom}(q)$ and q is a projective geometric quotient for the G-action.

We finish this section with a discussion of the blowing-up procedure presented in [3] and [21]. Let us restrict to the case of a $\hat{U}$-action. Then there exists a projective geometric $\hat{U}$-quotient of $X^{\hat{U}-s}$ if $\mathrm{Stab}_U(z)$ is trivial for all $z \in Z_{min}$ by Theorem 3.31, see also [21, Thm. 2.26]. Bérczi et al. describe in [3] a sequence of blow-ups of X respectively X_{min} such that this stabiliser condition can be satisfied for an appropriate choice of a linearisation on the blow-up. We give the following example which seems to be in contradiction with the claims in [3]. On the other hand, the blow-up construction given in [21] occurs to have the proposed properties in this particular example.

We consider the groups

$$\hat{U} := \left\{ \begin{pmatrix} t^{-1} & & \\ & t^{-1} & \\ u_1 & u_2 & t \end{pmatrix} \ : \ u_i \in k, t \in k^* \right\}, U := \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ u_1 & u_2 & 1 \end{pmatrix} \ : \ u_i \in k \right\}$$

and $\lambda(t) = \mathrm{diag}(t^{-1}, t^{-1}, t)$. The extension $\hat{U} = U \rtimes_\lambda \mathbb{G}_{m,k}$ acts on the space $V := k^{3\times n}$ of three by n matrices by left multiplication.

We write an element $v \in V$ as

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_{11} & \dots & v_{1n} \\ v_{21} & \dots & v_{2n} \\ v_{31} & \dots & v_{3n} \end{pmatrix}$$

and choose coordinates $\{x_{ij}\}$ dual to $\{v_{ij}\}$ on V. The $\hat{U}$-action on V induces a $\hat{U}$-action on $X := \mathbb{P}(V^*)$ which is canonically linearised by $\mathcal{O}(1)$. We will denote points $x \in X$ by $(v_1 : v_2 : v_3)$ where v_i denotes the ith row of $v \in V$.

The $\hat{U}$-action is adapted with

$$\begin{aligned} Z_{min} &= \{(v_1 : v_2 : v_3) \in X \mid v_3 = 0\} \\ X_{min} &= \{(v_1 : v_2 : v_3) \in X \mid v_1 \neq 0 \text{ or } v_2 \neq 0\}. \end{aligned}$$

There are two possibilities for $\dim \mathrm{Stab}_U(x)$ for $x = (v_1 : v_2 : v_3) \in X_{min}$: the stabiliser $\mathrm{Stab}_U(x)$ is trivial if v_1 and v_2 are linearly independent, and is one-dimensional otherwise. By Definition [3, Def. 8.4], Bérczi et al. blow-up X_{min} in the locus of points $x \in X_{min}$ for which $\dim \mathrm{Stab}_u(x)$ is maximal. In this example, this is the locus where v_1 and v_2 are linearly dependent.

Let I denote the ideal sheaf of the center of the blow-up $\pi\colon \widetilde{X_{min}} \to X_{min}$ and I' its inverse image ideal sheaf. Since $I(2)$ is globally generated by the maximal minors of the matrix

$$\begin{pmatrix} x_{11} & \dots & x_{1n} \\ x_{21} & \dots & x_{2n} \end{pmatrix}$$

the sheaf $\pi^*(\mathcal{O}_X(2)) \otimes I'$ is very ample. Its global sections

$$\pi^*(x_{ij}x_{kl}) \otimes (x_{1p}x_{2q} - x_{1q}x_{2p}) \in H^0(\widetilde{X_{min}}, \pi^*(\mathcal{O}_X(2)) \otimes I')$$

define an immersion

$$\widetilde{X_{min}} = \mathbf{Proj} \bigoplus_{d \geq 0} I^d \cong \mathbf{Proj} \bigoplus_{d \geq 0} I(2)^d \hookrightarrow X_{min} \times \mathbb{P}^N$$

such that $\mathrm{pr}_1^*(\mathcal{O}_X(2)) \otimes \mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^N}(1))$ pulls-back to $\pi^*(\mathcal{O}_X(2)) \otimes I'$.

We note that all x_{ij} with $i \leq 2$ are U-invariants. In particular, all minors

$$x_{1p}x_{2q} - x_{1q}x_{2p}$$

are U-invariants. As a consequence, the induced U-action on $\widetilde{X_{min}}$ as a subscheme of the product $X_{min} \times \mathbb{P}^N$ is

$$u.(x, y) = (u.x, y) \text{ for all } u \in U, (x, y) \in \widetilde{X_{min}}$$

Therefore, we must have $\mathrm{Stab}_U(x) = \mathrm{Stab}_U(\pi(x))$ for all $x \in \widetilde{X_{min}}$. This is a contradiction to [3, Prop. 8.8].

We have not checked the proofs in [21] for the blowing-up procedure presented there, but we did the calculations for this particular example.

Let Δ denote the U-orbit in X_{min} of the closed subset of X_{min} given by

$$Z_{min}^{d_{max}} := \{z \in Z_{min} \mid \dim \mathrm{Stab}_U(z) = 1\}.$$

This orbit is again closed in X_{min}. Let I be the ideal sheaf of Δ. As above, $I(2)$ is generated by global sections. But this time there are more minors involved, namely all maximal minors of the matrices

$$\begin{pmatrix} x_{11} & \dots & x_{1n} \\ x_{21} & \dots & x_{2n} \end{pmatrix}, \begin{pmatrix} x_{11} & \dots & x_{1n} \\ x_{31} & \dots & x_{3n} \end{pmatrix} \text{ and } \begin{pmatrix} x_{21} & \dots & x_{2n} \\ x_{31} & \dots & x_{3n} \end{pmatrix}.$$

Similary as before, we may embed $\widetilde{X_{min}}$

$$\widetilde{X_{min}} = \mathbf{Proj} \bigoplus_{d \geq 0} I^d \cong \mathbf{Proj} \bigoplus_{d \geq 0} I(2)^d \hookrightarrow X_{min} \times \mathbb{P}^N$$

by the collection of all global sections

$$x_{ij}x_{kl} \otimes m \in H^0(\widetilde{X_{min}}, \pi^*(\mathcal{O}_X(2)) \otimes I')$$

where m runs through all maximal minors of the three matrices above.

Perfoming this blow-up, the unipotent group U acts on the space of global sections of $I(2)$. In particular, for $(u_1, u_2) \in U$:

$$x_{1i}x_{3j} - x_{1j}x_{3i} \mapsto x_{1i}x_{3j} - x_{1j}x_{3i} - u_2(x_{1i}x_{2j} - x_{1j}x_{2i})$$
$$x_{2i}x_{3j} - x_{2j}x_{3i} \mapsto x_{2i}x_{3j} - x_{2j}x_{3i} - u_1(x_{2i}x_{1j} - x_{2j}x_{1i}).$$

Let as in [21],

$$\widetilde{Z_{min}} := \{x \in \widetilde{X_{min}}^{\mathbb{G}_{m,k}} \mid \operatorname{diag}(t^{-1}, t^{-1}, t) \text{ acts on } L_d^*|_x \text{ with weight } -d-2\}$$

which is in fact the strict transform of Z_{min}. It can now be easily seen, that every $x \in \widetilde{X_{min}}$ such that

$$\lim_{t \to 0} \lambda(t).x \in \widetilde{Z_{min}}$$

has a trivial stabiliser in U.

In a footnote to [21, Prop. 2.35], V. Hoskins and J. Jackson choose as linearisation on the blow-up a linearisation defined on $L_d := \pi^*(\mathcal{O}_X(d)) \otimes I'$ for d sufficietly large. They claim that the linearisation can be chosen to be adapted. In our particular example, at least the canonical linearisation on L_d is not adapted for all positive d. This can be seen as follows. The maximal weight of λ on $H^0(X, \mathcal{O}_X(1))$ is one. For every section $f \in H^0(X, \mathcal{O}_X(d))$ with maximal weight d, the sections

$$(x_{1i}x_{3j} - x_{1j}x_{3i}) \otimes \pi^*(f) \text{ and } (x_{2i}x_{3j} - x_{2j}x_{3i}) \otimes \pi^*(f) \in H^0(\widetilde{X_{min}}, I' \otimes \pi^*\mathcal{O}_X(d))$$

have weight d because the minors have weight zero. On the other hand, the sections

$$(x_{1i}x_{2j} - x_{1j}x_{2i}) \otimes \pi^*(f) \in H^0(\widetilde{X_{min}}, I' \otimes \pi^*\mathcal{O}_X(d))$$

have weight $d + 2$.

4. Matrix factorisations and Shamash's construction

D. Eisenbud introduced matrix factorisations in [14] over a regular local ring. We work analogously over an $\mathbb{N}_0$-graded ring

$$R = \bigoplus_{d \geq 0} R_d$$

with $R_0 = k$ and assume that R is finitely generated over R_0.

A finite free $\mathbb{Z}$-graded R-module is a finite direct sum of modules of the form $R(a)$ for various $a \in \mathbb{Z}$.

Definition 4.1. A graded matrix factorisation of a homogeneous element $f \in R_d$ of degree d is a pair (φ, ψ) of homogeneous maps of degree zero between finite free $\mathbb{Z}$-graded R-modules

$$F \xrightarrow{\varphi} G \xrightarrow{\psi} F(d)$$

such that $\psi \circ \varphi = f \cdot \mathrm{id}_F$ and $\varphi \circ \psi = f \cdot \mathrm{id}_G$. We define a morphism α between two graded matrix factorisations (φ_1, ψ_1) and (φ_2, ψ_2) as a commutative diagram

$$\begin{tikzcd}
F_1 \arrow[r, "\varphi_1"] \arrow[d, "\alpha_1"] & G_1 \arrow[r, "\psi_1"] \arrow[d, "\alpha_2"] & F_1(d) \arrow[d, "\alpha_1"] \\
F_2 \arrow[r, "\varphi_2"] & G_2 \arrow[r, "\psi_2"] & F_2(d)
\end{tikzcd}$$

where the vertical maps α_i are homogeneous morphism of degree zero.

The work [6] together with C. Böhning and H.-C. von Bothmer reviews Shamash's construction in the graded setting. As mentioned in the introduction, it is written mostly by myself, except for parts of the proofs of [6, Lem. 1.1] and [6, Thm. 3.4]. Here, we will state the results and only indicate the ideas. The construction is already contained in [31] and in [14].

We start with a free resolution

$$L_\bullet \colon \ \dots \longrightarrow 0 \longrightarrow L_m \xrightarrow{\partial_m} L_{m-1} \xrightarrow{\partial_{m-1}} \dots \xrightarrow{\partial_2} L_1 \xrightarrow{\partial_1} L_0$$

of finite free $\mathbb{Z}$-graded R-modules. Let $M := \operatorname{Coker} \partial_1$ be annihilated by some homogeneous non-zero divisor $f \in R_d$ of positive degree d. From this, we will construct a graded matrix factorisation of f over R.

It will be important to dinstinguish two gradings on $L_\bullet = \bigoplus_{i \in \mathbb{N}_0} L_i$. First, by definition, $L_\bullet$ is $\mathbb{N}_0$-graded because it is a complex bounded on the right. The degree i part of $L_\bullet$ with respect to this grading is L_i. Second, it is a $\mathbb{Z}$-graded R-module, since every L_i is one. We denote the shift with respect to the first grading by $L_\bullet[a]$, $a \in \mathbb{Z}$ and with respect to the second grading by $L_\bullet(a)$. The differential $\partial \colon L_\bullet \to L_\bullet[-1]$ is homogeneous of degree zero with respect to both gradings.

Lemma 4.2 ([6, Lem. 1.1])**.** *For every $k = 0, \dots, \lfloor \frac{m+1}{2} \rfloor$, there exists homogeneous maps of degree 0 with respect to both gradings*

$$s_k \colon L_\bullet \to L_\bullet(kd)[2k-1]$$

such that $s_0 = \partial$ and

$$\Theta := s_0 + s_1 + s_2 + s_3 + \ldots \in \mathrm{Hom}(L_\bullet, L_\bullet)$$

fullfills $\Theta^2 = f \cdot \mathrm{id}_{L_\bullet}$ as an endomorphism of the module $L_\bullet$.

Let us give the idea of the proof. We start with $s_0 = \partial$ and construct the higher s_k inductively. If we compare in the equation $\Theta^2 = f \cdot \mathrm{id}_{L_\bullet}$ every degree i part L_i separately, then we obtain

$$s_0 s_1 + s_1 s_0 = f \cdot \mathrm{id}_{L_\bullet} \text{ and } 0 = \sum_{i+j=k, i,j \geq 0} s_i s_j \text{ for all } k \geq 2.$$

Because $f \in \mathrm{ann}(M)$, multiplication by f on $L_\bullet$ is null-homotopic. This implies, that we may choose s_1 as a homotopy. Since f has degree d, $s_1 \colon L_\bullet \to L_\bullet(d)[1]$ is homogeneous of degree zero with respect to both gradings.

Let $k \geq 1$ such that all s_l for $l \leq k$ have already been constructed. The condition for s_{k+1} is

$$s_0 s_{k+1} + s_{k+1} s_0 = - \sum_{i+j=k+1, i \geq 1, j \geq 1} s_i s_j$$

where on the right hand side only s_l with $l \leq k$ are involved. One proves now, that

$$T_k := - \sum_{i+j=k+1, i \geq 1, j \geq 1} s_i s_j$$

as a homogeneous morphism $T_k \colon L_\bullet \to L_\bullet(kd)[2k-2]$ of degree zero is in fact a morphism of complexes. Hence, T_k induces a map

$$M = \mathrm{Coker}(L_1 \to L_0) \to \mathrm{Coker}(L_{2k-1} \to L_{2k-2}) \subseteq L_{2k-3}.$$

But the module on the right hand side is a submodule of a free module, hence torsion free. Whereas the module on the left has torsion. Hence, T_k induces the zero map on the zeroth homology modules. We may choose s_{k+1} as a null-homotopy for T_k.

We will now construct a matrix factorisation of f from the s_k.

Construction 4.3 ([6, Constr. 2.7])**.** *Let $F_\bullet = \bigoplus_{n \geq 0} F_n$ with*

$$F_n := \bigoplus_{2j+i=n,\ 0 \leq i \leq n,\ j \geq 0} L_i(-jd).$$

Then $\Theta := \sum_{k \in \mathbb{N}_0} s_k$ from Lemma 4.2 defines a homogeneous map of degree zero

$$\theta \colon F_\bullet \to F_\bullet[-1].$$

If m denotes the length of $L_\bullet$, then

$$F_m = \bigoplus_{j\geq 0} L_{m-2j}(-jd) \ \text{ and } \ F_{m+1} = \bigoplus_{j\geq 0} L_{m-2j+1}(-jd).$$

Note that $F_m(-d) = F_{m+2}$. Since $L_\bullet = F_m \oplus F_{m+1}$, we have the matrix decomposition

$$\Theta\colon \ L_\bullet = F_m(-d) \oplus F_{m+1} \xrightarrow{\left(\begin{smallmatrix} 0 & \varphi \\ \psi & 0 \end{smallmatrix}\right)} F_m \oplus F_{m+1} = L_\bullet$$

with $\varphi = \Theta|_{F_{m+1}}$ and $\psi = \Theta|_{F_m(-d)}$. The defining property $\Theta^2 = f \cdot \mathrm{id}_{L_\bullet}$ gives

$$\varphi\psi = f \cdot \mathrm{id}_{F_m} \ \text{ and } \ \psi\varphi = f \cdot \mathrm{id}_{F_{m+1}}.$$

Because φ and ψ are both homogeneous of degree zero as morphisms of graded R-modules, (φ, ψ) is a graded matrix factorisation of f over R.

Remark 4.4. Let $A := R/f$. It is shown in [6, Lem. 4.5] that the complex $S := F \otimes A$ with differential $\partial_S := \Theta \otimes A$ is a free resolution of $M = \operatorname{Coker} \partial_1$ over A. Note that the complex S becomes eventually two-periodic.

5. Recap on Quiver Representations

We recall some parts of the geometric invariant for quiver representations as contained in A. King's work [24].

A quiver Q consists of two finite sets, a set Q_0 of vertices and a set Q_1 of arrows as well as two maps $h, t\colon Q_1 \to Q_0$ which indicate for each arrow $a \in Q_1$ its head $h(a)$ and tail $t(a)$ in Q_0. If we assign to each arrow $a \in Q_1$ a finite-dimensional vector space V_a, we call Q weighted. A representation of a weighted quiver Q is given by a set of finite dimensional vector spaces $\{W_v\}_{v\in Q_0}$ and a set of linear maps $\{f_a\}_{a\in Q_1}$ such that $f_a\colon W_{t(a)} \otimes V_a \to W_{h(a)}$ for all $a \in Q_1$. The dimension vector $w \in \mathbb{N}^{Q_0}$ of a representation is defined as $w_v = \dim_k W_v$ for all $v \in Q_0$.

A subrepresentation consists of subvectorspaces $W'_v \subseteq W_v$ for all $v \in Q_0$ such that $f_a(W'_{t(a)} \otimes V_a) \subseteq W_{h(a)}$ for all $a \in Q_1$.

Let $\{U_v\}_{v\in Q_0}$ with linear maps $\{g_a\}_{a\in Q_1}$ be a second representation. A collection of linear maps $\varphi_v\colon W_v \to U_v$ such that

$$\varphi_{h(a)} \circ f_a = g_a \circ \varphi_{t(a)} \otimes \mathrm{id}_{V_a}$$

is called a morphism of represenstations.

We denote by

$$R(Q, w) = \bigoplus_{a\in Q_1} \mathrm{Hom}(W_{t(a)} \otimes V_a, W_{h(a)})$$

the space of representations of Q with fixed dimension vector w. The group

$$\mathrm{GL}(w) := \prod_{v\in Q_0} \mathrm{GL}_k(W_v)$$

acts on $R(Q, w)$ by the rule

$$(g.f)_a = g_{h(a)} \circ f_a \circ g^{-1}_{t(a)} \otimes \mathrm{id}_{V_a} .$$

The only line bundle on $R(Q, w)$ is $\mathcal{O}$. For every character $\chi \in X(\mathrm{GL}(w))$ we can consider the twist $\mathcal{O}(\chi)$ of the trivial linearisation with χ, see Definition 3.19. A coordinate function $f \in k[R(Q, w)]$ is an invariant section of $\mathcal{O}(\chi)$ if and only if for all $g \in \mathrm{GL}(w)$ and $x \in R(Q, w)$ we have $f(g.x) = \chi(g)f(x)$, i.e. f is a χ-semi-invariant function. We denote the subspace of χ^n-semi-invariants with $k[R(Q, w)]^{\mathrm{GL}(w),\chi^n}$.

Note that the diagonal group

$$\Delta = \{(t \cdot \mathrm{id}_{W_v})_{v\in Q_0} \in \mathrm{GL}(w) \mid t \in k^*\}$$

acts trivially on $R(Q, w)$. Hence, if there are any non-trivial invariant sections, then we must have $\chi(\Delta) = 1$. If this is the case, we may consider semi-stability and stability as in Definition 2.37 with respect to the group action by $\overline{G} := \mathrm{GL}(w)/\Delta$ linearised with the line bundle $\mathcal{O}(\chi)$.

For any sequence of integers $\theta = (e_v)_{v\in Q_0}$ with the property $\sum_{v\in Q_0} e_v w_v = 0$ we can associate the character

$$\chi_\theta(g) := \prod_{v\in Q_0} \det(g_v)^{e_v}$$

of $\mathrm{GL}(w)$. Then χ_θ restricted to Δ is trivial.

Definition 5.1. A representation $\{f_a\}_{a\in Q_1} \in R(Q,w)$ is called θ-semi-stable (respectively stable) if for all subrepresentations $\{W'_v\}_{v\in Q_0}$ which are neither the whole representation nor the zero representation:

$$\sum_{v\in Q_0} e_v \dim_k(W'_v) \geq 0 \quad \text{(respectively } > 0).$$

A. King proves in [24] the following fundamental result.

Theorem 5.2 ([24, Prop. 3.1])**.** *A representation $\{f_a\}_{a\in Q_1} \in R(Q,w)$ is semi-stable (respectively stable) with respect to the $\overline{G}$-action linearised by $\mathcal{O}(\chi_\theta)$ if and only if it is θ-semi-stable (respectively θ-stable).*

We will have to consider weighted quivers of A_4-type in the construction in the last chapter:

$$Q = (Q_0, Q_1)\colon \ \bullet \xrightarrow{V_3} \bullet \xrightarrow{V_2} \bullet \xrightarrow{V_1} \bullet$$

where each V_i is a finite dimensional vector space defining the weight of the quiver. We number the vertices from left to right by 3, 2, 1 and 0 (in the applications the quiver will come from a resolution of length 3, hence our choice of numbering).

Let $w \in \mathbb{N}^{Q_0}$ be a dimension vector and W_i a vector space of dimension w_i for all $i = 0, \dots, 3$. Let $\theta = (e_i)_{i\in Q_0} \in \mathbb{Z}^{Q_0}$ such that $e_0w_0 + \ldots + e_3w_3 = 0$.

We observe that a necessary condition for the existence of θ-semi-stable points is

$$e_0w_0 + \ldots + e_iw_i \geq 0 \text{ for all } i = 0, \dots, 2.$$

This is because we may choose as a subrepresentation $W'_j = W_j$ for all $j = 0, \dots i$ and $W'_j = 0$ otherwise.

Let χ_θ denote the character of $\mathrm{GL}(w)$ associated to θ.

Lemma 5.3. *If in the above situation $f \in k[R(Q,w)]^{\mathrm{GL}(w),\chi_\theta^n}$ is a semi-invariant, then it is a homogeneous polynomial of degree $n(-3e_3w_3 - 2e_2w_2 - e_1w_1)$.*

Proof. We consider the action of

$$g = (\mathrm{id}_{w_0}, t^{-1}\cdot \mathrm{id}_{w_1}, t^{-2}\cdot \mathrm{id}_{w_2}, t^{-3}\cdot \mathrm{id}_{w_3}) \in \mathrm{GL}(w)$$

for $t \in \mathbb{G}_{m,k}(k)$. If f is a semi-invariant, we must have $f(g.x) = \chi_\theta(g)^n f(x)$. By construction g acts as multiplication with t on any representation $x \in R(Q,w)$. On the other hand $\chi_\theta(g)^n = t^{n(-3e_3w_3-2e_2w_2-e_1w_1)}$. $\square$

Rather than working with the affine space $R(Q,w)$, we prefer the projectivisation $\mathbb{P}(R(Q,w)^*)$ as in the work of Drézet and Trautmann [12]. If we choose as $\overline{G}$-linearisation the twisted line bundle

$$L := \mathcal{O}(-3e_3w_3 - 2e_2w_2 - e_1w_1)(\chi_\theta),$$

then the affine cone over $\mathbb{P}(R(Q,w)^*)^{ss,\overline{G}}(L)$ is

$$R(Q,w)^{ss,\overline{G}}(\mathcal{O}(\chi_\theta)) \setminus \{0\}.$$

This follows since the action of $\overline{G}$ on

$$H^0(\mathbb{P}(R(Q,w)^*), L^n) = S^{n(-3e_3w_3-2e_2w_2-e_1w_1)}R(Q,w)^*$$

is given by $g.f(x) = \chi_\theta(g)^n f(g^{-1}.x)$ for all $g \in \overline{G}$, $x \in R(Q,w)$. Hence f is a χ_θ^n-semi-invariant in $k[R(Q,w)]$ if and only if f defines a $\overline{G}$-invariant global section in L^n.

6. Moduli spaces of matrix factorisations

6.1. **Moduli spaces of matrix factorisations of Shamash type.** Let V be a finite dimensional vector space and $\mathscr{F}$ be a coherent sheaf of codimension three on $\mathbb{P}(V)$. We denote with $S := S^{\bullet}V$ the symmetric algebra which is the homogeneous coordinate ring of $\mathbb{P}(V)$. The sheaf $\mathscr{F}$ is arithmetically Cohen-Macaulay if and only if the S-module $M := \Gamma_*(\mathscr{F}) = \bigoplus_{n\in\mathbb{Z}} H^0(\mathbb{P}(V), \mathscr{F}(n))$ is Cohen-Macaulay, see for example [1, Prop. 1.2]. A graded module over an $\mathbb{N}_0$-graded ring is Cohen-Macaulay if and only if

$$\operatorname{depth}_R(M) := \operatorname{depth}_R(\mathfrak{m}, M) = \dim M,$$

where $\mathfrak{m}$ denotes the irrelevant ideal in S. Hence we conclude by the Auslander-Buchsbaum theorem for graded modules, see [13, Exc. 19.8]

$$\operatorname{pdim}_R(M) = \operatorname{depth}_R(S) - \operatorname{depth}_R(M) = \dim V - (\dim V - 3) = 3.$$

We assume for simplicity that the graded free minimal resolution of M has the following form

$$0 \longrightarrow S^{n_3}(-a_3) \longrightarrow S^{n_2}(-a_2) \longrightarrow S^{n_1}(-a_1) \longrightarrow S^{n_0}$$

where $a_3 > a_2 > a_1 > 0$. If $f \in \operatorname{ann} M$ is homogeneous of degree d, we may apply Shamash's Construction 4.3 to obtain a matrix factorisation

$$(\varphi\colon F \to G, \psi\colon G \to F(d))$$

of f with

$$F = S^{n_3}(-a_3 - d) \oplus S^{n_1}(-a_1 - 2d) \text{ and } G = S^{n_2}(-a_2 - d) \oplus S^{n_0}(-2d).$$

Notation 6.1. $\operatorname{Hom}_S(M, N)$ denotes homogeneous morphisms between graded S-modules M and N of degree zero throughout this section.

We write φ and ψ as block matrices

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} \text{ and } \psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}$$

with respect to the direct sum decomposition of F and G above, i.e. for example

$$\varphi_{11} \in \operatorname{Hom}_S(S^{n_3}(-a_3 - d), S^{n_2}(-a_2 - d)).$$

Note that $\varphi_{21} = 0$ by Construction 4.3.

Definition 6.2. A matrix factorisation $(\varphi\colon F \to G, \psi\colon G \to F(d))$ of f is of Shamash type if $\varphi_{21} = 0$. Let $MF_f^{Sh}(F, G) \subseteq \operatorname{Hom}(F, G) \oplus \operatorname{Hom}(G, F(d))$ denote the subset of matrix factorisations of f of Shamash type.

Remark 6.3. If (φ, ψ) is of Shamash type, then we have a complex

$$0 \longrightarrow S^{n_3}(-a_3) \xrightarrow{\varphi_{11}} S^{n_2}(-a_2) \xrightarrow{\psi_{21}} S^{n_1}(-a_1) \xrightarrow{\varphi_{22}} S^{n_0} \tag{5}$$

which is in general not an exact complex.

Following J.-M. Drézet and G. Trautmann [12], we may think of φ_{ij} respectively ψ_{ij} as S-module homomorphisms as well as of quiver representations. Let $m, n \in \mathbb{N}$ and $\alpha \geq \beta \geq 0$. There is a canonical isomorphism

$$\mathrm{Hom}_S(S(-\alpha)^m, S(-\beta)^n) \cong \mathrm{Hom}_S(S^m, S^n \otimes \mathrm{Hom}(S(-\alpha), S(-\beta))).$$

If we restrict every S-module homomorphism to its degree zero part, we obtain

$$\mathrm{Hom}_S(S^m, S^n \otimes \mathrm{Hom}(S(-\alpha), S(-\beta))) \cong \mathrm{Hom}_k(k^m, k^n \otimes S^{\alpha-\beta}V).$$

Finally,

$$\mathrm{Hom}_k(k^m, k^n \otimes S^{\alpha-\beta}V) \cong \mathrm{Hom}_k(k^m \otimes (S^{\alpha-\beta}V)^*, k^n)$$

canonically. The right hand side is the representation space with dimension vector (m, n) of the weighted quiver

$$\bullet \xrightarrow{(S^{\alpha-\beta}V)^*} \bullet.$$

Furthermore, let

$$\mu \in \mathrm{Hom}_S(S(-\alpha)^m, S(-\beta)^n) \text{ and } \nu \in \mathrm{Hom}_S(S(-\beta)^n, S(-\gamma)^p)$$

for $\beta \geq \gamma \geq 0$. By abuse of notation let the images of μ respectively ν in

$$\mathrm{Hom}_k(k^m, k^n \otimes S^{\alpha-\beta}V) \text{ respectively } \mathrm{Hom}_k(k^n, k^p \otimes S^{\beta-\gamma}V)$$

also be denoted by μ respectively ν. We may then describe the composition of μ and ν as

$$k^m \xrightarrow{\mu} k^n \otimes S^{\alpha-\beta}V \xrightarrow{\nu \otimes \mathrm{id}} k^p \otimes S^{\beta-\gamma}V \otimes S^{\alpha-\beta}V \xrightarrow{\mathrm{id} \otimes m} k^p \otimes S^{\alpha-\gamma}V$$

where $m\colon S^{\beta-\gamma}V \otimes S^{\alpha-\beta}V \to S^{\alpha-\gamma}V$ denotes multiplication in S.

Let Q denote the weighted quiver:

$$\begin{tikzcd}
\bullet \arrow[r, shift left, "S^{a_3-a_2}V^*"] & \bullet \arrow[l, shift left, "S^{a_2-a_3+d}V^*"] \arrow[r, shift left, "S^{a_2-a_1}V^*"] & \bullet \arrow[l, shift left, "S^{a_1-a_2+d}V^*"] \arrow[r, shift left, "S^{a_1}V^*"] & \bullet \arrow[l, shift left, "S^{d-a_1}V^*"] \arrow[lll, bend right, "S^{2d-a_3}V^*"']
\end{tikzcd}$$

We number its edges from right to left by $0, 1, 2$ and 3. If w denotes the dimension vector (n_0, n_1, n_2, n_3), then the affine space of all $(\varphi, \psi) \in \mathrm{Hom}(F, G) \oplus \mathrm{Hom}(G, F(d))$ such that $\varphi_{21} = 0$ is isomorphic to the space of quiver representations

$$\begin{aligned} R(Q, w) = {} & \mathrm{Hom}(k^{n_0} \otimes S^{2d-a_3} V^*, k^{n_3}) \oplus \\ & \bigoplus_{3 \geq j > i \geq 0} \mathrm{Hom}(k^{n_j} \otimes S^{a_j - a_i} V^*, k^{n_i}) \oplus \\ & \bigoplus_{3 \geq i > j \geq 0} \mathrm{Hom}(k^{n_j} \otimes S^{a_j - a_i + d} V^*, k^{n_i}) \end{aligned}$$

by the foregoing isomorphism

$$\mathrm{Hom}_S(S(-\alpha)^m, S(-\beta)^n) \cong \mathrm{Hom}_k(k^m \otimes (S^{\alpha-\beta} V)^*, k^n).$$

We will pass freely back and forth between (φ, ψ) as S-module homomorphisms or as quiver representations.

Let us next describe the automorphism group of $MF_f^{Sh}(F, G)$ in the case

$$a_3 - a_1 \leq d \text{ and } a_2 \leq d.$$

In this situation, let H_F denote the group

$$\left\{ \begin{pmatrix} f_1 & u \\ 0 & f_2 \end{pmatrix} \in \mathrm{Aut}(F) \;\middle|\; f_1 \in \mathrm{GL}_{n_3}, f_2 \in \mathrm{GL}_{n_1}, u \colon S^{n_1}(-a_1 - d) \to S^{n_3}(-a_3) \right\}$$

and let H_G be the group

$$\left\{ \begin{pmatrix} g_1 & v \\ 0 & g_2 \end{pmatrix} \in \mathrm{Aut}(G) \;\middle|\; g_1 \in \mathrm{GL}_{n_2}, g_2 \in \mathrm{GL}_{n_0}, v \colon S^{n_0}(-d) \to S^{n_2}(-a_2) \right\}.$$

Then $H_F \times H_G$ acts on $R(Q, w)$ by the rule

$$(\gamma_1, \gamma_2).(\varphi, \psi) = (\gamma_2 \varphi \gamma_1^{-1}, \gamma_1 \psi \gamma_2^{-1}).$$

The action restricts to $MF_f^{Sh}(F, G)$.

Since the diagonal $\Delta = \{(\lambda \cdot \mathrm{id}_F, \lambda \cdot \mathrm{id}_G) \mid \lambda \in \mathbb{G}_{m,k}\}$ acts trivial, we pass to the quotient $\Gamma := (H_F \times H_G)/\Delta$. The unipotent radical of Γ is given by the product $R_u(H_F) \times R_u(H_G)$ where

$$R_u(H_F) = \left\{ \begin{pmatrix} f_1 & u \\ 0 & f_2 \end{pmatrix} \in H_F \;\middle|\; f_1 = \mathrm{id}_{n_3}, f_2 = \mathrm{id}_{n_1} \right\}$$

and

$$R_u(H_G) = \left\{ \begin{pmatrix} g_1 & v \\ 0 & g_2 \end{pmatrix} \in H_G \;\middle|\; g_1 = \mathrm{id}_{n_2}, g_2 = \mathrm{id}_{n_0} \right\}.$$

Note that in our situation $R_u(\Gamma)$ is abelian, namely given by several copies of $\mathbb{G}_{a,k}$.

We may choose as a Levi-factor the reductive subgroup

$$R := (\mathrm{GL}_{n_3} \times \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \mathrm{GL}_{n_0})/\Delta$$

embedded in Γ by putting $u = 0$ and $v = 0$. There is the following observation to make.

Lemma 6.4. *The action of the chosen Levi-factor R on $R(Q, w)$ is the action of the automorphism group of quiver representations with dimension vector w.* $\square$

Let us once write out the action of the unipotent radical on (φ, ψ):

$$\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ 0 & \varphi_{22} \end{pmatrix} \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \varphi_{11} & \varphi_{12} + v\varphi_{22} - \varphi_{11}u \\ 0 & \varphi_{22} \end{pmatrix}$$
$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \psi_{11} + u\psi_{21} & \psi_{12} + u\psi_{22} - \psi_{11}v - u\psi_{21}v \\ \psi_{21} & \psi_{22} - \psi_{21}v \end{pmatrix}$$

Proposition 6.5. *Assume that $(\varphi, \psi) \in MF_f^{Sh}(F, G)$ has the property such that the complex (5) in Remark 6.3 is exact. Then*

$$\mathrm{Stab}_{R_u(\Gamma)}(\varphi, \psi) \cong \mathrm{Hom}(S^{n_0}(-d), S^{n_3}(-a_3)).$$

Proof. If (φ, ψ) is a matrix factorisation, $\varphi = f^{-1}\psi^{-1}$ over the quotient field of S. In particular $(u, v) \in R_u(\Gamma)$ fixes (φ, ψ) if and only if it fixes ψ (respectively φ). Hence $(u, v) \in \mathrm{Stab}_{R_u(\Gamma)}(\varphi, \psi)$ if and only if

$$u\psi_{21} = 0,\ \psi_{21}v = 0,\ u\psi_{22} = \psi_{11}v.$$

Since the complex

$$0 \longrightarrow S^{n_3}(-a_3) \xrightarrow{\varphi_{11}} S^{n_2}(-a_2) \xrightarrow{\psi_{21}} S^{n_1}(-a_1) \xrightarrow{\varphi_{22}} S^{n_0}$$

is exact by assumption, there exists

$$\alpha\colon S^{n_0}(-d) \to S^{n_3}(-a_3) \text{ and } \beta\colon\ \mathrm{Coker}(\psi_{21}) \to S^{n_3}(-a_3 + d)$$

such that $u = \beta\varphi_{22}$ and $v = \varphi_{11}\alpha$. The exactness implies $\mathrm{Coker}(\psi_{21}) = \mathrm{Im}(\varphi_{22})$. We claim that $\beta = \alpha|_{\mathrm{Im}(\varphi_{22})}$. This follows since

$$\beta \circ (f \cdot \mathrm{id}) = \beta\varphi_{22}\psi_{22} = u\psi_{22} = \psi_{11}v = \psi_{11}\varphi_{11}\alpha = f \cdot \alpha$$

We note that $f \cdot S^{n_0}$ is contained in $\mathrm{Im}(\varphi_{22})$. Since f is a non-zero divisor we may cancel and obtain that β is the restriction of α. On the other hand any element $(\alpha\varphi_{22}, \varphi_{11}\alpha) \in R_u(\Gamma)$ fixes (φ, ψ). Since α is unique with $v = \varphi_{11}\alpha$, the map

$$\mathrm{Hom}(S^{n_0}(-d), S^{n_3}(-a_3)) \to \mathrm{Stab}_{R_u(\Gamma)}(\varphi, \psi), \alpha \mapsto (\alpha\varphi_{22}, \varphi_{11}\alpha)$$

is a bijection. $\square$

We have to quotient a projective scheme X in order to obtain a projective quotient by Theorem 3.20. Unfortunately $MF_f^{Sh}(F, G)$ is not stable under the action of $\mathbb{G}_{m,k}$ on $R(Q, w)$ by scalar multiplication and hence does not define a closed subscheme of

$\mathbb{P}(R(Q,w)^*)$. Therefore we will consider factorisations of any multiple of f and allow zero-factorisations too, i.e. (φ,ψ) such that $\varphi\psi = 0$ and $\psi\varphi = 0$.

Definition 6.6. Let $MF_f^{gen.Sh}(F,G)$ be the variety of generalised matrix factorisations of $f \in S^dV$ of Shamash type of format (F,G) defined as the closed subset of $R(Q,w)$ given by

$$\{(\varphi,\psi) \in R(Q,w) \mid \ \exists \lambda \in k : \varphi\psi = \lambda f \cdot \mathrm{id}_G, \psi\varphi = \lambda f \cdot \mathrm{id}_F\}$$

endowed with the reduced induced subscheme structure.

The quotient of $MF_f^{gen.Sh}(F,G) \setminus \{0\}$ by the action of $\mathbb{G}_{m,k}$ by scalar multiplication on $R(Q,w)$ is denoted by

$$\overline{MF}_f^{gen.Sh}(F,G) \subseteq \mathbb{P}(R(Q,w)^*).$$

We write (φ,ψ) for points in $R(Q,w)$ and $[\varphi,\psi]$ for points in $\mathbb{P}(R(Q,w)^*)$.

Remark 6.7. If f is irreducible, $\varphi\psi = f \cdot \mathrm{id}$ implies that $\det(\varphi)$ and $\det(\psi)$ are powers of f determined by the degrees of f and the entries of φ and ψ. This is not true for general f. Therefore the variety of generalised matrix factorisations of Shamash type as defined above might have different components corresponding to different values of $\det(\varphi)$ and $\det(\psi)$. Since we are not going to prove any geometric properties of the constructed quotients, we will ignore these subtleties and continue our study without any assumptions on f.

The group Γ acts on $\overline{MF}_f^{gen.Sh}(F,G)$ and the action is linearised by the restriction of $\mathcal{O}_{\mathbb{P}(R(Q,w)^*)}(1)$, call it L. Our aim is to apply Theorem 3.20. To ease the notation, let

$$\overline{\mathrm{MF}} := \overline{MF}_f^{gen.Sh}(F,G)$$

if f and the free S-modules F and G are clear from the context.

First, we need to choose an admissible positive grading $\lambda\colon \mathbb{G}_{m,k} \to Z(R)$. All possible gradings are given by

$$\lambda\colon t \mapsto \mathrm{diag}(t^A \cdot \mathrm{id}_{n_3}, t^B \cdot \mathrm{id}_{n_1}, t^C \cdot \mathrm{id}_{n_2}, t^D \cdot \mathrm{id}_{n_0})$$

for $A,B,C,D \in \mathbb{Z}$. The grading is positive if and only if $A > B$ and $C > D$. Next we have to find a character $\chi \in X(\Gamma)$ such that $\langle \chi, \lambda \rangle$ is the maximal weight of λ on $H^0(\overline{\mathrm{MF}}, L^m)$ for some $m > 0$. We note that the canonical map

$$H^0(\mathbb{P}(R(Q,w)^*), \mathcal{O}_{\mathbb{P}(R(Q,w)^*)}(m)) \to H^0(\overline{\mathrm{MF}}, L^m)$$

is surjective for all $m \gg 0$ since L is very ample. Furthermore, by Lemma 3.21 we know that the maximal weight of λ on $H^0(\overline{\mathrm{MF}}, L^m)$ is m times the maximal weight on $H^0(\overline{\mathrm{MF}}, L)$ and the analogous statement holds for the maximal weight on

$$H^0(\mathbb{P}(R(Q,w)^*), \mathcal{O}_{\mathbb{P}(R(Q,w)^*)}(m)) = S^m R(Q,w)^*.$$

Therefore it is enough to compute the maximal weight on $R(Q,w)^*$ and then to observe that its weight-vectors are not all mapped to zero in $H^0(\overline{\mathrm{MF}}, L)$.

The maximal weight on $R(Q,w)^*$ is the minimal weight on $R(Q,w)$. Let (φ,ψ) be a generalised matrix factorisation of Shamash type. Then

$$\lambda(t).\varphi = \begin{pmatrix} t^{C-A}\cdot\varphi_{11} & t^{C-B}\cdot\varphi_{12} \\ 0 & t^{D-B}\cdot\varphi_{22} \end{pmatrix} \text{ and } \lambda(t).\psi = \begin{pmatrix} t^{A-C}\cdot\psi_{11} & t^{A-D}\cdot\psi_{12} \\ t^{B-C}\cdot\psi_{21} & t^{B-D}\cdot\psi_{22} \end{pmatrix}$$

From $A > B$ and $C > D$ we conclude that the minimal weight is one of the weights $C-A$, $D-B$ or $B-C$. One of the principles in the work [3] by Bérczi et al. is, that the non-vanishing of global sections with maximal weight ought to force the stabilisers in the unipotent radical to be trivial. Therefore our strategy is to choose λ in such a way that the minimal weight space in $R(Q,w)$ is as large as possible. If we let a and α be integers and put

$$A := a, B := a + 2\alpha, C := a + \alpha, D := a + 3\alpha,$$

then φ_{11}, φ_{22} and ψ_{21} all have weight α. Hence $-\alpha$ is the maximal weight of λ on the global sections of L. Furthermore, λ is a positive grading for all $\alpha < 0$.

Finally, let θ be a sequence of integers $e_0, \ldots, e_3$ such that

$$e_0n_0 + \ldots + e_3n_3 = 0$$

and let χ_θ be the associated character as in Section 5. We calculate

$$\begin{aligned} \langle\chi_\theta, \lambda\rangle &= ae_3n_3 + (a+\alpha)e_2n_2 + (a+2\alpha)e_1n_1 + (a+3\alpha)e_0n_0 \\ &= a(e_3n_3 + e_2n_2 + e_1n_1 + e_0n_0) + \alpha(e_2n_2 + 2e_1n_1 + 3e_0n_0) \\ &= -\alpha(3e_3n_3 + 2e_2n_2 + e_1n_1). \end{aligned}$$

This shows, that if $m := 3e_3n_3 + 2e_2n_2 + e_1n_1$ is a positive number, we may take $K := L^m(-\chi_\theta)$ and apply Theorem 3.20 to calculate the locus of semi-stable points. Note that $m > 0$ for all θ with

$$e_0n_0 + \ldots + e_in_i < 0 \text{ for all } i = 0, \ldots, 2.$$

Let us summarize.

1. The group $\Gamma = R_u(\Gamma) \rtimes R$ with $R = (\mathrm{GL}_{n_3} \times \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \mathrm{GL}_{n_0})/\Delta$ acts on
$$\overline{\mathrm{MF}} = \overline{MF}_f^{gen.Sh}(F,G) \subseteq \mathbb{P}(R(Q,w)^*).$$
2. The action is linearised by L which is the restriction of $\mathcal{O}_{\mathbb{P}(R(Q,w)^*)}(1)$ to the space of generalised matrix factorisations.
3. We have chosen a positive grading $\lambda\colon \mathbb{G}_{m,k} \to Z(R)$ such that $-\alpha$ is the maximal weight of λ on the global sections of L and
$$\langle\chi_\theta, \lambda\rangle = -\alpha(3e_3n_3 + 2e_2n_2 + e_1n_1)$$

4. θ is a sequence of integers $e_0, \ldots, e_3$ with the property

$$e_0 n_0 + \ldots + e_3 n_3 = 0$$

and such that $e_0 n_0 + \ldots e_i n_i$ is negative for all $i = 0, \ldots, 2$.
5. The linearisation K is defined by $K := L^{3e_3n_3+2e_2n_2+e_1n_1}(-\chi_\theta)$.

As in Section 3, we write $X^{ss,R}(L_\infty) = X^{ss,R}(L_b)$ for $b \gg 0$ sufficiently large.

Theorem 6.8. *If* $\mathrm{Stab}_{R_u(\Gamma)}([\varphi, \psi])$ *is trivial for all points* $[\varphi, \psi] \in \overline{\mathrm{MF}}$ *that are R-semi-stable with respect to K, then the set of factorisations $[\varphi, \psi]$ such that the orbit $R_u(\Gamma).[\varphi, \psi]$ is contained in the R-semi-stable locus with respect to L_∞ is open and admits a projective good quotient Z for the Γ-action. Furthermore, the open set of Γ-stable factorisations $[\varphi, \psi] \in \overline{\mathrm{MF}}$ which are not zero-factorisations admits a geometric quotient which is open in Z.*

Proof. By Theorem 3.20, there exists a projective good quotient

$$\pi\colon\ \overline{\mathrm{MF}}^{ss,\Gamma}(K, L) \to Z$$

and $b \gg 0$ such that $(L_b)^a$ descends to an ample line bundle P on Z for some $a > 0$. It is left to prove the last statement.

Claim. There exists a Γ-invariant global section g in L^2 such that $[\varphi, \psi]$ is a zero-factorisation if and only if g vanishes at the point $[\varphi, \psi]$.

To prove the claim let $v_1, \ldots, v_{\dim V}$ be a basis of V and $(\varphi, \psi) \in R(Q, w)$. Then

$$\mathrm{tr}(\varphi\psi) = \sum_{|\alpha|=d} \lambda_\alpha(\varphi, \psi) v^\alpha$$

where λ_α are quadratic polynomials in the coordinates of φ and ψ. On the other hand, we may write $f = \sum_{|\alpha|=d} f_\alpha v^\alpha$. Hence, if $f_\alpha \neq 0$, then $g := \lambda_\alpha$ defines a Γ-invariant section in L^2 with the claimed property.

Let $f_1, \ldots, f_m$ be global R-invariant sections of K^l, $l > 0$, such that

$$\bigcup_{i=1}^{m} \overline{\mathrm{MF}}_{f_i} = \overline{\mathrm{MF}}^{ss,R}(K)$$

There exist $p, q > 0$ such that $f_i^p g^q$ descends for every i to a global section t_i of P^s over Y for some $s > 0$ by Remark 3.16. If Z' denotes as in Theorem 3.13 the base of the geometric quotient of the locus of Γ-stable points, it follows that the preimage of Z'_{t_i} is

$$\overline{\mathrm{MF}}^{s,\Gamma}(K, L)_{f_i^p g^q}.$$

Hence, $\bigcup_{i=1}^{m} Z'_{t_i}$ is a geometric quotient of the union of these principal open subsets. The theorem follows since those cover exactly the locus of Γ-stable factorisations $[\varphi, \psi]$ that are not zero-factorisations. $\square$

By Section 5, if $e_i > 0$ for all i, then the condition that $e_0 n_0 + \dots e_i n_i > 0$ for all $i \leq 2$ is a necessary condition for the representation space with weight vector (n_0, n_1, n_2, n_3) of the weighted A_4-quiver

$$\bullet \xrightarrow{S^{a_3-a_2}V^*} \bullet \xrightarrow{S^{a_2-a_1}V^*} \bullet \xrightarrow{S^{a_1}V^*} \bullet$$

to have $-\theta$-semi-stable points. As above, we numbered the edges from right to left by $0, 1, 2, 3$. The reader may have already observed that the minimal weight space of the action of λ on $R(Q, w)$ corresponds exactly to representations of this A_4-quiver.

Lemma 6.9. *A generalised matrix factorisation $[\varphi, \psi] \in \overline{\mathrm{MF}}$ is R-semi-stable with respect to K if and only if the representation of the weighted A_4-quiver defined by*

$$k^{n_3} \xrightarrow[S^{a_3-a_2}V^*]{\varphi_{11}} k^{n_2} \xrightarrow[S^{a_2-a_1}V^*]{\psi_{21}} k^{n_1} \xrightarrow[S^{a_1}V^*]{\varphi_{22}} k^{n_0}$$

is $(-\theta)$-semi-stable.

Proof. Let
$$\overline{K} := \mathcal{O}_{\mathbb{P}(R(Q,w)^*)}(3e_3 + 2e_2n_2 + e_1n_1)(-\chi_\theta),$$
then $\overline{K}$ restricts to K on $\overline{\mathrm{MF}}$. We may therefore compute semi-stability with respect to the action on $\mathbb{P}(R(Q, w)^*)$ linearised with $\overline{K}$ by Remark 2.39. The action of R on

$$g \in H^0(\mathbb{P}(R(Q, w)^*), \overline{K}^n) = S^{n(3e_3n_3+2e_2n_2+e_1n_1)} R(Q, w)^*$$

is given by $(r.g)(x) = (-\chi_\theta)^n(r) g(r^{-1}.x)$ for all $r \in R$, $x \in R(Q, w)$. Hence g is a $(-\chi_\theta)^n$-semi-invariant in $k[R(Q, w)]$ of degree $n(3e_3n_3 + 2e_2n_2 + e_1n_1)$ if and only if g defines an R-invariant global section in $\overline{K}^n$. Furthermore, any $(-\chi_\theta)^n$-semi-invariant g must have λ-weight $n \cdot \langle \chi_\theta, \lambda \rangle$. This is the maximal weight of λ on

$$S^{n(3e_3n_3+2e_2n_2+e_1n_1)} R(Q, w)^*.$$

Hence g is a function only depending on the components φ_{11}, ψ_{21} and φ_{22} and not on ψ_{11}, φ_{12}, ψ_{22} and ψ_{12}, i.e. g is a $(-\chi_\theta)^n$-semi-invariant for the weighted A_4 quiver above. On the other hand, any $(-\chi_\theta)^n$-semi-invariant g of this quiver must have degree $n(3e_3n_3 + 2e_2n_2 + e_1n_1)$ by Lemma 5.3. This shows that $[\varphi, \psi]$ is R-semi-stable with respect to K if and only if the associated representation of the weighted A_4-quiver is $-\chi_\theta$-semi-stable. That $-\chi_\theta$-semi-stability is equivalent to $-\theta$-semi-stability is King's Theorem, see Theorem 5.2. □

Theorem 6.8 and Lemma 6.9 reduce the problem of constructing projective quotients of generalised matrix factorisations to the study of quiver representations of the weighted A_4-quiver that is defined by the complex (5) in Remark 6.3. In view of Proposition 6.5 it is very tempting to connect exactness of this complex with $-\theta$-semi-stability of the quiver representation for a well chosen θ. Unfortunately we have not yet found any leverage to attack this problem in a satisfying generality.

6.2. **Matrix factorisations of elliptic quintic and twisted quartic type.** We show that we can apply Theorem 6.8 in two interesting cases. Let from now on $\dim V = 5$ and C denote a curve in $\mathbb{P}(V)$ with ideal sheaf I_C. We let $S_C := S/\Gamma_*(I_C)$ be the homogeneous coordinate ring of C. We call C arithmetically Cohen-Macaulay if S_C is a Cohen-Macaulay ring. Since C has codimension three, the graded minimal free resolution of S_C over S has length three in this case.

Let $X \subseteq \mathbb{P}(V)$ be a hypersurface of degree d with homogeneous coordinate ring $S_X = S/f$, $f \in S^d(V)$. If X contains C scheme-theoretically, i.e. $I_X \subseteq I_C$, then f annihilates S_C and Shamash's construction provides us with a matrix factorisation.

Elliptic Quintic Curves. Any line bundle L of degree $D \geq 3$ on a smooth irreducible curve C of genus one is very ample, see [15, Cor. 6.7]. Since it has positive degree, the dimension of the space of global sections of L is D by Riemann-Roch. Hence the curve is embedded in $\mathbb{P}(H^0(C, L)) = \mathbb{P}^{D-1}$ by the linear system $|L|$. An elliptic quintic curve is such an embedded curve C in the case $D = 5$. The resolution of S_C over S for an elliptic quintic curve C in $\mathbb{P}(V)$ has the form

$$0 \longrightarrow S(-5) \longrightarrow S^5(-3) \longrightarrow S^5(-2) \longrightarrow S \longrightarrow S_C \longrightarrow 0,$$

see for example [15, Thm. 6.26]. Applying Shamash's Construction 4.3, we obtain a matrix factorisation $(\varphi\colon F \to G, \psi\colon G \to F(d))$ of f of format given by

$$F = S(-5-d) \oplus S^5(-2-2d) \text{ and } G = S^5(-3-d) \oplus S(-2d).$$

Proposition 6.10. *Let $d = 3$. If we choose $\theta = (e_0, e_1, e_2, e_3)$ such that $e_1 < 0$ and $e_2 > 0$ then $\mathrm{Stab}_{R_u(\Gamma)}([\varphi, \psi])$ is trivial for all generalised matrix factorisations $[\varphi, \psi] \in \overline{\mathrm{MF}}$ that are R-semi-stable with respect to K.*

Proof. Since every unipotent group has trivial characters only, we can choose a lift $(\varphi, \psi) \in R(Q, w)$ of $[\varphi, \psi]$ and compute the stabiliser of the lift. Since $d = 3$, the entries of φ are all quadratic and the entries of ψ are all linear forms. The group Γ is a product of two parabolic subgroups of GL_6 quotient by the diagonal subgroup Δ.

We use Lemma 6.9. Let the representation of the A_4-quiver given by φ_{11}, ψ_{21} and φ_{22} be $(-\theta)$-semi-stable and let $(u, v) \in R_u(\Gamma)$ such that it stabilises (φ, ψ). This gives the identities

$$u\psi_{21} = 0 \text{ and } \psi_{21}v = 0.$$

Therefore, if $v \neq 0$, we may choose $W_2 := k \cdot v$ and $W_i := 0$ for all $i \neq 0$. Then the vector spaces $W_0, \ldots, W_3$ define a non-trivial quiver subrepresentation which is in contradiction to $(-\theta)$-semi-stability since $e_2 > 0$.

If $u \neq 0$, we may choose $W_1 := \ker(u\colon k^5 \to k)$ and $W_i = k^{n_i}$ else. Then $\{W_i\}_i$ defines a quiver subrepresentation which is in contradiction to semi-stability since

$$\sum_{i=0}^{3} e_i \dim W_i = e_0 + 4e_1 + 5e_2 + e_3 > 0.$$

□

We obtain for matrix factorisations associated to elliptic quintic curves lying on a cubic threefold defined by $f \in S^3V$ the final result:

Theorem 6.11. *Assume that $\theta = (e_0, e_1, e_2, e_3)$ is such that $e_1 < 0$ and $e_2 > 0$. The locus of semi-stable generalised matrix factorisations*

$$\overline{\mathrm{MF}}^{ss,\Gamma}(K,L) = \overline{MF}_f^{gen.Sh}(F,G)^{ss,\Gamma}(K,L)$$

admits a projective good quotient Z for the Γ-action which contains as an open subset a geometric quotient of the locus of Γ-stable factorisations which are not zero-factorisations. Furthermore,

$$[\varphi,\psi] \in \overline{\mathrm{MF}}^{ss,\Gamma}(K,L) \Longleftrightarrow R_u(\Gamma).[\varphi,\psi] \subseteq \overline{\mathrm{MF}}^{ss,R}(L_\infty).$$

Twisted Quartic Curves. A rational normal curve C of degree four or short a twisted quartic curve is a $\mathbb{P}^1$ embedded into $\mathbb{P}^4$ by the linear system $|\mathcal{O}_{\mathbb{P}^1}(4)|$. In this case, the resolution of S_C takes the form

$$0 \longrightarrow S^3(-4) \longrightarrow S^8(-3) \longrightarrow S^6(-2) \longrightarrow S \longrightarrow S_C \longrightarrow 0,$$

see for example [15, Cor. 6.9]. The matrix factorisation $(\varphi\colon F \to G, \psi\colon G \to F(d))$ of f, that we obtain from Shamash's Construction 4.3, has the format

$$F = S^3(-4-d) \oplus S^6(-2-2d) \text{ and } G = S^8(-3-d) \oplus S(-2d).$$

Proposition 6.12. *Let $d = 3$. If we choose $\theta = (e_0, e_1, e_2, e_3)$ such that*

$$e_1 < 0,\ e_2 > 0 \text{ and } e_1 + e_2 < 0,$$

then $\mathrm{Stab}_{R_u(\Gamma)}([\varphi,\psi])$ is trivial for all generalised matrix factorisations $[\varphi,\psi] \in \overline{\mathrm{MF}}$ that are R-semi-stable with respect to K.

Proof. Since $d = 3$, the entries of φ and ψ as morphisms of free modules have the following degrees

$$\deg(\varphi) = \begin{pmatrix} 1 & 2 \\ & 2 \end{pmatrix} \text{ and } \deg(\psi) = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}.$$

An element in the unipotent radical $R_u(\Gamma)$ is given by

$$\left(\begin{pmatrix} \mathrm{id}_3 & u \\ 0 & \mathrm{id}_6 \end{pmatrix}, \begin{pmatrix} \mathrm{id}_8 & v \\ 0 & 1 \end{pmatrix} \right) \in R_u(H_F) \times R_u(H_G)$$

with $u \in \mathrm{Hom}(S^6(-1), S^3)$ and $v \in \mathrm{Hom}(S, S^6)$, i.e. the entries of u are all linear and the entries of v are scalars. Let $[\varphi, \psi]$ be R-semi-stable with respect to K. As in the case of elliptic quintics we may choose a lift (φ, ψ) in $R(Q, w)$ of $[\varphi, \psi]$ to compute the stabiliser. We will use Lemma 6.9 in the following way. Under the assumption that $(u, v) \neq 0$, we will construct subvectorspaces

$$W_0 \subseteq k^{n_0} = k, W_1 \subseteq k^{n_1} = k^6, W_2 \subseteq k^{n_2} = k^8, W_3 \subseteq k^{n_3} = k^3$$

which define a subrepresentation of the A_4-quiver defined by φ_{11}, ψ_{21} and φ_{22}. We will call such a sequence $W_0, \ldots, W_3$ admissible. If

$$e_0 \dim W_0 + e_1 \dim W_1 + e_2 \dim W_2 + e_3 \dim W_3 > 0,$$

then we found a contradiction to $(-\theta)$-semi-stability.

Let us assume that (u, v) fixes (φ, ψ). Then we must have $\psi_{21} v = 0$. If $v \neq 0$, we set $W_2 := k \cdot v$ and $W_i = 0$ for all $i \neq 2$. The sequence $\{W_i\}_i$ is admissible and

$$e_0 \dim W_0 + e_1 \dim W_1 + e_2 \dim W_2 + e_3 \dim W_3 = e_2 > 0.$$

This shows $v = 0$.

Le us assume that $(u, 0)$ stabilises (φ, ψ). This implies the identities

$$u\psi_{21} = 0 \text{ and } \varphi_{11} u = 0.$$

We use the following notations

$$u = (l_1, \ldots, l_6) = \begin{pmatrix} l_{11} & \ldots & l_{16} \\ l_{21} & \ldots & l_{26} \\ l_{31} & \ldots & l_{36} \end{pmatrix}$$

with $l_i \in \mathrm{Hom}(S(-1), S^3)$ and $l_{ij} \in \mathrm{Hom}(S(-1), S)$. Further, let

$$n := \max_i \dim_k k\langle l_{i1}, \ldots, l_{i6}\rangle$$

Since $\dim V = 5$, there are six possible cases: $n = 0, \ldots, 5$. We need to prove that solely $n = 0$ occurs.

For any $r \in \mathrm{GL}_6$, we may replace u by ur^{-1} and ψ_{21} by $r\psi_{21}$. This does not change n and the quiver representation defined by φ_{11}, ψ_{21} and φ_{22} is $(-\theta)$-semi-stable if and only if the representation defined by $\varphi_{11} r^{-1}$, $r\psi_{21}$ and φ_{22} is $(-\theta)$-semi-stable.

$\boxed{n = 1}$ Without loss of generality let us assume that the first row of u is non-zero. After the actions ur^{-1} and $r\psi_{21}$ with some $r \in \mathrm{GL}_6$, we may assume that

$$l_{11} \neq 0 \text{ and } l_{12} = \ldots = l_{16} = 0.$$

We conclude from $u\psi_{21} = 0$ that the first row of ψ_{21} vanishes. The sequence defined by

$$W_1 := \{0\} \times k^5 \text{ and } W_i = k^{n_i} \text{ for all } i \neq 1$$

is then admissible and

$$e_0 \dim W_0 + e_1 \dim W_1 + e_2 \dim W_2 + e_3 \dim W_3 > 0$$

because $e_1 > 0$ by assumption.

$\boxed{n=2}$ As in the preceding case, we may assume without loss of generality that $l_{13} = \ldots = l_{16} = 0$, and that l_{11} and l_{12} are linearly independent. Then the columns of the first two rows of ψ_{21} are linear syzygies of the vector (l_{11}, l_{12}). There exists only one linear independent linear syzygy. Hence we may find an $r \in \mathrm{GL}_8$ such that

$$\psi_{21} \cdot r = \begin{pmatrix} 0 & \ldots & 0 & * \\ 0 & \ldots & 0 & * \\ * & \ldots & \ldots & * \\ \vdots & \ldots & \ldots & \vdots \\ * & \ldots & \ldots & * \end{pmatrix}.$$

The sequence defined by

$$W_1 := \{0\} \times \{0\} \times k^4, W_2 := k^7 \times \{0\} \text{ and } W_i = k^{n_i} \text{ for } i = 0, 3$$

is admissible and

$$\sum_{i=0}^{3} e_i \dim W_i = e_0 + 4e_1 + 7e_2 + 3e_3 = -e_2 - 2e_1 > -e_2 - e_1 > 0.$$

$\boxed{n=3}$ Without loss of generality we may assume that $l_{14} = l_{15} = l_{16} = 0$, and that l_{11}, l_{12} and l_{13} are linearly independent. As before, since there are exactly three linearly independent linear syzygies of the vector (l_{11}, l_{12}, l_{13}), there exists an $r \in \mathrm{GL}_8$ such that

$$\psi_{21} \cdot r = \begin{pmatrix} 0 & \ldots & 0 & * & * & * \\ 0 & \ldots & 0 & * & * & * \\ 0 & \ldots & 0 & * & * & * \\ * & \ldots & \ldots & \ldots & \ldots & * \\ * & \ldots & \ldots & \ldots & \ldots & * \\ * & \ldots & \ldots & \ldots & \ldots & * \end{pmatrix}.$$

In this case, we may choose as an admissible sequence

$$W_1 := \{0\}^3 \times k^3, W_2 := k^5 \times \{0\}^3 \text{ and } W_i = k^{n_i} \text{ for } i = 0, 3.$$

Then

$$\sum_{i=0}^{3} e_i \dim W_i = e_0 + 3e_1 + 5e_2 + 3e_3 = -3e_2 - 3e_1 > 0.$$

$\boxed{n=4}$ Without loss of generality we may assume that $l_{15} = l_{16} = 0$, and that l_{11}, l_{12}, l_{13} and l_{14} are linearly independent. There are six linearly independent linear syzygies of the vector $(l_{11}, l_{12}, l_{13}, l_{14})$. Therefore, there exists an $r \in \mathrm{GL}_8$ such that

$$\psi_{21} \cdot r = \begin{pmatrix} 0 & 0 & * & \dots & * \\ 0 & 0 & * & \dots & * \\ 0 & 0 & * & \dots & * \\ 0 & 0 & * & \dots & * \\ * & \dots & \dots & \dots & * \\ * & \dots & \dots & \dots & * \end{pmatrix}.$$

The sequence

$$W_1 := \{0\}^4 \times k^2, W_2 := k^2 \times \{0\}^6 \text{ and } W_i = k^{n_i} \text{ for } i = 0, 3$$

is admissible and

$$\sum_{i=0}^{3} e_i \dim W_i = e_0 + 2e_1 + 2e_2 + 3e_3 = -4e_1 - 6e_2 > -4e_1 - 4e_2 > 0.$$

$\boxed{n=5}$ Up to now, the identity $u\psi_{21} = 0$ was enough to derive a contradiction. We now consider in addition the second equation $\varphi_{11} u = 0$. Let

$$\varphi_{11} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ \vdots & \vdots & \vdots \\ m_{81} & m_{82} & m_{83} \end{pmatrix}$$

where m_{ij} is a linear form and set

$$m := \max_i \dim_k k\langle m_{i1}, m_{i2}, m_{i3} \rangle.$$

We have four cases: $m = 0, \dots, 3$. We must have $\varphi_{11} \neq 0$, i.e. $m = 0$ is not possible, because $e_3 > 0$ by our overall assumption that

$$e_0 n_0 + \ldots + e_i n_i < 0$$

for all $i \leq 2$.

Further, the columns $l_1, \dots, l_6$ span a vector space of dimension at least five and are syzygies of every row vector (m_{i1}, m_{i2}, m_{i3}) of φ_{11}. Therefore, $m \neq 3$, because a vector (m_{i1}, m_{i2}, m_{i3}) of three linearly independent linear forms has only three linearly independent linear syzygies.

$\boxed{m=2}$ Without loss of generality we may assume that

$$\dim k\langle m_{11}, m_{12}, m_{13} \rangle = 2.$$

There exists $r \in \mathrm{GL}_3$ such that after the actions $\varphi_{11} r^{-1}$ and ru, we achieved $m_{13} = 0$. We conclude from $\varphi_{11} u = 0$ that the columns of the first two rows of u

$$\begin{pmatrix} l_{11} \\ l_{21} \end{pmatrix}, \dots, \begin{pmatrix} l_{16} \\ l_{26} \end{pmatrix}$$

are syzygies of the vector (m_{11}, m_{12}). But there is only one linearly independent linear syzygy because m_{11} and m_{12} are linearly independent by assumption. Hence there exists $r \in \mathrm{GL}_6$ such that

$$ur = \begin{pmatrix} 0 & \dots & 0 & l_{16} \\ 0 & \dots & 0 & l_{26} \\ l_{31} & \dots & \dots & l_{36} \end{pmatrix}.$$

Note that by replacing u with ru, we might have changed

$$n = \max_i \dim_k k\langle l_{i1}, \dots, l_{i6} \rangle.$$

Now, either the already proven case $n = 1$ applies or there is an $i \leq 5$ such that $l_{3i} \neq 0$. If such an i exists, then $\varphi_{11} u = 0$ implies that the last column of φ_{11} vanishes. This is easily seen to be in contradiction to $(-\theta)$-semi-stability since $e_3 > 0$.

$\boxed{m = 1}$ This is the last case to be considered. Without loss of generality we may assume that

$$\dim k\langle m_{11}, m_{12}, m_{13} \rangle = 1.$$

There exists $r \in \mathrm{GL}_3$ such that if we replace φ_{11} by $\varphi_{11} r^{-1}$ and u by ru, we achieved

$$\varphi_{11} = \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ \vdots & \vdots & \vdots \\ * & * & * \end{pmatrix}.$$

Then $\varphi_{11} u = 0$ implies $l_{1j} = 0$ for all $j = 1, \dots, 6$.

Since $m = 1$, there exists $r \in \mathrm{GL}_3$ of the form

$$r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

such that

$$\varphi_{11} \cdot r = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \\ \vdots & \vdots & \vdots \\ * & * & * \end{pmatrix}.$$

Acting with r on u by ru leaves also the first row of u invariant. Hence, we may assume that φ_{11} is of the form above and the first row of u vanishes. If $m_{22} \neq 0$, then the second row of u must also vanish since $\varphi_{11}u = 0$. Because $n \neq 0$ by assumption, we conclude further that the last column in φ_{11} is zero. As we have already observed, this is a contradiction to $(-\theta)$-semi-stability. Therefore, $m_{22} = 0$. Now, we iterate the argument to conclude that the last two columns of φ_{11} vanish which is a contradiction to semi-stability. This proves the proposition. □

We obtain for matrix factorisations associated to twisted quartic curves lying on a cubic threefold defined by $f \in S^3V$ the final result:

Theorem 6.13. *Assume that* $\theta = (e_0, e_1, e_2, e_3)$ *is such that*

$$e_1 < 0, e_2 > 0 \textit{ and } e_1 + e_2 < 0.$$

The locus of semi-stable generalised matrix factorisations

$$\overline{\mathrm{MF}}^{ss,\Gamma}(K, L) = \overline{MF}_f^{gen.Sh}(F, G)^{ss,\Gamma}(K, L)$$

admits a projective good quotient Z *for the* Γ*-action which contains as an open subset a geometric quotient of the locus of* Γ*-stable factorisations which are not zero-factorisations. Furthermore,*

$$[\varphi, \psi] \in \overline{\mathrm{MF}}^{ss,\Gamma}(K, L) \Longleftrightarrow R_u(\Gamma).[\varphi, \psi] \subseteq \overline{\mathrm{MF}}^{ss,R}(L_\infty).$$

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