Quantum Gravity Effects in Rotating Black Hole Spacetimes

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Abstract

The aim of this work is to explore, within the framework of the presumably asymptotically safe Quantum Einstein Gravity, quantum corrections to black hole spacetimes, in particular in the case of rotating black holes. We have analysed this problem by exploiting the scale dependent Newton's constant implied by the renormalization group equation for the effective average action, and introducing an appropriate "cutoff identification" which relates the renormalization scale to the geometry of the spacetime manifold. We used these two ingredients in order to "renormalization group improve" the classical Kerr metric that describes the spacetime generated by a rotating black hole.

We have focused our investigation on four basic subjects of black hole physics. The main results related to these topics can be summarized as follows. Concerning the critical surfaces, i.e. horizons and static limit surfaces, the improvement leads to a smooth deformation of the classical critical surfaces. Their number remains unchanged. In relation to the Penrose process for energy extraction from black holes, we have found that there exists a non-trivial correlation between regions of negative energy states in the phase space of rotating test particles and configurations of critical surfaces of the black hole. As for the vacuum energy-momentum tensor and the energy conditions we have shown that no model with "normal" matter, in the sense of matter fulfilling the usual energy conditions, can simulate the quantum fluctuations described by the improved Kerr spacetime that we have derived. Finally, in the context of black hole thermodynamics, we have performed calculations of the mass and angular momentum of the improved Kerr black hole, applying the standard Komar integrals. The results reflect the antiscreening character of the quantum fluctuations of the gravitational field. Furthermore we calculated approximations to the entropy and the temperature of the improved Kerr black hole to leading order in the angular momentum. More generally we have proven that the temperature can no longer be proportional to the surface gravity if an entropy-like state function is to exist.

Zusammenfassung

Das Hauptziel dieser Arbeit ist die Untersuchung von Quanteneffekten in der Raumzeit schwarzer Löcher im Rahmen der vermutlich asymptotisch sicheren Quanten-Einsteingravitation, wobei insbesondere rotierende schwarze Löcher betrachtet werden. Grundlage der Untersuchungen ist die skalenabhängige Newton-Konstante, die sich aus der Renormierungsgruppengleichung der effektiven Mittelwertwirkung ergibt, sowie eine "Cutoff-Identifikation", die die Renormierungsskala zur Geometrie der Raumzeitmannigfaltigkeit in Beziehung setzt. In diesem Rahmen wird eine "Renormierungsgruppenverbesserung" der klassischen Kerr-Metrik durchgeführt, die die Raum
zeit eines rotierenden schwarzen Loches beschreibt.

Die Untersuchungen konzentrieren sich auf vier zentrale Fragestellungen der Physik schwarzer Löcher. Die jeweils wichtigsten Ergebnisse zu diesen Themen können folgendermaßen zusammengefasst werden. Hinsichtlich der kritischen Flächen, d.h. der Horizonte und statischen Grenzflächen, zeigt es sich, daß die Quanteneffekte zwar zu einer Deformation der entschprechenden klassischen Flächen führen, deren Art und Anzahl aber unverändert bleibt. Im Zusammenhang mit dem Penrose-Prozess zur Energieextraktion aus schwarzen Löchern wurde eine nichttriviale Korrelation zwischen den Parameterbereichen negativer Energie für rotierende Testteilchen und den kritischen Flächen gefunden. In Bezug auf den Energieimpulstensor des Vakuums und seiner Positivitätseigenschaften wurde gezeigt, daß es kein Modell mit "normaler" Materie, d.h. solcher, die die üblichen Energiebedingungen erfüllt, geben kann, dessen Materie die berücksichtigten Quanteneffekte simuliert. Umfangreiche Untersuchungen beschäftigen sich mit der Thermodynamik dieser schwarzen Löcher. Ihre Masse und ihr Drehimpuls wurden über die Komar-Integrale berechnet; die Ergebnisse spiegeln den anti-abschirmenden Charakter der Quantenfluktuationen der Metrik wider. Weiterhin wurden in führender Ordnung bzgl. des Drehimpulses Quantenkorrekturen zur Entropie und Temperatur schwarzer Löcher berechnet. Es wurde allgemein gezeigt, daß wenn man die Existenz einer Entropie-ähnlichen Zustandsfunktion fordert, nach der Renormierungsgruppenverbesserung die Temperatur nicht mehr in der üblichen Weise durch die Oberflächengravitation gegeben sein kann.

Contents

Chapter 1

Introduction

The aim of this work is to explore, within the framework of the presumably asymptotically safe Quantum Einstein Gravity, quantum corrections to black hole spacetimes, in particular in the case of rotating black holes. Having this in mind, this introductory chapter is intended to outline the procedure we implement in order to reach this goal. It is organized in the following way: The first section is a short overview of the above mentioned field of research. It is intended to acquaint the reader with the main results of the field without further explanation of its basic concepts. In sections 1.2, 1.3 and 1.4 we present more extensively those concepts and tools which are fundamental for our purposes. Also some relevant results from previous works are included and explained. In the discussion (section 1.5) we further analyse the results presented in the previous sections, we state in particular how they must be interpreted, and how they will be implemented in the next chapters.

1.1 Asymptotically Safe Quantum Einstein Gravity: An Overview

During the past decade a lot of efforts went into the exploration of the nonperturbative renormalization behavior of Quantum Einstein Gravity [1]-[16]. In [1] a functional renormalization group (RG) equation for gravity has been introduced; it defines a Wilsonian RG flow on the theory space consisting of all diffeomorphism invariant action functionals for the metric $g_{\mu\nu}$. In [1] it has been applied to the Einstein-Hilbert truncation which allows for an approximate calculation of the beta-functions of Newton's constant and the cosmological constant. The complete flow pattern was found in [4], and higher derivative truncations were analyzed in [3, 5, 10]. Matter fields were added in refs. [2, 9], and in [12] the beta-functions of [1] and [3] were used for finding optimized RG flows. The most remarkable result of these investigations is that the beta-functions of [1] predict a non-Gaussian RG fixed point [8]. After detailed studies of the reliability of the pertinent truncations [3, 4, 5, 12] it is now believed that it corresponds to a fixed point in the exact theory and is not an approximation artifact. It was found to possess all the necessary properties to make quantum gravity nonperturbatively renormalizable along the lines of Weinberg's "asymptotic safety" scenario [17, 18], thus overcoming the notorious problems related to its nonrenormalizability in perturbation theory. We shall refer to the quantum field theory of the metric tensor whose infinite cutoff limit is taken at the non-Gaussian fixed point as Quantum Einstein Gravity or "QEG". This theory should not be thought of as a quantization of classical general relativity. Its bare action is dictated by the fixed point condition and is therefore expected to contain more invariants than the Einstein-Hilbert term only. Independent evidence pointing towards a fixed point in the full theory came from the symmetry reduction approach of Ref. [19] where the 2-Killing subsector of the gravitational path integral was quantized exactly.

Except for the latter investigations, all recent studies of the asymptotic safety scenario in gravity made use of the approach outlined in [1]. It is based upon the concept of the effective average action [20, 21, 22], a specific continuum implementation of the Wilsonian renormalization group. In its original form for matter theories in flat spacetime it has been applied to a wide range of problems both in particle and statistical physics. As compared to alternative functional RG approaches in the continuum [23] the average action has various crucial advantages; the most important one is its similarity with the standard effective action Γ. In fact, the average action is a scale dependent functional Γ_k depending on a "coarse graining" scale k which approaches Γ in the limit $k \to 0$ and the bare action S in the limit $k \to \infty$. The close relationship of Γ_k and the standard Γ was often crucial for finding the right truncations of theory space encapsulating the essential physics.

Another advantage of the average action is that it defines a family of effective field theories $\{\Gamma_k, 0 \leq k < \infty\}$ labeled by the coarse graining scale k. If a physical situation involves only a single mass scale, then it is well described by a tree level evaluation of Γ_k, with k chosen to equal that scale. In particular, the stationary points of Γ_k have the interpretation of a k-dependent field average (approaching the standard 1-point function for $k \to 0$).

In gravity the effective average action of [1] is a diffeomorphism invariant functional of the metric: $\Gamma_k[g_{\mu\nu}]$. Here the analogous average field $\langle g_{\mu\nu}\rangle_k$ satisfies the "effective Einstein equations"

$$\frac{\delta\Gamma_k}{\delta g_{\mu\nu}(x)}\left[\langle g\rangle_k\right] = 0. \tag{1.1}$$

A given quantum state $|\Psi\rangle$ of the gravitational field implies an infinite family of average metrics: $\{\langle g_{\mu\nu}\rangle_k,\ 0 \leq k < \infty\}$. A scale dependence of the metric [29] has profound consequences since $\langle g_{\mu\nu}\rangle_k$ describes a geometry of spacetime which depends on the degree of "coarse graining", or the "resolving power" of the "microscope" with which it is looked upon. In the case of QEG, it has been shown [3, 5] that this scale dependence leads to fractal properties of spacetime, and that in the scaling regime of the non-Gaussian fixed point, corresponding to sub-Planckian distances, the fractal dimension of spacetime equals 2. In particular, making essential use of (1.1) and the effective field theory properties of Γ_k, the spectral dimension [24] has been calculated; it was found to interpolate between 4 at macroscopic, and 2 at microscopic distances [25]. In [26], Connes et al. speculated about the possible relevance of this dimensional reduction for the noncommutative geometry of the standard model. Remarkably, exactly the same dimensional reduction has been found in Monte Carlo simulations within the causal dynamical triangulation approach [24, 27, 28].

1.2 Renormalization Group Improvement of Black Hole Spacetimes: General Procedure

The procedure we implement in this work for finding the leading quantum corrections to black hole spacetimes is based on the following two key premises:

1. The Newton constant, as well as the cosmological constant, play the role of a coupling constant, in the sense used in effective field theories. They are given by the prefactors of the $\int d^4x\sqrt{-g}R$ and $\int d^4x\sqrt{-g}$-terms, respectively, in a derivative expansion of the effective action. As in any effective field theory,

such coupling constants run under renormalization group transformations. We shall use the scale dependent couplings implied by the RG equation for the effective average action.

2. Similarly as in the familiar renormalization group derivation of the Uehling correction to the Coulomb potential in massless QED [30, 32] where one identifies the RG scale k with the inverse of the radial distance, we introduce an analogous "cutoff identification", which relates k to the geometry of the spacetime manifold. The situation will be more complicated than in QED since this cutoff identification must be invariant under general coordinate transformations, as required by general relativity [30]. The key step consists in replacing in the black hole metric the classical Newton constant by the running coupling $G = G(k)$ where, in turn, k is expressed in geometrical terms via the cutoff identification.

The application of this "renormalization group improvement" to the Kerr metric for rotating black holes is presented in the next two sections. In section 1.3 we give a general presentation of the framework upon which this work is based. Its main conceptual tool is the effective average action and the asymptotic safety hypothesis for Quantum Einstein Gravity. More specifically, we derive an explicit formula for the running of the Newton constant, using the Einstein-Hilbert truncation of the space of action functionals in order to solve the exact renormalization group equation (ERGE) for the effective average action Γ_k. In section 1.4 we present the infrared cutoff identification, the second element of our improvement procedure.

1.3 The Running Newton Constant

The two objectives of this section are to give the theoretical basis for understanding the running of the Newton constant in the framework of Quantum Einstein Gravity, and to present the main steps of the derivation of a simple approximate formula for this running [30]. For this purpose we shall present concepts like the asymptotic safety scenario, the effective average action and the Einstein-Hilbert truncation. They are included in the respective subsections.

1.3.1 General Framework: Asymptotically Safe Quantum Einstein Gravity

The criterion of perturbative renormalizability has been a valuable key for understanding the structure of fully predictive quantum field theories [33, S. 12.3]. It requires that the theory has a finite set of terms in the (bare) Lagrangian, which are invariant under specific symmetries, and provides finite results for every physical quantity; every infinity that appears can be "absorbed" into a finite set of undetermined parameters whose values must be fixed by the experiment [33]. This demand applies no matter how the quantization of the specific theory is performed. If the quantization of the theory is performed using an expansion of the generating functional Z in some small coupling then it should be "renormalizable" according to the well known perturbative renormalization theory. For the case of general relativity (GR) derived from the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ -R + 2\Lambda \right\} \tag{1.2}$$

the perturbative renormalization by an expansion in G has failed. The theory is perturbatively non-renormalizable, since the coupling G has the dimension of a length squared [33].

Nevertheless, the possibility of absorbing all infinities can be recovered if the constraint of having only a finite set of terms in the Lagrangian is not required[33]. If one includes every possible term allowed by the symmetry conditions, renormalizability is again accomplished in a more general way, where infinitely many coupling constants are available to absorb every ultraviolet divergence coming from arbitrarily high order loop integrals [33, 17].

This generalized framework gives a new interpretation to physical theories like GR which are known to provide a good approximation within a specific ("classical") regime, in particular theories which work only at the low energy scales. At higher scales, other terms which are highly supressed at lower scales, but are present in the most general Lagrangian consistent with the symmetries, become relevant. This interpretation is the basis of the effective field theories (EFT) which are able to predict quantum corrections at low energies.

In every EFT there exists a fundamental scale[1] that defines in an absolute way the orders of magnitude of every term in the general Lagrangian and therefore how

[1] For quantum gravity it is the Planck scale ($\approx 10^{19}$GeV).

much they are suppressed at lower energies[2]. When this scale is reached the EFTs loose any predictability, since infinitely many terms must be taken into account and their respective renormalized couplings are unknown parameters of the theory. As a result, we would have an infinite set of unpredicted parameters, to be taken from the experiment.

A natural question turns out to be: what happens at the fundamental scale where the EFTs loose their predictability?

The asymptotic safety hypothesis proposes that the theory which would explain phenomena at this scale and above, continues to be a quantum field theory (no strings or anything else!). More especifically, it is "one in which the finite or infinite number of generalized couplings do not run off to infinity with increasing energy" [17] but hit a so-called fixed point in the space of coupling constants. It is a special point in this space that remains invariant under the renormalization group transformations.

For QCD the fixed point happens to be at zero values for all the essential couplings. This is called a Gaussian fixed point (GFP), which leads in this case to the familiar asymptotic freedom. Nevertheless, a fixed point could also be realised at non-zero values, in this case it is called a non-Gaussian fixed point (NGFP). The latter is expected to be the case for Quantum Einstein Gravity.

The theory space is the formal framework where the asymptotic safety hypothesis is implemented [14]. It is defined to be the space of all action functionals which depend on a given set of fields and are invariant under certain symmetries. In particular, for Einstein Gravity, the symmetry transformations are the general coordinate transformations, and the field is the metric tensor field. The infinitely many generalized couplings g_i needed to parametrize a general action functional are local coordinates on the theory space. This space carries an important geometrical structure, namely a vector field of beta functions $\beta_i\left(g_1, g_2 ...\right)$. It is necessary in order to describe the "RG running" of every g_i in the mass scale k. The evolution equation for $g_i\left(k\right)$ is given by

$$k\partial_k g_i\left(k\right) = \beta_i\left(g_1, g_2, \cdots\right) \tag{1.3}$$

By definition, the renormalization group (RG) trajectories, i.e. the solutions to the "exact renormalization group equations" (1.3) are the integral curves of the vector

[2] In fact these terms turn out to be supressed by factors of the ratio of the typical energy of some set of physical phenomena and the fundamental energy scale.

field β_i defining the “RG flow”.

The parameter k defines a lower limit for the integration of high energy modes in the generating functional of the specific quantum field theory to be studied. In this sense k must be intepreted as an infrared cutoff of the theory, and as a result, (1.3) must be understood as the equation that governs the dependence of the couplings g_i on the variation of the cutoff k, when the number of field-modes that are integrated out increases or decreases.

The asymptotic safety hypothesis assumes that the β_i's have a common zero at a point with coordinates g_{*i} not all of which are zero. Given such a NGFP of the RG flow one defines its ultraviolet critical surface $\mathcal{S}_{\mathcal{UV}}$ to consist of all points of theory space which are attracted into it in the limit $k \to \infty$. A specific quantum field theory is defined by a RG trajectory which exists globally, i.e. is well behaved all the way down from "$k = \infty$" in the UV to $k = 0$ in the IR. The key idea of asymptotic safety is to base the theory upon one of the trajectories running inside the hypersurface $\mathcal{S}_{\mathcal{UV}}$ since these trajectories are manifestly well-behaved and free from fatal singularities, blowing up couplings, etc, in the large-k limit. Moreover, a theory based upon a trajectory inside $\mathcal{S}_{\mathcal{UV}}$ can be predictive, the problem of an increasing number of undetermined parameters which plagues effective field theories does not arise [14, 17]. If $\Delta \equiv dim\ \mathcal{S}_{\mathcal{UV}}$ is finite, there exists only a Δ-parameter family of different quantum theories, i.e. there are only Δ undetermined parameters which must be taken from the experiment.

1.3.2 The Effective Average Action

As already mentioned in the last section, the couplings g_i parametrize a general action functional. In the framework of the effective average action, this action is defined to depend parametrically, via the running couplings, $g_i \equiv g_i(k)$ on the mass scale k which has the interpretation of an IR cutoff [22, 34]. We denote the effective average action by $\Gamma_k\left[g_{\mu\nu}, ...\right]$ where the dots stand for possible matter fields. The running with k results from adding a $k-$dependent IR cutoff term $\Delta_k S$ to the classical action in the standard Euclidean functional integral [1]. This term gives a momentum dependent mass square $R_k\left(p^2\right)$ to the field modes with momentum p. It is designed to vanish if $p^2 \gg k^2$, but suppresses the contribution of the modes with $p^2 < k^2$ to the path integral. When regarded as a function of k, Γ_k describes a curve in theory space that interpolates between the classical action $S = \Gamma_{k\to\infty}$

and the conventional effective action $\Gamma = \Gamma_{k=0}$. The change of Γ_k induced by an infinitesimal change of k is described by a functional differential equation, the exact RG equation or "ERGE". In a symbolic notation it reads

$$k\partial_k \Gamma_k = \frac{1}{2} Tr \left[k\partial_k R_k \left(\Gamma_k^{(2)} + R_k \right)^{-1} \right] \tag{1.4}$$

where $\Gamma_k^{(2)}$ is the Hessian of Γ_k with respect to the dynamical fields. For a detailed discussion of this equation we must refer to the literature [1]. Suffice it to say that expanding $\Gamma_k \left[g_{\mu\nu}, \cdots \right] = \sum_i g_i(k) I_k \left[g_{\mu\nu}, \cdots \right]$ in terms of diffeomorphism invariant field monomials $I_k \left[g_{\mu\nu}, ... \right]$ with coefficients $g_i(k)$, equation (1.4) assumes the component form (1.3). In the next subsection we illustrate this procedure within a simple approximation.

1.3.3 Einstein-Hilbert Truncation and the Running Newton Constant

In general it is impossible to find exact solutions to equation (1.4) and we are forced to rely upon approximations. A powerful nonperturbative approximation scheme is the truncation of theory space where the RG flow is projected onto a finite-dimensional subspace. In practice one makes an ansatz for Γ_k that comprises only a few couplings and inserts it into the RG equation. This leads to a, now finite, set of coupled differential equations of the form (1.3).

The simplest approximation one might try is the "Einstein-Hilbert truncation" [1, 3] defined by the ansatz

$$\Gamma_k \left[g_{\mu\nu} \right] = \left(16\pi G_k \right)^{-1} \int d^d x \sqrt{g} \left\{ -R(g) + 2\bar{\lambda}_k \right\} \tag{1.5}$$

It applies to a d-dimensional Euclidean spacetime and involves only the cosmological constant $\bar{\lambda}_k$ and the Newton constant G_k as running parameters. By inserting (1.5) into the RG equation (1.4) one obtains a set of two β−functions $\left(\beta_\lambda, \beta_g \right)$ for the dimensionless cosmological constant $\lambda_k \equiv k^{-2} \bar{\lambda}_k$ and the dimensionless Newton constant $g_k \equiv k^{d-2} G_k$, respectively. They describe a two-dimensional RG flow on the plane with coordinates $g_1 \equiv \lambda$ and $g_2 \equiv g$. The resulting equations (1.3) are given by [30, 1]:

$$\partial_t g = \left(d - 2 + \eta_N \right) g \tag{1.6}$$

and

$$\partial_t \lambda = -\left(2-\eta_N\right)\lambda + \frac{1}{2} g \left(4\pi\right)^{\left(1-\frac{d}{2}\right)} \times \tag{1.7}$$
$$\times \left[2d\left(d+1\right)\Phi^1_{\frac{d}{2}}\left(-2\lambda\right) - 8d\Phi^1_{\frac{d}{2}}\left(0\right) - d\left(d+1\right)\eta_N \tilde{\Phi}^1_{\frac{d}{2}}\left(-2\lambda\right)\right]$$

Here $t \equiv \ln k$ and

$$\eta_N\left(g,\lambda\right) = \frac{gB_1\left(\lambda\right)}{1-gB_2\left(\lambda\right)} \tag{1.8}$$

is the anomalous dimension of the operator $\sqrt{g}R$ and the functions $B_1\left(\lambda\right)$ and $B_2\left(\lambda\right)$ are given by

$$\begin{aligned} B_1\left(\lambda\right) &\equiv \frac{1}{3}\left(4\pi\right)^{\left(1-\frac{d}{2}\right)} \times \\ &\times \left[d\left(d+1\right)\Phi^1_{\frac{d}{2}-1}\left(-2\lambda\right) - 6d\left(d-1\right)\Phi^2_{\frac{d}{2}}\left(-2\lambda\right) - 4d\Phi^1_{\frac{d}{2}-1}\left(0\right) - 24\Phi^2_{\frac{d}{2}}\left(0\right)\right] \\ B_2\left(\lambda\right) &\equiv -\frac{1}{6}\left(4\pi\right)^{\left(1-\frac{d}{2}\right)}\left[d\left(d+1\right)\tilde{\Phi}^1_{\frac{d}{2}-1}\left(-2\lambda\right) - 6d\left(d-1\right)\tilde{\Phi}^2_{\frac{d}{2}}\left(-2\lambda\right)\right] \end{aligned} \tag{1.9}$$

with the cutoff-dependent "threshold functions" $(p = 1, 2, ...)$

$$\begin{aligned} \Phi^p_n\left(y\right) &= \frac{1}{\Gamma\left(n\right)}\int_0^\infty dz\, z^{n-1}\frac{R^{(0)}\left(z\right) - zR^{(0)\prime}\left(z\right)}{\left[z+R^{(0)}\left(z\right)+y\right]^p} \\ \tilde{\Phi}^p_n\left(y\right) &= \frac{1}{\Gamma\left(n\right)}\int_0^\infty dz\, z^{n-1}\frac{R^{(0)}\left(z\right)}{\left[z+R^{(0)}\left(z\right)+y\right]^p} \end{aligned} \tag{1.10}$$

They depend explicitly on the cutoff function $R^{(0)}\left(z\right)$ with $z \equiv p^2/k^2$. This function is related to the momentum dependent mass $R_k\left(p^2\right)$ and has to satisfy the conditions $R^{(0)}\left(0\right) = 1$ and $R^{(0)}\left(z\right) \to 0$ for $z \to \infty$. For explicit computations we use the exponential form

$$R^{(0)}\left(z\right) = \frac{z}{e^z - 1} \tag{1.11}$$

It is important to distinguish between R, R_k, and $R^{(0)}\left(z\right)$. R is the curvature scalar that defines together with the cosmological constant, the Einstein-Hilbert action. The operator R_k describes the transition from the high-momentum to the low-momentum regime. It depends on the dimensionless function $R^{(0)}$ which interpolates smoothly between $R^{(0)}\left(0\right) = 1$ and $\lim_{u\to\infty} R^{(0)}\left(u\right) = 0$. Except for these two conditions the function $R^{(0)}(z)$ is arbitrary. For further details about the effective average action in gravity and the derivation of the above results we refer to [1].

Considering General Relativity as the low energy effective field theory of Quantum Einstein Gravity we identify the standard Einstein-Hilbert action with the average action $\Gamma_{k_{obs}}$. Here k_{obs} is some typical "observational scale" at which the classical tests of general relativity have confirmed the validity of the Enstein-Hilbert action. In order to find an approximate solution to the flow equation we assume that also for $k > k_{obs}$ i.e. at higher momenta, Γ_k is well approximated by an action of the Einstein-Hilbert form. Concerning the observed values for the couplings, we set $G(k_{obs}) = G_0$ and $\bar{\lambda}(k_{obs}) \approx 0$. Furthermore, since at least within our approximation, there is essentially no running of these couplings between k_{obs} (the scale of the solar system, say) and cosmological scales ($k \approx 0$) we may set $k_{obs} = 0$ and identify the measured couplings with

$$G(0) \equiv G_0 \tag{1.12}$$

and

$$\bar{\lambda}(0) \approx 0 \tag{1.13}$$

Finally, since in the present work we are not dealing with cosmological scales, we assume that $\bar{\lambda} \ll k^2$ for all scales of interest, so that we may approximate $\lambda(k) \approx 0$. In that case we may also neglect the evolution equation (1.7) for λ , and we substitute $B_1(0)$ and $B_2(0)$ in (1.8) so that we have

$$\eta_N(g) = \frac{gB_1}{1 - gB_2} \tag{1.14}$$

with

$$B_1 \equiv B_1(0) = -\left(\frac{1}{3\pi}\right)\left[24\Phi_2^2(0) - \Phi_1^1(0)\right] \tag{1.15}$$

$$B_2 \equiv B_2(0) = -\left(\frac{1}{6\pi}\right)\left[18\tilde{\Phi}_2^2(0) - 5\tilde{\Phi}_1^1(0)\right] \tag{1.16}$$

Assuming $d = 4$ in the following (no extra dimensions!) we stay with (1.6) in the form

$$k\partial_k g \quad = \quad (2 + \eta_N)\, g = \beta(g(k)) \tag{1.17}$$

where the anomalous dimension is

$$\eta_N(g) = \frac{gB_1}{1 - gB_2} \tag{1.18}$$

and $\beta(g)$ is given by

$$\beta(g) = 2g\left(\frac{1 - \omega' g}{1 - B_2 g}\right) \tag{1.19}$$

We also defined

$$w \equiv -\frac{1}{2}B_1,\ \omega' = w + B_2 \tag{1.20}$$

For the exponential cutoff (1.11) we have explicitly

$$\begin{aligned}
\Phi_1^1(0) &= \frac{\pi^2}{6},\ \Phi_2^2(0) = 1 \\
\tilde{\Phi}_1^1(0) &= 1,\ \tilde{\Phi}_2^2(0) = \frac{1}{2}
\end{aligned}$$

and, as a result,

$$w = \frac{4}{\pi}\left(1 - \frac{\pi^2}{144}\right),\ B_2 = \frac{2}{3\pi}$$

The evolution equation (1.17) displays two fixed points g_* for which $\beta(g_*) = 0$ holds. There exists an infrared attractive (Gaussian) fixed point at $g_*^{\mathrm{IR}} = 0$ and an ultraviolet attractive (non-Gaussian) fixed point at

$$g_*^{\mathrm{UV}} = \frac{1}{\omega'}$$

The UV fixed point separates in a natural way a weak coupling regime $\left(g < g_*^{\mathrm{UV}}\right)$ from a strong coupling regime where $g_*^{\mathrm{UV}} < g$. Since the $\beta-$function in (1.19) is positive for $g \in \left[0, g_*^{\mathrm{UV}}\right]$ and negative otherwise, the renormalization group trajectories which result from (1.17) with (1.19) fall into the following three classes:

1. Trajectories with $g(k) < 0$ for all k. They are attracted towards g_*^{IR} for $k \to 0$.

2. Trajectories with $g(k) > g_*^{\mathrm{UV}}$ for all k. They are attracted towards g_*^{UV} for $k \to \infty$.

3. Trajectories with $g(k) \in \left[0, g_*^{\mathrm{UV}}\right]$ for all k. They are attracted towards g_*^{IR} for $k \to 0$ and towards g_*^{UV} for $k \to \infty$.

In the present work we deal exclusively with the third class. The first class implies a negative Newton constant, therefore we discard it, and the second class is not connected to a low energy regime where the Einstein-Hilbert truncation is known to be a good approximation. Therefore we discard it too.

The differential equation (1.17) with (1.19) can be integrated analytically to yield

$$\frac{g}{(1-\omega' g)^{\frac{w}{\omega'}}} = \frac{g(k_0)}{[1-\omega' g(k_0)]^{\frac{w}{\omega'}}}\left(\frac{k}{k_0}\right)^2 \tag{1.21}$$

This expression cannot be solved for $g = g(k_0)$ in closed form. However, it is obvious that this solution interpolates between the IR behavior $g(k) \propto k^2$ for $k \to 0$ and $g(k) \to \frac{1}{\omega'}$ for $k \to \infty$.

In order to obtain an approximate analytic expression for the running Newton constant we observe that the ratio $\frac{\omega'}{w}$ is actually very close to unity. Numerically one has $w \approx 1.2$, $B_2 \approx 0.21$, $\omega' \approx 1.4$, $g_*^{UV} \approx 0.71$ so that $\frac{\omega'}{w} \approx 1.18$ is indeed close to 1. Replacing $\frac{\omega'}{w} \to 1$ in eq. (1.21) yields a rather accurate approximation with the same general features as the exact solution. In this case we can easily solve (1.21) to get:

$$g(k) = \frac{g(k_0)\, k^2}{w g(k_0)\, k^2 + [1 - w g(k_0)]\, k_0^2} \tag{1.22}$$

This function is an *exact* solution to the renormalization group equation with the approximate anomalous dimension $\eta_N = -2wg + O(g^2)$ which is the first term in the perturbative expansion of eq. (1.18):

$$\eta_N = -2wg\left[1 + \sum_{n=1}^{\infty}(B_2 g)^n\right]$$

Remarkably, for the trajectory (1.22) the quantity $B_2\, g(k)$ remains negligibly small for all values of k. It assumes its largest value at the UV fixed point where $B_2\, g_*^{\mathrm{UV}} = 0.15$. Thus equation (1.22) provides us with a consistent approximation.

In terms of the dimensionful Newton constant $G(k) \equiv \frac{g(k)}{k^2}$ eq. (1.22) reads

$$G(k) = \frac{G(k_0)}{1 + wG(k_0)\,[k^2 - k_0^2]} \tag{1.23}$$

As mentioned before we set $k_0 = 0$ and $G(0) \equiv G_0$. Hence

$$G(k) = \frac{G_0}{1 + wG_0 k^2} \tag{1.24}$$

where G_0 has to be identified with the experimentally observed value of the Newton constant. On the other hand w is a constant which depends on the choice of the cutoff function $R^{(0)}$ (see (1.20) and (1.15)). In statistical mechanics language, the quantity w is "non-universal". As a result, $G(k)$ cannot be directly observable. On the way from $G(k)$ to an observable quantity a second source of non-universality ($R^{(0)}$-dependence) must show up which compensates for the $R^{(0)}$-dependence of w. We shall come back to this point in a moment.

We can use the parameter w to switch off quantum effects since:

1. It is proportional to $\hbar$
2. In the approximation $\frac{\omega'}{w} \to 1$ that we are assuming when we use expression (1.24), w is the only one constant related to the running, and it is a prefactor to k^2.
3. As mentioned before, within the approximation to RG-improvement of black hole spacetimes we are applying, we recover Classical General Relativity for $k = k_{obs} = 0$.

From (1.24) we see that when we go to higher momentum scales k, $G(k)$ decreases monotonically. For small k we have

$$G(k) = G_0 - wG_0^2k^2 + O\left(k^4\right) \tag{1.25}$$

while for $G_0^{-1} \ll k^2$ the fixed point behavior sets in and $G(k)$ "forgets" its infrared value:

$$G(k) \approx \frac{1}{wk^2}$$

Remarkably, for $k \to \infty$, i.e. at very high energies, Newton's constant vanishes! Whenever in our analysis of Kerr black holes we need the explicit form of the function $G(k)$ we shall employ the approximate formula (1.24).

1.4 Identification of the Infrared Cutoff

As already mentioned in section 1.2, we implement a diffeomorphism invariant cutoff identification, similar in spirit to the identification made in the renormalization group based derivation of the Uehling correction to the Coulomb potential in massless QED [30, 32]. In that derivation, one starts from the classical potential energy

$V_{cl}(r) = \frac{e^2}{4\pi r}$ and replaces e^2 by the running gauge coupling in the one-loop approximation:

$$e^2(k) = \frac{e^2(k_0)}{1 - b\ln\left(\frac{k}{k_0}\right)} \quad ; \quad b \equiv \frac{e^2(k_0)}{6\pi^2} \tag{1.26}$$

Since, in the massless theory, r is the only dimensionful quantity which could define a scale, it is reasonable to identify the renormalization point k with the inverse of the distance r. Thus we write

$$V(r) = \frac{e^2\left(\frac{1}{r_0}\right)}{4\pi r}\left[1 + b\ln\left(\frac{r_0}{r}\right) + O\left(e^2\right)\right] \tag{1.27}$$

which is the correct (one-loop, massless) Uehling potential, usually derived by more conventional perturbative methods [32]. We can interpret, in this case, the success of the improvement $e^2 \to e^2(k)\,,\ k(r) \propto \frac{1}{r}$ in recovering expression (1.27), by saying that it encapsulates the most important effects which the quantum fluctuations exert on the electric field produced by a point charge.

A similar process can be carried out in order to RG improve exact solutions of the Einstein field equations. We expect in this case, in analogy to QED, that for $k \approx k_{obs}$ the running of Newton's constant represents the most important quantum correction to the classical solutions. In order to respect diffeomorphism invariance, an invariant analogous to the identification $k(r) \propto \frac{1}{r}$ must be found. In this work we propose a position dependent IR-cutoff of the form

$$k(P) = \frac{\xi}{d(P)} \tag{1.28}$$

where $d(P)$ is a distance scale which provides the relevant cutoff for the Newton constant when a test particle is located at the point P of the black hole spacetime. Furthermore, ξ is a constant of order unity that accounts for our lack of knowledge about the exact physical mechanism of IR-cutoff.

A possibility of implementing the diffeomorphism invariance required by general relativity is to define $d(P)$ to be the proper distance from a fixed initial point P_0 to the final point P along some curve $\mathcal{C}$, with respect to the classical metric which we want to improve:

$$d(P) = \int_{\mathcal{C}} \sqrt{|ds^2|} \tag{1.29}$$

There is still some ambiguity as for the correct identification of the spacetime curve $\mathcal{C}$. However, certain universal features will be found when we analyse different curves in the relevant (asymptotic) regimes. Hence consistent approximations can be obtained. This analysis will be systematically carried out for the Kerr metric in chapter 3. For the Schwarzschild spacetime it has been performed in [30].

Concerning the intepretation of ξ the following remarks are in order. If we substitute (1.28) in (1.24) we get

$$G(P) = \frac{G_0 d^2(P)}{d^2(P) + G_0 \bar{w}} \tag{1.30}$$

with $\bar{w} = w\xi^2$. Eq. (1.30) represents the position dependence of the running G. (Implicitly $G(P)$ depends via $d(P)$ also on the parameters of our physical system such as the mass and the angular momentum of the black hole.) Eq. (1.30) shows that w and ξ occur exclusively in the form of their product that we have called $\bar{w} = w\xi^2$. The numerical value of this product cannot be obtained by renormalization arguments alone: w is a non-universal constant since its value depends on the shape of the function $R^{(0)}$ and also ξ is unknown as long as one does not identify the specific cutoff for a concrete process. In a sense, the product $\bar{w} = w\xi^2$ can be determined experimentally by measuring the asymptotic correction to Newton's law. As a result, the nonuniversalities of w and ξ^2 should cancel to some extent in a consistent ab initio calculation. (See [30] for a discussion of this point.)

1.5 Discussion

In section 1.2 we have presented the two guiding ideas of the method we will use in this work for RG improving solutions of the classical Einstein equations. In this sense equations (1.24) and (1.28) with (1.29) are the main results to be applied in the next chapters. We emphasize that, since we take only into account the running of Newton's constant among the infinitely many couplings in the most general diffeomorphism invariant Langrangian, our procedure is safe only if we don't go too far away from k_{obs} and if the RG corrections are not too big.

It is worthwhile, at this stage, to analyse the physical mechanism behind the scale dependence in (1.24) and its further implementation, via the cutoff identification (1.28)[35]. We can start again by analysing the easier case of the running electric charge. In this case, the combination of quantum mechanics and special relativity converts the vacuum of electrodynamics into a sea of virtual electron-positron pairs which are continuously created and anihilated. When we immerse an external test charge into this sea, it gets polarized in very much the same way as an ordinary dielectric. The polarization cloud of the virtual $e^+ + e^-$ pairs surrounding the test charge tends to screen it, and it appears to be larger at small distances and smaller at large distances. In an experiment which resolves length scales $l \equiv k^{-1}$ one measures the effective charge $e(k)$ which includes the effect of this polarization of the vacuum. As a consequence of this screening mechanism the classical Coulomb potential is replaced by the more complicated quantum corrected Uehling potential from (1.27). At least in the limit of massless electrons, this potential is directly related to the running charge.

Guided by the analogy with the running electric charge, it is tempting to speculate on how quantum gravitational effects might modify Newton's law and lead to a scale dependence of G. It is plausible to assume that in the large distance limit the leading quantum effects are described by quantizing the linear fluctuations of the metric, $g_{\mu\nu}$. One obtains a free field theory in a possibly curved background spacetime whose elementary quanta, the gravitons, carry energy and momentum.

The vacuum of this theory will be populated by virtual graviton pairs, and the central question is how these virtual gravitons respond to the perturbation by an external test body which we immerse in the vacuum. Assuming that also in this situation gravity is universally attractive, the gravitons will be attracted towards the test body. Hence it will become "dressed" by a cloud of virtual gravitons surrounding it so that its effective mass seen by a distant observer is larger than it would be in absence of any quantum effects. This means that while in QED the quantum fluctuations *screen* external charges, in quantum gravity they have an *antiscreening* effect on external test masses. This entails Newton's constant becoming a scale dependent quantity $G(k)$ which is small at small distances $l \equiv k^{-1}$, and which becomes large at larger distances. This behavior is similar to the running of nonabelian gauge coupling in Yang-Mills theory and it is exactly the behavior actually found within the Einstein-Hilbert truncation.

Chapter 2

The Kerr Metric: An Exact Solution of Einstein's Equation for Rotating Black Holes

In this chapter we present a historical introduction to the Kerr metric and its properties without giving rigorous proofs, leaving the ones which are needed for the purposes of this work to the subsequent chapters.

The Kerr metric in the $(\hat{t}, x, y, z)$-system of coordinates used in Kerr's original article from 1963 reads [36]

$$\begin{aligned} ds^2 &= -d\hat{t}^2 + dx^2 + dy^2 + dz^2 + \frac{2mr}{r^4 + a^2 z^2} \times \\ \times & \left[d\hat{t}^2 + \frac{r}{a^2 + r^2}(xdx + ydy) + \frac{a}{a^2 + r^2}(ydx - xdy) + \frac{z}{r}dz \right]^2 \end{aligned} \tag{2.1}$$

with r^2 defined as the solution to the equation

$$r^4 - r^2\left(\rho^2 - a^2\right) - a^2 z^2 = 0$$

where $\rho^2 \equiv x^2 + y^2 + z^2$. The metric (2.1) depends on the two independent parameters m and a. Written in this way (in the so-called Kerr-Schild coordinates) it shows almost directly its asymptotic flatness. For $a = 0$ it reduces to the Schwarzschild spacetime, which indicates that m is related to the mass of a gravitational source. Concerning a, already in the same article, Roy P. Kerr proposes this solution of the Einstein equations as representing the gravitational field outside a rotating source with an angular momentum given by $J \equiv ma/G_0$. The formal demonstration for this interpretation came in 1968 when Jeffrey M. Cohen calculated the angular momentum of the source by integrating over the whole Kerr metric field [37], leaving no doubt of the meaning of this solution. The importance of the Kerr metric in studying the phenomenon of gravitational collapse reached a fundamental level with the so-called “no hair” theorem by Brandon Carter in 1970 [38] which was inspired by earlier results from Werner Israel for the cases of the Schwarzschild and Reissner-Nordström spacetimes [39]. The no-hair theorem from Carter demonstrates that, under reasonable physical conditions, the Kerr spacetime is the final state of any realistic, asymptotically flat gravitational collapse configuration, and therefore the resulting black hole depends exclusively on two parameters, namely m and a. (For more details see [38].)

For a study of the symmetries of the Kerr spacetime, the form (2.1) is not quite appropiate, since its Cartesian-like character shadows the axial symmetry around the rotation axis of the source. In 1967, Robert Boyer and Richard Lindquist found a coordinate system which shows the axial symmetry more clearly. Besides, it reduces to the standard form of the Schwarzschild metric for $a = 0$. It contains only one mixing term (a $dtd\varphi$ term) so that this metric resembles the metric of a rotating Minkowski spacetime (see section 2.2 on frame dragging) [40]. The so-called Boyer-Lindquist (B-L) representation results from applying the following intermediate coordinate transformation to (2.1) [41]:

$$\begin{aligned} \cos\theta &= \frac{z}{r} \\ (r - ia)\, e^{i\hat{\varphi}} \sin\theta &= x + iy \\ u &= \hat{t} + r \end{aligned} \tag{2.2}$$

A further transformation is then needed to eliminate the resulting $dudr$ and $d\hat{\varphi}dr$ cross terms:

$$du = dt + \frac{r^2 + a^2}{\Delta} dr \tag{2.3}$$

$$d\varphi = d\hat{\varphi} - \frac{a}{\Delta} dr \tag{2.4}$$

Here we introduced

$$\Delta \equiv \Delta(r) \equiv r^2 + a^2 - 2mr$$

The final result in the (t, r, θ, φ)-coordinate system has the general form

$$ds^2 = g_{tt} dt^2 + 2g_{t\varphi} dt d\varphi + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\varphi\varphi} d\varphi^2 \tag{2.5}$$

with the following (covariant) components of the metric:

$$
\begin{aligned}
g_{tt} &= -\left(1-\frac{2MG_0 r}{\rho^2}\right) \ , \ g_{rr} = \frac{\rho^2}{\Delta} \ , \ g_{\varphi\varphi} = \frac{\Sigma \sin^2\theta}{\rho^2} \\
g_{\theta\theta} &= \rho^2 \ , \ g_{t\varphi} = -\frac{2MG_0 r a \sin^2\theta}{\rho^2}
\end{aligned} \tag{2.6}
$$

Here we introduced the abbreviations

$$\rho^2 \equiv r^2 + a^2\cos^2\theta \tag{2.7}$$

$$\Delta \equiv r^2 + a^2 - 2MG_0 r \tag{2.8}$$

$$\Sigma \equiv \left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta \tag{2.9}$$

and the definition of the "geometric mass",

$$m \equiv MG_0 \tag{2.10}$$

Inverting the metric we obtain the following contravariant components:

$$
\begin{aligned}
g^{tt} &= -\frac{\Sigma}{\rho^2\Delta} \ , \ g^{rr} = \frac{\Delta}{\rho^2} \ , \ g^{\varphi\varphi} = \frac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta} \\
g^{\theta\theta} &= \frac{1}{\rho^2} \ , \ g^{t\varphi} = -\frac{2MG_0 r a}{\rho^2\Delta}
\end{aligned} \tag{2.11}
$$

It is clear from (2.6) and (2.11) that the components of the Kerr metric in the B-L representation are symmetric under rotations in φ. For $a = 0$ we get directly from (2.6) the standard representation of the Schwarzschild metric [42]:

$$ds^2 = -\left(1-\frac{2MG_0}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2MG_0}{r}\right)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{2.12}$$

We will use the B-L respresentation (2.6) or (2.11) in most of the calculations of this thesis.

Another important representation of the Kerr and Schwarzschild spacetimes corresponds to the Eddington-Finkelstein (E-F) coordinate system(s). The E-F coordinates are the result of transforming the coordinates t and φ of the B-L coordinates according to one of the following two possibilities [60, 57]:

- Ingoing E-F coordinates ("Ingoing Patch")

$$dv = dt + dr^* \tag{2.13}$$

$$d\psi = d\varphi + dr^{\#} \tag{2.14}$$

- Outgoing E-F Coordinates ("Outgoing Patch")

$$
\begin{aligned}
du &= dt - dr^* && (2.15)\\
d\chi &= d\varphi - dr^{\#} && (2.16)
\end{aligned}
$$

The differentials of the functions $r^* = r^*(r)$ and $r^{\#} = r^{\#}(r)$ are defined as

$$
\begin{aligned}
dr^* &\equiv \left[\frac{r^2+a^2}{\Delta(r)}\right] dr\\
dr^{\#} &\equiv \left[\frac{a}{\Delta(r)}\right] dr && (2.17)
\end{aligned}
$$

The relations (2.17) can be integrated explicitly, with the result

$$
\begin{aligned}
r^*(r) &= \int \frac{(r^2+a^2)}{\Delta(r)} dr = r + \frac{mr_+}{\sqrt{m^2-a^2}} \ln\left|\frac{r}{r_+} - 1\right| - \frac{mr_-}{\sqrt{m^2-a^2}} \ln\left|\frac{r}{r_-} - 1\right|\\
r^{\#}(r) &= \int \frac{a}{\Delta(r)} dr = \frac{a}{2\sqrt{m^2-a^2}} \ln\left|\frac{r-r_+}{r-r_-}\right| && (2.18)
\end{aligned}
$$

where $r_\pm \equiv m \pm \sqrt{m^2-a^2}$. Thus the transition from the B-L to the ingoing E-F coordinates reads explicitly

$$
v = t + r^*(r) \ , \ r = r \ , \ \theta = \theta \ , \ \psi = \phi + r^{\#}(r) \qquad (2.19)
$$

while for the outgoing E-F coordinates

$$
v = t - r^*(r) \ , \ r = r \ , \ \theta = \theta \ , \ \chi = \phi - r^{\#}(r) \qquad (2.20)
$$

The sets of coordinates (u, χ) and (v, ψ) are in fact labels for outgoing and ingoing null geodesics in the Kerr spacetime, respectively [60]. Therefore the coordinate systems with $x^\mu = (u, r, \theta, \chi)$ and $x^\mu = (v, r, \theta, \psi)$ are called, respectively, the outgoing and ingoing E-F coordinate systems, or the outgoing and ingoing E-F patches. The components $g_{\alpha\beta}$ of the Kerr spacetime in E-F coordinates are given by

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{2G_0Mr}{\rho^2}\right) du^2 - 2drdu + 2a\sin^2\theta d\chi dr +\\
&\quad - \frac{4G_0Mar\sin^2\theta}{\rho^2} d\chi du + \frac{\Sigma\sin^2\theta}{\rho^2} d\chi^2 + \rho^2 d\theta^2 && (2.21)
\end{aligned}
$$

for the outgoing patch, and

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{2G_0Mr}{\rho^2}\right) dv^2 + 2drdv - 2a\sin^2\theta d\psi dr +\\
&\quad - \frac{4G_0Mar\sin^2\theta}{\rho^2} d\psi dv + \frac{\Sigma\sin^2\theta}{\rho^2} d\psi^2 + \rho^2 d\theta^2 && (2.22)
\end{aligned}
$$

for the ingoing patch. We will use the E-F coordinate systems, especially the ingoing patch, in cases where the B-L representation leads to coordinate singularities.

In the next sections we present the most important features of the Kerr metric which can be divided into the following four topics [53, 60]:

- Critical Surfaces.
- Frame Dragging.
- Energy Extraction.
- Thermodynamics.

We will come back in more detail to all of them in chapters 4, 5, 6 and 7, respectively, where we look at the consequences of including the running of Newton's constant.

2.1 Critical Surfaces

The coordinate singularity at $r = 2m$ in the Schwarzschild metric (2.12) defines a spherical surface with two important properties. A first characteristic is that proper distances ds^2 change their sign there. This can be seen by staying at some fixed radius r_0 :

$$ds^2\big|_{r_0} = g_{00}\big|_{r_0}\, dt^2 = -\left(1 - \frac{2m}{r_0}\right) dt^2 \tag{2.23}$$

ds^2 changes from negative values at $r_0 > 2m$, going through zero at $r_0 = 2m$, to positive values at $r_0 < 2m$. The zeros of g_{00} define infinite redshift points in spacetime, as will be explained in chapter 5. Therefore the Schwarzschild surface $r = 2m$ is also called an infinite redshift surface. The second property corresponds to a one way character of $r = 2m$ (see figure 2.1). Every massive or massless particle that crosses this surface from outside cannot come back to a $r_0 > 2m$ position. The term "event horizon" is related to this last property.

The Kerr metric splits the Schwarzschild surface $r = 2m$ into two different external surfaces, r_+ and $r_{S_+}(\theta)$ (see figure 2.2), one for each of the above mentioned properties. The infinite redshift surface $r_{S_+}(\theta)$ is also called the static limit. Inside of it, it is located the one-way surface r_+ or event horizon. This splitting causes interesting features of the rotating black hole, as it will be explained in the next paragraphs. From now on, we will call the two classes of surfaces critical surfaces (there are four in total) when we are refering to common properties of the both of them. In the next two subsections we give a qualitative description of these surfaces.

2.1.1 Event Horizon

The "one way" character is present when a (hyper-)surface in spacetime contains only light-like tangent vectors. Then this surface can be crossed by timelike or light-like trajectories in only one direction. This is called a null hypersurface (see figure 2.1). Every normal vector to this surface is also light-like.

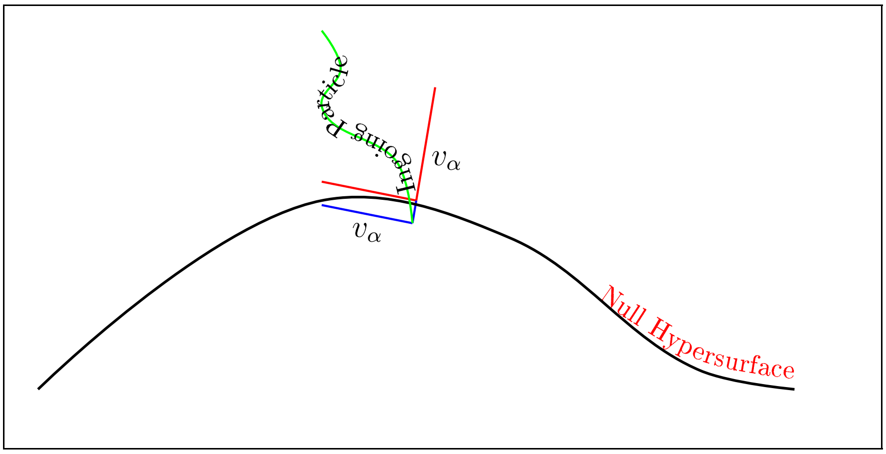


Fig. 2.1.
One-Way Surface: A particle crossing a null hypersurface is not able to cross it in the opposite direction, since it is bounded by a lightcone tangent to the surface.

Every hypersurface $\mathcal{S}$ in spacetime is determined by a function $\Phi(x)$ such that $\Phi(x) = const$ for every point in $\mathcal{S}$. A vector v_α tangent to $\mathcal{S}$ is defined by the gradient of Φ (except for a constant):

$$v_\alpha = A\Phi_{,\alpha} \tag{2.24}$$

Imposing the one-way condition of v_α being tangent to a null hypersurface means setting:

$$v_\alpha v^\alpha = 0 \tag{2.25}$$

Here $v^\alpha = g^{\alpha\beta} v_\beta$ and $g^{\alpha\beta}$ are the metric components of a specific spacetime the components (2.6) of the Kerr metric in our case. By substituting v_α and v^α in (2.25) one gets the following equation (For more details, see subsection 5.1.2):

$$g^{rr} = 0 \tag{2.26}$$

which, by using g^{rr} from (2.11), leads directly to:

$$\Delta(r) \equiv r^2 + a^2 - 2mr = 0 \tag{2.27}$$

Equation (2.27) is an algebraic second order equation in r with the following two solutions:

$$\begin{aligned} r_+ &= m + \sqrt{m^2 - a^2} \\ r_- &= m - \sqrt{m^2 - a^2} \end{aligned} \tag{2.28}$$

They are the radii of the internal and external event horizons $H_\pm$, respectively. The external r_+ has more physical relevance, since it is a direct boundary for points in spacetime which can still causally exert an influence to the observers at infinite r. The surface at r_- is also called the Cauchy horizon and it has relevance in the study of the internal structure of black holes [56]. One can easily see that for all values of a and m

$$r_- \leq m \leq r_+ \leq 2m \tag{2.29}$$

which means that rotation of the source ($a \neq 0$) shrinks the radius of the event horizon. On the other hand when $a = 0$, r_+ goes to $2m$ and the Cauchy horizon r_- coincides with the origin of coordinates.

2.1.2 Static Limit

The redshift of wavelengths for a light signal propagating in a gravitational field from a rest-source, at the point x_s in spacetime, to infinity, is given by (see subsection 5.1.1 and [41])

$$\lambda_\infty = \frac{\lambda_0}{\sqrt{g_{tt}(x_s)}} \tag{2.30}$$

where λ_0 denotes the proper wavelength of the signal and λ_∞, the wavelength of the signal observed at infinity. Then the condition for an infinite redshift is given by

$$g_{tt}(x_s) = 0 \tag{2.31}$$

which, using (2.6), turns out to be

$$r^2 + a^2 \cos^2\theta - 2mr = 0 \tag{2.32}$$

this is again an algebraic second order equation in r, which leads to the following solutions:

$$\begin{aligned} r_{S_+}(\theta) &= m + \sqrt{m^2 - a^2\cos^2\theta} \\ r_{S_-}(\theta) &= m - \sqrt{m^2 - a^2\cos^2\theta} \end{aligned} \tag{2.33}$$

The surfaces coming from (2.33) and (2.28) are shown in figure 2.2. From this figure we see that $r_{S_-}(\theta)$ and r_- are contained in r_+ and that $r_+ \leq r_{S_+}(\theta)$ where the equality holds at the poles. The fact that the outer infinite redshift surface is outside the outer event horizon means that particles can move inside the region bounded by these two surfaces and eventually come out again. It can be shown (see section 2.3 in this chapter and chapter 6) that precisely by going into this region, test particles can extract energy from the rotating black hole (the so-called Penrose process); therefore this region is called the active region of a black hole or the ergosphere.

The name “static limit” for the surfaces in (2.33) comes from the behavior of rotation frequencies for photons and massive particles (see also subsections 4.3.2 and 4.3.3). From the line element (2.5) we deduce that rotating photons at fixed r and θ fulfill the following general condition:

$$ds^2 = g_{tt}dt^2 + g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 \equiv 0 \tag{2.34}$$

After parametrizing spacetime curves fulfilling (2.34) by t we can obtain the rotation frequency Ω_{Light} from

$$\left(\frac{ds}{dt}\right)^2 = g_{tt} + g_{t\varphi}\Omega_{\text{Light}} + g_{\varphi\varphi}\Omega^2_{\text{Light}} \equiv 0 \tag{2.35}$$

where Ω_{Light} is defined by:

$$\Omega_{\text{Light}} = \left.\frac{d\varphi}{dt}\right|_{(\ r,\theta = \text{const}\ ,\ ds^2 = 0\ )}$$

The two solutions of (2.35) are given by $\Omega_{\text{Light}} = \Omega_\pm$ where

$$\Omega_\pm \equiv \omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \tag{2.36}$$

with the abbreviation

$$\omega \equiv -\frac{g_{t\varphi}}{g_{\varphi\varphi}} \tag{2.37}$$

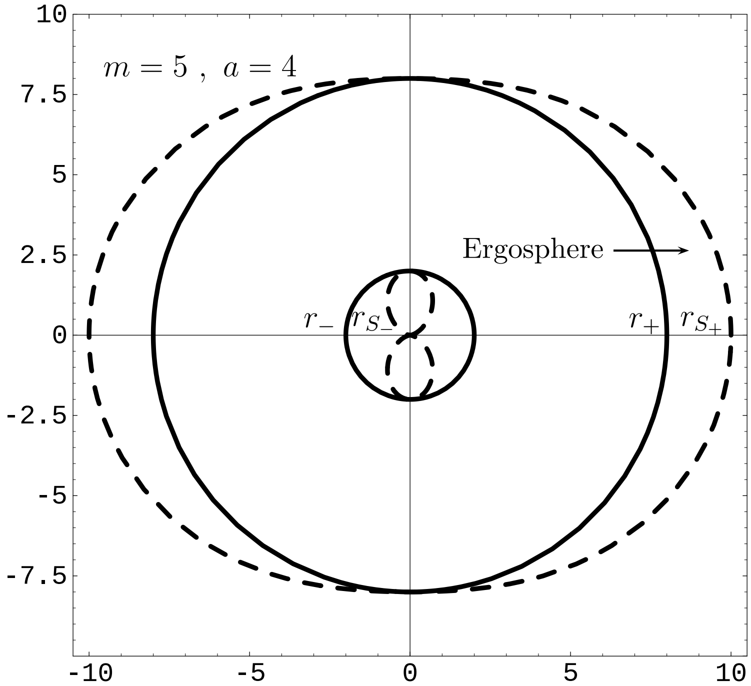


Fig. 2.2.

The four critical surfaces for the Kerr spacetime. The ergosphere, or active region, is the region bounded by the external static limit at $r_{S_+}(\theta)$ and the external event horizon at r_+.

The frequencies Ω_+ and Ω_- represent the rotation frequencies of photons flying in opposite directions; Ω_+ is related to photons that rotate in the same sense as the rotation of the black hole, if a is positive (co-rotating photons); and Ω_- is equivalently related to photons rotating in the opposite sense (counterrotating photons). On the other hand ω is called the dragging frequency and its interpretation will be given in the next section.

By definition, timelike trajectories are bounded by lightcones. For rotating particles this can be expressed, for the case of interest, by the following inequality (see also subsection 4.3.3):

$$\Omega_- < \Omega_{\text{t-like}} < \Omega_+ \tag{2.38}$$

For $g_{tt} = 0$, at the infinite redshift surface, we get $\Omega_- = 0$. This means that counterrotating photons are static at this surface as seen by an observer at infinity. As a consequence of (2.38), no massive particle can be static there. Going into the ergosphere reduces the size of the allowed interval of values in (2.38) even more and not even light can be static (see figure 2.3 and also section 4.3 for an extensive analysis).

Figure 2.3 shows the r-dependence of $\Omega_\pm$ for rotating photons in the Kerr spacetime at the equatorial plane ($\theta = \pi/2$). The allowed values of $\Omega_{\text{t-like}}$ for timelike particles can be read off as the vertical interval bounded by Ω_+ and Ω_-. Its size also depends on r. The frequency Ω_- goes through zero precisely at the static limit surface r_{S_+} where $g_{tt} \equiv 0$. The interval of allowed $\Omega_{t\text{-like}}$ values reduces to zero size at the event horizon r_+, where $\Omega_+ = \Omega_-$. We define $\Omega_\pm|_{r_+}$ as the frequency of rotation of the black hole and we call it simply Ω_{H}:

$$\Omega_{\mathrm{H}} \equiv \Omega_\pm|_{r_+} \tag{2.39}$$

Below r_+ no particle rotating at constant radius is allowed to exist anymore. Every particle or photon unavoidably falls towards the center of the black hole. Substituting the components (2.6) of the Kerr spacetime into the expression (2.36) for $\Omega_\pm$ when $\Omega_+ = \Omega_-$ leads to the following formula for Ω_{H}[1]:

$$\Omega_{\mathrm{H}} = \frac{a}{(r_+)^2 + a^2} \tag{2.40}$$

Notice that Ω_{H} in (2.40) is a function of the parameters of the black hole M and J, and the frequency of rotation is a constant on the whole event horizon.

[1]For more details see section 4.3.3 in chapter 4.

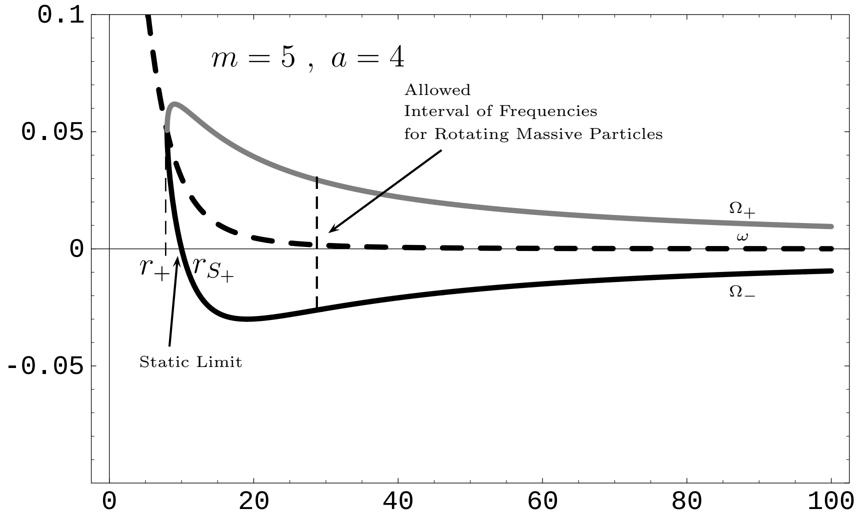


Fig. 2.3.

Radial dependence of rotational frequencies $\Omega_\pm$ and ω. Ω_- vanishes at the static limit r_{S_+}, the last radial distance at which a particle is allowed to be static. Rotation at constant radius is still possible until the event horizon r_+ is reached. At this point the three frequencies coalesce and every particle unavoidably falls towards $r = 0$.

2.1.3 Extremal Black Hole

A black hole with a maximum spinning rate $a = J/M$ is called an extremal black hole. For the Kerr spacetime, this is the maximum rate for $r_\pm$ in (2.28) to be still real-valued. From (2.28) we deduce that this is achieved for $a = M$ or $J = M^2$. As a result, we have $r_+^{\text{extr}} = r_-^{\text{extr}} = G_0 M$. The astrophysical importance of the extremal black hole relies on the fact that most of the spinning systems that can collapse to form a black hole, like massive stars, galactic nuclei, etc, are likely to be near to the maximum spinning rate [43].

2.2 Inertial Frame Dragging

> "An additional *tangential* rocket thrust is required to prevent orbiting, that is to keep the fixed stars in steady position overhead... Spacetime is swept around by the rotating black hole: Spacetime itself on the move [53, F-7]."
>
> Taylor & Wheeler.

The existence of a non-zero mixing component $g_{t\varphi}$ leads to an important characteristic of general axially symmetric spacetimes and in particular of the Kerr spacetime: the so called gravitational dragging of inertial frames. Such a mixing term appears even for the simple case when we let flat Minkowski spacetime rotate around one of its spatial axes, say the z axis, with a constant angular frequency ω_{Mink} (see figure 2.4). The coordinate transformation leading to this spacetime reads:

$$\begin{aligned} \bar{\varphi} &= \varphi + \omega_{\mathrm{Mink}} t \\ d\bar{\varphi} &= d\varphi + \omega_{\mathrm{Mink}} dt \end{aligned} \tag{2.41}$$

By substituting (2.41) in the Minkowski metric given by

$$ds^2 = -dt^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{2.42}$$

we get the following line element:

$$\begin{aligned} ds^2 = -\left(1 - r^2\omega_{\mathrm{Mink}}^2 \sin^2\theta\right) dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\bar{\varphi}^2 \\ -2\omega_{\mathrm{Mink}} r^2 \sin^2\theta dt d\bar{\varphi} \end{aligned} \tag{2.43}$$

The interpretation of this metric is the following. The equivalence principle tells us that (2.43) represents a gravitational field coming from a source whose potential, in the Newtonian limit, can be identified in the following way:

$$\Phi(r,\theta) \equiv -\frac{1}{2}\omega_{\mathrm{Mink}}^2 r^2 \sin^2\theta \tag{2.44}$$

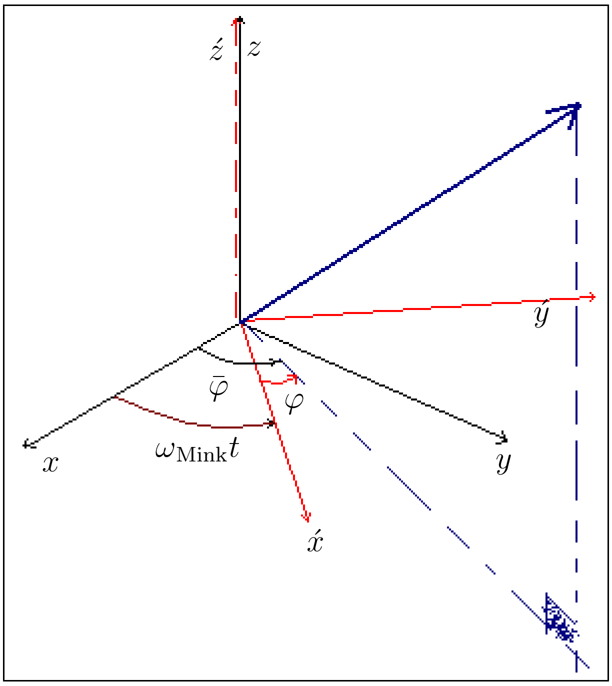


Fig. 2.4.
The coordinate transformation $\bar{\varphi} = \varphi + \omega_{\text{Mink}}t$ that goes from a Minkowski spacetime at rest to a rotating one with angular frequency ω_{Mink} generates a mixed component $g_{t\varphi} = -\omega_{\text{Mink}}r^2 \sin^2\theta$ in the line element, in analogy with the frame dragging term $g_{t\varphi}$ in the Kerr metric (2.6).

Using the same principle we can also identify the term $\omega_{\text{Mink}}r^2 \sin^2\theta$ with a function $\alpha(r, \theta)$ which has dimensions of angular momentum per unit mass and would be due to the rotation of the gravitational source:

$$\alpha(r, \theta) \equiv \omega_{\text{Mink}}r^2 \sin^2\theta \tag{2.45}$$

Then we can write (2.43) as follows:

$$ds^2 = -(1 + 2\Phi)\, dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\bar{\varphi}^2 - 2\alpha(r, \theta)\, dt d\bar{\varphi} \tag{2.46}$$

Though not exactly the same[2], (2.46) resembles the approximate solution found by Lense and Thirring in 1918 for the exterior field of a spinning sphere of constant density. It is the limit of the Kerr metric for small angular momentum and weak fields [44],[41]. As a conclusion of this analysis we can say that metric components mixing terms in time and angular coordinates can be interpreted, using the equivalence principle, either as terms due to rotating gravitational sources or as terms

[2]A radial gravitational term is missing, which would come, via the equivalence principle, from a radial acceleration, additional to the rotation of the Minkowski spacetime.

related to rotating flat spacetimes. This twofold interpretation is connected to the concept of dragging of inertial reference frames, which will be explained in the next subsection.

2.2.1 Inertial Frame Dragging

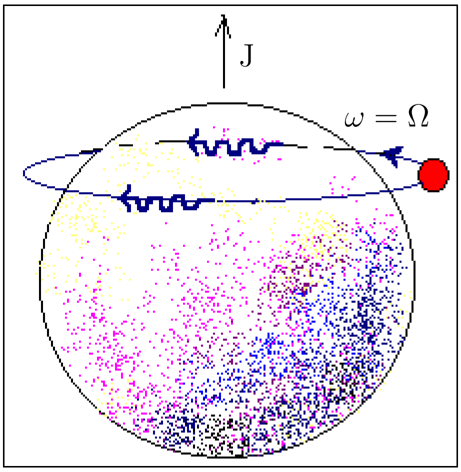


Fig. 2.5.
Two photons are simultaneously emitted in opposite directions by an observer which rotates with a frequency Ω near to a rotating gravitational source. If $\Omega = \omega$, the observer receives the photons back simultaneously, even if $\omega \neq 0$. This observer is thus defined to be in a dragged inertial frame, with dragging frequency ω.

An inertial reference frame at some point of the Kerr spacetime can be experimentally defined by performing the following experiment (see fig. 2.5). Two opposite light rays are emitted simultaneously by a rotating observer with frequency Ω from a point on the circumference defined by the values r and θ. In a flat spacetime the light rays will come back simultaneously only in the case when $\Omega = 0$, since the rotation frequencies Ω_- and Ω_+ are equal in magnitude and opposite in sign. But in a curved spacetime like (2.6) or (2.46) this is not necessarily the case. We define an inertial reference frame to be one in which simultaneously emitted opposite light rays come back simultaneously, no matter which is the value of the frequency of rotation of the emitter. From the definitions (2.36) and (2.37) for $\Omega_\pm$ and ω it can be shown that it is precisely the rotation frequency $\Omega = \omega$ that fulfills this requirement. In fact, the opposite traveling times are given by [57, Exercise 33.3, pag 895]:

$$\begin{aligned} \Delta t_+ &= \frac{2\pi}{\Omega_+ - \Omega} \\ \Delta t_- &= \frac{2\pi}{\Omega - \Omega_-} \end{aligned} \tag{2.47}$$

If they are equal then we have

$$\Omega = \frac{\Omega_+ + \Omega_-}{2} \tag{2.48}$$

which is precisely the case for $\Omega = \omega$ defined in (2.36) and (2.37). Since ω is in general $\neq 0$ we can say that an inertial reference frame is being dragged with a dragging frequency ω, as seen by an observer at infinity. For the case of the Kerr metric, ω has always a positive value and it is equal to $\Omega_+ = \Omega_-$ precisely at the event horizon (see figure 2.3). In chapter 4 (S. 4.3.1) we interpret ω also as a rotation frequency of zero angular momentum observers (ZAMO's) and we calculate it for the renormalization group improved Kerr metric.

2.3 Energy Extraction

The existence of negative energy states for test particles moving near to a rotating black hole leads, as we shall explain in more detail in chapter 6, to the possibility of extraction of energy from that black hole. Such a process of energy extraction is called a Penrose process.

Figure 2.6 describes schematically this process; it includes a composite system A of two particles B and C which crosses the static limit and reaches a negative energy state inside the ergosphere, near to the event horizon. Particle B with negative energy falls into the black hole decreasing in this way the total amount of its internal energy. The conservation of energy for the whole system (black hole + test particles) predicts therefore an equivalent increase of energy for particle C when it comes back to its final state at $r = \infty$.

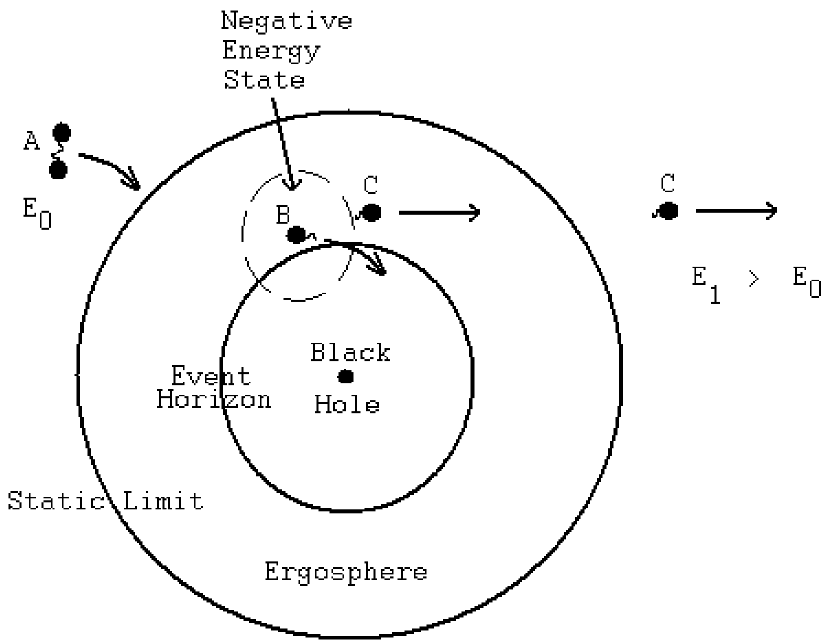


Fig. 2.6.
Penrose Process
The system A, composed by two particles B and C, enters the ergosphere with initial energy E_0 as measured at infinity. Near the event horizon, particle B reaches a negative energy state before crossing it. Particle C comes back to infinity with an energy $E_1 > E_0$.

We define the total energy E_{tot} before that particle B crosses the event horizon as

$$E_{\mathrm{tot}} = E_0 + E^0_{\mathrm{BH}} \tag{2.49}$$

where E_0 is the energy of the composite system A, and E^0_{BH} is the initial energy of the black hole. At the same time we define E_0 as the sum of initial energies of particles B and C:

$$E_0 = E^0_B + E^0_C \tag{2.50}$$

When the composite system enters the ergosphere and system B reaches a negative energy, we define

$$E^1_B < 0 \tag{2.51}$$

such that the energy of the black hole is reduced to

$$E^1_{\mathrm{BH}} = E^0_{\mathrm{BH}} - \left|E^1_B\right| \tag{2.52}$$

when particle B falls into it. The energy of particle C when it reaches infinity after leaving the black hole is given by

$$E_1 = E_{\text{tot}} - E^1_{\text{BH}} \tag{2.53}$$

where we have only used the conservation of energy. Substituting (2.52) in (2.53) and using (2.50) leads to:

$$E_1 = E_0 + \left|E^1_B\right| \tag{2.54}$$

The last equation shows that the final energy of particle C is greater than the initial energy of the composite system A, whereas (2.52) tells that the energy of the black hole has decreased. As a result we conclude that we have extracted energy from the black hole.

In 1970 D. Christodolou showed that the energy a black hole can provide through the Penrose process is only rotational [49]. He proved that there exists an irreducible mass M_{irr} given by the remnant mass that is left after the complete rotational energy has been delivered. The final state of the black hole corresponds to a Schwarzschild spacetime with mass M_{irr}. In this sense we say that a black hole is "alive" if it has an angular momentum $J \neq 0$. As long as an event horizon H_+ with $r_+ = MG_0 + \sqrt{(MG_0)^2 - (J/M)^2}$ exists, the highest amount of angular momentum a black hole is allowed to have is given by $J = G_0 M^2$, the extremal configuration[3]. Under this condition the extremal state is the most "alive" configuration of parameters J and M a black hole can assume.

In chapter 6 we describe the consequences for the process of energy extraction from the black hole of a renormalization group improvement of the Kerr metric via the running of the Newton constant.

2.4 Thermodynamics

The connection between the dynamics of black holes and the laws of thermodynamics was proposed at the beginnings of the 1970's in a series of articles that include, among others, the fundamental contributions of J. Bekenstein and S. Hawking

[3]To date, the available theoretical and experimental evidence points towards the validity of the so-called "cosmic censorship hypothesis" that tells that every gravitational collapse leads to a singularity surrounded by an event horizon. In other words, there exist no "naked singularities" in nature [52].

[45, 46]. Based upon previous results about the dynamics of black holes that interact with surrounding bodies [49], Bekenstein formulated in 1973 the generalized first and second laws of thermodynamics. They involve the entropy and the temperature of the black hole, where the entropy is identified (up to a constant) with the area A of the event horizon and the temperature with its surface gravity κ [45]. Almost simultaneously, J. M. Bardeen, B. Carter and S. Hawking formulated the four laws of black hole dynamics in agreement with the results of Bekenstein. In addition they presented the zeroth and third laws to complete the framework of the thermodynamics of black holes.

In 1975, S. Hawking gave a statistical interpretation of the temperature of the black hole by proving that quantum mechanical effects cause black holes to create and emit particles as if they were black bodies with a temperature proportional to κ [65]. It is thus possible for a black hole to be in thermal equilibrium with other thermodynamic systems. The laws of black-hole mechanics, therefore, are nothing but a description of the *ordinary* thermodynamics of black holes [60].

In this section we review the four laws of black hole mechanics, exploiting the Kerr spacetime as an illustrative example. We start with the definitions of area and surface gravity and their relation to the mass and angular momentum of a Kerr black hole through Smarr's formula.

2.4.1 Area and Surface Gravity

The area $\mathcal{A}$ of the event horizon H is defined as the surface integral in spacetime given by

$$\mathcal{A} = \oint_{\mathrm{H}} \sqrt{\sigma} d^2\theta \tag{2.55}$$

where σ is the determinant of σ_{ab} which is the metric induced from $g_{\alpha\beta}$ in the surface H, and $d^2\theta \equiv d\theta^1 d\theta^2$ with θ^a angular coordinates on H. The area of the Kerr event horizon is given by[4]

$$\mathcal{A} = 4\pi \left(r_+^2 + a^2\right) \tag{2.56}$$

Notice that $\mathcal{A}$ in (2.56) is a function of the parameters of the black hole, M and J, through the functions $r_+ (M, J)$ defined in (2.28) and $a = J/M$.

[4]For further details on definition (2.55) and on how the area is calculated for the Kerr black hole, see appendix I.

The surface gravity κ of a black hole is defined as the force required by an observer at infinity in order to hold (with an infinitely long, massless string) a particle of unit mass stationary at the event horizon. This can be expressed as follows:

$$\kappa \equiv \mathrm{acc}_{\infty}\left(r_{+}\right) \tag{2.57}$$

For a spherically symmetric spacetime with metric

$$ds^2 = -f\left(r\right)dt^2 + f^{-1}dr^2 + r^2 d\Omega^2 \quad , \tag{2.58}$$

definition (2.57) leads to the following result for κ:

$$\kappa = \frac{1}{2}f'\left(r_{+}\right) \tag{2.59}$$

In particular, for the Schwarzschild spacetime we have $\kappa = 1/4MG_0$.

The surface gravity can also be calculated by exploiting specific identities fulfilled by the Killing vectors of the respective spacetime at the event horizon, provided this event horizon has the property of being a Killing horizon, namely, a horizon generated by a Killing vector field.

The surface gravity of the Kerr black hole is given by[5]:

$$\kappa = \frac{\sqrt{\left(G_0 M\right)^2 - a^2}}{r_+^2 + a^2} \tag{2.60}$$

Notice that κ in (2.60) depends only on the parameters M and J. This means that it is constant on the whole event horizon. This fact is related to the zeroth law of black hole thermodynamics. On the other hand notice also that $\kappa = 0$ for the extremal Kerr black hole with $MG_0 = a$. This fact is related to the third law. We present these laws later in this section.

2.4.2 Smarr's Formula

Smarr's formula is an algebraic relationship satisfied by the parameters M and J of a classical black hole, its area $\mathcal{A}$, the surface gravity κ, and the angular frequency of the event horizon, Ω_{H}. It was discovered by Larry Smarr in 1973 [47]. For the classical Kerr black hole, Smarr's formula reads

$$M = 2\Omega_{\mathrm{H}}J + \frac{\kappa\mathcal{A}}{4\pi G_0} \tag{2.61}$$

[5]For further details on the mentioned identities, and how κ is calculated for the Kerr black hole, see appendix H.

Here we have

$$\begin{aligned} \Omega_{\mathrm{H}} &= \frac{a}{r_+^2 + a^2} \ , \ \kappa = \frac{(r_+ - m)}{r_+^2 + a^2} \ , \ \mathcal{A} = 4\pi(r_+^2 + a^2) \\ r_+ &= m + \sqrt{m^2 - a^2} \ , \ a = \frac{J}{M} \ , \ m = MG_0 \end{aligned} \tag{2.62}$$

It is a matter of simple algebra to show that the expressions in (2.62) satisfy the formula (2.61). Nevertheless, Smarr's formula is not only a property of Kerr black holes. It can be shown to hold true for *all* black hole spacetimes that are stationary and axially symmetric [60]. The proof does not require the explicit form of the solution.

2.4.3 The Zeroth Law

The zeroth law of black hole mechanics states that the surface gravity of a stationary black hole is uniform over the entire event horizon. We have already seen in subsection 2.4.1 that this statement is indeed true for the specific case of a Kerr black hole, but the scope of the zeroth is much wider: The black hole need not be isolated and its metric need not be the Kerr metric. The only requirement is the black hole to be stationary. For a rigorous demonstration of the zeroth law see [46, 48].

2.4.4 The First Law

The first law of classical ($G = G_0 = \mathrm{const}$) black hole thermodynamics states that if we consider a quasi-static process in which a stationary black hole of mass M, angular momentum J, and surface area $\mathcal{A}$ is transformed to a new stationary black hole with parameters $M + \delta M$, $J + \delta J$, $\mathcal{A} + \delta\mathcal{A}$, the changes in mass, angular momentum and surface area are related by

$$\delta M - \Omega_{\mathrm{H}} \delta J = \left(\frac{\kappa}{8\pi G_0}\right) \delta\mathcal{A} \tag{2.63}$$

The derivation of equation (2.63) does not make any reference to a specific solution of the Einstein field equations, like the Kerr spacetime. It relies only upon the stationary character of the black hole that performs the process, and the quasi-static property of the process itself. In particular, for the Kerr spacetime the functions $\mathcal{A} \equiv \mathcal{A}\,(M, J)$, $\Omega_{\mathrm{H}} \equiv \Omega_{\mathrm{H}}\,(M, J)$ and $\kappa \equiv \kappa\,(M, J)$ are given in (2.62). They do indeed satisfy (2.63).

Appealing to its analogy with the first law of standard thermodynamics, (2.63) is interpreted as

$$\delta M - \Omega_{\rm H}\delta J = T\delta S \tag{2.64}$$

where

$$T = \frac{\kappa}{2\pi} \tag{2.65}$$

and

$$S = \frac{\mathcal{A}}{4G_0} \tag{2.66}$$

are the black hole's temperature and entropy, respectively.

2.4.5 The Second Law

The second law of black-hole dynamics states that if the null energy condition is satisfied[6], then the surface area of a black hole can never decrease:

$$\delta\mathcal{A} \geq 0 \tag{2.67}$$

This area theorem was established by S. Hawking in 1971 with no mention of the analogy between area and entropy [50].

2.4.6 The Third Law

The third law of black hole dynamics states that if the stress-energy tensor is bounded and satisfies the weak energy condition, then the surface gravity of a black hole cannot be reduced to zero within a finite advanced time. A precise formulation of this law was given by Werner Israel in 1986 [51, 60].

2.5 Structure of this Thesis

In chapter 1 we have already presented the procedure of renormalization group improvement of black hole spacetimes. We will apply this procedure to the Kerr spacetime in the next chapters of this work. The motivation of the review in the present chapter has not only been to present the Kerr metric as the main subject of

[6]For a discussion of the energy conditions, see chapter 7.

study in this work, but also to give an organized presentation of its properties. This presentation is a guide for the organization of the next chapters. This organization is the following:

After explaining how to implement the cutoff identification in chapter 3 and presenting the improved Kerr metric and some general features in chapter 4, we move to the analysis of critical surfaces in chapter 5, the Penrose process in chapter 6, and the black hole thermodynamics in chapter 8. We also include an additional chapter concerning the energy conditions (chapter 7), briefly mentioned in this chapter. We dedicate the final Chapter 9 to the conclusions and an outlook.

Chapter 3

The Cutoff Identification

In chapter 1 we defined in (1.29) the distance $d\,(P)$ as

$$d\,(P) = \int_{\mathcal{C}} \sqrt{|ds^2|} \tag{3.1}$$

and a diffeomorphism invariant cutoff identification was proposed in the form of (1.28), given by

$$k\,(P) = \frac{\xi}{d\,(P)} \tag{3.2}$$

Since $d\,(P)$ depends on the choice of the path of integration, it is important to analyse various special cases in order to find universal characteristics and to discuss the advantages and drawbacks of the different choices. The two kinds of paths analysed in this chapter, namely the radial and circular paths, are by no means exhaustive, but at least representative, in the sense that a generic, exclusively r dependent asymptotic behavior is found for both of them. Finding (but not formally proving) this general asymptotic property of invariant distances for the Kerr metric and discussing its consequences for a RG improvement via the running Newton constant are the main goals of this chapter.

3.1 Radial Path for Schwarzschild and Kerr Metrics

A natural path $\mathcal{C}$ in (3.1) to start with is a straight line in space from the origin to the point P (See figure 3.1).

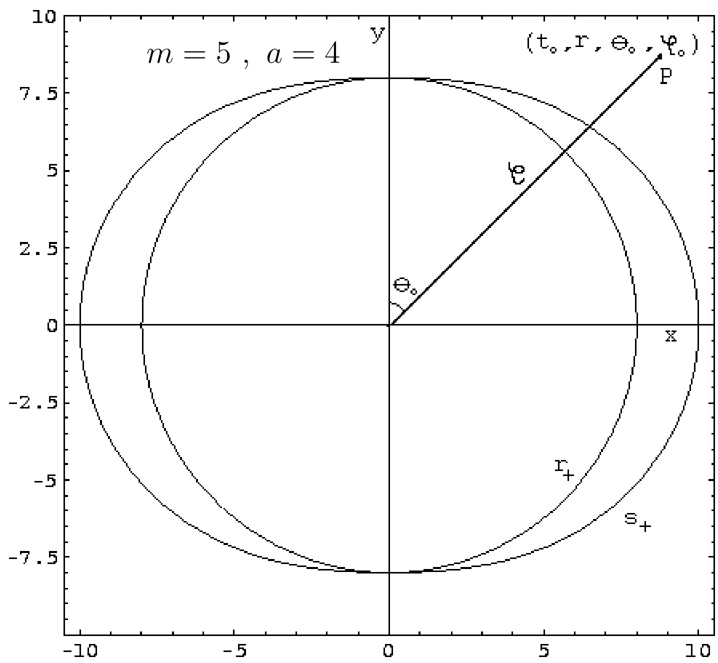


Fig. 3.1.

Straight line path from the origin to a point P in the Kerr spacetime ($t =$ constant).

In the B-L representation it can be parametrized by the r-coordinate together with $t = t_0,\ \varphi = \varphi_0,\ \theta = \theta_0$ where $(t_0, r, \theta_0, \phi_0)$ are the coordinates of the point P. From (2.6) we find for the line element along this path, since $dt = d\theta = d\varphi = 0$[1],

$$ds^2 = g_{rr}dr^2 = \left(\frac{\rho^2}{\Delta}\right) dr^2 = \left(\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2 - 2mr}\right) dr^2$$

so that

$$d\,(P) \equiv d\,(r,\theta) = \int_{\mathcal{C}} \sqrt{|ds^2|} = \int_0^r \sqrt{\left|\frac{r'^2 + a^2\cos^2\theta}{r'^2 + a^2 - 2mr'}\right|} dr' \tag{3.3}$$

The two roots of the discriminant $\Delta\,(r) \equiv r^2 + a^2 - 2mr$ appearing in the denominator of the integrand are given in (2.28) as follows:

$$\begin{aligned} r_+ &= m + \sqrt{m^2 - a^2} \\ r_- &= m - \sqrt{m^2 - a^2} \end{aligned} \tag{3.4}$$

[1]As a matter of convenience we use, only in this chapter and its related appendices, the geometrical mass $m = MG_0$ instead of the explicit product MG_0, because for the time being we are not dealing with any improvement of G_0.

These roots lead to the following three regions of integration (See fig. 3.2.):

$$\begin{cases} \text{Region 1:} & r < r_- \\ \text{Region 2:} & r_- < r < r_+ \\ \text{Region 3:} & r_+ < r \end{cases} \tag{3.5}$$

In figure 3.2 we see that the discriminant in (3.3) changes sign at r_- and r_+ and therefore the absolute value in the integrand is to be interpreted differently for every region in (3.5).

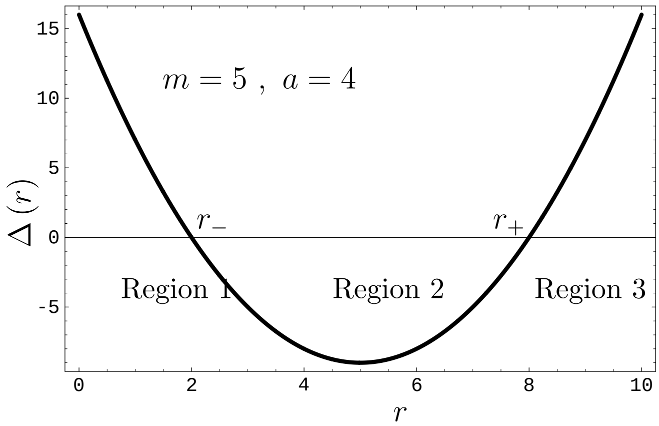


Fig. 3.2.
The quadratic function $\Delta(r)$ defines three separate regions for the integration in (3.3).

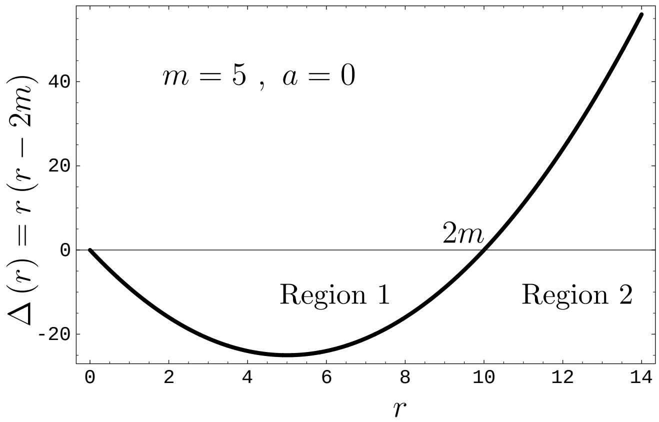


Fig. 3.3.
For the Schwarzschild spacetime, $\Delta(r)$ defines only two regions for the integration in (3.3) with $a = 0$.

Instead of attempting to integrate expression (3.3), which does not seem to be an easy task, we will restrict our analysis to certain special cases which, however, can give us insight into the main physical features of an improvement like (1.24) using (3.2) with (3.3).

3.1.1 $d(P)$ for the Schwarzschild Metric

The Schwarzschild metric is recovered from (2.6) by setting $a = 0$. For this case the roots in (3.4) reduce to:

$$\begin{aligned} r_+ &= 2m \\ r_- &= 0 \end{aligned} \tag{3.6}$$

Hence we have only two regions of integration as seen in figure 3.3, namely:

$$\begin{cases} \text{Region 1:} \quad r < 2m \\ \text{Region 2:} \quad 2m < r \end{cases} \tag{3.7}$$

In this case the invariant distance (3.3) is reduced to:

$$d\left(P\right) \equiv d\left(r\right) = \int_0^r \frac{r'}{\sqrt{\left|1 - \frac{2m}{r'}\right|}} \tag{3.8}$$

The integration in (3.8) gives the following results (See appendix B, S. B.1):

$$d\left(r\right) = \begin{cases} d_1\left(r\right) = -\left[\sqrt{2mr - r^2} + m \ \arctan\left(\frac{m-r}{\sqrt{2mr-r^2}}\right)\right] + \left(\frac{m\pi}{2}\right) & \text{for } r < 2m \\ d_2\left(r\right) = \pi m + \sqrt{r\left(r - 2m\right)} + m \ln\left(\frac{r-m+\sqrt{r^2-2mr}}{m}\right) & \text{for } 2m < r \end{cases} \tag{3.9}$$

For large values of r we can expand $d_2\left(r\right)$ of (3.9) to get the following asymptotic behavior:

$$d\left(r\right) = r - m + m\pi + m \ln\left[\frac{2\left(r-m\right)}{m}\right] + O\left(\frac{1}{r}\right) \tag{3.10}$$

Figure 3.4 shows the compound function $d\left(r\right)$ from (3.9). It can be seen that it is continuous at $r = 2m$ but with infinite slope as expected from the form of the integrands. The asymptotic behavior of (3.10) as r goes to infinity is also visible.

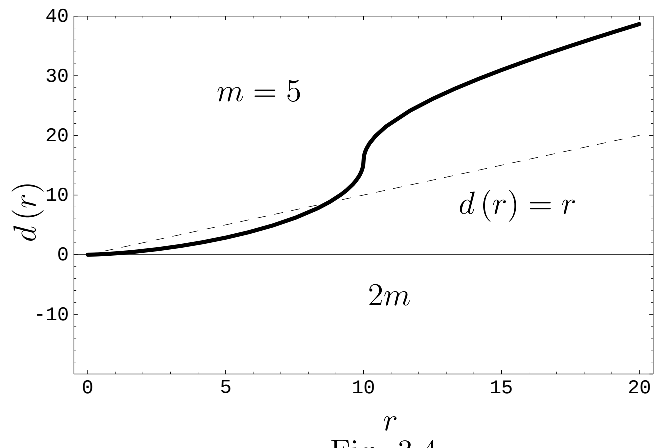


Fig. 3.4.
$d\left(r\right)$ vs. r plot for the invariant distance function (3.9) of the Schwarzschild spacetime.

3.1.2 $d(P)$ for the Kerr Metric Restricted to the Equatorial Plane

For the Kerr metric at the equatorial plane, the invariant distance integral (3.3) in B-L coordinates is reduced to the following form

$$d(r) \equiv d\left(r, \frac{\pi}{2}\right) \equiv \int_0^r \sqrt{\left|g_{rr}\left(r', \frac{\pi}{2}\right)\right|}dr' = \int_0^r \sqrt{\left|\frac{r'^2}{r'^2 + a^2 - 2mr'}\right|}dr' \tag{3.11}$$

More explicitly,

$$d(r) = \begin{cases} \int_0^r \frac{r'dr'}{\sqrt{r'^2+a^2-2mr'}} & \text{if } \Delta(r) > 0 \\ \int_0^r \frac{r'dr'}{\sqrt{2mr'-r'^2-a^2}} & \text{if } \Delta(r) < 0 \end{cases} \tag{3.12}$$

The behavior of the g_{rr} component in the equatorial plane through the three regions in (3.5) is depicted in figure 3.5. It changes its sign at r_- and r_+. As a consequence, the absolute value in the integrand (3.11) has to be interpreted differently from one region to another. Performing the integrations for every region separately we get three different expressions for $d(r)$. During the calculation it is useful to exploit that the radii $r_\pm$ of the horizons in (3.4) satisfy the algebraic equation (2.27), $r_\pm^2 + a^2 - 2mr_\pm = 0$.

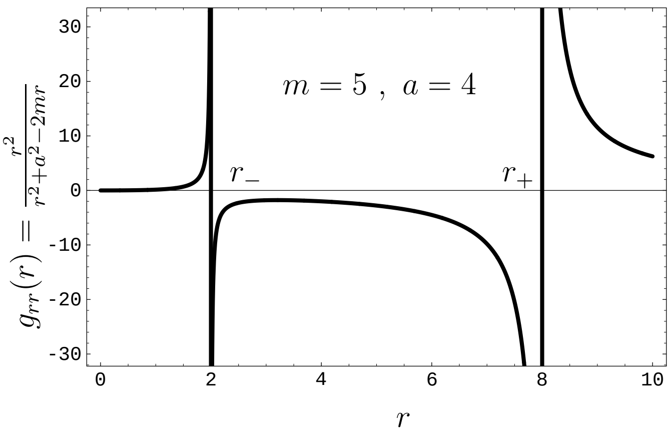


Fig. 3.5.
A g_{rr} vs. r plot. The roots of $\Delta(r) = r^2 + a^2 - 2mr$ define two vertical asymptotes $r = r_\pm$ that separate three different regions of integration in (3.11).

The results for every region are the following (see (B.10), (B.11) and (B.19) in appendix B):

$$d(r) = \begin{cases} d_1(r) & \text{if } r < r_- \\ d_2(r) & \text{if } r_- < r < r_+ \\ d_3(r) & \text{if } 0 < r_- < r_+ < r \end{cases} \tag{3.13}$$

where d_1, d_2 and d_3 are defined as:

$$d_1(r) = \sqrt{r^2 + a^2 - 2mr} + m \ln\left(\frac{-r + m - \sqrt{r^2 + a^2 - 2mr}}{|a - m|}\right) - a \tag{3.14}$$

$$\begin{aligned} d_2(r) = \frac{m}{2} \ln\left|\frac{m+a}{m-a}\right| - a - \sqrt{2mr - r^2 - a^2} \\ + m \arctan\left(\frac{r - m}{\sqrt{2mr - r^2 - a^2}}\right) + \frac{m\pi}{2} \end{aligned} \tag{3.15}$$

$$\begin{aligned} d_3(r) = \sqrt{r^2 + a^2 - 2mr} + m \ln\left(r - m + \sqrt{r^2 + a^2 - 2mr}\right) + \\ \pi m - a - m \ln|m - a| \end{aligned} \tag{3.16}$$

It is not difficult to see that the expressions (3.14), (3.15) and (3.16) reduce to the Schwarzschild expressions (3.9) when $a = 0$.
Figure 3.6 shows the dependence on r of $d(r)$ defined in (3.13), for a given value of m and different values of $a < m$ approaching m. The continuous and regular behavior at r_- and r_+ are preserved for this case as long as $a < m$. The slope at the two event horizons is infinite as expected. When $a \nearrow m$ the two event horizons coalesce at $r_- = r_+ = m$ and the expressions (3.14), (3.15) and (3.16) diverge. We discuss this special case of an extremal black hole at the end of this chapter.

3.1.3 Corrections Outside the Equatorial Plane

In this section we expand the integrand in (3.3) for small θ in order to see how much the results for $d(r)$ in (3.14) to (3.16) change when we move away from the equatorial plane at $\theta = \pi/2$. The resulting integrals are also carried out for the three regions 1, 2 and 3 specified in (3.5). (For more details see appendix B, section B.3). The results are the following:

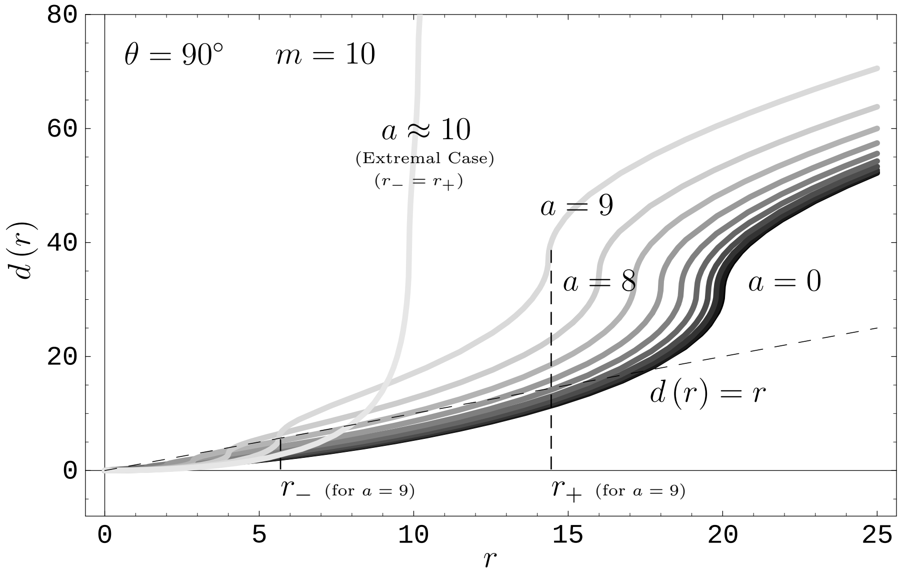


Fig. 3.6.
$d(r)$ vs. r plots for the composite distance function in (3.14), (3.15) and (3.16) of the Kerr spacetime at the equatorial plane. The "colors" run form black to gray for increasing a.

$$d(r,\theta) = \begin{cases} d_1(r,\theta) & \text{if } r < r_- \\ d_2(r,\theta) & \text{if } r_- < r < r_+ : \\ d_3(r,\theta) & \text{if } 0 < r_- < r_+ < r \end{cases} \tag{3.17}$$

Here $d_1(r,\theta)$, $d_2(r,\theta)$ and $d_3(r,\theta)$ have been evaluated to leading order $\cos^2\theta$:

$$\begin{aligned} d_1(r,\theta) \approx\ & \sqrt{r^2 - 2mr + a^2} + m \ln\left(-r + m - \sqrt{r^2 - 2mr + a^2}\right) \\ & + \frac{a\cos^2\theta}{2} \ln\left(\frac{r}{a^2 - mr + a\sqrt{r^2 - 2mr + a^2}}\right) - F_1(r_0, \theta, a, m) \end{aligned} \tag{3.18}$$

$$\begin{aligned} d_2(r,\theta) \approx\ & F_1(r_-, \theta, a, m) - F_1(r_0, \theta, a, m) \\ & - \sqrt{2mr - r^2 - a^2} + m \arctan\left(\frac{r - m}{\sqrt{2mr - r^2 - a^2}}\right) \\ & + \frac{a\cos^2\theta}{2} \arctan\left[\frac{mr - a^2}{a\sqrt{2mr - r^2 - a^2}}\right] \\ & - F_2(r_-, \theta, a, m) \end{aligned} \tag{3.19}$$

$$\begin{aligned} d_3(r,\theta) \approx\ & -F_1(r_0, \theta, a, m) + F_2(r_+, \theta, a, m) - F_2(r_-, \theta, a, m) \\ & + \sqrt{r^2 - 2mr + a^2} + m \ln\left(r - m + \sqrt{r^2 - 2mr + a^2}\right) \\ & + \frac{a\cos^2\theta}{2} \ln\left(\frac{-r}{a^2 - mr + a\sqrt{r^2 - 2mr + a^2}}\right) \end{aligned} \tag{3.20}$$

We define r_0 as a constant, arbitrarily near to zero. The r-independent functions $F_1(r_0, \theta, a, m)$, $F_1(r_-, \theta, a, m)$, $F_2(r_-, \theta, a, m)$, $F_2(r_+, \theta, a, m)$, $F_1(r_+, \theta, a, m)$ are given by (see appendix B, S.B.3):

$$\begin{aligned} F_1(r_0 < r_-, \theta, a, m) &= \sqrt{r_0^2 - 2mr_0 + a^2} + m \ln\left(-r_0 + m - \sqrt{r_0^2 - 2mr_0 + a^2}\right) \\ &\quad + \frac{a\cos^2\theta}{2} \ln\left(\frac{r_0}{a^2 - mr_0 + a\sqrt{r_0^2 - 2mr_0 + a^2}}\right) \end{aligned} \tag{3.21}$$

$$F_1(r_-, \theta, a, m) = F_1(r_+, \theta, a, m) = m \ln\left(\sqrt{m^2 - a^2}\right) + \frac{a\cos^2\theta}{2} \ln\left(\frac{1}{\sqrt{m^2 - a^2}}\right) \tag{3.22}$$

$$F_2(r_-, \theta, a, m) = -\frac{\pi m}{2} - \frac{\pi a \cos^2\theta}{4} \tag{3.23}$$

$$F_2(r_+, \theta, a, m) = \frac{\pi m}{2} + \frac{\pi a \cos^2\theta}{4} \tag{3.24}$$

From (3.21) we conclude that we cannot set $r_0 = 0$ since there would result a $\ln r_0$ divergence in the angular dependent term of $F_1(r_0 < r_-, \theta, a, m)$. However, this non-regular behavior does not represent any problem, since the difference $F_1(r, \theta, a, m) - F_1(r_0 < r_-, \theta, a, m)$ in (3.18) is finite and positive as long as $r_0 < r$, due to the monotonically increasing behavior of F_1 as a function of r. It is zero precisely for $r = r_0$. As a result we have $d_1(r) \searrow 0$ for $(r_0,\ r) \searrow 0$ such that always $r_0 < r$.

In figure 3.7 we plot the analytical results in (3.18), (3.19) and (3.20) for different values of a. We see that for all $\theta \approx \frac{\pi}{2}$, $\cos^2\theta \ll 1$, we get a similar behavior of $d(r, \theta)$ as in the equatorial plane (compare to figure 3.6). This indicates that the invariant distance (3.3) behaves smoothly near to $\theta = \frac{\pi}{2}$.

Furthermore, figure 3.8 shows that, for this approximation, varying θ near to $\frac{\pi}{2}$ means shifting $d(r, \theta)$ by a finite amount without changing its shape in a visible way.

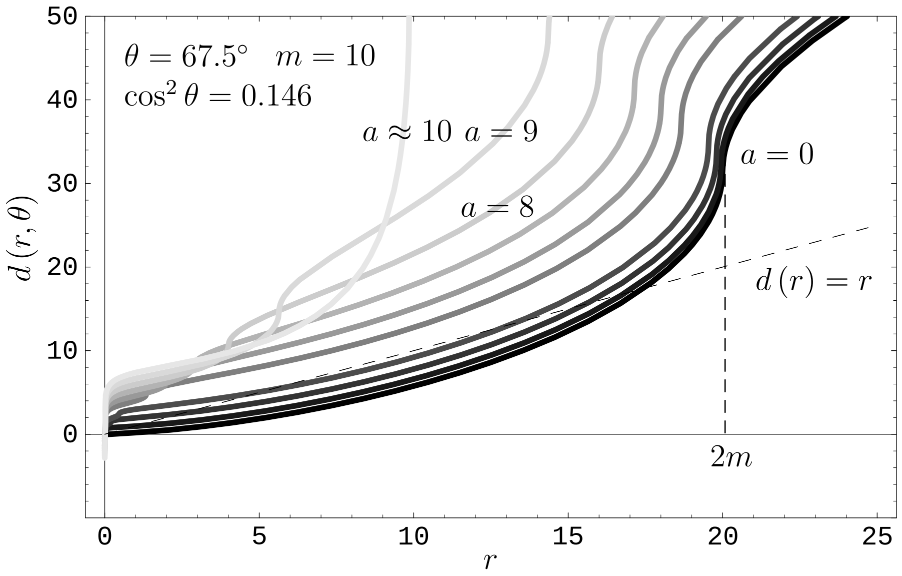


Fig. 3.7.
$d\,(r,\theta)$ vs. r plots for the composite approximate invariant distance function in (3.18), (3.19) and (3.20) from the Kerr spacetime near to the equatorial plane ($\cos^2\theta \ll 1$). We follow the same convention for the grayscale as in figure 3.6.

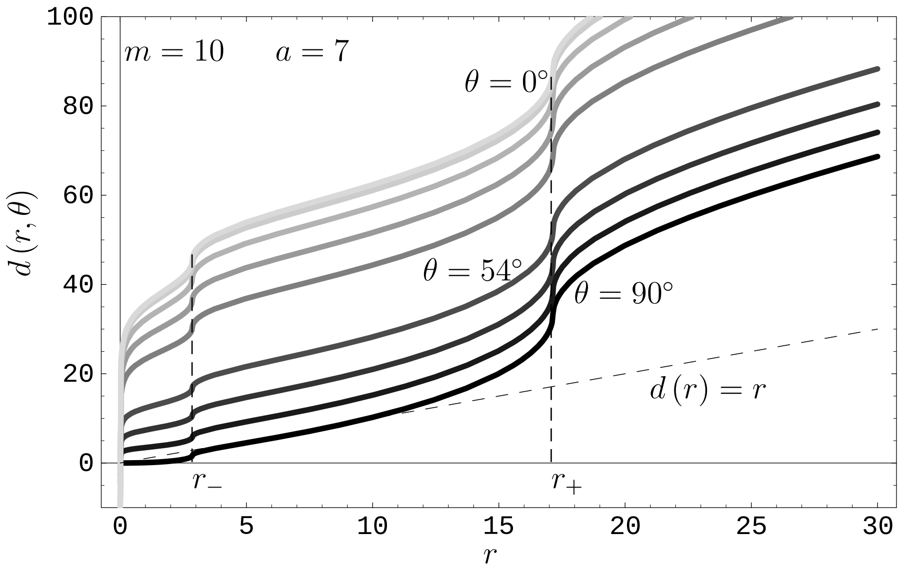


Fig. 3.8.
$d\,(r,\theta)$ vs. r plots for the composite approximate invariant distance function in (3.18), (3.19) and (3.20), from the Kerr spacetime. The “colors” run form black to gray for decreasing θ.

A remarkable fact which follows from the results (3.19) and (3.20) is that, similarly as in the equatorial plane, the distances they define are not regular for the extremal case $a = m$, as can be seen in (3.22). This behavior indicates that our cutoff identification should become unreliable when the angular momentum is too large.

There is another reason to expect that our treatment might become unreliable close to extremality. As the cutoff identification we are applying to the Kerr spacetime is inspired by the example of *massless* QED where the only dimensionful quantity is r we have to be careful in our case, since besides the radial distance we have two more possible dimensionful quantities on which the cutoff could depend, namely, M and a. (An analogous complication arises in *massive* QED.) while we believe on the basis of the analysis in [30] that $k = \xi/d\,(P)$ encapsulates the leading quantum effects for Schwarzschild black holes, this does not follow from simple dimensional analysis since the mass M sets a second scale, independent of $1/d\,(P)$. By continuity we may assume that also for Kerr black holes $k = \xi/d\,(P)$ is a sensible cutoff identification, provided their angular momentum is not too large. Close to extremality where the angular momentum defines a new independent scale this cutoff identification might break down, however. Therefore we shall analyze only that portion of the (m, a)-parameter space where $a \ll m$ so that we are far away from extremality. In this regime $k = \xi/d(P)$ should be the correct cutoff identification, to leading order.

3.1.4 The Asymptotic Regime $r \to \infty$

Expanding d_3 of (3.20) for $r \to \infty$ and neglecting terms of orders $\frac{1}{r}$ or smaller we get the following distance function (See Appendix B, section B.4):

$$d\,(r,\theta) \approx r - m + m \ln\left(2\,(r-m)\right) + F\,(\theta, a, m) \tag{3.25}$$

Here

$$\begin{aligned} F\,(\theta, a, m) \;\; = \;\; & \frac{a\cos^2\theta}{2}\ln\left(\frac{1}{m-a}\right) - F_1\,(r_0, \theta, a, m) \\ & + F_2\,(r_+, \theta, a, m) - F_2\,(r_-, \theta, a, m) \end{aligned}$$

comprises only r-independent terms. From (3.25) it is clear that as $r \to \infty$ every contribution coming from $F\,(\theta, a, m)$ is subdominant. (Except for the extremal case $m = a$, which we shall not consider.) As a result, we conclude that, *at large distances,* *the θ-dependence of $d\left(r,\theta\right)$ is of minor importance compared to its r-dependence.* In fact, in the following we shall often neglect the θ-dependence in a first aproximation.

3.2 Reduced Circumference

In this section we consider circular paths $\mathcal{C}$. The reduced circumference R of a circular path $\mathcal{C}$ is defined as:

$$R \equiv \frac{Perimeter}{2\pi} = \frac{1}{2\pi} \int_{\mathcal{C}} \sqrt{|ds^2|} \tag{3.26}$$

It can be used to define in an invariant way the size of a spherical object without refering directly to its radius. Since we are dealing with a black hole, this is quite a suitable definition which could eventually be used in our procedure of a cutoff identification, setting $k = \xi/(2\pi R)$. The parametrization of $\mathcal{C}$ can be done in spherical coordinates by putting the singularity at the origin and setting t and r fixed (see figure 3.9).

The orientation of the path of integration in (3.26) defines every time a new R for the same value of r, as seen in figure 3.9. By choosing different orientations we can see the axial dependence of the reduced circumference. For simplicity we choose an equatorial path and a meridian one to compare. As seen in the last section for the case of $d\left(r,\theta\right)$, we are interested in finding series expressions for the different invariant integrals we are analysing, in order to find common asymptotic behaviors. We do that in the next two sections for the above mentioned paths.

3.2.1 Reduced Circumference for the Kerr Spacetime (Equatorial Plane)

For the equatorial path $\mathcal{C}_1$ in figure 3.9 we use the parametrization $\varphi \mapsto x^\mu\left(\varphi\right) = (t = t_0, r, \theta = \frac{\pi}{2}, \varphi)$ leading to $dx^\mu\left(\varphi\right) = (0,0,0,1)d\varphi$. So the line element along $\mathcal{C}_1$ becomes $ds^2 = g_{\varphi\varphi}\left(r, \frac{\pi}{2}\right) d\varphi^2$. By integration we get the following length of the path $\mathcal{C}_1$:

$$R^{\mathrm{eq}}\left(r\right) = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\left| g_{\varphi\varphi}\left(r, \frac{\pi}{2}\right)\right|} d\varphi \tag{3.27}$$

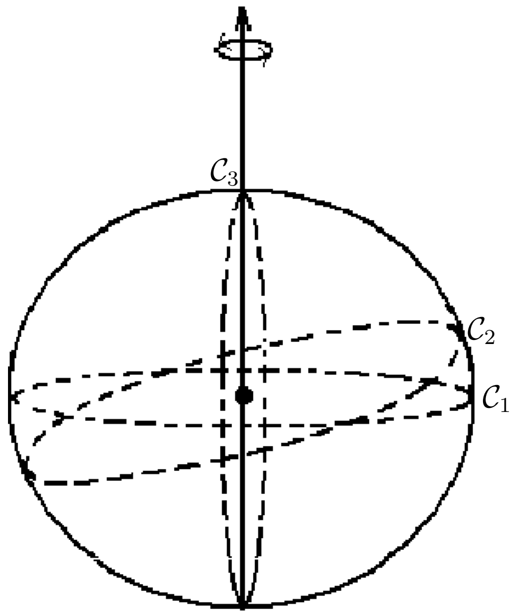


Fig. 3.9.
Three different circular paths, defining three different orientations of the reduced circumference.

This expression, in the case of a flat geometry and also for the Schwarzschild spacetime, corresponds to the radial coordinate, namely

$$R^{\rm eq}\left(r\right) = r$$

Nevertheless, this is not the case for general curved spaces and in particular not for the Kerr spacetime. By substituting in (3.27) the $g_{\varphi\varphi}\left(r, \frac{\pi}{2}\right)$ component from (2.6) we find:

$$\begin{aligned} R^{\rm eq}\left(r\right) &= \frac{1}{2\pi}\int_{\mathcal{C}_1}\sqrt{|ds^2|} = \frac{1}{2\pi}\int_0^{2\pi}\sqrt{\left|g_{\varphi\varphi}\left(r,\frac{\pi}{2}\right)\right|}d\varphi = \left|g_{\varphi\varphi}\left(r,\frac{\pi}{2}\right)\right| \\ &= \sqrt{\left|\frac{\Sigma\left(r,\frac{\pi}{2}\right)}{r^2}\right|} \end{aligned}$$

As a result,

$$R^{\rm eq}\left(r\right) = \sqrt{r^2 + a^2 + \frac{2ma^2}{r}} \tag{3.28}$$

This function is different from r precisely if $a \neq 0$. For large distances and small a, the leading terms of (3.28) are given by:

$$R^{\rm eq}\left(r\right) = r\left(1 + \frac{a^2}{2r^2} + \frac{ma^2}{r^3} + O\left(\frac{a^4}{r^4}\right)\right) \tag{3.29}$$

3.2.2 Reduced Circumference for the Kerr Spacetime (Meridian Plane)

For the meridian path $\mathcal{C}_3$ we use the parametrization $\theta \mapsto x^\mu(\theta) = (t_0, r, \theta, \varphi_0)$, where t_0,r and φ_0 are constants. Then ds^2 is simplified to $ds^2 = g_{\theta\theta} d\theta^2 = \left(r^2 + a^2 \cos^2\theta\right) d\theta^2$. The resulting reduced circumference reads

$$
\begin{aligned}
R^{\mathrm{me}}(r) &= \frac{1}{2\pi} \int_{\mathcal{C}_3} \sqrt{|ds^2|} = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{r^2 + a^2 \cos^2\theta}\, d\theta \\
&= \frac{4}{2\pi} \left(\sqrt{r^2 + a^2}\right) E\left(\frac{a^2}{a^2 + r^2}\right) \qquad (3.30)
\end{aligned}
$$

where $E(x)$ denotes a complete elliptic integral. The solution in (3.30) can be expanded for large r and small a, as follows (see appendix B, section B.5.):

$$
R^{\mathrm{me}} = r\left(1 + \frac{a^2}{4r^2} - \frac{a^4}{8r^4} + O\left(\frac{a^6}{r^6}\right)\right) \qquad (3.31)
$$

3.2.3 The Asymptotic Regime $r \to \infty$

By neglecting terms of order $\frac{1}{r}$ or smaller in (3.29) and (3.31) for R^{eq} and R^{me} respectively, we stay finally with the result $R = r$ both for the equatorial and the meridian path. It is to be expected that other paths with arbitrary orientation provide the same asymptotic result. At subdominant order differences will appear, but no major qualitative changes.

Figures 3.10 and 3.11 show the r dependence of R^{me} and R^{eq}, respectively, for different values of a. The asymptotic behavior $R \approx r$ for $r \to \infty$ is obvious in every plot. For the case $a = 0$, namely, the Schwarzschild metric the exact equality $R = r$ is recovered. It is also remarkable that the reduced circumference shows no special behavior at the event horizons r_- and r_+, since its r-derivative, for both cases, is regular at every $r > 0$. It can also be seen that R^{eq} and R^{me} are well behaved at $a = m$ in contrast to the integrals $d(r)$ or $d(r, \theta)$ analysed in the previous sections.

For the case of the equatorial plane, figure 3.11 shows that R^{eq} diverges as $r \to 0$ for any nonzero a, in contrast to the regular behavior for the meridian R^{me}. The eventual consequences of this behavior have to be seen with care since the reliability of our approach is restricted to distance scales which are not too far away from the range of validity of classical general relativity. Presumably, to be on the safe side, $d = R^{\mathrm{eq}}$ should not be used in the regime where it deviates too strongly from the linear behavior, i.e. for too small values of r.

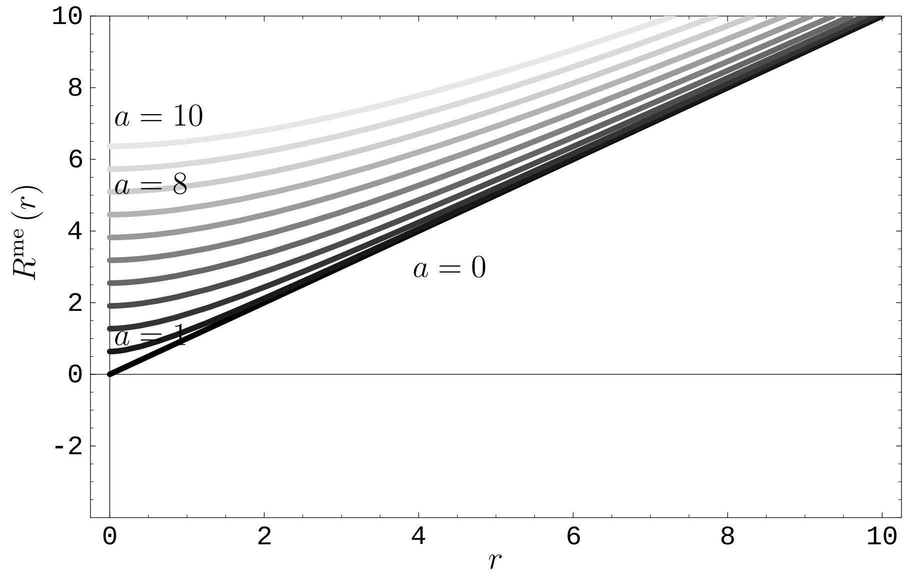


Fig. 3.10.
$R^{\mathrm{me}}(r)$ vs. r plots for the reduced circumference function in (3.30), using a meridian path in the Kerr spacetime like $\mathcal{C}_3$ in figure 3.9. The "colors" run from black to gray for increasing a.

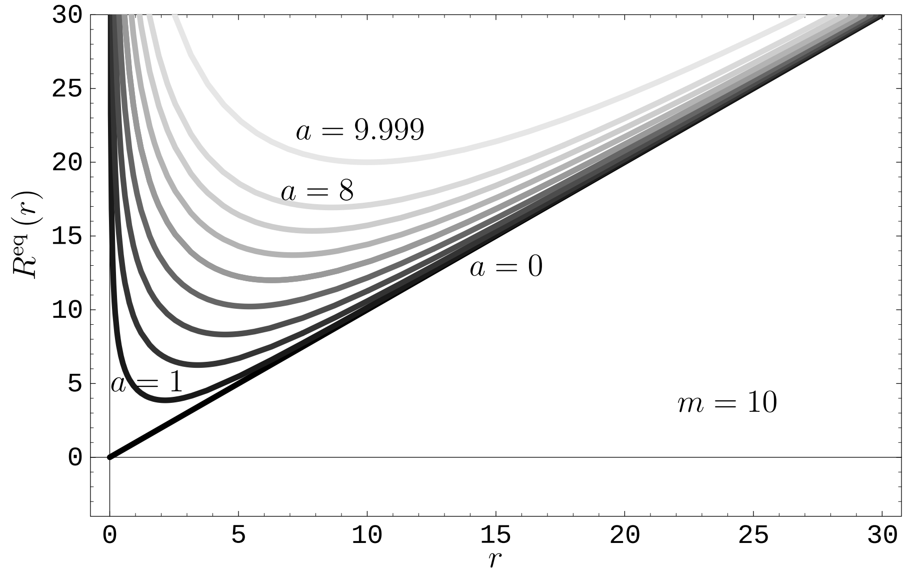


Fig. 3.11.
$R^{\mathrm{eq}}(r)$ vs. r plots for the reduced circumference function in (3.28), using an equatorial path in the Kerr spacetime like $\mathcal{C}_1$ in figure 3.9. The "colors" run from black to gray for increasing a.

3.3 Discussion

In chapter 1 we have emphasized that it is to be expected that the most important quantum correction to a classical solution of the Einstein equations near to the scale at which general relativity holds, would come from the running of the Newton constant. Nevertheless, the closer to the Planck scale we are, the more we need to take into account the contribution of more complicated invariant terms in the Lagrangian of Quantum Einstein Gravity. Therefore we should try, in our simplified approach, to stay in a safe region, not too close to the Planck scale where our assumption holds. This means that we have to choose large enough masses of our black holes, and stay at long enough distances to its center. This means also, after looking at the behavior of the integrals we have studied in this chapter, that we can work with an exclusively r-dependent cutoff identification, either coming from radial paths, from reduced circumferences, or from the proper length of any similar path. In fact, the θ-dependence of $d\,(r, \theta)$ is always comparatively weak and we do not believe that θ-dependent predictions can be made reliably within the present approach. As far as it is possible, we will do the analysis in the subsequent chapters by assuming a generic r-dependent Newton constant $G\,(r)$ without making any reference to the specific cutoff identification.

Concerning our ansatz $k = \xi/d(P)$ for the cutoff identification, we recall the analysis in subsection 3.1.3 related to its limitations for the extremal case $a = m$, where one more mass scale comes into play: In the next chapters we restrict our analysis to the regions of the (m, a)-parameter space where $a \ll m$, so that $k = \xi/d(P)$ is still the correct cutoff identification, to leading order.

We emphasize that the above mentioned limitations (staying away from the Planck scale, etc.) are limitations of the *improvement scheme* only, not to the underlying quantum theory of gravity, QEG. The later is likely to be valid for all distance scales, in particular beyond the Planck scale where the fixed point behavior sets in.

Chapter 4

General Features of the Improved Kerr Metric

In this chapter we present the improved Kerr metric and we extend to this case the analysis of the behavior of rotating test particles presented in chapter 2 for the classical Kerr spacetime. We get as a result from this analysis generalized formulas for the dragging frequency, frequencies of rotating photons and the frequency of rotation at the event horizon Ω_{H}. We find also generalizations of the equations for the critical surfaces. Furthermore we introduce the Killing vectors as an appropiate mathematical tool which simplifies many calculations and gives new formal interpretations to these calculations in relation to the symmetries of the improved Kerr spacetime.

4.1 The Improved Kerr Metric

Now we are already able to write down explicitly the improved Kerr metric in the B-L representation, using the running Newton constant (1.24) with the r dependent cutoff identification

$$k = \frac{\xi}{d\,(r)} \tag{4.1}$$

This is a spherically symmetric simplification of (1.28) that we are able to introduce only now, after the analysis from the last chapter. It leads to the following formula for $G\,(r)$ by substituting it in (1.24)

$$G\,(r) = \frac{G_0 d^2\,(r)}{d^2\,(r) + \bar{w} G_0} \tag{4.2}$$

The Kerr metric improved with the coupling $G(r)$ from (4.2) by substituting $G_0 \to G(r)$ into the classical components in (2.6), reads

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2MG(r)\,r}{\rho^2}\right) \ , \ g_{rr} = \frac{\rho^2}{\Delta_I(r)} \ , \ g_{\varphi\varphi} = \frac{\Sigma_I(r,\theta)\sin^2\theta}{\rho^2} \\ g_{\theta\theta} &= \rho^2 \ , \ g_{t\varphi} = -\frac{2MG(r)\,ra\sin^2\theta}{\rho^2} \end{aligned} \tag{4.3}$$

with the definitions

$$\rho^2 \equiv r^2 + a^2\cos^2\theta \tag{4.4}$$

$$\Delta_I(r) \equiv r^2 + a^2 - 2MG(r)\,r \tag{4.5}$$

$$\Sigma_I(r,\theta) \equiv \left(r^2 + a^2\right)^2 - a^2\Delta_I(r)\sin^2\theta \tag{4.6}$$

Similarly for the contravariant components we have from (2.11):

$$\begin{aligned} g^{tt} &= -\frac{\Sigma_I}{\rho^2\Delta_I} \ , \ g^{rr} = \frac{\Delta_I}{\rho^2} \ , \ g^{\varphi\varphi} = \frac{\Delta_I - a^2\sin^2\theta}{\rho^2\Delta_I\sin^2\theta} \\ g^{\theta\theta} &= \frac{1}{\rho^2} \ , \ g^{t\varphi} = -\frac{2MG(r)\,ra}{\rho^2\Delta_I} \end{aligned} \tag{4.7}$$

It is important to emphasize that the set of components (4.3) of the improved Kerr metric, or equivalently (4.7), are the main object of study all along this thesis.

The results from this chapter rely on the assumption of an exclusively r-dependent Newton's constant $G(r)$. In principle, the specific form of $G(r)$ might come from the improvement outlined in chapter 3, but how $G(r)$ arises from $G(k)$ plus a cutoff identification is irrelevant. Only G as a function of r will matter here. In this sense we have called these results "general features". They will be revisited in the next chapters for more specific purposes.

4.2 Killing Vectors

Killing vectors determine the symmetries on a Riemannian manifold, in our case, the spacetime. They are defined to fulfill the so-called Killing equation given by [59, P. 377]:

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0 \tag{4.8}$$

Represented in a system of coordinates, Killing vectors are related to cyclic coordinates x^κ of the metric tensor $g_{\mu\nu}$; they fulfill:

$$\frac{\partial g_{\mu\nu}}{\partial x^\kappa} = 0 \tag{4.9}$$

The Killing vector $X_{(k)}$ associated to x^κ is given by [57]

$$X_{(k)} = \frac{\partial}{\partial x^\kappa} \tag{4.10}$$

or written in components

$$X^\mu_{(k)} = \frac{\partial x^\mu}{\partial x^\kappa} = \delta^\mu_k \tag{4.11}$$

For the special case of the improved Kerr metric we know that, in the B-L representation, the coordinates t and φ are cyclic:

$$\frac{\partial g_{\mu\nu}}{\partial t} = 0 \; , \; \frac{\partial g_{\mu\nu}}{\partial \varphi} = 0 \tag{4.12}$$

Then we define as special cases of (4.10) the following Killing vectors (for a more formal proof see appendix G)

$$\boldsymbol{t} = \frac{\partial}{\partial t} \; , \; \boldsymbol{\varphi} = \frac{\partial}{\partial \varphi} \tag{4.13}$$

Although we are refering to B-L coordinates, it is clear that $\boldsymbol{t}$ and $\boldsymbol{\varphi}$ are coordinate independent vector fields. In fact we will represent them in other coordinates whenever this is necessary. The representation of $\boldsymbol{t}$ and $\boldsymbol{\varphi}$ in the B-L coordinates $x^\mu = (t, r, \theta, \varphi) \equiv (x^t, x^r, x^\theta, x^\varphi)$ reads

$$t^\mu \equiv \frac{\partial x^\mu}{\partial t} = \delta^\mu_t \; , \; \varphi^\mu \equiv \frac{\partial x^\mu}{\partial \varphi} = \delta^\mu_\varphi \tag{4.14}$$

Since t and φ are the only cyclic variables in $g_{\mu\nu}$ a general Killing vector of the Kerr spacetime can be written as a linear combination of the vectors in (4.13):

$$\boldsymbol{\eta} = \alpha \boldsymbol{t} + \beta \boldsymbol{\varphi} \tag{4.15}$$

Here α and β are constants (See Appendix G for a proof).

4.2.1 Representing Conserved Quantities with Killing Vectors

The four-velocity of a point particle is defined as follows:

$$u^\mu \equiv \frac{dx^\mu}{d\tau} \equiv \dot{x}^\mu \tag{4.16}$$

Using the $(- + ++)$ signature of the metric we can write

$$-d\tau^2 \;=\; ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{4.17}$$

which leads to the following relation for the components in (4.16)

$$-1 \;=\; u_\nu u^\nu$$

Quite generally, if ξ^μ is a Killing vector then for the motion along a geodesic the following conservation law holds true [60]

$$\xi_\mu u^\mu = \text{constant} \tag{4.18}$$

Using (4.18) we can relate the components of the Killing vectors to the conserved quantities in the specific spacetime. For the improved Kerr metric we have

$$\begin{aligned} E &= -t_\mu u^\mu \\ L &= \varphi_\mu u^\mu \end{aligned} \tag{4.19}$$

which are related, respectively, to the energy and angular momentum of test particles moving in the improved Kerr spacetime. These quantities will be important in chapter 6, when we deal with the possibility of extracting energy from the improved Kerr black hole.

4.3 Three Families of Observers

In chapter 2 we introduced the concepts of dragging of inertial reference frames, static limit and event horizon. These concepts are related to certain kinematical conditions for observers in the surroundings of the improved Kerr black hole [1]. In this section we reinterpret these concepts by setting the above mentioned kinematical conditions, namely by defining three different classes of observers, and we deduce generalizations of the formulas for the critical surfaces and the angular frequencies $\Omega_\pm$ and ω, presented in chapter 2, now for the improved Kerr spacetime. We close the section by showing that there is a distinguished angular frequency at the event horizon which depends exclusively on the parameters M and J and which plays, as we will see in chapter 8, an important role in black hole thermodynamics.

[1] We use the expression "observers" in the sense that any observer has the size of a test particle and is, as a consequence, subject to the kinematics of test particles. But, on the other hand, also in the sense that point-like systems can carry out measurements which define properties of spacetime.

4.3.1 Zero Angular Momentum Observers (ZAMOs) and Dragging Frequency

For this kind of observers it holds that $L = 0$. From (4.19) we have then:

$$0 = \varphi^{\mu} u_{\mu} = \varphi^{\mu} g_{\mu\nu} \frac{dx^{\nu}}{d\tau} \tag{4.20}$$

For the B-L coordinates we have:

$$\varphi^{\mu} \quad = \quad \delta^{\mu}_{\varphi} \tag{4.21}$$

Then, by substituting (4.21) in (4.20) we get

$$0 \quad = \quad g_{\varphi\varphi} \frac{d\varphi}{d\tau} + g_{\varphi t} \frac{dt}{d\tau} \tag{4.22}$$

Parametrizing (4.22) by the time coordinate t we have:

$$0 \quad = \quad g_{\varphi\varphi} \frac{d\varphi}{dt} + g_{\varphi t} \tag{4.23}$$

We define Ω to be the angular coordinate (or "bookkeeper") velocity. In the B-L representation we can write:

$$\Omega \equiv \frac{d\varphi}{dt} \tag{4.24}$$

Thus the ZAMO's angular velocity reads:

$$\Omega^{ZAMO}(r, \theta) \equiv \omega \equiv -\frac{g_{\varphi t}}{g_{\varphi\varphi}} \tag{4.25}$$

This is precisely the dragging frequency (2.37) from chapter 2. So we can interpret an observer in a dragged inertial reference frame also as an observer with zero angular momentum, measured at infinity.

Substituting the expressions for the improved Kerr metric components into (4.25) we get:

$$\omega(r, \theta) = \frac{2G(r) Mar}{\Sigma_I} \tag{4.26}$$

We can calculate the asymptotic behavior for $r \to \infty$. Using that $\Delta_I \approx r^2$, $\Sigma_I \approx r^4$ for $r \to \infty$ we obtain

$$\omega(r \to \infty, \theta) \approx \frac{2G(r \to \infty) J}{r^3} \tag{4.27}$$

with

$$J = aM$$

From (4.2) we see that $G(r \to \infty) = G_0$. As a result, we conclude from (4.27) that the dragging goes to zero at infinity as it happens before the improvement.

4.3.2 Static Observers

A point like observer following a given worldline is called static if there is no relative motion between him (or her) and the flat spacetime at infinity. The concept of a static limit is directly related to the existence of such observers. Therefore it's interesting for us to investigate under which conditions they are realized. By definition, the four-velocity of static observers is proportional to the Killing vector t^μ:

$$u^\mu \;=\; \gamma t^\mu$$

The factor γ is given by

$$\gamma \;\equiv\; \left(-g_{\mu\nu} t^\mu t^\nu\right)^{-\frac{1}{2}}$$

so that

$$u_\mu u^\mu \;=\; \gamma^2 t^\mu t_\mu = \left(-g_{\alpha\nu} t^\alpha t^\nu\right)^{-1} t^\mu t_\mu = -1$$

The motion of static observers is not geodesic. They must be held in place by a rocket engine, for example, which pulls the observer with a counterdragging frequency $-\omega$ given by (4.25).

Static observers exist only in those portions of the (improved) Kerr spacetime where t^μ is timelike. The "static limit" is reached when t^μ becomes a null vector, i.e. when

$$\gamma^{-2} \equiv -g_{\mu\nu} t^\mu t^\nu = -g_{tt} = 0 \tag{4.28}$$

Or more explicitly, in Boyer-Lindquist coordinates, when

$$r^2 - 2G(r)\,Mr + a^2 \cos^2\theta = 0 \tag{4.29}$$

This is precisely a generalization of equation (2.32) from chapter 2, with the identification $m \equiv G_0 M$. We denote the outer (inner) solution of (4.29) by $r^{\mathrm{I}}_{S\pm}$, in accordance with the definition of the classical static limit in chapter 2. The superscript "I" is for "improved". We will return in chapter 5 to (4.29), when we analyse the static limit surfaces for different versions of $d(r)$ in the cutoff identification (4.1).

4.3.3 Stationary Observers

A way of defining event horizons, different from the definition via one-way surfaces, is related to the existence of stationary observers. By definition, stationary observers move with a constant angular velocity $\Omega \equiv \frac{d\varphi}{dt}$ in the φ direction. They are termed "stationary" because they perceive no time variation in the gravitational field of an axially symmetric black hole. Their four-velocity involves a special case of (4.15) and is given by

$$u^\mu \;=\; \left[\gamma\left(t^\mu + \Omega\varphi^\mu\right)\right]\big|_{(r,\theta \text{ fixed})} \tag{4.30}$$

where the four vector defined as

$$\xi^\mu = t^\mu + \Omega\big|_{(r,\theta \text{ fixed})}\,\varphi^\mu \tag{4.31}$$

is a Killing vector evaluated at the location of the rotating particle. The factor γ is given by $\gamma \equiv \left(-g_{\mu\nu}\xi^\mu\xi^\nu\right)^{-\frac{1}{2}}$ or, more explicitly, by

$$\begin{aligned} \gamma^{-2}\big|_{(r,\theta \text{ fixed})} &= -g_{\mu\nu}\left(t^\mu + \Omega\varphi^\mu\right)\left(t^\nu + \Omega\varphi^\nu\right)\big|^{\text{B-L}}_{(r,\theta \text{ fixed})} \\ &= -g_{\varphi\varphi}\left(\frac{g_{tt}}{g_{\varphi\varphi}} - 2\Omega\omega + \Omega^2\right)\Bigg|^{\text{B-L}}_{(r,\theta \text{ fixed})} \end{aligned} \tag{4.32}$$

The Killing vector in (4.31) contains the same information about the motion as the four-velocity because γ is simply a constant normalization coefficient. As we shall see in chapter 8, this vector plays an important role in finding the surface gravity at the event horizon of the Kerr black hole.

Stationary observers exist only in those portions of the spacetime where $t^\mu + \Omega\big|_{(r,\theta \text{ fixed})}\,\varphi^\mu$ is timelike, that is where [2]:

$$\gamma^{-2} > 0 \tag{4.33}$$

By exploiting that $g_{\varphi\varphi} > 0\ ,\ \forall\, r > 0\ ,\ \forall\, \theta \neq 0, \pi$, we can conclude from (4.32) that the inequality

$$\frac{g_{tt}}{g_{\varphi\varphi}} - 2\Omega\omega + \Omega^2 < 0 \tag{4.34}$$

is fulfilled for stationary observers. It is equivalent to (4.33). This last inequality for Ω is satisfied if

$$\Omega_- < \Omega < \Omega_+ \tag{4.35}$$

with $\Omega_\pm$ given as in (2.36):

$$\Omega_\pm = \omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \tag{4.36}$$

[2] From now on we supress the specification "(r,θ fixed)" for the stationary observers.

Using (4.28), (4.29) and (4.36) we shall now analyse the behavior of these frequencies when a test particle crosses the static limit S^{I}_{+}. We perform this analysis by looking at the three cases $r > r^{\mathrm{I}}_{S_+}$, $r = r^{\mathrm{I}}_{S_+}$ and $r < r^{\mathrm{I}}_{S_+}$ in turn. In every case we use the fact that $G(r)$, $g_{\varphi\varphi}$ and ω are positive everywhere (to verify it see (4.2) and (C.4)).

1. **The Case $\boldsymbol{r > r^{\mathrm{I}}_{S_+}}$**

   Here it holds that

   $$g_{tt} = r^2 - 2G(r)\,Mr + a^2\cos^2\theta < 0 \tag{4.37}$$

   which implies the inequality:

   $$\sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} = \sqrt{\omega^2 + \left|\frac{g_{tt}}{g_{\varphi\varphi}}\right|} > \omega \tag{4.38}$$

   By applying (4.38) to (4.36) we infer that

   $$\Omega_- < 0 < \Omega_+ \tag{4.39}$$

   If we compare (4.39) with (4.35), $\Omega_- < \Omega < \Omega_+$, we conclude that $\Omega = 0$ is allowed and the observer can be static. (See Figure 4.1.)

2. **The Case $\mathbf{r = r^{\mathrm{I}}_{S_+}}$**

   For this case we have the static limit condition

   $$g_{tt} = r^2 - 2G(r)\,Mr + a^2\cos^2\theta = 0 \tag{4.40}$$

   which, by inserting (4.40) in (4.36), leads to

   $$\Omega_- = 0\ ,\ \Omega_+ = 2\omega$$

   We see that Ω_- changes sign at $r = r^{\mathrm{I}}_{S_+}$ where $g_{tt} = 0$. In this case only counterrotating light rays are seen static at infinity. (See figure 4.2.)

3. **The Case $\mathbf{r < r^{\mathrm{I}}_{S_+}}$**

   For this case we have already crossed the static limit surface and it holds that

   $$g_{tt} \;=\; r^2 - 2G(r)\,Mr + a^2\cos^2\theta > 0 \tag{4.41}$$

so that we can write

$$\sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} = \sqrt{\omega^2 - \left|\frac{g_{tt}}{g_{\varphi\varphi}}\right|} < \omega \tag{4.42}$$

and as a consequence we get

$$0 < \Omega_- < \Omega_+$$

Together with $\Omega_- < \Omega < \Omega_+$ this implies that $\Omega = 0$ cannot be realized: there exist no static observers anymore. No observers can avoid rotating in the same sense of the black hole. (See Figure 4.3.)

So far, we have done an analysis of the behavior of rotating observers that cross $r^{\mathrm{I}}_{S_+}$. If the observer stays inside and infinitesimally near to the static limit surface, it is expected to exist a finite sized interval (Ω_-, Ω_+) where stationary observers can exist. Once this interval is reduced to zero, by going further inside, we have no available stationary states for rotating observers. This is an indication that an event horizon is reached, namely, a surface which defines the end of the existence of stationary observers. This is a different definition as the one given in chapter 2, and is based on the concept of stationary observers. We analyse further this definition in the next paragraph.

Coalescence of Frequencies at the Event Horizon. We start finding a formula for $\Omega_\pm$ in terms of the B-L coordinates. By substituting the components of (4.3) and definition (4.25) in (4.36) we get the following expression (see appendix D):

$$\Omega_\pm = \omega \pm \frac{\rho^2\sqrt{\Delta_I}}{\Sigma_I \sin\theta} \tag{4.43}$$

By imposing $\Omega_- = \Omega_+$ to (4.43) we can find a formula for the event horizon's surface of the improved Kerr spacetime. From

$$\Omega_- = \Omega_+ \tag{4.44}$$

it follows that

$$\omega + \frac{\rho^2\sqrt{\Delta_I}}{\Sigma_I \sin\theta} = \omega - \frac{\rho^2\sqrt{\Delta_I}}{\Sigma_I \sin\theta}$$

and we get

$$\Delta_I = 0 \tag{4.45}$$

or more explicitly

$$r^2 + a^2 - 2G(r)\,Mr = 0 \tag{4.46}$$

We define the angular frequency Ω_{H} of the black hole's event horizon by

$$\Omega_{\mathrm{H}} \equiv \omega|_{\Delta_I=0} = \Omega_+|_{\Delta_I=0} = \Omega_-|_{\Delta_I=0} \tag{4.47}$$

As a consequence of (4.44), the accessible interval of frequencies for stationary observers given by (4.35) is reduced to a single value at the horizon, and only counter-rotating light is able to be stationary, being forced to move with the angular velocity Ω_{H}, see figure 4.4.

Notice that $\xi^\mu \xi_\mu = g_{tt}/g_{\varphi\varphi} - 2\omega\Omega + \Omega^2 = 0$ at the event horizon H. The Killing vector $\boldsymbol{\xi}$ becomes light-like. This means that $\boldsymbol{\xi}$ is a tangent vector to H. An event horizon with such a property of having Killing vectors as tangent vectors is called a **Killing horizon**.

Equations like (4.28) and (4.45) will be found again in chapter 5 for different physical considerations.

From (4.47) and (4.26) we can get an expression for the rotation frequency of light Ω_{H} at the radial value r_+^{I}, which, by definition is the radius of the outer event horizon and which solves equation (4.46)

$$\Omega_{\mathrm{H}} \equiv \omega\left(r_+^{\mathrm{I}}, \theta\right) = \frac{2G\left(r_+^{\mathrm{I}}\right) Mar_+^{\mathrm{I}}}{\Sigma_I\left(r_+^{\mathrm{I}}, \theta\right)} \tag{4.48}$$

with

$$\Sigma_I\left(r_+^{\mathrm{I}}, \theta\right) = \left[\left(r_+^{\mathrm{I}}\right)^2 + a^2\right]^2 - a^2 \Delta_I\left(r_+^{\mathrm{I}}\right)\sin^2\theta = \left[\left(r_+^{\mathrm{I}}\right)^2 + a^2\right]^2 \tag{4.49}$$

this equation turns out to be independent of θ. By inserting (4.49) in (4.48) we get

$$\Omega_{\mathrm{H}}(M, a) = \frac{a}{\left(r_+^{\mathrm{I}}\right)^2 + a^2} \tag{4.50}$$

where we have used the fact that r_+^{I} fulfills (4.46), namely

$$\left(r_+^{\mathrm{I}}\right)^2 + a^2 = 2G\left(r_+^{\mathrm{I}}\right) Mr \tag{4.51}$$

Eq. (4.50) is an important formula. It has the same appearance for the classical and the improved Kerr metric (see [60, P. 190]), but r_+ and r_+^{I} are different functions of M and a. In addition it is clearly coordinate independent since it is only a function of the parameters of the black hole a and M. Therefore we will use it no matter how we represent the improved Kerr spacetime.

We summarize this section by means of various figures. Figures 4.1 to 4.4 show several configurations of the roots $\Omega_\pm$ of the polynomial $\Omega^2 - 2\omega\Omega + g_{tt}/g_{\phi\phi}$. The angular frequency Ω of rotating particles is subject to $\Omega_- \leq \Omega \leq \Omega_\pm$. If $\Omega = \Omega\pm$, the rotating particles are photons. Since the angular frequency Ω of a timelike rotating observer is by definition bounded by $\Omega_\pm$, the configuration of these light frequencies determine their kinematical properties. We can distinguish the following cases:

Fig. 4.1: If $\Omega_- < 0 < \Omega_+$, a rotating observer can be in a static state with $\Omega = 0$.
Fig. 4.2: At the static limit $r = r^{\mathrm{I}}_{S_+}$ counterrotating photons reach zero coordinate tangential velocity. Every rotating observer is compelled to rotate with a positive angular frequency Ω such that $0 < \Omega < \Omega_+$.
Fig. 4.3: For $0 < \Omega_- < \Omega_+$ observers cannot remain static. These observers have crossed the static limit surface $r^{\mathrm{I}}_{S_+}$.
Fig. 4.4: At the event horizon the angular frequencies for photons Ω_- and Ω_+ coalesce and stationary observers are not allowed anymore. Light rotates with a frequency Ω_{H}.

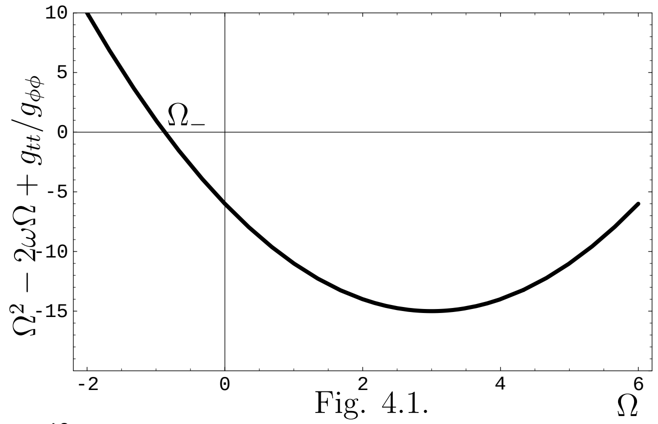


Fig. 4.1.

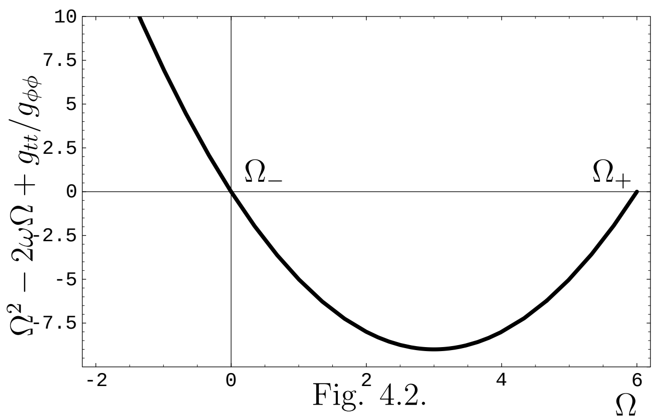


Fig. 4.2.

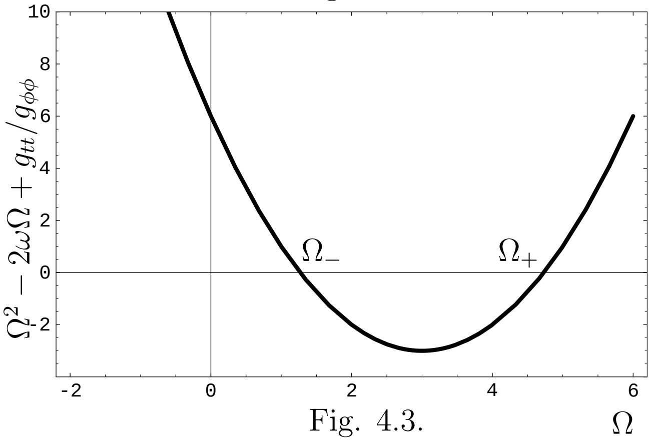


Fig. 4.3.

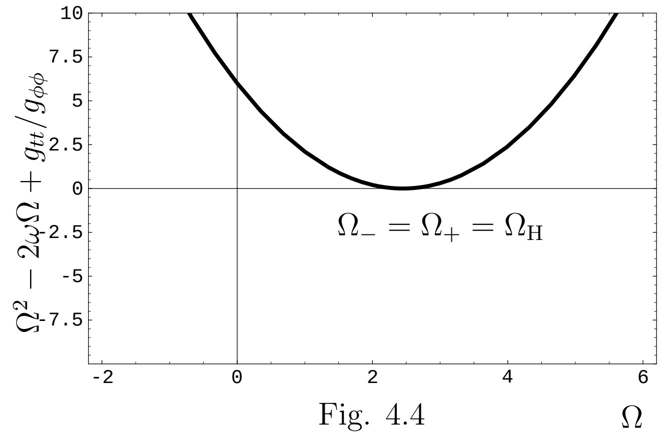


Fig. 4.4

Different configurations of the roots $\Omega_\pm$.

Chapter 5

Critical Surfaces of the Improved Kerr Metric

In this chapter we find equations for the critical surfaces of the improved Kerr metric, as generic expressions involving the distance function $d(r)$, and we solve them either analytically or numerically for specific choices of this distance function. Keeping in mind the limitations of our method we discuss the impact of quantum gravity on these critical surfaces, their shape, number and type. To this end we rely on the $d(r) = r$ approximation as a crucial tool of the analysis. The reasons are the relative simplicity of the equations it leads to, its property of being the first leading term of $d(r)$ in the $r \to \infty$ regime, and also, as we shall see in subsection 5.2.2, the fact that it leads to locally stable solutions of the equations for the critical surfaces. We shall be particularly interested in the transition from the clasical to the quantum regime and to see at least the onset of the new effects showing up at the Planck scale. The plots we present employ dimensionless quantities. As a result, the Planck scale is reached when the quantities approach the unity.

This chapter is divided into three sections, namely:

Section 5.1 *Derivation of the equations for critical surfaces, to be solved with a generic $d(r)$.*

Section 5.2 *Solution of the equations in section 5.1 and analysis of physical consequences, exploiting the properties of the $d(r) = r$ approximation.*

Section 5.3 *Discussion*

5.1 General Equations for Improved Critical Surfaces

5.1.1 Infinite Redshift Surfaces

In chapter 2 we presented the infinite redshift surfaces $S_{\pm}$ for the classical Kerr spacetime, given by

$$\begin{aligned} r_{S_+}(\theta) &= m + \sqrt{m^2 - a^2\cos^2\theta} \\ r_{S_-}(\theta) &= m - \sqrt{m^2 - a^2\cos^2\theta} \end{aligned} \tag{5.1}$$

We also emphasized there and in chapter 4, by analysing the angular frequencies of rotating test particles (or observers), that these surfaces are also a boundary for static observers, therefore they are called static limit surfaces. In this subsection we come back to the infinite redshift character of these surfaces and we deduce again the condition given in (4.29):

$$r^2 - 2G(r)Mr + a^2\cos^2\theta = 0 \tag{5.2}$$

We go further by applying the cutoff identification (4.1) and we establish a condition similar to (5.2) but depending on $d(r)$.

The redshift of signals due to gravity can be understood as a consequence of a position dependent proper time dilation. The proper time interval $d\tau(x^\mu)$ measured by a local observer at x^μ corresponds to a longer proper time dt measured by an observer at infinity in an asymptotically flat spacetime. The quantity t measured by this asymptotic observer is usually chosen as the time coordinate. The square root of the metric component g_{tt} relates both intervals [41]:

$$d\tau(x^\mu) = \sqrt{g_{tt}(x^\mu)}\, dt \tag{5.3}$$

Suppose now that n maxima of a wave of frequency ν_0 are emitted in proper time $d\tau_s(x^\mu)$ from a source at x^μ. Then

$$n = \nu_0 d\tau_s(x_s^\mu) \tag{5.4}$$

or using (5.3),

$$n = \nu_0\sqrt{g_{tt}(x_s^\mu)}\, dt \tag{5.5}$$

At infinity one certainly receives n maxima, but the frequency and time duration of the wave train have changed. There we have an equation similar to (5.4):

$$n = \nu_{\infty} dt \tag{5.6}$$

Thus by comparing (5.5) and (5.6) we get the following result for the relation of the two frequencies:

$$\nu_{\infty} = \nu_0 \sqrt{g_{tt}\left(x_s^{\mu}\right)} \tag{5.7}$$

For wavelengths we have an inverse relation to (5.7):

$$\lambda_{\infty} = \frac{\lambda_0}{\sqrt{g_{tt}\left(x_s^{\mu}\right)}} \tag{5.8}$$

Therefore it is clear that we have an infinite redshift if:

$$g_{tt}\left(x_s^{\mu}\right) = 0 \tag{5.9}$$

Applying (5.9) to the improved Kerr metric leads precisely to the condition (4.28) for the static limit surface. In B-L coordinates we have equation (5.2):

$$r^2 + a^2 \cos^2\theta - 2MG\left(r\right) \quad = \quad 0 \tag{5.10}$$

By using (4.2) for $G\left(r\right)$ we get:

$$d^2\left(r\right)\left(r^2 + a^2\cos^2\theta - 2MG_0 r\right) + G_0\bar{w}\left(r^2 + a^2\cos^2\theta\right) \quad = \quad 0 \tag{5.11}$$

We will return to this equation (more specifically, its dimensionless version) in section 5.2 when we implement specific expressions for $d\left(r\right)$, and we look for corrected static limit surfaces.

5.1.2 One Way Surface

In this subsection we come back to the one-way character of event horizons. By imposing the one-way condition (2.25) to the vectors tangent to a two-dimensional surface in the improved Kerr spacetime, we are able to reproduce equation (4.46) for the event horizon and also the equation (2.27) for the classical Kerr spacetime as a particular case. We confirm in this way two different properties of the same class of surfaces, namely, that the event horizon in the improved Kerr spacetime is a surface which can be crossed in one direction only, and also a surface which is a boundary for stationary observers.

Since the improved Kerr metric (4.3) is static and axially symmetric, any surface with these symmetries can be defined in B-L coordinates via a function $\Phi(r,\theta)$ by requiring:

$$\Phi(r,\theta) \;=\; \text{const} \tag{5.12}$$

A vector normal to the surface is obtained as the gradient of Φ, possibly up to a constant:

$$v_\alpha = A\partial_\alpha\Phi \tag{5.13}$$

In the B-L coordinates (t, r, θ, ϕ) we have explicitly

$$v_\alpha = A\left(0\ ,\ \frac{\partial\Phi}{\partial r}\ ,\ \frac{\partial\Phi}{\partial\theta}\ ,\ 0\right) \tag{5.14}$$

Raising the index of v_α according to

$$v^\alpha = g^{\alpha\beta}v_\beta = g^{\alpha t}v_t + g^{\alpha r}v_r + g^{\alpha\theta}v_\theta + g^{\alpha\varphi}v_\varphi \tag{5.15}$$

yields with (5.14):

$$\begin{aligned} v^\alpha \;&=\; A\left[g^{\alpha r}\left(\frac{\partial\Phi}{\partial r}\right) + g^{\alpha\theta}\left(\frac{\partial\Phi}{\partial\theta}\right)\right] && (5.16)\\ &=\; A\left(0\ ,\ g^{rr}\left(\frac{\partial\Phi}{\partial r}\right)\ ,\ g^{\theta\theta}\left(\frac{\partial\Phi}{\partial\theta}\right)\ ,\ 0\right) && (5.17)\end{aligned}$$

Let us now assume that v_α is a null vector, satisfying

$$v_\alpha v^a = 0$$

so that

$$g^{rr}\left(\frac{\partial\Phi}{\partial r}\right)^2 + g^{\theta\theta}\left(\frac{\partial\Phi}{\partial\theta}\right)^2 \;=\; 0 \tag{5.18}$$

Being null, v_α is both normal and tangent to the surface $\Phi = \text{const}$, which is a null hypersurface in this case.

Expression (5.18) is a partial differential equation that can be solved for $\Phi(r,\theta)$ by separation of variables only $g^{rr}/g^{\theta\theta}$ depends exclusively on r or θ. This is the case for the components in (4.7) because

$$\frac{g^{rr}}{g^{\theta\theta}} = \Delta_I(r) = a^2 + r^2 - 2G(r)Mr \tag{5.19}$$

which is indeed only r dependent. We propose then the separation ansatz

$$\Phi(r,\theta) = R(r)\,\Theta(\theta) \tag{5.20}$$

which leads to

$$\left(\frac{1}{\Theta}\frac{d\Theta}{d\theta}\right)^2 = -\frac{g^{rr}}{g^{\theta\theta}}\left(\frac{1}{R}\frac{dR}{dr}\right)^2 \tag{5.21}$$

after inserting (5.20) in (5.18) and dividing by Φ^2. As a result, every side of (5.21) is a constant, since each of them depends on only one variable. For the right hand side we write:

$$\left(\frac{1}{\Theta}\frac{d\Theta}{d\theta}\right)^2 = \lambda > 0 \tag{5.22}$$

Solving for Θ we find the following:

$$\frac{d\Theta}{d\theta} = \sqrt{\lambda}\,\Theta(\theta) \tag{5.23}$$

$$\Theta(\theta) = Be^{\sqrt{\lambda}\theta} \tag{5.24}$$

This solution is not periodic in θ. Therefore $\Phi(r,\theta)$ does not properly define a surface, unless $\lambda = 0$. Hence we conclude that our equations lead to a function Φ depending exclusively on r. We obtain the differential equation for $R(r)$ as follows:

$$\frac{g^{rr}}{g^{\theta\theta}}\left(\frac{1}{R}\frac{dR}{dr}\right)^2 = 0 \tag{5.25}$$

This equation leaves only two possibilities:

$$\frac{g^{rr}}{g^{\theta\theta}} = 0$$

$$\frac{dR}{dr} = 0$$

The second equation leads to a constant $\Phi = \text{const}$ which does not define a 2-dimensional surface. Therefore we turn to the first equation,

$$\frac{g^{rr}}{g^{\theta\theta}} = 0\ ,$$

which leads directly to:

$$r^2 + a^2 - 2MG(r)\,r = 0 \tag{5.26}$$

This is the equation of the event horizon, already given in (4.46). We have shown in this way that *the event horizon is a null hypersurface.*

Inserting the expression (4.2) for $G(r)$ in (5.26) leads to

$$d^2(r)\left[r^2+a^2-2MG_0r\right]+G_0\bar{w}\left(r^2+a^2\right) \quad = \quad 0 \tag{5.27}$$

Eq. (5.27) is a dimensionful equation for event horizon surfaces, where a generic $d(r)$ is included. In section 5.2 we will solve a dimensionles version of this equation and we look for corrected event horizons.

5.1.3 Dimensionless Variables and the Unified Equation for Critical Surfaces

As already mentioned at the beginning of this chapter, the use of dimensionless variables normalized with respect to the Planck scale helps in getting insight in how improvement changes the properties of black holes. Using the definitions in (A.7) we can transform the surface equations (5.11) and (5.27) to dimensionless equations. We also use the radial proper distance in Planck units

$$\tilde{d}(\tilde{r}) \equiv \frac{d(r)}{\sqrt{G_0}}$$

Hence

$$\tilde{d}^2(\tilde{r})\,G_0 = d^2(r)$$

so that we have for the static limit

$$\tilde{d}^2(\tilde{r})\left(\tilde{r}^2+\tilde{a}^2\cos^2\theta-2\tilde{m}\tilde{r}\right)+\bar{w}\left(\tilde{r}^2+\tilde{a}^2\cos^2\theta\right) \quad = \quad 0 \tag{5.28}$$

and similarly for the event horizon

$$\tilde{d}^2(\tilde{r})\left[\tilde{r}^2+\tilde{a}^2-2\tilde{m}\tilde{r}\right]+\bar{w}\left(\tilde{r}^2+\tilde{a}^2\right) \quad = \quad 0 \tag{5.29}$$

An obvious feature of equations (5.11) and (5.27), and likewise of (5.28) and (5.29) is that they differ only by the terms that include a or $\tilde{a}$. Thus we can define a constant b which unifies the equations for the static limit and the horizon:

$$b = \begin{cases} a\cos\theta & \text{for the static limit.} \\ a & \text{for the event horizon.} \end{cases} \tag{5.30}$$

In a similar way we define the dimensionless variable:

$$\tilde{b} = \begin{cases} \tilde{a}\cos\theta & \text{for the static limit.} \\ \tilde{a} & \text{for the event horizon.} \end{cases} \tag{5.31}$$

Now we can write one unified "master" equation replacing (5.11) and (5.27) or (5.28) and (5.29). Namely, for the dimensionful radius we have

$$d^2(r)\left(r^2 + b^2 - 2MG_0 r\right) + G_0\bar{w}\left(r^2 + b^2\right) = 0 \tag{5.32}$$

while for the dimensionless radius

$$\tilde{d}^2(\tilde{r})\left(\tilde{r}^2 + \tilde{b}^2 - 2\tilde{m}\tilde{r}\right) + \bar{w}\left(\tilde{r}^2 + \tilde{b}^2\right) = 0 \tag{5.33}$$

The dimensionless equation (5.33) yields solutions $\tilde{r}$ whose scale is normalized with respect to the Planck scale. We will concentrate primarily on solving this equation. It defines in a natural way the following 2-parameter family of functions whose zeros are to be found:

$$Q_{\tilde{b}}^{\bar{w}}(\tilde{r}) \equiv \tilde{d}^2(\tilde{r})\left(\tilde{r}^2 + \tilde{b}^2 - 2\tilde{m}\tilde{r}\right) + \bar{w}\left(\tilde{r}^2 + \tilde{b}^2\right) \tag{5.34}$$

Depending on the choice for $\tilde{b}$ given by (5.31), the roots of $Q_{\tilde{b}}^{\bar{w}}$ will define the corrected static limit or event horizon surfaces for the Kerr metric, respectively.

5.2 Solutions for the Critical Surfaces and Physical Consequences

We come now to the second part of this chapter where we assume a specific form of $d(r)$ and we look for solutions of the equations (5.28) and (5.29), for the static limit and the event horizon, respectively.

We start in subsection 5.2.1 with a general analysis of the equations (5.34) with $d(r) = r$. We present the solutions of these equations, when it is possible, and their properties. We also present special important cases, like the extremal black hole. In subsection 5.2.2 we present the stability properties of the related solutions. We conclude the section (subsections 5.2.3 and 5.2.4) with the analysis of several plots of the solutions and surfaces we have found.

5.2.1 Critical Surfaces for the Approximation $d(r) = r$: General Features

For $d(r) = r$ the cutoff identification is given by the simplest non-trivial expression:

$$k(r) = \frac{\xi}{r} \tag{5.35}$$

After substituting (5.35) in (4.2) we find the following running $G(r)$:

$$G(r) = \frac{G_0 r^2}{r^2 + \bar{w} G_0} \tag{5.36}$$

By substituting (5.36) in (5.32) and (5.33), respectively, we have

$$r^4 - 2MG_0 r^3 + r^2\left(b^2 + G_0\bar{w}\right) + G_0\bar{w}b^2 = 0 \tag{5.37}$$

$$\tilde{r}^4 - 2\tilde{m}\tilde{r}^3 + \tilde{r}^2\left(\tilde{b}^2 + \bar{w}\right) + \bar{w}\tilde{b}^2 = 0 \tag{5.38}$$

For $d(r) = r$ the equation of the critical surfaces is a polynomial in r. As a result, the respective function $Q_{\tilde{b}}^{\bar{w}}(r)$ is the following:

$$Q_{\tilde{b}}^{\bar{w}}(\tilde{r}) = \tilde{r}^4 - 2\tilde{m}\tilde{r}^3 + \tilde{r}^2\left(\tilde{b}^2 + \bar{w}\right) + \bar{w}\tilde{b}^2 \tag{5.39}$$

In the next two sections we describe the physical content of equation (5.38) by analysing several cases, corresponding to different values of the free parameters $\bar{w}$ and $\tilde{b}$. These cases are:

1. Classical Schwarzschild critical surfaces $\left(\bar{w} = 0\ ,\ \tilde{b} = 0\right)$.
2. Classical Kerr critical surfaces $\left(\bar{w} = 0\ ,\ \tilde{b} \neq 0\right)$.
3. Improved Schwarzschild critical surfaces $\left(\bar{w} \neq 0\ ,\ \tilde{b} = 0\right)$.
4. Improved Kerr critical surfaces $\left(\bar{w} \neq 0\ ,\ \tilde{b} \neq 0\right)$.

We shall refer to cases 1 to 3 as “particular cases” and the fourth case as the “general case”.

Particular Cases

1. The simplest case which results from setting $\bar{w} = 0$ and $\tilde{b} = 0$ leads to the well-known Schwarzschild singularities. Equation (5.38) is reduced to:

$$Q_0^0(\tilde{r}) \equiv \tilde{r}^3(\tilde{r} - 2\tilde{m}) = 0 \tag{5.40}$$

Clearly the solutions of (5.40) are either $\tilde{r} = 0$, the singularity at the origin, or $\tilde{r} = 2\tilde{m}$ the coordinate singularity that defines the event horizon.

2. The case 2 defined by $\bar{w} = 0$, $\tilde{b} \neq 0$ converts (5.38) to the factorized fourth-order equation given by:

$$Q_{\tilde{b}}^0(\tilde{r}) \equiv \tilde{r}^2\left(\tilde{r}^2 - 2\tilde{m}\tilde{r} + \tilde{b}^2\right) = 0 \tag{5.41}$$

The solutions of (5.41) are either $\tilde{r} = 0$ or the couple of solutions

$$\tilde{r}_{\tilde{b}_\pm} = \tilde{m} \pm \sqrt{\tilde{m}^2 - \tilde{b}^2} \tag{5.42}$$

Depending on the interpretation of $\tilde{b}$ we have either the solutions for the event horizon $\tilde{r}_\pm = \tilde{m} \pm \sqrt{\tilde{m}^2 - \tilde{a}^2}$, when $\tilde{b} = \tilde{a}$, or the static limit $\tilde{r}_{S_\pm} = \tilde{m} \pm \sqrt{\tilde{m}^2 - \tilde{a}^2\cos^2\theta}$, when $\tilde{b} = \tilde{a}\cos\theta$. Dimensionful versions of these solutions can be found by multiplying $\tilde{r}$ by $l_{\mathrm{Pl}} = \sqrt{G_0}$, namely:

$$\begin{aligned} r_\pm &= MG_0 \pm \sqrt{(MG_0)^2 - a^2} \\ r_{S_\pm} &= MG_0 \pm \sqrt{(MG_0)^2 - a^2\cos^2\theta} \end{aligned} \tag{5.43}$$

Or using $m = MG_0$ we have:

$$r_\pm = m \pm \sqrt{m^2 - a^2} \tag{5.44}$$

$$r_{S_\pm} = m \pm \sqrt{m^2 - a^2\cos^2\theta} \tag{5.45}$$

This is the unified way of presenting the critical surfaces of the Kerr spacetime, as already given in equations (2.28) and (2.33) of chapter 2.

3. The case 3 with $\tilde{b} = 0$, $\bar{w} \neq 0$ gives us the critical surfaces of the improved Schwarzschild spacetime [30]. In this case we have the polynomial equation (5.38) reduced to

$$Q_0^{\bar{w}}(\tilde{r}) \equiv \tilde{r}^2\left(\tilde{r}^2 - 2\tilde{m}\tilde{r} + \bar{w}\right) = 0 \tag{5.46}$$

which is equivalent to (5.41) if we perform the identification $\tilde{b}^2 \to \bar{w}$. As a result, the solutions of (5.46) have a similar form to the solutions of (5.41). In this case we have again the singularity at the origin $\tilde{r} = 0$ and two event horizons for the improved Schwarzschild spacetime given by:

$$\tilde{r}^{\mathrm{I}}_{\mathrm{Sch}_\pm} = \tilde{m} \pm \sqrt{\tilde{m}^2 - \bar{w}} \tag{5.47}$$

Now multiplying (5.47) by l_{pl} gives a dimensionful version for the radii of the event horizons:

$$r^{\mathrm{I}}_{\mathrm{Sch}_\pm} = MG_0 \pm \sqrt{(MG_0)^2 - G_0\bar{w}} \tag{5.48}$$

Or equivalently using $m = MG_0$:

$$r^{\mathrm{I}}_{\mathrm{Sch}_\pm} = m \pm \sqrt{m^2 - G_0\bar{w}} \tag{5.49}$$

The existence of *two different* radii $r^{\mathrm{I}}_{\mathrm{Sch}_\pm}$ represents the splitting of the Schwarz schild event horizon due to the RG-improvement into a set of two horizons which had been found and discussed in detail in Ref. [30].

Concerning the Planck scale, the value of M will be considered large in relation to a critical mass M_{cr} equal or near to the Planck mass M_{pl}. Following [30] we define M_{cr} to be the mass at which the two radii $r^{\mathrm{I}}_{\mathrm{Sch}+}$ and $r^{\mathrm{I}}_{\mathrm{Sch}-}$ merge to a unique value $r^{\mathrm{I}}_{\mathrm{Sch\text{-}cr}}$. From (5.48) we have [30]:

$$M_{\mathrm{cr}} \equiv \sqrt{\frac{\bar{w}}{G_0}} = \sqrt{\bar{w}}\; m_{\mathrm{pl}} \tag{5.50}$$

so that

$$G_0\,(M_{\mathrm{cr}})^2 = \bar{w} \tag{5.51}$$

M_{cr} is the lowest amount of mass a black hole configuration in the improved Schwarzschild spacetime is allowed to have. There exists no event horizon for $M < M_{\mathrm{crit}}$. We call the state with $M = M_{\mathrm{cr}}$ the "critical" quantum black hole [30].

The similarity of (5.48) with the radii $r^{\mathrm{RN}}_{\pm}$ of the Reissner-Nordström spacetime is clear if we identify the charge e of the black hole with $\sqrt{\bar{w}}$. In particular, the "critical" quantum black hole with $M = M_{\mathrm{cr}}$ corresponds to the extremal charged black hole with $e = \sqrt{G_0}M$.

We also define the related geometric critical mass to be,

$$m_{\text{cr}} \equiv G_0 M_{\text{cr}} \tag{5.52}$$

Thus the dimensionless ratio M_{cr}/M can be exploited as an expansion parameter when $M >> M_{\text{cr}}$. We denote this ratio as $\bar{m}$,

$$\bar{m} \equiv \frac{M_{\text{cr}}}{M} = \frac{m_{\text{cr}}}{m} \tag{5.53}$$

where $m \equiv MG_0$ is the geometrical mass of the black hole.

Since $\bar{w} \propto O(\hbar)$ the classical case (no improvement) corresponds to $M_{\text{cr}} = 0$ or $\bar{m} = 0$. As a result, the classical limit ($\hbar \to 0$) and the heavy mass limit ($M \to \infty$) will coincide for all results obtained in this work.

The radii $r^{\text{I}}_{\text{Sch}\pm}$ in (5.49) can be expressed as functions of $\bar{m}$ as follows:

$$r^{\text{I}}_{\text{Sch}\pm} = m \pm m\sqrt{1 - \frac{(m_{\text{cr}})^2}{m^2}} = m \pm m\sqrt{1 - \bar{m}^2} \tag{5.54}$$

Expanding (5.54) in $\bar{m}$ leads to

$$\begin{aligned} r^{\text{I}}_{\text{Sch}+} &= m\left[2 - \frac{\bar{m}^2}{2} + O\left(\bar{m}^4\right)\right] \\ r^{\text{I}}_{\text{Sch}-} &= m\left[\frac{\bar{m}^2}{2} + O\left(\bar{m}^4\right)\right] \end{aligned} \tag{5.55}$$

In the limit $M \gg M_{\text{cr}}$ we observe that $r^{\text{I}}_{\text{Sch}+} \to 2m$ and $r^{\text{I}}_{\text{Sch}-} \to 0$. This corresponds to the classical limit where the radius of the event horizon is the Schwarzschild radius $2m$ and the inner event horizon coincides with the origin of coordinates where the singularity of the Schwarzschild black hole is supposed to be. As $M \to M_{\text{cr}}$, $r^{\text{I}}_{\text{Sch}+}$ deviates smoothly from the classical value $2m$ and $r^{\text{I}}_{\text{Sch}-}$ grows out of the origin.

Analysis of the General Case $\bar{w} \neq 0$, $\tilde{b} \neq 0$

For the case $\bar{w} \neq 0$, $\tilde{b} \neq 0$ associated to the improved Kerr spacetime, we have to deal with a fourth order polynomial. In principle there could exist four complex solutions for (5.39) but we must restrict $\tilde{r}$ to the physical region given by $\tilde{r} \in \mathbb{R}$ and $\tilde{r} \geq 0$. An analysis of the extrema for $Q^{\bar{w}}_0(r)$ leads to an additional simplification.

For the special case $d(r) = r$ we are so far dealing with, these extrema can be found analytically. The condition for a critical point is:

$$\left.\frac{dQ_{\tilde{b}}^{\overline{w}}}{d\tilde{r}}\right|_{\tilde{r}_{crit}} = 2\tilde{r}\left[2\tilde{r}^2 - 3\tilde{m}\tilde{r} + \tilde{b}^2 + \overline{w}\right]\Big|_{\tilde{r}_{crit}} = 0 \tag{5.56}$$

The solutions of equation (5.56) are $\tilde{r}_{crit} = 0$ and:

$$\tilde{r}_{1,2} = \frac{3\tilde{m}}{4}\left[1 \pm \sqrt{1 - \frac{8}{9\tilde{m}^2}\left(\tilde{b}^2 + \overline{w}\right)}\right] \tag{5.57}$$

From the last expression we conclude that the positivity of the discriminant $1 - \frac{8}{9\tilde{m}^2}\left(\tilde{b}^2 + \overline{w}\right)$ gives a constraint on the existence of non-trivial extrema, namely:

$$0 \le 1 - \frac{8}{9\tilde{m}^2}\left(\tilde{b}^2 + \overline{w}\right) \le 1 \tag{5.58}$$

From now on we refer only to $\tilde{r}_{1,2}$ as real positive roots of (5.56), namely roots which fulfill (5.58). The maximum or minimum character of these extrema can also be analytically established. The second derivative of $Q_{\tilde{b}}^{\overline{w}}$ is given by:

$$\frac{d^2Q_{\tilde{b}}^{\overline{w}}}{d\tilde{r}^2} = 2\left[2\tilde{r}^2 - 3\tilde{m}\tilde{r} + \tilde{b}^2 + \overline{w}\right] + 2\tilde{r}\left[4\tilde{r} - 3\tilde{m}\right] \tag{5.59}$$

As a result we have for the stationary point at the origin:

$$\left.\frac{d^2Q_{\tilde{b}}^{\overline{w}}}{d\tilde{r}^2}\right|_{\tilde{r}_{\mathrm{crit}}=0} = \tilde{b}^2 + \overline{w} > 0 \tag{5.60}$$

As a consequence we have a local minimum at $\tilde{r}_{\mathrm{crit}} = 0$ [1]. On the other hand, by combining (5.57) and (5.59) we can write:

$$\left.\frac{d^2Q_{\tilde{b}}^{\overline{w}}}{d\tilde{r}^2}\right|_{\tilde{r}_{1,2}} = 2\tilde{r}_{1,2}\left[4\tilde{r}_{1,2} - 3\tilde{m}\right]$$

Here we have used the fact that $\tilde{r}_{1,2}$ fulfills eq. (5.56). As a result we have:

$$\left.\frac{d^2Q_{\tilde{b}}^{\overline{w}}}{d\tilde{r}^2}\right|_{\tilde{r}_{1,2}} = \begin{cases} \frac{9\tilde{m}^2}{2}\sqrt{1 - \frac{8}{9\tilde{m}^2}\left(\tilde{b}^2 + \overline{w}\right)}\left[\sqrt{1 - \frac{8}{9\tilde{m}^2}\left(\tilde{b}^2 + \overline{w}\right)} - 1\right] < 0 & \text{for } \tilde{r}_1 \\ \frac{9\tilde{m}^2}{2}\sqrt{1 - \frac{8}{9\tilde{m}^2}\left(\tilde{b}^2 + \overline{w}\right)}\left[1 + \sqrt{1 - \frac{8}{9\tilde{m}^2}\left(\tilde{b}^2 + \overline{w}\right)}\right] > 0 & \text{for } \tilde{r}_2 \end{cases} \tag{5.61}$$

[1] Here we assumed, as always, that $\overline{w} > 0$ as it is predicted by the renormalization group equation.

From (5.61) we conclude that $\tilde{r}_1$ is a local maximum and $\tilde{r}_2$ is a local minimum. Furthermore, since we have two minima and only one maximum at $\tilde{r} \geq 0$, and since the minimum at $r = 0$ is positive $\left(Q_{\tilde{b}}^{\bar{w}}(0) = \tilde{b}^2 \bar{w}\right)$ it follows that $Q_{\tilde{b}}^{\bar{w}}(\tilde{r})$ *has at most two real, strictly positive zeros*, see figure 5.1. We shall denote these zeros by $\tilde{r}_{\tilde{b}-}^{\mathrm{I}}$ and $\tilde{r}_{\tilde{b}+}^{\mathrm{I}}$, respectively.

The Quantum Extremality Condition

As already mentioned in chapter 2 the extremal Kerr event horizon occurs when the two solutions $r_\pm$ degenerate to just one, $r_+ = r_-$. This happens when $m = a$ and as a result $r_{\mathrm{extr}} = m = a$, as one can easily see from (5.44). We shall search for analogous extremality conditions in the new cases 3 and 4 of the improved Schwarzschild and Kerr spacetimes, respectively.

The extremal black hole of the improved Schwarzschild spacetime is reached for $\tilde{m}^2 = \bar{w}$, as one can derive from (5.47). Therefore we have $r_{\text{sc-extr}}^{\mathrm{I}} = M_{\mathrm{crit}} G_0 = \sqrt{G_0 \bar{w}}$. We call M_{crit} the critical mass and it plays an interesting role related to the problem of the final state of a black hole that evaporates due to the emmision of Hawking radiation [30].

The condition of extremality can be generalized to the improved Kerr metric by requiring that $\tilde{r}_{\tilde{b}+}^{\mathrm{I}} = \tilde{r}_{\tilde{b}-}^{\mathrm{I}}$ is a single double root of $Q_{\tilde{b}}^{\bar{w}}$. This double zero coincides with the minimum at $\tilde{r}_2$, see Fig. 5.2. By a straightforward algebraic process one can get to the following simplification of that condition (see appendix E)

$$\frac{\tilde{r}_2^2}{2}\left\{\tilde{b}^2 + \bar{w} - \tilde{m}\tilde{r}_2\right\} + \bar{w}\tilde{b}^2 = 0 \tag{5.62}$$

with

$$\tilde{r}_2 = \frac{3\tilde{m}}{4}\left[1 + \sqrt{1 - \frac{8}{9\tilde{m}^2}\left(\tilde{b}^2 + \bar{w}\right)}\right] \tag{5.63}$$

Eq. (5.62) with (5.63) defines a curve in the two dimensional $(\tilde{m}, \tilde{a})$ parameter space. All points $(\tilde{m}, \tilde{a})$ on this curve are extremal black holes; they have a degenerate horizon. We refer to (5.62) and (5.63) as the *quantum extremality condition*. We can see immediately that from (5.62) we recover the above mentioned extremal horizons as particular cases. In the general case we solve (5.62) for $\tilde{m}$ and describe the curve by a function $\tilde{m} = \tilde{m}(\tilde{a})$.

Knowing that $\bar{w}$ is a fixed parameter of the theory we plot the function $\tilde{m} = \tilde{m}(\tilde{a})$ obtained by numerically solving (5.62) and (5.63) in Fig. 5.3. We observe that there are deviations from the linear behavior of the classical extremal Kerr black hole given by $\tilde{m}\,(\tilde{a}) = \tilde{a}$, when $\tilde{a}$ goes to zero. For $\tilde{a} = 0$, $\tilde{m}$ reaches its minimum at $\sqrt{\bar{w}}$. On the other hand, for $\tilde{a} \to \infty$, $\tilde{m}\,(\tilde{a})$ approaches the classical behavior, namely $\tilde{m} \to \tilde{a}$.

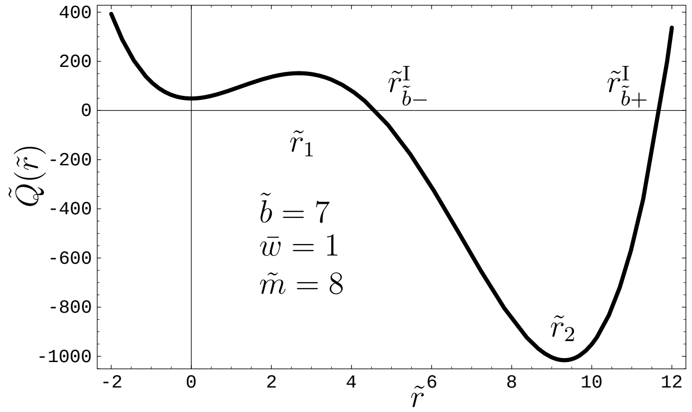


Fig. 5.1

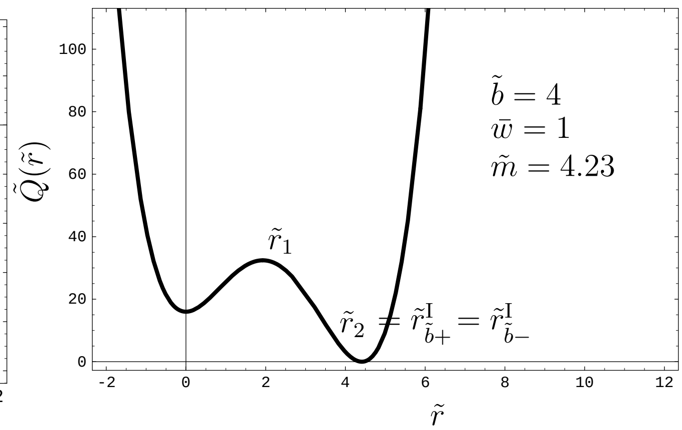


Fig. 5.2.

Figures 5.1 and 5.2 show two configurations of the fourth-order polynomial $\tilde{Q}(\tilde{r})$ defined in (5.39). Its roots represent radial coordinates of critical surfaces in the improved Kerr spacetime, assuming $d(r) = r$ and $k = \xi/d(r)$. Depending on the parameter values one has either two, one or no zeros in the positive real domain.

Fig. 5.1: Two roots, $\tilde{r}^{\mathrm{I}}_{\tilde{b}-}$ and $\tilde{r}^{\mathrm{I}}_{\tilde{b}+}$ of $\tilde{Q}(\tilde{r})$. They correspond to radial coordinates of two event horizons or two static limits, depending on the interpretation of $\tilde{b}$.
Fig. 5.2: Extremal case: The roots $\tilde{r}^{\mathrm{I}}_{\tilde{b}-}$ and $\tilde{r}^{\mathrm{I}}_{\tilde{b}+}$ degenerate to one value given by $\tilde{r}_2$ in (5.62) and (5.63).

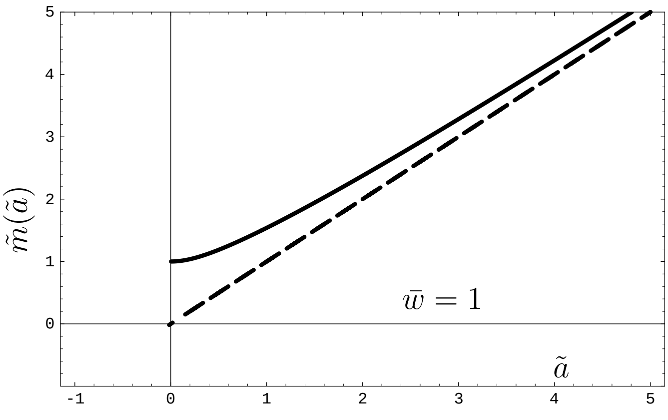


Fig. 5.3.
The $\tilde{m}\,(\tilde{a})$ dependence given by the "quantum extremality condition" for the extremal improved Kerr black hole with $d(r) = r$. The dashed line represents the $\tilde{m}\,(\tilde{a}) = \tilde{a}$ dependence of the classical Kerr spacetime. For $\tilde{a} \to 0$, $\tilde{m}$ assumes its minimum value at $\sqrt{\bar{w}}$, and for $\tilde{a} \to \infty$ $\tilde{m}$ approaches the classical behavior, namely $\tilde{m} \to \tilde{a}$.

Common Points of H-Surfaces and S-Surfaces

In this paragraph we prove a general property of both the classical and the improved event horizons and static limits, namely that, except at $\theta = 0$ and π, no S-surface intersects or touches any H-surface.

Assume a point (r, θ, ϕ) is on both a S and a H-surface. Hence r and θ satisfy both of the two equations in (5.38), for the event horizon with $b = a$ and the static limit with $b = a\cos\theta$. Subtracting the first from the second equation gives:

$$a^2 \sin^2\theta \left(r^2 + \bar{w}\right) = 0 \ , \tag{5.64}$$

For $a \neq 0$, $\theta \neq 0, \ \pi$ this equation has no solution. As a result, there exists no (r, θ, ϕ) lying on both an S and an H-surface except for the poles.

5.2.2 Structural Stability of the Polynomials $Q_{\bar{w}}^{b}(r)$ and Status of the $d(r) = r$ Approximation

We have done so far a first analysis of the behavior of critical surfaces in the approximation $d(r) = r$ which becomes exact asymptotically. Before going into further features of these solutions, it is important to clarify to what extent the results we are finding for this approximation have a general qualitative meaning and which properties change when we apply the exact form of $d(r)$ to equation (5.34) instead of $d(r) = r$. More generally, we would like to know how the results behave, when we slightly change the function $G = G(r)$. There are two important aspects related to this question that we must consider, namely:

- We would like to know whether the classical critical surfaces of the Kerr spacetime, defined by the solutions of the polynomial (5.41), are stable, or on the contrary the RG-improvement leads to a drastic change of their number and form. This question is related directly to the "structural stability" of the mentioned polynomial.

- On the other hand, it is also important to analyse the stability of solutions of the equation (5.39) when the approximation $d(r) = r$ is replaced by, more accurate functions $d(r)$.

In order to analyse these aspects we present in the next paragraphs a short introduction to the concept of structural stability of analytical functions in one variable

[54], as it is used in the context of catastrophe theory usually. It is important to emphasize that we are mainly interested in analyzing the stability of the zeros of functions like (5.39) or (5.41), which are directly related to the critical surfaces $H_\pm$ and $S_\pm$. Even though the definition of structural stability is based upon properties of the critical points of a function rather than upon its zeros, we can consider the zeros as critical points of the integrals of the functions in (5.39) or (5.41)[2]. We define $\bar{Q}(r)$ to be the definite integral in r of $Q(r)$:

$$\bar{Q}(r) \equiv \int_{r_0}^{r} Q(r')\, dr' \tag{5.65}$$

As a result, if r_1 is a zero of $Q(r)$, it is also a critical point of $\bar{Q}(r)$:

$$\left.\frac{d\bar{Q}}{dr}\right|_{r_1} = Q(r_1) = 0 \tag{5.66}$$

Once the concept of structural stability is presented we proceed to analyse the two above mentioned aspects in the subsequent paragraphs. We close this subsection with a summary and an analysis of the results obtained.

Structural Stability of Functions in One Variable

The concept of structural stability is introduced in the framework of catastrophe theory as a basic tool in the analysis of the behavior of critical points of analytical functions, under infinitesimal variation of these functions. We say that two functions $f_1(r)$ and $f_2(r)$ are of the same type, or equivalent, if they have the same configuration of critical points with the same properties, near $r = 0$. The analysis can be easily extended to other points $r \neq 0$ of the real line by performing coordinate translations. To determine whether or not a function $f(r)$ is stable we compare it with a generica neighbouring function f_α given by

$$f_\alpha(r) = f(r) + \alpha(r) \tag{5.67}$$

where $\alpha(r)$ is analytic and infinitesimally small, together with all its derivatives. We say that $f(r)$ is structurally stable at $r = 0$ if it is equivalent to $f_\alpha(r)$ for all sufficiently small, smooth functions $\alpha(r)$.

[2]We work in this subsection with polynomials like $Q_{\tilde{b}}^{\tilde{w}}(\tilde{r})$ that are defined for dimensionless variables. Nevertheless, from now on and until the end of this chapter, we omit for simplicity the tilde that denotes the dimensionless character.

The concepts of degeneracy and non-degeneracy of critical points give a useful tool to demonstrate the structural stability of analytical functions. We proceed now to explain it.

A critical point u of f_1 is called non-degenerate if its second derivative at u does not vanish. Thus we have for u

$$\left.\frac{df}{dr}\right|_{r=u} = 0 \tag{5.68}$$

This is the criticality condition. And additionally we have

$$\left.\frac{d^2 f}{dr^2}\right|_{r=u} \neq 0 \tag{5.69}$$

for the non-degeneracy property. Non-degeneracy turns out to be a very useful property of critical points. It can be shown, for example, that every non-degenerate critical point is isolated. This means that there exist no other such points in the infinitesimal vicinity of that point [54]. Other important properties of critical points depend on their degenerate or non-degenerate character [54]. In particular, in the vicinity of non-degenerate critical points, the function f is structurally stable [54]. In that case we simply say that the respective critical point is structurally stable. Furthermore, it can be proved that a critical point is structurally stable if and only if it is nondegenerate; hence every degenerate point is structurally unstable [54, 55]. As a result, it is sufficient to verify expression (5.69) in order to demonstrate the structural stability of f at u. We will use this method in the next paragraphs in order to check the stability of functions in the classical and improved cases. As mentioned before, we analyse the solutions of $Q(r) = 0$ as being critical points of $\bar{Q}(r)$, the integral of $Q(r)$. In addition we also study the critical points of $Q(r)$ themselves.

Stability of the Zeros of $Q_b^{\bar{w}}$

From the previous subsection we know that there exist at most two positive solutions $r^{\mathrm{I}}_{b\pm}$ of $Q_b^{\bar{w}}$ such that

$$Q_b^{\bar{w}}\left(r^{\mathrm{I}}_{b\pm}\right) = 0 \tag{5.70}$$

More explicitly we have

$$Q_b^{\bar{w}}\left(r^{\mathrm{I}}_{b\pm}\right) \equiv \left(r^{\mathrm{I}}_{b\pm}\right)^2 \left[\left(r^{\mathrm{I}}_{b\pm}\right)^2 + b^2 - 2mr^{\mathrm{I}}_{b\pm} + \bar{w}\right] + \bar{w}b^2 = 0 \tag{5.71}$$

The second derivative of $\bar{Q}_b^{\bar{w}}$ evaluated at $r_{b\pm}^{\mathrm{I}}$ is given by

$$\left.\frac{d^2\bar{Q}_b^{\bar{w}}}{dr^2}\right|_{r_{b\pm}^{\mathrm{I}}} = \left.\frac{dQ_b^{\bar{w}}}{dr}\right|_{r_{b\pm}^{\mathrm{I}}} = 2r_{b\pm}^{\mathrm{I}}\left[2\left(r_{b\pm}^{\mathrm{I}}\right)^2 - 3mr_{b\pm}^{\mathrm{I}} + b^2 + \bar{w}\right] \tag{5.72}$$

If we substitute $\left(r_{b\pm}^{\mathrm{I}}\right)^2 + b^2 - 2mr_{b\pm}^{\mathrm{I}} + \bar{w} = -\bar{w}b^2/\left(r_{b\pm}^{\mathrm{I}}\right)^2$ from (5.71) in (5.72) we find

$$\left.\frac{d^2\bar{Q}_b^{\bar{w}}}{dr^2}\right|_{r_{b\pm}^{\mathrm{I}}} = 2\left[\left(r_{b\pm}^{\mathrm{I}}\right)^3 - m\left(r_{b\pm}^{\mathrm{I}}\right)^2 - \frac{\bar{w}b^2}{r_{b\pm}^{\mathrm{I}}}\right] = \frac{2}{r_{b\pm}^{\mathrm{I}}}\left[\left(r_{b\pm}^{\mathrm{I}}\right)^4 - m\left(r_{b\pm}^{\mathrm{I}}\right)^3 - \bar{w}b^2\right] \neq 0 \tag{5.73}$$

This is different from zero except for a finite number of values for $r_{b\pm}^{\mathrm{I}}$, four at most, according to the fundamental theorem of algebra. As a result we can assert that the critical points $r_{b\pm}^{\mathrm{I}}$ are always stable except for a finite number of special cases.

Since the typical behavior of $Q_b^{\bar{w}}(r)$ is the one presented in figures 5.1 and 5.2, we conclude that the only possibility that $d^2\bar{Q}_b^{\bar{w}}/dr^2\big|_{r_{b\pm}^{\mathrm{I}}} = 0$, with $r_{b\pm}^{\mathrm{I}}$ real positive values, is the extremal case when $r_{b+}^{\mathrm{I}} = r_{b-}^{\mathrm{I}} \equiv r_{b(\mathrm{extr})}^{\mathrm{I}}$. In that case $r_{b(\mathrm{extr})}^{\mathrm{I}}$ is a critical point of $Q_b^{\bar{w}}$ and we have:

$$\left.\frac{dQ_b^{\bar{w}}}{dr}\right|_{r_{b(\mathrm{extr})}^{\mathrm{I}}} = \left.\frac{d^2\bar{Q}_b^{\bar{w}}}{dr^2}\right|_{r_{b(\mathrm{extr})}^{\mathrm{I}}} = 0 \tag{5.74}$$

As a consequence from (5.74) the degenerate critical point $r_{b(\mathrm{extr})}^{\mathrm{I}}$ in the extremal case is unstable.

Alternatively, we conclude from (5.73) that all $r_{b\pm}^{\mathrm{I}} \neq r_{b\mathrm{extr}}^{\mathrm{I}}$ are structurally stable (all non-extremal cases). This means that we do not expect any dramatic qualitative changes in the critical surfaces $\mathrm{H}_\pm$ and $\mathrm{S}_\pm$ for small changes of $\bar{Q}_b^{\bar{w}}$ which might come from corrections to $d(r) = r$, for example. More specifically, we only expect a small shifting of the solutions $r_{b\pm}^{\mathrm{I}}$ of figure 5.1.

Stability of Critical Points of $Q_b^{\bar{w}}$

In subsection 5.2.1 we have already found the critical points of the polynomial $Q_b^{\bar{w}} \equiv r^2\left(r^2 + b^2 - 2mr\right) + \bar{w}\left(r^2 + b^2\right)$. They solve equation (5.56)

$$\left.\frac{dQ_b^{\bar{w}}}{dr}\right|_{r_{crit}} = 2r\left[2r^2 - 3mr + b^2 + \bar{w}\right]\Big|_{r_{crit}} = 0$$

The solutions are given in (5.57) as follows:

$$r_{crit} = \begin{cases} r_0 = 0 \\ r_{1,2} = \frac{3m}{4}\left[1 \pm \sqrt{1 - \frac{8}{9m^2}\left(b^2 + \bar{w}\right)}\right] \end{cases} \tag{5.75}$$

We have also found the second derivative of $Q_b^{\bar{w}}$ at the critical points. At r_0 it is

$$\left.\frac{d^2 Q_b^{\bar{w}}}{dr^2}\right|_{r_0=0} = b^2 + \bar{w} > 0 \tag{5.76}$$

and at $r_{1,2}$ they read

$$\left.\frac{d^2 Q_b^{\bar{w}}}{dr^2}\right|_{r_{1,2}} = \begin{cases} \frac{9m^2}{2}\sqrt{1-\frac{8}{9m^2}\left(b^2+\bar{w}\right)}\left[\sqrt{1-\frac{8}{9m^2}\left(b^2+\bar{w}\right)}-1\right] < 0 & \text{for } r_1 \\ \frac{9m^2}{2}\sqrt{1-\frac{8}{9m^2}\left(b^2+\bar{w}\right)}\left[1+\sqrt{1-\frac{8}{9m^2}\left(b^2+\bar{w}\right)}\right] > 0 & \text{for } r_2 \end{cases} \tag{5.77}$$

From (5.76) and (5.77) we conclude that the polynomial $Q_b^{\bar{w}}$ is structurally stable at all critical points, $r_0 = 0$ and $r_{1,2}$.

The Polynomial $\bar{Q}_b^{\bar{w}=0}$ for the Classical Kerr Spacetime

The stability at $r_{b\pm}$ in the classical case is proved by substituting $\bar{w} = 0$ in (5.71) and (5.73), as follows. Equation (5.71) is reduced to equation (5.41) given by

$$Q_b^{\bar{w}=0}(r) \equiv r^2\left(r^2 - 2mr + b^2\right) = 0 \tag{5.78}$$

where its solutions are $r_0 = 0$, $r_{b\pm} = m \pm \sqrt{m^2 - b^2}$ as mentioned in subsection 5.2.1. The second derivative of $\bar{Q}_b^0$ at each of these solutions is given by

$$\left.\frac{d^2 \bar{Q}_b^0}{dr^2}\right|_{r_0=0} \equiv \left.\frac{dQ_b^0}{dr}\right|_{r_0=0} = 2\left(r_0\right)^2\left(r_0 - m\right) = 0 \tag{5.79}$$

$$\begin{aligned} \left.\frac{d^2 \bar{Q}_b^0}{dr^2}\right|_{r_{b\pm}} &\equiv \left.\frac{dQ_b^0}{dr}\right|_{r_{b\pm}} = 2\left(r_{b\pm}\right)^2\left(r_{b\pm} - m\right) \\ &= \pm 2\left(r_{b\pm}\right)^2\sqrt{m^2 - b^2} \neq 0 \end{aligned} \tag{5.80}$$

The stability implied by equation (5.80) tells us that the values $r_{b\pm}$ change smoothly by the improvement. They are identified with $r_{b\pm}^{\mathrm{I}}$ when $\bar{w} \neq 0$. If $r_{b\pm}$ were not stable, even their existence as real solutions of Q_b^0 could not be garanteed after small changes of $\bar{Q}_b^0$.

On the other hand from (5.79) we deduce that the solution $r_0 = 0$ is unstable. The specific consequence of this instability arising when we switch on $\bar{w}$ smoothly can be deduced by comparing $Q_b^{\bar{w}=0}$ with $Q_b^{\bar{w}}$ at $r_0 = 0$, as follows. From (5.71) and (5.78) we have

$$Q_b^{\bar{w}}(r) = Q_b^{\bar{w}=0}(r) + \bar{w}r^2 + \bar{w}b^2 \tag{5.81}$$

The polynomial $Q_b^{\bar{w}=0}$ is changed by two different terms $\bar{w}r^2$ and $\bar{w}b^2$. We analyse separately their impact on the properties at $r_0 = 0$. First, adding $\bar{w}r^2$ to $Q_b^{\bar{w}=0}$ shifts the coefficient b^2 of r^2 by an amount of $\bar{w}$, since we have

$$\begin{aligned} \left(Q_b^{\bar{w}}\right)^{(1)}(r) &\equiv Q_b^{\bar{w}=0}(r) + \bar{w}r^2 = r^4 - 2mr^3 + \left(b^2 + \bar{w}\right) r^2 \\ &= \left[r^2 - 2mr + \left(b^2 + \bar{w}\right)\right] r^2 \end{aligned} \tag{5.82}$$

This means that the structure of $Q_b^{\bar{w}=0}$ is not changed. The solutions of $\left(Q_b^{\bar{w}}\right)^{(1)} = 0$ are given by

$$\begin{aligned} r_0 &= 0 \\ r_{b\pm}^{(1)} &= m \pm \sqrt{m^2 - (b^2 + \bar{w})} \end{aligned}$$

As a result we only have a smooth shift of $r_{b\pm}$ depending on the value of $\bar{w}$. Furthermore, the number of zeros stays the same for all infinitesimal $\bar{w}$.

As a second instance, adding $\bar{w}b^2$ changes the original structure of $Q_b^{\bar{w}=0}$, since the polynomial $Q_b^{\bar{w}=0}$ has no constant term. The direct consequence of this is that $r_0 = 0$ is not a solution of $Q_b^{\bar{w}=0} = 0$ anymore. This can be considered an effect of the instability of $r_0 = 0$.

Since $\bar{w}$ is positive, no additional roots are expected to appear, so we stay with $r_{b\pm}^{\mathrm{I}}$, which are the result of a smooth shift of $r_{b\pm}$. Other consequences were conceivable if the improvement would lead to a negative value of $\bar{w}$, or if an additional term αr could be added. This is not the case, however, so the final conclusion of this analysis is that *the improvement with $d(r) = r$ causes only a smooth change of the solutions $r_{b\pm}$ for all infinitesimal $\bar{w}$.*

At the extremal case $m = b$ we have from (5.80)

$$\left.\frac{d^2 \bar{Q}_b^{\bar{w}=0}}{dr^2}\right|_{r_{b(extr)}} = 0 \tag{5.83}$$

As a result, the degenerate solution $r_{b(extr)}$ is again unstable.

The stability of the critical points of $Q_b^{\bar{w}=0}$ is proved by substituting $\bar{w} = 0$ in (5.76) and (5.77), as follows

$$\left.\frac{d^2 Q_b^{\bar{w}=0}}{dr^2}\right|_{r_0=0} = b^2 > 0 \tag{5.84}$$

$$\left.\frac{d^2Q_b^{\bar{w}=0}}{dr^2}\right|_{r_{1,2}} = \begin{cases} \frac{9m^2}{2}\sqrt{1-\frac{8b^2}{9m^2}}\left[\sqrt{1-\frac{8b^2}{9m^2}}-1\right] < 0 & \text{for } r_1 \\ \frac{9m^2}{2}\sqrt{1-\frac{8b^2}{9m^2}}\left[1+\sqrt{1-\frac{8b^2}{9m^2}}\right] > 0 & \text{for } r_2 \end{cases} \tag{5.85}$$

The nonzero expressions in (5.84) and (5.85) show that $Q_b^{\bar{w}=0}$ is stable in the vicinity of the critical points of the Kerr spacetime, r_0, r_1 and r_2, since for this case $b \neq 0$ strictly. It important to notice that $r_0 = 0$ is stable as critical point of $Q_b^{\bar{w}=0}$ but unstable as solution of $Q_b^{\bar{w}=0} = 0$.

The Improved Schwarzschild Spacetime

Concerning the structural stability, the improved Schwarzschild and the classical Kerr spacetimes are analogous since the polynomials $\bar{Q}_{b=0}^{\bar{w}}$ and $\bar{Q}_b^{\bar{w}=0}$ are equivalent when we identify b^2 with $\bar{w}$. As a result, for the improved Schwarzschild spacetime we find expressions analogous to (5.80) and (5.83). The stability is thus established at $r_{\mathrm{Sch}\pm}^{\mathrm{I}}$, too, except for the extremal case $m^2 = \bar{w}$, and for $r_0 = 0$.

Concerning the critical points of (5.46), we find expressions analogous to (5.84) and (5.85) when we identify b^2 with $\bar{w}$. As a result, $r_0 = 0$, r_1 and r_2 are also stable.

The Classical Schwarzschild Spacetime

The polynomial for the classical Schwarzschild spacetime ($\bar{w} = 0$ and $b = 0$) is given by:

$$Q_0^0(r) \equiv r^3(r-2m) = 0 \tag{5.86}$$

The solutions of (5.86) are $r_0 = 0$ and $r_{\mathrm{Sch}} = 2m$. Since $d^2\bar{Q}_0^0/dr^2\big|_{r_{\mathrm{Sch}}=2m} = 8m^3 > 0$ we conclude that the Schwarzschild event horizon r_{Sch} is also stable. On the contrary $r_0 = 0$ is unstable as in the previous cases.

This instability is responsible of the increasing number of critical surfaces when we switch on b, to come back to the Kerr spacetime. Comparing $Q_b^{\bar{w}=0}(r)$ and $Q_0^0(r)$ from (5.78) and (5.86) we have

$$Q_b^{\bar{w}=0}(r) = Q_0^0(r) + b^2r^2 \tag{5.87}$$

Equation (5.87) indicates that a small perturbation of the form b^2r^2 added to $Q_0^0(r)$

is enough to change its structure of zeros. In that case we have the three zeros

$$
\begin{aligned}
r_0 &= 0 \\
r_{b+} &= m + \sqrt{m^2 - b^2} \approx 2m \\
r_{b-} &= m - \sqrt{m^2 - b^2} \approx 0
\end{aligned}
\tag{5.88}
$$

Notice that r_{b-} approaches $r_0 = 0$ for $b^2 \to 0$. We say in this case that $r_0 = 0$ is unfolded into two solutions $r_0 = 0$ and r_{b-} when we turn on smoothly $b^2 = 0$ to $b^2 \neq 0$. On the contrary, since $r_{\mathrm{Sch}} = 2m$ is stable, it neither unfolds nor disappears, but changes smoothly from $2m$ to $r_{b+} = m + \sqrt{m^2 - b^2}$.

We find the critical points of $Q_{b=0}^{\bar{w}=0}$ by substituting $b = 0$ and $\bar{w} = 0$ into (5.75). As a result we have

$$
r_{crit}^{\mathrm{Sch}} = \begin{cases} r_0 = 0 \\ r_1 = 0 \\ r_2 = \frac{3m}{2} \end{cases}
\tag{5.89}
$$

Additionally, from (5.76) we find that the second derivative at $r_1 = r_0 = 0$ vanishes:

$$
\left. \frac{d^2 Q_{b=0}^{\bar{w}=0}}{dr^2} \right|_{r_0 = r_1 = 0} = 0
\tag{5.90}
$$

We conclude that we have a degenerate critical point at the origin. The instability is obvious in this case since the number of critical points changes when we move the parameters $\bar{w} = 0$ and $b = 0$ towards non-zero values. The critical point r_2 stays nevertheless stable, as its second derivative is always positive:

$$
\left. \frac{d^2 Q_{b=0}^{\bar{w}=0}}{dr^2} \right|_{r_2} = 9m^2 > 0
\tag{5.91}
$$

Summary

The goal of the present subsection has been to give a basis for the intepretation of the behavior of critical surfaces calculated by applying the approximation $d(r) = r$ in the cutoff identification $\kappa = \xi / d(r)$. Since this identification, entails an intrinsic undefiniteness in the choice of $d(r)$, this undefiniteness is tranported to the problem of finding critical surfaces of the improved Kerr spacetime, as one can notice from the generic character of equation (5.34). Nevertheless, we have chosen the approximation

$d(r) = r$ as the most appropiate due to its simplicity, and also because it entails the relevant qualitative features, that other more complicate expressions for $d(r)$ also reproduce. *The smooth shift of the classical critical Kerr surfaces and the conservation of their number for all infinitesimal* $\bar{w}$ are two remarkable examples of such qualitative features.

The stability of the solutions and critical points of the equation (5.38) coming from this approximation is to be considered only locally, according to its definition presented in this subsection. Radical changes, like the appeareance of degenerate roots or the non-existence of real positive solutions, appear when we abandon the asympotic region and get more near to the Planck or the extremal configuration. These are precisely the regions where our procedure looses its applicability. Examples of this "unreliable behavior" will be presented in the next subsection.

5.2.3 The Radius of Critical Surfaces as a Function of the Mass of the Black Hole

In this subsection we analyse the behavior of the solutions $r_{b\pm}$ and $r^{\mathrm{I}}_{b\pm}$ when we move the mass of the black hole from large values corresponding to macroscopic black holes to values near to the Planck scale. For a fixed b these radii are functions of m only. We start the analysis with the $r_{b\pm}$ of the classical Kerr spacetime.

The Radii $r_{b\pm}$ of the Classical Kerr Spacetime

In Figures 5.4 to 5.7 we show the m-dependence of $r_{\pm}$ and $r_{S\pm}$ for the classical Kerr spacetime, calculated by solving numerically the equations $r^2 - 2mr + b^2 = 0$, for $a = 5$ and different values of the polar angle θ. The extremal black hole is reached at $m = a = 5$. For $m < a$ the radii $r_{\pm}$ are not real-valued, and the surfaces $H_{\pm}$ do not exist. Nevertheless the static limits $S_{\pm}$ exist until the extremal surface is reached at $m = |a \cos\theta|$. At the poles where $\theta = 0,\ \pi$ (Fig. 5.4), $H_{\pm}$ and $S_{\pm}$ coincide. On the other hand, at the equator (Fig. 5.7), the radii of $S_{\pm}$ are $r_{S+} = 2m$ and $r_{S-} = 0$.

A different sort of r vs. m plot will prove interesting and useful later on. It is obtained when we constrain the fraction $\alpha \equiv a/m$ to be fixed. This fraction goes from $\alpha = 0$ for a Schwarzschild configuration with $m \equiv m_{\mathrm{irr}}$, the irreducible mass, to $\alpha = 1$ for the extremal state $m \equiv a$. The constant α defines how "alive" is the black hole in the terms explained in chapter 2. If $\alpha = \mathrm{const}$ we have for the Kerr black hole

$$r_{\pm}\left(m\right)=m\beta_{\pm}\ ,\ r_{S\pm}\left(m\right)=m\gamma_{\pm} \tag{5.92}$$

where $\beta_{\pm}\equiv 1\pm\sqrt{1-\alpha^2}$ and $\gamma_{\pm}\equiv 1\pm\sqrt{1-\alpha^2\cos^2\theta}$ are also constants. The $r_{b\pm}$ vs. m plots reduce, for the classical Kerr black hole, to straight lines crossing the origin, as one can corroborate in figures 5.8 to 5.11.

Since $0\leq\alpha\leq 1$, the radii $r_{b\pm}$ are defined for all $m\geq 0$ in such plots, even for values near to m_{pl}. This is an exclusive property of the functions $r_{b\pm}=m\pm\sqrt{m^2-b^2}$ of the classical Kerr spacetime. In this sense we say that the radii $r_{b\pm}$ are insensitive to the closeness of the Planck scale. The improvement changes the conditions for the existence of critical surfaces, and the respective functions $r_{b\pm}\left(m\right)$ for the improved Kerr spacetime are not defined for all $m\geq 0$, as we shall see in the next paragraphs.

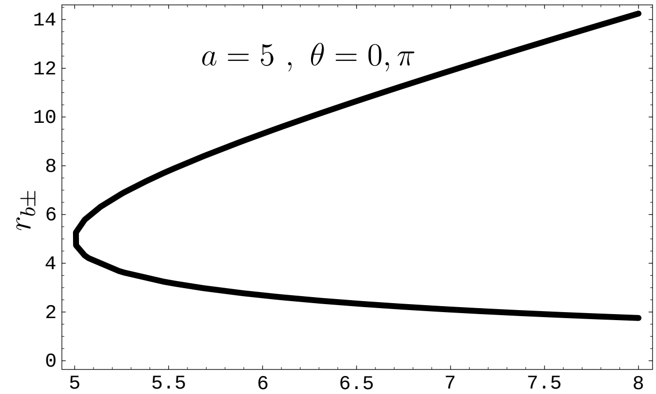


Fig. 5.4.

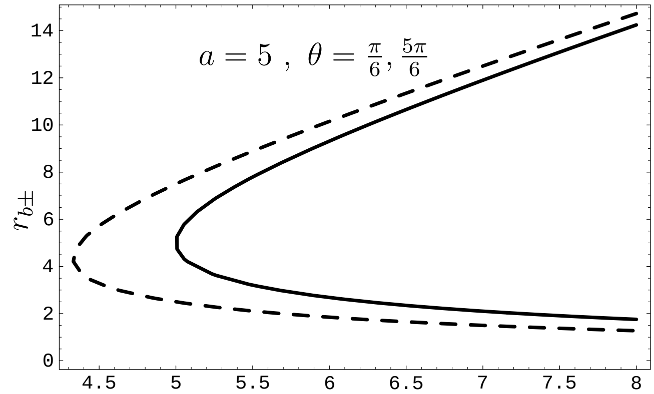


Fig. 5.5.

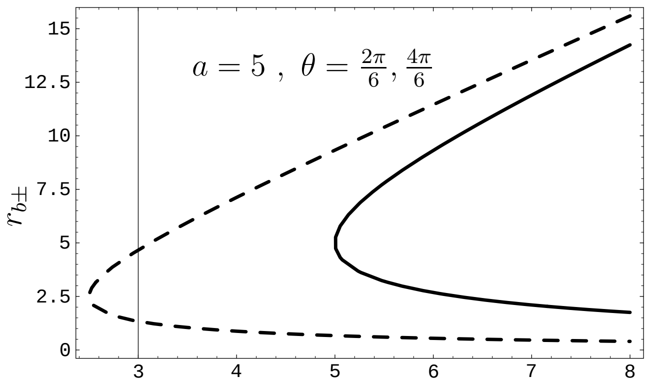


Fig. 5.6.

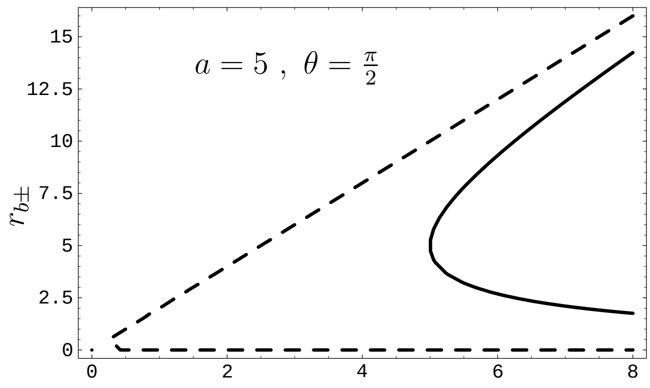


Fig. 5.7.

Figures 5.4 to 5.7 show for the classical Kerr black hole the m-dependence of the radii $r_{b\pm}$ for $a = 5$ and several values of θ. They exist until the extremal configurations are reached: $m = a = 5$ for $r_\pm$, and $m = 5\,|\cos\theta|$ for $r_{S\pm}$. Continuous lines represent $r_\pm$ whereas dashed lines represent $r_{S\pm}$.

Fig. 5.4: For $\theta = 0,\ \pi$, the radii $r_\pm$ and $r_{S\pm}$ coincide.

Figs. 5.5 and 5.6: For arbitrary values of $\theta \neq \frac{\pi}{2}$, $r_\pm$ and $r_{S\pm}$ are split into four different surfaces with $r_{S-} < r_- < r_+ < r_{S+}$.

Fig. 5.7: At the equatorial plane $r_{S-} = 0$ and $r_{S+} = 2m$.

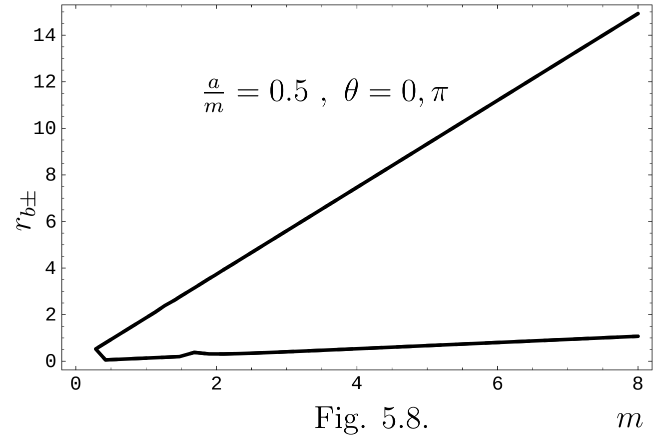


Fig. 5.8.

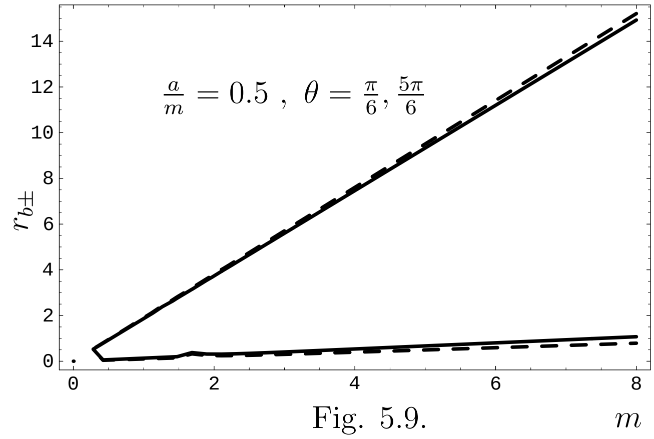


Fig. 5.9.

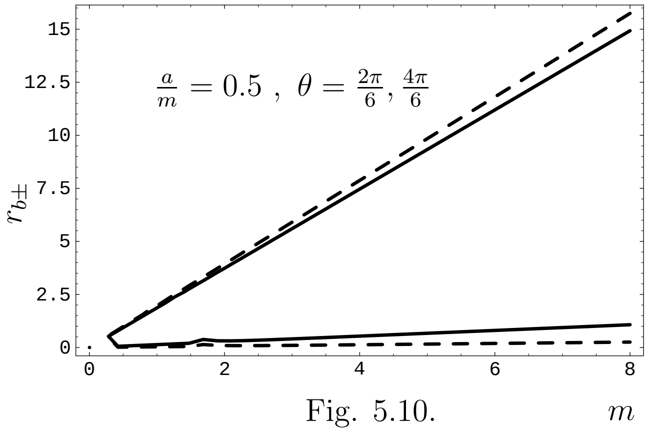


Fig. 5.10.

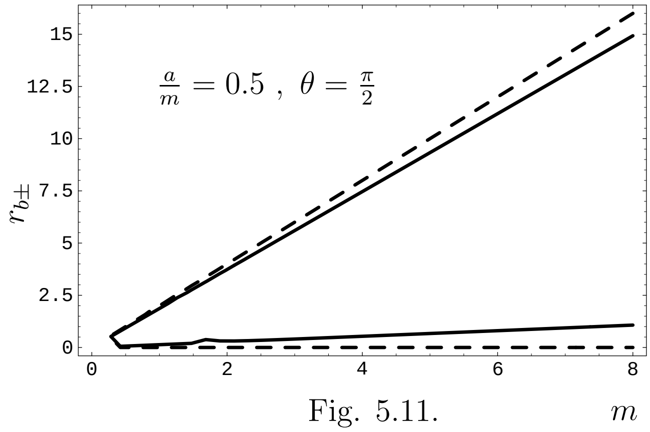


Fig. 5.11.

Figures 5.8 to 5.11 show the linear dependence of the radii $r_{b\pm}(m)$ of the classical Kerr metric $\alpha \equiv \frac{a}{m} = 0.5$ and several values of θ. Continuous lines represent $r_\pm$ and dashed lines represent $r_{S\pm}$.

Fig. 5.8: For $\theta = 0,\ \pi$, the radii $r_\pm$ and $r_{S\pm}$ coincide.
Figs. 5.9 and 5.10: The four surfaces $r_{S\pm}$ and $r_\pm$.
Fig. 5.11: At the equatorial plane, $r_{S-} = 0$ and $r_{S+} = 2m$.

Improved Metric with $d(r) = r$

We already discussed in subsection 5.2.1, the improvement with $d(r) = r$ leads to the equations (5.38) for the radii $r^{\mathrm{I}}_{b\pm}$. We have solved these equations numerically for m as independent variable. In figures 5.12 to 5.15 we display the improved radii $r^{\mathrm{I}}_{b\pm}$ together with the classical ones, $r_{b\pm}$, as functions of m, with a, $\bar{w}$ and θ fixed. The upper and lower branches of the curves correspond to $S^{\mathrm{class,I}}_{+}$, $H^{\mathrm{class,I}}_{+}$ and $S^{\mathrm{class,I}}_{-}$, $H^{\mathrm{class,I}}_{-}$, respectively. The continuous lines represent radii of H surfaces, the dashed ones represent radii of S surfaces.

We observe that for small enough m the radii $r^{\mathrm{class,I}}_{\pm}$ coalesce and then disappear. This coalescence occurs for the extremal black hole with $m = a$ for the classical case, and $m \approx a$ after the improvement. Since the radii $r^{\mathrm{class,I}}_{\pm}$ are θ-independent, we conclude that two spherical surfaces merge into one. For lower masses there exists no event horizon.

A similar coalescence of radii happens for $r^{\mathrm{class,I}}_{S\pm}$ at even lower masses. Since this coalescence occurs for configurations with no event horizon where our method becomes questionable, we don't consider it in the further for analysis. As predicted by the stability of $r_{b\pm}$, the improvement with $d(r) = r$ shifts $r_{b\pm}$ smoothly to the radii $r^{\mathrm{I}}_{b\pm}$ for $m_{\mathrm{pl}} \ll m$. The extremal points are also moved smoothly to higher values of m, as compared to the classical situation.

In figures 5.16 to 5.19 we present plots with the fraction a/m and $\bar{w}$ fixed. A marked separation from the linear dependence of $r_{b\pm}$ is present in the improved radii. This separation becomes important when $m \approx \sqrt{\bar{w}}$ until the extremal configuration is reached. We conclude that the extremal states observed in these plots are a consequence of the improvement, since they don't exist previously and $\bar{w}$ is the only fixed parameter.

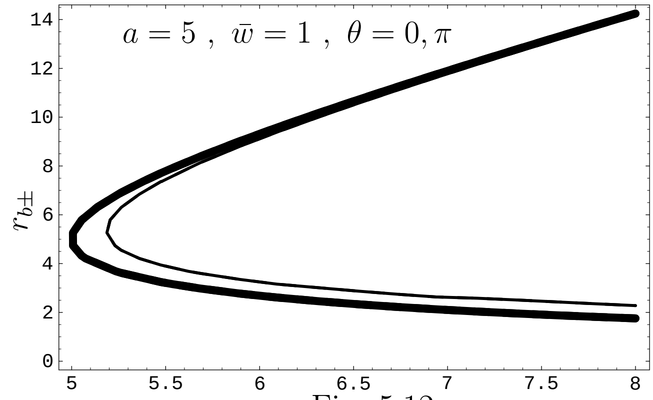


Fig. 5.12.

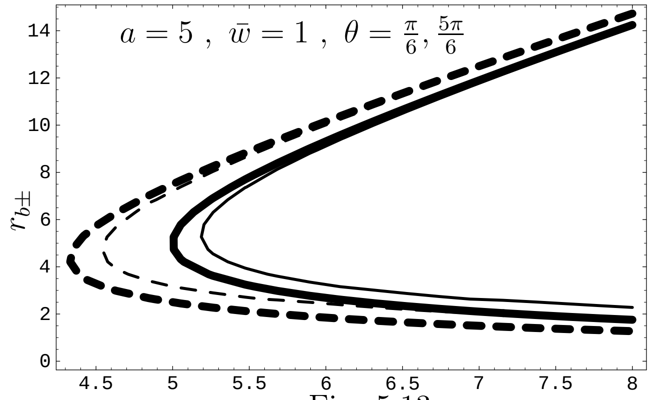


Fig. 5.13.

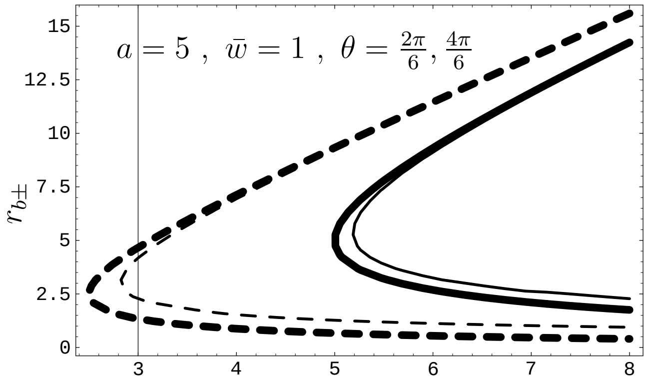


Fig. 5.14.

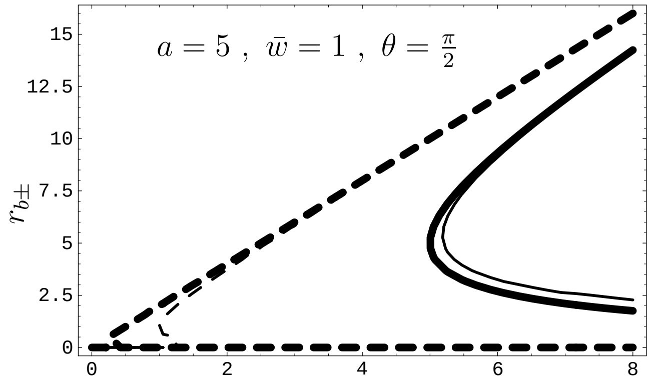


Fig. 5.15.

Figures 5.12 to 5.15 show the m-dependence of the radii $r_{b\pm}$ (thick lines) and $r^{\mathrm{I}}_{b\pm}$ (thin lines) for $\bar{w} = 1$, $a = 5$ and several values of θ. The continuous lines represent $r_{\pm}$ and $r^{\mathrm{I}}_{\pm}$. The dashed lines represent $r_{S\pm}$ and $r^{\mathrm{I}}_{S\pm}$. As predicted by the stability of $r_{b\pm}$, the improvement with $d\,(r) = r$ shifts $r_{b\pm}$ smoothly to the radii $r^{\mathrm{I}}_{b\pm}$ for $m_{\mathrm{pl}} \ll m$. The extremal points are also moved to higher values of m.

Fig. 5.12: For $\theta = 0,\ \pi$, event horizons and static limits coincide.
Figs. 5.13 and 5.14: For arbitrary values of $\theta \neq \frac{\pi}{2}$, there are eight different radii, the four classical $r_{b\pm}$, and the four quantum corrected $r^{\mathrm{I}}_{b\pm}$.
Fig. 5.15: At the equatorial plane, $r_{S-} = 0$ and $r_{S+} = 2m$.

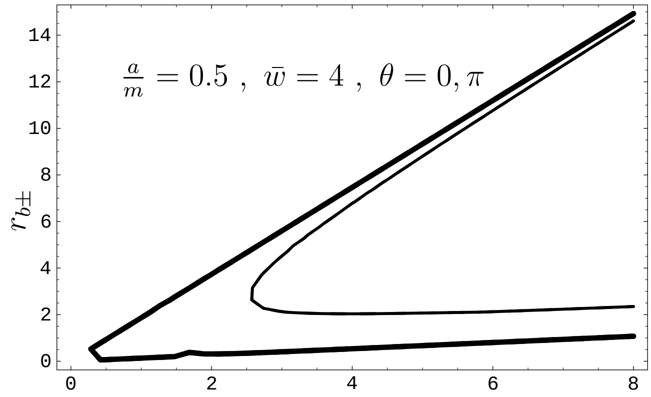


Fig. 5.16.

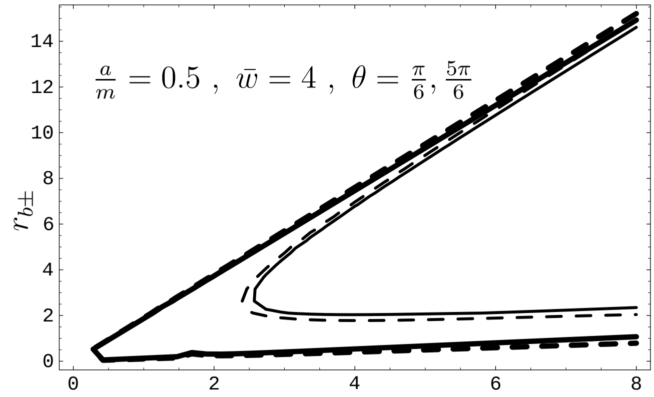


Fig. 5.17.

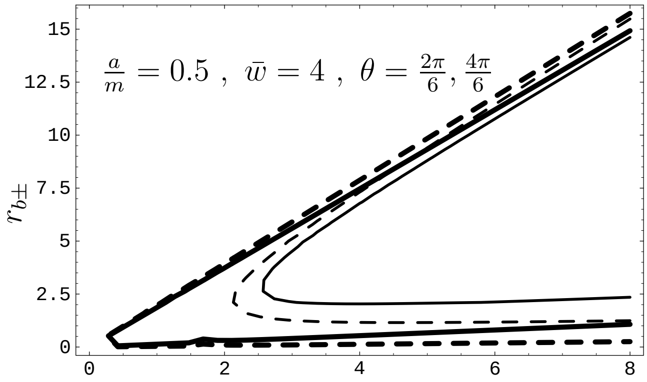


Fig. 5.18.

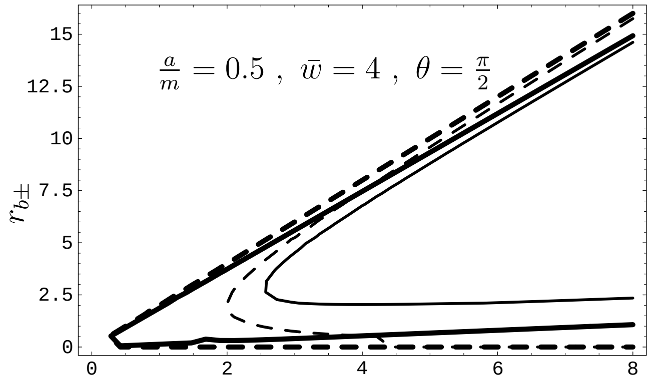


Fig. 5.19.

Figures 5.16 to 5.19 show the m-dependence of the classical radii $r_{b\pm}$ (thick lines) and improved $r^{\mathrm{I}}_{b\pm}$ (thin lines) for fixed $(a/m) = 0.5$, $\bar{w} = 4$ and several values of θ. The continuous lines represent $r_\pm$ and $r^{\mathrm{I}}_\pm$. The dashed lines represent $r_{S\pm}$ and $r^{\mathrm{I}}_{S\pm}$. As predicted by the stability of $r_{b\pm}$, the improvement with $d\,(r) = r$ shifts $r_{b\pm}$ smoothly to the radii $r^{\mathrm{I}}_{b\pm}$ for $m_{\mathrm{pl}} \ll m$. The linear m-dependence of the radii is seen to be violated by the improvement. An extremal configuration of the improved black hole is also visible near to $\sqrt{\bar{w}} = 2$.

Exact $d(r)$ from B-L Coordinates at the Equatorial Plane

We have tested the stability of the radii $r^{\mathrm{I}}_{b\pm}$ obtained from the $d\,(r) = r$ approximation by solving the general equation (5.33) for the critical surfaces with the exact form (3.13) of $d\,(r)$ at the equator. Since expression (3.13) is a complicated composite function, we have addressed the problem numerically. We developed a code in C language that calculates the solutions of (5.33) with $d\,(r)$ in (3.13) by means of the bisection method. Figures 5.20 to 5.23 present plots that were performed with data produced by the C program. These plots show the range from $m = a$ to $m = 20$ of the improved radii with the exact $d\,(r)$ at the equator, for $\bar{w} = 5$, and a increasing from $a = 0$ to $a = 15$. What we see is a rather similar m-dependence as in the approximation with $d\,(r) = r$. This confirms the stability of this approximation which we discussed above, and it justified our use of $d\,(r) = r$ in many of the later investigations.

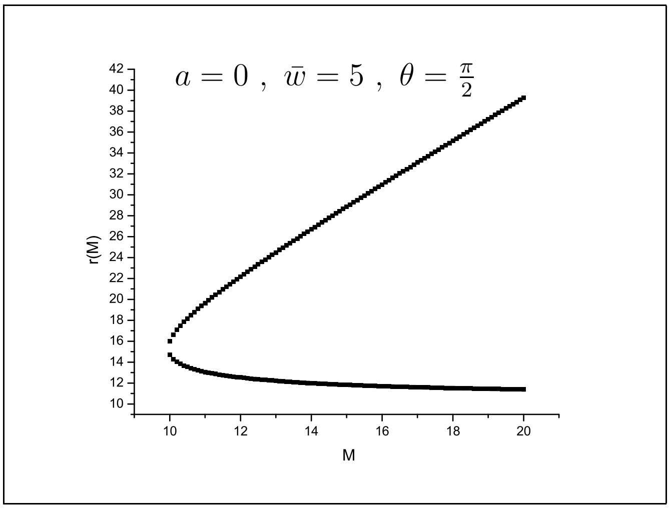


Fig. 5.20.

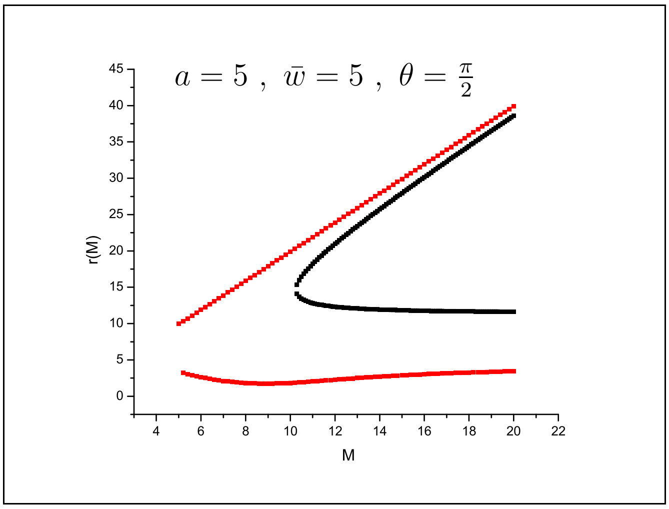


Fig. 5.21.

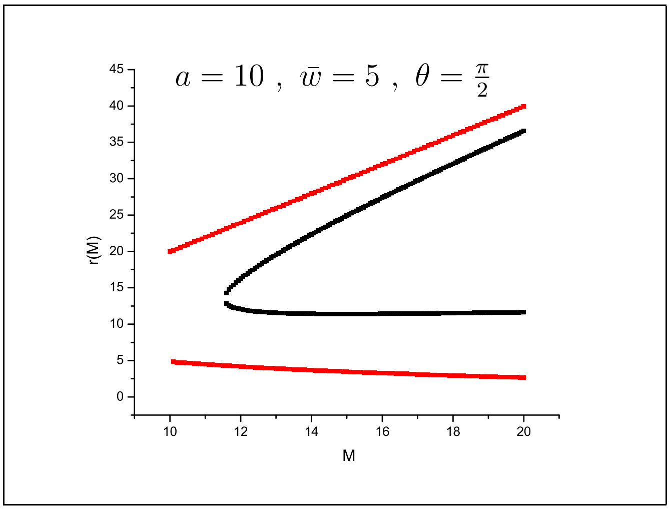


Fig. 5.22.

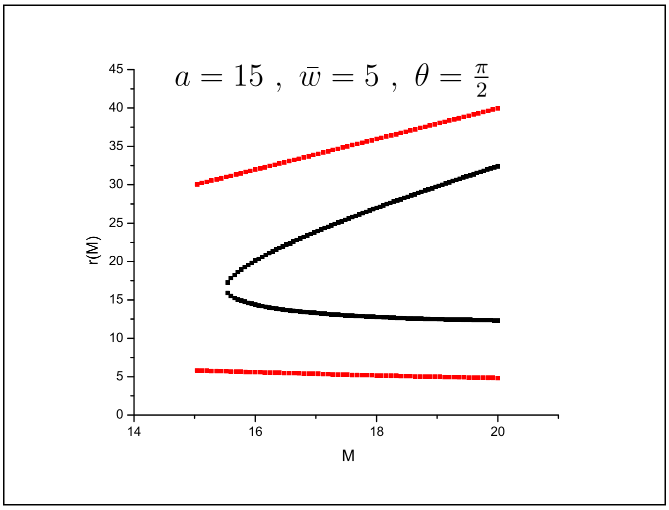


Fig. 5.23.

Figures 5.20 to 5.23 show the m-dependence of the improved radii $r^{\mathrm{I}}_{b\pm}$ obtained from the exact $d\,(r)$ given in (3.13) to (3.16) at $\theta = \frac{\pi}{2}$, for $\bar{w} = 5$ and several values of a. The gray curves are the improved static limits $r^{\mathrm{I}}_{S\pm}$, the black ones are the improved event horizons $r^{\mathrm{I}}_{\pm}$.

Fig. 5.20: For $a = 0$, H and S coincide.
An extremal configuration is visible in the improved Schwarzschild spacetime.
Figs. 5.21 to 5.23: The range from $m = a$ to $m = 20$ is shown for the improved H's and S's.
The pattern of curves is similar as in the $d\,(r) = r$ approximation.

5.2.4 2-dimensional Plots

In this subsection we explore the 2-dimensional shape of the critical surfaces improved with $d(r) = r$. The plots we present are build after solving equations (5.38) numerically, iterating θ from 0 to 2π. They represent cross sections through the black hole along a $\phi =$ constant plane.

Figures 5.24 and 5.25 show the critical surfaces of the classical and improved Kerr spacetimes, for $m = 6$, $a = 5$ and $\bar{w} = 4$. The continuous curves are event horizons, whereas the dashed ones are static limit. Figure 5.26 presents both the classical and improved surfaces combined in one plot, for an equal set of parameters. Thick curves belong to the classical black hole, thin lines to the improved one. The smooth displacement of the classical surfaces due to the improvement is noteworthy. Notice also that the intersection at the poles of event horizons and static limits, characteristic of the classical Kerr surfaces, is still present in the improved counterpart as we have proved in subsection 5.2.1.

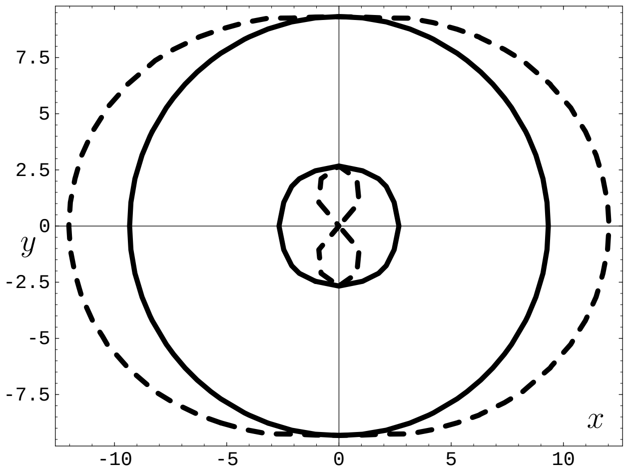


Fig. 5.24.
Classical critical surfaces for $m = 6$, $a = 5$, $\bar{w} = 0$.

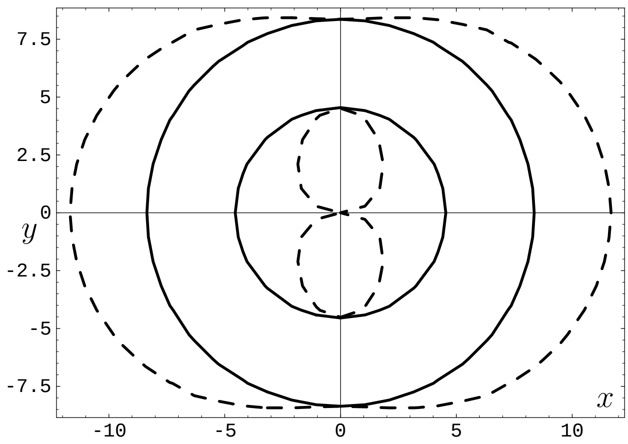


Fig. 5.25.

Improved critical surfaces for $m = 6$, $a = 5$, $\bar{w} = 4$.

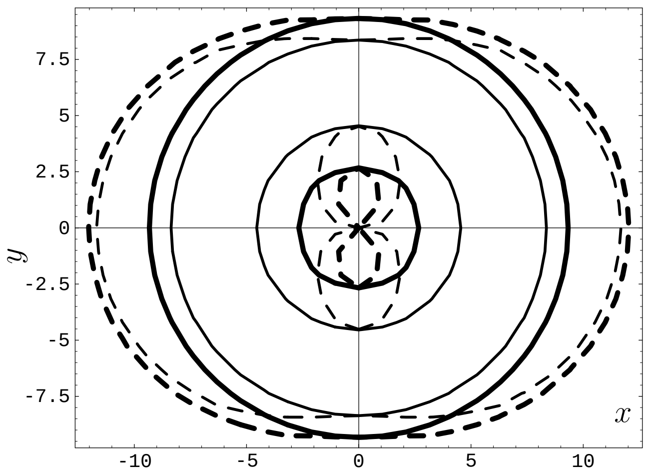


Fig. 5.26.

Classical plus improved critical surfaces for $m = 6$, $a = 5$, $\bar{w} = 4$.

5.3 Discussion

After the calculation and the analysis of the critical surfaces in this chapter, we come to the following two main conclusions:

- The improvement with the running Newton's constant leads, at least in the asymptotic regime where $d\,(r) \approx r$, only to a smooth displacement of the classical critical surfaces. Their number remains unchanged. This is a consequence of the local stability of the solutions $r_{b\pm}$, established in section 5.2.

- A new type of extremal configuration related to the parameter $\bar{w}$ appears as a consequence of the improvement. This configuration is reached at the Planck scale, with $m \approx \sqrt{\bar{w}}$, and it has to be distinguished from the extremal state reached at $m \approx a$ that is related to the classical extremal black hole. This separation of extremal states has been verified analytically with the $d\,(r) = r$ approximation and numerically by performing the r vs. m plots with a/m fixed. The general case is described by the *quantum extremality condition* given in (5.62) with (5.63). It is represented graphically in Fig. 5.3.

Chapter 6

Penrose Process for Energy Extraction

6.1 Introduction

In this chapter we concentrate first on finding the dynamical conditions for a test particle to reach the negative energy state which is needed for the Penrose process to be carried out. Then we determine the allowed negative energy kinematical region which results from bounding the rotation of massive test particles by the rotation of clockwise and counterclockwise light rays. The main consequence of a running Newton constant consists in a dependence of the topology of this allowed region on the mass of the black hole, in contrast to the monotonous behavior for the bare Kerr spacetime. This dependence is shown in a set of correlated graphics at the end of the chapter.

6.2 Conservation of Energy of Test Particles and Negative Energy Constraint

Since t is a cyclic variable for a test particle in the improved Kerr metric, the canonical momentum p_t is conserved. Correspondingly there exists a Killing vector t^μ that leads to the following expression for the conserved energy of a test particle moving around an improved Kerr black hole:

$$E \equiv -p_\mu t^\mu = -p_t \tag{6.1}$$

As mentioned in chapter 4. Using (6.1) we can write E explicitly using the B-L coordinates:

$$\begin{aligned} E &= -p_t = -m g_{\mu t}\frac{dx^\mu}{d\tau} \\ &= -m\left[g_{tt}\frac{dt}{d\tau} + g_{\varphi t}\frac{d\varphi}{d\tau}\right] \\ &= -m\left[-\left(1-\frac{2MG(r)\,r}{\rho^2}\right)\frac{dt}{d\tau} - \left(\frac{2MG(r)\,ra\sin^2\theta}{\rho^2}\right)\frac{d\varphi}{d\tau}\right] \end{aligned}$$

therefore we can write

$$\frac{E}{m} = \left(1-\frac{2MG(r)\,r}{\rho^2}\right)\frac{dt}{d\tau} + \left(\frac{2MG(r)\,ra\sin^2\theta}{\rho^2}\right)\frac{d\varphi}{d\tau} \tag{6.2}$$

where m is the mass of the test particle. Factorizing the differentials we obtain

$$\frac{E}{m} = \left[\left(1-\frac{2MG(r)\,r}{\rho^2}\right) + \left(\frac{2MG(r)\,ra\sin^2\theta}{\rho^2}\right)\Omega\right]\frac{dt}{d\tau} \tag{6.3}$$

where Ω is defined as:

$$\Omega \equiv \frac{d\varphi}{dt} \tag{6.4}$$

This is a general equation for the conserved energy of a test particle moving in the Kerr spacetime; it depends, among other variables, on the rotation frequency of the particle, Ω.

The result in (6.3) leads to a constraint on Ω if we want to reach zero or negative energy states for the test particle. This constraint,

$$E \leq 0\ ,$$

reads in explicit form

$$\left[\left(1-\frac{2MG(r)\,r}{\rho^2}\right) + \left(\frac{2MG(r)\,ra\sin^2\theta}{\rho^2}\right)\Omega\right] \leq 0$$

Since $g_{\varphi t} = \frac{2MG(r)ra\sin^2\theta}{\rho^2} > 0\ \forall\, r,\theta$ we can write:

$$\Omega \leq \Omega_0 \equiv -\frac{g_{tt}}{g_{\varphi t}} = \frac{2MG(r)\,r - \rho^2}{2MG(r)\,ra\sin^2\theta} \tag{6.5}$$

Similarly we have for the tangent "bookeeper velocity" [53]

$$\left.\left(\frac{ds}{dt}\right)\right|_{(r,\theta,t)=const} = g_{\varphi\varphi}\left(\frac{d\varphi}{dt}\right)^2 = R^2(r,\theta)\,\Omega^2$$

we define $v_{\tan}$ in terms of the reduced circumference $R(r,\theta)$ by

$$v_{\tan} \equiv R(r,\theta)\,\frac{d\varphi}{dt}\ ,\ R(r,\theta) > 0\ \ \forall\, r,\theta \tag{6.6}$$

Then the constraint (6.5) can equivalently be written as

$$v_{\tan} \le v_0 \quad\equiv\quad R(r,\theta)\left(\frac{2MG(r)\,r-\rho^2}{2MG(r)\,ra\sin^2\theta}\right) \tag{6.7}$$

with

$$R(r,\theta) = \sqrt{g_{\varphi\varphi}} = \sqrt{\frac{\Sigma_I \sin^2\theta}{\rho^2}} \tag{6.8}$$

The eqs. (6.5) and (6.7) are equivalent constraints, formulated in terms of angular frequencies or bookeeper velocities, respectively. In our following analysis we prefer (6.7) since we stay with (v,r) as the usual kinematical quantities for a description of the motion of point particles.

For a first discussion we restrict our analysis to the equatorial plane which simplifies the expressions and retains the behavior we are trying to describe. Then the reduced circumference $R\left(r,\frac{\pi}{2}\right)$ can be simplified to using

$$\Sigma_I\left(r,\frac{\pi}{2}\right) \quad=\quad \left(r^2+a^2\right)^2 - a^2\Delta = r^4 + r^2a^2 + 2Ma^2G(r)\,r$$

It assumes the form

$$R\left(r,\frac{\pi}{2}\right) \quad=\quad \sqrt{\frac{\Sigma_I\left(r,\frac{\pi}{2}\right)}{r^2}} = \sqrt{r^2+a^2+\frac{2Ma^2G(r)}{r}}$$

Hence the condition (6.7) is reduced to

$$v_{\tan} \le \frac{1}{a}\sqrt{r^2+a^2+\frac{2Ma^2G(r)}{r}}\left(1-\frac{r}{2MG(r)}\right) \equiv v_0^{eq} \tag{6.9}$$

where v_0^{eq} is defined to be the zero energy bookkeeper tangential velocity at the equator.

6.3 Negative Energy Kinematical Regions

As mentioned at the beginning of this chapter, the rotational motion of a material test particle is bounded by the rotation of light rays. The constraint given by (6.9) applies only to particles in the allowed kinematical region (no tachyons!). Its dependence on the radial coordinate r can be visualized by plotting the bookeeper tangential velocities of zero energy test particles together with the tangential velocities of light rays, in order to determine graphically the above mentioned allowed regions. For the light rays we can use the result in (4.36) in order to write:

$$v_{\pm}^{light} = R\left(r,\theta\right)\Omega_{\pm} = R\left(r,\theta\right)\left(\omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}}\right) \tag{6.10}$$

Combining $v_{\pm}^{\mathrm{light}}$ from (6.10) with v_0^{eq} (6.9) for the tangential velocity at the equator we obtain the plots shown in figures 6.1, 6.2 and 6.3. We have included also the dragging velocity $v_{\mathrm{dragging}} = R\left(r,\theta\right)\omega$. The graphics employ dimensionless quantities. For the respective definitions, see appendix A.

From the behavior of the functions displayed in Figs. (6.1), (6.2) and (6.3) we can conclude that there are negative energy regions which are characterized by two intersection points where the tangential velocities meet. Next we show that these points occur precisely at the static limits r_{S+}^{I} and the event horizon r_{+}^{I}. We start with the event horizon intersection point by looking at the definitions (4.25), (4.36) and (6.5) at r_{+}^{I}:

$$\omega|_{r_+^{\mathrm{I}}} = -\left.\frac{g_{\varphi t}}{g_{\varphi\varphi}}\right|_{r_+^{\mathrm{I}}} \ , \ \ \Omega_0|_{r_+^{\mathrm{I}}} = -\left.\frac{g_{tt}}{g_{\varphi t}}\right|_{r_+^{\mathrm{I}}} \ , \ \ \Omega_{\pm}|_{r_+^{\mathrm{I}}} = \left.\sqrt{\frac{g_{tt}}{g_{\varphi\varphi}}}\right|_{r_+^{\mathrm{I}}} \tag{6.11}$$

We recall also that the radius of the horizon r_{+}^{I} fulfills equation (5.27) [1]:

$$\left(r_+^{\mathrm{I}}\right)^2 + a^2 - 2MG\left(r_+^{\mathrm{I}}\right)r_+^{\mathrm{I}} \ \ = \ \ 0 \tag{6.12}$$

[1]We use already the dimensionless quantities defined in appendix A. For simplicity we supress the tildes $\sim$ used on this appendix.

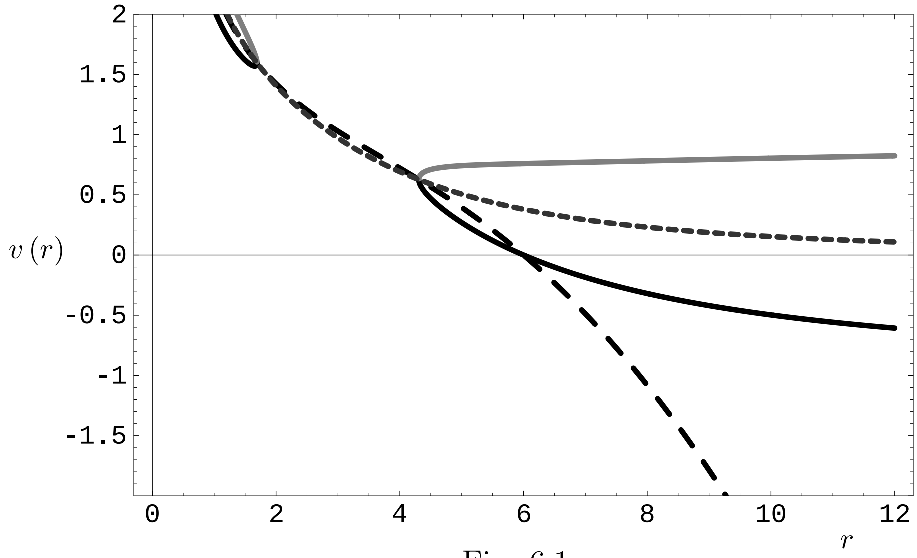


Fig. 6.1.

The $(r, v(r))$ kinematical configurations for test particles rotating around a Kerr black hole, inside the ergosphere and with negative energy, can be seen as the area in the $v(r)$ vs. r plot which is bounded by the $v_-(r)$ curve for the tangent velocity of counterrotating light rays and the $v_0^{eq}(r)$ curve for zero energy rotating test particles. The case shown corresponds to the classical Kerr metric at the equatorial plane.

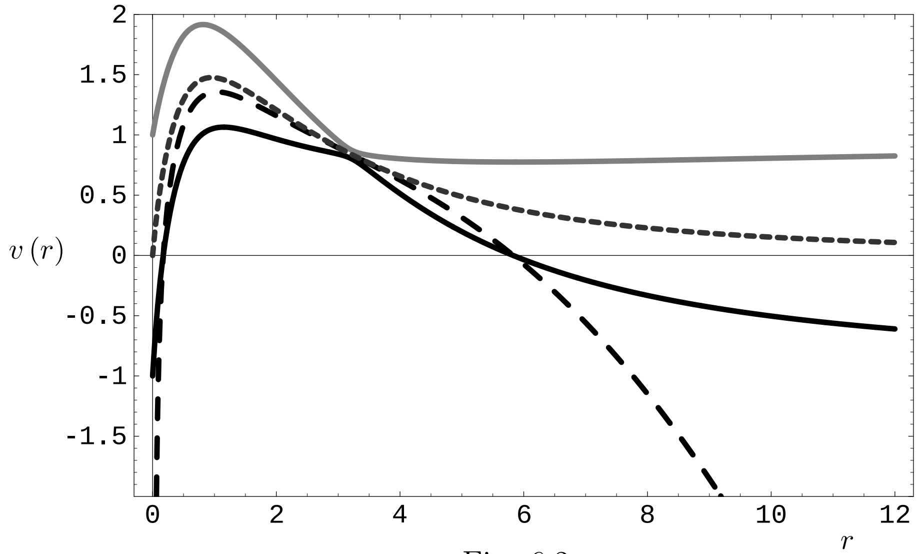


Fig. 6.2.

A similar plot as 6.1, but for the improved Kerr spacetime, using $d(r) = r$ and also restricted to the equatorial plane. For $r < r_+$ all curves differ substantially from their classical counterparts, but there still does exist a region of allowed negative energy states (r, θ).

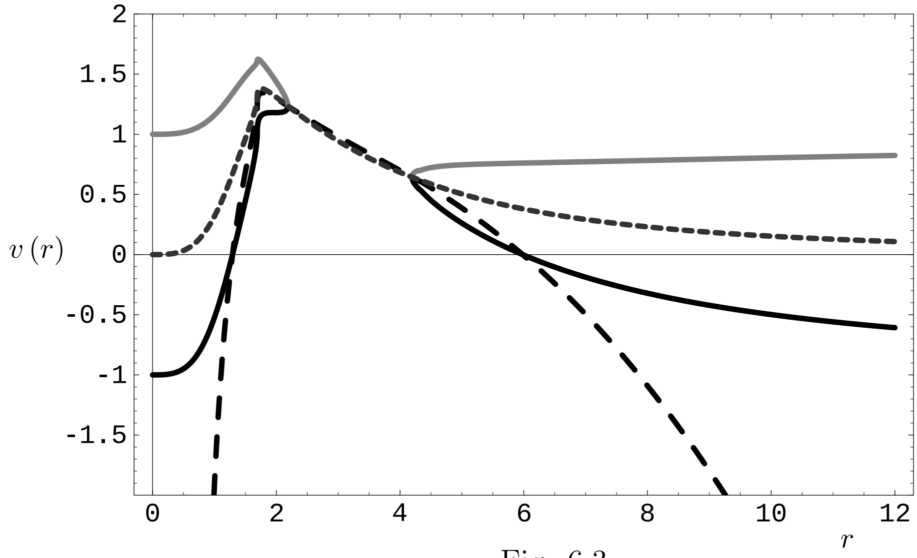


Fig. 6.3.

In this figure a more precise improvement is implemented. We use $d(r)$ coming from integrals (3.14), (3.15) and (3.16) calculated at the equatorial plane. We see that the shape of the curves change significantly, especially inside the external event horizon, but the negative energy region is still present for the chosen values $M = 3$ and $a = 2.7$.

The relevant components $g_{\mu\nu}$ at $r_+^{\rm I}$ in the B-L representation are given by:

$$g_{tt}|_{r_+^{\rm I}} = -\left(1 - \frac{2MG(r)\,r}{\rho^2}\right)\Bigg|_{r_+^{\rm I}} = \frac{a^2 \sin^2\theta}{\rho^2|_{r_+^{\rm I}}} \tag{6.13}$$

$$g_{\varphi\varphi}|_{r_+^{\rm I}} = \frac{\Sigma_I \sin^2\theta}{\rho^2}\Bigg|_{r_+^{\rm I}} = \frac{\left[\left(r_+^{\rm I}\right)^2 + a^2\right]^2 \sin^2\theta}{\rho^2|_{r_+^{\rm I}}} \tag{6.14}$$

$$g_{\varphi t}|_{r_+^{\rm I}} = -\frac{2MG\left(r_+^{\rm I}\right) r_+^{\rm I} a \sin^2\theta}{\rho^2|_{r_+^{\rm I}}} \tag{6.15}$$

Substituting (6.13), (6.14) and (6.15) in (6.11) we get explicitly

$$\begin{aligned}
\omega|_{r_+^{\rm I}} &= \frac{2MG\left(r_+^{\rm I}\right) r_+^{\rm I} a}{\left[\left(r_+^{\rm I}\right)^2 + a^2\right]^2} = \frac{2MG\left(r_+^{\rm I}\right) r_+^{\rm I} a}{\left[2MG\left(r_+^{\rm I}\right) r_+^{\rm I}\right]^2} = \frac{a}{2MG\left(r_+^{\rm I}\right) r_+^{\rm I}} \\
\Omega_\pm|_{r_+^{\rm I}} &= \sqrt{\frac{a^2}{\left[\left(r_+^{\rm I}\right)^2 + a^2\right]^2}} = \frac{a}{\left(r_+^{\rm I}\right)^2 + a^2} = \frac{a}{2MG\left(r_+^{\rm I}\right) r_+^{\rm I}} \\
\Omega_0|_{r_+^{\rm I}} &= \frac{a}{2MG\left(r_+^{\rm I}\right) r_+^{\rm I}}
\end{aligned}$$

Then all the frequencies in (6.11) are equal at r_+^{I}:

$$\omega|_{r_+^{\mathrm{I}}} = \Omega_\pm|_{r_+^{\mathrm{I}}} = \Omega_0|_{r_+^{\mathrm{I}}} = \frac{a}{2MG\left(r_+^{\mathrm{I}}\right) r_+^{\mathrm{I}}} = \frac{a}{\left(r_+^{\mathrm{I}}\right)^2 + a^2} \tag{6.16}$$

By multiplying with $R\left(r_+^{\mathrm{I}}, \theta\right)$, the equalities in (6.16) for the rotation frequencies imply directly the equality for the tangential velocities at r_+^{I}. In this way we have shown that *one of the intersection points which characterizes the negative energy region is an event horizon.*

For showing that the other intersecting point lies at the static limit r_{S+}^{I} we proceed as follows: In this case we concentrate only on the light and zero energy frequencies Ω_- and Ω_0 respectively, since ω and Ω_+, as seen in the figures 6.1 to 6.3, do not intersect them at r_{S+}^{I}, and we use the static limit condition (5.11) given by:

$$\left(r_{S+}^{\mathrm{I}}\right)^2 + a^2\cos^2\theta - 2MG\left(r_{S+}^{\mathrm{I}}\right) r_{S+}^{\mathrm{I}} = 0 \tag{6.17}$$

Then we obtain the metric components

$$g_{tt}|_{r_{S+}^{\mathrm{I}}} = -\left(1 - \frac{2MG(r)\,r}{\rho^2}\right)\bigg|_{r_{S+}^{\mathrm{I}}} = -\left(\frac{\left(r_{S+}^{\mathrm{I}}\right)^2 + a^2\cos^2\theta - 2MG\left(r_{S+}^{\mathrm{I}}\right) r_{S+}^{\mathrm{I}}}{\rho^2|_{r_{S+}^{\mathrm{I}}}}\right) = 0 \tag{6.18}$$

$$g_{\varphi\varphi}|_{r_{S+}^{\mathrm{I}}} = \frac{\Sigma_I \sin^2\theta}{\rho^2}\bigg|_{r_{S+}^{\mathrm{I}}} = \frac{\left\{\left[\left(r_{S+}^{\mathrm{I}}\right)^2 + a^2\right]^2 - a^2\,\Delta|_{r_{S+}^{\mathrm{I}}}\sin^2\theta\right\}\sin^2\theta}{\rho^2|_{r_{S+}^{\mathrm{I}}}} \tag{6.19}$$

$$g_{\varphi t}|_{r_{S+}^{\mathrm{I}}} = -\frac{2MG\left(r_{S+}^{\mathrm{I}}\right) r_{S+}^{\mathrm{I}}\, a\sin^2\theta}{\rho^2|_{r_{S+}^{\mathrm{I}}}} \tag{6.20}$$

As a consequence we get

$$\begin{aligned}
\omega|_{r_{S+}^{\mathrm{I}}} &= -\frac{g_{\varphi t}}{g_{\varphi\varphi}}\bigg|_{r_{S+}^{\mathrm{I}}} = -\frac{2MG\left(r_{S+}^{\mathrm{I}}\right) r_{S+}^{\mathrm{I}}\, a}{\Sigma_I|_{r_{S+}^{\mathrm{I}}}} = -\frac{2MG\left(r_{S+}^{\mathrm{I}}\right) r_{S+}^{\mathrm{I}}\, a}{\left[\left(r_{S+}^{\mathrm{I}}\right)^2 + a^2\right]^2 - a^2\,\Delta|_{r_{S+}^{\mathrm{I}}}\sin^2\theta} \\
\Omega_0|_{r_{S+}^{\mathrm{I}}} &= -\frac{g_{tt}}{g_{\varphi t}}\bigg|_{r_{S+}^{\mathrm{I}}} = 0 \\
\Omega_\pm|_{r_{S+}^{\mathrm{I}}} &= \omega|_{r_{S+}^{\mathrm{I}}} \pm \sqrt{\left(\omega|_{r_{S+}^{\mathrm{I}}}\right)^2 - \frac{g_{tt}}{g_{\varphi\varphi}}\bigg|_{r_{S+}^{\mathrm{I}}}} = \omega|_{r_{S+}^{\mathrm{I}}} \pm \omega|_{r_{S+}^{\mathrm{I}}}
\end{aligned}$$

These relations entail for Ω_- and Ω_+

$$\Omega_-|_{r_{S+}^{\mathrm{I}}} = 0\,,\ \Omega_+|_{r_{S+}^{\mathrm{I}}} = 2\,\omega|_{r_{S+}^{\mathrm{I}}}$$

Moreover, what is the relevant statement in the present context,

$$\Omega_0|_{r^{\mathrm{I}}_{S+}} = \Omega_-|_{r^{\mathrm{I}}_{S+}} = 0 \tag{6.21}$$

This is the final result we wanted to show, namely that *the zero energy frequency equals the frequency for counterrotating light rays at the static limit surface and they are zero there.* Again by multiplying with the reduced circumference R this result also establishes the equality for the respective tangential velocities.

The results in (6.16) and (6.21) tell us that the two intersection points that define the negative energy region, occur at the event horizon r^{I}_+ an the static limit r^{I}_{S+}, respectively.

6.4 (M, a)-Dependence of the Negative Energy Regions

In chapter 5 we saw the non-linear behavior of the $r^{\mathrm{I}}_\pm$ and $r^{\mathrm{I}}_{S\pm}$ vs. $\tilde{m}$ figures (with $\frac{a}{\tilde{m}}$ fixed) for the improved Kerr black hole in contrast to the linear shape for the classical Kerr metric. Since the negative energy region is directly related to r^{I}_+ and r^{I}_{S+}, it is to be expected that this region will also depend on $\tilde{m}$ similarly as r^{I}_+ and r^{I}_{S+} do, for each case, classical and improved. It can be concluded that in fact this is the case by analysing a set of representative figures that shows the m-dependence of this region. We present this set of figures at the end this section.

What we observe for the classical Kerr metric is that the negative energy region changes its size with m, but not its shape. This is due to the linear m-dependence of $r_\pm$ and $r_{S\pm}$. For the improved black hole the shape changes smoothly with m, the difference becoming dramatic compared to the classical case when $m \approx m_{\mathrm{Pl}}$.

The set of figures is divided in two sets of correlated plots for $\theta = \frac{\pi}{2}$ (Equator): the first one for the classical Kerr metric (Figures 6.4 to 6.7) and the second one for the $d(r) = r$ improvement (Figures 6.8 to 6.19). The respective left hand side and right hand side plots are correlated in the sense that they were computed with the same $a/\tilde{m}$ ratio, and that they show the same r^{I}_+ and r^{I}_{S+} values. By this arrangement we can see directly the relation between the shapes of the r^{I}_+ and r^{I}_{S+} vs. $\tilde{m}$ graphics and those ones of the negative energy regions. We have chosen $a/\tilde{m} = 0.9$ in all plots so that the negative energy regions are as visible as possible. The plots related to the improved Kerr spacetime were obtained with $\bar{w} = 1$.

The set of figures 6.4 to 6.7 shows clearly the above mentioned “invariance” with respect to m of the shape of negative energy regions in the classical Kerr spacetime. That means, no matter how small (or big) $\tilde{m}$ is, energy extraction will always be possible, provided $m \geq a$.

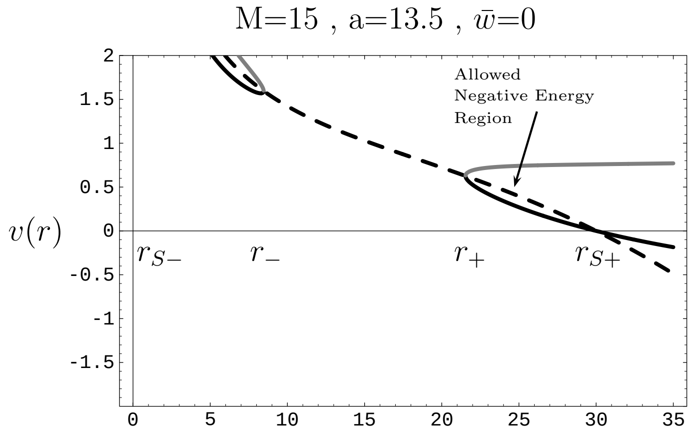
As for the improved spacetime, Fig. 6.8 shows the region of negative energy for $M = 5$, $a = 4.5$. Fig. 6.9 presents the respective set of radii $r_{\pm}^{\mathrm{I}}$ and $r_{S\pm}^{\mathrm{I}}$ connected by a dashed vertical line at $M = 5$. The distribution of these radii can also be observed in the horizontal axis of Fig. 6.8. Since we are still far away from $M = 1$ the shape of the “improved” negative energy region is not too much different from the classical one (compare, for example, with Fig 6.4). In figures 6.10 and 6.11 we have changed M from 5 to 4. We still have a similar distribution of radii but they are closer to each other. The shape of the negative energy region is almost unchanged. Figures 6.8 and 6.10 show an internal negative energy region bounded by r_{S-}^{I} and r_{-}^{I}. Since the posibility of extraction of energy relies on the existence of stationary states with negative energy outside r_{+}^{I}, the internal region cannot be considered as physically relevant.

Figures 6.12 to 6.19 were obtained for the regime $\tilde{m} \approx 1$ (the Planck scale). Drastic changes in the shape of negative energy regions are visible. Since the reliability of our method is questionable in this regime, any conclusions about the mentioned regions have to be considered with care. However we analyse these figures since they show interesting features.

In figures 6.12 and 6.13 the quantum extremal black hole with $M = M_{\mathrm{cr}}$ and $r_{-}^{\mathrm{I}} = r_{+}^{\mathrm{I}} = r_{\mathrm{extr}}^{\mathrm{I}}$ has been reached. The internal and external negative energy regions touch at $r_{\mathrm{extr}}^{\mathrm{I}}$.

Figures 6.14 and 6.15 show a hypothetical configuration for $M < M_{\mathrm{cr}}$ with two static limits $S_{\pm}^{\mathrm{I}}$ and no event horizon. The internal and external negative energy regions have been merged into just one. The existence of this region is determined by the static limit radii $r_{S_-}^{\mathrm{I}}$ and $r_{S_+}^{\mathrm{I}}$. In this case there exists an ergosphere from where energy can be extracted, but no horizons.

Figures 6.16 to 6.19 show configurations where no extraction of energy is allowed. The extremal static limit is shown in figure 6.17. For this configuration the negative energy region is reduced to zero. At the Planck mass with $M = 1$ (figures 6.10 and 6.19) the zero-energy curve is not even visible.



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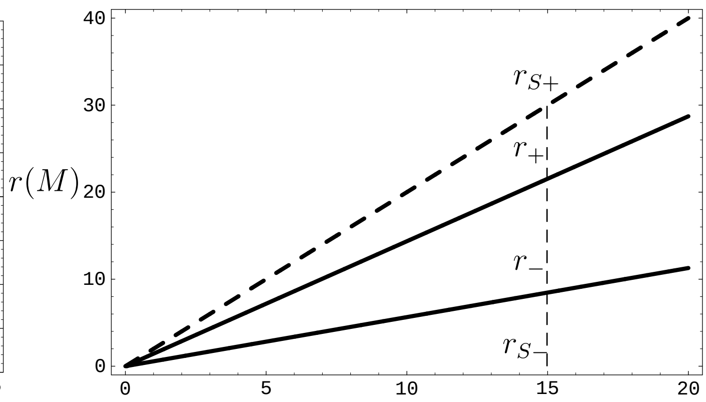

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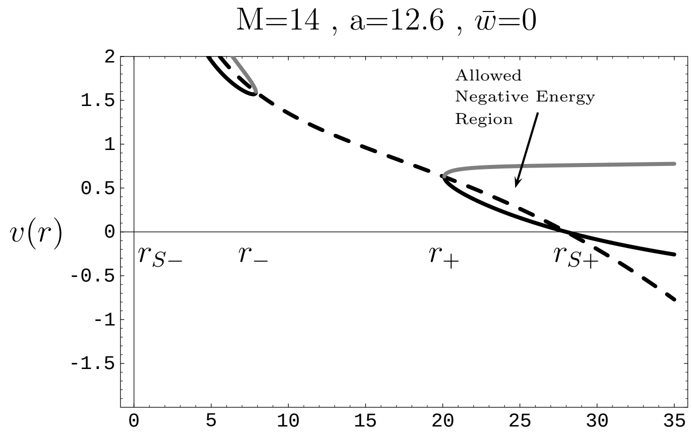

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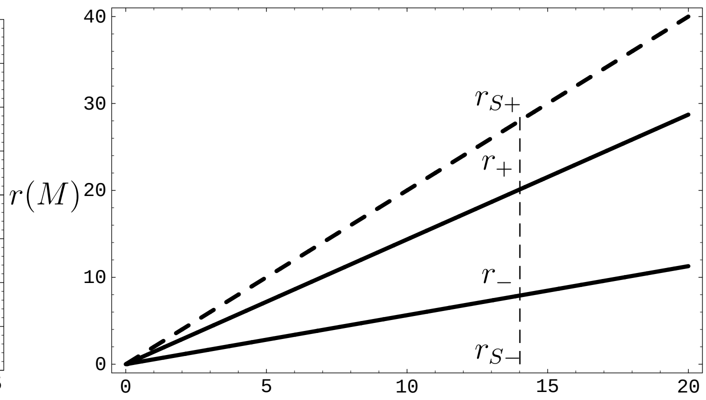

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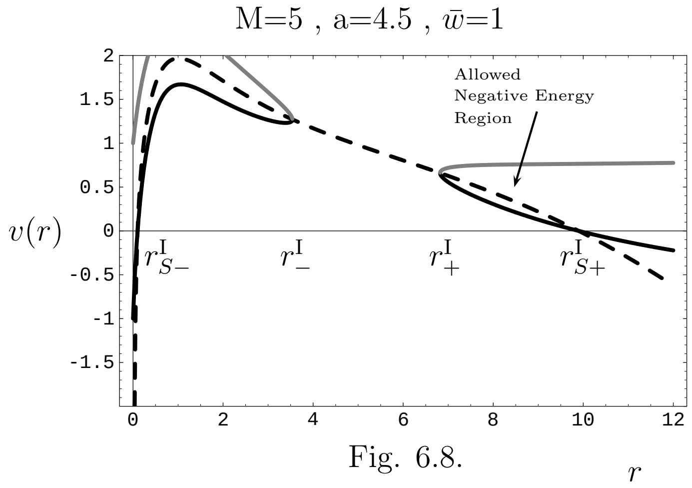

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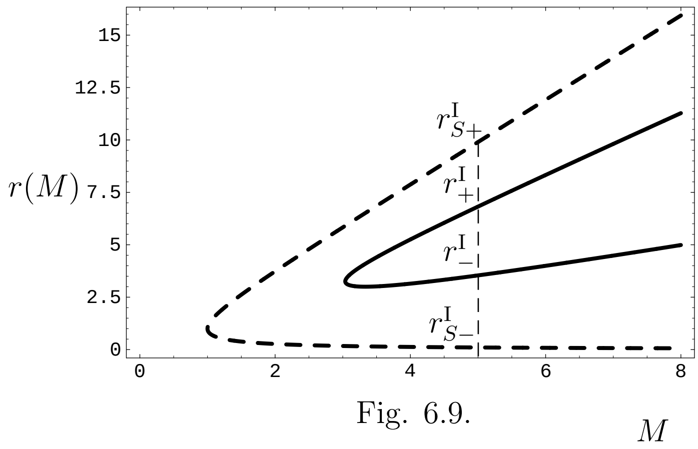

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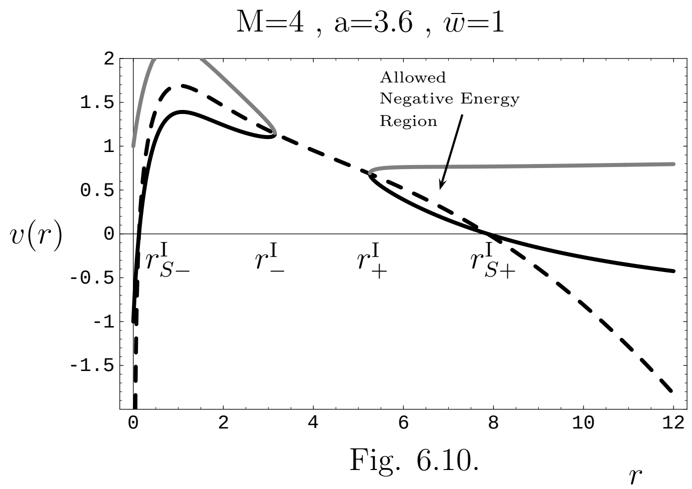

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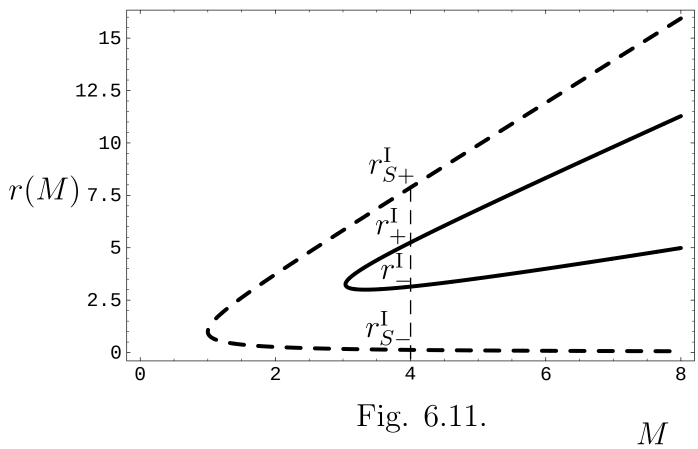

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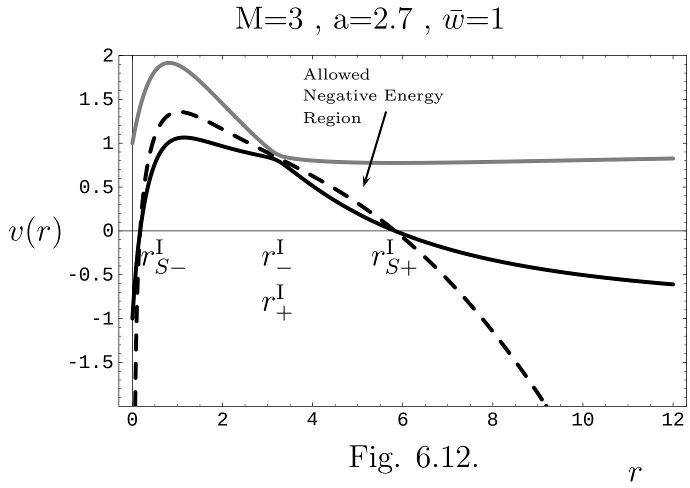

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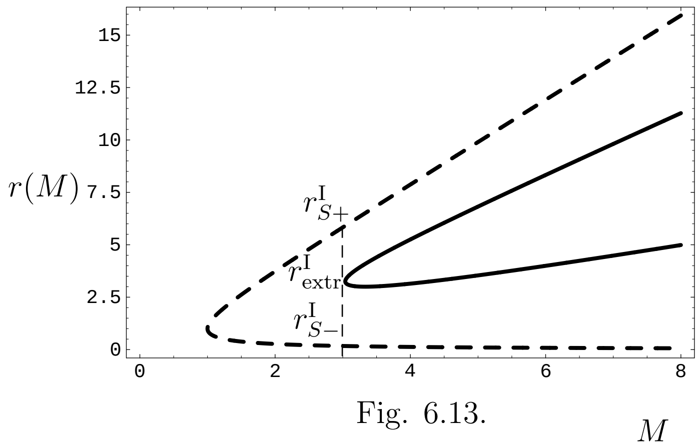

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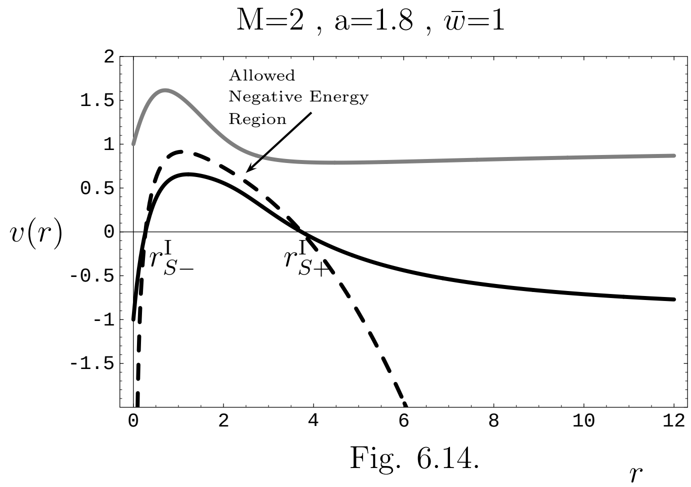
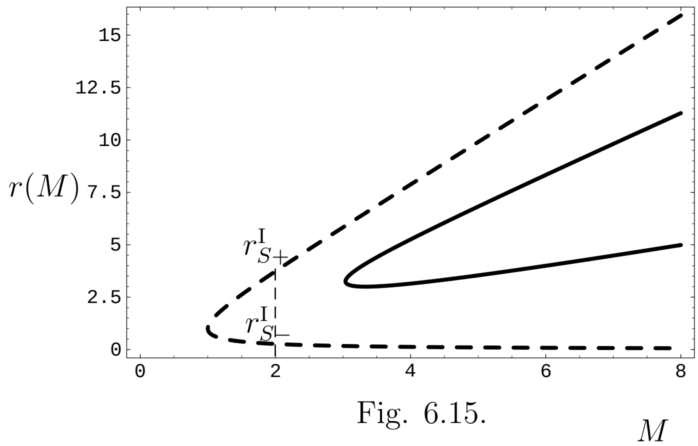


Fig. 6.14.



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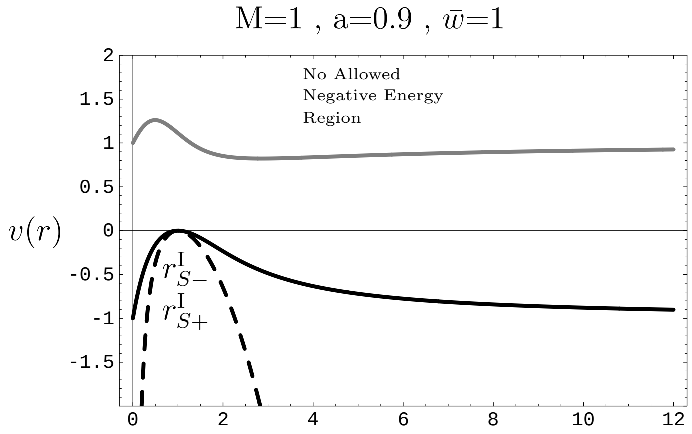

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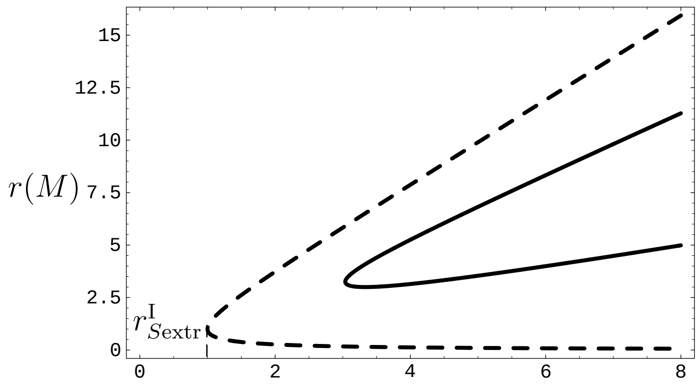

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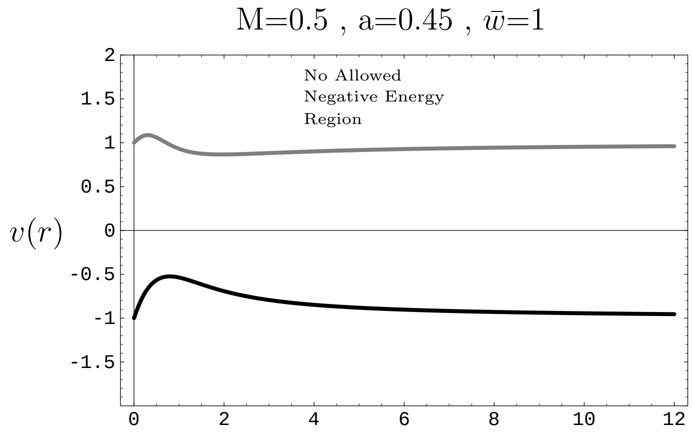


Fig. 6.18.

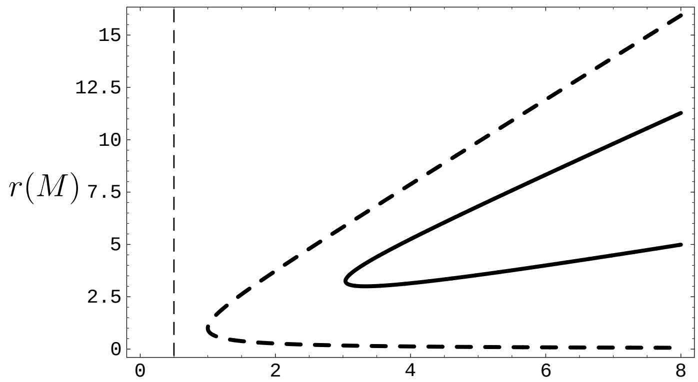


Fig. 6.19.

6.5 Conclusions

We can summarize the results of this chapter as follows. We have derived the key formula (6.3) for the energy of a test particle moving in the improved Kerr spacetime. With this formula we have found the condition (6.9) for the test particle to have a negative energy. We have shown in addition that this condition is fulfilled precisely inside the ergosphere.

Since the Penrose process is directly related to the negative energy states of test particles, we have investigated graphically the evolution of regions of such states in a space of configurations (r, v), when the mass of the black hole runs from 0 to ∞ and having a/m fixed. From this analysis we concluded that, while it is in principle possible to extract energy from classical black holes with arbitrarily small masses and angular momenta, there exists a lowest mass for the extraction in the improved Kerr spacetime. It is close to the Planck mass and defined by the extremal static limit. However since the reliability of our method is questionable in the regime $m \approx 1$ this result has to be considered with care.

Chapter 7

Vacuum Energy-Momentum Tensor and Energy Conditions

7.1 Introduction

The energy conditions are constraints imposed on the energy-momentum tensor based upon physically reasonable assumptions about the properties of matter. These assumptions have a general character, in the sense that they are supposed to be fulfilled by any "sensible" matter or field distribution. The importance of these conditions rests upon the fact, that with them, results of great generality can be derived, like the focusing or the singularity theorems of classical general relativity [60, 66].

Let us suppose that our quantum black hole has been generated via the *classical* Einstein equations, by an "effective" matter fluid that simulates the effect of the quantum fluctuations of the metric. We assume that this coupled gravity-"matter" system satisfies the conventional Einstein equations:

$$G_{\mu\nu} = 8\pi G_0 T^{\mathrm{Q}}_{\mu\nu} \tag{7.1}$$

As a result, the energy-momentum tensor $T^{\mathrm{Q}}_{\mu\nu}$ of the effective matter can be derived from (7.1) by calculating the Einstein tensor $G_{\mu\nu}$ for the improved Kerr spacetime. It is already established that quantum fields can violate the energy conditions [60]. The main goal of this chapter is to show that this is indeed the case for the energy-momentum tensor $T^{\mathrm{Q}}_{\mu\nu}$ derived from the improved Schwarzschild and Kerr spacetimes.

To put the energy conditions in a concrete form we assume that $T_{\mu\nu}$ admits the decomposition [60, 66]

$$T^{\mu\nu} = \rho e_0^\mu e_0^\nu + p_1 e_1^\mu e_1^\nu + p_2 e_2^\mu e_2^\nu + p_3 e_3^\mu e_3^\nu \tag{7.2}$$

in which the vectors $\boldsymbol{e}_\alpha$ form an orthonormal basis; they satisfy the relations

$$g_{\alpha\beta} e_\mu^\alpha e_\nu^\beta = \eta_{\mu\nu} \tag{7.3}$$

where $\eta_{\mu\nu}$ =diag$(-1, 1, 1, 1)$ is the Minkowski metric. Equations (7.2) and (7.3) imply that the quantities ρ and p_i are eigenvalues of $T^{\mu\nu}$ and components of a four-vector p_α, and $\boldsymbol{e}_\alpha$ are the normalized eigenvectors.

The decomposition (7.2) is one of four possible canonical forms [66]. It is the form assumed by the examples of quantum effective matter we are going to analyze. It represents the general case in which the energy momentum tensor has a timelike eigenvector $\boldsymbol{e}_0$. This eigenvector is unique unless $\rho = -p_i$ $(i = 1, 2, 3)$. The eigenvalue ρ represents the energy-density as measured by an observer whose world line at a point p of the spacetime manifold has unit tangent vector $\boldsymbol{e}_0$, and the eigenvalues p_i represent the three principal pressures in the three spacelike directions $\boldsymbol{e}_i$ $(i = 1, 2, 3)$. We shall formulate the energy conditions in terms of the quantitites ρ and p_i.

The organization of this chapter is the following: First we present the standard energy conditions, how they are related to each other, and their main consequences. After this, we present the energy-momentum tensor for the improved Schwarzschild and Kerr spacetimes and we show that they violate all the energy conditions previously defined. Finally we analyze the consequences of our results.

7.1.1 Weak Energy Condition

To an observer whose world-line at a point p in spacetime has unit tangent vector v^α, the local energy density appears to be $T_{\mu\nu} v^\mu v^v$. The weak energy condition states that the energy density of any matter distribution, as measured by any observer in spacetime, must be non-negative. As a result we must have

$$T_{\mu\nu} v^\mu v^v \geq 0 \tag{7.4}$$

for any future-directed timelike vector v^v. Such a vector can be represented as

$$v^\alpha = \gamma \left(e_0^\alpha + a e_1^\alpha + b e_2^\alpha + c e_3^\alpha \right) \ , \ \gamma = \left(1 - a^2 - b^2 - c^2\right)^{-\frac{1}{2}} \ , \tag{7.5}$$

where a, b, and c are arbitrary functions of the coordinates restricted by $a^2+b^2+c^2 < 1$. By substituting (7.5) and (7.2) in (7.4), and considering all independent possible choices of v^μ we end up with the following four inequalities [60, 66]:

$$\rho \geq 0 \ , \ \rho + p_i \geq 0 \ , \ i = 1, 2, 3 \tag{7.6}$$

7.1.2 Null Energy Condition

The null energy condition makes the same statement as the weak form, except that v^α is replaced by an arbitrary, future directed null vector k^α. Thus,

$$T_{\mu\nu} k^\mu k^v \geq 0 \tag{7.7}$$

is the statement of the null energy condition. We shall express k^α as

$$k^\alpha = e_0^\alpha + a' e_1^\alpha + b' e_2^\alpha + c' e_3^\alpha \tag{7.8}$$

where a', b', and c' are arbitrary functions of the coordinates restricted by $(a')^2 + (b')^2 + (c')^2 = 1$. A similar procedure as the previously carried out for the weak energy condition leads to the following three inequalities:

$$\rho + p_i \geq 0 \ , \ i = 1, 2, 3 \tag{7.9}$$

Notice that the weak energy condition implies the null condition.

7.1.3 Dominant Energy Condition

This is a slightly stronger condition than the weak energy condition. It is oriented to avoid the indiscriminate creation of particles, in the sense that spacetime must remain empty if it is empty at one time and no matter comes in from infinity. Conversely, matter present at one time cannot disappear and so must be present at another time [66]. It also embodies the notion that matter-energy should flow along timelike or null world lines [60]. It is never observed to be flowing faster than light (no tachyons).

The dominant energy condition is stated as follows. In addition to the non-negative character of the local energy density, we also require that the local energy flow vector is non-spacelike. As a result we have for every timelike vector v^α the following:

$$T_{\mu\nu} v^\mu v^v \quad \geq \quad 0 \tag{7.10}$$

with $T_{\mu\nu}v^{\mu}$ non-spacelike. In terms of the components of $T_{\mu\nu}$ in (7.2) the dominant energy condition takes the form

$$\rho \geq 0\ ,\ \rho \geq |p_i|\ ,\ i = 1, 2, 3 \tag{7.11}$$

as one can verify by substituting v^{μ} from (7.5) in $T_{\mu\nu}v^{\mu}$ and requiring it to be non-negative. By definition, the dominant energy condition implies the weak and the null energy conditions.

7.1.4 Strong Energy Condition

The strong energy condition has a much more technical motivation than the others. Namely, it is a basic requirement for the validity of the focusing theorem. This theorem establishes that gravity tends to focus geodesics, in the sense that initially diverging characteristic sets of geodesics called congruences[1], will be found to diverge less rapidly in the future. On the contrary if these geodesics are initially converging, they will converge more rapidly in the future [60].

The strong energy condition states that $T_{\mu\nu}$ should respect the following inequality

$$\left(T_{\mu\nu} - Tg_{\mu\nu}\right) v^{\mu}v^{v} \geq 0 \tag{7.12}$$

where $T = T_{\alpha}^{\alpha}$ is the trace of the momentum-energy tensor. Because $T_{\mu\nu} - Tg_{\mu\nu} = R_{\mu\nu}/(8\pi G_0)$ by virtue of the Einstein field equations, the strong energy condition is really a statement about the Ricci tensor:

$$R_{\mu\nu}v^{\mu}v^{v} \geq 0 \tag{7.13}$$

We can better understand the motivation of this condition if we analyse the evolution with the proper time of the cross section of a congruence about a reference geodesic. The focusing of geodesics can be measured by the decrease of the cross section. The equation that governs the evolution of the cross section of timelike geodesic congruences is called the Raychauduri's equation and is given by

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - R_{\alpha\beta}u^{\alpha}u^{\beta} \tag{7.14}$$

[1]A congruence in a given manifold is a family of curves such that through each point p of the manifold there passes precisely one curve of this family.

where θ is called the expansion parameter and it is equal to the fractional rate of change of δV, the congruence's cross-sectional volume at a point p in the spacetime manifold:

$$\theta(p) = \frac{1}{\delta V}\frac{d}{d\tau}\delta V(p) \tag{7.15}$$

As a result, if $\theta > 0$ at p, the congruence is diverging and if $\theta < 0$ the congruence is converging. Furthermore, $\sigma^{\alpha\beta}$ is the shear tensor and u^α is the tangent vector to the geodesic at p. Since θ^2 and $\sigma^{\alpha\beta}\sigma_{\alpha\beta}$ are positive, we have from the Raychauduri's equation (7.14)

$$\frac{d\theta}{d\tau} \leq 0 \tag{7.16}$$

if the strong energy condition holds. The expansion must therefore decrease during the congruence's evolution. This is precisely the statement of the focusing theorem: An initially diverging ($\theta > 0$) congruence will diverge less rapidly in the future, while an initially converging ($\theta < 0$) congruence will converge more rapidly in the future. The physical interpretation of the focusing theorem is that gravitation is an attractive interaction when the strong energy condition holds, and the geodesics get focused as a result of this attraction [60]. In terms of the components of $T_{\mu\nu}$ in (7.2) the strong energy condition is given by:

$$\rho + p_i \ \geq \ 0\ , \ \sum_{\alpha=0}^{3} p_\alpha \geq 0 \tag{7.17}$$

From equations (7.9) and (7.17) we conclude that the strong implies the null energy condition.

An overview of the relations among the different conditions is shown in figure 7.1 [67].

7.1.5 Violation of the Energy Conditions: Casimir Effect

While the energy conditions typically hold for classsical matter, they can be violated by quantized matter fields. A well-known example is the Casimir vacuum energy between two conducting plates separated by a distance d:

$$\rho = -\frac{\pi^2\hbar}{720 d^4} < 0 \tag{7.18}$$

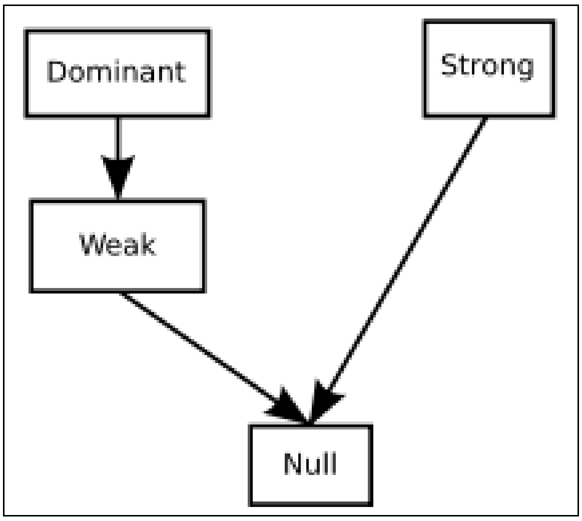


Fig. 7.1.
Relations of implication among the energy conditions.

Since ρ is negative, the Casimir vacuum energy density violates at least the dominant, the weak and the null conditions. It is therefore no surprise that the effective matter fluid associated to the quantum fluctuations of the improved spacetimes we deal with, also violates all the energy conditions. We dedicate the next two subsections to show this statement.

7.2 Improved Schwarzschild Spacetime

The Schwarzschild metric improved with a generic running Newton constant $G(r)$ is the result of replacing G_0 by $G(r)$ in (2.12). In that case we have:

$$ds^2 = -\left(1 - \frac{2MG(r)}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2MG(r)}{r}\right)} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{7.19}$$

Substituting (7.19) in the Einstein field equations (7.1) gives an energy-momentum tensor T^μ_ν of the form

$$T^\mu_\nu \equiv \mathrm{diag}\left(\ p_0, \ \ p_1, \ \ p_2, \ \ p_3 \ \right) = \mathrm{diag}\left(\ -\rho, \ \ p_r, \ \ p_\perp, \ \ p_\perp \ \right) \tag{7.20}$$

where the energy density ρ, the radial pressure p_r and the tangential pressure $p_\perp$ are given by

$$\begin{aligned} -\rho &= p_r = -\frac{MG'}{4\pi G_0 r^2} \qquad (7.21) \\ p_\perp &= -\frac{MG''}{8\pi G_0 r} \end{aligned}$$

Since $\rho = (MG') / (4\pi G_0 r^2)$, it obviously depends on the sign of G' whether the energy density is positive or not. Similarly we have for the sums $\rho + p_i$ and $\sum_{\alpha=0}^{3} p_\alpha$:

$$\rho + p_r = 0 \tag{7.22}$$

$$\rho + p_\perp = -\frac{MG''}{8\pi G_0 r} + \frac{MG'}{4\pi G_0 r^2} = \frac{M}{4\pi G_0 r}\left(\frac{G'}{r} - \frac{G''}{2}\right) \tag{7.23}$$

$$\sum_{\alpha=0}^{3} p_\alpha = -\frac{MG''}{4\pi G_0 r} \tag{7.24}$$

As a result, we need explicit formulas for G, G' and G'' in order to verify or falsify the energy conditions.

7.2.1 The Approximation $d(r) = r$

The asymptotic approximation $d(r) = r$ provides an important special case to be tested. The running $G(r)$ and its derivatives are given by

$$G(r) = \frac{G_0 r^2}{r^2 + G_0 \bar{w}} \tag{7.25}$$

$$G'(r) = \frac{2G_0 r G_0 \bar{w}}{\left[r^2 + G_0 \bar{w}\right]^2} \tag{7.26}$$

$$G''(r) = 2G_0^2 \bar{w} \left\{ \frac{G_0^2 \bar{w}^2 - 2G_0 \bar{w} r^2 - 3r^4}{\left[r^2 + G_0 \bar{w}\right]^4} \right\} \tag{7.27}$$

Substituting G' (7.26) in (7.21) gives the following for ρ:

$$\rho = \frac{MG'}{4\pi G_0 r^2} = \frac{MG_0 \bar{w}}{2\pi r \left[r^2 + G_0 \bar{w}\right]^2} \tag{7.28}$$

As a result, ρ is positive for all positive r. Nevertheless we should only believe in expression (7.28) when $l_{\mathrm{pl}} \ll r$. In particular the leading term of ρ for $r \to \infty$ is given by

$$\rho = \frac{MG_0 \bar{w}}{2\pi r^5} \left[1 - \frac{2G_0 \bar{w}}{r^2} + O\left(\frac{1}{r^2}\right)\right] \tag{7.29}$$

Thus, for $r \to \infty$, where $d(r) \to r$ the energy density remains positive.

For $\rho + p_\perp$ we substitute G' and G'' from (7.26) and (7.27) in (7.23). We get the following answer:

$$\rho + p_\perp = \frac{MG_0\bar{w}\left(6G_0\bar{w}r^2 + 5r^4 + G_0^2\bar{w}^2\right)}{4\pi r\left[r^2 + G_0\bar{w}\right]^4} > 0 \tag{7.30}$$

As a consequence we have $\rho + p_\perp > 0$. Combining this result and $\rho + p_r = 0$ and $\rho > 0$ from (7.22) and (7.28) *we conclude that for the asymptotic region with $d(r) \approx r$, the weak and the null energy conditions are fulfilled.*

In order to test the strong energy condition, we need to calculate, in particular, the sum $\sum_{\alpha=0}^{3} p_\alpha$ of all components p_α of T^μ_ν in (7.20), as follows

$$\sum_{\alpha=0}^{3} p_\alpha = -\frac{MG''}{4\pi G_0 r} = \frac{MG_0\bar{w}\left\{3r^4 + 2G_0\bar{w}r^2 - G_0^2\bar{w}^2\right\}}{2\pi r\left[r^2 + G_0\bar{w}\right]^4} \tag{7.31}$$

Expression (7.31) shows that for

$$3r^4 + 2G_0\bar{w}r^2 < G_0^2\bar{w}^2 \ ,$$

the sum of components $\sum_{\alpha=0}^{3} p_\alpha$ is negative. However this can happen only for values of r near to the Planck scale, where the $d(r) = r$ approximation is unreliable. In contrast we have for $r \to \infty$ that $\sum_{\alpha=0}^{3} p_\alpha > 0$. As a result *the strong energy condition is also satisfied for $r \gg l_{Pl}$.*

We will show now that *the dominant energy condition is violated for $r \gg l_{Pl}$.* We need to test whether $\rho \geq p_\perp$ or not. Substituting the second derivative (7.27) in $p_\perp$ from (7.21) leads to the following expression:

$$p_\perp = -\frac{MG''}{8\pi G_0 r} = \frac{MG_0\bar{w}}{4\pi r}\left\{\frac{3r^4 + 2G_0\bar{w}r^2 - G_0^2\bar{w}^2}{\left[r^2 + G_0\bar{w}\right]^4}\right\} \tag{7.32}$$

The leading term of (7.32) when $r \to \infty$ is

$$p_\perp = \frac{MG_0\bar{w}}{4\pi r^5}\left[3 + O\left(\frac{1}{r}\right)\right] \tag{7.33}$$

Thus, to $O\left(\frac{1}{r^5}\right)$ we have

$$\rho - p_\perp = -\frac{MG_0\bar{w}}{4\pi r^5} < 0 \tag{7.34}$$

As a result, for $r \to \infty$ the tranversal pressure $p_\perp$ is greater than the energy density. This violates the requirement (7.11) of the dominant energy condition. Since this condition is intended to avoid the anihilation or creation of particles, heuristically, its violation indicates the possibility of particle creation and anihilation, which is a typical feature of any relativistic quantum field theory.

7.2.2 The Exact $d(r)$

A numerical check of all the energy conditions can also be accomplished for regions near to the event horizon $r = 2MG(r)$ and even inside. In this case, we exploit the exact expression of $d(r)$ given in (3.9) for the Schwarzschild spacetime. Depending on whether $r < 2m$ or $r > 2m$ we substitute either d_1 or d_2 in the formula (4.2) for $G(r)$. The derivatives $G'(r)$, $G''(r)$ and the functions ρ and $p_\perp$ are calculated numerically.

Figures 7.2 to 7.5 show the radial dependence of ρ, $\rho + p_\perp$, $\sum_{\alpha=0}^{3} p_\alpha$ and $\rho - p_\perp$, respectively, for an improved Schwarzschild black hole with $m = 5$. The vertical line at $r = 2m = 10$ is an asymptote for every function plotted. This is because $d'(r)$ diverges at the event horizon (remember figure 3.4). In figure 7.2, the function ρ remains positive for all positive r. Figure 7.3 shows the existence of two regions inside the event horizon where a violation of the weak, null and strong conditions occurs since $\rho + p_\perp < 0$ in these regions. On the contrary, $\rho + p_\perp$ remains positive outside $r = 2m$ which is consistent with the previous analysis for the asymptotic region with $r \to \infty$. Figure 7.4 shows an r-dependence of $\sum_{\alpha=0}^{3} p_\alpha$ similar to that of the function $\rho + p_\perp$. This indicates the violation of the strong condition inside the event horizon; it is fulfilled outside. Figure 7.5 shows the r-dependence of $\rho - |p_\perp|$ which is crucial for testing the dominant condition. As verified in subsection 7.2.1 this condition is violated as $r \to \infty$, since $\rho - |p_\perp| < 0$ for $r > 2m$. Inside the event horizon the dominant condition is also violated in three separate regions.

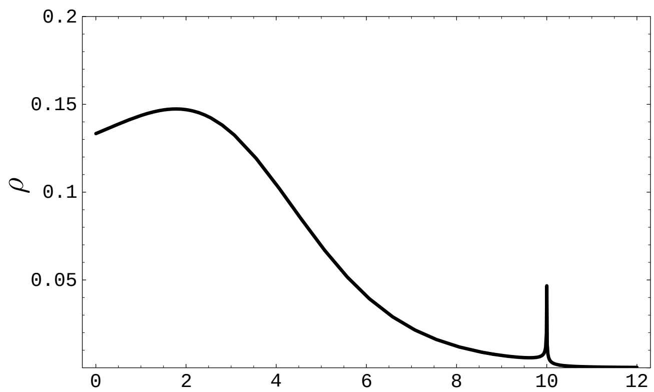
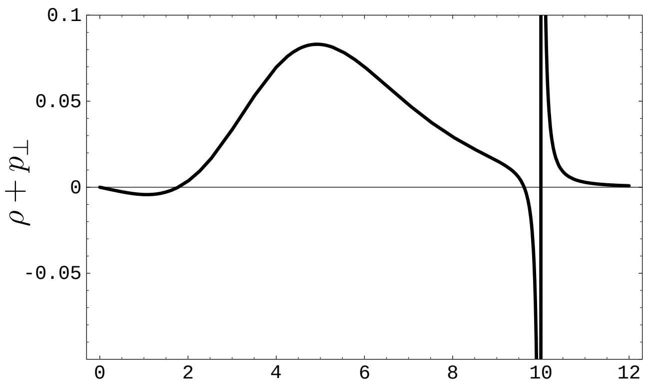


Fig. 7.2.



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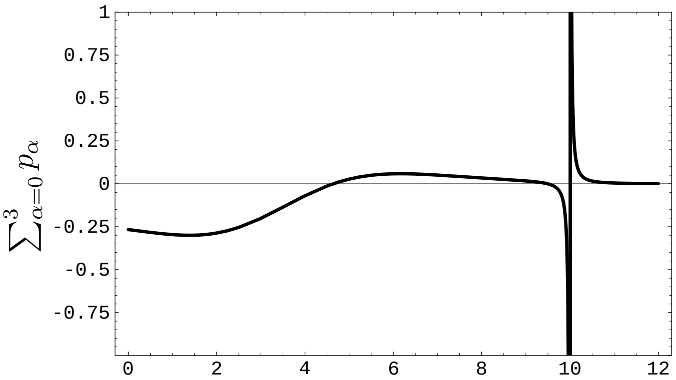

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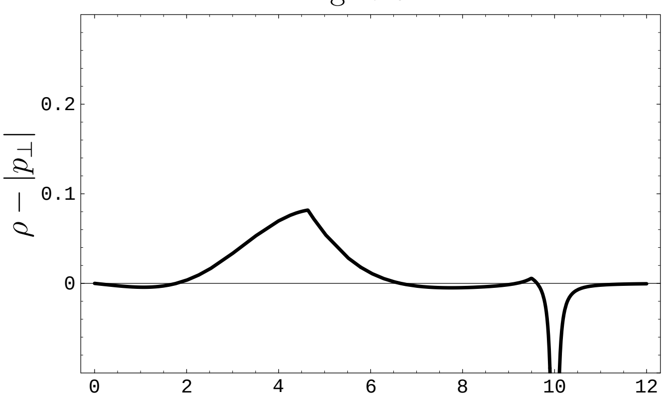

Fig. 7.5.

Figures 7.2 to 7.5 show the r-dependence of functions ρ, $\rho + p_\perp$, $\sum_{\alpha=0}^{3} p_\alpha$, and $\rho - |p_\perp|$, respectively, for an $m = 5$ improved Schwarzschild spacetime. The regions in the r-domain where they assume negative values, correspond to regions where some of the energy conditions are violated. The event horizon at $r = 2m$ defines an asymptote for every function, since $d'(r)|_{r=2m} = \infty$ (see figure 3.4).

Fig. 7.2: The energy density ρ is positive for all r.

Figs. 7.3 and 7.4: The functions $\rho + p_\perp$ and $\sum_{\alpha=0}^{3} p_\alpha$ are positive outside $r = 2m$ and negative at some regions inside. As a result in these regions, the strong, the weak and the null conditions are violated.

Fig. 7.5: The function $\rho - |p_\perp|$ is negative for $r > 2m$ and positive in a limited region inside the event horizon. The dominant energy condition is violated in the regions with negative $\rho - |p_\perp|$.

7.3 Improved Kerr Spacetime

In this section we present the energy momentum tensor $T_{\mu\nu}^{\text{Kerr-I}}$ for the "pseudo-matter" in the improved Kerr spacetime. It will be expressed as a function of a generic $G(r)$ and its derivatives $G'(r)$ and $G''(r)$. We show that its eigenvalues have in general non-zero values, and as a consequence $T_{\mu\nu}^{\text{Kerr-I}}$ can be diagonalized in the form (7.2). We shall conclude that the energy conditions are also violated by $T_{\mu\nu}^{\text{Kerr-I}}$.

Employing a combination of the symbolic computational programs Mathematica and Maple we have derived the following explicit result for $T_{\mu\nu}^{\text{Kerr-I}}$:

$$T_{\mu\nu}^{\text{Kerr-I}} = \frac{M}{4\pi G_0 \Delta A^3} \begin{pmatrix} p_1 & 0 & 0 & v \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ v & 0 & 0 & p_4 \end{pmatrix} \tag{7.35}$$

In writing down $T_{\mu\nu}^{\text{Kerr-I}}$ the following set of abbreviations has turned out convenient:

$$\begin{aligned} \Delta(r) &\equiv a^2 + r^2 - 2MrG(r) \ , \ \rho^2 \equiv r^2 + a^2\cos^2\theta \ , \ A(r,\theta) \equiv 2\rho^2 \\ B(r,\theta) &\equiv 8r^2\left(a^2+r^2\right) - a^4(\sin 2\theta)^2 \end{aligned}$$

$$\begin{aligned} p_1 &\equiv \alpha_1(r,\theta)\,G' + \beta_1(r,\theta)\,G'' \ , \ p_2 \equiv \alpha_2(r,\theta)\,G' + \beta_2(r,\theta)\,G'' \\ v &\equiv \alpha_\nu(r,\theta)\,G' + \beta_\nu(r,\theta)\,G'' \ , \ p_3 \equiv \alpha_3 G' + \beta_3 G'' \\ p_4 &\equiv \alpha_4(r,\theta)\,G' + \beta_4(r,\theta)\,G'' \end{aligned} \tag{7.36}$$

$$\begin{aligned} \beta_3(r,\theta) &= 4\Delta r\rho^2 \ , \beta_2(r,\theta) = 0 \ , \ \beta_1(r,\theta) = \beta_3 a^2\sin^2\theta \\ \beta_4(r,\theta) &\equiv \beta_3\csc^2\theta \ , \ \beta_\nu(r,\theta) \equiv -a\beta_3 \end{aligned} \tag{7.37}$$

$$\begin{aligned} \alpha_1(r,\theta) &\equiv -\left(a^2+r^2\right)\left[8r^2\left(a^2+r^2\right) - a^4(\sin 2\theta)^2\right] - 16ra^2MG\sin^2\theta\cos^2\theta \\ \alpha_2 &\equiv 8r^2\Delta^2 \ , \ \alpha_\nu \equiv 8ar^2\left(r^2+a^2\right) - a\alpha_3 \ , \alpha_3 \equiv 8\Delta a^2\cos^2\theta \\ \alpha_4 &\equiv \csc^2\theta\,\alpha_3 - 8a^2r^2 \end{aligned} \tag{7.38}$$

Given the enormous complexity of the intermediate expressions the simplicity of the final result for $T_{\mu\nu}^{\text{Kerr-I}}$ is rather surprising. The rows and columns of the matrix in eq. (7.35) are ordered according to $t - r - \theta - \varphi$. The matrix is diagonal except for the $t\varphi$ entry. Of course, a nonzero value of $T_{t\varphi}^{\text{Kerr-I}}$ was to be expected since it corresponds to (pseudo) matter rotating about the z-axis.

It is not difficult to diagonalize $T_{\mu\nu}^{\text{Kerr-I}}$. The eigenvalues of (7.35), without the overall coefficient $M/\left(4\pi G_0 \Delta A^3\right)$ are given by

$$
\begin{aligned}
l_1 &\equiv \frac{1}{2}\left[p_1 + p_4 + \sqrt{(p_1^2 - 2p_1p_4 + p_4^2 + 4v^2)}\right] \\
l_2 &\equiv p_2\ ,\ l_3 \equiv p_3 \\
l_4 &\equiv \frac{1}{2}\left[p_1 + p_4 - \sqrt{(p_1^2 - 2p_1p_4 + p_4^2 + 4v^2)}\right]
\end{aligned}
\tag{7.39}
$$

Hence the energy momentum tensor in its eigenbasis reads

$$
T_{\mu\nu}^{\text{Kerr-I}} = \frac{M}{4\pi G_0 \Delta A^3}\begin{pmatrix} l_1 & 0 & 0 & 0 \\ 0 & l_2 & 0 & 0 \\ 0 & 0 & l_3 & 0 \\ 0 & 0 & 0 & l_4 \end{pmatrix}
\tag{7.40}
$$

As a check we can set $a = 0$ in which case (7.40) reduces exactly to the corresponding energy momentum tensor of the improved Schwarzschild black hole:

$$
T_{\mu\nu}^{\text{Sch-I}} = \frac{1}{4\pi G_0 r^2}\begin{pmatrix} -\frac{MG'}{1-\frac{2MG(r)}{r}} & 0 & 0 & 0 \\ 0 & MG'\left(1 - \frac{2MG(r)}{r}\right) & 0 & 0 \\ 0 & 0 & \frac{MG''}{2r} & 0 \\ 0 & 0 & 0 & \frac{MG''}{2r}\left(\frac{1}{\sin^2\theta}\right) \end{pmatrix}
\tag{7.41}
$$

Since $T_{\mu\nu}^{\text{Sch-I}}$ is a special case of $T_{\mu\nu}^{\text{Kerr-I}}$, it is clear that the improved Kerr spacetime also must violate the various energy conditions in some portions of spacetime. Given our explicit results for the diagonal matrix elements $l_1 \cdots, l_4$ it is in principle straightforward to determine in which region of spacetime which one of the energy conditions is violated. We shall not perform this analysis here since for our present purposes it is enough to know that the conditions are violated *somewhere*. This observation implies that the quantum Kerr black hole cannot be thought of as a special solution of the coupled classical gravity + matter system since the "pseudo-matter" we encounter here has properties which are very different from ordinary matter.

7.4 Conclusion

In this chapter we have interpreted the RG-improved black hole as a special solution to the classical gravity + matter system and derived some properties of the effective fluid that is due to the quantum fluctuations of the metric. We have demonstrated in sections 7.2 and 7.3 that this effective fluid does not behave as an usual matter system that fulfills the energy conditions. The main result of this chapter can be summarized by saying that no classical gravity plus matter model with "normal" matter can simulate the quantum fluctuations implicit in the running of G. As a consequence we have to talk about a "quantum fluid" acting as a source on the right hand side of the Einstein field equations (7.1). The practical consequence of these findings is that in our analysis of improved black holes we cannot take advantage of the many results in the literature which concern classical black holes in the presence of matter. Almost all of these results rely on the energy conditions. As they are not satisfied by the "quantum fluid", the mechanics and thermodynamics of the improved black holes can be expected to show novel features not showed by the classical ones. In the next chapter we shall see that this is indeed the case.

Chapter 8

Thermodynamics of the Quantum Black Holes

8.1 Introduction

In this chapter we investigate the impact of the running Newton's constant on the thermodynamics of Kerr black holes. We concentrate our study on the following topics:

- Analysis of dynamical quantities like the Komar mass, the angular momentum and the surface gravity of the black hole.
- Analysis of the first law of black hole thermodynamics and the definitions of temperature and entropy related to the first law.

Taking into account these two main topics, the chapter is organized as follows. In section 8.2 we calculate the surface gravity of the general spherically symmetric and the improved Kerr black holes, exploiting the property that the event horizon is generated by a Killing vector field (Killing horizon). In sections 8.3 to 8.5 we calculate the mass and the angular momentum of the improved Kerr black hole, taking advantage of the Komar integrals that relate directly to these quantities when the spacetime is stationary and axially symmetric. We find that our results are consistent with a gravitational antiscreening due to the quantum gravity fluctuations described by the quantum effective matter discussed in chapter 7. In section 8.6 we show that the quantities we have calculated fulfill the Smarr's formula discussed in chapter 2. This is a consequence of the stationarity and axial symmetry of the improved Kerr spacetime. We dedicate section 8.7 to the study of the first law of black hole thermodynamics. We show that a modified first law can exist only when we give up the relationship $T = \kappa/2\pi$ which is at the heart of the classical thermodynamics of black holes. We also calculate $O\left(J^2\right)$ approximations to the first law, the temperature, and the entropy of the improved Kerr black hole, and we investigate the relation of these results to the corresponding quantities of the improved Schwarzschild spacetime. Finally in section 8.8 we present our conclusions about the possibility of formulating an "RG improved thermodynamics" of rotating black holes.

8.2 Surface Gravity

As mentioned in chapter 2, the surface gravity κ is the force required by an observer at infinity to hold a particle (of unit mass) stationary at the event horizon. We can define κ also in terms of the Killing vector $\boldsymbol{\xi}$ in (4.31) evaluated at the external event horizon H_+. We have seen in chapter 4 that $\xi^\mu \xi_\mu = 0$ at H_+. This means that $\boldsymbol{\xi}$ is orthogonal to itself, but since $\boldsymbol{\xi}$ is tangent to H_+, we conclude that $\boldsymbol{\xi}$ is also normal to the horizon. Thus ξ_α is proportional to $\partial_\alpha \Phi$. But $\Phi \equiv \xi^\mu \xi_\mu = 0$ on the horizon. As a result, there must exist a scalar κ such that

$$\partial_\alpha \left(-\xi^\beta \xi_\beta\right)\big|_{H_+} = 2\kappa \xi_\alpha|_{H_+} \tag{8.1}$$

This scalar is precisely the surface gravity.

In order to find κ it is necessary to use a coordinate system which is non-singular at the event horizon so that every quantity is well defined. Therefore we exploit in subsections 8.2.1 and 8.2.4 the "improved" version of the E-F coordinate systems presented in chapter 2. These transformations result from replacing G_0 by $G\left(r\right)$ in the differentials defined in (2.17), as follows:

$$\begin{aligned} dr_I^* &\equiv \left[\frac{r^2 + a^2}{\Delta_I\left(r\right)}\right] dr \\ dr_I^\# &\equiv \left[\frac{a}{\Delta_I\left(r\right)}\right] dr \end{aligned} \tag{8.2}$$

where $\Delta_I\left(r\right) \equiv r^2 + a^2 - 2MG\left(r\right) r$. The ingoing and outgoing patches are defined as follows:

- Ingoing E-F coordinates ("Ingoing Patch")

$$\begin{aligned} dv &= dt + dr_I^* & (8.3) \\ d\psi &= d\varphi + dr_I^{\#} & (8.4) \end{aligned}$$

- Outgoing E-F coordinates ("Outgoing Patch")

$$\begin{aligned} du &= dt - dr_I^* & (8.5) \\ d\chi &= d\varphi - dr_I^{\#} & (8.6) \end{aligned}$$

The two new sets of coordinates $x^\mu \equiv (v, r, \theta, \psi)$ and $x^\mu \equiv (u, r, \theta, \chi)$ are defined exactly as in Eqs. (2.19) and (2.20), respectively. The only difference as compared to the transformation in the classical case is the form of the functions $r^*(r)$ and $r^{\#}(r)$. They can be found by integrating (8.2),

$$\begin{aligned} r_I^* &= \int \left[\frac{r^2 + a^2}{\Delta_I(r)}\right] dr \\ r_I^{\#} &= \int \left[\frac{a}{\Delta_I(r)}\right] dr \end{aligned} \tag{8.7}$$

but now $\Delta_\mathrm{I}(r)$ and, as a consequence, the relationship between r^*, $r^{\#}$ and r, are functionally dependent on $G(r)$. For a general $G(r)$ the integrals (8.7) cannot be evaluated in closed form but, fortunately, all that is needed for transforming the metric are the differentials (8.2). In fact, applying the above transformations to the improved Kerr metric in B-L coordinates (2.6) gives the following line element (see appendix C):

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2G(r)Mr}{\rho^2}\right) du^2 - 2drdu + 2a\sin^2\theta d\chi dr + \\ &\quad - \frac{4G(r)Mar\sin^2\theta}{\rho^2} d\chi du + \frac{\Sigma_I \sin^2\theta}{\rho^2} d\chi^2 + \rho^2 d\theta^2 \end{aligned} \tag{8.8}$$

for the outgoing patch, and

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2G(r)Mr}{\rho^2}\right) dv^2 + 2drdv - 2a\sin^2\theta d\psi dr + \\ &\quad - \frac{4G(r)Mar\sin^2\theta}{\rho^2} d\psi dv + \frac{\Sigma_I \sin^2\theta}{\rho^2} d\psi^2 + \rho^2 d\theta^2 \end{aligned} \tag{8.9}$$

for the ingoing patch.

The procedure of the calculation of κ is the following. We find first an expression for $\boldsymbol{\xi}$ in E-F coordinates, after that we calculate the scalar $\xi^\mu \xi_\mu$. We find an expression of κ by identifying the coefficient of the right hand side of (8.1) after we substitute ξ_α and $\xi^\mu \xi_\mu$. We dedicate subsections (8.2.2) and (8.2.3) to the calculation of ξ_α and $\xi^\mu \xi_\mu$.

8.2.1 Surface Gravity of Spherically Symmetric Black Holes

As an illustration of how we use formula (8.1) to derive κ we calculate in this subsection the surface gravity of a spherically symmetric spacetime, and we apply the result to the improved Schwarzschild black hole.

The metric of a spherically symmetric and static spacetime is represented in Schwarzschild coordinates by [57, 59]

$$ds^2 = -f\left(r\right)dt^2 + \frac{dr^2}{f\left(r\right)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{8.10}$$

where $x^\mu = (t, r, \theta, \varphi)$. In this case the E-F coordinates result from transforming t of the Schwarzschild coordinates into one of the following two possible time coordinates [41, 57]

$$u = t - r^* \tag{8.11}$$

$$v = t + r^* \tag{8.12}$$

where r^* is defined by

$$r^*\left(r\right) \equiv \int \frac{dr}{f\left(r\right)} \tag{8.13}$$

In this case the coordinate systems with $x^\mu = (u, r, \theta, \varphi)$ and $x^\mu = (v, r, \theta, \varphi)$ are, respectively, the ordinary outgoing and ingoing E-F coordinates. The components $g_{\alpha\beta}$ of the spherically symmetric spacetime in E-F coordinates are given by (see appendix G)

$$ds^2 = -f\left(r\right)du^2 - 2dudr + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{8.14}$$

for the outgoing patch, and

$$ds^2 = -f\left(r\right)dv^2 + 2dvdr + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{8.15}$$

for the ingoing patch. Both of them lead to a representation of the spherically symmetric spacetime which is well behaved when $f\left(r_{\mathrm{H}}\right) = 0$ at a horizon.

From (8.10) we can see that the coordinate t is cyclic. This means from appendix F that we have a Killing vector given by

$$\boldsymbol{t} = \frac{\partial}{\partial t} \tag{8.16}$$

This is precisely the Killing vector to be used in the formula (8.1) for κ. Representing $\boldsymbol{t}$ with the ingoing E-F patch gives

$$\boldsymbol{t} \;=\; \frac{\partial}{\partial t} = \frac{dv}{dt}\frac{\partial}{\partial v} = \frac{\partial}{\partial v} \tag{8.17}$$

As a result, the components t^α are given by

$$t^\alpha \;=\; \frac{\partial x^\alpha}{\partial t} = \frac{\partial x^\alpha}{\partial u} = \delta^\alpha_u = \delta^\alpha_t \tag{8.18}$$

Lowering the index α gives

$$t_\alpha \;=\; g_{\alpha\beta}t^\beta = g_{\alpha\beta}\delta^\beta_v = g_{\alpha v} \tag{8.19}$$

More explicitly, the (v, r, θ, φ) components of t_α are the following:

$$t_\alpha = g_{\alpha u} \;=\; (g_{uu},\; g_{ru},\; g_{\theta u},\; g_{\varphi u}) = (-f\,(r)\,,\; 1,\; 0,\; 0) \tag{8.20}$$

Exploiting the expressions for t^α and t_α from (8.18) and (8.20) we find the following result for $-\partial_\beta\,(t^\alpha t_\alpha)$:

$$-\partial_\beta\,(t^\alpha t_\alpha) = \partial_\beta f\,(r) = f'\,(r)\,\partial_\beta r \tag{8.21}$$

Evaluating expression (8.21) at the radius of the event horizon $r = r_{\mathrm{H}}$ where $f\,(r_{\mathrm{H}}) = 0$ gives the following:

$$-\partial_\beta\,(t^\alpha t_\alpha)\big|_{r_{\mathrm{H}}} \;=\; f'\,(r_{\mathrm{H}})\,\partial_\beta r \tag{8.22}$$

On the other hand, t_α at $r = r_{\mathrm{H}}$ reads

$$t_\alpha\big|_{r_{\mathrm{H}}} \;=\; (-f\,(r_{\mathrm{H}})\,,\; 1,\; 0,\; 0) = (0,\; 1,\; 0,\; 0) = \partial_\beta r \tag{8.23}$$

Substitituting (8.22) and (8.23) in (8.1) leads to the final result for κ:

$$\kappa \;=\; \frac{f'\,(r_{\mathrm{H}})}{2} \tag{8.24}$$

This is the expression of κ for the spherically symmetric spacetime presented in chapter 2.

For the improved Schwarzschild spacetime we have $f(r) = 1 - 2MG(r)/r$. As a result, from (8.24) we deduce that κ is given by

$$\kappa = \left. \frac{M(G - rG')}{r^2} \right|_{r^{\mathrm{I}}_{\mathrm{Sch}+}} \tag{8.25}$$

Expression (8.25) is simplified to

$$\kappa = \frac{1}{4MG\left(r^{\mathrm{I}}_{\mathrm{Sch}+}\right)} - \frac{G'\left(r^{\mathrm{I}}_{\mathrm{Sch}+}\right)}{2G\left(r^{\mathrm{I}}_{\mathrm{Sch}+}\right)} \tag{8.26}$$

Here we have applied the equation of the improved Schwarzschild event horizon given by $r^{\mathrm{I}}_{\mathrm{Sch}+} = 2MG\left(r^{\mathrm{I}}_{\mathrm{Sch}+}\right)$. From (8.26) we recover the expression for the surface gravity of the classical Schwarzschild black hole $\kappa = 1/4MG_0$ that we obtain when $G(r) = G_0$.

Taking into account the proportionality between the surface gravity and the temperature of the improved Schwarzschild black hole [30], the formula (8.25) has been applied to study the impact of the running Newton constant in quantum-thermodynamical processes like the evaporation of black holes. For a detailed discussion see reference [30].

8.2.2 Two Representations for the Stationary Observer's Killing Vector ξ^μ

As seen in section 4.3.3 the Killing vector related to the stationary observers is given by (4.31)

$$\xi^\mu = t^\mu + \varphi^\mu \Omega \tag{8.27}$$

where Ω is the constant angular frequency of the observer. In this subsection we describe how to find representations of $\boldsymbol{\xi}$ in the B-L and E-F coordinates (see also appendix H).

The representation of (8.27) in B-L coordinates (t, r, θ, ϕ) is the following:

$$\xi^\mu \equiv \delta^\mu_t + \delta^\mu_\varphi \Omega \tag{8.28}$$

It is found by exploiting the expressions (4.14) for t^μ and φ^μ. Lowering the index μ gives

$$\xi_\mu = g_{\mu t} + \Omega g_{\mu\varphi} \tag{8.29}$$

With the transformation (8.3), (8.4) we can find the form of (8.27) in the ingoing E-F coordinates. If we represent the spacetime events as $x^\mu = (v, r, \theta, \psi)$ in these coordinates we find for the components ξ^μ the following expression:

$$\xi^\mu \quad = \quad \frac{\partial x^\mu}{\partial v} + \Omega \frac{\partial x^\mu}{\partial \psi} = \delta_v^\mu + \Omega \delta_\psi^\mu \tag{8.30}$$

With a lower index we have

$$\xi_\mu \quad = \quad g_{\mu v} + \Omega g_{\mu\psi} \ , \ \mu = v, r, \theta, \psi \tag{8.31}$$

Expressions (8.28) and (8.30) for the Killing vector $\boldsymbol{\xi}$ in B-L and E-F coordinates, respectively, are equivalent representations of this vector. Nevertheless (8.30) has the advantage of staying regular at the event horizon.

8.2.3 The Scalar Field $\xi^2(x)$

We proceed now to calculate the left hand side of (8.1), performing the scalar product of ξ^μ and ξ_μ. In E-F coordinates we multiply (8.30) and (8.31)

$$\xi_\mu \xi^\mu = g_{vv} + 2\Omega g_{v\psi} + \Omega^2 g_{\psi\psi} \tag{8.32}$$

The line element of the Kerr metric in the ingoing E-F coordinates is given by

$$\begin{aligned} ds^2 \quad = \quad & g_{vv} dv^2 + 2 g_{rv} dr dv + g_{\theta\theta} d\theta^2 + g_{\psi\psi} d\psi^2 + 2 g_{\psi r} d\psi dr \\ & + 2 g_{\psi v} d\psi dv \end{aligned} \tag{8.33}$$

where its non-zero components are given in (8.9). In particular the line element for stationary observers with r and θ fixed is given by

$$\left. ds^2 \right|_{(r,\theta \text{ fixed})} \quad = \quad g_{vv} dv^2 + g_{\psi\psi} d\psi^2 + 2 g_{\psi v} d\psi dv \tag{8.34}$$

A parametrization of the observer's world line by the temporal coordinate of (8.34) leads to

$$\left. \left(\frac{ds}{dv} \right)^2 \right|_{(r,\theta \text{ fixed})} \quad = \quad g_{vv} + g_{\psi\psi} \Omega d\psi^2 + 2 g_{\psi v} \Omega \tag{8.35}$$

Comparing the expression (8.32) with (8.35) we conclude that

$$\xi_\mu \xi^\mu = \left. \left(\frac{ds}{dv} \right)^2 \right|_{(r,\theta \text{ fixed})} \tag{8.36}$$

As a result, we can read off the scalar $\xi_\mu \xi^\mu$ directly from the metric components in (8.9) (See Appendix H):

$$\xi_\mu \xi^\mu = \left(\frac{ds}{dv}\right)^2\bigg|_{(r,\theta \text{ fixed})} = \frac{\Sigma_I \sin^2\theta}{\rho^2}(\omega - \Omega)^2 - \frac{\rho^2 \Delta_I}{\Sigma_I} \tag{8.37}$$

Here the improved dragging frequency ω is given by

$$\omega(r,\theta) = \frac{2G(r)\,Mar}{\Sigma_I}\ , \tag{8.38}$$

as already obtained in section 4.3.1.

8.2.4 κ for the Improved Kerr Metric

In the previous two subsections we have calculated the components of $\boldsymbol{\xi}$ and the scalar $\xi^\mu \xi_\mu$. We substitute now these results in the formula (8.1) in order to find an expression of the surface gravity of the improved Kerr black hole, using the ingoing E-F representation (8.9).

The *left hand side of (8.1)* can be found by differentiating expression in (8.37). The derivative of $-\boldsymbol{\xi}\cdot\boldsymbol{\xi}$ can be written at the horizon's radius r_+^{I} as follows:

$$\begin{aligned} -\partial_\alpha\left(\xi^\mu \xi_\mu\right)\big|_{r_+^{\mathrm{I}}} &= \partial_\alpha\left(\frac{\rho^2\Delta_I}{\Sigma_I} - \frac{\Sigma_I}{\rho^2}\sin^2\theta\,(\Omega-\omega)^2\right)\bigg|_{r_+^{\mathrm{I}}} \\ &= -\,\partial_\alpha(\Omega-\omega)^2\left(\frac{\Sigma_I}{\rho^2}\sin^2\theta\right)\bigg|_{r_+^{\mathrm{I}}} - 2\frac{\Sigma_I}{\rho^2}\sin^2\theta\,(\Omega-\omega)\,\partial_\alpha(\Omega-\omega)\bigg|_{r_+^{\mathrm{I}}} \\ &\quad + \frac{\rho^2}{\Sigma_I}\partial_\alpha\Delta_I\bigg|_{r_+^{\mathrm{I}}} + \Delta_I\partial_\alpha\left(\frac{\rho^2}{\Sigma_I}\right)\bigg|_{r_+^{\mathrm{I}}} \end{aligned} \tag{8.39}$$

As we already discussed in chapter 2, since all available frequencies coalesce to just one value, we have $\omega|_{r_+^{\mathrm{I}}} = \Omega|_{r_+^{\mathrm{I}}}$ at the horizon. In addition we have $\Delta_I|_{r_+^{\mathrm{I}}} = 0$. As a result, the expression in (8.39) is simplified to

$$-\partial_\alpha\left(\xi^\mu \xi_\mu\right)\big|_{r_+^{\mathrm{I}}} = \frac{\rho^2}{\Sigma_I}\partial_\alpha\Delta_I\bigg|_{r_+^{\mathrm{I}}} \tag{8.40}$$

Since $\Delta_I|_{r_+^{\mathrm{I}}} = 0$ we have the following expression for $\Sigma_I|_{r_+^{\mathrm{I}}}$:

$$\Sigma_I|_{r_+^{\mathrm{I}}} = \left[\left(r^2+a^2\right)^2 - a^2\Delta_I(r)\sin^2\theta\right]\Big|_{r_+^{\mathrm{I}}} = \left[\left(r_+^{\mathrm{I}}\right)^2 + a^2\right]^2 \tag{8.41}$$

The derivative $\partial_\alpha\Delta_I$ is given by

$$\partial_\alpha\Delta_I|_{r_+^{\mathrm{I}}} = 2\left[r_+^{\mathrm{I}} - M\,G'|_{r_+^{\mathrm{I}}}\,r_+^{\mathrm{I}} - MG\left(r_+^{\mathrm{I}}\right)\right]\partial_\alpha r|_{r_+^{\mathrm{I}}} \tag{8.42}$$

Finally substituting (8.41) and (8.42) in (8.40) gives

$$-\partial_\alpha\left(\xi^\mu\xi_\mu\right)\Big|_{r_+^{\rm I}} = \frac{\rho^2}{\Sigma_I}\partial_\alpha\Delta_I\Big|_{r_+^{\rm I}} = \frac{2\,\rho^2|_{r_+^{\rm I}}\left[r_+^{\rm I} - M\,G'|_{r_+^{\rm I}}\,r_+^{\rm I} - MG\left(r_+^{\rm I}\right)\right]}{\left[\left(r_+^{\rm I}\right)^2 + a^2\right]^2}\,\partial_\alpha r|_{r_+^{\rm I}} \tag{8.43}$$

Furthermore, for the *right hand side of (8.1)* we start finding the explicit form of $\boldsymbol{\xi}|_{r_+^{\rm I}}$ from(8.30) and (8.31):

$$\xi^\alpha|_{r_+^{\rm I}} = \delta_v^\mu + \Omega|_{\rm H}\,\delta_\psi^\mu \tag{8.44}$$

$$\xi_\alpha|_{r_+^{\rm I}} = g_{\alpha v}|_{r_+^{\rm I}} + \Omega_{\rm H}\,g_{\alpha\psi}|_{r_+^{\rm I}} \tag{8.45}$$

Here $\Omega_{\rm H}$ is defined in (4.48) or (4.50). We exploit for our current purposes the following expression for $\Omega_{\rm H}$ in E-F coordinates [1]:

$$\Omega_{\rm H} = \omega|_{r_+^{\rm I}} = -\frac{g_{v\psi}}{g_{\psi\psi}}\Big|_{r_+^{\rm I}} \tag{8.46}$$

By substituting (8.46) in (8.45) and using the E-F representation of $g_{\mu\nu}$ in (8.9) we can find the components $\xi_\alpha|_{r_+^{\rm I}}$ (see appendix H):

$$\begin{aligned}
\xi_v &= g_{vv}|_{r_+^{\rm I}} + \Omega_{\rm H}\,g_{v\psi}|_{r_+^{\rm I}} = g_{vv}|_{r_+^{\rm I}} - \frac{g_{v\psi}^2}{g_{\psi\psi}}\Big|_{r_+^{\rm I}} = 0\\
\xi_r &= g_{vr}|_{r_+^{\rm I}} + \Omega_{\rm H}\,g_{r\psi}|_{r_+^{\rm I}} = 1 - a\Omega_{\rm H}\sin^2\theta\\
\xi_\theta &= g_{v\theta}|_{r_+^{\rm I}} + \Omega_{\rm H}\,g_{\theta\psi}|_{r_+^{\rm I}} = 0\\
\xi_\psi &= g_{\psi v}|_{r_+^{\rm I}} + \Omega_{\rm H}\,g_{\psi\psi}|_{r_+^{\rm I}} = g_{\psi v}|_{r_+^{\rm I}} - \frac{g_{v\psi}}{g_{\psi\psi}}g_{\psi\psi}\Big|_{r_+^{\rm I}} = 0
\end{aligned}$$

Summarizing we have

$$\xi_\alpha|_{r_+^{\rm I}} = \left(1 - a\Omega_{\rm H}\sin^2\theta\right)\delta_\alpha^r = \left(1 - a\Omega_{\rm H}\sin^2\theta\right)\partial_\alpha r|_{r_+^{\rm I}} \tag{8.47}$$

Clearly the only nonzero component of (8.47) obtains for $\alpha = r$. To be more explicit we substitute $\Omega_{\rm H} = a/[\left(r_+^{\rm I}\right)^2 + a^2]$ to get:

$$\xi_r|_{r_+^{\rm I}} = 1 - a\Omega_{\rm H}\sin^2\theta = 1 - \frac{a^2\sin^2\theta}{\left(r_+^{\rm I}\right)^2 + a^2} = \frac{\left(r_+^{\rm I}\right)^2 + a^2\cos^2\theta}{\left(r_+^{\rm I}\right)^2 + a^2} = \frac{\rho^2|_{r_+^{\rm I}}}{\left(r_+^{\rm I}\right)^2 + a^2} \tag{8.48}$$

[1] The derivation of (8.46) in E-F coordinates is completely analogous to that one in the B-L representation presented in chapter 2. We omit it therefore.

This is our final result for the right hand side of (8.1).

Now we are ready to substitute (8.43) and (8.48) in (8.1) in order to find κ:

$$\frac{2\,\rho^2|_{r_+^{\mathrm{I}}}\left[r_+^{\mathrm{I}} - M\,\frac{dG}{dr}\big|_{r_+^{\mathrm{I}}}\,r_+^{\mathrm{I}} - MG\left(r_+^{\mathrm{I}}\right)\right]}{\left[\left(r_+^{\mathrm{I}}\right)^2 + a^2\right]^2}\,\partial_\alpha r|_{r_+^{\mathrm{I}}} \quad = \quad 2\kappa\frac{\rho^2|_{r_+^{\mathrm{I}}}}{\left(r_+^{\mathrm{I}}\right)^2 + a^2}\,\partial_\alpha r|_{r_+^{\mathrm{I}}} \tag{8.49}$$

Solving for κ in (8.49) gives the explicit formula

$$\kappa = \frac{\left[r_+^{\mathrm{I}} - M\,G'|_{r_+^{\mathrm{I}}}\,r_+^{\mathrm{I}} - MG\left(r_+^{\mathrm{I}}\right)\right]}{\left[\left(r_+^{\mathrm{I}}\right)^2 + a^2\right]} \tag{8.50}$$

Exploiting the derivative of Δ_I it can also be written as

$$\kappa = \frac{\Delta_I'\left(r_+^{\mathrm{I}}\right)}{2\left[\left(r_+^{\mathrm{I}}\right)^2 + a^2\right]} \quad \text{with} \quad \Delta_I' = \frac{d\Delta_I}{dr} \tag{8.51}$$

Expressions (8.50) and (8.51) are the main results of this subsection.

Two important remarks can be made concerning (8.51), namely:

1. κ has turned out to be independent of θ, therefore it is constant on H_+. This is a nontrivial result, since, contrary to the Schwarzschild black hole, the symmetry assumptions imply only t and φ-independence. The constancy of κ on H_+ is the contents of the zeroth theorem of classical black hole thermodynamics, which here is seen to generalize to the improved case.

2. κ vanishes for extremal black holes, for then $r_+^{\mathrm{I}} = r_-^{\mathrm{I}}$ meaning that Δ has a double zero:

$$\Delta_I\left(r_+^{\mathrm{I}} = r_-^{\mathrm{I}}\right) = \Delta_I'\left(r_+^{\mathrm{I}} = r_-^{\mathrm{I}}\right) = 0$$

   As a result, $\kappa = 0$ for extremal configurations.

Special Cases

1. For $G = G_0 =$ const the above equation leads to the correct result for the classical Kerr spacetime [60]

$$\kappa_{\text{class}} = \frac{[r_+ - MG_0]}{(r_+^2 + a^2)} = \frac{[r_+ - MG_0]}{2MG_0 r_+} \tag{8.52}$$

   where we have applied the equation $r_+^2 + a^2 = 2MG_0 r_+$ in the last equality.

2. For $a = 0$ we recover κ for the improved Schwarzschild black hole, from (8.50) we get

$$\begin{aligned}\kappa\left[r_+ = 2MG\left(r_+\right)\ ,\ a = 0\right] &= \frac{\left[2MG\left(r_+\right) - MG'\left(r_+\right)2MG\left(r_+\right) - MG\left(r_+\right)\right]}{4M^2G^2\left(r_+\right)} \\ &= \frac{1}{4MG\left(r_+\right)} - \frac{G'\left(r_+\right)}{2G\left(r_+\right)}\end{aligned}$$

which is precisely the expression in (8.26).

8.3 Komar Integrals

The purpose of this and the following three sections is to investigate the effect of the running Newton's constant on the mass and the angular momentum of the Kerr black hole [2]. Since the spacetimes we are dealing with are stationary and axially symmetric, we have already seen in chapter 4 that there exists a Killing vector associated to each one of these two symmetries. We have called them $\boldsymbol{t}$ and $\boldsymbol{\varphi}$, respectively. As a consequence, it is possible to define a mass M and an angular momentum J taking advantage of the so-called Komar integrals [61, 60]. These integrals define covariant conservation laws associated with every infinitesimal coordinate transformation [61]. The identification of the conserved quantity with energy or momentum or a similar quantity depends on the type of tranformation considered. In our case the transformations are those generated by $\boldsymbol{t}$ and $\boldsymbol{\varphi}$. As a result, the Komar formulae for the mass M_{Komar} and J_{Komar} read

$$M_{\text{Komar}} = -\frac{1}{8\pi G_0}\oint_S \nabla^\alpha t^\beta dS_{\alpha\beta} \tag{8.53}$$

$$J_{\text{Komar}} = \frac{1}{16\pi G_0}\oint_S \nabla^\alpha \varphi^\beta dS_{\alpha\beta} \tag{8.54}$$

Here S is a two-sphere at spatial infinity. The surface element $dS_{\alpha\beta}$ is given by

$$dS_{\alpha\beta} = -2n_{[\alpha}r_{\beta]}\sqrt{\sigma}d^2\theta \tag{8.55}$$

where n_α and r_α are the timelike and spacelike normals to S. σ is the determinant of σ_{ab}, the metric induced from $g_{\alpha\beta}$ in the 2-d surface H, and $d^2\theta \equiv d\theta^1 d\theta^2$ with θ^a angular coordinates on H (see appendix I). The integrals for M_{Komar} and J_{Komar} probe the metric under consideration only at spatial infinity. Since the improved

[2]In this section we omit the superscript in r_+^{I}, but we keep in mind that we deal with the corrected event horizon.

Kerr metric equals the classical one far away from the black hole, this implies that the values of $M_{\rm Komar}$ and $J_{\rm Komar}$ are not changed by the RG improvement. It is well known [60] that for the classical metric they coincide with the mass and angular momentum parameters which it contains:

$$M_{\rm Komar} = M \ , \ J_{\rm Komar} = J \tag{8.56}$$

Thus for S a surface at spatial infinity, 8.56 holds true also in the improved case.

The mass and angular momentum of spacetime as measured at infinity receives contributions from the pseudo-matter mimicking the quantum effects. Now we break up $M_{\rm Komar}$ and $J_{\rm Komar}$ into two pieces, one which contains ony the effect of the pseudo-matter within the (outer) horizon H, and one which is due to the matter distribution outside H. Including only the first contribution yields quantities M_H and J_H which we refer to as the mass and angular momentum of the black hole, meaning here only the portion of space bounded by H

The relation between the parameters M and J calculated at the spatial infinity and the quantities $M_{\rm H}$ and $J_{\rm H}$ calculated at the event horizon can be derived if we consider a spacelike hypersurface Σ extending from the event horizon to spatial infinity. Its inner boundary is H, a two dimensional cross section of the event horizon, and its outer boundary is S. Using Gauss' theorem, we find that M and J can be decomposed as:

$$M = M_{\rm H} + 2\int_\Sigma \left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right) n^\alpha t^\beta \sqrt{h} d^3y \tag{8.57}$$

$$J = J_{\rm H} - \int_\Sigma \left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right) n^\alpha \varphi^\beta \sqrt{h} d^3y \tag{8.58}$$

Here h_{ab} is the metric induced in Σ and y^a $(a = 1, 2, 3)$ are coordinates intrisic to the hypersurface (see appendix I). $M_{\rm H}$ and $J_{\rm H}$ are the black-hole mass and angular momentum, respectively. They are given by surface integrals over H:

$$M_{\rm H} = -\frac{1}{8\pi G_0}\int_{\rm H} \nabla^\alpha t^\beta ds_{\alpha\beta} \tag{8.59}$$

$$J_{\rm H} = \frac{1}{16\pi G_0}\int_{\rm H} \nabla^\alpha \varphi^\beta ds_{\alpha\beta} \tag{8.60}$$

with

$$ds_{\alpha\beta} = 2\xi_{[\alpha} N_{\beta]}\sqrt{\sigma} d^2\theta = \left(\xi_\alpha N_\beta - \xi_\beta N_\alpha\right)\sqrt{\sigma} d^2\theta \tag{8.61}$$

being $\xi_\alpha = t_\alpha + \Omega\varphi_\alpha$ the light-like Killing vector tangent and normal to H. N_α is an auxiliary null vector such that $N_\alpha\xi^\alpha = -1$ and $N_\alpha N^\alpha = 0$[3].

Equations (8.57) and (8.58) are interpreted as follows: The total mass M (angular momentum J) is given by a contribution M_H (J_H) from the black hole, plus a contribution from the matter distribution outside. If the black hole is in vacuum, then $M = M_H$ and $J = J_H$. According to the discussion of chapter 7 we expect that $M_H \neq M$ and $J_H \neq J$ when the contributions of the "quantum fluid" are taken into account. In the next two sections we corroborate this fact by explicitly calculating M_H and J_H, and we analyse the results obtained.

8.4 Mass of the Improved Kerr Black Hole

In order to calculate the mass of the improved black hole we start by transforming the Komar formula (8.59) of $M_{\rm H}$ to a more suitable form. Substituting the area element (8.61) in (8.59) leads to

$$\begin{aligned} M_{\rm H} &= -\frac{1}{8\pi G_0}\int_{\rm H} \nabla^\alpha t^\beta ds_{\alpha\beta} \\ &= -\frac{1}{8\pi G_0}\int_{\rm H} \nabla^\alpha t^\beta \left(\xi_\alpha N_\beta - \xi_\beta N_\alpha\right) ds \end{aligned} \tag{8.62}$$

Changing dummy indices in (8.62) gives the following:

$$M_{\rm H} = -\frac{1}{8\pi G_0}\int_{H} \xi_\alpha N_\beta \left(\nabla^\alpha t^\beta - \nabla^\beta t^\alpha\right) ds$$

At this point we exploit the Killing equation (4.8) applied to t^β:

$$\nabla^\alpha t^\beta + \nabla^\beta t^\alpha = 0 \tag{8.63}$$

As a result we have

$$M_{\rm H} = -\frac{1}{4\pi G_0}\int_{\rm H} \xi_\alpha N_\beta \left(\nabla^\alpha t^\beta\right) ds \tag{8.64}$$

Since B-L coordinates are non-regular at the event horizon we choose to evaluate $M_{\rm H}$ by performing the integral in (8.64) using the ingoing E-F coordinates. We start by calculating the α-sum in $\xi_\alpha N_\beta\left(\nabla^\alpha t^\beta\right)$ by evaluating the two contributing terms

[3]For more details about how to define hypersurface elements in a generic pseudo-Riemannian manifold see reference [60].

separately. In fact,

$$\begin{aligned}\xi_\alpha N^\beta \left(\nabla^\alpha t_\beta\right) &= \xi_r \delta^r_\alpha N^\beta \left(\nabla^\alpha t_\beta\right) \\ &= \xi_r \left(N^v \nabla^r t_v + N^r \nabla^r t_r\right)\end{aligned} \tag{8.65}$$

where we have used eq. (8.47) in the form $\xi_\alpha = \xi_r \delta^r_\alpha$. Now we find expressions for $\nabla^r t_v$ and $\nabla^r t_r$. From equation (F.5) we have

$$\nabla_\alpha t_\beta = \frac{1}{2}\left(\frac{\partial g_{\beta v}}{\partial x^\alpha} - \frac{\partial g_{v\beta}}{\partial x^\beta}\right) \tag{8.66}$$

and substituting $t_v = g_{vv}$ and $t_r = g_{vr}$ from (8.19) leads to

$$\begin{aligned}\nabla^r t_v &= g^{r\alpha}\nabla_\alpha t_v = \frac{g^{r\alpha}}{2}\left(\frac{\partial g_{vv}}{\partial x^\alpha} - \frac{\partial g_{\alpha v}}{\partial v}\right) \\ &= \frac{g^{r\alpha}}{2}\left(\frac{\partial g_{vv}}{\partial x^\alpha}\right)\end{aligned} \tag{8.67}$$

where we have used the time-independence of the metric in the last step. The only non-vanishing component of $g^{r\alpha}$ is g^{rr}, hence (8.67) turns out to be:

$$\nabla^r t_v = \frac{g^{rr}}{2}\left(\frac{\partial g_{vv}}{\partial r}\right) \tag{8.68}$$

Similarly we find the following expression for $\nabla^r t_r$:

$$\nabla^r t_r = -\frac{g^{rv}}{2}\left(\frac{\partial g_{vv}}{\partial r}\right) - \frac{g^{r\varphi}}{2}\left(\frac{\partial g_{\varphi v}}{\partial r}\right) \tag{8.69}$$

Now we are able to substitute (8.68) and (8.69) in (8.65), which gives

$$\xi_\alpha N^\beta \left(\nabla^\alpha t_\beta\right) = \xi_r \left[\left(\frac{\partial g_{vv}}{\partial r}\right)\left(N^v \frac{g^{rr}}{2} - N^r \frac{g^{rv}}{2}\right) - \frac{g^{r\varphi}}{2} N^r \left(\frac{\partial g_{\varphi v}}{\partial r}\right)\right] \tag{8.70}$$

And inserting (8.70) in (8.64) leads to

$$M_H = -\frac{1}{8\pi G_0}\int_H \xi_r \left[\left(\frac{\partial g_{vv}}{\partial r}\right)\left(N^v g^{rr} - N^r g^{rv}\right) - g^{r\varphi} N^r \left(\frac{\partial g_{\varphi v}}{\partial r}\right)\right] ds \tag{8.71}$$

In order to evaluate the integral of (8.71) at r_+ we need to find, in the E-F representation, the components $g^{rr}|_{r_+}$, $g^{rv}|_{r_+}$, $g^{r\varphi}|_{r_+}$, $N^v|_{r_+}$, $N^r|_{r_+}$, $\xi_r|_{r_+}$, and the derivatives $\frac{\partial g_{vv}}{\partial r}\Big|_{r_+}$ and $\frac{\partial g_{\varphi v}}{\partial r}\Big|_{r_+}$. By using the expressions in appendix I for each one of these terms, we can proceed as follows. First, substituting $g^{rr}|_{r_+} = 0$ and $\xi_r N^r = -1$ leads to

$$M_H = -\frac{1}{8\pi G_0}\int_H \left[\left(\frac{\partial g_{vv}}{\partial r}\right) g^{rv} + g^{r\varphi}\left(\frac{\partial g_{\varphi v}}{\partial r}\right)\right] ds \tag{8.72}$$

The components $g^{rv}|_{r_+}$ and $g^{r\varphi}|_{r_+}$ in (8.72) are given in (I.25) and (I.25) as $g^{rv}|_{r_+} = \left(r_+^2 + a^2\right) / \left.\rho^2\right|_{r_+}$ and , $g^{r\varphi}|_{r_+} = a/ \left.\rho^2\right|_{r_+}$ respectively. The derivatives $\left.\frac{\partial g_{vv}}{\partial r}\right|_{r_+}$ and $\left.\frac{\partial g_{\varphi v}}{\partial r}\right|_{r_+}$ are expressed in (I.28) and (I.29) as linear combinations of trigonometrical functions of θ, which will simplify the integration. With the usual abbreviation $\rho^2 = r^2 + a^2 \cos^2\theta$ we have explicitly

$$\begin{aligned}
\left.\frac{\partial g_{vv}}{\partial r}\right|_{r_+} &= 2M\frac{\partial}{\partial r}\left(\frac{G(r)\,r}{\rho^2}\right)\Bigg|_{r_+} \\
&= \frac{1}{\left(r^2+a^2\cos^2\theta\right)^2}\left(2Mr^2\right)\left[rG'(r) - G(r)\right]\Bigg|_{r_+} \\
&\quad + \frac{\cos^2\theta}{\left(r^2+a^2\cos^2\theta\right)^2}\left(2Ma^2\right)\left[rG'(r)+G(r)\right]\Big|_{r_+}
\end{aligned} \tag{8.73}$$

$$\begin{aligned}
\left.\frac{\partial g_{\varphi v}}{\partial r}\right|_{r_+} &= -2Ma\sin^2\theta\frac{\partial}{\partial r}\left(\frac{G(r)\,r}{\rho^2}\right)\Bigg|_{r_+} \\
&= -\frac{\sin^2\theta}{\left(r^2+a^2\cos^2\theta\right)^2}\left(2Mar^2\right)\left[rG'(r) - G(r)\right]\Big|_{r_+} \\
&\quad - \frac{\sin^2\theta\cos^2\theta}{\left(r^2+a^2\cos^2\theta\right)^2}\left(2Ma^3\right)\left[rG'(r)+G(r)\right]\Big|_{r_+}
\end{aligned} \tag{8.74}$$

Finally the area element $ds|_{r_+}$ needed in (8.72) is also derived in appendix I and is given in (I.12) as $ds|_{r_+} = \left(r_+^2 + a^2\right)\sin\theta d\theta d\varphi$. Substituting everything into (8.72) leads to

$$\begin{aligned}
M_H &= -\frac{1}{8\pi G_0}\int_0^{2\pi}\int_0^{\pi}\left(r_+^2+a^2\right)\sin\theta d\theta d\varphi \times \qquad (8.75)\\
&\times \left[\left(2M\frac{\partial}{\partial r}\left(\frac{G(r)\,r}{\rho^2}\right)\Bigg|_{r_+}\right)\left(\frac{r_+^2+a^2}{\left.\rho^2\right|_{r_+}}\right) - \frac{2Ma^2\sin^2\theta}{\left.\rho^2\right|_{r_+}}\frac{\partial}{\partial r}\left(\frac{G(r)\,r}{\rho^2}\right)\Bigg|_{r_+}\right]
\end{aligned}$$

Integration in φ gives a factor of 2π. We can also factorize $\left.\frac{\partial}{\partial r}\left(\frac{G(r)r}{\rho^2}\right)\right|_{r_+}$ to get

$$M_H = -\frac{M\left(r_+^2+a^2\right)}{2G_0}\int_0^{\pi}\sin\theta\frac{\partial}{\partial r}\left(\frac{G(r)\,r}{\rho^2}\right)\Bigg|_{r_+} d\theta \tag{8.76}$$

where we have used $\rho^2 = r^2 + a^2\cos^2\theta$.

We can go further towards the integration in θ by expanding the derivative in r:

$$M_H = -\frac{M\left(r_+^2+a^2\right)}{2G_0}\int_0^{\pi}\frac{\sin\theta}{\left.\rho^4\right|_{r_+}}\left\{\left.\rho^2\right|_{r_+}\left[G'(r_+)\,r_+ + G(r_+)\right] - 2G(r_+)\,r_+^2\right\}d\theta \tag{8.77}$$

Now substituting $\rho^2 = r^2 + a^2\cos^2\theta$ and separating different types of integrals in θ gives

$$\begin{aligned} M_H &= -\frac{r_+^2 M\left(r_+^2+a^2\right)}{2G_0}\left[G'\left(r_+\right)r_+ - G\left(r_+\right)\right]\int_0^\pi \frac{\sin\theta}{\left(r_+^2+a^2\cos^2\theta\right)^2}d\theta \\ &\quad -\frac{Ma^2\left(r_+^2+a^2\right)}{2G_0}\left[G'\left(r_+\right)r_+ + G\left(r_+\right)\right]\int_0^\pi \frac{\sin\theta\cos^2\theta}{\left(r_+^2+a^2\cos^2\theta\right)^2}d\theta \end{aligned} \tag{8.78}$$

At this point we define

$$I_1 = \int_0^\pi \frac{\sin\theta}{\left(r_+^2+a^2\cos^2\theta\right)^2}d\theta \tag{8.79}$$

and

$$I_2 = \int_0^\pi \frac{\sin\theta\cos^2\theta}{\left(r_+^2+a^2\cos^2\theta\right)^2}d\theta \tag{8.80}$$

so that (8.78) is reduced to

$$M_H = -\frac{M\left(r_+^2+a^2\right)}{2G_0}\left\{r_+^2\left[G'\left(r_+\right)r_+ - G\left(r_+\right)\right]I_1 + a^2\left[G'\left(r_+\right)r_+ + G\left(r_+\right)\right]I_2\right\} \tag{8.81}$$

The definite integrals I_1 and I_2 from (8.79) and (8.80) are analytically solvable, see appendix I). They are given by (I.34) and (I.37):

$$I_1 = \left(\frac{1}{ar_+^3}\right)\left\{\frac{r_+a}{r_+^2+a^2} + \arctan\left[\left(\frac{a}{r_+}\right)\right]\right\} \tag{8.82}$$

$$I_2 = \left(\frac{1}{a^3r_+}\right)\left\{\arctan\left[\left(\frac{a}{r_+}\right)\right] - \frac{r_+a}{\left(r_+^2+a^2\right)}\right\} \tag{8.83}$$

The substitution of (8.82) and (8.83) in (8.81) and further simplification leads to the final result. We find that the black-hole mass is given by

$$M_H = M\frac{G\left(r_+\right)}{G_0}\left\{1 - \left[\frac{\left(r_+^2+a^2\right)G'\left(r_+\right)}{aG\left(r_+\right)}\right]\arctan\left(\frac{a}{r_+}\right)\right\} \tag{8.84}$$

It is convenient to define the function $F_M\left(r\right)$ as follows:

$$F_M\left(r\right) \equiv \frac{G\left(r\right)}{G_0}\left\{1 - \left[\frac{\left(r^2+a^2\right)G'\left(r\right)}{aG\left(r\right)}\right]\arctan\left(\frac{a}{r}\right)\right\} \tag{8.85}$$

Obviously $F_M(r_+)$ is the factor multiplying M on the RHS of eq. (8.84). Hence the black hole mass reads

$$M_H = M \; F_M(r_+) \tag{8.86}$$

We have analysed graphically the r-dependence of $F_M(r)$ from a radius $r_0 < r_+$ to infinity. Figure 8.1 shows this dependence resulting from the running $G(r)$ with the $d(r) = r$ approximation. This plot indicates that $F_M(r) < 1$ and therefore $M_H < M$. M_H is the mass of the matter and pseudo-matter contained within the event horizon, whereas M is the mass measured at spatial infinity. Since M is equal to M_H plus a positive contribution of the pseudo-matter between H and spatial infinity, we conclude that this latter contribution **increases** the amount of mass that an asymptotic observer measures. This is precisely an **antiscreening** effect. Furthermore, from (8.84) we know that $M_H = M$ if the running is switched off. As a result, we conclude that the quantum fluctuations described by the effective matter show an antiscreening behavior: the mass of a gravitating body seems to be bigger at large distances than at small distances.

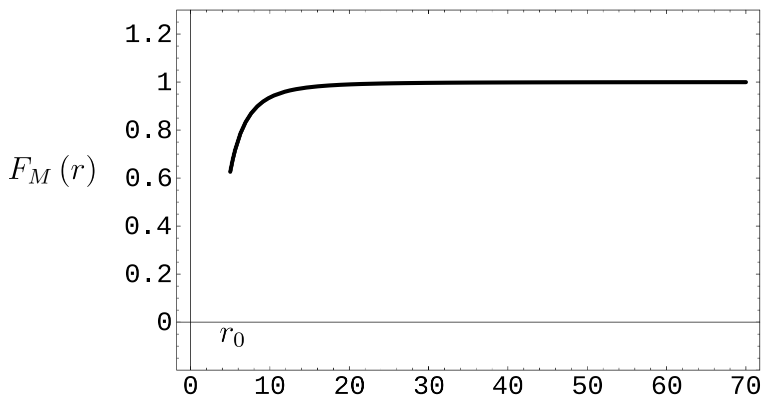

Fig. 8.1.

Radial dependence of the factor $F_M(r)$, for $M = 15$, $a = 5$, and $\bar{w} = 1$ employing the $d(r) = r$ approximation. The function is plotted from $r_0 = 5 < r_+ \approx 30$ to infinity. The plot shows that $F(r) < 1$ in the domain $[r_0\,,\,\infty)$, and $F(r) \to 1$ when $r \to \infty$.

8.5 Angular Momentum of the improved Kerr Black Hole

The evaluation of J_H in (8.60) is carried out in a similar way as in the case of M_H. We start by transforming (8.60) into

$$J_H = \frac{1}{8\pi G_0} \int_H \xi_\alpha N_\beta \nabla^\alpha \varphi^\beta ds \tag{8.87}$$

which results from changing dummy indices and applying the Killing equation for φ^β,

$$\nabla^\alpha \varphi^\beta + \nabla^\beta \varphi^\alpha = 0 \tag{8.88}$$

Inserting (8.47), $\xi_\alpha = \xi_r \delta^r_\alpha$, in the integrand of (8.87) leads to

$$\xi_\alpha N^\beta \nabla^\alpha \varphi_\beta = \xi_r \left(N^v \nabla^r \varphi_v + N^r \nabla^r \varphi_r \right) \tag{8.89}$$

From (F.5) we have for $\nabla^\alpha \varphi^\beta$ the following result

$$\nabla_\alpha \varphi_\beta = \frac{1}{2} \left(\frac{\partial g_{\beta\psi}}{\partial x^\alpha} - \frac{\partial g_{\alpha\psi}}{\partial x^\beta} \right)$$

Hence the derivatives in (8.89) are given by

$$\nabla^r \varphi_v = g^{r\alpha} \nabla_\alpha \varphi_v = \frac{g^{rr}}{2} \left(\frac{\partial g_{v\psi}}{\partial r} \right) \tag{8.90}$$

and

$$\nabla^r \varphi_r = -\frac{g^{rv}}{2} \left(\frac{\partial g_{v\psi}}{\partial r} \right) - \frac{g^{r\psi}}{2} \left(\frac{\partial g_{\psi\psi}}{\partial r} \right) \tag{8.91}$$

Here we wrote down only the non-zero components $g_{\alpha\beta}$ of the Kerr metric and we used that they depend on θ and r only. Now inserting (8.90), (8.91) in (8.89) leads to

$$\xi_\alpha N^\beta \nabla^\alpha \varphi_\beta = \frac{\xi_r}{2} \left[N^v g^{rr} \left(\frac{\partial g_{v\psi}}{\partial r} \right) - N^r g^{rv} \left(\frac{\partial g_{v\psi}}{\partial r} \right) - N^r g^{r\psi} \left(\frac{\partial g_{\psi\psi}}{\partial r} \right) \right] \tag{8.92}$$

After applying $g^{rr} = 0$ and $\xi_r N^r = -1$ to (8.92), the integral (8.87) turns out to be

$$J_H = \frac{1}{16\pi G_0} \int_H^\beta \left[g^{rv} \left(\frac{\partial g_{v\psi}}{\partial r} \right) + g^{r\psi} \left(\frac{\partial g_{\psi\psi}}{\partial r} \right) \right] ds \tag{8.93}$$

The various factors appearing in the integrand of (8.93) are given by (see appendix I)

$$
\begin{aligned}
\left.\frac{\partial g_{v\psi}}{\partial r}\right|_{r_+} &= -2Ma\sin^2\theta\frac{\partial}{\partial r}\left.\left(\frac{G(r)\,r}{\rho^2}\right)\right|_{r_+}, \quad \left.\frac{\partial g_{\psi\psi}}{\partial r}\right|_{r_+} = \sin^2\theta\frac{\partial}{\partial r}\left.\left(\frac{\Sigma_I}{\rho^2}\right)\right|_{r_+} \\
\left. g^{rv}\right|_{r_+} &= \frac{r_+^2+a^2}{\left.\rho^2\right|_{r_+}}, \quad \left. g^{r\psi}\right|_{r_+} = \frac{a}{\left.\rho^2\right|_{r_+}}
\end{aligned}
\tag{8.94}
$$

The area element is again defined by (I.12) as $\left. ds\right|_{r_+} = \left(r_+^2+a^2\right)\sin\theta d\theta d\varphi$. Substituting these terms in (8.93) gives

$$
\begin{aligned}
J_H &= \frac{1}{16\pi G_0}\int_0^{2\pi}\int_0^{\pi}\left(r_+^2+a^2\right)\sin\theta d\theta d\varphi \times \\
&\times\left[-2Ma\sin^2\theta\left(\frac{r_+^2+a^2}{\left.\rho^2\right|_{r_+}}\right)\left(\frac{\partial}{\partial r}\left.\left(\frac{G(r)\,r}{\rho^2}\right)\right|_{r_+}\right)+\right. \\
&\left.+\left(\frac{a}{\left.\rho^2\right|_{r_+}}\right)\left(\sin^2\theta\frac{\partial}{\partial r}\left.\left(\frac{\Sigma_I}{\rho^2}\right)\right|_{r_+}\right)\right]
\end{aligned}
\tag{8.95}
$$

The procedure of evaluation of (8.95) is carried out along similar lines as that for M_H. We start by performing the partial derivative of Σ_I/ρ^2 in the last term:

$$
\frac{\partial}{\partial r}\left.\left(\frac{\Sigma_I}{\rho^2}\right)\right|_{r_+} = \frac{\partial}{\partial r}\left.\left(\frac{\left(r^2+a^2\right)^2-a^2\sin^2\theta\Delta_I}{\rho^2}\right)\right|_{r_+}
\tag{8.96}
$$

Substituting $\Delta_I = r^2+a^2-2MrG(r)$ leads to

$$
\begin{aligned}
\frac{\partial}{\partial r}\left.\left(\frac{\Sigma_I}{\rho^2}\right)\right|_{r_+} &= \frac{\partial}{\partial r}\left.\left[r^2+a^2+2MrG(r)\,a^2\left(\frac{\sin^2\theta}{\rho^2}\right)\right]\right|_{r_+} \\
&= 2r_+ + 2Ma^2\sin^2\theta\,\frac{\partial}{\partial r}\left.\left[\left(\frac{rG(r)}{\rho^2}\right)\right]\right|_{r_+}
\end{aligned}
\tag{8.97}
$$

Now, after inserting (8.97) in (8.95), integrating over φ and simplifying, we find

$$
J_H = \frac{a\left(r_+^2+a^2\right)}{4G_0}\int_0^{\pi}\frac{\sin^3\theta d\theta}{\left.\rho^2\right|_{r_+}}\left\{r_+ - M\left.\rho^2\right|_{r_+}\frac{\partial}{\partial r}\left.\left(\frac{rG(r)}{\rho^2}\right)\right|_{r_+}\right\}
\tag{8.98}
$$

At this stage we have to perform the partial derivative in r of $rG(r)/\rho^2$, using $\rho^2 = r_+^2 + a^2\cos^2\theta$. We have then

$$
\frac{\partial}{\partial r}\left.\left(\frac{G(r)\,r}{\rho^2}\right)\right|_{r_+} = \frac{r_+^2\left[G'(r_+)\,r_+ - G(r_+)\right]+a^2\cos^2\theta\left[r_+G'(r_+)+G(r_+)\right]}{\left(r_+^2+a^2\cos^2\theta\right)^2}
\tag{8.99}
$$

Substituting (8.99) in (8.98) gives

$$\begin{aligned} J_H &= \left(\frac{ar_+}{4G_0}\right)\left(r_+^2+a^2\right)\int_0^\pi \frac{\sin^3\theta d\theta}{r_+^2+a^2\cos^2\theta} \\ &- \left(\frac{aMr_+^2}{4G_0}\right)\left(r_+^2+a^2\right)\left[G'\left(r_+\right)r_+-G\left(r_+\right)\right]\int_0^\pi \frac{\sin^3\theta d\theta}{\left(r_+^2+a^2\cos^2\theta\right)^2} \\ &- \left(\frac{Ma^3}{4G_0}\right)\left(r_+^2+a^2\right)\left[r_+G'\left(r_+\right)+G\left(r_+\right)\right]\int_0^\pi \frac{\sin^3\theta\cos^2\theta d\theta}{\left(r_+^2+a^2\cos^2\theta\right)^2} \end{aligned} \tag{8.100}$$

Here we are led to define the following integrals analogous to I_1 and I_2 used above:

$$I_3 = \int_0^\pi \frac{\sin^3\theta d\theta}{r_+^2+a^2\cos^2\theta} \tag{8.101}$$

$$I_4 = \int_0^\pi \frac{\sin^3\theta d\theta}{\left(r_+^2+a^2\cos^2\theta\right)^2} \tag{8.102}$$

$$I_5 = \int_0^\pi \frac{\sin^3\theta\cos^2\theta d\theta}{\left(r_+^2+a^2\cos^2\theta\right)^2} \tag{8.103}$$

Thus J_H of (8.100) can be written as

$$\begin{aligned} J_H &= \left(\frac{ar_+}{4G_0}\right)\left(r_+^2+a^2\right)I_3 - \left(\frac{aMr_+^2}{4G_0}\right)\left(r_+^2+a^2\right)\left[G'\left(r_+\right)r_+-G\left(r_+\right)\right]I_4 \\ &- \left(\frac{Ma^3}{4G_0}\right)\left(r_+^2+a^2\right)\left[r_+G'\left(r_+\right)+G\left(r_+\right)\right]I_5 \end{aligned} \tag{8.104}$$

Simplifying and factorizing $G'(r_+)$ and $G(r_+)$ reduces (8.104) to:

$$J_H = \frac{\left(r_+^2+a^2\right)a}{4G_0}\left\{r_+I_3 - Mr_+G'\left(r_+\right)I_{4+5} + G\left(r_+\right)MI_{4-5}\right\} \tag{8.105}$$

Here we defined I_{4+5} and I_{4-5} as the linear combinations

$$I_{4+5} = \left(r_+^2I_4 + a^2I_5\right) \tag{8.106}$$

$$I_{4-5} = \left(r_+^2I_4 - a^2I_5\right) \tag{8.107}$$

As in the case of I_1 and I_2, integrals I_3 to I_5 are also analytically solvable. They are given in appendix I by equations (I.46) to (I.48):

$$I_3 = -\frac{2}{a^2} + 2\left[\frac{\left(r_+^2+a^2\right)}{a^3r_+}\right]\arctan\left(\frac{a}{r_+}\right) \tag{8.108}$$

$$I_4 = \left(\frac{a^2 - r_+^2}{a^3 r_+^3}\right) \arctan\left(\frac{a}{r_+}\right) + \left[\frac{1}{a^2 r_+^2}\right] \tag{8.109}$$

$$I_5 = -\frac{3}{a^4} + \frac{\left(3r_+^2 + a^2\right)}{r_+ a^5} \arctan\left(\frac{a}{r_+}\right) \tag{8.110}$$

After inserting expressions (8.108) to (8.110) in (8.106) and (8.107), we find:

$$I_{4+5} = r_+^2 I_4 + a^2 I_5 = 2\left(a^2 + r_+^2\right) \frac{\arctan\left(\frac{a}{r_+}\right)}{r_+ a^3} - \frac{2}{a^2} \tag{8.111}$$

$$I_{4-5} = r_+^2 I_4 - a^2 I_5 = -\frac{4r_+}{a^3} \arctan\left(\frac{a}{r_+}\right) + \frac{4}{a^2} \tag{8.112}$$

Now substituting (8.108), (8.111) and (8.112) in (8.105) gives

$$J_H = \frac{\left(r_+^2 + a^2\right) a}{2G_0} \times$$

$$\times \left\{ -\frac{r_+}{a^2} + \frac{\left(r_+^2 + a^2\right)}{a^3} \arctan\left(\frac{a}{r_+}\right) - M r_+ G'(r_+) \left(\left(a^2 + r_+^2\right) \frac{\arctan\left(\frac{a}{r_+}\right)}{r_+ a^3} - \frac{1}{a^2} \right) \right.$$
$$\left. + G(r_+) M \left(-\frac{2r_+}{a^3} \arctan\left(\frac{a}{r_+}\right) + \frac{2}{a^2} \right) \right\} \tag{8.113}$$

the intermediate result (8.113) can be further simplified by factorizing $\arctan\left(\frac{a}{r_+}\right)$ and exploiting $r_+^2 + a^2 - 2MG(r_+) r_+ = 0$. Then we find

$$J_H = \frac{\left(r_+^2 + a^2\right)}{2G_0} \left\{ \left[r_+ - \frac{\left(a^2 + r_+^2\right)}{a} \arctan\left(\frac{a}{r_+}\right) \right] \left[\frac{MG'(r_+)}{a} \right] + \left[\frac{2G(r_+) M - r_+}{a} \right] \right\} \tag{8.114}$$

Two more applications of $r_+^2 + a^2 - 2MG(r_+) r_+ = 0$ turn out to be quite useful. The replacements

$$\begin{aligned} \frac{r_+^2 + a^2}{2G_0} &= \frac{MG(r_+) r_+}{G_0} \\ \frac{2G(r_+) M - r_+}{a} &= a \end{aligned} \tag{8.115}$$

transform (8.114) to an expression in which the classical result $J_H = J$ is isolated, and therefore the quantum correction can be easily identified. This brings us to the expression we were looking for:

$$J_H = \left\{ J + \left[1 - \frac{2MG(r_+)}{a} \arctan\left(\frac{a}{r_+}\right)\right] \left[\frac{M^2 G'(r_+) r_+^2}{a}\right]\right\} \frac{G(r_+)}{G_0} \tag{8.116}$$

We consider (8.116) as the final result of our calculation of J_H.

Eq. (8.116) can be written as a product $J_H = J F_J(r_+)$. We define $F_J \equiv F_J(r)$ as follows:

$$F_J(r) \equiv \frac{G(r)}{G_0} \left\{ 1 + \frac{r^2 M G'(r)}{a^2} \left[1 - \frac{2MG(r)}{a} \arctan\left(\frac{a}{r}\right)\right]\right\} \tag{8.117}$$

Figure 8.2 shows the r-dependence of $F_J(r)$ for $M = 15$, $a = 5$, $\bar{w} = 1$, employing the approximation $d(r) = r$. The similarity with $F_M(r)$ in figure 8.1 is clear: $F_J(r) < 1$ for $r_0 < r < \infty$ with $r_0 < r_+$. As a result, the conclusion is the same: The relationship (8.116) displays an **antiscreening of the angular momentum** of the black hole similar to the antiscreening of the mass. Since $J_H < J$, we conclude that the amount of angular momentum contained inside the horizon, J_H, is **increased** by the contribution of the quantum fluctuations or pseudo-matter between the event horizon and spatial infinity. This result obtains for any value of M and a.

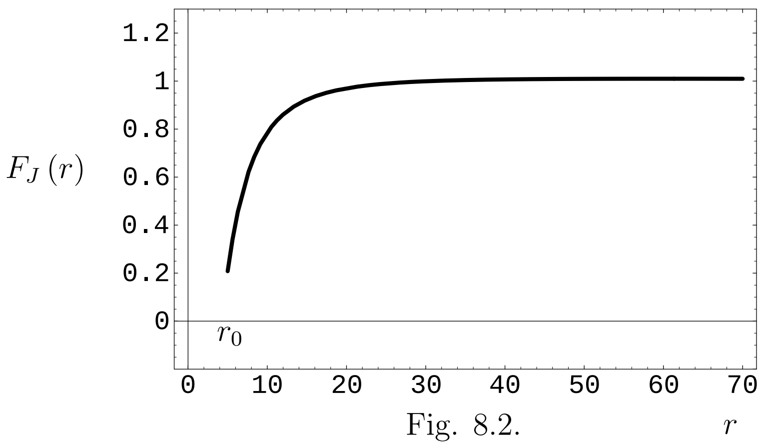

Fig. 8.2.

Radial dependence of the factor $F_J(r)$ for $M = 15$, $a = 5$, and $\bar{w} = 1$ employing the $d(r) = r$ approximation. The function is plotted from $r_0 = 5 < r_+ \approx 30$ to infinity. The plot shows that $F(r) < 1$ in the domain $[r_0\ ,\ \infty)$, and $F(r) \to 1$ when $r \to \infty$.

8.5.1 Low Angular Momentum Expansions for M_H and J_H

By performing expansions for low angular momentum J we are able to isolate the contributions to M_H and J_H independent of J if they exist. Employing a as the expansion parameter we have the following series:

$$\arctan\left(\frac{a}{r_+}\right)=\frac{a}{r_+}-\frac{a^3}{3r_+^3}+\frac{a^5}{5r_+^5}-\frac{a^7}{7r_+^7}+\cdots \tag{8.118}$$

Substituting eq. (8.118) in expression (8.84) for M_H gives

$$M_H=\frac{MG\left(r_+\right)}{G_0}\left\{1-\left[\frac{\left(r_+^2+a^2\right)G'\left(r_+\right)}{aG\left(r_+\right)}\right]\left(\frac{a}{r_+}-\frac{a^3}{3r_+^3}+\frac{a^5}{5r_+^5}-\frac{a^7}{7r_+^7}+\cdots\right)\right\}$$

Applying $2G\left(r_+\right)Mr_+=r_+^2+a^2$ in the term with G' leads to the desired expansion of M_H in powers of a:

$$M_H=\frac{MG\left(r_+\right)}{G_0}\left\{1-2MG'\left(r_+\right)\left(1-\frac{a^2}{3r_+^2}+\frac{a^4}{5r_+^4}-\frac{a^6}{7r_+^6}+\cdots\right)\right\} \tag{8.119}$$

The expansion (8.119) shows a leading contribution to M_H independent of J given by

$$\left.M_H\right|_{a=0}=\frac{MG\left(r^{\mathrm{I}}_{\mathrm{Sch}+}\right)}{G_0}\left[1-2MG'\left(r^{\mathrm{I}}_{\mathrm{Sch}+}\right)\right] \tag{8.120}$$

This is exactly the mass of the improved Schwarzschild black hole, as it should be.

In order to find the expansion of J_H we substitute (8.118) in (8.116) to have

$$J_H=\left\{J+\left[1-\frac{2MG\left(r_+\right)}{a}\left(\frac{a}{r_+}-\frac{a^3}{3r_+^3}+\frac{a^5}{5r_+^5}-\frac{a^7}{7r_+^7}+\cdots\right)\right]\left[\frac{M^2G'\left(r_+\right)r_+^2}{a}\right]\right\}\frac{G\left(r_+\right)}{G_0} \tag{8.121}$$

Expanding the term with G' leads to

$$\begin{aligned}J_H \;=\;& \frac{G\left(r_+\right)}{G_0}\left\{J+\frac{M^2G'\left(r_+\right)r_+^2}{a}+\right.\\ &\left.-\left[2G\left(r_+\right)Mr_+\right]M^2G'\left(r_+\right)\left(\frac{1}{a}-\frac{a}{3r_+^2}+\frac{a^3}{5r_+^4}-\frac{a^5}{7r_+^6}+\cdots\right)\right\}\end{aligned} \tag{8.122}$$

Now exploiting $2G\left(r_+\right)Mr_+=r_+^2+a^2$ in the same term and simplifying gives the final result

$$J_H=\frac{JG\left(r_+\right)}{G_0}\left\{1+MG'\left(r_+\right)\left[-\frac{2}{3}+\frac{2a^2}{15r_+^2}+O\left(\frac{a^4}{r_+^4}\right)\right]\right\} \tag{8.123}$$

Eq. (8.123) shows clearly that $\left.J_H\right|_{a=0}=0$ as expected.

8.6 Smarr's Formula

In chapter 2 we have already discussed the Smarr's formula in the context of the classical Kerr spacetime. We have mentioned that this formula is valid for all stationary and axially symmetric spacetimes. Now we show that our results (8.84) and (8.116) for M_H and J_H, together with the relations

$$\begin{aligned} \Omega_{\rm H} &= \frac{a}{\left(r_+^{\rm I}\right)^2 + a^2} = \frac{a}{2MG\left(r_+^{\rm I}\right) r_+^{\rm I}} \ , \ \mathcal{A} = 4\pi\left[\left(r_+^{\rm I}\right)^2 + a^2)\right] \\ \kappa &= \frac{\Delta'}{2\left[\left(r_+^{\rm I}\right)^2 + a^2\right]} \ , \ \Delta = \left(r_+^{\rm I}\right)^2 + a^2 - 2MG\left(r_+^{\rm I}\right) r_+^{\rm I} \end{aligned} \tag{8.124}$$

fulfill the Smarr's formula

$$M_H = 2\Omega_{\rm H} J_H + \frac{\kappa\mathcal{A}}{4\pi G_0} \tag{8.125}$$

This formula valid for the improved Kerr spacetime, has exactly the same structure as in the classical case.

To start, let us calculate $2\Omega_{\rm H} J_H$ from (8.124) and (8.116), as follows

$$2\Omega_{\rm H} J_H = \frac{1}{G_0}\left\{\frac{a^2}{r_+} + MG'\left(r_+\right) r_+ - \frac{2M^2 G\left(r_+\right) r_+ G'\left(r_+\right)}{a} \arctan\left(\frac{a}{r_+}\right)\right\} \tag{8.126}$$

From (8.84) we obtain

$$M_H G_0 - MG\left(r_+\right) = -\left[\frac{2M^2 G\left(r_+\right) r_+ G'\left(r_+\right)}{a}\right] \arctan\left(\frac{a}{r_+}\right) \tag{8.127}$$

where we have used

$$r_+^2 + a^2 = 2MG\left(r_+\right) r_+ \tag{8.128}$$

Substituting (8.127) in (8.126) leads to

$$2\Omega_{\rm H} J_H = \frac{1}{G_0}\left\{\frac{a^2}{r_+} + MG'\left(r_+\right) r_+ + M_H G_0 - MG\left(r_+\right)\right\} \tag{8.129}$$

On the other hand, using (8.124) we can represent $\kappa\mathcal{A}/4\pi G_0$ as follows

$$\frac{\kappa\mathcal{A}}{4\pi G_0} = \frac{\Delta'}{2G_0} \tag{8.130}$$

But from (8.124), we have for Δ'

$$\Delta' = 2r_+ - 2M\left[G'\left(r_+\right) r_+ + G\left(r_+\right)\right] \tag{8.131}$$

Now we can insert (8.131) in (8.130) to find:

$$\frac{\kappa\mathcal{A}}{4\pi G_0}=\frac{r_+}{G_0}-\frac{M}{G_0}\left[G'\left(r_+\right)r_++G\left(r_+\right)\right] \tag{8.132}$$

At this stage we are able to sum up (8.129) and (8.132) in order to find the right hand side of (8.125):

$$\begin{aligned}2\Omega_{\mathrm{H}}J_H+\frac{\kappa\mathcal{A}}{4\pi G_0} &= \frac{1}{G_0}\left\{\frac{a^2}{r_+}+MG'\left(r_+\right)r_++M_HG_0-MG\left(r_+\right)\right\}+\frac{r_+}{G_0}\\ &\quad -\frac{M}{G_0}\left[G'\left(r_+\right)r_++G\left(r_+\right)\right]\end{aligned} \tag{8.133}$$

Simplifying (8.133) leads directly to

$$2\Omega_{\mathrm{H}}J_H+\frac{\kappa\mathcal{A}}{4\pi G_0}=M_H+\frac{a^2+r_+^2-2MG\left(r_+\right)r_+}{G_0r_+}=M_H \tag{8.134}$$

where we have used again $r_+^2+a^2=2MG\left(r_+\right)r_+$. This completes the proof of Smarr's formula in (8.125).

8.7 The Modified First Law of Black Hole Thermodynamics

As discussed in chapter 2, eq. (2.63), i.e.,

$$\delta M-\Omega_{\mathrm{H}}\delta J=\left(\frac{\kappa}{8\pi G}\right)\delta\mathcal{A} \tag{8.135}$$

is interpreted as the first law of black hole thermodynamics, appealing to its analogy with the first law of standard thermodynamics. As a result, we interpret $\frac{\kappa}{2\pi}$ and $\frac{\mathcal{A}}{4G}$ as the temperature T and the entropy S of the black hole, respectively. Thus we read the relation (8.135) as

$$\delta M-\Omega_{\mathrm{H}}\delta J=T\delta S \tag{8.136}$$

Eq. (8.136) states that $\left(\delta M-\Omega_{\mathrm{H}}\delta J\right)/T$ is an exact differential, namely the "exterior derivative" of a state function $S\equiv S\left(M,J\right)$ which is interpreted as an entropy.

In this section we address the question of whether there exists a relationship analogous to (8.136) for the RG-improved Kerr black hole. In earlier sections we found that its surface gravity and angular momentum are given by[4]

$$\kappa\left(M,J\right) \equiv \frac{r_+^{\mathrm{I}} - M\left[r_+^{\mathrm{I}} G'\left(r_+^{\mathrm{I}}\right) + G\left(r_+^{\mathrm{I}}\right)\right]}{\left(r_+^{\mathrm{I}}\right)^2 + \left(\frac{J}{M}\right)^2} \tag{8.137}$$

$$\Omega_{\mathrm{H}}\left(M,J\right) \equiv \frac{\left(\frac{J}{M}\right)}{\left(r_+^{\mathrm{I}}\right)^2 + \left(\frac{J}{M}\right)^2} \tag{8.138}$$

where $r_+^{\mathrm{I}} \equiv r_+^{\mathrm{I}}\left(M,J\right)$ is a function of mass and angular momentum which can be determined numerically only. Given the results (8.137) and (8.138), the natural question to be asked is whether the resulting $\left(\delta M - \Omega_{\mathrm{H}}\delta J\right)/\kappa$ is an exact differential. Does there exist a state function $f \equiv f\left(M,J\right)$ such that

$$\left(\delta M - \Omega_{\mathrm{H}}\delta J\right)/\kappa \stackrel{?}{=} \delta f \tag{8.139}$$

If f exists one could again try to identify $T = \kappa/2\pi$ with the temperature and f with the entropy of the black hole. (As quantum effects typically lead to modifications of the classical relation $S = \mathcal{A}/4G_0$ [30, 31] it is not to be expected that f is simply proportional to the surface area.)

We dedicate subsection 8.7.4 to the analysis of this possibility. As a conclusion we shall find that no such f exists for any reasonable $G\left(r\right)$. This result entails that, in the improved case, there can be no first law of the form (8.136) in which the temperature T is proportional to the surface gravity κ.

It is, however, a logical possibility that the improvement modifies the classical relation $T = \kappa/2\pi$. Indeed in subsections 8.7.5 and 8.7.6 we shall construct state functions ("zero-forms") $T\left(M,J\right)$ and $S\left(M,J\right)$ which actually do satisfy

$$\delta M - \Omega_{\mathrm{H}}\left(M,J\right)\delta J = T\left(M,J\right)\delta S\left(M,J\right) \tag{8.140}$$

The function $T\left(M,J\right)$, tentatively to be regarded as the black hole's temperature, is not proportional to κ if $J \neq 0$ but equals $\kappa/2\pi$ if $J = 0$. Establishing (8.140) amounts to finding the "integrating factor" $1/T$ with which the differential $\delta M - \Omega_{\mathrm{H}}\left(M,J\right)\delta J$ must be multiplied to obtain an exact one.

Due to the algebraic difficulty of the problem we shall content ourselves with finding a consistent approximation to the integrating factor for low angular momentum J. We start the discussion in subsection 8.7.1 by presenting a general introduction

[4] In this section we write explicitly the fraction J/M instead of the parameter a in order to emphasize the dependence on J and M.

to differential forms and integrating factors. In subsection 8.7.5 we find approximate solutions to the partial differential equation for the integrating factor by using a power series ansatz. In subsection 8.7.6 we analyse the degree of validity of the solution we have found, and in subsection 8.7.7 we use the approximated integrating factor in order to derive $O\left(J^2\right)$ approximations to the first law, the temperature T, and the entropy S of the improved Kerr black hole. We also derive large-M expansions of T and S. In this way we establish a connection between the results for the improved Kerr spacetime achieved in this work and results from the literature, like the large M expansions for T and S of the improved Schwarzschild black hole [30].

8.7.1 Exact Differentials and Integrating Factors

The states an (improved) black hole can be in are labeled by the two parameter M and J. We visualize the corresponding state space as (part of) the 2-dimensional euclidean plane with cartesian coordinates $x^1 = M$, $x^2 = J$. Using the convenient language of differential forms, state functions are zero forms on this space, i.e. scalars $f = f(x) \equiv f(M, J)$.

Defining the exterior derivative as[5]

$$\delta = \delta M \frac{\partial}{\partial M} + \delta J \frac{\partial}{\partial J}$$

we say that a differential form $\boldsymbol{\alpha}$ is *closed* if $\delta\boldsymbol{\alpha} = 0$, and we say it is *exact* if $\boldsymbol{\alpha} = \delta\beta$ where β denotes a $(p-1)$-form when $\boldsymbol{\alpha}$ is a p-form. The state space being 2-dimensional, the only case of interest is $p = 1$. A general 1-form has the expansion

$$\boldsymbol{\alpha} = P(M, J)\,\delta M + N(M, J)\,\delta J \tag{8.141}$$

This 1-form is exact if there exists a zero-form $S(M, J)$ such that $\boldsymbol{\alpha} = \delta S$ or, in components,

$$P(M, J) = \frac{\partial S}{\partial M}\ ,\ N(M, J) = \frac{\partial S}{\partial J}$$

As a result we can write

$$\delta S = P(M, J)\,\delta M + N(M, J)\,\delta J = \left(\frac{\partial S}{\partial M}\right)\delta M + \left(\frac{\partial S}{\partial J}\right)\delta J$$

[5] To conform with the standard notation of thermodynamics we denote the exterior derivative by δ rather than d.

We assume that the states (M, J) form a simply connected subset of the euclidean plane so that $\delta\boldsymbol{\alpha} = 0$ is necessary and sufficient for the exactness of $\boldsymbol{\alpha}$.

It is a well-known result of the calculus of several variables that if $S\,(M, J)$ and its derivatives up to second order (including mixed second derivatives) are continuous then we have:

$$\frac{\partial S}{\partial J \partial M} = \frac{\partial S}{\partial M \partial J} \tag{8.142}$$

Hence (8.142) is a necessary condition for the exactness of (8.141). Or using P and M we write:

$$\frac{\partial P}{\partial J} = \frac{\partial N}{\partial M} \tag{8.143}$$

The demonstration of sufficiency of (8.143) on a simply connected domain is also a result from the elementary theory of ordinary differential equations [62]. As a consequence, the relation (8.143) is the basic equation to show the closedness, and hence in our case, exactness of a differential form. In the case when (8.143) is not fulfilled $\boldsymbol{\alpha}$ is a non-exact one-form.

If $\boldsymbol{\alpha}$ is not exact, one can try to find a function $\mu_\alpha\,(M, J)$ that, after being multiplied to $\boldsymbol{\alpha}$, converts it into an exact differential. In those cases we call $\mu_\alpha\,(M, J)$ an integrating factor of $\boldsymbol{\alpha}$ and we write

$$\delta S = \mu_\alpha\,(M, J)\,\boldsymbol{\alpha} = \mu_\alpha\,(M, J)\,P\,(M, J)\,\delta M + \mu_\alpha\,(M, J)\,N\,(M, J)\,\delta J \tag{8.144}$$

Since $\mu_\alpha \boldsymbol{\alpha}$ is exact by definition of $\mu_\alpha\,(M, J)$, eq. (8.143) is fulfilled with the substitutions $P\,(M, J) \to \mu_\alpha\,(M, J)\,P\,(M, J)$, $N\,(M, J) \to \mu_\alpha\,(M, J)\,N\,(M, J)$. As a result we have:

$$\frac{\partial}{\partial J}\,(\mu_\alpha P) = \frac{\partial}{\partial M}\,(\mu_\alpha N) \tag{8.145}$$

Eq. (8.145) is in fact a quasi-linear partial differential equation in M and J for $\mu_\alpha\,(M, J)$. This can be made more explicit by rewriting it, using the chain rule, as follows:

$$P\left(\frac{\partial \mu_\alpha}{\partial J}\right) - N\left(\frac{\partial \mu_\alpha}{\partial M}\right) = \mu_\alpha\left[\left(\frac{\partial N}{\partial M}\right) - \left(\frac{\partial P}{\partial J}\right)\right] \tag{8.146}$$

The theory of partial differential equations asserts that quasi-linear equations like (8.146) are solvable in general [63, 64]. How difficult it is in practice to find a solution μ_α as a function of M and J depends on the specific form of $P\,(M, J)$ and

$N(M,J)$, of course.

Let us finally discuss the application we are actually interested in. We investigate the 1-form

$$\boldsymbol{\alpha} = \left(\frac{1}{\kappa}\right)\delta M - \left(\frac{\Omega_{\rm H}}{\kappa}\right)\delta J \tag{8.147}$$

where the components $P(M,J) \equiv 1/\kappa$ and $N(M,J) \equiv -\Omega_{\rm H}/\kappa$ are given in terms of the surface gravity and angular velocity of the improved Kerr black hole presented in (8.137) and (8.138). At the classical level, we know that $\boldsymbol{\alpha}$ is exact: $\boldsymbol{\alpha} = \delta(S/2\pi)$ with $S = \mathcal{A}(M,J)/(4G_0)$. It is now a matter of straightforward (but lengthy!) differentiation to check whether the integrability condition (8.143) is fulfilled in the improved case. The result is that this condition is actually *not* satisfied in general, i.e. the 1-form (8.147) is not exact. Stated differently, κ is not an integrating factor for $\delta M - \Omega_{\rm H}\delta J$, and the temperature is not proportional to the surface gravity therefore.

In the next subsection we shall start the construction of an integrating factor for this 1-form, a generalization of κ in the classical case. In more physical terms this means that we are trying to find a function $T(M,J)$ which (at least as far as the first law is concerned) could be intepreted as the Bekenstein-Hawking temperature of the quantum corrected black hole.

8.7.2 Integrating Factor: General Setting

In this preparatory subsection we describe a convenient setting for finding the integrating factor and we test it in the classical case. It is already known that the classical Kerr spacetime fulfills a variation law of the form

$$\frac{\delta\mathcal{A}}{8\pi G_0} = \left(\frac{1}{\kappa}\right)\delta M - \left(\frac{\Omega_{\rm H}}{\kappa}\right)\delta J \tag{8.148}$$

with the following functions of M and J:

$$\begin{aligned} \kappa &\equiv \frac{r_+ - MG_0}{r_+^2 + \left(\frac{J}{M}\right)^2}\ ,\ \Omega_{\rm H} \equiv \frac{\left(\frac{J}{M}\right)}{r_+^2 + \left(\frac{J}{M}\right)^2}\ ,\ \mathcal{A} \equiv 4\pi\left[r_+^2 + \left(\frac{J}{M}\right)^2\right] \\ r_+ &= MG_0 + \sqrt{(MG_0)^2 - \left(\frac{J}{M}\right)^2} \end{aligned} \tag{8.149}$$

Substituting the functions (8.149) in the variation law leads to:

$$\frac{\delta\mathcal{A}}{8\pi G_0}=\left[\frac{r_+^2+\left(\frac{J}{M}\right)^2}{r_+-MG_0}\right]\delta M-\left[\frac{J}{M\left(r_+-MG_0\right)}\right]\delta J \tag{8.150}$$

After comparing (8.137) and (8.138) with (8.149) we see that the only explicit modification due to the r-dependence of $G(r)$ appears in the numerator of κ in (8.137) given by $r_+^{\mathrm{I}}-M\left[r_+^{\mathrm{I}}G'\left(r_+^{\mathrm{I}}\right)+G\left(r_+^{\mathrm{I}}\right)\right]$; it has an additional term involving $G'\left(r_+^{\mathrm{I}}\right)$. This numerator is reduced to r_+-MG_0 in the classical case. Taking advantage of this fact we propose the following differential form to be integrated in both cases, the classical one and the improved one:

$$\boldsymbol{\gamma}=\left[r_+^2+\left(\frac{J}{M}\right)^2\right]\delta M-\left(\frac{J}{M}\right)\delta J \tag{8.151}$$

It changes after the improvement only through the implicit dependence of r_+ on M and J, from r_+ to r_+^{I}. In that case we have:

$$\boldsymbol{\gamma}=\left[\left(r_+^{\mathrm{I}}\right)^2+\left(\frac{J}{M}\right)^2\right]\delta M-\left(\frac{J}{M}\right)\delta J \tag{8.152}$$

Ultimately we would like to know the integrating factor for the 1-form (8.147) with κ and Ω_{H} from (8.137) and (8.138) inserted. This form differs from $\boldsymbol{\gamma}$ by a scalar factor only:

$$\begin{aligned}\boldsymbol{\alpha}&=\left\{r_+^{\mathrm{I}}-M\left[r_+^{\mathrm{I}}G'\left(r_+^{\mathrm{I}}\right)+G\left(r_+^{\mathrm{I}}\right)\right]\right\}^{-1}\left\{\left[\left(r_+^{\mathrm{I}}\right)^2+\left(\frac{J}{M}\right)^2\right]\delta M-\left(\frac{J}{M}\right)\delta J\right\}\\&=h\left(M,J\right)\boldsymbol{\gamma}\end{aligned} \tag{8.153}$$

where

$$h\left(M,J\right)\equiv\left\{r_+^{\mathrm{I}}-M\left[r_+^{\mathrm{I}}G'\left(r_+^{\mathrm{I}}\right)+G\left(r_+^{\mathrm{I}}\right)\right]\right\}^{-1} \tag{8.154}$$

Thus, once we have managed to find an integrating factor μ_γ for the (simpler) 1-form $\boldsymbol{\gamma}$, we can obtain the corresponding factor for the much more complicated form $\boldsymbol{\alpha}$ simply by multiplication with the overall factor $1/h\left(M,J\right)$. By definition, μ_γ is such that $\mu_\gamma\boldsymbol{\gamma}$ is closed, i.e. $\delta\left(\mu_\gamma\boldsymbol{\gamma}\right)=0$ Using $\boldsymbol{\alpha}=h\boldsymbol{\gamma}$ this implies that $\delta\left[\left(\mu_\gamma/h\right)\boldsymbol{\alpha}\right]=0$. Hence

$$\mu_\alpha\equiv\mu_\gamma/h \tag{8.155}$$

is an integrating factor for $\boldsymbol{\alpha}$ if μ_γ is an integrating factor for $\boldsymbol{\gamma}$.

For the bare case, it is not difficult to see that $\boldsymbol{\gamma}$ of eq. (8.151) is non-exact. Since we know the explicit form of r_+ we can easily check the condition of exactness (8.143). In this case we have $r_+(M,J) = MG_0 + \sqrt{(MG_0)^2 - (J/M)^2}$ and the definitions $P = r_+^2 + (J/M)^2$, $N = -J/M$. As a result we get

$$\left.\frac{\partial P}{\partial J}\right|_{r_+} = -\frac{2J}{M\sqrt{(MG_0)^2 - \left(\frac{J}{M}\right)^2}} \ , \ \left.\frac{\partial N}{\partial M}\right|_{r_+} = \frac{J}{M^2} \tag{8.156}$$

namely

$$\left.\frac{\partial P}{\partial J}\right|_{r_+} \neq \left.\frac{\partial N}{\partial M}\right|_{r_+} \tag{8.157}$$

From (8.150) we deduce that the approppiate integrating factor μ_γ for the one-form $\boldsymbol{\gamma}$ of (8.151) is

$$\mu_\gamma^{\text{class}}(M,J) = \frac{1}{r_+ - MG_0} \tag{8.158}$$

This can be checked (see appendix J) by substituting (8.158) in (8.146) together with $P = r_+^2 + (J/M)^2$, $N = -J/M$ and the derivatives (8.156). From (8.158) we obtain for the integrating factor of $\boldsymbol{\alpha}$

$$\mu_\alpha^{\text{class}} = h_{\text{class}}^{-1}\mu_\gamma^{\text{class}} = 1 \tag{8.159}$$

Since $h_{\text{class}}^{-1} = r_+ - MG_0$. This is the expected result, of course.

The above procedure might appear rather indirect and cumbersome to deal with the classical case. In the improved situation it is a technically advantageous setting for the computation of μ_α, though.

8.7.3 Integrating Factor: Partial Differential Equation

Now we turn to the improved case and derive the explicit form of the partial differential equation (8.146) for the integrating factor μ_γ in the case of the 1-form $\boldsymbol{\gamma}$ of (8.152). For $\boldsymbol{\gamma}$, the component functions P and N are to be identified as:

$$P(M,J) = \left(r_+^{\text{I}}\right)^2 + \left(\frac{J}{M}\right)^2 \ , \ N = -\left(\frac{J}{M}\right) \tag{8.160}$$

As a result, the equation for $\mu_\gamma(M,J)$ reads

$$\left[\left(r_+^{\mathrm{I}}\right)^2+\left(\frac{J}{M}\right)^2\right]\left(\frac{\partial\mu}{\partial J}\right)+\left(\frac{J}{M}\right)\left(\frac{\partial\mu}{\partial M}\right)=-\mu\left\{\frac{\partial}{\partial M}\left(\frac{J}{M}\right)+\frac{\partial}{\partial J}\left[\left(r_+^{\mathrm{I}}\right)^2+\left(\frac{J}{M}\right)^2\right]\right\} \tag{8.161}$$

Here and in the following we simply write μ for the factor μ_γ. Performing the implicit derivatives in the right hand side of (8.161) gives:

$$\left[\left(r_+^{\mathrm{I}}\right)^2+\left(\frac{J}{M}\right)^2\right]\left(\frac{\partial\mu}{\partial J}\right)+\left(\frac{J}{M}\right)\left(\frac{\partial\mu}{\partial M}\right)=-\mu\left\{\frac{J}{M^2}+2r_+^{\mathrm{I}}\frac{\partial r_+^{\mathrm{I}}}{\partial J}\right\} \tag{8.162}$$

The partial derivative $\partial r_+^{\mathrm{I}}/\partial J$ can be found by differentiating the event horizon's equation, $\left(r_+^{\mathrm{I}}\right)^2+(J/M)^2-2MG\left(r_+^{\mathrm{I}}\right)r_+^{\mathrm{I}}=0$, with respect to J:

$$\frac{\partial}{\partial J}\left[\left(r_+^{\mathrm{I}}\right)^2+\left(\frac{J}{M}\right)^2-2MG\left(r_+^{\mathrm{I}}\right)r_+^{\mathrm{I}}\right]=0 \tag{8.163}$$

Solving for $\partial r_+^{\mathrm{I}}/\partial J$ from (8.163) gives

$$\frac{\partial r_+^{\mathrm{I}}}{\partial J}=-\frac{J}{M^2\left\{r_+^{\mathrm{I}}-M\left[r_+^{\mathrm{I}}G'\left(r_+^{\mathrm{I}}\right)+G\left(r_+^{\mathrm{I}}\right)\right]\right\}} \tag{8.164}$$

Substituting (8.164) in (8.162) leads to

$$\left[\left(r_+^{\mathrm{I}}\right)^2+\left(\frac{J}{M}\right)^2\right]\left(\frac{\partial\mu}{\partial J}\right)+\left(\frac{J}{M}\right)\left(\frac{\partial\mu}{\partial M}\right)=\mu\frac{J}{M^2}\left\{\frac{r_+^{\mathrm{I}}+M\left[r_+^{\mathrm{I}}G'\left(r_+^{\mathrm{I}}\right)+G\left(r_+^{\mathrm{I}}\right)\right]}{r_+^{\mathrm{I}}-M\left[r_+^{\mathrm{I}}G'\left(r_+^{\mathrm{I}}\right)+G\left(r_+^{\mathrm{I}}\right)\right]}\right\} \tag{8.165}$$

or, written in a more compact way,

$$f_1(M,J)\left(\frac{\partial\mu}{\partial J}\right)+f_2(M,J)\left(\frac{\partial\mu}{\partial M}\right)=-\mu f_3(M,J) \tag{8.166}$$

with the definitions:

$$f_1(M,J)\equiv\left(r_+^{\mathrm{I}}\right)^2+\left(\frac{J}{M}\right)^2\ ,\ f_2(M,J)\equiv\frac{J}{M}$$
$$f_3(M,J)\equiv-\frac{J}{M^2}\left\{\frac{r_+^{\mathrm{I}}+M\left[r_+^{\mathrm{I}}G'\left(r_+^{\mathrm{I}}\right)+G\left(r_+^{\mathrm{I}}\right)\right]}{r_+^{\mathrm{I}}-M\left[r_+^{\mathrm{I}}G'\left(r_+^{\mathrm{I}}\right)+G\left(r_+^{\mathrm{I}}\right)\right]}\right\} \tag{8.167}$$

Equation (8.166) is the main result of this subsection, it is the partial differential equation for the integrating factor μ_γ of the first law modified via the running Newtonś constant $G(r)$. It will be analysed in the next two subsections where we look for explicit solutions for the integrating factor.

8.7.4 Does the Modified First Law Preserve the Classical Form?

It is clear that the improvement we are implementing with the running $G(r)$ gives by itself no guarantee of preserving the classical form $\delta\mathcal{A}/(8\pi G_0) = (1/\kappa)\,\delta M - (\Omega_{\rm H}/\kappa)\,\delta J \equiv \boldsymbol{\alpha}$ of the first law. After the improvement, the quantities κ and $\Omega_{\rm H}$ are modified as presented in (8.137) and (8.138). If we assume that these corrected quantities still fulfill a first law of the classical form, there should exist a scalar function S, possibly different from $\mathcal{A}/4G$, such that $\delta S \stackrel{?}{=} h\boldsymbol{\gamma} \equiv \boldsymbol{\alpha}$ or explicitly,

$$\delta S \stackrel{?}{=} \left\{\frac{1}{r_+^I - M\left[r_+^I G'\left(r_+^I\right) + G\left(r_+^I\right)\right]}\right\}\left\{\left[\left(r_+^I\right)^2 + \left(\frac{J}{M}\right)^2\right]\delta M - \left(\frac{J}{M}\right)\delta J\right\} \tag{8.168}$$

Eq. (8.168), if satisfied, would mean that the form of (8.153) is exact as it stands, i.e. with a trivial, constant integrating factor $\mu_\alpha = 1$. Stated differently, eq. (8.168) proposes $\mu_\gamma = h$ as an integrating factor of $\boldsymbol{\gamma} = \left[\left(r_+^{\rm I}\right)^2 + (J/M)^2\right]\delta M - \left(\frac{J}{M}\right)\delta J$. As a result, the task of checking whether the original form (8.148) is preserved after the improvement is equivalent to checking whether $h \equiv 1/\left\{r_+^{\rm I} - M\left[r_+^{\rm I} G'\left(r_+^{\rm I}\right) + G\left(r_+^{\rm I}\right)\right]\right\}$ fulfills the differential equation (8.165) for the integrating factor $\mu \equiv \mu_\gamma$. Next we shall prove that , in general, this is actually *not* the case.
Substituting the above candidate $\mu = h$ into (8.165) gives:

$$\begin{aligned} &\left\{\left[\left(r_+^{\rm I}\right)^2 + \left(\frac{J}{M}\right)^2\right]\frac{\partial}{\partial J} + \left(\frac{J}{M}\right)\frac{\partial}{\partial M}\right\}\left(\frac{1}{r_+^{\rm I} - M\left[r_+^{\rm I} G'\left(r_+^{\rm I}\right) + G\left(r_+^{\rm I}\right)\right]}\right) \\ = \;& \frac{J}{M^2}\left\{\frac{r_+^{\rm I} + M\left[r_+^{\rm I} G'\left(r_+^{\rm I}\right) + G\left(r_+^{\rm I}\right)\right]}{\left\{r_+^{\rm I} - M\left[r_+^{\rm I} G'\left(r_+^{\rm I}\right) + G\left(r_+^{\rm I}\right)\right]\right\}}\right\}\left(\frac{1}{r_+^{\rm I} - M\left[r_+^{\rm I} G'\left(r_+^{\rm I}\right) + G\left(r_+^{\rm I}\right)\right]}\right) \end{aligned} \tag{8.169}$$

Now we have to check the equality in (8.169). For the left hand side we have the following identities[6](see appendix J)

$$\frac{\partial}{\partial J}\left(\frac{1}{r_+^{\rm I} - M\left[r_+^{\rm I} G' + G\right]}\right) = \frac{J\left[1 - M\left(2G' + r_+^{\rm I} G''\right)\right]}{M^2\left[r_+^{\rm I} - M\left(r_+^{\rm I} G' + G\right)\right]^3} \tag{8.170}$$

[6]To simplify these expressions we omit from now on, the $r_+^{\rm I}$ argument from $G\left(r_+^{\rm I}\right)$ and its derivatives when it does not lead to ambiguities. But we keep in mind that G should always be read as $G\left(r_+^{\rm I}\right)$.

$$\frac{\partial}{\partial M}\left(\frac{1}{r_+^{\mathrm{I}} - M\left[r_+^{\mathrm{I}} G' + G\right]}\right) = \frac{1}{\left[r_+^{\mathrm{I}} - M\left(r_+^{\mathrm{I}} G' + G\right)\right]^2} \times$$
$$\times \left\{ r_+^{\mathrm{I}} G' + G + \frac{\left[\left(\frac{J}{M}\right)^2 + M r_+^{\mathrm{I}} G\right]\left[M\left(2G' + r_+^{\mathrm{I}} G''\right) - 1\right]}{M\left[r_+^{\mathrm{I}} - M\left(r_+^{\mathrm{I}} G' + G\right)\right]} \right\} \tag{8.171}$$

After substituting (8.170) and (8.171) in the left hand side of (8.169), or similarly of (8.165), and simplifying we find:

$$\begin{aligned}
&\left[\left(r_+^{\mathrm{I}}\right)^2 + \left(\frac{J}{M}\right)^2\right]\left(\frac{\partial \mu}{\partial J}\right) + \left(\frac{J}{M}\right)\left(\frac{\partial \mu}{\partial M}\right) \\
= \;& \frac{J}{M^2\left[r_+^{\mathrm{I}} - M\left(r_+^{\mathrm{I}} G' + G\right)\right]^2} \times \\
&\times \left\{ \frac{\left[1 - M\left(2G' + r_+^{\mathrm{I}} G''\right)\right]\left\{\left(r_+^{\mathrm{I}}\right) - MG\right\} r_+^{\mathrm{I}}}{\left[r_+^{\mathrm{I}} - M\left(r_+^{\mathrm{I}} G' + G\right)\right]} + M\left(r_+^{\mathrm{I}} G' + G\right) \right\}
\end{aligned} \tag{8.172}$$

It is clear that (8.172) is not equal to the right hand side of (8.169). Thus we found that:

$$\begin{aligned}
&\frac{J}{M^2\left[r_+^{\mathrm{I}} - M\left(r_+^{\mathrm{I}} G' + G\right)\right]^2} \times \\
&\times \left\{ \frac{\left[1 - M\left(2G' + r_+^{\mathrm{I}} G''\right)\right]\left\{r_+^{\mathrm{I}} - MG\right\} r_+^{\mathrm{I}}}{\left[r_+^{\mathrm{I}} - M\left(r_+^{\mathrm{I}} G' + G\right)\right]} + M\left(r_+^{\mathrm{I}} G' + G\right) \right\} \\
&\neq \frac{J}{M^2}\left\{\frac{r_+^{\mathrm{I}} + M\left[r_+^{\mathrm{I}} G' + G\right]}{\left\{r_+^{\mathrm{I}} - M\left[r_+^{\mathrm{I}} G' + G\right]\right\}}\right\}\left(\frac{1}{r_+^{\mathrm{I}} - M\left[r_+^{\mathrm{I}} G' + G\right]}\right)
\end{aligned} \tag{8.173}$$

Nevertheless in the classical case $G(r) = G_0$, $G' = G'' = 0$ we have $\{J\left[r_+ + MG_0\right]\}$ / $\left\{M^2\left[r_+ - MG_0\right]^2\right\}$ for both sides. This confirms that $\mu = 1/\left[r_+ - MG_0\right]$ is an integrating factor of our chosen differential form (8.151) for the classical first law.

We can summarize the above result by saying that if there should exist a generalization of the first law at the improved level (which is something we cannot actually be sure of) then this generalization does *not* have the simple structure

$$\delta S^{\mathrm{I}} = \left(\frac{1}{\kappa}\right)\delta M - \left(\frac{\Omega_{\mathrm{H}}}{\kappa}\right)\delta J \tag{8.174}$$

with the improved κ and Ω_{H}, and with a quantum corrected entropy:

$$S^{\mathrm{I}} = \frac{\mathcal{A}}{4G} + \text{quantum corrections} \tag{8.175}$$

As a result, the temperature of the black hole (assuming this notion still makes sense in the improved case) cannot be proportional to the surface gravity. This is in contrast to the Schwarzschild black hole for which in Ref. [30] a first law of the form $\delta S^{\mathrm{I}} = (\delta M/\kappa)$ has been established and the quantum corrections in (8.175) were computed.

Up to now we tacitly assumed that $G = G\,(r)$ is an essentially arbitrary function of r, and correspondingly that the values of $G \equiv G\left(r_+^{\mathrm{I}}\right)$, $G' \equiv G'\left(r_+^{\mathrm{I}}\right)$ and $G'' \equiv G''\left(r_+^{\mathrm{I}}\right)$ are not subject to any special constraints. It is rather improbable that precisely the function $G\,(r)$ chosen by nature to define the running of the Newton constant happens to satisfy expression (8.173) with an equality sign, unless we believe in the proportionality of T and κ as a fundamental principle in black hole thermodynamics.

Nevertheless, if we impose the equality sign to hold in (8.173), we are able to find, after a straightforward simplification, a condition for integrability that relates r_+^{I}, $G\left(r_+^{\mathrm{I}}\right)$, $G'\left(r_+^{\mathrm{I}}\right)$ and $G''\left(r_+^{\mathrm{I}}\right)$ as functions of M and J. This condition is given by (see appendix J)

$$\left(\frac{J}{M}\right)^2 G'\left(r_+^{\mathrm{I}}\right) + G''\left(r_+^{\mathrm{I}}\right) r_+^{\mathrm{I}} \left[\left(\frac{J}{M}\right)^2 - M r_+^{\mathrm{I}} G\left(r_+^{\mathrm{I}}\right)\right] = 0 \tag{8.176}$$

If this condition is met, $\boldsymbol{\alpha}$ is exact, and there exists an entropy-like quantity S such that $\boldsymbol{\alpha} = \delta S$. Since eq. (8.176) represents a condition for $G\,(r)$ and its derivatives evaluated at r_+^{I}, there could exist in fact infinitely many different functions $G\,(r)$ that fulfill this condition but behave differently away from $r = r_+^{\mathrm{I}}$. With our present technology we cannot decide whether or not the condition (8.176) is actually satisfied or not. So we must consider it as a logical possibility that (8.176) indeed holds true for the "correct" function $G\,(r)$.

Let us now try to judge how plausible this scenario is. For this purpose we suppose that the ordinary differential equation

$$\left(\frac{J}{M}\right)^2 G'(r) + G''(r)\, r \left[\left(\frac{J}{M}\right)^2 - M r G(r)\right] = 0 \tag{8.177}$$

governs the behavior of $G\,(r)$ in the vicinity of r_+^{I}. In appendix J we have found several solutions for (8.177), where $G\,(r)$ is represented as a series in powers of $1/r$. We have chosen this representation hoping to find a $G\,(r)$ which, thanks to the "antiscreening", decreases with r. For instance, the asymptotic behavior of $G(k)$ for small k given in eq. (1.25) by

$$G(k) = G_0 - wG_0^2 k^2 + O\left(k^4\right) \tag{8.178}$$

After substituting the cutoff identification $k = \bar{w}/r$ we find

$$G(r) = G_0 - \frac{\bar{w}G_0^2}{r^2} + O\left(\frac{1}{r^4}\right) \tag{8.179}$$

We would consider the condition (8.177) plausible if its solution $G(r)$ would have at least qualitatively similar properties. In contrast to this, the solutions we found are far from recovering the behavior in (8.179). They are classified as follows (see appendix J)

1. $G(r) = \text{Const}$
2. $G(r) = \frac{\left(\frac{J}{M}\right)^2}{2Mr}$
3. $G(r) = \frac{\left(\frac{J}{M}\right)^2}{2Mr} + O\left(\frac{1}{r^2}\right)$

The behavior of cases 2 and 3 is similar for $r \to \infty$, namely, $G(r) \to 0$.
Comparing this set of solutions with (8.179) we can conclude that, at least in the vicinity of r_+^{I} these solutions do not behave as expected. Case 1 is simply the classical case that shows no running. Cases 2 and 3 tend to zero instead of tending to G_0 when $r \to \infty$. As a result, unless the behavior of $G(r)$ changes far away from r_+^{I} we consider the non-trivial solutions 2 and 3 as "exotic" possibilities.

As a conclusion of this subsection, it seems more plausible to believe that the first law in the improved case requires a non-trivial integrating factor for which $T \not\propto \kappa$. This brings us back to the difficult problem of solving the partial differential equation for $\mu(M, J)$, eq. (8.166). In the next section we address this task. Since the coefficients (8.167) of this differential equation are non-constant and rather complicated functions of M and J, we content ourselves with finding an approximation to order J^2 for the integrating factor.

8.7.5 $\mathrm{O}\left(J^2\right)$ Approximation to the Modified First Law

As a consequence of the previous analysis we come back to the differential equation (8.166). It is a quasi-linear, first order, partial differential equation for μ in the two variables M and J. The general form of such a differential equation is given by [63]

$$a_1 (M, J, \mu) \left(\frac{\partial \mu}{\partial J} \right) + a_2 (M, J, \mu) \left(\frac{\partial \mu}{\partial M} \right) = a_3 (M, J, \mu) \tag{8.180}$$

where $a_1 (M, J, \mu)$, $a_2 (M, J, \mu)$ and $a_3 (M, J, \mu)$ are continuously differentiable functions of M and J and they can eventually depend on μ also. It is clear that our main equation (8.166) with the functions f_1, f_2 and f_3 defined in (8.167) fits into this general definition. Even though a general theory for solving (8.180) is available and the existence and uniqueness of solutions is guaranteed [63], the calculational difficulty of finding particular solutions depends crucially on the specific form of the functions a_i. Having this in mind and knowing that the functions f_1 to f_3 are far from simple, we shall now construct an approximate solution to μ for low angular momentum, where the zeroth order approximation is already known [30], rather than addressing the task of looking for an exact solution. This approximate solution will be found using an ansatz in power series.

The various steps can be summarized as follows: The equation to be solved is given by (8.166),

$$f_1 (M, J) \left(\frac{\partial \mu}{\partial J} \right) + f_2 (M, J) \left(\frac{\partial \mu}{\partial M} \right) = -\mu f_3 (M, J) \tag{8.181}$$

with the definitions (8.167)

$$f_1 (M, J) \equiv \left(r_+^{\mathrm{I}} \right)^2 + \left(\frac{J}{M} \right)^2 \ , \ f_2 (M, J) \equiv \frac{J}{M} \ , \ f_3 (M, J) \equiv -\frac{J}{M^2} \left\{ \frac{r_+^{\mathrm{I}} + M \left[r_+^{\mathrm{I}} G' + G \right]}{r_+^{\mathrm{I}} - M \left[r_+^{\mathrm{I}} G' + G \right]} \right\} \tag{8.182}$$

We expand f_1, f_2, f_3 in power series with respect to J. Since these functions are known, we can, in principle, find explicitly the coefficients in their expansions by utilizing expressions (8.182). We also expand μ and its derivatives and we substitute them, along with the expansions of f_1 to f_3, into (8.181). The only unknowns are precisely the coefficients of μ. They are found by solving the recurrence relation that results after the insertion of all the power series.

Following the above mentioned steps, we start by expanding f_1, f_2, f_3 and μ as follows:

$$\begin{aligned} f_a (M, J) &= \sum_{l=0}^{\infty} f_a^l (M) J^l \ , \ a = 1 \ , 2 \ , 3 \\ \mu (M, J) &= \sum_{k=0}^{\infty} \mu_k (M) J^k \end{aligned} \tag{8.183}$$

The derivatives of μ are given by:

$$
\begin{aligned}
\frac{\partial \mu}{\partial J} &= \mu_1(M) + 2\mu_2(M)J + 3\mu_3(M)J^2 + \cdots = \sum_{k=0}^{\infty}(k+1)\mu_{k+1}(M)J^k \\
\frac{\partial \mu}{\partial M} &= \sum_{k=0}^{\infty}\left(\frac{d\mu_k}{dM}\right)J^k
\end{aligned}
\tag{8.184}
$$

Substituting (8.183) and (8.184) in our main equation (8.181) leads to:

$$
\begin{aligned}
&\left(\sum_{l=0}^{\infty} f_1^l(M)J^l\right)\left(\sum_{k=0}^{\infty}(k+1)\mu_{k+1}(M)J^k\right) \\
&\quad + \left(\sum_{l=0}^{\infty} f_2^l(M)J^l\right)\left(\sum_{k=0}^{\infty}\left(\frac{d\mu_k}{dM}\right)J^k\right) = -\left(\sum_{k=0}^{\infty}\mu_k(M)J^k\right)\left(\sum_{l=0}^{\infty} f_3^l(M)J^l\right)
\end{aligned}
\tag{8.185}
$$

The recurrence relation for $\mu_k(M)$ follows by rewriting eq. (8.185) in the form $\sum_{k=0}^{\infty} A_k(M)J^k = 0$ which implies that $A_k(M) = 0$ must be fulfilled for every k. Following this idea we write (8.185) in the form

$$
\sum_{m=0}^{\infty}\beta_m J^m + \sum_{m=0}^{\infty}\gamma_m J^m + \sum_{m=0}^{\infty}\alpha_m J^m = 0 \tag{8.186}
$$

where α_m, β_m and γ_m are defined via

$$
\sum_{m=0}^{\infty}\beta_m J^m \equiv \left(\sum_{l=0}^{\infty} f_1^l(M)J^l\right)\left(\sum_{k=0}^{\infty}(k+1)\mu_{k+1}(M)J^k\right) \tag{8.187}
$$

$$
\sum_{m=0}^{\infty}\gamma_m J^m \equiv \left(\sum_{l=0}^{\infty} f_2^l(M)J^l\right)\left(\sum_{k=0}^{\infty}\left(\frac{d\mu_k}{dM}\right)J^k\right) \tag{8.188}
$$

$$
\sum_{m=0}^{\infty}\alpha_m J^m \equiv \left(\sum_{k=0}^{\infty}\mu_k(M)J^k\right)\left(\sum_{l=0}^{\infty} f_3^l(M)J^l\right) \tag{8.189}
$$

As a result the recurrence relation can be written as

$$
\beta_m + \gamma_m + \alpha_m = 0 \tag{8.190}
$$

where β_m , γ_m and α_m have to be found from (8.187), (8.188) and (8.189), respectively. This is explained in more detail in appendix J. The results are the following:

$$
\beta_m = \sum_{l=0}^{m}(m-l+1)f_1^l\mu_{m-l+1}\ ,\ \gamma_m = \sum_{l=0}^{m} f_2^l\mu'_{m-l}\ ,\ \alpha_m = \sum_{l=0}^{m} f_3^l\mu_{m-l} \tag{8.191}
$$

Where $\mu'_k \equiv \frac{d}{dM}\mu_k(M)$. By substituting (8.191) into (8.190) we obtain the final form of the recurrence relation:

$$\sum_{l=0}^{m} \left\{ (m-l+1)\, f_1^l \mu_{m-l+1} + f_2^l \mu'_{m-l} + f_3^l \mu_{m-l} \right\} = 0 \tag{8.192}$$

We emphasize again that the only unknowns in (8.192) are the μ components, since the f components can in principle be found straightforwardly.

With this algorithm the integrating factor can be found at any desired order in J. Nevertheless the f_i^l component represents an l-th derivative which is not necessarily easy to carry out if l is large. In this work we calculate only those components which are needed in order to get an O(J^2) approximation to $\mu(M,J)$, for more details see appendix J. The approximation we have found is given by

$$\mu(M,J)|_{O(J^2)} = \mu_0(M) + \mu_2(M)\, J^2 \tag{8.193}$$

with

$$\mu_0(M) \;=\; \frac{1}{M\left(G - r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\right)} \,, \mu'_0(M) = \frac{\frac{M\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2 GG''}{r^{\mathrm{I}}_{\mathrm{Sch}_+} - M\left[r^{\mathrm{I}}_{\mathrm{sc}_+} G' + G\right]} + r^{\mathrm{I}}_{\mathrm{Sch}_+} G' - G}{M^2\left[G - r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\right]^2} \tag{8.194}$$

$$\mu_2 \;=\; \frac{\left[3G + r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\right] (\mu_0)^2 - \mu'_0}{2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2 M} \tag{8.195}$$

where $r^{\mathrm{I}}_{\mathrm{Sch}_+}$ is defined as $r^{\mathrm{I}}_{\mathrm{Sch}_+} = r^{\mathrm{I}}_{\mathrm{Sch}_+}(M) \equiv r^{\mathrm{I}}_+(0,M)$, and $dr^{\mathrm{I}}_{\mathrm{Sch}_+}/dM$ is given by:

$$\frac{dr^{\mathrm{I}}_{\mathrm{Sch}_+}}{dM} = \frac{2G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)}{\left[1 - 2MG'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]} \tag{8.196}$$

In the following subsections we analyse the consequences of the approximation (8.193).

8.7.6 Exactness in the O$\left(J^2\right)$ Approximation to $\mu(M,J)$

It can be shown that the $O(J^2)$ approximation in (8.193) to the integrating factor $\mu(M,J)$ in our main differential equation (8.145) is enough for satisfying it to order

$O(J)$, either for the classical and the improved case (see appendix J). Given an $O(J^n)$ approximation to $\mu(M,J)$, we ask to which order in J the equality in (8.145) holds true, actually. What we find is that an $O(J^n)$ approximation to μ satisfies the differential equation to order $O(J^{n-1})$. This result is reasonable since eq. (8.145) implies one order of derivation in J. As a result one order in J is decreased every time we include an approximation to the series of $\mu(M,J)$ and it can only influence the $O(J^{n-1})$ terms in this equation. For more details see appendix J.

8.7.7 Temperature and Entropy

So far we have developed a procedure for finding recursively any desired approximation to the integrating factor μ_γ by solving equations (8.192). Our final goal is to find approximations for the modified first law (8.140) and also for the temperature and the entropy of the improved Kerr spacetime. This subsection is divided into three paragraphs where we present the results for each of the above mentioned quantities.

In the first paragraph we present the $O(J^2)$ approximation for the first law that results after multiplying the 1-form $\boldsymbol{\alpha}$ to its integrating factor $\mu_\alpha = h^{-1}\mu_\gamma$. Since the first law of thermodynamics gives a definition of temperature, an expression for that quantity can be read off from equation (8.140). This will be done in the second paragraph where we present the $O(J^2)$ approximation to $T(M,J)$ in the form:

$$T(M,J)|_{O(J^2)} = T_0(M) + T_2(M)\,J^2 \tag{8.197}$$

Finally, concerning the entropy, its computation requires an integration in the (M,J) plane. It can be shown that, for the $O(J^2)$ approximation

$$S(M,J)|_{O(J^2)} = S_0(M) + S_2(M)\,J^2\ , \tag{8.198}$$

we need just one integration along the M axis in order to find the zeroth order coefficient S_0, since the second order coefficient S_2 requires no explicit integration. We do this in the third paragraph of this subsection where we find a set of two coupled equations for the $O(J^2)$ coefficients of the temperature and the entropy. The solutions we find for these coefficients are uniquely fixed in terms of the Schwarzschild quantities T_0 and $r^{\mathrm{I}}_{\mathrm{Sch}_+}$ and they are consistent with the expression (8.193) for μ_γ. In appendix J we verify this consistency, where we compare the results for $T(M,J)|_{O(J^2)}$ from both procedures, namely, applying μ_γ and solving the above mentioned set of two equations. In the appendix we also present in more detail the calculations described in this subsection.

$O\left(J^2\right)$ Approximation to the First Law

As already mentioned in subsection 8.7.2, once we have μ_γ we have immediately an expression for μ_α given by

$$\mu_\alpha = h^{-1}\mu_\gamma = \left\{r_+^{\mathrm{I}} - M\left[r_+^{\mathrm{I}}G' + G\right]\right\}\mu_\gamma \tag{8.199}$$

Since μ_α is by definition the integrating factor of $\boldsymbol{\alpha}$ this means that $\boldsymbol{\alpha}\mu_\alpha$ is an exact differential, thus we have

$$\delta\left(\frac{S}{2\pi}\right) = \boldsymbol{\alpha}\mu_\alpha = \frac{\mu_\alpha}{\kappa}\left(\delta M - \Omega_{\mathrm{H}}\delta J\right) \tag{8.200}$$

where we keep a 2π factor in the normalization of S as in the classical case [65, 60]. The $O\left(J^2\right)$ approximation to the first law can be presented as follows

$$\begin{aligned}\delta\left(\frac{S}{2\pi}\right)\bigg|_{O(J^2)} &= \bar{P}\big|_{O(J^2)}\,\delta M + \bar{N}\big|_{O(J^2)}\,\delta J \\ &= \left(\frac{\mu_\alpha}{\kappa}\right)\bigg|_{O(J^2)}\delta M - \left(\frac{\Omega_{\mathrm{H}}\mu_\alpha}{\kappa}\right)\bigg|_{O(J^2)}\delta J\end{aligned} \tag{8.201}$$

where $\bar{P}$ and $\bar{N}$ are the coefficients of δM and δJ respectively, already corrected by the integrating factor μ_γ. They are defined in the following way:

$$\bar{P} \equiv \mu_\gamma P = \frac{\mu_\alpha}{\kappa}\ ,\ \ \bar{N} \equiv \mu_\gamma N = -\frac{\Omega_{\mathrm{H}}\mu_\alpha}{\kappa}$$

After expanding $\bar{P}$ and $\bar{N}$ to $O\left(J^2\right)$, applying equations (8.160), (8.193), (8.137) and (8.138) for $P\left(M,J\right)$, μ_γ, κ and Ω_{H}, we find the following expression for the $O\left(J^2\right)$ approximation to the first law (for more details, see appendix J):

$$\delta\left(\frac{S}{2\pi}\right)\bigg|_{O(J^2)} = \left\{\mu_0\left(r_{\mathrm{Sch}+}^{\mathrm{I}}\right)^2 + \frac{J^2}{2M}\left[\frac{\mu_0}{M} - \mu_0'\right]\right\}\delta M - \left(\frac{J\mu_0}{M}\right)\delta J \tag{8.202}$$

It can be easily verified that the crossed derivatives of the coefficients in (8.202) are equal, showing in this way, the exactness of δS to $O\left(J^2\right)$.

$O\left(J^2\right)$ Approximation to the Temperature

From equation (8.200) for the first law we now define the temperature in the usual way as the coefficient of δS in $\delta M - \Omega_{\mathrm{H}}\delta J = T\delta S$. We obtain

$$T = \frac{\kappa}{2\pi\mu_\alpha} \tag{8.203}$$

Note that if $\mu_\alpha \neq 1$ this result differs from the familiar relationship $T = \kappa/(2\pi)$ of classical black hole thermodynamics. Substituting μ_α from (8.199) and κ from (8.137) in (8.203) leads to

$$T\left(M,J\right) = \frac{1}{2\pi\mu_\gamma\left[\left(r_+^{\mathrm{I}}\right)^2 + \left(\frac{J}{M}\right)^2\right]} \tag{8.204}$$

Expanding $T\left(M,J\right)$ in powers of J gives the following:

$$T\left(M,J\right) = \frac{1}{2\pi\mu_0\left(r_{\mathrm{Sch}+}^{\mathrm{I}}\right)^2} + \frac{J^2\left[\mu_0' - \frac{\mu_0}{M}\right]}{4\pi M\left(r_{\mathrm{Sch}+}^{\mathrm{I}}\right)^4\left(\mu_0\right)^2} + O\left(J^4\right) \tag{8.205}$$

Here we have also applied equations (8.160) and (8.193) for P and μ_γ respectively. For the calculational details see appendix J.

$O\left(J^2\right)$ Approximation to the Entropy

In order to find an $O\left(J^2\right)$-approximation to the entropy, we substitute in the first law (8.140), generic expressions of the approximations to $O\left(J^2\right)$ of T and S. We then obtain 2 independent equations after factorizing the coefficients of every order in the expansion, as follows. The generic expressions for T and S are given by (8.197) and (8.198):

$$\begin{aligned} T\left(M,J\right)\big|_{O\left(J^2\right)} &= T_0\left(M\right) + T_2\left(M\right)J^2 \\ S\left(M,J\right)\big|_{O\left(J^2\right)} &= S_0\left(M\right) + S_2\left(M\right)J^2 \end{aligned} \tag{8.206}$$

Substituting expressions (8.206) in the first law (8.140) leads to

$$\delta M - \Omega_{\mathrm{H}}\delta J = \left[T_0\left(M\right) + T_2\left(M\right)J^2\right]\delta\left[S_0\left(M\right) + S_2\left(M\right)J^2\right] \tag{8.207}$$

After expanding the right hand side of (8.207) we have

$$\delta M - \Omega_{\mathrm{H}}\delta J = T_0\delta S_0 + \delta S_0 T_2 J^2 + \delta\left(S_2 J^2\right)T_0 + O\left(J^3\right) \tag{8.208}$$

where we have ommited the argument M of $T_{0,2}$ and $S_{0,2}$. Since T_0 and S_0 correspond to the temperature and entropy of the Schwarzschild spacetime with $J = 0$ and $\Omega_{\mathrm{H}} = 0$, the first law for that case is given by:

$$\delta M = T_0 \delta S_0 \tag{8.209}$$

Substituting (8.209) in (8.208) gives the following:

$$-\Omega_{\mathrm{H}} \delta J = \delta S_0 T_2 J^2 + \delta \left(S_2 J^2 \right) T_0 + O\left(J^3 \right) \tag{8.210}$$

Expressing the variations in the right hand side of (8.210) in terms of δM and δJ leads to:

$$-\Omega_{\mathrm{H}} \delta J = T_2 J^2 \left(\frac{dS_0}{dM} \right) \delta M + T_0 \left[J^2 \left(\frac{dS_2}{dM} \right) \delta M + 2 J S_2 \delta J \right] + O\left(J^3 \right) \tag{8.211}$$

After factorizing and equating coefficients in δJ and δM, we find the following two coupled equations which determine S_2 and T_2:

$$T_2 \left(\frac{dS_0}{dM} \right) + T_0 \left(\frac{dS_2}{dM} \right) \quad = \quad 0 \tag{8.212}$$

$$2 J T_0 S_2 + \Omega_{\mathrm{H}} = 0 \tag{8.213}$$

We can find additional information about T_0 and S_0 from the first law for the Schwarzschild spacetime given in (8.209), as follows. Since these quantities depend exclusively on M, we can write

$$\delta M = T_0 \left(\frac{dS_0}{dM} \right) \delta M \tag{8.214}$$

Thus we deduce the following identity, usual in thermodynamics:

$$\frac{1}{T_0 \left(M \right)} = \frac{dS_0}{dM} \tag{8.215}$$

The integration of (8.215) gives an expression for $S_0 \left(M \right)$, namely

$$S_0 \left(M \right) = \int_{M_0}^{M} \frac{dM'}{T_0 \left(M' \right)} \tag{8.216}$$

This is the expression analized in reference [30] which deals with the improved Schwarzschild black hole. Since the functions $T_0 \left(M \right)$ and $S_0 \left(M \right)$ are already known, we conclude that expressions (8.212) and (8.213) define a system of two coupled equations which determine T_2 and S_2. We solve them as follows. Expanding Ω_{H} in (8.213) gives

$$\Omega_{\mathrm{H}}\left(M,J\right)=\frac{\left(\frac{J}{M}\right)}{\left(r_{+}^{\mathrm{I}}\right)^{2}+\left(\frac{J}{M}\right)^{2}}=\left(\frac{J}{M}\right)\frac{1}{\left(r_{\mathrm{Sch}_{+}}^{\mathrm{I}}\right)^{2}}+O\left(J^{3}\right) \tag{8.217}$$

Substituting (8.217) up to $O\left(J^2\right)$ in (8.213) gives the following final result for S_2:

$$S_{2}\left(M\right)=-\frac{1}{2T_{0}\left(M\right)M\left(r_{\mathrm{Sch}_{+}}^{\mathrm{I}}\right)^{2}} \tag{8.218}$$

On the other hand, substituting (8.215) in (8.212) and solving for T_2 gives an explicit expression for the temperature correction:

$$T_{2}\left(M\right)=-\left(T_{0}\right)^{2}\frac{dS_{2}}{dM}\left(M\right) \tag{8.219}$$

Here $S_2\left(M\right)$ on the RHS of 8.219 is explicitly given by (8.218).

It is important to remark that the results (8.218) and (8.219) for the entropy and temperature corrections are *uniquely fixed in terms of the Schwarzschild quantities.* Hence, within this approximation, the factorization of the 1-form $\delta M-\Omega_{\mathrm{H}}\delta J$ as $T\delta S$ is *unique.* A general 1-form $\boldsymbol{\alpha}$ can be expressed as $\boldsymbol{\alpha}=f_1\delta f_2$ in terms of two 0-forms f_1 and f_2 in many different ways; given $\boldsymbol{\alpha}$, the pair (f_1,f_2) is not unique. However, if we insist on recovering the Schwarzschild results for $J=0$ and work to order J^2 only, the identification of S_2 and T_2 is unambiguous.

It can be shown that the expression (8.205) for $T\left(M,J\right)|_{O(J^2)}$ which we have obtained with our general all-order formalism is consistent with equations (8.218) and (8.219). We demonstrate this in appendix J.

With expressions (8.216) and (8.218) we can write the $O\left(J^2\right)$ approximation to the entropy in the form

$$S\left(M,J\right)|_{O(J^{2})}=S_{0}-\frac{J^{2}}{2T_{0}M\left(r_{\mathrm{Sch}_{+}}^{\mathrm{I}}\right)^{2}}=S_{0}-\frac{\pi J^{2}}{M^{2}\left[G\left(r_{\mathrm{Sch}_{+}}^{\mathrm{I}}\right)-r_{\mathrm{Sch}_{+}}^{\mathrm{I}}G'\left(r_{\mathrm{Sch}_{+}}^{\mathrm{I}}\right)\right]} \tag{8.220}$$

with the following known functions of M:

$$S_{0}=\int_{M_{0}}^{M}\frac{dM'}{T_{0}\left(M'\right)}\ ,\ \mu_{0}=\frac{1}{M\left(G-r_{\mathrm{Sch}_{+}}^{\mathrm{I}}G'\right)}\ ,T_{0}=\frac{1}{2\pi\left(r_{\mathrm{Sch}_{+}}^{\mathrm{I}}\right)^{2}\mu_{0}}$$

Expresions (8.202), (8.205) and (8.220) for the $O\left(J^2\right)$ approximations to the first law, the temperature and the entropy are the main results of this subsection.

Large M Expansions for the $d(r) = r$ Approximation

The approximations (8.205) and (8.220) for the temperature and the entropy respectively, are expressions where the function $G(r)$ is not specified. To be more explicit, we exploit now the formula (5.36) for $G(r)$ in order to calculate quantum corrections to T and S in the asymptotic region where $d(r) = r$. This formula for $G(r)$ is given by

$$G(r) = \frac{G_0 r^2}{r^2 + G_0 \bar{w}} \tag{8.221}$$

In this way we establish a connection between the results for the improved Kerr spacetime achieved in this work and results from the literature, like the large M expansions for T and S of the improved Schwarzschild black hole [30].

The procedure for finding these expansions is the following. First we expand G and its derivatives G' and G'' for large M. After this, we find expressions for T and S in (8.205) and (8.220) as explicit functions of G and its derivatives. Finally we substitute the expansions of G, G' and G'' in T and S.

We expand $G(r^{\mathrm{I}}_{\mathrm{Sch}_+})$, $G'(r^{\mathrm{I}}_{\mathrm{Sch}_+})$ and $G''(r^{\mathrm{I}}_{\mathrm{Sch}_+})$ in the parameter $\bar{m} \equiv M_{\mathrm{cr}}/M$ defined in chapter 5. The function $G(r)$ from (8.221) and its two first derivatives as functions of m_{cr} are given by

$$G(r) = \frac{G_0 r^2}{r^2 + (m_{\mathrm{cr}})^2} \tag{8.222}$$

$$G'(r) = \frac{2G_0 r (m_{\mathrm{cr}})^2}{\left[r^2 + (m_{\mathrm{cr}})^2\right]^2} \tag{8.223}$$

$$G''(r) = 2G_0 (m_{\mathrm{cr}})^2 \left\{ \frac{(m_{\mathrm{cr}})^4 - 2(m_{\mathrm{cr}})^2 r^2 - 3r^4}{\left[r^2 + (m_{\mathrm{cr}})^2\right]^4} \right\} \tag{8.224}$$

Evaluating these functions at $r^{\mathrm{I}}_{\mathrm{Sch}_+}$ and substituting the $\bar{m}$-expansion (5.55) of this radius gives the following $1/M$-expansion:

$$G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) = G_0 \left[1 - \frac{\bar{m}^2}{4} - \frac{\bar{m}^4}{16} + O\left(\bar{m}^8\right)\right] \tag{8.225}$$

$$G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) = \left(\frac{2G_0}{m}\right) \left[\frac{\bar{m}^2}{8} + \frac{\bar{m}^4}{32} + O\left(\bar{m}^6\right)\right] \tag{8.226}$$

$$G''\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) = \frac{2G_0}{m^2} \left[-\frac{3\bar{m}^2}{16} - \frac{\bar{m}^4}{32} + O\left(\bar{m}^6\right)\right] \tag{8.227}$$

With these results we proceed to expand in $\bar{m}^2$ the formulas (8.205) and (8.220) for the temperature and the entropy in the next two paragraphs.

Expansion of the Temperature The temperature in (8.205) reads

$$T\left(M,J\right)\big|_{O(J^2)} \;=\; T_0 + T_2 J^2 \tag{8.228}$$

with T_0 and T_2 defined as

$$T_0 = \frac{1}{2\pi\mu_0\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2}\;,\; T_2 = \frac{\left[\mu_0' - \frac{\mu_0}{M}\right]}{4\pi M\left(\mu_0\right)^2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^4} \tag{8.229}$$

The expressions in (8.229) are functions of μ_0 and μ_0' given in (8.194):

$$\mu_0 \;=\; \frac{1}{M\left(G - r^{\mathrm{I}}_{\mathrm{Sch}_+}G'\right)} \tag{8.230}$$

$$\mu_0' \;=\; \frac{\frac{M\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2 GG''}{r^{\mathrm{I}}_{\mathrm{Sch}_+} - M\left[r^{\mathrm{I}}_{\mathrm{Sch}_+}G' + G\right]} + r^{\mathrm{I}}_{\mathrm{Sch}_+}G' - G}{M^2\left[G - r^{\mathrm{I}}_{\mathrm{Sch}_+}G'\right]^2} \tag{8.231}$$

Substituting $r^{\mathrm{I}}_{\mathrm{Sch}_+} = 2MG\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)$ in (8.230) and (8.231) simplifies μ_0 and μ_0' to the following expressions:

$$\begin{aligned}\mu_0 \;&=\; \frac{1}{MG\left(1-2MG'\right)} \\ \mu_0' \;&=\; \frac{4G''}{\left(1-2MG'\right)^3} - \frac{1}{M^2G\left(1-2MG'\right)}\end{aligned} \tag{8.232}$$

Substituting (8.232) in (8.229) leads to

$$T_0 = \frac{1}{2\pi\mu_0\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2} = \frac{\left(1-2MG'\right)}{8\pi MG} = \frac{1}{8\pi MG} - \frac{G'}{4\pi G} \tag{8.233}$$

$$T_2 = \frac{\left[2GG''M^2 + 4MG' - 4M^2G'^2 - 1\right]}{32\pi M^5G^3\left(1-2MG'\right)} \tag{8.234}$$

Expressions (8.233) and (8.234) present T_0 and T_2 as functions of M, G, and its derivatives, evaluated at $r^{\mathrm{I}}_{\mathrm{Sch}_+}$. As a result, we can exploit the expansions for $G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)$, $G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)$, $G''\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)$ given in (8.225), (8.226) and (8.227), as follows. For T_0 we have

$$T_0 = \frac{1}{8\pi G_0 M\left[1 - \frac{\bar{m}^2}{4} - \frac{\bar{m}^4}{16} + O\left(\bar{m}^8\right)\right]} - \frac{\left[\frac{\bar{m}^2}{8} + \frac{\bar{m}^4}{32} + O\left(\bar{m}^6\right)\right]}{2\pi G_0 M\left[1 - \frac{\bar{m}^2}{4} - \frac{\bar{m}^4}{16} + O\left(\bar{m}^8\right)\right]} \tag{8.235}$$

Expanding each term in (8.235) and simplifying gives

$$T_0 = \frac{1}{8\pi G_0 M}\left[1 - \frac{\bar{m}^2}{4} - \frac{\bar{m}^4}{8} + O\left(\bar{m}^6\right)\right] \tag{8.236}$$

Similarly for T_2 we have

$$T_2 = \frac{2GG''M^2 + 4MG' - 4M^2G'^2 - 1}{32\pi M^5 G^3\left(1 - 2MG'\right)} = -\frac{1}{32\pi G_0^3 M^5}\left[1 + \bar{m}^2 + \frac{15}{16}\bar{m}^4 + O\left(\bar{m}^6\right)\right] \tag{8.237}$$

Thus our final result for the large M expansion for the $O\left(J^2\right)$ approximation to $T\left(M, J\right)$ is given by

$$\begin{aligned} T\left(M, J\right)\big|_{O(J^2)} &= T_0 + T_2 J^2 \\ &= \frac{1}{8\pi G_0 M}\left[1 - \frac{\bar{m}^2}{4} - \frac{\bar{m}^4}{8} + O\left(\bar{m}^6\right)\right] \\ &\quad - \frac{J^2}{32\pi M^5 G_0^3}\left[1 + \bar{m}^2 + \frac{15}{16}\bar{m}^4 + O\left(\bar{m}^6\right)\right] \end{aligned} \tag{8.238}$$

Setting $\bar{m}$ to zero in this result, i.e. letting $M \to \infty$ takes us back to the classical Kerr spacetime:

$$T\left(M, J\right)\big|_{O(J^2)}^{\text{Kerr}} = \frac{1}{8\pi G_0 M} - \frac{J^2}{32\pi G_0^3 M^5} \tag{8.239}$$

This is precisely the $O\left(J^2\right)$ approximation for T_{Kerr}. To verify this statement we expand T_{Kerr} in (8.52):

$$\begin{aligned} T_{\text{Kerr}} \equiv \frac{\kappa}{2\pi} = \frac{r_+ - MG_0}{4\pi MG_0 r_+} &= \frac{\sqrt{\left(G_0\right)^2 M^2 - \frac{J^2}{M^2}}}{4\pi MG_0\left[G_0 M + \sqrt{\left(G_0\right)^2 M^2 - \frac{J^2}{M^2}}\right]} \\ &= \frac{1}{4\pi MG_0}\left[\frac{1}{2} - \frac{J^2}{8M^4\left(G_0\right)^2} + O\left(J^4\right)\right] \end{aligned} \tag{8.240}$$

On the other hand, setting $J = 0$ in (8.238) reproduces the large M expansion gives us the large expansion for the improved Schwarzschild spacetime (see Ref. [30]):

$$T\left(M\right) = \frac{1}{8\pi G_0 M}\left[1 - \frac{1}{4}\left(\frac{M_{\text{cr}}}{M}\right)^2 - \frac{1}{8}\left(\frac{M_{\text{cr}}}{M}\right)^4 + O\left(M^{-6}\right)\right] \tag{8.241}$$

Expansion of the Entropy For the $O\left(J^2\right)$ approximation to the entropy we have from (8.220), the following:

$$S\left(M,J\right)\big|_{O(J^2)} = S_0 - \frac{J^2}{2T_0 M\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2} = S_0 - \frac{\pi J^2}{M^2\left[G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) - r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]} \tag{8.242}$$

with

$$S_0 = \int_{M_0}^{M} \frac{dM'}{T_0\left(M'\right)}\ ,\ \mu_0 = \frac{1}{M\left(G - r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\right)}\ , T_0 = \frac{1}{2\pi\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2 \mu_0} \tag{8.243}$$

Expanding the $O\left(J^2\right)$ term yields

$$S_2 = -\frac{\pi}{M^2\left[G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) - r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]} = -\left(\frac{\pi}{M^2 G_0}\right)\left[1 + \frac{3}{4}\bar{m}^2 + \frac{5}{8}\bar{m}^4 + O\left(\bar{m}^6\right)\right]$$

As a result we have

$$S\left(M,J\right)\big|_{O(J^2)} = S_0 - \frac{J^2}{2T_0 M\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2} = S_0 - \left(\frac{\pi J^2}{M^2 G_0}\right)\left[1 + \frac{3}{4}\bar{m}^2 + \frac{5}{8}\bar{m}^4 + O\left(\bar{m}^6\right)\right] \tag{8.244}$$

For S_0 we can apply the result of Ref. [30] for the large M expansion of the entropy in the improved Schwarzschild spacetime[7], given by

$$S_0 = \frac{\mathcal{A}^{\mathrm{Sch}}_{\mathrm{class}}}{4G_0} + 2\pi\bar{w}\left[\frac{1}{2}\ln\left(\frac{2}{\bar{m}^2}\right) - \frac{3}{2} - \frac{3\bar{m}^2}{8} - \frac{5}{32}\bar{m}^4 + O\left(\bar{m}^6\right)\right] + S\left(M_{\mathrm{cr}}\right) \tag{8.245}$$

with the area of the Schwarzschild black hole given by

$$\mathcal{A}^{\mathrm{Sch}}_{\mathrm{class}} = 16\pi M^2\left(G_0\right)^2 \tag{8.246}$$

Thus we arrive at the following final answer for $S\left(M,J\right)\big|_{O(J^2)}$:

$$\begin{aligned} S\left(M,J\right)\big|_{O(J^2)} \quad = \quad & \frac{\mathcal{A}^{\mathrm{Sch}}_{\mathrm{class}}}{4G_0} + 2\pi\bar{w}\left[\frac{1}{2}\ln\left(\frac{2}{\bar{m}^2}\right) - \frac{3}{2} - \frac{3\bar{m}^2}{8} - \frac{5}{32}\bar{m}^4 + O\left(\bar{m}^6\right)\right] + S\left(M_{\mathrm{cr}}\right) \\ & - \left(\frac{\pi J^2}{M^2 G_0}\right)\left[1 + \frac{3}{4}\bar{m}^2 + \frac{5}{8}\bar{m}^4 + O\left(\bar{m}^6\right)\right] \end{aligned} \tag{8.247}$$

[7] In the mentioned reference we have to identify $\bar{m}^2$ with the parameter Ω.

As a check, we let $M \to \infty$ or, equivalently, $\hbar \to 0$. Setting $\bar{m} = 0$ gives

$$S(M,J)|_{O(J^2)} = \frac{\mathcal{A}^{\rm Sch}_{\rm class}}{4G_0} + S(M_{\rm cr}) - \frac{\pi J^2}{M^2 G_0} \tag{8.248}$$

This is precisely, up to a constant, the $O(J^2)$ approximation for the entropy in the classical Kerr spacetime:

$$\begin{aligned} S_{\rm Kerr} &= \frac{\mathcal{A}_{\rm Kerr}}{4G_0} = \frac{\pi}{G_0}\left[(r_+)^2 + \left(\frac{J}{M}\right)^2\right] = 2\pi M r_+ = 2\pi M\left[MG_0 + \sqrt{(MG_0)^2 - \left(\frac{J}{M}\right)^2}\right] \\ &= 4\pi M^2 G_0 - \frac{\pi J^2}{M^2 G_0} + O\left(J^4\right) \end{aligned} \tag{8.249}$$

where $4\pi M^2 G_0 = \mathcal{A}^{\rm Sch}_{\rm class}/4G_0$. The classical expression $S_{\rm Kerr}$ coincides with (8.248) if the undetermined constant of integration $S(M_{\rm cr})$ is chosen to vanish.

As a **summary** of this section we redisplay the large M expansions we have calculated for the $O(J^2)$ approximations for the temperature and the entropy of the RG-improved Kerr black hole:

$$\begin{aligned} T(M,J)|_{O(J^2)} &= \frac{1}{8\pi G_0 M}\left[1 - \frac{\bar{m}^2}{4} - \frac{\bar{m}^4}{8} + O\left(\bar{m}^6\right)\right] \\ &\quad - \frac{J^2}{32\pi M^5 G_0^3}\left[1 + \bar{m}^2 + \frac{15}{16}\bar{m}^4 + O\left(\bar{m}^6\right)\right] \end{aligned} \tag{8.250}$$

$$\begin{aligned} S(M,J)|_{O(J^2)} &= \frac{\mathcal{A}^{\rm Sch}_{\rm class}}{4G_0} + 2\pi\bar{w}\left[\frac{1}{2}\ln\left(\frac{2}{\bar{m}^2}\right) - \frac{3}{2} - \frac{3\bar{m}^2}{8} - \frac{5}{32}\bar{m}^4 + O\left(\bar{m}^6\right)\right] \\ &\quad - \left(\frac{\pi J^2}{M^2 G_0}\right)\left[1 + \frac{3}{4}\bar{m}^2 + \frac{5}{8}\bar{m}^4 + O\left(\bar{m}^6\right)\right] \end{aligned} \tag{8.251}$$

In writing down the result for the entropy we fixed the undetermined constant if integration such that $S = 0$ for $M = M_{\rm cr}$ and $J = 0$.

We observe that the angular momentum dependent terms in (8.250) and (8.251) *decrease* both the black hole's temperature and entropy as compared to the corresponding Schwarzschild quantities. We also see that the size of the J^2-corrections increases with $\bar{m}$, i.e. these corrections grow as the mass M of the black hole becomes smaller during the evaporation process.

As the most important aspect of the modified black hole thermodynamics we recall that $2\pi T$ does not agree with the surface gravity κ here as it does in the familiar (semi-) classical situation. We demonstrated that a modified first law can exist only when we give up the relationship $T = \kappa/2\pi$. we also showed that, to order J^2, there is a uniquely determined modification of this relationship which allows for the existence of a state function $S(M,J)$ with the interpretation of an entropy.

8.8 Conclusions

The results of this chapter can be summarized as follows. We have found exact expressions for the surface gravity κ, the mass M_H and the angular momentum J_H of the improved Kerr black hole. We have seen that the results for M_H and J_H reflect the antiscreening character of the quantum fluctuations of the gravitational field. Furthermore, we have studied how to construct a first law of improved black hole dynamics. In that way we have come to the conclusion that the temperature, considered as the integrating factor of the a priori inexact differential in the first law, cannot be proportional to the surface gravity as in the usual case. We have also computed $O\left(J^2\right)$ approximations to the first law, the entropy and the temperature of the improved Kerr black hole. These results are consistent with previous results for the improved Schwarzschild spacetime which they generalize in a nontrivial way.

Chapter 9

Conclusions and Outlook

In this work we presented a procedure for "RG improving" black hole spacetimes that is based upon Quantum Einstein Gravity, or "QEG", as a very promising candidate for a fundamental theory of quantum gravity. As we explained in the introduction, several results, obtained during the past 7 years indicate the existence of a non-Gaussian RG fixed point that would eventually "tame" the infinities in the ultraviolet limit which ruin the applicability of perturbation theory. We have considered the impact of QEG on black hole physics as one of the most important applications any theory of quantum gravity will have.

We have focused our investigation on four basic subjects of black hole physics. The main results related to these topics can be summarized as follows. Concerning the critical surfaces, i.e. horizons and static limit surfaces, the improvement leads to a smooth deformation of the classical critical surfaces. Their number remains unchanged. In relation to the Penrose process for energy extraction from black holes, we have found that there exists a (non-trivial) correlation between regions of negative energy states in the phase space of rotating test particles and configurations of critical surfaces of the black hole. As for the vacuum energy-momentum tensor and the energy conditions we have shown that no model with "normal" matter, in the sense of matter fulfilling the usual energy conditions, can simulate the quantum fluctuations described by the improved Kerr spacetime that we have derived. Finally, in the context of black hole thermodynamics, we have performed calculations of the mass and angular momentum of the improved Kerr black hole, applying the standard Komar integrals. The results reflect the antiscreening character of the quantum fluctuations of the gravitational field. Furthermore we calculated approximations to the entropy and the temperature of the improved Kerr black hole to leading order in the angular momentum. More generally we have proven that the temperature can no longer be proportional to the surface gravity, if an entropy-like state function is to exist.

We have tried to emphasize the limitations of the improvement procedure and the physical conditions that would eventually guarantee, if asymptotic safety is realized, the reliability of our results. Having these limitations in mind we now move on to formulate various related open questions and problems to be addressed in the future:

- Concerning the critical surfaces, it is important to clarify whether the disappearence of the event horizon and the static limit for small masses is a real phenomenon in the exact theory, or just an artifact result of our approach. In this direction, the inclusion of further gravitational couplings or geometric scales could eventually be helpful.

- In the analysis of the energy extraction from a black hole, it would be interesting to calculate the irreducible mass $M_{\rm irr}$, a concept that we have considered only briefly. Having a formula (even approximated) of $M_{\rm irr}$ as a function of $M_{\rm H}$ and $J_{\rm H}$ would give us a better idea of the efficiency of the process of energy extraction in the improved case.

- In classical general relativity the relation between irreducible mass, area and entropy of the black hole is very close. For the Kerr spacetime we have, in natural units, $M_{\rm irr}^2 = \mathcal{A}/16\pi = S/4\pi$. In chapter 8 we have only calculated an approximation to $O\left(J^2\right)$ of the entropy, but its relation to the area $\mathcal{A} = 4\pi\left[\left(r_+^{\rm I}\right)^2 + a^2\right]$ or to the irreducible mass $M_{\rm irr}$ is not known, if any.

- An investigation of the second and third laws of black hole dynamics is an important topic to be tackled, and an exact form of the first law would be desirable. Most probably progress in these directions will require new calculational schemes which allow for a more efficient extraction of physical information from QEG, its RG flow in particular.

The above list of issues is by no means complete, of course. The application of global techniques and the study of singularities is another important problem, for example. We hope nevertheless that this list will give rise to further interesting developments in the future.

Appendix A

Planck Units and Dimensionless Quantities

In this appendix we define dimensionless variables from the Planckian quantities. They are needed for many calculations or graphics throughout this work. Here we use the tilde superscript $\tilde{}$ for these quantities, even though it is sometimes supressed in the text, when it doesn't lead to ambiguities. (In those cases the dimensionless property of the respective quantities is explicitly mentioned).

With the fundamental constants given by

$$\begin{aligned} G_0 &= 6.67259 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2} \\ c &= 2.99792458 \times 10^{8}\,\mathrm{m}\,\mathrm{s}^{-1} \\ \hbar &= 1.05457266 \times 10^{-34}\,\mathrm{J}\,\mathrm{s} \end{aligned}$$

we construct the following physical values which define the Planck scale:

$$l_p = \sqrt{\frac{\hbar G_0}{c^3}} = 1.61605 \times 10^{-35}\,\mathrm{m} \tag{A.1}$$

$$m_p = \sqrt{\frac{\hbar c}{G_0}} = 2.17671 \times 10^{-8}\,\mathrm{kg} \tag{A.2}$$

$$t_p = \sqrt{\frac{\hbar G_0}{c^5}} = 5.39056 \times 10^{-44}\,\mathrm{s} \tag{A.3}$$

After setting $\hbar = c = 1$ we have for the Planck length, mass, and time, respectively,

$$l_p = \sqrt{G_0}\ ,\ m_p = \frac{1}{\sqrt{G_0}}\ ,\ t_p = \sqrt{G_0} \tag{A.4}$$

With the values in (A.4) we define the following dimensionless quantities:

$$\tilde{r} = \frac{r}{l_p} = \frac{r}{\sqrt{G_0}}\ ,\ \tilde{M} = \frac{M}{m_p} = \sqrt{G_0}M$$

For the case of the angular momentum J we have the dimensions

$$[J] = \frac{[M]\,[L]^2}{[t]}$$

which suggests the "Planck angular momentum"

$$J_p = \frac{m_p l_p^2}{t_p} = \frac{G_0}{G_0} = 1$$

As a result, the dimensionless $\tilde{J}$ agrees with $\tilde{J} = J/J_p = J$.

Proceeding similarly for the parameter a of the Kerr metric we write

$$\begin{aligned} [a] &= \left[\frac{J}{M}\right] = \frac{[L]^2}{[T]} \\ a_p &= \frac{l_p^2}{t_p} = \frac{\frac{\hbar G_0}{c^3}}{\sqrt{\frac{\hbar G_0}{c^5}}} = \sqrt{\frac{\hbar G_0}{c}} \\ \tilde{a} &= \frac{a}{a_p} = \frac{a}{\sqrt{\frac{\hbar G_0}{c}}} \end{aligned} \tag{A.5}$$

Then finally setting $\hbar = c = 1$ we have for a

$$\tilde{a} = \frac{a}{\sqrt{G_0}}$$

The canonical mass dimensions of the radial coordinate r, the angular momentum parameter a, the black hole mass M and the associated geometrical mass $m = MG_0$ are

$$[r] = -1\ ,\ [a] = -1\ ,\ [M] = 1\ ,\ [m] = -1 \tag{A.6}$$

As a result, they are related to their dimensionless counterparts $\tilde{r}$, $\tilde{a}$, $\tilde{M}$, and $\tilde{m}$ according to

$$r = \tilde{r}\sqrt{G_0}\ ,\ a = \tilde{a}\sqrt{G_0}\ ,\ M = \frac{\tilde{M}}{\sqrt{G_0}}\ ,\ m = MG_0 = \sqrt{G_0}\tilde{M}\ ,\ m = \tilde{m}\sqrt{G_0} \tag{A.7}$$

Of course it follows trivially that $\tilde{M} = \tilde{m}$.

Other important quantities which we need in dimensionless form are the following[1]:

[1]For an explanation of the various quantitities see the main text.

1. **Logarithmic Aproximation for $d\left(r\right)$**

   The dimensionful (approximate) distance function

$$d(r) = r + m\ln\left(\frac{r}{\sqrt{m^2 - a^2}}\right)$$

   has the dimensionless analog

$$\tilde{d}(r) \equiv \frac{d(r)}{\sqrt{G_0}} = \widetilde{r} + \tilde{M}\ln\left(\frac{\widetilde{r}}{\sqrt{\tilde{M}^2 - \tilde{a}^2}}\right) \tag{A.8}$$

2. **Running Newton Constant**

   In (4.2) we have defined $G\left(r\right)$ as:

$$G\left(r\right) = \frac{G_0 d\left(r\right)^2}{d\left(r\right)^2 + \bar{w}G_0}$$

   We can rewrite it using (A.8) or any other definition for $\tilde{d}\left(\tilde{r}\right)$:

$$G\left(r\right) = \frac{G_0^2\tilde{d}\left(\tilde{r}\right)^2}{G_0\tilde{d}\left(\tilde{r}\right)^2 + \bar{w}G_0} = G_0\frac{\tilde{d}\left(\tilde{r}\right)^2}{\tilde{d}\left(\tilde{r}\right)^2 + \bar{w}}$$

   Since $[G] = -2$ we define $\tilde{G} = G/G_0$ and obtain

$$\tilde{G}\left(\tilde{r}\right) \equiv \frac{G\left(r\right)}{G_0} = \frac{\tilde{d}\left(\tilde{r}\right)^2}{\tilde{d}\left(\tilde{r}\right)^2 + \bar{w}} \tag{A.9}$$

3. **Reduced Circumference (Equator)**

   Discussing circular paths we encounter the expression

$$\begin{aligned}
R_I\left(r, \frac{\pi}{2}\right) &= \sqrt{r^2 + a^2 + \frac{2Ma^2G\left(r\right)}{r}} \\
&= \sqrt{G_0\tilde{r}^2 + G_0\tilde{a}^2 + \frac{2\tilde{M}\left(G_0\tilde{a}^2\right)\tilde{G}\left(\tilde{r}\right)G_0}{\sqrt{G_0}\sqrt{G_0}\tilde{r}}} \\
&= \sqrt{G_0}\sqrt{\tilde{r}^2 + \tilde{a}^2 + \frac{2\tilde{M}\tilde{a}^2\tilde{G}\left(\tilde{r}\right)}{\tilde{r}}}
\end{aligned}$$

   As $[R_I] = -1$ we define $\tilde{R}_I\left(\tilde{r}\right)$ as follows:

$$\tilde{R}_I\left(\tilde{r}\right) \equiv \frac{R_I\left(r, \frac{\pi}{2}\right)}{\sqrt{G_0}} = \sqrt{\tilde{r}^2 + \tilde{a}^2 + \frac{2\tilde{M}\tilde{a}^2\tilde{G}\left(\tilde{r}\right)}{\tilde{r}}} \tag{A.10}$$

4. **Improved Metric Components (Equator)**

In the B-L representation of the improved Kerr metric we have the following expressions at $\theta = \frac{\pi}{2}$ for $g_{\varphi t}$:

$$g_{\varphi t} = -\frac{2G(r)\,Ma}{r} = -\frac{2G_0\tilde{G}(\tilde{r})\,\tilde{M}\tilde{a}\sqrt{G_0}}{\tilde{r}\sqrt{G_0}\sqrt{G_0}} = \left(-\frac{2\tilde{G}(\tilde{r})\,\tilde{M}\tilde{a}}{\tilde{r}}\right)\sqrt{G_0}$$

It entails the definition

$$\tilde{g}_{\varphi t} \equiv \frac{g_{\varphi t}}{\sqrt{G_0}} = -\frac{2\tilde{G}(\tilde{r})\,\tilde{M}\tilde{a}}{\tilde{r}} \tag{A.11}$$

Likewise $g_{\varphi\varphi} = R_I^2(r, \pi/2) = G_0\tilde{R}_I^2(\tilde{r})$ motivates us to set

$$\tilde{g}_{\varphi\varphi} \equiv \frac{g_{\varphi\varphi}}{G_0} = \tilde{R}_I^2(\tilde{r}) \tag{A.12}$$

For g_{tt} we have

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2G(r)\,M}{r}\right) = -\left(1 - \frac{2G_0\tilde{G}(\tilde{r})\,\tilde{M}}{\tilde{r}\sqrt{G_0}\sqrt{G_0}}\right) \\ &= -\left(1 - \frac{2\tilde{G}(\tilde{r})\,\tilde{M}}{\tilde{r}}\right) \end{aligned}$$

so that we define $\tilde{g}_{tt}$ as

$$\tilde{g}_{tt} \equiv g_{tt} = -\left(1 - \frac{2\tilde{G}(\tilde{r})\,\tilde{M}}{\tilde{r}}\right) \tag{A.13}$$

5. **Zero Energy Angular Frequency (Equator)**

For the frequency $\Omega_0\left(r, \frac{\pi}{2}\right)$ we perform the following substitutions

$$\begin{aligned} \Omega_0\left(r, \frac{\pi}{2}\right) &\equiv \frac{2MG(r)\,r - r^2}{2MG(r)\,ra} = \left(1 - \frac{r}{2MG(r)}\right)\frac{1}{a} \\ &= \left(1 - \frac{\tilde{r}}{2\tilde{M}\tilde{G}(\tilde{r})}\right)\frac{1}{\tilde{a}\sqrt{G_0}} \end{aligned}$$

so that we can define

$$\tilde{\Omega}_0\left(\tilde{r}, \frac{\pi}{2}\right) \equiv \sqrt{G_0}\,\Omega_0\left(r, \frac{\pi}{2}\right) = \left(1 - \frac{\tilde{r}}{2\tilde{M}\tilde{G}(\tilde{r})}\right)\frac{1}{\tilde{a}} \tag{A.14}$$

6. **Dragging Frequency** ω

From (2.37) we can write

$$\omega = -\frac{g_{\varphi t}}{g_{\varphi\varphi}} = -\frac{\sqrt{G_0}\tilde{g}_{\varphi t}}{G_0\tilde{g}_{\varphi\varphi}}$$

so that we come to the following definition of $\tilde{\omega}$:

$$\tilde{\omega} \equiv \omega\sqrt{G_0} = -\frac{\tilde{g}_{\varphi t}}{\tilde{g}_{\varphi\varphi}} \tag{A.15}$$

7. **Angular Frequency for Light Rays**

In a similar way as for (A.14) and (A.15) we can see that:

$$\tilde{\Omega}_{\pm} \equiv \Omega_{\pm}\sqrt{G_0} = \tilde{\omega} \pm \sqrt{\tilde{\omega}^2 - \frac{\tilde{g}_{tt}}{\tilde{g}_{\varphi\varphi}}} \tag{A.16}$$

8. **Tangential Velocities for Light Rays**

From (6.10), (A.10) and (A.16) we can write

$$v_{\pm}^{light} = R\left(r,\theta\right)\Omega_{\pm} = \sqrt{G_0}\tilde{R}\left(\tilde{r},\theta\right)\frac{\tilde{\Omega}_{\pm}}{\sqrt{G_0}} = \tilde{R}\left(\tilde{r},\theta\right)\tilde{\Omega}_{\pm}$$

Here we define $\tilde{v}_{\pm}^{light}$ as follows:

$$\tilde{v}_{\pm}^{light} \equiv v_{\pm}^{light} = \tilde{R}\left(\tilde{r},\theta\right)\tilde{\Omega}_{\pm} \tag{A.17}$$

In the main text we make frequent use of the above dimensionless expressions.

Appendix B

Proper Distance Integrals

B.1 $d\left(r\right)$ for the Schwarzschild Metric

The invariant distance $d\left(r\right)$ in (3.8) splits into the two following cases:

$$\begin{cases} \int_0^r \frac{dr'}{\sqrt{\left|1-\frac{2m}{r'}\right|}} = \int_0^r \frac{dr'}{\sqrt{\frac{2m}{r'}-1}} & \text{for } r < 2m \\ \int_0^r \frac{dr'}{\sqrt{\left|1-\frac{2m}{r'}\right|}} = \int_0^{2m} \frac{dr'}{\sqrt{\frac{2m}{r'}-1}} + \int_{2m}^r \frac{r'}{\sqrt{1-\frac{2m}{r'}}} & \text{for } 2m < r \end{cases} \tag{B.1}$$

The above integrals have the following primitives (see [58]):

$$\int \frac{dr}{\sqrt{1-\frac{2m}{r}}} = \sqrt{r\left(r-2m\right)} + m\ln\left(-m+r+\sqrt{r^2-2mr}\right) \tag{B.2}$$

$$\int \frac{dr}{\sqrt{\frac{2m}{r}-1}} = -\left[\sqrt{2mr-r^2} + m\ \arctan\left(\frac{m-r}{\sqrt{2mr-r^2}}\right)\right] \tag{B.3}$$

It's a matter of elementary analysis to verify the integrals in expressions (B.2) and (B.3). We omit therefore the proof (See [58]). We present instead the evaluation of these primitives for the relevant integration regions.

B.1.1 Definite Integrals for two Cases

Using the primitives in (B.2) and (B.3) we can find the expressions (3.9) used in the main text by inserting the limits for each case:

- **Region 1:** $r < 2m$

$$\begin{aligned} d(r) &= \int_0^r \frac{dr}{\sqrt{\left|1-\frac{2m}{r}\right|}} = \int_0^r \frac{dr}{\sqrt{\frac{2m}{r}-1}} \\ &= -\left[\sqrt{2mr-r^2} + m\ \arctan\left(\frac{m-r}{\sqrt{2mr-r^2}}\right)\right]\Bigg|_0^r \\ &= -\left[\sqrt{2mr-r^2} + m\ \arctan\left(\frac{m-r}{\sqrt{2mr-r^2}}\right)\right] + m\arctan(\infty) \\ &= -\left[\sqrt{2mr-r^2} + m\ \arctan\left(\frac{m-r}{\sqrt{2mr-r^2}}\right)\right] + \left(\frac{m\pi}{2}\right) \end{aligned} \tag{B.4}$$

- **Region 2:** $2m < r$

$$d(r) = \int_0^r \frac{dr}{\sqrt{\left|1-\frac{2m}{r}\right|}} = \int_0^{r_h} \frac{dr}{\sqrt{\frac{2m}{r}-1}} + \int_{r_h}^r \frac{dr}{\sqrt{1-\frac{2m}{r}}}$$

By using the primitives we get

$$\begin{aligned} d(r) &= -\left[\sqrt{2mr-r^2} + m\ \arctan\left(\frac{m-r}{\sqrt{2mr-r^2}}\right)\right]\Bigg|_0^{r_h} \\ &\quad + \left[\sqrt{r(r-2m)} + m\ln\left(-m+r+\sqrt{r^2-2mr}\right)\right]\Bigg|_{r_h}^r \\ d(r) &= -m\left[\arctan(-\infty) - \arctan(\infty)\right] + \\ &\quad \left[\sqrt{r(r-2m)} + m\ln\left(-m+r+\sqrt{r^2-2mr}\right)\right] - m\ln m \\ &= \pi m + \sqrt{r(r-2m)} + m\ln\left(\frac{r-m+\sqrt{r^2-2mr}}{m}\right) \end{aligned} \tag{B.5}$$

It can be easily checked that arguments and discriminants in (B.4) and (B.5) lead to well defined functions in the regions where they are to be applied. This is not so obvious for the next case, the Kerr metric at the equatorial plane.

B.2 $d(r)$ for the Kerr Metric at the Equatorial Plane.

In section 3.1.2 we saw that for the integration of (3.11) at the equator we get two different integrals given in (3.12) as

$$d(r) = \begin{cases} \int_0^r \frac{r'dr'}{\sqrt{r'^2+a^2-2mr'}} & \text{if } \Delta(r) > 0 \\ \int_0^r \frac{r'dr'}{\sqrt{2mr'-r'^2-a^2}} & \text{if } \Delta(r) < 0 \end{cases} \tag{B.6}$$

Primitives for each case in (B.6) are the following (see [58])

$$\int \frac{rdr}{\sqrt{r^2+a^2-2mr}} = \sqrt{r^2+a^2-2mr} + m\ln\left|r-m+\sqrt{r^2+a^2-2mr}\right| \tag{B.7}$$

$$\int \frac{rdr}{\sqrt{2mr-r^2-a^2}} = -\sqrt{2mr-r^2-a^2} + m\arctan\left(\frac{r-m}{\sqrt{2mr-r^2-a^2}}\right) \tag{B.8}$$

It is worthwhile, for the case of (B.7), to analyse the behavior of the argument in the logarithm. As seen in figure B.1, it has real values precisely for the regions at which this argument is to be applied, namely in the regions 1 and 3 as they are defined in (3.5).

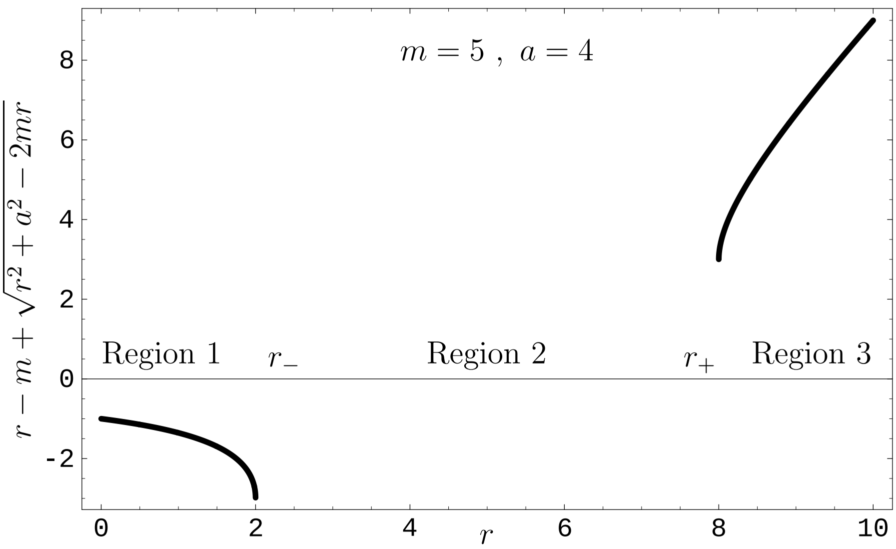

Fig. B.1.
r-dependence of the log-argument in (B.7) for a typical set of values of m and a. The change of sign from region 1 to region 3 has to be taken into account when calculating the absolute value in (B.7). Complex values are obtained for region 2, precisely where (B.7) does not apply.

Since the real Log function is only defined for positive arguments, one has to include an absolute value of this argument in order to guarantee that the primitive is well defined, even for region 1. It can be shown that (B.7) with the absolute value included is in fact a primitive:

$$f(r) = \int \frac{rdr}{\sqrt{r^2 + a^2 - 2mr}} = \sqrt{r^2 + a^2 - 2mr} + m \ln \left| r - m + \sqrt{r^2 + a^2 - 2mr} \right| \tag{B.9}$$

$$\left| r - m + \sqrt{r^2 + a^2 - 2mr} \right| = with \begin{cases} r - m + \sqrt{r^2 + a^2 - 2mr} & \text{For region 3} \\ -r + m - \sqrt{r^2 + a^2 - 2mr} & \text{For region 1} \end{cases}$$

Explicit derivatives read:

- **Region 1**

$$\begin{aligned} f(r) &= \sqrt{r^2 + a^2 - 2mr} + m \ln \left(-r + m - \sqrt{r^2 + a^2 - 2mr} \right) \\ \frac{df}{dr} &= \frac{r - m}{\sqrt{r^2 + a^2 - 2mr}} - \frac{m}{\left(r - m + \sqrt{r^2 + a^2 - 2mr} \right)} \left(-1 - \frac{r - m}{\sqrt{r^2 + a^2 - 2mr}} \right) \\ &= \frac{r}{\sqrt{r^2 + a^2 - 2mr}} \qquad \text{Q.E.D} \end{aligned}$$

- **Region 3**

$$\begin{aligned} f(r) &= \sqrt{r^2 + a^2 - 2mr} + m \ln \left(r - m + \sqrt{r^2 + a^2 - 2mr} \right) \\ \frac{df}{dr} &= \frac{r - m}{\sqrt{r^2 + a^2 - 2mr}} + \frac{m}{\left(r - m + \sqrt{r^2 + a^2 - 2mr} \right)} \left(1 + \frac{r - m}{\sqrt{r^2 + a^2 - 2mr}} \right) \\ &= \frac{r}{\sqrt{r^2 + a^2 - 2mr}} \qquad \text{Q.E.D} \end{aligned}$$

A last remark on this solution is that it can only be well defined for $a < m$. It is based on the existence of the two radii $r_\pm = m \pm \sqrt{m^2 - a^2}$. They are real valued provided that $m \geq a$. The cases $m = a$ and $m < a$ imply different integrations that should be considered independently. We don't perform this analysis since these cases are beyond the scope of our investigation. The arguments in (B.8) do not represent a major problem.

B.2.1 Definite Integrals for the Three Cases

Similarly as in the case for the Schwarzschild metric we evaluate the primitives (B.7) and (B.8) for the regions (3.5) in order to find expressions (3.14), (3.15) and (3.16).

- **Region 1:** $r < r_-$

$$
\begin{aligned}
d_1(r) &= \int_0^r \sqrt{\left|\frac{r^2}{r^2+a^2-2mr}\right|}dr = \int_0^r \frac{rdr}{\sqrt{r^2+a^2-2mr}} \\
&= \sqrt{r^2+a^2-2mr} + m\ln\left(\frac{-r+m-\sqrt{r^2+a^2-2mr}}{|a-m|}\right) - a
\end{aligned}
$$

- **Region 2:** $r_- < r < r_+$

$$
\begin{aligned}
d_2(r) &= \int_0^r \sqrt{\left|\frac{r^2}{r^2+a^2-2mr}\right|}dr \\
&= \int_0^{r_-} \frac{rdr}{\sqrt{r^2+a^2-2mr}} + \int_{r_-}^r \frac{rdr}{\sqrt{2mr-r^2-a^2}} \\
&= \left[\sqrt{r^2+a^2-2mr} + m\ln\left|r-m+\sqrt{r^2+a^2-2mr}\right|\right]\Big|_0^{r_-} + \\
&\quad \left[-\sqrt{2mr-r^2-a^2} + m\arctan\left(\frac{r-m}{\sqrt{2mr-r^2-a^2}}\right)\right]\Big|_{r_-}^r \\
&= m\ln\left|\frac{-\sqrt{m^2-a^2}}{a-m}\right| - a - \sqrt{2mr-r^2-a^2} \qquad \text{(B.10)} \\
&\quad +m\arctan\left(\frac{r-m}{\sqrt{2mr-r^2-a^2}}\right) - [m\arctan(-\infty)] \\
&= \frac{m}{2}\ln\left|\frac{m+a}{m-a}\right| - a - \sqrt{2mr-r^2-a^2} \\
&\quad +m\arctan\left(\frac{r-m}{\sqrt{2mr-r^2-a^2}}\right) + \frac{m\pi}{2} \qquad \text{(B.11)}
\end{aligned}
$$

- **Region 3:** $0 < r_- < r_+ < r$

$$
\begin{aligned}
d_3(r) &= \int_0^r \sqrt{\left|\frac{r^2}{r^2+a^2-2mr}\right|}dr \\
&= \int_0^{r_-} \frac{rdr}{\sqrt{r^2+a^2-2mr}} + \int_{r_-}^{r_+} \frac{rdr}{\sqrt{-r^2-a^2+2mr}} + \int_{r_+}^r \frac{rdr}{\sqrt{r^2+a^2-2mr}} \\
&= \left[\sqrt{r^2+a^2-2mr} + m\ln\left|r-m+\sqrt{r^2+a^2-2mr}\right|\right]\Big|_0^{r_-} \qquad \text{(B.12)} \\
&\quad + \left[-\sqrt{2mr-r^2-a^2} + m\arctan\left(\frac{r-m}{\sqrt{2mr-r^2-a^2}}\right)\right]\Big|_{r_-}^{r_+} \qquad \text{(B.13)}
\end{aligned}
$$

$$+\left[\sqrt{r^2+a^2-2mr}+m\ln\left|r-m+\sqrt{r^2+a^2-2mr}\right|\right]\Big|_{r_+}^{r} \tag{B.14}$$

Interval evaluations given by (B.12), (B.13) and (B.14) lead to the following expressions:

$$\begin{aligned}
&\left[\sqrt{r^2+a^2-2mr}+m\ln\left|r-m+\sqrt{r^2+a^2-2mr}\right|\right]\Big|_{0}^{r_-}\\
=\;& m\ln\left|-\sqrt{m^2-a^2}\right|-m\ln|a-m|-a\\
=\;& \frac{m}{2}\ln\left|\frac{m+a}{m-a}\right|-a
\end{aligned} \tag{B.15}$$

$$\left[-\sqrt{2mr-r^2-a^2}+m\arctan\left(\frac{r-m}{\sqrt{2mr-r^2-a^2}}\right)\right]\Big|_{r_-}^{r_+} \tag{B.16}$$

$$\begin{aligned}
=\;& m\arctan\left(\frac{\sqrt{m^2-a^2}}{\sqrt{2mr-r^2-a^2}\to 0^+}\right)-m\arctan\left(\frac{-\sqrt{m^2-a^2}}{\sqrt{2mr-r^2-a^2}\to 0^+}\right)\\
=\;& m\arctan(+\infty)-m\arctan(-\infty)=\pi m
\end{aligned} \tag{B.17}$$

$$\left[\sqrt{r^2+a^2-2mr}+m\ln\left|r-m+\sqrt{r^2+a^2-2mr}\right|\right]\Big|_{r_+}^{r} \tag{B.18}$$

$$=\sqrt{r^2+a^2-2mr}+m\ln\left(r-m+\sqrt{r^2+a^2-2mr}\right)-m\ln\left(\sqrt{m^2-a^2}\right)$$

Summing (B.15), (B.16) and (B.18) for the third case we get

$$\begin{aligned}
d_3(r) =\;& \sqrt{r^2+a^2-2mr}+m\ln\left(r-m+\sqrt{r^2+a^2-2mr}\right)\\
& -m\ln\left(\sqrt{m^2-a^2}\right)+\frac{m}{2}\ln\left|\frac{m+a}{m-a}\right|-a+m\pi\\
=\;& \sqrt{r^2+a^2-2mr}+m\ln\left(r-m+\sqrt{r^2+a^2-2mr}\right)+\\
& \pi m-a-m\ln|m-a|
\end{aligned} \tag{B.19}$$

Our final results (B.10), (B.11) and (B.19) yield in each region a well defined expression of the invariant distance $d(r)$ for the Kerr metric at the equatorial plane.

B.3 The Approximation of $d(r,\theta)$ for the Kerr Metric Outside the Equator

The absolute value in (3.3) can be separately expanded into the two following expressions to first order in $\alpha=\cos^2\theta$:

$$\sqrt{\frac{r^2+a^2\alpha}{r^2-2mr+a^2}}\approx\frac{r}{\sqrt{r^2+a^2-2mr}}+\frac{a^2\alpha}{2r\sqrt{r^2+a^2-2mr}} \tag{B.20}$$

$$\sqrt{\frac{r^2+a^2\alpha}{2mr-r^2-a^2}} \approx \frac{r}{\sqrt{2mr-r^2-a^2}} + \frac{a^2\alpha}{2r\sqrt{2mr-r^2-a^2}} \tag{B.21}$$

- **Case 1**

  Integrating (B.20) up to terms of order $\cos^2\theta$ we have

$$\int \sqrt{\frac{r^2+a^2\cos^2\theta}{r^2-2mr+a^2}}dr \approx \int \frac{rdr}{\sqrt{r^2-2mr+a^2}} + \frac{a^2\cos^2\theta}{2}\int \frac{dr}{r\sqrt{r^2-2mr+a^2}} \tag{B.22}$$

  Here the primitives are the following

$$\int \frac{rdr}{\sqrt{r^2-2mr+a^2}} = \sqrt{r^2-2mr+a^2} + m\ln\left(\left|r-m+\sqrt{r^2-2mr+a^2}\right|\right) \tag{B.23}$$

$$\int \frac{dr}{r\sqrt{r^2-2mr+a^2}} = \frac{1}{a}\ln\left(\left|\frac{r}{a^2-mr+a\sqrt{r^2-2mr+a^2}}\right|\right) \tag{B.24}$$

  Inserting (B.23) and (B.24) in (B.22) we find:

$$\begin{aligned}\int \sqrt{\frac{r^2+a^2\alpha}{r^2-2mr+a^2}}dr \approx\ & \sqrt{r^2-2mr+a^2} + m\ln\left(\left|r-m+\sqrt{r^2-2mr+a^2}\right|\right) \\ & + \frac{a\cos^2\theta}{2}\ln\left(\left|\frac{r}{a^2-mr+a\sqrt{r^2-2mr+a^2}}\right|\right)\end{aligned} \tag{B.25}$$

- **Case 2**

  Integrating now (B.21) up to terms of order $O(\cos^2\theta)$ we get

$$\int \sqrt{\frac{r^2+a^2\cos^2\theta}{2mr-r^2-a^2}}dr \approx \int \frac{rdr}{\sqrt{2mr-r^2-a^2}} + \frac{a^2\cos^2\theta}{2}\int \frac{dr}{r\sqrt{2mr-r^2-a^2}} \tag{B.26}$$

  The corresponding primitives are:

$$\int \frac{rdr}{\sqrt{2mr-r^2-a^2}} = -\sqrt{2mr-r^2-a^2} + m\arctan\left(\frac{r-m}{\sqrt{2mr-r^2-a^2}}\right) \tag{B.27}$$

$$\int \frac{dr}{r\sqrt{2mr - r^2 - a^2}} = \frac{1}{a} \arctan\left[\frac{mr - a^2}{a\sqrt{2mr - r^2 - a^2}}\right] \tag{B.28}$$

By substituting in (B.26) we get:

$$\begin{aligned} \int \sqrt{\frac{r^2 + a^2\cos^2\theta}{2mr - r^2 - a^2}} dr \quad \approx \quad & -\sqrt{2mr - r^2 - a^2} + m \arctan\left(\frac{r - m}{\sqrt{2mr - r^2 - a^2}}\right) \\ & + \frac{a\cos^2\theta}{2} \arctan\left[\frac{mr - a^2}{a\sqrt{2mr - r^2 - a^2}}\right] \end{aligned} \tag{B.29}$$

The approximate results in (B.25) and (B.29) are also to be used in regions 1, 2 and 3 from (3.5). For this case we have to deal with two Logs in (B.23) and (B.24). The first Log argument is the same as for (B.9), and the second one is plotted in figure B.2. Again the absolute value in the argument prevents the Log function from being ill-defined. The proof for the primitive in (B.24) runs along similar lines as that one for (B.9) and it doesn't represent any difficulty. The proofs for primitives (B.27) and (B.28) are also straightforward. We omit them therefore.

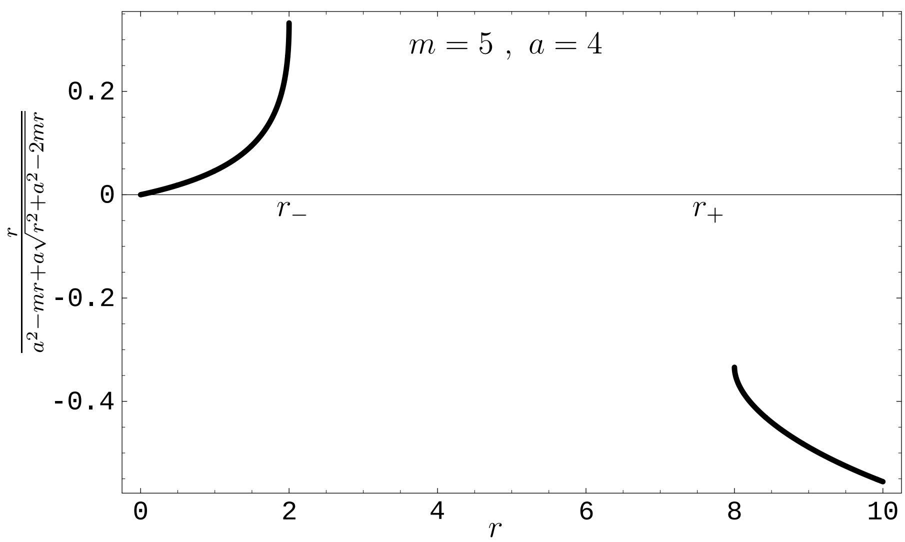

Fig. B.2.
r-dependence of the log-argument in (B.24) for a typical set of values of m and a. The change of sign from region 1 to region 3 has to be taken into account when calculating the absolute value in (B.24). Complex values are obtained for region 2, precisely where (B.24) does not apply.

B.3.1 Definite Integrals for three Cases

We evaluate now the primitives (B.25) and (B.29) for the regions defined in (3.5), namely:

- **Region 1:** $r_0 < r < r_-$

$$\begin{aligned} d_1(r,\theta) &= \int_{r_0}^{r}\sqrt{\left|\frac{r^2+a^2\cos^2\theta}{r^2-2mr+a^2}\right|}dr = \int_{r_0}^{r}\sqrt{\frac{r^2+a^2\cos^2\theta}{r^2-2mr+a^2}}dr \\ &\approx \left\{ \begin{array}{c} \sqrt{r^2-2mr+a^2}+m\ln\left(\left|r-m+\sqrt{r^2-2mr+a^2}\right|\right) \\ +\frac{a^2\cos^2\theta}{2a}\ln\left(\left|\frac{r}{a^2-mr+a\sqrt{r^2-2mr+a^2}}\right|\right) \end{array} \right\}\Bigg|_{r_0}^{r} \end{aligned}$$

Inserting the limits we get

$$d_1(r,\theta) \approx \begin{array}{c} \sqrt{r^2-2mr+a^2}+m\ln\left(-r+m-\sqrt{r^2-2mr+a^2}\right) \\ +\frac{a\cos^2\theta}{2}\ln\left(\frac{r}{a^2-mr+a\sqrt{r^2-2mr+a^2}}\right)-F_1(r_0,\theta,a,m) \end{array} \tag{B.30}$$

with

$$\begin{aligned} F_1(r_0<r_-,\theta,a,m) &= \sqrt{r_0^2-2mr_0+a^2}+m\ln\left(\left|r_0-m+\sqrt{r_0^2-2mr_0+a^2}\right|\right) \\ &\quad +\frac{a\cos^2\theta}{2}\ln\left(\left|\frac{r_0}{a^2-mr_0+a\sqrt{r_0^2-2mr_0+a^2}}\right|\right) \end{aligned}$$

Since we are in the region 1, we can transform the absolute values as follows:

$$\begin{aligned} F_1(r_0<r_-,\theta,a,m) &= \sqrt{r_0^2-2mr_0+a^2}+m\ln\left(-r_0+m-\sqrt{r_0^2-2mr_0+a^2}\right) \\ &\quad +\frac{a\cos^2\theta}{2}\ln\left(\frac{r_0}{a^2-mr_0+a\sqrt{r_0^2-2mr_0+a^2}}\right) \end{aligned} \tag{B.31}$$

- **Region 2:** $r_- < r < r_+$

$$\begin{aligned} d_2(r,\theta) &= \int_{r_0}^{r_-}\sqrt{\left|\frac{r^2+a^2\cos^2\theta}{r^2-2mr+a^2}\right|}dr \\ &= \int_{r_0}^{r_-}\sqrt{\frac{r^2+a^2\cos^2\theta}{r^2+a^2-2mr}}dr + \int_{r_-}^{r}\sqrt{\frac{r^2+a^2\cos^2\theta}{-r^2-a^2+2mr}}dr \end{aligned}$$

$$\approx \left\{ \begin{array}{c} \sqrt{r^2-2mr+a^2}+m\ln\left(\left|r-m+\sqrt{r^2-2mr+a^2}\right|\right) \\ +\frac{a\cos^2\theta}{2}\ln\left(\left|\frac{r}{a^2-mr+a\sqrt{r^2-2mr+a^2}}\right|\right) \end{array} \right\}\Bigg|_{r_0}^{r_-}$$

$$+\left\{ \begin{array}{c} -\sqrt{2mr-r^2-a^2}+m\arctan\left(\frac{r-m}{\sqrt{2mr-r^2-a^2}}\right) \\ +\frac{a\cos^2\theta}{2}\arctan\left[\frac{mr-a^2}{a\sqrt{2mr-r^2-a^2}}\right] \end{array} \right\}\Bigg|_{r_-}^{r}$$

Upon inserting the limits we have

$$\begin{aligned} d_2(r,\theta) \approx\ & F_1(r_-,\theta,a,m)-F_1(r_0,\theta,a,m) \\ & +\left\{ \begin{array}{c} -\sqrt{2mr-r^2-a^2}+m\arctan\left(\frac{r-m}{\sqrt{2mr-r^2-a^2}}\right) \\ +\frac{a\cos^2\theta}{2}\arctan\left[\frac{mr-a^2}{a\sqrt{2mr-r^2-a^2}}\right] \end{array} \right\} - F_2(r_-,\theta,a,m) \end{aligned} \tag{B.32}$$

where $F_1(r_0,\theta,a,m)$ is given by (B.31) and $F_1(r_-,\theta,a,m)$, $F_2(r_-,\theta,a,m)$ are given as follows:

$$F_1(r_-,\theta,a,m) = \left\{ \begin{array}{c} \sqrt{r^2-2mr+a^2}+m\ln\left(\left|r-m+\sqrt{r^2-2mr+a^2}\right|\right) \\ +\frac{a\cos^2\theta}{2}\ln\left(\left|\frac{r}{a^2-mr+a\sqrt{r^2-2mr+a^2}}\right|\right) \end{array} \right\}\Bigg|_{r_-}$$

$$F_1(r_-,\theta,a,m) = m\ln\left(\sqrt{m^2-a^2}\right)+\frac{a\cos^2\theta}{2}\ln\left(\frac{1}{\sqrt{m^2-a^2}}\right) \tag{B.33}$$

Here $r_- = m-\sqrt{m^2-a^2}$ was used in the last line. Similarly we can write for $F_2(r_-,\theta,a,m)$:

$$F_2(r_-,\theta,a,m) = \left\{ \begin{array}{c} -\sqrt{2mr-r^2-a^2}+m\arctan\left(\frac{r-m}{\sqrt{2mr-r^2-a^2}}\right) \\ +\frac{a\cos^2\theta}{2}\arctan\left[\frac{mr-a^2}{a\sqrt{2mr-r^2-a^2}}\right] \end{array} \right\}\Bigg|_{r_-}$$

And finally evaluating at $r_- = m-\sqrt{m^2-a^2}$ we find:

$$\begin{aligned} F_2(r_-,\theta,a,m) &= m\arctan(-\infty)+\frac{a\cos^2\theta}{2}\arctan(-\infty) \\ &= -\frac{\pi m}{2}-\frac{\pi a\cos^2\theta}{4} \end{aligned} \tag{B.34}$$

- **Region 3:** $r_- < r_+ < r$

$$d_3(r,\theta) = \int_{r_0}^{r}\sqrt{\left|\frac{r^2+a^2\cos^2\theta}{r^2-2mr+a^2}\right|}dr$$

$$= \int_{r_0}^{r_-}\sqrt{\frac{r^2+a^2\cos^2\theta}{r^2+a^2-2mr}}dr+\int_{r_-}^{r_+}\sqrt{\frac{r^2+a^2\cos^2\theta}{-r^2-a^2+2mr}}dr+\int_{r_+}^{r}\sqrt{\frac{r^2+a^2\cos^2\theta}{r^2+a^2-2mr}}dr$$

$$\approx \left\{ \begin{array}{c} \sqrt{r^2-2mr+a^2}+m\ln\left(\left|r-m+\sqrt{r^2-2mr+a^2}\right|\right) \\ +\frac{a\cos^2\theta}{2}\ln\left(\left|\frac{r}{a^2-mr+a\sqrt{r^2-2mr+a^2}}\right|\right) \end{array} \right\}\Bigg|_{r_0}^{r_-}$$

$$+\left\{ \begin{array}{c} -\sqrt{2mr-r^2-a^2}+m\arctan\left(\frac{r-m}{\sqrt{2mr-r^2-a^2}}\right) \\ +\frac{a\cos^2\theta}{2}\arctan\left[\frac{mr-a^2}{a\sqrt{2mr-r^2-a^2}}\right] \end{array} \right\}\Bigg|_{r_-}^{r_+}$$

$$+\left\{ \begin{array}{c} \sqrt{r^2-2mr+a^2}+m\ln\left(\left|r-m+\sqrt{r^2-2mr+a^2}\right|\right) \\ +\frac{a\cos^2\theta}{2}\ln\left(\left|\frac{r}{a^2-mr+a\sqrt{r^2-2mr+a^2}}\right|\right) \end{array} \right\}\Bigg|_{r_+}^{r}$$

Evaluating at the limits yields

$$\begin{aligned} d_3(r,\theta) \approx\ & -F_1(r_0,\theta,a,m)+F_1(r_-,\theta,a,m)+F_2(r_+,\theta,a,m)-F_2(r_-,\theta,a,m) \\ & +\left\{ \begin{array}{c} \sqrt{r^2-2mr+a^2}+m\ln\left(\left|r-m+\sqrt{r^2-2mr+a^2}\right|\right) \\ +\frac{a\cos^2\theta}{2}\ln\left(\left|\frac{r}{a^2-mr+a\sqrt{r^2-2mr+a^2}}\right|\right) \end{array} \right\}-F_1(r_+,\theta,a,m) \end{aligned}$$

Since we are in region 3 we can interpret the absolute values as follows

$$\begin{aligned} d_3(r,\theta) \approx\ & -F_1(r_0,\theta,a,m)+F_2(r_+,\theta,a,m)-F_2(r_-,\theta,a,m) \\ & +\left\{ \begin{array}{c} \sqrt{r^2-2mr+a^2}+m\ln\left(r-m+\sqrt{r^2-2mr+a^2}\right) \\ +\frac{a\cos^2\theta}{2}\ln\left(\frac{-r}{a^2-mr+a\sqrt{r^2-2mr+a^2}}\right) \end{array} \right\} \end{aligned} \tag{B.35}$$

with $F_1(r_0,\theta,a,m)$, $F_1(r_-,\theta,a,m)$ and $F_2(r_-,\theta,a,m)$ already given above. Furthermore, $F_2(r_+,\theta,a,m)$, $F_1(r_+,\theta,a,m)$ are given by

$$F_2(r_+,\theta,a,m) = \left\{ \begin{array}{c} -\sqrt{2mr-r^2-a^2}+m\arctan\left(\frac{r-m}{\sqrt{2mr-r^2-a^2}}\right) \\ +\frac{a\cos^2\theta}{2}\arctan\left[\frac{mr-a^2}{a\sqrt{2mr-r^2-a^2}}\right] \end{array} \right\}\Bigg|_{r_+}$$

Or, upon inserting $r_+=m+\sqrt{m^2-a^2}$,

$$F_2(r_+,\theta,a,m)=\frac{\pi m}{2}+\frac{\pi a\cos^2\theta}{4}$$

Likewise,

$$F_1(r_+,\theta,a,m) = \left\{ \begin{array}{c} \sqrt{r^2-2mr+a^2}+m\ln\left(\left|r-m+\sqrt{r^2-2mr+a^2}\right|\right) \\ +\frac{a\cos^2\theta}{2}\ln\left(\left|\frac{r}{a^2-mr+a\sqrt{r^2-2mr+a^2}}\right|\right) \end{array} \right\}\Bigg|_{r_+}$$

which becomes upon inserting inserting r_+:

$$\begin{aligned} F_1\left(r_+,\theta,a,m\right) &= m\ln\left(\sqrt{m^2-a^2}\right)+\frac{a\cos^2\theta}{2}\ln\left(\frac{1}{\sqrt{m^2-a^2}}\right) \\ &= F_1\left(r_-,\theta,a,m\right) \end{aligned}$$

Thus we see that F_1 has the same form at r_+ and r_-, respectively.

B.4 Series Expansion for $d_3\left(r,\theta\right)$

From equation (3.20) we have an expression for the external $d_3\left(r,\theta\right)$ approximation to $d\left(r,\theta\right)$ in (3.3), for $\theta\approx\frac{\pi}{2}$:

$$d_3\left(r,\theta\right) \approx \begin{cases} \sqrt{r^2-2mr+a^2} & \text{(A)} \\ +m\ln\left(r-m+\sqrt{r^2-2mr+a^2}\right) & \text{(B)} \\ +\frac{a\cos^2\theta}{2}\ln\left(\frac{-r}{a^2-mr+a\sqrt{r^2-2mr+a^2}}\right) & \text{(C)} \\ +F_2\left(r_+,\theta,a,m\right)-F_2\left(r_-,\theta,a,m\right)-F_1\left(r_0,\theta,a,m\right) \end{cases} \tag{B.36}$$

We expand now (B.36) term by term in powers of $\frac{1}{r}$:

- **Term (A)**

$$\sqrt{r^2+a^2-2mr}=r\sqrt{1+\frac{a^2}{r^2}-\frac{2m}{r}}=r\left[1-\frac{m}{r}+\frac{a^2-m^2}{2r^2}+\frac{m\left(a^2-m^2\right)}{2r^3}+O\left(\frac{1}{r^2}\right)\right] \tag{B.37}$$

- **Term (B)**

$$\begin{aligned} m\ln\left(r-m+\sqrt{r^2-2mr+a^2}\right) &= m\ln\left[\left(r-m\right)\left(1+\sqrt{1+\frac{a^2-m^2}{\left(r-m\right)^2}}\right)\right] \\ &= m\ln\left(r-m\right)+m\ln\left[\left(1+\sqrt{1+x}\right)\right] \end{aligned} \tag{B.38}$$

Here we introduce the quantity x, with the expansion

$$x=\frac{a^2-m^2}{\left(r-m\right)^2}=\left(\frac{a^2-m^2}{r^2}\right)\frac{1}{\left(1-\frac{m}{r}\right)^2}=\left(\frac{a^2-m^2}{r^2}\right)\left(1+\frac{2m}{r}+\frac{3m^2}{r^2}+\cdots\right)$$

By expanding

$$m\ln\left[\left(1+\sqrt{1+x}\right)\right]=m\left[\ln 2+\frac{x}{4}-\frac{3x^2}{32}+O\left(x^3\right)+\cdots\right]$$

and substituting into (B.38) we have

$$
\begin{aligned}
& m \ln\left(r-m+\sqrt{r^2-2mr+a^2}\right) = \\
& m \ln(r-m) + m\left[\ln 2 + \frac{a^2-m^2}{4(r-m)^2} - \frac{3}{32}\frac{\left(a^2-m^2\right)^2}{(r-m)^4} + \cdots\right] \\
& = m \ln(r-m) + m\left[\ln 2 + \left(\frac{a^2-m^2}{4r^2}\right)\left(1+\frac{2m}{r}+\frac{3m^2}{r^2}+\cdots\right) - \frac{3}{32}\frac{\left(a^2-m^2\right)^2}{(r-m)^4} + \cdots\right]
\end{aligned}
$$

Finally we can write:

$$
m \ln\left(r-m+\sqrt{r^2-2mr+a^2}\right) = m \ln(r-m) + m\left[\ln 2 + \left(\frac{a^2-m^2}{4r^2}\right) + O\left(\frac{1}{r^3}\right)\right] \tag{B.39}
$$

- **Term (C)**

$$
\frac{a\cos^2\theta}{2}\ln\left(\frac{-r}{a^2-mr+a\sqrt{r^2-2mr+a^2}}\right) = \frac{a\cos^2\theta}{2}\ln\left(\frac{1}{-\frac{a^2}{r}+m-a\sqrt{1-\frac{2m}{r}+\frac{a^2}{r^2}}}\right)
$$

$$
= \frac{a\cos^2\theta}{2}\left[\ln\left(\frac{1}{m-a}\right) - \frac{a}{r} - \left(\frac{am}{2r^2}\right) + O\left(\frac{1}{r^3}\right)\right] \tag{B.40}
$$

Summing (B.37), (B.39) and (B.40) we get

$$
\begin{aligned}
d_3(r,\theta) \quad \approx \quad & r-m+m\ln(2(r-m)) + \frac{a\cos^2\theta}{2}\ln\left(\frac{1}{m-a}\right) \\
& +F_2(r_+,\theta,a,m) - F_2(r_-,\theta,a,m) - F_1(r_0,\theta,a,m) \\
& +\frac{a^2\sin^2\theta-m^2}{2r} + m\frac{\left(a^2\sin^2\theta-m^2\right)}{4r^2} + O\left(\frac{1}{r^3}\right)
\end{aligned} \tag{B.41}
$$

After neglecting terms with orders higher or equal to $\frac{1}{r}$, we stay with

$$
\begin{aligned}
d_3(r,\theta) \quad \approx \quad & r-m+m\ln(2(r-m)) + \frac{a\cos^2\theta}{2}\ln\left(\frac{1}{m-a}\right) \\
& +F_2(r_+,\theta,a,m) - F_2(r_-,\theta,a,m) - F_1(r_0,\theta,a,m)
\end{aligned} \tag{B.42}
$$

This is the expression in eq. (3.25) of the main text.

B.5 Power Series for the Meridian Reduced Circumference

The indefinite integral for (3.30) is given by:

$$\int \sqrt{r^2 + a^2 \cos^2\theta}\, d\theta \;\; = \;\; \left(\sqrt{r^2 + a^2}\right) E\left(\theta \,\middle|\, \frac{a^2}{a^2 + r^2}\right) \tag{B.43}$$

where $E\left(\theta\,|x\,\right)$ denotes an elliptic integral of the second Kind. Evaluating the elliptic integral at the limits in (3.30) we get

$$E\left(0\,|x\,\right) \;\; = \;\; 0\;,\;\; E\left(2\pi\,|x\,\right) = 4E\left(x\right) \tag{B.44}$$

where $E\left(x\right)$ denotes a complete elliptic integral. For $a \ll r$ we can perform the following expansions:

$$\begin{aligned} E\left(x\right) &= \frac{\pi}{2} - \frac{\pi}{8}x - \frac{3\pi}{128}x^2 - \cdots \\ x &= \frac{a^2}{a^2 + r^2} = \frac{a^2}{r^2}\left(1 + \frac{a^2}{r^2} + \frac{a^4}{r^4} + \cdots\right) \\ \sqrt{r^2 + a^2} &= r\left(1 + \frac{a^2}{2r^2} + \cdots\right) \end{aligned} \tag{B.45}$$

Substituting (B.43), (B.44) and (B.45) in (3.30) we find

$$\begin{aligned} R^{\mathrm{Me}}_{\text{Kerr-I}} &= \frac{1}{2\pi}\int_0^{2\pi} \sqrt{r^2 + a^2\cos^2\theta}\, d\theta \\ &= \frac{1}{2\pi}\left(\sqrt{r^2 + a^2}\right) E\left(\theta \;\middle|\, \frac{a^2}{a^2 + r^2}\right)\Bigg|_0^{2\pi} \\ &= \frac{4}{2\pi}\left(\sqrt{r^2 + a^2}\right) E\left(\frac{a^2}{a^2 + r^2}\right) \qquad\qquad (\mathrm{B.46}) \\ &\approx \frac{r}{2\pi}\left(1 + \frac{a^2}{2r^2} + ..\right)\left[E\left(2\pi\;\middle|\frac{a^2}{r^2}\right) - E\left(0\;\middle|\frac{a^2}{r^2}\right)\right] \\ &\approx \frac{4r}{2\pi}\left(1 + \frac{a^2}{2r^2} + \cdots\right)\left[\frac{\pi}{2} - \frac{\pi}{8}\frac{a^2}{r^2} - \frac{3\pi}{128}\frac{a^4}{r^4} - \cdots\right] \\ &\approx r\left(1 + \frac{a^2}{4r^2} - \frac{a^4}{8r^4} + \cdots\right) \qquad\qquad (\mathrm{B.47}) \end{aligned}$$

The expression in (B.47) is precisely the expansion in eq. (3.31) of the main text.

Appendix C

Transformation from Boyer-Lindquist to Eddington-Finkelstein Coordinates

In chapters 2 and 8 we have pointed out the necessity of representing the Kerr spacetime in a system of coordinates that is well behaved at the event horizon in order to calculate physical quantites like the surface gravity or the temperature. The Eddington-Finkelstein coordinate systems (E-F) are an appropriate choice. In this appendix we calculate the representation of the improved Kerr spacetime in the ingoing E-F coordinates, starting from the Boyer-Lindquist (B-L) representation.

C.1 Some Useful Identities

In this section we demonstrate several identities that are needed later on in section C.2. Two equivalent forms for the Improved Kerr metric in Boyer-Lindquist coordinates are the following:

$$ds^2 = -\frac{\Delta_I}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\varphi - adt\right]^2 + \frac{\rho^2}{\Delta_I}dr^2 + \rho^2 d\theta^2 \tag{C.1}$$

This is the “squared” form ([57, 877]). The standard form is given by

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2G\left(r\right)Mr}{\rho^2}\right)dt^2 - \frac{4G\left(r\right)Mar\sin^2\theta}{\rho^2}d\varphi dt + \frac{\Sigma_I}{\rho^2}\sin^2\theta d\varphi^2 \\ &\quad + \frac{\rho^2}{\Delta_I}dr^2 + \rho^2 d\theta^2 \end{aligned} \tag{C.2}$$

with the following definitions:

$$\begin{aligned}
\Delta_I &= r^2 - 2G(r)\,Mr + a^2 \\
\Sigma_I &= \left(r^2 + a^2\right)^2 - a^2\Delta_I \sin^2\theta \\
\rho^2 &= r^2 + a^2\cos^2\theta \\
a &= \frac{J}{M}
\end{aligned} \tag{C.3}$$

From the first we can get almost directly the second:

$$\begin{aligned}
ds^2 &= -\frac{\Delta_I}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\varphi - adt\right]^2 + \frac{\rho^2}{\Delta_I}dr^2 + \rho^2 d\theta^2 \\
&= -\frac{\Delta_I}{\rho^2}dt^2 + \frac{2a\Delta_I\sin^2\theta}{\rho^2}d\varphi dt - \frac{\Delta_I a^2\sin^4\theta}{\rho^2}d\varphi^2 + \\
&\quad \frac{\sin^2\theta}{\rho^2}\left(r^2+a^2\right)^2 d\varphi^2 - \frac{2a\sin^2\theta\left(r^2+a^2\right)}{\rho^2}d\varphi dt + \frac{a^2\sin^2\theta}{\rho^2}dt^2 + \frac{\rho^2}{\Delta_I}dr^2 + \rho^2 d\theta^2 \\
&= \left(a^2\sin^2\theta - \Delta_I\right)\frac{dt^2}{\rho^2} + \frac{\sin^2\theta}{\rho^2}d\varphi^2\left[\left(r^2+a^2\right)^2 - \Delta_I a^2\sin^2\theta\right] \\
&\quad + \frac{2a\sin^2\theta}{\rho^2}\left[\Delta_I - \left(r^2+a^2\right)\right]d\varphi dt + \frac{\rho^2}{\Delta_I}dr^2 + \rho^2 d\theta^2
\end{aligned}$$

As a result we have:

$$ds^2 = -\left[1 - \frac{2G(r)\,Mr}{\rho^2}\right]dt^2 + \frac{\Sigma_I\sin^2\theta}{\rho^2}d\varphi^2 - \frac{4aMrG(r)\sin^2\theta}{\rho^2}d\varphi dt + \frac{\rho^2}{\Delta_I}dr^2 + \rho^2 d\theta^2$$

The positivity of Σ_I is also needed. It is straightforward to show that $\Sigma_I(r,\theta) > 0$ for all $r > 0$ and for all $\theta \neq 0, \pi$. It is enough to expand Σ_I as follows:

$$\begin{aligned}
\Sigma_I &= \left(r^2+a^2\right)^2 - a^2\Delta_I\sin^2\theta \\
&= r^4 + a^4\cos^2\theta + r^2a^2\left(2 - \sin^2\theta\right) \\
&\quad +2G(r)\,Mra^2\sin^2\theta
\end{aligned}$$

where every term is clearly greater than zero.

From the last proof we conclude immediately the following:

$$\begin{aligned}
g_{\varphi\varphi} &= \frac{\Sigma_I}{\rho^2}\sin^2\theta > 0 \quad \forall\, r > 0\ ,\ \forall\, \theta \neq 0, \pi \\
\omega &= \frac{2G(r)\,Mr}{\Sigma_I} > 0 \quad \forall\, r > 0\ ,\ \forall\, \theta
\end{aligned} \tag{C.4}$$

Some “trivial” useful identities are the following. From $\rho^2 = r^2 + a^2\cos^2\theta$ we deduce

$$r^2 + a^2 = \rho^2 + a^2\sin^2\theta = \rho^2 + f^2 \tag{C.5}$$

where $f^2 \equiv a^2 \sin^2\theta$. In terms of f the functions $\Delta_I(r) \equiv r^2 + a^2 - 2G(r) Mr$ and $\Sigma_I(r) \equiv \left(r^2 + a^2\right)^2 - a^2\Delta_I \sin^2\theta$ can be expressed as follows:

$$\Delta_I(r) = \rho^2 + f^2 - 2G(r) Mr \tag{C.6}$$

$$\Sigma_I = \left(\rho^2 + f^2\right)^2 - \Delta_I f^2 \tag{C.7}$$

C.2 Improved Eddington-Finkelstein Transformation

The improved E-F tranformations defined in chapter 8 comprise the ingoing coordinates given by

$$v = t + r_I^* \ , \ \psi = \varphi + r_I^\# \tag{C.8}$$

and the outgoing coordinates defined as

$$v = t - r_I^* \ , \ \chi = \varphi - r^\# \tag{C.9}$$

where r_I^* and $r_I^\#$ are the following integrals:

$$r_I^* = \int \left(\frac{r^2 + a^2}{\Delta_I}\right) dr \ , \ r_I^\# = \int \left(\frac{a}{\Delta_I}\right) dr$$

The differential form of these transformations is more advantageous when we wish to transform the metric components. The differential form of the ingoing transformation reads

$$dv = dt + dr_I^* = dt + \left(\frac{r^2 + a^2}{\Delta_I}\right) dr \ , \ d\psi = d\varphi + dr_I^\# = d\varphi + \left(\frac{a}{\Delta_I}\right) dr \tag{C.10}$$

For the outgoing coordinates we have

$$dv = dt - dr_I^* = dt - \left(\frac{r^2 + a^2}{\Delta_I}\right) dr \ , \ d\chi = d\varphi - dr_I^\# = d\varphi - \left(\frac{a}{\Delta_I}\right) dr \tag{C.11}$$

The improved Kerr spacetime in B-L coordinate reads

$$ds^2 = -\left(1 - \frac{2G(r) Mr}{\rho^2}\right) dt^2 - \frac{4G(r) Mar \sin^2\theta}{\rho^2} d\varphi dt + \frac{\Sigma_I}{\rho^2} \sin^2\theta d\varphi^2 + \frac{\rho^2}{\Delta_I} dr^2 + \rho^2 d\theta^2 \tag{C.12}$$

Substituting the ingoing E-F coordinates (C.10) in (C.12) yields

$$g_{tt} dt^2 = -\left(1 - \frac{2G(r) Mr}{\rho^2}\right) \left[dv^2 - 2\frac{\left(r^2 + a^2\right)}{\Delta_I} dv dr + \frac{\left(r^2 + a^2\right)^2}{\Delta_I^2} dr^2\right] \tag{C.13}$$

$$g_{\varphi t}d\varphi dt = -2\frac{2G(r)\,Mar\sin^2\theta}{\rho^2}\left[d\psi dv - \frac{a}{\Delta_I}drdv - \frac{(r^2+a^2)}{\Delta_I}d\psi dr + \frac{a\,(r^2+a^2)}{\Delta_I^2}dr^2\right] \tag{C.14}$$

$$g_{\varphi\varphi}d\varphi^2 = \frac{\Sigma_I\sin^2\theta}{\rho^2}\left[d\psi^2 - \frac{2a}{\Delta_I}d\psi dr + \frac{a^2}{\Delta_I^2}dr^2\right] \tag{C.15}$$

Factorizing differentials we get:

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + 2g_{\varphi t}d\varphi dt + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 = \qquad (C.16)\\ &\quad g_{tt}dv^2 + g_{rr}dr^2 + 2g_{rv}drdv + 2g_{r\psi}drd\psi \\ &\quad +2g_{\varphi t}d\varphi dt + 2g_{\varphi t}d\psi dv + g_{\varphi\varphi}d\psi^2 + g_{\theta\theta}d\theta^2 = \end{aligned}$$

$$-\left(1 - \frac{2G(r)\,Mr}{\rho^2}\right)dv^2 +$$

$$dr^2\left\{\begin{array}{c} -\left(1-\frac{2G(r)Mr}{\rho^2}\right)\left[\frac{\left(r^2+a^2\right)^2}{\Delta_I^2}\right] - 2\frac{2G(r)Mar\sin^2\theta}{\rho^2}\left[\frac{a\left(r^2+a^2\right)}{\Delta_I^2}\right] \\ +\frac{\Sigma_I\sin^2\theta}{\rho^2}\frac{a^2}{\Delta_I^2} + \frac{\rho^2}{\Delta_I^2} \end{array}\right\} + \tag{C.17}$$

$$drdv\left[2\frac{(r^2+a^2)}{\Delta_I}\left(1 - \frac{2G(r)\,Mr}{\rho^2}\right) + \frac{4G(r)\,Ma^2r\sin^2\theta}{\rho^2\Delta_I}\right] + \tag{C.18}$$

$$\left\{\frac{4G(r)\,(r^2+a^2)\,Mar\sin^2\theta}{\rho^2\Delta_I} - \frac{2a\Sigma_I\sin^2\theta}{\Delta_I\rho^2}\right\}d\psi dr + \tag{C.19}$$

$$-\frac{4G(r)\,Mar\sin^2\theta}{\rho^2}d\psi dv + \frac{\Sigma_I\sin^2\theta}{\rho^2}d\psi^2 + \frac{\rho^2}{\Delta_I}dr^2 + \rho^2 d\theta^2$$

By using the identities (C.5) to (C.7) we can simplify the partial results (C.17) to (C.19) as follows:

$$\begin{aligned} \rho^2\Delta_I^2 g_{rr}dr^2 &= dr^2\left\{\begin{array}{c} -\left[\rho^2 - 2G(r)\,Mr\right](r^2+a^2)^2 - 4G(r)\,Ma^2r\,(r^2+a^2)\sin^2\theta \\ +a^2\Sigma_I\sin^2\theta + \rho^4\Delta_I \end{array}\right\} \qquad (C.20)\\ &= dr^2\left\{\begin{array}{c} -\left[\Delta_I - f^2\right](\rho^2+f^2)^2 - 2\,(\rho^2+f^2-\Delta_I)\,f^2\,(\rho^2+f^2) \\ +f^2\left[(\rho^2+f^2)^2 - \Delta_I f^2\right] + \rho^4\Delta_I \end{array}\right\} \\ &= dr^2\left\{\begin{array}{c} -\left[\Delta_I - f^2\right](\rho^2+f^2)^2 + f^2\,(\rho^2+f^2)^2 - 2f^2\,(\rho^2+f^2)^2 \\ +2\Delta_I f^2\,(\rho^2+f^2) - \Delta_I f^4 + \rho^4\Delta_I \end{array}\right\} \\ &= dr^2\left\{\left[-\Delta_I + f^2 + f^2 - 2f^2\right]\left(\rho^2+f^2\right)^2 + 2\Delta_I f^2\left(\rho^2+f^2\right) - \Delta_I f^4 + \rho^4\Delta_I\right\} \\ &= dr^2\left\{-\Delta_I\rho^4 - 2\Delta_I\rho^2 f^2 - \Delta_I f^4 + 2\Delta_I f^2\rho^2 + 2\Delta_I f^4 - \Delta_I f^4 + \rho^4\Delta_I\right\} \\ &= 0 \end{aligned}$$

As a result, $g_{rr} = 0$ in the ingoing E-F coordinates. For g_{rv} we have

$$\begin{aligned}
2g_{rv}\Delta_I\rho^2 drdv &= 2drdv\left[\left(r^2+a^2\right)\left(\rho^2 - 2G\left(r\right)Mr\right) + 2G\left(r\right)Mrf^2\right] \\
&= 2drdv\left[\left(\rho^2+f^2\right)\left(\Delta_I - f^2\right) + \left(\rho^2+f^2-\Delta_I\right)f^2\right] \\
&= 2drdv\rho^2\Delta_I \qquad \text{(C.21)}
\end{aligned}$$

In this case we conclude $g_{rv} = 1$. Finally for $g_{r\psi}$ we find

$$\begin{aligned}
2g_{r\psi}\rho^2\Delta_I drd\psi &= -2a\sin^2\theta\left\{\Sigma_I - 2G\left(r\right)Mr\left(r^2+a^2\right)\right\}d\psi dr \qquad \text{(C.22)} \\
&= -2a\sin^2\theta\left\{\left(\rho^2+f^2\right)^2 - \Delta_I f^2 + \left(\Delta_I - f^2 - \rho^2\right)\left(\rho^2+f^2\right)\right\}d\psi dr \\
&= -2a\Delta_I\rho^2\sin^2\theta d\psi dr
\end{aligned}$$

which implies $g_{r\psi} = -a\sin^2\theta$.

Finally substituting results in (C.20) to (C.22) in the line element (C.16) we find:

$$\begin{aligned}
ds^2 &= -\left(1 - \frac{2G\left(r\right)Mr}{\rho^2}\right)dv^2 + 2drdv - 2a\sin^2\theta d\psi dr + \\
&\quad -\frac{4G\left(r\right)Mar\sin^2\theta}{\rho^2}d\psi dv + \frac{\Sigma_I\sin^2\theta}{\rho^2}d\psi^2 + \rho^2 d\theta^2 \qquad \text{(C.23)}
\end{aligned}$$

This is our final result. Replacing dr by $-dr$ in (C.23) gives the representation in the outgoing coordinates.

Appendix D

RG improved Rotation Frequency for Stationary Light Paths

In chapter 4 we have employed eq. (4.43) in order to find the condition $\Delta_I = 0$ of the event horizon. The equation is given by

$$\Omega_{\pm} = \omega \pm \frac{\rho^2 \sqrt{\Delta_I}}{\Sigma_I \sin\theta} \tag{D.1}$$

We now derive this relation. From (4.36) and (4.25) we have:

$$\begin{aligned} \Omega_{\pm} &= \omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \qquad \text{(D.2)} \\ \omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}} &= \left(\frac{1}{g_{\varphi\varphi}^2}\right)\left(g_{\varphi t}^2 - g_{\varphi\varphi} g_{tt}\right) \end{aligned}$$

Substituting the components (C.2) in (D.2) we have

$$\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}} = \left(\frac{\rho^4}{\Sigma_I^2 \sin^4\theta}\right)\left(g_{\varphi t}^2 - g_{\varphi\varphi} g_{tt}\right) = \left(\frac{\rho^4}{\Sigma_I^2 \sin^2\theta}\right) X$$

with X given by

$$X = \left(\frac{1}{\sin^2\theta}\right)\left(g_{\varphi t}^2 - g_{\varphi\varphi} g_{tt}\right)$$

Expanding X we have

$$\begin{aligned} X &= \left(\frac{1}{\sin^2\theta}\right)\left(g_{\varphi t}^2 - g_{\varphi\varphi} g_{tt}\right) \\ &= \left(\frac{1}{\rho^4 \sin^2\theta}\right)\left\{\left[2G(r)\,Mr\right]^2 a^2 \sin^4\theta + \left[\rho^2 - 2G(r)\,Mr\right]\Sigma_I \sin^2\theta\right\} \\ &= \left(\frac{1}{\rho^4}\right)\left\{\left[2G(r)\,Mr\right]^2 a^2 \sin^2\theta + \left[\rho^2 - 2G(r)\,Mr\right]\Sigma_I\right\} \end{aligned}$$

Employing identities (C.6) and (C.7) reduces X to

$$X = \left(\frac{1}{\rho^4}\right)\Big\{\left[\rho^2 + a^2\sin^2\theta - \Delta_I\right]^2 a^2\sin^2\theta \tag{D.3}$$

$$+ \left[\Delta_I - a^2\sin^2\theta\right]\left[\left(\rho^2 + a^2\sin^2\theta\right)^2 - a^2\Delta_I\sin^2\theta\right]\Big\} \tag{D.4}$$

Substituting $f^2 = a^2\sin^2\theta$ gives

$$\begin{aligned}
X &= \left(\frac{1}{\rho^4}\right)\Big\{\left[\rho^2 + f^2 - \Delta_I\right]^2 f^2 + \left[\Delta_I - f^2\right]\left[\left(\rho^2 + f^2\right)^2 - \Delta_I f^2\right]\Big\} \\
&= \left(\frac{1}{\rho^4}\right)\Big\{f^2\left(\rho^2 + f^2\right)^2 - 2\Delta_I f^2\left(\rho^2 + f^2\right) + \Delta_I^2 f^2 + \Delta_I\left(\rho^2 + f^2\right)^2 \qquad \text{(D.5)} \\
&\quad - f^2\left(\rho^2 + f^2\right)^2 - \Delta_I^2 f^2 + \Delta_I f^4\Big\} \\
&= \left(\frac{\Delta_I}{\rho^4}\right)\Big\{\left(\rho^2 + f^2\right)^2 - 2f^2\left(\rho^2 + f^2\right) + f^4\Big\} \\
&= \left(\frac{\Delta_I}{\rho^4}\right)\Big\{\left(\rho^2 + f^2 - f^2\right)^2\Big\} = \Delta_I
\end{aligned}$$

We have $X = \Delta_I$. As a result (D.2) is simplified to:

$$\Omega_\pm = \omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} = \omega \pm \sqrt{\left(\frac{\rho^4}{\Sigma_I^2\sin^2\theta}\right) X}$$

Substituting X we come to:

$$\Omega_\pm = \omega \pm \frac{\rho^2\sqrt{\Delta_I}}{\Sigma_I\sin\theta} \tag{D.6}$$

This is precisely the relation we wanted to proof.

Appendix E

Condition for Extremal Black Holes in the $d(r) = r$ Approximation

As remarked in subsection 5.2.1 the condition $Q_{\tilde{b}}^{\bar{w}}(\tilde{r}_2) = 0$ can be replaced by the simpler equation given in (5.62):

$$\frac{\tilde{r}_2^2}{2}\left\{\tilde{b}^2 + \bar{w} - \tilde{m}\tilde{r}_2\right\} + \bar{w}\tilde{b}^2 = 0 \tag{E.1}$$

This can be shown as follows.

We factorize the polynomial $Q_{\tilde{b}}^{\bar{w}}(\tilde{r}_2) = \tilde{r}^4 - 2\tilde{m}\tilde{r}^3 + \tilde{r}^2\left(\tilde{b}^2 + \bar{w}\right) + \bar{w}\tilde{b}^2$ to have

$$Q_{\tilde{b}}^{\bar{w}}(\tilde{r}_2) = \tilde{r}_2^2\left(\tilde{r}_2^2 - 2\tilde{m}\tilde{r}_2 + \tilde{b}^2 + \bar{w}\right) + \bar{w}\tilde{b}^2 \tag{E.2}$$

From (5.57) we have for $\tilde{r}_2$

$$\tilde{r}_2 = \frac{3\tilde{m}}{4}\left[1 + \sqrt{1 - \frac{8}{9\tilde{m}^2}\left(\tilde{b}^2 + \bar{w}\right)}\right] = \frac{1}{4}\left[3\tilde{m} + \sqrt{9\tilde{m}^2 - 8\left(\tilde{b}^2 + \bar{w}\right)}\right] \tag{E.3}$$

As a result, $\tilde{r}_2^2$ gives

$$\begin{aligned}\tilde{r}_2^2 &= \left(\frac{1}{16}\right)\left[3\tilde{m} + \sqrt{9\tilde{m}^2 - 8\left(\tilde{b}^2 + \bar{w}\right)}\right]^2 \\ &= \left(\frac{1}{8}\right)\left[9\tilde{m}^2 + 3\tilde{m}\sqrt{9\tilde{m}^2 - 8\left(\tilde{b}^2 + \bar{w}\right)} - 4\left(\tilde{b}^2 + \bar{w}\right)\right]\end{aligned} \tag{E.4}$$

Substituting (E.3) and (E.4) in (E.2) we get:

$$Q_{\tilde{b}}^{\bar{w}}\left(\tilde{r}_2\right) = \left(\frac{1}{16}\right)\left[3\tilde{m} + \sqrt{9\tilde{m}^2 - 8\left(\tilde{b}^2 + \bar{w}\right)}\right]^2 \times$$

$$\times\left\{\left(\frac{1}{8}\right)\left[9\tilde{m}^2 + 3\tilde{m}\sqrt{9\tilde{m}^2 - 8\left(\tilde{b}^2 + \bar{w}\right)} - 4\left(\tilde{b}^2 + \bar{w}\right)\right]\right.$$
$$\left. - \frac{2\tilde{m}}{4}\left[3\tilde{m} + \sqrt{9\tilde{m}^2 - 8\left(\tilde{b}^2 + \bar{w}\right)}\right] + \tilde{b}^2 + \bar{w}\right\} + \bar{w}\tilde{b}^2$$

$$= \left(\frac{1}{16}\right)\left[3\tilde{m} + \sqrt{9\tilde{m}^2 - 8\left(\tilde{b}^2 + \bar{w}\right)}\right]^2\left\{-\frac{3\tilde{m}^2}{8} - \frac{\tilde{m}}{8}\sqrt{9\tilde{m}^2 - 8\left(\tilde{b}^2 + \bar{w}\right)} + \frac{1}{2}\left(\tilde{b}^2 + \bar{w}\right)\right\} + \bar{w}\tilde{b}^2$$
$$= \left(\frac{1}{16}\right)\left[3\tilde{m} + \sqrt{9\tilde{m}^2 - 8\left(\tilde{b}^2 + \bar{w}\right)}\right]^2\left\{\frac{1}{2}\left(\tilde{b}^2 + \bar{w}\right) - \frac{\tilde{m}}{8}\left[3\tilde{m} + \sqrt{9\tilde{m}^2 - 8\left(\tilde{b}^2 + \bar{w}\right)}\right]\right\} + \bar{w}\tilde{b}^2$$

Remembering that

$$\tilde{r}_2 = \frac{3\tilde{m}}{4}\left[1 + \sqrt{1 - \frac{8}{9\tilde{m}^2}\left(\tilde{b}^2 + \bar{w}\right)}\right] \tag{E.5}$$

we come to the result:

$$Q_{\tilde{b}}^{\bar{w}}\left(\tilde{r}_2\right) = \frac{\tilde{r}_2^2}{2}\left\{\tilde{b}^2 + \bar{w} - \tilde{m}\tilde{r}_2\right\} + \bar{w}\tilde{b}^2$$

This is exactly the representation we wanted to derive.

Appendix F

Killing Vectors for the Improved Kerr Spacetime

By definition a Killing vector $\boldsymbol{\xi}$ fulfills the Killing equation:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{F.1}$$

Knowing that in some system of coordinates

$$\frac{\partial g_{\mu\nu}}{\partial t} = \frac{\partial g_{\mu\nu}}{\partial \varphi} = 0 \tag{F.2}$$

we can show that the vectors given by

$$\boldsymbol{t} = \frac{\partial}{\partial t} \ , \ \boldsymbol{\varphi} = \frac{\partial}{\partial \varphi} \tag{F.3}$$

are Killing vectors of the improved Kerr spacetime. In the B-L coordinates the components of $\boldsymbol{t} \equiv t^\mu \partial_\mu$ and $\boldsymbol{\varphi} \equiv \varphi^\mu \partial_\mu$ are

$$\begin{aligned} t^\nu &= \delta^\nu_t \ , \ \varphi^\nu = \delta^\nu_\varphi \\ t_\mu &= g_{\mu t} \ , \ \varphi_\mu = g_{\mu\varphi} \end{aligned} \tag{F.4}$$

Denoting any of the vectors in (F.4) by a generic vector $\boldsymbol{\eta}$ associated to the cyclic coordinate η we can directly check (F.1):

$$\begin{aligned} \nabla_\nu \eta_\mu &= g_{\mu\alpha} \nabla_\nu \eta^\alpha = g_{\mu\alpha} \left(\frac{\partial \eta^\alpha}{\partial x^\nu} + \Gamma^\alpha_{\nu\sigma} \eta^\sigma \right) = g_{\mu\alpha} \left[\frac{\partial \left(\delta^\alpha_\eta\right)}{\partial x^\nu} + \Gamma^\alpha_{\nu\sigma} \delta^\sigma_\eta \right] \\ &= g_{\mu\alpha} \Gamma^\alpha_{\nu\eta} = \Gamma_{\mu\nu\eta} = \frac{1}{2} \left(\frac{\partial g_{\mu\eta}}{\partial x^\nu} + \frac{\partial g_{\mu\nu}}{\partial \eta} - \frac{\partial g_{\nu\eta}}{\partial x^\mu} \right) \end{aligned}$$

Therefore we can write

$$\nabla_\nu \eta_\mu = \frac{1}{2}\left(\frac{\partial g_{\mu\eta}}{\partial x^\nu} - \frac{\partial g_{\nu\eta}}{\partial x^\mu}\right) \tag{F.5}$$

where we have used the cyclic character of the coordinates η. Then we see immediately that

$$\nabla_\nu \eta_\mu + \nabla_\mu \eta_\nu \;=\; \frac{1}{2}\left(\frac{\partial g_{\mu\eta}}{\partial x^\nu} - \frac{\partial g_{\nu\eta}}{\partial x^\mu} + \frac{\partial g_{\mu\eta}}{\partial x^\nu} - \frac{\partial g_{\nu\eta}}{\partial x^\mu}\right) = 0$$

Or in more detail,

$$\nabla_\nu t_\mu + \nabla_\mu t_\nu = 0$$
$$\nabla_\nu \varphi_\mu + \nabla_\mu \varphi_\nu = 0$$

This is the result we were looking for.

A similar result we get for any linear combination of $\boldsymbol{t}$ and $\boldsymbol{\varphi}$

$$\boldsymbol{\xi} \;=\; \alpha \boldsymbol{t} + \beta \boldsymbol{\varphi}$$

with α and β constant coeficients. By using (F.6) we have that trivially

$$\nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu = \alpha\left(\nabla_\nu t_\mu + \nabla_\mu t_\nu\right) + \beta\left(\nabla_\nu \varphi_\mu + \nabla_\mu \varphi_\nu\right) = 0 \tag{F.6}$$

so that any linear combination of $\boldsymbol{t}$ and $\boldsymbol{\varphi}$ is a Killing vector too.

Appendix G

Spherically Symmetric Space in E-F Coordinates

In this appendix we calculate the E-F representations of the static and spherically symmetric metric by performing the E-F transformations (8.11) and (8.12). The Schwarzschild representation of this metric is given in eq. (8.10) by

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{G.1}$$

We start with the ingoing coordinates by substituting the transformation (8.12) in (G.1). This transformation is given by

$$v = t + r^*\ ,\ r^* = \int \frac{dr}{f(r)}$$

The differential form of this transformation reads:

$$dv = dt + \frac{dr}{f(r)} \tag{G.2}$$

As a result, we have for the line element ds^2

$$\begin{aligned} ds^2 &= -f(r)\left(dv - \frac{dr}{f(r)}\right)^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \\ &= -f(r)\,dv^2 + 2dvdr - \frac{dr^2}{f(r)} + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \end{aligned} \tag{G.3}$$

so that finally

$$ds^2 = -f(r)\,dv^2 + 2dvdr + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{G.4}$$

This is exactly the result we wanted to derive (see eq. 8.15).

The representation with the outgoing coordinates is found after replacing in (G.4) the differential dr by $-dr$. As a result we have

$$ds^2 = -f(r)\,du^2 - 2dudr + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{G.5}$$

Eqs. (8.15) and (8.14) are the two representations discussed in subsection 8.2.1.

Appendix H

Calculation of the Surface Gravity

In this appendix we perform in detail some calculations related to the derivation of the surface gravity of the improved Kerr black hole. We find expressions for the Killing vector ξ in B-L and E-F coordinates, we evaluate this vector at the event horizon and finally we derive an expression of $\xi^\mu \xi_\mu$ in the ingoing E-F coordinates.

H.1 The Killing vector $\boldsymbol{\xi}$ in B-L and E-F Coordinates

Here we perform the calculation of the components ξ^μ in B-L and E-F Coordinates. These components are a linear combination of t^μ and φ^μ:

$$\xi^\mu = t^\mu + \varphi^\mu \Omega \tag{H.1}$$

Starting with the B-L coordinates we substitute the expressions (4.14) for t^μ and φ^μ in (H.1) and obtain

$$\xi^\mu \equiv \delta^\mu_t + \delta^\mu_\varphi \Omega \tag{H.2}$$

with $\xi^t = 1$, $\xi^r = 0$, $\xi^\theta = 0$, $\xi^\varphi = \Omega$. Lowering the index μ gives

$$\xi_\mu = g_{\mu t} + \Omega g_{\mu\varphi} \tag{H.3}$$

The transformation from B-L to E-F coordinates for the improved Kerr spacetime is given by

$$\begin{aligned} dr^*_I &\equiv \left(\frac{r^2+a^2}{\Delta_I}\right) dr \\ dr^\#_I &\equiv \left(\frac{a}{\Delta_I}\right) dr \end{aligned} \tag{H.4}$$

where $\Delta_I \equiv r^2 + a^2 - 2MG(r)\,r$. The ingoing and outgoing E-F coordinates, respectively, are defined as follows:

- Ingoing E-F Coordinates ("Ingoing Patch")

$$\begin{aligned} dv &= dt + dr_I^* && \text{(H.5)} \\ d\psi &= d\varphi + dr_I^{\#} && \text{(H.6)} \end{aligned}$$

- Outgoing E-F Coordinates ("Outgoing Patch")

$$\begin{aligned} du &= dt - dr_I^* && \text{(H.7)} \\ d\chi &= d\varphi - dr_I^{\#} && \text{(H.8)} \end{aligned}$$

We choose the *ingoing* E-F coordinates in order to find the form of (H.1). If we represent the spacetime events as $x^\mu = (v, r, \theta, \psi)$ in these coordinates [1], the Killing vectors $\boldsymbol{t}$ and $\boldsymbol{\varphi}$ have the components

$$t^\mu \equiv \frac{\partial x^\mu}{\partial v} = \left(\frac{\partial v}{\partial v}, \frac{\partial r}{\partial v}, \frac{\partial \theta}{\partial v}, \frac{\partial \psi}{\partial v}\right) = (1,\ 0,\ 0,\ 0) \tag{H.9}$$

$$\varphi^\mu \equiv \frac{\partial x^\mu}{\partial \psi} = \left(\frac{\partial v}{\partial \psi}, \frac{\partial r}{\partial \psi}, \frac{\partial \theta}{\partial \psi}, \frac{\partial \psi}{\partial \psi}\right) = (0,\ 0,\ 0,\ 1) \tag{H.10}$$

Exploiting the Kronecker deltas we have finally for t^μ and φ^μ the following result:

$$t^\mu = \delta^\mu_v\ ,\ \mu = v, r, \theta, \psi \tag{H.11}$$

$$\varphi^\mu = \delta^\mu_\psi\ ,\ \mu = v, r, \theta, \psi \tag{H.12}$$

The frequency Ω has completely equivalent definitions in both coordinate systems. Since $dt = dv$ and $d\psi = d\varphi$ we have

$$\Omega \equiv \frac{d\varphi}{dt} = \frac{d\psi}{dv} \tag{H.13}$$

Substituting (H.11), (H.12) and (H.13) in (H.1) leads to

$$\xi^\mu = \frac{\partial x^\mu}{\partial v} + \Omega \frac{\partial x^\mu}{\partial \psi} = \delta^\mu_v + \Omega \delta^\mu_\psi \tag{H.14}$$

With a lower index we have instead

$$\xi_\mu = g_{\mu v} + \Omega g_{\mu\psi}\ ,\ \mu = v, r, \theta, \psi \tag{H.15}$$

Expressions (H.2), (H.3) and (H.14), (H.15) are the respective expressions presented in subsection 8.2.2 for $\boldsymbol{\xi}$ in the B-L and E-F coordinates.

[1] Remember that we have $x^\mu = (t, r, \theta, \varphi)$ for the B-L coordinates.

H.2 The Killing vector ξ at the Kerr Event Horizon

In this subsection we evaluate at the event horizon, the components (H.15) of $\boldsymbol{\xi}$ in the ingoing E-F coordinates. We obtain, as a result, the expression (8.47) of chapter 8.

The component $\xi_v|_{r_+^{\mathrm{I}}}$ is given by

$$\xi_v = g_{vv}|_{r_+} + \Omega_{\mathrm{H}}\, g_{v\psi}|_{r_+} = g_{vv}|_{r_+} - \left.\frac{g_{v\psi}^2}{g_{\psi\psi}}\right|_{r_+} \tag{H.16}$$

Since at the event horizon all the rotation frequencies coalesce, in particular we have

$$\Omega_{\pm}|_{r_+} = \omega|_{r_+} \tag{H.17}$$

Substituting eq. (4.36) for $\Omega_{\pm}$ in (H.17) gives

$$\left.\left(\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}\right)\right|_{r_+} = 0$$

Substituting $\omega = -g_{\varphi t}/g_{\varphi\varphi}$ gives the following:

$$\left.\left(\frac{g_{t\varphi}^2}{g_{\varphi\varphi}} - g_{tt}\right)\right|_{r_+} = 0 \tag{H.18}$$

We have the following equivalent metric components in the B-L and the E-F representations:

$$g_{tt} = g_{vv}\ ,\ g_{v\psi} = g_{t\varphi}\ ,\ g_{\varphi\varphi} = g_{\psi\psi} \tag{H.19}$$

As a result eq. (H.18) is written in E-F coordinates as follows:

$$\left.\left(\frac{g_{t\varphi}^2}{g_{\varphi\varphi}} - g_{tt}\right)\right|_{r_+} = \left.\left(\frac{g_{v\psi}^2}{g_{\psi\psi}} - g_{vv}\right)\right|_{r_+} = 0$$

but from (H.16) we see that this is precisely ξ_v , namely:

$$\xi_v = g_{vv}|_{r_+} - \left.\frac{g_{v\psi}^2}{g_{\psi\psi}}\right|_{r_+} = 0$$

For the r and θ components we substitute $g_{vr} = 1$ and $g_{r\psi} = -a\sin^2\theta$

$$\begin{aligned} \xi_r &= g_{vr}|_{r_+} + \Omega_{\mathrm{H}}\, g_{r\psi}|_{r_+} = 1 - a\Omega_{\mathrm{H}}\sin^2\theta \\ \xi_\theta &= g_{v\theta}|_{r_+} + \Omega_{\mathrm{H}}\, g_{\theta\psi}|_{r_+} = 0 + \Omega_{\mathrm{H}} * 0 = 0 \end{aligned}$$

ξ_ψ is trivially equal to zero.

H.3 Simplification of ξ^2

As discussed in chapter 8 the scalar $\xi_\mu \xi^\mu$ can be read off directly from the metric components in (8.9). In the E-F representation we have:

$$\xi_\mu \xi^\mu = \left(\frac{ds}{dv}\right)^2 \Bigg|_{(r,\theta \text{ fixed})} \tag{H.20}$$

The metric components in the ingoing E-F representation read

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2G(r)Mr}{\rho^2}\right) dv^2 + 2drdv - 2a\sin^2\theta d\psi dr + \\ & -\frac{4G(r)Mar\sin^2\theta}{\rho^2} d\psi dv + \frac{\Sigma_I \sin^2\theta}{\rho^2} d\psi^2 + \rho^2 d\theta^2 \end{aligned} \tag{H.21}$$

Parametrizing the line element (H.21) in v at r and θ fixed, gives

$$\begin{aligned} \xi_\mu \xi^\mu &= -\left(1 - \frac{2G(r)Mr}{\rho^2}\right) - \frac{4G(r)Mar\sin^2\theta}{\rho^2}\Omega + \frac{\Sigma_I \sin^2\theta}{\rho^2}\Omega^2 \\ &= -\frac{1}{\rho^2}\left[\rho^2 - 2G(r)Mr\right] + \frac{\Sigma_I \sin^2\theta}{\rho^2}\left[\Omega^2 - \frac{4G(r)Mar}{\Sigma_I}\Omega\right] \\ &= -\frac{1}{\rho^2}\left[\rho^2 - 2G(r)Mr\right] + \frac{\Sigma_I \sin^2\theta}{\rho^2}\left[\Omega^2 - 2\omega\Omega + \omega^2\right] - \frac{\Sigma_I \sin^2\theta}{\rho^2}\omega^2 \\ &= -\frac{1}{\Sigma_I \rho^2}\left[\Sigma_I\left(\rho^2 - 2G(r)Mr\right) + \Sigma_I^2 \omega^2 \sin^2\theta\right] + \frac{\Sigma_I \sin^2\theta}{\rho^2}(\Omega - \omega)^2 \\ &= -\frac{1}{\Sigma_I \rho^2}\left[\Sigma_I\left(\rho^2 - 2G(r)Mr\right) + (2G(r)Mar)^2 \sin^2\theta\right] \\ & \quad + \frac{\Sigma_I \sin^2\theta}{\rho^2}(\Omega - \omega)^2 \end{aligned}$$

We define the function $F(r,\theta)$ to be

$$F(r,\theta) \equiv \left[\Sigma_I\left(\rho^2 - 2G(r)Mr\right) + (2G(r)Mar)^2 \sin^2\theta\right] \tag{H.22}$$

As a result $\xi^\mu \xi_\mu$ takes the form

$$\xi_\mu \xi^\mu = -\frac{F(r,\theta)}{\Sigma_I \rho^2} + \frac{\Sigma_I \sin^2\theta}{\rho^2}(\Omega - \omega)^2 \tag{H.23}$$

The function $F(r,\theta)$ can be simplified employing the identities (C.5) to (C.7)

$$\begin{aligned}
F(r,\theta) &= \left\{\Sigma_I\left[\rho^2-2G(r)Mr\right]+\left[2G(r)Mar\right]^2\sin^2\theta\right\} \\
&= \left\{\Sigma_I\left[\rho^2-2G(r)Mr\right]+\left[2G(r)Mr\right]^2 f^2\right\}\ ,\ (f^2=a^2\sin^2\theta) \\
&= \left[\left(\rho^2+f^2\right)^2-f^2\Delta_I\right]\left(\Delta_I-f^2\right)+\left(\rho^2+f^2-\Delta_I\right)^2 f^2 \\
&= \left(\rho^2+f^2\right)^2\left(\Delta_I-f^2\right)-f^2\Delta_I\left(\Delta_I-f^2\right)+\left(\rho^2+f^2\right)^2 f^2-2\left(\rho^2+f^2\right)\Delta_I f^2+\Delta_I^2 f^2 \\
&= \rho^4\Delta_I+2f^2\rho^2\Delta_I+f^4\Delta_I-2\rho^2\Delta_I f^2-2f^4\Delta_I-f^2\Delta_I^2+f^4\Delta_I+\Delta_I^2 f^2 \\
&= \rho^4\Delta_I
\end{aligned}$$

As a result the scalar $\xi^\mu\xi_\mu$ in eq. (H.23) is simplified to the final expression:

$$\xi_\mu\xi^\mu=\frac{\Sigma_I\sin^2\theta}{\rho^2}\left(\Omega-\omega\right)^2-\frac{\rho^2\Delta_I}{\Sigma_I} \qquad \text{(H.24)}$$

This is precisely eq. (8.37).

Appendix I

Calculation of M_H and J_H

I.1 Area of the Event Horizon

In order to calculate the area of a three dimensional hypersurface Σ in spacetime we need to define the metric intrinsic to Σ, namely the metric obtained by restricting the line element to displacements confined to the hypersurface. The parametric equations that define Σ are given by

$$x^\alpha = x^\alpha \left(y^a\right) \ , \ a = 1, 2, 3 \tag{I.1}$$

where y^a are coordinates intrinsic to Σ. The vectors $\boldsymbol{e}_a$ defined as

$$e^\alpha_a = \frac{dx^\alpha}{dy^a} \tag{I.2}$$

are tangent to curves contained in Σ. Now, for displacements within Σ we have

$$ds^2_\Sigma = g_{\alpha\beta} dx^\alpha dx^\beta = g_{\alpha\beta} \left(\frac{dx^\alpha}{dy^a} dy^a\right) \left(\frac{dx^\beta}{dy^b} dy^b\right) = h_{ab} dy^a dy^b \tag{I.3}$$

where the induced metric $h_{\alpha\beta}$ is defined as follows

$$h_{ab} \equiv g_{\alpha\beta} \left(\frac{dx^\alpha}{dy^a}\right) \left(\frac{dx^\beta}{dy^b}\right) = g_{\alpha\beta} e^\alpha_a e^\beta_b \tag{I.4}$$

By definition $h_{\alpha\beta}$ is the metric that determines distances inside Σ. Therefore, it relates the intrinsic line element with intrinsic displacements

$$ds^2_\Sigma = h_{ab} dy^a dy^b \tag{I.5}$$

as one can see from (I.3). When Σ is null, the following choice of coordinates takes advantage of the light-like character of the tangent vectors to Σ:

$$y^a \equiv \left(\lambda, \theta^A\right) \ , \ A = 1, 2 \tag{I.6}$$

where λ parametrizes the null geodesics in Σ, in such a way that a displacement from the point p along the geodesic γ is given by

$$dx^\beta = k^\beta d\lambda \tag{I.7}$$

and k^β is the tangent vector to γ at p. The coordinates θ^A change transversely to γ. In that sense we say that θ^A label the geodesics. The line element (I.5) takes the following form in terms of the coordinates (I.6)

$$ds^2_\Sigma = \sigma_{AB} d\theta^A d\theta^B \tag{I.8}$$

where we have used the null character of k^α, and σ_{AB} is given by

$$\begin{aligned} \sigma_{AB} &\equiv g_{\alpha\beta} e^\alpha_A e^\beta_B \\ e^\alpha_A &= \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_{\lambda \text{ fixed}} \end{aligned} \tag{I.9}$$

Here the induced metric is a two-tensor.

The area of the event horizon is defined by [60]

$$\mathcal{A} \equiv \oint_{\mathrm{H}} \sqrt{\sigma} d^2\theta \tag{I.10}$$

where H is a two-dimensional cross section of the event horizon, described by v =constant, $r = r^{\mathrm{I}}_+$, $0 \leq \theta \leq \pi$, $0 \leq \psi \leq 2\pi$[1]. The metric induced in H represented in the ingoing E-F coordinates results from fixing r and v in the line element given by:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2G(r)Mr}{\rho^2}\right) dv^2 + 2drdv - 2a\sin^2\theta d\psi dr + \\ &\quad - \frac{4G(r)Mar\sin^2\theta}{\rho^2} d\psi dv + \frac{\Sigma_I \sin^2\theta}{\rho^2} d\psi^2 + \rho^2 d\theta^2 \end{aligned} \tag{I.11}$$

As a result we have

$$\begin{aligned} ds^2_{\mathrm{H}} &= \sigma_{ab} d\theta^a d\theta^b \\ &= g_{\theta\theta} d\theta^2 + g_{\psi\psi} d\psi^2 \\ &= \rho^2 d\theta^2 + \frac{\Sigma_I}{\rho^2} \sin^2\theta d\psi^2 \end{aligned}$$

The determinant σ is given by $\sigma = \Sigma_I \sin^2\theta$ and its root is $\sqrt{\sigma} = \sqrt{\Sigma_I}\sin\theta$. As a result we have for (I.10)

$$\mathcal{A} = \int_0^{2\pi} \int_0^{\pi} \sqrt{\Sigma_I}\Big|_{r^{\mathrm{I}}_+} \sin\theta d\theta d\psi \tag{I.12}$$

[1]For a definition of event horizon in topological terms, see [68]

Since $\sqrt{\Sigma_I}\big|_{r_+^{\mathrm{I}}} = \left(r_+^{\mathrm{I}}\right)^2 + a^2$ is angular independent we can integrate (I.12) and find

$$\mathcal{A} = 4\pi\left[\left(r_+^{\mathrm{I}}\right)^2 + a^2\right] \tag{I.13}$$

This expression preserves the original form of the event horizon's area of the classical Kerr black hole. Nevertheless, the radius $r_+^{\mathrm{I}} \equiv r_+^{\mathrm{I}}(a, M)$ depends on the parameters a and M, and this relationship is modified by the renormalization effects.

I.2 Metric Components in the E-F Representation

In order to calculate M_H and J_H from (8.84) and (8.116) we need to know, besides the components $g_{\alpha\beta}$ of the improved Kerr metric in the E-F representation, also the components $g^{\alpha\beta}$ with upper indices, evaluated at the event horizon. In the ingoing E-F representation we have the following for $g_{\alpha\beta}$:

$$\begin{aligned}
g_{11} &= g_{vv} = -\left(1 - \frac{2G(r)\,Mr}{\rho^2}\right)\ ,\ g_{12} = g_{rv} = 1\ ,\ g_{24} = g_{r\psi} = -a\sin^2\theta \\
g_{41} &= g_{\psi v} = -\frac{2G(r)\,Mar\sin^2\theta}{\rho^2}\ ,\ g_{44} = g_{\psi\psi} = \frac{\Sigma\sin^2\theta}{\rho^2}\ ,\ g_{33} = g_{\theta\theta} = \rho^2 \\
g_{22} &= g_{rr} = 0
\end{aligned} \tag{I.14}$$

The rest are zero. Here we have chosen the ordering of coordinates as $(x^1, x^2, x^3, x^4) = (t, r, \theta, \varphi)$. The components $g^{\alpha\beta}$ are given by the inverse matrix of $g_{\alpha\beta}$, namely:

$$g^{\alpha\beta} = \frac{1}{\det g}\begin{bmatrix} g_{22}g_{44} - g_{24}^2 & -(g_{12}g_{44} - g_{24}g_{14}) & 0 & g_{12}g_{24} - g_{22}g_{14} \\ -(g_{12}g_{44} - g_{24}g_{14}) & g_{11}g_{44} - g_{14}^2 & 0 & -(g_{11}g_{24} - g_{12}g_{14}) \\ 0 & 0 & \frac{1}{g_{33}} & 0 \\ g_{12}g_{24} - g_{22}g_{14} & -(g_{11}g_{24} - g_{12}g_{14}) & 0 & g_{11}g_{22} - g_{12}^2 \end{bmatrix} \tag{I.15}$$

with $\det g$ given by

$$\begin{aligned}
\det g &= g_{11}g_{22}g_{44} - g_{11}g_{24}^2 - g_{12}^2g_{44} + 2g_{12}g_{24}g_{14} - g_{14}^2g_{22} \\
&= g_{vv}g_{rr}g_{\psi\psi} - g_{vv}g_{r\psi}^2 - g_{vr}^2g_{\psi\psi} + 2g_{vr}g_{r\psi}g_{v\psi} - g_{v\psi}^2g_{rr}
\end{aligned} \tag{I.16}$$

Inserting $g_{rr} = 0$ in the E-F representation, we find

$$\det g = -g_{vv}g_{r\psi}^2 - g_{vr}^2g_{\psi\psi} + 2g_{vr}g_{r\psi}g_{v\psi}$$

From (I.15) we see that the relevant expressions to be calculated are:

$$g^{rr} = \frac{g_{vv}g_{\psi\psi} - g_{v\psi}^2}{\det g}\ ,\ g^{rv} = \frac{g_{r\psi}g_{v\psi} - g_{vr}g_{\psi\psi}}{\det g}\ ,\ \ g^{r\psi} = \frac{g_{vr}g_{v\psi} - g_{vv}g_{r\psi}}{\det g} \tag{I.17}$$

They require the following components evaluated at the event horizon

$$\begin{aligned} g_{vv}|_{r_+} &= -\left(1 - \frac{2G\,(r)\,Mr}{\rho^2}\right)\Big|_{r_+} = -\left(\frac{\Delta_I - a^2\sin^2\theta}{\rho^2}\right)\Big|_{r_+} \\ &= \left(\frac{b^2}{\rho^2}\right)\Big|_{r_+} \end{aligned} \tag{I.18}$$

where

$$b^2 = a^2\sin^2\theta$$

and also

$$g_{vr}|_{r_+} = 1\ ,\ g_{v\psi}|_{r_+} = -\frac{2G\,(r_+)\,Mar_+\sin^2\theta}{\rho^2|_{r_+}}\ ,\ g_{r\psi}|_{r_+} = -a\sin^2\theta\ ,\ g_{rr}|_{r_+} = 0 \tag{I.19}$$

$$g_{\psi\psi}|_{r_+} = \frac{\left(r_+^2 + a^2\right)^2\sin^2\theta}{\rho^2|_{r_+}} = \frac{\left(\rho^2|_{r_+} + b^2\right)^2\sin^2\theta}{\rho^2|_{r_+}} \tag{I.20}$$

Here we have used the identities (C.5) to (C.7).

We need also to find $\det g|_{r_+}$. Inserting (I.18) to (I.20) in (I.16) we have

$$\det g|_{r_+} = 2g_{vr}g_{r\psi}g_{v\psi} - g_{vv}g_{r\psi}^2 - g_{vr}^2g_{\psi\psi} \tag{I.21}$$

$$\begin{aligned} &= \left(2a\sin^2\theta\right)\left(\frac{2G\,(r_+)\,Mar_+\sin^2\theta}{\rho^2|_{r_+}}\right) - \left(\frac{b^2a^2\sin^4\theta}{\rho^2}\right)\Big|_{r_+} \\ &\quad -\frac{\left(\rho^2|_{r_+} + b^2\right)^2\sin^2\theta}{\rho^2|_{r_+}} \end{aligned} \tag{I.22}$$

Simplifying (I.22) leads to

$$\det g|_{r_+} = \frac{\sin^2\theta}{\rho^2|_{r_+}}\left[4b^2G\,(r_+)\,Mr_+ - b^4 - \left(\rho^2\big|_{r_+} + b^2\right)^2\right] \tag{I.23}$$

Taking into account $r_+^2 + a^2 - 2G\,(r_+)\,Mr_+ = 0$ and $\rho^2|_{r_+} + b^2 = r_+^2 + a^2$ implies

$$\begin{aligned} \det g|_{r_+} &= \frac{\sin^2\theta}{\rho^2|_{r_+}}\left[2b^2\left(\rho^2\big|_{r_+} + b^2\right) - b^4 - \left(\rho^2\big|_{r_+} + b^2\right)^2\right] \\ &= -\sin^2\theta\,\rho^2\big|_{r_+} \end{aligned} \tag{I.24}$$

Now we are ready to find g^{rr} , g^{rv} and $g^{r\psi}$. Substituting in (I.17), expressions (I.18) to (I.20) and (I.24), leads to

$$\begin{aligned}
g^{rr}\big|_{r_+} &= \left.\frac{g_{vv}g_{\psi\psi}-g_{v\psi}^2}{\det g}\right|_{r_+} = \left(\frac{1}{-\rho^2\sin^2\theta\big|_{r_+}}\right)\times \\
&\quad \times\left[\left.\left(\frac{b^2}{\rho^2}\right)\right|_{r_+}\frac{\left(r_+^2+a^2\right)^2\sin^2\theta}{\rho^2\big|_{r_+}}-\left(\frac{2G\left(r_+\right)Mar_+\sin^2\theta}{\rho^2\big|_{r_+}}\right)^2\right] \\
&= \left(\frac{\sin^2\theta}{-\rho^2\sin^2\theta\big|_{r_+}}\right)\left.\left(\frac{b^2}{\rho^4}\right)\right|_{r_+}\times\left[\left(r_+^2+a^2\right)^2-\left(2G\left(r_+\right)Mr_+\right)^2\right] \\
&= 0
\end{aligned}$$

where we have used again $r_+^2+a^2-2G\left(r_+\right)Mr_+=0$. Then $g^{rr}\big|_{r_+}$ is identically zero. For $g^{rv}\big|_{r+}$ we proceed as follows: First substituting (I.24) and the components (I.18) to (I.20) into the expression for $g^{rv}\big|_{r+}$ in (I.17) gives rise to

$$\begin{aligned}
g^{rv}\big|_{r+} &= \left.\frac{g_{r\psi}g_{v\psi}-g_{vr}g_{\psi\psi}}{\det g}\right|_{r_+} = \\
&= \left.\frac{\left(-a\sin^2\theta\right)\left(-\frac{2G(r_+)Mar_+\sin^2\theta}{\rho^2|_{r_+}}\right)-\left[\frac{\left(r_+^2+a^2\right)^2\sin^2\theta}{\rho^2|_{r_+}}\right]}{-\sin^2\theta\,\rho^2\big|_{r_+}}\right|_{r_+}
\end{aligned}$$

and simplifying yields

$$g^{rv}\big|_{r+} = \left.\frac{2b^2G\left(r_+\right)Mr_+-\left(r_+^2+a^2\right)^2}{-\rho^4\big|_{r_+}}\right|_{r_+}$$

Now we apply $\rho^2+b^2=r^2+a^2$ to obtain

$$g^{rv}\big|_{r+} = \left.\frac{2b^2G\left(r_+\right)Mr_+-\left(\rho^2\big|_{+}+b^2\right)^2}{-\rho^4\big|_{r_+}}\right|_{r_+}$$

At this point we use $2MG\left(r_+\right)r_+=\rho^2\big|_{r_+}+b^2$, with the result

$$g^{rv}\big|_{r+} = \left.\frac{b^2\left(\rho^2\big|_{+}+b^2\right)-\left(\rho^2\big|_{+}+b^2\right)^2}{-\rho^4\big|_{r_+}}\right|_{r_+}$$

Finally simplifying we get to:

$$g^{rv}\big|_{r+} = \left.\frac{\left(\rho^2\big|_{+}+b^2\right)}{\rho^2\big|_{r_+}}\right|_{r_+}$$

The last component is $g^{r\psi}\big|_{r_+}$. First substituting components and simplifying yields

$$\begin{aligned} g^{r\psi}\big|_{r_+} &= \left.\frac{g_{vr}g_{v\psi}-g_{vv}g_{r\psi}}{\det g}\right|_{r_+} = \left.\frac{\left(-\frac{2G(r_+)Mar_+\sin^2\theta}{\rho^2|_{r_+}}\right)-\left.\left(\frac{b^2}{\rho^2}\right)\right|_{r_+}\left(-a\sin^2\theta\right)}{-\sin^2\theta\,\rho^2|_{r_+}}\right|_{r_+} \\ &= \left.\frac{a\,b^2|_{r_+}-2G\left(r_+\right)Mar_+}{-\,\rho^4|_{r_+}}\right|_{r_+} \end{aligned}$$

Applying $2G\left(r_+\right)Mar_+=\rho^2+b^2|_{r_+}$ leads to the final result

$$g^{r\psi}\big|_{r_+} = \left.\frac{a\,b^2|_{r_+}-\left(\rho^2+b^2|_{r_+}\right)a}{-\,\rho^4|_{r_+}}\right|_{r_+} = \frac{a}{\rho^2|_{r_+}} \tag{I.25}$$

I.3 Finding the Normal Vector N_α

N_α is a light like four-vector, orthogonal to the event horizon. It is an auxiliary vector, necessary to be included in order to isolate the part of the metric $g_{\mu\nu}$ that is transverse to ξ^μ, the generator of H [2]. It fulfills the following two conditions:

$$N_\mu\xi^\mu=-1\ ,\ N_\mu N^\mu=0$$

Exploiting equation (8.47) $\xi_\alpha|_{r_+}=\left(1-a\Omega_{\mathrm{H}}\sin^2\theta\right)\delta_{r\alpha}$ in E-F coordinates, we get the following for $N_\mu\xi^\mu=-1$

$$N^\mu\xi_\mu=N^r\left(1-a\Omega_{\mathrm{H}}\sin^2\theta\right)=-1$$

This implies

$$N^r=\frac{-1}{\left(1-a\Omega_{\mathrm{H}}\sin^2\theta\right)}$$

and by construction we have

$$N^r\xi_r|_{r_+}=-1 \tag{I.26}$$

From equations (8.72) and (8.93), where we have already applied (I.26), we see that this expression contains all the information about N^μ necessary in order to evaluate M_H and J_H from the Komar integrals (8.59) and (8.60). Therefore we do not need to go further into finding the rest of the components of N^μ.

[2]For more details about N^μ see Ref. [60]

I.4 Derivatives of Metric Components at r_+

Now we calculate the derivatives in (8.73) and (8.74). For $\left(\frac{\partial g_{vv}}{\partial r}\right)\Big|_{r_+}$ we can write

$$\begin{aligned}\left(\frac{\partial g_{vv}}{\partial r}\right)\Bigg|_{r_+} &= -\frac{\partial}{\partial r}\left(\frac{\rho^2 - 2G(r)\,Mr}{\rho^2}\right)\Bigg|_{r_+} \\ &= \left(\frac{2M}{\rho^4}\right)\left\{\rho^2\left[G'(r)\,r + G(r)\right] - 2r^2 G(r)\right\}\Bigg|_{r_+} \end{aligned}\tag{I.27}$$

Here we have used $\frac{\partial \rho^2}{\partial r} = 2r$. Now inserting $\rho^2 = r^2 + a^2\cos^2\theta$ leads to

$$\left(\frac{\partial g_{vv}}{\partial r}\right)\Bigg|_{r_+} = \left(\frac{2Mr^2}{\rho^4}\right)\left[rG'(r) - G(r)\right]\Bigg|_{r_+} + \left(\frac{2Ma^2\cos^2\theta}{\rho^4}\right)\left[rG'(r) + G(r)\right]\Bigg|_{r_+} \tag{I.28}$$

For $\left(\frac{\partial g_{\psi v}}{\partial r}\right)\Big|_{r_+}$ we have

$$\begin{aligned}\left(\frac{\partial g_{\psi v}}{\partial r}\right)\Bigg|_{r_+} &= -2Ma\,\frac{\partial}{\partial r}\left(\frac{G(r)\,r\sin^2\theta}{\rho^2}\right)\Bigg|_{r_+} \\ &= -\left(\frac{2Ma\sin^2\theta}{\rho^4}\right)\left[\left(rG'(r) + G(r)\right)\rho^2 - 2G(r)\,r^2\right]\Bigg|_{r_+}\end{aligned}$$

Now substituting $\rho^2 = r^2 + a^2\cos^2\theta$ leads to

$$\begin{aligned}\left(\frac{\partial g_{\psi v}}{\partial r}\right)\Bigg|_{r_+} = &-\left(\frac{2Mar^2\sin^2\theta}{\rho^4}\right)\left[rG'(r) - G(r)\right]\Bigg|_{r_+} \\ &-\left(\frac{2Ma^3\sin^2\theta\cos^2\theta}{\rho^4}\right)\left[rG'(r) + G(r)\right]\Bigg|_{r_+}\end{aligned}\tag{I.29}$$

I.5 Definite Integrals for M_H

Here we calculate the definite integrals necessary in order to get analytical expressions for M_H and J_H. All of them have a similar structure. They are carried out by applying the same kind of substitution. We start with:

$$I_1 = \int_0^\pi \frac{\sin\theta d\theta}{\left(r_+^2 + a^2\cos^2\theta\right)^2} \tag{I.30}$$

Factorizing $1/r_+^4$ leads to

$$I_1 = \left(\frac{1}{r_+^4}\right)\int_0^\pi \frac{\sin\theta d\theta}{\left(1 + \frac{a^2\cos^2\theta}{r_+^2}\right)^2} \tag{I.31}$$

Substituting $u = \left(\frac{a}{r_+}\right)\cos\theta$, $du = -\left(\frac{a}{r_+}\right)\sin\theta d\theta$ in (I.31) leads to

$$I_1 = -\left(\frac{1}{ar_+^3}\right)\int_{\frac{a}{r_+}}^{-\frac{a}{r_+}} \frac{du}{\left(1+u^2\right)^2}$$

The primitive for this integral is given by

$$\int \frac{du}{\left(1+u^2\right)^2} = \frac{1}{2}\left(\frac{u}{1+u^2} + \arctan u\right) \tag{I.32}$$

Substituting back $u = \left(\frac{a}{r_+}\right)\cos\theta$ gives

$$I_1 = -\left(\frac{1}{2ar_+^3}\right)\left(\frac{r_+ a\cos\theta}{r_+^2 + a^2\cos^2\theta} + \arctan\left[\left(\frac{a}{r_+}\right)\cos\theta\right]\right)\Bigg|_0^\pi \tag{I.33}$$

Now evaluating at the limits we come to the final result:

$$I_1 = \left(\frac{1}{ar_+^3}\right)\left\{\frac{r_+ a}{r_+^2 + a^2} + \arctan\left[\left(\frac{a}{r_+}\right)\right]\right\} \tag{I.34}$$

I_2 is defined as follows

$$I_2 = \int_0^\pi \frac{\cos^2\theta\sin\theta d\theta}{\left(r_+^2 + a^2\cos^2\theta\right)^2} \tag{I.35}$$

Again factorizing $1/r_+^4$ and substituting $u = \left(\frac{a}{r_+}\right)\cos\theta$, $du = -\left(\frac{a}{r_+}\right)\sin\theta d\theta$ gives

$$I_2 = \frac{1}{r_+^4}\int_0^\pi \frac{\cos^2\theta\sin\theta d\theta}{\left(1+\frac{a^2\cos^2\theta}{r_+^2}\right)^2} = -\frac{1}{a^3 r_+}\int_{\left(\frac{a}{r_+}\right)}^{-\left(\frac{a}{r_+}\right)} \frac{u^2 du}{\left(1+u^2\right)^2}$$

The indefinite integral in the new variable is given by

$$\int \frac{u^2 du}{\left(1+u^2\right)^2} = -\frac{u}{2\left(1+u^2\right)} + \left(\frac{1}{2}\right)\arctan u \tag{I.36}$$

Now substituting back $u = \left(\frac{a}{r_+}\right)\cos\theta$ leads to

$$I_2 = -\frac{1}{a^3 r_+}\left\{-\frac{r_+ a\cos\theta}{2\left(r_+^2 + a^2\cos^2\theta\right)} + \left(\frac{1}{2}\right)\arctan\left[\left(\frac{a}{r_+}\right)\cos\theta\right]\right\}\Bigg|_0^\pi$$

and finally, by applying the limits we find as a final result for I_2:

$$I_2 = \frac{1}{a^3 r_+}\left\{\arctan\left[\left(\frac{a}{r_+}\right)\right] - \frac{r_+ a}{\left(r_+^2 + a^2\right)}\right\} \tag{I.37}$$

I.6 Definite Integrals for J_H

The substitution we have used so far for I_1 and I_2 works also for I_3 to I_5. Since the procedure is almost identical for each integration, we perform every step simultaneously for the three of them. First we factorize $1/r_+^2$ or $1/r_+^4$ respectively

$$\begin{aligned} I_3 &= \frac{1}{r_+^2}\int_0^\pi \frac{\sin^3\theta d\theta}{1+\frac{a^2\cos^2\theta}{r_+^2}}\ ,\ I_4 = \frac{1}{r_+^4}\int_0^\pi \frac{\sin^3\theta d\theta}{\left(1+\frac{a^2\cos^2\theta}{r_+^2}\right)^2} \\ I_5 &= \frac{1}{r_+^4}\int_0^\pi \frac{\sin^3\theta\cos^2\theta d\theta}{\left(1+\frac{a^2\cos^2\theta}{r_+^2}\right)^2} \end{aligned} \tag{I.38}$$

Then substituting $u = \left(\frac{a}{r_+}\right)\cos\theta$, $du = -\left(\frac{a}{r_+}\right)\sin\theta d\theta$ leads to

$$I_3 = \left(\frac{r_+}{a^3}\right)\int_{\frac{a}{r_+}}^{-\frac{a}{r_+}} \frac{u^2}{1+u^2}du - \frac{1}{ar_+}\int_{\frac{a}{r_+}}^{-\frac{a}{r_+}} \frac{du}{1+u^2} \tag{I.39}$$

$$\begin{aligned} I_4 &= \left(\frac{1}{r_+a^3}\right)\int_{\frac{a}{r_+}}^{-\frac{a}{r_+}} \frac{u^2}{(1+u^2)^2}du - \left(\frac{1}{ar_+^3}\right)\int_{\frac{a}{r_+}}^{-\frac{a}{r_+}} \frac{du}{(1+u^2)^2} \\ I_5 &= -\frac{1}{a^3r_+}\int_{\frac{a}{r_+}}^{-\frac{a}{r_+}} \frac{u^2du}{(1+u^2)^2} + \frac{r_+}{a^5}\int_{\frac{a}{r_+}}^{-\frac{a}{r_+}} \frac{u^4du}{(1+u^2)^2} \end{aligned}$$

The set of primitives required by (I.39) includes (I.32), (I.36) and also

$$\int \frac{u^2du}{1+u^2} = u - \arctan u \tag{I.40}$$

$$\int \frac{du}{1+u^2} = \arctan u \tag{I.41}$$

$$\int \frac{u^4du}{(1+u^2)^2} = u + \frac{u}{2(1+u^2)} - \frac{3\arctan u}{2} \tag{I.42}$$

Then substituting (I.32), (I.36) and (I.40) to (I.42) in (I.39) gives

$$I_3 = \left(\frac{r_+}{a^3}\right)[u - \arctan u]\Big|_{\frac{a}{r_+}}^{-\frac{a}{r_+}} - \frac{1}{ar_+}\arctan u\Big|_{\frac{a}{r_+}}^{-\frac{a}{r_+}} \tag{I.43}$$

$$\begin{aligned} I_4 &= \left(\frac{1}{r_+a^3}\right)\left[-\frac{u}{2(1+u^2)} + \left(\frac{1}{2}\right)\arctan u\right]\Bigg|_{\frac{a}{r_+}}^{-\frac{a}{r_+}} \\ &\quad - \left(\frac{1}{ar_+^3}\right)\left[\frac{1}{2}\left(\frac{u}{1+u^2} + \arctan u\right)\right]\Bigg|_{\frac{a}{r_+}}^{-\frac{a}{r_+}} \end{aligned} \tag{I.44}$$

$$
\begin{aligned}
I_5 \quad = \quad & -\frac{1}{a^3 r_+}\left[-\frac{u}{2\left(1+u^2\right)}+\left(\frac{1}{2}\right)\arctan u\right]\Bigg|_{\frac{a}{r_+}}^{-\frac{a}{r_+}} \\
& +\frac{r_+}{a^5}\left[u+\frac{u}{2\left(1+u^2\right)}-\left(\frac{3}{2}\right)\arctan u\right]\Bigg|_{\frac{a}{r_+}}^{-\frac{a}{r_+}}
\end{aligned} \tag{I.45}
$$

Evaluating at the limits gives for I_3

$$
I_3 = -\frac{2}{a^2} + 2\left[\frac{\left(r_+^2+a^2\right)}{a^3 r_+}\right]\arctan\left(\frac{a}{r_+}\right) \tag{I.46}
$$

For I_4 we have

$$
I_4 = \left(\frac{a^2-r_+^2}{a^3 r_+^3}\right)\arctan\left(\frac{a}{r_+}\right) + \left[\frac{1}{a^2 r_+^2}\right] \tag{I.47}
$$

And finally for I_5 we find

$$
I_5 = -\frac{3}{a^4} + \frac{\left(3r_+^2+a^2\right)}{r_+ a^5}\arctan\left(\frac{a}{r_+}\right) \tag{I.48}
$$

Appendix J

Auxiliary Calculations related to the Modified First Law

For the sake of completeness of section 8.7 on the modified first law of the black hole thermodynamics, we present in this appendix all the intermediate steps for the various derivations sketched in that section. The appendix has three main goals. First, the ordinary differential equation governing the running Newton constant $G(r)$ in the vicinity of r_+^{I} , used in subsection 8.7.4, is discussed. Second, the $O(J^2)$ approximation to the integrating factor μ for the modified first law in subsection 8.7.5 is derived. And third, the degree of exactness of the mentioned approximation is discussed.

J.1 Differential Equation for $G(r)$

In this section we present the derivation of the condition (8.176) for $G(r)$ that comes from asuming that $T = \kappa/(2\pi)$ is valid in the improved case, i.e. that $\mu_\alpha = 1$, meaning that $\mu_\gamma = 1/\left[r_+^{\mathrm{I}} - M\left(G + r_+^{\mathrm{I}} G'\right)\right]$ is the correct integrating factor of $\boldsymbol{\gamma}$ in (8.152). We also find the solutions of the corresponding differential equation (8.177) by substituting an ansatz in form of a power series of $1/r$. As already mentioned in subsection 8.7.4, the solutions we find are not satisfactory in the sense that they don't fit with the expected behavior of a running Newton constant, at least in the vicinity of r_+^{I}, the region of validity of that equation.

We start by finding several identities that are basic for the calculations presented in that section.

J.1.1 Basic Identities

In this subsection we calculate the partial derivatives of $\mu_\gamma \equiv \mu = 1/\left[r_+^{\mathrm{I}} - M\left(G + r_+^{\mathrm{I}} G'\right)\right]$ to be inserted in the left hand side of the differential equation (8.169). We also need expressions for $\partial r_+^{\mathrm{I}}/\partial M$, $\partial r_+^{\mathrm{I}}/\partial J$ and $\partial^2 r_+^{\mathrm{I}}/\partial J^2$ as functions of J and M. We find the latter ones by differentiating the equation for the event horizon.

For $\partial r_+^{\mathrm{I}}/\partial M$ we have

$$\frac{\partial}{\partial M}\left[\left(r_+^{\mathrm{I}}\right)^2 + \left(\frac{J}{M}\right)^2 - 2Mr_+^{\mathrm{I}}G\right] = 0 \tag{J.1}$$

As a result, we find almost directly:

$$\frac{\partial r_+^{\mathrm{I}}}{\partial M} = \frac{\left(\frac{J^2}{M^3}\right) + Gr_+^{\mathrm{I}}}{\left[r_+^{\mathrm{I}} - Mr_+^{\mathrm{I}}G' - MG\right]} \tag{J.2}$$

Similarily for $\partial r_+^{\mathrm{I}}/\partial J$ we perform

$$\frac{\partial}{\partial J}\left[\left(r_+^{\mathrm{I}}\right)^2 + \left(\frac{J}{M}\right)^2 - 2MGr_+^{\mathrm{I}}\right] = 0 \tag{J.3}$$

After simplifying (J.3) we find:

$$\frac{\partial r_+^{\mathrm{I}}}{\partial J} = -\frac{J}{M^2\left[r_+^{\mathrm{I}} - M\left(G + r_+^{\mathrm{I}}G'\right)\right]} \tag{J.4}$$

For the second derivative $\partial^2 r_+^{\mathrm{I}}/\partial J^2$ we differentiate (J.4) as follows

$$\frac{\partial}{\partial J}\left\{\frac{\partial r_+^{\mathrm{I}}}{\partial J}\left[r_+^{\mathrm{I}} - M\left(G + r_+^{\mathrm{I}}G'\right)\right]\right\} = -\frac{\partial}{\partial J}\left[\frac{J}{M^2}\right] \tag{J.5}$$

The simplification of (J.5) and further factorization of $\partial^2 r_+^{\mathrm{I}}/\partial J^2$ lead to:

$$\frac{\partial^2 r_+^{\mathrm{I}}}{\partial J^2} = \frac{\left(\frac{\partial r_+^{\mathrm{I}}}{\partial J}\right)^2\left[M\left(\left[2G' + G''\right]\right) - 1\right] - \frac{1}{M^2}}{\left[r_+^{\mathrm{I}} - M\left(G + r_+^{\mathrm{I}}G'\right)\right]} \tag{J.6}$$

Knowing $\partial r_+^{\mathrm{I}}/\partial M$ and $\partial r_+^{\mathrm{I}}/\partial J$ we now proceed to find $\partial\mu/\partial M$ and $\partial\mu/\partial J$. For $\partial\mu/\partial M$ we have

$$\begin{aligned}\frac{\partial}{\partial M}\left\{\frac{1}{\left[r_+^{\mathrm{I}} - M\left(G + r_+^{\mathrm{I}}G'\right)\right]}\right\} &= -\frac{1}{\left[r_+^{\mathrm{I}} - M\left(G + r_+^{\mathrm{I}}G'\right)\right]^2}\times \\ &\times\left\{\frac{\partial r_+^{\mathrm{I}}}{\partial M}\left[1 - M\left(2G' + r_+^{\mathrm{I}}G''\right)\right] - \left(G + r_+^{\mathrm{I}}G'\right)\right\}\end{aligned} \tag{J.7}$$

Substituting $\partial r_+^{\mathrm{I}}/\partial M$ from (J.2) in (J.7) gives:

$$\frac{\partial}{\partial M}\left\{\frac{1}{\left[r_+^{\mathrm{I}}-M\left(G+r_+^{\mathrm{I}}G'\right)\right]}\right\}=\frac{1}{\left[r_+^{\mathrm{I}}-M\left(G+r_+^{\mathrm{I}}G'\right)\right]^2}\times$$
$$\times\left\{\frac{\left(\frac{J^2}{M^3}\right)+Gr_+^{\mathrm{I}}}{\left[r_+^{\mathrm{I}}-Mr_+^{\mathrm{I}}G'-MG\right]}\left[M\left(2G'+r_+^{\mathrm{I}}G''\right)-1\right]+\left(G+r_+^{\mathrm{I}}G'\right)\right\}$$

Similarly for $\partial\mu/\partial J$ we have

$$\frac{\partial}{\partial J}\left\{\frac{1}{\left[r_+^{\mathrm{I}}-M\left(G+r_+^{\mathrm{I}}G'\right)\right]}\right\}=-\frac{\left[1-M\left(2G'+r_+^{\mathrm{I}}G''\right)\right]}{\left[r_+^{\mathrm{I}}-M\left(G+r_+^{\mathrm{I}}G'\right)\right]^2}\frac{\partial r_+^{\mathrm{I}}}{\partial J}\tag{J.8}$$

Inserting $\partial r_+^{\mathrm{I}}/\partial J$ from (J.4) in (J.8) leads to:

$$\frac{\partial}{\partial J}\left\{\frac{1}{\left[r_+^{\mathrm{I}}-M\left(G+r_+^{\mathrm{I}}G'\right)\right]}\right\}=\frac{J\left[1-M\left(2G'+r_+^{\mathrm{I}}G''\right)\right]}{M^2\left[r_+^{\mathrm{I}}-M\left(G+r_+^{\mathrm{I}}G'\right)\right]^3}\tag{J.9}$$

Equations (J.8) and (J.9) for $\partial\mu/\partial M$ and $\partial\mu/\partial J$ are the final results of this subsection. It is easy to check that these expressions reduce to (J.114) and (J.111) for the classical case when setting $G=1,\ G'=G''=0$.

J.1.2 Derivation of the Integrability Condition

In section 8.7.4 we stated the possibility of finding a special function $G(r)$ that converts, when evaluated at the radius r_+^{I}, the expression (8.173) to an equality. Assuming this equality we can cancel common terms on each side of (8.173) to find

$$\begin{aligned}&\left\{1-M\left[2G'\left(r_+^{\mathrm{I}}\right)+r_+^{\mathrm{I}}G''\left(r_+^{\mathrm{I}}\right)\right]\right\}\left\{r_+^{\mathrm{I}}-MG\left(r_+^{\mathrm{I}}\right)\right\}\\&\qquad=r_+^{\mathrm{I}}-M\left[r_+^{\mathrm{I}}G'\left(r_+^{\mathrm{I}}\right)+G\left(r_+^{\mathrm{I}}\right)\right]\end{aligned}\tag{J.10}$$

Further simplification leads straightforwardly to

$$\left[2G'\left(r_+^{\mathrm{I}}\right)+r_+^{\mathrm{I}}G''\left(r_+^{\mathrm{I}}\right)\right]\left[G\left(r_+^{\mathrm{I}}\right)M-r_+^{\mathrm{I}}\right]+r_+^{\mathrm{I}}G'\left(r_+^{\mathrm{I}}\right)=0\tag{J.11}$$

After factorizing coefficients in every derivative we have

$$\left[2G\left(r_+^{\mathrm{I}}\right)M-r_+^{\mathrm{I}}\right]G'\left(r_+^{\mathrm{I}}\right)+r_+^{\mathrm{I}}G''\left(r_+^{\mathrm{I}}\right)\left[G\left(r_+^{\mathrm{I}}\right)M-r_+^{\mathrm{I}}\right]=0\tag{J.12}$$

An additional simplification can be done if we apply the event horizon equation $\left(r_+^{\mathrm{I}}\right)^2+(J/M)^2-2G\left(r_+^{\mathrm{I}}\right)Mr_+^{\mathrm{I}}=0$ in the form

$$\left(\frac{J}{M}\right)^2=\left[2G\left(r_+^{\mathrm{I}}\right)M-r_+^{\mathrm{I}}\right]r_+^{\mathrm{I}}\tag{J.13}$$

As a result we have for (J.12) the following condition:

$$\left(\frac{J}{M}\right)^2 \frac{G'\left(r^{\mathrm{I}}_+\right)}{r^{\mathrm{I}}_+} + G''\left(r^{\mathrm{I}}_+\right) r^{\mathrm{I}}_+ \left[\frac{J^2}{r^{\mathrm{I}}_+ M^2} - G\left(r^{\mathrm{I}}_+\right) M\right] = 0 \tag{J.14}$$

Aternatively by multiplying with r^{I}_+:

$$\left(\frac{J}{M}\right)^2 G'\left(r^{\mathrm{I}}_+\right) + r^{\mathrm{I}}_+ G''\left(r^{\mathrm{I}}_+\right) \left[\left(\frac{J}{M}\right)^2 - G\left(r^{\mathrm{I}}_+\right) M r^{\mathrm{I}}_+\right] = 0 \tag{J.15}$$

This is precisely the condition we wanted to derive.

The condition (J.15) is valid precisely at $r = r^{\mathrm{I}}_+ (M, J)$. We shall assume that in the vicinity of r^{I}_+ the function $G(r)$ is governed by the following nonlinear differential equation inspired by (J.15):

$$\left(\frac{J}{M}\right)^2 G'(r) + rG''(r) \left[\left(\frac{J}{M}\right)^2 - G(r) Mr\right] = 0 \tag{J.16}$$

J.1.3 Solution of the Differential Equation (J.16)

We proceed now to find solutions of (J.16). We summarize the steps as follows. First we change variables from r to $u \equiv 1/r$ in order to formulate a large r expansion of G in powers of u. After that, we apply the power series method for finding recurrence relations in the coefficients of the expansion. As a result we find the solutions presented in subsection 8.7.4.

We define

$$\tilde{G}(u) \equiv G(r = 1/u) \tag{J.17}$$

as the Newton constant in the new variable u. As a result we find the following expressions for $\tilde{G}'(u)$ and $G''(u)$

$$\tilde{G}'(u) = -\frac{G'(r)}{u^2}\ ,\ \tilde{G}''(u) = \frac{G''(r) + 2uG'(r)}{u^4} \tag{J.18}$$

Then for $G'(r)$ and $G''(r)$ as functions of $\tilde{G}'(u)$ and $\tilde{G}''(u)$ we have

$$G'(r) = -u^2\tilde{G}'(u)\ ,\ G''(r) = u^4\tilde{G}''(u) + 2u^3\tilde{G}'(u) \tag{J.19}$$

After substituting (J.17), (J.18) and (J.19) in (J.15) we find

$$-u^2\tilde{G}'(u)\left(\frac{J}{M}\right)^2 + (u)^2\left[u\tilde{G}''(u) + 2\tilde{G}'(u)\right]\left[\left(\frac{J}{M}\right)^2 - \frac{\tilde{G}(u) M}{u}\right] = 0 \tag{J.20}$$

Cancelling an $u \neq 0$ leads to

$$-u\tilde{G}'(u)\left(\frac{J}{M}\right)^2 + \left[u\tilde{G}''(u) + 2\tilde{G}'(u)\right]\left[u\left(\frac{J}{M}\right)^2 - \tilde{G}(u)M\right] = 0 \qquad \text{(J.21)}$$

Now we factor the coefficients for every order of derivation as follows

$$\tilde{G}'(u)\left[u\left(\frac{J}{M}\right)^2 - 2\tilde{G}(u)M\right] + u\tilde{G}''(u)\left[u\left(\frac{J}{M}\right)^2 - M\tilde{G}(u)\right] = 0 \qquad \text{(J.22)}$$

From now on we supress the tilde $\tilde{}$ and we present the new nonlinear differential equation for $\tilde{G}'(u)$ as:

$$G'(u)\left[u\left(\frac{J}{M}\right)^2 - 2G(u)M\right] + uG''(u)\left[u\left(\frac{J}{M}\right)^2 - MG(u)\right] = 0 \qquad \text{(J.23)}$$

As mentioned at the beginning of this section we will solve (J.23) by substituting a power series ansatz. The expansions for $G(u)$ and its derivatives are given by

$$\begin{aligned} G(u) &= G_0 + G_1u + G_2u^2 + \cdots = \sum_{k=0}^{\infty} u^k G_k \qquad \text{(J.24)} \\ G'(u) &= G_1 + 2G_2u + 3G_3u^2 + \cdots = \sum_{k=0}^{\infty} (k+1)\, u^k G_{k+1} \\ G''(u) &= 2G_2 + 2*3uG_3 + 3*4u^2G_4 + \cdots = \sum_{k=0}^{\infty} (k+1)(k+2)\, u^k G_{k+2} \end{aligned}$$

Now we substitute the expressions from (J.24) in (J.23) as follows

$$\begin{aligned} &\sum_{k=0}^{\infty} (k+1)\, u^k G_{k+1} \left[u\left(\frac{J}{M}\right)^2 - 2M\sum_{j=0}^{\infty} u^j G_j\right] \\ &+u\sum_{k=0}^{\infty} (k+1)(k+2)\, u^k G_{k+2} \left[u\left(\frac{J}{M}\right)^2 - M\sum_{j=0}^{\infty} u^j G_j\right] = 0 \qquad \text{(J.25)} \end{aligned}$$

Every term in (J.25) has to be transformed to a single power series. We treat them separately:

$$\mathrm{A} \equiv \left(\frac{J}{M}\right)^2 \sum_{k=0}^{\infty} (k+1)\, u^{k+1} G_{k+1} \qquad \text{(J.26)}$$

$$\mathrm{B} \equiv -2M\left[\sum_{k=0}^{\infty} (k+1)\, u^k G_{k+1}\right]\left[\sum_{j=0}^{\infty} u^j G_j\right] \qquad \text{(J.27)}$$

$$\mathrm{C} \equiv \left(\frac{J}{M}\right)^2 \sum_{k=0}^{\infty} (k+1)(k+2)\, u^{k+2} G_{k+2} \tag{J.28}$$

$$\mathrm{D} \equiv -M \left[\sum_{k=0}^{\infty} (k+1)(k+2)\, u^{k+1} G_{k+2}\right] \left[\sum_{j=0}^{\infty} u^j G_j\right] \tag{J.29}$$

Terms A and B can be easily transformed as follows

$$\begin{aligned} \mathrm{A} &\equiv \left(\frac{J}{M}\right)^2 \sum_{k=0}^{\infty} (k+1)\, u^{k+1} G_{k+1} = \left(\frac{J}{M}\right)^2 \left[uG_1 + 2u^2 G_2 + 3u^3 G_3 + \cdots\right] \\ &= \left(\frac{J}{M}\right)^2 \sum_{k=1}^{\infty} k u^k G_k \end{aligned} \tag{J.30}$$

$$\begin{aligned} \mathrm{C} &\equiv \left(\frac{J}{M}\right)^2 \sum_{k=0}^{\infty} (k+1)(k+2)\, u^{k+2} G_{k+2} = \left(\frac{J}{M}\right)^2 \left[(2*1)\, u^2 G_2 + (2*3)\, u^3 G_3 + \cdots\right] \\ &= \left(\frac{J}{M}\right)^2 \left[\sum_{k=2}^{\infty} k\,(k-1)\, u^k G_k\right] \end{aligned} \tag{J.31}$$

For terms B and D we factorize coefficients of the same power after expanding the products. For B we have

$$\begin{aligned} \mathrm{B} &\equiv -2M \left[\sum_{k=0}^{\infty} (k+1)\, u^k G_{k+1}\right] \left[\sum_{j=0}^{\infty} u^j G_j\right] \\ &= -2M \left[G_1 + 2uG_2 + 3u^2 G_3 + \cdots\right] \left[G_0 + uG_1 + u^2 G_2 + \cdots\right] \\ &= -2M \left[G_0 G_1 + 2uG_0 G_2 + 3u^2 G_3 G_0 + u\,(G_1)^2 + 2u^2 G_2 G_1 + 3u^3 G_3 G_1 \right. \\ &\qquad \left. + u^2 G_1 G_2 + 2u^3 (G_2)^2 + 3u^4 G_3 G_2 + \cdots\right] \end{aligned}$$

Factorizing powers of u leads to

$$\begin{aligned} \mathrm{B} \equiv -2M \Big\{ & G_0 G_1 + \left[2G_0 G_2 + (G_1)^2\right] u + \left[3G_3 G_0 + 2G_2 G_1 + G_1 G_2\right] u^2 \\ & + \left[3G_3 G_1 + 2\,(G_2)^2 + G_3 G_1 + 4G_4 G_0\right] u^3 + \cdots \Big\} \end{aligned}$$

or written in a more compact way

$$\mathrm{B} \equiv -2M \sum_{k=0}^{\infty} \gamma_k u^k \tag{J.32}$$

with γ_k defined as

$$\gamma_k \equiv \sum_{j=0}^{k} (k-j+1)\, G_j G_{k-j+1} \tag{J.33}$$

Similarily for D we write

$$\begin{aligned}
\mathrm{D} \equiv -M\left[\sum_{k=0}^{\infty}(k+1)(k+2)\,u^{k+1}G_{k+2}\right]\left[\sum_{j=0}^{\infty}u^{j}G_{j}\right] \\
= -M\left[(1*2)*u_{+}^{\mathrm{I}}G_2+(2*3)*u^2G_3+(3*4)\,u^3G_4+\cdots\right]\times \\
\times\left[G_0+uG_1+u^2G_2+\cdots\right] \\
= -M\left[(1*2)\,uG_2G_0+(1*2)\,u^2G_2G_1+(1*2)\,u^3(G_2)^2\right. \\
+(2*3)\,u^2G_3G_0+(2*3)\,u^3G_3G_1+(2*3)\,u^4G_3G_2 \\
\left.+(3*4)\,u^3G_4G_0+(3*4)\,u^4G_4G_1+(3*4)\,u^5G_4G_2+\cdots\right]
\end{aligned}$$

After factorizing powers of u we find

$$\begin{aligned}
\mathrm{D} = -M\left\{(1*2)\,uG_2G_0+\left[(1*2)\,G_2G_1+(2*3)\,G_3G_0\right]u^2\right. \\
+\left[(1*2)\,(G_2)^2+(2*3)\,G_3G_1+(3*4)\,G_4G_0\right]u^3 \\
\left.+\left[(1*2)\,G_2G_3+(2*3)\,G_3G_2+(3*4)\,G_4G_1+(4*5)\,G_5G_0\right]u^4+\cdots\right\}
\end{aligned} \tag{J.34}$$

Here we can simplify the expression (J.34) by defining the coefficients β_k as follows

$$\mathrm{D} = -M\sum_{k=2}^{\infty}\beta_k u^k \tag{J.35}$$

$$\beta_k = \sum_{j=0}^{k-1}(k-j+1)(k-j)\,G_jG_{k-j+1}$$

Having expressions for A, B, C and D in (J.30), (J.32), (J.31) and (J.35) respectively, we substitute them in (J.25) to have

$$\left(\frac{J}{M}\right)^2\sum_{k=1}^{\infty}ku^kG_k-2M\sum_{k=0}^{\infty}\gamma_ku^k+\left(\frac{J}{M}\right)^2\left[\sum_{k=2}^{\infty}k(k-1)\,u^kG_k\right]-M\sum_{k=2}^{\infty}\beta_ku^k=0 \tag{J.36}$$

We can factorize in (J.36) one summation from $k=2$ to ∞, as follows

$$\left[\left(\frac{J}{M}\right)^2G_1-2M\gamma_1\right]u-2M\gamma_0+\sum_{k=2}^{\infty}\left\{\left(\frac{J}{M}\right)^2k^2G_k-M\beta_k-2M\gamma_k\right\}u^k=0 \tag{J.37}$$

Equation (J.37) implies the cancellation of every coefficient for each power of u. As a result we have

$$2M\gamma_0=0 \tag{J.38}$$

$$\left(\frac{J}{M}\right)^2 G_1 - 2M\gamma_1 = 0 \tag{J.39}$$

$$\left(\frac{J}{M}\right)^2 k^2 G_k - M\beta_k - 2M\gamma_k = 0 \ \ , \ \ k = 2, 3 \ \cdots \tag{J.40}$$

with β_k and γ_k given in (J.35) and (J.33)

$$\begin{aligned} \beta_k \ &\equiv \ \sum_{j=0}^{k-1} (k-j+1)\,(k-j)\, G_j G_{k-j+1} \\ \gamma_k \ &\equiv \ \sum_{j=0}^{k} (k-j+1)\, G_j G_{k-j+1} \end{aligned} \tag{J.41}$$

After substituting these two last definitions in (J.38) to (J.40) we find the following infinite set of equations

$$2MG_0G_1 = 0 \tag{J.42}$$

$$\left(\frac{J}{M}\right)^2 G_1 - 2M\left[2G_0G_2 + (G_1)^2\right] = 0 \tag{J.43}$$

$$\begin{aligned} \left(\frac{J}{M}\right)^2 k^2 G_k - M\sum_{j=0}^{k-1} (k-j+1)\,(k-j)\, G_j G_{k-j+1} & \\ -2M\sum_{j=0}^{k} (k-j+1)\, G_j G_{k-j+1} &= 0 \\ k &= 2, 3 \ \cdots \end{aligned} \tag{J.44}$$

Eq. (J.44) can be reorganized with just one summation from $j = 0$ to $j = k-1$ as follows

$$G_k\left[\left(\frac{J}{M}\right)^2 k^2 - 2MG_1\right] - M\sum_{j=0}^{k-1} (k-j+1)\,(k-j+2)\, G_j G_{k-j+1} = 0 \tag{J.45}$$

We add the cases for $k = 2, 3$ as special examples of (J.45) to be used later, as follows

$$G_2\left(\frac{J}{M}\right)^2 - 2MG_1G_2 - 3MG_0G_3 = 0 \tag{J.46}$$

$$9G_3\left(\frac{J}{M}\right)^2 - 14MG_1G_3 - 20MG_0G_4 - 6M\left(G_2\right)^2 = 0 \tag{J.47}$$

Equation (J.42) implies three independent possibilities to be analysed, namely that either $G_0 = 0$ or $G_1 = 0$ or both. The most plausible option is $G_0 \neq 0$ and $G_1 = 0$ so that we can still recover the classical case. After analysing equations (J.43) and (J.45) one can deduce that every coefficient $G_k \neq G_0$ should vanish. This gives, as a result, a trivial case where no improvement is allowed. We show this as follows. For $G_1 = 0$ equation (J.43) is transformed to

$$4MG_0G_2 = 0 \tag{J.48}$$

This implies directly that $G_2 = 0$. Inserting $G_1 = G_2 = 0$ in (J.46) gives $G_0G_3 = 0$ which implies again $G_3 = 0$. If we proceed iteratively by substituting $G_1 = G_2 = \cdots = G_n = 0$ in the $k = n$ equation we find $G_{n+1} = 0$. By induction we conclude that if $G_1 = 0$ the $G_n = 0$ for all $n > 0$. As a result from the previous analysis we conclude that the case $G_0 = G_1 = 0$ is the most trivial one with every coefficient equal to zero.

Only one case is left to analyse, namely $G_1 \neq 0$ and $G_0 = 0$. We proceed in a similar way as in the previous case, as follows. Substituting $G_0 = 0$ in (J.43) gives the following condition

$$G_1\left[\left(\frac{J}{M}\right)^2 - 2MG_1\right] = 0 \tag{J.49}$$

Avoiding the already known case for $G_1 = 0$ we stay with the alternative solution to (J.50):

$$G_1 = \frac{\left(\frac{J}{M}\right)^2}{2M} \tag{J.50}$$

This defines a non-trivial expression for G_1, the coefficient of $u = 1/r$ in the expansion of $G(u)$ in (J.24).

We must go further in finding the rest of the coefficients for this case, as follows. Substituting $G_0 = 0$ and $G_1 = (J/M)^2/(2M)$ in equation (J.46) gives $G_2 * 0 = 0$ without any restriction to G_2. This means that we can choose either $G_2 = 0$ or $G_2 \neq 0$. Assuming the most general case for $G_2 \neq 0$ leads to the following expresions for G_3 and G_4

$$G_3 = \frac{3M\left(G_2\right)^2}{\left(\frac{J}{M}\right)^2}, G_4 = \frac{52M^2\left(G_2\right)^3}{5\left(\frac{J}{M}\right)^4} \tag{J.51}$$

Expressions in (J.51) for G_3 and G_4 lead us to propose an ansatz for the rest of G_j given by

$$G_j = \alpha_j \frac{\left(G_2\right)^{j-1}}{\left(\frac{J}{M}\right)^{2(j-2)}} \tag{J.52}$$

where α_j is a pure number to be found for each coefficient G_j. Exploiting this ansatz we perform the following substitutions in (J.45):

$$G_1 = \frac{\left(\frac{J}{M}\right)^2}{(2M)},\ G_k = \alpha_k \frac{\left(G_2\right)^{k-1}}{\left(\frac{J}{M}\right)^{2(k-2)}}, G_{k-j+1} = \alpha_{k-j+1} \frac{\left(G_2\right)^{k-j}}{\left(\frac{J}{M}\right)^{2(k-j-1)}} \tag{J.53}$$

The result reads:

$$\frac{\left(G_2\right)^{k-1}}{\left(\frac{J}{M}\right)^{2(k-3)}} \left[\left(k^2-1\right)\alpha_k - M\sum_{j=0}^{k-1}(k-j+1)(k-j+2)\,\alpha_j\alpha_{k-j+1}\right] = 0 \tag{J.54}$$

Equation (J.54) implies the following recurrence relation for α_k:

$$\alpha_k = \frac{M}{(k^2-1)}\sum_{j=0}^{k-1}(k-j+1)(k-j+2)\,\alpha_j\alpha_{k-j+1} \tag{J.55}$$

Summarizing the result for the solution to (J.23) with $G_0 = 0$ and $G_k \neq 0$ for $k = 1, 2, \cdots$, we write

$$G(u) = G_1 u + G_2 u^2 + \cdots$$

$$= \frac{\left(\frac{J}{M}\right)^2}{2M}u + G_2 u^2 + \sum_{k=3}^{\infty} u^k G_k = \frac{\left(\frac{J}{M}\right)^2}{2M}u + G_2 u^2 + \sum_{k=3}^{\infty} \frac{\alpha_k\left(G_2\right)^{k-1}}{\left(\frac{J}{M}\right)^{2(k-2)}} u^k$$

$$\alpha_k = \frac{M}{(k^2-1)}\sum_{j=0}^{k-1}(k-j+1)(k-j+2)\,\alpha_j\alpha_{k-j+1}$$

The value of G_2 remains undefined. Changing from u to r leads to

$$G(r) = \frac{\left(\frac{J}{M}\right)^2}{2Mr} + \frac{G_2}{r^2} + \sum_{k=3}^{\infty} \frac{\alpha_k\left(G_2\right)^{k-1}}{r^k\left(\frac{J}{M}\right)^{2(k-2)}} \tag{J.56}$$

We consider expression (J.56) as the final result for this case.

We can now put together the various results from this section as follows. We have found three possible independent solutions for the differential equation (J.23) or alternatively (J.16). We list them below:

1. $G_0 \neq 0$ and $G_k = 0$, $k = 1, 2, \cdots$. This is the classical case with no running Newton constant.

2. $G(r) = \frac{\left(\frac{J}{M}\right)^2}{2Mr}$

3. $G(r) = \frac{\left(\frac{J}{M}\right)^2}{2Mr} + \frac{G_2}{r^2} + \sum_{k=3}^{\infty} \frac{\alpha_k (G_2)^{k-1}}{r^k \left(\frac{J}{M}\right)^{2(k-2)}}$
   with $\alpha_k = \frac{M}{(k^2-1)} \sum_{j=0}^{k-1} (k-j+1)(k-j+2)\, \alpha_j \alpha_{k-j+1}$

The behavior of cases 2 and 3 is similar for $r \to \infty$, namely, $G(r) \to 0$.

J.2 Integrating Factor for the Modified First Law: $\mathrm{O}\left(J^2\right)$ Approximation

In this section we calculate the coefficients μ_0, μ_1 and μ_2 of the $\mathrm{O}(J^2)$ approximation $\mu_{O(J^2)}(M, J)$ presented in equation (8.193) of chapter 8. For that purpose we need explicit expressions for several coefficients in different series expansions, namely the coefficients α_m, β_m and γ_m in the recurrence relation (8.186), and the coefficients f_i^l for f_1 to f_3 in (8.183). They are calculated in subsections J.2.2 and J.2.3. In subsections J.2.4 and J.2.5 we find the integrating factor for the first law of thermodynamics in the case of the improved Schwarzschild spacetime, and we demonstrate that $\mu = 1/(r_+ - M)$ is the appropriate integrating factor in the case of the classical Kerr spacetime.

J.2.1 Solving the Recurrence Relation for $\mu_{O(J^2)}(M, J)$

In order to get an $\mathrm{O}(J^2)$ approximation to μ we need to calculate only a few f_i^l components. We can recognize these components by expanding (8.192) as follows

$$(m+1)\, f_1^0 \mu_{m+1} + f_2^0 \mu_m' + f_3^0 \mu_m + \cdots + f_1^m \mu_1 + f_2^m \mu_0' + f_3^m \mu_0 = 0 \tag{J.57}$$

From (J.57) we see that the μ-component of highest order in the recurrence relation is μ_{m+1}, which comes from $l = 0$ in the summation. On the other hand for $l = m$ we get the f_i of highest order, namely f_i^m. This means that for a $m + 1$-th order μ component we need at most the f_i components up to the m-th order.

In our case, since we calculate up to μ_2 we need only the f_i^0 and the f_i^1 coefficients. Those coefficients are given by (see subsection J.2.3):

$$
\begin{aligned}
f_1^0 &= \left[r_+^{\mathrm{I}}(0,M)\right]^2 \quad , \; f_1^1 = 0 \; , \; f_2^0 = 0 \; , \; f_2^1 = \frac{1}{m} \; , \; f_3^0 = 0 \\
f_3^1 &= -\frac{1}{M^2}\left.\left\{\frac{r_+^{\mathrm{I}} + M\left[r_+^{\mathrm{I}} G' + G\right]}{r_+^{\mathrm{I}} - M\left[r_+^{\mathrm{I}} G' + G\right]}\right\}\right|_{J=0}
\end{aligned} \tag{J.58}
$$

Here $r_+^{\mathrm{I}}(0,M)$ is the radius of the outer event horizon of the improved *Schwarzschild* metric. From now on we denote it $r_+^{\mathrm{I}}(0,M) \equiv r_{\mathrm{Sch}_+}^{\mathrm{I}}(M)$ or simply $r_{\mathrm{Sch}_+}^{\mathrm{I}}$. Then the coefficients of (J.58) can be written as

$$
\begin{aligned}
f_1^0 &= \left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)^2 \quad , \; f_1^1 = 0 \; , \; f_2^0 = 0 \; , \; f_2^1 = \frac{1}{m} \; , \; f_3^0 = 0 \\
f_3^1 &= -\left(\frac{1}{M^2}\right)\frac{r_{\mathrm{Sch}_+}^{\mathrm{I}} + M\left[r_{\mathrm{Sch}_+}^{\mathrm{I}} G'\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right) + G\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)\right]}{r_{\mathrm{Sch}_+}^{\mathrm{I}} - M\left[r_{\mathrm{Sch}_+}^{\mathrm{I}} G'\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right) + G\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)\right]}
\end{aligned} \tag{J.59}
$$

Setting $m = 0$ and $m = 1$ in (8.192) is enough for finding all the components we need. As a result we have, beginning with $m = 0$:

$$f_1^0 \mu_1 + f_2^0 \mu_0' + f_3^0 \mu_0 = 0 \tag{J.60}$$

Substituting the coefficients (J.58) in (J.60) leads directly to:

$$\mu_1 = 0 \tag{J.61}$$

As a result we have found that the first order component of μ vanishes. Next, for $m = 1$, we have:

$$2 f_1^0 \mu_2 + f_2^0 \mu_1' + f_3^0 \mu_1 + f_1^1 \mu_1 + f_2^1 \mu_0' + f_3^1 \mu_0 = 0 \tag{J.62}$$

Substituting again the coefficients in (J.58) gives:

$$2\left.\left(r_+^{\mathrm{I}}\right)^2\right|_{J=0} \mu_2 + \left(\frac{1}{M}\right)\mu_0' - \frac{1}{M^2}\left.\left\{\frac{r_+^{\mathrm{I}} + M\left[r_+^{\mathrm{I}} G'\left(r_+^{\mathrm{I}}\right) + G\left(r_+^{\mathrm{I}}\right)\right]}{r_+^{\mathrm{I}} - M\left[r_+^{\mathrm{I}} G'\left(r_+^{\mathrm{I}}\right) + G\left(r_+^{\mathrm{I}}\right)\right]}\right\}\right|_{J=0} \mu_0 = 0 \tag{J.63}$$

Or substituting $r_{\mathrm{Sch}_+}^{\mathrm{I}}$:

$$2\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)^2 \mu_2 + \left(\frac{1}{M}\right)\mu_0' - \left(\frac{1}{M^2}\right)\left[\frac{r_{\mathrm{Sch}_+}^{\mathrm{I}} + M\left[r_{\mathrm{Sch}_+}^{\mathrm{I}} G'\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right) + G\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)\right]}{r_{\mathrm{Sch}_+}^{\mathrm{I}} - M\left[r_{\mathrm{Sch}_+}^{\mathrm{I}} G'\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right) + G\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)\right]}\right]\mu_0 = 0 \tag{J.64}$$

Equation (J.63) gives an expression for μ_2 as a function of μ_0.

The zeroth component μ_0 can be easily found, knowing that it is the integrating factor for a variation law of the form $\kappa^{\mathrm{I}}_{\mathrm{Sch}}\delta s = \delta M$ with $J = 0$. It is given by (see subsection J.2.4):

$$\mu_0 = \frac{1}{M\left[G\left(r^{\mathrm{I}}_{+}\right) - r^{\mathrm{I}}_{+}G'\left(r^{\mathrm{I}}_{+}\right)\right]|_{J=0}} = \frac{1}{M\left[G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) - r^{\mathrm{I}}_{\mathrm{Sch}_+}G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]} \tag{J.65}$$

Its first derivative in M reads (see subsection J.2.4):

$$\mu_0' = -\frac{G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) - r^{\mathrm{I}}_{\mathrm{Sch}_+}G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) - Mr^{\mathrm{I}}_{\mathrm{Sch}_+}\frac{dr^{\mathrm{I}}_{\mathrm{Sch}_+}}{dM}G''\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)}{M^2\left[G\left(r^{\mathrm{I}}_{\mathrm{sc}_+}\right) - r^{\mathrm{I}}_{\mathrm{Sch}_+}G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]^2} \tag{J.66}$$

With $dr^{\mathrm{I}}_{\mathrm{Sch}_+}/dM$ defined as

$$\frac{dr^{\mathrm{I}}_{\mathrm{Sch}_+}}{dM} = \frac{2G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)}{\left[1 - 2MG'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]} \tag{J.67}$$

In principle we can now substitute (J.65) and (J.66) in (J.64), in order to find μ_2 explicitly as a function of $r^{\mathrm{I}}_{\mathrm{sc}_+}$, $G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)$ and its derivatives. It is not necessary to present this complete complicated expression. It will be more illuminating to write μ_2 as

$$\mu_2 = \frac{\mu_0}{2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2 M^2}\left\{\frac{r^{\mathrm{I}}_{\mathrm{Sch}_+} + M\left[r^{\mathrm{I}}_{\mathrm{Sch}_+}G' + G\right]}{r^{\mathrm{I}}_{\mathrm{Sch}_+} - M\left[r^{\mathrm{I}}_{\mathrm{Sch}_+}G' + G\right]}\right\} - \frac{\mu_0'}{2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2 M} \tag{J.68}$$

with μ_0 and μ_0' given in (J.65) and (J.66). Here G and its derivatives are evaluated at $r^{\mathrm{I}}_{\mathrm{Sch}_+}$. The radius $r^{\mathrm{I}}_{\mathrm{Sch}_+} \equiv r^{\mathrm{I}}_{+}(0, M)$ is to be obtained by solving the horizon condition for $J = 0$, i.e. $r^{\mathrm{I}}_{\mathrm{Sch}_+} = 2MG\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)$. Substituting this condition in (J.68) leads to a further simplification of μ_2 as follows:

$$\mu_2 = \frac{\left[3G + r^{\mathrm{I}}_{\mathrm{Sch}_+}G'\right](\mu_0)^2 - \mu_0'}{2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2 M} \tag{J.69}$$

We can now write, as a summary of this subsection, the O(J^2)-approximation to $\mu(M, J)$, namely

$$\mu(M, J)|_{O(J^2)} = \mu_0 + \mu_2 J^2 \tag{J.70}$$

with μ_0 and μ_2 defined in (J.65) and (J.69), respectively. They correspond, as expected, to the expressions in (8.194) of chapter 8.

J.2.2 Derivation of the Coefficients α_m, β_m and γ_m

In subsection 8.7.5 we presented the expressions (8.187), (8.188) and (8.189) for the coefficients α_m, β_m and γ_m, respectively. These expressions can be further simplified if one expands the products of summations in each of them, factorize coefficients of powers in J, and redefine a unique summation for every expression. Here we carry out this simplification for one of the coefficients, say β_m. The procedure is completely analogous for the other coeffcients. The expressions (8.187), (8.188) and (8.189) are given by

$$\sum_{m=0}^{\infty} \beta_m J^m \equiv \left(\sum_{l=0}^{\infty} f_1^l (M) J^l\right) \left(\sum_{k=0}^{\infty} (k+1) \mu_{k+1} (M) J^k\right) \tag{J.71}$$

$$\sum_{m=0}^{\infty} \gamma_m J^m \equiv \left(\sum_{l=0}^{\infty} f_2^l (M) J^l\right) \left(\sum_{k=0}^{\infty} \left(\frac{d\mu_k}{dM}\right) J^k\right) \tag{J.72}$$

$$\sum_{m=0}^{\infty} \alpha_m J^m \equiv \left(\sum_{k=0}^{\infty} \mu_k (M) J^k\right) \left(\sum_{l=0}^{\infty} f_3^l (M) J^l\right) \tag{J.73}$$

Expanding the summations for (J.71) gives[1]

$$\sum_{m=0}^{\infty} \beta_m J^m \equiv \left[f_1^0 + f_1^1 J + f_1^2 J^2 + \cdots\right] \left[\mu_1 + 2\mu_2 J + 3\mu_3 J^2 + \cdots\right] \tag{J.74}$$

After multiplying term by term we have

$$\begin{aligned} \sum_{m=0}^{\infty} \beta_m J^m \equiv\ & \mu_1 f_1^0 + \mu_1 f_1^1 J + \mu_1 f_1^2 J^2 \\ & +2\mu_2 f_1^0 J + 2\mu_2 f_1^1 J^2 + 2\mu_2 f_1^2 J^3 \\ & +3\mu_3 f_1^0 J^2 + 3\mu_3 f_1^1 J^3 + 3\mu_3 f_1^2 J^4 + \cdots \end{aligned} \tag{J.75}$$

Next we factorize coefficients of equal powers of J as follows

$$\begin{aligned} \sum_{m=0}^{\infty} \beta_m J^m \equiv\ & \mu_1 f_1^0 + \left(\mu_1 f_1^1 + 2\mu_2 f_1^0\right) J \\ & + \left(\mu_1 f_1^2 + 2\mu_2 f_1^1 + 3\mu_3 f_1^0\right) J^2 \\ & + \left(4\mu_4 f_1^0 + 3\mu_3 f_1^1 + 2\mu_2 f_1^2 + \mu_1 f_1^3\right) J^3 + \cdots \end{aligned} \tag{J.76}$$

[1]From now on we omit the argument M of $f_i^m (M)$ and $\mu_m (M)$. We continue to denote derivatives with respect to M by a prime.

From the expression (J.76) we can construct directly a general definition for the coefficient β_m, since the summation is already explicit for each power of J. As mentioned before, for γ_m and α_m the procedure is completely analogous. These definitions are the ones given in (8.191) as follows:

$$\beta_m \equiv \sum_{l=0}^{m} (m-l+1)\, f_1^l \mu_{m-l+1} \ , \ \gamma_m \equiv \sum_{l=0}^{m} f_2^l \mu'_{m-l} \ , \ \alpha_m \equiv \sum_{l=0}^{m} f_3^l \mu_{m-l} \tag{J.77}$$

It can be easily checked that each of these formulae really reproduce the expansions where they come from. For example, for the case of β_m, we expand the summation in (J.77) for each m to recover expression (J.76).

J.2.3 Series Expansions for the Functions $f_i\,(M, J)$

Here we calculate the expansions for the functions $f_i\,(J, M)$ in the partial differential equation (8.166). These functions are defined in (8.167) as follows

$$\begin{aligned} f_1\,(M,J) \equiv \left(r_+^{\mathrm{I}}\right)^2 + a^2 \ , \ f_2\,(M,J) \equiv \frac{J}{M} \ , \\ f_3\,(M,J) \equiv \ -\frac{J}{M^2}\left\{\frac{r_+^{\mathrm{I}} + M\left[r_+^{\mathrm{I}} G'\left(r_+^{\mathrm{I}}\right) + G\left(r_+^{\mathrm{I}}\right)\right]}{\left\{r_+^{\mathrm{I}} - M\left[r_+^{\mathrm{I}} G'\left(r_+^{\mathrm{I}}\right) + G\left(r_+^{\mathrm{I}}\right)\right]\right\}}\right\} \end{aligned} \tag{J.78}$$

For $f_2\,(M, J)$ we have simply

$$f_2\,(M,J) \equiv \sum_{l=0}^{\infty} f_2^l\,(M)\, J^l = \frac{J}{M} \tag{J.79}$$

As a result we have

$$f_2^1\,(M) \equiv \frac{1}{M} \ , \ f_2^l\,(M) = 0 \text{ for } l \neq 1 \tag{J.80}$$

For $f_1\,(M, J)$ and $f_3\,(M, J)$ we carry out Taylor series expansions centered in $J = 0$ as follows

$$\begin{aligned} f_1\,(M,J) &= \sum_{k=0}^{\infty} \left.\frac{\partial^k f_1}{\partial J^k}\right|_{J=0} \frac{J^k}{k!} \\ f_3\,(M,J) &= \sum_{k=0}^{\infty} \left.\frac{\partial^k f_3}{\partial J^k}\right|_{J=0} \frac{J^k}{k!} \end{aligned} \tag{J.81}$$

As a result the coefficients $f_{1,3}^l\,(M)$ can be expressed as

$$f_1^l\,(M) \equiv \frac{1}{k!} \left.\frac{\partial^k f_1}{\partial J^k}\right|_{J=0} \ , \ f_3^l\,(M) \equiv \frac{1}{k!} \left.\frac{\partial^k f_3}{\partial J^k}\right|_{J=0}$$

For finding the $O\left(J^2\right)$ approximation to $\mu\left(M,J\right)$ given in (J.70) we need only, as explained in subsection J.2.1, the coefficients $f_i^0\left(M\right)$ and $f_i^1\left(M\right)$ presented in (J.58). We calculate them as follows. $f_1\left(M,J\right)$ is given by

$$f_1\left(M,J\right) \equiv \left(r_+^{\mathrm{I}}\right)^2 + \left(\frac{J}{M}\right)^2 \tag{J.82}$$

Thus we have for $f_1^0\left(M\right)$ the following result:

$$f_1^0\left(M\right) \equiv f_1\left(M,0\right) = \left[r_+^{\mathrm{I}}\left(M,0\right)\right]^2 = \left[r_{\mathrm{Sch}+}^{\mathrm{I}}\left(M\right)\right]^2 \tag{J.83}$$

For $f_1^1\left(M\right)$ we have the derivative given by

$$f_1^1\left(M\right) \equiv \left.\frac{\partial f_1}{\partial J}\right|_{J=0} \tag{J.84}$$

Substituting in (J.84) the definition (J.82) of f_1 leads to

$$f_1^1\left(M\right) \equiv \left.\frac{\partial}{\partial J}\left[\left(r_+^{\mathrm{I}}\right)^2 + \left(\frac{J}{M}\right)^2\right]\right|_{J=0} = \left.\left[2r_+^{\mathrm{I}}\frac{\partial r_+^{\mathrm{I}}}{\partial J} + \frac{2J}{M^2}\right]\right|_{J=0} = \left.2r_+^{\mathrm{I}}\frac{\partial r_+^{\mathrm{I}}}{\partial J}\right|_{J=0} \tag{J.85}$$

Here we can exploit the expression (J.4) for $\partial r_+^{\mathrm{I}}/\partial J$ as follows

$$f_1^1\left(M\right) = \left.2r_+^{\mathrm{I}}\frac{\partial r_+^{\mathrm{I}}}{\partial J}\right|_{J=0} = -\frac{2Jr_+^{\mathrm{I}}}{M^2\left[r_+^{\mathrm{I}} - M\left(G + r_+^{\mathrm{I}}G'\right)\right]} = 0 \tag{J.86}$$

As a result we see that the component $f_1^1\left(M\right)$ is zero. Going further with f_1^2 we have to calculate the following expression

$$f_1^2\left(M\right) \equiv \left.\frac{1}{2}\frac{\partial^2 f_1}{\partial J^2}\right|_{J=0} \tag{J.87}$$

Exploiting the first derivative in (J.85) we find

$$\begin{aligned} f_1^2\left(M\right) &\equiv \left.\frac{1}{2}\frac{\partial}{\partial J}\left[2r_+^{\mathrm{I}}\frac{\partial r_+^{\mathrm{I}}}{\partial J} + \frac{2J}{M^2}\right]\right|_{J=0} \\ &= \left.\left\{\left(\frac{\partial r_+^{\mathrm{I}}}{\partial J}\right)^2 + r_+^{\mathrm{I}}\left(\frac{\partial^2 r_+^{\mathrm{I}}}{\partial J^2}\right)\right\}\right|_{J=0} + \frac{1}{M^2} \end{aligned} \tag{J.88}$$

Here we can substitute the derivatives $\partial r^{\mathrm{I}}_{+}/\partial J$ and $\partial^2 r^{\mathrm{I}}_{+}/\partial J^2$ in (J.4) and (J.6) respectively. The result is given by

$$
\begin{aligned}
f_1^2\left(M\right) = \left\{ \left(-\frac{J}{M^2\left[r^{\mathrm{I}}_{+} - M\left(G + r^{\mathrm{I}}_{+} G'\right)\right]} \right)^2 + \right. \\
\left. + \, r^{\mathrm{I}}_{+} \left(\frac{\left(\frac{\partial r^{\mathrm{I}}_{+}}{\partial J}\right)^2 \left[M\left(\left[2G' + G''\right]\right) - 1\right] - \frac{1}{M^2}}{\left[r^{\mathrm{I}}_{+} - M\left(G + r^{\mathrm{I}}_{+} G'\right)\right]} \right) \right\} \Bigg|_{J=0} + \frac{1}{M^2} \\
= - \left\{ \frac{r^{\mathrm{I}}_{+}}{M^2\left[r^{\mathrm{I}}_{+} - M\left(G + r^{\mathrm{I}}_{+} G'\right)\right]} \right\} \Bigg|_{J=0} + \frac{1}{M^2} \\
= -\frac{1}{M} \left[\frac{G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) + r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)}{r^{\mathrm{I}}_{\mathrm{Sch}_+} - M\left[G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) + r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]} \right]
\end{aligned}
\tag{J.89}
$$

Thus we can present the expansion in powers of J of $f_1\left(M, J\right)$ up to order $O\left(J^2\right)$, as:

$$
f_1\left(M, J\right) = \left[r^{\mathrm{I}}_{\mathrm{Sch}_+}\left(M\right)\right]^2 - \frac{1}{M} \left[\frac{G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) + r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)}{r^{\mathrm{I}}_{\mathrm{Sch}_+} - M\left[G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) + r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]} \right] J^2 + \cdots
\tag{J.90}
$$

For $f_3\left(M, J\right)$ we have

$$
f_3\left(M, J\right) \equiv -\frac{J}{M^2} \left\{ \frac{r^{\mathrm{I}}_{+} + M\left[r^{\mathrm{I}}_{+} G'\left(r^{\mathrm{I}}_{+}\right) + G\left(r^{\mathrm{I}}_{+}\right)\right]}{\left\{r^{\mathrm{I}}_{+} - M\left[r^{\mathrm{I}}_{+} G'\left(r^{\mathrm{I}}_{+}\right) + G\left(r^{\mathrm{I}}_{+}\right)\right]\right\}} \right\}
\tag{J.91}
$$

Thus $f_3\left(M, 0\right) = 0$. The first derivative is given by

$$
\begin{aligned}
f_3^1\left(M\right) &\equiv \frac{\partial f_3}{\partial J}\bigg|_{J=0} = -\frac{1}{M^2} \left\{ \frac{r^{\mathrm{I}}_{+} + M\left[r^{\mathrm{I}}_{+} G'\left(r^{\mathrm{I}}_{+}\right) + G\left(r^{\mathrm{I}}_{+}\right)\right]}{\left\{r^{\mathrm{I}}_{+} - M\left[r^{\mathrm{I}}_{+} G'\left(r^{\mathrm{I}}_{+}\right) + G\left(r^{\mathrm{I}}_{+}\right)\right]\right\}} \right\} \Bigg|_{J=0} \\
&= -\frac{1}{M^2} \left\{ \frac{r^{\mathrm{I}}_{\mathrm{Sch}_+} + M\left[r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) + G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]}{r^{\mathrm{I}}_{\mathrm{Sch}_+} - M\left[r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) + G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]} \right\}
\end{aligned}
\tag{J.92}
$$

As a consequence, up to order $O\left(J\right)$, $f_3\left(M, J\right)$ is given by

$$
f_3\left(M, J\right) = -\frac{J}{M^2} \left\{ \frac{r^{\mathrm{I}}_{\mathrm{Sch}_+} + M\left[r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) + G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]}{r^{\mathrm{I}}_{\mathrm{Sch}_+} - M\left[r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) + G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]} \right\} + \cdots
\tag{J.93}
$$

Expressions (J.80), (J.90) and (J.93) are the final results of this subsection and they correspond to the expressions presented in (J.59), in section 8.7.5.

J.2.4 Improved Schwarzschild Spacetime

The zeroth component μ_0 of the integrating factor μ in (8.183) can be easily found, knowing that it is the integrating factor for the 1-form $\boldsymbol{\gamma}$ in (8.152) when $J = 0$. On the other hand we already know that the first law of thermodynamics for the improved Schwarzschild spacetime is given by [30]

$$\kappa^{\mathrm{I}}_{\mathrm{Sch}} \delta f = \delta M \tag{J.94}$$

Here the surface gravity $\kappa^{\mathrm{I}}_{\mathrm{Sch}}$ of the improved Schwarzschild spacetime is a special case of the κ defined in (8.137), namely

$$\begin{aligned}\kappa^{\mathrm{I}}_{\mathrm{Sch}} \equiv \kappa^{\mathrm{I}}_{\mathrm{Kerr}}\big|_{J=0} &= \left.\frac{r^{\mathrm{I}}_{+} - M\left[r^{\mathrm{I}}_{+} G'\left(r^{\mathrm{I}}_{+}\right) + G\left(r^{\mathrm{I}}_{+}\right)\right]}{\left(r^{\mathrm{I}}_{+}\right)^2}\right|_{J=0} \\ &= \frac{r^{\mathrm{I}}_{\mathrm{Sch}_+} - M\left[r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) + G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]}{\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2}\end{aligned} \tag{J.95}$$

Or using $r^{\mathrm{I}}_{\mathrm{Sch}_+} = 2MG\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)$, the equation for the event horizon, which comes from setting $J = 0$ in $\left(r^{\mathrm{I}}_{+}\right)^2 + (J/M)^2 - 2MG\left(r^{\mathrm{I}}_{+}\right) r^{\mathrm{I}}_{+} = 0$, we can also write

$$\kappa^{\mathrm{I}}_{\mathrm{Sch}} = \left.\frac{M\left[G\left(r^{\mathrm{I}}_{+}\right) - r^{\mathrm{I}}_{+} G'\left(r^{\mathrm{I}}_{+}\right)\right]}{\left(r^{\mathrm{I}}_{+}\right)^2}\right|_{J=0} = \frac{M\left[G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) - r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]}{\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2} \tag{J.96}$$

Substituting (J.96) in (J.94) leads to

$$\delta f \left\{ \frac{M\left[G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) - r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]}{\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2} \right\} = \delta M \tag{J.97}$$

At this point we can evaluate the differential form $\boldsymbol{\gamma}$ given in (8.152) for the case $J = 0$ as follows

$$\boldsymbol{\gamma}\big|_{J=0} = \left(r^{\mathrm{I}}_{+}\right)^2 \delta M \tag{J.98}$$

After comparing (J.97) with (J.98) we conclude that the integrating factor for $\boldsymbol{\gamma}\big|_{J=0}$ is given by

$$\mu_0 = \frac{1}{M\left[G\left(r^{\mathrm{I}}_{+}\right) - r^{\mathrm{I}}_{+} G'\left(r^{\mathrm{I}}_{+}\right)\right]\big|_{J=0}} = \frac{1}{M\left[G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) - r^{\mathrm{I}}_{\mathrm{Sch}_+} G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]} \tag{J.99}$$

It converts $\boldsymbol{\gamma}|_{J=0}$ to the exact differential δf in (J.94).

We also need an expression for the first derivative $\mu_0' \equiv d\mu_0/dM$, in the formula (J.64) for the second order coefficient μ_2. For finding this derivative we proceed as follows:

$$\begin{aligned}
\frac{d\mu_0}{dM} &= \frac{d}{dM}\left\{\frac{1}{M\left[G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) - r^{\mathrm{I}}_{\mathrm{Sch}_+}G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]}\right\} \\
&= -\frac{\left[G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) - r^{\mathrm{I}}_{\mathrm{Sch}_+}G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right] + M\frac{d}{dM}\left[G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) - r^{\mathrm{I}}_{\mathrm{Sch}_+}G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]}{M^2\left[G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) - r^{\mathrm{I}}_{\mathrm{Sch}_+}G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]^2} \\
&= -\frac{G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) - r^{\mathrm{I}}_{\mathrm{Sch}_+}G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) - Mr^{\mathrm{I}}_{\mathrm{Sch}_+}\frac{dr^{\mathrm{I}}_{\mathrm{Sch}_+}}{dM}G''\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)}{M^2\left[G\left(r^{\mathrm{I}}_{\mathrm{sc}_+}\right) - r^{\mathrm{I}}_{\mathrm{Sch}_+}G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]^2}
\end{aligned} \tag{J.100}$$

Here we can exploit again the equation for the event horizon of the improved Schwarzschild $r^{\mathrm{I}}_{\mathrm{Sch}_+} = 2MG\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)$, in order to find $dr^{\mathrm{I}}_{\mathrm{Sch}_+}/dM$

$$\frac{dr^{\mathrm{I}}_{\mathrm{Sch}_+}}{dM} = 2\left\{G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) + M\frac{dr^{\mathrm{I}}_{\mathrm{Sch}_+}}{dM}G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right\}$$

After solving for $dr^{\mathrm{I}}_{\mathrm{Sch}_+}/dM$ we find

$$\frac{dr^{\mathrm{I}}_{\mathrm{Sch}_+}}{dM} = \frac{2G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)}{\left[1 - 2MG'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]} \tag{J.101}$$

J.2.5 Classical Kerr Spacetime

We have mentioned in subsection 8.7.2 that the function[2]

$$\mu = \frac{1}{r_+ - M} \tag{J.102}$$

is an integrating factor for the 1-form $\boldsymbol{\gamma}$ defined in (8.151)

$$\boldsymbol{\gamma} = P\delta M + N\delta J$$

with

$$P \equiv r_+^2 + \left(\frac{J}{M}\right)^2 \ , \ N = -\frac{J}{M} \tag{J.103}$$

[2]For simplicity we set in this section $G_0 = 1$.

This means that the 1-form $\mu\boldsymbol{\gamma}$, given by

$$\mu\boldsymbol{\gamma} = \frac{r_+^2 + \left(\frac{J}{M}\right)^2}{r_+ - M}\delta M - \frac{J}{M\left(r_+ - M\right)}\delta J \tag{J.104}$$

is an exact differential. As a result we can write $\delta S = \mu\boldsymbol{\gamma}$ with $S \equiv S\left(J, M\right)$ a state function as defined in (8.7). We show in this subsection that μ defined in (J.102) actually fulfills the partial differential equation for integrability (8.146) given by

$$P\left(\frac{\partial\mu}{\partial J}\right) - N\left(\frac{\partial\mu}{\partial M}\right) = \mu\left[\left(\frac{\partial N}{\partial M}\right) - \left(\frac{\partial P}{\partial J}\right)\right] \tag{J.105}$$

Starting with the right hand side of (J.105), we have

$$\mu\left[\left(\frac{\partial N}{\partial M}\right) - \left(\frac{\partial P}{\partial J}\right)\right] = \frac{1}{r_+ - M}\left[\frac{J}{M^2} + \frac{2J}{M\left(r_+ - M\right)}\right] \tag{J.106}$$

where we have utilized the derivatives in (8.156) and also the identity

$$r_+ - M = \sqrt{M^2 - \left(\frac{J}{M}\right)^2} \tag{J.107}$$

After simplifying (J.106) we obtain:

$$\mu\left[\left(\frac{\partial N}{\partial M}\right) - \left(\frac{\partial P}{\partial J}\right)\right] = \frac{J\left(r_+ + M\right)}{M^2\left(r_+ - M\right)^2} \tag{J.108}$$

For the left hand side we evaluate first the partial derivatives. For $\partial\mu/\partial J$ we have

$$\frac{\partial\mu}{\partial J} = \frac{\partial}{\partial J}\left(\frac{1}{r_+ - M}\right) = -\frac{1}{\left(r_+ - M\right)^2}\frac{\partial r_+}{\partial J} \tag{J.109}$$

The derivative $\partial r_+/\partial J$ can be calculated from (J.107) to give

$$\frac{\partial r_+}{\partial J} = -\frac{J}{M^2\sqrt{M^2 - \left(\frac{J}{M}\right)^2}} = -\frac{J}{M^2\left(r_+ - M\right)} \tag{J.110}$$

Substituting (J.110) in (J.109) leads to

$$\frac{\partial\mu}{\partial J} = \frac{J}{M^2\left(r_+ - M\right)^3} \tag{J.111}$$

On the other hand for $\partial\mu/\partial M$ we have:

$$\frac{\partial\mu}{\partial M} = \frac{\partial}{\partial M}\left(\frac{1}{r_+ - M}\right) = -\frac{1}{\left(r_+ - M\right)^2}\left(\frac{\partial r_+}{\partial M} - 1\right) \tag{J.112}$$

The derivative $\partial r_+/\partial M$ can also be found from (J.107) as follows

$$\frac{\partial r_+}{\partial M} \quad = \quad \frac{M+\frac{J^2}{M^3}}{\sqrt{M^2-\left(\frac{J}{M}\right)^2}}+1=\frac{M+\frac{J^2}{M^3}}{r_+-M}+1 \tag{J.113}$$

As the result of inserting (J.113) in (J.112) we find

$$\frac{\partial\mu}{\partial M}=-\frac{M+\frac{J^2}{M^3}}{\left(r_+-M\right)^3} \tag{J.114}$$

We can finally evaluate the left hand side, utilizing (J.103), (J.111), and (J.114) as follows

$$P\left(\frac{\partial\mu}{\partial J}\right)-N\left(\frac{\partial\mu}{\partial M}\right)=\left[r_+^2+\left(\frac{J}{M}\right)^2\right]\left(\frac{J}{M^2\left(r_+-M\right)^3}\right)-\left(\frac{J}{M}\right)\frac{M+\frac{J^2}{M^3}}{\left(r_+-M\right)^3}$$

After a straightforward simplification we find:

$$P\left(\frac{\partial\mu}{\partial J}\right)-N\left(\frac{\partial\mu}{\partial M}\right)=\frac{J\left(r_++M\right)}{M^2\left(r_+-M\right)^2} \tag{J.115}$$

The last result for the left hand side of eq. (J.106) is equal to the previous one for its right hand side in (J.108). This means that (J.106) is fulfilled by μ given in (J.102) with the definitions in (J.103), as we wanted to show.

J.3 Exactness in the O$\left(J^2\right)$ Approximation to $\mu\left(M,J\right)$

In this section we demonstrate that the $O\left(J^2\right)$ approximation to the integrating factor $\mu\left(M,J\right)$ in our main differential equation (8.145) is sufficient for satisfying this differential equation to order $O\left(J\right)$, both for the classical and the improved case.

Classical Case

We start with the classical case since it is simpler and we already know the correct integrating factor. For this case we have[3] $G=1$, $G'=0$, $r^{\mathrm{I}}_{\mathrm{Sch}_+}\Big|_{\bar{w}=0}=2M$. As a result (J.70), gets simplified to

$$\mu^{\text{Class}}\left(M,J\right)\Big|_{O(J^2)}=\frac{1}{M}\left(1+\frac{1}{2M^4}J^2\right) \tag{J.116}$$

[3]For simplicity we set $G_0=1$ in this paragraph.

which is precisely the approximation to $O\left(J^2\right)$ of the integrating factor $1/\left(r_+ - M\right)$ for the variations law fulfilled by the Kerr spacetime. In fact we have

$$r_+ = M + \sqrt{M^2 - \left(\frac{J}{M}\right)^2} \tag{J.117}$$

and as a result we can write

$$\frac{1}{r_+ - M} = \frac{1}{\sqrt{M^2 - \frac{J^2}{M^2}}} = \frac{1}{M}\left(1 + \frac{1}{2M^4}J^2 + \frac{3}{8}\frac{J^4}{M^8} + \cdots\right) \tag{J.118}$$

where we have also included the next term in the expansion which is $\mathrm{O}(J^4)$. Nevertheless we stay only with the approximation to $\mathrm{O}(J^2)$ in (J.116) in order to determine the error it introduces. More precisely, we have to check up to which order of J the difference

$$\frac{\partial}{\partial J}\left[\left.\mu^{\text{Class}}\right|_{O(J^2)} \left.P\right|_{O(J^2)}\right] - \frac{\partial}{\partial M}\left[\left.\mu^{\text{Class}}\right|_{O(J^2)} \left.N\right|_{O(J^2)}\right] \tag{J.119}$$

goes to zero.

We start by finding the products μP and μN in (J.119), by applying (J.116) and also the definitions $P \equiv r_+^2 + \left(\frac{J}{M}\right)^2$, $N \equiv -J/M$ for the Kerr spacetime. As a result we have

$$\begin{aligned} \left.\mu^{\text{Class}}\right|_{O(J^2)} \left.P\right|_{O(J^2)} &= \frac{1}{M}\left(1 + \frac{1}{2M^4}J^2\right)\left.\left(r_+^2 + \left(\frac{J}{M}\right)^2\right)\right|_{O(J^2)} \\ \left.\mu^{\text{Class}}\right|_{O(J^2)} \left.N\right|_{O(J^2)} &= -\frac{J}{M^2}\left(1 + \frac{1}{2M^4}J^2\right) \end{aligned} \tag{J.120}$$

Now we evaluate the derivatives as follows. For μN we have:

$$\begin{aligned} \frac{\partial}{\partial M}\left(\left.\mu^{\text{Class}}\right|_{O(J^2)} \left.N\right|_{O(J^2)}\right) &= -\frac{\partial}{\partial M}\left[\frac{J}{M^2} + \frac{J^3}{2M^6}\right] \\ &= \frac{2J}{M^3} + \frac{3J^3}{M^7} \end{aligned} \tag{J.121}$$

For μP we have first to expand $r_+^2 + \left(\frac{J}{M}\right)^2$. In order to do this we use the identity $r_+^2 + \left(\frac{J}{M}\right)^2 = 2Mr_+$, so that we can write:

$$r_+^2 + \left(\frac{J}{M}\right)^2 = 2Mr_+ = 2M\left[M + \sqrt{M^2 - \left(\frac{J}{M}\right)^2}\right]$$

Thus the J-expansion for $r_+^2 + \left(\frac{J}{M}\right)^2$ is given by

$$r_+^2 + \left(\frac{J}{M}\right)^2 = 4M^2 - \frac{J^2}{M^2} - \frac{J^4}{4M^6} + \cdots \tag{J.122}$$

Now we can perform the derivative of $\mu^{\text{Class}}P$, namely

$$\begin{aligned} \frac{\partial}{\partial J}\left[\mu^{\text{Class}}\big|_{O(J^2)}\, P\big|_{O(J^2)}\right] &= \frac{\partial}{\partial J}\left[\frac{1}{M}\left(1+\frac{J^2}{2M^4}\right)\left(4M^2 - \frac{J^2}{M^2}\right)\right] \\ &= \frac{2J}{M^3} - \frac{2J^3}{M^7} \end{aligned} \tag{J.123}$$

It is now clear from (J.121) and (J.123) that the subtraction in (J.119) cancels to order O(J) :

$$\frac{\partial}{\partial J}\left[\mu^{\text{Class}}\big|_{O(J^2)}\, P\big|_{O(J^2)}\right] - \frac{\partial}{\partial M}\left(\mu^{\text{Class}}\big|_{O(J^2)}\, N\big|_{O(J^2)}\right) = -\frac{5J^3}{M^7} \tag{J.124}$$

In this way we have shown that for the classical Kerr spacetime an expansion to $O\left(J^2\right)$ of μ^{Class} suffices to satisfy eq. (J.119) to $O\left(J\right)$.

Improved Case

Now we attempt to show that the approximation for $\mu\left(M,J\right)$ to $O\left(J^2\right)$ given in (J.70), with its components presented in (J.65) and (J.68), is enough to satisfy the exactness condition (8.145) to order $O\left(J\right)$. We have to evaluate again the difference

$$\frac{\partial}{\partial J}\left(\mu\big|_{O(J^2)}\, P\big|_{O(J^2)}\right) - \frac{\partial}{\partial M}\left(\mu\big|_{O(J^2)}\, N\big|_{O(J^2)}\right) \tag{J.125}$$

but now P, N and μ are given by:

$$\begin{aligned} P &\equiv \left(r_+^{\text{I}}\right)^2 + \left(\frac{J}{M}\right)^2,\ N \equiv -\frac{J}{M}\ ,\ \mu\left(M,J\right)\big|_{O(J^2)} = \mu_0 + \mu_2 J^2 \\ \mu_0 &= \frac{1}{M\left(G - r^{\text{I}}_{\text{Sch}_+}G'\right)}\ ,\mu_0' = \frac{\frac{M r_+^2 G G''}{r^{\text{I}}_{\text{Sch}_+} - M\left[r^{\text{I}}_{\text{Sch}_+}G' + G\right]} + r^{\text{I}}_{\text{Sch}_+}G' - G}{M^2\left[G - r^{\text{I}}_{\text{Sch}_+}G'\right]^2} \\ \mu_2 &= \left(\frac{1}{2\left(r^{\text{I}}_{\text{Sch}_+}\right)^2 M^2}\right)\left[\frac{r^{\text{I}}_{\text{Sch}_+} + M\left[r^{\text{I}}_{\text{sc}_+}G' + G\right]}{r^{\text{I}}_{\text{Sch}_+} - M\left[r^{\text{I}}_{\text{Sch}_+}G' + G\right]}\right]\mu_0 - \left(\frac{\mu_0'}{2\left(r^{\text{I}}_{\text{Sch}_+}\right)^2 M}\right) \end{aligned} \tag{J.126}$$

Concerning the difficulty of this demonstration we remark that, in addition to the notorious complexity of expressions in (J.126), we have to face the problem that $G\left(r\right)$ is an arbitrary unspecified function and that the resulting form of $r_+^{\text{I}}\left(M,J\right)$ is not known analytically. As a result we have to proceed in a more abstract way than in the previous demonstration for the bare Kerr black hole. We can summarize the whole procedure as follows.

First, using the chain rule, we expand derivatives in the difference (J.125) so that we can substitute independently every relevant expansion. We also expand $r_+^{\mathrm{I}}(M,J)$ in powers of J and we find explicitly some of its coefficients as functions of M, $r_{\mathrm{Sch}_+}^{\mathrm{I}}$ G and its derivatives. We do this by exploiting the equation (4.51) of the event horizon, namely $\left(r_+^{\mathrm{I}}\right)^2+(J/M)^2-2MG\left(r_+^{\mathrm{I}}\right)r_+^{\mathrm{I}}=0$. Having an expansion for $r_+^{\mathrm{I}}(M,J)$ we can also expand $P\equiv\left(r_+^{\mathrm{I}}\right)^2+(J/M)^2$ and insert it in (J.125) together with N and $\mu(M,J)|_{O(J^2)}$. After that, we compare the coefficients of equal powers of J from the terms $\partial(\mu P)/\partial J$ and $\partial(\mu N)/\partial M$ in eq. (J.125).

We start applying the chain rule to the derivatives in (J.125). For μN we have:

$$\frac{\partial}{\partial M}\left(\mu|_{O(J^2)}N\right)=-\left(\frac{J}{M}\right)\frac{\partial}{\partial M}\mu|_{O(J^2)}+\left(\frac{J}{M^2}\right)\mu|_{O(J^2)} \tag{J.127}$$

Similarly for μP we find:

$$\frac{\partial}{\partial J}\left(\mu|_{O(J^2)}P|_{O(J^2)}\right)=\left[\left(r_+^{\mathrm{I}}\right)^2+\left(\frac{J}{M}\right)^2\right]\frac{\partial\,\mu|_{O(J^2)}}{\partial J}+\mu|_{O(J^2)}\frac{\partial}{\partial J}\left[\left(r_+^{\mathrm{I}}\right)^2+\left(\frac{J}{M}\right)^2\right] \tag{J.128}$$

We substitute the following expressions into (J.127)

$$\begin{aligned}\mu|_{O(J^2)}(M,J) &= \mu_0(M)+\mu_2(M)J^2 \\ \frac{\partial\,\mu|_{O(J^2)}}{\partial M} &= \mu_0'+\mu_2'J^2\end{aligned} \tag{J.129}$$

and we factorize powers of J, as follows:

$$\frac{\partial}{\partial M}\left(\mu|_{O(J^2)}N\right) = \left[\frac{\mu_0}{M^2}-\left(\frac{1}{M}\right)\mu_0'\right]J+\left[\frac{\mu_2}{M^2}-\left(\frac{1}{M}\right)\mu_2'\right]J^3 \tag{J.130}$$

On the other hand we substitute the following expressions into (J.128):

$$\begin{aligned}\mu|_{O(J^2)}(M,J) &= \mu_0+\mu_2J^2 \\ \frac{\partial}{\partial J}\mu|_{O(J^2)} &= 2\mu_2J\end{aligned} \tag{J.131}$$

The result reads :

$$\frac{\partial}{\partial J}\left(\mu|_{O(J^2)}\, P\right) = \left[\left(r^{\rm I}_{+}\right)^2 + \left(\frac{J}{M}\right)^2\right] 2\mu_2 J + \left(\mu_0 + \mu_2 J^2\right)\frac{\partial}{\partial J}\left[\left(r^{\rm I}_{+}\right)^2 + \left(\frac{J}{M}\right)^2\right] \tag{J.132}$$

At this stage we cannot go further if we don't expand $r^{\rm I}_{+}(M,J)$ in powers of J. We make the following ansatz:

$$r^{\rm I}_{+}(M,J) = r^{\rm I}_{{\rm Sch}_+}(M,\bar{w}) + Jc_1(M,\bar{w}) + J^2 c_2(M,\bar{w}) + J^3 c_3(M,\bar{w}) + \cdots \tag{J.133}$$

Here every $c_i(M,\bar{w})$ depends on $G\left(r^{\rm I}_{+}\right)$. The O$(J^0)$ term in (J.133) is precisely $r^{\rm I}_{{\rm Sch}_+}$, the improved Schwarszchild external event horizon. Thus $\left(r^{\rm I}_{+}\right)^2$ can be expanded as (we omit the arguments M and $\bar{w}$):

$$\begin{aligned}\left[r^{\rm I}_{+}(M,J)\right]^2 &= \left(r^{\rm I}_{{\rm Sch}_+}\right)^2 + 2Jr^{\rm I}_{{\rm Sch}_+}c_1 + J^2\left[2r^{\rm I}_{{\rm Sch}_+}c_2 + (c_1)^2\right] + J^3\left(2c_1c_2 + 2r^{\rm I}_{{\rm Sch}_+}c_3\right)\\ &\quad + J^4\left[2c_1c_3 + (c_2)^2\right] + \cdots\end{aligned} \tag{J.134}$$

As a result, we have the following series for P:

$$\begin{aligned}P \equiv \left(r^{\rm I}_{+}\right)^2 + \left(\frac{J}{M}\right)^2 &= \left(r^{\rm I}_{{\rm Sch}_+}\right)^2 + 2Jr^{\rm I}_{{\rm Sch}_+}c_1 + J^2\left[2r^{\rm I}_{{\rm Sch}_+}c_2 + (c_1)^2 + \frac{1}{M^2}\right]\\ &\quad + J^3\left(2c_1c_2 + 2r^{\rm I}_{{\rm Sch}_+}c_3\right) + J^4\left[2c_1c_3 + (c_2)^2\right] + \cdots\end{aligned} \tag{J.135}$$

Or in a more compact form:

$$P = a_0 + a_1 J + a_2 J^2 + a_3 J^3 + a_4 J^4 + \cdots \tag{J.136}$$

with

$$\begin{aligned} a_0 &= \left(r^{\rm I}_{{\rm Sch}_+}\right)^2\ ,\ a_1 = 2r^{\rm I}_{{\rm Sch}_+}c_1\ ,\ a_2 = \left[2r^{\rm I}_{{\rm Sch}_+}c_2 + (c_1)^2 + \frac{1}{M^2}\right]\\ a_3 &= \left(2c_1c_2 + 2r^{\rm I}_{{\rm Sch}_+}c_3\right)\ ,\ a_4 = \left[2c_1c_3 + (c_2)^2\right]\ ,\ \cdots\end{aligned} \tag{J.137}$$

By differentiating (J.135) term by term with respect to J we find

$$\frac{\partial P}{\partial J} = b_0 + b_1 J + b_2 J^2 + b_3 J^3 + \cdots \tag{J.138}$$

with the coefficients

$$\begin{aligned} b_0 &= 2r^{\rm I}_{{\rm Sch}_+}c_1\ ,\ b_1 = 2\left[2r^{\rm I}_{{\rm Sch}_+}c_2 + (c_1)^2 + \frac{1}{M^2}\right]\ ,\ b_2 = 6\left(c_1c_2 + r^{\rm I}_{{\rm Sch}_+}c_3\right)\\ b_3 &= 4\left[2c_1c_3 + (c_2)^2\right]\end{aligned} \tag{J.139}$$

Now we substitute (J.138) and (J.135) in (J.132) to get:

$$\begin{aligned}\frac{\partial}{\partial J}\left(\mu|_{O(J^2)}P\right) &= \left(a_0 + a_1 J + a_2 J^2 + a_3 J^3 + a_4 J^4 + \cdots\right)(2\mu_2 J)\\ &+ \left(\mu_0 + \mu_2 J^2\right)\left(b_0 + b_1 J + b_2 J^2 + b_3 J^3 + \cdots\right)\end{aligned} \tag{J.140}$$

Factorizing powers of J in (J.140) leads to

$$\frac{\partial}{\partial J}\left(\mu|_{O(J^2)}P\right) = d_0 + d_1 J + d_2 J^2 + \cdots \tag{J.141}$$

with the coefficients

$$\begin{aligned} d_0 &= \mu_0 b_0 = 2r^{\mathrm{I}}_{\mathrm{Sch}_+}c_1\mu_0 \\ d_1 &= 2\mu_2 a_0 + \mu_0 b_1 = 2\mu_2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2 + 2\mu_0\left[2r^{\mathrm{I}}_{\mathrm{Sch}_+}c_2 + (c_1)^2 + \frac{1}{M^2}\right] \\ d_2 &= 2a_1\mu_2 + b_0\mu_2 + b_2\mu_0 = 4\mu_2 r^{\mathrm{I}}_{\mathrm{Sch}_+}c_1 + 2r^{\mathrm{I}}_{\mathrm{Sch}_+}c_1\mu_2 + 6\left(c_1c_2 + r^{\mathrm{I}}_{\mathrm{Sch}_+}c_3\right)\mu_0\end{aligned} \tag{J.142}$$

Now we have to find the explicit dependence on J of the coefficients c_i in the series (J.133) for $r^{\mathrm{I}}_+(M,J)$, so that we can substitute them into (J.142). As already mentioned, these coefficients can be found from the general equation for the event horizon (4.51) :

$$\left(r^{\mathrm{I}}_+\right)^2 + \frac{J^2}{M^2} - 2MG\left(r^{\mathrm{I}}_+\right)r^{\mathrm{I}}_+ = 0 \tag{J.143}$$

Expanding $G\left(r^{\mathrm{I}}_+\right)$ in J we have

$$\begin{aligned} G\left(r^{\mathrm{I}}_+\right) &= \overline{G}_0(M,\bar{w}) + \overline{G}_1(M,\bar{w})J + \overline{G}_2(M,\bar{w})J^2 + \cdots \\ &= \sum_{k=0}^{\infty}\overline{G}_k(M,\bar{w})J^k\end{aligned} \tag{J.144}$$

with

$$\overline{G}_k(M,\bar{w}) = \frac{1}{k!}\left.\frac{\partial^k G\left(r^{\mathrm{I}}_+\right)}{\partial J^k}\right|_{J=0} \tag{J.145}$$

We can find the first three coefficients in (J.145) as follows:

$$\begin{aligned} \overline{G}_0(M,\bar{w}) &= G\left(r^{\mathrm{I}}_+\right)\big|_{J=0} = G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right) \\ \overline{G}_1(M,\bar{w}) &= \left.\frac{\partial G\left(r^{\mathrm{I}}_+\right)}{\partial J}\right|_{J=0} = G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\left.\frac{\partial r^{\mathrm{I}}_+}{\partial J}\right|_{J=0} = -\left.\frac{JG'\left(r^{\mathrm{I}}_+\right)}{M^2\left[r^{\mathrm{I}}_+ - M\left(G'r^{\mathrm{I}}_+ + G\right)\right]}\right|_{J=0} = 0 \\ \overline{G}_2(M,\bar{w}) &= \frac{1}{2}\left.\frac{\partial^2 G\left(r^{\mathrm{I}}_+\right)}{\partial J^2}\right|_{J=0} = \frac{1}{2}\left.\left\{G''\left(r^{\mathrm{I}}_+\right)\left(\frac{\partial r^{\mathrm{I}}_+}{\partial J}\right)^2 + G'\left(r^{\mathrm{I}}_+\right)\left(\frac{\partial^2 r^{\mathrm{I}}_+}{\partial J^2}\right)\right\}\right|_{J=0} \\ &= -\frac{G'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)}{2M^2\left[r^{\mathrm{I}}_{\mathrm{Sch}_+} - M\left(G'r^{\mathrm{I}}_{\mathrm{Sch}_+} + G\right)\right]}\end{aligned} \tag{J.146}$$

Here we have used the following expressions (see subsection J.1.1 from this appendix):

$$\frac{\partial r_+^{\mathrm{I}}}{\partial J} = -\frac{JG'\left(r_+^{\mathrm{I}}\right)}{M^2\left[r_+^{\mathrm{I}} - M\left(G' r_+^{\mathrm{I}} + G\right)\right]}\,,\; \frac{\partial^2 G\left(r_+^{\mathrm{I}}\right)}{\partial J^2} = \frac{\left(\frac{\partial r_+^{\mathrm{I}}}{\partial J}\right)^2\left[M\left(G'' r_+^{\mathrm{I}} + 2G\right) - 1\right] - \frac{1}{M^2}}{\left[r_+^{\mathrm{I}} - M\left(G' r_+^{\mathrm{I}} + G\right)\right]} \tag{J.147}$$

These expressions are found by performing recursive derivatives on (J.143). Substituting in (J.143) the expansions (J.133) and (J.144) for r_+^{I} and $G\left(r_+^{\mathrm{I}}\right)$, respectively, leads to:

$$\begin{aligned} &a_0 + a_1 J + a_2 J^2 + a_3 J^3 + a_4 J^4 + \cdots \\ &-2M\left[\overline{G}_0 + \overline{G}_1 J + \overline{G}_2 J^2 + \cdots\right]\left[r_{\mathrm{Sch}_+}^{\mathrm{I}} + Jc_1 + J^2 c_2 + J^3 c_3 + \cdots\right] = 0 \end{aligned} \tag{J.148}$$

Here and in the following we suppress the $(M, \bar{w})$-arguments. Factorizing in (J.148) the powers of J yields:

$$\begin{aligned} &a_0 - 2M\overline{G}_0 r_{\mathrm{Sch}_+}^{\mathrm{I}} + \left\{a_1 - 2M\left[\overline{G}_1 r_{\mathrm{Sch}_+}^{\mathrm{I}} + c_1\overline{G}_0\right]\right\} J \\ &+\left\{a_2 - 2M\left[\overline{G}_2 r_{\mathrm{Sch}_+}^{\mathrm{I}} + \overline{G}_0 c_2 + \overline{G}_1 c_1\right]\right\} J^2 + \cdots = 0 \end{aligned} \tag{J.149}$$

As a result, the coefficients of every power should be independently equal to zero:

$$a_0 - 2M\overline{G}_0 r_{\mathrm{Sch}_+}^{\mathrm{I}} = 0 \tag{J.150}$$

$$a_1 - 2M\left[\overline{G}_1 r_{\mathrm{Sch}_+}^{\mathrm{I}} + c_1\overline{G}_0\right] = 0 \tag{J.151}$$

$$a_2 - 2M\left[\overline{G}_2 r_{\mathrm{Sch}_+}^{\mathrm{I}} + \overline{G}_0 c_2 + \overline{G}_1 c_1\right] = 0 \tag{J.152}$$

Substituting the expressions for $\overline{G}_i$ and a_i in (J.150) and factorizing gives:

$$r_{\mathrm{Sch}_+}^{\mathrm{I}}\left[r_{\mathrm{Sch}_+}^{\mathrm{I}} - 2MG\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)\right] = 0 \tag{J.153}$$

The term in parentheses is precisely the condition for the event horizon of the improved Schwarzschild spacetime. It is fulfilled by $r_{\mathrm{sc}_+}^{\mathrm{I}}$ by definition.

Performing similar substitutions in (J.151) leads to

$$2c_1\left[r_{\mathrm{Sch}_+}^{\mathrm{I}} - MG\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)\right] = 0 \tag{J.154}$$

Taking advantage of the event horizon equation $r^{\rm I}_{\rm Sch_+} = 2MG\left(r^{\rm I}_{\rm Sch_+}\right)$ we have

$$2c_1 MG\left(r^{\rm I}_{\rm Sch_+}\right) = 0 \tag{J.155}$$

This means that c_1 must be identically zero. We exploit this when simplifying (J.152) after substituting the $\overline{G}_i$'s and a_i's, namely

$$\left[2r^{\rm I}_{\rm Sch_+}c_2 + \frac{1}{M^2}\right] - 2M\left[G\left(r^{\rm I}_{\rm Sch_+}\right)c_2 - \frac{G'\left(r^{\rm I}_{\rm Sch_+}\right)r^{\rm I}_{\rm Sch_+}}{2M^2\left[r^{\rm I}_{\rm Sch_+} - M\left(G' r^{\rm I}_{\rm Sch_+} + G\right)\right]}\right] = 0$$

Solving for c_2 leads to

$$c_2 = -\frac{1}{2r^{\rm I}_{\rm Sch_+} - 2MG\left(r^{\rm I}_{\rm Sch_+}\right)}\left[\frac{1}{M^2} + \frac{G'\left(r^{\rm I}_{\rm Sch_+}\right)r^{\rm I}_{\rm Sch_+}}{M\left[r^{\rm I}_{\rm Sch_+} - M\left(G' r^{\rm I}_{\rm Sch_+} + G\right)\right]}\right] \tag{J.156}$$

At this point we use again the identity $r^{\rm I}_{\rm Sch_+} = 2MG\left(r^{\rm I}_{\rm Sch_+}\right)$ in order to simplify (J.156):

$$c_2 = -\frac{1}{2M^3\left[G\left(r^{\rm I}_{\rm Sch_+}\right) - G'\left(r^{\rm I}_{\rm Sch_+}\right)r^{\rm I}_{\rm Sch_+}\right]} \tag{J.157}$$

After comparing (J.157) with the expression of μ_0 in (J.126), we find

$$c_2 = -\frac{\mu_0}{2M^2} \tag{J.158}$$

Knowing the components c_1 and c_2 of $r^{\rm I}_+(M, J)$ we can come back to the first two coefficients (J.142) in the expansion (J.141) of the derivative of μP in J. They are[4]

$$d_0 = 2r^{\rm I}_{\rm Sch_+}c_1\mu_0 = 0 \tag{J.159}$$

$$d_1 = 2\mu_2\left(r^{\rm I}_{\rm Sch_+}\right)^2 + 2\mu_0\left[2r^{\rm I}_{\rm Sch_+}c_2 + (c_1)^2 + \frac{1}{M^2}\right] = 2\mu_2\left(r^{\rm I}_{\rm Sch_+}\right)^2 + \frac{2\mu_0}{M^2} - \frac{2r^{\rm I}_{\rm Sch_+}\mu_0^2}{M^2} \tag{J.160}$$

[4]Of course, further d's require the knowledge of more c's. We stop at order $O(1)$ which is our main goal.

As a result we have the following expression to $O(J)$ for (J.141)

$$\frac{\partial}{\partial J}\left(\mu|_{O(J^2)}\,P\right)=\left[2\mu_2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2+\frac{2\mu_0}{M^2}-\frac{2r^{\mathrm{I}}_{\mathrm{Sch}_+}\mu_0^2}{M^2}\right]J+O\left(J^2\right) \tag{J.161}$$

If our calculations are correct, the $O(J)$ component in (J.161) should be equal to the respective $O(J)$ component in (J.130). In order to show this we have to further simplify (J.161) by exploiting the expressions (J.126) and (J.153) for μ_2 and $r^{\mathrm{I}}_{\mathrm{Sch}_+}$, respectively

$$\frac{\partial}{\partial J}\left(\mu|_{O(J^2)}\,P\right)\bigg|_{O(J)}=\left(\frac{\mu_0}{M^2}\right)\left[\frac{3+2MG'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)}{1-2MG'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)}\right]-\frac{4G\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\mu_0^2}{M}+\frac{2\mu_0}{M^2}-\frac{\mu_0'}{M} \tag{J.162}$$

At this point we can apply to the first term in (J.162) the following identity for μ_0:

$$\mu_0=\frac{1}{MG\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\left[1-2MG'\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)\right]} \tag{J.163}$$

This identity can be found by substituting $r^{\mathrm{I}}_{\mathrm{Sch}_+}=2MG\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)$ into the μ_0 of (J.126). The result, after some trivial algebraic steps, is the following:

$$\frac{\partial}{\partial J}\left(\mu|_{O(J^2)}\,P\right)\bigg|_{O(J)}=\frac{\mu_0}{M^2}-\frac{\mu_0'}{M} \tag{J.164}$$

This is precisely the $O(J)$ component in (J.130). This completes the proof that *with the $O\left(J^2\right)$ approximation to μ the exactness condition is satisfied to order $O\left(J\right)$.*

J.4 Temperature and Entropy

In this section we calculate the $O\left(J^2\right)$ approximations to the first law and the associated temperature $T\left(M,J\right)$. In order to perform these calculations we exploit the $O\left(J^2\right)$ approximation for the integrating factor μ_γ given in (8.193). We also check the consistency of the result obtained with the general all order formalism with what was found using the direct derivation of the J^2 coefficients S_2 and T_2 presented in subsection 8.7.7.

J.4.1 Expansion of the First Law

In subsection 8.7.7 we have presented the $O\left(J^2\right)$ approximation to the first law, as follows:

$$\delta S|_{O(J^2)} = \left.\left(\frac{\mu_\alpha}{\kappa}\right)\right|_{O(J^2)} \delta M - \left.\left(\frac{\Omega_{\mathrm{H}}\mu_\alpha}{\kappa}\right)\right|_{O(J^2)} \delta J \tag{J.165}$$

We have also defined $\bar{P}\big|_{O(J^2)}$ and $\bar{N}\big|_{O(J^2)}$ to be the coefficients of δM and δJ in (J.165), respectively

$$\bar{P} = \frac{\mu_\alpha}{\kappa}\ ,\ \bar{N} = -\frac{\Omega_{\mathrm{H}}\mu_\alpha}{\kappa} \tag{J.166}$$

Substituting κ and Ω_{H} from (8.137) and (8.138), and $\mu_\alpha = h^{-1}\mu_\gamma$, in (J.166) leads to

$$\bar{P}\big|_{O(J^2)} = \mu_\gamma \left.\left[\left(r^{\mathrm{I}}_{+}\right)^2 + \left(\frac{J}{M}\right)^2\right]\right|_{O(J^2)} \tag{J.167}$$

$$\bar{N}\big|_{O(J^2)} = -\left.\left(\frac{J\mu_\gamma}{M}\right)\right|_{O(J^2)} \tag{J.168}$$

Calculating $\bar{N}\big|_{O(J^2)}$ is straightforward:

$$\bar{N}\big|_{O(J^2)} = -\left.\left[\frac{J\left(\mu_0 + \mu_2 J^2\right)}{M}\right]\right|_{O(J^2)} = -\frac{J\mu_0}{M} \tag{J.169}$$

whereas for $\bar{P}\big|_{O(J^2)}$ we apply the expansion (J.135) for P we have found in the previous section. It is given by

$$\begin{aligned} P \ \equiv\ & \left(r^{\mathrm{I}}_{+}\right)^2 + \left(\frac{J}{M}\right)^2 = \left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2 + 2Jr^{\mathrm{I}}_{\mathrm{Sch}_+}c_1 + J^2\left[2r^{\mathrm{I}}_{\mathrm{Sch}_+}c_2 + \left(c_1\right)^2 + \frac{1}{M^2}\right] \\ & + J^3\left(2c_1c_2 + 2r^{\mathrm{I}}_{\mathrm{Sch}_+}c_3\right) + J^4\left[2c_1c_3 + \left(c_2\right)^2\right] + \cdots \end{aligned} \tag{J.170}$$

Substituting $c_1 = 0$ and $c_2 = -\mu_0/\left(2M^2\right)$ from (J.155) and (J.157) into (J.170) gives the following expansion for P:

$$P \equiv \left(r^{\mathrm{I}}_{+}\right)^2 + \left(\frac{J}{M}\right)^2 = \left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2 + \left(\frac{J^2}{M^2}\right)\left[1 - r^{\mathrm{I}}_{\mathrm{Sch}_+}\mu_0\right] + O\left(J^4\right) \tag{J.171}$$

Now substituting (J.171) and $\mu_\gamma = \left(\mu_0 + \mu_2 J^2\right)$ in (J.167) yields

$$\begin{aligned} \mu_\gamma \left.\left[\left(r^{\mathrm{I}}_{+}\right)^2 + \left(\frac{J}{M}\right)^2\right]\right|_{O(J^2)} &= \left(\mu_0 + \mu_2 J^2\right) \left.\left[\left(r^{\mathrm{I}}_{+}\right)^2 + \left(\frac{J}{M}\right)^2\right]\right|_{O(J^2)} \\ &= \mu_0\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2 + J^2\left\{\mu_2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2 + \mu_0\left(\frac{1 - r^{\mathrm{I}}_{\mathrm{Sch}_+}\mu_0}{M^2}\right)\right\} \end{aligned} \tag{J.172}$$

Now we insert expression (J.69) for μ_2 into (J.172). Thus we find

$$\mu_\gamma \left[\left(r_+^{\mathrm{I}}\right)^2 + \left(\frac{J}{M}\right)^2 \right] \Bigg|_{O(J^2)} =$$
$$\mu_0 \left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)^2 + J^2 \left\{ \frac{\left[3G + r_{\mathrm{sc}_+}^{\mathrm{I}} G'\right] (\mu_0)^2 - \mu_0'}{2M} + \mu_0 \left(\frac{1 - r_{\mathrm{Sch}_+}^{\mathrm{I}} \mu_0}{M^2} \right) \right\} \tag{J.173}$$

Exploiting $r_{\mathrm{Sch}_+}^{\mathrm{I}} = 2MG\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)$ leads to

$$\mu_\gamma \left[\left(r_+^{\mathrm{I}}\right)^2 + \left(\frac{J}{M}\right)^2 \right] \Bigg|_{O(J^2)} = \mu_0 \left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)^2 + J^2 \left\{ \left[\frac{2MG' - 1}{2M}\right] G (\mu_0)^2 - \frac{\mu_0'}{2M} + \frac{\mu_0}{M^2} \right\} \tag{J.174}$$

Applying $G(1 - 2MG') = \frac{1}{M\mu_0}$ in the left hand side of (J.174) gives

$$\bar{P}\big|_{O(J^2)} = \mu_\gamma \left[\left(r_+^{\mathrm{I}}\right)^2 + \left(\frac{J}{M}\right)^2 \right] \Bigg|_{O(J^2)} = \mu_0 \left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)^2 + \frac{J^2}{2M} \left[\frac{\mu_0}{M} - \mu_0'\right] \tag{J.175}$$

Substituting (J.169) and (J.175) for $\bar{N}_{O(J^2)}$ and $\bar{P}_{O(J^2)}$ respectively, in (J.165) leads to

$$\begin{aligned} \delta\left(\frac{S}{2\pi}\right)\Bigg|_{O(J^2)} &= \bar{P}\big|_{O(J^2)} \delta M + \bar{N}\big|_{O(J^2)} \delta J \\ &= \left\{ \mu_0 \left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)^2 + \frac{J^2}{2M} \left[\frac{\mu_0}{M} - \mu_0'\right] \right\} \delta M - \frac{J\mu_0}{M} \delta J \end{aligned}$$

This is the final result for the $O(J^2)$ approximation to the first law presented in subsection 8.7.7.

J.4.2 Expansion of the Temperature

In subsection 8.7.7 we have found the following expression for $T(M, J)$:

$$T(M, J) = \frac{1}{2\pi\mu_\gamma \left[\left(r_+^{\mathrm{I}}\right)^2 + \left(\frac{J}{M}\right)^2 \right]} \tag{J.176}$$

From (J.176) we can calculate an $O(J^2)$ approximation to $T(M, J)$ by expanding the inverse of $\mu_\gamma P$ in a Taylor series about $J = 0$, as follows. The Taylor series about $J = 0$ of a generic function $F(J, M)$ is given by

$$F(J, M) = F_0(M) + F_1(M) J + F_2(M) J^2 + \cdots$$

with

$$F_k(M) = \frac{1}{k!}\left.\frac{\partial^k F}{\partial J^k}\right|_{J=0}$$

For the inverse of $F(J,M)$ we have

$$F(J,M)^{-1} \equiv \frac{1}{F(J,M)} = F_0^{(-1)}(M) + F_1^{(-1)}(M)\,J + F_2^{(-1)}(M)\,J^2 + \cdots \quad \text{(J.177)}$$

where the coefficients up to $O(J^2)$ are given by

$$\begin{aligned} F_0^{(-1)}(M) &= \frac{1}{F(0,M)} = \frac{1}{F_0}\ ,\ F_1^{(-1)}(M) = -\frac{1}{F^2(0,M)}\left.\frac{\partial F}{\partial J}\right|_{J=0} = -\frac{F_1}{(F_0)^2} \\ F_2^{(-1)}(M) &= \frac{1}{2F^2(0,M)}\left.\left[\frac{2}{F}\left(\frac{\partial F}{\partial J}\right)^2 - \frac{\partial^2 F}{\partial J^2}\right]\right|_{J=0} = \frac{1}{(F_0)^2}\left[\frac{(F_1)^2}{F_0} - F_2\right] \end{aligned} \quad \text{(J.178)}$$

As a result we have for the inverse of $\mu_\gamma(J,M)$ up to $O(J^2)$

$$\left.\frac{1}{\mu_\gamma(J,M)}\right|_{O(J^2)} = \frac{1}{\mu_0} - \frac{\mu_2 J^2}{(\mu_0)^2} \quad \text{(J.179)}$$

where μ_0 and μ_2 are the components of $\mu_\gamma(J,M)|_{O(J^2)}$. A similar expansion can be obtained for $1/P$. We start with the definition of $P(J,M)|_{O(J^2)}$ in (J.175) given by

$$\begin{aligned} P(J,M)|_{O(J^2)} &\equiv \left.\left(r_+^{\mathrm{I}}\right)^2 + \left(\frac{J}{M}\right)^2\right|_{O(J^2)} = P_0 + P_2 J^2 \\ &= \mu_0\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)^2 + \frac{J^2}{2M}\left[\frac{\mu_0}{M} - \mu_0'\right] \end{aligned} \quad \text{(J.180)}$$

Thus we can identify the following coefficients:

$$P_0 = \left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)^2\ ,\ P_1 = 0\ ,\ P_2 = \left(\frac{1}{M^2}\right)\left[1 - \mu_0 r_{\mathrm{Sch}_+}^{\mathrm{I}}\right] \quad \text{(J.181)}$$

We can apply the coefficients in (J.181) in order to find $1/P(J,M)$ up to $O(J^2)$. Substituting (J.181) in (J.178) leads to

$$\begin{aligned} P_0^{(-1)}(M) &= \frac{1}{P_0} = \frac{1}{\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)^2}\ ,\ P_1^{(-1)}(M) = -\frac{P_1}{(P_0)^2} = 0 \\ P_2^{(-1)}(M) &= \frac{1}{(P_0)^2}\left[\frac{(P_1)^2}{P_0} - P_2\right] = -\frac{P_2}{(P_0)^2} = \frac{\mu_0 r_{\mathrm{Sch}_+}^{\mathrm{I}} - 1}{M^2\left(r_{\mathrm{Sch}_+}^{\mathrm{I}}\right)^4} \end{aligned} \quad \text{(J.182)}$$

As a result, we have the following expression for $1/P\left(J,M\right)\big|_{O(J^2)}$:

$$\left.\frac{1}{P\left(J,M\right)}\right|_{O(J^2)} = \frac{1}{\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2} + \frac{J^2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\mu_0 - 1\right)}{M^2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^4} + \cdots \tag{J.183}$$

Now we can substitute expressions (J.179) and (J.183), for $1/\mu_\gamma\left(J,M\right)\big|_{O(J^2)}$ and $1/P\left(J,M\right)\big|_{O(J^2)}$, respectively, into the equation (J.176) for the temperature. This leads to

$$\begin{aligned}
T\left(M,J\right) &= \frac{1}{2\pi\mu_\gamma\left[\left(r^{\mathrm{I}}_{+}\right)^2 + \left(\frac{J}{M}\right)^2\right]} = \frac{1}{2\pi}\left[\frac{1}{\mu_0} - \frac{\mu_2 J^2}{\left(\mu_0\right)^2}\right]\left[\frac{1}{\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2} + \frac{J^2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\mu_0 - 1\right)}{M^2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^4}\right] \\
&= \frac{1}{2\pi\mu_0\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2} + \frac{J^2}{2\pi\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2\mu_0}\left[\frac{\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\mu_0 - 1\right)}{M^2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2} - \frac{\mu_2}{\mu_0}\right] + O\left(J^4\right) \qquad \text{(J.184)}
\end{aligned}$$

Thus we identify the coefficients T_0 and T_2 in (J.184) to be

$$T_0 = \frac{1}{2\pi\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2\mu_0} \tag{J.185}$$

$$T_2 = \frac{1}{2\pi\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2\mu_0}\left[\frac{\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\mu_0 - 1\right)}{M^2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2} - \frac{\mu_2}{\mu_0}\right] \tag{J.186}$$

T_2 can be further simplified by substituting μ_2 in (J.186) as follows:

$$\begin{aligned}
T_2 &= -\frac{1}{2\pi\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2\mu_0 M}\left[\frac{\left[3G + r^{\mathrm{I}}_{\mathrm{Sch}_+}G'\right]\mu_0 - \frac{\mu_0'}{\mu_0}}{2\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2} + \frac{\left(1 - r^{\mathrm{I}}_{\mathrm{Sch}_+}\mu_0\right)}{M\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2}\right] \qquad \text{(J.187)} \\
&= \frac{1}{4\pi M\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^4\left(\mu_0\right)^2}\left[\mu_0' - \left[3G + r^{\mathrm{I}}_{\mathrm{Sch}_+}G'\right]\left(\mu_0\right)^2 - \frac{2\left(1 - r^{\mathrm{I}}_{\mathrm{Sch}_+}\mu_0\right)\mu_0}{M}\right]
\end{aligned}$$

Substituting $r^{\mathrm{I}}_{\mathrm{Sch}_+} = 2MG$ and simplifying gives

$$T_2 = \frac{1}{4\pi M\left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^4\left(\mu_0\right)^2}\left[\mu_0' + \left(\mu_0\right)^2 G\left(1 - 2MG'\right) - \frac{2\mu_0}{M}\right] \tag{J.188}$$

We now substitute in (J.188) the identity

$$G\left(1 - 2MG'\right) = \frac{1}{M\mu_0}$$

which comes from using $r^{\mathrm{I}}_{\mathrm{Sch}_+} = 2MG$ in the expression for μ_0 given in (J.126). As a result we find

$$T_2 = \frac{\left[\mu_0' - \frac{\mu_0}{M}\right]}{4\pi M \left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^4 (\mu_0)^2} \tag{J.189}$$

After substituting expression (J.189) in (J.184) we find the final result for $T\left(M, J\right)$:

$$T\left(M, J\right) = \frac{1}{2\pi\mu_0 \left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2} + \frac{J^2\left[\mu_0' - \frac{\mu_0}{M}\right]}{4\pi M \left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^4 (\mu_0)^2} + O\left(J^4\right) \tag{J.190}$$

This is precisely the expression presented in equation (8.205) of the main text.

J.4.3 Direct Calculation of the Coefficients S_2 and T_2: a Consistency Check

In subsection 8.7.7 we also found from the simple "J^2-method" the leading coefficients T_2 and S_2. We check now that they lead to the result (J.189) for T_2, which we obtained using the general all-order method. We verify in this way the consistency of the two calculations of T_2.

The expressions from the simplified J^2-method are

$$S_2 \quad = \quad -\frac{1}{2T_0 M \left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2} \tag{J.191}$$

$$T_2 \quad = \quad -\left(T_0\right)^2 \left(\frac{dS_2}{dM}\right) \tag{J.192}$$

Substituting (J.191) in (J.192) gives the following:

$$T_2 \quad = \quad \left[\frac{-\left(T_0\right)^2}{-2}\right] \frac{d}{dM}\left(\frac{1}{T_0 M \left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2}\right) \tag{J.193}$$

We have already an expression of T_0 given in (J.185). It is the temperature of the Schwarzschild black hole:

$$T_0 = \frac{1}{2\pi \left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^2 \mu_0} \tag{J.194}$$

Substituting (J.194) in (J.193) leads to

$$
\begin{aligned}
T_2 &= \frac{1}{4\pi \left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^4 (\mu_0)^2} \left[\frac{M\mu_0' - \mu_0}{M^2}\right] && \text{(J.195)} \\
&= \frac{\left[\mu_0' - \frac{\mu_0}{M}\right]}{4\pi \left(r^{\mathrm{I}}_{\mathrm{Sch}_+}\right)^4 (\mu_0)^2 M} && \text{(J.196)}
\end{aligned}
$$

By comparing (J.196), coming from equations (8.212) and (8.213) for T_2 and S_2, with (J.189) calculated using μ_γ, we conclude that they are indeed equal.

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